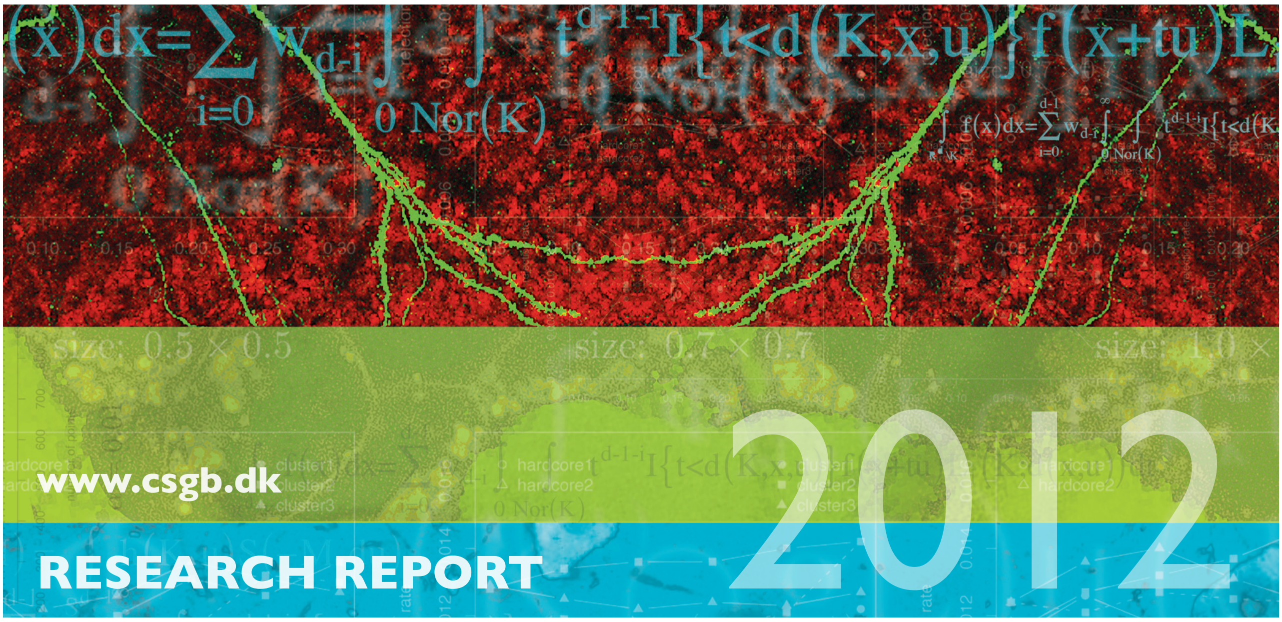
CENTRE FOR **STOCHASTIC GEOMETRY**
AND ADVANCED **BIOIMAGING**



Jérémy Auneau-Cognacq, Johanna Ziegel and Eva B. Vedel Jensen

Rotational integral geometry of tensor valuations

No. 02, March 2012

Rotational integral geometry of tensor valuations

Jérémy Auneau-Cognacq[a], Johanna Ziegel[b] and Eva B. Vedel Jensen[a]

[a]Centre for Stochastic Geometry and Advanced Bioimaging, Department of Mathematics, Aarhus University, Denmark; Email: manitou@imf.au.dk, eva@imf.au.dk
[b]Institute of Applied Mathematics, Heidelberg University, Germany; Email: j.ziegel@uni-heidelberg.de

Abstract

We derive a new rotational Crofton formula for Minkowski tensors. In special cases, this formula gives (1) the rotational average of Minkowski tensors defined on linear subspaces and (2) the functional defined on linear subspaces with rotational average equal to a Minkowski tensor. Earlier results obtained for intrinsic volumes appear now as special cases.

Keywords: Crofton formula; local stereology; Minkowski tensors; rotational integral geometry; tensor valuations

1 Introduction

Minkowski tensors are an important morphometric tool in the analysis of shape and orientation of spatial structures. They have been used with success, mainly in material science [1–3] but there are also examples from the biosciences [4]. In the imaging literature, Minkowski tensors are called moments; cf. [5].

These application examples have in common that the structure of interest can be studied directly. However, in some cases the structure is only available via sections or projections. In [6, 7], formulae of Crofton type are derived for Minkowski tensors. It is shown that integrals of Minkowski tensors on k-flats (k-dimensional affine subspaces) in $\mathbb{R}^d$ with respect to the motion invariant measure on such flats can be expressed as linear combinations of Minkowski tensors evaluated on the original set. These results build on the work of Alesker [8, 9] and its extension in [10].

In the present paper, we derive a rotational Crofton formula for Minkowski tensors. This formula concerns integrals of Minkowski tensors on k-subspaces (k-dimensional linear subspaces) in $\mathbb{R}^d$ with respect to the rotation invariant measure on such subspaces.

The motivation for deriving rotational formulae comes from local stereology [11], where the aim is estimating quantitative properties of spatial structures from sections passing through fixed points. Local stereology is widely used in the biosciences.

Corresponding author: Johanna Ziegel, j.ziegel@uni-heidelberg.de

The rotational Crofton formula for Minkowski tensors can be used to derive local stereological estimators of Minkowski tensors. Furthermore, the new identities generalize results in [12–14] obtained for Minkowski tensors of rank zero, i.e. intrinsic volumes.

The paper is organized as follows. In Section 2 we review the definition of Minkowski tensors. Integrated versions of Minkowski tensors are introduced in Section 3. In Section 4, it is shown that a genuine rotational Crofton formula holds for these integrated Minkowski tensors. This formula is an important tool in the derivation of the new geometric identities, also presented in Section 4. In Section 5, explicit formulae for the integrated Minkowski tensors are derived. Section 6 describes the use of the new geometric identities in local stereology while Section 7 summarizes the results for tensors of rank two in $\mathbb{R}^3$. The paper concludes with a discussion of the results and open problems.

2 Minkowski tensors

Let $X \subseteq \mathbb{R}^d$ be a convex and compact subset of $\mathbb{R}^d$. Let r be a non-negative integer. The *volume tensor of rank* r is then defined by

$$\Phi_{d,r,0}(X) = \frac{1}{r!} \int_X x^r \lambda_d(\mathrm{d}x), \tag{2.1}$$

where λ_d is the Lebesgue measure in $\mathbb{R}^d$. The notation x^r is short for the tensor $x \otimes \cdots \otimes x$ with r factors. Note that $\Phi_{d,r,0}(X) \propto \mathbb{E}(Y^r)$ where Y is a uniform random point in X.

