Introduction to the theory of elliptic algebras

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Abstract

The purpose of this course is to make an informal and nontechnical introduction to the theory of elliptic algebras. These algebras are associative $\mathbb{N}$-graded algebras presented by $n$ generators and $n(n - 1)/2$ quadratic relations and satisfying the so-called Poincare-Birkhoff-Witt condition (PBW-algebras). We will consider examples of such algebras depending on two continuous parameters (namely, on an elliptic curve and a point on this curve) which are flat deformations of the polynomial ring in $n$ variables. Diverse properties of these algebras will be described, together with their relations to integrable systems, deformation quantization, moduli spaces and other directions of modern investigations.
Appendix D.  

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Introduction

In this course we work over the base field $\mathbb{C}$. All manifolds are supposed to be algebraic as well as differential geometric structures on these manifolds. By algebra we always mean associative algebra if the opposite is not stated explicitly.

1 Some basic definitions

Algebra defined by generators and relations

Let $V$ be a linear space over the field $\mathbb{C}$. Let $L \subset T^*V$. By the algebra generated by $V$ with defined relations $L$ we mean the quotient algebra $A = T^*V/(L)$ where $(L)$ is the two sided ideal in the tensor algebra $T^*V$ generated by $L$. We will systematically omit $\otimes$ writing elements from $L$. Let $x_1, ..., x_n$ be some basis of $V$ and $r_1, ..., r_N$ be some basis of $L$. We could also say that $A$ is generated by $x_1, ..., x_n$ with relations $r_1 = 0, ..., r_N = 0$. For example, the polynomial algebra $\mathbb{C}[x_1, ..., x_n]$ is generated by $x_1, ..., x_n$ with relations $x_i x_j - x_j x_i = 0, i, j = 1, ..., n$. In this case $N = \dim L = n(n-1)/2$.

Universal enveloping algebra

Let $\mathfrak{g}$ be a Lie algebra of dimension $n$ with a basis $\{x_1, ..., x_n\}$. The universal enveloping algebra $U(\mathfrak{g})$ is the algebra defined by the generators $\{x_1, ..., x_n\}$ and relations $x_i x_j - x_j x_i - [x_i, x_j] = 0$ where $i, j = 1, ..., n$ and $[x_i, x_j]$ is the Lie bracket. A key property of the universal enveloping algebra is the so called Poincare-Birkhoff-Witt theorem which states that the ordered monomials $\{x_1^{\alpha_1} \ldots x_n^{\alpha_n}; \alpha_1, \ldots, \alpha_n \in \mathbb{Z}_{\geq 0}\}$ form a basis of this algebra.

PBW-algebra

Let $V$ be a linear space of dimension $n$ over the field $\mathbb{C}$. Let $L \subset V \otimes V$ be a subspace of dimension $\frac{n(n-1)}{2}$. Let us construct an algebra $A$ with the space
of generators $V$ and the space of defining relations $L$, that is, $A = T^*V/(L)$, where $T^*V$ is the tensor algebra of the space $V$ and $(L)$ is the two-sided ideal generated by $L$. It is clear that the algebra $A$ is $\mathbb{Z}_{\geq 0}$-graded because the ideal $(L)$ is homogeneous. We have $A = C \oplus A_1 \oplus A_2 \oplus \ldots$, where $A_1 = V$, $A_2 = V \otimes V/L$, $A_3 = V \otimes V \otimes V/(V \otimes L + L \otimes V)$, etc.

**Definition.** We say that $A$ is a PBW-algebra (or satisfies the Poincare-Birkhoff-Witt condition) if $\dim A_\alpha = n(n+1)/2\alpha!$.

Thus, a PBW-algebra is an algebra with $n$ generators and $n(n+1)/2$ quadratic relations for which the dimensions of the graded components are equal to those of the polynomial ring in $n$ variables.

**Theta functions**

See Appendix A for definition and main properties of theta functions in one variable and Appendix B about some theta functions in several variables.

### 2 Historical remarks

In the paper [45] devoted to study the so-called $XYZ$-model and the representations of the corresponding algebra of monodromy matrices, Sklyanin introduced the family of associative algebras with four generators and six quadratic relations which are nowadays called Sklyanin algebras (see also Appendix D.1). The algebras of this family are naturally indexed by two continuous parameters, namely, by an elliptic curve and a point on this curve, and each of them is a flat deformation of the polynomial ring in four variables in the class of $\mathbb{Z}_{\geq 0}$-graded associative algebras. On the other hand, a family of algebras with three generators (and three quadratic relations) with the same properties arose in [2], [34] (see also [52]). In what follows it turned out (see [10], [17]-[22], [32]-[38]) that such algebras exist for arbitrarily many generators.

Algebras of this kind arise in diverse areas of mathematics: in the theory of integrable systems [45], [46], [28], [9], moduli spaces [20], deformation quantization [12], [26], non-commutative geometry [2], [3], [11], [27], [47]-[49], [51], cohomology of algebras [8], [29], [41]-[44], [50], and quantum groups and $R$-matrices [45], [46], [25], [16], [14], [23], [31]. See Appendix D.
3 Poisson manifolds and deformation quantization

Let \( M \) be a manifold (\( C^\infty \), analytic, algebraic, etc.) and let \( \mathcal{F}(M) \) be the function algebra on \( M \). In the “physical” language, \( M \) is the state space of the system and \( \mathcal{F}(M) \) is the algebra of observables. In [5] the following approach to the quantization was suggested: the underlying vector space of the quantum algebra of observables coincide with that of \( \mathcal{F}(M) \), but the multiplication is deformed and is no longer commutative (though still associative). Moreover, the multiplication depends on the deformation parameter (Planck constant). For \( \hbar = 0 \) we have the ordinary commutative multiplication. Since the Planck constant is small, we do not notice that the observables in classical mechanics are non-commutative. Expanding the multiplication in the series in powers of \( \hbar \) we obtain
\[
 f \ast g = fg + \{f, g\} \hbar + o(\hbar).
\]
The operation \( \{\cdot, \cdot\} : \mathcal{F}(M) \otimes \mathcal{F}(M) \to \mathcal{F}(M) \) is bilinear, and, applying the gauge transformations, one can make it anticommutative, \( \{f, g\} = -\{g, f\} \). Moreover, since the multiplication \( \ast \) is associative, we see that \( \{f, gh\} = \{fg\} h + \{f, h\} g \) (the Leibniz rule) and \( \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0 \) (the Jacobi identity). The Leibniz rule means that \( \{f, g\} = \langle w, df \wedge dg \rangle \), where \( w \) is a bivector field on \( M \), and the Jacobi identity means that \( [w, w] = 0 \). Thus, every quantization defines a Poisson structure \( w \) (or \( \{\cdot, \cdot\} \)) on the manifold \( M \). The inverse problem arises: construct a quantization \( \ast \) from a manifold \( M \) with Poisson structure (a Poisson manifold). This problem was solved in [26] at the formal level. Namely, bidifferential operators \( B_n \) (\( n \geq 2 \)) were constructed on a Poisson manifold \( M \) in such a way that the formal series
\[
 f \ast g = fg + \{f, g\} \hbar + \sum_{n \geq 2} B_n(f, g) \hbar^n
\]
satisfies the condition \( (f \ast g) \ast h = f \ast (g \ast h) \). Moreover, the operators \( B_n \) are constructed from \( \{\cdot, \cdot\} \) by an explicit formula. Thus, the problem of formal quantization was solved; however, the problem on the convergence of the series for \( f \ast g \) and on its identification remains open. This problem seems to be very complicated. For instance, let \( M = \mathbb{C}^n \), \( \mathcal{F}(M) = S^*(\mathbb{C}^n) = \mathbb{C}[x_1, \ldots, x_n] \), and let the Poisson bracket be quadratic \( \{x_i, x_j\} = \sum_{\alpha, \beta} c^{\alpha\beta}_{ij} x_\alpha x_\beta \), where \( c^{\alpha\beta}_{ij} \in \mathbb{C} \) are symmetric with respect to \( \alpha, \beta \) and antisymmetric with respect to \( i, j \). In this case, the multiplication \( \ast \) must be homogeneous, that is, \( S^\alpha(\mathbb{C}^n) \ast S^\beta(\mathbb{C}^n) \subset S^{\alpha+\beta}(\mathbb{C}^n) \). Therefore, the structure constants of the multiplication \( \ast \) in the basis \( \{x^{\alpha_1}_1 \ldots x^{\alpha_n}_n; \alpha_1, \ldots, \alpha_n \in \mathbb{Z}_{\geq 0}\} \) turn out to be formal series in \( \hbar \) which baffle
the explicit evaluation even in the simplest case \( n = 2 \), \( \{x_1, x_2\} = \alpha x_1 x_2 \). In this case it is natural to assume that the quantum algebra must be defined by the relation \( x_1 x_2 = e^{-\alpha \hbar} x_2 x_1 \). On the other hand, the algebras \( Q_{n,k}(E, \eta) \) introduced below are examples of the quantization of \( M = \mathbb{C}^n \), where \( \eta \) plays the role of Planck constant because \( Q_{n,k}(E, 0) = \mathbb{C}[x_1, \ldots, x_n] \). Moreover, the structure constants of the algebra \( Q_{n,k}(E, \eta) \) turn out to be elliptic functions of \( \eta \). There are also rational and trigonometric limits of the algebras \( Q_{n,k}(E, \eta) \) in which the structure constants are rational (trigonometric) functions of \( \eta \) (see [32]).

4 Examples of PBW-algebras

Since there are no classification results in the theory of PBW-algebras (for \( n > 3 \)), we deal with specific examples only. The known examples can conditionally be divided into two classes, namely, rational and elliptic algebras.

Let us present examples of rational algebras.

1. Skew polynomials. This is the algebra with the generators \( \{x_i; i = 1, \ldots, n\} \) and the relations \( x_i x_j = q_{i,j} x_j x_i \), where \( i < j \) and \( q_{i,j} \neq 0 \).

One can readily see that the monomials \( \{x_1^{\alpha_1} \cdots x_n^{\alpha_n}; \alpha_1, \ldots, \alpha_n \in \mathbb{Z}_{\geq 0}\} \) form a basis of the algebra of skew polynomials, which implies the PBW condition. Since \( q_{i,j} \) are arbitrary non-zero numbers, we have obtained an \( n(n-1)/2 \)-parameter family of algebras.

2. Projectivization of Lie algebras. Let \( g \) be a Lie algebra of dimension \( n - 1 \) with a basis \( \{x_1, \ldots, x_{n-1}\} \). We construct an algebra with \( n \) generators \( \{c, x_1, \ldots, x_{n-1}\} \) and the relations \( c x_i = x_i c \) and \( x_i x_j - x_j x_i = c[x_i, x_j] \).

The condition PBW follows from the Poincare-Birkhoff-Witt theorem for the algebra \( g \).

3. Drinfeld algebra. A new realization of the quantum current algebra \( U_q(\widehat{sl}_2) \) was suggested in [13] (see also [25] for details and definitions). Namely, the generators \( x_k^\pm, h_k \ (k \in \mathbb{Z}) \) similar to the ordinary basis of the Lie algebra \( \widehat{sl}_2 \) were introduced. It is assumed that the elements \( x_k^+ \) satisfy the quadratic relations

\[
 x_{k+1}^+ x_k^+ - q^2 x_k^+ x_{k+1}^+ = q^2 x_k^+ x_{k+1}^+ - x_{k+1}^+ x_k^+ .
\]  

The elements \( x_k^- \) satisfy similar relations. The algebra \( \text{Dr}_n(q) \subset U_q(\widehat{sl}_2) \) generated by \( x_1^+, \ldots, x_n^+, n \in \mathbb{N}, q \in \mathbb{C}^* \), is a PBW-algebra.
5 Elliptic algebras

In the elliptic case the algebra depends on two continuous parameters, namely, an elliptic curve $E$ and a point $\eta \in E$. Just these algebras are the subject of our course. Their structure constants are elliptic functions of $\eta$ with modular parameter $\tau$. Our main example is given by the algebras $Q_{n,k}(E,\eta)$, where $n \geq 3$ is the number of generators, $k$ is a positive integer coprime to $n$, and $1 \leq k < n$. We define the algebra $Q_{n,k}(E,\eta)$ by the generators $\{x_i; i \in \mathbb{Z}/n\mathbb{Z}\}$ and the relations

$$
\sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+r(k-1)}(0)}{\theta_{kr}(\eta)\theta_{j-i-r}(-\eta)} x_{j-r} x_{i+r} = 0.
$$

(2)

See Appendix A for definition of functions $\theta_i(z)$. The structure of these algebras depends on the expansion of the number $n/k$ in the continued fraction, and therefore we first study the simplest case $k = 1$ and then pass to the general case. The fact that the algebra $Q_{n,k}(E,\eta)$ belongs to the class of PBW-algebras is proved only for generic parameters $E$ and $\eta$ (see §2.6 and §3). However, we conjecture that this holds for any $E$ and $\eta$. A possible way to prove this conjecture is to produce an analog of the functional realization (see §2.1) for arbitrary $k$ by using the constructions in §5.

One can say that the algebras $Q_{n,k}(E,\eta)$ are a typical example of elliptic algebras. However, they are far from exhausting the list of all elliptic algebras. The simplest example of an elliptic algebra that does not belong to this class (and even is not a deformation of the polynomial ring) can be constructed as follows. Let the group $(\mathbb{Z}/2\mathbb{Z})^2$ with the generators $g_1, g_2$ act by automorphisms on the algebra $Q_{4,1}(E,\eta)$ as follows: $g_1(x_i) = x_{i+2}$, $g_2(x_i) = (-1)^i x_i$. The same group acts on the algebra of $(2 \times 2)$ matrices, $g_1(\gamma) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}$, $g_2(\gamma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \gamma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}$. This gives an action on the tensor product of associative algebras $Q_4(E,\eta) \otimes \text{Mat}_2$. Let $\tilde{Q}_4(E,\eta) \subset Q_4(E,\eta) \otimes \text{Mat}_2$ consist of elements invariant with respect to the group action. One can readily see that the dimension of the graded components of $\tilde{Q}_4(E,\eta)$ coincide with those of $Q_4(E,\eta)$, and therefore $\tilde{Q}_4(E,\eta)$ is a PBW-algebra. For another example of PBW-algebra (with 3 generators), see the end of §1.
6 Functional construction of PBW-algebras

Let us now describe one of the main constructions of PBW-algebras. Let \( \lambda(x, y) \) be a meromorphic function of two variables. We construct an associative graded algebra \( \mathcal{F}_\lambda \) as follows. Let the underlying linear space of \( \mathcal{F}_\lambda \) coincide with

\[
\mathcal{F}_\lambda = \mathbb{C} \oplus F_1 \oplus F_2 \oplus \ldots,
\]

where \( F_1 = \{ f(u) \} \) is the space of meromorphic functions of one variable and \( F_\alpha = \{ f(u_1, \ldots, u_\alpha) \} \) is the space of symmetric meromorphic functions of \( \alpha \) variables. The space \( F_\alpha \) is a natural extension of the symmetric power \( S^\alpha F_1 \). The multiplication in the algebra \( \mathcal{F}_\lambda \) is defined as follows: for \( f \in F_\alpha \), and \( g \in F_\beta \) the product \( f \ast g \in F_{\alpha+\beta} \) is

\[
f \ast g(u_1, \ldots, u_{\alpha+\beta}) = \sum_{\sigma \in S_{\alpha+\beta}} f(u_{\sigma_1}, \ldots, u_{\sigma_\alpha}) g(u_{\sigma_{\alpha+1}}, \ldots, u_{\sigma_{\alpha+\beta}}) \prod_{1 \leq i \leq \alpha \atop \alpha+1 \leq j \leq \alpha+\beta} \lambda(u_{\sigma_i}, u_{\sigma_j}). \quad (3)
\]

In particular, if \( f, g \in F_1 \), then

\[
f \ast g(u_1, u_2) = f(u_1)g(u_2)\lambda(u_1, u_2) + f(u_2)g(u_1)\lambda(u_2, u_1). \quad (4)
\]

One can readily see that the multiplication \( \ast \) is associative for any \( \lambda(x, y) \).

We now assume that \( \lambda(x, y) = \frac{x-y}{x+y} \), where \( q \in \mathbb{C}^\ast \). Let \( F_1^{(n)} = \{ 1, u, \ldots, u^{n-1} \} \subset F_1 \) be the space of polynomials of degree less than \( n \). Let \( F_\alpha^{(n)} = S^\alpha F_1^{(n)} \subset F_\alpha \) be the space of symmetric polynomials in \( \alpha \) variables of degree less than \( n \) with respect to any variable. One can readily see that \( F_\alpha^{(n)} \ast F_\beta^{(n)} \subset F_{\alpha+\beta}^{(n)} \). Therefore, the algebra \( \mathcal{F}_\lambda^{(n)} = \oplus_\alpha F_\alpha^{(n)} \) is a subalgebra of \( \mathcal{F}_\lambda \). Moreover, for \( q = 1 \) the algebra \( \mathcal{F}_\lambda^{(n)} \) is the polynomial ring \( S^* F_1^{(n)} \) because \( \lambda(x, y) = 1 \) in this case. Therefore, the algebra \( \mathcal{F}_\lambda^{(n)} \) is a PBW-algebra for generic \( q \). This algebra is isomorphic to the Drinfeld algebra \( \text{Dr}_n(q) \), and an isomorphism is given by the rule \( u^k \mapsto x_{k+1}^k \). The algebra \( Q_n(\mathcal{E}, \eta) \) can be obtained in a similar way with the only modification that the polynomials are replaced by theta functions (see §2.1). A similar construction [38], [22] enables one to construct quantum moduli spaces \( \mathcal{M}(\mathcal{E}, B) \) (see Appendix D.3) for any Borel subgroup \( B \). The construction of algebras \( Q_{n,k}(\mathcal{E}, \eta) \) for \( k > 1 \) (and, more generally, quantum moduli spaces \( \mathcal{M}(\mathcal{E}, P) \) for a parabolic subgroup \( P \)) is more complicated and involves exchange algebras (see §5 and [21]) or elliptic \( R \)-matrices (see §4).
Let us now describe the contents of the course. In §1 we describe the simplest elliptic PBW-algebras, namely, algebras $Q_{3,1}(\mathcal{E}, \eta)$ with three generators. These algebras were studied in many papers, see, for instance, [2], [3]. The section is of illustrative nature; we intend to explain some methods of studying elliptic algebras by the simplest example. The main attention in the course is paid to the algebras $Q_n(\mathcal{E}, \eta)$, which are discussed in §2. We give an explicit construction of these algebras, present natural families of their representations (which are studied in [19] in more detail), and describe the symplectic leaves of the corresponding Poisson algebra (we recall that $Q_n(\mathcal{E}, 0)$ is the polynomial ring in $n$ variables).

