17. Riesz’ representation theorem

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We read the lecture notes on Riesz’ representation theorem, mostly stolen from Walter Rudin’s book: ’Real and complex analysis’.
The Riesz representation theorem - the setting

Throughout the note $X$ will be a fixed locally compact Hausdorff space. By $C_c(X)$ we denote the vector space of continuous compactly supported functions on $X$, i.e. a continuous function $f : X \rightarrow \mathbb{C}$ is in $C_c(X)$ if and only if

$$\text{supp } f = \{ x \in X : f(x) \neq 0 \}$$

is a compact subset of $X$.

A linear functional $\Lambda : C_c(X) \rightarrow \mathbb{C}$ is said to be positive when

$$f \in C_c(X), \quad f(x) \geq 0 \quad \forall x \in X \quad \Rightarrow \quad \Lambda(f) \geq 0.$$ 

Some simple examples of positive linear functionals on $C_c(X)$ are easy to come by: For example, one may choose a couple of points $x_1, x_2 \in X$ and two non-negative real numbers $\alpha_1, \alpha_2$, and define $\Gamma : C_c(X) \rightarrow \mathbb{C}$ such that

$$\Gamma(f) = \alpha_1 f(x_1) + \alpha_2 f(x_2).$$

Then $\Gamma$ is a positive linear functional on $C_c(X)$.
The Riesz representation theorem - the setting

More generally: Let $\mu$ be a measure defined on a $\sigma$-algebra of subsets of $X$ which (at least) contains the Borel sets in $X$. If $\mu$ is finite on every compact subset of $X$, i.e. if $\mu(K) < \infty$ when $K \subseteq X$ is compact, then any function $f \in C_c(X)$ is integrable with respect to $\mu$. (This follows from Proposition 15 ii) on p. 267 in Roydends book since $|f| \leq M 1_K$, where $M = \sup \{|f(x)| : x \in X\} < \infty$ and $1_K$ is the characteristic function of $K = \text{supp } f$.) We can therefore define $\Lambda_\mu : C_c(X) \to \mathbb{C}$ by integration with respect to $\mu$, i.e.

$$\Lambda_\mu(f) = \int_X f \, d\mu,$$

(sometimes written $\int_X f(x) \, d\mu(x)$ to emphasize the variable.) It follows from Proposition 15 i)+ ii) on p. 267 in Roydends book that $\Lambda_\mu$ is then a positive linear functional on $C_c(X)$. The main content of Riesz’s representation theorem is that every positive linear functional on $C_c(X)$ arises in this way.
Let $X$ be a locally compact Hausdorff space. Let $\Lambda : C_c(X) \to \mathbb{C}$ be a positive linear functional. Then there exists a σ-algebra $\mathcal{M}$ in $X$ which contains all Borel sets in $X$, and there exists a unique positive measure $\mu$ on $\mathcal{M}$ which represents $\Lambda$ in the sense that

(a) \[ \Lambda(f) = \int_X f \, d\mu \text{ for every } f \in C_c(X), \]

and which has the following additional properties:
The Riesz representation theorem for positive functionals

**Theorem**

(b) \( \mu(K) < \infty \) when \( K \subseteq X \) is compact.

(c) For every \( E \in \mathcal{M} \), we have

\[
\mu(E) = \inf \{ \mu(V) : E \subseteq V, \ V \text { open} \}.
\]

(d) The relation

\[
\mu(E) = \sup \{ \mu(K) : K \subseteq E, \ K \text { compact} \}
\]

holds for every open set \( E \), and every \( E \in \mathcal{M} \) with \( \mu(E) < \infty \).

(e) If \( E \in \mathcal{M} \), \( A \subseteq E \), and \( \mu(E) = 0 \), then \( A \in \mathcal{M} \).
The proof - uniqueness

The uniqueness part of the statement is that if $\mu_1$ and $\mu_2$ are both measures on $\mathcal{M}$ with the properties (a)-(d), then $\mu_1 = \mu_2$.

We start by proving this. For this we introduce the following notation: When $f \in C_c(X)$ is a function taking values in $[0, 1]$, and $E \subseteq X$ we shall write

$$f \prec E,$$

when $\text{supp} \ f \subseteq E$, and

$$E \prec f$$

when $f(x) = 1$ for $x \in E$.

