Algorithms to Estimate the Rose of Directions of a Spatial Fibre Process

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Quantifying Anisotropy of Stationary Fibre Processes

1. Roses of Directions and Intersections, the Associated Zonoid
2. Three nonparametric Estimation Methods
3. Convergence results

Applications: Carbon Fibres and Simulations

1. Carbon fibres
2. Simulated data

Conclusion
Stationary Fibre Processes

Stationary fibre process $X$: locally finite random collection of $C^1$-fibres in $\mathbb{R}^d$, $d \geq 2$, with translation invariant distribution.

Length density $\overline{L}$ of $X$: mean total fibre length per unit volume.
The Rose of Directions

Rose of directions $\mathcal{R}$:

distribution of the tangent in a *typical* point of $X$

$\mathcal{R}$ is an even measure on $S^{d-1}$.

(Euclidean) unit sphere in $\mathbb{R}^d$

Directional measure: $\eta = \overline{L} \cdot \mathcal{R}$.
The Rose of Intersections

Rose of intersections $\gamma$:

$$\gamma(u) = \text{intensity of } X \cap u^\perp$$

(mean number of intersection points per unit $(d-1)$-volume)

$\gamma$ is the Cosine transform of $\eta$:

$$\gamma(u) = \int_{S^{d-1}} |\cos \Theta(u, v)| \, d\eta(v) =: \mathcal{I}_\eta(u), \quad u \in S^{d-1}.$$ 

Thus, $\gamma \longleftrightarrow \eta$. 

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Estimating the Rose of Directions
The associated zonoid

Let $Z \subset \mathbb{R}^d$ be a compact, convex set with support function

$$h(Z, u) = \gamma(u)$$

is called the associated zonoid (Steiner compact) of $X$.

Thus, $Z \longleftrightarrow \gamma \longleftrightarrow \eta$:

All three quantities yield the same information on (an-)isotropy.
Estimation of $\eta$ from the Rose of Intersections.

Assume that there are given:

- a sequence of test planes $u_1^\perp, u_2^\perp, \ldots$, i.e. $u_1, u_2 \ldots \in S^{d-1}$,
- $k \in \mathbb{N}$ blurred measurements

$$y_i = \gamma(u_i) + \varepsilon_i, \quad i = 1, \ldots, k,$$

where $\varepsilon_i$ are independent random variables with mean 0 and variance $\sigma^2$.

**Problem:** Find an estimator $\hat{\eta}_k$ for $\eta$ from this data.

**Key idea:** Let $\hat{\eta}_k$ be an even measure on $S^{d-1}$, such that the cosine transform $T_{\hat{\eta}_k}$ of $\hat{\eta}_k$ in directions $u_1, \ldots, u_k$ best fits the measurements $y_1, \ldots, y_k$. 
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Least Squares Estimation

If “best fit” is understood in the sense of least squares:

Then \( \hat{\eta}_k \) is a solution of

\[
\text{minimize } \sum_{i=1}^{k} (T_{\mu}(u_i) - y_i)^2 \\
\text{subject to: } \mu \text{ is an even measure on } S^{d-1}.
\]

If \( \varepsilon_1, \ldots, \varepsilon_k \) are i.i.d. Gaussian, \( \hat{\eta}_k \) is a maximum likelihood estimator,

Using results from convex geometry, (LSQ) can be discretized loss-free and becomes a least squares problem with less than \( k^{d-1} \) unknowns.
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(LSQ)

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- Using results from convex geometry, (LSQ) can be discretized loss-free and becomes a least squares problem with less than $k^{d-1}$ unknowns.
“Best fit” in terms of the Kullback-Leibler divergence:

Then $\hat{\eta}_k$ is a solution of

$$\text{minimize } \sum_{i=1}^{k} \left( y_i \log I_{\mu}(u_i) - I_{\mu}(u_i) \right)$$

subject to: $\mu$ is an even measure on $S^{d-1}$.

If $y_1, \ldots, y_k$ are Poisson distributed, then $\hat{\eta}_k$ is a maximum likelihood estimator.

Again, (EM) can be discretized loss-free. Numerical solution using the iterative EM-algorithm or MCMC-methods.
Best Estimation in the Sense of Information Theory

“Best fit” in terms of the Kullback-Leibler divergence:

Then $\hat{\eta}_k$ is a solution of

\[
\text{minimize } \sum_{i=1}^{k} (y_i \log T_\mu(u_i) - T_\mu(u_i)) \\
\text{subject to: } \mu \text{ is an even measure on } S^{d-1}.
\]  

(EM)

- If $y_1, \ldots, y_k$ are Poisson distributed, then $\hat{\eta}_k$ is a maximum likelihood estimator.
- Again, (EM) can be discretized loss-free. Numerical solution using the iterative EM-algorithm or MCMC-methods.
“Best fit” in a geometric sense: Find the largest zonoid with support function $T(\hat{\eta}_k(\cdot))$ with $T(\hat{\eta}_k(u_i)) \leq y_i$, $i = 1, \ldots, k$.

