Canonical Kernels on Hermitian Symmetric Spaces

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Abstract

This thesis studies symmetric spaces G/H with G a semisimple Lie group and where the isotropy subgroup H has a non-discrete center; we will consider the cases when G/H is either a Hermitian, pseudohermitian, or parahermitian symmetric space. For non-compact G with has finite center and $H = K$ is a maximal compact subgroup, G/K is a Hermitian symmetric space of the non-compact type and the Harish-Chandra embedding realizes G/K as a bounded symmetric domain D. Clerc and Ørsted [CØ03] expressed the symplectic area of a geodesic triangle in terms of the Bergman kernel k_D of D. We prove a similar formula for the compact dual U/K using a slightly different kernel k_c. We give a geometric characterization of the zeroes of this kernel.

Semisimple parahermitian symmetric spaces are also studied using a generalized Borel embedding due to Kaneyuki [Kan87]. We introduce a suitable kernel function and relate it to the symplectic area of geodesic triangles. We also treat complex parahermitian symmetric spaces $G_{\mathbb{C}}/H_{\mathbb{C}}$ separately. Here $G_{\mathbb{C}}$ and $H_{\mathbb{C}}$ are complex Lie groups with $G_{\mathbb{C}}$ simple. In this case, we introduce a holomorphic kernel function $k_{\mathbb{C}}$ and calculate the (complex) symplectic area of geodesic triangles. Finally we show how the other kernels k_D and k_c may be recovered from the complex kernel $k_{\mathbb{C}}$ as suitable restrictions.

Dansk Resumé

Denne afhandling handler om symmetriske rum G/H, hvor G er en semisimpel Lie gruppe og hvor isotropigruppen H har et ikke-diskret center. Vi betragter tilfældene hvor G/H er Hermitisk, pseudohermitisk eller parahermitisk. Hvis G er en ikke-kompakt Lie gruppe med diskret center og $H = K$ er en maksimal kompakt undergruppe, så er G/K et Hermitisk symmetrisk rum af ikke-kompakt type, og Harish-Chandra indlejringen realiserer G/K som et begrænset symmetrisk område D. Clerc og Ørsted [CØ03] fandt et udtryk for det symplektiske areal af en geodætisk trekant ved brug af Bergmankernen k_D for D. Vi viser en tilsvarende formel for det kompakte duale rum U/K ved hjælp af en anderledes kerne k_c. Desuden giver vi en geometrisk karakterisering af nulpunkterne for k_c.

Vi studerer også semisimple parahermitiske symmetriske rum ved hjælp af en generaliseret Borelindlejring, der skyldes Kaneyuki [Kan87]. Vi introducerer en passende kernefunktion og relaterer den til det symplektiske areal geodætiske trekanter. Vi betragter de komplekse parahermitiske symmetriske rum $G_{\mathbb{C}}/H_{\mathbb{C}}$ separat. Her er $G_{\mathbb{C}}$ og $H_{\mathbb{C}}$ komplekse Liegrupper og $G_{\mathbb{C}}$ er simpel. I dette tilfælde definerer vi en holomorf kernefunktion $k_{\mathbb{C}}$ og udregner det (komplekse) symplektiske areal af geodætiske trekanter. Til sidst viser vi, hvordan de andre kernefunktioner k_D og k_c optræder som passende restriktioner af $k_{\mathbb{C}}$.

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CONTENTS

INTRODUCTION

If I had to boil it down to one sentence, I would say that this thesis concerns the relation between the geometry and the canonical kernels of Hermitian symmetric spaces. Of course, both of the terms 'canonical kernels' and 'Hermitian symmetric spaces' require further explanation to be given later. Suffice it to say for now that Hermitian symmetric spaces is too narrow a term; this thesis also deals with (semisimple) parahermitian spaces and to a lesser extent pseudohermitian symmetric spaces. This introduction is not a 'start at the beginning', but rather an overview of the contents of the thesis and my attempt to explain why these contents are included. Definitions of spaces and objects mentioned below are found in the subsequent chapters of the thesis. As such, this introduction is intended for readers already familiar with symmetric spaces.

Hermitian symmetric spaces are well-studied objects. I will not consider the Euclidean spaces here, so really I am talking of semisimple Hermitian symmetric spaces. There is a very rich structure theory coming from Harish-Chandra's embedding which realizes a Hermitian symmetric space of the non-compact type as a bounded symmetric domain. Using the embedding, one can construct holomorphic discrete series, classify boundary components, give further realizations of the Hermitian symmetric space as a Siegel domain, et cetera. As a by-product of this embedding, one also obtains useful coordinates for the compact dual spaces. In terms of bounded symmetric domains, Theorem 2.1 of [CØ03] can be formulated as follows:

Theorem (Clerc-Ørsted, Domic-Toledo) *Let D be a bounded symmetric domain in $\mathbb{C}^N$ and k_D the Bergman kernel of D. Denote by $\omega = i\partial\overline{\partial}\log k_D(z,z)$ the associated Kähler form. Let Δ be an oriented geodesic triangle in D with vertices z_0, z_1, and z_2. Then*

$$\int_{\Delta} \omega = -(\arg k_D(z_0,z_1) + \arg k_D(z_1,z_2) + \arg k_D(z_2,z_0)), \qquad \text{(A)}$$

where $\arg k_D$ is the continuous argument for k_D such that $\arg k_D(z,z) = 0$ for all $z \in D$. Furthermore, the right hand side of the formula is bounded as a function of $z_0, z_1, z_2 \in D$.

The details are given in Section 9 of this thesis. Specializing to the case of the unit disc $\mathbb{D}$ in the complex plane where $k_{\mathbb{D}}(z,w) = \pi^{-1}(1 - z\overline{w})^{-2}$, the formula reads

$$\int_{\Delta} \omega = -(\arg(1 - z_0\overline{z_1})^{-2} + \arg(1 - z_1\overline{z_2})^{-2} + \arg(1 - z_2\overline{z_0})^{-2}),$$

and here ω is the volume form corresponding to the usual hyperbolic metric with curvature -1 on $\mathbb{D}$. Thus the right-hand side is bounded by $\pm\pi$. This special case was already considered in [CØ01].

It seems reasonable then, given the similarities between compact and non-compact dual Riemannian symmetric spaces, that a similar 'area-formula' should hold for Hermitian symmetric spaces of the non-compact type. Naturally a first test case of such a conjecture would be the Riemann sphere $\mathbb{CP}^1$. Here we are really thinking of $\mathbb{CP}^1$ as $\mathbb{C} \cup \{\infty\}$ and using $\mathbb{C}$ as coordinates on $\mathbb{CP}^1$. This is the same as having choosen a specific base point in $\mathbb{CP}^1$. The compact version of the Bergman kernel is the function $k(z, w) = (1+z\overline{w})^2$ and the form $i\partial\overline{\partial} \log k(z, z)$ is the volume form corresponding to the usual metric of constant curvature $+1$. However, $k(z, w)$ has zeroes as z, w range over $\mathbb{C}$; hence, one has to be more careful when speaking of $\arg k$. Furthermore, the right-hand side of (A) depends only on the vertices of the geodesic triangle Δ. But three points in $\mathbb{C}$ are vertices of more than one geodesic triangle on the Riemann sphere, and in general these triangles have different area. I worked around these difficulties by considering only those pairs $(z, w) \in \mathbb{C}^2$ for which $1 + z\overline{w}$ is not a real number ≤ 0. Then it turns out that there is a unique shortest geodesic segment connecting z and w and this segment runs in $\mathbb{C}$, e.g. does not pass through ∞. Taking the base point 0 as the last vertex, it is now possible to construct an oriented geodesic triangle $\Delta \subset \mathbb{C}$ with vertices 0, z, and w and sides made up of distance-realizing geodesic segments in $\mathbb{C}$. It turns out that

$$\int_{\Delta(0,z,w)} i\partial\overline{\partial} \log k = -2\mathrm{Arg}(1 + z\overline{w}),$$

where Arg is the usual main argument. This formula leads to

$$\exp\left(\int_{\Delta(0,z,w)} \partial\overline{\partial} \log k\right) = \frac{1 + z\overline{w}}{1 + \overline{z}w},$$

and this holds for any pair (z, w) such that $1 + z\overline{w} \neq 0$ and *any* oriented geodesic triangle $\Delta \subset \mathbb{C}$ with vertices 0, z, and w. I have since learned that this result is already known; it is mentioned in the book [Per86] where the expression $1+z\overline{w}$ arises as the scalar product between so-called coherent states. Later, in [HM94], this result was generalized to the complex n-dimensional projective space $\mathbb{CP}^n$. These results are described in §4, chapter II.

In chapter IV, the results for $\mathbb{CP}^1$ are generalized to Hermitian symmetric spaces of the compact type. The setting is as follows: Let $M = U/K_0$ be an irreducible Hermitian symmetric space of the compact type; here U is a compact simple Lie group and K_0 the group of fixed points of an involution of U. The non-compact dual of M is realized as a bounded symmetric domain $D \subset \mathfrak{p}^+$, where $\mathfrak{p}^+$ is a complex vector space which is mapped holomorphically and injectively onto an open and dense subset of M. Letting k_D denote the Bergman kernel of D, we define $k_c(z,w) = k_D(z,-w)^{-1}$ for $z, w \in \mathfrak{p}^+$. Then $k(z,w)$ is a polynomial in z and w and $\omega = i\partial\overline{\partial} \log k_c(z,z)$ is the pull-back to $\mathfrak{p}^+$ of the Kähler form of M. The question of when $k(z,w)$ vanishes is studied by writing U as a product $U = K_0 A K_0$ which reduces the problem to the case of a *polysphere* $(\mathbb{CP}^1)^r$ where the results for $\mathbb{CP}^1$ can be applied. A continuous argument $\arg k_c(z,w)$ is defined for the subset $\mathcal{S}$ of $\mathfrak{p}^+ \times \mathfrak{p}^+$ consisting of pairs (z,w) which are connected by a geodesic segment $\gamma\colon [0,1] \to \mathfrak{p}^+$ which realizes the distance between z and w and it the unique geodesic segment doing so. This is the geometric argument for k_c.

Geodesic triangles $\Delta \subset \mathfrak{p}^+$ are also studied. As ω is exact, the problem of calculating $\int_\Delta \omega$ reduces to calculating path integrals $\int_\gamma \rho$ where $\gamma\colon [0,1] \to \mathfrak{p}^+$ is a geodesic segment and $\rho = -i(\partial - \overline{\partial}) \log k$ is a particular 1-form on $\mathfrak{p}^+$ such that $\omega = \frac{1}{2} d\rho$. Then, if γ realizes the distances between its endpoints and if it is the only geodesic segment doing so, we have (Theorem 12.5)

$$\frac{1}{2} \int_\gamma \rho = -\arg k_c(\gamma(0), \gamma(1)),$$

where $\arg k_c(\gamma(0), \gamma(1))$ is the previously defined geometric argument for k_c. Now Stoke's theorem gives a formula similar to (A). The chapter concludes with a discussion of another argument for k_c which may be called a spectral argument for k_c. I did not succeed in showing that the geometric and spectral arguments are identical.

Following a suggestion by my advisor, I also considered the two-sheeted hyperboloid $\Sigma = SL(2,\mathbb{R})/R^*$ with its constant curvature metric, and the results are described in §5. The space Σ is studied using paracomplex numbers

$$x + jy, \qquad x, y \in \mathbb{R},$$

where j is adjoined to $\mathbb{R}$ and satisfies $j^2 = 1$. Paracomplex coordinates on Σ given by the set B of paracomplex numbers $x + jy$ where $x^2 - y^2 \neq 1$. Then it turns out that a suitable kernel is given by the function $k(z,w) = (1 - z\overline{w})^2$ using paracomplex numbers z and w and an analogue of complex conjugation. Following a brief introduction to basic paracomplex differential geometry and the paracomplex version of the exponential map and logarithm, it is proved that, for a geodesic segment γ in B which does not pass through a pair of antipodal points of Σ, we have (Theorem 5.7)

$$\frac{1}{2} \int_\gamma \rho = -\arg k(\gamma(0), \gamma(1))$$

where $\rho = j(\partial - \overline{\partial})\log k$ is the paracomplex version of the 1-form studied for the Riemann sphere. The notational similarities are very pleasing in my opinion. However, the algebra of paracomplex numbers play no role elsewhere in this thesis.

The space $SL(2,\mathbb{R})/\mathbb{R}^*$ is a parahermitian symmetric space. This class of symmetric spaces was introduced by Kaneyuki and Kozai, [KK85], and the results of §5 are generalized to general semisimple parahermitian symmetric spaces in Chapter V, but I avoid the use of paracomplex numbers. The simply connected semisimple parahermitian symmetric spaces are in one to one correspondence with 3-graded semisimple Lie algebras $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}_0$ is the Lie algebra of H. Kaneyuki also introduced, [Kan87], a generalized Borel embedding (or compactification) of a parahermitian symmetric space G/H, embedding it as an open orbit in $G/P^+ \times G/P^-$ where $P^\pm$ are certain opposite parabolic subgroups. This provides coordinates on G/H from an open subset $\mathcal{M}$ in the vector space $\mathfrak{g}_1 + \mathfrak{g}_{-1}$ and Kaneyuki writes down a canonical kernel $k(x_1 + x_{-1})$, $x_{\pm 1} \in \mathfrak{g}_{\pm 1}$, which transforms suitably under the group G. I proceed to define a 'polarized' kernel $\kappa(x,y)$ for points $x = x_1 + x_{-1}$ and $y = y_1 + y_{-1}$ in $\mathfrak{g}_1 + \mathfrak{g}_{-1}$ as

$$\kappa(x,y) = \frac{k(x_1 + y_{-1})}{k(y_1 + x_{-1})}$$

whenever the right-hand side is defined. Now G/H has a G-invariant symplectic form ω given as $-\frac{1}{2} dd_J \log k$ on $\mathcal{M}$ where $d_j = Jd$, J denoting the paracomplex structure on G/H. Curve integrals of $d_J \log k$ along geodesics are related to the kernel $\kappa(x,y)$ in §16. If $\gamma\colon [0,1] \to \mathcal{M}$ is a geodesic segment and $g \in G$ is an element such that $g\gamma$ is another curve in $\mathcal{M}$, then (Lemma 16.1)

$$\exp\Big(\int_\gamma d_J \log k - \int_{g\gamma} d_J \log k\Big) = \frac{\kappa(\gamma(0),\gamma(1))}{\kappa(g\gamma(0), g\gamma(1))}$$

provided that $\kappa(\gamma(0),\gamma(1))$ is defined. If it happens that $g\gamma$ passes through the origin, then we simply get

$$\exp\Big(\int_\gamma d_J \log k\Big) = \kappa(\gamma(0),\gamma(1))^{-1},$$

which should be compared to the previously mentioned results.

The space $SL(2,\mathbb{C})/C^*$ is also parahermitian, but it is also a quotient of a complex Lie group by a subgroup fixed by a holomorphic involution and hence a complex symmetric space. Furthermore, all of the sample spaces $\mathbb{D}$, the Riemann sphere $\mathbb{CP}^1$, and the hyperboloid Σ are embedded as totally geodesic submanifolds of $SL(2,\mathbb{C})/\mathbb{C}^*$. Kaneyuki's compactification realizes $SL(2,\mathbb{C})/\mathbb{C}^*$ as the subset of pairs of distinct lines in $\mathbb{CP}^1 \times \mathbb{CP}^1$ and we can think of $\mathcal{M}$ as the set of pairs $(z,w) \in \mathbb{C}^2$ such that $1 - zw \neq 0$. Going through the construction of the canonical kernels for a parahermitian symmetric space while taking the complex structure into account yields a complex valued kernel $k_{\mathbb{C}}(z,w) = 1 - zw$ for $(z,w) \in \mathbb{C}^2$. The embeddings of the disc and the Riemann sphere correspond to two real forms of $\mathbb{C}^2$ given by all pairs of the form $(z,\overline{z})$ and $(z,-\overline{z})$ respectively. The restriction of $k_{\mathbb{C}}(z,w)$ to each of these subspaces allows one to recover the kernels k and k_c.

The last part of the thesis concerns the complex simple parahermitian symmetric spaces, among which $SL(2,\mathbb{C})/\mathbb{C}^*$ is the basic example. These spaces are in one to one correspondence with complex simple 3-graded Lie algebras $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$. The corresponding symmetric space can be taken as $\mathrm{Int}(\mathfrak{g})/C(Z_0)$ where $C(Z_0)$ is the centralizer in $\mathrm{Int}(\mathfrak{g})$ of a specific element Z_0 in the center of $\mathfrak{g}_0$. The procedures and the results are similar to that of Chapter V; I am simply redoing everything over the complex numbers. The complex kernel $k_{\mathbb{C}}$ and $\kappa_{\mathbb{C}}$ are introduced and related to curve integrals. Any real form of $\mathfrak{g}$, invariant under the involution defined by the grading, corresponds to a totally geodesic embedding of a symmetric space G/H into $\mathrm{Int}(\mathfrak{g})/C(Z_0)$. The space G/H may be either parahermitian, hermitian of the compact or non-compact type, or even pseudohermitian; another class of symmetric spaces, which are not studied separately in this thesis. In each case, the restriction of $k_{\mathbb{C}}$ and $\kappa_{\mathbb{C}}$ provide suitable kernels for these spaces. In fact, the theory developed here allows one to reprove formula (A).

Chapter I

SYMMETRIC SPACES

In this chapter we have collected some basic facts about symmetric spaces. The purpose is not to give a thorough exposition of this vast field. but merely to fix notation and definitions as well as to state some theorems which will be used throughout this text.

§1 *Affine Symmetric Spaces*

The main reference for general results regarding symmetric spaces is [KN96]. However, [Loo69] offers a more algebraic point of view.

Definition 1.1 *A symmetric space is a triple* (G, H, σ) *where*

1. G *is a connected Lie group,*
2. H *a Lie subgroup of* G*,*
3. *and* σ *is an involution of* G*, i.e.* $\sigma \neq \mathrm{id}$ *and* $\sigma^2 = \mathrm{id}$*, such that*

$$(G^\sigma)_0 \subset H \subset G^\sigma,$$

where G^σ *is the closed subgroup of* σ*-fixed elements in* G *and* $(G^\sigma)_0$ *its identity component.*

We say that the symmetric space (G, H, σ) *is effective (resp. almost effective) if* G *acts effectively (resp.) almost effectively) on* G/H*. The symmetric space is said to be semisimple if* G *is semisimple.*

Remark 1.2 The subgroup H is closed since it contains the identity component of the closed group G^σ and hence $M = G/H$ is a manifold with the quotient topology; $M = G/H$ is also referred to as a symmetric space (or affine symmetric space). The involution σ defines an involution s_o of M with an isolated fixed point at $o = eH$ by $s_o(gH) = \sigma(g)H$. The map s_o is called the symmetry at $o \in M$. It follows then that for every point $x \in M$ there is an involution s_x with x as an isolated fixed point. We may always assume that the symmetric space is effective by considering $(G/N, H/N, \sigma)$ instead, where $N \subset H$ is the largest normal subgroup of G contained in H. Then G/H and $(G/N)/(H/N)$ are equivariantly diffeomorphic with respect to the canonical homomorphism $G \to G/N$. $\diamondsuit$

Let (G, H, σ) be a symmetric space and let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively. Let $d\sigma\colon \mathfrak{g} \to \mathfrak{g}$ denote the differential of σ at $e \in G$. Then $d\sigma$ is an involution and an automorphism of $\mathfrak{g}$, $\mathfrak{h}$ is the 1-eigenspace, and we have a direct sum decomposition (*the canonical decomposition*)

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{q},$$

where $\mathfrak{q}$ is the (-1)-eigenspace of $d\sigma$ in $\mathfrak{g}$. Furthermore, we have the relations

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q},$$

in particular the adjoint representation $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$ preserves $\mathfrak{q}$. We will write $\mathrm{ad}_{\mathfrak{q}}\colon \mathfrak{h} \to \mathrm{End}(\mathfrak{q})$ for this representation of $\mathfrak{h}$. The symmetric space (G, H, σ) is almost effective if and only if this representation is faithful, or equivalently if $\mathfrak{h}$ does not contain any non-trivial ideal of $\mathfrak{g}$. Similarly, $\mathfrak{q}$ is invariant under $\mathrm{Ad}(h)$ for each $h \in H$. Here Ad denotes the adjoint representation of G For $h \in H$ we will write $\mathrm{Ad}_{\mathfrak{q}}(h)$ for the restriction of $\mathrm{Ad}(h)$ to $\mathfrak{q}$.

Theorem 1.3 *Let (G, H, σ) be a symmetric space and write $\pi\colon G \to G/H$ for the canonical quotient map. There is a unique G-invariant affine connection on $M = G/H$. This connection is invariant by the symmetries s_x, $x \in M$, of M, which are then affine symmetries. Furthermore*

1. *M is complete.*
2. *The torsion T vanishes and the curvature tensor R is parallel, i.e. $\nabla R = 0$. At $o = eH$ we have*

   $$R_o(X, Y)Z = -[[X, Y], Z],$$

   for $X, Y, Z \in \mathfrak{q}$ under the identification $T_o M = \mathfrak{q}$.
3. *For each $X \in \mathfrak{q}$, parallel transport along $\pi(\exp(tX))$ coincides with the differential of $\exp(tX)$ as a transformation of M.*
4. *The curve $\exp(tX).o = \pi(\exp(tX))$ is a geodesic for every $X \in \mathfrak{q}$ and every geodesic on M starting at o is of this form.*
5. *Any G-invariant pseudo-Riemannian metric on M (such metrics exist in particular for $\mathfrak{g}$ semisimple) induces the unique G-invariant connection on M.*

Proof. See [KN96, Chapter XI]. ∎

As this thesis only deals with semisimple Lie groups, we will almost exclusively discuss semisimple symmetric spaces in the following, even though we could define Hermitian, pseudohermitian, and parahermitian symmetric spaces without speaking about semisimple groups.

1.A Invariant Structure

If $\mathfrak{g}$ is semisimple, which will be the case in the later parts of this thesis, and B is its Killing form, then $\mathfrak{h}$ and $\mathfrak{q}$ are orthogonal under B. Hence, B restricted to $\mathfrak{q}$ is non-degenerate and $Q = \frac{1}{2}B_{\mathfrak{q}\times\mathfrak{q}}$ induces a G-invariant pseudoriemannian structure on G/H. We say that Q is *normalized* due to the fact that it equals the Ricci curvature of G/H. Regardless of the normalization constant chosen, G acts on $M = G/H$ by isometries and we have a converse result.

Proposition 1.4 *If (G, H, σ) is an effective semisimple symmetric space equipped with the pseudoriemannian structure given by the Killing form, then G is the identity component of the isometry group of $M = G/H$.*

Proof. The proof is similar to that of [Hel01, Chapter V,Theorem 4.1]. ∎

Proposition 1.5 *Let (G, H, σ) be a semisimple symmetric space and suppose that $J\colon \mathfrak{q} \to \mathfrak{q}$ is a linear map satisfying*

1. *J commutes with $Ad_{\mathfrak{q}}(h)$ for every $h \in H$,*
2. *and $B(JX, Y) + B(X, JY) = 0$ for all $X, Y \in \mathfrak{q}$.*

Then there exists a unique element $H_0 \in \mathfrak{h}$ of the Lie algebra of H such that $J = ad_{\mathfrak{q}} H_0$.

In particular, this H_0 lies in the center of the Lie algebra $\mathfrak{h}$. The proof is essentially similar to the proof of Theorem 5 in [Koh65].

Proof. We extend J to all of $\mathfrak{g}$ by putting $J = 0$ on $\mathfrak{h}$. We claim that J is a derivation of $\mathfrak{g}$. It suffices to show that

$$[JX, Y] + [X, JY] = 0$$

for all $X, Y \in \mathfrak{q}$. Let $Z \in \mathfrak{h}$ and observe that

$$\begin{aligned} B([JX, Y] + [X, JY], Z) &= B([JX, Y], Z) + B([X, JY], Z) \\ &= B(Y, [Z, JX]) + B(JY, [Z, X]) \\ &= B(Y, J[Z, X]) - B(Y, J[Z, X]) = 0, \end{aligned}$$

which proves the claim. Since $\mathfrak{g}$ is semisimple it follows that $J = \mathrm{ad}(H_0)$ for some $H_0 \in \mathfrak{g}$. Let H' be the $\mathfrak{q}$-part of H_0. For $X \in \mathfrak{q}$ we have $[H_0, X] = JX$ which again belongs to $\mathfrak{q}$, so $[H', X] = 0$. But as H_0 commutes with all of $\mathfrak{h}$ we also have $[H', \mathfrak{h}] = 0$. So H' is a central element in $\mathfrak{g}$, hence zero. ∎

There are two cases of special interest to us. An endomorphism $J\colon \mathfrak{q} \to \mathfrak{q}$ is said to be a *complex structure* on $\mathfrak{q}$ if

$$J^2 = -\mathrm{id}_{\mathfrak{q}}$$

and a *paracomplex structure* on $\mathfrak{q}$ if

$$J^2 = \mathrm{id}_{\mathfrak{q}}.$$

The main objects of study in this thesis will be semisimple symmetric spaces (G, H, σ) with a paracomplex or complex structure J on $\mathfrak{q}$ satisfying the conditions of Proposition 1.5. If this is the case we will simply refer to J as an *invariant para(complex) structure*, even though we have imposed additional requirements on J.

1.B Hermitian and Pseudohermitian Symmetric Spaces

We give the definition of Hermitian and pseudohermitian symmetric spaces from a common point of view. The paper [Sha71] discusses pseudohermitian symmetric spaces in detail. Hermitian symmetric spaces will be discussed in Chapters III and IV.

Definition 1.6 *Let (G, H, σ) be an effective semisimple symmetric space with an invariant complex structure J.*

1. *Then we say that (G, H, σ, J) is a semisimple Hermitian symmetric space if H is compact. We make the furhter distinctions that (G, H, σ) is of the non-compact type if σ is a Cartan involution and of the compact type if G is compact.*

2. *Otherwise, i.e. when H is not compact, we say that (G, H, σ, J) is a semisimple pseudohermitian space.*

A semisimple Hermitian symmetric space (G, H, σ, J) with its invariant pseudoriemannian structure Q is of the compact type if and only if Q is negative definite and of the non-compact type if and only if Q is positive definite.

If (G, H, σ) is not effective but has an invariant complex structure J, then, letting N denote the largest normal subgroup of G contained in H, $(G/N, GH/N, \sigma, J)$ is a semisimple pseudohermitian or Hermitian symmetric space with the same structure J. Let us also note that J induces a complex structure $M = G/H$ such that G acts by holomorphic transformations of M.

Proposition 1.7 *Let (G, H, σ, J) be a semisimple pseudohermitian or Hermitian symmetric space. Then $M = G/H$ is simply connected.*

Proof. See the remark in [Sha71] following Proposition 2.6. For semisimple Hermitian symmetric spaces see [Hel01, Chapter VIII,Theorem 4.6]. ∎

1.C Parahermitian Symmetric Spaces

Parahermitian symmetric spaces were introduced in [KK85].

Definition 1.8 *A semisimple parahermitian symmetric space is a semisimple symmetric space* (G, H, σ) *with an invariant paracomplex structure* J*.*

However, in this case $M = G/H$ need not be simply connected.

§2 Symmetric Lie Algebras

Symmetric Lie algebras are the infinitesimal versions of symmetric spaces.

Definition 2.1 *A symmetric Lie algebra is a triple* $(\mathfrak{g}, \mathfrak{h}, \sigma)$ *where* $\mathfrak{g}$ *is a Lie algebra,* σ *is an involutive automorphism of* $\mathfrak{g}$ *and* $\mathfrak{h}$ *is the subalgebra of* $\mathfrak{g}$ *consisting of the fixed points of* σ*.*

We say that a symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is *effective* if $\mathfrak{h}$ does not contain any non-trivial ideal of $\mathfrak{g}$ and *semisimple* (resp. *simple*) if $\mathfrak{g}$ is semisimple (resp. simple). We say that $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is *complex* if $\mathfrak{g}$ is a complex Lie algebra and σ is complex Linear. Simple symmetric Lie algebras were classified by Berger, [Ber57].

If G is a Lie group with Lie algebra $\mathfrak{g}$ such that σ defines an involution, which we will also denote by σ, of G, and if H is a subgroup of G with Lie algebra $\mathfrak{h}$ such that $H \subset G^\sigma$, then (G, H, σ) is a symmetric space. We say that (G, H, σ) is associated to $(\mathfrak{g}, \mathfrak{h}, \sigma)$. Conversely, any symmtric space (G, H, σ) has an associated symmetric Lie algebra as we have already seen.

Definition 2.2 *Let* $(\mathfrak{g}, \mathfrak{h}, \sigma)$ *be an effective semisimple symmetric Lie algebra and let* $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ *be the canonical decomposition with respect to* σ *and let* B *be the Killing form of* $\mathfrak{g}$*. Suppose that* $J \colon \mathfrak{q} \to \mathfrak{q}$ *is a linear map satisfying*

1. J *commutes with* $\mathrm{ad}_\mathfrak{q} H$ *for all* $H \in \mathfrak{h}$*,*
2. *and* $B(JX, Y) + B(X, JY) = 0$ *for all* $X, Y \in \mathfrak{q}$*,*

and suppose furthermore that J *is either a complex or paracomplex structure on* $\mathfrak{q}$*. Then we say that* J *is an invariant complex or paracomplex structure of the symmetric Lie algebra* $(\mathfrak{g}, \mathfrak{h}, \sigma)$*. To be more specific, we say that*

1. $(\mathfrak{g}, \mathfrak{h}, \sigma, J)$ *is a semisimple parahermitian symmetric Lie algebra if* J *is paracomplex,*
2. *and* $(\mathfrak{g}, \mathfrak{h}, \sigma, J)$ *is a semisimple Hermitian symmetric Lie algebra if* J *is complex and* $\mathfrak{h}$ *is compactly embedded in* $\mathfrak{g}$*. Then* $(\mathfrak{g}, \mathfrak{h}, \sigma, J)$ *is of the non-compact type if* σ *is a Cartan involution and of the compact type if* $\mathfrak{g}$ *is a compact Lie algebra.*

3. *Lastly, if J is complex and $\mathfrak{h}$ is not compactly embedded we say that $(\mathfrak{g}, \mathfrak{h}, \sigma, J)$ is a semisimple pseudohermitian symmetric Lie algebra.*

If $\mathfrak{g}$ is simple we will use this adjective rather than semisimple.

The proof of Proposition 1.5 tells us that an invariant paracomplex or complex structure J for a semisimple symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is of the form $J = \mathrm{ad}_{\mathfrak{q}} J_0$ for some J_0 in the center of $\mathfrak{h}$. Note also, that σ is determined by J or J_0 since $\sigma = \exp(\varepsilon\pi \mathrm{ad} J_0)$ where $\varepsilon = 1$ if J is complex and $\varepsilon = i$ when J is paracomplex. In the latter case, one has to work with the complexification $\mathfrak{g}^{\mathbb{C}}$ rather than $\mathfrak{g}$.

Remark 2.3 Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be an effective semisimple symmetric Lie algebra with invariant (para)complex structure J. Let

$$\mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_s,$$

be the decomposition of $\mathfrak{g}$ into simple ideals. Write J_0 for the element of $\mathfrak{h}$ such that $\mathrm{ad}_{\mathfrak{q}} J_0 = J$ and decompose $J_0 = J_1 + \cdots + J_s$ with each $J_k \in \mathfrak{g}_k$. Then each σ maps each $\mathfrak{g}_k$ to itself and J_k is a (para)complex structure on $\mathfrak{q} \cap \mathfrak{g}_k$. All the simple symmetric Lie algebras $(\mathfrak{g}_k, \mathfrak{h} \cap \mathfrak{g}_k, \sigma|_{\mathfrak{g}_k})$ have an invariant (para)complex structure given by J_k. ♢

Proposition 2.4 *Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be a simple symmetric Lie algebra. Then the center $\mathfrak{z}(\mathfrak{h})$ of $\mathfrak{h}$ has dimension 0, 1 or 2 over $\mathbb{R}$, and we have*

1. *If $\mathfrak{z}(\mathfrak{h})$ is 1-dimensional, then $(\mathfrak{g}, \mathfrak{h}, \sigma)$ has either an invariant paracomplex or an invariant complex structure. In either case the structure is unique up to a sign. Furthermore, the complexification $\mathfrak{g}^{\mathbb{C}}$ is simple.*

2. *If $\mathfrak{z}(\mathfrak{h})$ is 2-dimensional, then $\mathfrak{g}$ is a complex simple Lie algebra and σ is a complex linear involution. Furthermore, $(\mathfrak{g}, \mathfrak{h}, \sigma)$ has an invariant paracomplex structure J_0 and an invariant complex structure iJ_0.*

Thus J_0 is unique up to a sign.

