UNIVERSITY OF AARHUS
DEPARTMENT OF MATHEMATICS

ISSN: 1397–4076

DIOPHANTINE EXPONENTS FOR MILDLY RESTRICTED APPROXIMATION

by Yann Bugeaud and Simon Kristensen

Preprint Series No.: 9

September 2007

2007/09/11

Ny Munkegade, Bldg. 1530
DK-8000 Aarhus C, Denmark

http://www.imf.au.dk
institut@imf.au.dk

DIOPHANTINE EXPONENTS FOR MILDLY RESTRICTED APPROXIMATION

YANN BUGEAUD AND SIMON KRISTENSEN

Abstract. We are studying the Diophantine exponent $\mu_{n,\ell}$ defined for integers $1 \leq \ell < n$ and a vector $\alpha \in \mathbb{R}^n$ by letting $\mu_{n,\ell} = \sup\{\mu \geq 0 : 0 < \|\underline{x} \cdot \alpha\| < H(\underline{x})^{-\mu}$ for infinitely many $\underline{x} \in \mathcal{C}_{n,\ell} \cap \mathbb{Z}^n\}$, where $\cdot$ is the scalar product and $\|\cdot\|$ denotes the distance to the nearest integer and $\mathcal{C}_{n,\ell}$ is the generalised cone consisting of all vectors with the height attained among the first ℓ coordinates. We show that the exponent takes all values in the interval $[\ell + 1, \infty)$, with the value n attained for almost all α. We calculate the Hausdorff dimension of the set of vectors α with $\mu_{n,\ell}(\alpha) = \mu$ for $\mu \geq n$. Finally, letting w_n denote the exponent obtained by removing the restrictions on $\underline{x}$, we show that there are vectors α for which the gaps in the increasing sequence $\mu_{n,1}(\alpha) \leq \cdots \leq \mu_{n,n-1}(\alpha) \leq w_n(\alpha)$ can be chosen to be arbitrary.

1. Introduction

Throughout the present paper, $\|\cdot\|$ denotes the distance to the nearest integer. For a n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of real numbers, let denote by $w_n(\alpha)$ the supremum of the real numbers w such that the inequality

$$0 < ||x_1\alpha_1 + \ldots + x_n\alpha_n|| \leq H(\underline{x})^{-w},$$

has infinitely many solutions in integer n-tuples $\underline{x} = (x_1, \dots, x_n)$ of height $H(\underline{x})$, where $H(\underline{x}) = \max\{|x_1|, \dots, |x_n|\}$. This is the most classical exponent of Diophantine approximation. Further exponents have been introduced recently by Bugeaud and Laurent [9].

Approximation problems closely related to the study of the exponents w_n were considered by Jarník [17], Schmidt [23] and Thurnheer [26, 27, 28, 29, 30]. In these papers relatively mild restrictions are placed on the integer vectors $\underline{x}$. In Jarník's paper [17], the additional restriction was put on $\underline{x}$ that at least ℓ of its coordinates had to be non-zero. In the papers by Schmidt and Thurnheer, stronger restrictions were made, all of which can be viewed as special cases of the ones considered in the present paper, where we restrict the $\underline{x}$ to a rectangular cone (see below).

We introduce and study the following exponents of restricted approximation. Let $1 \leq \ell < n$ be integers and $\alpha = (\alpha_1, \dots, \alpha_n)$ be a real n-tuple. We denote by $\mu_{n,\ell}(\alpha)$ the supremum of the real numbers μ such that the inequality

$$0 < ||x_1\alpha_1 + \ldots + x_n\alpha_n|| \leq H(\underline{x})^{-\mu}$$

has infinitely many solutions in integer n-tuples $\underline{x} = (x_1, \dots, x_n)$ satisfying

$$\max\{|x_{\ell+1}|, \dots, |x_n|\} < \max\{|x_1|, \dots, |x_\ell|\}. \tag{1}$$

The present research is supported by a collaborative grant from the French Embassy in Denmark. SK is a Steno Research Fellow funded by the Danish Natural Science Research Council.

This simply means that we impose that the height of $\underline{x}$ is attained among its ℓ first coordinates. We write $\mu_{n,\ell}$ (resp. w_n) instead of $\mu_{n,\ell}(\alpha)$ (resp. $w_n(\alpha)$) when there is no confusion. By extension and for consistency of notation, we define $\mu_{n,n} = w_n$.

Since the α_i do not play the same role, the situation is not symmetrical, thus difficulties of a new kind occur. It is different from inhomogeneous approximation, since we have here less constraints. Geometrically, we are restricting the 'denominators' $\underline{x}$ to lie in some rectangular cone. In Schmidt's original paper [23], $n = 2$ and the $\underline{x}$ were restricted to the first quadrant. The rotated setting is better suited to our purposes, and it causes no loss of generality as was also remarked by Schmidt. Actually, our results remain valid if (1) is replaced by

$$\max\{|x_{\ell+1}|, \dots, |x_n|\} < C \max\{|x_1|, \dots, |x_\ell|\},$$

where C is an arbitrary given positive number.

The exponent μ defined by Schmidt [23] is simply $\mu_{2,1}$ with our notation. The exponents $\mu_{n,n-1}$ correspond to those introduced by Thurnheer [30]. One of our aims in the present psper is to show that, from a metric point of view, all the exponents $\mu_{n,\ell}$ with $\ell = 1, \dots, n$ have a similar behaviour (Theorem 4). In the opposite direction, we construct explicit examples of n-tuples α for which all the $\mu_{n,\ell}(\alpha)$, $1 \le \ell \le n$, are different (Theorem 2). We further investigate (Theorem 1) the set of values taken by the functions $\mu_{n,\ell}$.

2. Results

In the present paper, we are mainly concern with the *spectra* of the exponents of Diophantine approximation, that is, with the set of values taken by $\mu_{n,\ell}$ on the set of real n-tuples whose coordinates are, together with 1, linearly independent over the rationals. The reason for the latter restriction on the set of n-tuples is to avoid pathologies within the setup. Indeed, if we did have linear dependence, we would essentially be studying a lower dimensional problem, and the resulting spectrum would incorporate such lower dimensional phenomena. This would in turn obscure the nature of the exponent.

For convenience, unless the contrary is stated explicitly, we assume that the coordinates of the real n-tuples occurring from now on are, together with 1, linearly independent over the rationals.

Choosing $x_{\ell+1} = \cdots = x_n = 0$ and applying Dirichlet's *Schubfachprinzip*, we easily get that $\mu_{n,\ell} \ge \ell$, for any positive integers ℓ and n with $1 \le \ell < n$. Furthermore, since there are n free coefficients, namely $x_1, \dots, x_n$, in the definition of $\mu_{n,\ell}$, we can reasonably expect $\mu_{n,\ell}$ to be often at least equal to n. However, as noted by Schmidt [23], for any positive ϵ, there exist real n-tuples α with $\mu_{n,1}(\alpha) \le 2 + \epsilon$.

Theorem 1. *Let ℓ and n be positive integers with $1 \le \ell \le n$. Then, $\mu_{n,\ell}(\alpha) = n$ for almost all real n-tuples α. Furthermore, for any real number μ_ℓ with $\mu_\ell \ge \ell + 1$, there exist uncountably many n-tuples α having $\mu_{n,\ell}(\alpha) = \mu_\ell$.*

In fact, we prove a more precise result than the first assertion (see Theorem 4 and Theorem 6 in Section 3). Namely, we establish a zero–one law for Lebesgue measure and a zero-infinity law for Hausdorff measure, which implies that from the metrical point of view, all the exponents $\mu_{n,1}, \dots, \mu_{n,n-1}$ and $w_n = \mu_{n,n}$ have the same behaviour. However, in view of the remarks preceeding the statement of Theorem 1 together with the theorem, there is a distinct difference in the spectrum of these exponents. In particular, the fact (established in Section 4) that the spectrum of $\mu_{n,\ell}$ includes the interval $[\ell+1, n]$ is the most interesting part of Theorem 1.

For $(n, \ell) = (2, 1)$, the first assertion of Theorem 1 was proved by Thurnheer [29]. The main tool for the proof of Theorem 1 is the theory of Hausdorff measure. Consequently, it does not yield to *explicit* examples of n-tuples α with prescribed values for $\mu_{n,\ell}(\alpha)$. However, inspired by Schmidt's construction of T-numbers [21, 22], we give an effective construction of n-tuples α with prescribed exponents $\mu_{n,\ell}(\alpha)$, provided that these values are sufficiently large. This approach enables us also to prove that the difference between $w_n(\alpha)$ and $\mu_{n,\ell}(\alpha)$ can be arbitrarily large. In the statement of the next theorem, established in Section 5, we adopt the convention that $+\infty + x = +\infty$ for any real number x.

