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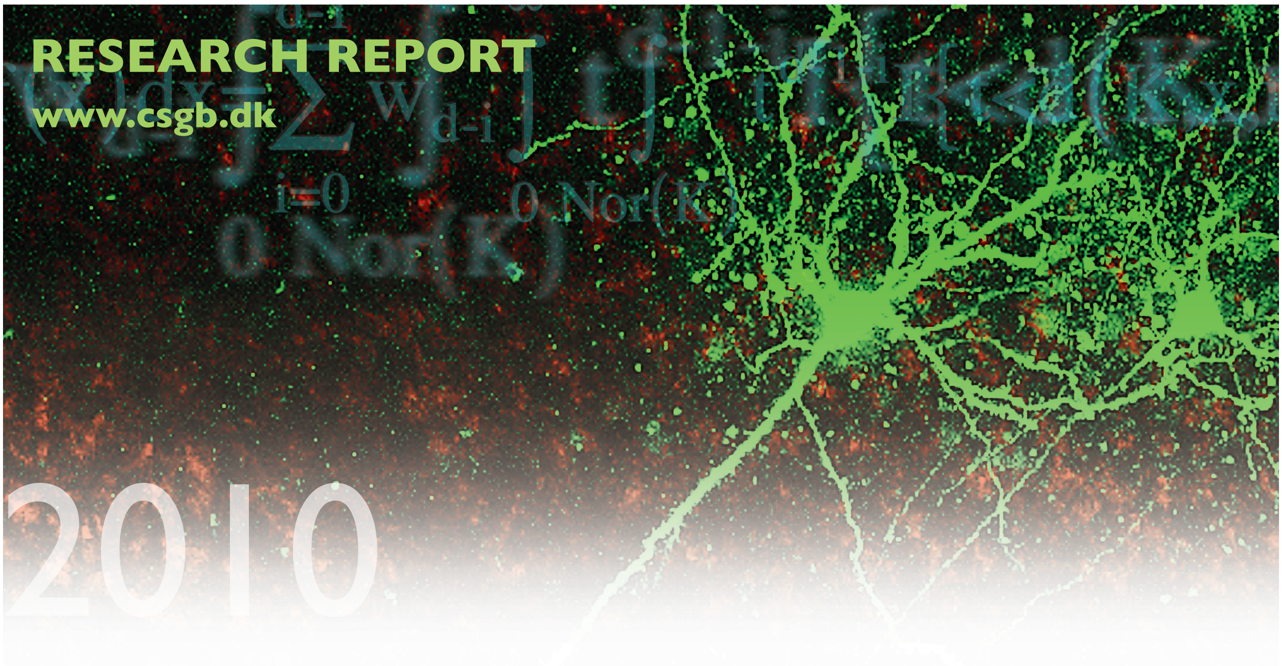

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Summary

The closed form of a rotational version of the famous Crofton formula is derived. In the case where the sectioned object is a compact d-dimensional C^2 manifold with boundary, the rotational average of intrinsic volumes measured on sections passing through a fixed point can be expressed as an integral over the boundary involving hypergeometric functions. In the more general case of a compact subset of $\mathbb{R}^d$ with positive reach, the rotational average also involves hypergeometric functions. For convex bodies, we show that the rotational average can be expressed as an integral with respect to a natural measure on supporting flats. It is an open question whether the rotational average of intrinsic volumes studied in the present paper can be expressed as a limit of polynomial rotation invariant valuations.

Keywords. Geometric measure theory, hypergeometric functions, integral geometry, intrinsic volume, stereology.

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1 Introduction

Local stereology is a collection of sampling designs based on sections through a reference point of the structure under study, cf. [14]. The majority of the local stereological methods have been derived in the eighties and the nineties, including methods of estimating number, length, surface area and volume. These methods have found numerous applications, in particular in the microscopic analysis of tissue samples, cf. [8, 13, 15, 16, 20, 26, 28] and references therein. As pointed out in [11], local stereology is closely related to geometric tomography, especially to central concepts of the dual Brunn-Minkowski theory, see also [10]. Up-to-date monographs on stereology are Baddeley and Jensen [6] and Beneš and Rataj [7].

Rotational integral formulae are the fundamental tool of local stereology. A theory of rotational integral geometry, dual to the theory of translative integral geometry [25], has evolved, including rotational integral formulae for number, length, surface area and volume [14]. A basic tool in these developments has been the generalized Blaschke-Petkantschin formula, see [17] and [31]. Only very recently, rotational integral formulae have been derived for intrinsic volumes in general, cf. [4, 12, 18]. These new formulae open up the possibility for developing local stereological methods of estimating curvature (for instance, integral of mean curvature).

One of these formulae shows how rotational averages of intrinsic volumes measured on sections are related to the geometry of the sectioned object $X \subset \mathbb{R}^d$. The rotational average considered is of the following form

$$\int_{\mathcal{L}_j^d} V_k(X \cap L_j)\, \mathrm{d}L_j^d, \tag{1}$$

$0 \leq k \leq j \leq d$, where $\mathcal{L}_j^d$ is the set of $j-$dimensional *linear* subspaces in $\mathbb{R}^d$, V_k is the kth intrinsic volume and $\mathrm{d}L_j^d$ is the element of the rotation invariant measure on $\mathcal{L}_j^d$ with total measure

$$\int_{\mathcal{L}_j^d} \mathrm{d}L_j^d = c_{d,j}.$$

Here,

$$c_{d,j} = \frac{\sigma_d \sigma_{d-1} \cdots \sigma_{d-j+1}}{\sigma_j \sigma_{j-1} \cdots \sigma_1},$$

where $\sigma_k = 2\pi^{\frac{k}{2}}/\Gamma(\frac{k}{2})$ is the surface area of the unit sphere in $\mathbb{R}^k$.

The rotational average (1) is an example of a rotational invariant valuation. Such valuations have been studied in recent years by Alesker [2] among others. For $k = j$, (1) is the jth dual elementary mixed volume, cf. e.g. [19], and we have

$$\int_{\mathcal{L}_j^d} V_j(X \cap L_j)\, \mathrm{d}L_j^d = c_{d-1,j-1} \int_X |x|^{-(d-j)}\, \mathrm{d}x^d,$$

where $\mathrm{d}x^d$ is the element of the d-dimensional Lebesgue measure, cf. e.g. [18, (9)].

The situation is more complicated for $k < j$. Assume (for simplicity) that $X \subset \mathbb{R}^d$ is a compact $d-$dimensional C^2 manifold with boundary. For a boundary point $x \in \partial X$, let $n(x)$ be the unit outer normal vector to X at x, let $\kappa_i(x)$, $i = 1, \ldots, d-1$, be the principal curvatures at $x \in \partial X$ and $a_i(x)$, $i = 1, \ldots, d-1$, the corresponding principal directions. In [18], it was shown under mild regularity conditions ($O \notin \partial X$ and for almost all $L_j \in \mathcal{L}_j^d$, there is no $x \in \partial X \cap L_j$ with $n(x) \perp L_j$) that the rotational average (1) is of the following form

$$\int_{\mathcal{L}_j^d} V_k(X \cap L_j)\, \mathrm{d}L_j^d = \int_{\partial X} \sum_{\substack{I \subseteq \{1,\ldots,d-1\} \\ |I| = j-1-k}} w_{I,j,k}(x) \prod_{i \in I} \kappa_i(x)\, \mathcal{H}^{d-1}(\mathrm{d}x), \tag{2}$$

provided the integral exists. In (2), $\mathcal{H}^k$ denotes the $k-$dimensional Hausdorff measure. The weight functions $w_{I,j,k}$ are non-negative functions defined on ∂X. The function $w_{I,j,k}(x)$ depends on the linear subspace spanned by the principal directions $a_i(x)$, $i \in I$.

If X is a ball, the function $w_{I,j,k}$ is constant and the rotational average is therefore proportional to the $(d-j+k)$th intrinsic volume of X which has the following integral representation

$$V_{d-j+k}(X) = \frac{1}{\sigma_{j-k}} \int_{\partial X} \sum_{|I|=j-1-k} \prod_{i \in I} \kappa_i(x)\, \mathcal{H}^{d-1}(\mathrm{d}x),$$

cf. [24, Section 13.6] or [27, Section V.3].

In the present paper, we derive a simple closed form expression of $w_{I,j,k}$, involving hypergeometric functions. We show that $w_{I,j,k}(x)$ depends on the norm of x and of two angles: the angle $\beta(x)$ formed by x and $n(x)$, and the angle $\alpha_I(x)$ formed by x and $\operatorname{span}\{a_i(x) : i \notin I\}$. This expression allows us to understand the geometric structure of the rotational average and derive a simplified form of the integrand at the right-hand side of (2) at locally spherical boundary points. Furthermore, it will be shown that for convex bodies the rotational average can be expressed as an integral with respect to a natural measure on supporting $(j-1-k)-$dimensional flats. This result gives new insight concerning the question of characterizing rotation invariant valuations [2, 3].