Proof. See Lemma 1 and Theorem 6 in [Koh65]. ■

2.A Classification

If $(\mathfrak{g}, \mathfrak{h}, \sigma, J)$ is a simple symmetric Lie algebra and J is a paracomplex or complex structure and the center of $\mathfrak{h}$ is 1-dimensional over $\mathbb{R}$, then $\mathfrak{g}^{\mathbb{C}}$ is simple. If we let $\sigma^{\mathbb{C}}$ denote the complex Linear extension of σ to $\mathfrak{g}^{\mathbb{C}}$ and let $\mathfrak{g}_0$ denote its fixed points, then $\mathfrak{h}$ is a real form of $\mathfrak{g}_0$ and $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}_0, \sigma^{\mathbb{C}})$ is a simple complex symmetric Lie algebra. If $J_0 \in \mathfrak{h}$ is the element such that $J = \mathrm{ad}_{\mathfrak{q}}(J_0)$, then J_0 and iJ_0 are invariant paracomplex and complex structures (depending on whether J_0 was originally paracomplex or complex) on $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}_0, \sigma^{\mathbb{C}})$. Assume for simplicity that J_0 is paracomplex. Then $\mathfrak{g}^{\mathbb{C}}$ splits into eigenspaces of $\mathrm{ad}(J_0)$ as

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1,$$

where $\mathfrak{g}_\lambda$ is the λ-eigenspace of $\mathrm{ad}(J_0)$. That is, $\mathfrak{g}^{\mathbb{C}}$ is a 3-graded Lie algebra in the sense that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$.

Conversely, if $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ is a complex simple 3-graded Lie algebra, then there exists a unique element $J_0 \in \mathfrak{g}_0$ such that $\mathfrak{g}_\lambda$ is the (λ)-eigenspace for $\mathrm{ad}(J_0)$. Furthermore, $\sigma = \exp(i\pi \mathrm{ad} J_0)$ is an involution of $\mathfrak{g}$ whose fixed point set is $\mathfrak{g}_0$ and is $-\mathrm{id}$ on both $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$. It follows that $(\mathfrak{g}, \mathfrak{g}_0, \sigma)$ is a complex simple symmetric Lie algebra and J_0 (resp. iJ_0) is an invariant paracomplex (resp. complex) structure.

Complex simple 3-graded Lie algebras are classified in [KN64]. They found all pairs $(\mathfrak{g}, J_0)$ where $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{R}$ or $\mathbb{C}$ and $\mathrm{ad}J_0$ has eigenvalues 0, ± 1 on $\mathfrak{g}$. They did not distinguish pairs $(\mathfrak{g}, J_0)$ and $(\mathfrak{g}, J_0')$ for which there exist an automorphism of $\mathfrak{g}$ which sends J_0 to J_0'. We give the table for the complex simple Lie algebras here and assume $p \leq q$, $p+q \geq 2$ and $n > 2$. We write $Z_{p,q}$ for the matrix

$$Z_{p,q} = \begin{pmatrix} aI_p & 0 \\ 0 & -bI_q \end{pmatrix}$$

with a, b chosen by $pa - bq = 0$ og $a + b = 1$. Thus if $(\mathfrak{g}, \mathfrak{h}, \sigma, J_0)$ is any

Table 1: Simple complex 3-graded Lie algebras.

Type	$\mathfrak{g}$	$\mathfrak{g}_0$	J_0 (classical $\mathfrak{g}$)
$I_{p,q}$	$\mathfrak{sl}(p+q, \mathbb{C})$	$\mathfrak{sl}(p, \mathbb{C}) + \mathfrak{sl}(q, \mathbb{C}) + \mathbb{C}$	$Z_{p,q}$
II_n	$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{sl}(\mathfrak{n}, \mathbb{C}) + \mathbb{C}$	$\frac{1}{2}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$
III_n	$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C}) + \mathbb{C}$	$Z_{n,n}$
IV_n	$\mathfrak{so}(2+n, \mathbb{C})$	$\mathfrak{so}(n, \mathbb{C}) + \mathbb{C}$	$Z_{2,n}$
V	$E_6^{\mathbb{C}}$	$\mathfrak{so}(10, \mathbb{C}) + \mathbb{C}$	
VI	$E_7^{\mathbb{C}}$	$E_6^{\mathbb{C}} + \mathbb{C}$	

simple Hermitian, parahermitian or pseudohermitian symmetric Lie algebra such that $\mathfrak{h}$ has 1-dimensional center, then $\mathfrak{g}^{\mathbb{C}}$ together with either J_0 or iJ_0 is isomorphic to one of the 3-graded Lie algebras above and $\mathfrak{g}$ is a real form of $\mathfrak{g}^{\mathbb{C}}$ which is invariant under $\sigma^{\mathbb{C}}$. The type of $(\mathfrak{g}, \mathfrak{h}, \sigma, J_0)$ will then be the type of the 3-graded $\mathfrak{g}^{\mathbb{C}}$. We gather the spaces in the classification tables in [Sha71], [KK85], and [Hel01] according to their type. Note that the center of $\mathfrak{h}$ is 1-dimensional in the following tables. It follows from Proposition 2.4 that the center of $\mathfrak{h}$ determines J_0 up to a sign and hence σ completely.

Table 2: Simple symmetric Lie algebras $(\mathfrak{g}, \mathfrak{h}, \sigma)$ which are either Hermitian, parahermitian or pseudohermitian of type $I_{p,q}$.

Compact type	$(\mathfrak{su}(p+q), \mathfrak{su}(p) + \mathfrak{su}(q) + \mathbb{R})$
Non-compact type	$(\mathfrak{su}(p,q), \mathfrak{su}(p) + \mathfrak{su}(q) + \mathbb{R})$
Parahermitian	$(\mathfrak{sl}(p+q,\mathbb{R}), \mathfrak{sl}(p,\mathbb{R}) + \mathfrak{sl}(q,\mathbb{R}) + \mathbb{R})$
	$(\mathfrak{su}^*(2p'+2q'), \mathfrak{su}^*(2p') + \mathfrak{su}^*(2q') + \mathbb{R})^\ddagger$
	$(\mathfrak{su}(n,n), \mathfrak{sl}(n,\mathbb{C}) + \mathbb{R})^\dagger$
Pseudohermitian	$(\mathfrak{su}(p+q-h-k, h+k), \mathfrak{su}(p-k,k) + \mathfrak{su}(q-h,h) + \mathbb{R})$
	$(\mathfrak{sl}(2n,\mathbb{R}), \mathfrak{sl}(n,\mathbb{C}) + \mathbb{R})^\dagger$
	$(\mathfrak{su}^*(2n), \mathfrak{sl}(n,\mathbb{C}) + \mathbb{R})^\dagger$

‡ When $p = 2p'$ and $q = 2q'$ are both even.
† When $p = q = n$.

Table 3: Simple symmetric Lie algebras $(\mathfrak{g}, \mathfrak{h})$ which are either Hermitian, parahermitian or pseudohermitian of type II_n.

Compact type	$(\mathfrak{so}(2n), \mathfrak{u}(n))$
Non-compact type	$(\mathfrak{so}^*(2n), \mathfrak{u}(n))$
Parahermitian	$(\mathfrak{so}(n,n), \mathfrak{sl}(n,\mathbb{R}) + \mathbb{R})$
	$(\mathfrak{so}^*(4m), \mathfrak{so}^*(2m) + \mathbb{R})^\flat$
Pseudohermitian	$(\mathfrak{so}(2(n-k), 2k), \mathfrak{u}(n-k,k))$
	$(\mathfrak{so}^*(2n), \mathfrak{u}(n-k,k))$

♭ When $n = 2m$ is even.

Table 4: Simple symmetric Lie algebras $(\mathfrak{g}, \mathfrak{h})$ which are either Hermitian, parahermitian or pseudohermitian of type III_n.

Compact type	$(\mathfrak{sp}(n), \mathfrak{u}(n))$
Non-compact type	$(\mathfrak{sp}(n,\mathbb{R}), \mathfrak{u}(n))$
Parahermitian	$(\mathfrak{sp}(n,\mathbb{R}), \mathfrak{sl}(n,\mathbb{R}) + \mathbb{R})$
	$(\mathfrak{sp}^*(m,m), \mathfrak{su}^*(m) + \mathbb{R})^\flat$
Pseudohermitian	$(\mathfrak{sp}(n-k,k), \mathfrak{u}(n-k,k))$

♭ When $n = 2m$ is even.

Table 5: Simple symmetric Lie algebras $(\mathfrak{g}, \mathfrak{h})$ which are either Hermitian, parahermitian or pseudohermitian of type IV_n.

Compact type	$(\mathfrak{so}(2+n), \mathfrak{so}(n) + \mathbb{R})$
Non-compact type	$(\mathfrak{so}(2,n), \mathfrak{so}(n) + \mathbb{R})$
Parahermitian	$(\mathfrak{so}(n-k+1, k+1), \mathfrak{so}(n-k) + \mathfrak{so}(k) + \mathbb{R})$
Pseudohermitian	$(\mathfrak{so}(n+2-k, k), \mathfrak{so}(n-k, k) + \mathbb{R})$ $(\mathfrak{so}^*(2m+2), \mathfrak{so}^*(2m) + \mathbb{R})^\flat$

$\flat$ When $n = 2m$ is even.

Table 6: Simple symmetric Lie algebras $(\mathfrak{g}, \mathfrak{h})$ which are either Hermitian, parahermitian or pseudohermitian of type V and VI.

	Type V	Type VI
Compact type	$(E_6, \mathfrak{so}(10) + \mathbb{R})$	$(E_7, E_6 + \mathbb{R})$
Non-compact type	$(E_6^3, \mathfrak{so}(10) + \mathbb{R})$	$(E_7^3, E_6 + \mathbb{R})$
Parahermitian	$(E_6^1, \mathfrak{so}(5,5) + \mathbb{R})$ $(E_6^4, \mathfrak{so}(1,9) + \mathbb{R})$	$(E_7^1, E_6^1 + \mathbb{R})$ $(E_7^3, E_6^4 + \mathbb{R})$
Pseudohermitian	$(E_6^2, \mathfrak{so}^*(10) + \mathbb{R})$ $(E_6^2, \mathfrak{so}(6,4) + \mathbb{R})$ $(E_6^3, \mathfrak{so}(8,2) + \mathbb{R})$ $(E_6^3, \mathfrak{so}^*(10) + \mathbb{R})$	$(E_7^1, E_6^2 + \mathbb{R})$ $(E_7^2, E_6^3 + \mathbb{R})$ $(E_7^2, E_6^3 + \mathbb{R})$ $(E_7^3, E_6^3 + \mathbb{R})$

Chapter II

FOUR ELEMENTARY SYMMETRIC SPACES

In this chapter we will discuss three fundamental examples of symmetric spaces: The unit disc $\mathbb{D}$ in the complex plane, the Riemann sphere $\mathbb{CP}^1$, and the one-sheeted hyperboloid. These spaces are easy to visualize and are the simplest instances of the general notions of non-compact Hermitian symmetric spaces, compact Hermititan symmetric spaces, and parahermitian symmetric spaces respectively. The purpose of treating all three examples now is to illustrate their similarities and differences before turning to each particular family of symmetric spaces. Lastly we will discuss a 'complexifcation' containing all three spaces, thus adding a fourth space to our list of examples. Some results in this chapter will be proven again later, although often by different methods.

§3 *The Unit Disc*

In this section we will cover some results obtained in [CØ01] for the unit disc

$$\mathbb{D} = \{z \in \mathbb{C} \colon \ |z| < 1\}$$

in the complex plane $\mathbb{C}$. We consider $\mathbb{D}$ together with the kernel function

$$k(z, w) = (1 - z\overline{w})^{-2}$$

defined for all $z, w \in \mathbb{D}$. The usual hyperbolic metric on $\mathbb{D}$ comes from k by defining the Hermitian form

$$H_z = 2\frac{d}{dz}\frac{d}{d\overline{z}} \log k(z, z)(dz \otimes d\overline{z})$$

at $z \in \mathbb{D}$. We can write $H = g - i\omega$ where g is the Riemannian metric

$$g = 4\frac{dx \otimes dx + dy \otimes dy}{(1 - |z|^2)^2},$$

which has constant curvature -1, and

$$\omega = 4\frac{dx \wedge dy}{(1 - |z|^2)^2} = i\partial\overline{\partial} \log k(z, z)$$

is the volume form corresponding to g.

It is well known that the geodesics in $\mathbb{D}$ are arcs of circles which intersect the boundary $\partial\mathbb{D}$ at right angles, including lines through 0. Two distinct points are connected by exactly one geodesic segment. Given three distinct points in $\mathbb{D}$, we may form a *geodesic triangle* Δ with these three points as vertices by defining Δ to be the set bounded by the three geodesic segments connecting the three points. We may give Δ an orientation by ordering the three vertices

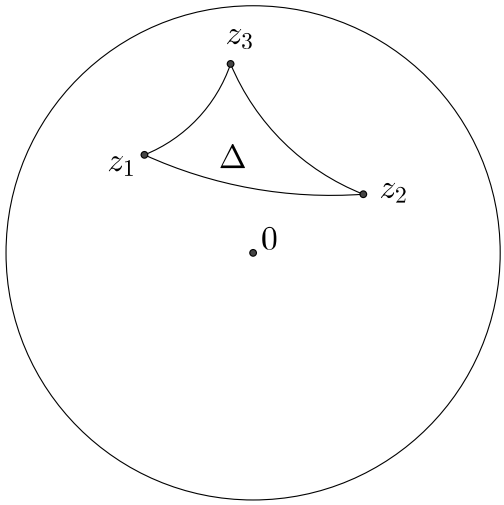

Figure 1: A geodesic triangle Δ with vertices z_0, z_1 and z_2.

cyclically. It is then clear that there is a one-to-one correspondence between ordered triples of pairwise distinct points and oriented geodesic triangles. With these conventions the following theorem holds:

Theorem 3.1 *[CØ01] Let z_0, z_1 and z_2 be the vertices of an oriented geodesic triangle Δ in $\mathbb{D}$. Then the (oriented) area of Δ is given by*

$$-(\arg k(z_0, z_1) + \arg k(z_1, z_2) + \arg k(z_2, z_0)),$$

where arg *is the unique argument for k satisfying* $\arg k(z, z) = 0$ *for $z \in \mathbb{D}$.*

Proof. Since $k(z, w)$ is never zero or a negative real number, we may simply use the main argument. The theorem can be proved by making use of the group

$$SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\},$$

which acts isometrically and transitively on the unit disc as fractional linear transformations, i.e.

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} .z = \frac{az + b}{\bar{b}z + \bar{a}}.$$

For each $g \in SU(1,1)$ and $z, w \in \mathbb{D}$, we have

$$1 - g(z)\overline{g(w)} = j(g,z)^{-1}(1 - z\bar{w})\overline{j(g,w)}^{-1},$$

with

$$j(g,z) = \bar{b}z + \bar{a}, \quad g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU(1,1).$$

Let us therefore define the expression

$$C(z_1, z_2, z_3) = \frac{1 - z_1\overline{z_2}}{1 - \overline{z_1}z_2} \cdot \frac{1 - z_2\overline{z_3}}{1 - \overline{z_2}z_3} \cdot \frac{1 - z_3\overline{z_1}}{1 - \overline{z_3}z_1}, \tag{3.1}$$

for $z_1, z_2, z_3 \in \mathbb{D}$. Then C invariant under the action of $SU(1,1)$, that is $C(g(z_1), g(z_2), g(z_3)) = C(z_1, z_2, z_3)$ for every $g \in SU(1,1)$. Since $1 - z\bar{w}$ and its inverse belongs to the right half plane for all z and w in the unit disc, we can define a continuous argument for c by adding the main argument of each of the factors in the expression for C, that is

$$\arg C(z_1, z_2, z_3) = 2\left(\mathrm{Arg}(1 - z_1\overline{z_2}) + \mathrm{Arg}(1 - z_2\overline{z_3}) + \mathrm{Arg}(1 - z_3\overline{z_1})\right). \tag{3.2}$$

This argument is also invariant under $SU(1,1)$, that is

$$\arg C(g(z_1), g(z_2), g(z_3)) = \arg C(z_1, z_2, z_3) \tag{3.3}$$

for each $g \in SU(1,1)$. Now it suffices to prove that the area of a geodesic

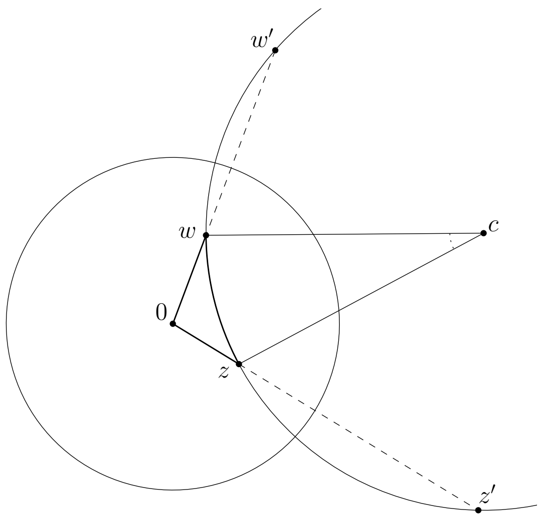

Figure 2: A geodesic triangle with vertices 0, z and w.

triangle with vertices 0, z_1 and z_2 is equal to $\arg C(0, z_1, z_2)$, which in turn is equal to $-\arg k(z_1, z_2)$. There is a geometric argument for this (see figure 2) using the classical formula giving the area of a geodesic triangle as the angular defect of the triangle. Using classical Euclidean geometry, it can be seen that this defect is equal to the angle between $z - c$ and $w - c$, where c is the center of the circle connecting z and w. This circle intersects the unit circle at two right angles. Hence, the points $z' = \overline{z}^{-1}$ and $w' = \overline{w}^{-1}$ also belong to this circle, so

$$\arg \frac{z-c}{w-c} = 2 \arg \frac{z-z'}{w-z'} = 2 \arg \frac{1}{1-\overline{z}w} = -\arg \frac{1}{(1-z\overline{w})^2},$$

which was the desired result. ■

Remark 3.2 The formula may be rewritten as

$$\int_\Delta \omega = \arg C(z_0, z_1, z_2), \tag{3.4}$$

where Δ is the oriented geodesic triangle with vertices z_0, z_1 and z_2. The formula also holds in the degenerate case when one or more of the z_i's are identical, in which case Δ has empty interior and C equals 1. ◊

3.A Ideal Triangles and the Maslov Index

We may extend our definition of an oriented geodesic triangle to include triangles with vertices on the boundary $\partial\mathbb{D}$. A geodesic triangle Δ whose vertices all lie on $\partial\mathbb{D}$ is called an *ideal triangle.* An ideal triangle is defined

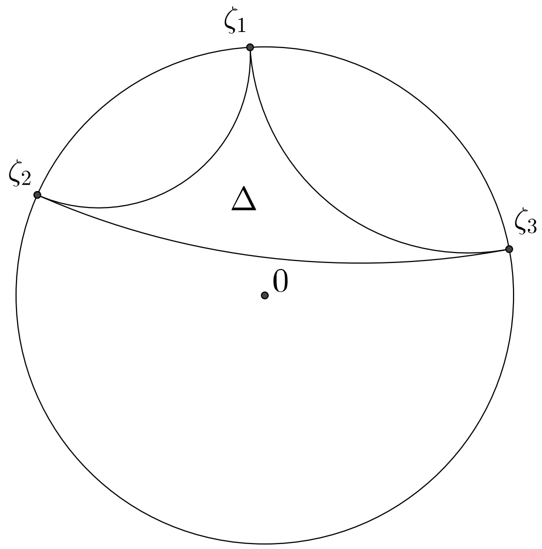

Figure 3: An ideal triangle Δ with vertices ζ_1, ζ_2, and ζ_3.

by three ordered distinct points $(\zeta_1, \zeta_2, \zeta_3)$ on $(\partial\mathbb{D})^3$ with edges made up of the infinite geodesics between these points. The oriented area of Δ is π or $-\pi$ depending on the orientation of the vertices; the area is π if one reaches ζ_2 before ζ_3 when traversing $\delta\mathbb{D}$ counterclockwise starting at ζ_1, and $-\pi$ otherwise. This gives an expression for the classical *Maslov index* $\iota(\zeta_1, \zeta_2, \zeta_3)$ as

$$\iota(\zeta_1, \zeta_2, \zeta_3) = \frac{1}{\pi} \int_\Delta \omega,$$

so ι takes the values ± 1. For the classical setting and definition of the Maslov index, see the book [GS77, Ch. IV].

Notice that we can extend C to triples of distinct points on $\partial\mathbb{D}$ (with value -1 for all such triples) and that $\arg C$ also extends continuously. By approaching the triple $(\zeta_1, \zeta_2, \zeta_3)$ from the inside of $\mathbb{D}$, we have

$$\arg C(\zeta_1, \zeta_2, \zeta_3) = \lim_{r \uparrow 1} \arg C(r\zeta_1, r\zeta_2, r\zeta_3),$$

hence $\arg C(\zeta_1, \zeta_2, \zeta_3)$ equals the oriented area of Δ. Putting this together we obtain the formula

$$\iota(\zeta_1, \zeta_2, \zeta_3) = \frac{1}{\pi} \lim_{r \uparrow 1} \arg C(r\zeta_1, r\zeta_2, r\zeta_3),$$

valid for any triple of pairwise distinct points ζ_1, ζ_2, and ζ_3 on $\partial\mathbb{D}$. From this formula we can deduce a number of properties of the Maslov index ι, such as its invariance under the action of $SU(1,1)$ on $\partial\mathbb{D}$

$$\iota(g(\zeta_1), g(\zeta_2), g(\zeta_3)) = \iota(\zeta_1, \zeta_2, \zeta_3),$$

and the cocycle property

$$\iota(\zeta_1, \zeta_2, \zeta_3) = \iota(\zeta_1, \zeta_2, \zeta_4) + \iota(\zeta_2, \zeta_3, \zeta_4) + \iota(\zeta_3, \zeta_1, \zeta_4),$$

for any quartuple of distinct points $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ on $\partial\mathbb{D}$.

§4 The Riemann Sphere

There are a number of similarities between the geometry of the unit disc $\mathbb{D}$ and that of the Riemann sphere $\mathbb{CP}^1$. One simply has to 'change a sign' to obtain the Riemannian metric $g^{(c)}$ on $\mathbb{C}$ given by

$$g_z^{(c)} = 4\frac{dx \otimes dx + dy \otimes dy}{(1+|z|^2)^2}, \tag{4.1}$$

which is (up to a factor 4) the standard Fubini-Study metric on $\mathbb{C}$. The metric $g^{(c)}$ is also the pull-back metric arising from inverse stereographic projection from $\mathbb{C}$ to the round sphere $S^2 \subset \mathbb{R}^3$ given by

$$\varphi \colon z = x + iy \mapsto \frac{1}{1+|z|^2}(2x, 2y, 1 - |z|^2),$$

so in particular $g^{(c)}$ has constant curvature +1. The volume form of $g^{(c)}$ is

$$\omega^{(c)} = \frac{4dx \wedge dy}{(1+|z|)^2},$$

and it should be observed that, because stereographic projection is orientation reversing, this form is computed using the standard orientation of $\mathbb{C}$. A similar sign change provides us with a kernel function

$$k_c(z,w) = (1 + z\overline{w})^2, \tag{4.2}$$

defined for all $z, w \in \mathbb{C}$. The power is changed from -2 to 2 in order to have the identity $\omega^{(c)} = i\partial\overline{\partial} \log k_c(z,z)$ hold. The rest of this section will now be devoted to proving an area formula similar to the formula from Theorem 3.1, but before we may state such a theorem, we have to investigate the geometry a little further.

Under stereographic projection, great circles on S^2 are mapped to circles and lines in $\mathbb{C}$ which intersect the unit circle $S^1 = \{z \in \mathbb{C} \colon |z| = 1\}$ at opposite points. Together with the unit circle itself, these circles and lines are the geodesics in $\mathbb{C}$ with the metric g_c. In particular, given a point ζ on the unit circle, there are infinitely many geodesics passing through both ζ and $-\zeta$ as shown in figure 4. The geodesics through 0 are straight lines.

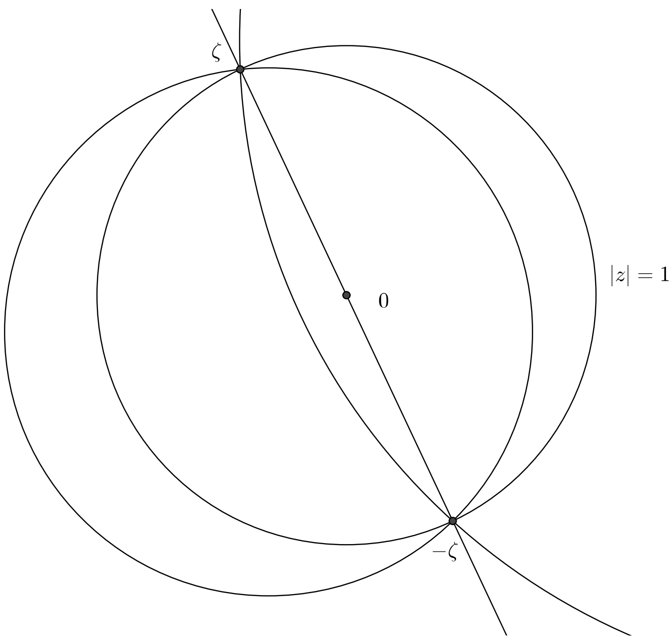

Figure 4: Several geodesics passing through the points ζ and $-\zeta$ on the unit circle.

In order to be able to define a continuous argument for k_c we introduce the set

$$\mathcal{S} = \{(z,w) \in \mathbb{C} \times \mathbb{C} \colon 1 + z\overline{w} \notin (-\infty, 0]\} \tag{4.3}$$

which is star-like with respect to $(0,0)$. Hence it is possible to define a continuous argument for the restriction of k_c to $\mathcal{S}$. We can choose this argument as twice the main argument of $1+z\overline{w}$, that is

$$\arg k_c(z,w) = 2\mathrm{Arg}(1+z\overline{w}), \quad (z,w) \in \mathcal{S}, \tag{4.4}$$

defines a continuous argument for k_c on $\mathcal{S}$. Since k_c is invariant under rotations we must have

$$argk_c(e^{it}z, e^{it}w) = \arg k_c(z,w),$$

for all $t \in \mathbb{R}$ and any pair $(z,w) \in \mathcal{S}$. Furthermore, for any z, w we see that $k_c(z,w) = \overline{k_c(w,z)}$ whence

$$\arg k_c(z,w) = -\arg k_c(w,z)$$

for any $(z,w) \in \mathcal{S}$.

The kernel k_c and the set $\mathcal{S}$ capture some of the behaviour of geodesics as shown below.

Proposition 4.1 *Let z and w be arbitrary complex numbers. Then*

1. *$k_c(z,w) = 0$ if and only if z and w are mapped to a pair of antipodal points under stereographic projection. Thus, in this case there are infinitely many distinct geodesic segments of equal length in $\mathbb{C}$ connecting z and w, each of which realizes the distance between z and w.*

2. *$1+z\overline{w}$ is real if and only if $\varphi(z)$ and $\varphi(w)$ lie on a great circle passing through the north pole $(0,0,1) \in S^2$ and the south pole $(0,0,-1) = \varphi(0)$.*

3. *The pair (z,w) belongs to $\mathcal{S}$ if and only there is a unique geodesic segment in $\mathbb{C}$ which connects z and w and realizes the distance between these two points.*

Proof. The proof of 1. consists of a straightforward calculation checking that z and $-\overline{z}^{-1}$ are mapped to antipodal points under stereographic projection.

Moving on to the statement in 2., we observe that $1+z\overline{w}$ is real if and only if z and w are linearly dependent over $\mathbb{R}$, and hence connected by a line through the origin. This line is mapped to a great circle on S^2 passing through $(0,0,1)$ and $(0,0,-1)$ as shown on figure 5.

To prove 3. we take two points $z, w \in \mathbb{C}$. We may assume without loss of generality that z and w are distinct. Furthermore, since $(z,w) \in \mathcal{S}$ if and only if $(w,z) \in \mathcal{S}$, we can take z to be non-zero. It follows from 1. that we may assume $k_c(z,w) \neq 0$. Hence there is a unique shortest great circle arc connecting $\varphi(z)$ and $\varphi(w)$ on S^2. Thus in order to prove 3. we have to prove that this arc passes through $(0,0,1)$ if and only if $(z,w) \notin \mathcal{S}$.

So assume that (z,w) does not belong to $\mathcal{S}$, i.e. that $s = 1+z\overline{w}$ is a real number < 0. This happens if and only if $w = (s-1)\overline{z}^{-1}$, so in particular w is

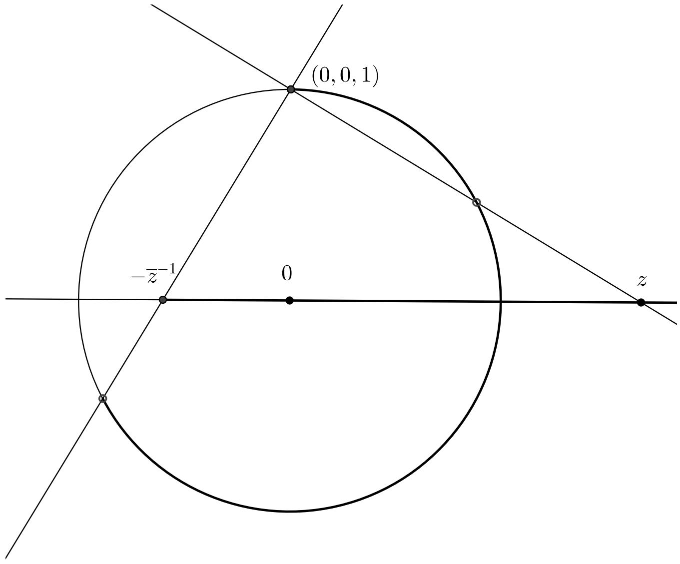

Figure 5: Stereographic projection restricted to the line through z and 0. Points on the thick half-line are connected to z by a shortest geodesic segment which avoids $(0, 0, 1)$.

lies on the line through 0 and z. We parametrize this line by $t \mapsto z_t = (t-1)\overline{z}^{-1}$ so that $1 + z\overline{z_t} = t$. Then z_0 is the 'antipodal point' of z. If t is < 0, the unique shortest geodesic segment on S^2 between z_t and z pass through the north pole $(0, 0, 1)$ as shown on figure 5. If, on the other hand, the arc through $\varphi(z)$ and $\varphi(w)$ passes through $(0, 0, 1)$, then $1 + z\overline{w}$ is real by 2., and hence w lies on the line through z and 0. Then $w = z_t$ as before, but t cannot be positive, for then $\varphi(z)$ and $\varphi(w)$ are connected by a shortest geodesic segment that does not pass through $(0, 0, 1)$. ∎

Remark 4.2 At this point we could follow the reasoning in the case of the unit disc $\mathbb{D}$ and introduce the group $SU(2)$ and its partial action on $\mathbb{C}$. Suppose g is an element of $SU(2)$, that is

$$g = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1.$$

Then we have a partially defined action of g on $\mathbb{C}$ via

$$g(z) = \frac{\alpha z + \beta}{-\overline{\beta} z + \overline{\alpha}},$$

which is isometric and defined everywhere except possibly for one point. If we define

$$j(g, z) = -\overline{\beta} z + \overline{\alpha}$$

we find as before

$$j(g,z)(1+g(z)\overline{g(w)})\overline{j(g,w)} = 1 + z\overline{w}$$

for all $z, w \in \mathbb{C}$ where $j(g,z)$ and $j(g,w)$ are non-zero. Given three points z_1, z_2 and z_3 such that all pairs (z_i, z_j) belong to $\mathcal{S}$, we can proceed to define the $SU(2)$-invariant expression

$$C(z_1,z_2,z_3) = \frac{1+z_1\overline{z_2}}{1+\overline{z_1}z_2} \cdot \frac{1+z_2\overline{z_3}}{1+\overline{z_2}z_3} \cdot \frac{1+z_3\overline{z_1}}{1+\overline{z_3}z_1}, \tag{4.5}$$

and define an argument for C by setting

$$\arg C(z_1,z_2,z_3) = \arg k_c(z_1,z_2) + \arg k_c(z_2,z_3) + \arg k_c(z_3,z_1), \tag{4.6}$$

but $\arg C$ is not necessarily invariant under elements of $SU(2)$. To see this, let $\zeta = \frac{1}{2}(-1+i\sqrt{3})$ and consider the triple $(1,\zeta,\zeta^2)$ of third roots of unity. All pairs of these three elements belong to $\mathcal{S}$ and

$$\arg k_c(1,\zeta) = -\frac{2\pi}{3}.$$

Using the invariance property of $\arg K$, we find

$$\arg k_c(\zeta,\zeta^2) = \arg k_c(1,\zeta) = -\frac{2\pi}{3}, \quad \arg k_c(\zeta^2,1) = \arg k_c(1,\zeta) = -\frac{2\pi}{3},$$

and thus $\arg C(1,\zeta,\zeta^2) = -2\pi$. Now consider the element

$$g = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

of $SU(2)$. This particular element acts on $\mathbb{C} \setminus \{0\}$ by inversion whence

$$(g(1), g(\zeta), g(\zeta^2)) = (1, \zeta^2, \zeta).$$

However $\arg C(1,\zeta^2,\zeta) = 2\pi$ so $\arg C$ is not $SU(2)$-invariant. $\Diamond$

Theorem 4.3 *Let (z,w) be a pair in $\mathcal{S}$. Construct an oriented geodesic triangle Δ with vertices $0, z, w$ by choosing the shortest geodesics connecting 0 and z as well as 0 and w, and finally choose the shortest geodesic connecting z and w. Then the oriented area of Δ is given by*

$$\int_\Delta \zeta^{(c)} = -\arg k_c(z,w), \tag{4.7}$$

where arg *is the argument for k_c defined on $\mathcal{S}$ such that $\arg k_c(z,z) = 0$.*

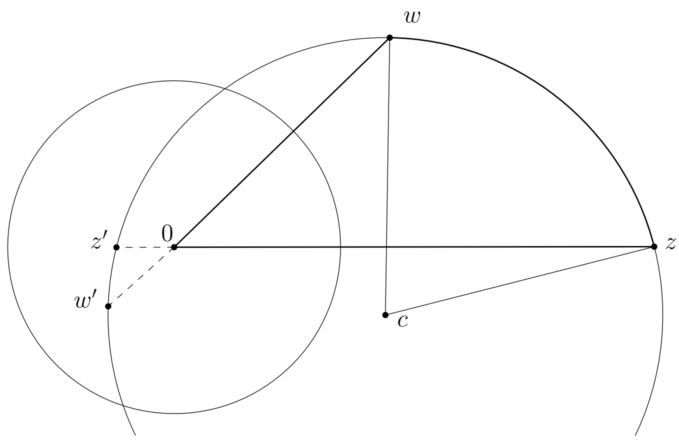

Figure 6: An oriented geodesic triangle with vertices 0, z and w.

