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Loss Rates and Structural Properties of Reflected Stochastic Processes

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Contents

Preface & acknowledgments

This thesis constitutes the result of my PhD studies at the department of mathematics, Aarhus university. My studies have been conducted under the careful supervision of Søren Asmussen, to whom I owe many thanks for his advice, both that of mathematical nature, and regarding more general questions on scientific research. My PhD studies were carried out during Jan. 2006 - May 2009, with the period Jan. 2008 - April 2008 spent at Stanford University, where I visited Michel Mandjes, to whom I also owe many thanks - both for professional collaboration and great hospitality.
Finally, I would like to thank all my colleagues at the Department of Mathematics, in particular my office-mate Andreas, and my girlfriend Anja for her love and support.

Introduction

In this introduction, we aim to give a description of the content of the thesis, intended for the reader with little or no mathematical background. Furthermore, we present results and conjectures, which did not make their way into any of the papers, and we provide the mathematical terminology needed for this presentation.

The main topic of this thesis is the study of various characteristics of reflected stochastic processes, in particular Lévy processes. Apart from its intrinsic mathematical interest, the study of reflected stochastic processes is motivated, by the fact that they arise naturally in mathematical models of real-life phenomena, in particular in queueing theory. One of the simplest set-ups in queueing theory concerns customers arriving at random times to a server, and upon arrival, presents the server with jobs of random length. We assume the server handles the requests one at a time. An obvious quantity of interest in such a system, is the *workload*, which is the amount of time needed for the server to clear the system, provided no new customers arrive. This quantity is also denoted the *virtual waiting time* as it represents the time needed to initiate service of a hypothetical customer arriving at time t. As we shall see, one sometimes imposes the requirement that the workload is restricted to be less than some $K > 0$. In this situation, it is natural to think of a buffer of size K, and that the work which exceeds the buffer size is in some sense lost. Measuring the size of, and approximating, this loss is the focal point of two of the included papers **Paper B** and **Paper C**. We refer to the case where we have no restrictions on the workload as the case of an infinite buffer.
The first paper, **Paper A**, examines the mean value and variance of reflected processes, which in the context of queueing theory tells us something about how the workload builds up over time. It is proved that the mean value of the workload is increasing and concave (its rate of increase is decreasing) both in the case finite and infinite buffer. These facts are fairly obvious in the infi-

nite buffer case, but not so obvious in the case of a finite buffer, where one could feasibly imagine the mean workload "overshoot" the stable mean, which would lead to a non-increasing function. In **Paper A** we also prove that the variance of the workload is increasing in the case of an infinite buffer, a fact which is also somewhat surprising as one could imagine the variance could peak in some finite time.

The papers **Paper B** and **Paper C** are dedicated to the study of the so-called loss rate. The loss rate is a measure of the amount of work lost in a finite buffer system. The background for both papers is the paper Asmussen and Pihlsgård [7], in which the loss rate is expressed in terms, which are easily interpretable from a modeling point of view. The expression derived in [7] is still somewhat inaccessible from a practical point of view, and we derive asymptotics, which are approximate expressions, for different cases, which are not covered in the original paper. These derivations takes us through various results, which are of independent interest.

The paper **Paper D** concerns an extreme value problem, which is derived from a parallel computing set-up in which we assume that jobs can fail, and have to be restarted. This study expands upon the work initiated in Asmussen et al. [9], where the set-up was examined in case of single processor. **Paper D** deals with the case where we imagine the job is distributed to multiple processors working in parallel, and we examine the mathematical implications of such a distribution. We quickly see that this leads us to a parameterized set-up, where some values of the parameters correspond to classical extreme value theory, while other values takes us beyond.

1 Reflected stochastic processes

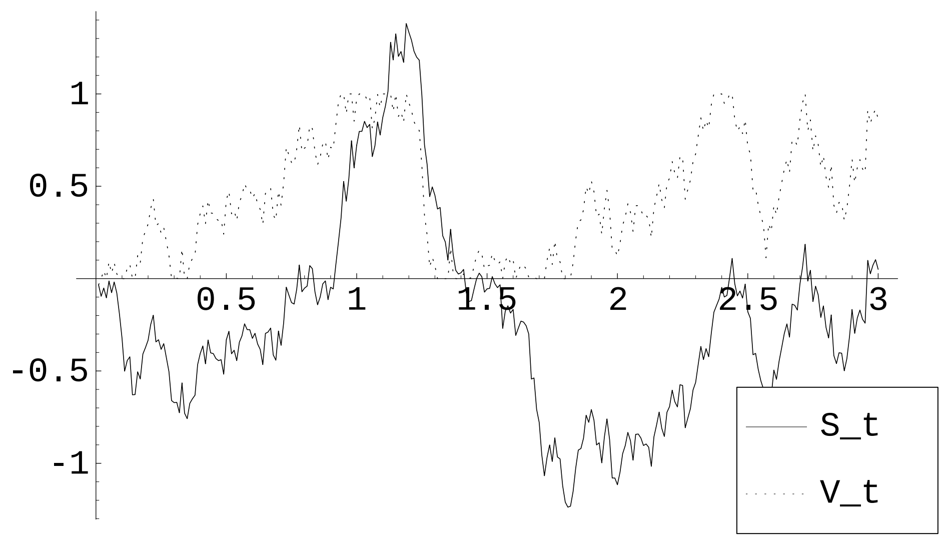

Figure 1: A sample path of a Brownian motion, and its reflected version.

We consider a stochastic process $\boldsymbol{S} = \{S_t\}_{t \in \mathbb{T}}$, and the cases of discrete time ($\mathbb{T} = \mathbb{N}_+$) and continuous time ($\mathbb{T} = [0, T]$ or $[0, \infty)$). The reflected version (at 0 and $K > 0$) of $\boldsymbol{S}$ is denoted $\boldsymbol{V} = \{V_t\}_{t \in \mathbb{T}}$. In the discrete-time case $\boldsymbol{V}$ is obtained through the recursion

$$V_{n+1} = 0 \vee [V_n + \Delta S_n] \wedge K \qquad K \in (0, \infty] \tag{1.1}$$

with initial value $V_0 \in [0, K]$ and $\Delta S_n = S_n - S_{n-1}$. We use the standard notation $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. We note in passing that "reflected" is somewhat of a misnomer, and a better term would be "constrained". However, we adhere to the standard terminology. In the case of one-sided reflection, i.e. $K = \infty$, the recursion (1.1) is often referred to as the "Lindley recursion", and in analogy with this, we refer to the case where $K < \infty$ as the "two-sided Lindley recursion".

In continuous time there are different but equivalent approaches. The reflected process may be defined as part of a Skorokhod Problem or it may be defined as a path transformation as defined in section 6 in **Paper A**. We elaborate on these approaches in section 2. The approaches are equivalent, and lead to a decomposition

$$V_t = y + S_t + L_t^0 - L_t^K \tag{1.2}$$

of the reflected process started at $y \in [0, K]$ where $\{L_t^0\}$ and $\{L_t^K\}$ are the local times at 0, K respectively ($L_t^K \equiv 0$ when $K = \infty$). In the enclosed papers, the process $\boldsymbol{S}$ will always a random walk or a Lévy process, unless explicitly stated. In this case, because of the regenerative structure of the reflected process, there exists a stationary distribution which satisfies

$$\overline{\pi}_K(y) = \pi_K[y, K] = \mathbb{P}\big(S_{\tau[y-K,y)} \geq y\big), \quad 0 \leq y \leq K \tag{1.3}$$

where $\tau[u, v) = \inf\big\{t > 0 \mid S_t \notin [u, v)\big\}$.
When $K = \infty$ we make assumptions which ensure $S_\infty := \lim_{t\to\infty} S_t = -\infty$, and (1.3) still holds in the sense

$$\overline{\pi}_\infty(y) = \mathbb{P}\big(\sup_{t\geq 0} S_t \geq y\big) = \mathbb{P}(\tau(y) < \infty) \tag{1.4}$$

where $\tau(y) = \inf\{t > 0 : S_t \geq y\}$.
The *loss rate* is defined as

$$\ell^K = \mathbb{E}_{\pi_K} L_1^K \tag{1.5}$$

where $\mathbb{E}_{\pi_K}$ refers to the stationary situation.

Lévy processes

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A *Lévy process* $\{S_t\}$ is a stochastic process on $\mathbb{R}$ with stationary independent increments which is continuous in probability with $S_0 = 0$ *a.s.* Every Lévy process $\{S_t\}_{t\geq 0}$ is associated with a unique *characteristic triplet* (θ, σ, ν), where $\theta \in \mathbb{R}$, $\sigma \geq 0$ and ν is a measure, the *Lévy measure*, on $\mathbb{R}$, which satisfies $\int_{-\infty}^{\infty}(1\wedge y^2)\nu(\mathrm{d}y) < \infty$ and $\nu(\{0\}) = 0$. The *Lévy exponent* is given by

$$\kappa(\alpha) = \theta\alpha + \frac{\sigma^2\alpha^2}{2} + \int_{-\infty}^{\infty} \big[e^{\alpha x} - 1 - \alpha I(|x| \leq 1)\big]\,\nu(\mathrm{d}x)$$

and is defined for α in $\Theta := \{\alpha \in \mathbb{C} \mid \mathbb{E}e^{\Re(\alpha)S_1} < \infty\}$. The Lévy exponent is the unique function κ satisfying $\mathbb{E}\mathrm{e}^{\alpha X_t} = \mathrm{e}^{t\kappa(\alpha)}$ and $\kappa(0) = 0$. One is often interested in Lévy processes which have no negative jumps, since many things simplify in this case. In terms of the Lévy measure, this is the requirement that $\nu((-\infty, 0]) = 0$, and we refer to such processes as *spectrally positive*.

Spectral positivity implies $(-\infty, 0] \subseteq \Theta$, and in this case we prefer to work with the *Laplace exponent*, defined by $\varphi(\alpha) = \kappa(-\alpha)$ for all real α such that $\mathbb{E}\mathrm{e}^{-\alpha S_1} < \infty$, which by the previous remark includes $[0, \infty)$. The function $\varphi(\cdot)$ is increasing on $[0, \infty)$ and hence its inverse, which we denote $\psi(\cdot)$ is well-defined.

Heavy tails, subexponentiality and integrated tails

We follow the standard definitions of the classes $\mathcal{L}, \mathcal{S}$ and $\mathcal{S}^*$ of distributions, that is, if B is a distribution on $[0, \infty)$ we have $B \in \mathcal{L}$ iff

$$\lim_{x\to\infty} \frac{\overline{B}(x+y)}{\overline{B}(x)} = 1, \qquad \text{for all } y$$

where $\overline{B}(x) = 1 - B(x)$. The class $\mathcal{S}$ is defined by the requirement

$$\lim_{x\to\infty} \frac{\overline{B^{*n}}(x)}{\overline{B}(x)} = n \qquad n = 2, 3, \cdots,$$

where B^{*n} denotes the nth convolution power of B. A subclass of $\mathcal{S}$ is $\mathcal{S}^*$, where we require that the mean μ_B of B is finite and

$$\lim_{x\to\infty} \int_0^x \frac{\overline{B}(x-y)}{\overline{B}(x)} \overline{B}(y)\mathrm{d}y = 2\mu_B.$$

The classes are related by:

$$\mathcal{S}^* \subseteq \mathcal{S} \subseteq \mathcal{L}.$$

We have the following definitions: For a random variable X with finite negative mean $\mathbb{E}X = \mu < \infty$, we set $\mathbb{E}X^+ = \mu^+$, $\overline{F}(x) = \mathbb{P}(X > x)$ and note $\mu^+ = \int_0^\infty \mathbb{P}(X > t)\mathrm{d}t$. Therefore, the function

$$F_e(x) := \begin{cases} \frac{1}{\mu^+} \int_0^x \mathbb{P}(X > t)\,\mathrm{d}t & x \geq 0 \\ 0 & x < 0 \end{cases}$$

defines a distribution function, which is absolutely continuous with density $\overline{F}(x)/\mu^+$. We frequently use the unnormed tail, and therefore define $\overline{F}_I := \mu^+ \overline{F}_e$ (the integrated tail). Note that

$$\mathbb{E}(X - x)^+ = \int_0^\infty \mathbb{P}((X - x)^+ > t)\,\mathrm{d}t = \int_x^\infty \mathbb{P}(X > t)\,\mathrm{d}t = F_I(x) \qquad (1.6)$$

and by l'Hospital: $\overline{F}(x) \sim \overline{F}^*(x) \Rightarrow \overline{F}_I(x) \sim \overline{F}^*_I(x)$ for distribution functions F and F^*.
The notion of heavy-tailedness carries over to Lévy processes through Theorem 1 in Embrechts et al. [16], which states states that if we assume ν is tail equivalent to a subexponential distribution, that is $\overline{\nu}(x) := \int_x^\infty \nu(\mathrm{d}y) \sim \overline{B}(x)$ for $B \in \mathcal{S}$, then

$$\overline{F}(x) \sim \overline{\nu}(x) \tag{1.7}$$

where $\overline{F}(x) := \mathbb{P}(S_1 > x)$. The main virtue of random walks with heavy-tailed increments or Lévy processes with heavy-tailed Lévy measure is that we have an asymptotic relationship for the tail of the overall supremum of the process, which, because of(1.4) gives us an asymptotic relation for the stationary distribution in the case of one-sided reflection. Using (1.4),(1.7) and applying Theorem 4.1 from Maulik and Zwart [41] we have

$$\overline{\nu}_I(K) := \int_K^\infty \overline{\nu}(y)\,\mathrm{d}y \sim |\mathbb{E}S_1|\overline{\pi}_\infty(K) \tag{1.8}$$

for Lévy processes if the integrated tail is subexponential, and by Theorem 9.1 p. 296 in [5]

$$\overline{F}_I(K) \sim |\mathbb{E}S_1|\overline{\pi}_\infty(K) \tag{1.9}$$

for random walks if $F_e \in \mathcal{S}$.

The rest of the introduction gives addendum and elaboration on the papers in the thesis.

Notes

The Lindley recursion appears in a queueing-theory setting in Lindley [39], and a recursion similar to the two-sided Lindley recursion appears in Daley [14] in a queueing theory setting. This recursion also appears in connection with finite capacity dam models, see Moran [42]. The two-sided Lindley recursion also appear in Phatarfod et al. [43].
The representation (1.3) is implicit in the discussion by Lindley [38] of a paper by C.B.Winsten (Winsten [55]), it appears explicitly in Ghosal [25], and in the generality needed for the thesis in Siegmund [50].
The literature on Lévy process is vast. Standard references include Sato [49], Bertoin [11] and Kyprianou [36]. The fact that spectral positivity implies $(-\infty, 0] \subseteq \Theta$ is mentioned in Example 25.11 in Sato [49], and that $\varphi(\cdot)$ is

increasing on $[0, \infty)$ is found in Bertoin [11] chap. VII.
The definitions of the classes of heavy-tailed random variables above are also standard, and are found in Asmussen [5],Asmussen [6] and Klüppelberg [33].

2 Structural properties of reflected Lévy processes

In **Paper A** we prove various structural properties of the functions $t \mapsto \mathbb{E}V_t$ and $t \mapsto \mathbb{V}\mathrm{ar}V_t$. Specifically, it is proved that the mean value function is increasing and concave, both in the case of one- and two-sided reflection, and that the variance function is increasing in the case of one-sided reflection. For two-sided reflection, the proof relies on a new representation of the two-sided reflected process, which is of independent interest. Structural properties of this kind were studied in Kella [29] and Kella and Sverchkov [31] in the case of one-sided reflection. In [29] the author assumes the Lévy process is spectrally positive and proves concavity by examining properties of the Laplace transform, which is particularly simple in this case. In [31] the authors prove the same result but in much greater generality, since they only assume that the involved processes are right-continuous and have stationary increments. The approach in [31] is based on explicit representations of the reflected process in terms of the original process. Both approaches are used in **Paper A** and similarly to the papers [29], [31], we see that we can obtain the most general results by using an approach based on an explicit representation. The approach using the Laplace transform is included, because it has a potential to be applicable in cases where explicit representations are not useful. The Laplace transform approach uses the concept of *complete monotonicity*. Complete monotonicity is defined in Definition 3.1 in **Paper A**, and the main virtue of this class of functions, is that they are the functions which can act as Laplace transforms for positive random variables (if properly normalized). Hence, we can prove monotonicity properties of functions by proving that the Laplace transforms of their derivatives are completely monotone. This approach has been successfully applied in Es-Saghouani and Mandjes [20]. Various explicit expressions for V_t in terms of S_t have appeared in the literature, recently in Kruk et al. [34] with a slight simplification in Kruk et al. [35]. In [34] the authors give a detailed description of the connection between their derived expression and other expressions in the literature.

In the paper **Paper A**, we prove that the variance of a one-sided reflected process is increasing. The proof relies on the concept of *concordance*, which was introduced by Lehmann [37]. We note that, for the purpose of **Paper A**, we could also have referred to Theorem 2.1 in Esary et al. [21], which states that a finite set of independent random variables $X_1, X_2, \cdots, X_n$ is *associated*, that is

$$\operatorname{Cov}\left[f(X_1, X_2, \cdots, X_n), g(X_1, X_2, \cdots, X_n)\right] \geq 0$$

for every increasing (in each coordinate) function f and g. We note that a single variable is associated, that is, $\operatorname{Cov}[f(X), g(X)] \geq 0$ for every increasing function. This is proved in Hardy et al. [26].

The paper **Paper A** also concerns the subject of how one should define a two-sided reflected processes in continuous time. In much of the literature this is done as a solution to a Skorokhod problem. Given a cadlag process $\{S_t\}$ we say a triplet $(\{V_t\}, \{L_t^0\}, \{L_t^K\})$ of processes is the *solution to the Skorokhod problem on* $[0, K]$ if $V_t = S_t + L_t^0 - L_t^K \in [0, K]$ for all t and

$$\int_0^T V_t \, \mathrm{d}L_t^0 = 0 \quad \forall T \quad \text{and} \quad \int_0^T (K - V_t) \, \mathrm{d}L_t^K = 0 \quad \forall T\,.$$

That is $\{L_t^0\}$ can only increase when $V_t = 0$ and $\{L_t^K\}$ can only increase when $V_t = K$. A proof of the uniqueness of such a solution is provided in the appendix, and the existence was proved in Tanaka [54] for continuous $\{S_t\}$ and in Anulova and Liptser [4] for cadlag $\{S_t\}$. Various explicit expressions appear in the literature and in Theorem 6.2 in **Paper A** we provide the following new expression:

$$V_t := \sup_{s \in [0,t]} \left[(S_t - S_s) \wedge \inf_{u \in [s,t]} (K + S_t - S_u) \right]. \tag{2.1}$$

In the remainder of this section we give some explicit results, where one can see the structural properties proved in **Paper A**. Furthermore, we know that under suitable stability conditions $\mathbb{E}V_t$ will converge to $\mathbb{E}V_\infty$ and we present a result which measure the rate of this convergence. Finally, we give some examples showing that the proved structural properties fail to hold under more general conditions.

Explicit results

We start by giving explicit expressions for $\mathbb{E}V_t$ in the few cases where these are available. From **Paper A** we have the formula

$$\int_0^\infty e^{-\vartheta t} \mathbb{E}V_t \mathrm{d}t = -\frac{\varphi'(0)}{\vartheta^2} + \frac{1}{\vartheta \psi(\vartheta)}\,. \tag{2.2}$$

Using formula (2.2), we are able to do calculations in the case where $\{S_t\}$ is a Brownian motion with drift. In this case we have $\varphi(\alpha) = -\alpha\mu + \frac{1}{2}\alpha\sigma^2$, and $\psi(\vartheta) = \mu/\sigma + \sqrt{\mu^2 + 2\sigma^2\vartheta}/\sigma^2$.
According to the formula (2.2) we have:

$$\int_0^\infty e^{-\vartheta t}\mathbb{E}V_t \mathrm{d}t = \frac{\mu}{\vartheta^2} + \frac{1}{\vartheta\left(\frac{\mu}{\sigma} + \frac{\sqrt{\mu^2+2\sigma^2\vartheta}}{\sigma^2}\right)}. \tag{2.3}$$

Recall the definition of the error function: $\mathrm{Erf}(t) = (2/\pi)\int_0^t \mathrm{e}^{-u^2}\mathrm{d}u$. Define $\kappa = \mu^2/(2\sigma^2)$. The Laplace transform of the error function is given by

$$\int_0^\infty \mathrm{e}^{-\vartheta t}\mathrm{Erf}\left(\sqrt{t\kappa}\right)\mathrm{d}t = \frac{\sqrt{\kappa}}{\vartheta(\kappa+\vartheta)^{\frac{1}{2}}}, \tag{2.4}$$

and by differentiating w.r.t. ϑ above, we obtain

$$\int_0^\infty \mathrm{e}^{-\vartheta t} t\mathrm{Erf}\left(\sqrt{t\kappa}\right)\mathrm{d}t = \frac{\sqrt{\kappa}}{2\vartheta\left(\kappa+\vartheta\right)^{\frac{3}{2}}} + \frac{\sqrt{\kappa}}{\vartheta^2\left(\kappa+\vartheta\right)^{\frac{1}{2}}}.$$

Furthermore, we have:

$$\int_0^\infty \mathrm{e}^{-\vartheta t}\sqrt{\frac{2t}{\pi}}\mathrm{e}^{-t\kappa}\mathrm{d}t = \frac{1}{\sqrt{2}\left(\kappa+\vartheta\right)^{\frac{3}{2}}}.$$

Using the decomposition:

$$\begin{aligned}
&\frac{\mu}{\vartheta^2} + \frac{1}{\vartheta\left(\frac{\mu}{\sigma} + \frac{\sqrt{\mu^2+2\sigma^2\vartheta}}{\sigma^2}\right)} \\
&= \frac{1}{2}\Big(\frac{\mu}{\vartheta^2} + \sigma\frac{1}{\sqrt{2}\left(\kappa+\vartheta\right)^{\frac{3}{2}}} - \mu\Big(\frac{\sqrt{\kappa}}{2\vartheta\left(\kappa+\vartheta\right)^{\frac{3}{2}}} + \frac{\sqrt{\kappa}}{\vartheta^2\left(\kappa+\vartheta\right)^{\frac{1}{2}}}\Big) \\
&- \frac{\sigma^2}{\mu}\frac{\sqrt{\kappa}}{\vartheta\sqrt{\kappa+\vartheta}}\Big),
\end{aligned}$$

we may invert the Laplace transform in (2.3) to obtain:

$$\begin{aligned}
\mathbb{E}V_t &= \frac{1}{2}\left(\mu t + \sigma\sqrt{\frac{2t}{\pi}}\mathrm{e}^{-t\kappa} - \mu t\mathrm{Erf}\left(\sqrt{t\kappa}\right) - \frac{\sigma^2}{\mu}\mathrm{Erf}\left(\sqrt{t\kappa}\right)\right) \\
&= \mu t + \frac{\sigma^2}{2\mu} + \frac{\sigma}{2}\sqrt{\frac{2t}{\pi}}\mathrm{e}^{-t\frac{\mu^2}{2\sigma^2}} - \left(t\mu + \frac{\sigma^2}{\mu}\right)\Phi\Big(\frac{\sqrt{t}|\mu|}{\sigma}\Big),
\end{aligned} \tag{2.5}$$

where $\Phi(\cdot)$ is the c.d.f. of the standard normal distribution.
Another case in which in which we can do explicit calculations is the compound Poisson case with exponential jumps. Consider a Lévy process $\{S_t\}$ with Laplace exponent

$$\varphi(\alpha) = \delta\left(\frac{\beta}{\beta+\alpha} - 1\right) + \alpha \tag{2.6}$$

and its reflected version $\{V_t\}$. We note that $\{V_t\}$ is the workload process in an $M/M/1$ queue with service intensity β and arrival intensity δ. We may invert (2.6) to obtain

$$\psi(\vartheta) = \frac{1}{2}\left(\vartheta - \beta + \delta + \sqrt{4\vartheta\beta + (-\vartheta+\beta-\delta)^2}\right), \tag{2.7}$$

and if we define $\xi(\vartheta) = \frac{1}{2\beta}\left(\vartheta + \delta + \beta - \sqrt{(\vartheta+\beta+\delta)^2 - 4\delta\beta}\right)$, we may write

$$\begin{aligned}\frac{1}{\psi(\vartheta)} &= \frac{1}{\frac{1}{2}\left(\vartheta - \beta + \delta + \sqrt{4\vartheta\beta + (-\vartheta+\beta-\delta)^2}\right)} \\ &= \frac{1 - \frac{1}{2\beta}\left(\vartheta + \delta + \beta - \sqrt{(\vartheta+\beta+\delta)^2 - 4\delta\beta}\right)}{\vartheta} \\ &= \frac{1-\xi(\vartheta)}{\vartheta}. \end{aligned} \tag{2.8}$$

Next, we note that by Proposition 8.10 p. 105 in Asmussen [5], or Theorem 7 in Takács [52], $\xi(\vartheta)$ is the Laplace transform of the busy period of an $M/M/1$ queue with with arrival intensity β and service intensity δ, i.e. with the roles of the parameters reversed compared to above. Furthermore, according to Corollary 8.7 p. 103 in [5] the density of G is

$$g(t) = \delta e^{-(\beta+\delta)t}[I_0(2\mu t) - I_2(2\mu t)] = \frac{\rho^{-1}}{t} e^{-(\delta+\beta)t} I_1(2\mu t)$$

where $\mu = \sqrt{\beta\delta}$ and $\rho = \beta/\delta$ and

$$I_n(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!(n+k)!}$$

is the *modified Bessel function of integer of integer order* n. Using Feller [22] p. 412 we recognize (2.8) as the Laplace Transform of $\overline{G}(x) := \int_x^\infty g(t)\mathrm{d}t$. We also know from Abate and Whitt [1] equation (34) that $1/\psi(\vartheta)$ is the Laplace

Transform of $t \mapsto P(V_t = 0 \mid V_0 = 0)$, which according to **Paper A** is equal to the derivative of $\mathbb{E}V_t$. By combining these facts, we obtain:

$$\mathbb{E}V_t = \int_0^t \int_x^\infty \frac{\rho^{-1}}{y} e^{-(\delta+\beta)y} I_1(2\mu y)\mathrm{d}y\mathrm{d}x\,. \tag{2.9}$$

Figure 2 displays some mean value functions. Let $\{V_t^1\}$ be the reflected version of a Lévy process with Laplace exponent

$$\varphi(\alpha) = \left(\frac{2}{2+\alpha} - 1\right) + \alpha + \frac{\alpha^2}{2}\,,$$

that is, a compound Poisson process with an added independent Brownian term. Furthermore we let $\{V_t^2\}$ be the reflected version of the Brownian motion with unit drift and variance, and let $\{V_t^3\}$ be the reflected version of the compound Poisson process from (2.6) with parameters $\beta = 2$ and $\delta = 1$.

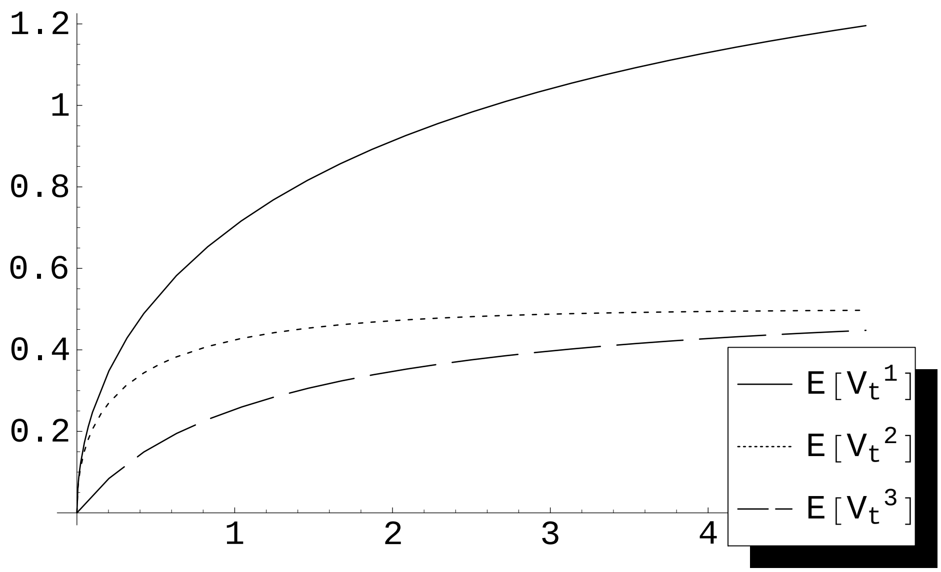

Figure 2: 3 mean value functions.

We note how the addition of a Brownian term manifests itself as an infinite derivative at 0.
Instead of using the Laplace transforms as above, one could derive the expression for the mean value function in the Brownian case, by exploiting the

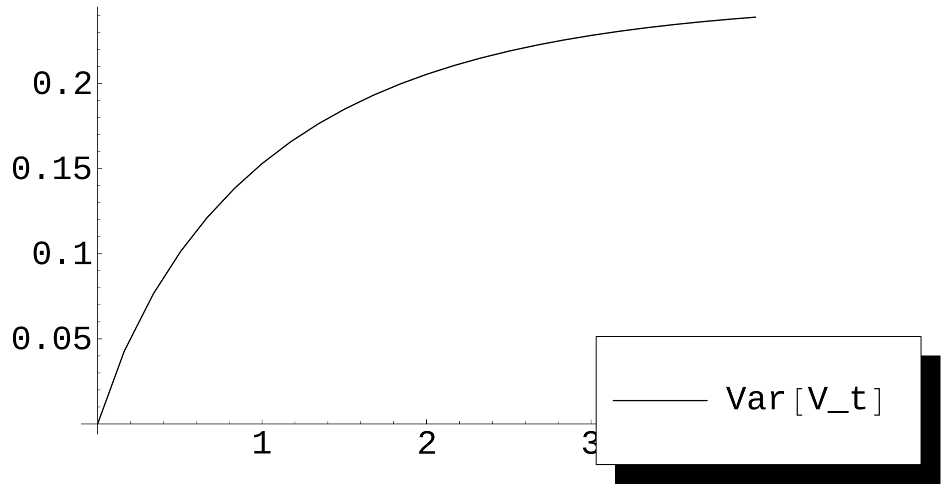

Figure 3: Plot of $t \mapsto \mathbb{V}\mathrm{ar}[V_t]$

fact the distribution of the supremum of a Brownian motion with drift is well-known. In fact:

$$\mathbb{P}(\sup_{s\in[0,t]} B_s - s \geq y) = 1 - \Phi(\frac{y}{\sqrt{t}} + \sqrt{t}) + \mathrm{e}^{-2y}\Phi(-\frac{y}{\sqrt{t}} + \sqrt{t}) \,, \qquad (2.10)$$

where $\{B_t\}$ is a standard Brownian motion. From Proposition 3 Chap. VI in Bertoin [11], we know that $\sup_{s\in[0,t]} B_s - s =_{\mathscr{D}} V_t$ where V_t is the reflection of $\{B_s - s\}$ started at 0. We can differentiate the above expression to obtain the density of V_t:

$$v(t) = \frac{\sqrt{2}}{\sqrt{\pi t}}\mathrm{e}^{-\frac{(y+t)^2}{2t}} + \mathrm{e}^{-2y}\Phi(\frac{-y+t}{\sqrt{t}}) \,, \qquad (2.11)$$

and then use the expression above to verify (2.5) in the case of unit mean and variance, and to obtain an expression for the variance of V_t. We abstain from writing down the somewhat complicated expression, and provide Figure 3 instead, which shows that the variance function is increasing as it is proved in **Paper A** Theorem 4.6.

Time-dependent results for two-sided reflection

As seen in the previous section, there are explicit results available in the case of one-sided reflection. With one particular exception, this is not the case for two-sided reflection. Not even Brownian motion, for which much is know in the case of one-sided reflection seems to be tangible. However, below we give a derivation of the k'th moment for a discrete-time two-sided reflected process. Since the two-sided reflection has compact support, this determines the distribution. Furthermore, we see a rate of convergence of the mean which, although still exponential, is slower than what we see in the case of one-sided reflection in the following section. This not surprising, since intuitively, the upper barrier "pushes down" the process.

Let $K > 0$ and let $X_1, X_2, \dots$ be an i.i.d. sequence of r.v.s which are uniform on $[a, b]$ for $a < -K$ and $K < b$. For $x \in [0, K]$ we have the following elementary calculation:

$$\begin{aligned}&\mathbb{E}[(0 \vee (x + X_1) \wedge K)^n] = \mathbb{E}[(x + X)^n, x + X_1 \in [0, K]] + K^n \mathbb{P}(X > K - x) \\ &= \int_{-x}^{K-x} (x + y)^n \frac{1}{b - a} \mathrm{d}y + K^n \frac{(b - (K - x))}{b - a} \\ &= \frac{K^{n+1}}{(b - a)(n + 1)} + K^n \frac{(b - (K - x))}{b - a} \,,\end{aligned}$$

and thus, if we define $G_n(x) := \mathbb{E}[(0 \vee (x + X_1) \wedge K)^n]$ and $\alpha_n := K^n/(b - a)$ and $\beta_n := K^n(b + bn - Kn)/((b - a)(n + 1))$, we have

$$G_n(x) = \alpha_n x + \beta_n \,. \qquad 0 \leq x \leq K$$

Now, let $V_1, V_2, \dots$ be the two-sided reflected process started from 0. We find that

$$\mathbb{E}[V_{k+1}] = \mathbb{E}[0 \vee (V_k + X_{k+1}) \wedge K] = \mathbb{E}[G_1(V_k)] = \alpha_1 \mathbb{E}[V_k] + \beta_1 \,.$$

Iterating the above equality, using $V_0 = 0$, we find:

$$\mathbb{E}[V_k] = \beta \sum_{i=1}^{k-1} \alpha^i = \beta_1 \frac{1 - \alpha_1^k}{1 - \alpha_1} \,.$$

Furthermore, using the above expression, we find:

$$\mathbb{E}[V_{k+1}^n] = \mathbb{E}[G_n(V_k)] = \alpha_n \mathbb{E}[V_k] + \beta_n = \alpha_n \beta_1 \frac{1 - \alpha_1^k}{1 - \alpha_1} + \beta_n \,. \tag{2.12}$$

Rate of convergence

In this section we examine the rate of convergence of the mean value at time t towards the mean value in stationarity. Specifically, we consider the continuous-time case, as the discrete-time case is already covered by Theorem 2.2 p. 356 in Asmussen [5]. From **Paper A** we know

$$\lim_{t\to\infty} \mathbb{E}V_t \uparrow \mathbb{E}V_\infty$$

and we want to asses the rate of this convergence. In the light-tailed spectrally positive case, we can do this by using the Heaviside Operational Principle (see Abate and Whitt [2] and Doetsch [15] p. 254) . Using (2.2) and the formula $\mathbb{E}V_\infty = -\varphi''(0)/(2\varphi'(0))$, we see that the Laplace transform of $\mathbb{E}V_\infty - \mathbb{E}V_t$ is

$$-\frac{\varphi''(0)}{2\varphi'(0)\vartheta} + \frac{\varphi'(0)}{\vartheta^2} - \frac{1}{\vartheta\psi(\vartheta)}.$$

We assume $\varphi(\alpha)$ is the Laplace exponent of a light-tailed Lévy process meaning that the equation $\varphi(\alpha) = 0$ has a non-zero solution. Since φ is strictly convex, this implies that the minimum of $\varphi(\alpha)$ is attained between this solution and 0. Define γ_0 to be solution of $\varphi'(\gamma_0) = 0$ and set $\theta_* = -\varphi(\gamma_0)$ and $\delta = \varphi''(\gamma_0)/2$. We may expand φ in its Taylor series around γ_0 by writing $\varphi(\alpha) = -\theta_* + \delta(\alpha-\gamma_0)^2 + O((\alpha-\gamma_0)^3)$, and hence we can expand $\psi(\vartheta) = \gamma_0 + \delta^{-1/2}\sqrt{(\vartheta+\theta_*)} + O(\vartheta+\theta_*)$. The Heaviside Operational Principle relies on expanding the Laplace Transform around its rightmost singularity, which in this case is $-\theta_*$. Using the expansion above, we obtain:

$$\begin{aligned}
&\int_0^\infty e^{-\vartheta t}(\mathbb{E}V_\infty - \mathbb{E}V_t)\mathrm{d}t \\
&= -\frac{\varphi''(0)}{2\varphi'(0)\theta_*} + \frac{\varphi'(0)}{\theta_*^2} - \frac{1}{\theta_*(\gamma_0 + \delta^{-1/2}\sqrt{(\vartheta+\theta_*)}) + O(\vartheta+\theta_*)} \\
&= -\frac{\varphi''(0)}{2\varphi'(0)\theta_*} + \frac{\varphi'(0)}{\theta_*^2} - \frac{\gamma_0 - \delta^{-1/2}\sqrt{(\vartheta+\theta_*)}}{\theta_*\gamma_0^2} + o(\vartheta+\theta_*)\,,
\end{aligned}$$

and we may apply the Heaviside Theorem to obtain

$$\mathbb{E}V_\infty - \mathbb{E}V_t \sim -\frac{\mathrm{e}^{-\theta_* t}}{t^{\frac{3}{2}}\sqrt{\delta}\theta_*\gamma_0^2\Gamma(-\frac{1}{2})} = \frac{\mathrm{e}^{-\theta_* t}}{t^{\frac{3}{2}}\theta_*\gamma_0^2\sqrt{\delta 2\pi}}\,. \tag{2.13}$$

If we compare the asymptotics above to those of Theorem 2.2 Asmussen [5] p. 356 we see very similar rates of convergence - slightly faster than exponential - but, for obvious reasons, different constants.

Counterexamples to structural properties

Paper A deals only with reflected processes started from 0. It is relevant to investigate if it is possible to prove structural properties when the processes are started from $x > 0$. Obviously the processes cannot be increasing for large enough x, and concavity cannot hold either. The following example shows that for general starting points, the mean value functions can display quite irregular behavior. First we note, that the formula (2.2) is easily extended to a general starting point $x \geq 0$ using Thm. IX.3.10 Asmussen [5]:

$$\int_0^\infty e^{-\vartheta t}\mathbb{E}_x V_t \mathrm{d}t = \frac{-\varphi'(0)}{\vartheta^2} + \frac{x}{\vartheta} + \frac{\mathrm{e}^{-\psi(\vartheta)x}}{\vartheta\psi(\vartheta)}\,. \tag{2.14}$$

We notice that the argument on page 43 in **Paper A** is equally valid for a general starting point $x \geq 0$, and in this case implies the formula

$$\mathbb{E}_x V_t = -\varphi'(0)t + x + \int_0^t \mathbb{P}(V_s = 0 \mid V_0 = x)\mathrm{d}s\,.$$

Hence, any irregular behavior of $t \mapsto \mathbb{P}(V_t = 0 \mid V_0 = x)$, will be carried over to $t \mapsto \mathbb{E}_x V_t$. A particular example of such irregular behavior arises when one considers the workload process of an $M/D/1$ queue with arrival rate 1 and deterministic job size equal to 1/2 started at, say, 1/2. Then $\mathbb{P}(V_t = 0 \mid V_0 = 1/2) = 0$ for $t < 1/2$ and $\mathbb{P}(V_{1/2} = 0 \mid V_0 = 1/2) = 1 - \mathrm{e}^{-1}$. By the remarks above, this implies that $t \mapsto \mathbb{E}[V_t]$ is not differentiable at 1/2 and as can be seen in Figure 4 below, $t \mapsto \mathbb{E}[V_t]$ has several local minima. Note that the apparent jumps are due to numerical issues.