It follows then from Urysohn’s lemma, Lemma 0.17 in ’Kommentarer’ that whenever $K$ is a compact subset of $X$, $V$ an open subset of $X$, and $K \subseteq V$, then there is an $f \in C_c(X)$ such that

$$K \prec f \prec V.$$
In particular, it follows that

\[
\mu_1(K) = \int_X 1_K \, d\mu_1 \leq \int_X f \, d\mu_1 = \int_X f \, d\mu_2 \leq \int_X 1_V \, d\mu_2 = \mu_2(V)
\]

(1)

in this situation, i.e. when \( K \) is compact, \( V \) is open and \( K \subseteq V \). Since \( \mu_2 \) satisfies (c) we deduce from (1) that \( \mu_1(K) \leq \mu_2(K) \) when \( K \subseteq X \) is compact. Since \( \mu_1 \) and \( \mu_2 \) both have property (d) we conclude that \( \mu_1(V) \leq \mu_2(V) \) when \( V \subseteq X \) is open. Since they also both have property (c) we conclude that \( \mu_1(E) \leq \mu_2(E) \) for all \( E \in \mathcal{M} \). By symmetry we must also have the reversed inequality, and we may therefore conclude that \( \mu_1 = \mu_2 \).
We turn now to the construction of the measure $\mu$. First we define $\mu(V)$ when $V \subseteq X$ is open:

$$\mu(V) = \sup \{ \Lambda(f) : f \prec V \}.$$  \hfill (2)

Since a part of the conditions in '$f \prec V$' is that $f(X) \subseteq [0, 1]$ it follows that $\mu(V) \geq 0$ because $\Lambda$ is a positive linear functional. Note that it may very well be that $\mu(V) = \infty$! Note also that when $V_1$ and $V_2$ are both open and $V_1 \subseteq V_2$, then it follows from (2) that $\mu(V_1) \leq \mu(V_2)$. Hence, when we set

$$\mu(E) = \inf \{ \mu(V) : E \subseteq V \}$$  \hfill (3)

for any subset $E \subseteq X$, we haven’t changed the definition on open sets. In other words, with (3) we have an extension of $\mu$ from open to arbitrary sets.
This extension will not in general give us a measure defined, as it is, on all subsets of \(X\), and a major part of the proof is to identify a \(\sigma\)-algebra of sets in \(X\) which contains the Borel sets, and on which \(\mu\) does give us a measure. (The trouble, of course, is the countable additivity of \(\mu\)!)

Let \(\mathcal{M}_F\) be the subsets \(E\) of \(X\) for which \(\mu(E) < \infty\) and

\[
\mu(E) = \sup \{ \mu(K) : K \subseteq E \text{ compact} \}.
\]

(4)

Let \(\mathcal{M}\) be the subsets \(E\) of \(X\) with the property that

\[
E \cap K \in \mathcal{M}_F
\]

(5)

when \(K \subseteq X\) is compact.
We prove that $\mu : \mathcal{M} \to [0, \infty]$ has the required properties:

First of all, observe that $\mu$ is *monotone*, in the sense that

$$A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$$

(6)

since $\{\mu(V) : B \subseteq V\} \subseteq \{\mu(V) : A \subseteq V\}$.

Next we establish the following

If $E_1, E_2, E_3, \ldots$, are arbitrary subsets of $X$,

$$\mu \left( \bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

(7)
Proof of (7)

We prove first that

\[ \mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2) \]  

(8)

when \( V_1 \) and \( V_2 \) are open.

To this end, let \( g \prec V_1 \cup V_2 \). It follows then from Theorem 0.18 in 'Kommentarer' - the theorem on partitions of unity - that there are functions \( h_i \in C_c(X) \), \( i = 1, 2 \), such that \( h_1(x) + h_2(x) = 1 \) when \( x \in \text{supp} \ g \), and \( g_i \prec V_i \), \( i = 1, 2 \).

It follows that \( g = gh_1 + gh_2 \) so the additivity of \( \Lambda \) shows that \( \Lambda(g) = \Lambda(gh_1) + \Lambda(gh_2) \).