For $k = 0, \ldots, d-1$ and non-negative integers r and s, let

$$\Phi_{k,r,s}(X) = \frac{\omega_{d-k}}{r!\, s!\, \omega_{d-k+s}} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} x^r u^s \Lambda_k(X, \mathrm{d}(x,u)), \tag{2.2}$$

where $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the $(d-1)$-dimensional unit sphere, $\mathbb{S}^{d-1}$, $x^r u^s$ is the symmetric tensor product of x^r and u^s, and $\Lambda_k(X, \cdot)$ is the kth support measure or generalized curvature measure of X, $k = 0, \ldots, d-1$; see [15, p. 253]. Note that for $s = 0$

$$\Phi_{k,r,0}(X) = \frac{1}{r!} \int_X x^r \Phi_k(X, \mathrm{d}x),$$

where $\Phi_k(X, \cdot)$ is the kth curvature measure of X, $k = 0, \ldots, d$. For $r = s = 0$, we have $\Phi_{k,0,0} = V_k$, where V_k denotes the kth intrinsic volume, $k = 0, \ldots, d$. For further details; see [16].

The measures $\Lambda_k(X, \cdot)$, $k = 0, \ldots, d-1$, are concentrated on the normal bundle $\mathrm{Nor}X$ of X which consists of all pairs (x, u) where $x \in \partial X$ and u is an outer unit normal vector of X at x. For this reason, the tensors defined in (2.2) are called *surface tensors*. In fact, $\Lambda_k(X, \cdot)$ has the following integral representation for a Borel set $A \subseteq \mathbb{R}^d \times \mathbb{S}^{d-1}$

$$\Lambda_k(X, A) = \frac{1}{\omega_{d-k}} \int_{A \cap \mathrm{Nor}X} \sum_{|I|=d-1-k} \frac{\prod_{i \in I} \kappa_i(x,u)}{\prod_{i=1}^{d-1} \sqrt{1 + \kappa_i^2(x,u)}} \mathcal{H}^{d-1}(\mathrm{d}(x,u)),$$

where the sum runs over all subsets $I \subset \{1, \dots, d-1\}$ with $d-1-k$ elements, $\kappa_i(x,u)$, $i = 1, \dots, d-1$, are the generalized curvatures at (x,u) and $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure.

If ∂X is a regular hypersurface of class C^2, then (2.2) reduces to

$$\Phi_{k,r,s}(X) = \frac{\omega_{d-k}}{r!\, s!\, \omega_{d-k+s}} \int_{\partial X} x^r u(x)^s \Phi_k(X, \mathrm{d}x),$$

where $u(x)$ is the unique outer unit normal vector of X at x. For such smooth X

$$\Phi_k(X, A) = \frac{1}{\omega_{d-k}} \int_{A \cap \partial X} \sum_{|I|=d-1-k} \prod_{i \in I} \kappa_i(x)\, \mathcal{H}^{d-1}(\mathrm{d}x), \tag{2.3}$$

where $A \subseteq \mathbb{R}^d$ is a Borel set and $\kappa_i(x)$, $i = 1, \dots, d-1$, are the principal curvatures of X at x. For $k = d-1$, (2.3) reduces to

$$\Phi_{d-1}(X, A) = \tfrac{1}{2}\mathcal{H}^{d-1}(A \cap \partial X). \tag{2.4}$$

Furthermore, $\Phi_{d-1,r,0}(X) \propto \mathbb{E}(Y^r)$ where Y is a uniform random point on ∂X and $\Phi_{d-1,0,s}(X) \propto \mathbb{E}(U^s)$ where U is the unique outer unit normal vector at a uniform random point on ∂X. The surface tensor $\Phi_{d-1,0,s}$ has been studied in a stereological context in [17].

The tensors defined at (2.1) and (2.2) constitute the Minkowski tensors. They are tensor-valued valuations, continuous with respect to the Hausdorff metric and isometry covariant. For the definition of isometry covariance and further results on Minkowski tensors; see [6, 7] and references therein.

3 Integrated Minkowski tensors

In this section, we will introduce integrated Minkowski tensors for which a genuine rotational Crofton formula holds. These tensors are weighted averages of tensors defined on j-flats. Integrated Minkowski tensors appear to be the natural tool in the development of rotational integral geometry for Minkowski tensors. Also, there are some interesting new tensors in the class of integrated Minkowski tensors, which are not isometry covariant.

Any j-flat (j-dimensional affine subspace) F_j in $\mathbb{R}^d$ can be written as $F_j = x + L_j$ where L_j is a j-subspace, i.e. a j-dimensional linear subspace, and $x \in L_j^\perp$. By $d(F_j, O)$ we denote the distance of F_j to the origin. We have $d(F_j, O) = d(x + L_j, O) = |x|$, where $|\cdot|$ is the Euclidean norm. The element of the motion invariant measure on the space $\mathcal{F}_j^d$ of j-flats in $\mathbb{R}^d$ can be decomposed as $\mathrm{d}F_j^d = \lambda_{d-j}(\mathrm{d}x)\mathrm{d}L_j^d$ where $\mathrm{d}L_j^d$ is the element of the rotation invariant measure on $\mathcal{L}_j^d$, the set of j-subspaces in $\mathbb{R}^d$, and, for given $L_j \in \mathcal{L}_j^d$, $\lambda_{d-j}(\mathrm{d}x)$ is the element of the Lebesgue measure in $L_j^\perp$. The total mass of $\mathrm{d}L_j^d$ is chosen to be

$$\int_{\mathcal{L}_j^d} \mathrm{d}L_j^d = c_{d,j},$$

where

$$c_{d,j} = \frac{\omega_d \omega_{d-1} \dots \omega_{d-j+1}}{\omega_j \omega_{j-1} \dots \omega_1}.$$

Definition 3.1. *For* $0 \le k < j < d$, $t > j - d$ *or* $0 \le k < j = d$, $t \ge 0$ *and non-negative integers* r *and* s, *the integrated Minkowski tensors are*

$$\Phi_{k,r,s}^{j,t}(X) := \int_{\mathcal{F}_j^d} \Phi_{k,r,s}^{(F_j)}(X \cap F_j) d(F_j, O)^t \mathrm{d}F_j^d,$$

and

$$\Phi_{j,r,0}^{j,t}(X) := \int_{\mathcal{F}_j^d} \Phi_{j,r,0}^{(F_j)}(X \cap F_j) d(F_j, O)^t \mathrm{d}F_j^d,$$

where the integrands $\Phi_{k,r,s}^{(F_j)}(X \cap F_j)$ *and* $\Phi_{j,r,0}^{(F_j)}(X \cap F_j)$ *are calculated relative to* F_j; *cf. [7, Sections 2 and 3].*

The condition $t > j - d$ ensures that $\Phi_{k,r,s}^{j,t}(X)$ is well-defined. In fact, if $t > j - d$ then

$$\int_{\mathcal{F}_j^d} \mathbb{1}\{X \cap F_j \neq \emptyset\} d(F_j, O)^t \mathrm{d}F_j^d < \infty$$

and $\Phi_{k,r,s}(X \cap F_j)$ is uniformly bounded as a function of F_j.