Finally, we note that while we prove in **Paper A** that the variance function is increasing in the case of one-sided reflection, this cannot be the case in general for two-sided reflection. As a counterexample, one can take a Poisson process with positive drift reflected at 0 and $K > 0$ and started a 0. Since this an increasing process, the lower barrier will not play a role, and since the process will eventually get stuck at the upper barrier, its variance will converge to 0. However since variance at time 0 is 0 and the variance is strictly positive for some $t > 0$, the variance function cannot be increasing. This is of course a somewhat pathological counterexample, and it appears reasonable to conjecture that the variance function is increasing provided the mean of the original process is negative.

Notes

The formula (2.10) is found Chapter 12 in Mandjes [40], and (2.4) is formula 16.2.1 in Roberts and Kaufman [47]. The formula (2.1) is new, but a discrete-

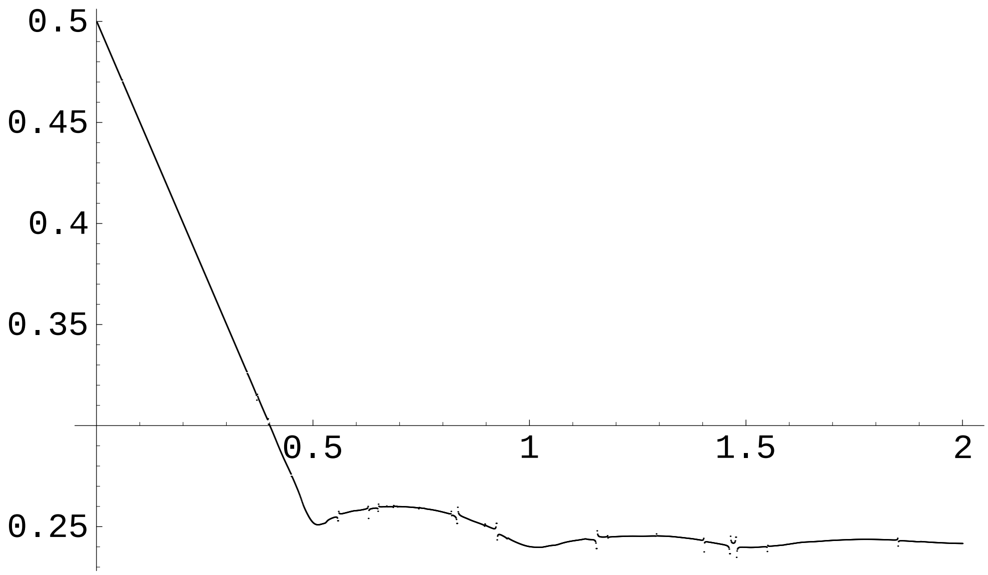

Figure 4: Plot of $t \mapsto \mathbb{E}[V_t]$, found by numerical inversion of the Laplace transform.

time version appears in Phatarfod et al. [43], which however, has been overlooked in the literature. Explicit expressions for the reflected process of the type (2.1) have appeared in the literature, see Borovkov [12], Cooper et al. [13], Kruk et al. [34] and Kruk et al. [35].

Formulas describing transient characteristics of $M/M/1$ queues tend to be complicated, involving for example infinite sums of modified Bessel functions (see e.g. Prabhu [46]). Sometimes integral representations are available, like the formula given for the mean queue length in Takács [52]. In view of this, the formula (2.9) is relatively simple. A moment based approach to the workload process is given in Abate and Whitt [1]. See also Abate and Whitt [3] for an overview of calculations of such characteristics.

3 Loss rate asymptotics

In the papers **Paper B** and **Paper C** we derive various asymptotics for the loss rate for two-sided reflected Lévy processes. At the heart of both derivations lies the expression for the loss rate derived in Asmussen and Pihlsgård

[7]:

$$\ell^K = \frac{\mathbb{E}S_1}{K}\int_0^K \overline{\pi}_K(x)\,\mathrm{d}x + \frac{\sigma^2}{2K} + \frac{1}{2K}\int_0^K \pi_K(\mathrm{d}x)\int_{-\infty}^{\infty}\varphi_K(x,y)\nu(\mathrm{d}y), \quad (3.1)$$

where

$$\varphi_K(x,y) = \begin{cases} -(x^2+2xy) & \textit{if } y \le -x \\ y^2 & \textit{if } -x < y < K-x \\ 2y(K-x)-(K-x)^2 & \textit{if } y \ge K-x\,. \end{cases} \quad (3.2)$$

The expression (3.1) is derived in Asmussen and Pihlsgård [7] by using Theorem 3.1 p. 255 in Asmussen [5] to establish the local martingale property of a certain stochastic process. The useful result in Theorem 3.1 was originally found in Kella and Whitt [32], and is also used in Proposition 4.3 in **Paper B** to establish a local martingale property. We note that [32] also plays a fundamental role in the paper Kella [30], which in turn plays an important role as background for **Paper A** and thereby establishes a connection between this paper and **Paper B** and **Paper C**.
Asymptotics of the loss rate were derived Jelenković [27] in the discrete-time case, and we provide a new proof of this result, where we exploit the representation (1.3). The discrete-time result provide a clue towards what to expect in the continuous-time case. In this section we collect the comments to the papers **Paper B** and **Paper C**, as both papers concern asymptotics of the same object, namely the loss rate of a Lévy process reflected in two barriers.

The Scale function

When the involved Lévy processes are spectrally positive, one may apply Theorem 8 Chap. VII of Bertoin [11] to obtain the stationary distribution, since the referred theorem may be restated as

Theorem 3.1. *Let $\{S_t\}$ be a Lévy process with no negative jumps and characteristic exponent $\kappa(\alpha)$. Then*

$$\overline{\pi}_K(y) = 1 - \frac{W(y)}{W(K)}$$

Where $W[0,\infty) \to W[0,\infty)$ is the unique continuous increasing function satisfying

$$\int_0^\infty e^{-\alpha x} W(x)\,\mathrm{d}x = \frac{1}{\kappa(-\alpha)}$$

The function $W(\cdot)$ is called the *scale function*.
We proceed to prove an interesting consequence of Theorem 8 in [11]. We consider a spectrally positive Lévy process $\{S_t\}$ with finite negative mean. Let V have the stationary distribution of the one-sided reflected version of $\{S_t\}$ and let V^K have the two-sided stationary distribution. Recall that $\tau[u,v) = \inf\{t > 0 \mid S_t \notin [u,v)\}$ and set $\tau(y-K,y] := \inf\{t > 0 \mid S_t \notin (u,v]\}$. We notice that

$$\begin{aligned}\mathbb{P}(V^K \leq y) &= \lim_{\epsilon \downarrow 0} = \mathbb{P}(V^K < y + \epsilon) \\ &= 1 - \lim_{\epsilon \downarrow 0} \mathbb{P}(S_{\tau[y-K+\epsilon, y+\epsilon)} \geq y + \epsilon) = 1 - \mathbb{P}(S_{\tau(y-K,y]} > y)\,. \end{aligned} \tag{3.3}$$

We know from Corollary 2.8 in [5] that $\mathbb{P}(V \geq x) = \mathbb{P}(\sup_{t \geq 0} S_t \geq x)$. Let $\tilde{\tau}[u,v) = \inf\{t > 0 \mid -S_t \notin [u,v)\}$, and let $\tilde{W}(\cdot)$ denote the scale function of $\{-S_t\}$. According to the proof of Theorem 8 Chap. VII in [11] we have $\mathbb{P}(-\inf_{t>0}(-S_t) \leq x) = c\tilde{W}(x)$ for some $c > 0$. Using the representation (1.3) and (3.3) we find:

$$\begin{aligned}\mathbb{P}(V^K \leq y) &= \mathbb{P}(S_{\tau(y-K,y]} \leq y - K) = \mathbb{P}(-S_{\tilde{\tau}[-y,K-y)} \geq K - y) \\ &= \frac{\tilde{W}(y)}{\tilde{W}(K)} = \frac{\mathbb{P}(-\inf_{t>0}(-S_t) \leq y)}{\mathbb{P}(-\inf_{t>0}(-S_t) \leq K)} \\ &= \frac{\mathbb{P}(V \leq y)}{\mathbb{P}(V \leq K)} = \mathbb{P}(V \leq y \mid V \leq K)\,.\end{aligned}$$

That is: In the spectrally positive case, we can obtain the stationary distribution of the two-sided reflected process, by conditioning the one-sided stationary distribution to be below the upper barrier.

Explicit examples

In this section, we calculate the stationary distribution and loss rate for some stochastic processes. When we are given a Lévy process $\{S_t\}$ with characteristic exponent $\kappa(\alpha)$ and characteristic triplet (θ, σ, ν), the first task is to compute the stationary distribution, which, because of (1.3) is a two-sided exit problem. Once the stationary distribution is obtained, the loss rate can in principle be calculated using (3.1). In practice, however, it is typically easier to use the remark in Asmussen and Pihlsgård [7], that if the continuous part of the local time at 0 or K disappears, then ℓ^K is available as the solution to some linear equations.
For spectrally one-sided processes, we can apply Theorem 3.1. When we have

both negative and positive jumps we apply optional stopping of the Wald martingale $\{Y_t\} = \left\{ \mathrm{e}^{\alpha S_t - t\kappa(\alpha)} \right\}$. Specifically, we find non-zero solutions to $\kappa(\alpha) = 0$ and plug these into the equation $1 = \mathbb{E}\mathrm{e}^{\alpha S_\tau - \tau\kappa(\alpha)}$ where τ is an optional stopping time. We note that in **Example 4** and **Example 5** we need to analytically extend the derived equations, from a strip in the complex plane, to the entire complex plane except singularities.

Example 1

Let us consider a compound Poisson process with exponential Lévy measure:

$$S_t := \sum_{i=0}^{N_t} U_i - \beta t.$$

Where (U_i) is an i.i.d. sequence, with U_1 having an exponential distribution with parameter γ, and (N_t) is an independent Poisson process with intensity λ. The parameters satisfy $\beta > \lambda/\gamma$.

We have

$$\kappa(\alpha) = \frac{\lambda\alpha}{\gamma - \alpha} - \beta\alpha.$$

Setting $\xi := \gamma - \lambda/\beta$, we have $\kappa(\xi) = 0$ and using optional stopping of the martingale $\left\{ e^{\xi S_t} \right\}$ with the stopping time $\tau[y-K, y)$, which is justified by Corollary 4.1 in [5], we have for $y > 0$:

$$\begin{aligned} 1 &= \mathbb{E}[e^{\xi S_{\tau[y-K,y)}}] \\ &= \overline{\pi}_K(y)\mathbb{E}[e^{\xi S_{\tau[y-K,y)}} \mid S_{\tau[y-K,y)} \geq y] + \pi_K(y)\mathbb{E}[e^{\xi S_{\tau[y-K,y)}} \mid S_{\tau[y-K,y)} < y] \end{aligned}$$

Since the process has no negative jumps, and $X =_{\mathscr{D}} E(\gamma) \Rightarrow X - y \mid X > y =_{\mathscr{D}} E(\gamma)$, we obtain

$$1 = \overline{\pi}_K(y) e^{\xi y} \frac{\gamma}{\gamma - \xi} + \pi_K(y) e^{\xi(y-K)}\,.$$

And since $\overline{\pi}_K(y) + \pi_K(y) = 1$, we may isolate $\overline{\pi}_K(y)$ and obtain:

$$\overline{\pi}_K(y) = \frac{e^{-\xi y} - e^{-\xi K}}{\frac{\gamma}{\gamma - \xi} - e^{-\xi K}} \qquad y > 0$$

or

$$1 - \overline{\pi}_K(y) = \frac{\frac{\gamma}{\gamma-\xi} - e^{-\xi y}}{\frac{\gamma}{\gamma-\xi} - e^{-\xi K}} \qquad y > 0\,.$$

In accordance with Theorem 3.1.
Using (3.1) or (3.4) in Asmussen and Pihlsgård [7] we obtain:

$$\ell^K = \frac{-\mathbb{E}S_1 e^{-\xi K}}{\frac{\gamma}{\gamma-\xi} - e^{-\xi K}} = \frac{\frac{\lambda\xi}{\mu} e^{\frac{\lambda}{\beta}K}}{\gamma e^{\gamma K} - \frac{\lambda}{\beta} e^{\frac{\lambda}{\beta}K}}.$$

If we let $\beta \to \lambda/\gamma$ the expression above tends to

$$\ell^K = \frac{\lambda}{\gamma^2}\frac{1}{\frac{1}{\gamma}+K} \sim_{K\to\infty} \frac{\lambda}{\gamma^2}\frac{1}{K} = \frac{1}{2K}\int_0^\infty y^2\lambda\gamma e^{-\gamma y}\,\mathrm{d}y$$

in accordance with Theorem 3.1 in **Paper C**.
Example 2
Next, we add an independent Brownian Motion:

$$S_t := \sigma B_t + \sum_{i=0}^{N_t} U_i - \beta t \qquad \sigma,\beta,\lambda,\gamma > 0, \beta > \frac{\lambda}{\gamma},$$

where (B_t) is a standard Brownian motion, which is independent of (U_i) and (N_t), where (U_i) and (N_t) are as in **Example 1**. Then

$$\kappa(\alpha) = \frac{\alpha^2\sigma^2}{2} + \frac{\lambda\alpha}{\gamma-\alpha} - \beta\alpha$$

The non-zero real solutions to the equation $\kappa(\alpha) = 0$ are

$$\begin{aligned}
\xi_1 &= \frac{2\beta + \gamma\sigma^2 + \sqrt{4\beta^2 - 4\beta\gamma\sigma^2 + 8\lambda\sigma^2 + \gamma^2\sigma^4}}{2\sigma^2} \\
&= \frac{\beta + \frac{\gamma\sigma^2}{2} + \sqrt{\left(\beta - \frac{\gamma\sigma^2}{2}\right)^2 + 2\lambda\sigma^2}}{\sigma^2} \\
\xi_2 &= \frac{2\beta + \gamma\sigma^2 - \sqrt{4\beta^2 - 4\beta\gamma\sigma^2 + 8\lambda\sigma^2 + \gamma^2\sigma^4}}{2\sigma^2} \\
&= \frac{\beta + \frac{\gamma\sigma^2}{2} - \sqrt{\left(\beta - \frac{\gamma\sigma^2}{2}\right)^2 + 2\lambda\sigma^2}}{\sigma^2}
\end{aligned}$$

and one can check that $\xi_1, \xi_2 > 0$. We note that we have the following curious formula for the mean of the one-sided reflected process:

$$\int_0^\infty y\pi_\infty(\mathrm{d}y) = -\frac{\mathbb{V}\mathrm{ar}(S_1)}{2\mathbb{E}S_1} = \frac{\sigma^2 + \frac{2\lambda}{\gamma^2}}{2\left(\beta - \frac{\lambda}{\gamma}\right)} = \frac{1}{\xi_1} + \frac{1}{\xi_2} - \frac{1}{\gamma}.$$

The stationary distribution and the loss rate are found to be:

$$1-\overline{\pi}_K(y)=\frac{\frac{\gamma}{\gamma-\xi_1}-\frac{\gamma}{\gamma-\xi_2}-\frac{\xi_1}{\gamma-\xi_1}e^{-\xi_2 y}+\frac{\xi_2}{\gamma-\xi_2}e^{-\xi_1 y}}{\frac{\gamma}{\gamma-\xi_1}-\frac{\gamma}{\gamma-\xi_2}-\frac{\xi_1}{\gamma-\xi_1}e^{-\xi_2 K}+\frac{\xi_2}{\gamma-\xi_2}e^{-\xi_1 K}},$$

$$\ell^K=\frac{-\mathbb{E}S_1\left(e^{-\xi_2 K}\frac{\xi_1}{\gamma-\xi_1}-e^{-\xi_1 K}\frac{\xi_2}{\gamma-\xi_2}\right)}{\frac{\gamma}{\gamma-\xi_1}-\frac{\gamma}{\gamma-\xi_2}-\frac{\xi_1}{\gamma-\xi_1}e^{-\xi_2 K}+\frac{\xi_2}{\gamma-\xi_2}e^{-\xi_1 K}}.$$

Example 3
Let (B_t) is a standard Brownian Motion, and we set

$$S_t:=\sigma B_t-\beta t\,.$$

Then $\xi:=2\beta/\sigma^2$ solves $\kappa(\alpha)=0$ and we have:

$$\overline{\pi}_K(y)=\frac{e^{-y\xi}}{1-e^{-\xi s}}\qquad\text{and}\qquad\ell^K=\frac{\beta e^{-\xi K}}{1-e^{-\xi K}}$$

Example 4
We consider a compound Poisson process with both positive and negative jumps:

$$S_t:=\sum_{i=0}^{N_t}U_i-\sum_{i=0}^{\tilde{N}_t}T_i-\beta t\quad U\overset{\mathcal{D}}{=}E(\gamma)\,,N_t\overset{\mathcal{D}}{=}\mathrm{po}(t\lambda)\,,T_i\overset{\mathcal{D}}{=}E(\theta)\,,\tilde{N}_t\overset{\mathcal{D}}{=}\mathrm{po}(t\xi)$$

where $\beta>\lambda/\gamma-\xi/\theta$. The Lévy exponent is

$$\kappa(\alpha)=\frac{\lambda\alpha}{\gamma-\alpha}-\frac{\xi\alpha}{\theta+\alpha}-\beta\alpha$$

and the solutions of $\kappa(\alpha)=0$ are

$$\xi_1=\frac{\beta\gamma-\beta\theta-\lambda-\xi+\sqrt{(\beta\gamma-\beta\theta-\lambda-\xi)^2+4\beta(\beta\gamma\theta-\theta\lambda+\gamma\xi)}}{2\beta}$$
$$\xi_2=\frac{\beta\gamma-\beta\theta-\lambda-\xi-\sqrt{(\beta\gamma-\beta\theta-\lambda-\xi)^2+4\beta(\beta\gamma\theta-\theta\lambda+\gamma\xi)}}{2\beta}$$

And we have, for $y>0$

$$\overline{\pi}_K(y)=\frac{\frac{\theta}{\theta+\xi_1}-\frac{\theta}{\theta+\xi_2}+e^{-\xi_2(y-K)}\frac{\xi_1}{\theta+\xi_1}-e^{-\xi_1(y-K)}\frac{\xi_2}{\theta+\xi_2}}{\frac{\theta}{\theta+\xi_1}-\frac{\theta}{\theta+\xi_2}+e^{\xi_2 K}\frac{\gamma}{\gamma-\xi_2}\frac{\xi_1}{\theta+\xi_1}-e^{\xi_1 K}\frac{\gamma}{\gamma-\xi_1}\frac{\xi_2}{\theta+\xi_2}}$$

and

$$\ell^K = \frac{\frac{\lambda}{\gamma}\frac{\xi_1}{\theta+\xi_1}\frac{\xi_2}{\gamma-\xi_2} - \frac{\lambda}{\gamma}\frac{\xi_2}{\theta+\xi_2}\frac{\xi_1}{\gamma-\xi_1}}{\frac{\theta}{\theta+\xi_1} - \frac{\theta}{\theta+\xi_2} + e^{\xi_2 K}\frac{\gamma}{\gamma-\xi_2}\frac{\xi_1}{\theta+\xi_1} - e^{\xi_1 K}\frac{\gamma}{\gamma-\xi_1}\frac{\xi_2}{\theta+\xi_2}}$$

Example 5
The most general example we consider is the compound Poisson process from **Example 4** with an added independent Brownian motion:

$$S_t := \sum_{i=0}^{N_t} U_i - \sum_{i=0}^{\tilde{N}_t} T_i + \sigma B_t - \beta t\,.$$

In this case the Lévy exponent is

$$\kappa(\alpha) = \frac{\lambda\alpha}{\gamma-\alpha} - \frac{\xi\alpha}{\theta+\alpha} + \frac{\sigma^2\alpha^2}{2} - \beta\alpha\,.$$

It is still possible to derive explicit expressions for the stationary distribution and the loss rate, by solving $\kappa(\alpha) = 0$, which now yields 3 non-zero solutions, and solving 4 equations in 4 unknowns. We shall refrain from stating the expressions as they become extremely lengthy. Instead, we plot the graphs of distribution functions for the stationary distribution in the case where $\lambda = \gamma = \xi = \theta = \beta = 1$, $K = 2$ for different values of the variance.

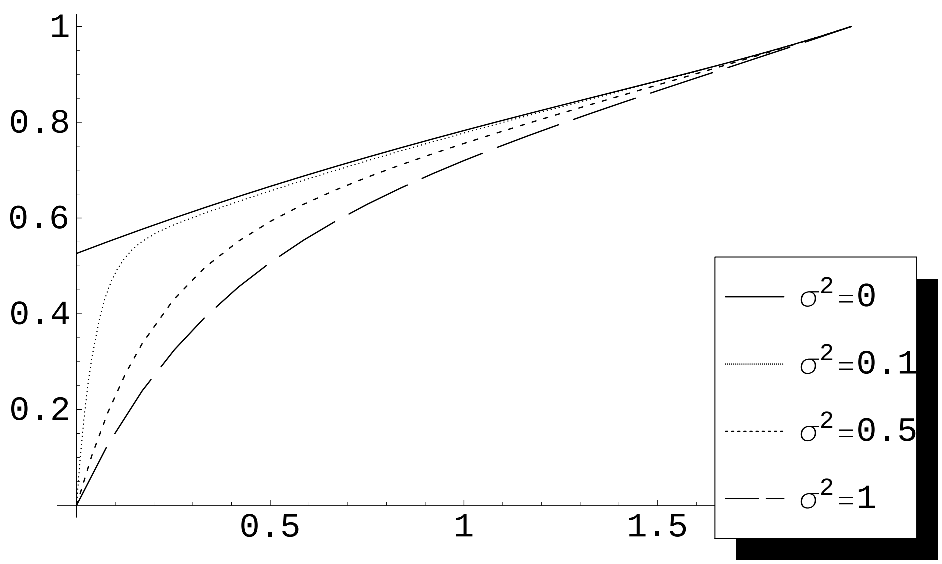

Figure 5: 4 values of the variance

Loss rate asymptotics for random walks

In this section we prove a proposition on the loss rate for a reflected random walk. This result was originally proved in Jelenković [27], but we provide a shorter proof by taking advantage of the representation of the stationary distribution provided by (1.3). Proposition 3.1 proved below is the discrete-time counterpart of Theorem 3.1 which is main theorem of **Paper B**.

Proposition 3.1. *Let $X_1, X_2, \cdots$ be an i.i.d. sequence with mean μ and let ℓ^K be the loss rate at K of the associated random walk, reflected in* 0 *and K as defined by* (1.5) *above. Assume $\overline{F}(x) \sim \overline{B}(x)$ for some distribution $B \in \mathcal{S}^*$. Then*

$$\ell^K \sim \int_K^\infty \overline{F}(y)\,\mathrm{d}y, \qquad K \to \infty.$$

Proof. By partial integration as in Pihlsgård [44], we have

$$\ell^K = \mathbb{E}(X-K)^+ + \int_0^K \mathbb{P}(X > K-y)\overline{\pi}_K(y)\,\mathrm{d}y. \tag{3.4}$$

Since $\overline{F}_I(K) = \mathbb{E}(X-K)^+$, we need to prove that

$$\limsup_K \int_0^K \frac{\mathbb{P}(X > K-y)\overline{\pi}_K(y)}{\overline{F}_I(K)}\,\mathrm{d}y = 0. \tag{3.5}$$

For any $A > 0$ we have:

$$\begin{aligned}
&\limsup_K \int_0^A \frac{\mathbb{P}(X > K-y)\overline{\pi}_K(y)}{\overline{F}_I(K)}\,\mathrm{d}y \\
&\leq \limsup_K \frac{\mathbb{P}(X > K-A)}{\overline{F}_I(K)} \int_0^A \overline{\pi}_K(y)\,\mathrm{d}y = 0
\end{aligned}$$

so therefore, for any $A > 0$:

$$\limsup_K \int_0^K \frac{\mathbb{P}(X > K-y)\overline{\pi}_K(y)}{\overline{F}_I(K)}\,\mathrm{d}y = \limsup_K \int_A^K \frac{\mathbb{P}(X > K-y)\overline{\pi}_K(y)}{\overline{F}_I(K)}\,\mathrm{d}y\,. \tag{3.6}$$

By Theorem 9.1 in [5], $\overline{\pi}_\infty(y)|\mu| \sim \overline{F}_I(y)$ so that for large A we have for $y > A$: $\overline{\pi}_\infty(y) \leq 2\overline{F}_I(y)/|\mu| = 2\mu^+\overline{F}_e(y)/|\mu|$. Note that Proposition 4.1 in

Paper B also holds for random walks. Using this, we have:

$$\begin{aligned}
&\limsup_K \int_A^{K-A} \frac{\mathbb{P}(X > K-y)\overline{\pi}_K(y)}{\overline{F}_I(K)}\,\mathrm{d}y \\
\leq &\limsup_K \int_A^{K-A} \frac{\mathbb{P}(X > K-y)\overline{\pi}_\infty(y)}{\overline{F}_I(K)}\,\mathrm{d}y \\
\leq &2\limsup_K \int_A^{K-A} \frac{\mu^+\mathbb{P}(X > K-y)\overline{F}_e(y)}{|\mu|\overline{F}_I(K)}\,\mathrm{d}y \\
= &2\limsup_K \int_A^{K-A} \frac{\mathbb{P}(X > K-y)\overline{F}_e(y)}{|\mu|\overline{F}_e(K)}\,\mathrm{d}y \\
= &2\limsup_K \frac{\overline{F}_e^{*2}(K)}{\overline{F}_e(K)} \int_A^{K-A} \frac{\mathbb{P}(X > K-y)\overline{F}_e(y)}{|\mu|\overline{F}_e^{*2}(K)}\,\mathrm{d}y \\
= &4\limsup_K \int_A^{K-A} \frac{\mathbb{P}(X > y)\overline{F}_e(K-y)}{|\mu|\overline{F}_e^{*2}(K)}\,\mathrm{d}y\,.
\end{aligned}$$

If we let U and V be independent with $U =_{\mathscr{D}} V =_{\mathscr{D}} F_e$ and use Lemma 1.2 of the appendix we obtain:

$$\begin{aligned}
&4\limsup_K \int_A^{K-A} \frac{\mathbb{P}(X > y)\overline{F}_e(K-y)}{|\mu|\overline{F}_e^{*2}(K)}\,\mathrm{d}y \\
&\limsup_K 4\frac{\mu^+}{|\mu|}\mathbb{P}(A < U \leq K-A \mid U+V > K) = \frac{2\mu^+}{|\mu|}\overline{F}_e(A)\,.
\end{aligned}$$

By combining the result above with (3.6) we have

$$\limsup_K \int_0^{K} \frac{\mathbb{P}(X > K-y)\overline{\pi}_K(y)}{\overline{F}_I(K)}\,\mathrm{d}y \tag{3.7}$$

$$\leq \frac{2\mu^+}{|\mu|}\overline{F}_e(A) + \limsup_K \int_{K-A}^{K} \frac{\mathbb{P}(X > K-y)\overline{\pi}_K(y)}{\overline{F}_I(K)}\,\mathrm{d}y\,. \tag{3.8}$$

We continue our calculation of the last integral above:

$$\begin{aligned}
&\limsup_K \int_{K-A}^{K} \frac{\mathbb{P}(X > K-y)\overline{\pi}_K(y)}{\overline{F}_I(K)}\,\mathrm{d}y \\
&= \limsup_K \int_0^{A} \frac{\mathbb{P}(X > y)\overline{\pi}_K(K-y)}{\overline{F}_I(K)}\,\mathrm{d}y \\
&\leq \limsup_K \frac{\overline{\pi}_K(K-A)}{\overline{F}_I(K)} \int_0^{A} \mathbb{P}(X > y)\,\mathrm{d}y\,.
\end{aligned}$$

If we define $\sigma_A = \inf\{y \geq 0 \mid S_y < -A\}$, $M_n = \max_{k \leq n} S_k$ and use (1.3) we have:

$$\overline{\pi}_K(K-A) = \mathbb{P}(M_{\sigma_A} > K-A)\,.$$

By Theorem 1 of Foss and Zachary [23] we have $\mathbb{P}(M_{\sigma_A} > K-A) \sim \mathbb{E}\sigma_A \overline{F}(K)$ and therefore:

$$\limsup_K \frac{\overline{\pi}_K(K-A)}{\overline{F}_I(K)} \int_0^A \mathbb{P}(X > y)\,\mathrm{d}y =$$
$$\limsup_K \mathbb{E}\sigma_A \frac{\overline{F}(K)}{\overline{F}_I(K)} \int_0^A \mathbb{P}(X > y)\,\mathrm{d}y = 0\,.$$

In view of (3.8) we have for large A:

$$\limsup_K \int_0^K \frac{\mathbb{P}(X > K-y)\overline{\pi}_K(y)}{\overline{F}_I(K)}\,\mathrm{d}y \leq \frac{2\mu^+}{|\mu|}\overline{F}_e(A)\,.$$

Letting A tend to infinity completes the proof. □

The Pollaczek-Khinchine Formula

An important part of **Paper B** is the formula (3.2) which states that for a Lévy process $\{S_t\}$ with negative mean we have:

$$\mathbb{E}\big[e^{\alpha V}\big] = -\frac{\alpha \mathbb{E}_{\pi_\infty} L_1^c + \mathbb{E}_{\pi_\infty}\big[\sum_{0\leq s\leq 1}(1-e^{-\alpha \Delta L_s})\big]}{\kappa(\alpha)}\,, \tag{3.9}$$

where V is a random variable, which has the stationary distribution of the one-sided reflected version of $\{S_t\}$, and L_1^c is the continuous part of the local time evaluated at 1. As it is noted in **Paper B**, when $\{S_t\}$ is spectrally positive, the jump part in (3.9) disappear have we have the following equation

$$\mathbb{E}\big[e^{\alpha V}\big] = -\frac{\alpha \mathbb{E}_{\pi_\infty} L_1}{\kappa(\alpha)} = \frac{\alpha \mathbb{E} S_1}{\kappa(\alpha)}\,.$$

The justification of the second equality is provided in Corollary 3.4 p. 257 in [5]. Now we assume $\{S_t\}$ is a compound Poisson process with a unit drift, where the jumps have distribution B and with intensity β. Set $\rho = \beta\mathbb{E}B$ and let $\hat{B}$ be the moment generating function of B. Then $\mathbb{E}S_1 = \rho - 1$ and $\kappa(\alpha) = \beta\hat{B}[\alpha] - \beta - \alpha$, and hence, the equation above takes the form

$$\mathbb{E}\big[e^{\alpha V}\big] = (1-\rho)\frac{\alpha}{\beta\hat{B}[\alpha] - \beta - \alpha} \tag{3.10}$$

and we recognize the classic version Pollaczek-Khinchine formula.

Continuity of the loss rate

A contribution of **Paper C** is Theorem 3.2 which states that the loss rate is continuous if considered a function of the Lévy process, when one makes an assumption of uniform integrability. We notice that the stated assumption cannot be relaxed in general: Let $\{S_t^{(n)}\}$ be a sequence of Lévy processes with characteristic triplet $(0, 0, \nu^{(n)})$ where $\nu^{(n)}(\{-1\}) = 1 - 1/(2n)$ and $\nu^{(n)}(\{n\}) = 1/(2n)$. Then $S_1^{(n)} \xrightarrow{\mathcal{D}} S_1^{(0)}$ where the Lévy measure of $S_1^{(0)}$ is $\nu^{(0)}(\{-1\}) = 1$. If the loss rate were continuous, we should have $\ell^{K(n)} \to 0$. However using (3.4) in Pihlsgård [44] we get

$$\ell^{K(n)} = \int_0^K \pi_K(\mathrm{d}x) \int_{K-x}^{\infty} (x + y - K)\nu^{(n)}(\mathrm{d}y) \geq$$
$$\int_K^{\infty} (y - K)\nu^{(n)}(\mathrm{d}y) = (n - K)^+ \frac{1}{2n}$$

Letting n tend to infinity we see that $\liminf_n \ell^{K(n)} \geq 1/2$.
When proving continuity of the loss rate, we need continuity of stationary distribution which is proved in Proposition 2.1 in **Paper C**. The uniform integrability is not needed for this result, a fact which is not entirely surprising given that the stationary distributions are uniformly bounded. The corresponding result for one-sided reflection in discrete time is proved in Theorem 6.1 p. 285 in [5] and this result needs an assumption of uniform integrability. We note that the discrete-time analogue of Theorem 3.2 can be proved using an approach similar to the one used in paper **Paper C**. Indeed, let $\{S_i^n\}$ be sequence of random walks, such that $\{S_1^n\}$ is uniformly integrable and $S_1^n \xrightarrow{\mathcal{D}} S_1^0$. The reflected process is obtained through the recursion (1.1), and the loss rate is given by

$$\ell^{K,n} = \int_0^K \int_{-\infty}^{\infty} \Psi(x, y) F_n(\mathrm{d}y) \pi_K(\mathrm{d}x)\,, \tag{3.11}$$

where F_n is the distribution function of S_1^n and $\Psi(x, y) = (x + y - K)^+$. The continuity of the stationary distribution follows precisely as in Proposition 4.1 with the Portmanteau Lemma playing the part of Theorem 13.17 in Kallenberg [28]. Similarly to the proof of Theorem 3.2 we can split the contribution from the integral in (3.11) into two parts:

$$\int_{[-a,a]^c} \Psi(x, y) F_n(\mathrm{d}y) + \int_{[-a,a]} \Psi(x, y) F_n(\mathrm{d}y) \tag{3.12}$$

for some $a > 0$, and use uniform integrability to make the contribution from the first integral arbitrarily small. Define a sequence of functions f_n by $f_n(x) = \int_{[-a,a]} \Psi(x,y) F_n(\mathrm{d}y)$. Equicontinuouity and uniform boundedness of (f_n) now follows, and we can conclude that $\ell^{K,n} \to \ell^{K,n}$ using the same reasoning as in the proof of Theorem 3.2.
In **Paper C** we also provide Theorem 3.3 which gives necessary and sufficient conditions for uniform integrability of a sequence of infinitely divisible distributions in terms of the characteristic triplet. The result, which does not seem to appear in the literature, is similar to Theorem 25.3 in Sato [49] which states that for a Lévy process $\{S_t\}$ with Lévy measure ν and a *submultiplicative* function g, that is a function g satisfying $0 \le g(x+y) \le ag(x)g(y)$ for some $a > 0$, we have $\mathbb{E}[g(S_t)] < \infty$ if an only if $\int_{[-1,1]^c} g(y)\nu(\mathrm{d}y) < \infty$. We observe that for $\alpha > 0$ we have that $x \mapsto |x|^\alpha \wedge 1$ is submultiplicative, and it is tempting to conjecture that Theorem 3.3 should hold for all submultiplicative g.

Notes

The result in Theorem 3.1 is derived in Takács [53] using combinatorial methods and later in Emery [17]. A short proof appear in Rogers [48].
The Pollaczek-Khinchine formula presented in (3.10) in the version based on moment generating functions. The formula typically refers to the mean steady-state waiting time in the $M/G/1$ queue and goes back to Pollaczek [45].
The loss rate considered here is only one of several interesting quantities, which occur in the context of reflected processes and loss probabilities. Other such quantities are the steady-state probability that a customer gets partially rejected (see Bekker and Zwart [10]) or the steady-state probability of a full buffer, which goes back to the work of Erlang (Erlang [18] and Erlang [19]).

4 Parallel computing, failure recovery, and extreme values

The paper **Paper D** concerns extreme value theory, and its content is disjoint from the rest of the thesis. In the paper we describe a mathematical model for a computer system, where the jobs have random length and the processor fails at random times. The specific model in the paper deals with the case where the processor has to restart from scratch, if the processor fails before the jobs is completed. We are interested in determining the total time needed to complete the job. This model was examined in Asmussen et al. [9] where tail asymptotics for the total job length were determined. These tail asymptotics revealed, that under quite natural assumptions, the total job length will be heavy tailed. This is fairly intuitive - sometimes the job lengths will be large, and when this happens, the processor will have to restart many times, before the job can be completed, which leads to a very long total job time. Since the tail asymptotics were made available in [9], we can use extreme value theory to determine the behavior of several independent copies of the total job length, which is a natural object to consider as a model for a parallel computing set-up, that is, a situation where the job is divided into smaller parts and then distributed to several processors. In **Paper D** we assume that the job is distributed to M processors and we allow the length of the job which is to be distributed, to depend on M. Our aim is then, to determine the asymptotic total execution time as $M \to \infty$. If the job length increases at exactly to the same rate as the number of processors, then the job length faced by each processor is independent of M, and we are in the setting of classical extreme value theory. However, in **Paper D** we also examine cases where the job length increases slower or faster than M, and this takes us to a triangular array setting, since the job length faced by each processor will depend on M. A general triangular array setting was studied in Freitas and Hüsler [24].

5 A conjecture on the stationary distribution of FBM

Before proceeding to the papers, we present a conjecture on the stationary distribution of reflected fractional Brownian Motion. We may define the reflected version of any stochastic process $\{S_t\}$ with paths in $D[0,\infty)$ by using the mapping (6.2) defined **Paper A**. Provided $\{S_t\}$ has stationary increments, there will still exists a limiting distribution as t tends to infinity, since by Corollary 7.6 in **Paper A** all moments of the reflected process will be increasing, and since they are bounded, they will be convergent, which implies the existence of weak limit of the reflected process.