Since \( gh_i \prec V_i \) we conclude that \( \Lambda(g) \leq \mu(V_1) + \mu(V_2) \). It follows then from (2) that \( \mu_2(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2) \), proving (8).
Proof of (7)

By induction it follows that

\[ \mu \left( \bigcup_{i=1}^{n} V_i \right) \leq \sum_{i=1}^{n} \mu(V_i) \]  \hspace{1cm} (9)

for any finite collection \( V_1, V_2, \ldots, V_n \) of open sets in \( X \).

This is then used to prove (7) in the following way: If \( \mu(E_i) = \infty \) for some \( i \), the inequality (7) is trivial, so we may assume that \( \mu(E_i) < \infty \) for all \( i \).

Let \( \epsilon > 0 \). It follows from the definition, (3), that there are open sets \( V_i, i = 1, 2, \ldots, \) in \( X \) such that \( E_i \subseteq V_i \) and

\[ \mu(V_i) \leq \mu(E_i) + 2^{-i} \epsilon \] for all \( i \).

Since \( \bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} V_i \) we conclude that

\[ \mu \left( \bigcup_{i=1}^{\infty} E_i \right) \leq \mu \left( \bigcup_{i=1}^{\infty} V_i \right) \]  \hspace{1cm} (10)
Proof of (7)

To estimate the right-hand side, let \( f \prec \bigcup_{i=1}^{\infty} V_i \). Then \( V_i, i = 1, 2, \ldots \), is an open cover of the compact set \( \text{supp} \ f \) so there is an \( n \in \mathbb{N} \) such that \( f \prec \bigcup_{i=1}^{n} V_i \).

Then

\[
\Lambda(f) \leq \mu \left( \bigcup_{i=1}^{\infty} V_i \right),
\]

and by use of (9) we find that

\[
\Lambda(f) \leq \sum_{i=1}^{n} \mu(V_i) \leq \sum_{i=1}^{n} (\mu(E_i) + 2^{-i} \epsilon) \leq \epsilon + \sum_{i=1}^{n} \mu(E_i) \leq \epsilon + \sum_{i=1}^{\infty} \mu(E_i).
\]

Since \( f \prec \bigcup_{i=1}^{\infty} V_i \) was arbitrary we conclude (by using (2)) that

\[
\mu \left( \bigcup_{i=1}^{\infty} V_i \right) \leq \epsilon + \sum_{i=1}^{\infty} \mu(E_i).
\]

Since \( \epsilon > 0 \) was arbitrary, we conclude that

\[
\mu \left( \bigcup_{i=1}^{\infty} V_i \right) \leq \sum_{i=1}^{\infty} \mu(E_i).
\]

In combination with (10), this proves (7).
The next step is to establish

If $K \subseteq X$ is compact, then $K \in \mathcal{M}_F$, and

$$
\mu(K) = \inf \{\Lambda(f) : K \prec f\}.
$$  \tag{11}

If $K \prec f$, and $0 < \alpha < 1$, set $V_\alpha = \{x \in X : f(x) > \alpha\}$. Then $K \subseteq V_\alpha$ since $f(x) = 1$ when $x \in K$, and when $g \in C_c(X)$ is any function such that $g \prec V_\alpha$, we have that $\alpha g \leq f$. Hence $\Lambda(\alpha g) = \alpha \Lambda(g) \leq \Lambda(f)$ because $\Lambda$ is linear and positive. It follows that

$$
\mu(K) \leq \mu(V_\alpha) = \sup \{\Lambda(g) : g \prec V_\alpha\} \leq \alpha^{-1} \Lambda(f).
$$
Proof of (11)

In particular we see that $\mu(K) < \infty$, and by letting $\alpha \to 1$ we obtain the conclusion that $\mu(K) \leq \Lambda(f)$.

Since $K \prec f$ was arbitrary, we see that $\mu(K) \leq \inf \{\Lambda(f) : K \prec f\}$. On the other hand, if we let $\epsilon > 0$, it follows from (3) that there is an open set $V \supseteq K$ such that $\mu(V) \leq \mu(K) + \epsilon$.

By Urysohn's lemma there is a function $K \prec f \prec V$, and for this function $\Lambda(f) \leq \mu(V)$ by (2). Since $K \prec f$, we conclude that $\inf \{\Lambda(f) : K \prec f\} \leq \mu(K) + \epsilon$.

Since $\epsilon > 0$ was arbitrary, we have established the equality in (11). As we saw, $\mu(K) < \infty$, so it follows from (6) and the compactness of $K$ that $K \in M_F$. - (11) is proved.
Proof of (12)

Every open set satisfies (4).