Proof. We will assume that Δ is positively oriented. Both the area of Δ and the argument of k_c is invariant under rotation, so we assume that z is real and positive. The geodesic segment connecting z and w is part of a circle that intersects the unit circle in two opposite points. The points $z' = -\overline{z}^{-1}$ and $w' = -\overline{w}^{-1}$ also belong to this circle as these points are the antipodes of z and w. Our assumption on the orientation of Δ implies that the shortest geodesic segment connecting z and w must lie in the upper half plane. In particular the imaginary part of w must be strictly positive.

The classical formula gives the area of Δ as the angular excess, i.e. $\alpha+\beta+\gamma-\pi$, where α, β and γ are the interior angles of Δ. Using classical geometry, this is seen to be equal to the angle between $z-c$ and $w-c$, which in turn is twice the angle between $z-z'$ and $w-z'$. This angle can be computed using the usual main argument. We get

$$\begin{aligned}\int_\Delta \omega^{(c)} &= 2\mathrm{Arg}\frac{w-z'}{z-z'} \\ &= 2\mathrm{Arg}(1+\overline{z}w) \\ &= -2\mathrm{Arg}(1+z\overline{w}) \\ &= -\arg k_c(z,w),\end{aligned}$$

as claimed. ∎

Given three points z_0, z_1 and z_2 such that each of the pairs (z_i, z_j) belongs to $\mathcal{S}$ we can construct an oriented geodesic triangle $\Delta = \Delta(z_0, z_1, z_2)$ as follows: Δ is the set bounded by the three unique shortest geodesic segments connecting the pairs (z_0, z_1), (z_1, z_2) and (z_2, z_0). The orientation is given by traversing the boundary in the order $z_0 \mapsto z_1 \mapsto z_2 \mapsto z_0$.

Theorem 4.4 *Let z_0, z_1 and z_2 be three points in $\mathbb{C}$ such that each of the pairs (z_i, z_j) belong to $\mathcal{S}$. Construct an oriented geodesic triangle $\Delta(z_0, z_1, z_2)$*

in the manner just described. Then

$$\int_{\Delta(z_0,z_1,z_2)} \omega^{(c)} = -(\arg k_c(z_0,z_1) + \arg k_c(z_1,z_2) + \arg k_c(z_2,z_0)), \tag{4.8}$$

where arg *is the argument for* k_c *on* $\mathcal{S}$ *satisfying* $\arg k_c(z,z) = 0$.

Proof. According to Theorem 4.3 we have

$$-\arg k_c(z_0,z_1) = \int_{\Delta(0,z_0,z_1)} \omega^{(c)},$$

and similarly for the other terms on the right hand side of (4.8). Introducing the notation Δ_{01}, Δ_{12} and Δ_{20} for the geodesic triangles $\Delta(0,z_0,z_1)$, $\Delta(0,z_1,z_2)$ and $\Delta(0,z_2,z_1)$ respectively, we have

$$-(\arg k_c(z_0,z_1)+\arg k_c(z_1,z_2)+\arg k_c(z_2,z_0)) = \int_{\Delta_{01}} \omega^{(c)}+\int_{\Delta_{12}} \omega^{(c)}+\int_{\Delta_{20}} \omega^{(c)},$$

but ω is exact so Stoke's theorem gives

$$\int_{\Delta_{01}} \omega^{(c)} + \int_{\Delta_{12}} \omega^{(c)} + \int_{\Delta_{20}} \omega^{(c)} = \int_{\Delta(z_0,z_1,z_2)} \omega^{(c)},$$

as claimed. ■

Remark 4.5 One can prove results similar to the above for a family k_n of functions

$$k_n(z,w) = (1+z\overline{w})^n,$$

with n any integer and with the corresponding form $i\partial\overline{\partial}\log k_n(z,z) = \frac{n}{2}\omega^{(c)}$. With the same definition of $\mathcal{S}$ we may define $\arg k_n = \frac{n}{2}\arg k_c$ and obtain the same results as above. ◊

In essence, theorem 4.3 gives a geometric interpretation of $\arg k_c(z,w)$ for $(z,w) \in \mathcal{S}$ by giving a specific choice of a geodesic triangle $\Delta(0,z,w)$ whose area equals $-\arg k_c(z,w)$. Any continuous variation of (z,w) within $\mathcal{S}$ gives a continuous deformation of $\Delta(0,z,w)$. But we could also start with a general oriented geodesic triangle Δ in $\mathbb{C}$ with vertices z_1, z_2, and z_3. Δ is bounded by geodesic segments connecting the vertices and the orientation is determined by the ordering (z_1,z_2,z_3) of the vertices as in the case of the unit disc.

Theorem 4.6 *Suppose that* $z,w \in \mathbb{C}$ *are points such that* $k_c(z,w) \neq 0$*, and that we are given an oriented geodesic triangle* $\Delta \subset \mathbb{C}$ *with vertices* 0*,* z*, and* w*, and orientation* $(0,z,w)$*. Then*

$$\exp\left(-i\int_{\Delta} \omega^{(c)}\right) = \frac{1+z\overline{w}}{1+\overline{z}w}, \tag{4.9}$$

holds.

Proof. The 'degenerate' case is when $(z,w) \notin \mathcal{S}$ where Δ has empty interior. Here both sides of (4.9) equals 1. Suppose then that (z,w) belongs to $\mathcal{S}$. Then there are two possible geodesic segments connecting z and w as seen on figure 6. There are thus two possible oriented geodesic triangles with vertices $(0,z,w)$. Let Δ' denote the oriented geodesic triangle with the shortest segment between z and w as an edge, and let Δ'' denote the oriented geodesic triangle that has the longest segment between z and w as an edge. The difference between the area of Δ' and the area of Δ'' is the area of the disc bounded by the geodesic through z and w, i.e.

$$\int_{\Delta'} \omega^{(c)} - \int_{\Delta''} \omega^{(c)} = 2\pi,$$

and hence

$$\exp\left(-i \int_{\Delta'} \omega^{(c)} \right) = \exp\left(-i \int_{\Delta''} \omega^{(c)} \right),$$

and now it suffices to prove (4.9) for the triangle Δ', but this is a consequence of (4.7). ∎

The formula (4.9) is mentioned in [Per86] and generalized to $\mathbb{CP}^n$ in [HM94]. The proof presented here seems to be new.

Corollary 4.7 *Let $\Delta \subset \mathbb{C}$ be an oriented geodesic triangle with vertices z_1, z_2, and z_3 and orientation corresponding to the ordering (z_1, z_2, z_3). Then we have*

$$\exp\left(-i \int_{\Delta} \omega^{(c)} \right) = C(z_1, z_2, z_3),$$

where C is the function defined by (4.5)*.*

Proof. The proof is similar to the proof of Theorem 4.4. Use the three vertices of Δ to construct three geodesic triangles Δ_j with vertices (and orientation) $(0, z_j, z_{j+1})$ where $1 \le j \le 3$ and $z_4 = z_1$. Then

$$\int_{\Delta} \omega^{(c)} = \sum_{j=1}^{3} \int \Delta_j \omega^{(c)},$$

by Stoke's theorem and the result follows from (4.9). ∎

4.A Maximal Argument

Let us compute the supremum of $\arg k_c(z,w)$ when (z,w) runs over all of $\mathcal{S}$. From a geometric point of view we are looking for the maximal area of a spherical triangle on the two-sphere whose sides are shortest geodesic arcs not passing through the antipodes of the vertices. Hence there is a sharp upper bound of 2π. For a more analytical approach, pick $(z, w) \in \mathcal{S}$. Assuming that z is non-zero we may use the rotational invariance of $\arg k_c$ to further assume that z is real and strictly positive. Then the $\mathcal{S}$-fiber over z consists of all $w \in \mathbb{C}$ which do not lie in $(-\infty, -z^{-1}]$. As we are looking for the maximal value of $\arg k_c(z, w)$ we further assume that w is not real and has negative imaginary part. We thus have to estimate $\operatorname{Arg}(1 + se^{i\theta})$ for $s > 0$ and $\theta \in (0, \pi)$. But this expression is increasing in s and $\lim_{s\to\infty} \operatorname{Arg}(1 + se^{i\theta}) = \theta$, so

$$\sup_{(z,w)\in\mathcal{S}} \arg k_c(z, w) = 2\pi$$

as claimed. To estimate the maximal area of a geodesic triangle $\Delta(z_0, z_1, z_2)$ constructed from a triple (z_0, z_1, z_2) where each of the pairs (z_i, z_j) belong to $\mathcal{S}$, first note that there is a simple upper bound given by the total surface area of the two-sphere, that is

$$\int_{\Delta(z_0,z_1,z_2)} \omega \in (-4\pi, 4\pi).$$

This bound is best possible as

$$\lim_{r\to\infty} \int_{\Delta(r,r\omega,r\omega^2)} \omega = -4\pi,$$

where $\omega = \frac{1}{2}(-1 + i\sqrt{3})$.

§5 *The Hyperboloid*

The third space we will consider is the two-dimensional hyperboloid Σ of one sheet given by

$$\Sigma = \Big\{(x, y, z) \in \mathbb{R}^3 \colon -x^2 + y^2 + z^2 = 1\Big\}$$

with the pseudo-Riemannian structure g induced by the Lorentz metric

$$-dx \otimes dx + dy \otimes dy + dz \otimes dz$$

on $\mathbb{R}^3$. It is well-known, see e.g. [Wol67, Theorem 2.4.4], that (Σ, g) has constant curvature $+1$ and g has signature $(1, 1)$. The geodesics on Σ are the intersections $\Sigma \cap P$ where P is a plane through 0. The full isometry group of Σ is the matrix group

$$O(1, 2) = \left\{ A \in GL(3, \mathbb{R}) \colon A^t \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} A = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right\}$$

and Σ is a homogeneous space under this group. It should be noted that Σ is also a homogeneous space under the adjoing action of $SL(2,\mathbb{R})$ using the linear isomorphism

$$(x,y,z) \mapsto \begin{pmatrix} z & y+x \\ y-x & -z \end{pmatrix}$$

between $\mathbb{R}^3$ and $\mathfrak{sl}(2,\mathbb{R})$. In this way Σ is the symmetric space $SL(2,\mathbb{R})/\mathbb{R}^*$, but we will not need this fact in this section.

We pick $p_0 = (0,0,-1)$ as our base point. Then it is easy to see that the space-like geodesics on Σ are exactly the closed geodesics, and that the null-lines are straight lines. We need some facts about the geometry of Σ from [Wol67, Lemma 11.2.1].

Proposition 5.1 *Let p be any point on Σ. Let ℓ_1 and ℓ_2 denote the null-lines through $-p$.*

1. *The set $\Sigma \setminus (\ell_1 \cup \ell_2)$ has three connected components and contains p.*
2. *Any point in the connected component of p is connected to p in $\Sigma \setminus \ell_1 \cup \ell_2$ by a unique geodesic segment lying entirely inside the connected component of p.*
3. *Any point in the other two components of $\Sigma \setminus (\ell_1 \cup \ell_2)$ is not connected to p by a geodesic segment.*
4. *The only point on $\ell_1 \cup \ell_2$ which is connected to p by a geodesic arc is $-p$.*

In particular, if $p = (x,y,z)$ be a point on Σ not equal to $-p_0$, then p_0 and p are connected by at least one geodesic if and only if $z < -1$.

As in the previous example, we will use stereographic projection to obtain some useful coordinates on Σ. This time however, we will think of the projection not as a map into $\mathbb{C}$ but as a map into the algebra $\mathbb{A}$ of paracomplex numbers which will be briefly introduced below.

5.A Paracomplex Numbers

The paracomplex numbers $\mathbb{A}$ is a two-dimensional associative unital algebra over $\mathbb{R}$ consisting of elements of the form

$$w = x + jy, \qquad x, y \in \mathbb{R},$$

where j is the imaginary paracomplex unit satisfying $j^2 = 1$. Define an $\mathbb{R}$-basis of $\mathbb{A}$ by

$$\overline{\mathscr{E}} = \frac{1}{2}(1+j), \qquad \mathscr{E} = \frac{1}{2}(1-j),$$

and observe that $\mathscr{E}^2 = \mathscr{E}$, $\overline{\mathscr{E}}^2 = \overline{\mathscr{E}}$ and $\mathscr{E}\overline{\mathscr{E}} = 0$. Hence we may also think of $\mathbb{A}$ as the vector space $\mathbb{R}^2$ equipped with coordinate-wise multiplication $(x, y) \cdot (u, v) = (xu, yv)$ as well as the usual topology. In the following we will mostly use the basis 1 and j, but nevertheless define the coordinate maps $()_\pm \colon \mathbb{A} \to \mathbb{R}$ by $w = w_+\overline{\mathscr{E}} + w_-\mathscr{E}$ for any $w \in \mathbb{A}$.

Paracomplex conjugation is defined by $\overline{w} = x - jy$ and the corresponding modulus is $|w|^2 = w\overline{w} = x^2 - y^2$. Notice that $|w|^2 = 0$ if and only if w lies on one of the null-lines $\mathbb{R}\overline{\mathscr{E}}$ or $\mathbb{R}\mathscr{E}$ and that any paracomplex number with non-zero modulus is invertible . The 'unit circle' of paracomplex numbers of norm 1 is a hyperbola. The paracomplex exponential function is defined using

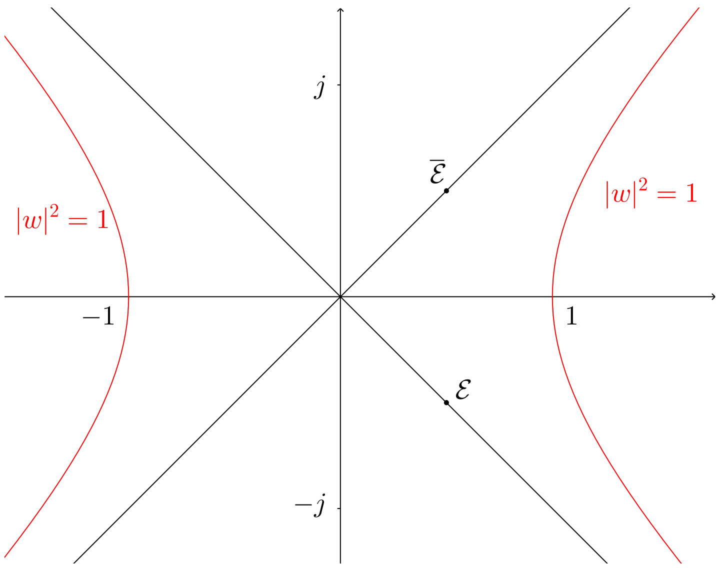

Figure 7: The plane of paracomplex numbers.

the analogue of Euler's formula

$$e^{x+jy} = e^x(\cosh y + j \sinh y), \qquad x, y \in \mathbb{R},$$

and the usual identity $e^{w_1+w_2} = e^{w_1}e^{w_2}$ holds for any $w_!, w_2 \in \mathbb{A}$. In terms of $()_\pm$-coordinates exponentiation is given by $(e^w)_\pm = e^{w_\pm}$. Hence for any $w \in \mathbb{A}$ with $w_\pm > 0$ we define the paracomplex logarithm $\log w$ by $(\log w)_\pm = \log(w_\pm)$.

Suppose that $U \subset \mathbb{A}$ is open and that $f \colon U \to \mathbb{A}$ is an $\mathbb{A}$-valued smooth function. Then, as $\mathbb{A}$-valued forms, $df = \frac{\partial f}{\partial w}dw + \frac{\partial f}{\partial \overline{w}}d\overline{w}$ where

$$\frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \overline{w}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - j\frac{\partial}{\partial y}\right)$$

and

$$dw = dx + j dy$$
$$d\overline{w} = dx - j dy.$$

We will write $\partial f = \frac{\partial f}{\partial w} dw$ and $\overline{\partial} f = \frac{\partial f}{\partial \overline{w}} d\overline{w}$. We say that f is paraholomorphic if $\overline{\partial} f = 0$. Observe that e^w and $\log w$ are both paraholomorphic and

$$\frac{\partial}{\partial w} e^w = e^w, \qquad \frac{\partial}{\partial w} \log w = w^{-1}.$$

For any paraholomorphic function f we use the notation $f' = \frac{\partial}{\partial w} f$.

5.B Paracomplex Structure of Σ

We define $\varphi\colon \Sigma \setminus \{z = 1\} \to \mathbb{A}$ by

$$\varphi(x, y, z) = \frac{1}{1 - z}(x + jy)$$

for $(x, y, z) \in \Sigma$, $z \neq 1$. The image of φ is the set

$$B = \left\{ w \in \mathbb{A}\colon \ |w|^2 \neq 1 \right\},$$

and the metric g and volume form ω on Σ are given by

$$g = 4 \frac{-dx \otimes dx + dy \otimes dy}{(1 - |w|^2)^2}$$

and

$$\omega = 4 \frac{dx \wedge dy}{(1 - |w|^2)^2}$$

in these coordinates and the orientation defined by $(1, j)$. Let us to introduce the kernel

$$k(z, w) = (1 - z\overline{w})^2$$

for any elements z and w in $\mathbb{A}$. The following result, which is straightforward to verify, shows that k captures some of the geometry of Σ.

Proposition 5.2 *Let w_0 be a point in B and let $p = \varphi^{-1}(w_0)$ be the corresponding point on Σ. If w is another point in B then $k(w_0, w)$ is invertible if and only if $\varphi^{-1}(w)$ does not lie on a null-line through $-p$.*

The situation is similar to what we found for the Riemann sphere. If $w = \varphi(p)$ is a point in B, then w is invertible if and only if $-p$ lies in the domain of φ, and in this case $\varphi(-p) = \overline{w}^{-1}$. We refer to this point as the *antipodal* point of w or p, where it is understood that if w is not invertible it only has an antipodal point when considered as point on Σ.

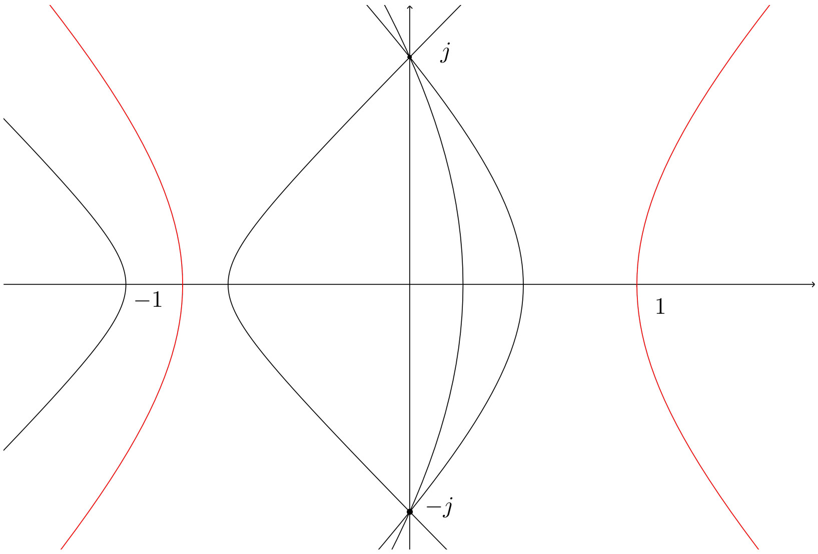

Figure 8: Several geodesics connecting j and $-j$.

The geodesics on Σ are mapped under φ to hyperbolas in $\mathbb{A}$. In particular, geodesics through p_0 are mapped to straight lines through 0.

Inspired by the geometry of the unit disc in $\mathbb{C}$ we define the group

$$G = \left\{ \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \in M_2(\mathbb{A}) \colon \ |a|^2 - |b|^2 = 1 \right\}$$

consisting of 2×2 matrices with paracomplex entries. This is a Lie group under the usual matrix multiplication rules, and we define a partial action of G on $\mathbb{D}$ by

$$g(w) = \frac{aw + b}{\overline{b}w + \overline{a}}, \qquad g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \in G,$$

wherever $(\overline{b}w + \overline{a})$ is invertible. Note that the transformation defined by g is paraholomorphic and

$$g'(w) = \frac{\partial}{\partial w} g(w) = (\overline{b}w + \overline{a})^{-2}. \tag{5.1}$$

Suppose that $g(w_1)$ and $g(w_2)$ are both defined. Then

$$1 - g(w_1)\overline{g(w_2)} = (\overline{b}w_1 + \overline{a})^{-1}(1 - w_1\overline{w_2})(b\overline{w_2} + a)^{-1},$$

and we have the following transformation rule

$$k(g(w_1), g(w_2)) = g'(w_1)k(w_1, w_2)\overline{g'(w_2)}, \tag{5.2}$$

which together with the observation that

$$g = -2\frac{dw \otimes d\overline{w} + d\overline{w} \otimes dw}{(1 - |w|^2)^2}$$

shows that elements of G act on B as isometries.

Remark 5.3 The group G is actually $SL(2,\mathbb{R})$ in disguise. For if $\left(\begin{smallmatrix} a & b \\ \overline{b} & \overline{a} \end{smallmatrix}\right) \in G$ we may write

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} = \begin{pmatrix} a_+ & b_+ \\ b_- & a_- \end{pmatrix} \overline{\mathscr{E}} + \begin{pmatrix} a_- & b_- \\ b_+ & a_+ \end{pmatrix} \mathscr{E}$$

using the $()_\pm$ coordinates $a = a_+\overline{\mathscr{E}} + a_-\mathscr{E}$ and $b = b_+\overline{\mathscr{E}} + b_-\mathscr{E}$. Then $\left(\begin{smallmatrix} a_+ & b_+ \\ b_- & a_- \end{smallmatrix}\right)$ is an element of $SL(2,\mathbb{R})$ and the action of g on $w \in \mathbb{A}$ may be written as

$$g(w) = \frac{a_+w_+ + b_+}{b_-w_+ + a_-}\overline{\mathscr{E}} + \frac{a_-w_- + b_-}{b_+w_- + a_+}\mathscr{E}$$

at every point $w \in \mathbb{A}$ where $g(w)$ is defined. Conversely, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an element of $SL(2,\mathbb{R})$, then

$$g = \begin{pmatrix} a\overline{\mathscr{E}} + d\mathscr{E} & b\overline{\mathscr{E}} + c\mathscr{E} \\ c\overline{\mathscr{E}} + b\mathscr{E} & d\overline{\mathscr{E}} + a\mathscr{E} \end{pmatrix}$$

is an element of G and

$$g(x\overline{\mathscr{E}} + y\mathscr{E}) = \frac{ax + b}{cx + d}\overline{\mathscr{E}} + \frac{dy + c}{by + a}\mathscr{E}$$

shows how the action of G is just two different linear fractional transformations of $\mathbb{R}$. ◊

5.C Area of Geodesic Triangles

We now return to the volume form ω. It is related to the kernel k via

$$\omega_w = -j\frac{\partial}{\partial w}\frac{\partial}{\partial \overline{w}} \log k(w,w) dw \wedge d\overline{w}$$

at every point $w \in B$. Note that $k(z,w)$ is a square in $\mathbb{A}$ and that $\log k(z,w)$ thus makes sense whenever $|k(z,w)|^2 \neq 0$. We introduce the operator $d_{\mathbb{A}} = j(\partial - \overline{\partial})$. Then

$$\omega_z = \frac{1}{2} dd_{\mathbb{A}} \log k(z,z)$$

holds for all $z \in B$.

Definition 5.4 *Let $\gamma\colon [a,b] \to B$ be a smooth curve segment such that $k(\gamma(a),\gamma(b))$ is invertible. Then we define a cocycle α by*

$$\alpha(\gamma) = \arg k(\gamma(a),\gamma(b)) + \frac{1}{2}\int_\gamma d_{\mathbb{A}} \log k,$$

where $\arg k(\gamma(a),\gamma(b))$ denotes the paracomplex imaginary part of $\log k(\gamma(a),\gamma(b))$.

The main feature of the cocycle α is that it is G-invariant.

Proposition 5.5 *Suppose that $\gamma\colon [a,b] \to B$ is a smooth curve segment connecting the points $w_1 = \gamma(a)$ and $w_2 = \gamma(b)$ such that $k(w_1, w_2)$ is invertible. Assume that $g \in G$ is an element such that the action of g is defined on all points of γ. Then $\alpha(\gamma) = \alpha(g\gamma)$.*

Proof. It follows from (5.2) that α is defined on the curve $g\gamma$. Using (5.1) we see that $\log g'(w_1)$ and $\log g'(w_2)$ are defined, whence

$$\log k(g(w_1), g(w_2)) = \log g'(w_1) + \log k(w_1, w_2) + \log \overline{g'(w_2)}$$

holds. Comparing the paracomplex imaginary parts yields

$$\arg k(g(w_1), g(w_2)) = \arg g'(w_1) + \arg k(w_1, w_2) - \arg g'(w_2).$$

Furthermore, for all $w \in \mathbb{A}$ where $g(w)$ is defined

$$\begin{aligned} \log k(g(w), g(w)) &= \log(|g'(w)|^2\, k(w,w)) \\ &= \log |g'(w)|^2 + \log k(w,w), \end{aligned}$$

and thus

$$\begin{aligned} \int_{g\gamma} d_{\mathbb{A}} \log k &= \int_\gamma d_{\mathbb{A}} g^* \log k \\ &= \int_\gamma d_{\mathbb{A}} \log |g'(w)|^2 + \int_\gamma d_{\mathbb{A}} \log k, \end{aligned}$$

as $g^* d_{\mathbb{A}} \log k = d_{\mathbb{A}} g^* \log k$. The real part of the paraholomorphic function $\log g'(w)$ is $\frac{1}{2} \log |g'(w)|^2$ and

$$d_{\mathbb{A}} \log |g'(w)|^2 = 2d \arg g'(w)$$

since $d = \partial + \overline{\partial}$. Thus

$$\int_\gamma d_{\mathbb{A}} \log |g'(w)| = 2(\arg g'(w_2) - \arg g'(w_1))$$

and putting the above together we find

$$\begin{aligned} \alpha(g\gamma) &= \arg k(g(w_1), g(w_2)) + \frac{1}{2} \int_{g\gamma} d_{\mathbb{A}} \log k \\ &= \arg g'(w_1) + \arg k(w_1, w_2) - \arg g'(w_2) \\ &\quad + \int_\gamma d_{\mathbb{A}} \log k + \arg g'(w_2) - \arg g'(w_1) \\ &= \alpha(\gamma) \end{aligned}$$

as claimed. ∎

Remark 5.6 Let w be a point in B with $|w|^2 < 1$. Consider the matrix

$$g_w = \begin{pmatrix} \sqrt{1-|w|^2}^{-1} & w\sqrt{1-|w|^2}^{-1} \\ \overline{w}\sqrt{1-|w|^2}^{-1} & \sqrt{1-|w|^2}^{-1} \end{pmatrix},$$

which is an element of G. Then g_w maps 0 to w so its inverse, which is the element g_{-w}, maps w to 0. Notice that g_{-w} is defined for exactly those $z \in \mathbb{A}$ where $1 - z\overline{w}$ is invertible. Note also that the inversion map $w \mapsto -w^{-1}$ coincides with the action of

$$g = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} \in G,$$

on the invertible elements of $\mathbb{A}$.

Suppose now that $\gamma\colon [a,b] \to B$ is a smooth curve with the property that $k(\gamma(a), \gamma(t))$ is invertible for all t in $[a,b]$. Then if $|\gamma(a)| < 1$ it follows that $g_{-\gamma(a)}\gamma$ is defined and starts at 0. If on the other hand $\gamma(a)$ has modulus greater than 1, then so does $\gamma(t)$ for all t and thus $g\gamma(t) = -\gamma(t)^{-1}$ is a well-defined smooth curve of modulus less than 1. This curve can be mapped to a curve starting at 0 using G as we have just seen. We conclude that given any curve $\gamma\colon [a,b] \to B$ satisfying the assumption that $k(\gamma(a), \gamma(t))$ is invertible for all $t \in [a,b]$, there exists an element g of G such that $g\gamma$ is defined and has starting point at 0. $\diamondsuit$

Theorem 5.7 *Let $\gamma\colon [a,b] \to B$ be a geodesic segment and suppose that γ does not pass through the antipodal point of $\gamma(a)$. Then $\alpha(\gamma)$ is defined and vanishes, that is*

$$\frac{1}{2}\int_\gamma d_{\mathbb{A}} \log k = -\arg k(\gamma(a), \gamma(b)).$$

Proof. We first show that $k(\gamma(a), \gamma(t))$ is invertible for every t in $[a,b]$. Let $\tilde{\gamma} = \varphi^{-1} \circ \gamma\colon [a,b] \to \Sigma$ denote the lift of γ to Σ. By assumption $\tilde{\gamma}$ does not pass through the antipodal point of $\tilde{\gamma}(a)$ regardless of wether $\gamma(a)$ is invertible or not. And since $\tilde{\gamma}$ is a geodesic it follows from Proposition 5.1 that $\tilde{\gamma}$ cannot pass through the null-lines at the antipodal point of $\tilde{\gamma}(a)$. Thus $k(\gamma(a), \gamma(t))$ is invertible for all t in $[a,b]$ by Proposition 5.2. Hence, as we have already remarked, it is possible to find an element g of G such that the action of g is defined on all points of γ and $g\gamma(a) = 0$. Now $g\gamma$ is a geodesic starting at 0 so it is a line segment. From

$$d_{\mathbb{A}} \log k(w,w) = 2j\frac{w d\overline{w} - \overline{w} dw}{1 - |w|^2}$$

it is easy to see that the curve integral of $d_{\mathbb{A}} \log k$ vanishes along any segment of a line containing 0. This concludes the proof. ∎

Corollary 5.8 *Let $\gamma\colon [a,b] \to B$ be a segment of a null or time-like geodesic. Then $\alpha(\gamma) = 0$.*

Proof. Indeed, it follows from Proposition 5.1 that a null or time-like γ does not pass through a pair of antipodal points. ∎

Using the above results we may easily prove an area formula for geodesic triangles in B. To be precise, an oriented geodesic triangle $\Delta(w_0, w_1, w_2)$ in B with vertices w_0, w_1, and w_2 is a domain Δ bounded by a piecewise smooth simple closed curve consisting of three geodesic segments in B connecting the vertices. The ordering (w_0, w_1, w_2) of the vertices determines the orientation of Δ.