Note that the classical Minkowski tensors are integrated Minkowski tensors with $t = 0$ and $j = d$. More generally, for $t = 0$ these integrated Minkowski tensors are simply averages of classical Minkowski tensors defined on j-flats. Such averages have been studied in depth in [7] where it is shown that for $t = 0$ the integrated Minkowski tensors are linear combinations of the classical Minkowski tensors. In particular, it follows for $t = 0$ and $0 \le k < j \le d - 1$ that, cf. [7, Theorem 2.4 and 2.5],

$$\Phi_{k,r,s}^{j,0}(X) = a_{d,j,k,s} \Phi_{d+k-j,r,s}(X), \quad s \in \{0, 1\}, \tag{3.1}$$

$$\Phi_{j,r,0}^{j,0}(X) = c_{d,j} \Phi_{d,r,0}(X). \tag{3.2}$$

The constant of proportionality $a_{d,j,k,s}$ is for $0 \le k < j \le d - 1$ and $s \in \{0, 1\}$ given by

$$a_{d,j,k,s} = c_{d-1,j-1} \frac{(j-1)!(d+k-j)!}{k!(d-1)!} \frac{\omega_{d+k-j+s+2}}{\omega_{k+s+2}}.$$

Note also that (3.1) with $s = 0$ and (3.2) are direct consequences of the Crofton formula for curvature measures; see [15, Theorem 5.3.3].

The integrated Minkowski tensors fulfil

$$\Phi_{k,r,s}^{j,t}(RX) = R\Phi_{k,r,s}^{j,t}(X), \quad \text{for all } R \in O_d,$$

where O_d is the orthogonal group in $\mathbb{R}^d$, and the right hand side involves the natural operation of O_d on the space of symmetric tensors of rank $r + s$. However, they are not isometry covariant in general. For instance, for $r = s = 0$,

$$\Phi_{k,0,0}^{j,t}(X) = \int_{\mathcal{F}_j^d} V_k(X \cap F_j) d(F_j, O)^t \mathrm{d}F_j^d$$

depends for $t \neq 0$ on the choice of the origin and is therefore not translation invariant. Note also that

$$\Phi_{k,r,s}^{j,t}(\alpha X) = \alpha^{k+r+t+d-j} \Phi_{k,r,s}^{j,t}(X),$$

so $\Phi_{k,r,s}^{j,t}$ is homogeneous of degree $k + r + t + d - j$.

4 A rotational Crofton formula

We will now show a genuine rotational Crofton formula for the integrated Minkowski tensors.

Proposition 4.1. *For $0 \le k < j < p \le d$, $t > j - d$ and non-negative integers r and s, we have*

$$\Phi_{k,r,s}^{j,t}(X) = \frac{1}{c_{d-j-1,p-j-1}} \int_{\mathcal{L}_p^d} \Phi_{k,r,s}^{j,d-p+t}(X \cap L_p) \mathrm{d}L_p^d. \tag{4.1}$$

For $j = k$, (4.1) holds for $s = 0$.

Proof. We use the following decomposition

$$\mathrm{d}F_j^d = \frac{d(F_j, O)^{d-p}}{c_{d-j-1,p-j-1}} \mathrm{d}F_j^p \, \mathrm{d}L_p^d, \tag{4.2}$$

$0 < j < p \le d$; see [15, p. 285]. We find

$$\begin{aligned}
\Phi_{k,r,s}^{j,t}(X) &= \int_{\mathcal{F}_j^d} \Phi_{k,r,s}^{(F_j)}(X \cap F_j) d(F_j, O)^t \mathrm{d}F_j^d \\
&= \frac{1}{c_{d-j-1,p-j-1}} \int_{\mathcal{L}_p^d} \int_{\mathcal{F}_j^p} \Phi_{k,r,s}^{(F_j)}(X \cap F_j) d(F_j, O)^{d-p+t} \mathrm{d}F_j^p \, \mathrm{d}L_p^d \\
&= \frac{1}{c_{d-j-1,p-j-1}} \int_{\mathcal{L}_p^d} \Phi_{k,r,s}^{j,d-p+t}(X \cap L_p) \mathrm{d}L_p^d.
\end{aligned}$$

The second statement is proved in exactly the same manner. □

Proposition 4.1 does not directly provide rotational formulae for the classical Minkowski tensors since the maximal possible value of j in Proposition 4.1 is $d - 1$. However, by combining Proposition 4.1 with equations (3.1) and (3.2), we can derive rotational Crofton formulae for all classical Minkowski tensors $\Phi_{k,r,s}$ with $s \in \{0, 1\}$. In this sense, the situation is not more complicated than in [7], where affine averages of classical Minkowski tensors are considered. Here, these affine averages are again classical Minkowski tensors only for $s \in \{0, 1\}$.

First, we derive a result for a rotational average of a classical Minkowski tensor.

Corollary 4.2. *For $s \in \{0, 1\}$ and $t = p - d$, the result in Proposition 4.1 reduces to*

$$\int_{\mathcal{L}_p^d} \Phi_{m,r,s}^{(L_p)}(X \cap L_p) \, \mathrm{d}L_p^d = \frac{c_{d-(p-q)-1,q-1}}{a_{p,p-q,m-q,s}} \Phi_{m-q,r,s}^{p-q,p-d}(X), \tag{4.3}$$

for $0 < q \le m < p \le d$.
If $m = p$ and $s = 0$, we get for $0 < q < p \le d$

$$\int_{\mathcal{L}_p^d} \Phi_{p,r,0}^{(L_p)}(X \cap L_p) \, \mathrm{d}L_p^d = \frac{c_{d-(p-q)-1,q-1}}{c_{p,p-q}} \Phi_{p-q,r,0}^{p-q,p-d}(X). \tag{4.4}$$

Proof. Combining Proposition 4.1 with equation (3.1), we find

$$\begin{aligned}\int_{\mathcal{L}_p^d} \Phi_{m,r,s}^{(L_p)}(X \cap L_p)\,\mathrm{d}L_p^d &= \frac{1}{a_{p,p-q,m-q,s}} \int_{\mathcal{L}_p^d} \Phi_{m-q,r,s}^{p-q,0}(X \cap L_p)\,\mathrm{d}L_p^d \\ &= \frac{c_{d-(p-q)-1,q-1}}{a_{p,p-q,m-q,s}}\, \Phi_{m-q,r,s}^{p-q,p-d}(X).\end{aligned}$$

The second statement is proved in exactly the same manner. □

Example 4.3. For $d = 3$ and $p = 2$, we obtain

$$\int_{\mathcal{L}_2^3} \Phi_{2,r,0}^{(L_2)}(X \cap L_2)\mathrm{d}L_2^3 = \frac{1}{\pi}\, \Phi_{1,r,0}^{1,-1}(X), \tag{4.5}$$

and

$$\int_{\mathcal{L}_2^3} \Phi_{1,r,s}^{(L_2)}(X \cap L_2)\mathrm{d}L_2^3 = \frac{1}{\sqrt{\pi}} \frac{\Gamma((s+3)/2)}{\Gamma((s+2)/2)}\, \Phi_{0,r,s}^{1,-1}(X), \tag{4.6}$$

for $s \in \{0, 1\}$. Explicit forms of the integrated Minkowski tensors appearing on the right-hand sides of (4.5) and (4.6) are given below in Example 5.2 and Example 5.4 for $s = 0$. ■