The paper is organized as follows. In Section 2, we provide background knowledge on hypergeometric functions and angles of subspaces. In Section 3, the closed form expression of $w_{I,j,k}$ is presented. The proof of the result is deferred to the Section 7. In Section 4, further simplifications are derived for locally spherical boundary points. In Section 5, a reformulation of (2) is derived in terms of an integral with respect to a natural measure on supporting flats. In Section 6, we discuss the possibilities for expressing the rotation average as a limit of polynomial rotation invariant valuations.

2 Preliminaries

2.1 Hypergeometric functions

A hypergeometric function can be represented by a series of the following form

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!}.$$

When $a=0$ or $b=0$, the hypergeometric function is identically equal to 1. The series converges absolutely for $|z|<1$. In case $0<b<c$, we can also represent the hypergeometric series by an integral

$$F(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 (1-zy)^{-a} y^{b-1} (1-y)^{c-b-1} dy. \tag{3}$$

Here $B(s,t) = \Gamma(s)\Gamma(t)/\Gamma(s+t)$ is the Beta function. When $z=1$, the extra assumption $c-a-b>0$ is necessary. Transformation formulae for hypergeometric functions are often useful. In particular, we shall use the following formulae, cf. [1, (15.2.17), (15.2.20) and (15.2.24)],

$$(c-a-1)F(a,b;c;z) + aF(a+1,b;c;z) = (c-1)F(a,b;c-1;z), \tag{4}$$

$$c(1-z)F(a,b;c;z) + (c-b)zF(a,b;c+1;z) = cF(a-1,b;c;z), \tag{5}$$

$$bF(a,b+1;c;z) = (c-1)F(a,b;c-1;z) - (c-b-1)F(a,b;c;z). \tag{6}$$

We shall also use the following integral representation which can be obtained from (3), using the substitution $y = r^2/(1+r^2)$

$$\int_0^\infty \left(1-z+\frac{z}{1+r^2}\right)^{-a} (r^2)^{b-\frac{1}{2}} (1+r^2)^{-c} \, \mathrm{d}r = \tfrac{1}{2} B(b,c-b) F(a,b;c;z).,\tag{7}$$

This integral representation is valid under the same assumptions as in (3).

2.2 Angle of subspaces

For $x_1, \dots, x_p \in \mathbb{R}^d$, $p \le d$, we let $P(x_1, \dots, x_p)$ be the parallelotope spanned by $x_1, \dots, x_p$,

$$P(x_1, \dots, x_p) = \{\lambda_1 x_1 + \cdots + \lambda_p x_p : 0 \le \lambda_i \le 1, i = 1, \dots, p\}.$$

We denote its p-dimensional volume by

$$\nabla_p(x_1, \dots, x_p) = \mathcal{H}^p(P(x_1, \dots, x_p)).$$

This quantity equals the norm of the corresponding p-vector $x_1 \wedge \cdots \wedge x_p$.

Definition ([30, p. 532])**.** Let $L_p \in \mathcal{L}_p^d$ and $L_q \in \mathcal{L}_q^d$. Choose an orthonormal basis of $L_p \cap L_q$ and extend it to an orthonormal basis of L_p and an orthonormal basis of L_q. Then, $\mathcal{G}(L_p, L_q)$ is the d-dimensional volume of the parallelotope spanned by these vectors. □

For any two linear subspaces L_p and L_q, $\mathcal{G}(L_p, L_q)$ can be regarded as a generalized sinus of the angle between L_p and L_q. In particular, for $d = 3$ and $0 < p, q < d$, it is easy to show that $\mathcal{G}(L_p, L_q)$ is simply $|\sin\alpha|$ where α is the angle between L_p and L_q.

If $\dim(L_p + L_q) < d$ then $\mathcal{G}(L_p, L_q) = 0$. In the case $\dim(L_p + L_q) = d$ and either $p = 0$ or $q = 0$, then $\mathcal{G}(L_p, L_q) = 1$. Finally, if $\dim(L_p + L_q) = d$ and $0 < p, q < d$, we can choose orthonormal bases for

$$\begin{aligned} L_p \cap L_q &: a_1, \dots, a_{p+q-d} \\ L_p \cap (L_p \cap L_q)^\perp &: b_1, \dots, b_{d-q} \\ L_q \cap (L_p \cap L_q)^\perp &: c_1, \dots, c_{d-p}. \end{aligned}$$

Then,

$$\begin{aligned} \mathcal{G}(L_p, L_q) &= \nabla_d\left(a_1, \dots, a_{p+q-d}, b_1, \dots, b_{d-q}, c_1, \dots, c_{d-p}\right) \\ &= \nabla_{d-q}\left(p(b_1|L_q^\perp), \dots, p(b_{d-q}|L_q^\perp)\right) \\ &= \nabla_{d-p}\left(p(c_1|L_p^\perp), \dots, p(c_{d-p}|L_p^\perp)\right), \end{aligned}$$

cf. [14, Proposition 2.13 and 2.14].

If both L_p and L_q are contained in a subspace L_r of $\mathbb{R}^d$, we can consider the $\mathcal{G}$-function relatively in L_r. This will be denoted by $\mathcal{G}^{(L_r)}(L_p, L_q)$. If $L_p \perp x$ ($x \in \mathbb{R}^d$, $x \neq O$), then the following identity holds, cf. [14, Proposition 5.1],

$$\mathcal{G}(L_p, L_q) = \cos\angle(x, L_q)\mathcal{G}^{(x^\perp)}(L_p, L_q \cap x^\perp), \tag{8}$$

where $x^\perp$ is the orthogonal complement to the linear subspace spanned by x.

The integral over the whole Grassmannian of the squared $\mathcal{G}$-function is constant

$$\int_{\mathcal{L}_i^d} \mathcal{G}(L_i, L_j)^2 \, \mathrm{d}L_i^d = K_{ij}^d c_{d,i}, \tag{9}$$

where $K_{ij}^d = \frac{i!j!}{d!(i+j-d)!}$ if $i + j \ge d$ and 0 otherwise, see e.g. [21, Lemma 4.3].

3 The closed form of $w_{I,j,k}$

We shall formulate the main result, the closed form of $w_{I,j,k}$, for a compact set X with positive reach, in order to cover both important applications, convex bodies and sets with C^2 smooth boundary. The reader can for his/her convenience always imagine one of these two particular cases.

Let $X \subset \mathbb{R}^d$ be a compact set with positive reach and let $\operatorname{nor} X$ denote its unit normal bundle. Let $\kappa_1(x,n), \dots, \kappa_{d-1}(x,n)$ be the principal curvatures and $a_1(x,n), \dots, a_{d-1}(x,n)$ the corresponding principal directions defined almost everywhere on $(x,n) \in \operatorname{nor} X$, see [18] for further details. If X is smooth, then the unit normal $n = n(x)$ is a function of $x \in \partial X$ and

$$\operatorname{nor} X = \{(x, n(x)) : x \in \partial X\}.$$

Hence, all the functions defined on the unit normal bundle $\operatorname{nor} X$ can be considered as functions on ∂X.

In [18, Theorem], it was shown for $0 \le k < j \le d$ that, under the following assumptions

(A1) $O \notin \partial X$,

(A2) for almost all $L_j \in \mathcal{L}_j^d$, there is no $(x,n) \in \operatorname{nor} X$ with $x \in L_j$ and $n \perp x$,

the rotational integral equals

$$\int_{\mathcal{L}_j^d} V_k(X \cap L_j)\, \mathrm{d}L_j^d$$
$$= \int_{\operatorname{nor} X} \sum_{\substack{I \subseteq \{1,\dots,d-1\} \\ |I| = j-1-k}} w_{I,j,k}(x,n) \frac{\prod_{i \in I} \kappa_i(x,n)}{\prod_{i=1}^{d-1} \sqrt{1+\kappa_i^2(x,n)}}\, \mathcal{H}^{d-1}(\mathrm{d}(x,n)), \tag{10}$$

provided that the integral exists. Here,

$$w_{I,j,k}(x,n) = \sigma_{j-k}^{-1} |x|^{j-d} Q_j(x,n,A_I(x,n)), \tag{11}$$

where

$$A_I(x,n) = \operatorname{span}\{a_i(x,n) : i \notin I\} \in \mathcal{L}_{d-1-|I|}^d.$$

For $A_q \in \mathcal{L}_q^d$, the function Q_j is defined as the following integral

$$Q_j(x,n,A_q) = \int_{\mathcal{L}_{j(1)}^d} \frac{\mathcal{G}(L_j, A_q)^2}{|p(n|L_j)|^{d-q}}\, \mathrm{d}L_{j(1)}^d, \tag{12}$$

where $\mathcal{L}_{j(1)}^d$ is the set of j-dimensional subspaces containing the line spanned by x and $p(\cdot|L_j)$ indicates orthogonal projection onto L_j. If $j = 1$ and $x \perp n$ we set $Q_j(x,n,A_q) := 0$.