Theorem 2. *Let $n \geq 2$ be an integer. Let $\delta_1, \ldots, \delta_{n-1}$ be elements of $\mathbb{R}_{\geq 0} \cup \{+\infty\}$. Then, there exist uncountably many n-tuples α having $w_n(\alpha) = \mu_{n,n-1}(\alpha) + \delta_{n-1}$ and $\mu_{n,\ell}(\alpha) = \mu_{n,\ell-1}(\alpha) + \delta_{\ell-1}$, for $\ell = 2, \ldots, n-1$.*

The proof of Theorem 2 rests on the effective construction of real numbers ξ for which the n-tuples $(\xi^n, \ldots, \xi)$ have the required properties.

The above theorems say nothing on the values of the spectrum of $\mu_{n,\ell}$ belonging to the interval $[\ell, \ell+1)$. Schmidt proved that $\mu_{2,1}(\alpha) \geq (1+\sqrt{5})/2$, a result extended to the exponents $\mu_{n,n-1}$ by Thurnheer [30]. In particular, the following problems remain open.

Problem 1. Let ℓ and n be integers with $1 \leq \ell < n$. To prove or to disprove that there exist α such that

$$\mu_{n,\ell}(\alpha) < \ell + 1.$$

For $n = 2$, Problem 1 was previously posed by Schmidt [23, 24].

Problem 2. To establish a uniform lower bound for $\mu_{n,1}$ that tends to 2 when n tends to infinity.

We have been unable to make any progress on these questions (see however Section 7). Nevertheless, for sake of completeness, we restate the lower bounds obtained by Schmidt and Thurnheer, by making use of the exponents of Diophantine approximation w_n and $\hat{w}_n$, the latter being defined as follows. For an integer $n \geq 1$ and a real n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$, we denote by $\hat{w}_n(\alpha)$ the supremum of the real numbers $\hat{w}$ such that, for any real number $X > 1$, the inequality

$$0 < ||x_1\alpha_1 + \ldots + x_n\alpha_n|| \leq X^{-\hat{w}}$$

has an integer solution $\underline{x} = (x_1, \ldots, x_n)$ satisfying $H(\underline{x}) \leq X$.

Theorem 3. *Let ℓ and n be integers with $1 \leq \ell < n$. For any real n-tuple α, we have*

$$\mu_{n,\ell}(\alpha) \geq \frac{\ell\, \hat{w}_n(\alpha)}{\hat{w}_n(\alpha) - n + \ell} \tag{2}$$

and

$$\mu_{n,n-1}(\alpha) \geq \hat{w}_n(\alpha) - 1 + \frac{\hat{w}_n(\alpha)}{w_n(\alpha)}.$$

The second inequality from Theorem 3 implies that $\mu_{n,n-1} \geq \hat{w}_n - 1$. Combined with (2), this yields the lower bound

$$\mu_{n,n-1}(\alpha) \geq \frac{n-1+\sqrt{n^2+2n-3}}{2}, \tag{3}$$

for any α in $\mathbb{R}^n$. This was established in 1976 by Schmidt [23] for $n=2$ and in 1990 by Thurnheer [30] for arbitrary n. Observe that the right hand side of (3) is greater than $n - 1/n$.

Furthermore, it should be noted that Schmidt's and Thurnheer's results are slightly more precise, since they assert the existence of a positive constant C such that, for any real n-tuple α (whose coordinates, together with 1, are linearly independent over the rationals) there exist infinitely many integer n-tuples $\underline{x} = (x_1, \ldots, x_n)$ satisfying

$$0 < ||x_1\alpha_1 + \ldots + x_n\alpha_n|| \leq CH(\underline{x})^{-(n-1+\sqrt{n^2+2n-3})/2}$$

and

$$|x_n| < \max\{|x_1|, \ldots, |x_{n-1}|\}.$$

Although Theorem 3 is essentially proved in the papers by Schmidt and Thurnheer, we include a proof of (2), postponed to Section 6.

Throughout, we use the Vinogradov notation and write $a \ll b$ if there is a constant $C > 0$ such that $a \leq Cb$. If $a \ll b$ and $b \ll a$, we write $a \asymp b$. Furthermore, $\dim(E)$ denotes the Hausdorff dimension of the set E.

3. The metrical theory for the exponents $\mu_{n,\ell}$

It is the purpose of the present section to show that the exponent $\mu_{n,\ell}$ takes all values in the interval $[n, +\infty)$. This follows from a more general metrical result, which gives a complete metrical description of the sets

$$\begin{aligned}\mathcal{L}_{n,\ell}(\psi) = \big\{(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n : \|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \leq \psi(H(x))\\ \text{for infinitely many } \underline{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n\\ \text{with } \max\{|x_{\ell+1}|, \ldots, |x_n|\} < \max\{|x_1|, \ldots, |x_\ell|\}\big\}\end{aligned}$$

Our result also includes the case $\ell = n$, where the last condition on the integer vectors $\underline{x}$ is empty and Theorem 4 below reduces to a classical result of Groshev [15].

Theorem 4. *Let $\psi : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ be a non-increasing function, let n, ℓ be integers with $1 \leq \ell \leq n$. Then, $\mathcal{L}_{n,\ell}$ is null (resp. full) according to the convergence (resp. divergence) of the series*

$$\sum_{n=1}^{\infty} h^{n-1}\psi(h).$$

Proof of Theorem 4 (convergence part). The case of convergence is a consequence of the usual Khintchine–Groshev theorem (see, *e.g.*, [6]), since $\mathcal{L}_{n,\ell}$ is a subset of the corresponding set without restrictions on points $\underline{x}$. Since the measure of the larger set is zero in the case of convergence, the convergence half follows. □

The case of divergence will be derived from the following result, which is the simplest version of the divergence part of the Borel–Cantelli Lemma.

Lemma 5. *Let (Ω, A, μ) be a probability space and $(E_n)_{n\geq 1}$ be a sequence of μ-measurable sets such that $\sum_{n=1}^{\infty} \mu(E_n) = \infty$. Suppose that whenever $m \neq n$,*

$$\mu(E_m \cap E_n) = \mu(E_m)\mu(E_n).$$

Then,

$$\mu(\limsup_{n\to\infty} E_n) = 1$$

Proof of Theorem 4 (divergence part). In order to prove that our set has full measure, we note that $\mathcal{L}_{n,\ell}$ is invariant under translation by integer vectors. Hence it suffices to show that $\mathcal{L}_{n,\ell} \cap [0,1]^n$ has measure 1. We consider the sets

$$B(x_1, \dots, x_n) = \{(\alpha_1, \dots, \alpha_n) \in [0,1]^n : \|x_1\alpha_1 + \cdots + x_n\alpha_n\| \leq \psi(H(\underline{x}))\},$$

where $\underline{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n$. It is easy to see that

$$|B(x_1, \dots, x_n)| \asymp \psi(H(\underline{x})), \tag{4}$$

where $|B|$ denotes the Lebesgue measure of the set B. Furthermore, if $(x_1, \dots, x_n)$ and $(x'_1, \dots, x'_n)$ are linearly independent, then

$$|B(x_1, \dots, x_n) \cap B(x'_1, \dots, x'_n)| = |B(x_1, \dots, x_n)| \cdot |B(x'_1, \dots, x'_n)|. \tag{5}$$

This is proved in, *e.g.*, [14].

We will impose further restrictions on the $\underline{x}$'s in order to ensure that (5) holds for any pair of distinct vectors. Any vector α satisfying infinitely many of the further restricted inequalities automatically lies within $\mathcal{L}_{n,\ell}$. Hence, a lower bound on the estimate on the measure of the further restricted set implies a lower bound on the measure of the original set.

We define

$$\begin{aligned} P_N = \big\{(x_1, \dots, x_n) \in \mathbb{Z}^n_{\geq 0} \ : \ & H(\underline{x}) = N, x_n \geq 1 \\ & \gcd(x_1, \dots, x_n) = 1 \text{ and} \\ & 2\max\{|x_{\ell+1}|, \dots, |x_n|\} \leq \max\{|x_1|, \dots, |x_\ell|\}\big\}. \end{aligned}$$

If $\underline{x} \in P_N$ and $\underline{x}' \in P_{N'}$ are linearly dependent, then for some integer $r \in \mathbb{Z}$, $\underline{x} = r\underline{x}'$ or $r\underline{x} = \underline{x}'$. In either case, by assumption of coprimality, $r = \pm 1$, and since the last coordinates are assumed to be positive, $r = 1$, whence $\underline{x} = \underline{x}'$. Hence, (5) holds for any pair of distinct vectors $\underline{x} \in P_N$ and $\underline{x}' \in P_{N'}$.

