Digital Stereology

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How to determine the quartz content in a block of granite?
Delesse (1847):

\[ \text{volume fraction in 3D} \approx \text{area fraction in a planar section} \]
A problem from Geology III

Rosiwal (1898):

volume fraction in 3D \approx \text{length fraction in linear sections}

Glagolev (1933):

volume fraction in 3D \approx \text{relative number of points in } X
Let $X$ be the phase of interest (quartz)

$$\int_{-\infty}^{\infty} \text{Area}(X \cap L_z) \, dz = \text{Vol}(X)$$

Random sampling: $z =$ uniform random “height” $\xi \in [0, 1]$:

$$\mathbb{E}_\xi \text{Area}(X \cap L_\xi) = \text{Vol}(X).$$

expectation w.r.t. $\xi$
Two basic approaches

- **Design based approach:**
  The *sampling* is done in a random, homogenous way, the set \( X \) is deterministic.

- **Model based approach:**
  No assumptions on the sampling procedure, the set \( X \) is “stochastically homogenous”

(\( \rightsquigarrow \) stochastic geometry: stationary random set)

We will only use the design based approach here!
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- **Design based approach:**
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- **Model based approach:**
  No assumptions on the sampling procedure
  the set $X$ is “stochastically homogenous”
  (⇝ stochastic geometry: stationary random set)

We will only use the design based approach here!
Stereology is a subarea of stochastic geometry dealing with the estimation of geometric characteristics (volume, area, boundary length, particle number, . . .) of structures from random samples.

Sampling schemes can be
- sections with lower dimensional test planes (Delesse, Rosival),
- sections with full-dimensional test windows,
- sections with point lattices (Glagolev).

Digital stereology deals with point lattice samples.
Outline of the Talk

- Stereology
  - Geometric characteristics
  - Sampling with planes and full dimensional sets

- Digital Stereology
  - Digitization of sets
  - Digitization of characteristics
  - Estimation of the surface area measure
The convex ring

\[ X \in \mathcal{R} = \{ \text{finite unions of convex bodies} \subset \mathbb{R}^d \}, \quad d \geq 1. \]

\( \mathcal{R} \) is called convex ring.

We will need Minkowski addition of \( X, X' \in \mathcal{R} \):

\[ X \oplus X' := \{ x + x' \mid x \in X, x' \in X' \}. \]
Let $B^d$ be the (Euclidean) unit ball in $\mathbb{R}^d$ and $\kappa_d$ its volume.

Jakob Steiner (1840): If $X$ is a convex body, then

$$\text{Vol}(X \oplus \epsilon B^d) = \sum_{j=0}^{d} \kappa_{d-j} V_j(X) \epsilon^{d-j}, \quad \epsilon \geq 0.$$ 

$V_j(X) =: j$-th intrinsic volume of $X$. 

\[ X \oplus \epsilon B^2 \]
**Intrinsic Volumes**

**Properties of $V_j$:**

1. **Motion-invariant:** $V_j(\vartheta(X + x)) = V_j(X)$,
   - Translation vector $x \in \mathbb{R}^d$
   - Rotation $\vartheta \in SO_d := \text{rotation group}$

2. **Monotone:** $X \subset X' \Rightarrow V_j(X) \leq V_j(X')$

3. **Additive:** $V_j(X \cup X') = V_j(X) + V_j(X') - V_j(X \cap X')$
   - (where $X$, $X'$ and $X \cup X'$ are convex bodies)

**Hadwiger (1957):**

Any motion-inv., monotone, additive functional on the convex bodies is a nonneg. linear combination of $V_0, \ldots, V_d$. 
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Additive Extensions of $V_j$

$V_j(X) := \text{additive extension of } V_j \text{ on } R.$

Geometric interpretation:

$V_d(X) = \text{Vol}(X)$ is the volume (Lebesgue measure) of $X$, $2V_{d-1}(X)$ = surface area of $X$ 

($= (d-1)$-dim. Hausdorff measure $\mathcal{H}^{d-1}(\partial X)$),

$V_j(X) : \text{connected to curvatures in boundary points}$

$V_0(X) = \text{Euler-Poincaré-characteristic of } X.$

$V_0(X) = 4 - 1 = 3.$
Let $L$ be a fixed $k$-dimensional linear subspace in $\mathbb{R}^d$. 

arbitrary movement of $L$: $\vartheta(L + y)$, $y \in L^\perp$, $\vartheta \in SO_d$.

Crofton's formula for $X \in \mathcal{R}$

$$\int_{SO_d} \int_{L^\perp} V_j(X \cap \vartheta(L + y)) \, d\mathcal{H}^{d-k}(y) d\nu(\vartheta) =$$

$$\text{const}_{d,j,k} \, V_{d+j-k}(X).$$

Here: $\nu = \text{invariant probability measure on } SO_d$.