A process with stationary increments which has drawn much interest in recent years is fractional Brownian motion. We say $\{S_t\}$ is a standard fractional Brownian motion iff it is a centered Gaussian processes with correlation function

$$\rho(s,t) = Cov(S_s^{2H}, S_t^{2H}) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \qquad 0 < H < 1$$

The parameter H is called the *Hurst parameter*. We can simulate this process using the software provided by Ton Dieker. Plotted below are empirical distribution functions and qqplots for two values of H namely $H = 1/3$ and $H = 2/3$. Recall that the beta distribution $\mathrm{B}(s,t)$ is the distribution on $[0,1]$ with density proportional to $x \mapsto x^{s-1}(1-x)^{t-1}$. The plots below suggest that the limiting distribution for $H = 1/3$ is a beta distribution with parameters $s = t = 2$, and the limiting distribution for $H = 2/3$ is $\mathrm{B}(1/2, 1/2)$. Furthermore, since $H = 1/2$ corresponds to the standard Brownian motion, for which we know that the stationary distribution is uniform ($\mathrm{B}(1,1)$) it appears reasonable to conjecture that the limiting distribution for general H is a $\mathrm{B}(1/H-1, 1/H-1)$ distribution. Since by [8] the representation (1.3) still holds for processes with stationary increments (in discrete time, for continuous time see [51]), the truth of the conjecture should in turn imply the following statement about the two-sided exit probability for fractional Brownian motion:

Let $\{S_t\}$ be a fractional Brownian motion with Hurst parameter H, and set $\tau[y-1,y) = \inf\{t > 0 \mid S_t \notin [y-1,y)\}$. Furthermore, let B denote the distribution function of a $\mathrm{B}(1/H-1, 1/H-1)$ distribution. We conjecture that

$$\mathbb{P}\big(S_{\tau[y-1,y)} \geq y\big) = 1 - B(y)\,.$$

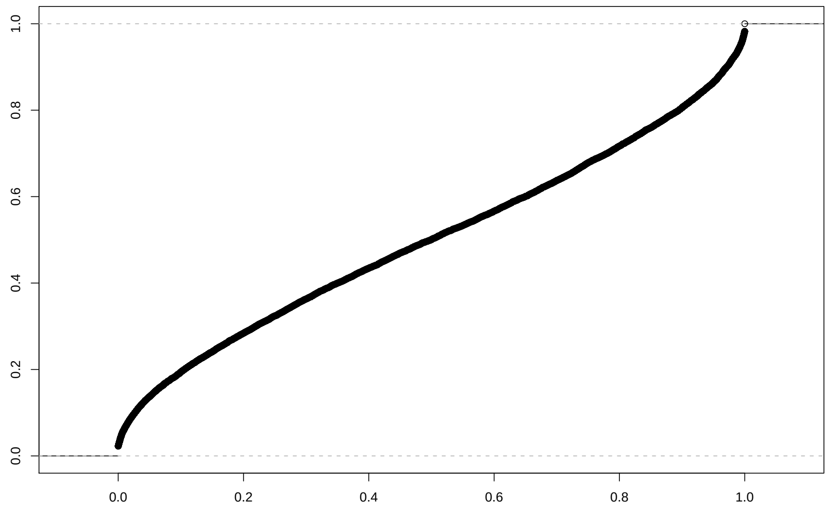

Figure 6: Empirical distribution function for simulated values, $H = 2/3$

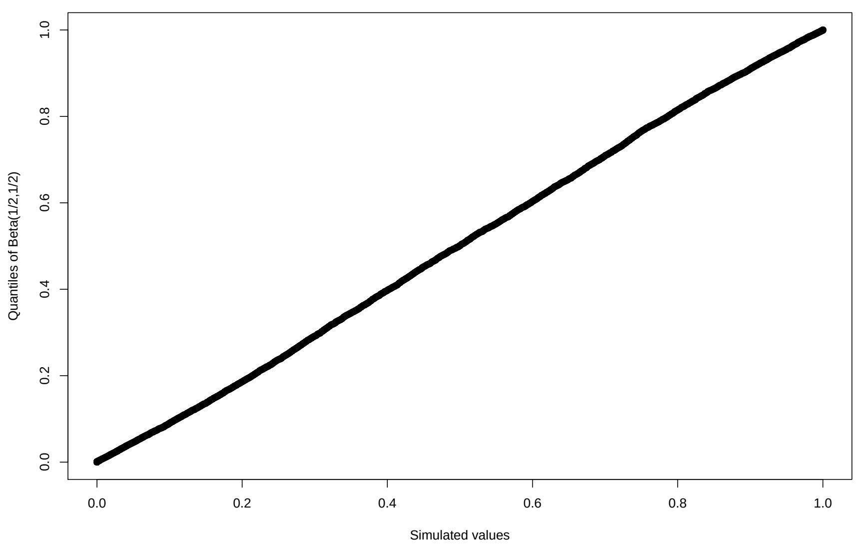

Figure 7: qqplot, $H = 2/3$

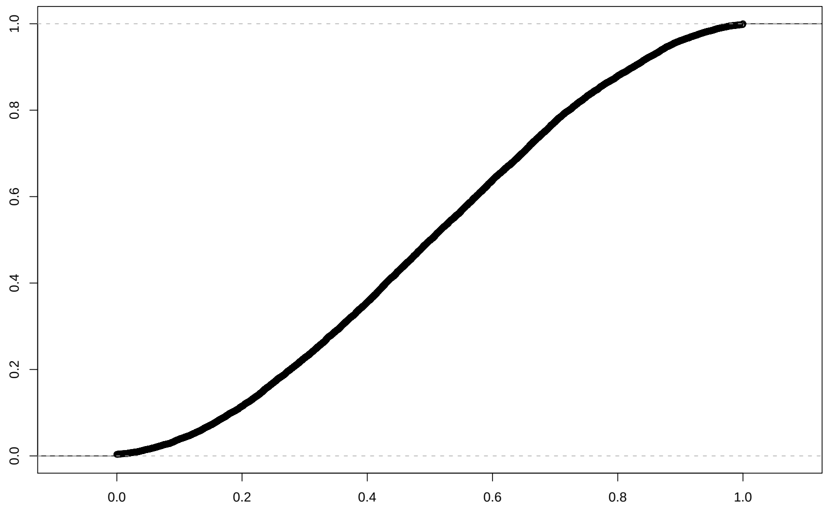

Figure 8: Empirical distribution function for simulated values, $H = 1/3$

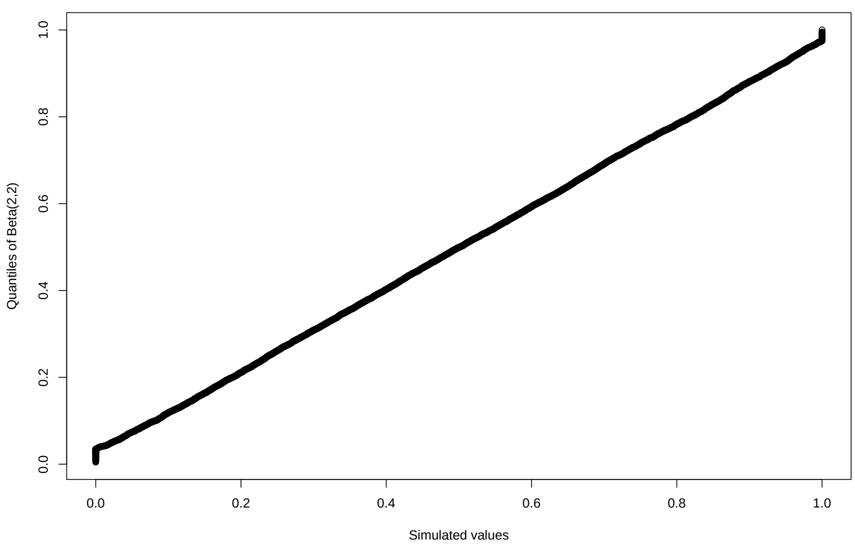

Figure 9: qqplot, $H = 1/3$

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PAPER A

Structural Properties of Reflected Lévy Processes

Lars Nørvang Andersen & Michel Mandjes

Abstract

This paper considers a number of structural properties of reflected Lévy processes, where both one-sided reflection (at 0) and two-sided reflection (at both 0 and $K > 0$) are examined. With V_t being the position of the reflected process at time t, we focus on the analysis of $\zeta(t) := \mathbb{E}V_t$ and $\xi(t) := \mathbb{V}\mathrm{ar}V_t$. We prove that for the one- and two-sided reflection we have $\zeta(t)$ is increasing and concave, whereas for the one-sided reflection, $\xi(t)$ is increasing. In most proofs we first establish the claim for the discrete-time counterpart (that is, a reflected random walk), and then use a limiting argument. A key step in our proofs for the two-sided reflection is a new representation of the position of the reflected process in terms of the driving Lévy process.
Keywords Complete monotonicity, Lévy processes, One/Two-sided reflection, Mean function, Variance function, Stationary increments, concordance.
Mathematics Subject Classification (2000) Primary 60K25 Secondary 60F05 90B22

1 Introduction

In this paper we consider structural properties of reflected Lévy processes, where both one-sided reflection (at 0) and two-sided reflection (at both 0 and $K > 0$) are examined. We assume throughout that the reflected process is started at 0, and we have that in the case of one-sided reflection, the position of the reflected process V_t is given by $S_t + L_t$, where $\{S_t\}_{t\geq 0}$ is the driving Lévy process, and $\{L_t\}_{t\geq 0}$ is the local time at 0, which can be written as $-\inf_{0\leq s\leq t} S_s$. In case of two-sided reflection, we have a similar construction in the sense that V_t can be decomposed as $S_t + L_t - \bar{L}_t$, with $\bar{L}_t$ the local time at K, given as part of the solution to a Skorokhod problem, but finding explicit solutions for V_t, in terms of S_s with $0 \leq s \leq t$, is rather involved; recently, such expressions have appeared in Kruk et al. [10] and Kruk et al. [11].

More precisely, we focus in this work on the analysis of two objects, viz. $\zeta(t) := \mathbb{E}V_t$ and $\xi(t) := \mathbb{V}\mathrm{ar}V_t$. Our goal is to prove a number of structural properties regarding the shape of these two functions. For the one-sided reflection, the function $\zeta(\cdot)$ was already examined in detail before. In Kella [7] it was shown that $\zeta(\cdot)$ is concave as long as the underlying Lévy process does not have any positive jumps, relying on martingale techniques. This result was generalized by Kella and Sverchkov [8] to general Lévy processes (in fact, even just stationary increments are needed), with an elementary proof that uses stochastic monotonicity. To our best knowledge, however, there are no results for the two-sided counterpart, nor any results for the variance function $\xi(\cdot)$.

The contributions of this paper are the following. In the first part of the paper we consider the case of one-sided reflection.

- In Section 3 we consider the special case of a spectrally positive Lévy process, that is, a Lévy process without negative jumps. We present an elementary proof of the fact that the expected value of the position at time t is concave in t. Although this result was already covered by [7], we included it because we believe the proof technique is interesting, and may be of use in other situations as well. More particularly, the proof relies on the concept of complete monotonicity to show that the desired property holds in the special case of a compound Poisson Lévy process, and then uses a limiting argument (approximating any spectrally positive Lévy process by a suitable sequence of compound Poisson processes).

- Section 4 focuses on one-sided reflection, but now we treat the case of general Lévy input, roughly as follows. First we prove the desired result for the discrete-time version of the Lévy process (which is a random walk) by means of an extremely short and insightful argument. Then a limiting procedure ensures that the concavity is preserved in continuous time, thus reestablishing the result by [8]. Importantly, the same method (that is, first proving the desired property for the random walk, and then a limiting argument) can be followed to prove the new result that the variance curve, i.e., $\xi(t)$, is increasing in t; the proof relies on the concept of 'concordance'.

The second part of the paper concentrates on similar issues, but now in the setting of a two-sided reflected Lévy process.

- As mentioned above, new explicit formulae for V_t (in terms of S_s for $0 \le s \le t$) have appeared recently. We derive in Section 5 a new explicit representation, which is similar to the one found in [10], but somewhat shorter. This alternative representation carries over to continuous time, as argued in Section 6.
- Relying on the new representation for V_t for the case of two-sided reflection, as presented in Section 6, in Section 7 we prove the new result that $\zeta(t)$ is an increasing concave function of t. We do this by first proving the desired result for the discrete-time counterpart, that is, a random walk reflected at 0 and K, and then we use a limiting argument. We finish this second part with the observation that the results carry over to the situation in which we just assume stationary increments (rather than stationary independent increments).

The paper now continues with a section in which the model and some preliminaries are given.

2 Model, notation, and preliminaries

In this paper we study reflected versions of the Lévy process $\{S_t\}_{t\geq 0}$. We distinguish between one-sided and two-sided reflection.

- *One-sided reflection* (at 0). The reflection of $\{S_t\}_{t\geq 0}$ at 0, which we denote by $\{V_t\}_{t\geq 0}$, can be formally introduced as follows (see for instance [2, Ch. IX]). Define the increasing process $\{L_t\}_{t\geq 0}$ by $L_t = -\inf_{0\leq s\leq t} S_s$; this process is commonly referred to as the local time at 0. Then the reflected process (or: workload process, queueing process) $\{V_t\}_{t\geq 0}$ is given through
$$V_t := S_t + \max\{L_t, V_0\};$$

observe that $V_t \geq 0$ for all $t \geq 0$. Throughout this paper the focus lies on the special case that $V_0 = 0$, and hence $V_t = S_t + L_t$. It is straightforward that $\zeta(t)$ increases in t, using Proposition 3 p. 158 in [4].

- *Two-sided reflection* (at 0 and $K > 0$). Again starting off at 0, we now have that the position of the reflected process at time t, i.e. V_t, is given by $V_t = S_t + L_t - \bar{L}_t$, with the increasing process $\{\bar{L}_t\}_{t\geq 0}$ denoting the local time at K, given as part of the solution to a Skorokhod problem. In [10] an explicit expression for L_t and $\bar{L}_t$ (in terms of S_s with $0 \leq s \leq t$) is given. In particular,

$$V_t = S_t - \sup_{s\in[0,t]} \left[\left((S_s - K)) \vee \inf_{u\in[0,t]} S_u \right) \wedge \inf_{u\in[s,t]} S_u \right].$$

We recall that we denote $\zeta(t) := \mathbb{E}V_t$ and $\xi(t) := \mathbb{V}\mathrm{ar}V_t$.

In Section 3 we consider the case in which the underlying Lévy process does not have negative jumps (i.e. is spectrally positive), and in which there is just reflection at 0. Assuming stability (i.e. $\mathbb{E}S_1 < 0$), the Laplace exponent $\varphi(\alpha) := \log \mathbb{E}e^{-\alpha S_1}$ is given by a function $\varphi(\cdot) : [0,\infty) \mapsto [0,\infty)$ that is increasing and convex on $[0,\infty)$, with slope $\varphi'(0) = -\mathbb{E}S_1$ in the origin. Therefore the inverse $\psi(\cdot)$ of $\varphi(\cdot)$ is well defined on $[0,\infty)$. In the sequel we rule out the trivial case that $\{S_t\}_{t\geq 0}$ is a (downward) *subordinator*, i.e., a monotone (decreasing) process. Throughout, we assume that $\varphi''(0)$ is finite (unless stated otherwise).

Important examples of spectrally positive Lévy processes are the following. (1) *Brownian motion with drift*, where $\varphi(\alpha) = -\alpha\mu + \frac{1}{2}\alpha^2\sigma^2$. (2) *Compound Poisson with drift*. Jobs arrive according to a Poisson process of rate λ; the jobs $B_1, B_2, \ldots$ are i.i.d. samples from a distribution with Laplace transform $\beta(\alpha) := \mathbb{E}e^{-\alpha B}$; the storage system is continuously depleted at a rate $-M < 0$ (where M is often referred to as the *drift*). It can be verified that $\varphi(\alpha) = M\alpha - \lambda + \lambda\beta(\alpha)$.

Using [2, Thm. IX.3.10] or [9], it is straightforward to prove that, as long as the Lévy process is spectrally positive, $\mu_V := \mathbb{E}V_\infty = -\varphi''(0)/(2\varphi'(0))$, and

$$\rho(\vartheta) := \int_0^\infty e^{-\vartheta t}\mathbb{E}V_t \mathrm{d}t = \int_0^\infty e^{-\vartheta t}\zeta(t)\mathrm{d}t = -\frac{\varphi'(0)}{\vartheta^2} + \frac{1}{\vartheta\psi(\vartheta)}. \tag{2.1}$$

3 One-sided reflection: spectrally positive input

This section focuses on establishing a number of structural properties of $\zeta(\cdot)$ for the case of spectrally positive Lévy input. As mentioned above, it is evident that $\zeta(\cdot)$ is positive and increasing; in this section we prove that it is concave as well. We do this by extensively using the concept of completely monotonous functions [3, 13]. The desired result is first proven for the case of compound Poisson input; then we show how to construct a sequence of compound Poisson processes approximating any spectrally positive Lévy process arbitrarily closely, which allows us to prove the claim. The class $\mathscr{C}$ of completely monotone functions is defined as follows.

Definition 3.1. *A function $f(\alpha)$ on $[0,\infty)$ is completely monotone if for all $n \in \mathbb{N}$*

$$(-1)^n \frac{d^n}{d\alpha^n} f(\alpha) \geq 0.$$

We write $f(\alpha) \in \mathscr{C}$.

The following deep and powerful result is due to Bernstein [3]. It says that there is equivalence between $f(\alpha)$ being completely monotone, and the possibility of writing $f(\alpha)$ as a Laplace transform. For more background and basic properties of completely monotone functions, see [6, pp. 439-442].

Theorem 3.2. *[Bernstein] A function $f(\alpha)$ on $[0,\infty)$ is the Laplace transform of a non-negative random variable if and only if (i) $f(\alpha) \in \mathscr{C}$, and (ii) $f(0) = 1$.*

In the M/G/1 setting, $\psi(\vartheta) = \lambda + \vartheta - \lambda\pi(\vartheta)$, where $\pi(\vartheta)$ is the Laplace transform of the busy period, and the deterministic service rate has value 1; it is assumed that $-\lambda\beta'(0) < 1$. In our decomposition $V_t = S_t + L_t$, we have L_t can increase only when $V_t = 0$ and then with a unit drift, hence:

$$L_t = \int_0^t I(V_s = 0)\mathrm{d}s\,.$$

Taking means and using 'Tonelli' yields $\zeta(t) = -\varphi(0)t + \int_0^t \mathbb{P}(V_s = 0)\mathrm{d}s$, and we obtain $\zeta'(t) = -\varphi(0) + \mathbb{P}(V_s = 0)$. Furthermore, we observe that $\zeta''(t)$ exists: Using $M_t := \sup_{0<s\leq t} S_s =_{\mathscr{D}} V_t$, and letting Δ_t denote the event that S_t makes at least on jump in $(\epsilon, t+\epsilon)$ we obtain:

$$\begin{aligned}
&\epsilon^{-1}\left(\mathbb{P}(V_t = 0) - \mathbb{P}(V_{t+\epsilon} = 0)\right) = \epsilon^{-1}\left(\mathbb{P}(M_t = 0) - \mathbb{P}(M_{t+\epsilon} = 0)\right) \\
&= \epsilon^{-1}\mathbb{P}(M_t = 0, M_{t+\epsilon} > 0) = \epsilon^{-1}\mathbb{P}(M_t = 0, M_{t+\epsilon} > 0|\Delta_t)(1 - e^{\lambda\epsilon}) \\
&\sim \mathbb{P}(M_t = 0, M_{t+\epsilon} > 0|\Delta_t)\lambda\,.
\end{aligned}$$

Since $\epsilon \mapsto \mathbb{P}(M_t = 0, M_{t+\epsilon} > 0|\Delta_t)$ is decreasing, we obtain that $\zeta''(t)$ exists.

Let us consider the transforms of $\zeta'(t)$ and $\zeta''(t)$. Using integration by parts, it is readily checked that

$$\int_0^\infty e^{-\vartheta t}\zeta'(t)\mathrm{d}t = -\frac{\varphi'(0)}{\vartheta} + \frac{1}{\psi(\vartheta)}.$$

Applying integration by parts once again yields that

$$\int_0^\infty e^{-\vartheta t}\zeta''(t)\mathrm{d}t = -\zeta'(0) - \varphi'(0) + \frac{\vartheta}{\psi(\vartheta)} = -\left(1 - \frac{\vartheta}{\psi(\vartheta)}\right),$$

using that

$$\zeta'(0) = \lim_{\vartheta\to\infty}\int_0^\infty \vartheta e^{-\vartheta t}\zeta'(t)\mathrm{d}t = 1 - \varphi'(0);$$

notice that the transform of $\zeta''(t)$ is only well defined when $\vartheta/\psi(\vartheta)$ has a finite limit as $\vartheta \to \infty$, which is indeed the case for compound Poisson input. The transform can further be simplified to

$$-\frac{\lambda(1-\pi(\vartheta))}{\lambda(1-\pi(\vartheta))+\vartheta}. \tag{3.1}$$

Observe that Lemma 4.1 (item 1) of [13] entails that the negative of (3.1) is in $\mathscr{C}$, thus proving that indeed in the M/G/1 context $\zeta''(t)$ is negative, i.e., $\zeta(t)$ is concave. We have proved the following

Lemma 3.3. *$\zeta(t)$ is concave for compound Poisson input processes with negative drift (with one-sided reflection).*

We now consider the context of a general spectrally positive Lévy process, and use Lemma 3.3 to prove that also in this setting $\zeta(t)$ is concave. We first recall that the Laplace exponent $\varphi(\alpha)$ of a spectrally positive Lévy process can be written as [4, Section VII.1], with $M \in \mathbb{R}$, $\sigma^2 > 0$, and measure $\Pi_\varphi(\cdot)$ such that $\int_{(0,\infty)} \min\{1, x^2\}\Pi_\varphi(\mathrm{d}x) < \infty$,

$$\varphi(\alpha) = \alpha M + \frac{1}{2}\alpha^2\sigma^2 + \int_{(0,\infty)} (e^{-\alpha x} - 1 + \alpha x 1_{(0,1)})\Pi_\varphi(\mathrm{d}x).$$

The idea is now to approximate the spectrally positive Lévy process arbitrarily closely by a sequence of compound Poisson processes. To this end, let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of numbers in $(0,1]$, such that $\varepsilon_n \downarrow 0$. Then we can rewrite $\varphi(\alpha) = \varphi_n(\alpha) + \bar{\varphi}_n(\alpha)$, with

$$\varphi_n(\alpha) = \left(M + \int_{\varepsilon_n}^1 x\Pi_\varphi(\mathrm{d}x) + \frac{\sigma^2}{\varepsilon_n}\right)\alpha + \frac{\sigma^2}{\varepsilon_n^2}\left(e^{-\alpha\varepsilon_n} - 1\right) + \int_{\varepsilon_n}^\infty \left(e^{-\alpha x} - 1\right)\Pi_\varphi(\mathrm{d}x);$$

$$\bar{\varphi}_n(\alpha) = \sigma^2 \left(\frac{1}{2}\alpha^2 - \frac{e^{-\alpha\varepsilon_n} + \alpha\varepsilon_n - 1}{\varepsilon_n^2} \right) + \int_0^{\varepsilon_n} \left(e^{-\alpha x} - 1 + \alpha x\right) \Pi_\varphi(\mathrm{d}x).$$

Let $\psi_n(\cdot)$ denote the inverse of $\varphi_n(\cdot)$.

Lemma 3.4. *(i) For all $\alpha \geq 0$, $\varphi_n(\alpha) \to \varphi(\alpha)$ as $n \to \infty$.*
(ii) For all $\alpha \geq 0$, $\varphi_n'(\alpha) \to \varphi'(\alpha)$ as $n \to \infty$.
(iii) For all $n \in \mathbb{N}$, $\varphi_n'(0) = \varphi'(0)$.

Proof. Straightforward calculations. □

It is important to notice that, for any $n \in \mathbb{N}$, $\varphi_n(\cdot)$ can be interpreted as the Laplace exponent of a compound Poisson process (with negative drift), say $\{S_{n,t}\}_{t\geq 0}$. This is seen as follows. The drift term is

$$\left(M + \int_{\varepsilon_n}^1 x\Pi_\varphi(\mathrm{d}x) + \frac{\sigma^2}{\varepsilon_n} \right),$$

which is positive for n sufficiently large. Then, the term $(\sigma^2/\varepsilon_n^2) \cdot (e^{-\alpha\varepsilon_n} - 1)$ can be interpreted as the contribution of a Poisson stream (arrival rate σ^2/ε_n^2) of jobs of deterministic size ε_n. Also,

$$\int_{\varepsilon_n}^\infty \left(e^{-\alpha x} - 1\right) \Pi_\varphi(\mathrm{d}x) = \Pi_\varphi([\varepsilon_n, \infty)) \int_{\varepsilon_n}^\infty \left(e^{-\alpha x} - 1\right) \frac{\Pi_\varphi(\mathrm{d}x)}{\Pi_\varphi([\varepsilon_n, \infty))},$$

which is the contribution of a Poisson stream (arrival rate $\Pi_\varphi([\varepsilon_n, \infty))$) of jobs, whose sizes are i.i.d. samples from a 'truncated distribution' with density $\Pi_\varphi(\mathrm{d}x)/\Pi_\varphi([\varepsilon_n, \infty))$, for $x \geq \varepsilon_n$.

Just as we introduced the reflected version $\{V_t\}_{t\geq 0}$ of $\{S_t\}_{t\geq 0}$, we can construct the reflected version $\{V_{n,t}\}_{t\geq 0}$ of $\{S_{n,t}\}_{t\geq 0}$. Analogously to $\zeta(t)$, we denote $\zeta_n(t) := \mathbb{E}V_{n,t}$. Note that, due to Lemma 3.4.(iii), the queueing processes $\{V_{n,t}\}_{t\geq 0}$ are stable (recall that we assumed $\varphi'(0) > 0$). From (2.1), we have that for any $n \in \mathbb{N}$,

$$\rho_n(\vartheta) := \int_0^\infty e^{-\vartheta t}\, \zeta_n(t)\mathrm{d}t = -\frac{\varphi'(0)}{\vartheta^2} + \frac{1}{\vartheta\psi_n(\vartheta)}. \tag{3.2}$$

Corollary 3.5. *For all $n \in \mathbb{N}$, $\zeta_n(t)$ is positive (that is, larger than or equal to 0), increasing (non-strictly), and concave (non-strictly).*

Lemma 3.6. *For all $\vartheta \geq 0$, $\psi_n(\vartheta) \to \psi(\vartheta)$ as $n \to \infty$.*

Proof. First observe that $\varphi_n(\alpha) \to \varphi(\alpha)$ (Lemma 3.4) entails that, as $n \to \infty$,

$$| \psi_n(\varphi(\alpha)) - \psi_n(\varphi_n(\alpha)) | \leq \left| \sup_{\vartheta \geq 0} \psi_n'(\vartheta) \right| \cdot | \varphi(\alpha) - \varphi_n(\alpha) | \to 0,$$

where we used that $\psi_n(\cdot)$ is concave with slope $1/\varphi'(0)$ in 0. Hence it also holds that $\psi_n(\varphi(\alpha))$ converges, as $n \to \infty$, to $\alpha = \psi(\varphi(\alpha))$. But as $\varphi(\alpha)$ is a bijection of $[0, \infty)$ onto $[0, \infty)$, this proves the claim. □

Proposition 3.7. *$\zeta(t)$ is concave for spectrally positive Lévy processes (with one-sided reflection).*

Proof. Our proof consists of the following steps.

(1) Using (3.2) and Lemma 3.6, we see that, for all $\vartheta \geq 0$,

$$\lim_{n\to\infty} \rho_n(\vartheta) = \rho(\vartheta) = \int_0^\infty e^{-\vartheta t} \zeta(t) \mathrm{d}t.$$

(2) Realize that, as $\zeta_n(\cdot)$ is positive (that is, larger than or equal to 0), increasing (non-strictly), and concave (non-strictly) due to Lemma 3.3, $\lim_{n\to\infty} \zeta_n(\cdot)$ (given it exists) inherits these properties.

(3) Because of dominated convergence (use that $\zeta_n(t)$ increases in t, and that $\mu_{V,n} := \zeta_n(\infty) \to \zeta(\infty) = \mu_V$ as $n \to \infty$; these observations immediately yield an integrable majorizing function),

$$\lim_{n\to\infty} \rho_n(\vartheta) = \lim_{n\to\infty} \int_0^\infty e^{-\vartheta t} \zeta_n(t) \mathrm{d}t = \int_0^\infty e^{-\vartheta t} \lim_{n\to\infty} \zeta_n(t) \mathrm{d}t.$$

(4) The uniqueness of the Laplace transform, together with Steps (1) and (3), now implies that we have $\lim_{n\to\infty} \zeta_n(t) = \zeta(t)$. Then Step (2) yields the stated.

This finishes the proof. □

4 One-sided reflection: general Lévy input

In this section we prove that for the one-sided reflection we have $\zeta(t)$ is increasing and concave, and that $\xi(t)$ is increasing.

Discrete-time case

Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables, and define $S_0 := 0$, $S_n := X_1 + X_2 + \cdots + X_n$, its associated random walk. Define the convex function $\Psi(x) := \max(0, x) = x^+$ and let $\{V_n\}_{n=0}^{\infty}$ denote the *reflected version* of $\{S_n\}_{n=0}^{\infty}$, that is, V_n is given by the Lindley recursion $V_{n+1} := \Psi(V_n + X_{n+1})$, initialized by $V_0 := 0$. By [2, Cor. III.6.4], $V_n =_{\mathscr{D}} M_n$, where M_n denotes the 'running maximum', i.e., $\max_{0\leq k\leq n} S_k$.

We say a sequence $(a_n)_{n=0}^{\infty}$ is *concave* if $a_{n+2} + a_n \leq 2a_{n+1}\ \forall n$, that is, if $a_{n+1} - a_n$ is decreasing. We now give an extremely short proof of the fact that $\zeta(n) := \mathbb{E}V_n$ is a concave sequence.

Proposition 4.1. *$\zeta(n)$ is concave for random walks (with one-sided reflection).*

Proof. According to [2, Prop. VIII.4.5], we have that $\zeta(n) - \zeta(n-1) = \mathbb{E}S_n^+/n$. Furthermore, using $(X_i, S_n) =_{\mathscr{D}} (X_1, S_n)$ we have

$$S_n = \mathbb{E}[S_n \mid S_n] = \sum_{i=1}^{n} \mathbb{E}[X_i \mid S_n] = \sum_{i=1}^{n} \mathbb{E}[X_1 \mid S_n] = n\mathbb{E}[X_1 \mid S_n] \qquad a.s.$$

which implies $\mathbb{E}[X_1 \mid S_n] = S_n/n$ $a.s$, which in turn implies $\mathbb{E}[S_n/n \mid S_{n+1}] = S_{n+1}/(n+1)$ $a.s.$ and applying the conditional Jensen's inequality to the convex function $\Psi(\cdot)$, we conclude

$$\frac{S_{n+1}^+}{n+1} = \Psi\left(\mathbb{E}\left[\frac{S_n}{n} \mid S_{n+1}\right]\right) \leq \mathbb{E}\left[\Psi\left(\frac{S_n}{n}\right) \mid S_{n+1}\right] = \mathbb{E}\left[\frac{S_n^+}{n} \mid S_{n+1}\right] \quad a.s.,$$

and taking means on both sides yields the desired result. □

Our next goal is to prove that for the random walk introduced above $\xi(n) = \mathbb{V}\text{ar}(S_n)$ increases in n. We do so by using the concept of *concordance*, cf. the results of [12]. Here, a pair of random variables (X, Y) or its distribution function F is said to be *positively quadrant dependent* if

$$\mathbb{P}(X \leq x, Y \leq y) \geq \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y) \qquad \forall x, y.$$

According to Lemma 3 of [12] it holds that positively quadrant dependence implies that the covariance between X and Y is non-negative: $\mathbb{C}ov(X, Y) \geq 0$. Furthermore, we define two real-valued functions r, s to be *concordant* for the ith coordinate if, considered as functions of the ith coordinate (with all other coordinates held fixed) they are either both non-decreasing or both non-increasing. The main result in [12], which we will use below, is the following.

Theorem 4.2. *[Lehmann] Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independent with distribution functions $F_1, \ldots, F_n$. Assume F_i is positively quadrant dependent and let r and s be concordant for the ith coordinate. Set*

$$X := r(X_1, \cdots, X_n), \qquad Y := s(Y_1, \cdots, Y_n).$$

Then (X, Y) is positively quadrant dependent.

In particular, since (X, X) is positively quadrant dependent, we have

$$\mathbb{C}ov(r(X_1, X_2, \cdots, X_n), s(X_1, X_2, \cdots, X_n)) \geq 0$$

if the X_i's are independent and r and s are concordant for all coordinates. Using this insight, we can prove the following result.

Theorem 4.3. *$\xi(n)$ is increasing for random walks (with one-sided reflection).*

Proof. Using the identity

$$\begin{aligned} \mathbb{V}\mathrm{ar}(X+Y) &= \mathbb{C}ov(X,X) + 2\mathbb{C}ov(X,Y) + \mathbb{C}ov(Y,Y) \\ &= \mathbb{C}ov(X,X) + \mathbb{C}ov(2X+Y,Y) \end{aligned}$$

with $X \equiv M_{n-1}$ and $Y \equiv (S_n - M_{n-1}) \cdot I(M_{n-1} < S_n)$ (where $I(A)$ is the indicator function of the event A), we obtain that $\mathbb{V}\mathrm{ar}(M_n)$ equals $\mathbb{V}\mathrm{ar}(M_{n-1}) + j_n$, where

$$j_n := \mathbb{C}ov(2M_{n-1} + (S_n - M_{n-1}) \cdot I(M_{n-1} < S_n), (S_n - M_{n-1}) \cdot I(M_{n-1} < S_n)),$$

and therefore the proof is complete if we can show that $j_n \geq 0$. For $\underline{x} := (x_1, \cdots, x_n) \in \mathbb{R}^n$, we set $s_n \equiv s_n(\underline{x}) := x_1 + \cdots + x_n$ and $m_n \equiv m_n(\underline{x}) = \max(0, s_1, \cdots, s_n)$, and we define functions

$$r_n(\underline{x}) = 2m_{n-1} + (s_n - m_{n-1}) \cdot I(s_n > m_{n-1}), t_n(\underline{x}) = (s_n - m_{n-1}) \cdot I(s_n > m_{n-1}),$$

so that we have that $j_n = \mathbb{C}ov(r_n(\underline{X}), t_n(\underline{X}))$. Hence, we wish to prove that r_n and t_n are concordant in all coordinates. We shall show that both functions are increasing in all their coordinates. To this end first rewrite $t_n(\underline{x})$ as $\max(\hat{t}_n(\underline{x}), 0)$, where

$$\hat{t}_n(\underline{x}) = \min(x_1 + \cdots + x_{n-1}, x_2 + \cdots + x_{n-1}, \cdots, 0) + x_n, \tag{4.1}$$

which is evidently increasing in all its coordinates. Finally, regarding r_n, we notice that since $r_n(\underline{x}) = 2m_{n-1} + t_n(\underline{x})$ and the fact that the term $2m_{n-1} = 2\max(x_1 + \cdots + x_{n-1}, \cdots, 0)$ is increasing, we see that so is r_n, and we are done. $\square$

Continuous-time case

We now consider a Lévy process $\{S_t\}_{t\geq 0}$, as well as its reflection at 0, denoted by $\{V_t\}_{t\geq 0}$. We wish to extend Proposition 4.1 and Theorem 4.3 to Lévy processes, that is, we wish to prove that $\zeta(\cdot)$ is concave, and $\xi(\cdot)$ is increasing. We prove the former by showing that for given $0 \leq x < y < z$ we have

$$\frac{\zeta(y) - \zeta(x)}{y - x} \geq \frac{\zeta(z) - \zeta(x)}{z - x},$$

which is an alternative characterization of concavity. Throughout, we assume that $\mathbb{E}S_1 < 0$ and $\mathbb{E}S_1^2 < \infty$, which is a natural assumption, since it implies that $\lim_{t\to\infty} \zeta(t) < \infty$, as proven in [1, Cor. 4.1].

Let $0 \leq x < y < z$ be given, and let $T \in \mathbb{R}$ be any number larger than z. In the sequel we use bold fonts to denote the corresponding process between 0 and T; for instance, $\boldsymbol{S} := \{S_t\}_{0\leq t\leq T}$. Define the one-sided reflection mapping $\mathscr{S} : D[0, T] \to D[0, T]$ by

$$\mathscr{S}[\boldsymbol{x}](t) := x(t) - \inf_{s\leq t} x(s) \qquad \text{for } \boldsymbol{x} \in D[0, T].$$

This means that the value of the reflected process at time t, that is, V_t, is alternatively written as $\mathscr{S}[\boldsymbol{S}](t)$.

We define the sequence $\boldsymbol{S}^n := \{S_t^n\}_{t\geq 0}$ by $S_t^n = S_{\lfloor nt\rfloor/n}$, $n \in \mathbb{N}, 0 \leq t \leq T$, which, as shown below, approximates the Lévy process $\boldsymbol{S}$ sufficiently well for our purposes. We also introduce the reflected version $V_t^n = \mathscr{S}[\boldsymbol{S}^n](t)$ of the elements of the sequence $\boldsymbol{S}^n$. Let $\zeta^n(\cdot)$ and $\xi^n(\cdot)$ be defined in a self-evident manner as piecewise constant functions.

We prove our claims on $\zeta(\cdot)$ and $\xi(\cdot)$ by first showing that $\mathscr{S}[\boldsymbol{S}^n]$ converges weakly to $\mathscr{S}[\boldsymbol{S}]$ in the Skorokhod topology, by which we mean the J_1-topology on $D[0, T]$; see [15] for background on the J_1-topology. This result will be used to prove uniform convergence of the $\zeta^n(\cdot)$ and $\xi^n(\cdot)$ functions, which is needed in order to extend our discrete-time results to continuous time.

Lemma 4.4. *$V_t^n \xrightarrow{\mathcal{D}} V_t$, as $n \to \infty$.*

Proof. First we prove $\boldsymbol{S}^n \xrightarrow{\mathcal{D}} \boldsymbol{S}$, $n \to \infty$ in $D[0, T]$ equipped with the Skorokhod topology, under the assumption that $\mathbb{E}S_1 = 0$ (which we later generalize to any value of $\mathbb{E}S_1 \neq 0$). To this end, we need to prove convergence of the corresponding finite-dimensional distributions, as well as tightness. We notice that there is pointwise convergence, i.e. $S_t^n = S_{\lfloor nt\rfloor/n} \to S_t$ *a.s.*

as $n \to \infty$, as a direct consequence of the fact that $\boldsymbol{S}$ is right-continuous. Furthermore, for $s < t$,

$$(S_t^n - S_s^t, S_s^n) = (S_{\lfloor nt \rfloor/n} - S_{\lfloor ns \rfloor/n}, S_{\lfloor ns \rfloor/n}) \xrightarrow{\mathcal{D}} (S_t - S_s, S_s),$$

applying (i) $S_t^n \to S_t$ $a.s.$, (ii) independence of the components of this random vector, and (iii) [5, Thm. 3.2]. The case with more than two time points is dealt with analogously, and we have thus proved convergence of the finite-dimensional distributions. Regarding tightness, we have, for $t_1 \le t \le t_2$ and $\sigma^2 := \mathbb{V}\mathrm{ar}(S_1)$,

$$\mathbb{E}(S_t^n - S_{t_1}^n)^2 (S_{t_2}^n - S_t^n)^2 = \frac{\sigma^4}{n^2}(\lfloor nt \rfloor - \lfloor nt_1 \rfloor)(\lfloor nt_2 \rfloor - \lfloor nt \rfloor) \le \sigma^4 (t_2 - t_1)^2 ,$$

where the last inequality is due to [5, Eqns. (16.4)-(16.5)]. Tightness now follows as a direct application of [5, Thm. 15.6].