Let $\mu : \mathbb{Z}_{\geq 0} \to \{-1, 0, 1\}$ denote the Möbius function, *i.e.*,

$$\mu(n) = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ has } k \text{ distinct prime factors.} \end{cases}$$

We use the identity

$$\sum_{d\mid n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

With this identity, we can extimate the number of elements of P_N as follows,

$$\begin{aligned}
\#P_N &= \sum_{\substack{H(\underline{x})=N \\ \gcd(x_1,\dots,x_n)=1 \\ 2\max\{|x_{\ell+1}|,\dots,|x_n|\}\leq\max\{|x_1|,\dots,|x_\ell|\}}} 1 \\
&= \sum_{\substack{H(\underline{x})=N \\ \gcd(x_1,\dots,x_n)=k \\ 2\max\{|x_{\ell+1}|,\dots,|x_n|\}\leq\max\{|x_1|,\dots,|x_\ell|\}}} \sum_{d|k} \mu(d) \\
&= \sum_{d|N} \mu(d) \sum_{\substack{H(\underline{x}')=N/d \\ 2\max\{|x_{\ell+1}|,\dots,|x_n|\}\leq\max\{|x_1|,\dots,|x_\ell|\}}} 1 \\
&\asymp \sum_{d|N} \mu(d) \left(\frac{N}{d}\right)^{n-1} \ell.
\end{aligned}$$

If $n = 2$, then

$$\sum_{d|N} \mu(d) \left(\frac{N}{d}\right) \ell = \ell\phi(N),$$

where ϕ denotes the Euler totient function. If $n > 2$,

$$\sum_{d|N} \mu(d) \left(\frac{N}{d}\right)^{n-1} \ell = \ell N^{n-1} \sum_{d|N} \frac{\mu(d)}{d^{n-1}}.$$

In order to show that this is comparable to N^{n-1}, it suffices to note that

$$\begin{aligned}
\frac{6}{\pi^2} = \frac{1}{\zeta(2)} \leq \frac{1}{\zeta(n-1)} = \prod_{\text{all primes } p} \left(1 - \frac{1}{p^{n-1}}\right) \\
< \prod_{\substack{p|N \\ p \text{ is prime}}} \left(1 - \frac{1}{p^{n-1}}\right) = \sum_{d|N} \frac{\mu(d)}{d^{n-1}} < 1. \quad (6)
\end{aligned}$$

where ζ denotes the Riemann ζ-function. Note that we have used the Euler product formula for this function in order to make the argument completely clear.

In order to prove Theorem 4, it suffices to prove that if the series $\sum h^{n-1}\psi(h)$ is divergent, then

$$\sum_{N=1}^{\infty} \sum_{x \in P_N} \psi(N) = \infty. \tag{7}$$

In view of the above, this is immediate when $n \geq 3$. When $n = 2$, we use the identity

$$\sum_{r=1}^{N} \phi(r) = \frac{3}{\pi^2} N^2 + O(N \log N),$$

from elementary number theory. We split the sum (7) into dyadic blocks to get

$$\begin{aligned}\sum_{N=1}^{\infty}\sum_{x\in P_N}\psi(N) &= \sum_{k=0}^{\infty}\psi(2^{k+1})\sum_{2^k\le r<2^{k+1}}\phi(r)\\ &= \sum_{k=0}^{\infty}\psi(2^{k+1})\left(\frac{9}{2\pi^2}2^{2(k+1)}+O(k2^k)\right)=\infty.\end{aligned}$$

The final equality follows by condensation and assumption of divergence. Hence, Lemma 5 applies, and the theorem follows. □

We now turn our attention to Hausdorff measures. We have the following theorem.

Theorem 6. *Let $\psi:\mathbb{Z}_{\ge 0}\to\mathbb{R}_{>0}$ be non-increasing, let $f:\mathbb{R}\to\mathbb{R}$ be a dimension function such that $r\mapsto r^{-n}f(r)$ is monotonically increasing and such that $g: r\mapsto r^{-(n-1)}f(r)$ is also a dimension function. Then,*

$$\mathcal{H}^f(\mathcal{L}_{n,\ell}(\psi))=\begin{cases}0 & \text{whenever } \sum_{r=1}^{\infty} r^n g\left(\frac{\psi(r)}{r}\right)<\infty,\\ \infty & \text{whenever } \sum_{r=1}^{\infty} r^n g\left(\frac{\psi(r)}{r}\right)=\infty.\end{cases}$$

Proof. To prove the convergence result, we cover each $B(x_1,\dots,x_n)$ by no more than some constant times $H(\underline{x})^n\psi(H(\underline{x}))^{-(n-1)}$ balls of width $\asymp \psi(H(\underline{x}))/H(\underline{x})$. Using this cover to bound the Hausdorff f-measure, we get for any N,

$$\begin{aligned}\mathcal{H}^f(\mathcal{L}_{n,\ell}(\psi)) &\ll \sum_{r\ge N}\sum_{H(\underline{x})=r} f\left(\frac{\psi(r)}{r}\right)r^n\psi(r)^{-(n-1)}\\ &\ll \sum_{r\ge N} r^n\left(\frac{\psi(r)}{r}\right)^{-(n-1)} f\left(\frac{\psi(r)}{r}\right)\\ &= \sum_{r\ge N} r^n g\left(\frac{\psi(r)}{r}\right)\to 0.\end{aligned}$$

To get the divergence case, we apply result of Beresnevich and Velani [4], which combines the Hausdorff and Lebesgue theory for lim sup sets of the type considered here in one package. With reference to their setup, we let $\mathcal{R}$ be the collection of hyperplanes in $\mathbb{R}^n$ given by the equations

$$R_{(x_1,\dots,x_n,y)}=\{\alpha\in\mathbb{R}^n : x_1\alpha_1+\cdots+x_n\alpha_n=y\},$$

where $(x_1,\dots,x_n)\in\mathbb{Z}^n$ satisfies

$$2\max\{|x_{\ell+1}|,\dots,|x_n|\}\le\max\{|x_1|,\dots,|x_\ell|\},$$

and $y\in\mathbb{Z}$. Also, let $\Upsilon(x_1,\dots,x_n,y)=\psi(H(\underline{x}))/(nH(\underline{x}))$ and let

$$\begin{aligned}&\Delta(R_{(x_1,\dots,x_n,y)},\Upsilon(x_1,\dots,x_n,y))\\ &\qquad=\{x\in\mathbb{R}^n : \operatorname{dist}(x,R_{x_1,\dots,x_n,y})\le\Upsilon(x_1,\dots,x_n,y)\}.\end{aligned}$$

It is an easy exercise to show that

$$\limsup\Delta(R_{(x_1,\dots,x_n,y)},\Upsilon(x_1,\dots,x_n,y))\subseteq\limsup B(x_1,\dots,x_n).$$

In order to invoke the main result of [4], we need a line in $\mathbb{R}^n$, such that the angle between the hyperplanes in $\mathcal{R}$ and the line is bounded away from zero. Note that the line $V = \operatorname{span}\{(0,\dots,0,1)\}$ has this property. It now follows from [4, Theorem 3] together with Theorem 4 that the divergence part holds. □

Note that the same result holds for $\ell = n$, so the above result contains the classical result of Jarník [16] and its extension to arbitrary Hausdorff measure in [13], where the more general problem of systems of linear forms is considered. In addition, the result shows that the metrical theory is indifferent to restrictions of the form studied in this paper. As a consequence of Theorem 6, we see that the dimension result valid for exact order sets [2] in the classical case remains valid under mild restrictions.

Corollary 7. *Let $\mu > n$. Then,*

$$\dim\{\alpha \in \mathbb{R}^n : \mu_{n,\ell}(\alpha) = \mu\} = n - 1 + \frac{n+1}{\mu+1}.$$

In particular, the exponent $\mu_{n,\ell}$ attains all values between n and ∞.

This result is an exact order version of a previous result of Rynne [20], who calculated the Hausdorff dimension of sets of vectors for which the Diophantine exponent obtained by restricting the $\underline{x}$ to arbitrary subsets of $\mathbb{Z}^n$ is upper bounded. In our setting, Rynne's result would give

$$\dim\{\alpha \in \mathbb{R}^n : \mu_{n,\ell}(\alpha) \le \mu\} = n - 1 + \frac{n+1}{\mu+1}.$$

Clearly, the present result is stronger in the present setup, although Rynne's result is applicable to a wider class of restrictions.