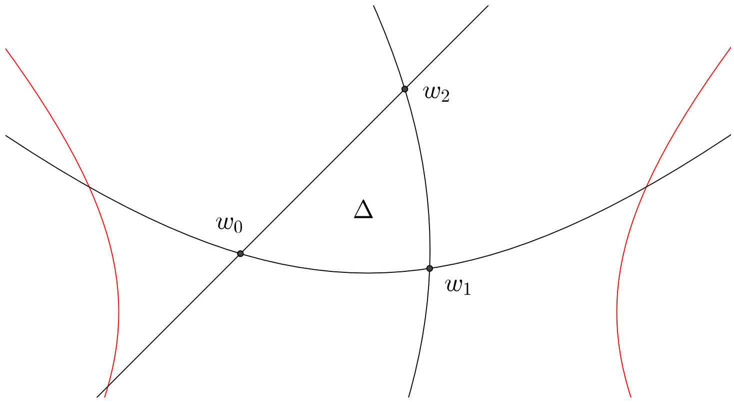

Figure 9: A time-like, a space-like, and a null geodesic bounding a geodesic triangle Δ with vertices w_0, w_1, and w_2.

Theorem 5.9 *Let Δ be an oriented geodesic triangle in B with vertices w_0, w_1, and w_2 and assume that the vertices are traversed in that order. Assume furthermore that each of the segments satisfy the conditions of Theorem 5.7. Then the oriented area of Δ is given by*

$$\int_{\Delta} \omega = -(\arg k(w_0, w_1) + \arg k(w_1, w_2) + \arg k(w_2, w_0)).$$

Proof. Let γ_0, γ_1, and γ_2 denote the geodesic segments that form the boundary of Δ, enumerated so that γ_0 starts at w_0 and ends at w_1 from where γ_1 starts and so on. Then it follows from Stoke's theorem and 5.7 that

$$\begin{aligned}\int_{\Delta} \omega &= \frac{1}{2} \int dd_{\mathbb{A}} \log k \\ &= \frac{1}{2} \sum_{i=0}^{2} \int_{\gamma_i} d_{\mathbb{A}} \log k \\ &= -(\arg k(w_0, w_1) + \arg k(w_1, w_2) + \arg k(w_2, w_0)),\end{aligned}$$

as claimed. ■

Remark 5.10 At this point we have seen three similar theorems (3.1, 4.4, and 5.9) on geodesic triangles in three different geometries. In the case of the unit disc the theorem was proved using group invariance of both sides of the formula (3.4) and the Gauss-Bonnet theorem. That same theorem was used to prove Theorem 4.3 for spherical triangles and this latter theorem may be viewed as an analogue of Theorem 5.7 concerning the cocycle α. The vanishing of α was of course instrumental in the proof of Theorem 5.9, which did not rely on the Gauss-Bonnet theorem even though versions of this theorem for two dimensional spacetimes such as Σ exists, see e.g. [BN84].

The common trend in these examples is the kernel functions k that transform in a suitable manner under the involved groups. From these k's we constructed the volume forms which were all of the form $\frac{1}{2}d\rho$ where ρ is a 1-form related to $\log k$ in each case. After applying Stoke's theorem everything comes down to relating $\arg k$ to path integrals of ρ over geodesic segments. This was done explicitly in the case of the cocycle α. This last approach focused more on group theory and less on the geometry, in particular the constant curvature, of the space Σ. In the next section we will explore this point of view further for all three spaces. ◊

§6 The Complex Picture

In the preceeding sections we have studied three homogeneous spaces

$$SU(1,1)/U(1), \quad SU(2)/U(1), \quad SL(2,\mathbb{R})/\mathbb{R}^*, \tag{6.1}$$

and in each case the relation between the geometry of the space and a suitable kernel function. This section is devoted to an attempt at unifying these three examples by considering the complex homogeneous space

$$\mathbf{X} = SL(2,\mathbb{C})/K_{\mathbb{C}},$$

where we use $K_{\mathbb{C}}$ to denote the subgroup of diagonal matrices in $SL(2,\mathbb{C})$. As we have $SU(2)\cap K_{\mathbb{C}} = U(1) = SU(1,1)\cap K_{\mathbb{C}}$ and $SL(2,\mathbb{R})\cap K_{\mathbb{C}} = \mathbb{R}^*$, we may view each of the spaces in (6.1) as subspaces of $\mathbf{X}$ by considering the orbits of $SU(2)$, $SU(1,1)$ and $SL(2,\mathbb{R})$ through the identity coses $o = K_{\mathbb{C}} \in \mathbf{X}$.

Another way of thinking about this is to consider the spaces (6.1) as surfaces in $\mathbb{R}^3$ given by the equations $x^2+y^2+z^2=1, -x^2+y^2+z^2=1$, and $x^2-y^2-z^2=1, x>0$. We can find each of these spaces in the *complex* quadratic variety given by $z_1^2+z_2^2+z_3^2=1$. And this space is an orbit of the adjoint group of $SL(2,\mathbb{C})$ acting on the three dimensional complex vector space $\mathfrak{sl}(2,\mathbb{C})$.

Notice that $\mathbf{X}$ is a symmetric space with respect to the involution $\sigma\colon SL(2,\mathbb{C}) \to SL(2,\mathbb{C})$ given by

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

whose fixed point set is $K_{\mathbb{C}}$. The differential of σ is $d\sigma\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{sl}(2,\mathbb{C})$, but we will also denote this map by σ. Let us introduce some notation

$$\begin{aligned} \mathfrak{k} = \text{Lie algebra of } K_{\mathbb{C}} &= \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in \mathbb{C} \right\}, \\ \mathfrak{q} = -1 \text{ eigenspace of } \sigma &= \left\{ \begin{pmatrix} 0 & z \\ w & 0 \end{pmatrix} : z, w \in \mathbb{C} \right\}, \\ \mathfrak{p}^+ &= \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} : z \in \mathbb{C} \right\}, \\ \mathfrak{p}^- &= \left\{ \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix} : w \in \mathbb{C} \right\}, \end{aligned}$$

and

$$P^{\pm} = \exp(\mathfrak{p}^{\pm}),$$

such that $\mathfrak{sl}(2,C) = \mathfrak{k} + \mathfrak{q}$ and $\mathfrak{q} = \mathfrak{p}^+ + \mathfrak{p}^-$. Note that $\mathfrak{k}$ normalizes both $\mathfrak{p}^+$ and $\mathfrak{p}^-$. There is a $SL(2,\mathbb{C})$-invariant complex-valued tensor Q on $\mathbf{X}$ given a the identity coset $o = K_{\mathbb{C}}$ by

$$Q_o(X,Y) = 4\operatorname{tr}(XY), \quad X, Y \in \mathfrak{q},$$

where we have identified q and $T_o\mathbf{X}$. Under this identification, Q is the Killing form of $\mathfrak{sl}(2,\mathbb{C})$ times a factor 1/2. Restricting Q to one of the three subspaces (6.1) yields (up to a sign in the case of the Riemann sphere) the invariant metric on each of these spaces. The real part of Q provides a pseudo-Riemannian structure on $\mathbf{X}$.

Lemma 6.1 *Define a map $m\colon P^+ \times K_{\mathbb{C}} \times P^- \to SL(2,\mathbb{C})$ by*

$$m(p,k,q) = pkq$$

for $p \in P^+, k \in K_{\mathbb{C}}, q \in P^-$. Then m is a diffeomorphism onto its image which is open and dense in $SL(2,\mathbb{C})$. The same holds for the map $\check{m}$ defined by multiplying the elements (p,k,q) in reverse order.

Proof. Since $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{k} + \mathfrak{p}^+ + \mathfrak{p}^-$, it follows that m is everywhere regular and hence has open image. The image of m is also dense because we have

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & \beta\delta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha - \beta\delta^{-1}\gamma & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta^{-1}\gamma & 1 \end{pmatrix}$$

whenever $\delta \neq 0$ and the claim for $\check{m}$ follows if we transpose and invert the above formula. To see that m and $\check{m}$ is injective it is enough to observe that $P^+ \cap K_{\mathbb{C}} P^- = \{I\}$. ∎

We introduce coordinates on $\mathbf{X}$ by first embedding it into the cartesian product of one-dimensional complex projective space with itself $\mathbb{CP}^1 \times \mathbb{CP}^1$; under the usual homographic $SL(2,\mathbb{C})$-action we have $\mathbb{CP}^1 = SL(2,\mathbb{C})/K_{\mathbb{C}}P^-$ or $\mathbb{CP}^1 = SL(2,\mathbb{C})/K_{\mathbb{C}}P^+$ depending on the choice of base-point. As $K_{\mathbb{C}}P^- \cap K_{\mathbb{C}}P^+ = K_{\mathbb{C}}$ we may now view $\mathbf{X}$ as the orbit of $SL(2,\mathbb{C})$ on the coset pair $(K_{\mathbb{C}}P^-, K_{\mathbb{C}}P^+)$ in $SL(2,\mathbb{C})/K_{\mathbb{C}}P^- \times SL(2,\mathbb{C})/K_{\mathbb{C}}P^+ = \mathbb{CP}^1 \times \mathbb{CP}^1$. This orbit is precisely the set of pairs (ℓ_1, ℓ_2) of distinct complex lines. In terms of homogeneous coordinates we have realized our space $\mathbf{X}$ as

$$\mathbf{X} = \left\{ ([z_0 \colon z_1], [w_0 \colon w_1]) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \colon \det \begin{pmatrix} z_0 & z_1 \\ w_0 & w_1 \end{pmatrix} \neq 0 \right\}. \tag{6.2}$$

Map $\mathfrak{p}^+ \times \mathfrak{p}^-$ into $SL(2,\mathbb{C})/K_{\mathbb{C}}P^- \times SL(2,\mathbb{C})/K_{\mathbb{C}}P^+$ by

$$\left(\left(\begin{smallmatrix} 0 & z \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ w & 0 \end{smallmatrix}\right) \right) \mapsto \Big(\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) K_{\mathbb{C}}P^-, \left(\begin{smallmatrix} 1 & 0 \\ w & 1 \end{smallmatrix}\right) K_{\mathbb{C}}P^+ \Big),$$

and this map is an embedding of $\mathfrak{p}^+ \times \mathfrak{p}^-$ into an open and dense subset of $SL(2,\mathbb{C})/K_{\mathbb{C}}P^- \times SL(2,\mathbb{C})/K_{\mathbb{C}}P^+$ by 6.1. In order to ease the notation and rewrite this embedding in terms of homogeneous coordinates we introduce the map $\xi \colon \mathbb{C}^2 \to \mathbb{CP}^1 \times \mathbb{CP}^1$ given by

$$\xi(z,w) = ([z \colon 1], [1 \colon w])$$

for $(z,w) \in \mathbb{C}^2$. From the description of $\mathbf{X}$ given by (6.2) we are led to define

$$\mathcal{X} = \Big\{ (z,w) \in \mathbb{C}^2 \colon 1 - zw \neq 0 \Big\},$$

and one can think of the restriction $\xi \colon \mathcal{X} \to \mathbf{X}$ as inverse stereographic projection. The partial action of $SL(2,\mathbb{C})$ on $\mathcal{X}$ is then given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} .(z,w) = \left(\frac{\alpha z + \beta}{\gamma z + \delta}, \frac{\gamma + \delta w}{\alpha + \beta w} \right), \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{C}),$$

for all points $(z,w) \in \mathcal{X}$ where this expression makes sense.

Inside $\mathcal{X}$ we find the planar models of (6.1) discussed in the previous sections. Namely, the unit disc

$$\begin{aligned} D &= \Big\{ (z, \overline{z}) \in \mathbb{C}^2 \colon \ |z| < 1 \Big\} \\ &= \xi^{-1}(SU(1,1).o), \end{aligned}$$

the complex plane

$$\begin{aligned} C &= \Big\{ (z, -\overline{z}) \in \mathbb{C}^2 \colon z \in \mathbb{C} \Big\} \\ &= \xi^{-1}(SU(2).o), \end{aligned}$$

and the space

$$\begin{aligned} B &= \Big\{(x,y) \in \mathbb{C}^2 \colon x, y \in \mathbb{R}, xy \neq 1\Big\} \\ &= \xi^{-1}(SL(2,\mathbb{R}).o), \end{aligned}$$

where $(x,y) \in B$ corresponds to the paracomplex number $x\overline{\mathscr{E}} + y\mathscr{E}$. Define a 'metric' on $\mathfrak{X}$ by

$$H_{(z,w)} = \frac{4}{(1-zw)^2} dz \otimes dw.$$

Then each of the constant curvature metrics on D, C and B in $\mathfrak{X}$ are just the restrictions of H (up to a sign on C).

Definition 6.2 (A common kernel) *Define a function* $k_{\mathbb{C}} \colon \mathfrak{X} \to \mathbb{C}$ *by*

$$k_{\mathbb{C}}(z,w) = 1 - zw,$$

and another function $\kappa_{\mathbb{C}}$ *by*

$$\kappa_{\mathbb{C}}[(z_1,w_1),(z_2,w_2)] = \frac{k_{\mathbb{C}}(z_1,w_2)}{k_{\mathbb{C}}(z_2,w_1)} = \frac{1-z_1w_2}{1-z_2w_1},$$

whereever this makes sense.

The restriction of $k_{\mathbb{C}}$ to B, C, and D is the kernels of the past sections up to a power of 2. For points $(z,\overline{z})$ and $(w,\overline{w})$ in D we find

$$\kappa_{\mathbb{C}}[(z,\overline{z}),(w,\overline{w})] = \frac{1-z\overline{w}}{1-\overline{z}w}$$

and similarly

$$\kappa_{\mathbb{C}}[(z,-\overline{z}),(w,-\overline{w})] = \frac{1+z\overline{w}}{1+\overline{z}w}$$

for points in C and

$$\kappa_{\mathbb{C}}[(x,y),(u,v)] = \frac{1-xv}{1-uy}$$

for points in B. A straightforward computation gives

$$1 - \frac{\alpha z+\beta}{\gamma z+\delta}\frac{\gamma+\delta w}{\alpha+\beta w} = (\gamma z+\delta)^{-1}(1-zw)(\alpha+\beta w)^{-1} \tag{6.3}$$

whenever $\alpha\delta - \beta\gamma = 1$. So if

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is an element of $SL(2,\mathbb{C})$, then

$$\begin{aligned} \kappa_{\mathbb{C}}[g(z_1,w_1),g(z_2,w_2)] = (\gamma z_1+\delta)^{-1}(\gamma z_2+\delta)(\alpha+\beta w_1)(\alpha+\beta w_2)^{-1} \\ \cdot \kappa_{\mathbb{C}}[(z_1,w_1),(z_2,w_2)], \end{aligned} \tag{6.4}$$

if the action of g is defined on (z_1, w_1) and (z_2, w_2). The differential

$$dg(z,w)\colon \mathbb{C}^2 \to \mathbb{C}^2$$

of the action of g at (z, w) is given by

$$dg(z,w)\colon (u,v) \mapsto ((\gamma z + \delta)^{-2}u, (\alpha + \beta w)^{-2}v),$$

for $u, v \in \mathbb{C}$. Put together with (6.3), this shows that H is invariant under the action of g. The alternating part of H is the form Ω given as

$$\Omega = \frac{2}{(1-zw)^2} dz \wedge dw,$$

at a point $(z, w) \in \mathfrak{X}$. This form is also invariant under the $SL(2, \mathbb{C})$ action. Define

$$\rho = \frac{zdw - wdz}{1 - zw}, \tag{6.5}$$

which is a complex-valued, holomorphic 1-form on $\mathfrak{X}$ satisfying $d\rho = \Omega$. The parahermitian structure on $\mathbf{X}$ is pulled back under ξ to an $SL(2, \mathbb{C})$-invariant complex $(1, 1)$-tensor field J on $\mathfrak{X}$ given at the origin by the linear map $(z, w) \mapsto (z, -w)$ and everywhere else by the same formula after identifying $T_{(z,w)}\mathfrak{X} = \mathbb{C}^2$ in the usual fashion. Thus J acts on the holomorphic 1-forms by $J(dz) = dz$ and $J(dw) = -dw$. Given a point $(z, w) \in \mathfrak{X}$ we find that

$$\rho_{(z,w)} = J d \log k_{\mathbb{C}}(z, w)$$

for any choice of continuous logarithm of $k_{\mathbb{C}}$ at (z, w).

Lemma 6.3 *Let I be some compact interval and let $\gamma\colon I \to \mathfrak{X}$ be a smooth curve with starting point $p_1 = (z_1, w_1)$ and endpoint $p_2 = (z_2, w_2)$. Suppose that $g \in SL(2, \mathbb{C})$ is an element such that the action of g is defined on all points of γ. Assume furthermore that $\kappa_{\mathbb{C}}(p_1, p_2)$ is defined and non-zero. Then $\kappa_{\mathbb{C}}(gp_1, gp_2)$ is defined and*

$$\exp\left(\int_{g\gamma} \rho - \int_{\gamma} \rho\right) = \frac{\kappa_{\mathbb{C}}(p_1, p_2)}{\kappa_{\mathbb{C}}(gp_1, gp_2)} \tag{6.6}$$

holds.

Proof. It follows from (6.4) that $\kappa_{\mathbb{C}}(gp_1, gp_2)$ is defined and that

$$\frac{\kappa_{\mathbb{C}}(p_1, p_2)}{\kappa_{\mathbb{C}}(gp_1, gp_2)} = \frac{(cz_1 + d)(a + bw_2)}{(cz_2 + d)(a + bw_1)},$$

where we have written g as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Write $\gamma(t) = (\gamma_+(t), \gamma_-(t))$. By our assumption on g, neither of $(c\gamma_+(t) + d)^{-1}$ and $(a + b\gamma_-(t))^{-1}$ vanishes as t runs in I. Therefore, we choose continuous logarithms for $(cz + d)^{-1}$ along γ_+ and for $(a+bz)^{-1}$ along γ_-. Finally, pick a logarithm of k along γ. From (6.3) we may define

$$\log k_{\mathbb{C}}(g\gamma(t)) = \log(c\gamma_+(t)+\delta)^{-1} + \log k_{\mathbb{C}}(\gamma(t)) + \log(a+b\gamma_-(t))^{-1},$$

as a logarithm of $k_{\mathbb{C}}$ along $g\gamma$. Now we compute

$$\begin{aligned}
\int_{g\gamma} \rho &= \int_{\gamma} g^*\rho \\
&= \int_{\gamma} Jdg^* \log k_{\mathbb{C}} \\
&= \int_{\gamma} Jd\log k_{\mathbb{C}} + \int_{\gamma} \left[-\frac{\partial}{\partial z}\log(cz+d)dz + \frac{\partial}{\partial w}\log(a+bw)dw\right] \\
&= \int_{\gamma} \rho + \int_{\gamma} d(\log(a+bw) - \log(cz+d)) \\
&= \int_{\gamma} \rho + \log(a+bw_2) + \log(cz_1+d) - \log(a+bw_1) - \log(cz_2+d),
\end{aligned}$$

and the claim follows. ∎

Lemma 6.4 *Let γ be a segment of a geodesic passing through $(0,0) \in \mathfrak{X}$. Then $\rho(\dot\gamma)$ vanishes everywhere.*

Proof. Geodesics through $(0,0)$ are all of the form $t \mapsto \exp(tX)(0,0)$ for some $X \in \mathfrak{q}$. Write

$$X = \begin{pmatrix} 0 & z \\ w & 0 \end{pmatrix}$$

for some $z, w \in \mathbb{C}$. Then

$$\exp(tX) = \begin{pmatrix} e_1(t) & e_2(t)z \\ e_2(t)w & e_1(t) \end{pmatrix}$$

where $e_1(t) = \sum_{n=0}^{\infty} t^{2n}\frac{(zw)^n}{(2n)!}$ and $e_2(t) = \sum_{n=0}^{\infty} t^{2n+1}\frac{(zw)^n}{(2n+1)!}$ for all $t \in \mathbb{R}$. Thus

$$\exp(tX)(0,0) = \frac{e_2(t)}{e_1(t)}(z,w)$$

whenever $e_1(t)$ is non-zero. So the image of the geodesic is a line through $(0,0)$. Now it is straightforward to see that $\rho(\dot\gamma)$ vanishes along γ. ∎

Definition 6.5 (Antipodal points and null-planes) *Given two distinct lines $(\ell_1,\ell_2) \in \mathbb{CP}^1 \times \mathbb{CP}^1$ we define the antipodal point of (ℓ_1,ℓ_2) to be (ℓ_2,ℓ_1). Let Q_1 and Q_2 denote the stabilizers of ℓ_1 and ℓ_2 for the $SL(2,\mathbb{C})$ action on $\mathbb{CP}^1$. Then we define the null-planes through (ℓ_1,ℓ_2) as the orbits $Q_1(\ell_1,\ell_2) = \{\ell_1\} \times Q_1\ell_2$ and $Q_2(\ell_1,\ell_2) = Q_2\ell_1 \times \{\ell_2\}$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$.*

Remark 6.6 When (ℓ_1, ℓ_2) is considered as a pair $(g_1 K_{\mathbb{C}} P^-, g_2 K_{\mathbb{C}} P^+)$ of cosets, the antipodal point is $(g_2 \varpi K_{\mathbb{C}} P^-, g_1 \varpi K_{\mathbb{C}} P^+)$ where ϖ is the matrix

$$\varpi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which normalizes $K_{\mathbb{C}}$ and conjugates P^+ to P^- and vice versa. In coordinates, the antipodal point of $(z, w) \in \mathfrak{X}$ is (w^{-1}, z^{-1}) when both z and w are non-zero. For a point $(z, -\overline{z}) \in C$ we see that its antipodal point is $(-\overline{z}^{-1}, z^{-1})$ in accordance with our previous use of this term, and similarly for $(x, y) \in B$. If $(\ell_1, \ell_2) = \xi(z, w)$ for some $(z, w) \in \mathfrak{X}$, then the preimages of the null-planes through (ℓ_1, ℓ_2) consist of all points in $\mathfrak{X}$ of the form $(z + \eta, w)$ or $(z, w + \eta)$ for some $\eta \in \mathbb{C}$.

Lastly, if (z_0, w_0) is a point in $\mathfrak{X}$ with both z_0 and w_0 non-zero, then $\kappa_{\mathbb{C}}[(z_0, w_0), (z, w)]$ is defined and non-zero precisely when (z, w) is not of the form $(w^{-1}+\eta, z^{-1})$ or $(w^{-1}, z^{-1}+\eta)$ for some $\eta \in \mathbb{C}$. That is, $\kappa_{\mathbb{C}}[(z_0, w_0), (z, w)]$ is defined and non-zero whenever (z, w) does not lie on one of the null-planes through the antipodal point of (z_0, w_0). ♢

Lemma 6.7 *Let $(\ell_1, \ell_2) \in \mathbf{X}$ be a pair of distinct lines and let N_1, N_2 denote the null-planes through the antipodal point (ℓ_2, ℓ_1). There is at most one point on $N_1 \cup N_2$ which is connected to (ℓ_1, ℓ_2) by a geodesic in $\mathbf{X}$ and this point is (ℓ_2, ℓ_1).*

Proof. Using $SL(2, \mathbb{C})$-invariance it suffices to prove this claim when $(\ell_1, \ell_2) = ([0 : 1], [1 : 0]) = \xi(0, 0)$. In this case $N_1 \cup N_2$ is consists of all pairs of lines of the form $([1 : 0], [\eta : 1])$ or $([1 : \eta], [0 : 1])$ and a geodesic $\gamma \colon \mathbb{R} \to \mathbf{X}$ through $\xi(0, 0)$ may be written as

$$\gamma(t) = ([e_2(t)z : e_1(t)], [e_1(t) : e_2(t)w]), \qquad t \in \mathbb{R},$$

for some $z, w \in \mathbb{C}$ and with e_1, e_2 given as in the proof of Lemma 6.4. If this curve passes through $N_1 \cup N_2$ at some time t_0, then $e_1(t_0) = 0$. But then $\gamma(t_0)$ equals $([1 : 0], [0 : 1])$ which is the antipodal point of $\xi(0, 0)$. ■

Theorem 6.8 *Let $\gamma \colon [a, b] \to \mathfrak{X}$ be a geodesic segment such that γ does not pass through the anitpodal point of $\gamma(a)$. Then*

$$\exp \int_\gamma \rho = \kappa_{\mathbb{C}}(\gamma(a), \gamma(b))^{-1}.$$

Proof. This result will follow from Lemmas 6.3 and 6.4 once we show that there exists a $g \in SL(2, \mathbb{C})$ such that the action of g is defined on all points of γ and $g\gamma(a) = (0, 0)$. Write $\gamma(a) = (z_0, w_0)$ and define g_0 by

$$g_0 = \begin{pmatrix} 1 & -z_0 \\ -w_0(1 - z_0 w_0)^{-1} & (1 - z_0 w_0)^{-1} \end{pmatrix}$$

and observe that g_0 belongs to $SL(2, \mathbb{C})$. Furthermore, g_0 maps (z_0, w_0) to $(0, 0)$. If (z, w) is any other point in $\mathfrak{X}$, then it is easy straightforward to check that $g_0(z, w)$ is defined if and only if $\kappa_{\mathbb{C}}[(z_0, w_0), (z, w)]$ is defined and non-zero. But this holds whenever (z, w) is a point on γ by our assumption and Lemma 6.7. ■

Remark 6.9 Let us apply this theorem to reprove Theorem 3.1. Let $\iota\colon \mathbb{D} \to \mathfrak{X}$ be the embedding $\iota(z) = (z, \overline{z})$ of the unit disc into $\mathfrak{X}$. The map ι is merely the embedding of $SU(1, 1)/U(1)$ into $SL(2, \mathbb{C})/K_{\mathbb{C}}$ written in terms of coordinates given by ξ and hence totally geodesic. So let z_0, z_1 and z_2 be points in $\mathbb{D}$ and consider the oriented geodesic triangle $\Delta = \Delta(z_0, z_1, z_2)$ in $\mathbb{D}$. Rewriting the volume form ω on $\mathbb{D}$ as

$$\omega = \frac{-2i}{(1 - |z|^2)^2} dz \wedge d\overline{z},$$

it becomes clear that the pull-back $\iota^*\Omega$ equals $-i\omega$. Thus

$$\begin{aligned} \int_\Delta \omega &= i \sum_{k=1}^{3} \int_{\gamma_k} \iota^* \rho \\ &= i \sum_{k=1}^{3} \int_{\iota\gamma_k} \rho, \end{aligned}$$

where γ_1, γ_2 and γ_3 are the geodesic segments constituting the sides of Δ. Now we apply Theorem 6.8 and obtain

$$\exp\left(\sum_{k=1}^{3} \int_{\iota\gamma_k} \rho \right) = C(z_0, z_1, z_2)^{-1},$$

with C defined by (3.1). Since C takes values of modulus one, the conclusion is that the area of $\Delta(z_0, z_1, z_2)$ is an argument for $C(z_0, z_1, z_2)$. But this argument vanishes whenever the z_i's are not distinct and hence it follows by continuity that it must be the argument defined by (3.2). ♢

Chapter III

HERMITIAN SYMMETRIC SPACES OF THE NON-COMPACT TYPE

As previously mentioned, the unit disc $\mathbb{D}$ is the basic example of a Hermitian symmetric space of the non-compact type. In [CØ03], the results of §1 were generalized to this class of spaces, expanding on previous work in [DT87]. The main technical tool to be used here is Harish-Chandra's embedding which realizes a Hermitian symmetric spaces of the non-compact type as a bounded symmetric domain.

§7 Structure Theory

In order to be able to explain the results in [CØ03], and because we will need the notation and the technical machinery later, we give an overview of Harish-Chandra's embedding including some detailed structure theory. Proofs for these classical results will be omitted, but can be found in the books [Hel01] and [Sat80] which have slightly different points of view. The article of A. Korányi in [FKK$^+$00] contains a very thorough exposition as well. A good overview of the classical examples of bounded symmetric domains, among many other results, appear in [Wol72].

7.A Bounded Symmetric Domains

Definition 7.1 *A bounded symmetric domain Ω is an open and connected (i.e. a domain) subset of a complex vector space V such that each point $z \in \Omega$ is an isolated fixed point of an involutive holomorphic map $s_z : \Omega \to \Omega$.*

It follows from a classical theorem of H. Cartan that there is at most one involution s_z at $z \in \Omega$. A bounded symmetric domain Ω is a priori a symmetric space in the sense of [Loo69]. However, once a non-trivial Lebesgue measure $d\lambda$ on V is chosen, Ω becomes equipped with a *Bergman kernel* $k : \Omega \times \Omega \to \mathbb{C}$ with the properties

1. For every fixed w the function $k_w(z) = k(z, w)$ is an element of $\mathscr{H}^2(\Omega)$; the space of holomorphic square-integrable functions on Ω.

2. For every $f \in \mathscr{H}^2(\Omega)$ and $w \in \Omega$ we have $f(w) = \int_\Omega f(z)\overline{k_w(z)}d\lambda(z)$.

The existence and uniqueness of a function with these properties follows at once from the fact that pointwise evaluation $\mathrm{ev}_w\colon \mathscr{H}^2(\Omega) \to \mathbb{C}$ given by $\mathrm{ev}_w(f) = f(w)$ is continuous and that $\mathscr{H}^2(\Omega)$ is a closed subspace of $L^2(\Omega)$. Furthermore it may be proved that $k(z,w) = \overline{k(w,z)}$ and that $k(z,z) > 0$. It follows that $k(z,\overline{w})$ is holomorphic. If $\phi\colon \Omega \to \Omega$ is an automorphism, i.e. a holomorphic bijection with holomorphic inverse, the change of variables formula gives

$$k(\phi(z),\phi(w)) = j(\phi,z)^{-1}k(z,w)\overline{j(\phi,w)}^{-1}, \qquad z,w \in \Omega$$

with $j(\phi,z)$ being the Jacobian determinant of ϕ at z. If the measure $d\lambda$ is scaled by a factor c, the Bergman kernel is scaled by c^{-1}.

Picking coordinates $z_1,\ldots,z_N$ on V, we define a tensor H_z for $z \in \Omega$ by

$$H_z = \sum_{i,j} 2\frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log k(z,z) dz_i \otimes d\overline{z_j},$$

and H is independent of the choice of Lebesgue measure. Then we may write $H = h - i\omega$ where h is a Riemannian structure on Ω and induces the *Bergman metric* on Ω. The imaginary part of H is a Kähler form ω and is given by

$$\begin{aligned}\omega_z &= i\partial\overline{\partial}\log k(z,z)\\ &= i\sum_{i,j}\frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log k(z,z) dz_i \wedge d\overline{z_j}\end{aligned}$$

or simply $\omega_z = \frac{1}{2}dd_{\mathbb{C}}\log k(z,z)$ where $d_{\mathbb{C}} = -i(\partial - \overline{\partial})$. Any automorpism $\phi\colon \Omega \to \Omega$ preserves H; that is, $\phi^*H = H$. Thus Ω is a Hermitian symmetric space and it may furthermore be shown to be of the non-compact type. Conversely, the Harish-Chandra embedding shows that any Hermitian symmetric space of the non-compact type is holomorphically isometric to a bounded symmetric domain with the Bergman metric.

7.B Harish-Chandra Realization

Let M be a Hermitian symmetric space of the non-compact type. Let $(\mathfrak{g}_0, \mathfrak{k}_0, \theta)$ denote the associated effective symmetric Lie algebra. We will assume that

$\mathfrak{g}_0$ is simple. We introduce the following notation:

$\mathfrak{k}_0$: 1-eigenspace of θ,
$\mathfrak{p}_0$: (-1)-eigenspace of θ,
J: $\mathrm{ad}_{\mathfrak{p}_0}\mathfrak{k}_0$-invariant complex structure on $\mathfrak{p}_0$,
H_0: unique central element of $\mathfrak{k}_0$ such that $\mathrm{ad}_{\mathfrak{p}_0}H_0 = J$,
$\mathfrak{h}_0$: maximal abelian subspace of $\mathfrak{k}_0$,
$\mathfrak{g}$: complexification of $\mathfrak{g}_0$,
B: Killing form of $\mathfrak{g}$,
$\mathfrak{k}, \mathfrak{p}, \mathfrak{h}$: complex subalgebras of $\mathfrak{g}$ spanned by $\mathfrak{k}_0, \mathfrak{p}, \mathfrak{h}_0$,
$\mathfrak{p}^{\pm}$: $(\pm i)$-eigenspace of $\mathrm{ad}_{\mathfrak{p}}H_0$,
$\mathfrak{u} = \mathfrak{k}_0 + i\mathfrak{p}_0$: compact real form of $\mathfrak{g}$,
τ, σ: conjugations of $\mathfrak{g}$ with respect to $\mathfrak{u}$ and $\mathfrak{g}_0$ respectively,

and it is known that

1. $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$,
2. $\mathfrak{p}^{\pm}$ are abelian,
3. the Riemannian structure on $\mathfrak{p}_0$ coming from M is proportional to the killing form B restricted to $\mathfrak{p}_0 \times \mathfrak{p}_0$,
4. $\mathfrak{g}$ is simple and the center of $\mathfrak{k}_0$ is spanned by H_0.