The case where $\mathbb{E}S_1 =: \mu \neq 0$ follows by defining processes $\hat{\boldsymbol{S}}^n$ through

$$\hat{S}_t^n := S_{\lfloor nt \rfloor/n} - \frac{\mu \lfloor nt \rfloor}{n},$$

and using the above to conclude that $\hat{\boldsymbol{S}}^n \xrightarrow{\mathcal{D}} \hat{\boldsymbol{S}}$. Furthermore, $\mu \lfloor nt \rfloor / n \to \mu t$ uniformly, and therefore also in the Skorokhod topology. Since $\{\mu t\}$ is continuous, the functions $\{S_t + \mu t\}$ and $\{\mu t\}$ have no common discontinuity points and therefore

$$\boldsymbol{S}^n = \left\{ S_{\lfloor nt \rfloor/n} - \frac{\mu \lfloor nt \rfloor}{n} \right\} + \left\{ \frac{\mu \lfloor nt \rfloor}{n} \right\} \xrightarrow{\mathcal{D}} \{S_t - \mu t\} + \{\mu t\} = \boldsymbol{S}.$$

This completes to proof of the weak convergence $\boldsymbol{S}^n \xrightarrow{\mathcal{D}} \boldsymbol{S}$.

Next, we use the Skorokhod Representation Theorem, i.e. Thm. 3.2.2 in [15], to construct a sequence of processes

$$\tilde{\boldsymbol{S}}^n = \left\{ \tilde{S}_s^n \right\}_{s \ge 0} \quad n \in \mathbb{N},$$

with $\tilde{\boldsymbol{S}}^n =_{\mathscr{D}} \boldsymbol{S}^n$ such that

$$\lim_{n \to \infty} \tilde{\boldsymbol{S}}^n = \tilde{\boldsymbol{S}} \quad a.s. \text{ in the Skorokhod topology on } D[0,T],$$

where $\tilde{\boldsymbol{S}} =_{\mathscr{D}} \boldsymbol{S}$. Since $\mathscr{S}$ is continuous in the Skorokhod topology in [15, Thm. 13.5.1], we have

$$\lim_{n \to \infty} \left\{ \mathscr{S}[\tilde{\boldsymbol{S}}^n] \right\} = \left\{ \mathscr{S}[\tilde{\boldsymbol{S}}] \right\} \quad a.s.$$

Furthermore, since $\mathbb{P}(\Delta\mathscr{S}[\tilde{\boldsymbol{S}}^n](t) \neq 0) \leq \mathbb{P}(\Delta\tilde{S}_t \neq 0) = 0$ for all $t \geq 0$ we conclude, relying on [5, p. 121], that $\mathscr{S}[\tilde{\boldsymbol{S}}^n](t) \to \mathscr{S}[\tilde{\boldsymbol{S}}](t)$ $a.s.$ as $n \to \infty$, $0 \leq t \leq T$. Since almost sure convergence implies weak convergence, it holds that $\mathscr{S}[\tilde{\boldsymbol{S}}^n](t) \xrightarrow{\mathcal{D}} \mathscr{S}[\tilde{\boldsymbol{S}}](t)$ which together with $\tilde{\boldsymbol{S}} =_{\mathscr{D}} \boldsymbol{S}$ implies $\mathscr{S}[\boldsymbol{S}^n](t) \xrightarrow{\mathcal{D}} \mathscr{S}[\boldsymbol{S}](t)$, $0 \leq t \leq T$, or, in other words, $V_t^n \xrightarrow{\mathcal{D}} V_t$ for all $0 \leq t \leq T$. □

Lemma 4.5. *As $n \to \infty$,*

$$\sup_{0 \leq y < \infty} |\zeta^n(y) - \zeta(y)| \to 0.$$

As $n \to \infty$, for $a, b \geq 0$,

$$\sup_{a \leq y \leq b} |\xi^n(y) - \xi(y)| \to 0\,.$$

Proof. $\zeta^n(t) \to \zeta(t)$ follows from Lemma 4.4 and dominated convergence, using that $V_t^n \leq \sup_{t \leq T} S_t - \inf_{t \leq T} S_t$; here realize that $\mathbb{E}[\sup_{t \leq T} S_t] < \infty$ (because of the fact that $\mathbb{E}[\sup_{t \leq T} S_t] \leq \lim_{t \to \infty} \mathbb{E}V_t < \infty$), and that we also have $\mathbb{E}[-\inf_{t \leq T} S_t] < \infty$ (due to [2, Lemma IX.3.3]).

The stated uniform convergence is now a consequence of Corollary 1.7, after extending the function $\zeta^n(t)$ and $\zeta(t)$ to the negative half-line by equating them to 0 for $t < 0$, and by noticing that $\mathbb{E}V_t^n \leq \mathbb{E}V_t \leq \lim_{t \to \infty} \mathbb{E}V_t < \infty$.

Similarly, the result for $\xi(t)$ is a consequence of Corollary 1.7, after observing that both $\mathbb{E}(\sup_{t \leq T} S_t)^2$ and $\mathbb{E}[(\inf_{t \leq T} S_t)^2]$ are finite, as follows from [2, Lemma IX.3.3]. □

To prove that the function $\zeta(\cdot)$ is concave, we have to circumvent the difficulty that the functions $\zeta^n(\cdot)$, being piecewise constant, *themselves* are not concave. This is done by defining a linear interpolation, which *is* concave, see (4.2) below.

Note that, with a slight abuse of notation, from now on we allow the one-sided reflection mapping to be applied to sequences, so that

$$\mathscr{S}[\boldsymbol{a}](n) = a_n - \min_{0 \leq i \leq n} a_i \qquad \boldsymbol{a} = (a_i)_{i=0}^{\infty} \in \mathbb{R}^{\infty}.$$

Theorem 4.6. *$\zeta(t)$ is concave, and $\xi(t)$ is increasing for Lévy processes (with one-sided reflection).*

Proof. We begin by proving the claimed concavity of $\zeta(\cdot)$. Fix $n \in \mathbb{N}$ and consider a sequence of i.i.d. random variables $\{Y_i^n\}_{i=1}$ such that $Y_1^n =_{\mathscr{D}} S_{1/n}$. Then

$$\boldsymbol{S}^n \overset{\mathscr{D}}{=} \left\{ \sum_{i=1}^{\lfloor nt \rfloor} Y_i^n \right\}_{t \geq 0},$$

defining empty sums as 0. Now consider the random walk $T_m^n := \sum_{i=1}^m Y_i^n$, and its reflected version $(\mathscr{S}[\boldsymbol{T}^n](k))_{k=1}^\infty$, and set $s^n(k) := \mathbb{E}[\mathscr{S}[\boldsymbol{T}^n](k)]$. We know from Proposition 4.1 that the sequence $(s^n(m))_{m=1}^\infty$ is concave, and hence so is the following function (which linearly interpolates):

$$\bar{\zeta}^n(t) := n(s^n(\lfloor nt\rfloor + 1) - s^n(\lfloor nt\rfloor))t + (\lfloor nt\rfloor + 1)s^n(\lfloor nt\rfloor) - s^n(\lfloor nt\rfloor + 1)\lfloor tn\rfloor. \tag{4.2}$$

Note that $\bar{\zeta}^n(\lfloor nt\rfloor/n) = s^n(\lfloor nt\rfloor) = \mathbb{E}[V_t^n]$ and $\bar{\zeta}^n((\lfloor nt\rfloor + 1)/n) = s^n(\lfloor nt\rfloor + 1) = \mathbb{E}[V_{t+\frac{1}{n}}^n]$, the latter being seen by realizing that

$$\begin{aligned} s^n(\lfloor nt\rfloor + 1) &= \mathbb{E}\mathscr{S}[\boldsymbol{T}^n](\lfloor nt\rfloor + 1) = \mathbb{E}\mathscr{S}[\boldsymbol{T}^n](\lfloor n(t+1/n)\rfloor) \\ &= \mathbb{E}\mathscr{S}\left[\boldsymbol{S}^n\right](t+1/n) = \mathbb{E}[V_{t+\frac{1}{n}}^n]. \end{aligned}$$

By concavity of $\bar{\zeta}^n(\cdot)$, we have, for $x < y < z$ and any $n \in \mathbb{N}$,

$$\frac{\bar{\zeta}^n(y) - \bar{\zeta}^n(x)}{y - x} \geq \frac{\bar{\zeta}^n(z) - \bar{\zeta}^n(x)}{z - x}.$$

Since n was arbitrary, we may let n approach infinity to obtain

$$\frac{\zeta(y) - \zeta(x)}{y - x} \geq \frac{\zeta(z) - \zeta(x)}{z - x},$$

using $\zeta^n(t) = \bar{\zeta}^n(\lfloor nt\rfloor)/n) \leq \bar{\zeta}^n(t) \leq \bar{\zeta}^n(\lceil nt\rceil)/n) = \zeta^n(t + 1/n)$ and the uniform convergence established in Lemma 4.5.

Next, we define $\xi^n(t) := \mathbb{V}\mathrm{ar}V_t^n$, and $v^n(k) = \mathbb{V}\mathrm{ar}(\mathscr{S}[T^n])$. From Proposition 4.1 we have, for $t_1 \leq t_2$, $\xi^n(t_1) = v^n(\lfloor nt_1\rfloor) \leq v^n(\lfloor nt_2\rfloor) = \xi^n(t_2)$, and letting n tend to infinity and invoking the convergence of ξ^n, as given by Lemma 4.5, we conclude that $\xi(t_1) \leq \xi(t_2)$. □

5 Two-sided reflection: solution of Lindley recursion in discrete time

Let, as before, $\boldsymbol{X} = (X_n)_{n=1}^\infty \in \mathbb{R}^\infty$ be an i.i.d. sequence, and $S_0 := 0$, $S_n = X_1 + \cdots X_n$, for $n \geq 1$. Where the previous sections studied the one-sided Lindley recursion, we now consider a variant in which there is reflection at $K > 0$ as well:

$$V_{n+1} = 0 \vee (V_n + X_{n+1}) \wedge K;$$

we say that the random walk has two reflecting barriers, viz. 0 and K. We write the V_n obtained through this procedure as $\mathscr{D}[\boldsymbol{S}](n)$ (analogously to $\mathscr{S}[\boldsymbol{S}](n)$).

In the discrete-time, one-sided case, as mentioned before, the Lindley recursion was solved through

$$\mathscr{S}[\boldsymbol{s}](n) = s_n - \min_{0\le i\le n} s_i,$$

for $\boldsymbol{s} = (s_i)_{i=0}^{\infty} \in \mathbb{R}^{\infty}$. Our first goal is to find the counterpart of this solution for the case of two-sided reflection. This is done in the following result. We denote, for a finite index set A,

$$\min_{j\in A}(a_j, b_k) := \min_{j\in A} a_j \wedge b_k.$$

Proposition 5.1. *The solution of the two-sided reflection is given by*

$$\mathscr{D}[\boldsymbol{s}](n) = \max_{k\in\{0,\cdots,n\}} \left(\min_{j\in\{k,\cdots,n\}} (s_n - s_k, K + s_n - s_j) \right) \tag{5.1}$$

Proof. We prove the claim by induction. For $n = 1$ we indeed have

$$\begin{aligned} &\max_{k\in\{0,1\}} \big(\min_{j\in\{k,1\}} (s_1 - s_k \wedge K + s_1 - s_j) \big) \\ =\ & \max(\min(s_1, K + s_1, K), \min(0, K)) = 0 \vee x_1 \wedge K = \mathscr{D}[\boldsymbol{s}](1). \end{aligned}$$

Now, assume (5.1) holds for some n. We first focus on the case $x_{n+1} \le 0$. Then we have that

$$\begin{aligned} &\mathscr{D}[\boldsymbol{s}](n+1) = v_{n+1} = 0 \vee (v_n + x_{n+1}) \wedge K = 0 \vee (v_n + x_{n+1}) \\ =\ & 0 \vee \left(\max_{k\in\{0,\cdots,n\}} \left(\min_{j\in\{k,\cdots,n\}} (s_n - s_k, K + s_n - s_j) \right) + x_{n+1} \right) = \\ =\ & 0 \vee \left(\max_{k\in\{0,\cdots,n\}} \left(\min_{j\in\{k,\cdots,n\}} (s_{n+1} - s_k, K + s_{n+1} - s_j) \right) \right). \end{aligned} \tag{5.2}$$

Since $x_{n+1} \le 0$, we have

$$\min_{j\in\{k,\cdots,n+1\}} s_{n+1} - s_j = \min_{j\in\{k,\cdots,n\}} s_{n+1} - s_j,$$

so that (5.2) equals

$$\begin{aligned} &0 \vee \left(\max_{k\in\{0,\cdots,n\}} \left(\min_{j\in\{k,\cdots,n+1\}} (s_{n+1} - s_k, K + s_{n+1} - s_j) \right) \right) = \\ =\ & \max_{k\in\{0,\cdots,n+1\}} \left(\min_{j\in\{k,\cdots,n+1\}} (s_{n+1} - s_k, K + s_{n+1} - s_j) \right), \end{aligned} \tag{5.3}$$

as desired. Similarly, when $x_{n+1} > 0$ we have:

$$
\begin{aligned}
v_{n+1} &= 0 \vee (v_n + x_{n+1}) \wedge K = (v_n + x_{n+1}) \wedge K \\
&= \left(\max_{k \in \{0, \cdots, n\}} \left(\min_{j \in \{k, \cdots, n\}} (s_n - s_k, K + s_n - s_j) \right) + x_{n+1} \right) \wedge K \\
&= \max_{k \in \{0, \cdots, n\}} \left(\min_{j \in \{k, \cdots, n\}} (s_{n+1} - s_k, K + s_{n+1} - s_j) \wedge K \right),
\end{aligned}
$$

which equals (5.3) as well, as desired. This finishes the proof. □

Remark 5.2. To see why the doubly-reflected process has the particular form (5.1), we may, for $n \geq k$, define w_n^k to be the value obtained by applying the recursion $w_{n+1}^k = (w_n^k + x_{n+1}) \wedge K$ to the increments $x_{k+1}, x_{k+2}, \ldots$, with $w_k^k = 0$. Let v_n be the sequence of outcomes of the two-sided reflection.

Then $w_n^k = \min_{j \in \{k, \cdots, n\}} (s_n - s_k, K + s_n - s_j)$, and obviously $w_n^k \leq v_n$. But v_n has to be one of the w_n^k for some $k \in \{0, \cdots, n\}$, namely the largest i such that $v_i = 0$. Therefore $v_n = \max_{k \in \{0, \cdots, n\}} w_n^k$, so that we obtain (5.1). This explains why this specific expression comes out. ♢

Next, we present an alternative expression for $\mathscr{D}[\boldsymbol{s}]$, which we will need when we treat the continuous-time case.

Proposition 5.3. *The solution of the two-sided reflection is given by*

$$
\mathscr{D}[\boldsymbol{s}](n) = \min_{k \in \{0, \ldots, n\}} \left[\left((s_n - s_k + K) \wedge \max_{i \in \{0, \ldots, n\}} (s_n - s_i) \right) \vee \max_{i \in \{k, \ldots, n\}} (s_n - s_i) \right] \tag{5.4}
$$

Proof. The proof is again by induction. The case $n = 1$ is a matter of straightforward verification. Next, assume the stated holds for some n. Then we have

$$
\begin{aligned}
&= 0 \vee \Big(\min_{k\in\{0,\dots,n\}} \big[\big((s_n - s_k + K) \wedge \max_{i\in\{0,\dots,n\}} (s_n - s_i) \big) \\
&\qquad\qquad \vee \max_{i\in\{k,\dots,n\}} (s_n - s_i) \big] + x_{n+1} \Big) \wedge K \\
&= 0 \vee \min_{k\in\{0,\dots,n\}} \Big[\big((s_{n+1} - s_k + K) \wedge \max_{i\in\{0,\dots,n\}} (s_{n+1} - s_i) \big) \\
&\qquad\qquad \vee \max_{i\in\{k,\dots,n\}} (s_{n+1} - s_i) \Big] \wedge K \\
&= \min_{k\in\{0,\dots,n\}} \Big[\big((s_{n+1} - s_k + K) \wedge \max_{i\in\{0,\dots,n\}} ((s_{n+1} - s_i) \vee 0) \big) \\
&\qquad\qquad \vee \max_{i\in\{k,\dots,n\}} ((s_{n+1} - s_i) \vee 0) \Big] \wedge K \\
&= \min_{k\in\{0,\dots,n\}} \Big[\big((s_{n+1} - s_k + K) \wedge \max_{i\in\{0,\dots,n+1\}} (s_{n+1} - s_i) \big) \\
&\qquad\qquad \vee \max_{i\in\{k,\dots,n+1\}} (s_{n+1} - s_i) \Big] \wedge K. \qquad (5.5)
\end{aligned}
$$

We notice that

$$
\begin{aligned}
&\big((s_{n+1} - s_k + K) \wedge \max_{i\in\{0,\dots,n+1\}} (s_{n+1} - s_i) \big) \vee \max_{i\in\{k,\dots,n+1\}} (s_{n+1} - s_i) \\
&\qquad = \begin{cases} \max_{i\in\{0,\dots,n+1\}} (s_{n+1} - s_i) & \text{if } k = 0; \\ \max_{i\in\{0,\dots,n+1\}} (s_{n+1} - s_i) \wedge K & \text{if } k = n+1, \end{cases}
\end{aligned}
$$

so that (5.5) equals

$$
\min_{k\in\{0,\dots,n+1\}} \big[\big((s_{n+1} - s_k + K) \wedge \max_{i\in\{0,\dots,n+1\}} (s_{n+1} - s_i) \big) \vee \max_{i\in\{k,\dots,n+1\}} (s_{n+1} - s_i) \big]
$$

This proves the claim. □

The expressions (5.1) and (5.4) provide two solutions to the two-sided Lindley recursion. Since the latter is a discrete-time analogue of the two-sided reflection mapping found in [10], Proposition 5.1 suggests an alternative expression for the two-sided reflection mapping. Our next goal is to formulate and prove this. We do this in the next section.

6 Two-sided reflection: solution of Lindley recursion in continuous time

The starting point of two-sided reflection in 0 and $K > 0$ in the continuous-time case, is the Skorokhod problem. Given $\psi \in D[0,\infty)$ there exists a functional $\mathscr{D}[\boldsymbol{\psi}]$ taking only values in $[0, K]$ and non-decreasing functions $\boldsymbol{\eta}_\ell$ and $\boldsymbol{\eta}_u$ such that $\mathscr{D}[\boldsymbol{\psi}] = \boldsymbol{\psi} + \boldsymbol{\eta}_\ell - \boldsymbol{\eta}_u$ and

$$\int_0^\infty I(\mathscr{D}[\boldsymbol{\psi}](s) > 0)\mathrm{d}\eta_\ell(s) = 0, \quad \int_0^\infty I(\mathscr{D}[\boldsymbol{\psi}](s) < K)\mathrm{d}\eta_u(s) = 0.$$

The triple $(\mathscr{D}[\boldsymbol{\psi}], \eta_\ell, \eta_u)$ is said to *solve the Skorokhod problem* for $\boldsymbol{\psi}$ on $[0, K]$, and we think of $\mathscr{D}[\boldsymbol{\psi}]$ as $\boldsymbol{\psi}$ reflected at 0 and K. The existence and uniqueness of such a triple was established in [14], and explicit solutions were given in [11] and [10], the simplest of which is

$$\mathscr{D}[\boldsymbol{\psi}](t) = \psi(t) - \sup_{s\in[0,t]} \left[\left((\psi(s) - K)) \vee \inf_{u\in[0,t]} \psi(u) \right) \wedge \inf_{u\in[s,t]} \psi(u) \right], \quad (6.1)$$

where we assume $\psi(0) = 0$; notice that this is the continuous-time counterpart of (5.4). In view of Propositions 5.1 and 5.3 it seems reasonable to conjecture that $\mathscr{D} = \mathscr{M}$, where

$$\mathscr{M}[\psi](t) := \sup_{s\in[0,t]} \left[(\psi(t) - \psi(s)) \wedge \inf_{u\in[s,t]} (K + \psi(t) - \psi(u)) \right]. \quad (6.2)$$

We prove this by first showing that $\mathscr{M}$ is Lipschitz-continuous in the J_1 topology.

Lemma 6.1. *The mapping $\mathscr{M}$ is Lipschitz-continuous in the uniform and J_1 metrics as a mapping from $D[0, T]$ for $T \in [0, \infty]$, with constant 2.*

Proof. We follow the proof of Corollary 1.5 in [11] closely. Fix $T < \infty$. We begin by proving Lipschitz-continuity in the uniform metric. Define

$$R_t[\boldsymbol{\psi}](s) := \left[(-\psi(s)) \wedge \inf_{u\in[s,t]} (K - \psi(u)) \right]; \quad S[\boldsymbol{\psi}](t) := \sup_{s\in[0,t]} R_t[\boldsymbol{\psi}](s). \quad (6.3)$$

For $\psi_1, \psi_2 \in D[0, T]$ we have

$$\begin{aligned} S[\boldsymbol{\psi_1}](t) - S[\boldsymbol{\psi_2}](t) &\le \sup_{s\in[0,t]} (R_t[\boldsymbol{\psi_1}](s) - R_t[\boldsymbol{\psi_2}](s)) \\ &\le \sup_{s\in[0,t]} \left[|-\psi_1(s) - (-\psi_2(s))| \vee \left| \inf_{u\in[s,t]} (K - \psi_1(u)) - \inf_{u\in[s,t]} (K - \psi_2(u)) \right| \right] \\ &\le \| \boldsymbol{\psi_1} - \boldsymbol{\psi_2} \|_T . \end{aligned}$$

The same inequality applies to $S[\boldsymbol{\psi_2}](t) - S[\boldsymbol{\psi_2}](t)$, so that taking the supremum leads to

$$\| S[\boldsymbol{\psi_1}] - S[\boldsymbol{\psi_2}] \|_T \le \| \boldsymbol{\psi_1} - \boldsymbol{\psi_2} \|_T ,$$

and this proves Lipschitz-continuity, with constant 2:

$$\| \mathscr{M}[\boldsymbol{\psi_1}] - \mathscr{M}[\boldsymbol{\psi_2}] \|_T \leq \| \boldsymbol{\psi_1} - \boldsymbol{\psi_2} \| + \| S[\boldsymbol{\psi_1}] - S[\boldsymbol{\psi_t}] \|_T \leq 2 \; \| \boldsymbol{\psi_1} - \boldsymbol{\psi_2} \|_T \; .$$

We now turn to the J_1-metric, and we let $\mathcal{M}$ denote the class of strictly increasing continuous functions from $[0, T]$ onto itself with continuous inverse. An elementary verification yields that for $\boldsymbol{\psi} \in D[0, T]$ and $\boldsymbol{\lambda} \in \mathcal{M}$ we have $\mathscr{M}[\psi \circ \boldsymbol{\lambda}] = \mathscr{M}[\psi] \circ \boldsymbol{\lambda}$. With $\boldsymbol{e}$ being the identity, this leads to

$$\begin{aligned} d_{J_1}(\mathscr{M}[\boldsymbol{\psi_1}], \mathscr{M}[\boldsymbol{\psi_2}]) &= \inf_{\boldsymbol{\lambda} \in \mathcal{M}} \{ \| \; \mathscr{M}[\boldsymbol{\psi_1}] \circ \boldsymbol{\lambda} - \mathscr{M}[\boldsymbol{\psi_2}] \; \|_T \vee \| \; \boldsymbol{\lambda} - \boldsymbol{e} \; \|_T \} \\ &= \inf_{\boldsymbol{\lambda} \in \mathcal{M}} \{ \| \; \mathscr{M}[\boldsymbol{\psi_1} \circ \boldsymbol{\lambda}] - \mathscr{M}[\boldsymbol{\psi_2}] \; \|_T \vee \| \; \boldsymbol{\lambda} - \boldsymbol{e} \; \|_T \} \\ &\leq \inf_{\boldsymbol{\lambda} \in \mathcal{M}} \{ 2 \; \| \; \boldsymbol{\psi_1} \circ \boldsymbol{\lambda} - \boldsymbol{\psi_2} \; \|_T \vee \| \; \boldsymbol{\lambda} - \boldsymbol{e} \; \|_T \} \\ &\leq 2 d_{J_1}(\boldsymbol{\psi_1}, \boldsymbol{\psi_2}), \end{aligned}$$

where we used the Lipschitz-continuity in the uniform metric. This proves Lipschitz-continuity in the J_1 metric, again with constant 2; it is valid for every $T < \infty$ and hence also for $T = \infty$. □

We are now ready to prove that $\mathscr{D} = \mathscr{M}$.

Theorem 6.2. *For $\boldsymbol{\psi} \in D[0, \infty)$ we have $\mathscr{D}[\boldsymbol{\psi}](t) = \mathscr{M}[\boldsymbol{\psi}](t)$.*

Proof. Let $\boldsymbol{\psi} \in D[0, \infty)$ be given, and define $\boldsymbol{\gamma}_n$ and $\boldsymbol{\psi}_n$ by $\gamma_n(t) := \lfloor nt \rfloor / n$, $\psi_n(t) := \psi(\gamma_n(t))$. Since $\boldsymbol{\gamma}_n \to \boldsymbol{e}$ in the uniform topology, we have $\boldsymbol{\gamma}_n \to_{d_{J_1}} \boldsymbol{e}$ and hence $(\boldsymbol{\psi}, \boldsymbol{\gamma}_n) \to (\boldsymbol{\psi}, \boldsymbol{e})$ in the strong version of the J_1 topology (see p. 83 in [15]). Since $\boldsymbol{e}$ is strictly increasing we may apply Theorem 13.2.2 in [15] to obtain $\boldsymbol{\psi}_n \to_{d_{J_1}} \boldsymbol{\psi}$. Fix $t < T$, and consider $\boldsymbol{\psi}$ as element of $D[0, T]$. Since the image $\psi_n([0, T])$ is finite, we may apply Propositions 5.1 and 5.3, in conjunction with (6.1), to obtain $\mathscr{D}[\boldsymbol{\psi}_n] = \mathscr{M}[\boldsymbol{\psi}_n]$. Next, we let $n \to \infty$ and use the J_1-continuity of the $\mathscr{D}$ mapping proved in [11], and the J_1-continuity of $\mathscr{M}$ proved to obtain Lemma 6.1. We thus establish the stated. □

Remark 6.3. Letting $K \to \infty$ yields $\sup_{s \in [0,t]} \left[(\psi(t) - \psi(s)) \right]$, which is indeed the standard one-sided reflection, $\mathscr{S}$. ♢

7 Two-sided reflection: structural properties

In this section, we use the results proved in Sections 5 − 6 to prove that the mean value of the position of a reflected Lévy process, on which a double reflection is imposed, is an increasing and concave function. We thus establish the 'two-sided counterpart' of the result presented in [8].

Lemma 7.1. *Let $\boldsymbol{x} \in \mathbb{R}^\infty$ be a sequence of real numbers, with cumulative sums $\boldsymbol{s} \in \mathbb{R}^\infty$. Define, for a given $m \in \mathbb{N}$, $\boldsymbol{s}_m = (s_{n,m})_{n\geq 0}$, where $s_{n,m} := s_{m+n} - s_m$. For $n \in \mathbb{N}$ we have*

$$\mathscr{D}[\boldsymbol{s}](m+n) - \mathscr{D}[\boldsymbol{s}_m](n) \geq 0, \tag{7.1}$$

and for $n_1, n_2 \in \mathbb{N}$,with $n_1 \leq n_2$,

$$\mathscr{D}[\boldsymbol{s}](m+n_2) - \mathscr{D}[\boldsymbol{s}_m](n_2) \leq \mathscr{D}[\boldsymbol{s}](m+n_1) - \mathscr{D}[\boldsymbol{s}_m](n_1). \tag{7.2}$$

Proof. By (5.1) we have

$$\begin{aligned}\mathscr{D}[\boldsymbol{s}_m](n) = & \max_{k\in\{0,\cdots,n\}} \left(\min_{j\in\{k,\cdots,n\}} (s_{n,m} - s_{k,m}, K + s_{n,m} - s_{j,m}) \right) \\ = & \quad s_{n+m} + \max_{k\in\{0,\cdots,n\}} \left(\min_{j\in\{k,\cdots,n\}} (-s_{k+m}, K - s_{j+m}) \right) \\ = & \quad s_{n+m} + \max_{k\in\{0,\cdots,n\}} \left(\min_{j\in\{k+m,\cdots,n+m\}} (-s_{k+m}, K - s_j) \right) \\ = & \quad s_{n+m} + \max_{m\leq k\leq n+m} \left(\min_{j\in\{k,\cdots,n+m\}} (-s_k, K - s_j) \right),\end{aligned}$$

so that $\mathscr{D}[\boldsymbol{s}](m+n) - \mathscr{D}[\boldsymbol{s}_m](n)$ equals

$$\begin{aligned}&\max_{k\in\{0,\ldots,n+m\}} \big(\min_{j\in\{k,\ldots,n+m\}} (-s_k, K-s_j)\big) - \max_{k\in\{m,\ldots,n+m\}} \big(\min_{j\in\{k,\ldots,n+m\}} (-s_k, K-s_j)\big) \\ &\qquad \geq 0,\end{aligned}$$

which proves (7.1). Turning to (7.2), we first notice that it is enough to prove the statement for $n_2 = n_1 + 1$, and using the notation $v_n := \mathscr{D}[\boldsymbol{s}](n)$, $v_n^m := \mathscr{D}[\boldsymbol{s_m}](n)$ we find

$$\begin{aligned}&v_{m+n_1+1} - v_{n_1+1}^m - (v_{n_1+m} - v_{n_1}^m) \\ &\quad = \quad 0 \vee v_{m+n_1} + x_{m+n_1+1} \wedge K - 0 \vee v_{n_1}^m + x_{m+n+1} \wedge K - (v_{n_1+m} - v_{n_1}^m),\end{aligned}$$

which equals

$$\begin{cases} -(v_{n_1+m} - v^m_{n_1}) & \text{if } v_{m+n_1} + x_{m+n_1+1} < 0 \\ \left.\begin{aligned} &v_{m+n_1} + x_{m+n_1+1} - (0 \vee v^m_{n_1} + x_{m+n_1+1}) \\ &\quad - (v_{n_1+m} - v^m_{n_1}) = (v^m_{n_1} + x_{m+n_1+1}) \wedge 0 \end{aligned}\right\} & \text{if } v_{m+n_1} + x_{m+n_1+1} \in [0, K] \\ \left.\begin{aligned} &K - ((v^m_{n_1} + x_{m+n_1+1}) \wedge K) - (v_{n_1+m} - v^m_{n_1}) \\ &\quad = K + (-(v^m_{n_1} - x_{m+n_1+1}) \vee (-K)) \\ &\qquad - (v_{n_1+m} - v^m_{n_1}) \\ &\quad = K + (-x_{m+n_1+1} \vee -K + v^m_{n_1}) - v_{n_1+m} \\ &\quad = (K - x_{m+n_1+1} \vee v^m_{n_1}) - v_{n_1+m} \\ &\quad = (K - x_{m+n_1+1} - v_{n_1+m}) \vee (v^m_{n_1} - v_{n_1+m}) \end{aligned}\right\} & \text{if } v_{m+n_1} + x_{m+n_1+1} > K. \end{cases}$$

Now (7.2) follows, since $-(v_{n_1+m} - v^m_{n_1}) \leq 0$. □

The results of Lemma 7.1 are easily extended to a class of piecewise constant functions.

Lemma 7.2. *Let $\boldsymbol{\psi} \in D[0, \infty)$ be of the form*

$$\psi(t) = \sum_{i=0}^{\infty} s_i I([ai, a(i+1)))(t)$$

for $\boldsymbol{s} := (s_i)_{i=0}^{\infty} \in \mathbb{R}^{\infty}$, with $s_0 \equiv 0$, $a > 0$. Define $\boldsymbol{\psi}_r \in D[0, \infty)$ by $\psi_r(t) := \psi(r+t) - \psi(r)$. Then

$$\mathscr{D}[\boldsymbol{\psi}](r+t) - \mathscr{D}[\boldsymbol{\psi}_r](t) \geq 0, \tag{7.3}$$

and, for $t_1 \leq t_2$,

$$\mathscr{D}[\boldsymbol{\psi}](r+t_2) - \mathscr{D}[\boldsymbol{\psi}_r](t_2) \leq \mathscr{D}[\boldsymbol{\psi}](r+t_1) - \mathscr{D}[\boldsymbol{\psi}_r](t_1). \tag{7.4}$$

Proof. Assume $a = 1$, and write $r = m + q$ for $q \in [0, 1)$ and $m = \lfloor r \rfloor$. Recall from Lemma 7.1 the definition of $\boldsymbol{s}_m$, viz. $s_{n,m} := s_{m+n} - s_m$. Then $\psi(t) = s_{\lfloor t \rfloor}$ and

$$\psi_r(t) = \psi(\lfloor r+t \rfloor) - \psi(\lfloor r \rfloor) = \psi(\lfloor q+t \rfloor + m) - \psi(m) = s_{m+\lfloor q+t \rfloor} - s_m = s_{\lfloor t+q \rfloor, m},$$

so that $\mathscr{D}[\boldsymbol{\psi}](t) = \mathscr{D}[\boldsymbol{s}](\lfloor t \rfloor)$ and $\mathscr{D}[\boldsymbol{\psi}_r](t) = \mathscr{D}[\boldsymbol{s}_m](\lfloor t+q \rfloor)$ where $m = \lfloor r \rfloor$ (which can be verified by making an elementary picture). Using that $\lfloor r+t \rfloor = \lfloor r \rfloor + \lfloor t+q \rfloor$, we find that

$$\mathscr{D}[\boldsymbol{\psi}](r+t) - \mathscr{D}[\boldsymbol{\psi}_r](t) = \mathscr{D}[\boldsymbol{s}](\lfloor r \rfloor + \lfloor t+q \rfloor) - \mathscr{D}[\boldsymbol{s}_m](\lfloor t+q \rfloor) \geq 0$$

and

$$\begin{aligned}\mathscr{D}[\boldsymbol{\psi}](r+t_2) - \mathscr{D}[\boldsymbol{\psi}_r](t_2) &= \mathscr{D}[\boldsymbol{s}](\lfloor r\rfloor + \lfloor t_2+q\rfloor) - \mathscr{D}[\boldsymbol{s}_m](\lfloor t_2+q\rfloor)\\ \le\ & \mathscr{D}[\boldsymbol{s}](\lfloor r\rfloor + \lfloor t_1+q\rfloor) - \mathscr{D}[\boldsymbol{s}_m](\lfloor t_1+q\rfloor) = \mathscr{D}[\boldsymbol{\psi}](r+t_1) - \mathscr{D}[\boldsymbol{\psi}_r](t_1).\end{aligned}$$

Now choose an $a \neq 1$ arbitrarily. Define $\tilde{\psi}(t) := \psi(at)$. Then $\tilde{\psi}_r(t) = \psi_{ar}(at)$, and $\mathscr{D}[\tilde{\boldsymbol{\psi}}](t) = \mathscr{D}[\boldsymbol{\psi}](at)$, and $\mathscr{D}[\tilde{\boldsymbol{\psi}}_r](t) = \mathscr{D}[\boldsymbol{\psi}_{ar}](at)$. Since (7.3) and (7.4) hold for $\tilde{\boldsymbol{\psi}}$ for any $r, t \geq 0$ we find for the given $r \geq 0$, that

$$\mathscr{D}[\boldsymbol{\psi}](r+t) - \mathscr{D}[\boldsymbol{\psi}_r](t) = \mathscr{D}[\tilde{\boldsymbol{\psi}}](r/a+t/a) - \mathscr{D}[\tilde{\boldsymbol{\psi}}_{r/a}](t/a) \geq 0$$

and similarly

$$\begin{aligned}\mathscr{D}[\boldsymbol{\psi}](r+t_2) - \mathscr{D}[\boldsymbol{\psi}_r](t_2) &= \mathscr{D}[\tilde{\boldsymbol{\psi}}](r/a+t_2/a) - \mathscr{D}[\tilde{\boldsymbol{\psi}}_{r/a}](t_2/a)\\ \le\ & \mathscr{D}[\tilde{\boldsymbol{\psi}}](r/a+t_1/a) - \mathscr{D}[\tilde{\boldsymbol{\psi}}_{r/a}](t_1/a) = \mathscr{D}[\boldsymbol{\psi}](r+t_1) - \mathscr{D}[\boldsymbol{\psi}_r](t_1).\end{aligned}$$

This proves the claim. □

We can now prove the continuous-time version of Lemma 7.1.

Lemma 7.3. *Let $\boldsymbol{\psi} \in D[0,\infty)$ and define $\boldsymbol{\psi}_r \in D[0,\infty)$ by $\psi_r(t) := \psi(r+t) - \psi(r)$. Then*

$$\mathscr{D}[\boldsymbol{\psi}](r+t) - \mathscr{D}[\boldsymbol{\psi}_r](t) \geq 0, \tag{7.5}$$

and, for $t_1 \leq t_2$,

$$\mathscr{D}[\boldsymbol{\psi}](r+t_2) - \mathscr{D}[\boldsymbol{\psi}_r](t_2) \leq \mathscr{D}[\boldsymbol{\psi}](r+t_1) - \mathscr{D}[\boldsymbol{\psi}_r](t_1). \tag{7.6}$$

Proof. Define $\gamma^n(t) := \lfloor nt\rfloor/n$, and $\psi^n(t) = \psi(\gamma^n(t))$. Then Lemma 7.2 applies to $\boldsymbol{\psi}^n$, and hence

$$\mathscr{D}[\boldsymbol{\psi}^n](r+t) - \mathscr{D}[\boldsymbol{\psi}^n_r](t) \geq 0, \tag{7.7}$$

$$\mathscr{D}[\boldsymbol{\psi}^n](r+t_2) - \mathscr{D}[\boldsymbol{\psi}^n_r](t_2) \leq \mathscr{D}[\boldsymbol{\psi}^n](r+t_1) - \mathscr{D}[\boldsymbol{\psi}^n_r](t_1). \tag{7.8}$$

Using the same argument as in the proof of Theorem 6.2, we have $\boldsymbol{\psi}^n \to_{d_{J_1}} \boldsymbol{\psi}$. We also have $\boldsymbol{\psi}^n_r \to_{d_{J_1}} \boldsymbol{\psi}_r$, since $\psi^n_r(t) = \psi^n(t+r) - \psi^n(r)$ and we regard $-\psi^n(r)$ as a constant function, which converges uniformly, and hence in the J_1-topology as well, to $-\psi(r)$. In general, addition is not continuous in the J_1 topology, but since $-\psi(r)$ is a constant function, $t \mapsto \psi(r+t)$ and $-\psi(r)$ have no common discontinuity points (where both are considered as function of

t), and we have $\boldsymbol{\psi}_r^n \to_{d_{J_1}} \boldsymbol{\psi}_r$. We wish to let n tend to infinity in (7.7) and (7.8), and we therefore assume that $r+t_1$ and $r+t_1$ are both continuity points for ψ, which implies that they are continuity points for $\mathscr{D}[\boldsymbol{\psi}]$, and also that t_1 and t_2 are continuity points for $\mathscr{D}[\boldsymbol{\psi}_r]$. Under this assumption, we let $n \to \infty$, and thus obtain (7.5) and (7.6) when r, t_1, t_2 are continuity points. However, since $\mathscr{D}$ maps càdlàg functions to càdlàg functions, we have that (7.5) and (7.6) hold for all r, t_1, t_2 whenever $\boldsymbol{\psi} \in D[0, \infty)$, as claimed. □

We can now prove the main results.