Proof. Let $\psi(r) = r^{-\mu}$ and $\psi_0(r) = r^{-\mu}/\log^2 r$. We consider the set

$$\mathcal{L}_{n,\ell}(\psi) \setminus \mathcal{L}_{n,\ell}(\psi_0).$$

This set is certainly contained in the set of the corollary. We show that the dimension of this set satisfies the corresponding lower bound. This in turn follows if we show that for $s = n - 1 + \frac{n+1}{\mu+1}$,

$$\mathcal{H}^s(\mathcal{L}_{n,\ell}(\psi)) = \infty \text{ and } \mathcal{H}^s(\mathcal{L}_{n,\ell}(\psi_0)) = 0.$$

But this follows from Theorem 6, since on inserting all definitions and reducing,

$$\sum_{r=1}^{\infty} r^n g\left(\frac{\psi(r)}{r}\right) = \sum_{r=1}^{\infty} \frac{1}{r} = \infty,$$

whereas

$$\sum_{r=1}^{\infty} r^n g\left(\frac{\psi_0(r)}{r}\right) = \sum_{r=1}^{\infty} \frac{1}{r\log^2 r} < \infty.$$

This completes the proof of Corollary 7. □

4. Small values of the exponents $\mu_{n,\ell}$

As noted just above Theorem 1, Schmidt [23] proved that for any positive ϵ and any integer $n \ge 2$ there are n-tuples α such that $\mu_{n,1}(\alpha) \le 2 + \epsilon$. His proof can be easily modified to assert the existence of α with

$$\mu_{n,1}(\alpha) \le 2. \tag{8}$$

The purpose of the present section is to prove something more, namely the following theorem.

Theorem 8. *Let $1 \le \ell < n$ and let $\ell + 1 \le \mu \le n$. Then there are continuum many $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $1, \alpha_1, \dots, \alpha_n$ linearly independent over $\mathbb{Q}$ such that*

$$\mu_{n,\ell}(\alpha) = \mu.$$

The proof is an extension of the method employed by Schmidt to prove (8) in [23]. We need a lemma from the metrical theory of Diophantine approximations. We first define an auxiliary Diophantine exponent. Let $1 \le \ell < n$ and let $(\alpha_{\ell+1}, \dots, \alpha_n) \in \mathbb{R}^{n-\ell}$ be fixed. We define

$$\begin{aligned}\tilde{\nu}_{n,\ell}(\alpha_1, \dots, \alpha_\ell) = \sup \Big\{ \nu > 0 : \min_{1 \le i \le n-\ell} &\|x_1\alpha_1 + \cdots + x_\ell\alpha_\ell + x_{\ell+i}\alpha_{\ell+i}\| < H(\underline{x})^{-\nu} \\ &\text{for infinitely many } \underline{x} \in \mathbb{Z}^n \\ &\text{with } \max\{|x_1|, \dots, |x_\ell|\} > \max\{|x_{\ell+1}|, \dots, |x_n|\} \Big\}.\end{aligned}$$

We use a metrical result for this exponent.

Lemma 9. *Let $1 \le \ell < n$ and let $(\alpha_{\ell+1}, \dots, \alpha_n) \in \mathbb{R}^{n-\ell}$ be fixed. Then, for $\nu \ge \ell+1$,*

$$\dim \left\{ (\alpha_1, \dots, \alpha_\ell) \in \mathbb{R}^\ell : \tilde{\nu}_{n,\ell}(\alpha_1, \dots, \alpha_\ell) = \nu \right\} = \ell - 1 + \frac{\ell+2}{\nu+1}.$$

Proof. Let $\psi : \mathbb{Z}_{\ge 0} \to \mathbb{R}_{>0}$ be non-increasing. Note first that

$$\begin{aligned}\mathcal{E}(\psi) &= \Big\{ (\alpha_1, \dots, \alpha_\ell) \in \mathbb{R}^\ell : \min_{1 \le i \le n-\ell} \|x_1\alpha_1 + \cdots + x_\ell\alpha_\ell + x_{\ell+i}\alpha_{\ell+i}\| < \psi(H(\underline{x})) \\ &\qquad \text{for infinitely many } \underline{x} \in \mathbb{Z}^n \\ &\qquad \text{with } \max\{|x_1|, \dots, |x_\ell|\} > \max\{|x_{\ell+1}|, \dots, |x_n|\} \Big\} \\ &\subseteq \bigcup_{1 \le i \le n-\ell} \Big\{ (\alpha_1, \dots, \alpha_\ell) \in \mathbb{R}^\ell : \|x_1\alpha_1 + \cdots + x_\ell\alpha_\ell + x_{\ell+i}\alpha_{\ell+i}\| < \psi(H(\underline{x})) \\ &\qquad \text{for infinitely many } \underline{x} \in \mathbb{Z}^n, \text{ with } \max\{|x_1|, \dots, |x_\ell|\} > |x_{\ell+i}| \Big\} \\ &= \bigcup_{1 \le i \le n-\ell} \mathcal{E}_i(\psi),\end{aligned}$$

where $\mathcal{E}_i(\psi)$ is defined by the last equality. Furthermore, as the minimum in the definition of $\mathcal{E}$ can only be attained for finitely many values of i, there exists i_0 such that $1 \le i_0 \le n - \ell$ and $\mathcal{E}_{i_0}(\psi) \subseteq \mathcal{E}(\psi)$. The upshot is that

$$\min_{1 \le i \le n-\ell} \dim(\mathcal{E}_i(\psi)) \le \dim \mathcal{E}(\psi) \le \max_{1 \le i \le n-\ell} \dim(\mathcal{E}_i(\psi)).$$

We calculate the dimension of a generic $\mathcal{E}_i$, say of $\mathcal{E}_1$.

As in the proof of Theorem 4, we restrict ourselves to the unit cube and consider the set $\mathcal{E}^* = \mathcal{E}_1 \cap [0,1]^\ell$. In analogy with the proof of Theorem 4, let

$$\begin{aligned}&B(x_1, \dots, x_\ell, x_{\ell+1}) \\ &\qquad = \left\{ (\alpha_1, \dots, \alpha_\ell) \in [0,1]^\ell : \|x_1\alpha_1 + \cdots + x_\ell\alpha_\ell + x_{\ell+1}\alpha_{\ell+1}\| \le \psi(H(\underline{x})) \right\},\end{aligned}$$

As in that proof, we find that

$$|B(x_1, \dots, x_\ell, x_{\ell+1})| \asymp \psi(H(\underline{x})). \tag{9}$$

Also, by the same argument as the one used in [14], if $(x_1, \ldots, x_\ell)$ and $(x'_1, \ldots, x'_\ell)$ are linearly independent, then for any $x_{\ell+1}, x'_{\ell+1}$,

$$\begin{aligned} &\left|B(x_1, \ldots, x_\ell, x_{\ell+1}) \cap B(x'_1, \ldots, x'_\ell, x'_{\ell+1})\right| \\ &\qquad = |B(x_1, \ldots, x_\ell, x_{\ell+1})| \cdot \left|B(x'_1, \ldots, x'_\ell, x'_{\ell+1})\right|. \end{aligned} \tag{10}$$

Finally, standard arguments from the proof of the one-dimensional Khintchine's Theorem (see *e.g.* [10]) show that if $(x_1, \ldots, x_\ell)$ and $(x'_1, \ldots, x'_\ell)$ are linearly dependent, and $(x_{\ell+1}, x'_{\ell+1}) = 1$, then

$$\begin{aligned} &\left|B(x_1, \ldots, x_\ell, x_{\ell+1}) \cap B(x'_1, \ldots, x'_\ell, x'_{\ell+1})\right| \\ &\qquad \ll |B(x_1, \ldots, x_\ell, x_{\ell+1})| \cdot \left|B(x'_1, \ldots, x'_\ell, x'_{\ell+1})\right|. \end{aligned} \tag{11}$$

Now, let

$$\begin{aligned} P_N = \big\{(x_1, \ldots, x_\ell, x_{\ell+1}) \in \mathbb{Z}_{\geq 0}^{\ell+1} \ &: H(x) = N, \\ &\gcd(x_1, \ldots, x_\ell, x_{\ell+1}) = 1 \text{ and} \\ &x_{\ell+1} \text{ is prime with } x_{\ell+1} \leq N/2\big\}. \end{aligned}$$

Let $\pi(x)$ denote the prime counting function, *i.e.*,

$$\pi(x) = \{p \leq x : p \text{ is prime}\}.$$

Arguing again as in the proof of Theorem 4, we find that

$$\# P_N \asymp \sum_{d|N} \mu(d) \left(\frac{N}{d}\right)^{\ell-1} \pi(N/(2d)) \asymp \sum_{d|N} \mu(d) \left(\frac{N}{d}\right)^{\ell} \frac{1}{\log(N/(2d))}.$$

The last asymptotic equality comes from the Prime Number Theorem. It is straightforward to check that if $\underline{x}, \underline{x}' \in \cup_{N \geq N_0} P_N$, then either (10) or (11) holds.