In the case where X is smooth, (10) reduces to (2) with $w_{I,j,k}(x) = w_{I,j,k}(x,n(x))$ and $\kappa_i(x) = \kappa_i(x,n(x))$.

Our main result formulated in Theorem 1 below follows from an expression of Q_j as a linear combination of hypergeometric functions. We use the notation

$$\beta(x,n) := \angle(x,n) \in [0,\pi], \quad \alpha_I(x,n) := \angle(x, A_I(x,n)) \in [0,\pi/2].$$

Note that $x \neq O$ if $(x,n) \in \operatorname{nor} X$ by (A1). The proof of Theorem 1 is deferred to the last section of the paper. The cases $j=1$ and $j=d$ are treated separately in Remark 2 below.

Theorem 1. *Let $0 \le k < j < d$, $j \ge 2$ and let I be a subset of $\{1,\dots,d-1\}$ with $|I| = j-1-k$ elements. Let $X \subset \mathbb{R}^d$ be a set with positive reach. If (A1) and (A2) are satisfied, then* (10) *holds with*

$$w_{I,j,k}(x,n) = C_{d,k,j}|x|^{j-d}\left[f_1(\beta(x,n)) + f_2(\beta(x,n))\cos^2\alpha_I(x,n)\right]$$

and

$$C_{d,k,j} = \sigma_{j-k}^{-1} c_{d-1,j-1}\frac{(j-1)!(d+k-j-1)!}{(d-1)!k!}.$$

The functions f_1 and f_2, defined on $[0,\pi]$, are given by

$$f_1(\beta) = (d+k-j)F\left(\tfrac{j-k}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; \sin^2\beta\right),$$

$$f_2(\beta) = \begin{cases} 0, & \beta = 0,\pi, \\ j-d, & \beta = \frac{\pi}{2}, \\ (j-d-(d-1)\cot^2\beta)F\left(\frac{j-k}{2}, \frac{d-j}{2}; \frac{d+1}{2}; \sin^2\beta\right) & \\ \quad + (d-1)\cot^2\beta F\left(\frac{j-k}{2}, \frac{d-j}{2}; \frac{d-1}{2}; \sin^2\beta\right), & \beta \neq 0, \frac{\pi}{2}, \pi. \end{cases}$$

Remark 2. Note that if $j=1$ then, necessarily, $k=0$, $I=\emptyset$ and $Q_1(x,n,n^\perp) = |\cos\beta(x,n)|$; hence, $w_{\emptyset,1,0}(x,n) = \frac{1}{2}|x|^{1-d}|\cos\beta(x,n)|$. If $j=d$ then no integration is carried out in (1) and we have $w_{I,d,k} = \sigma_{d-k}^{-1}$. These two particular cases are not included in Theorem 1. □

Two special cases were already derived in [18]. Let $k=0$ and $j=d-1$. Let $A_I(x,n) = \operatorname{span}\{a\}$. Assume that $\alpha_I(x,n) = \angle(x,a) > 0$ and $0 < \beta(x,n) = \angle(x,n) < \pi$. Let $\theta(x,n)$ be the angle formed by the projections $p(n|x^\perp)$ and $p(a|x^\perp)$ ($\cos\theta = \frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}$, see the end of Section 7). Then, we find, using (6),

$$\begin{aligned} &w_{I,d-1,0}(x,n) \\ &\quad = \frac{1}{2(d-1)}|x|^{-1}\sin^2\alpha_I(x,n)\Big[\sin^2\theta(x,n)F\left(\tfrac{d-1}{2}, \tfrac{1}{2}; \tfrac{d+1}{2}; \sin^2\beta(x,n)\right) \\ &\qquad\qquad + \cos^2\theta(x,n)F\left(\tfrac{d-1}{2}, \tfrac{3}{2}; \tfrac{d+1}{2}; \sin^2\beta(x,n)\right)\Big]. \end{aligned} \tag{13}$$

This agrees with the result presented in [18, Section 4.2].

When $k=j-1$, we have $I=\emptyset$, $A_I(x,n) = \operatorname{span}\{a_i(x,n) : i=1,\dots,d-1\} = n^\perp$ and $\angle(x, A_I(x,n)) = \frac{\pi}{2} - \angle(x,n)$; hence, $\cos\alpha_I(x,n) = \sin\beta(x,n)$. Then, by applying (5), we obtain

$$w_{I,j,j-1}(x,n) = \frac{c_{d-1,j-1}}{2}|x|^{-(d-j)}F\left(-\tfrac{1}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2\beta(x,n)\right). \tag{14}$$

Combining (2) and (14), we find in case of C^2 smooth boundary (cf. [18, Section 4.1])

$$\int_{\mathcal{L}_j^d} V_{j-1}(X \cap L_j)\, \mathrm{d}L_j^d$$
$$= \frac{c_{d-1,j-1}}{2} \int_{\partial X} |x|^{-(d-j)} F\left(-\tfrac{1}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2 \beta(x)\right) \mathrm{d}x^{d-1}.$$

4 Further simplifications

At locally spherical boundary points, the rotational formula may be further simplified. First, we derive a simple expression for the sum of $w_{I,j,k}(x, n)$.

Lemma 3. *For $0 \le k < j \le d$,*

$$\sum_{\substack{I \subseteq \{1,\dots,d-1\} \\ |I|=j-1-k}} w_{I,j,k}(x, n)$$
$$= \frac{c_{d-1,j-1}}{\sigma_{j-k}} \binom{j-1}{k} |x|^{-(d-j)} F\left(\tfrac{j-k-2}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2 \beta(x, n)\right).$$

Proof. Recall that $A_I(x, n) = \operatorname{span}\{a_i(x, n) : i \notin I\}$ and $\alpha_I(x) = \angle(x, A_I(x, n))$. We find

$$\sum_{|I|=j-1-k} \cos^2 \alpha_I(x, n)$$
$$= \sum_{|I|=j-1-k} |p(x|A_I(x, n))|^2 = \sum_{|I|=j-1-k} \sum_{i \notin I} (x \cdot a_i(x, n))^2$$
$$= \sum_{|I|=d-j+k} \sum_{i \in I} (x \cdot a_i(x, n))^2 = \sum_{i=1}^{d-1} \binom{d-2}{d-j+k-1} (x \cdot a_i(x, n))^2$$
$$= \binom{d-2}{j-k-1} |p(x| \operatorname{span}\{a_1(x, n), \dots, a_{d-1}(x, n)\}|^2$$
$$= \binom{d-2}{j-k-1} |p(x|n^\perp)|^2 = \binom{d-2}{j-k-1} \sin^2 \beta(x, n).$$

Using Theorem 1 and (5), we arrive at the following formula

$$\sigma_{j-k} |x|^{d-j} \sum_{|I|=j-1-k} w_{I,j,k}(x, n)$$
$$= c_{d-1,j-1} \frac{(j-1)!(d+k-j-1)!}{(d-1)!k!} \binom{d-2}{j-k-1}$$
$$\times \Big[(j-1) \sin^2 \beta(x, n) F\left(\tfrac{j-k}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; \sin^2 \beta(x, n)\right)$$
$$+ (d-1) \cos^2 \beta(x, n) F\left(\tfrac{j-k}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2 \beta(x, n)\right) \Big]$$
$$= c_{d-1,j-1} \binom{j-1}{k} F\left(\tfrac{j-k-2}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2 \beta(x, n)\right).$$

□

In case $k = 0$ and $j = d - 1$, the expression above reduces to the one in [18], namely

$$\sum_{|I|=d-2} w_{I,d-1,0}(x,n) = \frac{c_{d-1,d-2}}{\sigma_{d-1}}|x|^{-1}F\left(\tfrac{d-3}{2}, \tfrac{1}{2}; \tfrac{d-1}{2}; \sin^2\beta(x,n)\right).$$

Let us look at the simplifications of the rotational formula implied by Lemma 3 in the case where X is a compact $d-$dimensional C^2 manifold with boundary. In this case, there is a unique unit normal $n(x)$ at each $x \in \partial X$. The weight functions $w_{I,j,k}$ and the curvatures κ_i can be regarded as functions of x only.