Theorem 7.4. *$\zeta(n)$ is increasing and concave for random walks (with two-sided reflection).*

Proof. Set $S_n^1 := S_{n+1} - S_1$, $\boldsymbol{S}^1 := \left\{S_n^1\right\}_{n=0}^{\infty}$, and $V_n^1 = \mathscr{D}[\boldsymbol{S}^1](n)$. By stationarity of the increments we have $\{S_n\} =_{\mathscr{D}} \left\{S_n^1\right\}$ and $\{V_n\} =_{\mathscr{D}} \left\{V_n^1\right\}$. Using (7.1) with $m = 1$ we have $V_{n+1} - V_n^1 \geq 0$, and we see that $\zeta(n)$ is increasing by taking means. Furthermore, by (7.2) we have $n \mapsto V_{n+1} - V_n^1$ is decreasing, and taking means implies that $\zeta(n)$ is concave. □

Theorem 7.5. *$\zeta(t)$ is increasing and concave for Lévy processes (with two-sided reflection).*

Proof. Define $\boldsymbol{S}^r$ by $S_t^r = S_{t+r} - S_r$. By the stationary increments we have $\boldsymbol{S}^r =_{\mathscr{D}} \boldsymbol{S}$ and $\mathscr{D}[\boldsymbol{S}^r] =_{\mathscr{D}} \mathscr{D}[\boldsymbol{S}]$. Set $V_t^r = \mathscr{D}[\boldsymbol{S}^r](t)$ and $V_t = \mathscr{D}[\boldsymbol{S}](t)$. According to (7.5) and (7.6) we have that $V_{r+t} - V_t^r \geq 0$ and also that $t \mapsto V_{r+t} - V_t^r$ is decreasing. Taking means yields the desired result. □

The following statement follows immediately from the facts that $V_{n+1} - V_n^1 \geq 0$ and $V_{s+t} - V_t^s \geq 0$.

Corollary 7.6. *For any $q \geq 0$, we have $n \mapsto \mathbb{E}V_n^q$ and $t \mapsto V_t^q$ are increasing, both for random walks and Lévy processes (with two-sided reflection).*

Remark 7.7. The reader can verify that in the argumentation of this section, we *did not use that increments are independent* — in fact all results, in particular Thms. 7.4 and 7.5, hold under the assumption of just stationary increments. We conclude that we have, in passing, extended the result by Kella and Sverchkov [8], who considered processes with stationary increments reflected at 0, to the case of two-sided reflection. ♢

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PAPER B

Subexponential Loss Rate Asymptotics for Lévy Processes

Lars Nørvang Andersen

Abstract

We consider a Lévy process reflected in barriers at 0 and $K > 0$. The loss rate is the mean time spent at the upper barrier K at time 1 when the process is started in stationarity, and is a natural continuous-time analogue of the stationary expected loss rate for a reflected random walk. We derive asymptotics for the loss rate when K tends to infinity, when the mean of the Lévy process is negative and the positive jumps are subexponential. In the course of this derivation, we achieve a formula, which is a generalization of the celebrated Pollaczeck-Khinchine formula.
Keywords finite buffer, heavy tails, Lévy process, local times, loss rate, Pollaczeck-Khinchine formula, reflection, subexponential distributions.

1 Introduction

In the papers Jelenković [12] and Pihlsgård [17], the authors examine the loss rate associated with a stochastic process obtained by reflecting a random walk in two barriers at 0 and $K > 0$, and derive asymptotic expressions for the loss rate as K tends to infinity. In particular, Jelenković [12] derives the asymptotics of the loss rate in the case of heavy tails. The continuous-time analogue of the loss rate associated with a reflected random walk, is the loss rate associated with a reflected Lévy process which is examined in Asmussen and Pihlsgård [3], where an explicit expression for the loss rate in terms of the characteristic triplet of the Lévy process is provided. Furthermore, [3] gives the asymptotic behavior of the loss rate as K tends to infinity in the case where the mean of the Lévy process is positive as well the case where the mean is negative and the jumps of the process are light-tailed, and in the Andersen and Asmussen [1] the authors examine loss rate asymptotics for centered Lévy processes. In this paper we derive asymptotics where the mean is negative and the process is heavy-tailed.

Reflected processes may be used to model waiting time processes in queues with finite capacity (Cohen [7], Cooper et al. [8], Bekker and Zwart [4], Daley [9]).

It may be used to model a finite dam or fluid model (Asmussen [2],Moran [16], Stadje [20]). Furthermore, it is used in models of network traffic or telecommunications systems involving a finite buffer (Jelenković [12], Zwart [21], Kim and Shroff [13]) and in this context the loss rate can be interpreted as the bit loss rate in a finite data buffer.

The main contribution of this paper is Theorem 3.1 which provides an asymptotic expression for the loss rate in the heavy-tailed case. In the course of the derivation of this expression, we also obtain a formula, (3.2), which is a generalization of the celebrated Pollaczeck-Khinchine formula.

The outline of the paper is as follows: In Section 2 we provide the essential background on Lévy processes, and give the formal definition of the loss rate. With the definitions and previous results settled we can state the main results in Section 3. The proofs are given in Section 4.

2 Preliminaries

Lévy Processes and the Loss rate

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A *Lévy process* $\mathbf{S} := \{S_t\}$ is a real-valued stochastic process on $\mathbb{R}$ with stationary independent increments which is continuous in probability and with $S_0 = 0$ $\mathbb{P} - a.s.$ Every Lévy process $\mathbf{S}$ is associated with a unique *characteristic triplet* (θ, σ, ν), where $\theta \in \mathbb{R}$, $\sigma \geq 0$ and ν is a measure (*the Lévy measure*) with $\int_{-\infty}^{\infty}(1 \wedge y^2)\nu(\mathrm{d}y) < \infty$ and $\nu(\{0\}) = 0$. The *Lévy exponent* is given by

$$\kappa(\alpha) = \theta\alpha + \frac{\sigma^2\alpha^2}{2} + \int_{-\infty}^{\infty} \left[e^{\alpha x} - 1 - \alpha I(|x| \leq 1)\right] \nu(\mathrm{d}x)$$

and is defined for α in $\Theta := \{\alpha \in \mathbb{C} \mid \mathbb{E}e^{\Re(\alpha)S_1} < \infty\}$. The Lévy exponent is the unique function satisfying $\mathbb{E}e^{\alpha X_t} = e^{t\kappa(\alpha)}$ and $\kappa(0) = 0$. We assume throughout this paper that $\mathbb{E}|S_1| < \infty$. We use the cadlag version of $\mathbf{S}$, which exists because of stochastic continuity. We note that this implies that $\Delta S_t := S_t - S_{t-}$ is well-defined. Standard references for Lévy processes are Bertoin [5], Kyprianou [14] and Sato [19].
We are given a Lévy process through its characteristic triplet, and reflect it in barriers at 0 and $K > 0$. The reflected process is given as part of the solution to a Skorokhod problem and is denoted $\mathbf{V^K}$. We have a decomposition

$$V_t^K = y + S_t + L_t^0 - L_t^K \tag{2.1}$$

of the reflected process started at $y \in [0, K]$ where $\mathbf{L^0} := \{L_t^0\}$ and $\mathbf{L^K} := \{L_t^K\}$ are the local times at 0, K respectively. Note that the reflected process and the local times are cadlag, so that objects such as $\Delta L_t^0 := L_t^0 - L_{t-}^0$ are well-defined and by way of being increasing, the local times are of bounded variation which allow is to decompose them into a continuous part and a jump part. For more information on Skorokhod problems, see Asmussen [2], Asmussen and Pihlsgård [3] and Andersen and Asmussen [1].

Because of the independent, identically distributed increments of $\mathbf{S}$, $\mathbf{V^K}$ has a regenerative structure which yields a stationary distribution denoted π_K. The stationary distribution satisfies:

$$\overline{\pi}_K(y) = \pi_K[y, K] = \mathbb{P}\big(S_{\tau[y-K,y)} \geq y\big), \quad 0 \leq y \leq K \tag{2.2}$$

where $\tau[u, v) = \inf\big\{t > 0 \mid S_t \notin [u, v)\big\}$. See Asmussen [2] pp. 393-394 for a derivation of this representation. When $K = \infty$, we have one-sided reflection (see Asmussen [2], IX 2a). In this case $L_t^K \equiv 0$, and $L_t^0 := (-\inf_{0 \leq v \leq t} S_v - y)^+$, and we have a result similar to (2.2) of the one-sided stationary distribution which follows from Cor. 2. IX p. 253 in Asmussen [2]:

$$\overline{\pi}_\infty(y) = \mathbb{P}\big(\sup_{t \geq 0} S_t \geq y\big) = \mathbb{P}(\tau(y) < \infty) \tag{2.3}$$

where $\tau(y) = \inf\{t > 0 : S_t \geq y\}$. Furthermore, for notational convenience we set $L_t^0 := L_t$ when $K = \infty$.
We follow the standard definitions of the classes $\mathcal{S}$ and $\mathcal{S}^*$ of distribution functions. The class $\mathcal{S}$ is defined by the requirement that $\overline{F^{*n}}(x) \sim n\overline{F}(x)$ ($F^{*n} = n$th convolution power), and $\mathcal{S}^*$ by

$$\lim_{x\to\infty} \frac{1}{\mu} \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)} \overline{F}(y) \mathrm{d}y = 2$$

where μ is the first moment of F. It is well-known that $\mathcal{S}^* \subseteq \mathcal{S}$ and using (2.3) we may apply Theorem 4.1 from Maulik and Zwart [15] to get

$$\overline{\nu}_I(K) := \int_K^\infty \overline{\nu}(y)\, \mathrm{d}y \sim |\mathbb{E}S_1| \overline{\pi}_\infty(K) \tag{2.4}$$

when $\mathbb{E}S_1 < 0$ and $\overline{\nu}_I(x) \sim \overline{F}(x)$ for some $F \in \mathcal{S}$. The latter condition is ensured by requiring that $\overline{\nu}(x) \sim \overline{F}(x)$ for some $\overline{F}(x) \in \mathcal{S}^*$. .

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The loss rate is defined as

$$\ell^K = \mathbb{E}_{\pi_K} L_1^K\,, \tag{2.5}$$

that is, as the mean of L_1^K when the process is started in stationarity.

According to Theorem 3.6 in Asmussen and Pihlsgård [3] we have the following expression of the loss rate, in terms of the characteristic triplet of the Lévy processes:

$$\ell^K = \frac{\mathbb{E}S_1}{K} \int_0^K \overline{\pi}_K(x)\, \mathrm{d}x + \frac{\sigma^2}{2K} + \frac{1}{2K} \int_0^K \pi_K(\mathrm{d}x) \int_{-\infty}^\infty \varphi_K(x,y)\nu(\mathrm{d}y), \tag{2.6}$$

where

$$\varphi_K(x,y) = \begin{cases} -(x^2 + 2xy) & \text{if } y \leq -x \\ y^2 & \text{if } -x < y < K - x \\ 2y(K-x) - (K-x)^2 & \text{if } y \geq K - x\,. \end{cases} \tag{2.7}$$

3 Main results

We start by stating the main results. The first result provides the asymptotics in the case of heavy tails and negative drift.

Theorem 3.1. *Let* $\mathbf{S}$ *be a Lévy process with Lévy measure* ν *such that* $\overline{\nu}_I(x) \sim \overline{B}(x)$ *for some* $B \in \mathcal{S}$*, and with finite negative mean:* $\mathbb{E}S_1 = \mu < 0$*. Define the conditions*

(I) $\mathbb{E}S_1^2 < \infty$ *and* $\int_K^\infty \overline{\nu}_I(y)\,\mathrm{d}y / \overline{\nu}_I(K) \in O(K)$ *.*

(II) $\overline{\nu}(K) \sim L(K)K^{-\alpha}$ *where* L *is a locally bounded slowly varying function and* $1 < \alpha < 2$*.*

If either (I) or (II) holds, then

$$\ell^K \sim \int_K^\infty \overline{\nu}(y)\mathrm{d}y \tag{3.1}$$

We remark that the requirement $\int_K^\infty \overline{\nu}_I(y)\,\mathrm{d}y/\overline{\nu}_I(K) \in O(K)$ in Theorem 3.1 is very weak. Indeed, suppose $\overline{\nu}_I(x) \sim \overline{B}(x)$ where B is either lognormal, Benktander or heavy-tailed Weibull. Then we recognize $a(x) := \int_x^\infty \overline{B}(y)\,\mathrm{d}y/\overline{B}(x)$ as the mean-excess function and it is known (see Goldie and Klüppelberg [11]), that $a(x) \in o(x)$. Furthermore, it is easily checked that the condition is satisfied when B is a Pareto or Burr distribution, provided that the second moment is finite.

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We also derive the following theorem, giving an expression for the moment generating function of the stationary distribution in the case of one-sided reflection. Recall our decomposition of the one-sided reflected process $V_t(x) = x + S_t - L_t(x)$ and that $\{L_t^c\}$ is the continuous part of the local time.

Theorem 3.2. *Suppose* $-\infty < \mu = \mathbb{E}S_1 < 0$ *then* $V := \lim_t V_t$ *exists in distribution and for* $\alpha \in \Theta$ *with* $\kappa(\alpha) < \infty$ *we have:*

$$\mathbb{E}\big[e^{\alpha V}\big] = -\frac{\alpha \mathbb{E}_{\pi_\infty} L_1^c + \mathbb{E}_{\pi_\infty}\big[\sum_{0\le s\le 1}(1 - e^{-\alpha \Delta L_s})\big]}{\kappa(\alpha)} \tag{3.2}$$

If $\mathbf{S}$ has no negative jumps, the term $\mathbb{E}_{\pi_\infty}\big[\sum_{0\le s\le 1}(1 - e^{-\alpha\Delta L_s})\big]$ disappears, and $\mathbb{E}_{\pi_\infty} L_1^c = \mathbb{E}_{\pi_\infty} L_1 = \mu$, and we see that Theorem 3.2 indeed is a generalization of Corollary 3.4 in [2] which is itself a generalization of the Pollaczeck-Khinchine formula.

4 Loss rate asymptotics in the case of negative drift and heavy tails

In this section we prove Theorem 3.1 and in the pursuit of this, we prove Theorem 3.2. We first prove Proposition 4.1, which is a set of inequalities which allow us to compare the stationary distributions in the cases of one and two-sided reflection. Next, we prove Proposition 4.2 showing that 1 is a lower bound for $\liminf_K \ell^K/\overline{\nu}_I(K)$, which is essentially half of Theorem 3.1. Lemma 4.1 and Proposition 4.3 provide a martingale, and using optional stopping of this martingale yields Theorem 3.2, which gives the m.g.f. $\mathbb{E}[e^{\alpha V}]$. We differentiate this transform in Corollary 4.1 to give us the mean of V, which is needed in the proof of Theorem 3.1.

Proposition 4.1. *Let* $\mathbf{S}$ *be a Lévy process, and let* $\overline{\pi}_K(y), \overline{\pi}_\infty(y)$ *be the tails of the reflected (one/two-sided) distributions. Then we have the following inequalities for* $x > 0, K > 0$

$$0 \le \overline{\pi}_\infty(x) - \overline{\pi}_K(x) \le \overline{\pi}_\infty(K)\,. \tag{4.1}$$

Proof. The inequalities are trivial for $x > K$. Let $0 \le x \le K$. The inequality $\overline{\pi}_K(x) \le \overline{\pi}_\infty(x)$ follows from the representations (2.2) and (2.3). The inequality $\overline{\pi}_\infty(x) - \overline{\pi}_K(x) \le \overline{\pi}_\infty(K)$, follows by dividing the sample paths of $\mathbf{S}$ which cross above x into those which do so by first passing below $K - x$, and those which stay above $K - x$. To be precise, define $\tau(y) := \inf\{t > 0 : S_t \ge y\}$ and $\sigma(y) := \inf\{t > 0 : S_t < y\}$. Then, since any path which passes below $K - x$ and then above x must pass an interval of length at least K, we have by the strong Markov property:

$$\begin{aligned}\mathbb{P}(\sigma(x-K) < \tau(x) < \infty) &\le \mathbb{P}\big(\sup_{t>0} S_{\sigma(x-K)+t} - S_{\sigma(x-K)} > K\big)\\ &= \mathbb{P}(\tau(K) < \infty) = \overline{\pi}_\infty(K)\,.\end{aligned}$$

And therefore:

$$\begin{aligned}&\overline{\pi}_\infty(x) = \mathbb{P}(\tau(x) < \infty) =\\ &\mathbb{P}(\sigma(x-K) < \tau(x) < \infty) + \mathbb{P}(\tau(x) < \sigma(x-K) < \infty) \le\\ &\overline{\pi}_K(x) + \overline{\pi}_\infty(K)\,.\end{aligned}$$

□

In our effort to prove that $\ell^K \sim \overline{\nu}_I(K)$ we need to prove that 1 is a lower bound for $\liminf_K \ell^K/\overline{\nu}_I(K)$ and an upper bound for $\limsup_K \ell^K/\overline{\nu}_I(K)$. The former holds without any regularity conditions.

Proposition 4.2. *For any Lévy process we have*

$$1 \le \liminf_K \frac{\ell^K}{\overline{\nu}_I(K)}$$

Proof. We have

$$\int_0^K \pi_k(\mathrm{d}x) \int_K^\infty (y-K+x)\nu(\mathrm{d}y) \le \ell^K$$

since the left hand side is the contribution to the local time by the jumps larger than K. Since

$$\begin{aligned}\overline{\nu}_I(K) &\le \int_K^\infty (y-K)\nu(\mathrm{d}y) + \int_0^K x\pi_k(\mathrm{d}x)\\ &= \int_0^K \pi_k(\mathrm{d}x)\int_K^\infty (y-K+x)\nu(\mathrm{d}y)\end{aligned}$$

we are done. □

Recall our decomposition $V_t(x) = x + S_t - L_t(x)$ of the one-sided reflection of the Lévy process started at x and reflected in 0 and we let $L_t^c(x)$ and $L_t^j(x)$ denote the continuous and jump parts of the local time respectively. We suppress the x's for ease of notation.

Lemma 4.1. *For $\alpha \in \Theta$ and $t > 0$ we have*

$$\mathbb{E}\Big[\sum_{0\le s\le t} \big|1-e^{-\alpha\Delta L_s}\big|\Big] < \infty \tag{4.2}$$

Proof. Setting $\underline{\Delta} L_s = \Delta L_s I(L_s \le 1)$ and $\overline{\Delta} L_s = \Delta L_s I(\Delta L_s > 1)$ we can split the sum into the contribution from the jumps of size ≤ 1 and those of size > 1 by writing

$$\mathbb{E}\Big[\sum_{0\le s\le t} \big|1-e^{-\alpha\Delta L_s}\big|\Big] = \mathbb{E}\Big[\sum_{0\le s\le t} \big|1-e^{-\alpha\underline{\Delta} L_s}\big|\Big] + \mathbb{E}\Big[\sum_{0\le s\le t} \big|1-e^{-\alpha\overline{\Delta} L_s}\big|\Big],$$

and we note that first sum on the r.h.s. is bounded, since there exists a constant c such that $|1-e^{\alpha x}| \le c|\alpha|x$ for $x \in [0,1]$ and therefore

$$\mathbb{E}\Big[\sum_{0\le s\le t} \big|1-e^{-\alpha\underline{\Delta} L_s}\big|\Big] \le c|\alpha|\mathbb{E}\Big[\sum_{0\le s\le t} \underline{\Delta} L_s|\Big] \le c|\alpha|\mathbb{E}L_t < \infty,$$

where the last inequality follows from Lemma 3.3 p. 256 in Asmussen [2]. Since

$$|1-e^{-\alpha\overline{\Delta} L_s}| = \big|I(\overline{\Delta} L_s > 0) - e^{-\alpha\overline{\Delta} L_s}\big| \le I(\overline{\Delta} L_s > 0) + e^{-\Re(\alpha)\overline{\Delta} L_s} I(\overline{\Delta} L_s > 0)$$

we have

$$\mathbb{E}\Big[\sum_{0\le s\le t}\big|1-e^{-\alpha\overline{\Delta}L_s}\big|\Big]\le$$
$$\mathbb{E}\Big[\sum_{0\le s\le t}I(\overline{\Delta}L_s>0)\Big]+\mathbb{E}\Big[\sum_{0\le s\le t}e^{-\Re(\alpha)\overline{\Delta}L_s}I(\overline{\Delta}L_s>0)\Big]$$

A jump of size of >1 at time s of the local time can only occur if the process itself makes a negative jump of size >1, and therefore $I(\overline{\Delta}L_s>0)\le I(\overline{\Delta}S_s<0)$, where $\overline{\Delta}S_s:=\Delta S_s I(\Delta S_s<-1)$, which implies

$$\mathbb{E}\Big[\sum_{0\le s\le t}I(\overline{\Delta}L_s>0)\Big]\le\mathbb{E}\Big[\sum_{0\le s\le t}I(\overline{\Delta}S_s<0)\Big]=t\int_{-\infty}^{-1}\nu(\mathrm{d}y)<\infty$$

where the last number is finite because $\mathbb{E}|S_1|<\infty$. Regarding the remaining sum, we observe that if $\Re(\alpha)\ge 0$ we have

$$\mathbb{E}\Big[\sum_{0\le s\le t}e^{-\Re(\alpha)\overline{\Delta}L_s}I(\overline{\Delta}L_s>0)\Big]\le\mathbb{E}\Big[\sum_{0\le s\le t}I(\overline{\Delta}L_s>0)\Big]$$

and the sum is finite by the inequalities above. If $\Re(\alpha)<0$ we have

$$\mathbb{E}\Big[\sum_{0\le s\le t}e^{-\Re(\alpha)\overline{\Delta}L_s}I(\overline{\Delta}L_s>0)\Big]$$
$$\le\mathbb{E}\Big[\sum_{0\le s\le t}e^{\Re(\alpha)\overline{\Delta}S_s}I(\overline{\Delta}S_s<0)\Big]=t\int_{-\infty}^{-1}e^{\Re(\alpha)y}\nu(\mathrm{d}y)<\infty\,,$$

where the last inequality follows from Theorem 25.3 i Sato [19] and the fact that $\alpha\in\Theta$. Putting everything together we have that (4.2) is finite. □

The lemma above is used in the following generalization of Cor. 3.2 p 256 in Asmussen [2].

Proposition 4.3. *Consider a Lévy process* $\mathbf{S}$*, and let* $\mathbf{V}$ *be the process reflected at* 0 *and let* $\mathbf{L^c}:=\{L_t^c\}$ *and* $\mathbf{L^j}:=L_t^j$ *be the continuous and jump part of the local time* $\mathbf{L}$*. Then for* $\alpha\in\Theta$ *and* $x\ge 0$

$$M_t:=\kappa(\alpha)\int_0^t e^{\alpha V_s(x)}\,\mathrm{d}s+e^{\alpha x}-e^{\alpha V_t(x)}+\alpha L_t^c(x)+\sum_{0\le s\le t}(1-e^{-\alpha\Delta L_s(x)}) \tag{4.3}$$

is a martingale.

Proof. For notational convenience, we write $V_s := V_s(x)$ and $L_s^c = L_s^c(x)$. Since the local time is of bounded variation, we may apply Theorem 3.1 p. 255 in Asmussen [2], to obtain that

$$\kappa(\alpha)\int_0^t e^{\alpha V_s}\,\mathrm{d}s + e^{\alpha x} - e^{\alpha V_t} + \alpha\int_0^t e^{\alpha V_s}\,\mathrm{d}L_s^c + \sum_{0\leq s\leq t} e^{\alpha V_s}(1-e^{-\alpha\Delta L_s})$$

is a local martingale. Since L_t^c can only increase when $V_t = 0$ and $\Delta L_t > 0 \Rightarrow V_t = 0$, the expression above is equal to M_t, so that M_t is a local martingale. According to Lemma 3.3 p. 35 of Protter [18] it will be a martingale if we can prove that $\mathbb{E}\sup_{s\leq t}|M_s| < \infty$. But this follows from

$$\mathbb{E}[\sup_{0\leq s\leq t}|M_s|] \leq$$
$$\kappa(\alpha)t\mathbb{E}\sup_{0\leq s\leq t}|e^{\alpha V_s}| + |e^{\alpha x}| + \mathbb{E}|e^{\alpha V_t}| + |\alpha|\mathbb{E}[L_t^c] + \mathbb{E}\sum_{0\leq s\leq t}|(1-e^{-\alpha\Delta L_s})|$$

which is finite according to lemma 3.3 of Asmussen [2] and Lemma 4.1 above. □

We are now ready to prove Theorem 3.2

Proof. The existence of V follows from Cor. 2.6 p. 253 in Asmussen [2]. Let V_0 be a r.v. independent of $\mathbf{S}$ and distributed as V, and set $x = V_0, t = 1$ in (4.3). Then $\mathbf{V}$ is stationary and by taking expectation we get

$$0 = \kappa(\alpha)\mathbb{E}_{\pi_\infty}[\int_0^1 e^{\alpha V_s}\,\mathrm{d}s] + \alpha\mathbb{E}_{\pi_\infty}L_1^c + \mathbb{E}_{\pi_\infty}\Big[\sum_{0\leq s\leq 1}(1-e^{-\alpha\Delta L_s})\Big] \Rightarrow$$
$$\kappa(\alpha)\int_0^1 \mathbb{E}_{\pi_\infty}\big[e^{\alpha V}\big]\,\mathrm{d}s + \alpha\mathbb{E}_{\pi_\infty}L_1^c + \mathbb{E}_{\pi_\infty}\Big[\sum_{0\leq s\leq 1}(1-e^{-\alpha\Delta L_s})\Big] \Rightarrow$$
$$\mathbb{E}\big[e^{\alpha V}\big] = -\frac{\alpha\mathbb{E}_{\pi_\infty}[L_1^c] + \mathbb{E}_{\pi_\infty}\big[\sum_{0\leq s\leq 1}(1-e^{-\alpha\Delta L_s})\big]}{\kappa(\alpha)}.$$

□

Next, we use the results above to obtain an expression for the mean of the stationary distribution in the case of one-sided reflection.

Corollary 4.1. *If* $\mathbf{S}$ *is square integrable then* V *is integrable and we have*

$$\mathbb{E}[V] = \frac{\mathbb{E}_{\pi_\infty}[\sum_{0\leq s\leq 1}\Delta L_s^2] - Var(S_1)}{2\mathbb{E}S_1} \tag{4.4}$$
$$= \frac{\int_{-\infty}^{\infty} y^2\nu(\mathrm{d}y) + \sigma^2 - \int_0^\infty\int_{-\infty}^{-x}(x+y)^2\nu(\mathrm{d}y)\pi_\infty(\mathrm{d}x)}{2|\mathbb{E}S_1|} \tag{4.5}$$

Proof. Since S_1 is non-degenerate, we have by Lemma 4 in Feller [10] that there exists $\epsilon > 0$ such that $\kappa(\mathrm{i}t) \neq 0$ for $t \in (-\epsilon, \epsilon) \setminus \{0\}$, and we may use (3.2) to obtain the characteristic function φ of V and we wish to show that φ is differentiable at 0. Define $g(t) := \mathbb{E}_{\pi_\infty}\big[\sum_{0\leq s\leq 1}(1 - e^{-\mathrm{i}t\Delta L_s})\big]$ and set $\ell_1 := \mathbb{E}_{\pi_\infty} L_1^c$. By Doob's inequality, we have that $\mathbb{E}S_1^2 < \infty$ implies $\mathbb{E}L_1^2 < \infty$ and therefore $\mathbb{E}_{\pi_\infty} L_1^2 < \infty$, and this implies that g is twice differentiable at 0 and we see that $g'(0) = \mathrm{i}\mathbb{E}_{\pi_\infty}\big[\sum_{0\leq s\leq t}\Delta L_s\big] = \mathrm{i}\mathbb{E}_{\pi_\infty} L_t^j$, $g''(0) = \mathbb{E}_{\pi_\infty}\big[\sum_{0\leq s\leq t}\Delta L_s^2\big]$ and $\mathrm{i}\ell_1 + g'(0) = \mathrm{i}\mathbb{E}_{\pi_\infty} L_1 = -\mathrm{i}\mathbb{E}S_1$.
Applying Proposition 4.2 and using l'Hospital's rule twice (see Prop. 4.1 the in Appendix), we have:

$$\begin{aligned}
\lim_{t\to 0}\frac{\mathbb{E}e^{\mathrm{i}tV}-1}{t} &= \lim_{t\to 0}\frac{-t\mathrm{i}\ell_1 - g(t) - \kappa(\mathrm{i}t)}{t\kappa(\mathrm{i}t)} \\
&= \lim_{t\to 0}\frac{-\mathrm{i}\ell_1 - g'(t) - \mathrm{i}\kappa'(\mathrm{i}t)}{\kappa(\mathrm{i}t) + t\mathrm{i}\kappa'(t\mathrm{i})} \\
&= \lim_{t\to 0}\frac{-g''(t) + \kappa''(\mathrm{i}t)}{\mathrm{i}\kappa'(\mathrm{i}t) + \mathrm{i}\kappa'(t\mathrm{i}) - t\kappa''(t\mathrm{i})} \\
&= \frac{-g''(0) + \kappa''(0)}{2\mathrm{i}\kappa'(0)}.
\end{aligned}$$

We see that φ is differentiable, and since V is positive, we have that V is integrable (see Prop. 1.2 in the Appendix). The first moment is

$$\mathbb{E}V = \frac{-g''(0) + \kappa''(0)}{2(-1)\kappa'(0)}$$

which is (4.4). We obtain (4.5) by conditioning on the value of the process prior to a jump. □

We are now ready for the proof of Theorem 3.1. The proof has two distinct cases, depending on whether or not the Lévy process is square integrable. If this is the case we require only mild regularity conditions. However, if the Lévy process has infinite variance, we impose stronger regularity conditions.

Proof. Because of Proposition 4.2, we only need to prove

$$\limsup_K \ell^K/\overline{\nu}_I(K) \leq 1\,.$$

Define the following:

$$\mathcal{I}_1 := \frac{\mathbb{E}S_1}{K}\int_0^K x\pi_K(\mathrm{d}x)$$
$$\mathcal{I}_2 := \frac{\sigma^2}{2K}$$
$$\mathcal{I}_3 := \frac{1}{2K}\int_0^K \pi_K(\mathrm{d}x)\int_{-\infty}^{\infty}\varphi_K(x,y)\nu(\mathrm{d}y)\,.$$

Then, because of the expression for the loss rate given by (2.6) and the inequality from Proposition 4.1 we have the following inequality:

$$\ell^K \leq \frac{\mathbb{E}S_1}{K}\int_0^K \overline{\pi}_\infty(x)\,\mathrm{d}x - \mathbb{E}S_1\overline{\pi}_\infty(K) + \mathcal{I}_2 + \mathcal{I}_3\,. \tag{4.6}$$

First, we assume (I) holds. By (2.4) we have

$$\lim_K \frac{-\mathbb{E}S_1\overline{\pi}_\infty(K)}{\overline{\nu}_I(K)} = 1\,, \tag{4.7}$$

so we will be done, if we can show

$$\limsup_K \frac{1}{\overline{\nu}_I(K)}\left[\frac{\mathbb{E}S_1}{K}\int_0^K \overline{\pi}_\infty(y)\,\mathrm{d}y + \mathcal{I}_2 + \mathcal{I}_3\right] = 0\,. \tag{4.8}$$

We start by rewriting the term in the brackets above. Using Cor. 4.1 and the assumption that $\mathbb{E}S_1^2 < \infty$ we have that $\int_0^\infty \overline{\pi}_\infty(y)\mathrm{d}y < \infty$ and using (4.4)

$$\begin{aligned}
&\frac{\mathbb{E}S_1}{K}\int_0^K \overline{\pi}_\infty(y)\,\mathrm{d}y \\
&= \frac{\mathbb{E}S_1}{K}\int_0^\infty \overline{\pi}_\infty(y)\,\mathrm{d}y - \frac{\mathbb{E}S_1}{K}\int_K^\infty \overline{\pi}_\infty(y)\,\mathrm{d}y \\
&= \frac{\mathbb{E}_{\pi_\infty}[\sum_{0\leq s\leq 1}\Delta L_s^2] - \mathbb{V}\mathrm{ar}(S_1)}{2K} + \frac{|\mathbb{E}S_1|}{K}\int_K^\infty \overline{\pi}_\infty(y)\,\mathrm{d}y\,.
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
&\mathcal{I}_2+\mathcal{I}_3\\
&=\frac{\sigma^2}{2K}+\frac{1}{2K}\int_0^K\pi_K(\mathrm{d}x)\Big(\int_{-\infty}^{-x}-(x^2+2xy)\nu(\mathrm{d}y)+\frac{1}{2K}\int_{-x}^{K-x}y^2\nu(\mathrm{d}y)\\
&+\frac{1}{2K}\int_{K-x}^{\infty}\big[2y(K-x)-(K-x)^2\big]\nu(\mathrm{d}y)\Big)\\
&=\frac{\sigma^2}{2K}+\frac{1}{2K}\int_{-\infty}^{\infty}y^2\nu(\mathrm{d}y)+\frac{1}{2K}\int_0^K\pi_K(\mathrm{d}x)\int_{-\infty}^{-x}\big[-(x^2+2xy)-y^2\big]\nu(\mathrm{d}y)\\
&+\frac{1}{2K}\int_0^K\pi_K(\mathrm{d}x)\int_{K-x}^{\infty}\big[2y(K-x)-(K-x)^2-y^2\big]\nu(\mathrm{d}y)\\
&=\frac{\sigma^2}{2K}+\frac{1}{2K}\int_{-\infty}^{\infty}y^2\nu(\mathrm{d}y)-\frac{1}{2K}\int_0^K\pi_K(\mathrm{d}x)\int_{-\infty}^{-x}(x+y)^2\nu(\mathrm{d}y)\\
&-\frac{1}{2K}\int_0^K\pi_K(\mathrm{d}x)\int_{K-x}^{\infty}(y-(K-x))^2\nu(\mathrm{d}y)\\
&=\frac{\mathbb{V}\mathrm{ar}(S_1)-\mathbb{E}_{\pi_K}[\sum_{0\le s\le 1}\Delta L_s^2]}{2K}-\frac{1}{2K}\int_0^K\pi_K(\mathrm{d}x)\int_{K-x}^{\infty}(y-(K-x))^2\nu(\mathrm{d}y).
\end{aligned}
$$

We note the fact that

$$
\mathbb{E}_{\pi_\infty}[\sum_{0\le s\le 1}\Delta L_s^2]\le\mathbb{E}_{\pi_K}[\sum_{0\le s\le 1}\Delta L_s^2]
$$

which can be verified using partial integration and (4.1). Using this in the last equation above, we may continue our calculation and obtain:

$$
\begin{aligned}
\mathcal{I}_2+\mathcal{I}_3\le{}&\frac{\mathbb{V}\mathrm{ar}(S_1)-\mathbb{E}_{\pi_\infty}[\sum_{0\le s\le 1}\Delta L_s^2]}{2K}\\
&-\frac{1}{2K}\int_0^K\pi_K(\mathrm{d}x)\int_{K-x}^{\infty}(y-(K-x))^2\nu(\mathrm{d}y)\,.
\end{aligned}
$$

Comparing the expressions above we see that fractions cancel, and the expression in the brackets in (4.8) is less than

$$
\frac{|\mathbb{E}S_1|}{K}\int_K^{\infty}\overline{\pi}_\infty(y)\,\mathrm{d}y-\frac{1}{2K}\int_0^K\int_{K-x}^{\infty}(y-(K-x))^2\nu(\mathrm{d}y)\pi_K(\mathrm{d}x)\,.
$$

Applying partial integration

$$\frac{|\mathbb{E}S_1|}{K}\int_K^\infty \overline{\pi}_\infty(y)\,\mathrm{d}y - \frac{1}{2K}\int_0^K\int_{K-x}^\infty (y-(K-x))^2\nu(\mathrm{d}y)\pi_K(\mathrm{d}x)$$
$$= \frac{|\mathbb{E}S_1|}{K}\int_K^\infty \overline{\pi}_\infty(y)\,\mathrm{d}y - \frac{1}{2K}\int_K^\infty (y-K)^2\nu(\mathrm{d}y) - \frac{1}{K}\int_0^K \pi_K(x)\overline{\nu}_I(K-x)\mathrm{d}x$$
$$\leq \frac{|\mathbb{E}S_1|}{K}\int_K^\infty \overline{\pi}_\infty(y)\,\mathrm{d}y - \frac{1}{2K}\int_K^\infty (y-K)^2\nu(\mathrm{d}y)$$
$$= \frac{|\mathbb{E}S_1|}{K}\int_K^\infty \overline{\pi}_\infty(y)\,\mathrm{d}y - \frac{1}{K}\int_K^\infty \overline{\nu}_I(y)\,\mathrm{d}y.$$

