Limits of tangent spaces, separating sets and exceptional tangents

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Plan

\[ V, 0 \leq C^n, 0 \text{ surface (mostly)} \]

1) \((C_x)\) tangent cone approx's \(V\) near \(0\). Not always well

Complex-analytic geom (Le. Teissier thy)

2) \((R_e)\) metric cone approx's \(V\) near \(0\). Not always we

Lipschitz geom

3) What is the relation between 1) and 2)?

Intersection of real alg + cx anal geom
Tangent Cone

\[ C(V, 0) = \{ v \in \mathbb{C}^n : \overline{OV} = \{ tv, t \in \mathbb{C} \} \text{ is a limit at } 0 \} \]

often denoted \( CV \)

\( CV \) algebraic

\( \dim CV = \dim V \) , but \( CV \) homog

Dfn: \( \delta \)-conical nbhd of \( CV \)

\[ N_\delta(CV) = \{ w \in \mathbb{C}^n : \exists v \in CV \text{ such that } \angle (\overline{OW}, \overline{OV}) < \delta \} \]

Classical Thm: \( V \) approximates \( CV \) well at \( 0 \).

Given any \( \delta > 0 \), \( \exists \varepsilon \) such that \( V \cap B_\varepsilon \subseteq N_\delta(CV) \land \)

where

\[ B_\varepsilon = \{ x \in \mathbb{C}^n : ||x|| < \varepsilon \} \]
REAL TANGENT CONE

$W, 0 \in \mathbb{R}^m, 0$ semi-algebraic

$C(W,0) = \{ w \in \mathbb{R}^m : \text{Ow} = \{ tw, t > 0 \} \text{ is a limit at 0} \}$ of rays thru $W$ based at 0

often denoted $CW$ or $C_m(W,0)$

$CW$ semi-alg

$\dim CW \leq \dim W$

Thm: $W$ gets arbitrarily close to $CW$ near 0.

$W = \{ x^2 + y^2 - z^3 = 0 \}$

$CW = \{ z\text{-axis} \}$

$\dim CW = 1 < \dim W$
CV is a very radical simplification of V

Ex: \( V = \{ x^2 + y^2 = 0 \ \text{with} \ \mu + \delta \geq 3 \} \) has \( CV = \{ x = 0 \} \)

\[ V = \{ x^2 - y^2 + z^2 = 0 \} \]

Consider: \( D(V,0) = \{ T \in \mathbb{C}^n: T \text{ is a limit of tangent spaces to } V \text{ at smooth pts of } V \text{ near } 0 \} \)

Thm (Whitney, Mironako): \( D(CV,0) \subset D(V,0) \), often \( \subset \)

\( D(V,0) \) has more info than \( CV,0 \)
What is extra stuff in $D(v,0)$ when $v,0 \in \mathbb{C}^n$ surface?

**STRUCTURE THM (Lê, Teissier, Henry):** \exists finite (possibly \#)

set of lines $l_1, \ldots, l_r \subset \mathbb{C}(v,0)$, $0 \in l_i$, (called exceptional lines)

such that

$$D(v,0) = D(v,0) \cup K(l_1) \cup \cdots \cup K(l_r)$$

where

$$K(l_i) = \{ \text{planes in } \mathbb{C}^n \text{ containing } l_i \}.$$

$D(v,0)$ ~ $$ \rightarrow \text{ "Auréole" } = (C, v, \text{ except lines})$
CHARACTERIZATION OF EXCEPTIONAL LINES (La Teissier)

Choose projection $\pi : \mathbb{C}^n \rightarrow \mathbb{C}^2$

Set $\Gamma = \text{crit} (\pi |_{V-\Sigma})$  "polar curve rel $\pi$"

Consider $C(\Gamma, 0)$ union of lines

As $\pi$ varies, some lines in $C(\Gamma, 0)$ vary, others stay fixed.

Exceptional lines = fixed lines

Remark: If $C^3 \ni V, 0$ isolated, ${\lambda}_{1, \ldots, 4} = \bigcap_{\pi : \mathbb{C}^3 \rightarrow \mathbb{C}^2} C(\Gamma, 0)$

$\text{C}(\Gamma_{\text{ex}}, 0) = \text{C}(\Gamma_{\text{ex}}, 0) \rightarrow \text{exceptional line}$

$\text{C}(\Gamma_{\text{ex}}, 0) = \Gamma_{\text{ex}} \neq \Gamma_{\text{ex}} = \text{C}(\Gamma_{\text{ex}}, 0) \rightarrow \text{no except line}$
II. METRIC CONES

Notation: \( K \subseteq \mathbb{C}^n \) or \( \mathbb{R}^m \) any subset

\[ \text{Cone } K = \{ rv : v \in K, \, 0 \leq r \leq 1 \} \]

Thm: \( V, 0 \subseteq \mathbb{C}^n \) algebraic \( \Rightarrow V \) is homeomorphic to the cone over its link.