We let Δ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. From the relations $[\mathfrak{k}, \mathfrak{p}^{\pm}] \subset \mathfrak{p}^{\pm}$ it follows that for every $\alpha \in \Delta$ the root space $\mathfrak{g}^{\alpha}$ is contained in either $\mathfrak{p}^{+}$, $\mathfrak{p}^{-}$, or $\mathfrak{k}$; this partitions the roots into compact and non-compact roots according to wether $\mathfrak{g}^{\alpha}$ lies in $\mathfrak{k}$ or not. By choosing a suitable ordering we can assume that $\mathfrak{p}^{+}$ is the sum of the positive non-compact root spaces.

Definition 7.2 *Two roots $\alpha, \beta \in \Delta$ are strongly orthogonal if neither $\alpha + \beta$ or $\alpha - \beta$ is a root.*

It is possible to choose a maximal set $\Gamma = \{\gamma_1, \dots, \gamma_r\}$ of positive non-compact roots, by taking γ_k to be the lowest compat positive root which is strongly orthogonal to all of $\gamma_1, \dots, \gamma_{k-1}$. Then

$$\mathfrak{a} = \sum_{k=1}^{r} \mathbb{C}(X_{\gamma_k} + X_{-\gamma_k}),$$

is a maximal abelian subspace of $\mathfrak{p}$ for any choice of non-zero elements $X_{\pm\gamma_k} \in \mathfrak{g}^{\pm\gamma_k}$. We make a specific choice of elements $X_{\pm k} \in \mathfrak{g}^{\pm\gamma_k}$ satisfying

$$X_k - X_{-k} \in \mathfrak{u}, \qquad i(X_k + X_{-k}) \in \mathfrak{u}, \qquad [X_k, X_{-k}] = \frac{2}{\gamma_k(H_k)} H_k,$$

where $H_k \in \mathfrak{h}$ satisfying $B(H_k, H) = \gamma_k(H)$ for all $H \in \mathfrak{h}$. In this case, if $\mathfrak{a}$ is defined as above, the space

$$\mathfrak{a}_0 = \sum_{k=1}^{r} \mathbb{R}(X_k + X_{-k}),$$

equals $\mathfrak{a} \cap \mathfrak{p}_0$ and is thus a maximal abelian subspace of $\mathfrak{p}_0$.

Now let G be the simply connected complex Lie group with Lie algebra $\mathfrak{g}$, and let G_0, K_0, K, P^+, P^-, and U denote the analytic subgroups of G with Lie algebras $\mathfrak{g}_0, \mathfrak{k}_0, \mathfrak{k}, \mathfrak{p}^+, \mathfrak{p}^-$, and $\mathfrak{u}$ respectively. We will abuse notation and write σ and τ for the involutions of G whoose differentials at the identity are σ and τ. With the complex structure J and Riemannian structure from M on $\mathfrak{p}_0$, G_0/K_0 is holomorphically isometric to M.

Lemma 7.3 *The following holds:*

1. *The exponential map of G induces a diffeomorphism of $\mathfrak{p}^+$ onto P^+ and similarily for $\mathfrak{p}^-$ and P^-.*

2. *The map $P^+ \times K \times P^- \to G$ given by $(p^+, k, p^-) \mapsto p^+kp^-$ is injective, holomorphic and has a dense open image in G containing G_0.*

3. *G_0KP^- is open and contained in P^+KP^- and $G_0 \cap KP^- = K_0$. Furthermore, KP^- is a parabolic subgroup of G.*

The Harish-Chandra embedding is the holomorphic map $\xi\colon G_0/K_0 \to \mathfrak{p}^+$ given by

$$\exp \xi(gK_0) \in gKP^-, \qquad g \in G_0.$$

The Borel embedding (see [Wol72]) $G_0/K_0 \to G/KP^-$ is then given by $gK_0 \mapsto gKP^-$. The compact group U may be shown to act transitively on G/KP^- with $U \cap KP^- = K_0$; this gives a way of embedding G_0/K_0 into its compact dual U/K_0 as an open subset. Define $\Xi\colon \mathfrak{p}^+ \to G/KP^-$ by

$$\Xi(X) = \exp(X)KP^-, \qquad X \in \mathfrak{p}^+,$$

and note that the Borel embedding is given as the composition of Ξ with the Harish-Chandra embedding ξ.

We consider the domain $D = \xi(G_0/K_0) \subset \mathfrak{p}^+$. Equivalently, D is the preimage under Ξ of the G_0-orbit G_0KP^- in G/KP^-. We use the decomposition $G_0 = K_0 \exp(\mathfrak{a}_0)K_0$ to show that $\mathbb{D}$ is bounded in $\mathfrak{p}^+$. Every $g \in G_0$ may be written as $g = k_1 \exp(Z)k_2$ with $k_1, k_2 \in K_0$ and $Z \in \mathfrak{a}_0$, and since K_0 normalizes P^+ we find

$$\xi(gK_0) = \mathrm{Ad}_{\mathfrak{p}^+}(k_1)\xi(\exp(Z)K_0).$$

If we write $Z = \sum_{k=1}^{r} t_k(X_k + X_{-k})$ with t_k real, then it follows from $SL(2,\mathbb{C})$-computations that

$$\exp(Z) = \exp(X)\exp(H)\exp(Y),$$

where

$$\begin{aligned} X &= \sum_{k=1}^{r} \tanh(t_k) X_k, \\ H &= \sum_{k=1}^{r} -\log(\cosh t_i)[X_k, X_{-k}], \\ Y &= \sum_{k=1}^{r} \tanh(t_k) X_{-k}, \end{aligned}$$

are elements of $\mathfrak{p}^+, \mathfrak{k}$, and $\mathfrak{p}^-$ respectively. Hence

$$\xi(\exp(Z)K_0) = \sum_{k=1}^{r} \tanh(t_k) X_k,$$

so that

$$D = \left\{ \mathrm{Ad}(k) \sum_{k=1} \tanh(t_k) X_k \colon k \in K_0, t_1, \ldots, t_r \in \mathbb{R} \right\}, \tag{7.1}$$

and thus D is bounded. G_0 acts on D as automorphisms, and in particular $\mathrm{Ad}_{\mathfrak{p}^+} \exp(\pi H_0)$ equals $-\mathrm{id}_{\mathfrak{p}^+}$, whence it follows that D is a bounded symmetric domain. What is more, D is star-like with respect to the origin in $\mathfrak{p}^+$ and circular because $\mathrm{Ad}\exp(tH_0)$ acts as multiplication by e^{it}, $t \in \mathbb{R}$.

7.C Polydisc Embedding

Using the $K_0 \exp(\mathfrak{a}_0) K_0$-decomposition of G_0 we saw, (7.1), that D was swept out by K_0 acting on the 'cube' in $\mathfrak{p}^+$ consisting of all elements of the form $\sum_{k=1}^{r} s_k X_k$ with $s_k \in (-1, 1)$. Let

$$\mathfrak{g}_0(\gamma_k) = \mathbb{R}(X_k + X_{-k}) + \mathbb{R}(iX_k - iX_{-k}) + i\mathbb{R}H_k$$

and

$$\mathfrak{g}_0(\Gamma) = \sum_{k=1}^{r} \mathfrak{g}_0(\gamma_k),$$

so that $\mathfrak{g}_0(\Gamma)$ is a sum of commuting subalgebras of $\mathfrak{g}_0$ each of which is isomorphic to $\mathfrak{su}(1,1)$. Let $G_0(\Gamma)$ denote the analytic subgroup of G_0 with Lie algebra $\mathfrak{g}_0(\Gamma)$. This group is a covering group of $PSU(1,1)^r$ and the orbit of $0 \in \mathfrak{p}^+$ is

$$G_0(\Gamma).0 = \left\{ \sum_{k=1} z_k X_k \colon \ |z_k| < 1 \right\},$$

by elementary $SL(2,\mathbb{C})$-computations.

The differential of the Harish-Chandra embedding ξ at $eK_0 \in G_0/K_0$ is given by

$$d_{eK_0}\xi(X) = \frac{1}{2}(X - i[H_0, X]), \qquad X \in \mathfrak{p}_0,$$

and this expression is a real linear, $\mathrm{Ad}(K_0)$-equivariant isomorphism between $\mathfrak{p}_0$ and $\mathfrak{p}^+$. We let $\mathfrak{a}_0^+$ denote the image of $\mathfrak{a}_0$ in $\mathfrak{p}^+$, and we let $\mathfrak{a}^+$ denote the complex subspace spanned by $\mathfrak{a}_0^+$, that is

$$\mathfrak{a}_0^+ = \sum_{k=1}^{r} \mathbb{R}X_k,$$
$$\mathfrak{a}^+ = \sum_{k=1}^{r} \mathbb{C}X_k.$$

Note that

$$D \cap \mathfrak{a}^+ = \left\{ \sum_{k=1} z_k X_k \colon\ |z_k| < 1 \right\} = G_0(\Gamma).0$$

is a product of r unit discs $\mathbb{D}$. The holomorphic embedding $f\colon \mathbb{D}^r \to D$ given by

$$f(z_1, \ldots, z_r) = \sum_{k=1} z_k X_k, \tag{7.2}$$

is called the polydisc embedding. Putting this together yields

Proposition 7.4 (Polydisc Embedding Theorem) *The embedding*

$$f\colon \mathbb{D}^r \to D$$

is totally geodesic. What is more, there exists a surjective homomorphism of $G_0(\Gamma)$ *onto* $PSU(1,1)^r$ *such that* f *is equivariant with respect to the action of* $G_0(\Gamma)$ *on* $\mathbb{D}^r$ *and* D*. For any* $z \in \mathfrak{p}^+$ *there exists a* $k \in K_0$ *such that* $Ad(k)(z)$ *lies in* $\mathfrak{a}_0^+ \cap D$*.*

7.D Šilov Boundary

The action of G_0 extends to the topological closure $\overline{D}$ of D and the decomposition of the boundary ∂D into G_0-orbits was found by Wolf and Korányi in [WK65]. There are r orbits and only one of them is closed; this is the orbit of

$$X^r = X_1 + \cdots X_r \in \partial D,$$

and in fact $G_0(X^r) = \mathrm{Ad}K_0(X^r)$. This is the only G_0-orbit which is also a K_0-orbit. The remaining orbits are the G_0 orbits of

$$X^j := \sum_{k=1}^{r} X_k,$$

for $1 \leq j < r$.

Definition 7.5 *Let $\Omega \subset V$ be a bounded domain in the complex vector space V. The Šilov boundary of Ω is the smallest closed subset S of $\partial\Omega$ such that*

$$\max_{z\in\overline{\Omega}} |f(z)| \leq \max_{s\in S} |f(s)| \,,$$

for every continuous function $f\colon \overline{\Omega} \to \mathbb{C}$ which is holomorphic on Ω.

It is clear from the definition that the Šilov boundary of D is made up of G_0-orbits. This leads to

Theorem 7.6 *The Šilov boundary S of $D \subset \mathfrak{p}^+$ is the G_0-orbit of X^r, i.e $S = G_0(X^r) = AdK_0(X^r)$. In particular, S contains the r-torus $G_0(\Gamma)(X^r)$ consisting of the points $\sum_{k=1}^r \zeta_k X_k$, $|\zeta_k| = 1$.*

7.E Restricted Roots and Cayley Transform

We return to the strongly orthogonal roots $\Gamma = \{\gamma_1, \ldots, \gamma_r\}$ and define

$$x_k = X_k + X, \qquad y_k = Jx_k = iX_k - iX_{-k},$$

which allows us to write $\mathfrak{a}_0$ as the $\mathbb{R}$-span of the x_k and $\mathfrak{g}_0(\gamma_k)$ as the $\mathbb{R}$-span of x_k, y_k and $[x_k, y_k]$. Next write

$$\mathfrak{h}_0^- = [\mathfrak{a}_0, J\mathfrak{a}_0] = \sum_{k=1}^r i\mathbb{R}H_k,$$

and

$$\mathfrak{h}_0+ = \{H \in \mathfrak{h}_0\colon [H, \mathfrak{a}_0] = 0\} \,,$$

so that $\mathfrak{h}_0$ is the orthogonal direct sum of $\mathfrak{h}_0^-$ and $\mathfrak{h}_0^+$. Observe that the strongly orthogonal roots vanish on $\mathfrak{h}_0^+$ and are as such determined by their restriction to $\mathfrak{h}_0^-$. Let $\pi(\Delta)$ denote the set of roots restricted to $\mathfrak{h}_0^-$ with the strongly orthogonal roots identified with their restriction.

Theorem 7.7 *[Restricted Roots] There are only two possibilities for the restricted roots $\pi(\Delta)$:*

1. *The restricted roots are $\left\{\pm\frac{1}{2}\gamma_s \pm \frac{1}{2}\gamma_t\colon 1 \leq s,t \leq r\right\}$, this is the C_r-case,*
2. *or the restricted roots are $\left\{\pm\frac{1}{2}\gamma_s \pm \frac{1}{2}\gamma_t, \frac{1}{2}\gamma_t\colon 1 \leq s,t \leq r\right\}$, the BC_r-case.*

In both cases, the Weyl group of the restricted root system consists of all signed permutations of the strongly orthogonal roots Γ. The multiplicity of γ_k is 1, and the multiplicity of $\frac{1}{2}(\gamma_t \pm \gamma_s)$, $s \neq t$, is a, and the multiplicity of $\frac{1}{2}\gamma_t$ is $2b$.

Given a root $\gamma \in \Gamma$, the Cayley element $c_{\gamma_k} \in U$ is given by

$$c_{\gamma_k} = \exp\left(i\frac{\pi}{4}y_k\right),$$

and the *Cayley transform* is the element

$$c = \prod_{k=1}^{r} c_{\gamma_k} = \exp\left(i\frac{\pi}{4}\sum_{k=1}^{r} y_k\right),$$

or rather, its adjoint action $\mathrm{Ad}(c)$. Another routine $SL(2,\mathbb{C})$-calculation shows that $\mathrm{Ad}(c)$ maps $i\mathfrak{h}_0^-$ to $\mathfrak{a}_0$ while fixing $\mathfrak{h}_0^+$. Theorem 7.7 now gives the structure of the restricted roots $\Delta(\mathfrak{g}_0, \mathfrak{a}_0)$. Now, the order of c is either 4 or 8 and $\sigma c = c^{-1}$, so $\mathrm{Ad}(c^4)$ is an involution (possibly the identity) which preserves $\mathfrak{g}_0$. Furhtermore, $\mathrm{Ad}(c^4)$ commutes with the Cartan involution θ so that the fixed point set $\mathfrak{g}_{0,T} \subset \mathfrak{g}_0$ of $\mathfrak{g}_0$ decomposes

$$\mathfrak{g}_{0,T} = \mathfrak{k}_{0,T} + \mathfrak{p}_{0,T},$$

where $\mathfrak{k}_{0,T} = \mathfrak{k}_0 \cap \mathfrak{g}_{0,T}$ and $\mathfrak{p}_{0,T} = \mathfrak{p}_0 \cap \mathfrak{g}_{0,T}$. The subalgebra $\mathfrak{g}_{0,T}$ contains $\mathfrak{h}_0$ and $\mathfrak{a}_0$. Hence, if $G_{0,T}$ and $K_{0,T}$ denote the subgroups of G_0 corresponding to $\mathfrak{g}_{0,T}$ and $\mathfrak{k}_{0,T}$ respectively, the space $G_{0,T}/K_{0,T}$ is Hermitian symmetric and the inclusion of $G_{0,T}$ into G_0 gives a totally geodesic embedding of $G_{0,T}/K_{0,T}$ into G_0/K_0. Both $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are invariant under $\mathrm{Ad}(c^4)$ and we denote the 1-eigenspaces by $\mathfrak{p}_T^{\pm}$. The image of $G_{0,T}/K_{0,T}$ under the Harish-Chandra embedding is $D_T = D \cap \mathfrak{p}_T^+$.

Proposition 7.8 *The following are equivalent:*

1. *$Ad(c^4)$ is the identity.*
2. *$H_0 = \sum_{k=1}^{r} H_k \in i\mathfrak{h}_0^-$.*
3. *The restricted roots of $\mathfrak{h}_0^-$ form a root system of type C_r.*
4. *The real dimension of the manifold S equals the complex dimension of $\mathfrak{p}^+$.*
5. *D is biholomorphically equivalent to a tube-domain $T = V + iC$, where V is a real vector space and C is a symmetric cone in V.*

If either of the above conditions are fulfilled, we say that G_0/K_0 and D is of tube-type.

It may be shown that $G_{0,T}/K_{0,T} = D_T$ is of the non-compact type and a tube-type domain.

§8 Kernel Functions

We keep the notation introduced in the previous section. The following material is from [Sat80].

Definition 8.1 (The P^+KP^- decomposition) *For any $g \in P^+KP^-$ we define elements g_+, g_0, g_- in P^+, K and P^- respectively by*

$$g = g_+ g_0 g_-$$

i.e. we factorize g into components according to the inverse map $P^+KP^- \to P^+ \times K \times P^-$.

With this definition, we can define the partial action of G on $\mathfrak{p}^+$ by

$$g(z) = (g \exp z)_+,$$

meaning that $g(z)$ is defined whenever $g \exp(z) \in P^+KP^-$. For $g \in G_0$ and $z \in D$, $g(z)$ equals $\xi(g\xi^{-1}(z))$.

Definition 8.2 (Canonical automorphy factor) *For $g \in G$ and $z \in \mathfrak{p}^+$ such that $g(\exp z) \in P^+KP^-$ we define the canonical factor of automorphy $J(g, z)$ by*

$$J(g, z) = (g \exp z)_0.$$

Proposition 8.3 *For each $g \in G$, the Jacobian (i.e. complex linear differential) at $z_0 \in \mathfrak{p}^+$ of the map $z \mapsto g(z)$ is given by $Ad_{\mathfrak{p}^+} J(g, z_0)$. If h is another element of G and both $g(z_0)$ and $h(g(z_0))$ are defined, then $(hg)(z_0)$ is defined and equals $h(g(z_0))$. Furthermore*

$$J(hg, z_0) = J(h, g(z_0))J(g, z_0).$$

Definition 8.4 (Kernel of automorphy) *For $z, w \in \mathfrak{p}^+$ such that*

$$\exp(-\sigma w) \exp(z) \in P^+KP^-$$

we define the canonical automorphy kernel $K(z, w)$ as

$$K(z, w) = ((\exp(-\sigma w) \exp(z))_0)^{-1}, \tag{8.1}$$

i.e. $K(z, w) = J(\exp(-\sigma w), z)^{-1}$.

We list some properties of the kernel:

Proposition 8.5 *The automorphy kernel K satisfies*

1. *$K(z, w)$ is defined for all $z, w \in D$*

2. $K(w,z) = \sigma K(z,w)^{-1}$

3. *If* $g \in G$ *and* $z, w \in \mathfrak{p}^+$ *are elements such that* $g(z), (\sigma g)(w)$ *and* $K(z,w)$ *is defined, then* $K(g(z), (\sigma g)(w))$ *is defined and equals*
$$K(g(z), (\sigma g)(w)) = J(g,z)K(z,w)\sigma J(\sigma g, w)^{-1}. \tag{8.2}$$

4. *If* $z = \sum_{i=1}^r z_i X_i$ *and* $w = \sum_{i=1}^r w_i X_i$ *for complex* z_k *and* w_k *of modulus* < 1*, then*
$$K(z,w) = \exp(\sum_{k=1}^r \log(1 - z_k\overline{w_k})[X_k, X_{-k}]). \tag{8.3}$$

For $g \in G$ and $z \in \mathfrak{p}^+$ such that $g(z)$ is defined, we introduce
$$j(g,z) = \det \mathrm{Ad}_{\mathfrak{p}^+} J(g,z),$$
and if $K(z,w)$ is defined for $z, w \in \mathfrak{p}^+$ we let
$$k(z,w) = \det \mathrm{Ad}_{\mathfrak{p}^+} K(z,w). \tag{8.4}$$
Now if $g \in G_0$ then (8.2) says
$$k(g(z), g(w)) = j(g,z)k(z,w)\overline{j(g,w)}, \tag{8.5}$$
and hence the restriction of $k(z,w)^{-1}$ to $D \times D$ obeys the same transformation rule as the Bergman kernel of D. Since $k(z,0) = 1$ it follows that $k(z,w)^{-1}$ is the Bergman kernel k_D of D with respect to the Lebesgue measure λ on $\mathfrak{p}^+$ normalized by $\lambda(D) = 1$.

If z and w are points in $\mathfrak{p}^+$ we define the *Bergman operator* $b(x,y) \in \mathrm{End}(\mathfrak{p}^+)$ by
$$b(z,w) = \mathrm{id} - \mathrm{ad}[z, \sigma w] + \frac{1}{4}(\mathrm{ad} z)^2(\mathrm{ad}\sigma w)^2, \tag{8.6}$$
where the right-hand side is restricted to $\mathfrak{p}^+$. It is straightforward to check that $b(z,w)$ maps $\mathfrak{p}^+$ to itself.

Proposition 8.6 *Let* z, w *be points in* $\mathfrak{p}^+$ *and suppose that* $K(z,w)$ *is defined. Then*
$$Ad_{\mathfrak{p}^+} K(z,w) = b(z,w) \tag{8.7}$$
holds.

Proposition 8.7 *The Hermitian form on* D *coming from the Bergman kernel is given by*
$$\langle X, Y \rangle_z = -B(AdK(z,z)^{-1}X, \tau Y)$$
for $z \in D$ *and* $X, Y \in T_z D = \mathfrak{p}^+$*. The corresponding* G_0*-invariant Riemannian structure on* $M = G_0/K_0$ *is given by* $\frac{1}{2}B$ *on* $\mathfrak{p}_0$*.*

The corresponding Kähler form ω is given by

$$\omega_z = -i\partial\overline{\partial}\log k(z,z), \qquad z \in D, \tag{8.8}$$

and we have $\omega_z = -\frac{1}{2}dd_{\mathbb{C}}\log k(z,z)$ where $d_{\mathbb{C}} = -i(\partial - \overline{\partial})$.

Remark 8.8 Using the above results, the Bergman kernel $k_{\mathbb{D}}$ of the unit disc is easily computed to be

$$k_{\mathbb{D}}(z,w) = (1 - z\overline{w})^{-2}, \qquad z, w \in \mathbb{D},$$

and hence the Bergman kernel of the polydisc $\mathbb{D}^r$ is

$$k_{\mathbb{D}^r}(z,w) = \prod_{k=1}^{r}(1 - z_k\overline{w_k})^{-2},$$

for $z = (z_1, \dots, z_r)$ and $w = (w_1, \dots, w_r)$ in $\mathbb{D}^r$. Applying the structure of the restricted roots given by Theorem 7.7 we then find

$$k_D(f(z), f(w))^2 = k_{\mathbb{D}^r}(z,w)^p, \qquad z, w \in \mathbb{D}^r,$$

where f is the polydisc embedding (7.2) and $p = (r-1)a + b + 2$ as in [FKK$^+$00, pp 237-238]. Furthermore, there exists an $\mathrm{Ad}(K_0)$-invariant polynomial $h\colon \mathfrak{p}^+ \times \mathfrak{p}^+ \to \mathbb{C}$ such that

$$h(z,w)^p = k(z,w), \qquad z, w \in \mathfrak{p}^+,$$

and h is given explicitly by

$$h(z,w) = \prod_{k=1}^{r}(1 - z_k\overline{w_k}),$$

for $z = \sum_{k=1}^r z_k X_k$ and $w = \sum_{k=1}^r w_k X_k$ in $\mathfrak{a}^+$. We define the *the normalized Bergman kernel* $\widetilde{k_D}(z,w)$ as

$$\widetilde{k_D}(z,w) = h(z,w)^{-2}, \qquad z, w \in D,$$

even though it is not a Bergman kernel in the strict sense of the term. This does not prevent us from defining a Kähler form

$$\widetilde{\omega} = i\partial\overline{\partial}\log\widetilde{k_D}(z,z),$$

on D. Now $\widetilde{\omega} = \frac{2}{p}\omega$, and a computation in [CØ03] proves that the minimal holomorphic sectional curvature of D equipped with $\widetilde{\omega}$ is -1. This explains the use of the word 'normalized'. ◊

§9 Symplectic Area of Geodesic Triangles

Since D is simply connected and $k\colon D \times D \to \mathbb{C}$ has no zeroes and is strictly positive on the diagonal, we may define a continuous logarithm $\log k\colon D\times D \to \mathbb{C}$ satisfying $\log k(z,z) = 0$ for all $z \in D$. The logarithm of the Bergman kernel k_D is thus $-\log k$. Following [Wie04] we introduce the singular cochain

$$\alpha(\gamma) = \arg k_D(\gamma(0),\gamma(1)) + \frac{1}{2}\int_\gamma d_{\mathbb{C}} \log k_D(z,z),$$

where $\gamma\colon [0,1] \to D$ is a C^1 curve segment. We claim that α is a G_0-invariant cochain. To see this let $g \in G_0$ and observe that

$$\int_{g\gamma} d_{\mathbb{C}} \log k_D(z,z) = \int_\gamma d_{\mathbb{C}} g^* \log k_D(z,z),$$

and apply (8.5) to see

$$\begin{aligned} g^* \log k_D(z,z) &= \log k_D(gz,gz) \\ &= -\log|j(g,z)|^2 + \log k_D(z,z), \end{aligned}$$

which, after fixing some continuous logarithm of $j(g,z)$ along γ, gives

$$\begin{aligned} \frac{1}{2}\int_{g\gamma} d_{\mathbb{C}} \log k_D(z,z) - \frac{1}{2}\int_\gamma d_{\mathbb{C}} \log k_D(z,z) &= -\int_\gamma d_{\mathbb{C}} \log|j(g,z)|\, k_D(z,z) \\ &= -\int_\gamma d_{\mathbb{C}} \Re \log j(g,z) \\ &= -\int_\gamma d \Im \log j(g,z) \\ &= \arg j(g,\gamma(0)) - \arg j(g,\gamma(1)). \end{aligned}$$

However (8.5) also says

$$\arg k_D(g\gamma(1), g\gamma(0)) = -\arg j(g,\gamma(1)) + \arg k_D(\gamma(0),\gamma(1)) + \arg j(g,\gamma(0)),$$

using the same argument for $j(g,z)$ along γ, and this shows that $\alpha(g\gamma) = \alpha(\gamma)$. From a cohomolgy point of view, we have proven the following: If Σ is a C^1 simplex in D with vertices z_0, z_1, and z_2, then the G_0-invariant cocycles

$$c_1(\Sigma) = \int_\Sigma \omega$$

and

$$c_2(\Sigma) = -(\arg k_D(z_0,z_1) + \arg k_D(z_1,z_2) + \arg k_D(z_2,z_0))$$

define the same G_0-invariant singular cohomology class, as their difference if the boundary of α.

By an oriented geodesic triangle $\Delta = \Delta(z_0, z_1, z_2)$ in D with vertices z_0, z_1, and z_2 in D we mean the broken geodesic curve consisting of the three geodesic segments connecting the vertices z_0, z_1 and z_2 and traversed in that order. If $\Sigma \subset D$ is a smooth oriented surface with Δ as boundary, we may define the integral $\int_\Delta \omega$ to be $\int_\Sigma \omega$. This is well-defined since ω is closed, and $\int_\Delta \omega$ is called the symplectic area of Δ.

Theorem 9.1 ([DT87],[CØ03]) *Let $\Delta(z_0, z_1, z_2)$ be a geodesic triangle with vertices z_0, z_1, and z_2. Then*

$$\int_{\Delta(z_0,z_1,z_2)} \omega = -(\arg k_D(z_0, z_1) + \arg k_D(z_1, z_2) + \arg k_D(z_2, z_0)). \tag{9.1}$$

Proof. It follows from Stoke's theorem that is suffices to show $\alpha(\gamma) = 0$ whenever $\gamma\colon [0,1] \to D$ is a geodesic segment in D. As α is G_0-invariant it also suffices to consider the case when γ is a geodesic segment through 0. Using a suitable element of K_0, we may further assume that $\dot\gamma(0)$ lies in the totally geodesic subspace $\mathfrak{a}_0^+$. Then $\gamma = f(\sigma_1, \dots, \sigma_r)$ where f is the polydisc embedding (7.2) and $\sigma\colon [0,1] \to \mathbb{D}^r$ is a geodesic segment starting at the origin. It follows that

$$\begin{aligned} d_{\mathbb{C}} \log k_D(\dot\gamma) &= f^*(d_{\mathbb{C}} \log k_D)(\dot\sigma) \\ &= (d_{\mathbb{C}} f^* \log k_D)(\dot\sigma) \\ &= \frac{p}{2} d_{\mathbb{C}} \log k_{\mathbb{D}^r}(\dot\sigma) \\ &= \frac{p}{2} \sum_{k=1}^{r} d_{\mathbb{C}} \log k_{\mathbb{D}}(\dot\sigma_k), \end{aligned}$$

whence it suffices to check that α vanishes in the special case when D is the unit disc. This is a special case of Lemma 6.4. ∎

Remark 9.2 Iin the proof of Theorem 9.1 it would also have been sufficient to prove that

$$\int_{\Delta(0,z_1,z_2)} \omega = -\arg k_D(z, w),$$

for any z and w in D, and this result is stated in a somewhat different form in [DG78]. ◊

Multiplying both sides of (9.1) by $\frac{2}{p}$ leads to

$$\int_{\Delta(z_0,z_1,z_2)} \widetilde{\omega} = -(\arg \widetilde{k_D}(z_0, z_1) + \arg \widetilde{k_D}(z_1, z_2) + \arg \widetilde{k_D}(z_2, z_0)), \tag{9.2}$$

where we use $-2/p \log k(z, w)$ as a logarithm for $\widetilde{k_D}$.

Theorem 9.3 ([DT87],[CØ03]) *The normalized symplectic area is bounded,*

$$\int_{\Delta(z_0,z_1,z_2)} \widetilde{\omega} \in (-r\pi, r\pi)\,,$$

and this bound is optimal in the sense that

$$\inf_{(z_0,z_1,z_2)\in D^3} \int_{\Delta(z_0,z_1,z_2)} \widetilde{\omega} = -r\pi, \qquad \sup_{(z_0,z_1,z_2)\in D^3} \int_{\Delta(z_0,z_1,z_2)} \widetilde{\omega} = r\pi,$$

holds.

This result shows that the normalized Kähler form $\widetilde{\omega}$ defines a bounded continuous 2-cocycle on the group G_0. This topic is discussed in [Wie04] and will not play a role in this thesis.

9.A Ideal Triangles

It is also possible to consider triangles whose vertices lie on ∂D. For the unit disc $\mathbb{D}$, three pairwise distinct points on the boundary ∂D determined a unique oriented ideal triangle with area $\pm\pi$. Since $k_{\mathbb{D}}(z,w)$ is well-defined for distinct points on the boundary it was also possible to extend the continuous argument $\arg k_{\mathbb{D}}(z,w)$ to distinct points $z, w \in \partial\mathbb{D}$. These ideas also work for the bounded symmetric domain D and its boundary ∂D, albeit with some necessary modifications.

Three pairwise distinct points $\zeta_0, \zeta_1, \zeta_2$ on ∂D form an ideal triangle Δ if there exists three geodesics γ_0, γ_1 and γ_2 in D connecting ζ_1, ζ_2 and ζ_3. To be precise, we require that $\zeta_k = \lim_{t\to\infty} \gamma_k(-t)$ and $\zeta_{k+1} = \lim_{t\to\infty} \gamma_k(t)$ with the index k taken mod 3. In general, given three boundary points, these geodesics may neither exists nor be unique. For ζ_0 and ζ_1 to be connected by a geodesic it is necessary that both points lie in the same G_0-orbit on the boundary. However, if ζ_0 and ζ_1 are connected by a geodesic, then $h(\zeta_0, \zeta_1)$ is non-zero.

Definition 9.4 *Two points z and w in $\mathfrak{p}^+$ are said to be transverse if $h(z,w) \neq 0$. This relation is symmetric and will be denoted $z \top w$.*

If we let $\overline{D}^2_\top$ denote the set of transverse pairs in $\overline{D} \times \overline{D}$, then it is clear that $\overline{D}^2_\top$ is star-shaped with respect to 0 and hence that the argument $\arg h(z,w)$ extends to $\overline{D}^2_\top$. Then we may of course also extend $\arg \widetilde{k_D}$ to $\overline{D}^2_\top$. We therefore define the symplectic area of an ideal triangle $\Delta(\zeta_0, \zeta_1, \zeta_2)$ as

$$\int_\Delta \widetilde{\omega} = -(\arg \widetilde{k_D}(\zeta_0,\zeta_1) + \arg \widetilde{k_D}(\zeta_1,\zeta_2) + \arg \widetilde{k_D}(\zeta_2,\zeta_0)),$$

which makes sense because the vertices of an ideal triangle are pairwise transversal.