The structure of the algebras $Q_{n,k}(\mathcal{E}, \eta)$, $k > 1$, is more complicated, and the detailed description of their properties is beyond the framework of the course (see [35], [20]). The main properties of these algebras are described in §3. In §4 we explain the relationship between these algebras and Belavin’s elliptic $R$-matrices. In §5 we establish a relation of the algebras $Q_{n,k}(\mathcal{E}, \eta)$ to the so-called exchange algebras (see (24), (25), and also [36], [24], [33]).

In Appendices A, B, C we present the notation we need and the properties of theta functions of one and several variables. Appendix D contains a brief review of relations of elliptic algebras with other areas of mathematics. We tried to make this part independent of the main text.

In conclusion we say a few words concerning the facts that remain outside the course but are immediately connected with its topic. In [37] the algebras $Q_{n,k}(\mathcal{E}, \eta)$ are studied provided that $\eta \in \mathcal{E}$ is a point of finite order. In this case the properties of the algebras $Q_{n,k}(\mathcal{E}, \eta)$ are similar to those of quantum groups if $q$ is a root of unity; in particular, these algebras are finite-dimensional over the centre. In [32] we study rational degenerations of the algebras $Q_{n,k}(\mathcal{E}, \eta)$ occurring if the elliptic curve $\mathcal{E}$ degenerates into the union of several copies of $\mathbb{CP}^1$ or into $\mathbb{CP}^1$ with a double point.

The algebras $Q_{n,k}(\mathcal{E}, \eta)$ are obtained when quantizing the components of the moduli spaces $\mathcal{M}(P, \mathcal{E})$ (see Appendix D.3) that are isomorphic to $\mathbb{CP}^{n-1}$. The quantization of other components leads to elliptic algebras of more general form. These algebras were constructed in [38], [22] if $P$ is a Borel subgroup of an arbitrary group $G$. The case in which $P \subseteq GL_m$ is an arbitrary parabolic subgroup of $GL_m$ is studied in [21].

The symplectic leaves of a Poisson manifold corresponding to the family of algebras $Q_{n,k}(\mathcal{E}, \eta)$ in a neighbourhood of $\eta = 0$ and for a fixed elliptic
The corresponding Poisson algebras belong to the class of algebras with regular structure of symplectic leaves; these algebras were studied in [39].

§1 Algebras with three generators

In this section we consider the simplest examples of elliptic PBW-algebras, namely, the algebras with three generators. Let us first study the quadratic Poisson structures on $\mathbb{C}^3$. Let $x_0, x_1, x_2$ be the coordinates on $\mathbb{C}^3$ and let there be a Poisson structure that is quadratic in these coordinates. We construct the polynomial $C = x_0\{x_1, x_2\} + x_1\{x_2, x_0\} + x_2\{x_0, x_1\}$. This is a homogeneous polynomial of degree three because the Poisson structure is quadratic. It is clear that the form of this polynomial is preserved under linear changes of coordinates (up to proportionality). Let us restrict ourselves to the non-degenerate case in which the equation $C = 0$ defines a non-singular projective manifold. It is clear that this is an elliptic curve. Moreover, by a linear change of variables one can reduce the polynomial $C$ to the form

$$C = x_0^3 + x_1^3 + x_2^3 + 3k x_0 x_1 x_2,$$

where $k \in \mathbb{C}$. In this case, as one can see by direct computations using the definition of $C$ and the Jacobi identity, the Poisson structure must be of the form (up to proportionality):

$$\{x_0 x_1\} = x_2^2 + k x_0 x_1, \quad \{x_1 x_2\} = x_0^2 + k x_1 x_2, \quad \{x_2 x_0\} = x_1^2 + k x_2 x_0. \quad (5)$$

Moreover, $\{x_i, C\} = 0$, and every central element is a polynomial in $C$.

We recall that each Poisson manifold can be partitioned into the so-called symplectic leaves, which are Poisson submanifolds, and the restrictions of the Poisson structure to these submanifolds are non-degenerate. In our case, the symplectic leaves are as follows:

1) the origin $x_0 = x_1 = x_2 = 0$;
2) the homogeneous manifold $C = 0$ without the origin;
3) the manifolds $C = \lambda$, where $\lambda \in \mathbb{C}$, $\lambda \neq 0$.

It is clear that our Poisson structure admits the automorphisms $x_i \mapsto \varepsilon^i x_i$ and $x_i \mapsto x_{i+1}$, where $\varepsilon^3 = 1$, $i \in \mathbb{Z}/3\mathbb{Z}$. It is natural to assume that the quantization of the Poisson structure (see Appendix D.2) is the family of associative algebras with the generators $x_0, x_1, x_2$ and three quadratic relations admitting the same automorphisms. However, each generic three-dimensional space of quadratic relations which is invariant with respect to
These automorphisms is of the form
\begin{align*}
x_0x_1 - qx_1x_0 &= px_2^2, \\
x_1x_2 - qx_2x_1 &= px_0^2, \\
x_2x_0 - qx_0x_2 &= px_1^2,
\end{align*}
(6)
where \( p, q \in \mathbb{C} \) are complex numbers. We denote by \( A_{p,q} \) the algebra with the generators \( x_0, x_1, x_2 \) and the defining relations (6). It is clear that the algebra \( A_{p,q} \) is \( \mathbb{Z}_{\geq 0} \)-graded, that is, \( A_{p,q} = \mathbb{C} \oplus F_1 \oplus F_2 \oplus \ldots \), where \( F_\alpha F_\beta \subseteq F_{\alpha+\beta} \). Here \( F_\alpha \) stands for the linear space spanned by the (non-commutative) monomials in \( x_0, x_1, x_2 \) of degree \( \alpha \). It is natural to expect that the dimension of \( F_\alpha \) is equal to that of the space of polynomials in three variables of degree \( \alpha \), that is, \( \dim F_\alpha = \binom{\alpha+1}{2} \).

Moreover, the Poisson algebra (5) has a central function \( C = x_0^3 + x_1^3 + x_2^3 + 3k x_0 x_1 x_2 \), and the centre is generated by the element \( C \). Therefore, it is natural to expect that for generic \( p \) and \( q \) the algebra \( A_{p,q} \) has a central element of the form \( C_{p,q} = \varphi x_0^3 + \psi x_1^3 + \mu x_2^3 + \lambda x_0 x_1 x_2 \), where \( \varphi, \psi, \mu, \) and \( \lambda \) are functions of \( p \) and \( q \) (one can verify the existence of an element \( C_{p,q} \) by the immediate calculation), and the centre is generated by \( C_{p,q} \).

The standard technique of proving such assertions (for instance, the Poincare-Birkhoff-Witt theorem for Lie algebras) makes use of the filtration on an algebra and the study of the graded adjoint algebra. In our case the algebra is already graded, and one cannot proceed by the ordinary induction on the terms of lesser filtration; therefore we use another technique. Namely, we shall study a certain class of modules over the algebra \( A_{p,q} \) and try to obtain results on the algebra \( A_{p,q} \) by using an information on the modules. The following class of modules is useful for our purposes.

**Definition.** A module over a \( \mathbb{Z}_{\geq 0} \)-graded algebra \( A \) is said to be linear if it is \( \mathbb{Z}_{\geq 0} \)-graded as an \( A \)-module, generated by the space of degree 0, and the dimensions of all components are equal to 1.

Let us study the linear modules over the algebra \( A_{p,q} \). By definition, a linear module \( M \) admits a basis \( \{v_\alpha, \alpha \geq 0\} \) with the following action of the generators:
\begin{align*}
x_0v_\alpha &= x_\alpha v_{\alpha+1}, \\
x_1v_\alpha &= y_\alpha v_{\alpha+1}, \\
x_2v_\alpha &= z_\alpha v_{\alpha+1},
\end{align*}
where \( x_\alpha, y_\alpha, z_\alpha \) are sequences, and \( x_\alpha, y_\alpha, z_\alpha \) do not vanish simultaneously for any \( \alpha \) (we want \( M \) be generated by \( v_0 \)). A change of the basis of the
form $v_\alpha \rightarrow \lambda_\alpha v_\alpha$ multiplies the triple $(x_\alpha, y_\alpha, z_\alpha) \in \mathbb{C}^3$ by $\frac{\lambda_{\alpha+1}}{\lambda_\alpha}$, that is, the module $M$ is defined by the sequence of points $(x_\alpha : y_\alpha : z_\alpha) \in \mathbb{CP}^2$ uniquely up to isomorphism of graded modules. It is clear that a sequence of points $(x_\alpha : y_\alpha : z_\alpha) \in \mathbb{CP}^2$ defines a module over the algebra $A_{p,q}$ if and only if the relations (6) hold for the operators on $M$ corresponding to this sequence. This is equivalent to the following relations:

$$
\begin{align*}
x_{\alpha+1} y_\alpha - q y_{\alpha+1} x_\alpha &= p z_{\alpha+1} z_\alpha, \\
y_{\alpha+1} z_\alpha - q z_{\alpha+1} y_\alpha &= p x_{\alpha+1} x_\alpha, \\
z_{\alpha+1} x_\alpha - q x_{\alpha+1} z_\alpha &= p y_{\alpha+1} y_\alpha.
\end{align*}
$$

The relations (7) form a system of linear equations for $x_\alpha, y_\alpha, z_\alpha$ which has a non-zero solution (by the assumption on the module $M$), and therefore the determinant

$$
\begin{vmatrix}
-q y_{\alpha+1} & x_{\alpha+1} & -p z_{\alpha+1} \\
-p x_{\alpha+1} & -q z_{\alpha+1} & y_{\alpha+1} \\
z_{\alpha+1} & -p y_{\alpha+1} & -q x_{\alpha+1}
\end{vmatrix}
$$

must vanish. Similarly, the relations (7) form a system of linear equations on $x_{\alpha+1}, y_{\alpha+1}, z_{\alpha+1}$ that has a non-zero solution, and therefore

$$
\begin{vmatrix}
y_{\alpha} & -q x_\alpha & -p z_\alpha \\
p x_\alpha & z_\alpha & -q y_\alpha \\
-q z_\alpha & -p y_\alpha & x_\alpha
\end{vmatrix} = 0.
$$

One can readily see that these determinants give the same cubic polynomial in three variables. We see that for any $\alpha \geq 0$ the point with the coordinates $(x_\alpha : y_\alpha : z_\alpha)$ belongs to the cubic

$$
x_\alpha^3 + y_\alpha^3 + z_\alpha^3 + \frac{p^3 + q^3 - 1}{pq} x_\alpha y_\alpha z_\alpha = 0.
$$

Moreover, if a point $(x_\alpha : y_\alpha : z_\alpha)$ belongs to this cubic, then, solving the system of linear equations (7) with respect to $x_{\alpha+1}, y_{\alpha+1}, z_{\alpha+1}$, we obtain a new point $(x_{\alpha+1} : y_{\alpha+1} : z_{\alpha+1})$ on the same cubic (because the determinant of the system (7) must be equal to 0). Thus, the system (7) defines an automorphism of the projective manifold (8). Let us choose some $k = \frac{p^3 + q^3 - 1}{pq}$. Then, varying $q$, we obtain a one-parameter family of automorphisms of the projective curve in $\mathbb{CP}^2$ given by the equation $x^3 + y^3 + z^3 + kxyz = 0$. As is known, for generic $k$ this equation defines an elliptic curve. Let this curve be $E = \mathbb{C}/\Gamma$, where $\Gamma$ is an integral lattice generated by 1 and $\tau$, where $\text{Im}\ \tau > 0$. The parameter $k$ is a function of $\tau$. If $k$ is chosen, then, passing to the limit as $q \rightarrow 1$, we see that $p \rightarrow 0$, and the automorphism defined by (7) tends to the identity automorphism. Therefore, our family of automorphisms of
the elliptic curve $E$ given by the equation (8) is a deformation of the identity automorphism. Thus, every automorphism of this family is a translation, of the form $u \to u + \eta$, where $u, \eta \in E = \mathbb{C}/\Gamma$. Let $u_\alpha \in E = \mathbb{C}/\Gamma$ be a point with the coordinates $(x_\alpha : y_\alpha : z_\alpha)$. We see that $u_{\alpha+1} = u_\alpha + \eta$, where $\eta$ depends only on the algebra, that is, on $p$ and $q$. Hence, $u_\alpha = u + \alpha \eta$, where $u \in E$ is the parameter of the module $M$. We have obtained the following result.

**Proposition 1.** The linear modules over the algebra $A_{p,q}$ are indexed by a point of the elliptic curve $E \subset \mathbb{CP}^2$ given by the equation $x^3 + y^3 + z^3 + k_{p,q}xyz = 0$, where $k_{p,q} = \frac{p^3 + q^3 - 1}{pq}$. The module $M_u$ corresponding to a point $u \in E$ is given by the formulas

$$x_0v_\alpha = x_\alpha v_{\alpha+1}, \quad x_1v_\alpha = y_\alpha v_{\alpha+1}, \quad x_2v_\alpha = z_\alpha v_{\alpha+1},$$

where $(x_\alpha : y_\alpha : z_\alpha)$ are the coordinates of the point $u + \alpha \eta \in E$. Here the shift $\eta$ is determined by $p$ and $q$.

We note that, when studying linear modules, for an algebra $A_{p,q}$ we have constructed both an elliptic curve $E \subset \mathbb{CP}^2$ and a point $\eta \in E$. In what follows we shall see that, conversely, the algebra $A_{p,q}$ can be reconstructed from $E$ and $\eta$. Thus, two continuous parameters, $E$ (that is $\tau$) and $\eta$, give a natural parameterization of the algebras $A_{p,q}$. Therefore, we change the notation and denote the algebra $A_{p,q}$ by $Q_3(E,\eta)$.

Let us now apply a uniformization of the elliptic curve $E \subset \mathbb{CP}^2$ given by the equation (8) by theta functions of order three (see Appendix A). A point $u \in E = \mathbb{C}/\Gamma$ has the coordinates $(\theta_0(u) : \theta_1(u) : \theta_2(u)) \in \mathbb{CP}^2$. In this notation, the module $M_u$ is given by the formulas

$$x_0v_\alpha = \theta_0(u + \alpha \eta)v_{\alpha+1}, \quad x_1v_\alpha = \theta_1(u + \alpha \eta)v_{\alpha+1}, \quad x_2v_\alpha = \theta_2(u + \alpha \eta)v_{\alpha+1}.$$  

Let $e$ be the linear operator in the space with basis $\{v_\alpha, \alpha \geq 0\}$ given by the formula $ev_\alpha = v_{\alpha+1}$. Let $u$ be the diagonal operator in the same space such that $eu = (u - \eta)e$. We have $uv_\alpha = (u_0 + \alpha \eta)v_\alpha$ for some $u_0 \in \mathbb{C}$. It is clear that the generators of the algebra $Q_3(E,\eta)$ in the representation $M_u$ become

$$x_0 = \theta_0(u)e, \quad x_1 = \theta_1(u)e, \quad x_2 = \theta_2(u)e.$$  

This gives the following reformulation of the description of linear modules.
Proposition 2. Let us consider the $\mathbb{Z}_{\geq 0}$-graded algebra $B(\eta) = C \oplus B_1 \oplus B_2 \oplus \ldots$, where $B_\alpha = \{f(u)e^\alpha\}$, $f$ ranges over all holomorphic functions, and the multiplication is given by the formula: $f(u)e^\alpha \cdot g(u)e^\beta = f(u)g(u - \alpha \eta)e^{\alpha + \beta}$. Then there is an algebra homomorphism $\varphi : Q_3(\mathcal{E}, \eta) \rightarrow B(\eta)$ such that $x_0 \rightarrow \theta_0(u)e$, $x_1 \rightarrow \theta_1(u)e$, $x_2 \rightarrow \theta_2(u)e$.

Proposition 2 provides a lower bound for the dimension $\dim F_\alpha$ of the graded components of the algebra $Q_3(\mathcal{E}, \eta)$. Really, the homomorphism $\varphi$ preserves the grading, that is, $\varphi(F_\alpha) \subset B_\alpha$. We have

$$\varphi(x_{i_1} \ldots x_{i_\alpha}) = \theta_{i_1}(u)e \ldots \theta_{i_\alpha}(u)e = \theta_{i_1}(u)\theta_{i_2}(u - \eta) \ldots \theta_{i_\alpha}(u - (\alpha - 1)\eta)e^\alpha.$$ 

Thus, $\varphi(F_\alpha)$ is the linear space (of holomorphic functions) spanned by the functions $\{\theta_{i_1}(u), \ldots, \theta_{i_\alpha}(u - (\alpha - 1)\eta)\}; i_1, \ldots, i_\alpha = 0, 1, 2$. It is clear that all these functions are theta functions of order 3 and belong to the space $\Theta_{3\alpha, \frac{\alpha(\alpha - 1)}{2}3\eta}(\Gamma)$. One can readily prove that the image $\varphi(F_\alpha)$ coincides with the entire space $\Theta_{3\alpha, \frac{\alpha(\alpha - 1)}{2}3\eta}(\Gamma)$, and hence $\dim \varphi(F_\alpha) = 3\alpha$. We have obtained the bound $\dim F_\alpha \geq 3\alpha$. On the other hand, we know that $\dim F_\alpha \leq \frac{(\alpha + 1)(\alpha + 2)}{2}$ because the relations in $Q_3(\mathcal{E}, \eta)$ are deformations of the relations in the polynomial ring in three variables. We expect that the equality $\dim F_\alpha = \frac{(\alpha + 1)(\alpha + 2)}{2}$ holds for generic $\tau$ and $\eta$. Let us compare these numbers:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dim F_\alpha$ (conjecture)</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15</td>
</tr>
<tr>
<td>$\dim \varphi(F_\alpha)$</td>
<td>3</td>
<td>6</td>
<td>9</td>
<td>12</td>
</tr>
</tbody>
</table>

We see that the first discrepancy holds for $\alpha = 3$; possibly $\varphi$ has a one-dimensional kernel on the space $F_3$. It can be shown that, really, there is a cubic element $C \in Q_3(\mathcal{E}, \eta)$ such that $C \neq 0$ and $\varphi(C) = 0$. The element $C$ turns out to be central, that is, $x_\alpha C = C x_\alpha$ for $\alpha = 0, 1, 2$. Passing to the limit as $\eta \rightarrow 0$ (for a fixed $\tau$), we see that $C \rightarrow x_0^3 + x_1^3 + x_2^3 + kx_0x_1x_2$ because the $\theta_i(u)s$ uniformize the elliptic curve, that is, $\theta_0^3 + \theta_1^3 + \theta_2^3 + k\theta_0\theta_1\theta_2 = 0$. Further, if $C$ is central and is not a zero divisor (the latter obviously holds for generic $\tau$ and $\eta$), then every element $\ker \varphi$ must be divisible by $C$ according to the dimensional considerations. Really, the graded linear space $\oplus_{\alpha \geq 0} F_\alpha$ turns out to be not smaller than $\left(\oplus_{\alpha \geq 0} \Theta_{3\alpha, \frac{\alpha(\alpha - 1)}{2}3\eta}\right) \otimes \mathbb{C}[C]$, where $\deg C = 3$. One can readily see that the component of degree $\alpha$ of this tensor product of graded linear spaces is of dimension $\frac{(\alpha + 1)(\alpha + 2)}{2}$. However, we know that
\[ \dim F_{\alpha} \leq \frac{(\alpha+1)(\alpha+2)}{2}, \] which implies \( \dim F_{\alpha} = \frac{(\alpha+1)(\alpha+2)}{2} \). We have obtained the following result.