(for all \( \varepsilon > 0 \) sufficiently small, \( S_\varepsilon \subseteq V \) and

\[ V \cap B_\varepsilon, 0 \cong \text{Cone}(V \cap S_\varepsilon), 0 \text{ homom} \]

where \( S_\varepsilon = \varepsilon B_\varepsilon \))

? How "good" is this approx? "Metrically similar"?

\( V \) inherits two metrics

- Outer metric (distance in \( \mathbb{C}^n \))
- Inner metric (arc length in \( V \)... from Riemannian metric)

Prop: Inner metric determined by outer metric (although they can differ).
**Dfn:** The Lipschitz category

A map \( f: Y \to Z \) of metric spaces is **Lipschitz**, if \( \exists K > 0 \)

\[
\frac{1}{K} d_Y(u,v) \leq d_Z(fu,fv) \leq Kd_Y(u,v)
\]

**Bi-Lipschitz** \( \iff \) bijective and Lipschitz.

**Two metrics on** \( V \) **are Lipschitz equivalent** if the

identity map: \( (V,d_1) \to (V,d_2) \)

is **bi-Lipschitz**. In Lipschitz category we say \( d_1 \) and \( d_2 \) are the same.

**Prop.** (up to Lip equiv), Inner and outer metrics on \( V \) indep of choices of gens of \( G(V,0) \).

**Ex:** inner \( \neq \) outer metric for \( V = \{x^2 - y^3 = 0 \} \subseteq \mathbb{C}^2 \)

**Ex:** \( V,0 \) homog \( \Rightarrow \) inner = outer (\( = \) cone(\( V \cap S^2 \)) all \( \varepsilon \))
**METRIC TRIVIALITY**

**Defn** Say $(V,0)$ is metrically conical (or trivial) if the minor metric is the same as that on $\text{Cone } K$ for some $K \subseteq S_\mathbb{R} \subseteq \mathbb{C}^n$.

(i.e., $\exists \varepsilon > 0$ such that $V \cap B_\varepsilon(0)$ is lipschitz equivalent to cone $K$)

**Examples:** The following are metrically conical:

- $W,0$ with $0$ smooth
- $C,0$ a curve singularity (Fernandes / C)
- $W,0$ homogeneous
- $V,0$ where $V = \{ x^a + y^b + z^a = 0 \}, a < b$

Over $\mathbb{R}$, the surface ($\beta$-horn with $\beta = 1/\beta$)

$$(x^2 + y^2)^{\frac{\beta}{2}} = z^{2\beta}$$

is not metrically conical when $\beta > 1$.

Reason: $\sup\{ r > 0 : \lim_{\varepsilon \to 0} \frac{\text{Vol}(V \cap B_\varepsilon)}{\varepsilon^2} = 0 \}$

exists (non-Robin) and is bi-lip invariant (Brasselet-Birbrair) and varies as $\beta$ does.

Over $\mathbb{C}$, examples abound (Birbrair, Fernandes, Neumann) but were (initially) harder to find.
Dfn \( W, 0 \subseteq \mathbb{R}^m \) semi alg, the \( k \)-density of \( W \) at \( 0 \) is

\[
\Theta^k(W,0) = \lim_{\varepsilon \to 0} \frac{\mathcal{H}^k(W \cap B^m_\varepsilon)}{\text{vol}(B^m_\varepsilon)}
\]

Here, \( \mathcal{H}^k \) is \( k \)-dim' Hausdorff measure, \( B^m_\varepsilon = \) \( k \)-dim' ball with radius \( \varepsilon \).

Limit exists in semi-alg context. (More generally, use \( \lim \inf \) and \( \lim \sup \) to define lower + upper \( k \)-density)

Examples

- \( W, 0 \) smooth and \( \dim_{tr} W = k \) \( \Rightarrow \) \( \Theta^k(W,0) = 1 \)

We use convention that \( \mathcal{H}^k \) coincides with \( k \)-dim' Lebesgue measure \( \lambda^k \) for Borel sets. Other common convention: \( \lambda^k = \frac{\text{vol}(B^m_\varepsilon)}{\varepsilon^k} \mathcal{H}^k \) where \( \text{vol}(B^m_\varepsilon) = \frac{\pi^{m/2}}{\Gamma((m+1)/2)} \varepsilon^m \)

- \( W, 0 \) metric cone over smooth \( k \)-dim \( K \subseteq S^1 \subseteq \mathbb{R}^m \), then \( \Theta^k(W,0) > 0 \)