It follows from Lemma 3 that at locally spherical boundary points $x \in \partial X$ where $\kappa_i(x) = \kappa(x)$, $i = 1, \ldots, d-1$, the integrand of (2) simplifies to

$$\begin{aligned}\sum_{\substack{I\subseteq\{1,\ldots,d-1\}\\|I|=j-1-k}} & w_{I,j,k}(x)\prod_{i\in I}\kappa_i(x) \\ &= \frac{c_{d-1,j-1}}{\sigma_{j-k}}\binom{j-1}{k}|x|^{-(d-j)}F\left(\tfrac{j-k-2}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2\beta(x)\right)\kappa(x)^{j-1-k}.\end{aligned}$$

For $k = j - 2$, this hypergeometric function is identically equal to 1. If almost all boundary points are locally spherical, we get

$$\int_{\mathcal{L}_j^d} V_{j-2}(X \cap L_j)\,\mathrm{d}L_j^d = \frac{(j-1)c_{d-1,j-1}}{2\pi}\int_{\partial X}|x|^{-(d-j)}\kappa(x)\,\mathrm{d}x^{d-1}.$$

Unfortunately, locally spherical boundary points are rare unless X is a finite union of disjoint balls.

5 The rotational average as valuation on convex bodies

In this section, we will show for convex bodies that the rotational averages of intrinsic volumes can represented as integrals with respect to natural measures on supporting flats.

For this purpose, let X be a convex body in $\mathbb{R}^d$. For convenience, we introduce the following short notation for the rotational average

$$\Phi_{k,j}(X) := \int_{\mathcal{L}_j^d} V_k(X \cap L_j)\,\mathrm{d}L_j^d,$$

$0 \le k < j \le d$. According to [18, Propositions 2], $\Phi_{k,j}(X) < \infty$ whenever X is a convex body. Clearly, $\Phi_{k,j}$ is a valuation which is continuous with respect to the Hausdorff metric on convex bodies and $O(d)$-invariant (see [2]).

We shall find an expression of $\Phi_{k,j}$ as an integral with respect to a certain measure $\Gamma_{d+k-j}(X;\cdot)$ associated with the convex body X. This measure is supported by $(j-k-1)$-dimensional affine subspaces "locally colliding" with X. The measure has been introduced by Firey [9], see also Weil [29], and, independently, and in different settings, in connection with absolute curvature measures by Baddeley [5] for smooth bodies and by Rother and Zähle [23] for sets with positive reach.

Given a convex body X and $0 \le i \le d-1$, let

$$\mathcal{F}_i^d(X) = \{(x, n, L_i) : (x, n) \in \operatorname{nor} X,\ L_i \in \mathcal{L}_i^{d-1}(n^\perp)\}.$$

Note that for $(x, n, L_i) \in \mathcal{F}_i^d(X)$, $x + L_i$ is an i-dimensional affine subspace that supports X at x. The projection

$$f : (x, n, L_i) \mapsto (p(x|L_i^\perp), L_i)$$

maps $\mathcal{F}_i^d(X)$ into the set of i-dimensional affine subspaces in $\mathbb{R}^d$ supporting X. The image of f will be denoted by $\mathcal{A}_i^d(X)$. Consider the following natural invariant measure $\mu_i(X; \cdot)$ on $\mathcal{A}_i^d(X)$, defined by the following equation for an arbitrary nonnegative measurable function on $\mathcal{A}_i^d(X)$,

$$\int_{\mathcal{A}_i^d(X)} h(z, L_i)\, \mu_i(X; \mathrm{d}(z, L_i)) = \int_{\mathcal{L}_i^d} \int_{\{z\colon (z,L_i) \in \mathcal{A}_i^d(X)\}} h(z, L_i)\, \mathcal{H}^{d-i-1}(\mathrm{d}z)\ \mathrm{d}L_i^d.$$

Then, the measure $\Gamma_{d-1-i}(X; \cdot)$ on $\mathcal{F}_i^d(X)$ is defined as

$$\begin{aligned}&\int_{\mathcal{F}_i^d(X)} g(x, n, L_i)\, \Gamma_{d-1-i}(X; \mathrm{d}(x, n, L_i)) \\ &\qquad = \int_{\mathcal{A}_i^d(X)} \sum_{(x,n,L_i) \in f^{-1}(z,L_i)} g(x, n, L_i)\, \mu_i(X; \mathrm{d}(z, L_i)),\end{aligned}$$

where g is now any nonnegative measurable function on $\mathcal{F}_i^d(X)$.

The following integral representation for $\Gamma_{d-1-i}(X; \cdot)$ was derived in [23]

$$\begin{aligned}&\int_{\mathcal{F}_i^d(X)} g(x, n, L_i)\, \Gamma_{d-1-i}(X; \mathrm{d}(x, n, L_i)) \qquad (15)\\ &\quad = \binom{d-1}{i} \sigma_{i+1}^{-1} \int_{\operatorname{nor} X} \sum_{|I|=i} \frac{\prod_{i \in I} \kappa_i(x, n)}{\prod_{i=1}^{d-1} \sqrt{1 + \kappa_i^2(x, n)}} \\ &\qquad \times \int_{\mathcal{L}_i^{d-1}(n^\perp)} g(x, n, L_i)\, \mathcal{G}^{(n^\perp)}(L_i, A_I(x, n))^2\ \mathrm{d}L_i^{d-1}\, \mathcal{H}^{d-1}(\mathrm{d}(x, n)).\end{aligned}$$

The result of this section follows:

Theorem 4. *Let X be a convex body in $\mathbb{R}^d$ with $O \notin \partial X$. If $0 \le k < j < d$, $j \ge 2$, then*

$$\Phi_{k,j}(X) = \int_{\mathcal{F}_{j-1-k}^d(X)} g(x, n, L_{j-1-k})\, \Gamma_{d+k-j}(X; \mathrm{d}(x, n, L_{j-1-k}))$$

with

$$g(x, n, L_{j-1-k}) = C'_{d,k,j} |x|^{j-d} \left[g_1(\beta(x, n)) + g_2(\beta(x, n)) |p(\bar{x}|L_{j-1-k})|^2 \right].$$

Here $\bar{x} = x/|x|$,

$$C'_{d,k,j} = C_{d,k,j}\sigma_{j-k}\binom{d-1}{j-1-k}^{-1},$$
$$g_2(\beta) = \frac{1}{M^{d-1}_{j-1-k} - N^{d-1}_{j-1-k}} f_2(\beta)$$

and

$$g_1(\beta) = \frac{\binom{d-1}{j-1-k}}{c_{d-1,d+k-j}}\left(f_1(\beta) - \frac{N^{d-1}_{j-1-k}}{M^{d-1}_{j-1-k} - N^{d-1}_{j-1-k}}\sin^2\beta f_2(\beta)\right).$$

The functions f_1 *and* f_2 *are given in Theorem 1 while the constants* M^{d-1}_{j-1-k} *and* N^{d-1}_{j-1-k} *are given in Lemma 7 in the Appendix.*

Proof. Let us decompose $\bar{x}$ as

$$\bar{x} = p(\bar{x}|A_I) + p(\bar{x}|\operatorname{span}\{n\}) + p(\bar{x}|A_I^\perp \cap n^\perp).$$

If $x \not\perp A_I$, i.e. if $\alpha \neq \pi/2$, we can write $p(\bar{x}|A_I) = (\cos\alpha)\pi(\bar{x}|A_I)$, with the spherical projection

$$y := \pi(\bar{x}|A_I) = p(\bar{x}|A_I)/|p(\bar{x}|A_I)|.$$

(Note that we can use this identity even when $\alpha = \pi/2$, choosing any unit vector for y.) Analogously, we denote $z = \pi(\bar{x}|A_I^\perp \cap n^\perp)$ and we get

$$\bar{x} = (\cos\alpha)\, y + (\cos\beta)\, n + \sqrt{1-\cos^2\alpha - \cos^2\beta}\, z.$$

For $L = L_{j-1-k} \perp n$, we have

$$\begin{aligned}|p(\bar{x}|L)|^2 = {} & \cos^2\alpha |p(y|L)|^2 + (1-\cos^2\alpha-\cos^2\beta)|p(z|L)|^2 \\ & + 2\cos\alpha\sqrt{1-\cos^2\alpha-\cos^2\beta}\, p(y|L)\cdot p(z|L).\end{aligned}$$

Integrating with respect to $\mathrm{d}L^{d-1}_{j-1-k}$, the last summand vanishes. Thus, we get, using Lemma 7,

$$\begin{aligned}&\int_{\mathcal{L}^{d-1}_{j-1-k}(n^\perp)} |p(\bar{x}|L_{j-1-k})|^2 \mathcal{G}^{(n^\perp)}(L_{j-1-k}, A_I(x,n))^2 \,\mathrm{d}L^{d-1}_{j-1-k} \\ &\quad = \cos^2\alpha M^{d-1}_{j-1-k} + (1-\cos^2\alpha-\cos^2\beta)N^{d-1}_{j-1-k} \\ &\quad = \cos^2\alpha(M^{d-1}_{j-1-k} - N^{d-1}_{j-1-k}) + \sin^2\beta\, N^{d-1}_{j-1-k}.\end{aligned} \tag{16}$$