One way to produce ideal triangles whose symplectic area reach the maximal values of $\pm r\pi$ is to consider the diagonal polydisc embedding $\rho\colon \mathbb{D} \to D$

$$\begin{aligned}\rho(z) &= z\sum_{k=1}^{r} X_k \\ &= f(z,\ldots,z), \qquad z \in \mathbb{D},\end{aligned}$$

which is holomorphic and totally geodesic and extends to a complex linear map of $\mathbb{C}$ into $\mathfrak{p}^+$. Now the image of an ideal triangle $\Delta(\zeta_0,\zeta_1,\zeta_2)$ in $\mathbb{D}$ under ρ is an ideal triangle $\rho(\Delta) = \Delta(\rho(\zeta_0),\rho(\zeta_1),\rho(\zeta_2))$ in D. Using Remark 8.8 it is easy to prove that

$$\int_{\rho(\Delta)} \widetilde{\omega} = \pm r\pi,$$

where the sign depends on the orientation of $\Delta(\zeta_0,\zeta_1,\zeta_2)$. Note that the vertices of $\rho(\Delta)$ lie on the Šilov boundary of D. There is a converse statement for ideal triangles with maximal symplectic area.

Theorem 9.5 ([CØ03]) *Let $\Delta \subset D$ be an ideal triangle with vertices $\zeta_0,\zeta_1,\zeta_2 \in \partial D$, and suppose that $\int_\Delta \widetilde{\omega} = r\pi$. Then it is possible to move Δ with an element of $g \in G_0$ in such a way that*

$$g(\zeta_0) = \rho(1), \qquad g(\zeta_1) = \rho(-1), \qquad g(\zeta_2) = \rho(-i).$$

The stabilizer of the three vertices is a compact subgroup of G_0 whose fixed points are exactly $g^{-1}(\rho(\mathbb{D}))$.

In particular, ζ_0,ζ_1,ζ_2 lie on the Šilov boundary. It follows from [Wie04, Chp. 4,Lemma 4.5] that any triple $(\zeta_0,\zeta_1,\zeta_2)$ or pairwise transverse points ζ_0,ζ_1,ζ_2 on the Šilov boundary S are the vertices of some ideal triangle $\Delta \subset D$. We conclude with a result about ideal triangles with vertices on the Šilov boundary.

Proposition 9.6 ([Wie04]) *As $(\zeta_0,\zeta_1,\zeta_2)$ varies over all triples of pairwise transversal points on S, the corresponding symplectic area $\int_{\Delta(\zeta_0,\zeta_1,\zeta_2)} \widetilde{\omega}$ takes the values $\pi(r-2k)$, $k = 0,1,\ldots,r$, if D is of tube-type, or all values in the interval $[-r\pi, r\pi]$ if D is not of tube-type.*

Chapter IV

HERMITIAN SYMMETRIC SPACES OF THE COMPACT TYPE

We turn our attention to Hermitian symmetric spaces of the compact type with the goal of generalizing the results obtained for the Riemann sphere $\mathbb{CP}^1$ in §2. We will keep the notation from the previous chapter and consider the compact Hermitian symmetric space U/K_0 where U is simply connected, simple, and compact with Lie algebra $\mathfrak{u} = \mathfrak{k}_0 + i\mathfrak{p}_0$. We let $\mathfrak{g} = \mathfrak{u}^{\mathbb{C}}$ denote the complexification and G the associated simply connected complex Lie group. There are complex subalgebras $\mathfrak{k}$, $\mathfrak{p}^+$ and $\mathfrak{p}^-$ of $\mathfrak{g}$ and corresponding subgroups K, P^+, and P^- of G . Furthermore, U may be considered as a subgroup of G and $U/K_0 = G/KP^-$. The map $\Xi\colon \mathfrak{p}^+ \to G/KP^-$ given by $\Xi(x) = \exp(x)KP^-$ is an embedding with dense image.

We will use $\mathfrak{p}^+$ to provided coordinates for almost all of U/K_0 and to prove generalizations of Theorems 4.3 and 4.4.

§10 Polysphere Embedding

We will need the compact analogue of the polydisc embedding. Recall that we picked a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{k}$ and chose a maximal set $\Gamma = \{\gamma_1, \dots, \gamma_r\}$ of strongly orthogonal non-compact positive roots together with representatives $X_{\pm k} \in \mathfrak{g}^{\pm\gamma_k}$ satisfying

$$X_k - X_{-k} \in \mathfrak{u}, \quad i(X_k + X_{-k}) \in \mathfrak{u}, \quad [X_k, X_{-k}] = \frac{2}{\gamma_k(H_k)} H_k,$$

where $H_k \in \mathfrak{h}$ satisfies $B(H_k, H) = \gamma_k(H)$ for all $H \in \mathfrak{h}$. Thus each of the r subalgebras

$$\mathfrak{g}_c(\gamma_k) = \mathbb{R}(X_k - X_{-k}) + i\mathbb{R}(X_k + X_{-k}) + i\mathbb{R}[X_k, X_{-k}]$$

is isomorphic to $\mathfrak{su}(2)$ and any two of them commute. Define

$$\mathfrak{g}_c(\Gamma) = \sum_{k=1}^{r} \mathfrak{g}_c(\gamma_k)$$

and let $G_c(\Gamma)$ denote the corresponding connected subgroup of U. Then $G_c(\Gamma)$ is covered by r copies of $SU(2)$. We define the subspaces $\mathfrak{a}^+$ and $\mathfrak{a}^+_{\mathbb{R}}$ of $\mathfrak{p}^+$ by

$$\mathfrak{a}^+ = \sum_{k=1}^{r} \mathbb{C}X_k,$$

and

$$\mathfrak{a}_{\mathbb{R}}^{+} = \sum_{k=1}^{r} \mathbb{R} X_k.$$

Note that $\mathfrak{a}_0$ and $\mathfrak{a}_{\mathbb{R}}^{+}$ are isomorphic under the restriction of the $\mathrm{Ad}(K_0)$-equivariant $\mathbb{R}$-isomorphism $X \mapsto \frac{1}{2}(X - i\mathrm{ad}(H_0)X)$ between $\mathfrak{p}_0$ and $\mathfrak{p}^{+}$. If $Z = \sum_{k=1}^{r} it_k(X_k + X_{-k})$ is an element of $i\mathfrak{a}_0$ and $t_k \notin \frac{\pi}{2} + \pi\mathbb{Z}$, then it follows from $SL(2,\mathbb{C})$-computations that

$$\exp(Z) = \exp(X)\exp(H)\exp(Y),$$

where

$$X = \sum_{k=1}^{r} \tan(t_k) X_k,$$

$$H = \sum_{k=1}^{r} -\log(\cos t_i)[X_k, X_{-k}],$$

$$Y = \sum_{k=1}^{r} \tan(t_k) X_{-k},$$

are elements of $\mathfrak{p}^{+}, \mathfrak{k}$, and $\mathfrak{p}^{-}$ respectively.

The following theorem from [Wol72] is the compact analogue of the polydisc embedding.

Theorem 10.1 *The orbit of $G_c(\Gamma)$ through $o = eK_0$ is a submanifold of U/K_0 and it is the image of a holomorphic and totally geodesic embedding of the product $(\mathbb{CP}^1)^r$ of r copies of the Riemann sphere. This embedding is equivariant with respect to the covering $SU(2)^r \to G_c(\Gamma)$ and the action of $SU(2)^r$ on $(\mathbb{CP}^1)^r$.*

If we write $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, then the embedding $F\colon (\mathbb{CP}^1)^r \to G/KP^{-}$ is given by

$$F(z_1, \ldots, z_r) = \Xi\Big(\sum_{k=1}^{r} z_k X_k\Big) \tag{10.1}$$

for $z_k \in \mathbb{C}$.

If $A_c = \exp(i\mathfrak{a}_0)$, then we have $U = K_0 A_c K_0$ and the orbit of A_c through $0 \in \mathfrak{p}^{+}$ is $\mathfrak{a}_{\mathbb{R}}^{+}$. We will need this decomposition several times as it allows us to move any pair of points on U/K_0 into the embedded polysphere $G_c(\Gamma).o$.

Lemma 10.2 (Polysphere Lemma) *Let p and q be arbitrary points in U/K_0 and γ any geodesic segment connecting these two points. Then there exist an element of U that maps p, q, and γ into $\Xi(\mathfrak{a}^{+})$.*

Proof. We can use U to move p to $\Xi(0)$ and then use K_0 to make sure the tangent of γ at $\Xi(0)$ lies in $\mathfrak{a}^+$. Thus γ lies in $G_c(\Gamma).o$, whence $q \in G_c(\Gamma).o$ as well. We write $p = F(0, \ldots, 0)$ and $q = F(z_1, \ldots, z_r)$ with $z_i \in \mathbb{C} \cup \{\infty\}$. If we write γ in coordinates as $F(\gamma_1, \ldots, \gamma_r)$ where each γ_k is a geodesic on $\mathbb{CP}^1$ connecting p_k and q_k. Now there is an element of $SU(2)$ which maps p_k, q_k, and γ_k to the 'equator' of $\mathbb{CP}^1$. Repeating this procedure for each of the factors of $(\mathbb{CP}^1)^r$ shows that we can arrange for p, q and γ to lie in $\Xi(\mathfrak{a}^+)$. ■

Corollary 10.3 *Given any two points x and y on $G_c(\Gamma).o$, there exists a geodesic in the embedded polysphere $G_c(\Gamma).o$ which realizes the distance between x and y considered as points in U/K_0.*

Proof. Suppose that $\gamma\colon [0,1] \to U/K_0$ realizes the distance between x and y. By the previous lemma there exists a $u \in U$ such that $u(x)$, $u(y)$, and $u(\gamma(t))$ lie in $G_c(\Gamma).o$. Now, since the embedding is SU^r-equivariant, there exists an element u' of $G_c(\Gamma)$ such that $x = u'u(x)$ and $y = u'u(y)$. It follows that $u'u(\gamma(t))$ is a geodesic in $G_c(\Gamma).o$ which realizes the distance between x and y. ■

§11 The Compact Kernel

We will introduce the compact analogue of the kernel K and show that it carries some information about the geometry of U/K_0. We will make heavy use of the same ideas that worked in the non-compact case. Recall that $\tau\colon \mathfrak{g} \to \mathfrak{g}$ is the conjugation of $\mathfrak{g}$ with respect to the compact real form $\mathfrak{u}$.

Definition 11.1 (Compact kernel of automorphy) *Let z and w be points in $\mathfrak{p}^+$ such that $\exp(-\tau w)\exp(z)$ belongs to $P^+KP^- \subset G$. Then we define the compact automorphy kernel $K_c(z,w)$ as*

$$K_c(z,w) = ((\exp(-\tau w)\exp(z))_0)^{-1}, \tag{11.1}$$

and we denote by $\mathrm{Dom}(K_c)$ the set of pairs $(z,w) \in \mathfrak{p}^+ \times \mathfrak{p}^+$ where K_c is defined.

Proposition 11.2 *The kernel K_c satisfies*

1. *$K_c(z,w) = K(z,-w)$, where K is the kernel of automorphy defined by (8.4).*

2. *For all $(z,w) \in \mathrm{Dom}(K_c)$*

$$Ad_{\mathfrak{p}^+} K_c(z,w) = \mathrm{id}_{\mathfrak{p}^+} - ad_{\mathfrak{p}^+}[z,\tau w] + \frac{1}{4}(adz)^2(ad\tau w)^2|_{\mathfrak{p}^+},$$

3. *If* $(z,w) \in \mathrm{Dom}(K_c)$*, then* $(w,z) \in \mathrm{Dom}(K_c)$ *and*

$$K_c(z,w) = \tau K_c(w,z)^{-1}.$$

4. *If* $z = \sum_{k=1}^r z_k X_k$ *and* $w = \sum_{k=1}^r w_k X_k$ *are points in* $\mathfrak{a}^+$*, then* $(z,w) \in \mathrm{Dom}(K_c)$ *if and only if* $1 + z_k \overline{w_k} \neq 0$ *for each* k*. If this holds, then* $K_c(z,w)$ *is given by*

$$K_c(z,w) = \exp\left(\sum_{k=1}^{r} \log(1 + z_k\overline{w_k})[X_k, X_{-k}]\right) \tag{11.2}$$

regardless of the choice of logarithms.

5. *For every* $z \in \mathfrak{p}^+$ *we have* $(z,z) \in \mathrm{Dom}(K_c)$ *and* $K_c(z,z)$ *is an element of* $\exp(i\mathfrak{k}_0)$*,*

6. *Let* $g \in G$ *and* $(z,w) \in \mathrm{Dom}(K_c)$ *and suppose that both* $g(z)$ *and* $(\tau g)(w)$ *are defined. Then* $(g(z), (\tau g)(w)) \in \mathrm{Dom}(K_c)$ *and the transformation rule*

$$K_c(g(z), (\tau g)(w)) = J(g,z) K_c(z,w) \tau J(\tau g, w)^{-1} \tag{11.3}$$

holds.

7. *As a special case of* 6.*,*

$$K_c(Ad(k)z, Ad(k)w) = k K_c(z,w) k^{-1}$$

for all $k \in K_0$ *and* $(z,w) \in \mathrm{Dom}(K_c)$*.*

Proof. To see that the first statement holds, observe that the two conjugations σ and τ differ by a sign on $\mathfrak{p}^{\mathbb{C}}$. Then 2. follows from the corresponding formula (8.7) for $\mathrm{Ad}_{\mathfrak{p}^+} K(z,w)$. And if $(z,w) \in \mathrm{Dom}(K_c)$, then we write

$$\exp(-\tau w)\exp(z) = p K_c(z,w)^{-1} q$$

for some $p \in P^+$ and $q \in P^-$. Then

$$\begin{aligned} \exp(-\tau z)\exp(w) &= \tau(\exp(-\tau w)\exp(z))^{-1} \\ &= \tau q^{-1} \tau K_c(z,w) \tau p^{-1}, \end{aligned}$$

and now 3. follows since $\tau p \in P^-$ and $\tau q \in P^+$. Strong orthogonality reduces 4. to an $SL(2,\mathbb{C})$ computation.

In order to prove 6., we assume that $(z,w) \in \mathrm{Dom}(K_c)$ and that $g \in G$ is an element such that $g(z)$ and $(\tau g)(w)$ are both defined. Put

$$\begin{aligned} g\exp(z) &= \exp(g(z)) J(g,z) p \\ \tau(g)\exp(w) &= \exp((\tau g)(w)) J(\tau g, w) p' \end{aligned}$$

with $p, p' \in P^-$. Then

$$\begin{aligned}\exp(-\tau w)\exp(z) &= \tau(\tau(g)\exp(w))^{-1} g\exp(z)\\ &= (\tau p')^{-1}\tau J(\tau g, w)^{-1}\exp(-\tau(\tau(g)(w)))\exp(g(z))J(g,z)p.\end{aligned}$$

As $\exp(-\tau w)\exp(z)$ belongs to P^+KP^- we must have

$$P^+K\exp(-\tau w)\exp(z)KP^- \subset P^+KP^-$$

and thus $\exp(-\tau(\tau(g)(w)))\exp(g(z))$ must belong to P^+KP^- as well. Using the fact that K normalizes both P^+ and P^- we compare K-parts and find

$$K_c(z,w)^{-1} = \sigma J(\tau g, w)^{-1} K_c(g(z), (\tau g)(w))^{-1} J(g,z),$$

which leads to the desired identity. Now 5. follows from 7. and (11.2). ∎

The kernel K_c stores some geometric information about the space U/K_0 as shown below. The main tool is Lemma 10.2 together with (11.3) and (11.2), which allows us to simply consider the polysphere $(\mathbb{CP}^1)^r$ where the results of §4 can be applied.

Proposition 11.3 *Let x, y be points in $\mathfrak{p}^+$. $K_c(x,y)$ is defined if and only if there is a unique shortest geodesic curve between $\Xi(x)$ and $\Xi(y)$.*

Proof. Suppose first that $K_c(x,y)$ is defined. It follows from Lemma 10.2 and the transformation rule (11.3) that we may assume that x and y belong to $\mathfrak{a}^+$. So write $x = \sum_{j=1}^r x_j X_j$ and $y = \sum_{j=1}^r y_j X_j$ where x_j and y_j are complex numbers. As $K_c(x,y)$ is defined we have $1 + x_j\overline{y_j} \neq 0$ for each j. This implies, see Proposition 4.1, that there is a unique shortest geodesic path γ_j in $\mathbb{CP}^1$ between x_j and y_j. Hence the path $(\gamma_1, \ldots, \gamma_r)$ is a shortest path between x and y considered as points in $(\mathbb{CP}^1)^r$. Then it follows from Corollary 10.3 that this path is also a shortest path in U/K_0.

On the other hand, if x and y are connected by a unique shortest geodesic path γ in U/K_0, we may once again use Lemma 10.2 and assume that x and y lie in $\mathfrak{a}^+$ and that γ runs in $G_c(\Gamma)K_0$. Thus we view γ as a shortest geodesic in $(\mathbb{CP}^1)^r$ connecting the points $(x_1, \ldots, x_r)$ and $(y_1, \ldots, y_r)$. If $1 + x_j\overline{y_j} = 0$ for some j then γ would not be unique and hence $K_c(x,y)$ is defined.

There is another interpretation of $\mathrm{Dom}(K_c)$ in terms of *cut points*. Given a point x on a Riemannian manifold and a geodesic starting at x, the cut point y is the point on the geodesic where it ceases to be the shortest path to x. On a compact manifold, any geodesic starting at x has a cut point. We write $C(x)$ for the set of cut points of x and $C(x)$ is also called the cut locus of x.

Proposition 11.4 *Let x, y be points in $\mathfrak{p}^+$. Then $K_c(x,y)$ is defined if and only if y is not a cut point of x.*

Proof. Suppose that γ is a geodesic segment between x and y. Then we may once again assume that x and y and γ lie on the polysphere $(\mathbb{CP}^1)^r$. Write $x = (x_1, \dots, x_r)$ and $y = (y_1, \dots, y_r)$ where $x_j, y_j \in \mathbb{C}$. If y is the cut point it follows that $1 + x_j\overline{y_j} = 0$ for at least one j, $1 \leq j \leq r$. Hence $K_c(x, y)$ is not defined. If, on the other hand, y is not the cut point, then $1 + x_j\overline{y_j} \neq 0$ for all $1 \leq j \leq r$ and it follows that $K_c(x, y)$ is defined. ■

Given a point $z \in \mathfrak{p}^+$, we construct an element g_z of $\exp(i\mathfrak{p}_0) \subset U$ such that $g_z(0) = z$ by setting

$$g_z = \exp(z)K_c(z, z)^{\frac{1}{2}} \exp(\tau z), \tag{11.4}$$

where the square root is well-defined since $K_c(z, z) \in \exp(i\mathfrak{k}_0)$. A priori g_z is only an element of G, but in the case $z = \sum_{i=1}^r z_i X_i \in \mathfrak{a}^+$ the element g_z is given explicitly by

$$g_z = \exp\left(\sum_{i=1}^r \frac{z_i}{|z_i|} \arctan(|z_i|)X_i - \frac{\overline{z_i}}{|z_i|} \arctan(|z_i|)X_{-i}\right) \in \exp(i\mathfrak{a}_0),$$

and as the definition of g_z shows that $kg_zk^1 = g_{\mathrm{Ad}(k)z}$ for all $k \in K_0$ we have $g_z \in \exp(i\mathfrak{p}_0)$ in general.

Lemma 11.5 *Let $x \in \mathfrak{p}^+$ be an arbitrary point. Define $g_x \in U$ by* (11.4)*. Then the action of g_x on a point $y \in \mathfrak{p}^+$ is defined if and only if $(-x, y) \in \mathrm{Dom}(K_c)$.*

Proof. Given x and y in $\mathfrak{p}^+$ we calculate

$$g_x \exp(y) = \exp(x)K_c(x, x)^{1/2} \exp(\tau x) \exp(y)$$

and since $\exp(x)K_c(x, x)^{1/2}$ is in P^+K, it follows that $g_x \exp(y)$ belongs to P^+KP^- if and only if $\exp(\tau x)\exp(y)$ belongs to P^+KP^-. In other words $g_x(y)$ is defined if and only if $K_c(y, -x)$ is defined and the claim follows since $(y, -x) \in \mathrm{Dom}(K_c)$ if and only if $(-x, y) \in \mathrm{Dom}(K_c)$. ■

Definition 11.6 (The compact kernel function) *For $(z, w) \in \mathrm{Dom}(K_c)$ we define the compact kernel k_c as*

$$k_c(z, w) = \det Ad_{\mathfrak{p}^+} K_c(z, w), \quad z, w \in \mathfrak{p}^+ \tag{11.5}$$

where det *denotes the complex-valued determinant of the complex linear map $Ad_{\mathfrak{p}^+} K_c(z, w) \colon \mathfrak{p}^+ \to \mathfrak{p}^+$.*

Note that $k_c(z, w)$ is the determinant of $b(z, -w)$ and hence defined on all of $\mathfrak{p}^+ \times \mathfrak{p}^+$. From the transformation rule (11.3) we get the formula

$$k_c(g(z), g(w)) = j(g, z)k_c(z, w)\overline{j(\tau g, w)}, \tag{11.6}$$

for any $g \in G$ and $z, w \in \mathfrak{p}^+$ such that $g(z)$ and $(\tau g)(w)$ are both defined. Recall that $k\colon \mathfrak{p}^+ \times \mathfrak{p}^+ \to \mathbb{C}$ denotes the kernel function defined by (8.4). By Proposition 11.2 we have

$$k_c(z, w) = k(z, -w)$$

for all $z, w \in \mathfrak{p}^+$. Recall the $\mathrm{Ad}(K_0)$-invariant polynomial $h(z, w)$ determined by

$$h(z, w) = \prod_{k=1}^{r} (1 - z_k \overline{w_k})$$

for points $z = \sum_{k=1}^{r} z_k X_k$ and $w = \sum_{k=1}^{r} w_k X_k$ in $\mathfrak{a}^+$. Then, for $z, w \in \mathfrak{a}^+$,

$$k_c(z, w) = h(z, -w)^p$$

with the positive integer $p = a(r-1) + b + 2$ given by the theorem of the restricted roots, 7.7.

There is a U-invariant Kä hler form on $\mathfrak{p}^+$ given by

$$\omega_z = i\partial\overline{\partial} \log k_c(z, z)$$

for $z \in \mathfrak{p}^+$. Furthermore, $\omega = \frac{1}{2} d\rho$ where

$$\rho_z = d_{\mathbb{C}} \log k_c(z, z) \tag{11.7}$$

for $z \in \mathfrak{p}^+$.

§12 *Geodesics and Triangles in* $\mathfrak{p}^+$

Inspired by Proposition 4.1 we define a certain subset $\mathcal{S}$ of $\mathfrak{p}^+ \times \mathfrak{p}^+$. We will consider the pairs of points which, when considered as points in U/K_0, are connected by a unique shortest geodesic segment in U/K_0 and we will furthermore require that this segment is contained in $\Xi(\mathfrak{p}^+)$.

Definition 12.1 *Let $\mathcal{S}$ be the set of all pairs $(x, y) \in \mathfrak{p}^+ \times \mathfrak{p}^+$ for which there exists a unique geodesic segment $\gamma\colon [0,1] \to \mathfrak{p}^+$ such that $\gamma(0) = x$, $\gamma(1) = y$, and such that γ realizes the distance between x and y.*

Proposition 12.2 *Let $(x, y) \in \mathcal{S}$. Then*

1. *$(x, y) \in \mathrm{Dom}(K_c)$.*
2. *The pair (y, x) belongs to $\mathcal{S}$.*
3. *If $k \in K_0$ then $(Ad(k)x, Ad(k)y) \in \mathcal{S}$.*
4. *For any $z \in \mathfrak{p}^+$ we have $(0, z) \in \mathcal{S}$.*

5. *If x and y are connected by $\gamma\colon \mathbb{R} \to U/K_0$, then any pair of points on γ lying between x and y also belong to $\mathcal{S}$.*

Proof. The first statement follows from Proposition 11.3. The second follows since elements of K_0 acts as isometries on all of $\mathfrak{p}^+$. Then 3. follows by reduction to the case $z \in \mathfrak{a}^+$. The fourth statement follows from the definition of $\mathcal{S}$. ∎

It follows that $\mathcal{S}$ is a contractible subset of $\mathfrak{p}^+ \times \mathfrak{p}^+$ with respect to the origin $(0,0)$. Since $\mathcal{S}$ is a subset of $\mathrm{Dom}(K_c)$ it follows that k_c is non-vanishing on $\mathcal{S}$. Thus there is a unique continuous logarithm $\log k_c(z,w)$ defined on $\mathcal{S}$ which is real when $z = w$.

Now we return to the 1-form ρ defined by (11.7). As our main interest will be path integrals of ρ we need the following lemmas.

Lemma 12.3 *Let $\gamma\colon [a,b] \to \mathfrak{p}^+$ be a smooth curve segment, and suppose that $k_c(\gamma(a),\gamma(b))$ is defined. Assume that $g \in U$ is an element such that the action of g is defined on all points of γ, that is, $g\gamma$ is another smooth curve segment in $\mathfrak{p}^+$. Then we have*

$$\frac{k_c(g\gamma(a), g\gamma(b))}{k_c(g\gamma(b), g\gamma(a))} \exp i \int_{g\gamma} \rho = \frac{k_c(\gamma(a), \gamma(b))}{k_c(\gamma(b), \gamma(a))} \exp i \int_{\gamma} \rho \tag{12.1}$$

Proof. Let z be any point on γ. It follows from (11.6) that

$$k_c(g(z), g(z)) = j(g,z) k_c(z,z) \overline{j(g,z)}$$

and hence

$$\log k_c(g(z), g(z)) = \log |j(g,z)|^2 + \log k_c(z,z).$$

Now from $d_{\mathbb{C}} \log k(z,z) = \rho_z$ and the above we see that

$$\begin{aligned} \int_{g\gamma} \rho &= \int_{\gamma} d_c g^* \log k_c(z,z) \\ &= \int_{\gamma} d_c \log |j(g,z)|^2 + \int_{\gamma} d_c \log k(z,z), \end{aligned}$$

and since the action of g is defined along γ we may choose a holomorphic logarithm of $z \mapsto j(g,z)$ along γ. This logarithm, denoted $\log j(g,z)$, has real part $\log |j(g,z)|$ and it follows from the Cauchy-Riemann equations that $d_{\mathbb{C}} \log |j(g,z)| = d\Im \log j(g,z)$. Hence, $d_{\mathbb{C}} \log |j(g,z)|^2 = 2d\Im \log j(g,z)$ and

$$\int_{g\gamma} \rho = 2\Im \log j(g, \gamma(b)) - 2\Im \log j(g,\gamma(a)) + \int_{\gamma} \rho$$

follows. After taking exponentials we obtain

$$\exp i \int_{g\gamma} \rho = \frac{j(g,\gamma(b))}{\overline{j(g,\gamma(b))}} \frac{\overline{j(g,\gamma(a))}}{j(g,\gamma(a))} \exp i \int_{\gamma} \rho. \tag{12.2}$$

Using (11.6) again, we see that

$$\frac{k_c(g\gamma(a), g\gamma(b))}{k_c(g\gamma(b), g\gamma(a))} = \frac{j(g,\gamma(a))}{\overline{j(g,\gamma(a))}} \frac{k_c(\gamma(a),\gamma(b))}{k_c(\gamma(b),\gamma(a))} \frac{\overline{j(g,\gamma(b))}}{j(g,\gamma(b))} \tag{12.3}$$

and upon combining (12.2) with (12.3) we obtain (12.1). ∎

Lemma 12.4 *Let* $\gamma\colon [a,b] \to \mathfrak{p}^+$ *be a geodesic segment passing through* 0*. Then* $\rho_{\gamma(t)}(\dot\gamma(t)) = 0$ *for all* t*.*

Proof. This proof is a variation of the proof of Theorem 9.1 in which a similar statement played a key role. Since $k_c(z,w)$ is $\mathrm{Ad}(K_0)$-invariant, so if ρ and we may therefore assume that γ runs in $\mathfrak{a}^+$. Write $\gamma(t) = \sum_{k=1}^r \gamma_k(t) X_k$ where $\gamma_k\colon [a,b] \to \mathbb{C}$ is a geodesic in the Riemann sphere $\mathbb{CP}^1$. Put

$$k_{\mathbb{C}^r}(z,w) = \prod_{k=1}^r (1 + z_k \overline{w_k})^2$$

for $z = (z_1, \ldots, z_r), w = (w_1, \ldots, w_r)$ in $\mathbb{C}^r$. Then

$$\begin{aligned}(d_{\mathbb{C}} \log k_c)(\dot\gamma) &= \frac{p}{2}(d_{\mathbb{C}} \log k_{\mathbb{C}^r})(\dot\gamma_1, \ldots, \dot\gamma_r)\\ &= \frac{p}{2}\sum_{k=1}^r (d_{\mathbb{C}} \log k_{\mathbb{C}})(\dot\gamma_k)\end{aligned}$$

and the calculation reduces to the situation $\mathbb{CP}^1$. Here each γ_k is a line through the origin and $k_{\mathbb{C}}(z,z) = (1+|z|^2)^2$ and it is straightforward to verify that $(d_{\mathbb{C}} \log k_{\mathbb{C}}(z,z))(\dot\gamma)$ vanishes. ∎

Theorem 12.5 *Let* $(z,w) \in \mathcal{S}$ *and let* $\gamma\colon [0,1] \to \mathfrak{p}^+$ *denote the geodesic segment in* $\mathfrak{p}^+$ *connecting* $z = \gamma(0)$ *and* $w = \gamma(1)$*. Then*

$$\exp \frac{1}{i} \int_{\gamma} \rho = \frac{k_c(z,w)}{k_c(w,z)} \tag{12.4}$$

and

$$\frac{1}{2}\int_{\gamma} \rho = -\arg k_c(z,w), \tag{12.5}$$

where $\arg k_c(z,w)$ *is the imaginary part of the continuous logarithm of* k_c *defined on* $\mathcal{S}$ *by* $\log k_c(0,0) = 0$*.*

Proof. Since (z, w) belongs to $\mathcal{S}$ it follows from Proposition 12.2(5) and Proposition 11.3 that $(z, \gamma(t)) \in \mathrm{Dom}(K_c)$ for all $t \in [0, 1]$. Thus g_{-z} is defined on all points of γ by Lemma 11.5. As $g_{-z}\gamma$ starts at 0 we combine Lemma 12.4 and 12.3 to obtain

$$1 = \frac{k_c(z, w)}{k_c(w, z)} \exp i \int_\gamma \rho,$$

proving (12.4). As $k_c(w, z) = \overline{k_c(z, w)}$ we have

$$\frac{k_c(z, w)}{k_c(w, z)} = \exp 2i \arg k_c(z, w),$$

where arg is the imaginary part of $\log k_c$ defined on $\mathcal{S}$. Since both sides of (12.4) depend continuously on $(z, w) \in \mathcal{S}$ we conclude that (12.5) holds. ■

Now, if we are given a triple (z_0, z_1, z_2) of points in $\mathfrak{p}^+$ such that each of the pairs (z_i, z_j) belong to $\mathcal{S}$, then we may form an oriented geodesic triangle $\Delta = \Delta(z_0, z_1, z_2)$ as follows: The triangle Δ is made up of the three unique shortest geodesic segments connecting the three vertices z_0, z_1, and z_2 with orientation given by traversing the boundary in the order $z_0 \to z_1 \to z_2 \to z_0$. If Σ is a smooth surface in $\mathfrak{p}^+$ with $\Delta = \partial\Sigma$, then $\int_\Sigma \omega$ only depends on the boundary Δ and therefore we will not specify any particular 'interior' of Δ; the triangle is just a broken geodesic loop.

Theorem 12.6 *Let (z_0, z_1, z_2) be a triple of points in $\mathfrak{p}^+$ and suppose that each pair (z_i, z_j) belongs to $\mathcal{S}$. Construct the oriented geodesic triangle $\Delta(z_0, z_1, z_2)$ as above. Then*

$$\int_\Sigma \omega = -\big(\arg k_c(z_0, z_1) + \arg k_c(z_1, z_2) + \arg k_c(z_2, z_0)\big)$$

holds for any smooth surface $\Sigma \subset \mathfrak{p}^+$ with Δ as its boundary.