**Proposition 3.** For generic \( \tau \) and \( \eta \) the algebra \( Q_3(\mathcal{E}, \eta) \) has a cubic central element \( C \). The quotient algebra \( Q_3(\mathcal{E}, \eta)/(C) \) is isomorphic to \( \bigoplus_{\alpha \geq 0} \Theta_{3\alpha, \frac{\alpha(\alpha-1)}{2}+\eta+\frac{\alpha}{2}}(\Gamma) \), where the product of elements \( f \in \Theta_{3\alpha, \frac{\alpha(\alpha-1)}{2}+\eta+\frac{\alpha}{2}}(\Gamma) \) and \( g \in \Theta_{3\beta, \frac{\beta(\beta-1)}{2}+\eta}(\Gamma) \) is given by the formula \( f \ast g(u) = f(u)g(u-3\alpha \eta) \).

It follows from our description of \( Q_3(\mathcal{E}, \eta)/(C) \) that this algebra is centre-free for generic \( \eta \). Therefore, the centre of the algebra \( Q_3(\mathcal{E}, \eta) \) is generated by the element \( C \).

Let us now find the relations in the algebra \( Q_3(\mathcal{E}, \eta) \), that is, let us express \( p \) and \( q \) in terms of \( \tau \) and \( \eta \). We have \( x_i x_{i+1} - qx_{i+1} x_i - px_{i+2}^2 = 0 \) (these are the relations in (6)). Applying the homomorphism \( \varphi \), we obtain
\[ \theta_i(u)\theta_{i+1}(u-\eta) - q\theta_{i+1}(u)\theta_i(u-\eta) - p\theta_{i+2}(u)\theta_{i+2}(u-\eta) = 0. \]

Hence (see (28) in Appendix A), \( q = \frac{\theta_1(\eta)}{\varphi'_{2}(\eta)} \), \( p = \frac{\theta_0(\eta)}{\varphi'_{2}(\eta)} \).

The similar investigation of the Sklyanin algebra with four generators (see Appendix D.1) gives the following result.

**Proposition 4.** For a generic Sklyanin algebra \( S \) with four generators and the relations (39) one can find an elliptic curve \( \mathcal{E} = \mathbb{C}/\Gamma \) defined by two quadrics in \( \mathbb{CP}^3 \) and a point \( \eta \in \mathcal{E} \) such that:

1) There is a graded algebra homomorphism \( \varphi: S \to B(\eta); \)
2) The image of this homomorphism in \( B_\alpha \) is \( \Theta_{4\alpha, \frac{\alpha(\alpha-1)}{2}+\eta+\frac{\alpha}{2}}(\Gamma) \);
3) The kernel of this homomorphism is generated by two quadratic elements \( C_1 \) and \( C_2 \).

Thus, \( S/(C_1, C_2) = \bigoplus_{\alpha \geq 0} \Theta_{4\alpha, \frac{\alpha(\alpha-1)}{2}+\eta+\frac{\alpha}{2}}(\Gamma) \).

The Sklyanin algebra \( S \) can be reconstructed from \( \mathcal{E} \) and \( \eta \). Let us denote this algebra by \( Q_4(\mathcal{E}, \eta) \).

The following natural question arises: Does there exist a similar algebra \( Q_n(\mathcal{E}, \eta) \) for any \( n \)?

To answer this question, the information concerning linear modules is insufficient because these modules are too small to reconstruct the algebra \( Q_n(\mathcal{E}, \eta) \) for any \( n \). Really, the algebra \( Q_n(\mathcal{E}, \eta) \) must have the functional dimension \( n \), whereas the linear modules are of dimension one. Therefore, these modules can be used only when reconstructing a quotient algebra of
To overcome these difficulties, it is natural to study more general modules. Namely, let us study modules over the algebra $Q_3(E, \eta)$ with a basis $\{v_{i,j}; i, j \in \mathbb{Z}_{\geq 0}\}$ and such that the generators of the algebra $Q_3(E, \eta)$ take any element $v_{ij}$ to a linear combination of $v_{i+1,j}$ and $v_{i,j+1}$. Calculations show that every such module is of the form

$$x_iv_{\alpha,\beta} = \frac{\theta_i(u_1 + (\alpha - 2\beta)\eta)}{\theta(u_1 - u_2 + 3(\alpha - \beta)\eta)}v_{\alpha+1,\beta} + \frac{\theta_i(u_2 + (\beta - 2\alpha)\eta)}{\theta(u_2 - u_1 + 3(\beta - \alpha)\eta)}v_{\alpha,\beta+1},$$

where $i \in \mathbb{Z}/3\mathbb{Z}$, $\alpha, \beta \in \mathbb{Z}_{\geq 0}$, and $u_1, u_2 \in \mathbb{C}$. Thus, the modules of this kind are indexed by a pair of points $u_1, u_2 \in E$. If we now assume that the algebra $Q_3(E, \eta)$ has analogous modules (see (15)), then the above information uniquely defines the algebra $Q_3(E, \eta)$.

Remarks. 1. One can pose the following more general problem. Let $M \subset \mathbb{CP}^{n-1}$ be a projective manifold and let $T$ be an automorphism of $M$. For a point $u \in M$ we denote by $z_i(u)$ (where $i = 0, \ldots, n-1$) the homogeneous coordinates of $u$. Does there exist a PBW-algebra with $n$ generators $\{x_i; i = 0, \ldots, n-1\}$ that has a linear module $L_u$ (for any point $u \in M$) given by the formula $x_i v_{\alpha} = z_i(T^u) v_{\alpha+i}$? Here $T^u$ stands for $T(T(\ldots T(u) \ldots))$. The algebras $Q_{n,k}(E, \eta)$ are a solution of this problem for some $M$ and $T$, namely, if $M = E^p$ is a power of a curve $E$ and $T$ a translation (see §5, Proposition 12). Here $p$ stands for the length of the expansion of $n/k = n_1 - n_2 - \ldots - 1/p$ in the continued fraction.

2. Let $A_3$ be the algebra with the generators $x, y, z$ and the relations $\varepsilon zx + \varepsilon^5 y^2 + xz = 0$, $\varepsilon^2 z^2 + yx + \varepsilon^4 xy = 0$, and $zy + \varepsilon^7 yz + \varepsilon^8 x^2 = 0$, where $\varepsilon^9 = 1$. This PBW-algebra corresponds to the case in which $M \subset \mathbb{CP}^2$ is an elliptic curve given by the equation $x^3 + y^3 + z^3 = 0$ and $T$ is an automorphism corresponding to the complex multiplication on $M$. The algebra $A_3$ is not a quantization of any Poisson structure on $\mathbb{C}^3$.

§2 Algebra $Q_n(E, \eta)$

1 Construction

For any $n \in \mathbb{N}$, any elliptic curve $E = \mathbb{C}/\Gamma$, and any point $\eta \in E$ we construct a graded associative algebra $Q_n(E, \eta) = \mathbb{C} \oplus F_1 \oplus F_2 \oplus \ldots$, where $F_1 = \Theta_{n,\varepsilon}(\Gamma)$

\footnote{This example was communicated to the author by Oleg Ogievetsky [1], [15], [40].}
and \( F_\alpha = S^\alpha \Theta_{n,\epsilon+(\alpha-1)n}(\Gamma) \). By construction, \( \dim F_\alpha = \frac{n(n+1)...(n+\alpha-1)}{\alpha!} \). It is clear that the space \( F_\alpha \) can be realized as the space of holomorphic symmetric functions of \( \alpha \) variables \( \{f(z_1, \ldots, z_\alpha)\} \) such that

\[
\begin{align*}
  f(z_1 + 1, z_2, \ldots, z_\alpha) &= f(z_1, \ldots, z_\alpha), \\
  f(z_1 + \tau, z_2, \ldots, z_\alpha) &= (-1)^n e^{-2\pi i (nz_1 - c - (\alpha-1)n)} f(z_1, \ldots, z_\alpha).
\end{align*}
\]

(9)

For \( f, g \in F_\alpha \) we define the symmetric function \( f * g \) of \( \alpha + \beta \) variables by the formula

\[
f * g(z_1, \ldots, z_{\alpha+\beta}) = \frac{1}{\alpha!\beta!} \sum_{\sigma \in S_{\alpha+\beta}} f(z_{\sigma_1}, \ldots, z_{\sigma_\alpha}) g(z_{\sigma_{\alpha+1}} - 2\alpha \eta, \ldots, z_{\sigma_{\alpha+\beta}} - 2\alpha \eta) \times \\
\prod_{1 \leq i \leq \alpha} \prod_{\alpha+1 \leq j \leq \alpha+\beta} \frac{\theta(z_{\sigma_i} - z_{\sigma_j} - n\eta)}{\theta(z_{\sigma_i} - z_{\sigma_j})}.
\]

In particular, for \( f, g \in F_1 \) we have

\[
f * g(z_1, z_2) = f(z_1) g(z_2 - 2\eta) \frac{\theta(z_1 - z_2 - n\eta)}{\theta(z_1 - z_2)} + f(z_2) g(z_1 - 2\eta) \frac{\theta(z_2 - z_1 - n\eta)}{\theta(z_2 - z_1)}.
\]

Here \( \theta(z) \) is a theta function of order one (see Appendix A).

**Proposition 5.** If \( f \in F_\alpha \) and \( g \in F_\beta \), then \( f * g \in F_{\alpha+\beta} \). The operation \( * \) defines an associative multiplication on the space \( \bigoplus_{\alpha \geq 0} F_\alpha \).

**Proof.** Let us show that \( f * g \in F_{\alpha+\beta} \). It immediately follows from the assumptions (9) concerning \( f \) and \( g \) and also from the properties of \( \theta(z) \) (see Appendix A) that every summand in the formula for \( f * g \) satisfies condition (9) for \( F_{\alpha+\beta} \). Hence, \( f * g \) is a meromorphic symmetric function satisfying condition (9). This function can have a pole of order not exceeding one on the diagonals \( z_i - z_j = 0 \) and also for \( z_i - z_j \in \Gamma \) because \( \theta(z) \) has zeros for \( z \in \Gamma \). However, the order of a pole of a symmetric function on the diagonal must be even. This implies that the function \( f * g \) is holomorphic for \( z_i = z_j \), and it follows from (9) that \( f * g \) is holomorphic for \( z_i - z_j \in \Gamma \) as well.

One can immediately see that the multiplication \( * \) is associative. \( \square \)
2 Main properties of the algebra $Q_n(\mathcal{E}, \eta)$

By construction, the dimensions of the graded components of the algebra $Q_n(\mathcal{E}, \eta)$ coincide with those for the polynomial ring in $n$ variables. For $\eta = 0$ the formula for $f \ast g$ becomes

$$f \ast g(z_1, \ldots, z_{\alpha+1}) = \frac{1}{\alpha! \beta!} \sum_{\sigma \in S_{\alpha+\beta}} f(z_{\sigma_1}, \ldots, z_{\sigma_\alpha}) g(z_{\sigma_{\alpha+1}}, \ldots, z_{\sigma_{\alpha+\beta}}).$$

This is the formula for the ordinary product in the algebra $S^\ast \Theta_{n,c}(\Gamma)$, that is, in the polynomial ring in $n$ variables. Therefore, for a fixed elliptic curve $\mathcal{E}$ (that is, for a fixed modular parameter $\tau$) the family of algebras $Q_n(\mathcal{E}, \eta)$ is a deformation of the polynomial ring. In particular (see Appendix D.2), there is a Poisson algebra, which we denote by $q_n(\mathcal{E})$. One can readily obtain the formula for the Poisson bracket on the polynomial ring from the formula for $f \ast g$ by expanding the difference $f \ast g - g \ast f$ in the Taylor series with respect to $\eta$. It follows from the semicontinuity arguments that the algebra $Q_n(\mathcal{E}, \eta)$ with generic $\eta$ is determined by $n$ generators and $n(n - 1)/2$ quadratic relations. One can prove (see §2.6) that this is the case if $\eta$ is not a point of finite order on $\mathcal{E}$, that is, $N\eta \notin \Gamma$ for any $N \in \mathbb{N}$.

The space $\Theta_{n,c}(\Gamma)$ of the generators of the algebra $Q_n(\mathcal{E}, \eta)$ is endowed with an action of a finite group $\tilde{\Gamma}_n$ which is a central extension of the group $\Gamma/n\Gamma$ of points of order $n$ on the curve $\mathcal{E}$ (see Appendix A). It immediately follows from the formula for the product $\ast$ that the corresponding transformations of the space $F_\alpha = S^\alpha \Theta_{n,c}(\Gamma)$ are automorphisms of the algebra $Q_n(\mathcal{E}, \eta)$.

3 Bosonization of the algebra $Q_n(\mathcal{E}, \eta)$

The main approach to obtain representations of the algebra $Q_n(\mathcal{E}, \eta)$ is to construct homomorphisms from this algebra to other algebras with simple structure (close to Weil algebras) which have a natural set of representations. These homomorphisms are referred to as bosonizations, by analogy with the known constructions of quantum field theory.

Let $B_{p,n}(\eta)$ be a $\mathbb{Z}^p$-graded algebra whose space of degree $(\alpha_1, \ldots, \alpha_p)$ is of the form $\{f(u_1, \ldots, u_p)e_1^{\alpha_1} \cdots e_p^{\alpha_p}\}$, where $f$ ranges over the meromorphic functions of $p$ variables and $e_1, \ldots, e_p$ are elements of the algebra $B_{p,n}(\eta)$. Let $B_{p,n}(\eta)$ be generated by the space of meromorphic functions $f(u_1, \ldots, u_p)$ and
by the elements $e_1, \ldots, e_p$ with the defining relations

\[ e_\alpha f(u_1, \ldots, u_p) = f(u_1 - 2\eta, \ldots, u_\alpha + (n - 2)\eta, \ldots, u_p - 2\eta) e_\alpha, \]
\[ e_\alpha e_\beta = e_\beta e_\alpha, \quad f(u_1, \ldots, u_p) g(u_1, \ldots, u_p) = g(u_1, \ldots, u_p) f(u_1, \ldots, u_p) \]  

We note that the subalgebra of $B_{p,n}(\eta)$ consisting of the elements of degree $(0, \ldots, 0)$ is the commutative algebra of all meromorphic functions of $p$ variables with the ordinary multiplication.

**Proposition 6.** Let $\eta \in \mathcal{E}$ be a point of infinite order. For any $p \in \mathbb{N}$ there is a homomorphism $\varphi_p : Q_n(\mathcal{E}, \eta) \to B_{p,n}(\eta)$ that acts on the generators of the algebra $Q_n(\mathcal{E}, \eta)$ by the formula:

\[ \varphi_p(f) = \sum_{1 \leq \alpha \leq p} f(u_\alpha) \frac{1}{\theta(u_\alpha - u_1) \cdots \theta(u_\alpha - u_p)} e_\alpha. \]  

Here $f \in \Theta_{n,c}(\Gamma)$ is a generator of $Q_n(\mathcal{E}, \eta)$ and the product in the denominator is of the form $\prod_{i \neq \alpha} \theta(u_\alpha - u_i)$.

**Proof.** We write $\xi_\alpha = \frac{1}{\theta(u_\alpha - u_1) \cdots \theta(u_\alpha - u_p)} e_\alpha$. It is clear that the elements $\xi_1, \ldots, \xi_p$ together with the space of meromorphic functions $\{f(u_1, \ldots, u_p)\}$ generate the algebra $B_{p,n}(\eta)$. The relations (10) become

\[ \xi_\alpha f(u_1, \ldots, u_p) = f(u_1 - 2\eta, \ldots, u_\alpha + (n - 2)\eta, \ldots, u_p - 2\eta) \xi_\alpha \]
\[ \xi_\alpha \xi_\beta = -\frac{e^{2\pi i (u_\beta - u_\alpha)} \theta(u_\alpha - u_\beta + n\eta)}{\theta(u_\beta - u_\alpha + n\eta)} \xi_\beta \xi_\alpha \]

The formula (11) can be represented as

\[ \varphi_p(f) = \sum_{1 \leq \alpha \leq p} f(u_\alpha) \xi_\alpha. \]  

Using (12) and the formula for the multiplication in the algebra $Q_n(\mathcal{E}, \eta)$ and assuming that $\varphi_p$ is a homomorphism, one can readily evaluate the extension of the map $\varphi_p$ to the entire algebra. For instance, in the grading 2 we have

\[ \varphi_p(f * g) = \sum_{1 \leq \alpha \leq p} f(u_\alpha) \xi_\alpha \cdot \sum_{1 \leq \beta \leq p} g(u_\beta) \xi_\beta = \sum_{1 \leq \alpha, \beta \leq p} f(u_\alpha) \xi_\alpha f(u_\beta) \xi_\beta = \]
\[ = \sum_{1 \leq \alpha, \beta \leq p, \alpha \neq \beta} f(u_\alpha) g(u_\beta - 2\eta) \xi_\alpha \xi_\beta + \sum_{1 \leq \alpha \leq p} f(u_\alpha) g(u_\alpha + (n - 2)\eta) \xi_\alpha^2. \]
The first sum is
\[ \sum_{1 \leq \alpha < \beta \leq p} (f(u_\alpha)g(u_\beta - 2\eta)\xi_\alpha\xi_\beta + f(u_\beta)g(u_\alpha - 2\eta)\xi_\beta\xi_\alpha) = \]
\[ = \sum_{1 \leq \alpha < \beta \leq p} \left( f(u_\alpha)g(u_\beta - 2\eta)\xi_\alpha\xi_\beta - f(u_\beta)g(u_\alpha - 2\eta)\frac{e^{2\pi i (u_\alpha - u_\beta)}\theta(u_\beta - u_\alpha - n\eta)}{\theta(u_\alpha - u_\beta + n\eta)}\xi_\alpha\xi_\beta \right) = \]
\[ = \sum_{1 \leq \alpha < \beta \leq p} \frac{\theta(u_\alpha - u_\beta)}{\theta(u_\alpha - u_\beta - n\eta)} \times \]
\[ \times \left( f(u_\alpha)g(u_\beta - 2\eta)\frac{\theta(u_\alpha - u_\beta - n\eta)}{\theta(u_\alpha - u_\beta)} + f(u_\beta)g(u_\alpha - 2\eta)\frac{\theta(u_\beta - u_\alpha - n\eta)}{\theta(u_\beta - u_\alpha)} \right) \xi_\alpha\xi_\beta = \]
\[ = \sum_{1 \leq \alpha < \beta \leq p} \frac{\theta(u_\alpha - u_\beta)}{\theta(u_\alpha - u_\beta - n\eta)} f^*(u_\alpha, u_\beta)\xi_\alpha\xi_\beta \]

Moreover, \( f(u_\alpha)g(u_\alpha + (n - 2)\eta) = \frac{\theta(-n\eta)}{\theta(-2n\eta)} f^*g(u_\alpha, u_\alpha + n\eta) \). We finally obtain
\[ \varphi_p(f^*g) = \]
\[ = \sum_{1 \leq \alpha < \beta \leq p} \frac{\theta(u_\alpha - u_\beta)}{\theta(u_\alpha - u_\beta - n\eta)} f^*g(u_\alpha, u_\beta)\xi_\alpha\xi_\beta + \frac{\theta(-n\eta)}{\theta(-2n\eta)} \sum_{1 \leq \alpha \leq p} f^*g(u_\alpha, u_\alpha + n\eta)\xi_\alpha^2 \]

We see that the map \( \varphi_p \) can be extended to the quadratic part of the algebra \( Q_n(\mathcal{E}, \eta) \) because the right-hand side of (13) depends on \( f^*g \) only but not on \( f \) and \( g \) separately. Thus implies the assertion for generic \( \eta \) because in this case the algebra \( Q_n(\mathcal{E}, \eta) \) is defined by quadratic relations. To prove a more exact assertion (for the case in which \( \eta \) is a point of infinite order), one must continue the above calculation. We obtain the following formula: if \( f \in F_\alpha \),
then

$$
\varphi_p(f) = \sum_{i_1, \ldots, i_p \geq 0, \ i_1 + \ldots + i_p = 0} B_{i_1, \ldots, i_p} f(u_1, u_1 + n\eta, \ldots, u_1 + (i_1 - 1)n\eta, u_2, u_2 + n\eta, \ldots, u_2 + (i_2 - 1)n\eta, \ldots) \xi_1^{i_1} \ldots \xi_p^{i_p},
$$

where

$$
B_{i_1, \ldots, i_p} = \prod_{1 \leq \lambda \leq \lambda' \leq p} \frac{\theta(u_\lambda + n\mu\eta - u_\lambda' - n\mu'\eta)}{\theta(u_\lambda + n\mu\eta - u_\lambda' - n\mu'\eta - n\eta)},
$$

This product can be represented as \( \prod_{1 \leq i < j < p} \frac{\theta(v_i - v_j)}{\theta(v_i - v_j - n\eta)} \), where \((v_1, \ldots, v_p) = (u_1, u_1 + n\eta, \ldots)\) are the arguments of the function \( f \) in the formula (14) for \( \xi_1^{i_1} \ldots \xi_p^{i_p} \).