Returning to (4.8) and applying the results above we get

$$\limsup_K \frac{1}{\overline{\nu}_I(K)}\left[\frac{\mathbb{E}S_1}{K}\int_0^K \overline{\pi}_\infty(y)\,\mathrm{d}y + \mathcal{I}_2 + \mathcal{I}_3\right]$$
$$\leq \limsup_K \frac{1}{\overline{\nu}_I(K)}\left[\frac{|\mathbb{E}S_1|}{K}\int_K^\infty \overline{\pi}_\infty(y)\,\mathrm{d}y - \frac{1}{K}\int_K^\infty \overline{\nu}_I(y)\,\mathrm{d}y\right]$$
$$= \limsup_K \frac{\int_K^\infty \overline{\nu}_I(y)\,\mathrm{d}y}{K\overline{\nu}_I(K)}\left[\frac{\int_K^\infty |\mathbb{E}S_1|\overline{\pi}_\infty(y)\,\mathrm{d}y}{\int_K^\infty \overline{\nu}_I(y)\,\mathrm{d}y} - 1\right] = 0,$$

where the last equality follows since the term in the brackets tends to 0, and the fraction outside it is bounded by assumption. This proves that (3.1) holds under condition (I).
We now assume condition (II)
We start by noticing the following consequences of the assumptions:

$$\int_K^\infty \overline{\nu}(y)\,\mathrm{d}y \sim \int_K^\infty \frac{L(y)}{y^\alpha}\,\mathrm{d}y \sim \frac{K^{-\alpha+1}L(K)}{\alpha-1} \qquad K\to\infty \tag{4.9}$$

where the last equivalence follows by Proposition 1.5.10 of Bingham et al. [6] and the fact that $\alpha > 1$. Since by Proposition 1.3.6 of Bingham et al. [6], we have $K^{-\alpha+2}L(K)\to\infty$, (4.9) implies $K\overline{\nu}_I(K)\to\infty$.
The inequality (4.6) still holds, as does the limit in (4.7), so we proceed to analysis of $\mathbb{E}S_1\int_0^K \overline{\pi}_\infty(y)\mathrm{d}y/(\overline{\nu}_I(K)K)$
Since $K\overline{\nu}_I(K)\to\infty$ $K\to\infty$ we see that for any A we have

$$\lim_{K\to\infty}\frac{\mathbb{E}S_1}{K\overline{\nu}_I(K)}\int_0^A \overline{\pi}_\infty(y)\,\mathrm{d}y = 0\,. \tag{4.10}$$

Because of the result above we have for any A

$$\lim_{K\to\infty}\frac{\mathbb{E}S_1}{K\overline{\nu}_I(K)}\int_0^K \overline{\pi}_\infty(y)\,\mathrm{d}y = \lim_{K\to\infty}\frac{\mathbb{E}S_1}{K\overline{\nu}_I(K)}\int_A^K \overline{\pi}_\infty(y)\,\mathrm{d}y$$

and using $|\mathbb{E}S_1|\overline{\pi}_\infty(K) \sim \overline{\nu}_I(K) \sim K^{-\alpha+1}L(K)/(\alpha-1)$ we have

$$\begin{aligned}
&\lim_{K\to\infty} \frac{\mathbb{E}S_1}{K\overline{\nu}_I(K)} \int_A^K \overline{\pi}_\infty(y)\,\mathrm{d}y = \lim_{K\to\infty} -\frac{1}{K\overline{\nu}_I(K)} \int_A^K \overline{\nu}_I(y)\,\mathrm{d}y \\
&= -\lim_{K\to\infty} \frac{1}{K\overline{\nu}_I(K)} \int_A^K \frac{y^{-\alpha+1}L(y)}{(\alpha-1)}\,\mathrm{d}y
\end{aligned}$$

in the sense that if either limit exits so does the other and they are equal. Furthermore, since $-\alpha+1 > -1$ and L is locally bounded, we may apply Proposition 1.5.8 in Bingham et al. [6] to obtain

$$\begin{aligned}
&-\lim_{K\to\infty} \frac{1}{K\overline{\nu}_I(K)} \int_A^K \frac{y^{-\alpha+1}L(y)}{(\alpha-1)}\,\mathrm{d}y \\
&= -\lim_{K\to\infty} \frac{1}{K\overline{\nu}_I(K)} \frac{K^{-\alpha+2}L(K)}{(-\alpha+2)(\alpha-1)} = -\frac{1}{-\alpha+2}.
\end{aligned}$$

That is, we obtain

$$\lim_{K\to\infty} \frac{\mathbb{E}S_1}{K\overline{\nu}_I(K)} \int_0^K \overline{\pi}_\infty(y)\,\mathrm{d}y = -\frac{1}{-\alpha+2}. \tag{4.11}$$

Returning to (4.6) we have

$$\begin{aligned}
&\limsup_K \frac{\ell^K}{\overline{\nu}_I(K)} \\
&= \limsup_K \frac{\mathbb{E}S_1}{K\overline{\nu}_I(K)} \int_0^K \overline{\pi}_\infty(y)\,\mathrm{d}y - \frac{\mathbb{E}S_1\overline{\pi}_\infty(K)}{\overline{\nu}_I(K)} + \frac{\mathcal{I}_2}{\overline{\nu}_I(K)} + \frac{\mathcal{I}_3}{\overline{\nu}_I(K)} \\
&= -\frac{1}{-\alpha+2} + 1 + \limsup_K \frac{\mathcal{I}_2}{\overline{\nu}_I(K)} + \frac{\mathcal{I}_3}{\overline{\nu}_I(K)}. \qquad (4.12)
\end{aligned}$$

Since $K\overline{\nu}_I(K) \to \infty$ we have

$$\mathcal{I}_2/\overline{\nu}_I(K) = \frac{\sigma^2}{2K\overline{\nu}_I(K)} = 0$$

and we may continue our calculation from (4.12)

$$-\frac{1}{-\alpha+2} + 1 + \limsup_K \frac{\mathcal{I}_2}{\overline{\nu}_I(K)} + \frac{\mathcal{I}_3}{\overline{\nu}_I(K)} = -\frac{1}{-\alpha+2} + 1 + \limsup_K \frac{\mathcal{I}_3}{\overline{\nu}_I(K)} \tag{4.13}$$

So we turn our attention to $\mathcal{I}_3$. First we divide the integral into two:

$$2K\mathcal{I}_3 = \tag{4.14}$$

$$\underbrace{\int_0^K \pi_K(\mathrm{d}x) \int_{-\infty}^{-x} -(x^2+2xy)\nu(\mathrm{d}y) + \int_{-x}^{0} y^2\nu(\mathrm{d}y)}_{A(K)} \tag{4.15}$$

$$\underbrace{\int_0^K \pi_K(\mathrm{d}x) \int_{0}^{K-x} y^2\nu(\mathrm{d}y) + \int_{K-x}^{\infty} 2(K-x)y-(K-x)^2\nu(\mathrm{d}y)}_{B(K)}\,. \tag{4.16}$$

We may assume ν is bounded from below, otherwise we may truncate ν at $-L$ for some $L > 0$ which is chosen large enough to ensure that the mean of S_1 remains negative. This truncation may increase the loss rate, which is not a problem, since we are proving an upper bound. Thus, we may assume that $A(K)$ is bounded:

$$A(K) \le \int_0^K \pi_K(\mathrm{d}x) \int_{-\infty}^{0} y^2\nu(\mathrm{d}y) \le \int_{-\infty}^{0} y^2\nu(\mathrm{d}y) < \infty$$

And therefore, since $K\overline{\nu}_I(K) \to \infty$, we have

$$\frac{A(K)}{2K\overline{\nu}_I(K)} \to 0\,. \tag{4.17}$$

Turning to $B(K)$, we first perform partial integration

$$\begin{aligned}
B(K) &= \int_0^K y^2\nu(\mathrm{d}y) + \int_K^\infty 2Ky - K^2\nu(\mathrm{d}y)\\
&\quad - \int_0^K \overline{\nu}_I(K-x)\overline{\pi}_K(x)\,\mathrm{d}x\\
&\le \int_0^K y^2\nu(\mathrm{d}y) + \int_K^\infty 2Ky - K^2\nu(\mathrm{d}y)\\
&= \int_0^K 2y\overline{\nu}(y)\,\mathrm{d}y - K^2\overline{\nu}(K) + \int_K^\infty 2Ky - K^2\nu(\mathrm{d}y)\\
&= \int_0^K 2y\overline{\nu}(y)\,\mathrm{d}y + 2K\int_K^\infty \overline{\nu}(y)\,\mathrm{d}y\,.
\end{aligned}$$

Since $y\overline{\nu}(y) \sim y^{-\alpha+1}L(y)$ way may apply Proposition 1.5.8 from [6]:

$$\int_0^K 2y\overline{\nu}(y)\,\mathrm{d}y \sim 2\frac{L(K)K^{-\alpha+2}}{2-\alpha}$$

and therefore:

$$\lim_K \frac{1}{2K\overline{\nu}_I(K)} \int_0^K 2y\overline{\nu}(y)\,\mathrm{d}y = \frac{\alpha-1}{2-\alpha}\,.$$

Combining this with our inequality for $B(K)$ above, we have:

$$\limsup_K \frac{B(K)}{2K\overline{\nu}_I(K)} \leq \frac{\alpha-1}{2-\alpha} + 1 = \frac{1}{2-\alpha}\,.$$

Finally, by combining this with (4.12), (4.17) and (4.13) we have get

$$\limsup_K \frac{\ell^K}{\overline{\nu}_I(K)} \leq 1$$

and we are done. □

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PAPER C

Local Time Asymptotics for Centered Lévy Processes with Two-Sided Reflection

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Abstract

The present paper is concerned with the local times of a Lévy process reflected at two barriers 0 and $K > 0$. The reflected process is decomposed into the original process plus local times at 0 and K and a starting condition, and we study ℓ^K, the mean rate of increase of the local time at K when the reflected process is started in stationarity. We derive asymptotics ($K \to \infty$) for ℓ^K when the Lévy process has mean zero. The precise form of the asymptotics depends on the existence or non-existence of a finite second moment, paralleling the difference between the normal and the stable central limit theorem. To achieve the asymptotic results, we prove a uniform integrability criterion for Lévy processes and a continuity result for ℓ^K, which are of independent interest.

Keywords continuity of the local time, finite buffer, Lévy process, reflection, loss rate, Skorokhod problem, stable central limit theorem, stable distribution, uniform integrability.

1 Introduction

A *Lévy process* $S = \{S_t\}_{t\geq 0}$ is a real-valued stochastic process on $\mathbb{R}$ with stationary independent increments which is continuous in probability and has $S_0 = 0$ a.s. We reflect the Lévy process at barriers 0 and $K > 0$. The reflected process $V^K = \{V_t^K\}_{t\geq 0}$ can be constructed as part of the solution to a two-sided Skorokhod problem, which yields a representation:

$$V_t^K = y + S_t + L_t^0 - L_t^K \tag{1.1}$$

of the reflected process started at $y \in [0, K]$, where $L^0 = \{L_t^0\}$ and $L^K = \{L_t^K\}$ are the local times at 0, K respectively. More precisely, (V^K, L^0, L^K) is a triplet of processes such that $V_t^K \in [0, K]$ and

$$\int_0^T V_t^K \, \mathrm{d}L_t^0 = 0 \quad \forall T \quad \text{and} \quad \int_0^T (K - V_t^K) \, \mathrm{d}L_t^K = 0 \quad \forall T\,. \tag{1.2}$$

The process V^K is regenerative (as a cycle, take e.g. an excursion from 0 to K followed by an excursion from K to 0). Such a cycle clearly has an absolutely continuous distribution, and it follows by general theory (Asmussen [3] VI.1) that there exists a unique stationary distribution π^K such that the distribution of V_t^K converges to π^K weakly and in total variation. The object of the present paper is to study asymptotic properties as $K \to \infty$ of the stationary rate of growth $\ell^K := \mathbb{E}_{\pi^K} L_1^K$ of the local time

Besides its intrinsic probabilistic interest, this problem has a long applied motivation. Two-sided reflected processes may be used to model waiting time processes in queues with finite capacity (Bekker and Zwart [5], Cohen [9], Cooper et al. [10], Daley [11]), or a finite dam or fluid model (Asmussen [3] , Moran [23], Stadje [28]). Furthermore, they are used in models of network traffic or telecommunications systems involving a finite buffer (Jelenković [15], Kim and Shroff [17] , Zwart [31]), and in this context the loss rate can be interpreted as the bit loss rate in a finite data buffer.

In view of this applied literature, we shall henceforth refer to ℓ^K as the *loss rate* (at the upper barrier K). In the Lévy process context, it is the object of study of the recent papers Asmussen and Pihlsgård [4] and Andersen [1]. In [4], an explicit expression for ℓ^K in terms of the characteristic triplet of the Lévy process is provided and used to derive the asymptotic behavior of ℓ^K as K tends to infinity in the case where the Lévy process is light-tailed and the mean is either strictly positive or strictly negative. Furthermore, in [4] the loss rate of a strictly stable Lévy process is explicitly calculated. The case of negative mean and heavy tails case is treated in Andersen [1]. In this paper we derive loss rate asymptotics when the mean is zero, i.e. $\mathbb{E}S_1 = 0$.

The main contribution of this paper is Theorem 3.1 which provides an asymptotic expression as $K \to \infty$ for the loss rate in the zero-mean case. The basic intuition behind this is simple: $\mathbb{E}S_1 = 0$ implies that the Lévy process after appropriate scaling and time change has a limit which is Brownian motion in the case of finite variance and (subject to a condition on regular variation) is stable in the case of infinite variance. For these limits, explicit expressions for the asymptotic loss rate have been derived in Asmussen and Pihlsgård [4], so the main technical problems becomes to establish continuity of $\ell^K = \ell^K(S)$ as function of S. This is of some of independent interest and is formulated in Theorem 3.2. A uniform integrability property is required, and conditions for this are given as Theorem 3.3.

The paper is organized as follows: In Section 2, we give some background on Lévy processes, the Skorokhod problem, and the stationary distribution. In Section 3 we state the main results of the paper, and the proofs are given in Sections 4, 5 and 6 .

2 Preliminaries

To every Lévy process $S = \{S_t\}_{t\geq 0}$ is associated a unique *characteristic triplet* (θ, σ, ν), where $\theta \in \mathbb{R}$, $\sigma \geq 0$ and ν is a measure (*the Lévy measure*) with $\int_{-\infty}^{\infty}(1 \wedge y^2)\nu(\mathrm{d}y) < \infty$ and $\nu(\{0\}) = 0$. The *Lévy exponent* is defined by

$$\kappa(s) := \theta s + \frac{\sigma^2 s^2}{2} + \int_{-\infty}^{\infty} \left[e^{sx} - 1 - sI(|x| \leq 1)\right] \nu(\mathrm{d}x)$$

and is defined for s in $\Theta := \{s \in \mathbb{C} \mid \mathbb{E}e^{\Re(s)S_1} < \infty\}$. The Lévy exponent is the unique function satisfying $\mathbb{E}e^{sS_t} = e^{t\kappa(s)}$ and $\kappa(0) = 0$, and we have

$$\mathbb{E}S_1 \ = \ \kappa'(0) \ = \ \theta + \int_{|y|>1} y\,\nu(\mathrm{d}y) \tag{2.1}$$

(the mean is assumed to be well-defined and finite for all Lévy processes encountered in the paper). We use the cadlag version of $\{S_t\}$, which exists because of stochastic continuity. Standard references for Lévy processes are Bertoin [6], Kyprianou [21] and Sato [26].

We will also need weak convergence properties:

Proposition 2.1. *Let $S^0, S^1, S^2, \ldots$ be Lévy processes with characteristic triplet $(\theta_n, \sigma_n, \nu_n)$ for S^n. Then the following properties are equivalent:*
(i) $S^n_t \xrightarrow{\mathcal{D}} S^0_t$ for some $t > 0$;

(ii) $S_t^n \xrightarrow{\mathcal{D}} S_t^0$ *for all* t*;*
(iii) $\{S_t^n\} \xrightarrow{\mathcal{D}} \{S_t^0\}$ *in* $D[0,\infty)$*;*
(iv) $\tilde{\nu}_n \to \tilde{\nu}_0$ *weakly, where* $\tilde{\nu}_n$ *is the bounded measure*

$$\tilde{\nu}_n(\mathrm{d}y) := \sigma_n\delta_0(\mathrm{d}y) + \frac{y^2}{1+y^2}\nu_n(\mathrm{d}y) \tag{2.2}$$

and $c_n \to c_0$ *where*

$$c_n := \theta_n + \int \left(\frac{y}{1+y^2} - yI(|y| \le 1)\right)\nu_n(\mathrm{d}y)$$

See e.g. Kallenberg [16] pp. 244–248, in particular Lemma 13.15 and 13.17. If one of (i)–(iv) hold, we write simply $S^n \xrightarrow{\mathcal{D}} S^0$.

The existence and uniqueness of a solution to the Skorokhod problem is proved in Tanaka [29] and in a more pragmatic manner in Asmussen [3] XIV.3. Verbally, the condition (1.2) states that $\{L_t^0\}$ can only increase when $V_t = 0$ and $\{L_t^K\}$ can only increase when $V_t = K$, which supports our interpretation of $\ell^K = \mathbb{E}_{\pi^K} L_1^K$ as a loss rate in a system where the "free traffic" is modeled by $\{S_t\}$.

The stationary distribution has the representation

$$\overline{\pi}_K(y) = \pi^K[y,K] = \mathbb{P}\big(S_{\tau[y-K,y)} \ge y\big), \quad 0 \le y \le K \tag{2.3}$$

where $\tau[u,v) = \inf\{t > 0 \mid S_t \notin [u,v)\}$, see Asmussen [3] pp. 393-394 as well as Lindley [22] and Siegmund [27]. This implies that the Laplace transform of π^K can be found in closed form whenever the scale function of S is explicitly available. For examples of this, see Hubalek and Kyprianou [14].

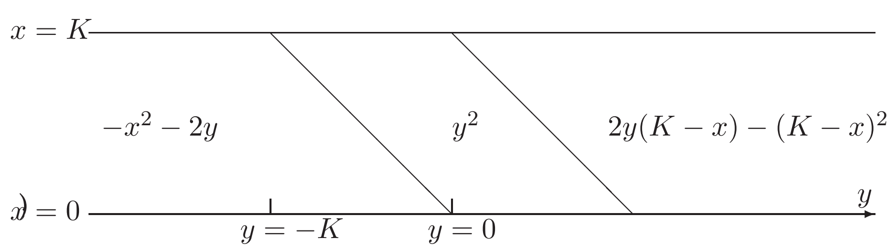
From Theorem 3.6 in Asmussen and Pihlsgård [4], we have the following expression for the loss rate, in terms of the characteristic triplet of the Lévy process and the stationary distribution:

$$\ell^K = \frac{\mathbb{E}S_1}{K}\int_0^K \overline{\pi}_K(x)\,\mathrm{d}x + \frac{\sigma^2}{2K} + \frac{1}{2K}\int_0^K \pi^K(\mathrm{d}x)\int_{-\infty}^{\infty} \varphi_K(x,y)\nu(\mathrm{d}y), \tag{2.4}$$

where

$$\varphi_K(x,y) = \begin{cases} -(x^2+2xy) & \textit{if } y \le -x \\ y^2 & \textit{if } -x < y < K-x \\ 2y(K-x) - (K-x)^2 & \textit{if } y \ge K-x\,. \end{cases} \tag{2.5}$$

Figure 1: The function $\varphi(x, y)$



For a graphical illustration, see Fig. 1 that depicts $\varphi(x, y)$ in the region $(x, y) \in [0, K] \times \mathbb{R}$ relevant for (2.4) (note that y is on the horizontal axis and x on the vertical).

One should note that various explicit expression for L_t^0 and L_t^K have been derived (in part independently) by a number of authors, see Andersen and Mandjes [2], Borovkov [8], Cooper et al. [10], Kruk et al. [18] and Kruk et al. [19]. However, they all have a form that is so complicated that they do not appear to be of use neither for deriving (2.4), (2.5) nor for the present purposes.

3 Main Results

Our main result provides the asymptotics in the case $\mathbb{E}S_1 = 0$ of zero drift.

Theorem 3.1.
a) Let $\{S_t\}$ be a Lévy process with $\mathbb{E}S_1 = 0$ and characteristic triplet (θ, σ, ν) which satisfies $\int_{-\infty}^{\infty} x^2\, \nu(\mathrm{d}x) < \infty$, Then

$$\ell^K \sim \frac{1}{2K} \int_{-\infty}^{\infty} y^2 \nu(\mathrm{d}y) + \frac{\sigma^2}{2K}\,, \qquad K \to \infty\,. \tag{3.1}$$

b) Let $\{S_t\}$ be an Lévy process with characteristic triplet (θ, σ, ν). Assume $\mathbb{E}S_1 = 0$ and that for some $1 < \alpha < 2$, there exists slowly varying functions $L_0(x), L_1(x)$ and $L_2(x)$ such that for $L(x) := L_1(x) + L_2(x)$ we have

$$\overline{\nu}(x) = x^{-\alpha} L_1(x) \qquad\qquad \nu(-x) = |x|^{-\alpha} L_2(x) \tag{3.2}$$

$$\lim_{x \to \infty} \frac{L_2(x)}{L(x)} = \frac{\beta + 1}{2} \qquad\qquad \lim_{x \to \infty} L_0(x)^\alpha L(x) = 1 \tag{3.3}$$

Then, setting $\rho = 1/2 + (\pi\alpha)^{-1} \arctan(\beta \tan(\pi\alpha/2))$, $d = (\beta + 1)/2$ and $c = (1 - \beta)/2$ we have $\ell^K \sim \gamma/(K^{\alpha-1} L_0^\alpha(K))$ where

$$\gamma \;=\; \frac{cB(2 - \alpha\rho, \alpha\rho) + dB(2 - \alpha(1 - \rho), \alpha(1 - \rho))}{B(\alpha\rho, \alpha(1 - \rho))(\alpha - 1)(2 - \alpha)}$$

The parameter ρ defined in Theorem 3.1 is known as the *positivity parameter* as it satisfies $\rho = \mathbb{P}(S_t > 0)$ when S is a strictly α-stable Lévy process, see Zolotarev [30].

We note incidentally that Theorem 3.1 also gives the asymptotics of $\ell^0 = \mathbb{E}_{\pi^K} L_1^0$ because a balance argument together with (1.1) gives $0 = \mathbb{E}S_1 + \ell^0 - \ell^K$ so that $\ell^0 = \ell^K$ in the mean zero case $\mathbb{E}S_1 = 0$.

To prove Theorem 3.1, we will use the fact that by properly scaling our Lévy process we may construct a sequence of Lévy processes which converges weakly to either a Brownian Motion or a stable process. Since ℓ^K has been calculated for both Brownian Motion and stable processes in Asmussen and Pihlsgård [4], we may use this convergence to obtain loss rate asymptotics in the case of zero drift, provided that the loss rate is continuous in the sense that weak convergence (in the sense of Proposition 2.1) of the involved processes implies convergence of the associated loss rates. To state our result:

Theorem 3.2. *Let $\{S^n\}_{n=0,1,\ldots}$ be a sequence of Lévy processes with associated loss rates $\ell^{K,n}$. Suppose $S^n \xrightarrow{\mathcal{D}} S^0$ and that the family $(S_1^n)_{n=1}^\infty$ is uniformly integrable. Then $\ell^{K,n} \to \ell^{K,0}$ as $n \to \infty$.*

We shall also need:

Theorem 3.3. *Let $\{X_n\}_{n=1,2,\ldots}$ be a sequence of weakly convergent infinitely divisible random variables, with characteristic triplets $(\theta_n, \sigma_n, \nu_n)$. Then for $\alpha > 0$:*

$$\lim_{a\to\infty} \sup_n \int_{[-a,a]^c} |y|^\alpha \nu_n(\mathrm{d}y) = 0 \Leftrightarrow \{|X_n|^\alpha \mid n \geq 1\} \text{ is uniformly integrable}$$

The result is certainly not unexpected, but does not appear to be in the literature; the closest we could find is Theorem 25.3 in Sato [26].

4 Proof of Theorem 3.2

We consider a sequence of Lévy process $\{S^n\}$ such that $S^n \xrightarrow{\mathcal{D}} S^0$ and use obvious notation like $\ell^{K,n}, \pi^{K,n}$ etc. Furthermore, we let $\tau^n(A)$ denote the first exit time of S^n from A. Here A will always be an interval.

We first show that weak convergence of S_1^n implies weak convergence of the stationary distributions.

Proposition 4.1. $S^n \xrightarrow{\mathcal{D}} S^0 \Rightarrow \pi^{K,n} \xrightarrow{\mathcal{D}} \pi^{K,0}$.

Proof. According to Theorem 13.17 in Kallenberg [16] we may assume $\Delta_{n,t} := \sup_{v\le t} |S^n(v) - S^0(v)| \xrightarrow{\mathbb{P}} 0$. Then

$$\begin{aligned}&\mathbb{P}\big(S^0_{\tau^0[y+\epsilon-K,y+\epsilon)} \ge y+\epsilon,\, \tau^0[y+\epsilon-K,y+\epsilon) \le t\big)\\ &\le \mathbb{P}\big(S^n_{\tau^n[y-K,y)} \ge y,\, \tau^n[y-K,y) \le t\big) + \mathbb{P}(\Delta_{n,t} > \epsilon)\\ &\le \mathbb{P}\big(S^n_{\tau^n[y-K,y)} \ge y\big) + \mathbb{P}(\Delta_{n,t} > \epsilon)\,.\end{aligned}$$

Letting first $n \to \infty$ gives

$$\liminf_{n\to\infty} \overline{\pi}^{K,n}(y) \;\ge\; \mathbb{P}\big(S^0_{\tau^0[y+\epsilon-K,y+\epsilon)} \ge y+\epsilon,\, \tau^0[y+\epsilon-K,y+\epsilon) \le t\big)\,,$$

and letting next $t \to \infty$, we obtain

$$\liminf_{n\to\infty} \overline{\pi}^{K,n} \;\ge\; \overline{\pi}^{K,0}(y+\epsilon)\,. \tag{4.1}$$

Similarly,

$$\begin{aligned}&\mathbb{P}\big(S^n_{\tau^n[y-K,y)} \ge y,\, \tau^n[y-K,y) \le t\big) \;\le\; \mathbb{P}\big(S^0_{\tau^0[y-\epsilon-K,y-\epsilon)} \ge y\big) + \mathbb{P}(\Delta_{n,t} > \epsilon)\,,\\ &\limsup_{n\to\infty} \mathbb{P}\big(S^n_{\tau^n[y-K,y)} \ge y,\, \tau^n[y-K,y) \le t\big) \;\le\; \overline{\pi}^{K,0}(y-\epsilon)\,.\end{aligned} \tag{4.2}$$

However,

$$\mathbb{P}\big(\tau^n[y-K,y) > t\big) \;\le\; \mathbb{P}\big(\tau^0[y-\epsilon-K,y+\epsilon) > t\big) + \mathbb{P}(\Delta_{n,t} > \epsilon)\,,$$

so that

$$\limsup_{n\to\infty} \mathbb{P}\big(\tau^n[y-K,y) > t\big) \;\le\; \mathbb{P}\big(\tau^0[y-\epsilon-K,y+\epsilon) > t\big)\,.$$

Since the r.h.s. can be chosen arbitrarily small, it follows by combining with (4.2) that

$$\limsup_{n\to\infty} \overline{\pi}^{K,n}(y) \;=\; \limsup_{n\to\infty} \mathbb{P}\big(S^n_{\tau^n[y-K,y)} \ge y\big) \;\le\; \overline{\pi}^{K,0}(y-\epsilon)\,.$$

Combining with (4.1) shows that $\overline{\pi}^{K,n}(y) \to \overline{\pi}^{K,0}(y)$ at each continuity point y of $\overline{\pi}^{K,0}$, which implies convergence in distribution. □

We will need the following lemma.

Lemma 4.1. *The function $\varphi(x,y)$ is continuous in the region $(x,y) \in [0,K] \times \mathbb{R}$ and satisfies $0 \le \varphi(x,y) \le 2y^2 \wedge 2K|y|$.*

Proof. By elementary calculus. For continuity, check that the expressions for $\varphi(x, y)$ on the regions $x+y \leq 0$ and $x+y \geq K$ equal y^2 on the lines $x+y=0$ and $x+y=K$. The claimed inequality is clear for $0 \leq x+y \leq K$. Consider $x+y<0$. Then $\varphi(x, y) \leq -2xy \leq 2y^2$ and $\varphi(x, y) \leq -2xy \leq 2K|y|$. Similarly for $x+y>K$, we have $\varphi(x, y) \leq 2y(K-x)$ which yields $\varphi(x, y) \leq 2y^2$ and $\varphi(x, y) \leq -2xy \leq 2Ky$. ☐

We are now ready to prove Theorem 3.2.

Proof. Recall the definition (2.2) of the bounded measure $\tilde{\nu}$ and let $\tilde{\varphi}_K(x, y) := \varphi_K(x, y)(1+y^2)/y^2$ for $y \neq 0$, $\tilde{\varphi}_K(x, 0) = 1$. The continuity of φ implies $\varphi(x, y) \sim y^2$ as $y \to 0$ and it easily follows that $\tilde{\varphi}(x, y)$ is continuous jointly in x, y. We also get

$$\int_{-\infty}^{\infty} \tilde{\varphi}(x, y)\tilde{\nu}_n(\mathrm{d}y) = \sigma_n^2 + \int_{-\infty}^{\infty} \varphi(x, y)\nu_n(\mathrm{d}y)$$

so that

$$\begin{aligned} a_n &:= \sigma_n^2 + \int_0^K \pi^{K,n}(\mathrm{d}x) \int_{-\infty}^{\infty} \varphi(x, y)\nu_n(\mathrm{d}y) \\ &= \int_0^K \pi^{K,n}(\mathrm{d}x) \int_{-\infty}^{\infty} \tilde{\varphi}(x, y)\tilde{\nu}_n(\mathrm{d}y)\,. \end{aligned}$$

Let $\tilde{\nu}_n^1, \tilde{\nu}_n^2$ denote the restrictions of $\tilde{\nu}_n$ to the sets $|y| \leq a$, resp. $|y| > a$. Then $0 \leq \varphi(x, y) \leq 2K|y|$, and uniform integrability (Theorem 3.3) imply that we can choose a such that

$$0 \leq \int_{[-a,a]^c} \tilde{\varphi}(x, y)\tilde{\nu}_n^2(\mathrm{d}y) < \epsilon$$

for all x and n (note that $\tilde{\nu}_n \leq \nu_n$ on $\mathbb{R}\setminus\{0\}$). We may also further assume that a and $-a$ are continuity points of ν_0 which implies $\tilde{\nu}_n^1 \to \tilde{\nu}_0^1$ weakly. In particular,

$$\sup_n \tilde{\nu}_n^1([-a, a]) < \infty. \tag{4.3}$$

Define

$$f_n(x) = \int_{-a}^{a} \varphi(x, y)\nu_n(\mathrm{d}y) + \sigma_n^2 = \int_{-a}^{a} \tilde{\varphi}(x, y)\tilde{\nu}_n^1 \mathrm{d}y$$

so that $f_n(x) \to f_0(x)$. Being continuous on the compact set $[0, K] \times [-a, a]$, $\tilde{\varphi}_K(x, y)$ is uniformly continuous. Together with (4.3) this implies that given

ϵ_1, there exists ϵ_2 such that $|f_n(x') - f_n(x'')| < \epsilon_1$ for all n whenever $|x' - x''| < \epsilon_2$. I.e., the family $(f_n)_0^\infty$ is equicontinuous and uniformly bounded. In particular, the convergence$f_n(x) \to f_0(x)$ is uniform in $x \in [0, K]$. Together with $\int f_0 \, \mathrm{d}\pi^{K,n} \to \int f_0 \, \mathrm{d}\pi^{K,0}$ this implies $\int f_n \, \mathrm{d}\pi^{K,n} \to \int f_0 \, \mathrm{d}\pi^{K,0}$ (see also Pollard [24] Example 19 p. 73 for related arguments). Putting this together with the uniform integrability estimate above and letting $\epsilon \to 0$ gives $a_n \to a_0$.

By uniform integrability $\mathbb{E}S_1^n \to \mathbb{E}S_1^0$, and further $\pi^{K,n} \xrightarrow{\mathcal{D}} \pi^{K,0}$ implies $\int_0^K \overline{\pi}^{K,n} \to \int_0^K \overline{\pi}^{K,0}$. Remembering $a_n \to a_0$ and inspecting the expression (2.4) for the loss rate shows that indeed $\ell^{K,n} \to \ell^{K,0}$. □

5 Proof of Theorem 3.3

The following proposition is standard:

Proposition 5.1. *Let $p > 0$ and let $X_n \in L^p$, $n = 0, 1, \ldots$, such that $X_n \xrightarrow{\mathcal{D}} X_0$. Then $\mathbb{E}|X_n|^p \to \mathbb{E}|X_0|^p$ if and only if the family $\{|X_n|^p\}_{n \geq 1}$ is uniformly integrable.*

Theorem 3.3 is proved through several preliminary results. First, we prove Lemma 5.1 which essentially states we may disregard the behavior of the Lévy measures on the interval $[-1, 1]$ in questions regarding uniform integrability. It is therefore sufficient to prove Theorem 3.3 for compound Poisson distributions, which is done in Proposition 5.2 and Proposition 5.3.

We start by examining the case where the Lévy measures have uniformly bounded support, i.e., there exists $A > 0$ such that $\nu_n([-A, A]^c) = 0$ for all n. We know from Lemma 25.6 and Lemma 25.7 in Sato [26] that this implies the existence of finite exponential moments for X_n and therefore $\mathbb{E}X_n^m$ exists and is finite as well for all $n, m \in \mathbb{N}$.

Lemma 5.1. *Suppose $X_n \xrightarrow{\mathcal{D}} X_0$ and the Lévy measures have uniformly bounded support. Then $\mathbb{E}X_n^m \to \mathbb{E}X_0^m$ for $m = 1, 2, \cdots$. In particular (cf. Proposition 5.1) the family $\{|X_n|^\alpha\}_{n \geq 1}$ is uniformly integrable for all $\alpha > 0$.*

Proof. By Lemma 25.6 of [26], the characteristic exponent $\kappa_n(s)$ of X_n is defined for all $s \in \mathbb{C}$, and we can work with the moment generating function $\mathbb{R} \ni t \to \mathbb{E}\mathrm{e}^{tX} \in \mathbb{R}$, which the by the Levy-Khinchine representation can be written as $\mathbb{E}\mathrm{e}^{tX_n} = e^{\kappa_n(t)}$ where

$$\kappa_n(t) = \theta_n t + \sigma_n^2 t^2/2 + \int_{-A}^{A} \big(\mathrm{e}^{ty} - 1 - tyI(|y| \leq 1)\big)\nu_n(\mathrm{d}y) \tag{5.1}$$

With the aim of applying Lemma 13.15 in Kallenberg [16], we rewrite (5.1) as

$$\kappa_n(t) = c_n t + \int_{-A}^{A} \left(\mathrm{e}^{ty} - 1 - \frac{ty}{1+y^2} \right) \frac{1+y^2}{y^2} \tilde{\nu}_n(\mathrm{d}y) \tag{5.2}$$

where $\tilde{\nu}_n$ is as above and

$$c_n = \theta_n + \int_{-A}^{A} \left(\frac{y}{1+y^2} - yI(|y| \leq 1) \right) \nu_n(\mathrm{d}y) \,.$$

According to Lemma 13.15 in [16], the weak convergence of $\{X_n\}_{n\geq 1}$ implies $c_n \to c_0$ and $\tilde{\nu}_n \xrightarrow{\mathcal{D}} \tilde{\nu}$. Since the integrand in (5.2) is bounded and continuous, this implies that $\kappa_n(t) \to \kappa_0(t)$, which in turn implies that all exponential moments converge. In particular, the family $\left\{ \mathrm{e}^{X_n} + \mathrm{e}^{-X_n} \right\}_{n\geq 1}$ is uniformly integrable, which implies that $\{|X_n|^\alpha\}_{n\geq 1}$ is so. □

Next, we express the condition of uniform integrability using the tail of the involved distributions. We will need the following lemma on weakly convergent compound Poisson distributions.

Lemma 5.2. *Let $U_0, U_1, \ldots$ be a sequence of positive random variables such that $U_n > 1$, and let $N_0, N_1, \ldots$ be Poisson random variables with rates $\lambda_0, \lambda_1 \ldots$ Set $X_n := \sum_1^{N_n} U_{i,n}$ (empty sum = 0) with the $U_{i,n}$ being $i.i.d$ for fixed n with $U_{i,n} =_{\mathscr{D}} U_n$. Then $X_n \xrightarrow{\mathcal{D}} X_0$ if and only if $U_n \xrightarrow{\mathcal{D}} U_0$ and $\lambda_n \to \lambda_0$.*

Proof. We use the continuity theorem for characteristic functions. The characteristic function of X_n is $\mathbb{E}^{\mathrm{i}sX_n} = \exp\{\lambda_n(\mathbb{E}^{\mathrm{i}sU_n} - 1\}$. From this the 'if' part is immediately clear. For the converse, we observe that $\exp(-\lambda_n) \to \exp(-\lambda_0) = \mathbb{P}(X_0 \leq 1/2)$ since $1/2$ is a continuity point of X_0 (note that $\mathbb{P}(X_0 \leq x) = \mathbb{P}(X_0 = 0)$ for all $x < 1$). Taking logs yields $\lambda_n \to \lambda_0$ and the necessity of $U_n \xrightarrow{\mathcal{D}} U_0$ then is obvious from the continuity theorem for characteristic functions. □

Using the previous result, we are ready to prove part of our main result for a class of compound Poisson distributions:

Proposition 5.2. *Let $U_0, U_1, \ldots, N_0, N_1, \ldots,$ and $X_0, X_1, \ldots$ be as in Lemma 5.2. Assume $X_n \xrightarrow{\mathcal{D}} X_0$. Then for $\alpha > 0$.*

$$\lim_{a\to\infty} \sup_n \mathbb{E}\big[X_n^\alpha I(X_n > a)\big] = 0 \Rightarrow \lim_{a\to\infty} \sup_n \mathbb{E}\big[U_n^\alpha I(U_n > a)\big] = 0 \,.$$

Proof. Let $G_n(x) = \mathbb{P}(X_n \le x)$,$F_n(x) = \mathbb{P}(U_n \le x)$,$\overline{F}_n(x) = 1 - F_n(x)$,$\overline{G}_n(x) = 1 - G_n(x)$, and let $F_n^{*m}(x), G_n^{*m}(x)$ denote the m'th fold convolutions. Then

$$\overline{G}_n(x) = \sum_{m=1}^{\infty} \frac{\lambda_n^m}{m!} \mathrm{e}^{-\lambda_n} \overline{F}_n^{*m}(x) \qquad x > 0$$

which implies $\overline{G}_n(x) \ge \lambda_n \mathrm{e}^{-\lambda_n} \overline{F}_n(x)$. Letting $\beta = \sup_n \mathrm{e}^{\lambda_n}/\lambda_n$, which is finite by Lemma 5.2, we get: $\overline{F}_n(x) \le \beta \overline{G}_n(x)$. Therefore:

$$\begin{aligned}
\mathbb{E}[U_n^\alpha I(U_n > a)] &= \int_0^\infty \alpha t^{\alpha-1} \mathbb{P}(U_n > a \vee t) \mathrm{d}t \\
&= a^\alpha \overline{F}_n(a) + \alpha \int_a^\infty t^{\alpha-1} \overline{F}_n(t) \mathrm{d}t \\
&\le \beta a^\alpha \overline{G}_n(a) + \beta\alpha \int_a^\infty t^{\alpha-1} \overline{G}_n(t) \mathrm{d}t \\
&= \beta \mathbb{E}[X_n^\alpha I(X_n > a)] \,.
\end{aligned}$$

Taking supremum and limits completes the proof. □

Next, we prove the converse of Proposition 5.2.