- \( V, 0 \subseteq C^1 \), \( \dim_{C^1} V = k \) \( \Rightarrow \) \( \Theta^{2k}(V,0) > 0 \)

Salient Fact: \( W, 0 \subseteq \mathbb{R}^m \) with \( \dim W = k \) and \( \dim CW < k \) then \( k \)-density of \( W \) is 0:

\( \dim CW < \dim W = k \) \( \Rightarrow \) \( \Theta^k(W,0) = 0 \)

Moreover, \( \dim CW = \dim W = k \) and \( \Theta^k(CW,0) > 0 \) \( \Rightarrow \) \( \Theta^k(W,0) > 0 \)
SEPARATING SETS

\(W, 0 \subseteq \mathbb{R}^m\) semi-\(k\) with \(\dim W = k\)

A separating set \(Y, 0 \subseteq W, 0\) is a subset with 

\((k-1)\)-density zero that separates \(W, 0\) into pieces with positive \(k\)-deni

Schematically:

Illuminating non-example:

\[ W = \mathbb{C}^3, \quad Y = \left\{ (x, y, z) \in \mathbb{R}^3 : |x|^2 + |y|^2 - |z|^2 = 0 \right\} \]

\[ \dim Y = 5 \]
\[ \dim CY = 2 \]
\[ \Theta^5(Y, 0) = 0 \]

\(Y\) separates \(W\) into two pieces \(h < 0\) and \(h > 0\) but first has 5-density 0.

Thm: In semi-\(k\)-category, separating sets are preserved by bi-Lipschitz maps

Pt. Separating sets can be defined in inner metric (sep set is set of zero upper \((k-1)\)-density that locally divides \(W, 0\) into sets of pos lower \(k\)-deni)

Dnu is same as long as things are semi-\(k\)-alg which we can guarantee by Kurdyka’s Pancake Thm: A semi-\(k\)-alg set has a finite semi-\(k\)-alg decomp into pieces whose inner + outer metrics are equivalent.

Thm If \(K \subseteq \mathbb{S}^{\infty - 1} \subseteq \mathbb{R}^m\) a compact Lipschitz manifold (even with bdry), Cone \(K\) does not have a separating set.
**Singular Varieties Often Have Separating Sets**

**Prop. example:** \( V_0 \subseteq \mathbb{C}^3 \) \( V = \{ xy = 0 \} \)

2-axis separates. To get a dim. 3 set, let

\( K = \{ h = 0 \} \) where \( h = \| x \|^2 + \| y \|^2 - \| z \|^2 \).

\( Y = V \cap K \) is a 3-dim. separating set

\( V - Y \) has 3 pieces

\( V_0 = V \cap \{ h < 0 \} \) \( \Theta^4(V_0, 0) = 0 \)

\( V_1 = \{ x = 0, h > 0 \} \) \( \Theta^4(V_1, 0) = 1 \)

\( V_2 = \{ y = 0, h > 0 \} \) \( \Theta^4(V_2, 0) = 1 \)

**Example:** \( V = \{ xy + z^{k+1} = 0 \} \)

\( Y = V \cap K \) is a separating set if \( k > 2 \)

Main point is to show \( V \cap \{ h < 0 \} \) is non-empty, which is the case because \( \{ x = y \} \cap V - 0 \subseteq \{ h < 0 \} \) when \( k > 1 \).

\( V = \{ h < 0 \} \) has two components \( V_1 \) with \( CV_1 = \{ x = 0 \} \) and \( V_2 \) with \( CV_2 = \{ y = 0 \} \); hence both with positive 4-density.

If we take \( K = \{ x^4 + y^2 - 3z^2 = 0 \} \) or \( K = \{ x^4 + y^2 - 12z^2 = 0 \} \),
then \( V \cap K \) is a separating set when \( k = 2 \) as well.

**Prop:** \( A_k \) is not metrically conical for \( k \geq 2 \)

\( A_k \) is isolated, hence its link is a manifold. If it were metrically conical it could not have a separating set.
Ex 1. \( V = \{ x^a + y^b + z^b = 0 \} \) with \( a < b \) is metrically conical.

Ex 2. \( V = \{ x^a + y^b + z^c = 0 \} \) has a separating set if \( a < b < c \) and \( \gcd(a, b) > 1 \).

Thm. \( V, 0 \in \mathbb{C}^3 \), an isolated quasi-homog with wts \( w_1 > w_2 > w_3 \) if \( V \cap \mathbb{R}^3 \neq \emptyset \) is reducible, then \( V \) has a separating set.

Pf: Write \( V \cap \mathbb{R}^3 = A \cup B \) disjoint closed sets.