In particular $j = k$: "Fubini’s theorem", 

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As $X$ is bounded, we may exclude planes lying "far out" (i.e. restrict to $|y| \leq M$ for some $M > 0$.)

\[
\mathbb{E}_\vartheta \mathbb{E}_{|y| \leq M} V_j(X \cap \vartheta(L + y)) = c \cdot V_{d+j-k}(X).
\]

- Unbiased estimator for $V_{d+j-k}$ from a random, $k$-dim. section.
- For $j = 0$, $k = 0$, \ldots, $d - 1$, this yields all intrinsic volumes except $V_0(X)$. 

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Sampling with full dimensional sets

For two convex bodies $X$ and $W$ consider

$$f_W(t) = \int_{\mathbb{R}^d} \mathbf{1}\{(y + tW) \cap X \neq \emptyset\} \, dy.$$ 

$W = B^d$:}

Steiner’s formula $\Rightarrow$

$$f_{B^d}(0) = V_d(X), \quad f'_{B^d}(0) = 2V_{d-1}(X), \quad f''_{B^d}(0) = 2\pi V_{d-2}(X), \ldots$$

For non-convex $X$ (most of) the formulae are wrong!
An Extension

A general Steiner formula \( (\text{Hug, 2000}) \)
\[ X \in \mathcal{R} \text{ and arbitrary convex bodies } W \Rightarrow \]

\[ f'_W(0) = \int_{S^{d-1}} h_W(-u) S(X,\, du). \]

\( f'_W(0) \): unit sphere in \( \mathbb{R}^d \)
\( h_W \): cont. function depending on \( W \)
\( S(X,\, du) \): surface area measure

Digital Stereology: Can we replace \( W \) by a finite set (points of a sampling grid)?
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A Simple Digitization Model

Regular lattice $t\mathbb{Z}^d$ with $t > 0$ being the lattice distance.

Digitization of $X$: $\hat{X}_t := X \cap (t\mathbb{Z}^d)$.

To avoid "lower dimensional parts", assume from now on

$$X \in \mathcal{R}_{reg} = \{X \in \mathcal{R} \mid X = \text{cl int } X\}.$$
Random digitization

We randomize the sampling scheme

- randomly translated lattice:
  Choose $\xi$ uniformly in $[0, 1]^d$ and consider $t(\xi + \mathbb{Z}^d)$.

- randomly rotated lattice:
  Choose $\vartheta$ uniformly in $SO_d$ and consider $\vartheta(t\mathbb{Z}^d)$.

In both cases: $\hat{X}_t$ becomes a random (finite) set.

We will only work with randomly translated lattices here.
Digitization of Characteristics: Definition

Assumptions:

- $\mathcal{M} \subset \mathcal{R}_{\text{reg}}$ is a family of sets,
- $\varphi : \mathcal{M} \to \mathbb{R}$ is a geometric characteristic (e.g. $V_j$),
- $X \in \mathcal{M}$ is a measurable, real-valued $\hat{\varphi}$ on digitized sets with

$$\lim_{t \to 0^+} \hat{\varphi}(\hat{X}_t) = \varphi(X) \text{ a.s.},$$

then $\varphi$ is consistently digitalizable on $\mathcal{M}$,
- $\lim_{t \to 0^+} \mathbb{E}\hat{\varphi}(\hat{X}_t) = \varphi(X)$, then $\varphi$ is digitalizable in mean on $\mathcal{M}$.

$\hat{\varphi}$ is called digital algorithm for $\varphi$ (Serra, 1982, Heijmans, 1992, K. 2005).
Examples

- $V_d = \text{Vol}$ is consistently digitalizable on $\mathcal{R}_{\text{reg}}$ with
  
  $$\hat{V}_d(\hat{X}_t) = t^d \cdot \text{card}(\hat{X}_t).$$

- $V_j$ is consistently digitalizable on the convex bodies in $\mathcal{R}_{\text{reg}}$
  
  $$\hat{V}_j(\hat{X}_t) = V_j(\text{conv} \, \hat{X}_t).$$

But: There are no digitization results for $V_j$ ($j < d$) on $\mathcal{R}_{\text{reg}}$! We discuss in the following candidates $\hat{V}_j$ used in practice.
The Euler-Poincaré-characteristic I

Digitization of $V_0$: polygonal approximation using $\hat{X}_t$:

Here elementary polygons are $\mathcal{E} = \{\cdot, -, |, \square\}$ (and similar for $d \neq 2$).

Then put $\hat{V}_0(\hat{X}_t) := V_0(P)$. 

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The Euler-Poincaré-characteristic II

Properties of $\hat{V}_0$:

- $V_0$ additive $\Rightarrow$ $\hat{V}_0$ locally computable
  "marching square/cube algorithms"
- $d = 1$: consistent digitization on $\mathcal{R}_{\text{reg}}$

\[ \lim_{t \to 0^+} \hat{V}_0(\hat{X}_t) = V_0(X). \]

No digitization on $\mathcal{R}_{\text{reg}}$ for $d > 1$!