Proposition 5.3. *Under the assumptions of Proposition 5.2 we have, for $\alpha > 0$:*

$$\lim_{a\to\infty} \sup_n \mathbb{E}\big[U_n^\alpha I(U_n > a)\big] = 0 \Rightarrow \lim_{a\to\infty} \sup_n \mathbb{E}\big[X_n^\alpha I(X_n > a)\big] = 0 \,.$$

Proof. We use the notation of Proposition 5.2. By Lemma 5.2 we have $F_n^{*1} \xrightarrow{\mathcal{D}} F_0^{*1}$ and by the Portmanteau lemma $F_n^{*m} \xrightarrow{\mathcal{D}} F_0^{*m}$. We note that the assumption of uniform integrability of the U_n^α implies that $\mathbb{E}\big(\sum_{i=1}^m U_{i,n}\big)^\alpha \to \mathbb{E}\big(\sum_{i=1}^m U_{i,0}\big)^\alpha$, since the $U_{i,n}$ are i.i.d in i and $U_{i,n} =_{\mathscr{D}} U_n$. Fix $m \in \mathbb{N}$. Since $\big(\sum_{i=1}^m U_{i,n}\big)^\alpha \le m^\alpha \sum_{i=1}^m U_{i,n}^\alpha$ and the family $\big(m^\alpha \sum_{i=1}^m U_{i,n}^\alpha\big)_{n\ge1}$ is uniformly integrable, we have that also the family $\big(\sum_{i=1}^m U_{i,n}\big)_{n\ge1}^\alpha$ is uniformly integrable. As noted above we have $\sum_{i=1}^m U_{i,n} \xrightarrow{\mathcal{D}} \sum_{i=1}^m U_{i,0}$, so Proposition 5.1 implies $\mathbb{E}\big(\sum_{i=1}^m U_{i,n}\big)^\alpha \to \mathbb{E}\big(\sum_{i=1}^m U_{i,0}\big)^\alpha$.

We next show $\mathbb{E}X_n^\alpha \to \mathbb{E}X_0^\alpha$ and thereby the assertion of the proposition. We have:

$$\begin{aligned}\lim_n \mathbb{E}X_n^\alpha &= \lim_n \sum_{m=0}^{\infty} \mathbb{E}\Big(\sum_{i=1}^{m} U_{i,n}\Big)^\alpha \frac{\lambda_n^m}{m!}\mathrm{e}^{-\lambda_n} \\ &= \sum_{m=0}^{\infty} \lim_n \mathbb{E}\Big(\sum_{i=1}^{m} U_{i,n}\Big)^\alpha \frac{\lambda_n^m}{m!}\mathrm{e}^{-\lambda_n} \\ &= \sum_{m=0}^{\infty} \mathbb{E}\Big(\sum_{i=1}^{m} U_{i,0}\Big)^\alpha \frac{\lambda_0^m}{m!}\mathrm{e}^{-\lambda_0} = \mathbb{E}X_0^\alpha\,,\end{aligned}$$

where we used dominated convergence with the bound

$$\mathbb{E}\Big(\sum_{i=1}^{m} U_{i,n}\Big)^\alpha \frac{\lambda_n^m}{m!}\mathrm{e}^{-\lambda_n} \le \gamma m^{\alpha+1}\beta^m/m!$$

with $\gamma = \sup_n \mathbb{E}U_n^\alpha$ and $\beta = \sup_n \lambda_n$. □

Proof of Theorem 3.3. Using the Lévy -Khinchine representation, we may write

$$X_n = X_n^{(1)} + X_n^{(2)} + X_n^{(3)} \tag{5.3}$$

where the $\big(X_n^{(i)}\big)_{n\ge1}$ are sequences of infinitely divisible distributions having characteristic triplets $(0, 0, [\nu]_{\{y<-1\}})$, $(\theta_n, \sigma_n, [\nu_n]_{\{|y|\le1\}})$ and $(0, 0, [\nu_n]_{\{y>1\}})$, respectively. Assume the family $\big(|X_n|^\alpha\big)_{n\ge1}$ is uniformly integrable. We wish to apply Proposition 5.2 to the family $\big((X_n^{(3)})^\alpha\big)$, and therefore we need to show that this family is uniformly integrable. First, we we rewrite (5.3) as $X_n - X_n^{(2)} = X_n^{(1)} + X_n^{(3)}$ and use Lemma 5.1 together with the inequality $|x-y|^\alpha \le 2^\alpha(|x|^\alpha + |y|^\alpha)$ to conclude that the family $\big(|X_n - X_n^{(2)}|^\alpha\big)_{n\ge1}$ is uniformly integrable, which in turn implies that the family $\big(|X_n^{(1)} + X_n^{(3)}|^\alpha\big)_{n\ge1}$ is uniformly integrable.

Assuming w.l.o.g. that 1 is a continuity point of ν_0, we have that $X_n^{(1)}$ is weakly convergent and therefore tight. This implies that there exists $r > 0$ such that $\mathbb{P}\big(|X_n^{(1)}| \le r\big) \ge 1/2$ for all n, which implies that for all n and for all t so large that $(t^{1/\alpha} - r)^\alpha > t/2$, we have:

$$\begin{aligned}&(1/2)\mathbb{P}\big((X_n^{(3)})^\alpha > t\big) \le \mathbb{P}\big(|X_n^{(1)}| \le r\big)\mathbb{P}\big(X_n^{(3)} > t^{1/\alpha}\big) \\ &= \mathbb{P}\big(|X_n^{(1)}| \le r, X_n^{(3)} > t^{1/\alpha}\big) \le \mathbb{P}\big(X_n^{(1)} + X_n^{(3)} > t^{1/\alpha} - r\big) \\ &= \mathbb{P}\big(|X_n^{(1)} + X_n^{(3)}|^\alpha > (t^{1/\alpha} - r)^\alpha\big) \le \mathbb{P}\big(|X_n^{(1)} + X_n^{(3)}|^\alpha > t/2\big)\,.\end{aligned}$$

This implies that $\big((X_n^{(3)})^\alpha\big)$ is uniformly integrable, since $\big(|X_n^{(1)}+X_n^{(3)}|^\alpha\big)$ is so. Applying Proposition 5.2 yields

$$\lim_a \sup_n \int_a^\infty y^\alpha \nu_n(\mathrm{d}y) = 0 \tag{5.4}$$

Together with a similar relation for $\int_{-\infty}^{-a}$ this gives

$$\lim_{a\to\infty} \sup_n \int_{[-a,a]^c} |y|^\alpha \nu_n(\mathrm{d}y) = 0\,.$$

For the converse, we assume $\lim_a \sup_n \int_{[-a,a]^c} |y|^\alpha \nu_n(\mathrm{d}y) = 0$, and return to our decomposition (5.3). As before, we apply Lemma 5.1 to obtain that the family $\big(X_n^{(2)}\big)$ is uniformly integrable. Furthermore, applying Proposition 5.3, we obtain that the families $\big(|X_n^{(1)}|^\alpha\big)$ and $\big(|X_n^{(3)}|^\alpha\big)$ are uniformly integrable, and since $|X_n|^\alpha \le 3^\alpha\big(|X_n^1|^\alpha + |X_n^2|^\alpha + |X_n^3|^\alpha\big)$, the proof is complete. □

6 Proof of Theorem 3.1

First we note the effect that scaling and time-changing a Lévy process has on the loss rate:

Proposition 6.1. *Let $\beta,\delta > 0$ and define $S_t^{\beta,\delta} = S_{\delta t}/\beta$. Then the loss rate $\ell^{K/\beta}(S^{\beta,\delta})$ for $S^{\beta,\delta}$ equals δ/β times the loss rate $\ell^K(S) = \ell^K$ for S.*

Proof. It is clear that scaling by β results in the same scaling of the loss rate. For the effect of δ, note that the loss rate is the expected local time in stationarity per unit time and that one unit of time for $S^{\beta,\delta}$ corresponds to δ units of time for S. □

Proof of Theorem 3.1 a). Define $S_t^K := S_{tK^2}/K$. Then by Proposition 6.1 we have

$$K\ell^K(S) = \ell^1(S^K)$$

By the central limit theorem we have $S_1^K \xrightarrow{\mathcal{D}} N(0,\psi^2)$ as $K\to\infty$, where

$$\psi^2 = \mathbb{V}\mathrm{ar}(S_1^1) = \sigma^2 + \int_{-\infty}^{\infty} y^2 \nu(\mathrm{d}y)\,.$$

By Proposition 2.1, this is equivalent to $S^K \xrightarrow{\mathcal{D}} \psi B$ where B is standard Brownian motion. We may apply Theorem 3.2, since

$$\mathbb{E}\big[(S_1^K)^2\big] = \mathbb{V}\mathrm{ar}(S_1^1)\,,$$

that is, $\{S_1^K\}_{K=1}^{\infty}$ is bounded in L^2 and therefore uniformly integrable, and we obtain $\lim_K K\ell^K(S) = \lim_K \ell^1(S^K) = \ell^1(\psi B) = \psi^2/2$, where the last equality follows directly from the expression for the loss rate given by (2.4).
□

Proof of Theorem 3.1 b). First we note that the stated conditions implies that the tails of ν are regularly varying, and therefore they are subexponential. Then by Embrechts et al. [12] we have that the tails of $P(S_1 < x)$ are equivalent to those of ν and hence we may write $P(S_1 > x) = x^{-\alpha}L_1(x)g_1(x)$, and $P(S_1 < -x) = x^{-\alpha}L_2(x)g_2(x)$ where $\lim_{x\to\infty} g_i(x) = 1$. $i = 1, 2$. The next step is to show that the fact that the tails of the distribution function is regularly varying allows us to apply the stable central limit theorem. Specifically, we show that the assumptions of Theorem 1.8.1 in Samorodnitsky and Taqqu [25] are fulfilled.

We notice that if we define $M(x) := L_1(x)g_1(x) + L_2(x)g_2(x)$ then $M(x)$ is slowly varying and

$$x^{\alpha}(P(S_1 < -x) + P(S_1 > x)) = M(x)\,. \tag{6.1}$$

Furthermore:

$$\frac{P(S_1 > x)}{P(S_1 < -x) + P(S_1 > x)} = L_2(x)g_2(x)/M(x) \sim L_2(x)/L(x) \to \frac{\beta+1}{2}\,, \tag{6.2}$$

as $x \to \infty$ since $L(x) \sim M(x)$. Let $L_0^{\#}(x)$ denote the de Bruin conjugate of L_0 (cf. Bingham et al. [7] p. 29) and set $f(n) := n^{(1/\alpha)}L_0^{\#}(n^{(1/\alpha)})$. Let $f^{\leftarrow}$ be the generalized inverse of f. By asymptotic inversion of regularly varying functions (p. 28-29 [7]) we have $f^{\leftarrow}(n) \sim (nL_0(n))^{\alpha}$ and using (3.3) we have

$$\frac{f^{\leftarrow}(n)L(n)}{n^{\alpha}} \sim \frac{(nL_0(n))^{\alpha}L(n)}{n^{\alpha}} = L_0(n)^{\alpha}L(n) \to 1$$

and since $f^{\leftarrow}(f(n)) \sim n$ we have

$$\frac{nM(f(n))}{f(n)^{\alpha}} \sim \frac{nL(f(n))}{f(n)^{\alpha}} \sim \frac{f^{\leftarrow}(f(n))L(f(n))}{f(n)^{\alpha}} \to 1 \tag{6.3}$$

and therefore, if we define $\sigma = (-\Gamma(1-\alpha)\cos(\alpha\pi/2))^{1/\alpha}$.

$$\frac{nM(\sigma^{-1}f(n))}{(\sigma^{-1}f(n))^{\alpha}} \sim \frac{nM(f(n))}{(\sigma^{-1}f(n))^{\alpha}} \to \sigma^{\alpha} \tag{6.4}$$

using slow variation of M. By combining (6.1), (6.2) and (6.4) we may apply the stable CLT Theorem 1.8.1 [25][1] to obtain $S_K/f(K) \xrightarrow{\mathcal{D}} X$ where X is a r.v. with c.h.f. φ, where

$$\varphi(t) = \exp(-|\sigma t|^{\alpha}(1 - \mathrm{i}\beta \operatorname{sgn}(t) \tan(\alpha\pi/2))$$

Recalling that κ is the characteristic exponent of S_1, this is equivalent to

$$\mathrm{e}^{\kappa(t/f(K))}K \to \varphi(t)$$

and therefore

$$\mathrm{e}^{\kappa(t/f(f^{\leftarrow}(K)))}(KL_0(K))^{\alpha} \sim \mathrm{e}^{\kappa(t/f(f^{\leftarrow}(K)))}f^{\leftarrow}(K) \to \varphi(t)$$

that is, for $S_t^K = S_{t(KL_0(K))^{\alpha}}/f(f^{\leftarrow}(K))$ we have $S_1^K \xrightarrow{\mathcal{D}} X$. Setting $d = (\beta+1)/2$ and $c = (\beta-1)/2$ we may use formula (3.37.13) in Hoffmann-Jørgensen [13] to obtain

$$-|\sigma t|^{\alpha}(1 - \mathrm{i}\beta \operatorname{sgn}(t) \tan(\alpha\pi/2) = \tag{6.5}$$

$$-|\sigma t|^{\alpha}(1 + \mathrm{i}(d-c) \operatorname{sgn}(t) \tan(\alpha\pi/2) = \tag{6.6}$$

$$d\alpha \int_{-\infty}^{0} (\mathrm{e}^{\mathrm{i}vt} - 1 - ivt)(-t)^{-\alpha-1}\,\mathrm{d}t + \tag{6.7}$$

$$c\alpha \int_{0}^{\infty} (\mathrm{e}^{\mathrm{i}vt} - 1 - ivt)t^{-\alpha-1}\,\mathrm{d}t\,. \tag{6.8}$$

That is, the characteristic triplet of X is $(\tau, 0, \nu)$, where

$$\nu(\mathrm{d}t) = \begin{cases} \frac{\alpha c}{(-t)^{\alpha+1}}\,\mathrm{d}t & t < 0 \\ \frac{\alpha d}{t^{\alpha+1}}\,\mathrm{d}t & t > 0 \end{cases} \tag{6.9}$$

and τ is a centering constant. We wish to use Theorem 3.2 and have to prove uniform integrability. Since $f(f^{\leftarrow}(K)) \sim K$ it is enough to prove uniform integrability of $\tilde{S}^K := S_{(KL_0(K))^{\alpha}}/K$ for large enough K. Note that by combining Proposition 11.10 and Corollary 8.3 in [26], we have that the Lévy measure of $\tilde{S}^K$ is ν_K, where $\nu_K(B) = (KL_0(K))^{\alpha}\nu(\{x : K^{-1}x \in B\})$. Using the assumptions in (3.2) this implies $\overline{\nu}_K(a) = (KL_0(K))^{\alpha}\overline{\nu}(aK) = L_0(K)^{\alpha}a^{-\alpha}L_1(aK)$ and $\nu_K(-a) = L_0(K)^{\alpha}a^{-\alpha}L_2(aK)$. Using Theorem 3.3, we see that it is enough to show that

$$\lim_{a\to\infty} \sup_{K>\gamma} \int_{[-a,a]^c} |y|\nu_K(\mathrm{d}y) = 0$$

[1]Note that the constants there should be replaced by their inverses.

for some $\gamma > 0$, which we specify later.
Using partial integration and the remark above, we find:

$$\int_{[-a,a]^c} |y|\nu_K(\mathrm{d}y) =$$
$$a\overline{\nu}_K(a) + \int_a^\infty \overline{\nu}_K(t)\mathrm{d}t + a\nu_K(-a) + \int_{-\infty}^{-a} \nu_K(t)\mathrm{d}t =$$
$$a^{-\alpha+1}L_0(K)^\alpha L(aK) + \int_a^\infty t^{-\alpha}L_0(K)^\alpha L(tK)\mathrm{d}t$$

Because of assumptions (3.3) we have $L_0(K)^\alpha L(K)$ is convergent, and in particular, we have $\beta := \sup_{K>\gamma} L_0(K)^\alpha L(K) < \infty$. Furthermore, using Potter's Theorem (Theorem 1.5.6 in [7]) we have that for $\delta > 0$ such that $1 + \delta < \alpha$ there exists $\gamma > 0$ such that

$$\frac{L(aK)}{L(K)} \le 2\max(a^\delta, a^{-\delta}) \qquad aK > \gamma, K > \gamma$$

Using these remarks, we see that

$$\lim_a \sup_{K>\gamma} a^{-\alpha+1}L_0(K)^\alpha L(aK) \le$$
$$\beta \lim_a \sup_{K>\gamma} a^{-\alpha+1}\frac{L(aK)}{L(K)} \le$$
$$2\beta \lim_a a^{-\alpha+1}\max(a^\delta, a^{-\delta}) = 0$$

and similarly for the integral:

$$\lim_a \sup_{K>\gamma} \int_a^\infty t^{-\alpha}L_0(K)^\alpha L(tK)\mathrm{d}t \le$$
$$\beta \lim_a \int_a^\infty t^{-\alpha} \sup_{K>\gamma} \frac{L(tK)}{L(K)}\mathrm{d}t \le$$
$$2\beta \lim_a \int_a^\infty t^{-\alpha}\max(t^\delta, t^{-\delta})\mathrm{d}t = 0\,.$$

We may therefore apply Theorem 3.2 and Proposition 6.1 to obtain

$$K^{\alpha-1}L_0(K)^\alpha \ell^K(S) \sim K^{\alpha-1}L_0(K)^\alpha \ell^{f(f^{\leftarrow}(K)}(S) = \ell^1\big(S^K\big)\,.$$

Letting $K \to \infty$ and using the expression for the loss rate of a stable distribution which is calculated in Example 3.2 in Asmussen and Pihlsgård [4] (see also Kyprianou [20]), yields the desired result. □

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PAPER D

Parallel Computing, Failure Recovery, and Extreme Values

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Abstract

A task of random size T is split into M subtasks of lengths $T_1, \ldots, T_M$, each of which is sent to one out of M parallel processors. Each processor may fail at a random time before completing its allocated task, and then has to restart it from the beginning. If $X_1, \ldots, X_M$ are the total task times at the M processors, the overall total task time is then $Z_M = \max_{1,\ldots,M} X_i$. Limit theorems as $M \to \infty$ are given for Z_M, allowing the distribution of T to depend on M. In some cases the limits are classical extreme value distributions, in others they are of a different type.
Keywords failure recovery, Fréchet distribution, geometric sums, Gumbel distribution, heavy tails, logarithmic asymptotics, mixture distribution, power tail, RESTART, triangular array.

1 Introduction

Consider a job that ordinarily would take a time T to be executed on some system (e.g., CPU). If at some time $U < T$ the processor fails, the job may take a *total time* $X \geq T$ to complete. We let F, G be the distributions of T, U with $H = H_{F,G}$ the distribution of X, which in addition to F, G will depend on the failure recovery scheme.

Many papers discuss methods of failure recovery and analyze their complexity, like *restartable processors* in Chlebus *et al.* [6], or *stage checkpointing* in De Prisco *et al.* [14], etc. There are many specific and distinct failure recovery schemes, but they can be grouped into three broad classes:

RESUME, also referred to as preemptive resume;
REPLACE, also referred to as preemptive repeat different;
RESTART, also referred to as preemptive repeat identical.

In the RESUME scenario, if there is a processor failure while a job is being executed, after repair is implemented the job can continue where it left off. All that is required mathematically is to remember the state of the system when failure occurred. In the REPLACE situation, if a job fails, it is replaced by a different job having the same distribution. Here, no details concerning the previous job are necessary in order to continue.

The analysis of the distribution function $H(x) = \mathbb{P}(X \leq x)$ when the policy is RESUME or REPLACE was carried out by Kulkarni *et al.* [10], [11] (see also Bobbio & Trivedi [4], Castillo & Siewiorek [16] and Chimento & Trivedi [5]). The RESTART policy had resisted detailed analysis until the recent papers by Sheahan *et al.* [15], Asmussen *et al.* [2], Jelenkovic & Tan [9], where the tail asymptotics of H was found under a variety of assumptions on F and G. The setting of [9] is file transfer problems and involves an on-off model that incorporates what in the present setting corresponds to repairs. In contrast, [2] has its background in the computer science literature on failure recovery in the execution of a program on a computer.

For many systems failure is sufficiently rare to be ignored, or dealt with as an afterthought. For other systems, failure is common enough that the design choice of how to deal with it may have a significant impact on the performance of the system. One such example arises in parallel computing, where the probability of failure of a single processor in isolation may be small, but the number of processors is so large (in practice, often hundreds or thousands) that the probability that one or more processors fail cannot be neglected. The present paper studies the implications of the analysis of [2] for this situation. To formalize the set-up, assume that the job is split into M parts of lengths

$S_1, \dots, S_M$, which are executed on M parallel processors. The total times on the processors, including restarts, are denoted $X_1, \dots, X_M$. Thus the total time for the whole job is $Z = \max_{i=1,\dots,M} X_i$. What can then be said about the distribution of Z? For example, assume there is given a cost function of the type $a + bM + cZ_M$ where a is a set-up cost, b a cost per processor and c the cost per unit time. One would then want to choose M to minimize the expected cost $a + bM + c\mathbb{E}Z_M$ (note that one expects $\mathbb{E}Z_M$ to be a decreasing function of M).

The reason for using parallel processors will often be that the job is large. For example, the job could consist in generating R replicates of a Monte Carlo estimator for some large R. On the other hand, there may be situations where speed is an essential factor when executing a job of small or moderate size, i.e. the cost function has a large c. For example this could occur in filtering a noisy signal or in option price calculations based upon high-frequency input. This suggests considering a general triangular array situation where the total job size $T = T_M$ and hence $F = F_M$, the distribution of the job time faced by a single processor, varies with the number M of processors. We then write $S_1^{(M)}, \dots, S_M^{(M)}$, $X_1^{(M)}, \dots, X_M^{(M)}$,

$$Z_M \;=\; \max_{i=1,\dots,M} X_i^{(M)} \,,$$

and $H_M(x) = \mathbb{P}(Z_M \le x)$. We will consider two scenarios:

(D) $T = T_M = t_M$ for some deterministic t_M and $S_M = s_M = t_M/M$; then F_M is the one-point distribution at s_M;

(Γ) F_M is Gamma(α_M, λ) with density

$$f_M(t) \;=\; \frac{\lambda^{\alpha_M}}{\Gamma(\alpha_M)} t^{\alpha_M - 1} \mathrm{e}^{-\lambda t} \,.$$

Further, $S_1^{(M)}, \dots, S_M^{(M)}$ are independent. Thus the distribution of the total job size is a Gamma$(M\alpha_M, \lambda)$ distribution.

A random total job size arises in situations where the run length of the job sent to parallel processing will not be known in advance but is random. An example is Monte Carlo simulations involving random number generation by acceptance-rejection or more complicated stopping times such as cycles in regenerative simulation (see [1]). Note that since for fixed λ, the Gamma(α, λ) distributions form a convolution semigroup in α, assumption (Γ) is a natural stochastic extension of the deterministic set-up (D) (α_M corresponds to s_M). For example, in the Monte Carlo setting each replication could take a

Gamma(α, β) time, and each processor would be asked to perform R_M replications. Then $\alpha_M = \beta R_M$. Of course, the Gamma case is only one among many where the total job size is infinitely divisible, and independence among subjobs is a reasonable assumption (such independence may certainly fail in some situations, but we do not consider this possibility here).

In scenario (D), we sometimes assume that $t_M = t_1 M^p$, i.e. $s_M = s_1 M^{p-1}$ for some $p \geq 0$. Here $p = 1$ could be relevant for the Monte Carlo example and $p = 0$ for the filtering example, though clearly in both situations intermediate values could also arise. The cases $p < 1$, $p = 1$ and $p > 1$ are qualitatively different since in the first $s_M \to 0$ and in the third $s_M \to \infty$ subject to (D), whereas s_M is constant when $p = 1$; analogous remarks apply to the Gamma case with the α_M taking the roles of the s_M.

We will assume throughout the paper that the failure time distribution G is independent of M and, except for Section 4, that G is exponential, with rate parameter μ.

The paper starts in Section 3 by an analysis of the case $p = 1$. This is fairly easy, because then S does not depend on M and the $X_{i,M}$ are i.i.d. random variables with a distribution not depending on M. Given the results from [2] on the tail of H, classical extreme value theory ([12]) can therefore be easily translated into a limit theorem for Z_M.

If $p \neq 1$, the $X_{i,M}$ have a distribution depending on M, so that we are beyond classical extreme value theory and have to consider a triangular array setting. This is carried out in Section 4 for $p < 1$ and Section 5 for $p > 1$. Finally, the Gamma case with $\alpha_M \to \infty$ is treated is Section 6 (the case $\alpha_M \to 0$ is non-trivial and is not included here).

2 Preliminaries

We first recall some background material from Asmussen *et al.* [2] for the RESTART setting with F independent of M. The key to the analysis in this work is the fact that given $T = t$, X is distributed as

$$t + S(t) \quad \text{where} \quad S(t) = \sum_{i=1}^{N(t)} U_i(t)\,, \tag{2.1}$$

where the $U_i(t)$ are i.i.d. distributed as U conditioned on $U \leq t$, i.e.

$$\mathbb{P}\big(U_i(t) \leq y\big) = \begin{cases} G(y)/G(t) & y \leq t \\ 1 & y > t \end{cases},$$

and $N(t)$ is an independent geometric r.v. with success parameter $\overline{G}(t) = 1 - G(t)$, that is, $\mathbb{P}\big(N(t) = n\big) = \overline{G}(t)G(t)^n$. The following result plays a key role in [2] as well as the present paper:

Lemma 2.1. *Assume $T \equiv t_1$ and $\overline{G}(t_1) > 0$. Then*

$$\overline{H}(x) \;\sim\; C(t_1)\mathrm{e}^{\gamma(t_1)t_1}\mathrm{e}^{-\gamma(t_1)x} \tag{2.2}$$

where $\gamma(t) > 0$ is the solution of $\int_0^t \mathrm{e}^{\gamma(t)y} G(\mathrm{d}y) \;=\; 1$ and $C(t) \;=\; \overline{G}(t)/\gamma(t)B(t)$ where $B(t) \;=\; \int_0^t y\mathrm{e}^{\gamma(t)y} g(y)\,\mathrm{d}y$. Further,

$$\mathrm{e}^{-\gamma(t_1)x} \;\le\; \overline{H}(x) \;\le\; \mathrm{e}^{\gamma(t_1)t_1}\mathrm{e}^{-\gamma(t_1)x} \tag{2.3}$$

A common terminology refers to (2.2) as the *Cramér-Lundberg approximation* and to (2.3) as *Lundberg's inequality*.

We shall also use the following obvious consequence of the representation (2.1) of the conditional distribution of X given $T = t$:

$$\overline{H}(x) \;=\; \int_0^\infty \mathbb{P}\big(t + S(t) > x\big)\, F(\mathrm{d}t)\,. \tag{2.4}$$

For Scenario (Γ), the relevant result from [2] is the following (with some typos in [2] corrected here):

Lemma 2.2. *Consider Scenario (Γ) with $\alpha_M \equiv \alpha$ independent of M. Then*

$$\overline{H}(x) \;\sim\; C\frac{\log^{\alpha-1} x}{x^{\lambda/\mu}} \;\; x \to \infty, \;\; where \;\; C \;=\; \frac{\Gamma\left(\frac{\lambda}{\mu}\right)}{\mu^{\frac{\lambda}{\mu}}}\frac{\lambda^\alpha}{\Gamma(\alpha)}\frac{1}{\mu^\alpha}\,.$$

We shall also need:

Lemma 2.3. *Let K be a distribution function such that $M\overline{H}_M\big(a_M y + b_M\big) \to \log -K(y)$ for all y. Then the distribution of $(Z_M - b_M)/a_M$ converges to K.*

The lemma is standard in extreme value theory when H_M is independent of M and follows from the fact that

$$\mathbb{P}\big((Z_M - b_M)/a_M \le y\big) \;=\; H_M\big(a_M y + b_M\big)^M \;=\; \big(1 - M\overline{H}_M\big(a_M y + b_M\big)/M\big)^M,$$

by taking logs and expanding in a Taylor series. The classical limits relevant for this paper are the Gumbel distribution with $K(y) = \mathrm{e}^{-\mathrm{e}^{-y}}$ and the Fréchet distribution with parameter $\beta > 0$ where $K(y) = \mathrm{e}^{-y^{-\beta}}$ (a Weibull limit may also occur in the classical setting, but is not relevant for RESTART because it requires bounded support). However, in a triangular setting other types of K's may occur, of which we will later see examples. A general reference on extreme value theory for triangular arrays is Valente Freitas & Hüsler [7]. However, this reference basically covers only a neighborhood of classical extreme value theory (i.e., S_M not too varying with M so that non-classical limits are not covered), and further, it requires a differentiability condition on H_M which fails at $s_M, 2s_M, \ldots$.

An important feature worth stressing is that extreme value statements deal with *typical values* of Z_M (of the form $b_M + a_M y$ in the setting of Lemma 2.3), not with tail behavior.

3 The case $p = 1$: classical extreme values

Assume that F_M does not depend on M.

Proposition 3.1. *Consider the case $s_M \equiv s_1$ in Scenario (D). Let γ denote the solution of $1 = \int_0^{s_1} \mathrm{e}^{\gamma y} \mu e^{-\mu y} \mathrm{d}y$ and set $C = \mathrm{e}^{-\mu s_1}/\gamma B$, where $B = \int_0^{s_1} y \mathrm{e}^{\gamma y} \mu \mathrm{e}^{-\mu y} \mathrm{d}y$. Then $\gamma\big(Z_M - s_1\big) - \log(MC)$ has a limiting Gumbel distribution as $M \to \infty$.*

Proof. Note that $Z_M - s_1$ is distributed as the maximum of M independent copies of $S(s_1)$ and that

$$\mathbb{P}\big(S(s_1) > x\big) \sim C\mathrm{e}^{-\gamma x}, \; x \to \infty,$$

by Lemma 2.1. An asymptotic exponential tail is a standard sufficient condition in extreme value theory for the random variable to be in the maximum domain of attraction of the Gumbel distribution, and the form of the normalizing constants also follows from this theory. A direct proof from Lemmas 2.1 and 2.3 is straightforward: with $a_M = 1/\gamma$, $b_M = s_1 + \log(MC)/\gamma$, one gets

$$M\overline{H}\big(a_M y + b_M\big) \sim MC\mathrm{e}^{\gamma s_1}\mathrm{e}^{-\gamma(a_M y + b_M)} = \mathrm{e}^{-y}.$$

□

The implication is that Z_M is of order $\log M/\gamma$. For example, since $-\log\log 2$ is the median in the Gumbel distribution, we obtain the approximation $s_1 - \log\log 2/\gamma + \log(MC)/\gamma$ for the median of Z_M; note that, as remarked at the end of Section 2, this is not a tail approximation but telling information about the typical values of Z_M. Similarly, since the Euler constant $\varphi \approx 0.577$ is the mean of the Gumbel distribution, one obtains the approximation $\varphi/\gamma + s_1 + \log(MC)/\gamma$ for $\mathbb{E}Z_M$ (for verification of the required uniform integrability, see Pickands [13]).

Proposition 3.2. *Consider Scenario* (Γ) *with* $\alpha_M \equiv \alpha$ *independent of* M*, and define*

$$a_M \;=\; \frac{C^{\mu/\lambda}}{(\lambda/\mu)^{(\alpha-1)\mu/\lambda}} M^{\mu/\lambda} \log^{((\alpha-1)\mu/\lambda} M\,,$$

where C *is defined in Lemma 2.2. Then* Z_M/a_M *has an approximate Fréchet distribution with parameter* $\beta = \lambda/\mu$.

Proof. The result again follows from the standard extreme value characterization of the maximum domain of attraction of the Fréchet distribution and Lemma 2.2. Again, a direct proof from Lemma 2.3 is easy: with $b_M = 0$, one gets

$$MH(a_M y) \;\sim\; \frac{1}{y^{\frac{\lambda}{\mu}}}\left(\frac{\log(a_M y)}{\log(M)\frac{\mu}{\lambda}}\right)^{\alpha-1} \;\to\; \frac{1}{y^{\frac{\lambda}{\mu}}},.$$

□

Again using the median as an example, the approximation for the median of Z_M becomes $a_M/\log^{1/\beta} 2$. The mean of the Fréchet distribution is finite if and only if $\beta > 1$ and then equals $\Gamma(1-1/\beta)$. This suggests the approximation $a_M\Gamma(1-1/\beta)$ for $\mathbb{E}Z_M$ when $\lambda > \mu$. Since a_M is roughly of order $M^{\mu/\lambda}$ which increases much faster than the $\log M$ occurring in Scenario (D), this shows the dramatic effect of randomness on the total job size.

4 Scenario (D) with $p < 1$

We now assume in Scenario (D) that $t_M = t_1 M^p$ for some $0 \le p < 1$ and $t_1 > 0$ so that $s_M = t_1 M^{p-1}$. We will work with the following condition on G:

$$G(x) = x^\alpha L(x) \tag{4.1}$$

with $\alpha > 0$ and L slowly varying at 0, so that $\lim_{x\to\infty} L((tx)^{-1})/L(x^{-1}) = 1$ $t > 0$. In particular, this covers a Gamma G where $L(x)$ has a limit as $x \downarrow 0$ (in the exponential set-up, $\alpha = 1$ and $L(x) \to \mu$).

We note the following consequence of (4.1):

$$\lim_M \mathbb{P}(s_M^{-1}U \le x \mid U \le s_M) = \lim_M \frac{G(s_M x)}{G(s_M)} = \lim_M x^\alpha \frac{L(s_M x)}{L(s_M)} = x^\alpha, \tag{4.2}$$

where $0 \le x \le 1$. We define $U^{(\alpha)}$ to be a random variable with distribution function

$$\mathbb{P}(U^{(\alpha)} \le x) = \begin{cases} 0 & x \le 0 \\ x^\alpha & 0 < x \le 1 \\ 1 & x > 1 \end{cases},$$

and because of (4.2) we have

$$s_M^{-1}U \le x \mid U \le s_M \xrightarrow{\mathcal{D}} U^{(\alpha)} .$$

Theorem 4.1.
I) Assume $p \neq (k\alpha - 1)/k\alpha$ *for any* $k \in \mathbb{N}$. *Set* $p^* = \lfloor 1/(\alpha(1-p)) \rfloor$. *Then*

$$t_1^{-1}M^{1-p}Z_M - 1 \xrightarrow{\mathbb{P}} p^* . \tag{4.3}$$

II) Assume $p = (k\alpha - 1)/k\alpha$ *for some* $k \in \mathbb{N}$, *and also that* $\lim_{x\downarrow 0} L(x) = \gamma \in (0, \infty]$ *exists. Then*

$$t_1^{-1}M^{1-p}Z_M - 1 \xrightarrow{\mathcal{D}} V , \tag{4.4}$$

where V *is distributed as*

$$\max_{1\le j\le N}\left(k-1, \sum_{i=1}^{k} U_i^j\right)$$

with the U_i^j *being i.i.d.* $U^{(\alpha)}$ *r.v.'s,* N *is an independent Poisson r.v. with mean* $\gamma^k t_1^{\alpha k}$ *when* $\gamma < \infty$, *and* $N = \infty$ *a.s. when* $\gamma = \infty$.

For the proof, we denote by $R_i^{(M)}$ the number of restarts of the ith processor, and let $V_k^{(M)}$ be the number of processors, with k restarts, so that

$$V_k^{(M)} = \sum_{i=1}^{M} I(R_i^{(M)} = k) .$$

Let $\rho_M = G(s_M)$ and define

$$\Theta_{M,k} = \rho_M^k(1-\rho_M).$$

We have $I(R_i^{(M)} = k) =_{\mathscr{D}} \mathrm{Bin}(1, \Theta_{M,k})$ and $V_k^{(M)} =_{\mathscr{D}} \mathrm{Bin}(M, \Theta_{M,k})$.

As a first step in the proof of Theorem 4.1, we examine the limit possibilities for $V_k^{(M)}$:

Proposition 4.1.
I) If $k < 1/(\alpha(1-p))$, *then, setting* $\sigma_M = \sqrt{M\Theta_{M,k}(1-\Theta_{M,k})}$, *we have*

$$\frac{1}{\sigma_M}\left(V_k^{(M)} - M\Theta_{M,k}\right) \xrightarrow{\mathcal{D}} N(0,1), \qquad M \to \infty. \tag{4.5}$$

II) If $k = 1/\big(\alpha(1-p)\big)$, and $\lim_{x\downarrow 0} L(x) = \gamma \in (0,\infty]$ exists, then

$$V_k^{(M)} \xrightarrow{\mathcal{D}} Po(t_1^{\alpha k}\gamma^k) \qquad M \to \infty\,, \tag{4.6}$$

where $\gamma = \infty$ corresponds to the degenerate case at ∞.
III) If $k > 1/\big(\alpha(1-p)\big)$, then

$$V_k^{(M)} \xrightarrow{\mathbb{P}} 0\,, \qquad M \to \infty. \tag{4.7}$$

Proof. First we notice that since $s_M = t_1 M^{p-1}$, then for all $k \in \mathbb{N}$

$$M\Theta_{M,k} = M^{1+k\alpha(p-1)} M^{-k\alpha(p-1)} G(s_M)^k \big(1 - G(s_M)\big) \tag{4.8}$$
$$\sim M^{1+k\alpha(p-1)} L(t_1 M^{p-1})^k t_1^{\alpha k} \tag{4.9}$$

Now, for the proof of I) assume $k < 1/\big(\alpha(1-p)\big)$. We need to prove that $M\Theta_{M,k} \to \infty$. This is seen by defining $H(y) = L(t_1/y)^k$. Then H is slowly varying at infinity, and we have:

$$M\Theta_{M,k} = M^{1+k\alpha(p-1)} L(t_1 M^{p-1})^k \;=\; M^{1+k\alpha(p-1)} H(M^{1-p})\,.$$

Substituting $x_M = M^{1-p}$ in this expression yields

$$x_M^{\frac{1}{1-p}-\alpha k} H(x_M)\,,$$

which tends to infinity by Proposition 1.3.6(v) in [3]. This implies that $\sigma_M \to \infty$, and therefore the normal approximation of the binomial distribution (e.g. (5.33.1) in [8]) implies

$$\frac{1}{\sigma_M}\sum_{i=1}^{M}\Big(I(R_i^{(M)} = k) - \Theta_{M,k}\Big) \xrightarrow{\mathcal{D}} N(0,1),$$

thus proving I).