Hence $V \in M_F$ when $V$ is open and $\mu(V) < \infty$. \hfill (12)

Let $V \subseteq X$ be open, and let $\alpha$ be a real number such that $\alpha < \mu(V)$. By definition of $\mu(V)$, cf. (2), there is an $f \in C_c(X)$ such that $f \prec V$ and $\Lambda(f) \geq \alpha$.

Then $K = \text{supp } f$ is a compact subset of $V$, and when $W$ is an open subset containing $K$, viz. $K \subseteq W$, then $f \prec W$ and hence $\mu(W) \geq \Lambda(f)$.

It follows from (3) that $\mu(K) \geq \Lambda(f) > \alpha$. Since $\alpha < \mu(V)$ was arbitrary, we conclude that $\sup \{\mu(K) : K \subseteq V\} \geq \mu(V)$. The reversed inequality is trivial, thanks to (6), so (12) has been established.
Proof of (13)

Suppose \( E = \bigcup_{i=1}^{\infty} E_i \), where \( E_1, E_2, E_3, \ldots \), are pairwise disjoint members of \( \mathcal{M}_F \). Then

\[
\mu(E) = \sum_{i=1}^{\infty} \mu(E_i). 
\]

If, in addition, \( \mu(E) < \infty \), then also \( E \in \mathcal{M}_F \).

We prove first that

\[
\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2) \quad (14)
\]

when \( K_i, i = 1, 2 \), are compact and disjoint.
Proof of (13)

By Urysohn’s lemma there is an \( f \in C_c(X) \) such that \( 0 \leq f \leq 1 \), \( f(x) = 1 \), \( x \in K_1 \), and \( f(x) = 0 \) when \( x \in K_2 \).

Let \( \varepsilon > 0 \). It follows from (11) that there is \( g \in C_c(X) \) such that \( K_1 \cup K_2 \prec g \) and \( \Lambda(g) \leq \mu(K_1 \cup K_2) + \varepsilon \).

Then \( K_1 \prec fg \) and \( K_2 \prec (1 - f)g \). Hence \( \mu(K_1) \leq \Lambda(fg) \) and \( \mu(K_2) \leq \Lambda((1 - f)g) \) by (11).

Since \( \Lambda \) is linear, we find that

\[
\mu(K_1) + \mu(K_2) \leq \Lambda(fg) + \Lambda((1 - f)g) = \Lambda(g) \leq \mu(K_1 \cup K_2) + \varepsilon.
\]

Since \( \varepsilon > 0 \) was arbitrary we conclude that \( \mu(K_1) + \mu(K_2) \leq \mu(K_1 \cup K_2) \), and then (14) follows from (7).
To prove the equality in (13) we pick compact subsets $K_i \subseteq E_i$ such that $\mu(K_i) \geq \mu(E_i) - 2^{-i}\epsilon$. This is possible since $E_i \in \mathcal{M}_F$.

For each $n$, $\bigcup_{i=1}^{n} K_i$ is a compact subset of $E$, and it follows from (14) that

$$
\mu(E) \geq \mu \left( \bigcup_{i=1}^{n} K_i \right) = \sum_{i=1}^{n} \mu(K_i) \geq \sum_{i=1}^{n} \mu(E_i) - 2^{-i}\epsilon \geq \sum_{i=1}^{n} \mu(E_i) - \epsilon.
$$

Since $n$ and $\epsilon > 0$ are arbitrary here, we conclude that

$$
\mu(E) \geq \sum_{i=1}^{\infty} \mu(E_i).
$$

Combined with (7) this yields the equality in (13).
Proof of (13)

Returning to (15) we conclude then that if $\mu(E) < \infty$, $K_n = \bigcup_{i=1}^{n} K_i$ is a compact subset of $E$ such that

$$\mu(K) \geq \sum_{i=1}^{n} \mu(E_i) - \epsilon = \mu(E) - \sum_{j=n+1}^{\infty} \mu(E_j) - \epsilon.$$ 

Since $\sum_{i=1}^{\infty} \mu(E_i) = \mu(E) < \infty$ we have that $\sum_{j=n+1}^{\infty} \mu(E_j) < \epsilon$ if $n$ is large enough. So for $n$ large, $K_n$ is a compact subset of $E$ such that $\mu(K_n) \geq \mu(E) - 2\epsilon$.