(Serra, 1982 ($d = 2$), Nagel et al., 2001 ($d = 3$))

\[ \lim_{t \to 0^+} \hat{V}_0(\hat{X}_t) = V_0(X) \text{ a.s.} \]

if $X$ is "morphologically open and closed".

Condition very restrictive; necessary cond. unknown.
**Intrinsic volumes** $V_j$ with $0 < j < d$

**Crofton's formula for** $X \in \mathcal{R}_{reg}$

$$V_{d-k}(X) = c \int_{SO_d} \int_{L^\perp} V_0(X \cap \vartheta(L + y)) \, dH^{d-k}(y) \, d\nu(\vartheta).$$

**Example** $d = 2$ (and $k = 1 \Rightarrow \text{"boundary length"})$:

1. **Bad approximation!**
   - up to 20% error!

2. **o.k:**
   - error $\leq 5\%$
Surface Area and Fit-and-miss Events

**Goal:** use $\hat{X}_t$ to estimate $S(X, \cdot) = \text{local counterpart of } V_{d-1}$.
Assume $d = 2$.

$$R, B = \text{finite, non-empty test sets in } \mathbb{Z}^d \ (\text{"red"}, \text{"black"})$$

$(R, B) := \text{is called configuration, e.g. } (R, B) \hat{=} \left( \begin{array}{c} \cdot \\ \cdot \end{array} \right)$

$\#(R, B) := \text{number of occurrences of } t(k + (R, B)), k \in \mathbb{Z}^d \text{ in } \hat{X}_t.$

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$\#(R, B) = 2$
(Jensen, K., 2003)

$$t \mathbb{E} \#(R, B) \rightarrow \int_{S^1} h_{(R,B)}(-u) \, dS(X, u), \quad t \rightarrow 0^+.$$  

The function $h_{(R,B)}(\cdot)$ is explicitly known.

Choice of the Configurations $(R, B)$

- $B \cup W$ should have small diameter,
- "marching square"-algorithms should be applicable.

$$(R, B) \overset{\hat{=}}{=} \left( \begin{array}{c} \cdot \vline \cdot \\ \cdot \vline \cdot \\ \cdot \vline \cdot \end{array} \right) \quad \text{"}2 \times 2\text{-configurations"}$$

$$(R, B) \overset{\hat{=}}{=} \left( \begin{array}{c} \cdot \vline \cdot \vline \cdot \\ \cdot \vline \cdot \vline \cdot \\ \cdot \vline \cdot \vline \cdot \end{array} \right) \quad \text{"}3 \times 3\text{-configurations"}$$
### 2 × 2-Configurations I

All 2 × 2-configurations that yield non-vanishing integrals:

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Application: For sufficiently small \( t > 0 \) the counts

\[
\#(R, B) \approx t \int_{S^1} h_{(R,B)}(-u) dS(X, u)
\]

lead to (estimates of) 8 different integrals of \( S(X, \cdot) \).

Model: \( \hat{S}(X, \cdot) = \sum_{i=1}^{8} \alpha_i \delta_{u_i} \) with \( \alpha_1, \ldots, \alpha_8 \geq 0 \).

Approach: Determine \( \alpha_1, \ldots, \alpha_8 \geq 0 \) in such a way that

\[
t \int_{S^1} h_{(R,B)}(-u) d\hat{S}(X, u) \text{ is "close to" } \#(R, B).
\]
Application example: Rolled Steel

The digital image of a rolled steel (black phase = $X$).

The estimated masses of $S(X, \cdot)$ from $2 \times 2$-configurations.

The total mass of this estimator also yields an estimator for $V_1(X)$. 
Objection!

"I think that he is simplifying things far too much..."
The considerations are restricted

1. to the planar case \((d = 2)\),
2. to \(\mathcal{R}_{\text{reg}}\) (might be too restrictive for applications),
3. to a very simple digitization model for \(\tilde{X}_t\),

Alternative: the threshold digitization \(\tilde{X}_t(\theta)\), \(0 < \theta \leq 1\):

\[
\text{Lattice point } x:\quad x \in \tilde{X}_t(\theta) : \iff \text{Vol}(\text{Cell}_x \cap X) \geq \theta \cdot \text{Vol}(\text{Cell}_x) .
\]

\[\theta = 1/2\]
A Far-reaching Generalization

(Rataj, K., 2005+)

\[ t \mathbb{E} \#(R, B) \rightarrow \int_{S^{d-1}} h(R, B)(-u) \, dS(X, u), \quad t \rightarrow 0+. \]

holds for

- \( d \geq 1 \),
- \( X \) in a \textbf{very general set-class}
  (full-dimensional finite unions of sets of positive reach),
- \( \hat{X}_t \) or \( \tilde{X}_t(\theta) \) as underlying digitization.