Omitting for brevity the indexes at M and N, we get from (15), (9) and (16)

$$\begin{aligned}&\int g \,\mathrm{d}\Gamma_{j-1-k}(X,\cdot) \\ &\quad = C_{d,k,j}|x|^{j-d}\left[c_{d-1,j-1-k}K^{d-1}_{j-k-1,d+k-j}g_1 + ((M-N)\cos^2\alpha + N\sin^2\beta)g_2\right].\end{aligned}$$

The proof is finished by comparing the last expression with Theorem 1. Note that the assumptions (A1) and (A2) of Theorem 1 are fulfilled. In particular, (A2) is fulfilled for convex bodies, as shown in [18, Proposition 1]. □

6 An open question

An $O(d)$-invariant valuation Φ on the set of convex bodies is called polynomial if $x \mapsto \Phi(K+x)$ is a polynomial for any convex body K. Alesker [2] showed for $d \geq 3$ that any continuous polynomial $O(d)$-invariant valuation Φ can be expressed in the form

$$\Phi(X) = \sum_{i=0}^{d-1} \int_{\operatorname{nor} X} p_i(|x|^2, x \cdot n)\, \Theta_i(X, \mathrm{d}(x,n)),$$

where $p_1, \dots, p_{d-1}$ are polynomials in two variables and $\Theta_i(X, \cdot)$ are the (extended) curvature measures of X defined as

$$\begin{aligned}\int & h(x,n)\, \Theta_i(\mathrm{d}(x,n)) \\ &= \sigma_{d-i}^{-1} \int_{\operatorname{nor} X} h(x,n) \sum_{|I|=d-1-i} \frac{\prod_{i \in I} \kappa_i(x,u)}{\prod_{i=1}^{d-1} \sqrt{1+\kappa_i^2(x,u)}}\, \mathcal{H}^{d-1}(\mathrm{d}(x,n)).\end{aligned}$$

He also showed that any continuous $O(d)$-invariant valuation is a locally uniform limit of continuous polynomial $O(d)$-invariant valuations, but he later found a gap in the proof (see [3]) and the validity of this assertion remained open. It seems plausible to expect that if this conjecture was true, then every continuous $O(d)$-invariant valuation could be expressed as an integral over the unit normal bundle. The valuations $\Phi_{k,j}$ given by rotational integrals are expressed as integrals over the larger flag manifolds $\mathcal{S}^d_{j-1-k}(X)$ and we doubt that they could be given as integrals over $\operatorname{nor} X$ only if $k < j-1$. This leads us to conjecture that these continuous $O(d)$-invariant valuations cannot be approximated by polynomial valuations.

7 Proof of Theorem 1

Let X be a set with positive reach fulfilling assumptions (A1) and (A2), let $(x,n) \in \operatorname{nor} X$, $0 \leq k < j < d$, $j \geq 2$, and let A_q be a subspace perpendicular to n and of dimension

$$q = d - 1 - (j - 1 - k) = d - j + k.$$

(Note that $j + q \geq d$.) We will derive a closed form of $Q_j(x, n, A_q)$ from (12) that will prove Theorem 1.

We first rewrite the integrand in (12), using subspaces in $x^\perp$. We use here and in the following the notation $\mathcal{L}^s_r(M)$ for the set of r-dimensional linear subspaces contained in $M \in \mathcal{L}^d_s$. Recall that $\beta = \beta(x,n) = \angle(x,n)$ and $\alpha = \angle(x, A_q)$.

Lemma 5. *Let $A_q \in \mathcal{L}^{d-1}_q(n^\perp)$ and let $L_j = L_{j-1} \oplus \operatorname{span}\{x\}$, where $L_{j-1} \in \mathcal{L}^{d-1}_{j-1}(x^\perp)$. Then,*

$$\mathcal{G}(L_j, A_q)^2 = \sin^2\alpha\, \mathcal{G}^{(x^\perp)}(L_{j-1}, p(A_q|x^\perp))^2 + \cos^2\alpha\, \mathcal{G}^{(x^\perp)}(L_{j-1}, A_q \cap x^\perp)^2. \qquad (17)$$

Proof. If $\alpha = \pi/2$ then $A_q \perp x$, $p(A_q|x^\perp) = A_q$ and (17) is obvious. If $\alpha = 0$ then (17) follows from (8). It is thus sufficient to consider the case $0 < \alpha < \pi/2$.

Consider first the case $j+q=d$. Then,

$$\dim(L_{j-1}+A_q\cap x^\perp)<d-1$$

and the second summand of (17) vanishes because $\mathcal{G}^{(x^\perp)}(L_{j-1},A_q\cap x^\perp)=0$. In order to prove (17) in the case $j+q=d$, first notice that if $\dim(L_j+A_q)<d$, then left- and right-hand sides of (17) are both zero. If $\dim(L_j+A_q)=d$, we can proceed as follows. Let $\{a_1,\dots,a_q\}$ be an orthonormal basis of A_q such that $a_1=\pi(x|A_q)$ and $a_i\perp x$, $i=2,\dots,q$. Then we have

$$\begin{aligned}\mathcal{G}(L_j,A_q)&=\nabla_q(p(a_1|L_j^\perp),p(a_2|L_j^\perp),\dots,p(a_q|L_j^\perp))\\&=\nabla_q(p(p(a_1|x^\perp)|L_j^\perp),p(a_2|L_j^\perp),\dots,p(a_q|L_j^\perp))\\&=|p(a_1|x^\perp)|\nabla_q(p(\pi(a_1|x^\perp)|L_j^\perp),p(a_2|L_j^\perp),\dots,p(a_q|L_j^\perp))\\&=|p(a_1|x^\perp)|\nabla_q(p(\pi(a_1|x^\perp)|L_{j-1}^\perp),p(a_2|L_{j-1}^\perp),\dots,p(a_q|L_{j-1}^\perp))\\&=|\sin\angle(x,A_q)|\mathcal{G}^{(x^\perp)}(L_{j-1},p(A_q|x^\perp)).\end{aligned}$$

Let now $j+q>d$ and choose an orthonormal basis $\{u_1,\dots,u_{j-1}\}$ of L_{j-1}. Given an index set $I\subseteq\{1,\dots,j-1\}$, we shall write L_I for the linear hull of $\{u_i|i\in I\}$. We have by [18, Lemma 1],

$$\mathcal{G}(L_j,A_q)^2=\sum_{|I|=d-q}\mathcal{G}(L_I,A_q)^2+\sum_{|I|=d-q-1}\mathcal{G}(L_I+\operatorname{span}\{x\},A_q)^2.$$

By applying (8) to each summand in the first sum and by repeating the above procedure from the case $q+j=d$ to each summand of the second sum, we obtain

$$\begin{aligned}&\mathcal{G}(L_j,A_q)^2\\&\quad=\sum_{|I|=d-q}\cos^2\alpha\,\mathcal{G}^{(x^\perp)}(L_I,A_q\cap x^\perp)^2+\sum_{|I|=d-q-1}\sin^2\alpha\,\mathcal{G}^{(x^\perp)}(L_I,p(A_q|x^\perp))^2\\&\quad=\cos^2\alpha\,\mathcal{G}^{(x^\perp)}(L_{j-1},A_q\cap x^\perp)^2+\sin^2\alpha\,\mathcal{G}^{(x^\perp)}(L_{j-1},p(A_{j-1}|x^\perp))^2.\end{aligned}$$

□

The case $x\|n$ will be treated separately, thus, we assume that $\beta=\beta(x,n)\in(0,\pi)$.

We introduce a function of a unit vector and a linear subspace in $\mathbb{R}^{d-1}\cong x^\perp$ which will be needed for the computation of Q_j. Let $d-j\le p\le d-1$, $B_p\in\mathcal{L}_p^{d-1}$ and $m\in S^{d-2}$. Define

$$I_{j-1}^{d-1}(m,B_p)=\int_{\mathcal{L}_{j-1}^{d-1}}\frac{\mathcal{G}(L_{j-1},B_p)^2}{(\cos^2\beta+|p(m|L_{j-1})|^2\sin^2\beta)^{\frac{d-q}{2}}}\,\mathrm{d}L_{j-1}^{d-1}. \tag{18}$$

Note that using Lemma 5, we have by (12)

$$Q_j(x,n,A_q)=\sin^2\alpha\,I_{j-1}^{d-1}(m,p(A_q|x^\perp))+\cos^2\alpha\,I_{j-1}^{d-1}(m,A_q\cap x^\perp), \tag{19}$$

where $m=\pi(n|x^\perp)$. We thus need to evaluate the integral (18) which is done in the following lemma. Recall that the constants $K_{i,j}^d$ are defined after (9).