The proof of III) uses the same calculation as above, where the assumption $k > 1/(\alpha(1-p))$ implies $M\Theta_{M,k} \to 0$ (again, using Proposition 1.3.6 in [3]), that is $\mathbb{E}V_k^{(M)} \to 0$, and since $V_k^{(M)} \geq 0$ we have $V_k^{(M)} \to 0$ in L^1, which proves III). Regarding II), we see that (4.6) follows from (4.9) and the Law of Small Numbers if $\gamma \in (0,\infty)$. If $\gamma = \infty$ then we may use (4.9) to conclude that $M\Theta_{M,k} \to \infty$. Using Chebycheff's inequality we have that $\mathbb{P}(V_k^M \leq \frac{M\Theta_{M,k}}{2}) \to 0$, and therefore $\lim_M \mathbb{P}(V_k^M \leq x) = 0$ for all x, which proves II). □

Corollary 4.1. *If $k < 1/\big(\alpha(p-1)\big)$, then $\lim_M \mathbb{P}(V_k^{(M)} \geq x) \to 1$ for all x.*

Proof. Since $M\Theta_{M,k}/\sigma_M \to \infty$

$$\lim_M \mathbb{P}(V_k^{(M)} \geq x) \ = \ \lim_M \mathbb{P}\left(\frac{V_k^{(M)} - M\Theta_{M,k}}{\sigma_M} \geq \frac{x - M\Theta_{M,k}}{\sigma_M}\right) \to 1\,.$$

□

We are now ready to prove Theorem 4.1:

Proof. In order for $t_1^{-1}M^{p-1}Z_M - 1 = s_M^{-1}(Z_M - s_M)$ to be greater than p^*, we must have at least one processor with $p^* + 1$ restarts. Using Proposition 4.1 III), we obtain

$$\limsup_M \mathbb{P}(t_1^{-1}M^{1-p}Z_M - 1 > p^*) \ \leq \ \limsup_M \mathbb{P}(V_{p^*+1} > 0) = 0\,.$$

Let $\epsilon, \epsilon_1 > 0$ be given. We wish to show that

$$\liminf_M \mathbb{P}(t_1^{-1}M^{1-p}Z_M - 1 \geq p^* - \epsilon) \geq 1 - \epsilon_1\,.$$

Let Z_M^* denote the random variable similar to Z_M, but where we only take the maximum over the processors with exactly p^* restarts, that is:

$$Z_M^* \ = \ \max_{1\leq i\leq M} X_i^{(M)} I\big(R_i^{(M)} = p^*\big)\,.$$

We see that

$$t_1^{-1}M^{1-p}Z_M^* - 1 \ =_{\mathscr{D}} \ \max_{1\leq i\leq V_{p^*}^{(M)}} \sum_{j=1}^{p^*} t_1^{-1}M^{p-1}U_j^{(M),i}$$

where the $U_j^{(M),i}$ are independent and distributed as $s_M^{-1}U \mid U < s_M$. Since $p^* < 1/\big(\alpha(1-p)\big)$, we have by Corollary 4.1 that $V_{p^*}^{(M)} \xrightarrow{\mathbb{P}} \infty$, and therefore, for any $K \in \mathbb{N}$

$$\liminf_M \mathbb{P}\Big(\max_{1\leq i\leq V_{p^*}^{(M)}} \sum_{j=1}^{p^*} t_1^{-1}M^{1-p}U_j^{(M),i} \geq p^* - \epsilon\Big) \ \geq \tag{4.10}$$

$$\liminf_M \mathbb{P}\Big(\max_{1\leq i\leq K} \sum_{j=1}^{p^*} t_1^{-1}M^{1-p}U_j^{(M),i} \geq p^* - \epsilon\Big)\,. \tag{4.11}$$

Furthermore, since

$$\max_{1\le i\le K} \sum_{j=1}^{p^*} t_1^{-1} M^{1-p} U_j^{(M),i} \xrightarrow{\mathcal{D}} \max_{1\le i\le K} \sum_{j=1}^{p^*} U_j^i$$

where the U_j^i are i.i.d. and are distributed as U^α, we may complete the proof of I) by choosing K so large that

$$\mathbb{P}\Big(\max_{1\le i\le K} \sum_{j=1}^{p^*} U_j^i \ge p^* - \epsilon\Big) \ge 1 - \epsilon_1 \,.$$

Regarding II), we see that if $k = 1/\big(\alpha(1-p)\big)$ then by (4.6) we have asymptotically N processors which have k restarts, where $N =_{\mathscr{D}} \mathrm{Po}(t_1^{\alpha k}\gamma^k)$; by (4.5) we have infinitely many processors with $k-1$ restarts, and by (4.7) we have 0 processors with $k+1$ restarts. Define the following r.v.'s:

$$Z_{M,1} = \max_{1\le i\le M} X_i^{(M)} I(R_i^{(M)} < k)$$
$$Z_{M,2} = \max_{1\le i\le M} X_i^{(M)} I(R_i^{(M)} = k)$$
$$Z_{M,3} = \max_{1\le i\le M} X_i^{(M)} I(R_i^{(M)} > k)$$

Then clearly $Z_M = \max(Z_{M,1}, Z_{M,2}, Z_{M,3})$ and since $t_1^{-1}M^{1-p}Z_{M,1} - 1 \xrightarrow{\mathbb{P}} k-1$, $t_1^{-1}M^{1-p}Z_{M,3} - 1 \xrightarrow{\mathbb{P}} 0$ and $t_1^{-1}M^{1-p}Z_{M,2} - 1 \xrightarrow{\mathcal{D}} \sum_{i=1}^k U_i^j$, where (U_j^j) is an i.i.d. sequence of r.v.'s distributed as $U^{(\alpha)}$, the proof is complete. □

5 Scenario (D) with $s_M \to \infty$

We now consider Scenario (D) with $s_M \to \infty$ (for example, $t = t_M = t_1 M^p$ with $p > 1$; equivalently, M grows with t like $t^{1/p}$, i.e. at a rate somewhat slower than t). That is, there is significant but not massive parallelization. Let $\gamma_M = \gamma(s_M)$ in the notation of Lemma 2.1. We shall prove:

Theorem 5.1. *Consider Scenario (D) with $s_M \to \infty$. Then $\mu \mathrm{e}^{-\mu s_M} Z_M - \log M$ has a limiting Gumbel distribution as $M \to \infty$.*

This means that Z_M is of order $\mathrm{e}^{\mu s_M} \log M/\mu = \mathrm{e}^{\mu t_M/M} \log M/\mu$.

Lemma 5.1. *Let $\gamma_M = \gamma(s_M)$ in the notation of Lemma 2.1. Then $\gamma_M - \mu \mathrm{e}^{-\mu s_M} = \mathrm{O}\big(s_M \mathrm{e}^{-2\mu s_M}\big)$.*

Proof. Evaluating the integral in the defining equation

$$1 \;=\; \int_0^{s_M} \mathrm{e}^{\gamma_M y} \mu \mathrm{e}^{-\mu y}\, \mathrm{d}y$$

explicitly, one gets

$$1 \;=\; \frac{\mu}{\mu - \gamma_M}\big(1 - \mathrm{e}^{-(\mu-\gamma_M)s_M}\big)\,,$$

which can be rewritten as

$$\gamma_M \;=\; \mu \mathrm{e}^{-(\mu-\gamma_M)s_M}\,. \tag{5.1}$$

This shows that γ_M is of first order $\mu \mathrm{e}^{-\mu s_M}$ (as is shown already in [2]). In particular, $\gamma_M s_M \to 0$ so that by Taylor expansion of (5.1),

$$\gamma_M \;\approx\; \mu \mathrm{e}^{-\mu s_M}(1 + \gamma_M s_M)\,.$$

This proves the assertion. □

Proof of Theorem 5.1. Let F_M denote the distribution of $X_i^{(M)}$. Then by Lundberg's inequality,

$$\mathrm{e}^{-\gamma_M x} \;\le\; \overline{H}_M(x) \;\le\; \mathrm{e}^{\gamma_M s_M}\mathrm{e}^{-\gamma_M x}\,.$$

Let $b_M = \log M/\gamma_M$, $a_M = 1/\gamma_M$. Then

$$M\overline{H}_M(a_M y + b_M) \;\ge\; M\mathrm{e}^{-\gamma_M(a_M y + b_M)} \;=\; M\mathrm{e}^{-y+\log M} \;=\; \mathrm{e}^{-y}\,.$$

Similarly,

$$M\overline{H}_M(a_M y + b_M) \;\le\; M\mathrm{e}^{\gamma_M s_M}\mathrm{e}^{-\gamma_M(a_M y + b_M)} \;\to\; 1\cdot \mathrm{e}^{-y}\,.$$

Thus $M\overline{H}_M(a_M y + b_M) \;\to\; \mathrm{e}^{-y}$ for all y, which implies that

$$\gamma_M Z_M - \log M \;=\; \frac{Z_M - b_M}{a_M}$$

has a Gumbel limit. It then follows that Z_M is roughly of order $1/\gamma_M$ or equivalently $\mathrm{e}^{\mu s_M}$. To replace γ_M by $\mu\mathrm{e}^{-\mu s_M}$ in the limit statement for Z_M, one therefore needs

$$(\gamma_M - \mu\mathrm{e}^{-\mu s_M})\mathrm{e}^{\mu s_M} \;\to\; 0\,,$$

which follows by Lemma 5.1. □

6 The Gamma case

We now consider Scenario (Γ) with $\alpha_M \to \infty$.

Theorem 6.1. *Consider the Gamma case with $\alpha_M \to \infty$ and let $r = \mu/\lambda$. Assume in addition that $\alpha_M/\log M \to \infty$. Then Z_M is of logarithmic order $\mathrm{e}^{r\alpha_M}$ in the sense that $\log Z_M/\alpha_M \xrightarrow{\mathbb{P}} r$ as $M \to \infty$.*

For the proof, define $x_M = \mathrm{e}^{r_1\alpha_M}$. We shall show that

$$M\overline{H}_M(x_M) \;\to\; \begin{cases} \infty & \text{if } r_1 < r \\ 0 & \text{if } r_1 > r. \end{cases} \tag{6.1}$$

Indeed, if $r_1 < r$ then (6.1) shows that the expected number of processors i with $X_{i,M} > x_M$ tends to ∞ and hence the probability that one $X_{i,M} > x_M$ tends to 1. Similarly, if $r_1 > r$ then (6.1) shows that the expected number of processors i with $X_{i,M} > x_M$ tends to 0, and hence so does the probability that one $X_{i,M} > x_M$.

Lemma 6.1. *Define*

$$I_M \;=\; \int_0^{cx_M} a^{\lambda/\mu-1}\mathrm{e}^{-a}\varphi_M(a)\,\mathrm{d}a \;=\; \int_0^{cx_M} a^{1/r-1}\mathrm{e}^{-a}\varphi_M(a)\,\mathrm{d}a$$

where $0 < c \le \mu'$ is a constant and

$$\varphi_M(a) \;=\; \Bigl(1 + \frac{\log\mu' - \log a}{\log x_M}\Bigr)^{\alpha_M-1}.$$

Then $I_M \to {\mu'}^{1/r_1}\Gamma(1/r - 1/r_1)$ as $M \to \infty$ when $r_1 > r$, whereas

$$\liminf_{M\to\infty} \frac{I_M}{\alpha_M^{1/2}\exp\bigl\{(\delta - \log\delta - 1)\alpha_M\bigr\}} \;>\; 0$$

when $r_1 < r$, where $\delta = r_1/r$.

Note that the convexity of the log implies that $\delta - \log\delta - 1 > 0$ when $\delta \ne 1$.

Proof. We split I_M up into the contributions I'_M and I''_M from $a < \mu'$ and $\mu' < a < cx_M$, respectively. For $a < \mu'$, $\varphi_M(a) \uparrow {\mu'}^{1/r_1}a^{-1/r_1}$, and hence by monotone convergence,

$$I'_M \;\uparrow\; \int_0^{\mu'} a^{1/r-1/r_1-1}\mathrm{e}^{-a}\,\mathrm{d}a\,.$$

When $r_1 > r$, we thus need in addition to show that

$$I''_M \;\to\; {\mu'}^{1/r_1} \int_{\mu'}^{\infty} a^{1/r-1/r_1-1} \mathrm{e}^{-a}\, \mathrm{d}a\,.$$

This follows by dominated convergence since $\varphi_M(a)$ is dominated by 1 on (μ', ∞) and has the limit ${\mu'}^{1/r_1} a^{-1/r_1}$.

Consider now the case $r_1 < r$. Substituting

$$y \;=\; 1 + (\log \mu' - \log a)/\log x_M \;=\; 1 + (\log \mu' - \log a)/r_1\alpha_M\,,$$

we have

$$\log a = \log \mu' + (1-y)r_1\alpha_M, \;\; \frac{1}{a}\mathrm{d}a = -r_1\alpha_M\,\mathrm{d}y, \;\; a = \mu'\mathrm{e}^{r_1\alpha_M}\mathrm{e}^{-r_1\alpha_M y}\,.$$

Thus, bounding e^{-a} below by $c_1 = \mathrm{e}^{-\mu'}$, we get

$$\begin{aligned} I_M \;\ge\; I'_M \;&\ge\; c_2\alpha_M\mathrm{e}^{\delta\alpha_M} \int_1^{\infty} y^{\alpha_M-1}\mathrm{e}^{-y\delta\alpha_M}\,\mathrm{d}y \\ &=\; c_2\mathrm{e}^{\delta\alpha_M}\delta^{-\alpha_M}\alpha_M^{1-\alpha_M} \int_{\delta\alpha_M}^{\infty} z^{\alpha_M-1}\mathrm{e}^{-z}\,\mathrm{d}z\,. \end{aligned}$$

The last integral divided by $\Gamma(\alpha_M)$ is the probability that a Gamma$(\alpha_M, 1)$ r.v. exceeds $\delta\alpha_M$. Since this probability goes to 1 when $\delta < 1$, we get

$$I_M \;\ge\; c_3\mathrm{e}^{\delta\alpha_M}\delta^{-\alpha_M}\alpha_M^{1-\alpha_M}\Gamma(\alpha_M)\,.$$

Using Stirling's approximation

$$\Gamma(\alpha_M) \;\sim\; \mathrm{e}^{-\alpha_M}\alpha_M^{\alpha_M-1}\sqrt{2\pi\alpha_M}$$

completes the proof. □

Proof of (6.1). Let first $r_1 < r$. By the Lundberg lower bound, we have for any $\mu' > \mu$ and some t_0 that

$$\overline{H}_M(x_M) \;\ge\; \int_{t_0}^{\infty} \mathrm{e}^{-\mu'\mathrm{e}^{-\mu t}x_M} \frac{\lambda^{\alpha_M}}{\Gamma(\alpha_M)} t^{\alpha_M-1}\mathrm{e}^{-\lambda t}\,\mathrm{d}t\,. \tag{6.2}$$

Substituting $a = \mu' \mathrm{e}^{-\mu t} x_M$, we have

$$t \;=\; \frac{1}{\mu}(\log \mu + \log x_M - \log a)\,, \quad \mathrm{d}t \;=\; -\frac{1}{\mu a}\,\mathrm{d}a\,,$$

and thus (6.2) becomes

$$\begin{aligned}
&\frac{\lambda^{\alpha_M}}{\Gamma(\alpha_M)\mu^{\alpha_M} x_M^{\lambda/\mu}} \int_0^{\mu' \mathrm{e}^{-\mu t_0} x_M} a^{\lambda/\mu - 1} \mathrm{e}^{-a} \big(\log \mu' + \log x_M - \log a\big)^{\alpha_M - 1}\,\mathrm{d}a \\
&\quad = \; \frac{1}{\Gamma(\alpha_M) r^{\alpha_M} \mu'^{\mu/\lambda}} \frac{\log^{\alpha_M - 1} x_M}{x_M^{1/r}} \int_0^{\mu' \mathrm{e}^{-\mu t_0} x_M} a^{\lambda/\mu - 1} \mathrm{e}^{-a} \varphi_M(a)\,\mathrm{d}a\,.
\end{aligned}$$

By Lemma 6.1, this implies that $M\overline{H}_M(x_M)$ is of larger order than

$$M \frac{1}{\Gamma(\alpha_M) r^{\alpha_M}} \frac{\log^{\alpha_M - 1} x_M}{x_M^{1/r}} \alpha_M^{1/2} \exp\big\{(\delta - \log\delta - 1)\alpha_M\big\}\,.$$

By Stirling's approximation, this is in turn of order

$$\begin{aligned}
&\frac{M \mathrm{e}^{\alpha_M}}{\alpha_M^{1/2} \alpha_M^{\alpha_M - 1} r^{\alpha_M}} \frac{\log^{\alpha_M - 1} x_M}{x_M^{1/r}} \\
&\quad = \; \frac{M \mathrm{e}^{\alpha_M}}{\alpha_M^{1/2} \alpha_M^{\alpha_M - 1} r^{\alpha_M}} \frac{\alpha_M^{\alpha_M - 1} r_1^{\alpha_M - 1} \alpha_M^{\alpha_M - 1}}{\mathrm{e}^{(r_1/r)\alpha_M}} \alpha_M^{1/2} \exp\big\{(\delta - \log\delta - 1)\alpha_M\big\} \\
&\quad = \; M \exp\big\{(1 - \delta + \log\delta)\alpha_M\big\}/\alpha_M^{1/2} \cdot \alpha_M^{1/2} \exp\big\{(\delta - \log\delta - 1)\alpha_M\big\} \\
&\quad = \; M \;\to\; \infty\,.
\end{aligned}$$

Now let $r_1 > r$. Choose μ' such that $\mu' \mathrm{e}^{-\mu t} \le \gamma(t)$ for $t \ge 1$ and let $c_4 = \mu' \sup_{t \ge 1} t\mathrm{e}^{-\mu t}$. Then by the upper Lundberg bound,

$$\overline{H}_M(x_M) \;\le\; \mathbb{P}(T_M \le 1) \;+\; c_4 \int_1^\infty \mathrm{e}^{-\mu' \mathrm{e}^{-\mu t} x_M} \frac{\lambda^{\alpha_M}}{\Gamma(\alpha_M)} t^{\alpha_M - 1} \mathrm{e}^{-\lambda t}\,\mathrm{d}t\;.$$

Here $\mathbb{P}(T_M \le 1)$ goes to 0 at least exponentially fast in α_M. Using the same substitution as when $r_1 < r$, the integral becomes

$$\begin{aligned}
&\frac{\lambda^{\alpha_M}}{\Gamma(\alpha_M)\mu^{\alpha_M} x_M^{\lambda/\mu}} \int_0^{\mu' \mathrm{e}^{-\mu} x_M} a^{\lambda/\mu - 1} \mathrm{e}^{-a} \big(\log \mu' + \log x_M - \log a\big)^{\alpha_M - 1}\,\mathrm{d}a \\
&\quad = \; \frac{1}{\Gamma(\alpha_M) r^{\alpha_M} \mu'^{\mu/\lambda}} \frac{\log^{\alpha_M - 1} x_M}{x_M^{1/r}} \int_0^{\mu' \mathrm{e}^{-\mu} x_M} a^{\lambda/\mu - 1} \mathrm{e}^{-a} \varphi_M(a)\,\mathrm{d}a. \qquad (6.3)
\end{aligned}$$

Here the last integral is $\mathrm{O}(1)$ by Lemma 6.1, and using Stirling's approximation as above shows that (6.3) is of order

$$M \exp\bigl\{(1-\delta+\log\delta)\alpha_M\bigr\}/\alpha_M^{1/2}\,.$$

Putting these estimates together, recalling that $\alpha_M/\log M \to \infty$ and that $1-\delta+\log\delta < 0$ for all $\delta \neq 1$ we see that $M\overline{H}_M(x_M) \to 0$. □

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Appendix

The two-sided Skorokhod problem

In Proposition [1] p.251 we have the one-sided reflected process and local time characterized as a solution to a Skorokhod problem. The following proposition implies a similar characterization in the case of two-sided reflection.

Proposition 1.1.
Let $\{L_t^{0,*}\}$ and $\{L_t^{K,*}\}$ be any non-decreasing right-continuous processes such that the process $\{V_t^*\}$ given by $\{V_0^*\} = x, V_t^* = S_t + L_t^{0,*} - L_t^{K,*}$ satisfies $0 \leq V_t^* \leq K$ for all t, $\int_0^T V_t^* \,\mathrm{d}L_t^{0,*} = 0 \;\forall T$ and $\int_0^T (K - V_t^*) \,\mathrm{d}L_t^{K,*} = 0 \;\forall T$ then $L_t^{0,*}(x) = L_t^0(x)$, $L_t^{K,*}(x) = L_t^K(x)$ and $V_t^* = V_t(x)$.

Proof. Mimicking the calculation in Proposition 2.2 p. 251 in [1] we set $D_t = L_t^0 - L_t^K - \left(L_t^{0,*} - L_t^{K,*}\right)$ and using integration-by-parts of this right-continuous process of bounded variation yields

$$
\begin{aligned}
D_t^2 &= 2\int_0^t D_s \,\mathrm{d}D_s - \sum_{s\leq t} (\Delta D_s)^2 \\
&= 2\int_0^t \left(V_s - V_s^*\right) \mathrm{d}D_s - \sum_{s\leq t} (\Delta D_s)^2 \\
&= 2\int_0^t \left(V_s - V_s^*\right) \mathrm{d}L_t^0 - 2\int_0^t \left(V_s - V_s^*\right) \mathrm{d}L_t^K \\
&\quad - 2\int_0^t \left(V_s - V_s^*\right) \mathrm{d}L_s^{0,*} + 2\int_0^t \left(V_s - V_s^*\right) \mathrm{d}L_s^{K,*} - \sum_{s\leq t} (\Delta D_s)^2 \\
&= -2\int_0^t V_s^* \,\mathrm{d}L_t^0 - 2\int_0^t \left(K - V_s^*\right) \mathrm{d}L_t^K \\
&\quad - 2\int_0^t V_s \,\mathrm{d}L_s^{0,*} - 2\int_0^t (K - V_s) \,\mathrm{d}L_s^{K,*} - \sum_{s\leq t} (\Delta D_s)^2 .
\end{aligned}
$$

Since the right hand side is non-positive we have $D_t = 0$ and by subtraction, this implies that $V_t = V_t^*$. Furthermore

$$
L_t^{0,*} - L_t^0 = L_t^K - L_t^{K,*} \qquad \forall t
$$

Since the left hand side can only change when $V_t = V_t^* = 0$ and the right hand side can only change when $V_t = V_t^* = K$, both sides must be constant, and therefore equal to 0 which proves the statement. □

Miscellaneous results

The following result is used in the proof of Proposition 3.1. A similar result is found in Proposition 1.2 of [2]

Lemma 1.2. *For i.i.d. random variables X, Y, with $X \in \mathcal{S}$ we have for any A*

$$\mathbb{P}(A < X \leq K - A | X + Y > K) \to \frac{1}{2}\overline{F}(A), \quad K \to \infty, \tag{1.4}$$

where F is the c.d.f. of X

Proof. We start by observing that for $K > 2A$ we have:

$$\mathbb{P}(A < X \leq K - A, X + Y > A) = \mathbb{P}(X + Y > K, A < X, X \leq K - A) = \underbrace{\mathbb{P}(X + Y > K)}_{\mathcal{A}(K)} - \underbrace{\mathbb{P}(X + Y > K, A \geq X)}_{\mathcal{B}(A,K)} - \underbrace{P(X + Y > K, X > K - A)}_{\mathcal{C}(A,K)}$$

and we have

$$\frac{\mathcal{A}(K)}{\mathbb{P}(X + Y > K)} = 1\,. \tag{1.5}$$

Using Proposition 1.2 in [2]:

$$\frac{\mathcal{B}(A, K)}{\mathbb{P}(X + Y > K)} = \mathbb{P}(X \leq A \mid X + Y > K) \to \frac{1}{2}F(A)\,. \qquad K \to \infty \tag{1.6}$$

For $\mathcal{C}(A, K)$ we have:

$$\begin{aligned}
\mathcal{C}(A, K) &= \mathbb{P}\big(X > K - \min(A, Y)\big) \\
&= \mathbb{P}(X > K - A, Y > A) + \mathbb{P}(X > K - Y, Y \leq A) \\
&= \mathbb{P}(X > K - A)\mathbb{P}(Y > A) + \mathbb{P}(X > K - Y, Y \leq A)\,.
\end{aligned}$$

Using that $\mathcal{S}$ is closed under convolution, we have:

$$\begin{aligned}
&\frac{\mathcal{C}(A, K)}{\mathbb{P}(X + Y > K)} \\
&= \frac{\mathbb{P}(X > K - A)}{\mathbb{P}(X + Y > K)}\overline{F}(A) + \mathbb{P}(Y \leq A \mid X + Y > K) \to \\
&\frac{1}{2}\overline{F}(A) + \frac{1}{2}F(A) = \frac{1}{2}\,. \qquad K \to \infty
\end{aligned}$$

The proof is finished by combining the result above with (1.5) and (1.6). ☐

It is well-known that if the characteristic function of a random variable X is twice differentiable then $\mathbb{E}X^2 < \infty$ but that there exists random variables with differentiable characteristic function but without finite first moment. The following result shows that differentiability and $\mathbb{E}X^- < \infty$ ensures finite first moment.

Proposition 1.2. *Let X be a r.v. with $\mathbb{E}X^- < \infty$ and characteristic function φ which is differentiable at 0. Then $\mathbb{E}|X| < \infty$*

Proof. Assume $\mathbb{E}|X| = \infty$ and let $\mathbb{C} \ni a := \lim_{x\to 0}(\varphi_X(x)-1)/x$ which exists, since φ is differentiable at 0. Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with the same distribution as X. Let $S_n := \sum_{i=1}^n X_i$ be the partial sums, and let γ denote the characteristic function of S_n/n. Then we have

$$\gamma(t) = \big(\varphi(t/n)\big)^n = \left(1 + \frac{n(\varphi(\frac{t}{n}) - 1)}{n}\right)$$

and since $n(\varphi_X(t/n)-1) = t(\varphi(t/n)-1)/(t/n) \to_{n\to\infty} at$ we have, according to (5.16.5) of [4] that $\gamma(t) \to e^{at}$ and since $|\gamma(t)| = e^{\Re(a)t}$ we must have $\Re(a) = 0$ and can therefore write $a = ib$ for $b \in \mathbb{R}$. By the continuity theorem this implies $S_n/n \xrightarrow{\mathcal{D}} b$, but according to the law of large numbers (eg. (4.12.1) in [4]) we have $S_n/n \to \infty$ almost surely. ☐

Lemma 1.3. *Assume that $s_\cdot, \tilde{s}_\cdot$ and $x_\cdot$ are functions in $D[0,\infty)$ such that $t \to x_t$ is increasing, $x_\cdot \geq 0$, and $s_t = \tilde{s}_t - x_t$ for $t \geq 0$. Let l_t^K and $\tilde{l}_t^K$ denote the local times at K. Let $v_\cdot^K$ and $\tilde{v}_\cdot^K$ be the two-sided reflected functions. Then for $t \geq 0$ we have $v_t^K \leq \tilde{v}_t^K$ and $l_t^K \leq \tilde{l}_t^K$.*

Proof. First, we note that if v_t and $\tilde{v}_t$ are the one-sided reflected functions and l_t and $\tilde{l}_t$ is the local time at 0 for the one-sided reflection of $s_\cdot$, $\tilde{s}_t$ respectively then $v_t \leq \tilde{v}_t$ and $l_t \geq \tilde{l}_t$. This is immediate, since for $v \geq 0$ we have $-s_v = -\tilde{s}_v + x_v \Rightarrow \sup_{v\leq t} -s_v \leq \sup_{v\leq t} -\tilde{s}_v + x_t \Rightarrow l_t \leq \tilde{l}_t + x_t$. Therefore we have

$$v_t = s_t + l_t = \tilde{s}_t - x_t + l_t \leq \tilde{s}_t + \tilde{l}_t = \tilde{v}_t\,.$$

Furthermore, the fact that $l_t \geq \tilde{l}_t$ is immediate from $\sup_{v\leq t} -s_v \geq \sup_{v\leq t} \tilde{s}_v$. So, assume that for some $t_0 > 0$ we have $v_{t_0}^K > \tilde{v}_{t_0}^K$. Consider $u := \inf\{t > 0 \mid v_t^K > \tilde{v}_t^K\}$, which is finite by assumption, and due to right-continuity, we have $v_u^K \geq \tilde{v}_u^K$, But $v_u^K > \tilde{v}_u^K$ is impossible since for $t < u$ we have $v_t^K \leq \tilde{v}_t^K$ so $v_u^K > \tilde{v}_u^K$ would imply a positive jump of $s_\cdot$ which does not correspond to a jump of $\tilde{s}_\cdot$. This contradicts $s_t = \tilde{s}_t - x_t$. This implies that $v_u^K = \tilde{v}_u^K$, and $v_t^K > \tilde{v}_t^K$ for $u < t < u + \epsilon$. But this contradicts $s_t = \tilde{s}_t - x_t$. By defining $v := \inf\{t > 0 \mid l_t^K > \tilde{l}_t^K\}$, we can reach a contradiction by the same arguments. □

Proposition 1.3. *Let $\{S_t\}$ be a Lévy process with associated characteristic triplet (θ, σ, ν), and let $\{S_t^*\}$ be the Lévy process with characteristic triplet $(\theta, \sigma, \nu_{[-L,\infty)})$, $L > 0$ that is, the Lévy process obtained by restricting the Lévy measure to $[-L, \infty)$ for some $L > 0$. Then, if ℓ^K is the loss rate if $\{S_t\}$ and $\ell^{K,*}$ the loss rate of $\{S_t^*\}$, we have*

$$\ell^K \le \ell^{K,*}.$$

Furthermore, if $\mathbb{E}S_1 < 0$, then L can be chosen large enough to ensure that $\mathbb{E}S_1^* < 0$

Proof. Let $L > 0$ be fixed. We may assume $\{S_t\}$ and $\{S_t^*\}$ are defined on the same probability space, and so, by the Lévy -Itô decomposition of [5] we have

$$S_t(\omega) = S_t^*(\omega) + X_t(\omega) \tag{1.7}$$

where $\{X_t\}$ is a compound Poisson consisting of the jumps $< -L$. By applying Lemma 1.3 we obtain

$$L_t(\omega) \le L_t^*(\omega) \tag{1.8}$$

for $t > 0$. The Lévy-Ito decomposition also implies a stochastic ordering between the stationary distributions, which is seen by using the representation of the stationary distribution from (1.3) because (1.7) implies

$$\mathbb{P}\big(S_{\tau[y-K,y)} \ge y\big) \le \mathbb{P}\big(S^*_{\tau[y-K,y)} \ge y\big)$$

so that if V_0 and V_0^* denote the random variables with the stationary distributions of $\{S_t\}$ and $\{S_t^*\}$ respectively, $V_0 \le V_0^*$. By combining this with (1.8) we have

$$\ell^K = \mathbb{E}_{V_0} L_1 \le \mathbb{E}_{V_0^*} L_1 \le \mathbb{E}_{V_0^*} L_1^* = \ell^{K,*}\,.$$

The last part of the statement is a simple consequence of the fact that

$$\mathbb{E}S_1 = \int_{\{|y|>1\}} y\nu(\mathrm{d}y) \qquad \mathbb{E}S_1^* = \int_{\{|y|>1,-L>y\}} y\nu(\mathrm{d}y)\,,$$

and that

$$\int_{\{|y|>1,-L>y\}} y\nu(\mathrm{d}y) \to \int_{\{|y|>1\}} y\nu(\mathrm{d}y) < 0 \qquad L \to \infty$$

so that for L large enough we have

$$\int_{\{|y|>1,-L>y\}} y\nu(\mathrm{d}y) < 0$$

□

Remark 1.4. *Because of Proposition 1.3 we have that for any Lévy process with negative mean and loss rate ℓ^K we have $\ell^K \le \ell^{K,*}$ where $\ell^{K,*}$ is the loss rate of of a Lévy process with negative mean and such that the right tail of the Lévy measure is identical to the right tail of the original Lévy measure, and bounded from below.*

Remark 1.5. *As it is noted in [3] l'Hospital's rule does not in general apply to complex-valued functions, and some care must be taken. The following proposition covers the case needed in Corollary 4.1.*

Proposition 1.4. *Assume $n(t) = u_1(t) + \mathrm{i}v_1(t)$ and $d(t) = u_2(t) + \mathrm{i}v_2(t)$ are complex-valued functions with $\lim_{t\to 0} n(t) = 0$ and $\lim_{t\to 0} d(t) = 0$, and assume $u_i(t)$ and $v_i(t)$ $i = 1, 2$ are differentiable for $t \in (-\epsilon, \epsilon)$ and the derivatives are continuous. Then we have*

$$\lim_{t\to 0} \frac{n(t)}{d(t)} = \frac{u_1'(0) + \mathrm{i}v_1'(0)}{u_2'(0) + \mathrm{i}v_2'(0)}$$

Proof. We may apply l'Hospital's rule for real functions to the obtain:

$$\frac{u_2(t) + \mathrm{i}v_2(t)}{u_1(t)} = \frac{u_2(t)}{u_1(t)} + \mathrm{i}\frac{v_2(t)}{u_1(t)} \to_{t\to 0} \frac{u_2'(0)}{u_1'(0)} + \mathrm{i}\frac{v_2'(0)}{u_1'(0)}$$
$$\frac{u_2(t) + \mathrm{i}v_2(t)}{v_1(t)} = \frac{u_2(t)}{v_1(t)} + \mathrm{i}\frac{v_2(t)}{v_1(t)} \to_{t\to 0} \frac{u_2'(0)}{v_1'(0)} + \mathrm{i}\frac{v_2'(0)}{v_1'(0)}$$

and therefore

$$\frac{n(t)}{d(t)} = \frac{u_1(t)}{u_2(t) + \mathrm{i}v_2(t)} + \mathrm{i}\frac{v_1(t)}{u_2(t) + \mathrm{i}v_2(t)} \to_{t\to 0} \frac{u_1'(0)}{u_2'(0) + \mathrm{i}v_2'(0)} + \mathrm{i}\frac{v_1'(0)}{u_2'(0) + \mathrm{i}v_2'(0)}$$

□

The following Lemma is used in **Paper A**.

Lemma 1.6. *Let $F_n(\cdot)$, $n = 1, 2, \ldots$, be a sequence of uniformly bounded increasing functions, such that $F_n(x) \to F_0(x)$ $\forall x \in \mathbb{R}$, where F_0 is continuous, and $\lim_{x\to-\infty} F_0(x) =: F_0(-\infty) \le F_n(x)$ and $F_n(x) \le F(\infty) := \lim_{x\to\infty} F_0(x)$ for all n and x. Then*

$$\sup_{-\infty<y<\infty} |F_n(y) - F_0(y)| \to 0\,.$$

Proof. Without loss of generality, we may assume that $0 \leq F_n(x) \leq 1$ for all x and n. F_0 is increasing, so the limits $a := F(-\infty)$ and $b := F(\infty)$ exist, and are finite, and we may assume $a = 0$ and $b = 1$. Set $F_0^{-1}(y) := \inf\{x \in \mathbb{R} \mid F(x) = y\}$ for $0 < y < 1$ and $F_0^{-1}(0) = -\infty$ and $F_0^{-1}(1) = \infty$. Let $k \in \mathbb{N}$, and set $x_j^k := F_0^{-1}(j/k)$ $j = 0, 1, \ldots k$. Then for $0 \leq j < k$ and $x_j^k < x < x_{j+1}^k$

$$\begin{aligned} F_n(x_j^k) - F_0(x_j^k) - \frac{1}{k} &= F_n(x_j^k) - F_0(x_{j+1}^k) \leq F_n(x) - F_0(x) \\ \leq \quad F_n(x_{j+1}^k) - F_0(x_j^k) &= F_n(x_{j+1}^k) - F_0(x_{j+1}^k) + \frac{1}{k}, \end{aligned}$$

since F_n and F are increasing, and F is continuous. Continuing our calculation, we obtain

$$\begin{aligned} |F_n(x) - F(x)| &= \max(F_n(x) - F_0(x), F_0(x) - F_n(x)) \\ &\leq \max_{j \in \{0, \cdots, k-1\}} (F_n(x_{j+1}^k) - F_0(x_{j+1}^k) + \frac{1}{k}, F_0(x_j^k) - F_n(x_j^k)) + \frac{1}{k}) \\ &= \frac{1}{k} + \max_{j \in \{0, \cdots, k-1\}} (F_n(x_{j+1}^k) - F_0(x_{j+1}^k), F_0(x_j^k) - F_n(x_j^k)) \\ &\leq \frac{1}{k} + \max_{j \in \{0, \cdots, k\}} |F_n(x_j^k) - F_0(x_j^k)|, \end{aligned}$$

and therefore

$$\sup_{-\infty < x < \infty} |F_n(x) - F(x)| \leq \frac{1}{k} + \max_{j \in \{0, \cdots, k\}} |F_n(x_j^k) - F_0(x_j^k)|.$$

Using that $0 \leq \lim_n F_n(-\infty) \leq F(y)$ for all $y \in \mathbb{R}$ we see that $\lim_n F_n(-\infty) = 0$, and similarly, that $\lim_n F_n(\infty) = 1$, we obtain

$$\lim_{n \to \infty} \sup_{-\infty \leq j \leq \infty} |F_n(x) - F(x)| \leq \frac{1}{k}$$

Since k was arbitrary, the proof is complete. □

Corollary 1.7. *Let $F_n(\cdot)$, $n = 1, 2, \ldots$, be a sequence of increasing functions, such that for some $K > 0, a, b \in \mathbb{R} : \sup_{x \in [a,b]} |F_n(x)| \leq K$ for all n, and $F_n(x) \to F_0(x)$ $\forall x \in [a, b]$, where F_0 is continuous. Then*

$$\sup_{a \leq y \leq b} |F_n(y) - F_0(y)| \to 0$$

Proof. Define $\tilde{F}_n$ for $n = 0, 1, \ldots$ by $\tilde{F}_n(t) := F_n(t)$ for $t \in [a, b]$, $\tilde{F}_n(t) = F_n(a)$ for $t < a$, and $\tilde{F}_n(t) = F_n(b)$ for $t > b$. By applying Lemma 1.6, we obtain

$$\sup_{a \leq y \leq b} |F_n(y) - F_0(y)| = \sup_{a \leq y \leq b} |\tilde{F}_n(y) - \tilde{F}_0(y)| \leq \sup_{-\infty \leq y \leq \infty} |\tilde{F}_n(y) - \tilde{F}_0(y)| \to 0$$

□

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