Let \( K_0 = \{ v \in \mathbb{R}^3 : d(v, A) = d(v, B) \} \)

Set \( K = \mathbb{R}^+ K_0 \cup \{ 0 \} \) using \( \mathbb{R}^+ \) action on \( \mathbb{C}^3 \) action.

\( K \) divides \( V \) into pieces \( Y, Z \)

\( C_K \) is a \( \mathbb{R}^3 \)-axis; its dim \( C_K \leq 2 \) so \( \Theta^3(K, 0) = 0 \).

\( C_Y \) and \( C_Z \) each contain a complex plane, so \( \Theta^4(Y, 0) > 0 \) and \( \Theta^4(Z, 0) > 0 \).

Ex. \( BS_t = x^3 + txy^3 + y^4z + z^9 \). \( (\text{Brieskorn - Speder, } \mu(BS_t) = t \neq 0, \ BS_t \cap \mathbb{R}^3 = 0 \} \) is curve \( x(x^2 + ty^3) = 0 \) with two components. Therefore has separating set, so is not metrically conical.

When \( t = 0 \), \( BS_0 = x^3 + y^4z + z^9 \) is metrically conical.

Cor. \( BS_0 \) not bi-Lip equiv to \( BS_t \) (so \( \mu \)-const \( \neq \) lip-triv).
III

**EXCEPTIONAL LINES + SEPARATING SETS**

**Thm.** $V, O \in C^n, 0$ isolated, surface singularity

1. If $W \subset V$ is a separating set at $O$, then $CW \cup \text{exceptional lines}$
2. No exceptional lines $\implies V$ metrically convex
3. $\{l_1, \ldots, l_k\} \subset CV$ exceptional lines
   $$\implies CV - \bigcup_{i=1}^{k} N_{\epsilon}(l_i) \text{ metrically convex.}$$

(Thm continues to hold if $O$ is not an isolated singularity, except that $CW$ could belong to $C \Sigma$ where $\Sigma = \text{sing } V$)

"Conjecture": Arrangement + incidence data of polar curves determine bi-Lipschitz type
View CV as object obtained by dilating V, 0

That is: consider varieties \( \frac{1}{t} V \) as \( t \to 0 \)

Eqns: For case, let \( V = \{ f = 0 \} \) h.surf.
\[
x \in \frac{1}{t} V \iff t x \in V \iff f(t x) = 0
\]
Write \( f = f_1 + f_2 + \ldots + f_m \) \( \text{homo, deg } k \)
\[
f(tx) = 0 \iff t^k (f_1(x) + t f_2(x) + \ldots + t^{m-k} f_m(x)) = 0
\]
\[
F(x, t) = 0
\]

Set \( W = \{ (x, t) \in C^1 \times C \mid F(x, t) = 0 \} \)
\( W_t = \{ x \mid (x, t) \in W \} = \frac{1}{t} V \)
\( W_0 = CV, W_1 = V, W_t \xrightarrow{\text{homo}} V \ t \neq 0 \).

Exceptional lines = \( \{ x_0 \in CV \mid \exists \text{ seq } x_{t_i} \in W_{t_i} \to x_0 \in W_0 \text{ with } \lim T(W_{t_i}, x_{t_i}) \neq T(W_0, x_0) \} \)

Note: \( t \) sufficiently small. Then CVS \( \to CV_0 \)
CONCLUSION

Deformation to normal cone gives a somewhat sharper statement:

Thus \( V_t, 0 \subseteq C^3, 0 \) surface, isolosing, \( l_1, \ldots, l_6 \) exceptional lines

1) For each \( l_i \), either
   a) \( \exists \) fast loop with horn tangent to \( l_i \)
   b) \( \exists \) separating set with tangent \( l_i \)
   c) \( l_i \) is tangent to discriminant of "projectum" of \( V \to CV \)

2) Every fast loop \( \times \) every separating set has tangent an except line

3) Only fast loops \( \times \) separating sets are obstructions to metric triviality

Return to Brauçon-Speddor

\( V_t \): \( x^3 + txy^3 + y^4z + z^8 = 0 \)

\( W_t = x^6 + txy^6 + y^8z + z^{15} \)

\( t = 0 \) \( V_0 = x^3 + y^4z + z^9 \) \( CV_0 = \{ x = 0 \} \)

\( V_0 \) is 3-fold cover of \( CV_0 = \{ x = 0 \} \)

branched over \( y^4z + z^9 = 0 = z(y^4 + z^8) = z \sum (y + z^8) \)

\( l_1 = \{ x = 0, y = 0 \} \)

\( l_2 = \{ x = 0, y = 0 \} \)

\( t \neq 0 \) \( CV_0 = \{ x = 0 \} \)

\( I_1 = \{ x = 0, y = 0 \} \)

\( V_t \) still covers \( CV_0 = \{ x = 0 \} \)