The formula (14) makes sense if \( \eta \) is a point of infinite order, and in this case the direct calculation shows that \( \varphi_p \) is a homomorphism.

### 4 Representations of the algebras \( Q_n(\mathcal{E}, \eta) \)

The formula (14) shows that the image of the homomorphism \( \varphi_p \) is contained in the subalgebra \( B^r_{p,n}(\eta) \subset B_{p,n}(\eta) \) consisting of the elements \( \sum_{a_1, \ldots, a_p} f_{a_1, \ldots, a_p} e^{a_1} \ldots e^{a_p} \), where the functions \( f_{a_1, \ldots, a_p} \) are holomorphic outside the divisors of the form \( u_i - u_j - \lambda n\eta \in \Gamma \), \( \lambda \in \mathbb{Z} \).

Let \( v_1, \ldots, v_p \in \mathbb{C} \) be such that \( v_i - v_j - \lambda n\eta \notin \Gamma \) for \( \lambda \in \mathbb{Z} \). We construct a representation \( M_{v_1, \ldots, v_p} \) of the algebra \( B^r_{p,n}(\eta) \) as follows. Let the representation \( M_{v_1, \ldots, v_p} \) have a basis \( \{ w_{a_1, \ldots, a_p}; a_1, \ldots, a_p \in \mathbb{Z}_{\geq 0} \} \) in which the elements \( e_1, \ldots, e_p \) act by the rule \( e_i w_{a_1, \ldots, a_p} = w_{a_1, \ldots, a_i + 1, \ldots, a_p} \). Thus, \( w_{a_1, \ldots, a_p} = e_1^{a_1} \ldots e_p^{a_p} w \), where \( w = w_{0, \ldots, 0} \). The action of the commutative subalgebra of \( B^r_{p,n}(\eta) \) consisting of the elements of degree 0 is diagonal in this basis. We set \( f w = f(v_1 - (n - 2)\eta, \ldots, v_p - (n - 2)\eta) w \), and hence \( f w_{a_1, \ldots, a_p} = f(v_1 + (2a_1 + \ldots + 2a_p - n a_1 - (n - 2))\eta, \ldots, v_p + (2a_1 + \ldots + 2a_p - n a_p - (n - 2))\eta) w_{a_1, \ldots, a_p} \). It is clear that these formulas really define a representation of the algebra \( B^r_{p,n}(\eta) \) in the space \( M_{v_1, \ldots, v_p} \), and, thanks to the homomorphism \( \varphi_p \), we have a representation of the algebra \( Q_n(\mathcal{E}, \eta) \) as well. One can readily see that the space \( M_{v_1, \ldots, v_p} \) admits a basis \( \{ v_{a_1, \ldots, a_p}; a_1, \ldots, a_p \in \mathbb{Z}_{\geq 0} \} \), in which the action of the generators of the algebra \( Q_n(\mathcal{E}, \eta) \) can be represented in the following form: if \( f \in \Theta_n(\Gamma) \),
then
\[ f_{\alpha_1, \ldots, \alpha_p} = \sum_{1 \leq i \leq p} f(v_i + (2\alpha_1 + \ldots + 2\alpha_p - n\alpha_i)\eta) \theta(v_i - v_1 - n(\alpha_i - \alpha_1)\eta) \ldots \theta(v_i - v_p - n(\alpha_i - \alpha_p)\eta) v_{\alpha_1, \ldots, \alpha_i+1, \ldots, \alpha_p}. \] (15)

The vectors \( v_{\alpha_1, \ldots, \alpha_p} \) are proportional to the vectors \( w_{\alpha_1, \ldots, \alpha_p} \). In particular, for \( p = 1 \) we obtain modules \( M_v \) with a basis \( \{ v_\alpha; \alpha \in \mathbb{Z}_{\geq 0} \} \) and the action
\[ f_{\alpha} = f(v - (n-2)\alpha\eta)v_{\alpha+1}. \] Thus, the algebra \( Q_n(\mathcal{E}, \eta) \) has a family of linear modules parametrized by the elliptic curve \( \mathcal{E} \subset \mathbb{C}P^{n-1} \), where the embedding is carried out by theta functions of order \( n \).

5 Symplectic leaves

We recall that \( Q_n(\mathcal{E}, 0) \) is the polynomial ring \( S^*\Theta_{n,c}(\Gamma) \). For a fixed elliptic curve \( \mathcal{E} = \mathbb{C}/\Gamma \) we obtain the family of algebras \( Q_n(\mathcal{E}, \eta) \), which is a flat deformation of the polynomial ring. We denote the corresponding Poisson algebra by \( q_n(\mathcal{E}) \). We obtain a family of Poisson algebras, depending on \( \mathcal{E} \), that is, on the modular parameter \( \tau \). Let us study the symplectic leaves of this algebra. To this end, we note that, when passing to the limit as \( \eta \to 0 \), the homomorphism \( \varphi_p \) of associative algebras gives a homomorphism of Poisson algebras. Namely, let us denote by \( b_{p,n} \) the Poisson algebra formed by the elements \( \sum_{\alpha_1, \ldots, \alpha_p \geq 0} f_{\alpha_1, \ldots, \alpha_p}(u_1, \ldots, u_p)e^\alpha \), where \( f_{\alpha_1, \ldots, \alpha_p} \) are meromorphic functions and the Poisson bracket is
\[ \{ u_\alpha, u_\beta \} = \{ e_\alpha, e_\beta \} = 0; \quad \{ e_\alpha, u_\beta \} = -2e_\alpha; \quad \{ e_\alpha, u_\alpha \} = (n-2)e_\alpha, \]
where \( \alpha \neq \beta \).

The following assertion results from Proposition 6 in the limit as \( \eta \to 0 \).

**Proposition 7.** There is a Poisson algebra homomorphism \( \psi_p: q_n(\mathcal{E}) \to b_{p,n} \) given by the following formula: if \( f \in \Theta_n(\Gamma) \), then \( \psi_p(f) = \sum_{1 \leq \alpha \leq p} f(u_\alpha) \frac{\theta(u_\alpha - u_1) \ldots \theta(u_\alpha - u_p)\eta^{\alpha}}{\theta(u_\alpha - u_1) \ldots \theta(u_\alpha - u_p)\eta^{\alpha}}. \)

Let \( \{ \theta_i(z); i \in \mathbb{Z}/n\mathbb{Z} \} \) be a basis of the space \( \Theta_{n,c}(\Gamma) \) and let \( \{ x_i; i \in \mathbb{Z}/n\mathbb{Z} \} \) be the corresponding basis in the space of elements of degree one in the algebra \( Q_n(\mathcal{E}, \eta) \) (this space is isomorphic to \( \Theta_{n,c}(\Gamma) \)). For an elliptic curve \( \mathcal{E} \subset \mathbb{C}P^{n-1} \) embedded by means of theta functions of order \( n \) (this is the set of points with the coordinates \( (\theta_0(z) : \ldots : \theta_{n-1}(z)) \)) we denote by \( C_p\mathcal{E} \) the variety of \( p \)-chords, that is, the union of projective spaces of dimension
$p - 1$ passing through $p$ points of $\mathcal{E}$. Let $K(C_p\mathcal{E})$ be the corresponding homogeneous manifold in $\mathbb{C}^n$. It is clear that $K(C_p\mathcal{E})$ consists of the points with the coordinates $x_i = \sum_{1 \leq \alpha \leq p} \frac{\delta_i(u_\alpha)}{(u_\alpha - u_{i1}) \ldots (u_\alpha - u_{ip})} e_\alpha$, where $u_\alpha, e_\alpha \in \mathbb{C}$.

Let $2p < n$. Then one can show that $\dim K(C_p\mathcal{E}) = 2p$ and $K(C_{p-1}\mathcal{E})$ is the manifold of singularities of $K(C_p\mathcal{E})$. It follows from Proposition 7 and from the fact that the Poisson bracket is non-degenerate on $C_{p,n}$ for $2p < n$ and $e_\alpha \neq 0$ that the non-singular part of the manifold $K(C_p\mathcal{E})$ is a $2p$-dimensional symplectic leaf of the Poisson algebra $q_n(\mathcal{E})$.

Let $n$ be odd. One can show that the equation defining the manifold $K(C_{n/2}\mathcal{E})$ is of the form $C = 0$, where $C$ is a homogeneous polynomial of degree $n$ in the variables $x_i$. This polynomial is a central function of the algebra $q_n(\mathcal{E})$.

Let $n$ be even. The manifold $K(C_{n/2}\mathcal{E})$ is defined by equations $C_1 = 0$ and $C_2 = 0$, where $\deg C_1 = \deg C_2 = n/2$. The polynomials $C_1$ and $C_2$ are central in the algebra $q_n(\mathcal{E})$.

### 6 Free modules, generations, and relations

Let $\eta$ be a point of infinite order.

**Proposition 8.** Let numbers $v_1, \ldots, v_n \in \mathbb{C}$ be in general position. Then the module $M_{v_1,\ldots,v_n}$ is generated by $v_0, \ldots, 0$ and is free over $Q_n(\mathcal{E}, \eta)$.

**Proof.** By construction, the dimensions of graded components of $M_{v_1,\ldots,v_n}$ coincide with those of the algebra $Q_n(\mathcal{E}, \eta)$. Let us show that the module is generated by the vector $v = v_0, \ldots, 0$. Let

\[ f_i = \prod_{\alpha \neq i} \theta(z - v_\alpha) \cdot (\theta(z + v_1 + \ldots + v_n - v_i - c). \]

It is clear that $f_i \in \Theta_{n,c}(\Gamma)$ for $1 \leq i \leq n$. Therefore, the $f_i$s are elements of degree 1 of the algebra $Q_n(\mathcal{E}, \eta)$. It follows from the formula (15) that $f_iv$ is non-zero and proportional to $v_{0,\ldots,1,\ldots,0} = e_iv$. Similarly, one can readily construct elements $f_{i;\alpha_1,\ldots,\alpha_n} \in \Theta_{n,c}(\Gamma)$ such that $f_{i;\alpha_1,\ldots,\alpha_n}v_{\alpha_1,\ldots,\alpha_n}$ is non-zero and proportional to $v_{\alpha_1,\ldots,\alpha_i+1,\ldots,\alpha_n}$. Namely, $f_{i;\alpha_1,\ldots,\alpha_n} = \prod_{\beta \neq i} \theta(z - v_\beta - (2\alpha_1 + \ldots + 2\alpha_n - n\alpha_\beta)\eta) \cdot (\theta(z + v_1 + \ldots + v_n - v_i + \ldots + \alpha_n)\eta - c).$ Thus, all elements $v_{\alpha_1,\ldots,\alpha_n}$ are obtained from $v$ by the action of elements of degree one in $Q_n(\mathcal{E}, \eta)$. \qed
Proposition 9. The algebra $Q_n(\mathcal{E}, \eta)$ is presented by $n$ generators and $\frac{n(n-1)}{2}$ quadratic relations.

Proof. It follows from the proof of Proposition 8 that the algebra $Q_n(\mathcal{E}, \eta)$ is generated by the elements of degree one. It is clear from the construction of the elements $f_{i;\alpha_1,\ldots,\alpha_n}$ that these elements admit quadratic relations of the form

$$f_{j;\alpha_1,\ldots,\alpha_i+1,\ldots,\alpha_n} = c_{i,j;\alpha_1,\ldots,\alpha_n} f_{i;\alpha_1,\ldots,\alpha_j+1,\ldots,\alpha_n} f_{j;\alpha_1,\ldots,\alpha_n},$$

(16)

where $c_{i,j;\alpha_1,\ldots,\alpha_n} \in \mathbb{C}^*$. To prove this relation, one must apply it to the vector $v_{\alpha_1,\ldots,\alpha_n}$. Let us show that these quadratic relations imply the other ones. Let a relation be of the form

$$\sum_{r \in \mathbb{Z}/n\mathbb{Z}} a_r b_r t_{i_r} = 0.$$ 

We expand the element $a_r$ in the basis $\{f_i\}$. The relation becomes

$$\sum_{r \in \mathbb{Z}/n\mathbb{Z}} b_r t_{i_r} f_i = 0.$$ 

Let us now expand $b_r$ in the basis $\{f_{i;0,\ldots,1,\ldots,0}\}$, where 1 stands at the $i_r$th place. Continuing this procedure, we eventually represent the relation in the form

$$\sum c_{i_1,\ldots,i_r} f_{i_1;\alpha_1,\ldots,\alpha_n} f_{i_2;\alpha_1,\ldots,\alpha_{i_2-1},\ldots,\alpha_n} \ldots f_{i_r} = 0.$$ 

It is clear that this relation follows from the relations (16).

Proposition 10. The relations in the algebra $Q_n(\mathcal{E}, \eta)$ are of the form

$$\sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{1}{\theta_{j-i-r}(-\eta)\theta_r(\eta)} x_{j-r} x_{i+r} = 0, \quad i \neq j; \quad i, j \in \mathbb{Z}/n\mathbb{Z}.$$ 

(17)

Proof. Let us apply the formula for the multiplication in the algebra $Q_n(\mathcal{E}, \eta)$ (see §2.1). Since $x_i = \theta_i(z)$, the relations (17) becomes

$$\sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{1}{\theta_{j-i-r}(-\eta)\theta_r(\eta)} \left( \theta_{j-r}(z_1)\theta_{i+r}(z_2-2\eta) \frac{\theta(z_1-z_2-n\eta)}{\theta(z_1-z_2)} + \theta_{j-r}(z_2)\theta_{i+r}(z_1-2\eta) \frac{\theta(z_2-z_1-n\eta)}{\theta(z_2-z_1)} \right) = 0.$$ 

This relation immediately follows from the relation (30) (see Appendix A).

§3 Main properties of the algebra $Q_{n,k}(\mathcal{E}, \eta)$

We again assume that $\mathcal{E} = \mathbb{C}/\Gamma$ is an elliptic curve and $\eta \in \mathcal{E}$. Let $n$ and $k$ be coprime positive integers such that $1 \leq k < n$. Let us present the algebra...
\(Q_{n,k}(E, \eta)\) by the generators \(\{x_i; i \in \mathbb{Z}/n\mathbb{Z}\}\) and the relations

\[
\sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+r(k-1)}(0)}{\theta_{kr}(\eta)\theta_{j-i-r}(-\eta)} x_{j-r} x_{i+r} = 0.
\] (18)

As is known (see §4), this is a PBW-algebra for generic \(E\) and \(\eta\). We conjecture that this holds for any \(E\) and \(\eta\). For generic \(E\) and \(\eta\) the centre of the algebra \(Q_{n,k}(E, \eta)\) is the polynomial ring in \(c = \gcd(n, k+1)\) elements of degree \(n/c\) (see [20]). Hypothetically, this is the case for any \(E\) and \(\eta\), where \(\eta\) is a point of infinite order. If \(\eta \in \mathcal{E}\) is a point of finite order, then the algebra \(Q_{n,k}(E, \eta)\) is finite-dimensional over its centre (see [37]). The following properties can readily be verified:

1) \(Q_{n,k}(E, 0) = \mathbb{C}[x_1, \ldots, x_n]\) is commutative;
2) \(Q_{n,n-1}(E, \eta) = \mathbb{C}[x_1, \ldots, x_n]\) is commutative for any \(\eta\);
3) \(Q_{n,k}(E, \eta) \simeq Q_{n,k}(E, \eta')\), where \(kk' \equiv 1 \pmod{n}\);
4) the maps \(x_i \mapsto x_{i+1}\) and \(x_i \mapsto \varepsilon x_i\) (where \(\varepsilon\) is a primitive root of unity of degree \(n\)) define automorphisms of the algebra \(Q_{n,k}(E, \eta)\).

It follows from the results of §5 (see Proposition 11) that the space of generators of the algebra \(Q_{n,k}(E, \eta)\) is naturally isomorphic to the space of theta functions \(\Theta_{n/k}(\Gamma)\) (see Appendix B). Moreover, this space of generators is dual to the space \(\Theta_{n/n-k}(\Gamma)\) (see Proposition 14). For a description of this duality between the spaces of theta functions, see Appendix C.

The algebra \(Q_{n,k}(E, \eta)\) is not a Hopf algebra and admits no comultiplications. However, there are homomorphisms of the algebra \(Q_{n,k}(E, \eta)\) to tensor products of other algebras of this kind (see [35; §3]). To describe these homomorphisms, we need the notation of Appendix B. Moreover, we denote by \(L_m(E, \eta) = \mathbb{C} \oplus \Theta_{m,0}(\Gamma) \oplus \Theta_{2m,m\eta}(\Gamma) \oplus \ldots\) the \(\mathbb{Z}_{\geq 0}\)-graded algebra with the multiplication * given by the formula \(f * g(z) = f(z + \beta \eta)g(z)\), where \(\beta\) is a power of \(g\). As we know from §2, \(L_n(E, (n-2)\eta)\) is a quotient algebra of \(Q_n(E, \eta)\).