If $\ell > 1$, as before by (6)

$$\# P_N \gg \frac{N^\ell}{\log N} \sum_{d|N} \mu(d) \left(\frac{1}{d}\right)^{\ell} \gg \frac{N^\ell}{\log N}, \tag{12}$$

and we find from usual arguments that

$$|\mathcal{E}^*| = 1$$

whenever

$$\sum_{h=1}^{\infty} \frac{h^\ell}{\log h} \psi(h) = \infty. \tag{13}$$

When $\ell = 1$, the same conclusion is ensured by summing (13) over dyadic blocks exactly as in the proof of Theorem 4.

On the other hand, it is a straigthforward consequence of (9) and the Borel–Cantelli Lemma that

$$|\mathcal{E}^*| = 0$$

whenever

$$\sum_{h=1}^{\infty} h^\ell \psi(h) < \infty. \tag{14}$$

As in the proof of Theorem 6, we obtain an analogous Hausdorff measure result by invoking the slicing technique of [4]. In the case $\ell = 1$, we use the one-dimensional version, known as the Mass Transference Principle from [3].

Let $f : \mathbb{R} \to \mathbb{R}$ is a dimension function with $r \mapsto r^{-\ell} f(r)$ monotonically increasing and such that $g(r) = r^{-(\ell-1)} f(r)$ is also a dimension function. We proceed to get upper and lower bounds on the Hausdorff f-measure of $\mathcal{E}^*$.

The covering argument from the proof of Theorem 6 gives that

$$\mathcal{H}^f(\mathcal{E}^*) = 0,$$

whenever

$$\sum_{h=1}^{\infty} h^{\ell+1} g\left(\frac{\psi(h)}{h}\right) < \infty. \tag{15}$$

For the divergence case, an argument similar to that of the proof of Theorem 6 gives that

$$\mathcal{H}^f(\mathcal{E}^*(\psi)) = \infty,$$

whenever

$$\sum_{h=1}^{\infty} h^{\ell+1} g\left(\frac{\psi(h)}{h \log h}\right) = \infty, \tag{16}$$

where we have used the divergence condition (13).

Now, consider the dimension function $f(r) = r^s$, where $s = \ell - 1 + (l+2)/(\nu+1)$. We immediately see that for $\psi(h) = h^{-\nu} \log h$,

$$\sum_{h=1}^{\infty} h^{\ell+1} g\left(\frac{\psi(h)}{h \log h}\right) = \infty = \sum_{h=1}^{\infty} \frac{1}{h} = \infty,$$

so that by (16), $\mathcal{H}^s(\mathcal{E}^*(\psi)) = \infty$. On the other hand, letting

$$\psi_0(r) = h^{-\nu} (\log h)^{-2(\nu+1)/(\ell+2)},$$

we have,

$$\sum_{h=1}^{\infty} h^{\ell+1} g\left(\frac{\psi(h)}{h}\right) = \sum_{h=1}^{\infty} \frac{1}{h(\log h)^2} < \infty,$$

so that by (15), $\mathcal{H}^s(\mathcal{E}^*(\psi_0)) = 0$. Plainly, the set we are estimating is a subset of $\mathcal{E}^*(\psi) \setminus \mathcal{E}^*(\psi_0)$, so it has the required dimension. $\square$

Note that the proof of Lemma 9 contains a result somewhat weaker than the zero-one law of Theorem 4. Indeed, there is a gap between the series required for the measure zero and the measure one case. We have no doubt that this gap can be closed, and that the logarithmic factors in (13) and 16 can be removed. Nonetheless, we do not consider the exponent defined here to be of enough interest on its own to warrant a more detailed calculation. Furthermore, the present result is sufficient for the purposes of this paper.

Proof of Theorem 8. As in [23], we take $\alpha_{\ell+1}, \dots, \alpha_n \in \mathbb{R}$ with $1, \alpha_{\ell+1}, \dots, \alpha_n$ linearly independent over $\mathbb{Q}$ and such that for every N large enough, there is an integer q with $1 \leq q \leq N$ and

$$\|q\alpha_{\ell+i}\| < e^{-N}, \tag{17}$$

with the possible exception of one value of i, say $i_0 = i_0(N)$. This is possible by Theorem 2 of [11].

With $\alpha_{\ell+1}, \dots, \alpha_n$ fixed, we take $\alpha_1, \dots, \alpha_\ell$ such that $1, \alpha_1, \dots, \alpha_n$ are linearly independent over $\mathbb{Q}$ and such that

$$\tilde{\nu}_{n,\ell}(\alpha_1, \dots, \alpha_\ell) = \mu.$$

This is possible by Lemma 9. Let ϵ be a positive real number. Then,

$$\|y_1\alpha_1 + \cdots + y_\ell\alpha_\ell + y_{\ell+i}\alpha_{\ell+i}\| > H(\underline{y})^{-\mu-\epsilon/3}$$

holds for any choice of integers $y_1, \ldots, y_\ell, y_{\ell+i}$ and any $i = 1, \ldots, n-\ell$, if $\max\{|y_1|, \ldots, |y_\ell|, |y_{\ell+i}|\}$ is large enough.

We show that

$$\|x_1\alpha_1 + \cdots + x_n\alpha_n\| > H(\underline{x})^{-\mu-\epsilon}$$

whenever $H(\underline{x})$ is large and $\underline{x}$ is in the appropriate range. This immediately implies that $\mu_{n,\ell}(\alpha) \leq \mu + \epsilon$. Let $N = [\log H(\underline{x})]^2$ and choose an integer q such that (17) holds for all but one i. Suppose without loss of generality that $i_0(N) = 1$. Arguing in analogy with [23], recalling that $H(\underline{x})$ is attained among the first ℓ coordinates of $\underline{x}$, we get

$$\begin{aligned}
\|x_1\alpha_1 + \cdots + x_n\alpha_n\| &\geq q^{-1}\,\|x_1 q\alpha_1 + \cdots x_n q\alpha_n\| \\
&\geq q^{-1}\big(\,\|x_1 q\alpha_1 + \cdots + x_\ell q\alpha_\ell + x_{\ell+1}q\alpha_{\ell+1}\| \\
&\qquad - H(\underline{x})\left(\|q\alpha_{\ell+2}\| + \cdots + \|q\alpha_n\|\right)\big) \\
&> q^{-1}\left((qH(\underline{x}))^{-\mu-\epsilon/3} - (n-\ell-1)H(\underline{x})e^{-N}\right) \\
&> q^{-1}\left(H(\underline{x})^{-\mu-(2\epsilon/3)} - (n-\ell-1)H(\underline{x})e^{-[\log H(\underline{x})]^2}\right) \\
&> H(\underline{x})^{-\mu-\epsilon},
\end{aligned}$$

when $H(\underline{x})$ is large enough.

On the other hand, it is clear from the definition of the exponent $\tilde{\nu}_{n,\ell}$ that, for α chosen as above,

$$\mu_{n,\ell}(\alpha) \geq \tilde{\nu}_{n,\ell}(\alpha_1, \ldots, \alpha_\ell) = \mu.$$

Since ϵ is arbitrary, this gives the result.

Since there are continuum many choices for $\alpha_1, \ldots, \alpha_\ell$ by Lemma 9, we have completed the proof. □

We conclude this section by assembling all the pieces required for a proof of Theorem 1.

Proof of Theorem 1. The first part is an immediate consequence of Theorem 4. The second part follows from Corollary 7 when $\mu_\ell \geq n$ and from Theorem 8 when $\mu_\ell \in [\ell+1, n)$. □

5. On the difference between $\mu_{n,\ell}$ and w_n

We now turn to the proof of Theorem 2. It depends on earlier work by Bugeaud [7], which is related to Schmidt's proof of the existence of T-numbers [21, 22]. In order to set the scene for the argument, we give some background on these numbers. For additional details, the reader is referred to [8].

In his 1932 classification of real numbers, Mahler [19] introduced for each positive integer n a Diophantine exponent for a real number ξ by letting

$$w_n(\xi) = \sup\big\{w > 0 : 0 < |P(\xi)| < H(P)^{-w} \quad \text{for infinitely many } P(X) \in \mathbb{Z}[X], \deg(P) \leq n\big\},$$

where $H(P)$ is the naïve height of the polynomial $P(X)$, *i.e.*, the maximum of the absolute values among the coefficients of $P(X)$. Observe that $w_n(\xi)$ equals $w_n(\alpha)$, for the n-tuple $\alpha = (\xi^n, \dots, \xi)$. A related quantity is

$$w(\xi) = \limsup_{n\to\infty} \frac{w_n(\xi)}{n}.$$

Using these quantities, Mahler classified the real numbers in four classes.