Lemma 6. *Let p, q, B_p, m, β be as above, and denote $\theta = \angle(m, B_p)$. If $\beta \neq \pi/2$ then,*

$$I_{j-1}^{d-1}(m, B_p) = \tfrac{1}{p} c_{d-1,j-1} K_{j-1,p}^{d-1} \Big[(p-(d-1)\cos^2\theta) F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; \sin^2\beta\right) + (d-1)\cos^2\theta F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2\beta\right) \Big].$$

If $\beta = \theta = \pi/2$ then

$$I_{j-1}^{d-1}(m, B_p) = c_{d-1,j-1} K_{j-1,p}^{d-1} F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; 1\right).$$

Proof. We shall treat only the case $\beta \in (0, \pi/2) \cup (\pi/2, \pi)$; the case $\beta = \theta = \pi/2$ is similar but simpler.

The first step will be to transform the integral over a Grassmannian into an integral over a sphere. To achieve this, we apply the coarea formula to the mapping $g : L_{j-1} \mapsto \pi(m|L_{j-1}^{\perp})$ defined on $\mathcal{L}_{j-1}^{d-1} \cap \{L : m \notin L\}$ with Jacobian $J_{d-2}g(L_{j-1}) = \tan^{d-j-1}\zeta$, where $\zeta = \angle(m, L_{j-1}^{\perp}) = \angle(m, u)$ and $u = \pi(m|L_{j-1}^{\perp})$ (cf. [21, Lemma 4.2]; note that $L_{j-1} \mapsto L_{j-1}^{\perp}$ is an isometry). The range of g is the semisphere $S_+^{d-2} := \{y \in S^{d-1} | \, y \cdot m > 0\}$. Using that

$$g^{-1}(u) = \{L_{j-2} \oplus \operatorname{span}\{v\} | \, L_{j-2} \in \mathcal{L}_{j-2}^{d-3}(v^{\perp} \cap m^{\perp})\}, \quad u \in S_+^{d-2},$$

where $v = \pi(m|L_{j-1})$, we get

$$\begin{aligned} & I_{j-1}^{d-1}(m, B_p) \\ & = \int_{S_+^{d-2}} \int_{g^{-1}(v)} \frac{\mathcal{G}(L_{j-1}, B_p)^2}{(\cos^2\beta + \sin^2\zeta \sin^2\beta)^{\frac{d-q}{2}}} \frac{1}{J_{j-2}g(L_{j-1})} \, \mathrm{d}L_{j-2}^{d-3} \mathcal{H}^{d-2}(\mathrm{d}u) \\ & = \int_{S_+^{d-2}} \frac{1}{(\cos^2\beta + \sin^2\zeta \sin^2\beta)^{\frac{d-q}{2}} \tan^{d-j-1}\zeta} \\ & \quad \times \int_{\mathcal{L}_{j-2}^{d-3}(v^{\perp} \cap m^{\perp})} \mathcal{G}(L_{j-2} \oplus \operatorname{span}\{v\}, B_p)^2 \, \mathrm{d}L_{j-2}^{d-3} \mathcal{H}^{d-2}(\mathrm{d}u). \end{aligned}$$

In order to evaluate the inner integral, we first apply (8):

$$\mathcal{G}(L_{j-1}, B_p)^2 = \cos^2\angle(u, B_p) \mathcal{G}^{(u^{\perp})}(L_{j-1}, B_p \cap u^{\perp})^2,$$

and then, we use Lemma 5 for the decomposition of $\mathcal{G}^{(u^{\perp})}(L_{j-1}, B_p \cap u^{\perp})^2$:

$$\begin{aligned} \mathcal{G}^{(u^{\perp})}(L_{j-1}, B_p \cap u^{\perp})^2 = {} & \sin^2\angle(v, B_p \cap u^{\perp}) \mathcal{G}^{(u^{\perp} \cap v^{\perp})}(L_{j-2}, p(B_p \cap u^{\perp} | v^{\perp}))^2 \\ & + \cos^2\angle(v, B_p \cap u^{\perp}) \mathcal{G}^{(u^{\perp} \cap v^{\perp})}(L_{j-2}, B_p \cap u^{\perp} \cap v^{\perp})^2. \end{aligned}$$

Note that the second term vanishes when $d = p + j$. Using now the identity (9), we obtain

$$\begin{aligned} & \int_{\mathcal{L}_{j-2}^{d-3}} \mathcal{G}(L_{j-2} \oplus \operatorname{span}\{v\}, B_p)^2 \, \mathrm{d}L_{j-2}^{d-3} = \cos^2\angle(u, B_p) \\ & \quad \times \left(\sin^2\angle(v, B_p \cap u^{\perp}) c_{d-3,j-2} K_{j-2,p-1}^{d-3} + \cos^2\angle(v, B_p \cap u^{\perp}) c_{d-3,j-2} K_{j-2,p-2}^{d-3}\right) \\ & = \cos^2\angle(u, B_p) c_{d-3,j-2} K_{j-2,p-1}^{d-3} \left(1 - \tfrac{d-j-1}{p-1} \cos^2\angle(v, B_p \cap u^{\perp})\right). \end{aligned}$$

Due to the definitions of u and v, we can write

$$m = p(m|v) + p(m|v^{\perp}) = v \sin\zeta + u \cos\zeta,$$

hence

$$\cos^2 \angle(v, B_p \cap u^{\perp}) = \frac{|p(m|B_p \cap u^{\perp})|^2}{\sin^2\zeta}.$$

Consequently, we obtain

$$I_{j-1}^{d-1}(m, B_p) = c_{d-3,j-2} K_{j-2,p-1}^{d-3} \left(G_1 - \tfrac{d-j-1}{p-1} G_2 \right), \tag{20}$$

where

$$G_1 = \int_{S_+^{d-2}} \frac{|p(u|B_p)|^2}{(\cos^2\beta + \sin^2\zeta \sin^2\beta)^{\frac{d-q}{2}} \tan^{d-j-1}\zeta} \mathcal{H}^{d-2}(\mathrm{d}u) \tag{21}$$

and

$$G_2 = \int_{S_+^{d-2}} \frac{|p(m|B_p \cap u^{\perp})|^2 |p(u|B_p)|^2 \cos^{d-j-1}\zeta}{(\cos^2\beta + \sin^2\zeta \sin^2\beta)^{\frac{d-q}{2}} \sin^{d-j+1}\zeta} \mathcal{H}^{d-2}(\mathrm{d}u). \tag{22}$$

Using the coarea formula with $\varphi : S_+^{d-2} \setminus \mathrm{span}(m) \to S^{d-3}(m^{\perp})$ defined by $\varphi(u) = \pi(u|m^{\perp}) =: u_0$ and with $J_{d-3}\varphi(u) = (\sin\angle(u,m))^{-(d-3)}$, we obtain (recall that $\zeta = \angle(u,m)$)

$$\begin{aligned} G_1 &= \int_{S^{d-3}(m^{\perp})} \int_{\varphi^{-1}(u_0)} \frac{|p(u|B_p)|^2 J_{d-3}^{-1}\varphi(u)}{(\cos^2\beta + \sin^2\zeta\sin^2\beta)^{\frac{d-q}{2}} \tan^{d-j-1}\zeta} \mathcal{H}^1(\mathrm{d}u)\mathcal{H}^{d-3}(\mathrm{d}u_0) \\ &= \int_{S^{d-3}(m^{\perp})} \int_{\varphi^{-1}(u_0)} \frac{|p(u|B_p)|^2 \cos^{d-j-1}\zeta \sin^{j-2}\zeta}{(\cos^2\beta + \sin^2\zeta\sin^2\beta)^{\frac{d-q}{2}}} \mathcal{H}^1(\mathrm{d}u)\mathcal{H}^{d-3}(\mathrm{d}u_0). \end{aligned}$$

Define $\xi : \mathbb{R}_+ \to \varphi^{-1}(u_0)$ by $\xi(r) = \frac{u_0 + rm}{|u_0 + rm|} = u$ with $J_1\xi(r) = \frac{1}{1+r^2}$. Note that $\cos^2\zeta = \cos^2\angle(\xi(r), m) = \frac{r^2}{1+r^2}$ and $\sin^2\zeta = \frac{1}{1+r^2}$. The area formula implies

$$G_1 = \int_{S^{d-3}(m^{\perp})} \int_0^{\infty} \frac{|p(\xi(r)|B_p)|^2 (\frac{r^2}{1+r^2})^{\frac{d-j-1}{2}} (\frac{1}{1+r^2})^{\frac{j-2}{2}}}{(\cos^2\beta + \frac{\sin^2\beta}{1+r^2})^{\frac{d-q}{2}}} \frac{\mathrm{d}r}{1+r^2} \mathcal{H}^{d-3}(\mathrm{d}u_0).$$