Let \(A\) be an associative algebra and let \(G \subset \text{Aut} A\). We denote by \(A^G \subset A\) the subalgebra consisting of the elements invariant with respect to \(G\).

There are the following algebra homomorphisms:

a) \(Q_{n,k}(E, \eta) \to (L_{kn}(E, n-k+1, \eta) \otimes Q_{k,l}(E, n-k+1, \eta))^{\Gamma_k}\), where \(l = d(n_3, \ldots, n_p)\) and the generators are taken to elements of bidegree \((1, 1)\).

b) \(Q_{n,k}(E, \eta) \to (L_{nk'}(E, n-k'+1, \eta) \otimes Q_{k',l'}(E, n-k'+1, \eta))^{\Gamma_{k'}}\), where \(l' = d(n_1, \ldots, n_{p-2})\) and the generators are taken to elements of bidegree \((1, 1)\).
c) $Q_{n,k}(E, \eta) \to (Q_{\alpha,\alpha}(E, \eta) \otimes L_{\beta\gamma}(E, \eta) \otimes Q_{\beta,\beta}(E, \eta))_{\Gamma_{\alpha\beta}}$, where $a = d(n_1, \ldots, n_{i-1})$, $b = d(n_{i+1}, \ldots, n_p)$, $\alpha = d(n_1, \ldots, n_{i-2})$, and $\beta = d(n_{i+2}, \ldots, n_p)$ for some $i$; the generators are taken to elements of multi-degree $(1, 1, 1)$.

Let us describe the map c) geometrically (the description of the maps a) and b) is the same). Let $f(z_1, \ldots, z_p) \in \Theta_{n/k}(\Gamma)$. For some $i$ ($1 < i < p$) we choose a $z_i$, then $f$ (regarded as a function of $z_1, \ldots, z_{i-1}$) belongs to a space isomorphic to $\Theta_{a/\alpha}(\Gamma)$. Similarly, when regarded as a function of $z_{i+1}, \ldots, z_p$, $f$ belongs to a space isomorphic to $\Theta_{b/\beta}(\Gamma)$. Thus, for a fixed $z_i$ we have $f \in \Theta_{a/\alpha}(\Gamma) \otimes \Theta_{b/\beta}(\Gamma)$. A family of linear maps $\Theta_{n/k}(\Gamma) \to \Theta_{a/\alpha}(\Gamma) \otimes \Theta_{b/\beta}(\Gamma)$ arises. With regard to the dependence on $z_i$, we obtain a linear map $\Theta_{n/k}(\Gamma) \to \Theta_{a/\alpha}(\Gamma) \otimes \Theta_{nab,0}(\Gamma) \otimes \Theta_{b/\beta}(\Gamma)$. The homomorphism c) corresponds to this map (the space of generators of the algebra $Q_{n,k}(E, \eta)$ is $\Theta_{n/k}(\Gamma)$, the space of generators of the algebra $L_{nab}(E, \eta)$ is $\Theta_{nab,0}(\Gamma)$, etc.).

§4 Belavin elliptic $R$-matrix and the algebra $Q_{n,k}(E, \eta)$

Let $V$ be a vector space of dimension $n$. For each $u \in \mathbb{C}$ we denote by $V(u)$ a vector space canonically isomorphic to $V$. Let $R$ be a meromorphic function of two variables with values in $\text{End}(V \otimes V)$. It is convenient to regard $R(u, v)$ as a linear map

$$R(u, v): V(u) \otimes V(v) \to V(v) \otimes V(u).$$

We recall that by the Yang-Baxter equation one means the condition that the following diagram is commutative:

\[
\begin{array}{ccc}
V(v) \otimes V(u) \otimes V(w) & \xrightarrow{R(u,v) \otimes 1} & V(u) \otimes V(w) \otimes V(v) \\
\downarrow{1 \otimes R(v,w)} & & \downarrow{R(v,w) \otimes 1} \\
V(u) \otimes V(w) \otimes V(v) & \xrightarrow{R(u,w) \otimes 1} & V(v) \otimes V(u) \otimes V(w) \\
\end{array}
\]
A solution of the Yang-Baxter equation is called R-matrix.

Let \( \{ x_i; i = 1, \ldots, n \} \) be a basis in the space \( V \) and let \( \{ x_i(u) \} \) be the corresponding basis in the space \( V(u) \). Let \( \mathcal{R}^\alpha_{\beta ij} (u, v) \) be an entry of an R-matrix \( \mathcal{R}(u, v) \), that is, \( \mathcal{R}(u, v) : x_i(u) \otimes x_j(v) \rightarrow \mathcal{R}^\alpha_{\beta ij} (u, v)x_\beta(v) \otimes x_\alpha(u) \).

Let an R-matrix \( \mathcal{R}(u, v) \) satisfy the relation \( \mathcal{R}(u, v)\mathcal{R}(v, u) = 1 \). By the Zamolodchikov algebra \( Z_R \) one means the algebra with the generators \( \{ x_i(u); i = 1, \ldots, n; u \in \mathbb{C} \} \) and the defining relations

\[
x_i(u)x_j(v) = R^\alpha_{\beta ij} (u, v) x_\beta(v)x_\alpha(u).
\]  

It is clear that the elements \( \{ x_{i_1}(u_1) \ldots x_{i_m}(u_m); 1 \leq i_1, \ldots, i_m \leq n \} \) of the Zamolodchikov algebra are linearly independent for generic \( u_1, \ldots, u_m \). Thus, Zamolodchikov algebras are infinite-dimensional analogues of PBW-algebras. We recall that by the classical r-matrix one means a Poisson structure of the form \( \{ x_i(u), x_j(v) \} = \mathcal{R}^\alpha_{\beta ij} (u, v)x_\alpha(u)x_\beta(v) \). It is clear that the Zamolodchikov algebra is a quantization of this Poisson structure if the R-matrix depends on an additional parameter \( \hbar \) and the relations (19) are of the form

\[
x_i(u)x_j(v) = x_j(v)x_i(u) + \hbar \mathcal{R}^\alpha_{\beta ij} (u, v)x_\beta(v)x_\alpha(u) + o(\hbar).
\]

The Yang-Baxter equation has elliptic solutions, which are referred to as Belavin-R-matrices. Let \( n, k \in \mathbb{N} \) be coprime and \( 1 \leq k < n \). For any \( n \) and \( k \) there is a two-parameter family of R-matrices \( R_{n,k}(\mathcal{E}, \eta) \) depending on an elliptic curve \( \mathcal{E} = \mathbb{C}/\Gamma \) and a point \( \eta \in \mathcal{E} \). Namely,

\[
R_{n,k}(\mathcal{E}, \eta)(u-v)(x_i(u) \otimes x_j(v)) = \frac{1}{p(u-v)} \sum_{r \in \mathbb{Z}/n\mathbb{Z}} \theta_{j-i+r(k-1)}(v-u+\eta) \frac{\theta_{kr(\eta)}\theta_{j-i-r}(v-u)}{\theta_{0(\eta)} \ldots \theta_{n-1(\eta)} \theta_{0(v-u)} \ldots \theta_{n-1(v-u)}} x_{j-r}(v) \otimes x_{i+r}(u),
\]

where \( p(u-v) = \cdots \theta_{n-1(0)} \theta_{0(v-u)} \) \( \theta_{0(v-u)} \). One can readily see that \( \det R_{n,k}(\mathcal{E}, \eta)(u-v) = \cdots \). Thus, the operator \( \mathcal{R}_{n,k}(\mathcal{E}, \eta)(-\eta) \) has a kernel. Let \( L_{n,k}(\mathcal{E}, \eta) \subset V \otimes V \) and \( L_{n,k}(\mathcal{E}, \eta) = \ker \mathcal{R}_{n,k}(\mathcal{E}, \eta)(-\eta) \). According to [10] we have \( \dim(L_{n,k}(\mathcal{E}, \eta)) = \frac{n(n-1)}{2} \), and \( L_{n,k}(\mathcal{E}, 0) \cong \Lambda^2 V \) for \( \eta = 0 \). Let \( Q_{n,k}(\mathcal{E}, \eta) = T^*V/(L_{n,k}(\mathcal{E}, \eta)) \) be the algebra with the generators \( \{ x_i; i \in \mathbb{Z}/n\mathbb{Z} \} \) and the defining relations \( L_{n,k}(\mathcal{E}, \eta) \). The dimensions of the graded components of the algebra \( Q_{n,k}(\mathcal{E}, \eta) \) coincide with those of the polynomial ring \( S^*V \) (see [10]). It follows from the formula for \( R_{n,k}(\mathcal{E}, \eta)(u-v) \) that the defining relations in the algebra \( Q_{n,k}(\mathcal{E}, \eta) \) are

\[
\sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+r(k-1)(0)}}{\theta_{kr(\eta)}\theta_{j-i-r}(-\eta)} x_{j-r}x_{i+r} = 0.
\]
In particular, we see that \( Q_{n,1}(\mathcal{E}, \eta) = Q_{n}(\mathcal{E}, \eta) \) if \( \eta \in \mathcal{E} \) is a point of infinite order.

It follows from the relations (20) that \( Q_{n,k}(\mathcal{E}, \eta) \simeq Q_{n,k'}(\mathcal{E}, \eta) \), where \( kk' \equiv 1 \pmod{n} \). Moreover, \( Q_{n,n^{-1}}(\mathcal{E}, \eta) \) is commutative for any \( \mathcal{E}, \eta \). It is also clear that \( Q_{n,k}(\mathcal{E}, 0) \) is commutative.

§5 Algebras \( Q_{n,k}(\mathcal{E}, \eta) \) and the exchange algebras

1 Homomorphisms of algebras \( Q_{n,k}(\mathcal{E}, \eta) \) into dynamical exchange algebras

The algebras \( Q_{n,k}(\mathcal{E}, \eta) \) for arbitrary \( k \) have representations similar to the homomorphisms \( \varphi_p \) related to the case \( k = 1 \) (see §2.3, (11)). However, the structure of the algebra similar to \( B_{p,n}(\eta) \) for \( k = 1 \) turns out to be more complicated for \( k > 1 \). Let

\[
n = \frac{n_1}{n_2} \cdots \frac{n_q}{n_q} = \frac{n_1}{n_2} - \frac{n_2}{n_3} - \cdots - \frac{n_{q-1}}{n_q} \quad \text{be the expansion of the number } \frac{n}{k} \text{ in the continued fraction in which } n_\alpha \geq 2 \text{ for } 1 \leq \alpha \leq q.
\]

It is clear that such an expansion exists and is unique. We recall that \( 1 \leq k < n \), where \( n \) and \( k \) are coprime. Let

\[
d(\{n_1, \ldots, n_q\}) = \det(M), \quad M = (m_{ij}),
\]

be a \((t \times t)\) matrix with the entries \( m_{ii} = m_i, \; m_{i,i+1} = m_{i+1,i} = -1, \; \text{and } m_{ij} = 0 \) for \(|i - j| > 1\). For \( t = 0 \) we set \( d(\emptyset) = 1 \). It is clear that \( n = d(n_1, \ldots, n_q) \) and \( k = d(n_2, \ldots, n_q) \).

**Proposition 11.** There is an algebra homomorphism of \( Q_{n,k}(\mathcal{E}, \eta) \) into the algebra \( C_{n,k}(\eta) \) generated by the commutative subalgebra \( \{ f(y_1, \ldots, y_q) \} \) of holomorphic functions of \( q \) variables and by an element \( e \) with the defining relations of the form \( ef(y_1, \ldots, y_q) = f(y_1 + \eta_1, \ldots, y_q + \eta_q)e, \) where \( \eta_\alpha = (d(n_1, \ldots, n_q) - d(n_1, \ldots, n_{\alpha-1}) - d(n_{\alpha+1}, \ldots, n_q)) \eta \). Moreover, \( x_i \to w_i(y_1, \ldots, y_q)e, \) where \( w_i \in \Theta_{n/k}(\Gamma) \) (see Appendix B).

This is a special case of Proposition 13 below.

The algebra \( C_{n,k}(\eta) \) has a family of modules \( L_{u_1, \ldots, u_q} \) with a basis \( \{ v_{\alpha}; \alpha \in \mathbb{Z}_{\geq 0} \} \) and with the action given by \( ev_{\alpha} = v_{\alpha+1} \) and \( f(y_1, \ldots, y_q)v_{\alpha} = f(v_1 - \alpha \eta_1, \ldots, v_q - \alpha \eta_q)v_{\alpha} \). This implies the following assertion.

**Proposition 12.** The algebra \( Q_{n,k}(\mathcal{E}, \eta) \) has a family of modules \( L_{v_1, \ldots, v_q} \) with a basis \( \{ v_{\alpha}; \alpha \in \mathbb{Z}_{\geq 0} \} \) and with the action : \( x_i v_{\alpha} = w_i(u_1 - \alpha \eta_1, \ldots, u_q - \alpha \eta_q)v_{\alpha} \)
relations in "general position" are as follows:

where \( \alpha \in \Theta_{n/k}(\Gamma) \) and \( \eta_j = (d(n_1, \ldots, n_q) - d(n_1, \ldots, n_{j-1}) - d(n_{j+1}, \ldots, n_q)) \eta. \)

In particular, we see that the algebra \( Q_{n,k}(E, \eta) \) has a family of linear modules depending on the point of \( E^d \), and the space of generators of the algebra \( Q_{n,k}(E, \eta) \) is isomorphic to \( \Theta_{n/k}(\Gamma) \).

Let \( C_{m_1, \ldots, m_q; n,k}(E, \eta) \) be the algebra generated by the commutative sub-algebra \( \{ \varphi(y_1, \ldots, y_{m_1}; \ldots; y_1,q, \ldots, y_{m_q},q) \} \), where \( \varphi \) are the meromorphic functions with the ordinary multiplication, and by elements \( \{ e_{\alpha_1, \ldots, \alpha_q}; 1 \leq \alpha_1 \leq m_t \} \). The defining relations for the algebra \( C_{m_1, \ldots, m_q;n,k}(E, \eta) \) look as follows:

\[
e_{\alpha_1, \ldots, \alpha_q} e_{1, \beta, \nu} = (y_{\beta, \nu} - (d(n_1, \ldots, n_{\nu-1}) + d(n_{\nu+1}, \ldots, n_q)) \eta) e_{\alpha_1, \ldots, \alpha_q}, \quad \alpha_\nu \neq \beta,
\]

\[
e_{\alpha_1, \ldots, \alpha_q} e_{\alpha_\nu, \nu} = (y_{\alpha_\nu, \nu} + (n - d(n_1, \ldots, n_{\nu-1}) - d(n_{\nu+1}, \ldots, n_q)) \eta) e_{\alpha_1, \ldots, \alpha_q}.
\]

These relations mean that the \( y_{\beta,\nu} \)s are dynamical variables. This immediately implies the relations between the elements \( e_{\alpha_1, \ldots, \alpha_q} \) and the meromorphic functions of the variables \( y_{\beta,\nu} \). The remaining relations are quadratic in \( e_{\alpha_1, \ldots, \alpha_q} \) with the coefficients depending on the dynamical variables \( y_{\beta,\nu} \). The relations in "general position" are as follows:

\[
e_{\alpha_1, \ldots, \alpha_q} e_{1, \beta_1, \ldots, \beta_q} = \Lambda e_{\beta_1, \ldots, \beta_q} e_{\alpha_1, \ldots, \alpha_q} + \sum_{1 \leq t \leq q-1} \Lambda_{t,t+1} e_{\beta_1, \ldots, \beta_t, \alpha_{t+1}, \ldots, \alpha_q} e_{\alpha_1, \ldots, \alpha_t, \beta_{t+1}, \ldots, \beta_q},
\]

where \( \alpha_1 \neq \beta_1, \ldots, \alpha_q \neq \beta_q \) and

\[
\Lambda = \frac{e^{-2\pi i m n}(y_{\beta,1} - y_{\alpha,1})(y_{\beta,q} - y_{\alpha,q} + n\eta)}{\theta(y_{\beta,1} - y_{\alpha,1} - n\eta) \theta(y_{\beta,q} - y_{\alpha,q})},
\]

\[
\Lambda_{t,t+1} = \frac{e^{-2\pi i m n}(y_{\beta,1} - y_{\alpha,1}) \theta(y_{\beta,1} - y_{\alpha,1}) \theta(y_{\beta,q} - y_{\alpha,q})}{\theta(y_{\beta,1} - y_{\alpha,1} - n\eta) \theta(y_{\beta,t} - y_{\alpha,t}) \theta(y_{\beta,t+1} - y_{\alpha,t+1}) \theta(y_{\beta,1},t+1 - y_{\alpha,1},t+1)}.
\]

The relations of non-general position occur if some \( \alpha_\nu \)s are equal to \( \beta_\nu \)s. These relations exist for any subset of the form \( \{ \psi + 1, \ldots, \psi + \varphi \} \), where \( 0 \leq \psi, \psi + \varphi \leq q, \) and \( \alpha_\psi = \beta_\psi \) (if \( 0 < \psi \)), \( \alpha_{\psi + \varphi + 1} = \beta_{\psi + \varphi + 1} \) (if \( \psi + \varphi < q \),
and \( \alpha_{\psi+1} \neq \beta_{\psi+1}, \ldots, \alpha_{\psi+q} \neq \beta_{\psi+q} \). These relations are of the form

\[
eq \sum_{1 \leq t < \varphi} \Lambda_{t,t+1} e_{\mu_1,\ldots,\mu_q,\beta_1,\ldots,\beta_q,\gamma_1,\ldots,\gamma_p}^t + e_{\mu_1,\ldots,\mu_q,\beta_1,\ldots,\beta_q,\gamma_1,\ldots,\gamma_p}^t \]

(22)

Here \( \alpha_1 \neq \beta_1, \ldots, \alpha_q \neq \beta_q \), and \( \varphi + \psi + p = q \). The coefficients \( \Lambda, \Lambda_{t,t+1} \) are defined by (21). If \( \psi = p = 0 \), then this relation coincides with (21), that is, becomes a relation in general position.