Note that for $r = s = 0$, the left-hand side of (4.3) takes the form of a rotational average of an intrinsic volume

$$\int_{\mathcal{L}_p^d} V_m(X \cap L_p)\mathrm{d}L_p^d, \quad m = 1, \ldots, p-1.$$

These rotational integrals have been studied in detail in [12, 14]. The main result in [12] is that under mild regularity conditions

$$\begin{aligned}\int_{\mathcal{L}_p^d} V_m(X \cap L_p)\,\mathrm{d}L_p^d = \frac{1}{\omega_{p-m}} \int_{\mathrm{Nor}X} |x|^{-(d-p)} \sum_{|I|=p-1-m} Q_p(x, u, A_I) \\ \times \frac{\prod_{i\in I} \kappa_i(x,u)}{\prod_{i=1}^{d-1} \sqrt{1+\kappa_i^2(x,u)}}\, \mathcal{H}^{d-1}(\mathrm{d}(x,u)),\end{aligned} \tag{4.7}$$

where the sum in (4.7) is over all subsets of $\{1, \ldots, d-1\}$ with $p-1-m$ elements and A_I is the subspace spanned by the principal directions $a_i(x, u)$ with $i \notin I$. The function Q_p is defined via an integral over all p-subspaces containing the line through the origin spanned by x. Later, in [14], it was shown that Q_p is a linear combination of hypergeometric functions. In Section 5 below, we extend the result (4.7) to the case of tensors with an arbitrary non-negative integer r and $s = 0$.

From an applied point of view, it is in fact more interesting to find the functional defined on the subspace L_p whose rotational average equals a given classical Minkowski tensor. This problem can again be solved for $s \in \{0, 1\}$ by combining Proposition 4.1 with equations (3.1) and (3.2).

Corollary 4.4. *For $s \in \{0,1\}$ and $t = 0$, the result in Proposition 4.1 reduces to*

$$\Phi_{d+m-p,r,s}(X) = \frac{1}{a_{d,p-q,m-q,s}\, c_{d-(p-q)-1,q-1}} \int_{\mathcal{L}_p^d} \Phi_{m-q,r,s}^{p-q,d-p}(X \cap L_p) \mathrm{d}L_p^d, \tag{4.8}$$

for $0 < q \leq m < p \leq d$.
If $m = p$ and $s = 0$, we get for $0 < q < p \leq d$

$$\Phi_{d,r,0}(X) = \frac{1}{c_{d,p-q}\, c_{d-(p-q)-1,q-1}} \int_{\mathcal{L}_p^d} \Phi_{p-q,r,0}^{p-q,d-p}(X \cap L_p) \mathrm{d}L_p^d. \tag{4.9}$$

Proof. Combining Proposition 4.1 with equation (3.1), we find

$$\begin{aligned}\int_{\mathcal{L}_p^d} \Phi_{m-q,r,s}^{p-q,d-p}(X \cap L_p) \mathrm{d}L_p^d &= c_{d-(p-q)-1,q-1} \Phi_{m-q,r,s}^{p-q,0}(X) \\ &= a_{d,p-q,m-q,s} c_{d-(p-q)-1,q-1}\, \Phi_{d+m-p,r,s}(X).\end{aligned}$$

The second statement is proved in exactly the same manner. □

Example 4.5. For $d = 3$ and $p = 2$, we obtain

$$\Phi_{3,r,0}(X) = \frac{1}{2\pi} \int_{\mathcal{L}_2^3} \Phi_{1,r,0}^{1,1}(X \cap L_2) \mathrm{d}L_2^3, \tag{4.10}$$

and

$$\Phi_{2,r,s}(X) = \frac{1}{2\pi}(s+2) \int_{\mathcal{L}_2^3} \Phi_{0,r,s}^{1,1}(X \cap L_2) \mathrm{d}L_2^3, \tag{4.11}$$

for $s \in \{0,1\}$. Explicit forms of the integrated Minkowski tensors appearing on the right-hand sides of (4.10) and (4.11) are given below in Example 5.2 and Example 5.6 for $s = 0$. ■

If we let $r = s = 0$ in Corollary 4.4, then (4.8) reduces to

$$V_{d+m-p}(X) = \frac{1}{a_{d,p-q,m-q,0}\, c_{d-(p-q)-1,q-1}} \int_{\mathcal{L}_p^d} \Phi_{m-q,0,0}^{p-q,d-p}(X \cap L_p) \mathrm{d}L_p^d.$$

In particular, for $q = 1$ we get for $m > 1$

$$V_{d+m-p}(X) = \frac{1}{a_{d,p-1,m-1,0}} \int_{\mathcal{L}_p^d} \Phi_{m-1,0,0}^{p-1,d-p}(X \cap L_p) \mathrm{d}L_p^d,$$

where

$$\Phi_{m-1,0,0}^{p-1,d-p}(X \cap L_p) = \int_{\mathcal{F}_{p-1}^p} V_{m-1}(X \cap F_{p-1}) d(F_{p-1}, O)^{d-p} \mathrm{d}F_{p-1}^p.$$

This is the main result in [13] where explicit expressions for $\Phi_{m-1,0,0}^{p-1,d-p}$ are given for $m = p, p-1$. In Section 5 below, we extend these results to $\Phi_{m-1,r,0}^{p-1,d-p}$ for an arbitrary non-negative integer r.

One of the advantages of using the integrated Minkowski tensors and the rotational Crofton formula that holds for these tensors is that, the problem of finding rotational averages of Minkowski tensors (Corollary 4.2) and the problem of finding functionals with rotational average equal to Minkowski tensors (Corollary 4.4) can be given a common formulation.

5 Explicit expressions for integrated Minkowski tensors

5.1 Volume tensors

In this subsection, we will indicate what kind of geometric information the integrated volume tensors carry about the original set X.