We now use that

$$|p(\xi(r)|B_p)|^2 = \frac{|p(u_0|B_p)|^2 + r^2|p(m|B_p)|^2 + 2r p(u_0|B_p)\cdot p(m|B_p)}{1+r^2}$$

which, using the equality $\int_{S^{d-3}(m^{\perp})} p(u_0|B_p)\cdot p(m|B_p)\mathcal{H}^{d-3}(du_0) = 0$ and (7), lead us to the following expression

$$\begin{aligned} G_1 &= \int_{S^{d-3}(m^{\perp})} \int_0^{\infty} \frac{(|p(u_0|B_p)|^2 + r^2|p(m|B_p)|^2)(r^2)^{\frac{d-j-1}{2}}}{(\cos^2\beta + \frac{1}{1+r^2}\sin^2\beta)^{\frac{d-q}{2}} (1+r^2)^{\frac{d+1}{2}}} \mathrm{d}r \mathcal{H}^{d-3}(\mathrm{d}u_0) \\ &= \tfrac{1}{2} B\left(\tfrac{d-j}{2}, \tfrac{j+1}{2}\right) F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; \sin^2\beta\right) H_1 \\ &\quad + \tfrac{1}{2} B\left(\tfrac{d-j+2}{2}, \tfrac{j-1}{2}\right) F\left(\tfrac{d-q}{2}, \tfrac{d-j+2}{2}; \tfrac{d+1}{2}; \sin^2\beta\right) |p(m|B_p)|^2 \sigma_{d-2} \end{aligned} \tag{23}$$

with

$$H_1 = \int_{S^{d-3}(m^\perp)} |p(u_0|B_p)|^2 \mathcal{H}^{d-3}(\mathrm{d}u_0).$$

The convergence criterion in (7) are satisfied since $1 < j < d$ and $0 < \beta < \frac{\pi}{2}$ by assumption.

Note that the differences between G_1 and G_2 are the extra terms $\sin^2 \zeta$ and

$$|p(m|B_p \cap u^\perp)|^2 = |p(m|B_p)|^2 \frac{|p(u|B_p \cap m^\perp)|^2}{|p(u|B_p)|^2}.$$

Hence, G_2 can be rewritten as

$$G_2 = |p(m|B_p)|^2 \int_{S_+^{d-2}} \frac{|p(u|B_p \cap m^\perp)|^2 \cos^{d-j-1} \zeta}{(\cos^2 \beta + \sin^2 \zeta \sin^2 \beta)^{\frac{d-q}{2}} \sin^{d-j+1} \zeta} \mathcal{H}^{d-2}(\mathrm{d}u).$$

By applying the area formula for the mappings $\varphi : u \mapsto \pi(u|m^\perp)$ and $\xi : r \mapsto \frac{u_0 + rm}{|u_0 + rm|}$, the integral above becomes

$$\begin{aligned}
&\int_{S^{d-3}(m^\perp)} \int_{\varphi^{-1}(u_0)} \frac{|p(u|B_p \cap m^\perp)|^2 \cos^{d-j-1} \zeta \sin^{j-4} \zeta}{(\cos^2 \beta + \sin^2 \zeta \sin^2 \beta)^{\frac{d-q}{2}}} \mathcal{H}^1(\mathrm{d}u) \mathcal{H}^{d-3}(\mathrm{d}u_0) \\
&= \int_{S^{d-3}(m^\perp)} |p(u_0|B_p \cap m^\perp)|^2 \int_0^\infty \frac{(r^2)^{\frac{d-j-1}{2}} (1+r^2)^{-\frac{d-1}{2}}}{(\cos^2 \beta + \frac{\sin^2 \beta}{1+r^2})^{\frac{d-q}{2}}} \mathrm{d}r \mathcal{H}^{d-3}(\mathrm{d}u_0),
\end{aligned}$$

where we used $|p(\xi(r)|B_p \cap m^\perp)|^2 = \frac{|p(u_0|B_p \cap m^\perp)|^2}{1+r^2}$ for the last equality. Using (7), we obtain

$$G_2 = \frac{|p(m|B_p)|^2}{2} B\left(\tfrac{d-j}{2}, \tfrac{j-1}{2}\right) F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2 \beta\right) H_2 \tag{24}$$

with

$$H_2 = \int_{S^{d-3}(m^\perp)} |p(u_0|B_p \cap m^\perp)|^2 \mathcal{H}^{d-3}(\mathrm{d}u_0).$$

Note that $|p(u_0|B_p \cap m^\perp)|^2 = \mathcal{G}(u_0^\perp, B_p \cap m^\perp)^2$ and, since the integration over $S^{d-3}(m^\perp)$ is, up to a factor 2, the integration over the Grassmannian $\mathcal{L}_{d-3}^{d-2}(m^\perp)$, we can apply (9) and obtain

$$H_2 = \sigma_{d-3} K_{d-3,p-1}^{d-2} = \omega_{d-2}(p-1), \tag{25}$$

where $\omega_k = \sigma_k / k$ is the volume of the unit ball in $\mathbb{R}^k$. In order to calculate H_1, we use the decomposition

$$|p(u_0|B_p)|^2 = |p(u_0|B_p \cap m^\perp)|^2 + (u_0 \cdot m_0)^2,$$

where $m_0 = \pi(m|B_p)$. We have $u_0 \cdot m_0 = \sin \theta (u_0 \cdot m_1)$ with $m_1 = \pi(m_0|m^\perp)$. Since, again by (9),

$$\int_{S^{d-3}(m^\perp)} (u_0 \cdot m_1)^2 \, \mathcal{H}^{d-3}(\mathrm{d}u_0) = \omega_{d-2},$$

we get

$$H_1 = \omega_{d-2}(p-1) + \sin^2\theta\omega_{d-2} = \omega_{d-2}(p-\cos^2\theta). \tag{26}$$

By inserting (26) into (23) and (25) into (24) we get

$$\begin{aligned} G_1 = {} & \frac{\omega_{d-2}(p-\cos^2\theta)}{2} B\left(\tfrac{d-j}{2}, \tfrac{j+1}{2}\right) F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; \sin^2\beta\right) \\ & + \frac{\sigma_{d-2}\cos^2\theta}{2} B\left(\tfrac{d-j+2}{2}, \tfrac{j-1}{2}\right) F\left(\tfrac{d-q}{2}, \tfrac{d-j+2}{2}; \tfrac{d+1}{2}; \sin^2\beta\right), \end{aligned}$$

and

$$G_2 = \frac{\omega_{d-2}(p-1)\cos^2\theta}{2} B\left(\tfrac{d-j}{2}, \tfrac{j-1}{2}\right) F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2\beta\right),$$

which, in combination with (20), implies

$$\begin{aligned} I_{j-1}^{d-1}(m, B_p) = {} & \tfrac{1}{2} c_{d-3,j-2} K_{j-2,p-1}^{d-3} \omega_{d-2} B\left(\tfrac{d-j}{2}, \tfrac{j-1}{2}\right) \\ & \times \Big[(p-\cos^2\theta)\tfrac{j-1}{d-1} F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; \sin^2\beta\right) \\ & \quad + (d-2)\cos^2\theta \tfrac{d-j}{d-1} F\left(\tfrac{d-q}{2}, \tfrac{d-j+2}{2}; \tfrac{d+1}{2}; \sin^2\beta\right) \\ & \quad - \tfrac{(d-j-1)}{p-1}(p-1)\cos^2\theta F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2\beta\right) \Big]. \end{aligned}$$

Applying (6) to the middle hypergeometric function, the expression above can be rewritten

$$\begin{aligned} I_{j-1}^{d-1}(m, B_p) = {} & \tfrac{1}{2} c_{d-3,j-2} K_{j-2,p-1}^{d-3} \omega_{d-2} B\left(\tfrac{d-j}{2}, \tfrac{j-1}{2}\right) \\ & \times \Big[\tfrac{j-1}{d-1}(p-\cos^2\theta - (d-2)\cos^2\theta) F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; \sin^2\beta\right) \\ & \quad + (j-1)\cos^2\theta F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2\beta\right) \Big]. \end{aligned}$$

Use

$$\tfrac{1}{p} c_{d-1,j-1} K_{j-1,p}^{d-1} = \frac{(j-1) c_{d-3,j-2} K_{j-2,p-1}^{d-3} \omega_{d-2} B(\tfrac{d-j}{2}, \tfrac{j-1}{2})}{2(d-1)}$$

and the proof is complete. □

Proof of Theorem 1. If $\beta = 0$ or π then $x \| n$, $\alpha = \pi/2$, $p(A_q|x^\perp) = A_q$, $p(n|L_j) = n$ and Q_j can be obtained by using Lemma 5:

$$Q_j(x, n, A_q) = \int \mathcal{G}^{(x^\perp)}(L_{j-1}, A_q)\, \mathrm{d}L_{j-1}^{d-1}.$$

The result is then obtained using (9).