Then $\hat{\eta}_k$ is a solution of

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{k} (y_i - T(\mu)(u_i)) \\
\text{subject to:} & \quad T(\mu)(u_i) \leq y_i, \quad i = 1, \ldots, k, \\
\text{and} & \quad \mu \text{ is an even measure on } S^{d-1}.
\end{align*}
\]

\(\text{ (LP)}\)

- No natural interpretation as maximum likelihood estimator.
- After loss-free discretization, (LP) becomes a linear program.
“Best fit” in a **geometric sense**: Find the largest zonoid with support function $\mathcal{T}_{\hat{\eta}_k}(\cdot)$ with $\mathcal{T}_{\hat{\eta}_k}(u_i) \leq y_i$, $i = 1, \ldots, k$.

Then $\hat{\eta}_k$ is a solution of

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{k} (y_i - \mathcal{T}_\mu(u_i)) \\
\text{subject to:} & \quad \mathcal{T}_\mu(u_i) \leq y_i, \quad i = 1, \ldots, k, \\
& \text{and} \quad \mu \text{ is an even measure on } S^{d-1}.
\end{align*}
\]  

(LP)

- **No** natural interpretation as maximum likelihood estimator.
- **After** loss-free discretization, (LP) becomes a linear program.
Consistency of the LSQ–estimator I

Assume that

- the measurement normals $u_1, u_2, \ldots$ are “nicely spread”,
- the measurement errors $\varepsilon_1, \varepsilon_2, \ldots$ are i.i.d. Gaussian,
- the directional measure $\eta$ is not degenerate.

Strong consistency: the LSQ–estimator $\hat{\eta}_k$ satisfies

$$\lim_{k \to \infty} \hat{\eta}_k = \eta,$$

almost surely,

in the weak sense.
Consistency of the LSQ–estimator II

Stronger assumption; \((u_i)\) is “uniformly spread”:
\[
\max_{u \in S^{d-1}} \min_{1 \leq i \leq k} \{ \|u - u_i\|, \|u - (-u_i)\| \} = O(k^{-1/(d-1)}).
\]

**Theorem (Speed of Convergence; Gardner, Milanfar & K.)**

Let \(\hat{\eta}_k\) be the LSQ–estimator and \(\varepsilon > 0\).
Almost surely, \(\exists c > 0, \ N \in \mathbb{N}\) such that for all \(k \geq N\):
\[
d_P(\hat{\eta}_k, \eta) \leq \begin{cases} 
  c \cdot k^{-1/15+\varepsilon} & d = 2, \\
  c \cdot k^{-(d+2)/(2(d+4)(2d+1))} + \varepsilon & d = 3, 4, \\
  c \cdot k^{-1/((d-1)(d+4))} + \varepsilon & d \geq 5.
\end{cases}
\]

\(d_P\) is the Prohorov–distance.
Consistency of the EM– and LP–estimators

- The EM-estimator is strongly consistent. (Assumptions like those for the LSQ–estimator.)
- The LP-estimator is weakly consistent. (Assumptions on \((\varepsilon_i)\) stronger than for the LSQ–estimator.)

Estimation of the associated zonoid:
- The associated zonoid can estimated by the zonoid \(\hat{Z}_k\) with
  \[
h(\hat{Z}_k, \cdot) = T_{\hat{\eta}_k}.
\]
- Consistency results for \(\hat{\eta}_k\) carry over to \(\hat{Z}_k\).
Planar sections type A (left) and type B (right). Polarized light micrographs of size $150\mu m \times 150\mu m$.

(A lot of) work done by Andreas Pfrang, University of Karlsruhe!
Estimators for type A fibre architectures

**LSQ**–estimator $\hat{\eta}_k$ from $k = 10$ (!) measurements. ($\cong EM$– and LP–estimator)

Estimator $\hat{Z}_k$ of the associated zonoid derived from $\hat{\eta}_k$. 
Estimators for type B fibre architectures I

$mm/mm^2$

$135$

$100$

$50$

$0$

$LSQ$–estimator $\hat{\eta}_k$ from $k = 11$ measurements.

Estimator $\hat{Z}_k$ of the associated zonoid derived from $\hat{\eta}_k$. 
Estimators for type B fibre architectures II

Estimator \( \hat{Z}_k \) derived from the EM–estimator.
(Same \( y_1, \ldots, y_{11} \) as before)

Estimator \( \hat{Z}_k \) derived from the LP–estimator.
LSQ–estimator $\hat{\eta}_k$ from isotropic, exact measurements.

$(y_1, \ldots, y_{10}$ as for type A)

Estimator $\hat{Z}_k$ of the associated zonoid derived from $\hat{\eta}_k$. 
LP-estimator for exact isotropic data

LP-estimator $\hat{\eta}_k$ from isotropic, exact measurements.
($y_1, \ldots, y_{10}$ as for type A)

Estimator $\hat{Z}_k$ of the associated zonoid derived from $\hat{\eta}_k$. 
The following conclusions can be drawn:

- **large $k$:** All three estimators are consistent; the LP-estimator being the most unstable,
- **small $k$:** (applications!): EM– and LSQ–estimator behave very similar, LP–estimator is poorest, expressed anisotropy can be detected by all three,
- **recommendation:** use of the LSQ-estimator (maximum likelihood, easy to implement),
- **associated zonoid:** an intuitive tool to illustrate anisotropy.