Proposition 5.1. *For $0 \le j \le d$, $t > j - d$ and r a non-negative integer,*

$$\Phi_{j,r,0}^{j,t}(X) = \frac{c_{d,j}}{r!} \frac{\Gamma(\frac{t+d-j}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{t+d}{2})\Gamma(\frac{d-j}{2})} \int_X x^r |x|^t \lambda_d(\mathrm{d}x).$$

Proof. Applying [11, Proposition 3.9] componentwise, we find

$$\begin{aligned}
\Phi_{j,r,0}^{j,t}(X) &= \int_{\mathcal{F}_j^d} \Phi_{j,r,0}^{(F_j)}(X \cap F_j) d(F_j, O)^t \mathrm{d}F_j^d \\
&= \frac{1}{r!} \int_{\mathcal{F}_j^d} \int_{X \cap F_j} x^r d(F_j, O)^t \lambda_j(\mathrm{d}x) \mathrm{d}F_j^d \\
&= \frac{1}{r!} \int_{\mathcal{L}_j^d} \int_{L_j^\perp} \int_{X \cap (y + L_j)} x^r |y|^t \lambda_j(\mathrm{d}x) \lambda_{d-j}(\mathrm{d}y) \mathrm{d}L_j^d \\
&= \frac{1}{r!} \int_{\mathcal{L}_j^d} \int_X x^r |p(x|L_j^\perp)|^t \lambda_d(\mathrm{d}x) \mathrm{d}L_j^d \\
&= \frac{1}{r!} \int_X x^r |x|^t \left[\int_{\mathcal{L}_j^d} \frac{|p(x|L_j^\perp)|^t}{|x|^t} \mathrm{d}L_j^d \right] \lambda_d(\mathrm{d}x) \\
&= \frac{c_{d,j}}{r!} \frac{\Gamma(\frac{t+d-j}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{t+d}{2})\Gamma(\frac{d-j}{2})} \int_X x^r |x|^t \lambda_d(\mathrm{d}x).
\end{aligned}$$

□

Example 5.2. For $d = 3$, $j = 1$, and $t = -1$, Proposition 5.1 gives an explicit expression for the integrated Minkowski tensor $\Phi_{1,r,0}^{1,-1}(X)$ appearing in (4.5). We find

$$\Phi_{1,r,0}^{1,-1}(X) = \frac{\pi^2}{r!} \int_X \frac{x^r}{|x|} \lambda_3(\mathrm{d}x).$$

For $d = 2$, $j = t = 1$, Proposition 5.1 gives an explicit expression for the integrated Minkowski tensor $\Phi_{1,r,0}^{1,1}(X \cap L_2)$ appearing in (4.10). We find

$$\Phi_{1,r,0}^{1,1}(X \cap L_2) = \frac{2}{r!} \int_{X \cap L_2} x^r |x| \lambda_2(\mathrm{d}x).$$

■

5.2 Surface tensors

In the proposition below, we give an explicit expression for the integrated Minkowski tensors

$$\Phi_{m-q,r,s}^{p-q,p-d}(X) \tag{5.1}$$

appearing in Corollary 4.2 in the case $m < p$. These tensors depend on a parameter q that can be chosen freely within the range $0 < q \leq m$. In the proposition below, we take $q = 1$. The result in Proposition 5.3 holds for $s = 0$ and is a generalization of the main result in [12].

Proposition 5.3. *Suppose that $O \notin \partial X$ and that for almost all $L_p \in \mathcal{L}_p^d$,*

$$(x, u) \in \operatorname{Nor} X,\ x \in L_p \Rightarrow u \not\perp L_p.$$

Then for $1 \leq m < p \leq d$, we have

$$\begin{aligned}\int_{\mathcal{L}_p^d} \Phi_{m,r,0}^{(L_p)}(X \cap L_p)\, \mathrm{d}L_p^d &= \frac{1}{a_{p,p-1,m-1,0}} \Phi_{m-1,r,0}^{p-1,p-d}(X) \\ &= \frac{1}{r!\omega_{p-m}} \int_{\operatorname{Nor} X} \frac{x^r}{|x|^{d-p}} \sum_{|I|=p-1-m} Q_p(x, u, A_I) \\ &\qquad \times \frac{\prod_{i\in I} \kappa_i(x, u)}{\prod_{i=1}^{d-1} \sqrt{1 + \kappa_i^2(x, u)}} \mathcal{H}^{d-1}(\mathrm{d}(x, u)),\end{aligned}$$

where the sum runs over all subsets I of $\{1, \ldots, d-1\}$ with $p-1-m$ elements and Q_p is explicitly given in [14].

Proof. Under the stated regularity conditions, we can apply a local version of the main result in [12, p. 558]. We have for any integrable function h

$$\begin{aligned}&\omega_{p-m} \int_{\mathcal{L}_p^d} \int_{L_p} h(x)\, \Phi_m^{(L_p)}(X \cap L_p, \mathrm{d}x)\, \mathrm{d}L_p^d \\ &\quad = \int_{\operatorname{Nor} X} \frac{h(x)}{|x|^{d-p}} \sum_{|I|=p-1-m} Q_p(x, u, A_I) \frac{\prod_{i\in I} \kappa_i(x, u)}{\prod_{i=1}^{d-1} \sqrt{1 + \kappa_i^2(x, u)}} \mathcal{H}^{d-1}(\mathrm{d}(x, u)).\end{aligned}$$

Since

$$\int_{\mathcal{L}_p^d} \Phi_{m,r,0}^{(L_p)}(X \cap L_p)\, \mathrm{d}L_p^d = \frac{1}{r!} \int_{\mathcal{L}_p^d} \int_{L_p} x^r \Phi_m^{(L_p)}(X \cap L_p, \mathrm{d}x)\, \mathrm{d}L_p^d,$$

the result of the proposition follows by choosing each element of x^r restricted to X as $h(x)$. □

Example 5.4. Using Proposition 5.3 with $d = 3$, $p = 2$ and $m = 1$, we obtain for $s = 0$ an explicit expression for the integrated Minkowski tensor $\Phi_{0,r,s}^{1,-1}(X)$ appearing in (4.6). We have $I = \emptyset$, $A_I = u^\perp$ and

$$Q_p(x, u, A_I) = \pi F(-\tfrac{1}{2}, \tfrac{1}{2}; 1; \sin^2 \angle(x, u)) = 2E(|\sin \angle(x, u)|, \tfrac{\pi}{2}),$$

where $\angle(x, u) \in [0, \pi]$ is the angle between x and u, $F(a, b; c; z)$ is the hypergeometric function defined at [18, eq. 15.2.1], and $E(\varphi, k)$ is Legendre's incomplete elliptic integral of second kind; see [18, eq. 19.2.5]. See also [12, p. 541]. It follows from Proposition 5.3 that, when X is smooth,

$$\Phi_{0,r,0}^{1,-1}(X) = \frac{2}{r!} \int_{\partial X} \frac{x^r}{|x|} E(|\sin \angle(x, u)|, \tfrac{\pi}{2}) \mathcal{H}^2(\mathrm{d}x). \qquad \blacksquare$$

In Corollary 4.4, we are interested in an explicit expression for

$$\Phi^{p-q,d-q}_{m-q,r,s}(X \cap L_p), \tag{5.2}$$

where again q is an integer that can be chosen freely in the range $0 < q \le m$. The difference between (5.1) and (5.2) is that the t-parameter is non-positive in the former, whereas it is positive in the latter. The case $m = p$ in equation (5.2) has been treated in the previous subsection, so we focus here on the case $m < p$. It suffices to find an explicit expression for $\Phi^{d-1,t}_{m-1,r,s}(X)$, where t is a positive integer. In the proposition below, we give the solution for $m = d - 1$ and $s = 0$.