In the case $\beta = \pi/2$ we have $m = n$, hence, $\theta = \pi/2$, and the result follows using (19) and Lemma 6.

Assume in the following that $\beta \in (0, \pi/2) \cup (\pi/2, \pi)$. We use the form of $Q_j = Q_j(x, n, A_q)$ given in (19). In the first summand in (19) we have a factor of the form $I_{j-1}^{d-1}(m, B_p)$ with $p = \dim p(A_q|x^\perp) = q$, unless $\alpha = 0$. Assume thus that $\alpha > 0$. We shall show that

$$\theta := \angle(m, p(A_q|x^\perp)) = \frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}. \tag{27}$$

Since

$$m = \frac{n - p(n|x)}{|p(n|x^\perp)|} = \frac{1}{\sin\beta}\left(n - \frac{\cos\beta}{|x|}x\right),$$

we have

$$p(m|p(A_q|x^\perp)) = \frac{1}{\sin\beta}p(n|p(A_q|x^\perp)).$$

If $\alpha + \pi/2$ then $p(A_q|x^\perp) = A_q \perp n$ and we get $\cos\theta = 0$, verifying (27) in this particular case. Assume now that $\alpha \in (0, \pi/2)$. By using the decomposition $A_q = \mathrm{span}\{a\} \oplus (A_q \cap x^\perp)$, where $a = \pi(x|A_q)$, and that $n \perp A_q$, we get

$$p(n|p(A_q|x^\perp)) = p(n|\pi(a|x^\perp)),$$

where

$$\pi(a|x^\perp) = \frac{a - p(a|x)}{|p(a|x^\perp)|} = \frac{1}{\sin\alpha}\left(a - \cos\alpha\frac{x}{|x|}\right).$$

Since $n \perp a$, we obtain

$$\cos\theta = |p(m|p(A_q|x^\perp))| = \frac{|\pi(a|x^\perp)\cdot n|}{\sin\beta} = \frac{1}{\sin\alpha\sin\beta}\left(\cos\alpha\frac{x\cdot n}{|x|}\right),$$

verifying (27) again.

In the second summand we have a similar factor with $p = \dim(A_q \cap x^\perp) = q - 1$ and $\theta = \frac{\pi}{2}$, i.e $\cos\theta = 0$. Lemma 6 together with the identity

$$K^{d-1}_{j-1,q-1} = \tfrac{j+q-d}{q}K^{d-1}_{j-1,q}$$

imply

$$\begin{aligned} Q_j = \frac{c_{d-1,j-1}K^{d-1}_{j-1,q}}{q}\Big\{ &\sin^2\alpha\big[(q-(d-1)\cos^2\theta)F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; \sin^2\beta\right) \\ &\qquad + (d-1)\cos^2\theta F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2\beta\right)\big] \\ &+ \cos^2\alpha\frac{j+q-d}{q-1}(q-1)F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; \sin^2\beta\right)\Big\}. \end{aligned}$$

The result now follows by using (27). $\square$

Appendix

The following lemma gives the values of the constants appearing in Theorem 4.

Lemma 7. *Let $A \in \mathcal{L}^d_{d-i}$, $y \perp A$, $z \in A$, $|y| = |z| = 1$. Then*

$$M^d_i := \int \mathcal{G}(A, V_i)^2 |p(y|V_i)|^2 \, dV^d_i = \frac{c_{d-1,i-1}}{2\sigma_{d-i}}\binom{d-1}{i-1}^{-1} B\left(\tfrac{i+4}{2}, \tfrac{d-i}{2}\right),$$

$$N^d_i := \int \mathcal{G}(A, V_i)^2 |p(z|V_i)|^2 \, dV^d_i = c_{d,i}\binom{d}{i}^{-1} - M_{d,d-i}.$$

Proof. First, we apply the coarea formula with

$$\varphi : \, V_i \mapsto \pi(y|V_i) =: v, \quad J_{d-1}\varphi(V_i) = \tan^{i-1}\gamma,$$

with $\gamma := \angle(y,v) = \angle(y,V_i)$ (cf. the beginning of the proof of Lemma 6). We get

$$M_i^d = \int_{S_+^{d-1}} \frac{\cos^2\gamma}{\tan^{i-1}\gamma} \int_{\varphi^{-1}\{v\}} \mathcal{G}(A,V_i)^2 \, \mathrm{d}V_i^d \, \mathcal{H}^{d-1}(\mathrm{d}v) \tag{28}$$

($S_+^{d-1} = \{v \in S^{d-1} : \, v \cdot y > 0\}$). Since $y \perp A$ and $v \perp V_i \cap y^\perp$, we obtain using twice (8):

$$\begin{aligned}\mathcal{G}(A,V_i)^2 &= \cos^2\gamma \mathcal{G}^{(y^\perp)}(A, V_i \cap y^\perp)^2 \\ &= \cos^2\gamma |p(v|A)|^2 \mathcal{G}^{(y^\perp \cap v^\perp)}(A \cap v^\perp, V_i \cap y^\perp)^2.\end{aligned}$$

Further, $\varphi^{-1}\{v\} = \{V'_{i-1} \oplus \mathrm{span}\{v\} : \, V'_{i-1} \in \mathcal{L}_{i-1}^{d-2}(y^\perp \cap v^\perp)\}$, thus

$$\begin{aligned}\int_{\varphi^{-1}\{v\}} \mathcal{G}(A,V_i)^2 \, \mathrm{d}V_i &= \cos^2\gamma |p(v|A)|^2 \int \mathcal{G}^{(y^\perp \cap v^\perp)}(A \cap v^\perp, V_i \cap y^\perp)^2 \, \mathrm{d}V_{i-1}^{d-2} \\ &= \cos^2\gamma |p(v|A)|^2 c_{d-2,i-1} \binom{d-2}{i-1}^{-1}\end{aligned}$$

by (9). Inserting this into (28), we get

$$M_i^d = c_{d-2,i-1} \binom{d-2}{i-1}^{-1} \int_{S_+^{d-1}} \frac{\cos^{i+3}\gamma}{\sin^{i-1}\gamma} |p(v|A)|^2 \, \mathcal{H}^{d-1}(\mathrm{d}v).$$

We apply now the coarea formula with $\psi : \, v \mapsto v \cdot y$, $J_1\psi(v) = \sin\gamma$, $\psi^{-1}\{r\} \cong \sqrt{1-r^2} S^{d-2}$:

$$M_i^d = c_{d-2,i-1} \binom{d-2}{i-1}^{-1} \int_0^1 \frac{r^{i+3}}{(1-r^2)^{i/2}} \int_{\psi^{-1}\{r\}} |p(v|A)|^2 \, \mathcal{H}^{d-2}(\mathrm{d}v) \, \mathrm{d}r,$$

where

$$\begin{aligned}\int_{\psi^{-1}\{r\}} |p(v|A)|^2 \, \mathcal{H}^{d-2}(\mathrm{d}v) &= (1-r^2)^{(d-2)/2} \int_{S^{d-2}} |p(v|A)|^2 \, \mathrm{d}v \\ &= (1-r^2)^{(d-2)/2} \int_{S^{d-2}} \mathcal{G}(A, v^\perp)^2 \, \mathrm{d}v \\ &= (1-r^2)^{(d-2)/2} \sigma_{d-1} K_{d-1,d-i-1}^{d-i,d-2} \\ &= \sigma_{d-1} \frac{d-i}{d-1} (1-r^2)^{(d-2)/2}\end{aligned}$$

(we used (9) in the last but one equality). Hence,

$$M_i^d = c_{d-2,i-1} \sigma_{d-1} \binom{d-2}{i-1}^{-1} \frac{d-i}{d-1} \int_0^1 r^{i+3} (1-r^2)^{(d-i-2)/2} \, \mathrm{d}r.$$

After routine calculation of the integral we finally arrive at

$$M_i^d = \frac{c_{d-1,i-1}}{\sigma_{d-i}} \binom{d-2}{i-1}^{-1} B\left(\tfrac{i+4}{2}, \tfrac{d-i}{2}\right).$$

For the second integral, we use:

$$\begin{aligned} N_i^d &= \int \mathcal{G}(A^\perp, V_i^\perp)(1 - |p(z|V_i^\perp)|^2) \, \mathrm{d}V_i^\perp \\ &= \int \mathcal{G}(A^\perp, V_i^\perp)^2 \, \mathrm{d}V_i^\perp - \int \mathcal{G}(A^\perp, V_i^\perp)^2 |p(z|V_i^\perp)|^2 \, \mathrm{d}V_i^\perp \\ &= c_{d,i} \binom{d}{i}^{-1} - M_i^d. \end{aligned}$$

□

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