It follows that $E$ satisfies (4) when $\mu(E) < \infty$. Thus $E \in \mathcal{M}_F$ in this case.
Proof of (17)

If $E \in \mathcal{M}_F$, and $\epsilon > 0$, there is a compact subset $K \subseteq X$ and an open subset $V \subseteq X$ such that $K \subseteq E \subseteq V$, and $\mu(V \setminus K) \leq \epsilon$. (17)

It follows from (3) that there is an open set $V \supseteq E$ such that $\mu(V) \leq \mu(E) + \frac{\epsilon}{2}$ and since (4) holds and $\mu(E) < \infty$ there is a compact subset $K \subseteq E$ such that $\mu(K) \geq \mu(E) - \frac{\epsilon}{2}$.

Note that $V = K \cup (V \setminus K)$, and that $V \setminus K$ is open. It follows from (12) that $V \setminus K \in \mathcal{M}_F$ since $\mu(V \setminus K) \leq \mu(V) < \infty$, and from (11) that $K \in \mathcal{M}_F$, so we conclude from (13) that

$$
\mu(V) = \mu(K) + \mu(V \setminus K),
$$

which implies that

$$
\mu(V \setminus K) = \mu(V) - \mu(K) \leq \mu(E) + \frac{\epsilon}{2} - \mu(E) + \frac{\epsilon}{2} = \epsilon.
$$
Proof of (18)

If \( A \in \mathcal{M}_F, \ B \in \mathcal{M}_F \), then \( A \setminus B, \ A \cup B, \) and \( A \cap B \) belong to \( \mathcal{M}_F \).

(18)

Since \( \mu(A \cup B) \leq \mu(A) + \mu(B) < \infty \) (using (7)), it follows from (6) that \( \mu(A \setminus B) < \infty, \ \mu(A \cup B) < \infty \) and \( \mu(A \cap B) < \infty \).

So it remains just to show that all three sets in the statement satisfy (4).

To this end, let \( \epsilon > 0 \). By (17) there are compact sets \( K_i, i = 1, 2 \), and open sets \( V_i, i = 1, 2 \), such that \( K_1 \subseteq A \subseteq V_1, \ K_2 \subseteq B \subseteq V_2 \), and \( \mu(V_i \setminus K_i) \leq \epsilon, i = 1, 2 \).

Note that

\[
A \setminus B \subseteq V_1 \setminus K_2 \subseteq (V_1 \setminus K_1) \cup (K_1 \setminus V_2) \cup (V_2 \setminus K_2).
\]
Proof of (18)

Of the three last sets two are open and one is compact. (12) and (11) they are all in $\mathcal{M}_F$ and hence (7) implies that

$$\mu(A \setminus B) \leq \mu(V_1 \setminus K_1) + \mu(K_1 \setminus V_2) + \mu(V_2 \setminus K_2) \leq \epsilon + \mu(K_1 \setminus V_2) + \epsilon.$$  

Then $K_1 \setminus V_2$ is a compact subset of $A \setminus B$ such that

$$\mu(K_1 \setminus V_2) \geq \mu(A \setminus B) - 2\epsilon.$$  

Since $\epsilon > 0$ was arbitrary, this shows that $A \setminus B \in \mathcal{M}_F$. Once this is established it follows from (13) that $A \cup B = (A \setminus B) \cup B$ and $A \cap B = A \setminus (A \setminus B)$ are both in $\mathcal{M}_F$.  

Proof of (19)

\( \mathcal{M} \) is a \( \sigma \)-algebra in \( X \) which contains all Borel sets. \hspace{1cm} (19)

In the following \( K \) is an arbitrary compact subset of \( X \). Recall that \( K \in \mathcal{M}_F \) by Observation 11.

Let \( A \in \mathcal{M} \), and consider the complement \( A^c = X \setminus A \). By definition of \( \mathcal{M} \), \( A \cap K \in \mathcal{M}_F \), and \( A^c \cap K = K \setminus (A \cap K) \) is therefore the set-theoretic difference of two sets from \( \mathcal{M}_F \). Hence \( A^c \cap K \in \mathcal{M}_F \) by (18). This shows that \( A^c \in \mathcal{M} \) since \( K \) was arbitrary.