Proposition 5.5. *For a non-negative integer r and a positive integer t*

$$\begin{aligned}\Phi^{d-1,t}_{d-2,r,0}(X) &= \frac{\omega_{d-1}}{2r!}\frac{\Gamma(\frac{t+1}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{t+1+d}{2})}\\ &\times \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} x^r|x|^t F\left(-\tfrac{t}{2},-\tfrac{1}{2};\tfrac{d-1}{2};\sin^2\angle(x,u)\right)\Lambda_{d-1}(X,\mathrm{d}(x,u)).\end{aligned} \tag{5.3}$$

Proof. Suppose first that X is a regular hypersurface of class C^2. Using [11, Proposition 2.10] componentwise, we find

$$\begin{aligned}\Phi^{d-1,t}_{d-2,r,0}(X) &= \int_{\mathcal{F}^d_{d-1}} \Phi^{(F_{d-1})}_{d-2,r,0}(X\cap F_{d-1})d(F_{d-1},O)^t\mathrm{d}F^d_{d-1}\\ &= \frac{1}{2r!}\int_{\mathcal{F}^d_{d-1}}\int_{\partial X\cap F_{d-1}} x^r d(F_{d-1},O)^t\mathcal{H}^{d-2}(\mathrm{d}x)\mathrm{d}F^d_{d-1}\\ &= \frac{1}{r!}\int_{\mathcal{L}^d_{d-1}}\int_{L^\perp_{d-1}}\int_{\partial X\cap(L_{d-1}+y)} x^r|y|^t\mathcal{H}^{d-2}(\mathrm{d}x)\lambda_1(\mathrm{d}y)\mathrm{d}L^d_{d-1}\\ &= \frac{1}{r!}\int_{\mathcal{L}^d_{d-1}}\int_{L^\perp_{d-1}}\int_{\partial X\cap(L_{d-1}+y)} x^r|p(x|L^\perp_{d-1})|^t\mathcal{H}^{d-2}(\mathrm{d}x)\lambda_1(\mathrm{d}y)\mathrm{d}L^d_{d-1}\\ &= \frac{1}{2r!}\int_{\mathcal{L}^d_{d-1}}\int_{\partial X} x^r|p(x|L^\perp_{d-1})|^t|p(u(x)|L_{d-1})|\mathcal{H}^{d-1}(\mathrm{d}x)\mathrm{d}L^d_{d-1}\\ &= \frac{1}{2r!}\int_{\partial X} x^r\left[\int_{\mathcal{L}^d_{d-1}}|p(x|L^\perp_{d-1})|^t|p(u(x)|L_{d-1})|\mathrm{d}L^d_{d-1}\right]\mathcal{H}^{d-1}(\mathrm{d}x).\end{aligned}$$

Using [13, Proposition 4] on the inner integral, we find

$$\begin{aligned}&\Phi^{d-1,t}_{d-2,r,0}(X)\\ &\quad= \frac{\omega_{d-1}}{4r!}\frac{\Gamma(\frac{t+1}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{t+1+d}{2})}\int_{\partial X} x^r|x|^t F\left(-\tfrac{t}{2},-\tfrac{1}{2};\tfrac{d-1}{2};\sin^2\angle(x,u(x))\right)\mathcal{H}^{d-1}(\mathrm{d}x).\end{aligned}$$

By (2.4) this yields the claim for smooth X. The quantities appearing on both sides of equation (5.3) are continuous with respect to the Hausdorff metric; see [15, Theorem 14.2.2]. Any convex compact set X can be approximated by smooth convex compact sets, hence the result follows. □

Using Proposition 5.5, we find for smooth X, that

$$\Phi_{p-2,0,0}^{p-1,d-p}(X \cap L_p) = \frac{\omega_{p-1}}{4} \frac{\Gamma(\frac{d-p+1}{2})\Gamma(\frac{p}{2})}{\Gamma(\frac{d+1}{2})} \times \int_{\partial(X\cap L_p)} |x|^{d-p} F\left(-\tfrac{d-p}{2}, -\tfrac{1}{2}; \tfrac{p-1}{2}; \sin^2 \angle(x, u(x))\right) \mathcal{H}^{p-1}(\mathrm{d}x).$$

This is one of the main results in [13, p. 6].

Example 5.6. Using Proposition 5.5 for $d = 2$ and $t = 1$, we obtain for $s = 0$ an explicit expression for the integrated Minkowski tensor $\Phi_{0,r,s}^{1,1}(X \cap L_2)$ appearing in (4.11). We find

$$\Phi_{0,r,0}^{1,1}(X \cap L_2) = \frac{1}{r!} \int_{\mathbb{R}^2 \times \mathbb{S}^1} x^r |x| F(-\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}; \sin^2 \angle(x, u)) \Lambda_1(X \cap L_2, \mathrm{d}(x, u)).$$

Here,

$$F(-\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}; \sin^2 \gamma) = \cos\gamma + \gamma \sin\gamma;$$

cf. [11, Example 5.10].

6 Applications to stereological particle analysis

In this section, we will briefly discuss how the new geometric identities can be applied in the stereological analysis of particle populations. The geometric identities are all of the form

$$\int_{\mathcal{L}_p^d} \alpha(X \cap L_p) \mathrm{d}L_p^d = \beta(X), \tag{6.1}$$

where either α or β may be a Minkowski tensor; see Corollaries 4.2 and 4.4. In the case where β is a Minkowski tensor, the geometric identity gives the measurement α to be determined in the section $X \cap L_p$ in order to estimate the Minkowski tensor specified by β.

These results can be used to estimate the distribution of a Minkowski tensor in a particle population from sectional data, thereby providing information about the orientation and shape of the particles. Let us assume that the particles are a realization of a marked point process $\Psi = \{[x_i; \Xi_i]\}$ where the x_is are the points in $\mathbb{R}^d$ and the marks Ξ_i are convex and compact subsets of $\mathbb{R}^d$. For simplicity, we assume that the marks are independent and identically distributed according to some probability distribution P_m on the space, $\mathcal{K}^d$, of convex, compact subsets of $\mathbb{R}^d$. The ith particle of the process is represented by $X_i = x_i + \Xi_i$. Let Ξ_0 have distribution P_m.

Our aim is to estimate the distribution of $\beta(\Xi_0)$ from sectional data. Available for observation is a sample of particles $\{x_i + \Xi_i : x_i \in W\}$ collected in a sampling window. It is possible to perform measurements on any virtual section $\Xi_i \cap L_p$. If L_p is an isotropic section, then

$$E(\alpha(\Xi_i \cap L_p)|\Xi_i) = \int_{\mathcal{L}_p} \alpha(\Xi_i \cap L_p) \frac{\mathrm{d}L_p^d}{c_{d,p}} = \frac{1}{c_{d,p}} \beta(\Xi_i).$$

The distribution of $\beta(\Xi_0)$ can now be estimated by the empirical distribution of $\{\hat{\beta}(\Xi_i) : x_i \in W\}$, where

$$\hat{\beta}(\Xi_i) = \frac{c_{d,p}}{N} \sum_{j=1}^{N} \alpha(\Xi_i \cap L_{p,j})$$

and $L_{p,j}$, $j = 1, \ldots, N$, are replicated virtual isotropic sections. It is worth noting that it may not be informative to estimate the mean tensor of a particle population. For instance, for any isotropic distribution P_m the tensor $E\beta(\Xi_0)$ will be a multiple of Q, which does not contain information about the shape of Ξ_0.