We note that, if \( \eta = 0 \), then the algebra \( C_{m_1,\ldots,m_q,n,k}(E,0) \) is commutative and does not depend on the elliptic curve \( E \). For \( q = 1 \) and \( q = 2 \) the algebra \( C_{m_1,\ldots,m_q,n,k}(E,0) \) is the polynomial ring in the variables \( \{ e_{\alpha_1,\ldots,\alpha_q} \} \) over the field of meromorphic functions of the variables \( \{ y_{\alpha,\beta} \} \). For \( q > 2 \) the algebra \( C_{m_1,\ldots,m_q,n,k}(E,0) \) is no longer a polynomial ring because additional relations occur. Namely, the relations (22) for \( \eta = 0 \) become

\[
eq e_{\mu_1,\ldots,\mu_q,\alpha_1,\ldots,\alpha_q,\gamma_1,\ldots,\gamma_p}^t + e_{\mu_1,\ldots,\mu_q,\alpha_1,\ldots,\alpha_q,\gamma_1,\ldots,\gamma_p}^t \]

(23)

The algebra \( C_{m_1,\ldots,m_q,n,k}(E,\eta) \) is a flat deformation of the function algebra on the manifold defined by the relations (23). Moreover, \( y_{\alpha,\beta} \) are dynamical variables. It is clear that the manifold defined by the equations (23) is rational, and the general solution of these equations is \( e_{\alpha_1,\ldots,\alpha_q} = e_{(1)}^{(t)} e_{(2)}^{(t)} \ldots e_{(q-1)}^{(t)} e_{(q)}^{(t)} \), where \( \{ e_{(t)}^{(t)} \} \) are independent variables.

**Proposition 13.** There is an algebra homomorphism \( \varphi: Q_{n,k}(E,\eta) \rightarrow C_{m_1,\ldots,m_q,n,k}(E,\eta) \) that acts on the generators of the algebra \( Q_{n,k}(E,\eta) \) as follows:

\[
\varphi(x_i) = \sum_{1 \leq \alpha_1 \leq m_1 \atop 1 \leq \alpha_2 \leq m_2} w_i (y_{\alpha_1,1}, \ldots, y_{\alpha_2,q}) e_{\alpha_1,\ldots,\alpha_q},
\]

(24)

Here \( w_i \in \Theta_{n,k}(\Gamma) \).

**Proof.** The algebra \( Q_{n,k}(E,\eta) \) is defined by the relations (18). Let us show
that the images $\varphi(x_i)$ satisfy the same relations. We have

$$\sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+r(k-1)}(0)}{\theta_{kr}(\eta)\theta_{j-i-r}(-\eta)} \varphi(x_{j-r})\varphi(x_{i+r}) =$$

$$= \sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+r(k-1)}(0)}{\theta_{kr}(\eta)\theta_{j-i-r}(-\eta)} w_{j-r}(y_{\alpha_1,1}, \ldots, y_{\alpha_q,q}) e_{\alpha_1,\ldots,\alpha_q} w_{i+r}(y_{\beta_1,1}, \ldots, y_{\beta_q,q}) e_{\beta_1,\ldots,\beta_q}.$$ 

Using the relations (21) and (22), we obtain an expression of the form

$$\sum_{a_1 \leq \beta_1; \ a_2 \leq \beta_2} \psi_{a_1,\ldots,a_q,\beta_1,\ldots,\beta_q} e_{a_1,\ldots,a_q} e_{\beta_1,\ldots,\beta_q}.$$

We must prove that the coefficients are equal to 0. Let us restrict ourselves to the case $\alpha_1 \neq \beta_1, \ldots, \alpha_q \neq \beta_q$. In this case we have by direct calculation

$$\psi_{a_1,\ldots,a_q,\beta_1,\ldots,\beta_q} = \sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+r(k-1)}(0)}{\theta_{kr}(\eta)\theta_{j-i-r}(-\eta)} \left( w_{j-r}(y_{\alpha_1,1}, \ldots, y_{\alpha_q,q}) w_{i+r}(y_{\beta_1,1}, \ldots, y_{\beta_q,q}) + \right.$$

$$\left. + \frac{e^{-2\pi in\eta}(y_{\alpha_1,1} - y_{\beta_1,1})}{\theta(y_{\alpha_1,1} - y_{\beta_1,1} - m\eta)} \frac{y_{\alpha_1,q} - y_{\beta_1,q} + m\eta}{\theta(y_{\alpha_1,q} - y_{\beta_1,q})} w_{j-r}(y_{\alpha_1,1}, \ldots, y_{\alpha_q,q}) \times \right.$$

$$\left. \times w_{i+r}(y_{\alpha_1,1} + t_1, \ldots, y_{\alpha_q,q} + t_q) + \right.$$

$$\left. + \sum_{1 \leq t \leq q-1} e^{-2\pi in\eta}(\eta) \frac{(y_{\alpha_1,1} - y_{\beta_1,1})}{\theta_{\alpha_1,1} - y_{\beta_1,1} - m\eta} \frac{y_{\alpha_1,t} + y_{\alpha_1,t+1} - y_{\beta_1,t} - y_{\alpha_1,t+1}}{\theta(y_{\alpha_1,t} - y_{\beta_1,t}) \theta(y_{\alpha_1,t+1} - y_{\alpha_1,t+1})} \times \right.$$

$$\left. \times w_{j-r}(y_{\beta_1,1}, \ldots, y_{\beta_1,t}, y_{\alpha_1,t+1}, \ldots, y_{\alpha_q,q}) \times \right.$$

$$\left. \times w_{i+r}(y_{\alpha_1,1} + t_1, \ldots, y_{\alpha_1} + t_t, y_{\beta_1,1} + t_{t+1}, \ldots, y_{\beta_q,q} + t_q) \right).$$

Here $t'_a = -(d(n_1, \ldots, n_{a-1}) + d(n_{a+1}, \ldots, n_q))\eta$. The equality $\psi_{a_1,\ldots,a_q,\beta_1,\ldots,\beta_q} = 0$ immediately follows from the identity (35) proved in Appendix B. \square

2 Homomorphism of the exchange algebra into the algebra $Q_{n,k}(\mathcal{E}, \eta)$

Let $q' \in \mathbb{N}$ and $\mu_1, \ldots, \mu_{q'}, \mu \in \mathbb{C}$. We define an associative algebra $Y_{q'}(\mathcal{E}, \mu; \mu_1, \ldots, \mu_{q'})$ as follows. The algebra $Y_{q'}(\mathcal{E}, \mu; \mu_1, \ldots, \mu_{q'})$ is presented
by the generators \( \{ e(u_1, \ldots, u_{q'}) ; u_1, \ldots, u_{q'} \in \mathbb{C} \} \) and the defining relations

\[
\frac{\theta(v_1 - u_1 + \mu)}{\theta(v_1 - u_1)} e(u_1, \ldots, u_{q'}) e(v_1 + \mu_1, \ldots, v_{q'} + \mu_{q'}) = \\
\sum_{1 \leq t < q'} \frac{\theta(\mu)}{\theta(v_t - u_t) \theta(u_{t+1} - v_{t+1})} \times \\
\times e(v_1, \ldots, v_t, u_{t+1}, \ldots, u_{q'}) e(u_1 + \mu_1, \ldots, u_t + \mu, v_{t+1} + \mu_{t+1}, \ldots, v_{q'} + \mu_{q'}) + \\
\frac{\theta(v_{q'} - u_{q'})}{\theta(v_{q'} - u_{q'})} e(v_1, \ldots, v_{q'}) e(u_1 + \mu_1, \ldots, u_{q'} + \mu_{q'}). 
\]

For \( \mu = \mu_1 = \ldots = \mu_{q'} = 0 \) the algebra \( Y_{q'}(E, 0; 0, \ldots, 0) \) is the polynomial ring in infinitely many variables \( \{ e(u_1, \ldots, u_{q'}) ; u_1, \ldots, u_{q'} \in \mathbb{C} \} \). One can show that the algebra \( Y_{q'}(E, \mu; \mu_1, \ldots, \mu_{q'}) \) is a flat deformation of this polynomial ring.

Let \( \frac{n}{n-k} = n_1' - \frac{1}{n_2'-\ldots-n_{q'}'} \) be an expansion in the continued fraction in which \( n'_\alpha \geq 2 \) for \( 1 \leq \alpha \leq q' \). It is clear that such an expansion exists and unique. For the relationship between the expansions in continued fractions of the numbers \( \frac{n}{k} \) and \( \frac{n}{n-k} \), see Appendix C.

**Proposition 14.** There is an algebra homomorphism

\[
\psi : Y_{q'}(E, \mu; \mu_1, \ldots, \mu_{q'}) \rightarrow Q_{n,k}(E, \eta),
\]

where \( \mu = n\eta \) and \( \mu_\alpha = (d(n'_1, \ldots, n'_{\alpha-1}) - d(n'_{\alpha+1}, \ldots, n'_{q'}))\eta. \) This homomorphism is of the form:

\[
\psi : e(u_1, \ldots, u_{q'}) \rightarrow \sum_{\alpha \in \mathbb{Z}/n\mathbb{Z}} w_\alpha(u_1, \ldots, u_{q'}) x_{1-\alpha}, \tag{25}
\]

where \( \{ x_i ; i \in \mathbb{Z}/n\mathbb{Z} \} \) are the generators of the algebra \( Q_{n,k}(E, \eta) \), and \( \{ w_\alpha ; \alpha \in \mathbb{Z}/n\mathbb{Z} \} \) is a basis in the space of theta functions \( \Theta_{n/n-k}(\Gamma) \) (see Appendix B).

**Proof.** Let us apply the map \( \psi \) to the difference between the left- and right-hand sides of the relations in the algebra \( Y_{q'}(E, \mu; \mu_1, \ldots, \mu_{q'}) \). We must verify
the resulting relation in the algebra $Q_{n,k}(E, \eta)$. We have

\[
\frac{\theta(v_1 - u_1 + \mu)}{\theta(v_1 - u_1)} \psi(e(u_1, \ldots, u_{q'})) \psi(e(v_1 + \mu_1, \ldots, v_{q'} + \mu_{q'})) - \\
- \sum_{1 \leq t < q'} \frac{\theta(\mu) \theta(v_t - u_t + u_{t+1} - v_{t+1})}{\theta(v_t - u_t) \theta(u_{t+1} - v_{t+1})} \times \\
\times \psi(e(v_1, \ldots, v_t, u_{t+1}, \ldots, u_{q'})) \psi(e(u_1 + \mu_1, \ldots, u_t + \mu, v_{t+1} + \mu_{t+1}, \ldots, v_{q'} + \mu_{q'})) - \\
- \frac{\theta(v_{q'} - u_{q'} + \mu)}{\theta(v_{q'} - u_{q'})} \psi(e(v_1, \ldots, v_{q'})) \psi(e(u_1 + \mu_1, \ldots, u_{q'} + \mu_{q'})) = \\
= \sum_{\alpha, \beta \in \mathbb{Z}/n\mathbb{Z}} x_{1-\alpha} x_{1-\beta} \times \\
\times \left( \frac{\theta(v_1 - u_1 + \mu)}{\theta(v_1 - u_1)} w_\alpha(u_1, \ldots, u_{q'}) w_\beta(v_1 + \mu_1, \ldots, v_{q'} + \mu_{q'}) - \\
- \sum_{1 \leq t < q'} \frac{\theta(\mu) \theta(v_t - u_t + u_{t+1} - v_{t+1})}{\theta(v_t - u_t) \theta(u_{t+1} - v_{t+1})} \times \\
\times w_\alpha(v_1, \ldots, v_t, u_{t+1}, \ldots, u_{q'}) w_\beta(u_1 + \mu_1, \ldots, u_t + \mu, v_{t+1} + \mu_{t+1}, \ldots, v_{q'} + \mu_{q'}) - \\
- \frac{\theta(v_{q'} - u_{q'} + \mu)}{\theta(v_{q'} - u_{q'})} w_\alpha(v_1, \ldots, v_{q'}) w_\beta(u_1 + \mu_1, \ldots, u_{q'} + \mu_{q'}) \right). 
\]

By using the identity (35) in Appendix B together with the relations (18) in the algebra $Q_{n,k}(E, \eta)$, one can readily see that this expression is equal to 0. \qed

33
Appendix A

Theta functions of one variable

Let $\Gamma \subset \mathbb{C}$ be an integral lattice generated by 1 and $\tau \in \mathbb{C}$, where $\text{Im} \tau > 0$. Let $n \in \mathbb{N}$ and $c \in \mathbb{C}$. We denote by $\Theta_{n,c}(\Gamma)$ the space of the entire functions of one variable satisfying the following relations:

$$f(z + 1) = f(z), \quad f(z + \tau) = (-1)^n e^{-2\pi i (nz - c)} f(z).$$

As is known [30], $\dim \Theta_{n,c}(\Gamma) = n$, every function $f \in \Theta_{n,c}(\Gamma)$ has exactly $n$ zeros modulo $\Gamma$ (counted according to their multiplicities), and the sum of these zeros modulo $\Gamma$ is equal to $c$. Let $\theta(z) = \sum_{\alpha \in \mathbb{Z}} (-1)^\alpha e^{2\pi i (\alpha z + \alpha^2/2) \tau}$. It is clear that $\theta(z) \in \Theta_{1,0}(\Gamma)$. It follows from what was said above that $\theta(0) = 0$, and this is the only zero modulo $\Gamma$. One can readily see that $\theta(-z) = -e^{-2\pi iz} \theta(z)$. Moreover, as is known, the function $\theta(z)$ can be expanded as the infinite product as follows:

$$\theta(z) = \prod_{\alpha \geq 1} (1 - e^{2\pi i \alpha \tau}) \cdot (1 - e^{2\pi i z}) \cdot \prod_{\alpha \geq 1} (1 - e^{2\pi i (z + \alpha \tau)})(1 - e^{2\pi i (\alpha \tau - z)}).$$

Let us introduce the following linear operators $T_{\frac{1}{n}}$ and $T_{\frac{1}{n} \tau}$ acting on the space of functions of one variable:

$$T_{\frac{1}{n}} f(z) = f \left( z + \frac{1}{n} \right), \quad T_{\frac{1}{n} \tau} f(z) = e^{2\pi i \left( z + \frac{1}{n^2} - \frac{1}{2n^2} \tau \right)} f \left( z + \frac{1}{n} \tau \right).$$

One can readily see that the space $\Theta_{n, \frac{1}{n}}(\Gamma)$ is invariant with respect to the operators $T_{\frac{1}{n}}$ and $T_{\frac{1}{n} \tau}$. Moreover, $T_{\frac{1}{n}} T_{\frac{1}{n} \tau} = e^{2\pi i} T_{\frac{1}{n} \tau} T_{\frac{1}{n}}$. The restriction of these operators to the space $\Theta_{n, \frac{1}{n}}(\Gamma)$ satisfy the relations $T_{\frac{1}{n}}^n = T_{\frac{1}{n} \tau}^n = 1$.

Let $\tilde{\Gamma}_n$ be the group with the generators $a, b, \varepsilon$ and the defining relations $ab = \varepsilon ba$, $a\varepsilon = \varepsilon a$, $b\varepsilon = \varepsilon b$, and $a^n = b^n = \varepsilon^n = 1$. The group $\tilde{\Gamma}_n$ is a central extension of the group $\Gamma_n = \Gamma / n \Gamma \simeq (\mathbb{Z}/n\mathbb{Z})^2$, namely, the element $\varepsilon$ generates a normal subgroup $C_n = \mathbb{Z}/n\mathbb{Z}$, and $\tilde{\Gamma}_n / C_n = \Gamma_n$. The formulas $a \mapsto T_{\frac{1}{n}}$, $b \mapsto T_{\frac{1}{n} \tau}$, and $\varepsilon \mapsto$ (multiplication by $e^{2\pi i}$) define an irreducible representation of the group $\tilde{\Gamma}_n$ in the space $\Theta_{n, \frac{1}{n}}(\Gamma)$. Let us choose a basis $\{\theta_{\alpha}; \alpha \in \mathbb{Z}/n\mathbb{Z}\}$ in the space $\Theta_{n, \frac{1}{n}}(\Gamma)$ in which our operators act as follows:
\[ T_{\frac{1}{2}} \theta_{-\alpha} = e^{2\pi i \frac{\alpha}{n}} \theta_{\alpha}, \text{ and } T_{\frac{1}{2}} \theta_{\alpha} = \theta_{\alpha+1}. \] It is clear that this choice can be carried out uniquely up to multiplication by a common constant. The functions \( \theta_{\alpha}(z) \) are of the form

\[ \theta_{\alpha}(z) = \theta \left( z + \frac{\alpha}{n} \right) \theta \left( z + \frac{1}{n} + \frac{\alpha}{n} \right) \ldots \theta \left( z + \frac{n-1}{n} + \frac{\alpha}{n} \right) e^{2\pi i (\alpha z + \frac{n(n-1)\tau}{2n})}. \]

One can readily see that \( \theta_{\alpha}(z) \in \Theta_{n, \frac{n-1}{2n}}(\Gamma) \), \( \theta_{\alpha+n}(z) = \theta_{\alpha}(z) \), and

\[ \begin{align*}
\theta_{\alpha} \left( z + \frac{1}{n} \right) &= e^{2\pi i \frac{\alpha}{n}} \theta_{\alpha}(z), \\
\theta_{\alpha} \left( z + \frac{1}{n} \tau \right) &= e^{-2\pi i (z + \frac{1}{n} - \frac{n-1}{2n}\tau/ \theta_{\alpha+1}(z)}.
\end{align*} \] (26)

It is clear that the functions \( \{ \theta_{\alpha} (z - \frac{1}{n} c - \frac{n-1}{2n}) ; \alpha \in \mathbb{Z}/n\mathbb{Z} \} \) form a basis in the space \( \Theta_{n,c}(\Gamma) \).