Proof. We have $\omega = \frac{1}{2}\rho$ so the result follows after an application of Stoke's theorem and (12.5) on each of the three geodesic segments of Δ. ■

In particular, for a geodesic triangle $\Delta(0, z_1, z_2)$ with $(z_1, z_2) \in \mathcal{S}$ we see that the symplectic area of Δ is given by $-\arg k_c(z_1, z_2)$. This result essentially appears in several articles by S. Berceanu, see e.g. [Ber99] and [Ber04]. It is proven by direct calculation for the complex Grassmannian. See also [BS00] where the authors use an embedding of U/K_0 into projective space $\mathbb{CP}^N$ and give an interpretation of the argument of k_c in terms of the symplecitc area of geodesic triangles in the ambient space $\mathbb{CP}^N$. Hangan and Masala [HM94] already proved an exponentiated version of Theorem 12.6.

§13 A Spectral Argument

It would be interesting to give a description of $\mathcal{S}$ similar to the way we introduced this set for $\mathbb{CP}^1$. In the hopes of doing so we will now describe how to construct a set $\mathcal{R} \subset \mathfrak{p}^+ \times \mathfrak{p}^+$ with certain similarities to $\mathcal{S}$.

To begin this construction, let s_j be the j'th elementary symmetric polynomial in r variables, i.e. s_j, $1 \le j \le r$ are determined by the relation

$$\prod_{j=1}^{r}(X-\lambda_j) = \sum_{j=0}^{r}(-1)^j s_j(\lambda_1,\dots,\lambda_r)X^j,$$

in the polynomial ring $\mathbb{C}[X]$ for any $\lambda_1,\dots,\lambda_r \in \mathbb{C}$. Now we define polynomials $h_j\colon \mathfrak{a}^+_{\mathbb{R}} \to \mathbb{R}$ by

$$h_j(z) = s_j(1+t_1^2,\dots,1+t_r^2),$$

where $z = \sum_{k=1}^{r} t_k X_k$ is an arbitrary element of $\mathfrak{a}^+_{\mathbb{R}}$. Chevalley's theorem ensures that h_j is the restriction to $\mathfrak{a}^+_{\mathbb{R}}$ of an $\mathrm{Ad}(K_0)$-invariant real-valued polynomial on $\mathfrak{p}^+$ which we will also denote h_j. We may then polarize h_j to obtain a polynomial $h_j(z,w)$ on $\mathfrak{p}^+ \times \mathfrak{p}^+$ such that $h_j(z,z) = h_j(z)$ for all $z \in \mathfrak{p}^+$. Furthermore, $h_j(z,w)$ is holomorphic and antiholomorphic as a function of z and w, respectively. When $z = \sum_{k=1}^{r} z_k X_k$ and $w = \sum_{k=1}^{r} w_k X_k$ are in $\mathfrak{a}^+$ we have

$$h_j(z,w) = s_j(1+z_1\overline{w_1},\dots,1+z_r\overline{w_r}),$$

and hence

$$\prod_{k=1}^{r}(\mathrm{X}-1-z_k\overline{w_k}) = \sum_{k=0}^{r}(-1)^k h_k(z,w)\mathrm{X}^k,$$

and the right-hand side makes sense for any $(z,w) \in \mathfrak{p}^+ \times \mathfrak{p}^+$. In the case when z,w both belong to $\mathfrak{a}^+$ Proposition 4.1 and Corollary 10.3 say (z,w) lies in $\mathcal{S}$ if and only if all roots of the polynomial $\sum_{k=0}^{r}(-1)^k h_k(z,w)X^k \in \mathbb{C}[X]$ lie in $\mathbb{C} \setminus (-\infty,0]$.

Definition 13.1 *Let $\mathcal{R}$ be the subset of $\mathfrak{p}^+ \times \mathfrak{p}^+$ consisting of all pairs (x,y) such that the polynomial*

$$p_{x,y}(X) = \sum_{k=1}^{r}(-1)^k h_k(z,w)X^k \in \mathbb{C}[X],$$

has all of its roots in $\mathbb{C} \setminus (-\infty,0]$.

As we have just seen,

$$\mathcal{R} \cap (\mathfrak{a}^+ \times \mathfrak{a}^+) = \mathcal{S} \cap (\mathfrak{a}^+ \times \mathfrak{a}^+).$$

Using the $\mathrm{Ad}(K_0)$-invariance of each h_j it is easy to prove that for each $x \in \mathfrak{p}^+$ both (x,x) and $(x,0)$ belong to $\mathscr{R}$. Since $h_j(y,x) = \overline{h_j(x,y)}$ we also see that (x,y) is in $\mathscr{R}$ if and only if (y,x) is in $\mathscr{R}$. It should also be observed that $k_c(x,y) = h_0(x,y)^p$ and hence that there is a continuous logarithm of k_c on $\mathscr{R}$ defined as follows: Let $\lambda_1, \dots, \lambda_r$ be the r roots of $p_{x,y}(X)$ counted with multiplicity. We define

$$\arg h_0(x,y) = \sum_{k=1}^{r} \mathrm{Arg}\lambda_k$$

and

$$\arg k_c(x,y) = p \arg h_0(x,y)$$

using the usual main argument. Since h_0 is the product of r roots, its argument lies in the interval $(-r\pi/2, r\pi/2)$.

The polynomial $p_{x,y}(X)$ appears in [Loo75, §16] as the *generic minimum polynomial* for the Jordan Pair $(\mathfrak{p}^+, \mathfrak{p}^-)$. If we write $y = \tau(y')$ where $y' \in \mathfrak{p}^-$ we have $m(X+1, x, y') = p_{x,y}(X)$. The polynomial $p_{x,y}(X)$ is in some sense determined completely by the maximal tube-type subdomain $\mathfrak{p}_T^+$. Namely, the restriction of Ξ to $\mathfrak{p}_T^+$ gives the Harish-Chandra embedding of $\mathfrak{p}_T^+$ into the compact Hermitian symmetric space $U_T/K_{0,T}$.

Proposition 13.2 *Suppose that $z_T \in \mathfrak{p}_T^+$ lies in the maximal tube-type subdomain and $w \in \mathfrak{p}^+$ is any point. Let $P \colon \mathfrak{p}^+ \to \mathfrak{p}_T^+$ denote the orthogonal projection. Then*

$$p_{z_T,w}(X) = p_{z_T,Pw}(X),$$

holds.

Proof. The coefficients of $p_{x,y}(X)$ are K_0-invariant polynomials on $\mathfrak{p}^+ \times \mathfrak{p}^+$ which are holomorphic and anti-holomorphic in the first and second variable respectively. Now the claim follows from [CØ03, Lemma 1.2]. ∎

13.A Example: Complex Projective Space

Let us compare the sets $\mathscr{R}$ and $\mathscr{S}$ for the complex projective space $\mathbb{CP}^n = SU(n+1)/S(U(n) \times U(1))$ where the rank r is 1. In this case, $\mathfrak{u}$ is $\mathfrak{su}(n+1)$ and $i\mathfrak{p}_0$ consists of all matrices of block form

$$\begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix},$$

where z is an n-dimensional complex column vector and z^* is the transpose conjugate. Then

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} : z \in M_{n,1}(\mathbb{C}) \right\},$$

and if we think of elements of $\mathbb{C}^n$ as column vectors we make the identification

$$\mathbb{C}^n \ni z \mapsto \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix},$$

between $\mathbb{C}^n$ and $\mathfrak{p}^+$. The map Ξ is then given by

$$\Xi(z) = \left[\begin{pmatrix} z \\ 1 \end{pmatrix}\right], \quad z \in \mathbb{C}^n,$$

where, if $w \in \mathbb{C}^{n+1}$, $[w]$ denotes the line $\mathbb{C}w$ in $\mathbb{CP}^n$. Next set $X_1 = (1, 0, \ldots, 0)^t \in \mathbb{C}^n$,

$$\mathfrak{a}^+ = \left\{(z, 0, \ldots, 0)^t \in \mathbb{C}^n \colon z \in \mathbb{C}\right\},$$

and define $h\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ by

$$h(z, w) = 1 + w^* z, \quad z, w \in \mathbb{C}^n,$$

still thinking of z and w as column vectors. Now h is invariant under the adjoint action of $K_0 = S(U(n) \times U(1))$ on $\mathbb{C}^n$ and $h(xX_1, yX_1) = 1 + x\overline{y}$ for $x, y \in \mathbb{C}$. Hence $\mathcal{R}$ consists of those pairs $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$ where $1 + w^* z \in \mathbb{C} \setminus (-\infty, 0]$.

Let us show that $\mathcal{R} = \mathcal{S}$. Let γ be a geodesic through 0 in $\mathbb{C}^n$. Then γ is a line through 0 and is given by

$$\gamma(t) = \frac{\tan(\|z\|t)}{\|z\|} z,$$

for some $z \in \mathbb{C}^n$ and for t in the interval $I = (-\frac{\pi}{2\|z\|}, \frac{\pi}{2\|z\|})$. Now if w is an arbitrary element in $\mathbb{C}^n$ and $g_w \in SU(n+1)$ is defined by (11.4), then $(g_w(\gamma(t_0)), w)$ belongs to $\mathcal{S}$ if and only if the action of g_w is defined on $\gamma(t)$ for all $t \in [0, t_0]$[1]. In this setting FiXme Note!

$$g_w = \begin{pmatrix} \sqrt{I_n + ww^*}^{-1} & \sqrt{1 + \|w\|^2}^{-1} w \\ -\sqrt{1 + \|w\|^2}^{-1} w^* & \sqrt{1 + \|w\|^2}^{-1} \end{pmatrix} \in SU(n+1),$$

so that

$$g_w(\gamma(t)) = \frac{\sqrt{1 + \|w\|^2}}{1 - w^*\gamma(t)} (\sqrt{I_n + ww^*}^{-1} \gamma(t) + \sqrt{1 + \|w\|^2}^{-1} w)$$

for all $t \in I$ where $w^*\gamma(t) \neq 1$. Hence

$$\begin{aligned} h(g_w(\gamma(t)), w) &= 1 + \frac{w^*\gamma(t) + \|w\|^2}{1 - w^*\gamma(t)} \\ &= \frac{1 + \|w\|^2}{1 - w^*\gamma(t)}, \end{aligned}$$

[1] FiXme Note: this could be a general lemma

and we may argue as follows: If $h(g_w(\gamma(t_0)), w) \notin (-\infty, 0]$ for some $t_0 \geq 0$ then $1 - w^*\gamma(t_0) \notin (-\infty, 0]$. But as we have $w^*\gamma(t) = \|z\|^{-1}\tan(\|z\|t)w^*z$, this implies that $1 - w^*\gamma(t) \notin (-\infty, 0]$ for all $t \in [0, t_0]$, in particular $g_w(\gamma(t))$ is defined for $t \in [0, t_0]$. If, on the other hand, $g_w(\gamma(t))$ is defined for all t in $[0, t_0]$, then $1 - w^*\gamma(t)$ does not vanish. Since $1 - w^*\gamma(0) = 1$ we see that $1 - w^*\gamma(t)$ avoids $(-\infty, 0]$ for $t \in [0, t_0]$ and hence so does $h(g_w(\gamma(t_0)), w)$.

From Proposition 13.2 together with the above we learn the following: Suppose z and w are two points in $\mathbb{C}^n$ and w' is the orthogonal projection of w onto z. Then (z, w) belongs to $\mathcal{S}$ if and only if (z, w') belongs to $\mathcal{S}$.

13.B The Other Rank 1 Space

We consider the space $SO(6)/U(3)$. It was already known to E. Cartan that this space is isomorphic to $\mathbb{CP}^3$, see [Hel01, p. 519]. Indeed, $SU(4)$ is a double cover of $SO(6)$ and $S(U(1) \times U(3))$ is isomorphic to $U(3)$. We give the calculation that $\mathcal{R} = \mathcal{S}$ nonetheless.

In geometric terms $SO(6)/U(3)$ is the space of all 3-dimensional complex subspaces of $\mathbb{C}^6$ that are isotropic under a fixed symmetric bilinear form S. It is convenient to assume that

$$S(z, w) = \sum_{k=1}^{3} (z_k w_{6+k} + w_k z_{6+k}), \quad z, w \in \mathbb{C}^6$$

rather than taking the usual symmetric form on $\mathbb{C}^n$. Then S has matrix

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $G = SO(6, \mathbb{C})$ is realized as the group

$$SO(6, \mathbb{C}) = \Big\{ g \in SL(6, \mathbb{C}) \colon g^t S g = S \Big\}$$

for which $G \cap SU(6)$ is a maximal compact subgroup isomorphic to $SO(6)$. With these choice we have

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} : z \in M_3(\mathbb{C}), z^t = -z \right\}$$

and we will identify $\mathfrak{p}^+$ with the complex vector space $A_3(\mathbb{C})$ of skew-symmetric complex 3×3 matrices. The isotropy subgroup $U(3)$ acts on $A_3(\mathbb{C})$ by

$$U.z = UzU^t$$

for $U \in U(3)$ and $z \in A_3(\mathbb{C})$. The function

$$h(z, w) = 1 + \frac{1}{2}\mathrm{tr}(zw^*), \quad z, w \in A_3(\mathbb{C}),$$

is invariant under this action. We put

$$E = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in A_3(\mathbb{C}),$$

and define

$$\mathfrak{a}^+ = \{zE \colon z \in \mathbb{C}\},$$

and observe that

$$h(zE, wE) = 1 + z\overline{w},$$

for $z, w \in \mathbb{C}$. Thus $\mathscr{R}$ consists of those pairs matrices z and w for which $1 + \frac{1}{2}\mathrm{tr}(zw^*)$ lies in $\mathbb{C} \setminus (-\infty, 0]$. We will prove that $\mathscr{R} = \mathscr{S}$ using the same approach that worked for $\mathbb{CP}^n$.

Let γ be a geodesic through 0. There is no loss of generality in assuming that γ runs in $\mathfrak{a}^+_{\mathbb{R}}$ and is, possibly after reparametrization, given by

$$\gamma(t) = \tan(t)E$$

where t runs in the interval $I = (-\frac{\pi}{2}, \frac{\pi}{2})$. If $w = (w_{ij}) \in A_3(\mathbb{C})$ is any matrix, then $g_w \in SO(6, \mathbb{C}) \cap SU(6)$ is given as

$$g_w = \begin{pmatrix} \sqrt{I_n + ww^*}^{-1} & \sqrt{1 + \|w\|^2}^{-1} w \\ -\sqrt{1 + \|w\|^2}^{-1} w^* & \sqrt{1 + \|w\|^2}^{-1} \end{pmatrix},$$

and the action $g_w(\gamma(t))$ is defined for those $t \in I$ where $1 - w^*\gamma(t)$ is invertible. Now some elementary matrix computations show that

$$h(g_w(\gamma(t)), w) = \frac{1 + \mathrm{tr}(ww^*)}{1 - \frac{1}{2}\mathrm{tr}(\gamma(t)w^*)}$$

and that the denominator is a square root of the determinant of $1 - \gamma(t)w^*$. Since

$$1 - \frac{1}{2}\mathrm{tr}(\gamma(t)w^*) = 1 - \tan(t)\overline{w_{12}}$$

we can argue just as in the previous example that $h(g_w(\gamma(t), w)$ is negative for some $t \in I$ if and only if $1 - \frac{1}{2}\mathrm{tr}(\gamma(t_0)w^*)$ vanishes at some t_0 between 0 and t.

Chapter V

PARAHERMITIAN SYMMETRIC SPACES

S. Kaneyuki has studied semisimple parahermitian symmetric spaces in great detail in [Kan03], [Kan87], and [Kan85]. The article [KK85] with M. Kozai is a well-written introduction to these spaces. Using ideas found in these papers we will give a generalization of the results obtained in §2 for the two-dimensional hyperboloid of one sheet.

§14 A Parahermitian Kernel Function

Let $M = G/H$ be a semisimple parahermitian symmetric space. A canonical kernel function for M is introduced and studied in [Kan85] as well as a generalized Borel embedding for M. To M corresponds an effective semisimple symmetric triple $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ where $\mathfrak{g}$ is a semisimple Lie algebra with an involution σ whose fixed points in $\mathfrak{g}$ is exactly $\mathfrak{h}$. Furthermore, $\mathfrak{h}$ contains a unique element Z_0 such that the centralizer of Z_0 in $\mathfrak{g}$ is $\mathfrak{h}$ and that $\mathrm{ad}(Z_0)$ is semisimple with $0, \pm 1$ as eigenvalues. Now M is a symmetric coset space G/H where

1. G is a connected Lie group with Lie algebra $\mathfrak{g}$ such that σ extends to an involution of G, and

2. H is an open subgroup of $C(Z_0) \cap G^\sigma$ where G^σ is the group of fixpoints of σ and $C(Z_0)$ is the centralizer of Z_0 in G.

Two examples to keep in mind are the hyperboloid $SL(2,\mathbb{R})/R^*$, which we have already met, and its double cover $SL(2,\mathbb{R})/R^*_{>0}$.

We let $\mathfrak{m}^\pm$ denote the (± 1)-eigenspaces of $\mathrm{ad}(Z_0)$, hence

$$\mathfrak{g} = \mathfrak{m}^+ + \mathfrak{h} + \mathfrak{m}^-,$$

is a direct sum decomposition of $\mathfrak{g}$ and $\mathfrak{q} = \mathfrak{m}^+ + \mathfrak{m}^-$ is the (-1)-eigenspace of σ. Now the closed and simply connected subgroups $\exp \mathfrak{m}^\pm$ are normalized by H, so we define

$$U^\pm = H \exp \mathfrak{m}^\pm,$$

which are parabolic subgroups of G. It is well-known that $U^+ \cap U^- = H$. Furthermore, the multiplication map

$$\exp \mathfrak{m}^+ \times H \times \exp \mathfrak{m}^- \to G$$

is everywhere regular and has open image in G, and similarly if one interchanges the $\exp \mathfrak{m}^{\pm}$ factors. We will let $\Omega = \exp \mathfrak{m}^{+} H \exp \mathfrak{m}^{-}$ and $\Omega' = \exp \mathfrak{m}^{-} H \exp \mathfrak{m}^{+}$ denote the open and dense subsets. If $g \in \Omega$ then g may be written in a unique way as a product $g = g_{+} g_0 g_{-}$ where $g_0 \in H$ and $g_{\pm} \in \exp \mathfrak{m}^{\pm}$. If g lies in Ω' we write $g = g'_{-} g'_0 g'_{+}$ for the corresponding decomposition.

Consider now the compact homogeneous spaces $M^{\pm} = G/U^{\pm}$. Let $o^{\pm}$ denote the identity cosets in $M^{\pm}$ and consider the product $M^{-} \times M^{+}$. The group G acts on this product by acting simultaneous on each factor and the G-orbit of (o^{-}, o^{+}) is G/H. Thus $M^{-} \times M^{+}$ becomes a G-equivariant compactification of M. Next we define embeddings

$$\xi_1 \colon \mathfrak{m}^{+} \to M^{-}, \qquad \xi_2 \colon \mathfrak{m}^{-} \to M^{+},$$

by

$$\xi_1(x) = \exp(x).o^{-}, \qquad x \in \mathfrak{m}^{+},$$
$$\xi_2(y) = \exp(y).o^{+}, \qquad y \in \mathfrak{m}^{-}.$$

And it follows from the preceeding discussion that both ξ_1 and x_2 have open images. These embeddings define partial actions of G on $\mathfrak{m}^{\pm}$. If $x \in \mathfrak{m}^{+}$ and $g \in G$ are such that $g \exp(x) \in \Omega$, then $g(x) \in \mathfrak{m}^{+}$ is defined by

$$\exp g(x) \in g \exp(x) U^{-},$$

or equivalently by $g(x) = \xi_1^{-1}(g \xi_1(x))$. The partial action of G on $\mathfrak{m}^{-}$ is defined in similar fashion by $g(y) = \xi_2^{-1}(g \xi_2(y))$ for all $g \in G$ and $y \in \mathfrak{m}^{-}$ such that $g \xi_2(y)$ lies in the image of ξ_2.

Remark 14.1 Carrying out this construction for $SL(2,\mathbb{R})/\mathbb{R}^{*}$ gives an embedding into two copies of real projective space $\mathbb{RP}^1 \times \mathbb{RP}^1$. The open orbit of $SL(2,\mathbb{R})$ are all pairs of distinct lines. The covering space $SL(2,\mathbb{R})/\mathbb{R}^{*}_{>0}$ is embedded into two copies of the space of rays in $\mathbb{R}^2$ emanating at the origin. The orbit of $SL(2,\mathbb{R})$ are the pairs of distinct rays. ♢

Definition 14.2 (Automorphy factors) *Let $x \in \mathfrak{m}^{+}$ and $g \in G$. Suppose that $g(x)$ is defined. Then we define the canonical automorphy factor $J^{+}(g,x) \in H$ by*

$$J^{+}(g,x) = (g \exp(x))_0. \tag{14.1}$$

If $y \in \mathfrak{m}^{-}$ is an element such that $g(y)$ is defined, we define $J^{-}(g,y) \in H$ by

$$J^{-}(g,y) = (g \exp(y))'_0, \tag{14.2}$$

and we define $j^{+}(g,x)$ and $j^{-}(g,y)$ by

$$j^{+}(g,x) = \det Ad_{\mathfrak{m}^{+}} J^{+}(g,x),$$
$$j^{-}(g,y) = \det Ad_{\mathfrak{m}^{+}} J^{-}(g,y),$$

for all g and where $J^{+}(g,x)$, $J^{-}(g,y)$ are defined.

Remark 14.3 The adjoint action of H leaves each $\mathfrak{m}^{\pm}$ invariant and hence $\det \mathrm{Ad}_{\mathfrak{m}^{\pm}}$ defines characters of H. It follows from $\mathfrak{h} = [\mathfrak{m}^+, \mathfrak{m}^-]$ that these two characters are inverses of each other. ♢

Proposition 14.4 *Let $x \in \mathfrak{m}^+$ and $g \in G$. Suppose that $g(x)$ is defined. Then the action of g maps a neighborhood of x into $\mathfrak{m}^+$ and its differential $d_x g \colon \mathfrak{m}^+ \to \mathfrak{m}^+$ is given by*

$$d_x(g) = Ad_{\mathfrak{m}^+} J^+(g, x).$$

If the action of g is defined on some $y \in \mathfrak{m}^-$, then the differential $d_y g \colon \mathfrak{m}^- \to \mathfrak{m}^-$ is given by

$$d_y g = Ad_{\mathfrak{m}^-} J^-(g, y).$$

Proof. The proof is the same as the one given in [Sat80, page 65]. ■

Taking both ξ_1 and ξ_2 together we obtain an embedding

$$\xi \colon \mathfrak{m}^+ + \mathfrak{m}^- \to M^- \times M^+,$$

defined by

$$\xi(z^+ + z^-) = (\xi_1(z^+), \xi_2(z^-)), \qquad z^{\pm} \in \mathfrak{m}^{\pm}.$$

The image of ξ is open and dense, but does not in general contain the orbit $G.(o^-, o^+)$. In fact, $g(o^-, o^+)$ lies in the image of ξ if and only if g lies in both $\exp \mathfrak{m}^+ H \exp \mathfrak{m}^-$ and $\exp \mathfrak{m}^- H \exp \mathfrak{m}^+$. We put

$$\mathscr{M} = \left\{ z = z^+ + z^- \in \mathfrak{m}^+ + \mathfrak{m}^- \colon (\xi_1(z^+), \xi_2(z^-)) \in G(o^-, o^+) \right\}, \qquad (14.3)$$

and the set $\mathscr{M}$ is the truncated orbit of $0 \in \mathfrak{m}^+ + \mathfrak{m}^-$ under G.

Definition 14.5 (Parahermitian automorphy kernel) *Let $z = z^+ + z^-$ be an element of $\mathfrak{m}^+ + \mathfrak{m}^-$ and suppose that $\exp(-z^-)\exp(z^+)$ is in Ω. Then we define the automorphy kernel $K(z) \in H$ by*

$$K(z)^{-1} = (\exp(-z^-)\exp(z^+))_0. \qquad (14.4)$$

The kernel function $k(z)$ is then given by

$$k(z) = \det Ad_{\mathfrak{m}^+} K(z) \qquad (14.5)$$

for all $z \in \mathfrak{m}^+ + \mathfrak{m}^-$ where $K(z)$ is defined. Lastly, we define the 'mixed kernel' $\kappa(z, w) \in H$ as

$$\kappa(z, w) = k(z^+ + w^-) k(w^+ + z^-)^{-1}, \qquad (14.6)$$

for all points $z = z^+ + z^-$ and $w = w^+ + w^-$ where the right hand side is defined.

Proposition 14.6 *Let $z = z^+ + z^-$ be an element of $\mathfrak{m}^+ + \mathfrak{m}^-$ and let $g \in G$. If $g(z)$ and $K(z)$ are both defined, then $K(g(z))$ is defined and*

$$K(g(z)) = J^+(g, z^+)K(z)J^-(g, z^-)^{-1}, \tag{14.7}$$

and, as a consequence,

$$k(g(z)) = j^+(g, z^+)k(z)j^-(g, z^-)^{-1}, \tag{14.8}$$

holds.

Proof. Write

$$g\exp(z^+) = \exp g(z^+)J^+(g, z^+)p^-,$$
$$g\exp(z^-) = \exp g(z^-)J^-(g, z^-)p^+,$$

where $p^\pm \in \exp \mathfrak{m}^\pm$. Then

$$\exp(-z^-)\exp(z^+) = (p^+)^{-1}J^-(g, z^-)^{-1}\exp(-g(z^-))\exp(g(z^+))p^-,$$

where $\exp(-z^-)\exp(z^+) \in \Omega$ by assumption. Hence $\exp(-g(z^-))\exp(g(z^+))$ is in Ω and the H factor is

$$(\exp(-g(z^-))\exp(g(z^+)))_0 = J^-(g, z^-)K(z)^{-1}J^+(g, z^+)^{-1},$$

and (14.7) follows immediately. ∎

Definition 14.7 *For $z = z^+ + z^-$ in $\mathfrak{m}^+ + \mathfrak{m}^-$ we define the Bergman operator $b(z) \in \mathrm{End}(\mathfrak{m}^+)$ by*

$$b(z) = \mathrm{id} - ad_{\mathfrak{m}^+}[z^+, z^-] + \frac{1}{4}(adz^+)^2(adz^-)^2_{|\mathfrak{m}^+},$$

so that $b(z)$ is an endomorphism of $\mathfrak{m}^+$.

Lemma 14.8 *If z such that $K(z)$ is defined, then*

$$b(z) = Ad_{\mathfrak{m}^+}K(z),$$

holds. In particular

$$\det b(z) = k(z)$$

and we may therefore extend the kernel function k to all of $\mathfrak{m}^+ + \mathfrak{m}^-$.

Proof. This follows by the same argument as in [Sat80]. ∎

§15 Parahermitian Geometry

The space $M = G/H$ has a pseudoriemannian structure Q given by

$$Q_o(X,Y) = \frac{1}{2}B(X,Y), \qquad X,Y \in \mathfrak{q},$$

at the base point $o = eH$ and under the usual identification $T_oM = \mathfrak{q}$. The associated parakähler form ω is given by

$$\omega(X,Y) = \frac{1}{2}B(X,[Z_0,Y]), \qquad X,Y \in \mathfrak{q},$$

at the point $o \in M$. Taking the pull-back of ω under $\xi\colon \mathcal{M} \to G(o^-,o^+) = G/H$ yields a 2-form on $\mathcal{M}$ and we shall also refer to this form as ω.

Let $z \in \mathcal{M}$. Then (14.7) implies that $K(z)$ is defined since $z = g(0)$ for some $g \in G$. Hence $k(z)$ is defined and non-zero. For $X \in \mathfrak{q}$ we let D_X denote the directional derivative, i.e.

$$D_X f(p) = \lim_{t\to 0} \frac{f(p+tX) - f(p)}{t} \qquad p \in \mathfrak{q},$$

for a differentiable function $f\colon \mathfrak{q} \to \mathbb{C}$. Consider then, as an analogue of a Hermitian form, the form

$$H_z(U,V) = 2D_{U^+}D_{V^-}\log k(z), \qquad U,V \in \mathfrak{q},$$

where $\log k(z)$ is a logarithm of k near z; H_z does not depend on the particular choice. Suppose that $U = U^+ \in \mathfrak{m}^+$ and $V = V^- \in \mathfrak{m}^-$. Then we have at $z = 0$

$$\begin{aligned} H_0(U,V) &= 2\lim_{t,s\to 0} \frac{\log\det(\mathrm{id} - (st)\mathrm{ad}_{\mathfrak{m}^+}[U,V] + \frac{(st)^2}{4}(\mathrm{ad}U)^2(\mathrm{ad}V)^2_{|\mathfrak{m}^+})}{st} \\ &= -2\mathrm{tr}\,(\mathrm{ad}_{\mathfrak{m}^+}[U,V]) \\ &= -B(U,V), \end{aligned}$$

and it follows that for general $U,V \in \mathfrak{q}$ we have

$$H_0(U,V) = -B(U^+,V^-) = -(Q_o(U,V) - \omega_o(U,V)), \qquad (15.1)$$

since both $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are isotropic subspaces under B.

Proposition 15.1 *The form H is invariant under the partial action of G and is given by*

$$H_z(U,V) = -B(Ad_{\mathfrak{m}^+}K(z)^{-1}(U^+),V^-), \qquad U,V \in \mathfrak{q}, \qquad (15.2)$$

at the point $z \in \mathcal{M}$.

Proof. Let $z \in \mathcal{M}$ and $g \in G$ and assume that $g(z)$ is defined. Let $U, V \in \mathfrak{q}$ be tangent vectors at z. Since g acts as a parahermitian transformation we have $d_z g(U)^{\pm} = d_z g(U^{\pm})$. Thus

$$\begin{aligned} H_{g(z)}(d_z g(U), d_z g(V)) &= D_{d_z g(U^+)} D_{d_z g(V^-)} \log k(g(z)) \\ &= D_{U^+} D_{V^-} (g^* \log k)(z) \end{aligned}$$

by the chain rule. It follows from the transformation rule (14.8) that

$$D_{U^+} D_{V^-} (g^* \log k)(z) = D_{U^+} D_{V^-} \log k(z) = H_z(U, V)$$

and this shows the invariance of H.

By (15.1), H_0 provides a perfect pairing of $\mathfrak{m}^+$ with $\mathfrak{m}^-$. As each $z \in \mathcal{M}$ is of the form $z = g(0)$ for some $g \in G$, it follows from the invariance that H_z provides a perfect pairing of $\mathfrak{m}^+$ and $\mathfrak{m}^-$. Hence we may write

$$H_z(U^+, V^-) = -B(\Phi(z)U^+, V^-), \qquad U^+ \in \mathfrak{m}^+, V^- \in \mathfrak{m}^-,$$

where $\Phi(z)$ is a uniquely determined linear endomorphism of $\mathfrak{m}^+$ where $\Phi(0) = \mathrm{id}$. Proposition 14.4 says that

$$d_z g(U^+) = \mathrm{Ad} J^+(g, z^+)(U^+),$$

and similarly $d_z g(V^-) = \mathrm{Ad} J^-(g, z^-)(V^-)$. Therefore, the invariance of H forces

$$\Phi(g(z)) = \mathrm{Ad}_{\mathfrak{m}^+} J^-(g, z^-)^{-1} \Phi(z) \mathrm{Ad}_{\mathfrak{m}^+} J^+(g, z^+) \tag{15.3}$$

for any $g \in G$ and $z \in \mathcal{M}$ where $g(z)$ is defined. Comparing (15.3) and (14.7) we find

$$\Phi(z) = \mathrm{Ad}_{\mathfrak{m}^+} K(z)^{-1}$$

for all $z \in \mathcal{M}$ since this equation holds at $z = 0$ and both sides satisfy the same transformation rule under the action of G. ∎

Let J denote the paracomplex structure on M. We introduce the operator $d_J \colon C^\infty(M) \to \Omega^1(M)$ by

$$(d_J f)(X) = df(JX),$$

where $f \in C^\infty(M)$ is a smooth function and $X \in \mathfrak{X}(M)$ is a vector field on M. In terms of the coordinates given by $\mathcal{M}$ we have at any point $p \in \mathcal{M}$

$$(d_J f)_p(Z) = (df)_p([Z_0, Z]),$$

for $Z \in \mathfrak{q} = T_p(\mathcal{M})$.

Proposition 15.2 *The G-invariant symplectic form ω on $\mathcal{M}$ satisfies*

$$\omega_z = -\frac{1}{2} d d_J \log k(z), \tag{15.4}$$

where $z \in \mathcal{M}$.