7 Rank two tensors in $\mathbb{R}^3$

Let us summarize the results obtained for tensors of rank two in $\mathbb{R}^3$. It is known that a basis of all continuous, isometry covariant tensor valuations of rank two in $\mathbb{R}^3$ is

$$\begin{aligned} &QV_j, && j = 0, 1, 2, 3, \\ &\Phi_{j,2,0}, && j = 0, 1, 2, 3, \\ &\Phi_{j,0,2}, && j = 1, 2, \end{aligned} \tag{7.1}$$

where Q is the metric tensor; see e.g. [7, p. 485]. For $d = 3$ and $p = 2$, Corollary 4.4 provides identities of the form (6.1) for the following elements of the basis: QV_2, QV_3, $\Phi_{2,2,0}$ and $\Phi_{3,2,0}$. The explicit formulae for these identities are given in the first four rows of Table 1. The remaining tensors in the basis (7.1) depend on local curvatures except for $\Phi_{2,0,2}$. They do not appear to be accessible on sections passing through the origin; see also Section 8 below. However, as we shall see now, it turns out to be possible to derive an identity of the form (6.1) for $\Phi_{2,0,2}$.

By Alesker's theorem [9, Theorem 2.2] and homogeneity considerations we find that

$$\int_{\mathcal{F}_1^3} \Phi_{0,0,2}^{(F_1)}(X \cap F_1) \mathrm{d}F_1^3 = \beta_1 QV_2(X) + \beta_2 \Phi_{0,2,0}(X) + \beta_3 \Phi_{2,0,2}(X)$$

for some coefficients $\beta_i \in \mathbb{R}$, $i = 1, 2, 3$. The coefficients turn out to be $\beta_1 = 1/16$, $\beta_2 = 0$, and $\beta_3 = \pi/4$. We derived them solving the system of linear equations resulting from Examples 7.1 and 7.2 below. In principle, the coefficients are also given in [7], but we found it instructive to determine them through explicit examples. By (4.2) we have

$$\int_{\mathcal{F}_1^3} \Phi_{0,0,2}^{(F_1)}(X \cap F_1) \mathrm{d}F_1^3 = \int_{\mathcal{L}_2^3} \int_{\mathcal{F}_1^2} \Phi_{0,0,2}^{(F_1)}(X \cap F_1) d(F_1, O) \mathrm{d}F_1^2 \mathrm{d}L_2^3,$$

and hence the formula in the last row of Table 1 follows using the first line of Table 1.

Example 7.1. If $X = \mathbb{B}^3$ is the unit ball in $\mathbb{R}^3$, then $V_2(X) = 2\pi$, $\Phi_{0,2,0}(X) = \Phi_{2,0,2}(X) = (1/6)Q$, and

$$\int_{\mathcal{F}_1^3} \Phi_{0,0,2}^{(F_1)}(X \cap F_1) \mathrm{d}F_1^3 = \frac{\pi}{6} Q. \qquad \blacksquare$$

Table 1: Rotational integral expressions for Minkowski tensors of rank two in $\mathbb{R}^3$. Here, $\beta(X) = \int_{\mathcal{L}_2^3} \alpha(X \cap L_2) dL_2^3$.

$\beta(X)$	$\alpha(X \cap L_2)$
$QV_2(X)$	$\frac{1}{\pi} \int_{\mathcal{F}_1^2} QV_0(X \cap F_1) d(F_1, O) \mathrm{d}F_1^2$
$QV_3(X)$	$\frac{1}{2\pi} \int_{\mathcal{F}_1^2} QV_1(X \cap F_1) d(F_1, O) \mathrm{d}F_1^2$
$\Phi_{2,2,0}(X)$	$\frac{1}{\pi} \int_{\mathcal{F}_1^2} \Phi_{0,2,0}^{(F_1)}(X \cap F_1) d(F_1, O) \mathrm{d}F_1^2$
$\Phi_{3,2,0}(X)$	$\frac{1}{2\pi} \int_{\mathcal{F}_1^2} \Phi_{1,2,0}^{(F_1)}(X \cap F_1) d(F_1, O) \mathrm{d}F_1^2$
$\Phi_{2,0,2}(X)$	$\int_{\mathcal{F}_1^2} \Big(\frac{4}{\pi} \Phi_{0,0,2}^{(F_1)}(X \cap F_1) - \frac{1}{4\pi^2} QV_0(X \cap F_1) \Big) d(F_1, O) \mathrm{d}F_1^2$

Example 7.2. Let $X = L_0 \cap \mathbb{B}^3$, where L_0 is the plane spanned by the first two standard basis vectors e_1, e_2 of $\mathbb{R}^3$. So X is a 2-dimensional unit disk in $\mathbb{R}^3$ with normal e_3. We obtain $V_2(X) = \pi$, $\Phi_{0,2,0}(X) = (1/4)(Q - e_3^2)$, $\Phi_{2,0,2}(X) = (1/4)e_3^2$, and

$$\int_{\mathcal{F}_1^3} \Phi_{0,0,2}^{(F_1)}(X \cap F_1) \mathrm{d}F_1^3 = \frac{\pi}{16}(Q + e_3^2). \qquad \blacksquare$$

8 Discussion

In the present paper, we have derived new rotational Crofton formulae for Minkowski tensors. The dimension of the rotating subspace appearing in these formulae is 2 or higher, see Section 4. In the case of volume tensors, it is also possible to derive a rotational Crofton formula, involving rotating lines. Using polar decomposition in $\mathbb{R}^d$, we have

$$\int_{\mathcal{L}_1^d} \alpha(X \cap L_1) \mathrm{d}L_1^d = \beta(X), \tag{8.1}$$

with

$$\beta(X) = \Phi_{d,r,0}(X)$$

and

$$\alpha(X \cap L_1) = \frac{1}{r!} \int_{X \cap L_1} x^r |x|^{d-1} \lambda_1(\mathrm{d}x).$$

As explained in Section 6, the measurement α can be used to estimate the functional β. A natural question to ask is whether α is unique. This is indeed the case in (8.1) if α is rotation invariant; cf. [19]. Whether this holds for the general geometric identity (6.1) with $p > 1$ is the object of future research.