We need some identities:

\[ \theta(nz) = \frac{n\theta_0(z) \ldots \theta_{n-1}(z) e^{-2\pi i n(n-1)\tau}}{\theta_1(0) \ldots \theta_{n-1}(0) \theta (\frac{1}{n}) \ldots \theta (\frac{n-1}{n})}. \] (27)

**Proof.** One can readily see by using relations (26) that the functions on both sides of the equation belong to the space \( \Theta_{n^2, n(n-1)\tau}(\Gamma) \). Moreover, it is clear that the zeros of both functions coincide, namely, these are \( n^2 \) points \( \{ \frac{\alpha}{n} + \frac{\beta}{n}\tau; \alpha, \beta \in \mathbb{Z} \} \) modulo \( \Gamma \). Hence, the functions on the left- and right-hand sides of the equation differ by a constant multiple, which can be evaluated by dividing (27) by \( \theta(z) \) and passing to the limit as \( z \to 0 \). \( \square \)

Let \( \theta_0, \theta_1, \theta_2 \in \Theta_{\mathbb{Z}, 0}(\Gamma) \). For \( z, \eta \in \mathbb{C} \) and \( \alpha \in \mathbb{Z}/3\mathbb{Z} \) we have

\[ \theta_0(\eta)\theta_{\alpha}(z + \eta)\theta_{\alpha}(z) + \theta_1(\eta)\theta_{\alpha+2}(z + \eta)\theta_{\alpha+1}(z) + \theta_2(\eta)\theta_{\alpha+1}(z + \eta)\theta_{\alpha+2}(z) = 0. \] (28)

**Proof.** It is clear that \( \theta_{\alpha}(z + \eta)\theta_{\beta}(z) \in \Theta_{\mathbb{Z} - 3\eta}(\Gamma) \) as a function of the variable \( z \). There must be three linear relations among these nine functions in a six-dimensional space. With regard to the action of the group \( \Gamma_3 \), we see that the relations must be of the form \( a(\eta)\theta_{\alpha}(z + \eta)\theta_{\alpha}(z) + b(\eta)\theta_{\alpha+1}(z + \eta)\theta_{\alpha+2}(z) + c(\eta)\theta_{\alpha+2}(z + \eta)\theta_{\alpha+1}(z) = 0 \). Really, every three-dimensional space of relations invariant with respect to the translations \( z \to z + \frac{1}{3} \) and \( z \to z + \frac{1}{3}\tau \) (see
(26)) is of this form, where \( a, b, c \) do not depend on \( \alpha \). By setting \( \alpha = 1 \) and \( z = 0 \), we obtain \( \frac{b(\eta)}{a(\eta)} = \frac{\theta(\eta)}{\theta(\eta)} \). By setting \( \alpha = 2 \) and \( z = 0 \), we obtain

\[
\frac{b(\eta)}{a(\eta)} = \frac{\theta(\eta)}{\theta(\eta)} \]

Let \( \theta_\alpha \in \Theta_{n,c}(\Gamma) \). Then

\[
\frac{\theta(y - z + nv - nu)}{\theta(y - z)\theta(nv - nu)} \theta_\alpha(y + u)\theta_\beta(z + v + \eta) + \frac{\theta(z - y + n\eta)}{\theta(z - y)\theta(n\eta)} \theta_\alpha(z + u)\theta_\beta(y + v + \eta) =
\]

\[
= \frac{1}{\eta} \left( \frac{1}{\eta} \right) \ldots \frac{1}{\eta} \left( \frac{n - 1}{\eta} \right) \sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{\beta - \alpha}(v - u + \eta)}{\theta_r(\eta)} \theta_{\beta - \alpha - r}(v - u) \theta_{\beta - r}(y + v)\theta_{\alpha + r}(z + u + \eta).
\]

(29)

**Proof.** This is a special case of the relation (31) (for \( p = 1 \)) proved in Appendix B.

By setting \( u = v + \eta \) in the relation (29) and making the change of variables \( y + v \to y \), \( z + v \to z \), we obtain

\[
\frac{\theta(z - y + n\eta)}{\theta(z - y)\theta(n\eta)} (\theta_\alpha(z + \eta)\theta_\beta(y + \eta) - \theta_\alpha(y + \eta)\theta_\beta(z + \eta)) =
\]

\[
= \frac{1}{\eta} \left( \frac{1}{\eta} \right) \ldots \frac{1}{\eta} \left( \frac{n - 1}{\eta} \right) \theta_{\beta - \alpha}(0) \sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{1}{\theta_r(\eta)} \theta_{\beta - \alpha - r}(-\eta) \theta_{\beta - r}(y)\theta_{\alpha + r}(z + 2\eta).
\]

(30)

**Appendix B**

Some theta functions of several variables associated with a power of an elliptic curve

Let \( n \) and \( k \) be coprime positive integers such that \( 1 \leq k < n \). We expand the ratio \( \frac{\theta_k}{\theta_n} \) in a continued fraction of the form:

\[
\frac{\theta_k}{\theta_n} = n_1 - \frac{1}{n_2 - \frac{1}{n_3 - \ldots - \frac{1}{n_p}}},
\]

where \( n_\alpha \geq 2 \) for any \( \alpha \). It is clear that such an expansion exists and is unique. We denote by \( d(m_1, \ldots, m_q) \) the determinant of the \((q \times q)\) matrix \((m_{\alpha\beta})\),
where \( m_{aa} = m_a, m_{a,a+1} = m_{a+1,a} = -1 \), and \( m_{a,\beta} = 0 \) for \(|\alpha - \beta| > 1\). For \( q = 0 \) we set \( d(\emptyset) = 1 \). It follows from the elementary theory of continued fractions that \( n = d(n_1, \ldots, n_p) \) and \( k = d(n_2, \ldots, n_p) \).

Let \( \Gamma \subset \mathbb{C} \) be an integral lattice generated by 1 and \( \tau \) again, where \( \text{Im} \tau > 0 \).

We denote by \( \Theta_{n/k}(\Gamma) \) the space of all functions of \( p \) variables satisfying the following relations:

\[
\begin{align*}
    f(z_1, \ldots, z_{\alpha} + 1, \ldots, z_p) &= f(z_1, \ldots, z_p), \\
    f(z_1, \ldots, z_{\alpha} + \tau, \ldots, z_p) &= (-1)^{n_{\alpha}} e^{-2\pi i(n_{\alpha}z_1 - z_{\alpha-1} - z_{\alpha+1}^\delta)} f(z_1, \ldots, z_p).
\end{align*}
\]

Here \( 1 \leq \alpha \leq p \) and \( z_0 = z_{p+1} = 0 \), and \( \delta_{1,\alpha} \) stands for the Kronecker delta. Thus, the functions \( f \in \Theta_{n/k}(\Gamma) \) are periodic with respect to each of the variables with period 1 and quasiperiodic with period \( \tau \). By the periodicity, each function in the space \( \Theta_{n/k}(\Gamma) \) can be expanded in a Fourier series of the form \( f(z_1, \ldots, z_p) = \sum_{\alpha_1, \ldots, \alpha_p \in \mathbb{Z}} a_{\alpha_1 \ldots \alpha_p} e^{2\pi i(\alpha_1 z_1 + \ldots + \alpha_p z_p)} \). By the quasiperiodicity, the coefficients satisfy the system of linear equations

\[
a_{\alpha_1 \ldots \alpha_{p-1} - 1, \alpha_p + n_{\alpha}, \alpha_{p+1} - 1, \ldots, \alpha_p} = (-1)^{n_{\alpha}} e^{2\pi i(\alpha_1 + \delta_{1,\alpha}-1)\tau} a_{\alpha_1 \ldots \alpha_p}.
\]

This system clearly has \( n = d(n_1, \ldots, n_p) \) linearly independent solutions each defining (for \( \text{Im} \tau > 0 \); \( n_1, \ldots, n_p \geq 2 \)) a function in the space \( \Theta_{n/k}(\Gamma) \).

For \( k = 1 \) we have the space of functions of one variable \( \Theta_n(\Gamma) = \Theta_n,0(\Gamma) \) (see Appendix A) with a basis \( \{ w_\alpha(z) = \theta_\alpha (z + \frac{n-1}{2}) \mid \alpha \in \mathbb{Z}/n\mathbb{Z} \} \). A similar basis can be constructed in the space \( \Theta_{n/k}(\Gamma) \) for an arbitrary \( k \). Let us define the operators \( T^1_n, T^1_n, T^1_n, T^1_n, T^1_n, T^1_n \) in the space of functions of \( p \) variables as follows:

\[
\begin{align*}
    T^1_n f(z_1, \ldots, z_p) &= f(z_1 + r_1, \ldots, z_p + r_p), \\
    T^1_n f(z_1, \ldots, z_p) &= e^{2\pi i(z_1 + \varphi)} f(z_1 + r_1 \tau, \ldots, z_p + r_p \tau).
\end{align*}
\]

Here \( r_\alpha = \frac{d(n_{\alpha+1}, \ldots, n_p)}{d(n_1, \ldots, n_p)} \) and \( \varphi \in \mathbb{C} \) is a constant.

It is clear that \( T^1_n T^1_n = e^{2\pi i} T^1_n T^1_n \). As in the case of theta functions of one variable, the space \( \Theta_{n/k}(\Gamma) \) is invariant with respect to the operators \( T^1_n \) and \( T^1_n, \) and the restriction of these operators to \( \Theta_{n/k}(\Gamma) \) satisfies the relations \( T^1_n = 1 \) and \( T^1_n = \mu \), where \( \mu \in \mathbb{C} \). Let us choose a \( \varphi \) in such a way that \( \mu = 1 \); clearly, this can be done uniquely up to multiplication of \( T^1_n \) by a root of unity of degree \( n \).
Proposition 15. There is a basis \( \{ w_\alpha(z_1, \ldots, z_p); \alpha \in \mathbb{Z}/n\mathbb{Z} \} \) in \( \Theta_{n/k}(\Gamma) \) such that
\[
T_\frac{1}{n} w_\alpha = e^{2\pi i \frac{k}{n} \alpha} w_\alpha, \quad T_\frac{1}{n} \tau w_\alpha = w_{\alpha+1}.
\]
This basis is defined uniquely up to multiplication by a common constant.

Proof. Let \( f \in \Theta_{n/k}(\Gamma) \) be an eigenvector of the operator \( T_\frac{1}{n} \) with an eigenvalue \( \lambda \). Since \( T_\frac{1}{n} T_\frac{1}{n} = 1 \) on the space \( \Theta_{n/k}(\Gamma) \), we have \( \lambda^n = 1 \). Moreover,
\[
T_\frac{1}{n} T_\frac{1}{n} \tau f = e^{2\pi i \frac{k}{n} \lambda} T_\frac{1}{n} \tau f,
\]
and hence \( T_\frac{1}{n} \tau f \) is also an eigenvector with the eigenvalue \( e^{2\pi i \frac{k}{n} \lambda} \). Since \( n \) and \( k \) are coprime, \( e^{2\pi i \frac{k}{n} \lambda} \) is a primitive root of unity of degree \( n \). Thus, the vectors \( \{ T_\frac{1}{n} \tau f; \alpha = 0, 1, \ldots, n-1 \} \) are eigenvectors for the operator \( T_\frac{1}{n} \) with different eigenvalues, and every root of unity of degree \( n \) is an eigenvalue for some \( T_\frac{1}{n} \tau f \). Let \( w_0 \) be such that \( T_\frac{1}{n} w_0 = w_0 \). We set \( w_\alpha = T_\frac{1}{n} \tau w_0 \). It is clear that \( T_\frac{1}{n} w_\alpha = e^{2\pi i \frac{k}{n} \alpha} w_\alpha \) and \( T_\frac{1}{n} \tau w_\alpha = w_{\alpha+1} \). Moreover, \( w_{\alpha+n} = w_\alpha \) because \( T_\frac{n}{n} \tau = 1 \) on the space \( \Theta_{n/k}(\Gamma) \).

Remark. Let \( L \) be the group of linear automorphisms on the space of functions of \( p \) variables of the form
\[
gf(z_1, \ldots, z_p) = e^{2\pi i (\varphi_1 z_1 + \ldots + \varphi_p z_p + \lambda)} f(z_1 + \psi_1, \ldots, z_p + \psi_p)
\]
for \( g \in L \). It is clear that \( L \) is a \((2p + 1)\)-dimensional Lie group. Let \( L' \subset L \) be the subgroup of transformations preserving the space \( \Theta_{n/k}(\Gamma) \), that is, \( L' = \{ g \in L; g(\Theta_{n/k}(\Gamma)) = \Theta_{n/k}(\Gamma) \} \). Let \( L'' \subset L' \) consist of the elements preserving each point of \( \Theta_{n/k}(\Gamma) \), that is, \( L'' = \{ g \in L'; gf = f \text{ for any } f \in \Theta_{n/k}(\Gamma) \} \). One can see that the quotient group \( L'/L'' = \tilde{G}_n \) is generated by the elements \( T_\frac{1}{n} \) and \( T_\frac{1}{n} \tau \) and by the multiplications by constants.

We shall use the notation \( w_{\alpha/n/k}(z_1, \ldots, z_p) \) if it is not clear from the context what are the theta functions in use.

We need the following identity relating theta functions in the spaces
\[ \Theta_{1,0}(\Gamma), \Theta_{n,\varphi-1}(\Gamma), \text{ and } \Theta_{n/k}(\Gamma): \]

\[
\frac{\theta(y_1 - z_1 + n\nu - nu)}{\theta(n\nu - nu)\theta(y_1 - z_1)} \times w_\alpha(y_1 + m_1 u, \ldots, y_p + m_p u) w_\beta(z_1 + m_1 v + l_1, \ldots, z_p + m_p v + l_p)
\]

\[
+ \sum_{1 \leq r \leq p} \frac{\theta(z_1 - y_1 + y_{t+1} - z_{t+1})}{\theta(z_1 - y_t)\theta(y_{t+1} - z_{t+1})} \times w_\alpha(z_1 + m_1 u, \ldots, z_p + m_p u)
\]

\[
+ \frac{\theta(z_p - y_p + \nu)}{\theta(z_p - y_p)\theta(\nu)} w_\alpha(y_1 + m_1 u, \ldots, y_p + m_p u)
\]

\[
+ w_\beta(y_1 + m_1 v + l_1, \ldots, y_p + m_p v + l_p)
\]

\[
= \frac{1}{n} \theta\left(\frac{1}{n}\right) \ldots \theta\left(\frac{n-1}{n}\right) \times \sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{\beta-\alpha+r}(k-1)(v - u + \nu)}{\theta_{\iota k}(\nu)\theta_{\beta-\alpha-r}(v - u)} w_{\beta-r}(y_1 + m_1 v, \ldots, y_p + m_p v)
\]

\[
\times w_{\alpha+r}(z_1 + m_1 u + l_1, \ldots, z_p + m_p u + l_p). \tag{31}
\]

Here \( m_\alpha = d(n_{\alpha+1}, \ldots, n_p) \) and \( l_\alpha = d(n_1, \ldots, n_{\alpha-1})\eta. \)

**Proof.** We denote by \( \varphi_{\alpha,\beta}(\eta, u, v, y_1, \ldots, y_p, z_1, \ldots, z_p) \) the difference between the right- and left-hand sides of the formula (31). The calculation shows that this function satisfies the following relations:

\[
\varphi_{\alpha,\beta}(\eta_1, \ldots, \eta_a + 1, \ldots, z_p) = \varphi_{\alpha,\beta}(\eta_1, \ldots, z_p),
\]

\[
\varphi_{\alpha,\beta}(\eta_1, \ldots, \eta_a + \tau, \ldots, z_p) = -e^{-2\pi i(n_{\alpha,\eta_a-\eta_{a-1}}-\eta_{a+1}+\delta_{\alpha,\eta_a} + \delta_{\eta_a,1})} \varphi_{\alpha,\beta}(\eta_1, \ldots, z_p),
\]

\[
\varphi_{\alpha,\beta}(\eta_1, \ldots, z_a + 1, \ldots, z_p) = \varphi_{\alpha,\beta}(\eta_1, \ldots, z_p),
\]

\[
\varphi_{\alpha,\beta}(\eta_1, \ldots, z_a + \tau, \ldots, z_p) = -e^{-2\pi i(n_{\alpha,z_a-z_{a-1}}-z_{a+1}+\delta_{\alpha,1}u+\delta_{\alpha,p}\eta)} \varphi_{\alpha,\beta}(\eta_1, \ldots, z_p). \tag{32}
\]

Here \( y_0 = y_{p+1} = 0 = z_{p+1} = 0, \) and \( \delta_{\alpha,\beta} \) stands for the Kronecker delta. Moreover, an evaluation shows that there are no poles on the divisors \( nu -\nu, y_1 - z_1, \ldots, y_p - z_p, y_{p+1}, z_{p+1} \in \Gamma, \) and hence the function \( \varphi_{\alpha,\beta} \) is holomorphic everywhere on \( \mathbb{C}^{2p+3}. \) However, it is clear that the functions \( \{ w_\lambda(y_1 + m_1 v, \ldots, y_p + m_p v)w_\nu(z_1 + m_1 u + l_1, \ldots, z_p + m_p u + l_p) ; \lambda, \nu \in \mathbb{Z}/n\mathbb{Z} \} \) form a basis in the space of holomorphic functions (of the variables
\( y_1, \ldots, y_p, z_1, \ldots, z_p \) satisfying the conditions (32). Therefore, the function \( \varphi_{\alpha, \beta} \) is of the form

\[
\varphi_{\alpha, \beta}(\eta, u, v, y_1, \ldots, z_p) = \sum_{\lambda, \nu \in \mathbb{Z}/n\mathbb{Z}} \psi_{\lambda, \nu}(\eta, u, v) w_{\lambda}(y_1 + m_1 v, \ldots, y_p + m_p v) \times w_{\nu}(z_1 + m_1 u + l_1, \ldots, z_p + m_p u + l_p).
\] (33)

Here the functions \( \psi_{\lambda, \nu}(\eta, u, v) \) are holomorphic and satisfy the relations

\[
\begin{align*}
\psi_{\lambda, \nu}(\eta + 1, u, v) &= \psi_{\lambda, \nu}(\eta, u + 1, v) = \psi_{\lambda, \nu}(\eta, u, v + 1) = \psi_{\lambda, \nu}(\eta, u, v), \\
\psi_{\lambda, \nu}(\eta + \tau, u, v) &= e^{-2\pi i n(v-u)} \psi_{\lambda, \nu}(\eta, u, v), \\
\psi_{\lambda, \nu}(\eta, u + \tau, v) &= e^{2\pi i n\eta} \psi_{\lambda, \nu}(\eta, u, v), \\
\psi_{\lambda, \nu}(\eta, u, v + \tau) &= e^{-2\pi i n\eta} \psi_{\lambda, \nu}(\eta, u, v).
\end{align*}
\] (34)

These relations are verified by the immediate calculation, namely, one must compare the multipliers at the translations by 1 and \( \tau \) in the formulas (32) and (33).

However, every holomorphic function of the variables \( \eta, u \) and \( v \) that satisfies relations (34) is vanishes. Really, since this function is periodic, it admits the expansion in the Fourier series

\[
\psi_{\lambda, \nu}(\eta, u, v) = \sum_{\alpha, \beta, \gamma \in \mathbb{Z}} a_{\lambda, \nu, \alpha, \beta, \gamma} e^{2\pi i (\alpha \eta + \beta u + \gamma v)}.
\]

Further, it follows from the quasiperiodicity that the coefficients \( a_{\lambda, \nu, \alpha, \beta, \gamma} \) are equal to 0.