- ξ is an A-number if $w(\xi) = 0$ (equivalently if ξ is algebraic).
- ξ is an S-number if $w(\xi) < \infty$.
- ξ is a T-number if $w(\xi) = \infty$ but $w_n(\xi) < \infty$ for all n.
- ξ is a U-number if $w(\xi) = \infty$ and $w_n(\xi) = \infty$ for some n.

An elementary covering argument shows that almost all numbers are S-numbers. Additionally, it is easy to see that Liouville numbers such as $\sum 10^{-n!}$ are U-numbers. By contrast, it is very difficult to prove that T-numbers exist. This was not accomplished until 1970, when Schmidt showed how to construct examples [21, 22] of such numbers.

In order to study the finer arithmetical properties of T-numbers, and in particular to study the relation between Mahler's classification and the related classification of Koksma [18], Bugeaud [7] refined Schmidt's construction. In the process, the following result was obtained.

Theorem 10 (Theorem 3' of [7])**.** *Let $n \geq 3$ be an integer, let $\mu \in [0,1]$ and let $\nu > 1$. Let $G(n) = 2n^3 + 2n^2 + 2n + 1$ and let $\chi > G(n)$. Then there is a number $\lambda \in (0, 1/2)$, prime numbers $g_1, g_2, \dots,$ with $g_1 \geq 11$ and integers $c_1, c_2 \dots,$ such that for $\gamma_j = 2^{1/n}[g_j^\mu]$, the following conditions are satisfied:*

(I_j) g_j does not divide the norm of $c_j + \gamma_j$ for any j.
(II_1) $\xi_1 = (c_1 + \gamma_1)/g_1 \in (1, 2)$.
(II_j) For any $j \geq 2$,

$$\xi_j = \frac{c_j + \gamma_j}{g_j} \in \left(\xi_{j-1} - \tfrac{1}{2} g_{j-1}^{-\nu}, \xi_{j-1} + \tfrac{3}{4} g_{j-1}^{-\nu}\right)$$

(III_1) For any algebraic number $\alpha \neq \xi_1$ of degree $\leq n$,

$$|\xi_1 - \alpha| \geq 2\lambda H(\alpha)^{-\chi}.$$

(III_j) For any $j \geq 2$ and any algebraic number $\alpha \notin \{\xi_1, \dots, \xi_j\}$ of degree $\leq n$,

$$|\xi_j - \alpha| \geq \lambda H(\alpha)^{-\chi}.$$

It is a modification of this theorem, which will enable us to prove Theorem 2. Rather than giving a complete proof (which would be quite long), we choose to outline a few explanations, based on Theorem 10. We refer to the original paper [7] for the proof of this theorem.

Proof of Theorem 2. We will be working with a number and the powers of it. Hence, our goal consists in finding real numbers ξ such that $\mu_{n,\ell}(\xi^n, \dots, \xi)$ takes a prescribed (large) value for $\ell = 1, \dots, n$. We will use a construction analogous to the one of Theorem 10.

Let n be an integer with $n \geq 2$. Let γ be a real algebraic number of degree n. The general approach consists in contructing inductively a rapidly increasing sequence $(c_j)_{j\geq 1}$ of integers and a rapidly increasing sequence $(g_j)_{j\geq 1}$ of prime numbers such that, besides various technical conditions, the sequence $(\xi_j)_{j\geq 1}$, where

$\xi_j = (c_j + \gamma)/g_j$, is rapidly convergent to a real number ξ. We do this in ensuring that the best algebraic approximants to ξ of degree at most n belong to the sequence $(\xi_j)_{j\geq 1}$ and, moreover, we control the differences $|\xi - \xi_j|$ in terms of the height $H(\xi_j)$ of ξ_j, that is, the maximal of the absolute values of the coefficients of its minimal polynomial.

More precisely, if λ is a sufficiently large real number, the construction gives that $|\xi - \xi_j| \asymp g_j^{-\lambda}$ and the height of ξ_j is exactly known in terms of g_j. In particular, if λ_1 and λ_2 are sufficiently large (for technical reasons) real numbers with $\lambda_1 < \lambda_2$, we are able to construct ξ such that $|\xi - \xi_j| \asymp H(\xi_j)^{-\lambda_2}$ for any j (see Condition (II_{j+1}) in Theorem 10 with $\nu = \lambda_2$), while $|\xi - \theta| \gg H(\theta)^{-\lambda_1}$ for any algebraic number θ of degree at most n which is not in the sequence $(\xi_j)_{j\geq 1}$ (see Condition (III_j) in 10 with $\chi = \lambda_1$).

Actually, the construction of [7] is flexible enough to give even more. Take $\lambda, \lambda_1, \ldots, \lambda_n$ real numbers with $\lambda \leq \lambda_1 \leq \ldots \leq \lambda_n$. For $k = 1, \ldots, n$, we are able indeed to construct ξ such that $|\xi - \xi_j| \asymp H(\xi_j)^{-\lambda_k}$ for any j congruent to k modulo n, while $|\xi - \theta| \geq H(\theta)^{-\lambda}$ for any algebraic number θ of degree at most n which is not in the sequence $(\xi_j)_{j\geq 1}$.

Since we are here concerned with linear forms in $1, \xi, \ldots, \xi^n$, we are not interested in the differences $|\xi - \theta|$, but merely in the values taken at ξ by integer polynomials of degree at most n. Denote by $P(X)$ the minimal defining polynomial of γ. Then, provided that g_j does not divide the norm of $c_j+\gamma$ (see Condition (I_j) of Theorem 10), the integer polynomial $Q_j(X) = P(g_jX - c_j)$ is the minimal defining polynomial of ξ_j. The construction allows us to control precisely the smallness of $|Q_j(\xi)|$, and to prove that $|Q(\xi)|$ is not too small when $Q(X)$ is not a multiple of some polynomial $Q_j(X)$.

Another important feature of this construction is that we do not have to use the same algebraic number γ at each step j of the process. Instead, we can work with a given sequence $(\gamma_j)_{j\geq 1}$ of real algebraic numbers of degree at most n. Furthermore, it has been heavily used in [7] that for $j \geq 1$ the algebraic number γ_j may depend on g_j, as in Theorem 10. This remark introduces a flexibility that is crucial for the present proof.

We now outline the difference between the proof of Theorem 10 found in [7] and the present proof.

Let $\ell = 1, \ldots, n$. For $j \geq 1$, we select $P_j(X)$, the minimal polynomial of γ_j, in such a way that the height of the polynomial $P_j(gX - c) - P_j(-c)$ is equal to the coefficient of X^ℓ, where ℓ is congruent to j modulo n. This means that on evaluating the polynomial $P_j(gX - c)$ at ξ, we get a linear form in the powers of ξ, say $a_n\xi^n + \ldots + a_1\xi + a_0$, where $|a_\ell| > \max\{|a_n|, \ldots, |a_{\ell-1}|, |a_{\ell+1}|, \ldots, |a_1|\}$. Choosing $\ell = 1$, this allows us to control precisely the small values of the linear form $||x_n\xi^n + \ldots + x_1\xi||$ subject to the condition $|x_1| > \max\{|x_2|, \ldots, |x_n|\}$. This corresponds exactly to the exponent $\mu_{n,1}(\xi^n, \ldots, \xi)$. Similarly, we can control the exponents $\mu_{n,2}(\xi^n, \ldots, \xi), \ldots, \mu_{n,n}(\xi^n, \ldots, \xi)$ by selecting ℓ appropriately. As we are controlling each exponent in a fixed residue class modulo n, we control simultaneously all exponents.

We slightly modify the construction given in [7]. Namely, we choose c_j at each step in order to ensure

$$2^{2n+2}c_j \leq g_j \leq 2^{2n+3}c_j. \tag{18}$$

With this choice, the resulting real number ξ is lying in the interval $(2^{-2n-4}, 2^{-2n})$.

Now, we give explicitly suitable minimal polynomials $P_j(X)$ of the numbers γ_j. For $\ell = n$, that is, for j divisible by n, we set $P_j(X) = X^n - 2g_j^n$. Therefore, the minimal polynomial of ξ_j is $(g_jX - c_j)^n - 2g_j^n$, and, by (18), its largest coefficient is, besides the constant coefficient, equal to the coefficient of X^n. Note that we work here with the same polynomial as in the proof of Theorem 10 with the parameter $\mu = 1$.