In Section 7, we have studied rank two tensors in $\mathbb{R}^3$ and found that QV_2, QV_3, $\Phi_{2,2,0}$, $\Phi_{3,2,0}$ and $\Phi_{2,0,2}$ can be expressed as rotational averages with respect to planes. The remaining rank two tensors in the basis (7.1) are QV_0, QV_1, $\Phi_{0,2,0}$, $\Phi_{1,2,0}$, and $\Phi_{1,0,2}$. For convex bodies, which is the set class considered in the present paper, QV_0 is identically equal to the metric tensor Q and therefore not of interest in relation to describing a convex body. The other rank two tensors depend on local curvatures and do not appear to be accessible on sections passing through the origin. Modified functionals may, however, be accessible. Let $X \subset \mathbb{R}^3$ be convex and compact with smooth boundary. If we consider more general functionals of the form

$$\widetilde{V}_1(X) = \frac{1}{\pi} \int_{\partial X} \sum_{i=1}^{2} w_i(x)\kappa_i(x)\mathcal{H}^2(\mathrm{d}x),$$

instead of

$$V_1(X) = \frac{1}{\pi} \int_{\partial X} \tfrac{1}{2}[\kappa_1(x) + \kappa_2(x)]\mathcal{H}^2(\mathrm{d}x),$$

where $w_i(x)$, $i = 1, 2$, are weight functions, summing to 1, then the weight functions may be chosen such that $\widetilde{V}_1$ is accessible on sections through the origin, cf. [12, p. 547–548]. Note that

$$\widetilde{V}_1(X) = V_1(X) + \frac{1}{\pi} \int_{\partial X} \left(w_1(x) - \tfrac{1}{2}\right) \Delta\kappa(x)\, \mathcal{H}^2(\mathrm{d}x),$$

where $\Delta\kappa(x) = \kappa_1(x) - \kappa_2(x)$ is the deviatoric curvature at $x \in \partial X$; see also [20]. It would be interesting to investigate whether a similar route could be pursued for the Minkowski tensors mentioned above.

A second type of integrals that are of vital interest in integral geometry besides Crofton integrals are kinematic integrals. In rotational integral geometry the analogous notion to kinematic integrals are principle rotational integrals. To the best of our knowledge, a principal rotational formula is still not available in the literature. Focusing on intrinsic volumes, such a formula involves integrals of the form

$$\int_{SO_d} V_k(X \cap \rho Y)\nu(\mathrm{d}\rho), \tag{8.2}$$

$k = 0, \dots, d$, where SO_d is the special orthogonal group in $\mathbb{R}^d$, X and Y are convex and compact subsets of $\mathbb{R}^d$, and ν is the unique rotation invariant probability measure on SO_d. From an applied point of view such a formula is interesting. Here, X is the unknown spatial structure of interest while Y is a known *test set* constructed by the observer. The aim is to get information about X from observation of the intersection of X with a randomly rotated version of Y. For $k = d$, (8.2) is, up to a known constant, equal to

$$\int_0^\infty r^{-(d-1)}\mathcal{H}^{d-1}(X \cap r\mathbb{S}^{d-1})\mathcal{H}^{d-1}(Y \cap r\mathbb{S}^{d-1})\,\mathrm{d}r.$$

A result of a similar form involving two terms can be obtained for $k = d - 1$.

Acknowledgements

The authors want to sincerely thank Markus Kiderlen for many fruitful discussions. This research has been supported by the Centre for Stochastic Geometry and Advanced Bioimaging, funded by the Villum Foundation.

References

[1] C. Beisbart, R. Dahlke, K. R. Mecke, H. Wagner, Vector- and tensor-valued descriptors for spatial patterns, in: Morphology of Condensed Matter, Vol. 600 of Lecture Notes in Physics, Springer, Berlin, 2002, pp. 249–271.

[2] G. E. Schröder-Turk, W. Mickel, S. C. Kapfer, M. A. Klatt, F. M. Schaller, M. J. F. Hoffmann, N. Kleppmann, P. Armstrong, A. Inayat, D. Hug, M. Reichelsdorfer, W. Peukert, W. Schwieger, K. Mecke, Minkowski tensor shape analysis of cellular, granular and porous structures, Adv. Mater. 23 (2011) 2535–2553.

[3] G. E. Schröder-Turk, S. C. Kapfer, B. Breidenbach, C. Beisbart, K. Mecke, Tensorial Minkowski functionals and anisotropy measures for planar patterns, J. Microsc. 238 (2011) 57–74.

[4] C. Beisbart, M. S. Barbosa, H. Wagner, L. da F. Costa, Extended morphometric analysis of neuronal cells with Minkowski valuations, Eur. Phys. J. B 52 (2006) 531–546.

[5] R. J. Prokop, A. P. Reeves, A survey of moment-based techniques for unoccluded object representation and recognition, Graph. Model. Im. Proc. 5 (1992) 438–460.

[6] R. Schneider, R. Schuster, Tensor valuations on convex bodies and integral geometry, ii, Rend. Circ. Mat. Palermo (2) Suppl. 70 (2002) 295–314.

[7] D. Hug, R. Schneider, R. Schuster, Integral geometry of tensor valuations, Adv. Appl. Math. 41 (2008) 482–509.

[8] S. Alesker, Continuous rotation invariant valuations on convex sets, Ann. Math. 149 (1999) 977–1005.

[9] S. Alesker, Description of continuous isometry covariant valuations on convex sets, Geom. Dedicata 74 (1999) 241–248.

[10] D. Hug, R. Schneider, R. Schuster, The space of isometry covariant tensor valuations, St. Petersburg Math. J. 19 (2008) 137–158.

[11] E. B. V. Jensen, Local Stereology, World Scientific, London, 1998.

[12] E. B. V. Jensen, J. Rataj, A rotational integral formula for intrinsic volumes, Adv. Appl. Math. 41 (2008) 530–560.

[13] J. Auneau, E. B. V. Jensen, Expressing intrinsic volumes as rotational integrals, Adv. Appl. Math. 45 (2010) 1–11.

[14] J. Auneau-Cognacq, J. Rataj, E. B. V. Jensen, Closed from of the rotational Crofton formula, Math. Nachr. 285 (2012) 164–180.

[15] R. Schneider, W. Weil, Stochastic and Integral Geometry, Springer, Berlin, 2008.

[16] R. Schneider, Convex bodies: the Brunn-Minkowski Theory., Cambridge University Press, Cambridge, 1993.

[17] R. Schneider, R. Schuster, Particle orientation from section stereology, Rend. Circ. Mat. Palermo (2) Suppl. 77 (2006) 623–633.

[18] Digital Library of Mathematical Functions, Release date 2011-08-29., National Institute of Standards and Technology from `http://dlmf.nist.gov/`, 2011.

[19] M. Kiderlen, E. B. V. Jensen, On uniqueness of rotational formulae, In preparation.

[20] H. Hansen-Goos, K. Mecke, Tensorial density functional theory for non-spherical hard-body fluids, J. Phys.-Condens. Matter 22 (2010) 1–16.