Consider then a sequence \( A_i, i = 1, 2, 3, \ldots \), of sets from \( \mathcal{M} \). Put \( B_1 = A_1 \cap K \), and \( B_n = (A_n \cap K) \setminus (B_1 \cup B_2 \cup \cdots \cup B_{n-1}) \), \( n \geq 2 \).

Then the \( B_i \)'s are disjoint subsets of \( X \) and
\[
K \cap (\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} B_n.
\]
It follows from (18) that \( B_i \in \mathcal{M}_F \) for all \( i \).

Since \( (K \cap (\bigcup_{n=1}^{\infty} A_n)) \leq \mu(K) < \infty \) by (6) and (11), we deduce from (13) that \( \bigcup_{n=1}^{\infty} B_n \in \mathcal{M}_F \). It follows that \( \bigcup_{n=1}^{\infty} A_n \in \mathcal{M} \) since \( K \) was arbitrary.

It follows that \( \mathcal{M} \) is a \( \sigma \)-algebra.
If $C$ is a closed subset of $X$, the intersection $C \cap K$ is compact and hence an element of $\mathcal{M}_F$ by Observation 11. Thus $C \in \mathcal{M}$ since $K$ was arbitrary, and we see that $\mathcal{M}$ contains all closed subsets of $X$. Being a $\sigma$-algebra it must therefore contain all Borel subsets of $X$.

$$\mathcal{M}_F = \{ A \in \mathcal{M} : \mu(A) < \infty \}.$$  

(20)

If $A \in \mathcal{M}_F$, $\mu(A) < \infty$ by definition. Furthermore, it follows from (11) and (18) that $A \cap K \in \mathcal{M}_F$ for every compact subset $K$. Hence $A \in \mathcal{M}$. This proves one of the desired inclusions.

To prove the other, assume that $A \in \mathcal{M}$ and that $\mu(A) < \infty$. To prove that $A \in \mathcal{M}_F$, let $\epsilon > 0$. By definition of $\mu$ there is an open set $V \supseteq A$ such that $\mu(V) < \infty$. By (12) $V \in \mathcal{M}_F$, and by (17) we can find a compact subset $K \subseteq V$ such that $\mu(V \setminus K) \leq \epsilon$. 

Klaus Thomsen  
17. Riesz’ representation theorem
Since $A \cap K \in \mathcal{M}_F$, there is a compact subset $H \subseteq A \cap K$ such that

$$\mu(H) \geq \mu(A \cap K) - \epsilon.$$

Since $A \subseteq (A \cap K) \cup (V \setminus K)$, it follows that

$$\mu(A) \leq \mu(A \cap K) + \mu(V \setminus K) \leq \mu(H) + \epsilon + \mu(V \setminus K) \leq \mu(H) + 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we see from this that $A \in \mathcal{M}_F$, as desired.
Proof of (21)

\( \mu \) is a measure on \( \mathcal{M} \). \hspace{1cm} (21)

Let \( E_1, E_2, \ldots \) be a sequence of mutually disjoint elements of \( \mathcal{M} \).

If \( \mu(E_i) = \infty \) for some \( i \), it is clear that

\[ \mu(\bigcup_{i=1}^{\infty} E_i) = \infty = \sum_{i=1}^{\infty} \mu(E_i). \]

Otherwise, it follows from (20) that \( E_i \in \mathcal{M}_F \) and hence

\[ \mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i) \text{ by (13)}. \]

The reader is now asked to observe that the properties of \( \mu \)
stipulated in (b)-(e) all hold: (b) follows from (11), (c) follows from
the definition of \( \mu \), (d) follows from (20) and (12), while (e) follows
from (20) since the set \( A \) of (e) is in \( \mathcal{M}_F \) be definition of \( \mathcal{M}_F \).
Proof of a)

It remains to check that condition (a) holds. To this end it suffices to check that

$$\Lambda(f) \leq \int_X f \, d\mu$$  \hspace{1cm} (22)

for every real-valued element of $C_c(X)$.

Indeed, it follows then that also $\Lambda(-f) \leq \int_X -f \, d\mu$ holds, which implies first that $\Lambda(f) \geq \int_X f \, d\mu$ and then that $\Lambda(f) = \int_X f \, d\mu$.

By linearity this yields (a).