For any integer $\ell = 1, \ldots, n-1$ and any positive integer a, the polynomial $X^n - 2(aX-1)^\ell$ is irreducible, by Eisenstein's criterion applied with the prime 2. If j is congruent to ℓ modulo n, we set

$$P_j(X) = X^n - 2([g_j^{(n-\ell)/\ell}]X - 1)^\ell, \tag{19}$$

where $[\cdot]$ denotes the integer part. Therefore, the minimal polynomial of ξ_j is

$$Q_j(X) := P_j(g_jX - c_j) = (g_jX - c_j)^n - 2([g_j^{(n-\ell)/\ell}](g_jX - c_j) - 1)^\ell,$$

and, by (18), its largest coefficient is equal to the coefficient of X^ℓ. Note that we work here with a family of polynomials of a similar shape to the one defined in Lemma 3 of [7]. In particular, it is easily shown that $P_j(X)$ as in (19) has exactly ℓ roots very close to $1/[g_j^{(n-\ell)/\ell}]$ and that its other roots are not too close to each other.

It remains for us to explain how one proceeds to control $|\xi - \xi_j|$. Let $\mu_{n,1}, \ldots, \mu_{n,n}$ be real numbers with $\mu_{n,1}$ sufficiently large and $\mu_{n,1} \leq \ldots \leq \mu_{n,n}$. Instead of working with a single ν as in the proof of Theorem 10, we work with a sequence $(\nu_j)_{j\geq 1}$. Observe that our choice for the polynomials $P_j(X)$ implies $H(\xi_j) \asymp g_j^n$ for $j \geq 1$. Let $\lambda_1, \ldots, \lambda_n$ be (large) real numbers to be chosen later on, and set $\nu_{jn+\ell} = \lambda_\ell$ for any $\ell = 1, \ldots, n$ and any $j \geq 1$. Then $|\xi - \xi_j| \asymp H(\xi_j)^{-\lambda_\ell/n}$ if j is congruent to ℓ modulo n.

We proceed as on page 101 of [7]. Suppose that $j \equiv \ell \pmod n$. Then, $P_j(X)$ has exactly ℓ roots very close to each other with γ_j being one of them. Let $\xi_j = \beta_{j1}, \beta_{j2}, \ldots, \beta_{jn}$ denote the roots of $Q_j(X) = P_j(g_jX - c_j)$. We order these so that $\beta_{j1}, \ldots, \beta_{j\ell}$ correspond to the roots $\gamma_1, \ldots, \gamma_\ell$ of $P_j(X)$ which are close. It follows that $|\xi - \beta_{ji}| \asymp g_j^{-n^2/\ell^2}$ for $i = 2, \ldots, \ell$. Denote by $\gamma_{\ell+1}, \ldots, \gamma_n$ the remaining roots of $P_j(X)$. Now, arguing as in [7], we get

$$\begin{aligned}
|Q_j(\xi)| &= g_j^n\, |\xi - \xi_j| \prod_{2\leq i\leq \ell} |\xi - \beta_{ji}| \prod_{\ell+1\leq i\leq n} |\xi - \beta_{ji}| \\
&\asymp g_j^\ell H(\xi_j)^{-\lambda_\ell/n} g_j^{-(\ell-1)n^2/\ell^2} \prod_{\ell+1\leq i\leq n} \left| \frac{1}{\left[g_j^{(n-\ell)/\ell}\right]} - \gamma_j \right| \\
&\asymp H(\xi_j)^{-\lambda_\ell/n} g_j^{n-n^2(\ell-1)/\ell^2} \asymp H(\xi_j)^{-\delta_\ell-\lambda_\ell/n} \asymp H(Q_j)^{-\delta_\ell-\lambda_\ell/n}
\end{aligned}$$

for $\delta_\ell := 1 - n(\ell-1)/\ell$. Here, we have used Lemma 6 of [7] in order to control the product over the last $n - \ell$ roots. Note that for $\ell > 1$, δ_ℓ is a negative number. However, this is of no importance for the approximation properties studied here, since we still have freedom to choose the λ_ℓ.

The fact that η_ℓ depends only on ℓ is a consequence of the particular shape of the polynomials $Q_j(X)$. It is now sufficient to select λ_ℓ in such a way that

$\delta_\ell + \lambda_\ell/n = \mu_{n,\ell}$. With this choice, we get

$$\mu_{n,\ell}(\xi^n, \ldots, \xi) \geq \mu_{n,\ell}, \quad \ell \geq 1, \ldots, n, \tag{20}$$

as expected.

The final estimate needed is a lower bound for $|Q(\xi)|$ when $Q(X) \neq Q_j(X)$ for any j. In order to obtain such a bound, we argue again as in [7]. Let $Q(X) = aR_1(X) \cdots R_p(X)$ be a factorisation of $Q(X) \neq Q_j(X)$, a polynomial of degree at most n, into primitive irreducibles. Using the property analogous to (III$_j$) of Theorem 10 with the present polynomials, we find in analogy with equation (24) of [7], that for $1 \leq i \leq p$,

$$\begin{aligned}|R_i(\xi)| &\gg H(R_i)^{2-\deg(R_i)} |\xi - \alpha| \gg H(R_i)^{-\lambda-\deg(R_i)+2} \\ &\gg H(R_i)^{-\lambda-n+2} \gg H(R_i)^{-\mu_{n,1}}.\end{aligned}$$

The last inequality follows on insisting that $\mu_{n,1}$ is large enough. Using the so-called Gelfond-inequality,

$$|Q(\xi)| \gg (H(R_1) \cdots H(R_p))^{-\mu_{n,1}} \gg H(Q)^{-\mu_{n,1}}.$$

It immediately follows that every polynomial taking $(\xi^n, \ldots, \xi)$ close sufficiently close to zero is found among the Q_j, so that

$$\mu_{n,\ell}(\xi^n, \ldots, \xi) \leq \mu_{n,\ell}, \quad \ell = 1, \ldots, n.$$

Together with (20), this completes the proof that the tuple $\alpha = (\xi^n, \xi^{n-1}, \ldots, \xi)$ satisfies all the desired equalities. Additionally, there is still enough flexibility in the construction to ensure that there are continuum many such ξ. This completes the proof.

To conclude, we point out that we can construct a suitable α with $\mu_{n,1}(\alpha) \ll n^3$, which ensures that the exponents are not all infinite. Our process is, like in [7], effective. □

6. Lower bounds for the exponents $\mu_{n,\ell}$

Using the exponents w_2 and $\hat{w}_2$, it is easily seen that Lemma 1 of Schmidt [4] can be rewritten as

$$\mu_{2,1} \geq \frac{\hat{w}_2}{\hat{w}_2 - 1}.$$

Its proof can be straightforwardly extended to arbitrary n and ℓ. This was already done by Thurnheer for $\ell = n - 1$.

Proposition 11. *Let $n \geq 2$ be an integer. For any n-tuple α and any integer $\ell = 1, \ldots, n$, we have*

$$\mu_{n,\ell}(\alpha) \geq \frac{\ell\, \hat{w}_n(\alpha)}{\hat{w}_n(\alpha) - n + \ell}. \tag{21}$$

Proof. For simplicity, we write $\mu_{n,\ell}$ for $\mu_{n,\ell}(\alpha)$ and $\hat{w}_n$ for $\hat{w}_n(\alpha)$. Without loss of generality, we may asume that $\ell < \mu_{n,\ell} < n$. Let $\eta \geq 1$ be a real number. Consider the convex body $\mathcal{B}$ given by the equations

$$\begin{aligned}|x_1|, \ldots, |x_\ell| &\leq N^\eta, \\ |x_{\ell+1}|, \ldots, |x_n| &\leq N, \\ |x_1\alpha_1 + \ldots + x_n\alpha_n + x_0| &\leq N^{-\ell\eta-n+\ell} = (N^\eta)^{-(\ell\eta+n-\ell)/\eta}.\end{aligned}$$

By Minkowski's theorem, it contains a non-zero point with integer coordinates. Let ϵ be a positive real number with $\epsilon < \mu_{n,\ell} - \ell$ and $\epsilon < n - \mu_{n,\ell}$. The definition of the exponent $\mu_{n,\ell}$ implies that, when N is sufficiently large, the system of equations

$$\begin{aligned} |x_1|, \dots, |x_n| &\le N^\eta, \\ \max\{|x_1|, \dots, |x_\ell|\} &> \max\{|x_{\ell+1}|, \dots, |x_n|\}, \\ |x_1\alpha_1 + \dots + x_n\alpha_n + x_0| &\le (N^\eta)^{-(\mu_{n,\ell}+\epsilon)}, \end{aligned}$$

has no solution. Consequently, if η is defined by

$$\mu_{n,\ell} + \epsilon = \frac{\ell\eta + n - \ell}{\eta},$$

then, for large N, any non-zero integer point $(x_0, x_1, \dots, x_n)$ in $\mathcal{B}$ satisfies

$$\max\{|x_1|, \dots, |x_\ell|\} \le \max\{|x_{\ell+1}|, \dots, |x_n|\} \le N.$$

This shows in turn that

$$\hat{w}_n \ge \ell\eta + n - \ell = (n-\ell)\left(1 + \frac{\ell}{\mu_{n,\ell} + \epsilon - \ell}\right),$$

which gives the desired inequality when ϵ tends to zero. □

Since w_n is almost always equal to n (see [25]) and $\mu_{n,\ell} \le w_n$, it immediately follows from Proposition 11 that the exponent $\mu_{n,\ell}$ is almost always equal to n. This gives an alternative proof of the first assertion of Theorem 1.