By setting \( u = v + \eta \) in the identity (31) and making the change of variables \( y_1 \to y_1 - m_1 v, z_1 \to z_1 - m_1 v, \ldots, y_p \to y_p - m_p v, z_p \to z_p - m_p v, \)
we obtain

\[
\frac{\theta(y_1 - z_1 - n\eta)}{\theta(-n\eta)\theta(y_1 - z_1)} w_\alpha(y_1 + m_1\eta, \ldots, y_p + m_p\eta)\theta_\beta(z_1 + l_1, \ldots, z_p + l_p) + \\
+ \sum_{1 \leq t < p} \frac{\theta(z_t - y_t + y_{t+1} - z_{t+1})}{\theta(z_t - y_t)\theta(y_{t+1} - z_{t+1})} w_\alpha(z_1 + m_1\eta, \ldots, z_t + m_t\eta, y_{t+1} + m_{t+1}\eta, \ldots, y_p + m_p\eta) \times \\
\times w_\beta(y_1 + l_1, \ldots, y_t + l_t, z_{t+1} + l_{t+1}, \ldots, z_p + l_p) + \\
+ \frac{\theta(z_p - y_p + n\eta)}{\theta(z_p - y_p)\theta(n\eta)} w_\alpha(z_1 + m_1\eta, \ldots, z_p + m_p\eta) w_\beta(y_1 + l_1, \ldots, y_p + l_p) = \\
= \frac{1}{n} \theta \left( \frac{1}{n} \right) \ldots \theta \left( \frac{n-1}{n} \right) \times \\
\times \sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{\beta - \alpha + r(k-1)}(0)}{\theta_{\beta - \alpha}(0)} w_{\beta - r}(y_1, \ldots, y_p) w_{\alpha + r}(z_1 + m_1\eta + l_1, \ldots, z_p + m_p\eta + l_p).
\]

(35)

Appendix C

Duality between the spaces $\Theta_{n/k}(\Gamma)$ and $\Theta_{n/n-k}(\Gamma)$

Let us construct a canonical element $\Delta_{n,k} \in \Theta_{n/k}(\Gamma) \otimes \Theta_{n/n-k}(\Gamma)$ carrying out the duality between these spaces (see (36)).

Proposition 16. Let

\[
\frac{n}{k} = n_1 - \frac{1}{n_2 - \ldots - \frac{1}{n_p}}, \quad \frac{n}{n-k} = n'_1 - \frac{1}{n'_2 - \ldots - \frac{1}{n'_{p'}}}
\]

be the expansions in continued fractions, where $n_\alpha \geq 2$ and $n'_\beta \geq 2$ for $1 \leq \alpha \leq p$ and $1 \leq \beta \leq p'$, respectively. Here $p$ and $p'$ stand for the lengths of the continued fractions. Then $p' = n_1 + \ldots + n_p - 2p + 1$ and $n'_1 + \ldots + n'_{p'} = 2(n_1 + \ldots + n_p) - 3p + 1$. Moreover, $n'_1 + \ldots + n'_\alpha = 2\alpha + \beta$ for $n_1 + \ldots + n_\beta - 2\beta + 1 \leq \alpha \leq n_1 + \ldots + n_{\beta+1} - 2\beta - 2$. In other words, the Young diagrams for the partitions $(n_1 - 1, n_1 + n_2 - 3, \ldots, n_1 + \ldots + n_\alpha - 2\alpha + 1, \ldots)$ and $(n'_1 - 1, n'_1 + n'_2 - 3, \ldots, n'_1 + \ldots + n'_{\beta} - 2\beta + 1, \ldots)$ are dual to each other.
Remark. For $k = 1$, $p = 1$, and $n_1 = n$ we have $p' = n - 1$ and $n'_1 = \ldots = n'_{n-1} = 2$. For $p > 1$, if $n_2, \ldots, n_{p-1} \geq 3$, then the sequence $(n'_1, \ldots, n'_p)$ becomes $(2^{(n_1-2)}, 3, 2^{(n_2-3)}, 3, \ldots, 3, 2^{(n_{p-1}-3)}, 3, 2^{(n_p-2)})$. Here $2^{(k)}$, $t \geq 0$, stands for a sequence of $t$ twos. This formula remains valid without the assumption that $n_2, \ldots, n_p \geq 3$ if we agree that the sequence $(m_1, 2^{(-1)}, m_2)$ is of length 1 and is equal to $(m_1 + m_2 - 2)$. This rule must be applied in succession to all occurrences $n_\alpha = 2$ for $2 \leq \alpha \leq p - 1$.

Proof. The proof can be carried out by induction on $\min(p, p')$. For $p = 1$, one must prove that $\frac{n}{n-1} = 2 - \frac{1}{2-\ldots-\frac{1}{2}}$ is of length $n - 1$. Let $p, p' > 1$ and let, say, $n_1 > 2$. We have $\frac{k}{k - d(n_3, \ldots, n_p)} = n_2 - \frac{1}{n_3 - \ldots - \frac{1}{n_p}}$. By assumption,

$$k = \frac{k - d(n_3, \ldots, n_p)}{n'_{n_1-1} - 1 - \frac{1}{n'_{n_1} - \frac{1}{n'_{n_1+1} - \ldots - \frac{1}{n'}_{p'}}}},$$

and the sequence $(n'_1, \ldots, n'_{n_1-2})$ is $(2^{(n_1-2)})$. Further, one must show that $n'_1 - \frac{1}{n'_2 - \ldots - \frac{1}{n'}} = \frac{n}{n-k}$. Here it is used that $d(n_1, \ldots, n_p) = n, d(n_2, \ldots, n_p) = k, \frac{d(n_1, \ldots, n_p)}{d(\ldots, n_p)} = n_1 - \frac{1}{n_2 - \ldots - \frac{1}{n_p}}$, and $d(n_1, \ldots, n_p) = n_1d(n_2, \ldots, n_p) - d(n_3, \ldots, n_p)$. \qed

Proposition 17. Let a function $\Delta_{n,k}(z_1, \ldots, z_p, z'_1, \ldots, z'_{p'})$ of $p + p'$ variables $z_1, \ldots, z_p, z'_1, \ldots, z'_{p'}$ be defined by the formula

$$\Delta_{n,k}(z_1, \ldots, z_p, z'_1, \ldots, z'_{p'}) = e^{2\pi i z'_{1}}\theta(z_1 - z'_1)\theta(z_p + z'_{p'}) \prod_{1 \leq \alpha \leq p'-1} \theta(z'_\alpha - z_{\alpha+1} + z_{n'_1 + \ldots + n'_a - 2\alpha + 1}) \times \prod_{1 \leq \beta \leq p-1} \theta(z_\beta - z_{\beta+1} + z'_{n_1 + \ldots + n_\beta - 2\beta + 1}).$$

This function satisfies the following relations:

$$\Delta_{n,k}(z_1, \ldots, z_\alpha + 1, \ldots, z'_{p'}) = \Delta_{n,k}(z_1, \ldots, z'_\beta + 1, \ldots, z'_{p'}) = \Delta_{n,k}(z_1, \ldots, z'_{p'}),$$

$$\Delta_{n,k}(z_1, \ldots, z_\alpha + \tau, \ldots, z'_{p'}) = (-1)^{n_\alpha}e^{-2\pi i (n_\alpha z_\alpha - z_{\alpha-1} - z_{\alpha+1} - (\delta_{\alpha,1} - 1)\tau)} \Delta_{n,k}(z_1, \ldots, z'_{p'}),$$

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Δ_{n,k}(z_1, \ldots, z_p + \tau, \ldots, z_p') = (-1)^{r_1} e^{-2\pi i (n'_{\beta}z_{\beta}' - z_{\beta+1} - (\delta_{\beta,1}-1)\tau)} \Delta_{n,k}(z_1, \ldots, z_p').

Here \( z_0 = z_{p+1} = z_0' = z_{p'+1} = 0 \) and \( \delta_{\alpha,1} \) stands for the Kronecker delta.

The proof immediately follows from our description of the duality between the sequences \((n_1, \ldots, n_p)\) and \((n'_1, \ldots, n'_{p'})\).

Proposition 18.

\[ \Delta_{n,k}(z_1, \ldots, z_p; z'_1, \ldots, z'_p) = c_{n,k} \sum_{\alpha \in \mathbb{Z}/n\mathbb{Z}} w_{\alpha}^{n/k}(z_1, \ldots, z_p)w_{\alpha}^{n/n-k}(z'_1, \ldots, z'_p). \]  

(36)

Here \( c_{n,k} \in \mathbb{C} \) is a constant.

Proof. It follows from the previous proposition that the function \( \Delta_{n,k} \) belongs to the space \( \Theta_{n/k}(\Gamma) \) when regarded as a function of the variables \( z_1, \ldots, z_p \). Similarly, \( \Delta_{n,k} \) belongs to \( \Theta_{n/n-k}(\Gamma) \) as a function of \( z'_1, \ldots, z'_{p'} \). Therefore,

\[ \Delta_{n,k}(z_1, \ldots, z_p; z'_1, \ldots, z'_{p'}) = \sum_{\alpha, \beta \in \mathbb{Z}/n\mathbb{Z}} \lambda_{\alpha,\beta} w_{\alpha}^{n/k}(z_1, \ldots, z_p)w_{\beta}^{n/n-k}(z'_1, \ldots, z'_{p'}). \]

However, one can readily see that

\[ \Delta_{n,k}(z_1 + r_1, \ldots, z_p + r_p; z'_1 + r'_1, \ldots, z'_p + r'_p) = e^{\frac{2\pi i}{n} r_1 - z_p - z'_p} \Delta_{n,k}(z_1, \ldots, z'_p), \]

where \( r_\alpha = \frac{d(n_1, \ldots, n_{\alpha-1})}{n} \) and \( r'_\beta = \frac{d(n'_1, \ldots, n'_{\beta-1})}{n} \). Hence, \( \lambda_{\alpha,\beta} = 0 \) for \( \alpha + \beta \neq 1 \mod n \) (because \( w_{\alpha}(z_1 + r_1, \ldots, z_p + r_p) = e^{2\pi i \frac{r_1}{n}} w_{\alpha}(z_1, \ldots, z_p) \)) and \( w_{\beta}(z'_1 + r'_1, \ldots, z'_{p'} + r'_{p'}) = e^{2\pi i \frac{r'_1}{n}} w_{\beta}(z'_1, \ldots, z'_{p'}) \)). Thus, \( \lambda_{\alpha,\beta} = \lambda_{\alpha} \delta_{\alpha+\beta,1} \).

Similarly,

\[ \Delta_{n,k}(z_1 + r_1 \tau, \ldots, z_p + r_p \tau; z'_1 + r'_1 \tau, \ldots, z'_p + r'_p \tau) = e^{2\pi i \frac{r_1}{n} \tau} \Delta_{n,k}(z_1, \ldots, z'_p). \]

Hence, \( \lambda_{\alpha} = \lambda_{\alpha+1} \), that is, \( \lambda_{\alpha} \) does not depend on \( \alpha \). \( \square \)
1 Integrable system, quantum groups, and $R$-matrices

One of the main methods in the investigation of exactly solvable models [6] in quantum and statistical physics is the inverse problem method (see [45]). This method leads to the study of representations of algebras of monodromy matrices, that is, to the study of meromorphic matrix functions $L(u)$ satisfying the relations

$$R(u - v)L^1(u)L^2(v) = L^2(v)L^1(u)R(u - v).$$

(37)

Here $R(u)$ is a chosen solution of the Yang-Baxter equation in the class of meromorphic matrix-valued functions,

$$R^{12}(u - v)R^{13}(u)R^{23}(v) = R^{23}(v)R^{13}(u)R^{12}(u - v).$$

(38)

We note that $R(u)$ takes the values in $(n^2 \times n^2)$ matrices with a fixed decomposition $\text{Mat}_{n^2} = \text{Mat}_n \otimes \text{Mat}_n$. We use the standard notation, namely, $L^1 = L \otimes 1$, $L^2 = 1 \otimes L$, $R^{12} = R \otimes 1$, etc. (see [45]).

In [45] Sklyanin studies the solutions of the equation (37) for the simplest elliptic solution of the equation (38), that is, for the so-called Baxter $R$-matrix, which is of the form $R(u) = 1 + \sum_{\alpha=1}^3 W_\alpha(u)\sigma_\alpha \otimes \sigma_\alpha$, where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices, and the coefficients $W_\alpha(u)$ can be expressed in terms of the Jacobi elliptic functions as follows:

$$W_1(u) = \frac{\text{sn}(i\eta, k)}{\text{sn}(u + i\eta, k)}, \quad W_2(u) = \frac{\text{dn}(u + i\eta, k)}{\text{sn}(u + i\eta, k)} \frac{\text{sn}(i\eta, k)}{\text{dn}(i\eta, k)},$$

$$W_3(u) = \frac{\text{cn}(u + i\eta, k)}{\text{sn}(u + i\eta, k)} \frac{\text{sn}(i\eta, k)}{\text{cn}(i\eta, k)}.$$

The functions $W_\alpha(u)$ uniformize the elliptic curve $\frac{W_2^2 - W_3^2}{W_2^2 - 1} = J_{\alpha,\beta}$, where the $J_{\alpha,\beta}$s do not depend on $u$ and satisfy the relation $J_{12} + J_{23} + J_{31} + J_{12}J_{23}J_{31} = 0$. Here $\alpha$, $\beta$, and $\gamma$ are pairwise distinct and $J_{\beta,\alpha} = -J_{\alpha,\beta}$. 

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We note that this elliptic curve is the complete intersection of two quadrics, for instance, \( w_1^2 - w_2^2 = J_{12}(w_3^2 - 1) \) and \( w_2^2 - w_3^2 = J_{23}(w_1^2 - 1) \).

Sklyanin discovered that the equation (37) for the Baxter \( R \)-matrix has a solution of the form
\[
L(u) = S_0 + \sum_{\alpha=1}^{3} W_\alpha(u) S_\alpha,
\]
where \( S_0 \) and \( S_\alpha \) are matrices that do not depend on \( u \) and satisfy the following relations:
\[
[S_\alpha, S_0]_+ = -iJ_\beta \gamma [S_\beta, S_\gamma]_+,
\]
\[
[S_\alpha, S_\beta]_+ = i[S_0, S_\gamma]_+ + (39)
\]
where \([a, b]_\pm = ab \pm ba\).

Sklyanin further studies the algebra with the generators \( S_0, S_\alpha \) and the relations (39); he denotes this algebra by \( \mathcal{F}_{\eta,k} \). The main assumption concerning this algebra is that it satisfies the PBW condition. Moreover, Sklyanin finds the quadratic central elements of the algebra \( \mathcal{F}_{\eta,k} \) and the finite-dimensional representations of the algebra \( \mathcal{F}_{\eta,k} \) by difference operators in some function space (see [46]).

In our notation, the Sklyanin algebra \( \mathcal{F}_{\eta,k} \) is the algebra \( Q_4(\mathcal{E}, \eta) \), where \( \mathcal{E} \) is an elliptic curve given by the functions \( W_\alpha(u) \), that is, a complete intersection of two quadrics in \( \mathbb{C}^3 \).

The Yang-Baxter equation has other elliptic solutions generalizing the Baxter solution (see [7]). The result of [7] can be described as follows: for any pair of positive integers \( n \) and \( k \) such that \( 1 \leq k < n \) and \( n \) and \( k \) are coprime there is a family of solutions \( R_{n,k}(\mathcal{E}, \eta)(u) \) of the equation (38). Here \( \mathcal{E} \) is an elliptic curve and \( \eta \in \mathcal{E} \), as above. The Baxter solution is obtained for \( n = 2 \) and \( k = 1 \).

According to [10], the Sklyanin result can be generalized to an arbitrary solution \( R_{n,k}(\mathcal{E}, \eta)(u) \). In our notation, the results of [10] look as follows: there is a homomorphism of the algebra of monodromy matrices for the \( R \)-matrix \( R_{n,k}(\mathcal{E}, \eta) \) into the algebra \( Q_{n^2,nk-1}(\mathcal{E}, \eta) \). Correspondingly, the algebra \( Q_{n^2,nk-1}(\mathcal{E}, \eta) \) is a deformation of the projectivization of the Lie algebra \( \mathfrak{sl}_n \). Moreover, there is a homomorphism of the algebra of monodromy matrices into the algebra \( Q_{dn^2,dnk-1}(\mathcal{E}, \eta) \) for any \( d \in \mathbb{N} \). It can be conjectured that every finite-dimensional representation of the algebra of monodromy matrices can be obtained from a representation of the algebra \( Q_{dn^2,dnk-1}(\mathcal{E}, \eta) \).

Another relationship between the elliptic solutions of the Yang-Baxter equation and the elliptic algebras follows from the results of [10]. The multiplication in the algebra \( Q_{n,k}(\mathcal{E}, \eta) \) is defined by the so-called Young projections \( S^\alpha V \otimes S^\beta V \to S^{\alpha+\beta} V \) corresponding to \( R_{n,k}(\mathcal{E}, \eta)(u) \) (see [10]). Moreover, \( Q_{n,k}(\mathcal{E}, \eta) = \sum_\alpha S^\alpha V \).
We also note that the study of algebras $Q_{n,k}(\mathcal{E}, \eta)$ and their representations led to deeper understanding of the structure of $R$-matrices $R_{n,k}(\mathcal{E}, \eta)(\nu)$ and of the corresponding algebraic objects (the Zamolodchikov algebra and the algebra of monodromy matrices). For this topic, see [33].

2 Moduli spaces

Let $G$ be a semisimple Lie group and let $P \subset G$ be a parabolic subgroup. Let $\mathcal{M}(\mathcal{E}, P)$ be the moduli space of the holomorphic $P$-bundles over an elliptic curve $\mathcal{E}$ [4]. According to [20], every connected component of the space $\mathcal{M}(\mathcal{E}, P)$ admits a natural Poisson structure. The main property of this structure is as follows: the preimages of the natural map $\mathcal{M}(\mathcal{E}, P) \rightarrow \mathcal{M}(\mathcal{E}, G)$ corresponding to forgetting the $P$-structure are symplectic leaves of the structure. The quantization problem for the Poisson manifold $\mathcal{M}(\mathcal{E}, P)$ arises. The solution of this problem could establish a relationship between the natural algebro-geometric problem of studying $P$-bundles (and the corresponding $G$-bundles) and the problem to study representations of the quantum function algebra on $\mathcal{M}(\mathcal{E}, P)$ because the representations correspond to symplectic leaves.

In [38], [22] these quantum algebras were constructed provided that $P = B$ is a Borel subgroup of an arbitrary group $G$. In [21] the quantum algebras were constructed in the case of $G = GL_m$ and an arbitrary parabolic subgroup $P$. Here the algebra $Q_n(\mathcal{E}, \eta)$ corresponds to the case $G = GL_2$, and the algebra $Q_{n,k}(\mathcal{E}, \eta)$ to the case $G = GL_{k+1}$, where $P$ consists of upper block triangular matrices of the form

\[
\begin{pmatrix}
  \ast & \ast \\
  0 & \ast \\
  k & 1
\end{pmatrix}.
\]

3 Non-commutative algebraic geometry

One of the main ideas of algebraic geometry is to study the geometry of a manifold by using the algebraic properties of a ring of functions on this manifold. The non-commutative algebraic geometry extends these methods
and the geometric intuition to an appropriate class of non-commutative rings. In [51], [48] the non-commutative algebraic geometry is developed in small dimensions. From this point of view, the algebras $Q_{n,k}(E, \eta)$ give examples of non-commutative vector spaces. Similar examples of non-commutative Grassmannians and other varieties are also known [11], [19], [22], [49].

4 Cohomology of algebras

Cohomology properties of quadratic algebras are studied in [29], [41]-[44], [50]. For generic $\eta$, the algebras $Q_{n,k}(E, \eta)$ are examples of Koszul algebras. One can readily prove this fact for $k = 1$ by using the construction of a free module in §2.6. The constructions of dual algebras $Q_{n,k}^!(E, \eta)$ are given in [36].

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