Proof. Indeed, let $X, Y \in \mathfrak{q}$ be tangent vectors at z. Then

$$\begin{aligned} dd_J \log k(z) &= -2D_{X^+}D_{Y^-} \log k(z) + 2D_{Y^+}D_{X^-} \log k(z) \\ &= -H_z(X,Y) + H_z(Y,X), \end{aligned}$$

whence it follows that $dd_J \log k(z)$ is invariant under G and that $dd_J \log k(0) = -2\omega_0$ by (15.1). This proves the claim. ■

§16 Symplectic Area

We can now state and prove a result connecting the mixed kernel κ defined by (14.6) with curve integrals of the 1-form $d_J \log k$.

Lemma 16.1 *Let $\gamma\colon [0,1] \to \mathscr{M}$ be a smooth curve segment and suppose that $\kappa(\gamma(0), \gamma(1))$ is defined. If g is any element of G such that the action of g is defined on all points of γ, then*

$$\exp\left(\int_\gamma d_J \log k - \int_{g\gamma} d_J \log k \right) = \frac{\kappa(\gamma(0), \gamma(1))}{\kappa(g\gamma(0), g\gamma(1))} \tag{16.1}$$

holds.

Proof. Let us write $\gamma(t) = \gamma(t)^+ + \gamma(t)^-$ where $\gamma(t)^\pm \in \mathfrak{m}^\pm$ for every $t \in [0,1]$. It follows from Proposition 14.6 that $\kappa(g\gamma(0), g\gamma(1))$ is defined and equals

$$\kappa(g\gamma(0), g\gamma(1)) = \frac{j^+(g, \gamma(0)^+)}{j^+(g, \gamma(1)^+)} \frac{j^-(g, \gamma(0)^-)}{j^-(g, \gamma(1)^-)} \kappa(\gamma(0), \gamma(1)).$$

On the other hand we have

$$\begin{aligned} \int_{g\gamma} d_J \log k &= \int_\gamma g^* d_J \log k \\ &= \int_\gamma d_J g^* \log k, \end{aligned}$$

as the action of g is a paraholomorphic map. The transformation rule for k gives

$$d_J g^* \log k(\gamma(t)) = d_J \log j^+(g, \gamma(t)^+) + d_J \log k(\gamma(t)) - d_J \log j^-(g, \gamma(t)^-)$$

for every $t \in [0,1]$. As the functions $\log j^\pm(g, z^\pm)$ only depend on $\mathfrak{m}^\pm$ respectively, we have $d_J \log j^-(g, z^-) = -d \log j^-(g, z^-)$ and $d_J \log j^+(g, z^+) = d \log j^+(g, z^+)$, thus

$$\begin{aligned} \int_\gamma d_J \log k - \int_{g\gamma} d_J \log k = &- \int_\gamma \left[d \log j^+(g, z^+) + d \log j^-(g, z^-) \right] \\ = &\log j^+(g, \gamma(0)^+) - \log j^+(g, \gamma(1)^+) \\ &+ \log j^-(g, \gamma(0)^-) - \log j^-(g, \gamma(0)^+) \end{aligned}$$

and (18.2) follows. ■

Theorem 16.2 *Let $\gamma\colon I \to \mathcal{M}$ be a geodesic curve passing through* 0*. Then $d_J \log k(\dot\gamma)$ vanishes.*

Proof. We may assume that γ is of the form $\gamma(t) = \exp(tX).0$ for some $X \in \mathfrak{q}$ and for t in some interval I containing 0. We will write $\gamma(t) = \gamma(t)^+ + \gamma(t)^-$ where $\gamma(t)^\pm \in \mathfrak{m}^\pm$. Let us put $g_t = \exp(tX)$ and decompose

$$g_t = p_t h_t q_t, \tag{16.2}$$

where $p_t \in \mathfrak{m}^+$, $h_t \in H$ and $q_t \in \mathfrak{m}^-$. By definition $p_t = \exp(\gamma(t)^+)$ and if we apply σ and take inverses on both sides of (16.2) we obtain

$$g_t = q_t h_t^{-1} p_t,$$

whence $q_t = \exp(\gamma(t)^-)$. For $n \in \mathbb{N}$, the symbol $O_\mathfrak{a}(t^n)$ denotes a smooth curve, defined for t close to 0, in the subspace $\mathfrak{a} \subset \mathfrak{g}$ with the property that $t^{-n} O_\mathfrak{a}(t^n)$ is bounded. With this in mind we have for small t

$$\begin{aligned} p_t &= \exp(tX^+ + O_{\mathfrak{m}^+}(t^2)), \\ h_t &= \exp(O_\mathfrak{h}(t^2)), \\ q_t &= \exp(tX^- + O_{\mathfrak{m}^-}(t^2)), \end{aligned} \tag{16.3}$$

as the derivative of g_t at 0 is $X = X^+ + X^-$. Next define the family δ_{t_0} of curves

$$\begin{aligned} \delta_{t_0}(t) &= \gamma(t_0 + t)^+ + \gamma(t_0 - t)^- \\ &= \xi_1^{-1}(g_{t_0+t}(o^-)) + \xi_2^{-1}(g_{t_0-t}(o^+)) \end{aligned}$$

for $t_0 \in I$ and for all t sufficiently close to 0. The curves δ_{t_0} satisfy

$$\frac{d}{dt}\delta_{t_0}(t) = [Z_0, \dot\gamma(t)]$$

whence

$$\frac{d}{dt}_{|t=t_0} \log k(\delta_{t_0}(t)) = (d_J \log k)(\dot\gamma(t_0)),$$

for all $t_0 \in I$. Hence we are going to consider $k(\delta_{t_0}(t))$ and its derivative with respect to t at 0. As $\delta_{t_0}(t) = g_{t_0}(\delta_0(t))$ we get

$$k(\delta_{t_0}(t)) = j^+(g_{t_0}, g_t(0)^+) k(\delta_0(t)) j^-(g_{t_0}, g_{-t}(0)^-)^{-1}, \tag{16.4}$$

and the chain rule for derivatives show

$$\begin{aligned} j^+(g_{t_0}, g_t(0)^+) &= j^+(g_{t_0+t}, 0) j^+(g_t, 0)^{-1}, \\ j^-(g_{t_0}, g_{-t}(0)^-)^{-1} &= j^-(g_{t_0-t}, 0)^{-1} j^-(g_{-t}, 0), \end{aligned}$$

and hence

$$j^+(g_{t_0}, g_t(0)^+) = \det \mathrm{Ad}_{\mathfrak{m}^+} h_{t_0+t} \det \mathrm{Ad}_{\mathfrak{m}^+} h_t^{-1},$$
$$j^-(g_{t_0}, g_{-t}(0)^-)^{-1} = \det \mathrm{Ad}_{\mathfrak{m}^+} h_{t_0-t} \det \mathrm{Ad}_{\mathfrak{m}^+} h_{-t}^{-1},$$

since $(g_t)_0 = h_t^{-1}$. From (16.3) we see that

$$\det \mathrm{Ad}_{\mathfrak{m}^+} h_t^{-1} = 1 + O(t^2), \qquad \det \mathrm{Ad}_{\mathfrak{m}^+} h_{-t}^{-1} = 1 + O(t^2),$$

and thus taking d/dt of both sides of (16.4) gives

$$\begin{aligned} \frac{d}{dt}_{|t=0} k(\delta_{t_0}(t)) = & \frac{d}{dt}_{|t=0} \det \mathrm{Ad}_{\mathfrak{m}^+} h_{t_0+t} + \frac{d}{dt}_{|t=0} \det \mathrm{Ad}_{\mathfrak{m}^+} h_{t_0-t} \\ & + (\det \mathrm{Ad}_{\mathfrak{m}^+} h_{t_0})^2 \frac{d}{dt}_{|t=0} k(\delta_0(t)) \\ =& (\det \mathrm{Ad}_{\mathfrak{m}^+} h_{t_0})^2 \frac{d}{dt}_{|t=0} k(\delta_0(t)). \end{aligned}$$

Now $k(\delta_0(t)) = k(\gamma(t)^+ + \gamma(-t)^-)$. As previously noted $\exp(\gamma(t)^+) = p_t$ and $\exp(\gamma(-t)^-) = q_{-t}$, so the Baker-Campbell-Hausdorff formula gives

$$\begin{aligned} \exp(-\gamma(-t)^-) \exp(\gamma(t)^+) &= q_{-t}^{-1} p_t \\ &= \exp(tX^- + O_{\mathfrak{m}^-}(t^2)) \exp(tX^+ + O_{\mathfrak{m}^+}(t^2)) \\ &= \exp(tX + \tfrac{t^2}{2}[X^-, X^+] + O_{\mathfrak{g}}(t^3)), \end{aligned}$$

for small t. Decomposing with respect to $\exp \mathfrak{m}^+ \times H \times \exp \mathfrak{m}^-$ shows that

$$K(\delta_0(t)) = \exp\left(\frac{t^2}{2}[X^+, X^-] + O_{\mathfrak{h}}(t^3)\right),$$

and thus $\frac{d}{dt} k(\delta_0(t))$ vanishes at $t = 0$. This completes the proof of the theorem.■

Corollary 16.3 *Let $\gamma\colon [0,1] \to \mathcal{M}$ be a geodesic segment. Suppose that there exists an element $g \in G$ such that the action of g is defined on all points of γ and such that $g\gamma$ passes through* 0. *Then*

$$c(\gamma(0), \gamma(1)) = -\int_\gamma d_J \log k,$$

defines a logarithm of $\kappa(\gamma(0), \gamma(1))$.

An oriented geodesic triangle $\Delta \subset \mathcal{M}$ with vertices z_0, z_1 and z_2 consists of three geodesic segments in $\mathcal{M}$ connecting the three vertices. The orientation is given by the ordering of the vertices. Since ω is exact, we may speak of the integral $\int_\Delta \omega$ instead of $\int_\Sigma \omega$ where $\Sigma \subset \mathcal{M}$ is any smooth surface with boundary $\partial\Sigma = \Delta$.

Corollary 16.4 *Suppose that Δ is an oriented geodesic triangle in $\mathscr{M}$ with vertices and orientation (z_0, z_1, z_2) and whose sides satisfy the conditions of Corollary 16.3. Then*

$$\exp\left(-2 \int_{\Delta} \omega \right) = \kappa(z_0, z_1)\kappa(z_1, z_2)\kappa(z_2, z_0),$$

holds.

Chapter VI

COMPLEX HERMITIAN SYMMETRIC SPACES

The kernel and the generalized embedding in the past chapter can also be constructed in the complex setting, that is, when the symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$ associated with a parahermitian symmetric space is complex. The ideas are the same as in Chapter II where $SL(2,\mathbb{C})/C^*$ is an example of this type of parahermitian space. An alternative formulation would be to say that we are dealing with a complex semisimple symmetric space $G_{\mathbb{C}}/H_{\mathbb{C}}$ which is both para- and pseudohermitian. In particular, the results of the previous chapter applies to the space but taking the complex structure into account will yield further results.

§17 The Complex Kernel

We consider a complex simple Lie algebra $\mathfrak{g}$ which is 3-graded:

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1,$$

where each $\mathfrak{g}_\lambda$ is a complex subspace of $\mathfrak{g}$ and $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$. Let Z_0 denote the unique element of $\mathfrak{g}_0$ such that $\mathrm{ad}(Z_0)X_\lambda = \lambda X$ for $X_\lambda \in \mathfrak{g}_\lambda$, and let σ denote the involutive automorphism of $\mathfrak{g}$ given by $\sigma(X_\lambda) = (-1)^\lambda X_\lambda$, $X_\lambda \in \mathfrak{g}_\lambda$. Then $(\mathfrak{g}, \mathfrak{g}_0, \sigma)$ is a symmetric Lie algebra which is both parahermitian and pseudohermitian, the corresponding structure is given by Z_0 and iZ_0 respectively. The (-1)-eigenspace of σ is $\mathfrak{q} = \mathfrak{g}_{-1} + \mathfrak{g}_1$. Let $G_{\mathbb{C}}$ be any complex Lie group with Lie algebra $\mathfrak{g}$ and let $H_{\mathbb{C}}$ be a Lie subgroup of $G_{\mathbb{C}}$ which centralizes Z_0 and has Lie algebra $\mathfrak{g}_0$. Then $G_{\mathbb{C}}/H_{\mathbb{C}}$ is a complex manifold in two ways: It has a pseudohermitian structure induced by iZ_0 and a complex structure as a coset space of a complex Lie group. When we speak of a complex structure in the following, we shall always mean the latter of these two.

Remark 17.1 We remark that $H_{\mathbb{C}}$ is connected and contains the center of $G_{\mathbb{C}}$. For $H_{\mathbb{C}}$ contains a maximal torus of $G_{\mathbb{C}}$; pick any maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{g}_0$. Then Z_0 is contained in $\mathfrak{t}$ and hence $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. It follows that $T = \exp(\mathfrak{t})$ contains the center of $G_{\mathbb{C}}$. The semidirect product $H_{\mathbb{C}} \exp(\mathfrak{g}_1)$ is a parabolic subgroup of $G_{\mathbb{C}}$ and hence connected. But this implies that $H_{\mathbb{C}}$ is connected. ◊

We will now use the embedding considered in Section 14, but we will include the complex structure. The results of that section provide us with an embedding

$G_{\mathbb{C}}/H_{\mathbb{C}} \subset G_{\mathbb{C}}/P^- \times G_{\mathbb{C}}/P^+$. Here

$$P^\pm = H_{\mathbb{C}} \exp(\mathfrak{g}_{\pm 1}),$$

are parabolic subgroups of $G_{\mathbb{C}}$ and hence $M^\pm = G_{\mathbb{C}}/P^\pm$ are homogeneous compact Kähler manifolds. Furthermore, $P^+ \cap P^- = H_{\mathbb{C}}$ and thus $G_{\mathbb{C}}/H_{\mathbb{C}}$ is holomorphically embedded as a $G_{\mathbb{C}}$-orbit in $M^- \times M^+$. There is a holomorphic embedding

$$\xi\colon \mathfrak{g}_1 \times \mathfrak{g}_{-1} \to M^- \times M^+,$$

given by $\xi(x, y) = (\exp(x)P^-, \exp(y)P^+)$ for $x \in \mathfrak{g}_1, y \in \mathfrak{g}_{-1}$. As in Section 14, the image of ξ is open, but in this case the image is also dense, see [Bor91, Proposition 14.21]. The partial actions of $G_{\mathbb{C}}$ on both $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are now holomorphic. We define $\mathscr{M} \subset \mathfrak{g}_1 + \mathfrak{g}_{-1}$ by (14.3) and

$$\Omega = \exp(\mathfrak{g}_1)P^-, \quad \Omega' = \exp(\mathfrak{g}_{-1})P^+,$$

are open and dense subsets of $G_{\mathbb{C}}$.

The next definition repeats the definitions of $K(z)$ and $J^\pm(g, x)$ given by (14.1), (14.2), and (14.4).

Definition 17.2 (Complex Automorphy Factor & Kernel) *Let x and y be points in $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ respectively and put $z = x + y$. Let g be an element of $G_{\mathbb{C}}$. If $g(x)$ is defined, we put*

$$J^+(g, x) = (g \exp(x))_0 \in H_{\mathbb{C}}. \tag{17.1}$$

If $g(y)$ is defined, we put

$$J^-(g, y) = (g \exp(y))'_0, \tag{17.2}$$

and if $\exp(-y)\exp(x)$ belongs to $\exp(\mathfrak{g}_1)H_{\mathbb{C}}\exp(\mathfrak{g}_{-1})$, we define $K(z) \in H_{\mathbb{C}}$ by

$$K(z)^{-1} = (\exp(-y)\exp(x))_0. \tag{17.3}$$

The complex structure comes enters the picture via the complex linear action of $\mathrm{Ad}(H_{\mathbb{C}})$ on each $\mathfrak{g}_{\pm 1}$ and the *holomorphic* characters $\det_{\mathbb{C}} \mathrm{Ad}_{\mathfrak{g}_{\pm 1}}\colon H_{\mathbb{C}} \to \mathbb{C}^*$, which we will use to define the complex kernel functions $k_{\mathbb{C}}$ and $\kappa_{\mathbb{C}}$.

Definition 17.3 (The complex kernel functions) *Suppose that $z = z^+ + z^- \in \mathfrak{g}_1 + \mathfrak{g}_{-1}$ is a point such that $K(z)$ is defined. Then we define the complex automorphy kernel $k_{\mathbb{C}}$ by*

$$k_{\mathbb{C}}(z) = \det_{\mathbb{C}} Ad_{\mathfrak{g}_1} K(z), \tag{17.4}$$

and if $w = w^+ + w^-$ is another point of $\mathfrak{g}_1 + \mathfrak{g}_{-1}$, we put

$$\kappa_{\mathbb{C}}(z, w) = \frac{k_{\mathbb{C}}(z^+ + w^-)}{k_{\mathbb{C}}(w^+ + z^-)}, \tag{17.5}$$

if the right-hand side is defined. We also define

$$j_{\mathbb{C}}^{\pm}(g,z^{\pm}) = \det_{\mathbb{C}} Ad_{\mathfrak{g}_{\pm 1}} J^{\pm}(g,x^{\pm}),$$

for all $g \in G_{\mathbb{C}}$ and $z = z^+ + z^-$ where $J^{\pm}(g,z^{\pm})$ is defined.

It follows from Proposition 14.4 that $j_{\mathbb{C}}^{\pm}(g,x^{\pm})$ is the complex Jacobian of the differential $d_{x^{\pm}}g\colon \mathfrak{g}_{\pm 1} \to \mathfrak{g}_{\pm 1}$ at $x^{\pm} \in \mathfrak{g}_{\pm 1}$. From Proposition 14.6 we obtain the transformation rule

$$k_{\mathbb{C}}(g(z)) = j_{\mathbb{C}}^{+}(g,z^{+}) k_{\mathbb{C}}(z) j_{\mathbb{C}}^{-}(g,z^{-}), \qquad (17.6)$$

valid for any $g \in G_{\mathbb{C}}$ and $z = z^+ + z^- \in \mathfrak{g}_1 + \mathfrak{g}_{-1}$ such that $k_{\mathbb{C}}(z)$ and $g(z)$ are both defined. It should be noted that $\kappa_{\mathbb{C}}(z,w)$ is holomorphic in both variables.

Remark 17.4 When applied to the complex symmetric space $SL(2,\mathbb{C})/\mathbb{C}^*$, this embedding is the same as the one considered in §6, but the kernels differ slightly. The kernels $k_{\mathbb{C}}$ and $\kappa_{\mathbb{C}}$ introduced above are the squares of k and K from §6. ♢

§18 Complex Symplectic Area

We wil also introduce a $G_{\mathbb{C}}$-invariant complex valued holomorphic 2-form Ω on $M_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ by

$$\Omega_o = \frac{1}{2} B(X,[Z_0,Y]), \quad X, Y \in \mathfrak{q},$$

at the identity coset $o = eH_{\mathbb{C}}$. Here B is the (complex-valued) Killing form of $\mathfrak{g}$. The pull-back to $\mathscr{M}$ is also denoted by Ω. Recall the operator d_J from §15. We may extend the definition of d_J to complex-valued smooth functions f on $M_{\mathbb{C}}$ by

$$(d_J f)(X) = df(JX),$$

where J is the parahermitian structure and X is a complex vector field on $M_{\mathbb{C}}$.

Proposition 18.1 *Let $z \in \mathscr{M}$ be an arbitrary point. Then*

$$\Omega_z = -\frac{1}{2} dd_J \log k_{\mathbb{C}}(z), \qquad (18.1)$$

where the right-hand side is independent of the particular choice of logarithm of $k_{\mathbb{C}}$ near z.

Proof. The proof is similar to the proof of Proposition 15.2. One proves that the right-hand side of (18.1) is invariant under the partial action of $G_{\mathbb{C}}$ and that equality holds in (18.1) at $0 \in \mathscr{M}$. ∎

The analogues of Lemma 16.1 and Theorem 16.2 as well as its corollaries hold in this setting.

Lemma 18.2 *Let $\gamma\colon [0,1] \to \mathcal{M}$ be a smooth curve segment and suppose that $\kappa_{\mathbb{C}}(\gamma(0), \gamma(1))$ is defined. If g is any element of $G_{\mathbb{C}}$ such that the action of g is defined on all points of γ, then*

$$\exp\Big(\int_{\gamma} d_J \log k_{\mathbb{C}} - \int_{g\gamma} d_J \log k_{\mathbb{C}}\Big) = \frac{\kappa_{\mathbb{C}}(\gamma(0), \gamma(1))}{\kappa_{\mathbb{C}}(g\gamma(0), g\gamma(1))} \tag{18.2}$$

holds.

Proof. The proof is the same as the proof of Lemma 16.1. ∎

Theorem 18.3 *Let $\gamma\colon I \to \mathcal{M}$ be a geodesic curve passing through 0. Then $d_J \log k_{\mathbb{C}}(\dot{\gamma})$ vanishes.*

Proof. The proof of Theorem 16.2 works here as well. ∎

Corollary 18.4 *Let $\gamma\colon [0,1] \to \mathcal{M}$ be a geodesic segment. Suppose that there exists an element $g \in G_{\mathbb{C}}$ such that the action of g is defined on all points of γ and such that $g\gamma$ passes through 0. Then*

$$c(\gamma(0), \gamma(1)) = -\int_{\gamma} d_J \log k_{\mathbb{C}},$$

defines a logarithm of $\kappa_{\mathbb{C}}(\gamma(0), \gamma(1))$.

Corollary 18.5 *Suppose that Δ is an oriented geodesic triangle in $\mathcal{M}$ with vertices and orientation (z_0, z_1, z_2) and whose sides satisfy the conditions of Corollary 18.4. Then*

$$\exp\Big(-2\int_{\Delta} \Omega\Big) = \kappa_{\mathbb{C}}(z_0, z_1)\kappa_{\mathbb{C}}(z_1, z_2)\kappa_{\mathbb{C}}(z_2, z_0), \tag{18.3}$$

holds.

We will proceed to show how this result is related to the previous computations of symplectic area of geodesic triangles.

§19 Restriction to Real Forms

Consider now a real form $\mathfrak{g}_r$ of $\mathfrak{g}$ such that $\mathfrak{g}_r$ is invariant under σ. The decomposition of $\mathfrak{g}_r$ under σ is

$$\mathfrak{g}_r = \underbrace{\mathfrak{g}_0 \cap \mathfrak{g}_r}_{\mathfrak{h}_r} + \underbrace{\mathfrak{q} \cap \mathfrak{g}_r}_{\mathfrak{q}_r},$$

and $(\mathfrak{g}_r, \mathfrak{h}_r, \sigma|_{\mathfrak{g}_r})$ is a simple symmetric Lie algebra.

Lemma 19.1 *Let $\mathfrak{g}_r$ be a real form of $\mathfrak{g}$ such that $\sigma(\mathfrak{g}_r) = \mathfrak{g}_r$. The simple symmetric Lie algebra $(\mathfrak{g}_r, \mathfrak{h}_r, \sigma|_{\mathfrak{g}_r})$ is either parahermitian or pseudohermitian.*

Proof. It suffices to show that $\mathfrak{h}_r$ contains either Z_0 or iZ_0, but not both. Now $\mathfrak{h}_r$ is a real form of $\mathfrak{g}_0$ and hence has one-dimensional center over $\mathbb{R}$ and contains an element of the form αZ_0 with α a non-zero complex number. As $\mathfrak{q}_r$ is a real form of the complex vector space $\mathfrak{q}$ and invariant under $\alpha(\operatorname{ad} Z_0)$ we must have $\alpha^2 \in \mathbb{R}$. After scaling and changing signs if needed, we find that either iZ_0 or Z_0 belongs to $\mathfrak{h}_r$. Consequently, $(\mathfrak{g}_r, \mathfrak{h}_r, \sigma|_{\mathfrak{g}_r})$ is eiher pseudohermitian or parahermitian. ∎

Now let us suppose that the conjugation $\tau_r \colon \mathfrak{g} \to \mathfrak{g}$ of $\mathfrak{g}$ with respect to $\mathfrak{g}_r$ defines an involution of $G_{\mathbb{C}}$ which we also denote by τ_r. Let G_r be the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_r$ and put $H_r = H_{\mathbb{C}} \cap G_r$. Then $M_r = G_r/H_r$ is a symmetric space which is either pseudohermitian or parahermitian and the inclusion $G_r \subset G_{\mathbb{C}}$ induces a totally geodesic embedding $G_r/H_r \to G_{\mathbb{C}}/H_{\mathbb{C}}$.

Proposition 19.2 *The preimage $\xi^{-1}(G_r/H_r)$ in $\mathcal{M}$ is contaied in $\mathfrak{q}_r$.*

Proof. If $z = z^+ + z^-$ belongs to $\xi^{-1}(G_r/H_r)$ then $z = g(0)$ for some $g \in G_r$. Thus g belongs to both $\exp(\mathfrak{g}_1) H_{\mathbb{C}} \exp(\mathfrak{g}_{-1})$ and $\exp(\mathfrak{g}_{-1}) H_{\mathbb{C}} \exp(\mathfrak{g}_1)$ and we have two decompositions

$$\begin{aligned} g &= \exp(z^+) h \exp(w^-) \\ &= \exp(z^-) h' \exp(w^+), \end{aligned}$$

where $h, h' \in H_{\mathbb{C}}$ and $w^{\pm} \in \mathfrak{g}_{\pm 1}$. Applying τ_r to g we find

$$\begin{aligned} g &= \tau_r g \\ &= \exp(\tau_r z^+) \tau_r h \exp(\tau_r w^-), \end{aligned}$$

and

$$g = \exp(\tau_r z^-) \tau_r h' \exp(\tau_r w^+),$$

and there are now two cases to consider. If $(\mathfrak{g}_r, \mathfrak{h}_r, \sigma|_{\mathfrak{g}_r})$ is pseudohermitian then τ_r maps $\mathfrak{g}_1$ to $\mathfrak{g}_{-1}$ and vice versa. The uniqueness of the decompositions of g implies that $\tau z^+ = z^-$ and hence that $z = z^+ + \tau_r z^+$ is fixed by τ_r. Thus z belongs to $\mathfrak{q}_r$. If $(\mathfrak{g}_r, \mathfrak{h}_r, \sigma|_{\mathfrak{g}_r})$ is parahermitian we have $\tau_r \mathfrak{g}_{\pm 1} = \mathfrak{g}_{\pm 1}$ instead, and it follows that $\tau_r z^+ = z^+$ and similarly $\tau_r z^- = z^-$. Hence $z \in \mathfrak{q}_r$. ∎

It is well-known, see e.g. [KN64, Lemma 4], that there exists a compact real form $\mathfrak{u}$ of $\mathfrak{g}$ such that $\mathfrak{u}$ is invariant under σ. Let θ denote the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{u}$. Put $\mathfrak{k} = \mathfrak{g}_0 \cap \mathfrak{u}$. Then $\theta Z_0 = -Z_0$ as $\mathrm{ad}_{\mathfrak{q}} Z_0$ has real eigenvalues and thus $(\mathfrak{u}, \mathfrak{k}, \sigma|_{\mathfrak{u}})$ is a simple Hermitian symmetric Lie algebra of the compact type. Let $\mathfrak{g}_{\mathrm{nc}}$ be the non-compact dual and let U, G be the corresponding subgroups of $G_{\mathbb{C}}$ and put $K = U \cap H_{\mathbb{C}}$. Both U/K and G/K has a complex structure induced by iZ_0. In terms of the notation of Chapter III we have $\mathfrak{g}_1 = \mathfrak{p}^+$. Now recall the Harish-Chandra embedding $\xi_{\mathrm{HC}} \colon G/K \to \mathfrak{g}_1$ and the compact version $\Xi \colon \mathfrak{g}_1 \to G_{\mathbb{C}}/P^- = U/K$ given by

$$\exp \xi_{\mathrm{HC}}(gK) \in gP^-$$

and

$$\Xi(x) = \exp(x) P^-,$$

for $g \in G$ and $x \in \mathfrak{g}_1$.

Proposition 19.3 *Let $\pi^+ \colon \mathfrak{q} \to \mathfrak{g}_1$ denote the projection along $\mathfrak{g}_{-1}$. Then $\pi^+ \circ \xi^{-1}$ restricted to $G/K \subset G_{\mathbb{C}}/H_{\mathbb{C}}$ is the Harish-Chandra embedding. Restricting ξ to $\mathfrak{g}_1$ and projecting onto the first factor of $M^- \times M^+$ yields the embedding Ξ of Chapter IV.*

Proof. This is a straightforward verification. ∎

It should be observed that the action of $G_{\mathbb{C}}$ on $\mathfrak{g}_1$ is the same as the action defined in Chapter III.

Put $\mathfrak{p} = \mathfrak{q} \cap \mathfrak{g}_{\mathrm{nc}}$. Then $\mathfrak{p}$ consists of all elements of the form $x - \theta x$ where x is an element of $\mathfrak{g}_1$ and $i\mathfrak{p}$ consists of all elements of the form $x + \theta x$, x again in $\mathfrak{g}_1$. Now let us write $K_{\mathbb{C}}(z)$ for the canonical kernel defined by (17.3) and compute its restriction to $\mathfrak{p}$ and $i\mathfrak{p}$. We find

$$K_{\mathbb{C}}(x - \theta x)^{-1} = (\exp(\theta x) \exp(x))_0 = K(x, x)^{-1},$$
$$K_{\mathbb{C}}(x + \theta x)^{-1} = (\exp(-\theta x) \exp(x))_0 = K_c(x, x)^{-1},$$

for $x \in \mathfrak{g}_1$ and by (8.4) and (11.1). Recall that $k(z, w)$ and $k_c(z, w)$ are the kernels defined on $\mathfrak{g}_1 = \mathfrak{p}^+$ by (8.4) and (11.5) respectively. From the above it is immediate that

$$k_{\mathbb{C}}(x - \theta x) = k(x, x)$$

and

$$k_{\mathbb{C}}(x + \theta x) = k_c(x, x)$$

for $x \in \mathfrak{g}_1$. If z, w are points in $\mathfrak{g}_1$, then

$$\kappa_{\mathbb{C}}(z - \theta z, w - \theta w) = \frac{k(z,w)}{k(w,z)}$$

and

$$\kappa_{\mathbb{C}}(z + \theta z, w + \theta w) = \frac{k_c(z,w)}{k_c(w,z)}$$

whenever $k(z,w)$ resp. $k_c(z,w)$ are defined. Notice that $\kappa_{\mathbb{C}}(x \pm \theta x, w \pm \theta w)$ has modulus one.

Let us now think of $\mathfrak{p}^+$ as $\mathfrak{g}_1$ together with the non-compact Kähler form $\omega = \frac{1}{2} dd_{\mathbb{C}} \log k(z,z)$ for $z \in D$, compare (8.8). Let $\iota\colon \mathfrak{p}^+ \to \mathfrak{p}$ denote the $\mathbb{R}$-linear map

$$\iota(z) = z - \theta z$$

for $z \in \mathfrak{p}^+ = \mathfrak{g}_1$. We will compute the pull-back $\iota^*\Omega$. Let $X, Y \in \mathfrak{p}^+$ and consider them as tangent vectors at the origin $0 \in \mathfrak{p}^+$. Then

$$\begin{aligned} \iota^*\Omega_0(X,Y) &= \Omega_0(X - \theta X, Y - \theta Y) \\ &= \frac{1}{2} B(X - \theta X, Y + \theta Y) \\ &= \frac{1}{2} \big(B(X, \theta Y) - B(Y, \theta X)\big) \\ &= -i \Im \langle X, Y \rangle_0 \\ &= i\omega_0(X,Y) \end{aligned}$$

in terms of the Hermitian form $\langle\,,\rangle$ on $\mathfrak{p}^+$ given in Proposition 8.7. It follows from G-equivariance that $\iota^*\Omega = i\omega$. If $\Delta \subset D$ is a geodesic triangle with vertices z_0, z_1 and z_2, then

$$\begin{aligned} \exp\Bigg(-2i \int_\Delta \omega \Bigg) &= \kappa_{\mathbb{C}}(\iota(z_0), \iota(z_1)) \kappa_{\mathbb{C}}(\iota(z_1), \iota(z_2)) \kappa_{\mathbb{C}}(\iota(z_2), \iota(z_0)) \\ &= \frac{k(z_0,z_1)}{\overline{k(z_0,z_1)}} \frac{k(z_1,z_2)}{\overline{k(z_1,z_2)}} \frac{k(z_2,z_0)}{\overline{k(z_2,z_0)}} \end{aligned}$$

by (18.3). As both sides of the above depend continuously on z_0, z_1 and z_2 we conclude that

$$\int_\Delta \omega = -(\arg k(z_0,z_1) + \arg k(z_1,z_2) + \arg k(z_0,z_2))$$

where $\arg k(z,z) = 0$ for $z \in \mathfrak{p}^+$. We have thus proven Theorem 9.1 once more.

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