When we follow the second part of the proof of Theorem 1 of Schmidt [23], we see that he actually established the inequality

$$\mu_{2,1} \ge \hat{w}_2 - 1 + \frac{\hat{w}_2}{w_2},$$

although he only used the (often) weaker inequality $\mu_{2,1} \ge \hat{w}_2 - 1$.

Likewise, Thurnheer [30] extended in 1990 Schmidt's result by proving that $\mu_{n,n-1} \ge \hat{w}_n - 1$, but his paper contains the proof of the lower bound

$$\mu_{n,n-1} \ge \hat{w}_n - 1 + \frac{\hat{w}_n}{w_n},$$

as given in Theorem 3.

7. Concluding remarks

It is interesting to note that while the results proved by metrical methods, *i.e.*, Theorem 1, are results in all of $\mathbb{R}^n$, the explicit constructions of Theorem 2 are carried out on the Veronese curve $(\xi^n, \xi^{n-1}, \dots, \xi) \subseteq \mathbb{R}^n$. For the classical exponent $w_n = \mu_{n,n}$, the metrical theory for Lebesgue measure is the same in the two settings, as shown by Beresnevich [1] for the Veronese curves and more generally for non-degenerate manifolds by Beresnevich, Bernik, Kleinbock and Margulis [5].

We have not been able to show that the metrical theory remains the same when restricted to non-degenerate curves and manifolds for $\mu_{n,\ell}$ with $\ell < n$. Nonetheless, it remains of interest whether the results of the present paper may be extended or improved on such sets.

For what it is worth, if $\alpha = (\xi^n, \xi^{n-1}, \ldots, \xi)$, then $\hat{w}_n(\alpha)$ is at most $2n-1$, as established by Davenport and Schmidt [12]. Hence, using Theorem 3,

$$\mu_{n,\ell}(\alpha) \geq 2\ell - \frac{\ell(2\ell-1)}{n-1+\ell}. \tag{22}$$

Inserting $\ell = 1$ and letting n increase, this provides a positive answer to Problem 2 along Veronese curves.

References

[1] V. Beresnevich, *On approximation of real numbers by real algebraic numbers*, Acta Arith. **90** (1999), no. 2, 97–112.

[2] V. Beresnevich, D. Dickinson, and S. Velani, *Sets of exact 'logarithmic' order in the theory of Diophantine approximation*, Math. Ann. **321** (2001), no. 2, 253–273.

[3] V. Beresnevich and S. Velani, *A mass transference principle and the Duffin-Schaeffer conjecture for Hausdorff measures*, Ann. of Math. (2) **164** (2006), no. 3, 971–992.

[4] ———, *Schmidt's theorem, Hausdorff measures, and slicing*, Int. Math. Res. Not. (2006), Art. ID 48794, 24 pages.

[5] V. V. Beresnevich, V. I. Bernik, D. Y. Kleinbock, and G. A. Margulis, *Metric Diophantine approximation: the Khintchine-Groshev theorem for nondegenerate manifolds*, Mosc. Math. J. **2** (2002), no. 2, 203–225.

[6] V. I. Bernik and M. M. Dodson, *Metric Diophantine approximation on manifolds*, Cambridge Tracts in Mathematics, vol. 137, Cambridge University Press, Cambridge, 1999.

[7] Y. Bugeaud, *Mahler's classification of numbers compared with Koksma's*, Acta Arith. **110** (2003), no. 1, 89–105.

[8] ———, *Approximation by algebraic numbers*, Cambridge Tracts in Mathematics, vol. 160, Cambridge University Press, Cambridge, 2004.

[9] Y. Bugeaud and M. Laurent, *On exponents of homogeneous and inhomogeneous Diophantine approximation*, Mosc. Math. J. **5** (2005), no. 4, 747–766, 972.

[10] J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 45, Cambridge University Press, New York, 1957.

[11] H. Davenport and W. M. Schmidt, *A theorem on linear forms*, Acta Arith. **14** (1967/1968), 209–223.

[12] ———, *Approximation to real numbers by algebraic integers*, Acta Arith. **15** (1968/1969), 393–416.

[13] D. Dickinson and S. Velani, *Hausdorff measure and linear forms*, J. Reine Angew. Math. **490** (1997), 1–36.

[14] M. M. Dodson, *Geometric and probabilistic ideas in the metric theory of Diophantine approximations*, Uspekhi Mat. Nauk **48** (1993), no. 5(293), 77–106.

[15] A. V. Groshev, *A thoerem on systems of linear forms*, Dokl. Akad. Nauk SSSR **19** (1938), 151–152.

[16] V. Jarník, *Über die simultanen diophantischen Approximationen*, Math. Z. **33** (1931), no. 1, 505–543.

[17] ———, *Über die angenäherte Lösung der Gleichung $x_1\theta_1 + \cdots + x_n\theta_n + x_0 = 0$ in ganzen Zahlen*, Casopis Mat. Fys. **66** (1937), 192–205.

[18] J. F. Koksma, *Über die Mahlersche Klasseneinteilung der transzendenten Zahlen und die Approximation komplexer Zahlen durch algebraische Zahlen*, Monatsh. Math. Phys. **48** (1939), 176–189.

[19] K. Mahler, *Zur Approximation der Exponentialfunktionen ond des Logarithmus. I, II*, J. Reine. Angew. Math. **166** (1932), 118–150.

[20] B. P. Rynne, *The Hausdorff dimension of certain sets arising from Diophantine approximation by restricted sequences of integer vectors*, Acta Arith. **61** (1992), no. 1, 69–81.

[21] W. M. Schmidt, *T-numbers do exist*, Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69), Academic Press, London, 1970, pp. 3–26.

[22] ———, *Mahler's T-numbers*, 1969 Number Theory Institute (Proc. Sympos. Pure Math., Vol. XX, State Univ. New York, Stony Brook, N.Y., 1969), Amer. Math. Soc., Providence, R.I., 1971, pp. 275–286.

[23] ———, *Two questions in Diophantine approximation*, Monatsh. Math. **82** (1976), no. 3, 237–245.

[24] ———, *Open problems in Diophantine approximation*, Diophantine approximations and transcendental numbers (Luminy, 1982), Progr. Math., vol. 31, Birkhäuser Boston, Boston, MA, 1983, pp. 271–287.

[25] V. G. Sprindžuk, *Mahler's problem in metric number theory*, Translated from the Russian by B. Volkmann. Translations of Mathematical Monographs, Vol. 25, American Mathematical Society, Providence, R.I., 1969.

[26] P. Thurnheer, *Un raffinement du théorème de Dirichlet sur l'approximation diophantienne*, C. R. Acad. Sci. Paris Sér. I Math. **293** (1981), no. 14, 623–624.

[27] ———, *Eine Verschärfung des Satzes von Dirichlet über diophantische Approximation*, Comment. Math. Helv. **57** (1982), no. 1, 60–78.

[28] ———, *Zur diophantischen Approximation von zwei reellen Zahlen*, Acta Arith. **44** (1984), no. 3, 201–206.

[29] ———, *Approximation diophantienne par certains couples d'entiers*, C. R. Math. Rep. Acad. Sci. Canada **7** (1985), no. 1, 51–53.

[30] ———, *On Dirichlet's theorem concerning Diophantine approximation*, Acta Arith. **54** (1990), no. 3, 241–250.

Y. Bugeaud, Université Louis Pasteur, Mathématiques, 7, rue René Descartes, F-67084 Strasbourg cedex, France

S. Kristensen, Department of Mathematical Sciences, Faculty of Science, University of Aarhus, Ny Munkegade, Building 530, DK-8000 Aarhus C, Denmark