We prove (22): Choose $a < b$ in $\mathbb{R}$ such that $f(X) \subseteq [a, b]$, and let $\epsilon > 0$. Choose $y_0 < y_1 < y_2 < \cdots < y_n$ such that $y_i - y_{i-1} < \epsilon$ for all $i$, $y_0 < a$ and $y_n = b$. Put

$$E_i = \{x \in \mathbb{R} : y_{i-1} < f(x) \leq y_i\} \cap K,$$

$i = 1, 2, \ldots, n$, where $K = \text{supp} \, f$.

Since $f$ is continuous and hence Borel measurable, the $E_i$'s are Borel sets. They are mutually disjoint.
It follows from the definition of $\mu$ that there are open sets $W_i \supseteq E_i$ such that $\mu(W_i) < \mu(E_i) + \frac{\epsilon}{n}$. Set $V_i = W_i \cap f^{-1}(\left(-\infty, y_i + \epsilon\right])$. Then $V_i$ is an open set such that $V_i \supseteq E_i$, $f(x) < y_i + \epsilon$, $x \in V_i$, and

$$\mu(V_i) \leq \mu(W_i) < \mu(E_i) + \frac{\epsilon}{n}. \tag{23}$$

Note that $K \subseteq \bigcup_{i=1}^{n} V_i$. It follows from Theorem 0.18 in 'Kommentarer' that there are continuous functions $h_i : X \to [0, 1]$ such that $h_i \prec V_i$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} h_i(x) = 1$, $x \in K$. Then $f = \sum_{i=1}^{n} h_if$, and it follows from (19) that
Proof of a)

\[ \mu(K) \leq \Lambda \left( \sum_{i=1}^{n} h_i \right) = \sum_{i=1}^{n} \Lambda(h_i). \quad (24) \]

Since \( h_i f \leq (y_i + \epsilon)h_i \), and since \( y_i - \epsilon < f(x) \) on \( E_i \), we find that

\[ \Lambda(f) = \sum_{i=1}^{n} \Lambda(h_i f) \leq \sum_{i=1}^{n} (y_i + \epsilon)\Lambda(h_i) \]

\[ = \sum_{i=1}^{n} (|a| + y_i + \epsilon)\Lambda(h_i) - \sum_{i=1}^{n} |a|\Lambda(h_i) \quad (25) \]

\[ \leq \sum_{i=1}^{n} (|a| + y_i + \epsilon)\mu(V_i) - \sum_{i=1}^{n} |a|\Lambda(h_i) \quad \text{(using (2))} \]
Proof of a)

\[ \leq \sum_{i=1}^{n} (|a| + y_i + \epsilon) \left[ \mu(E_i) + \frac{\epsilon}{n} \right] - \sum_{i=1}^{n} |a| \Lambda(h_i) \quad \text{(using (23))} \]

\[ \leq \sum_{i=1}^{n} (|a| + y_i + \epsilon) \left[ \mu(E_i) + \frac{\epsilon}{n} \right] - |a| \mu(K) \quad \text{(using (24))} \]

\[ = \sum_{i=1}^{n} (y_i + \epsilon) \left[ \mu(E_i) + \frac{\epsilon}{n} \right] + \sum_{i=1}^{n} |a| \frac{\epsilon}{n} \quad \text{(since} \sum_{i=1}^{n} \mu(E_i) = \mu(K)) \]

\[ = \sum_{i=1}^{n} (y_i - \epsilon) \mu(E_i) + 2\epsilon \mu(K) + \sum_{i=1}^{n} (|a| + y_i + \epsilon) \frac{\epsilon}{n} \]

\[ \quad \text{(since} \sum_{i=1}^{n} \mu(E_i) = \mu(K)) \]

(26)
Proof of a)

\[
\leq \int_X f \, d\mu + 2\epsilon \mu(K) + \sum_{i=1}^n (|a| + y_i + \epsilon) \frac{\epsilon}{n} \quad \text{(since } \sum_{i=1}^n (y_i - \epsilon)1_{E_i} \leq f) \]
\[
\leq \int_X f \, d\mu + 2\epsilon \mu(K) + \epsilon (|a| + b + \epsilon) \quad \text{(since } y_i \leq b \text{ for all } i). \]

(27)
Proof of a)

Since \( \epsilon > 0 \) was arbitrary here, this yields (22).