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Abstract

This paper analyzes the first order behavior (that is, the right sided derivative) of the volume of the dilation $A \oplus tQ$ as t converges to zero. Here A and Q are subsets of n-dimensional Euclidean space, A has finite perimeter and Q is finite. If Q consists of two points only, x and $x+u$, say, this derivative coincides up to sign with the directional derivative of the covariogram of A in direction u. By known results for the covariogram, this derivative can therefore be expressed by the cosine transform of the surface area measure of A. We extend this result to finite sets Q and use it to determine the derivative of the contact distribution function with finite structuring element of a stationary random set at zero. The proofs are based on approximation of the characteristic function of A by smooth functions of bounded variation and showing corresponding formulas for them.

Keywords: bounded variation; contact distribution function; dilation volume; directional variation; sets of finite perimeter; stationary random set; surface area measure

1 Introduction

Assume that $A \subset \mathbb{R}^n$ has regular boundary in the sense that the $(n-1)$-dimensional Hausdorff measure of its boundary, $\mathcal{H}^{n-1}(\partial A)$, is finite and that for $\mathcal{H}^{n-1}$ almost all $a \in \partial A$, there exists a unique outer unit normal vector $\nu_A(a) \in S^{n-1}$. This is the case e.g. if A is a topologically regular convex or polyconvex set, n-dimensional compact Lipschitz manifold with boundary or a “full-dimensional $\mathcal{U}_{\rm PR}$ set” ([18]). Then, the surface area measure of A is defined naturally as

$$S_{n-1}(A;\,\cdot\,) = \mathcal{H}^{n-1}\{a \in \partial A :\, \nu_A(a) \in \,\cdot\,\}.$$

The surface area measure is an important quantity in stochastic geometry and its estimation is a frequent task. Various integral formulas are used in this context. It is well-known that the intersection density of ∂A with lines of direction $u \in S^{n-1}$ is

$$\int_{S^{n-1}} |u \cdot v|\, S_{n-1}(A; dv),$$

and that these integrals (called cosine transform) determine only the symmetrized form of the surface area measure, $S_{n-1}(A;\,\cdot\,) + S_{n-1}(-A;\,\cdot\,)$. The cosine transform appears also in the directional derivative of the covariogram of A,

$$C(A, y) = \lambda_n(A \cap (A + y)), \qquad y \in \mathbb{R}^n,$$

(λ_n is Lebesgue-measure in $\mathbb{R}^n$), as

$$\lim_{r\to 0+} \frac{C(A, ru) - C(A, 0)}{r} = -\frac{1}{2} \int_{S^{n-1}} |u \cdot v|\, S_{n-1}(A; dv), \tag{1.1}$$

when $u \in S^{n-1}$ and A has finite volume. This was shown by Matheron [15] for convex bodies and extended considerably by Galerne [10].

Note that the covariogram can be expressed by means of dilation volumes with two-point test sets, namely

$$C(A, y) = 2\lambda_n(A) - \lambda_n(A \oplus \{0, y\}).$$

A natural extension is to consider the dilation volume $\lambda_n(A \oplus Q)$ for a general compact test set $Q \subset \mathbb{R}^n$. Generalizing (1.1), we have

$$\lim_{r\to 0+} \frac{\lambda_n(A \oplus rQ) - \lambda_n(A)}{r} = \int_{S^{n-1}} h(Q, v)\, S_{n-1}(A; dv), \tag{1.2}$$

where $h(Q,\,\cdot\,)$ is the support function of conv Q. This was shown in [13, Corollary 2] under the assumption that A is a compact gentle set. Besides a technical regularity condition this means that for $\mathcal{H}^{n-1}$-almost all points $a \in \partial A$ there are non-degenerate osculating balls containing a, one completely contained in A and the other in the closure of A^C. While the right hand side of (1.1) (known for all u) determines only the symmetrized form of the surface area measure, the right hand side of (1.2) determines $S_{n-1}(A;\cdot)$ itself, when the integrals are known for all sets Q that are congruent to a fixed triangle having at least one angle that is an irrational multiple of π. This was shown by Schneider [20]; see also [21, p. 283 and (5.1.18)]. In particular, for the determination of $S_{n-1}(A;\,\cdot\,)$ it is enough to know the right hand side of (1.2) for all three-point test sets Q; cf. [18] for a related result.

Although the class of gentle sets is reasonably large (it contains for instance all topologically regular polyconvex sets) this condition for the derivation of (1.2) seems to be rather artificial and its purpose is to make the proofs work. A different approach is based on the theory of sets with finite perimeter which are, by definition, sets $A \subset \mathbb{R}^n$ whose indicator function $\mathbf{1}_A$ has distributional derivative representable as a Radon measure $D\mathbf{1}_A$. (In other words, $\mathbf{1}_A$ has bounded variation.) The notion of sets with finite perimeter goes back to Caccioppoli [3] and De Giorgi [5, 6, 7]. We note that (poly)convex sets, compact $\mathcal{U}_{\mathrm{PR}}$-sets as well as compact gentle sets, or compact Lipschitz domains are sets of finite perimeter, simply as any set whose boundary has finite $\mathcal{H}^{n-1}$-measure has also finite perimeter.

In the following we describe, how the notion of surface area measure can be extended to sets of finite perimeter. The *essential boundary* $\partial^* A$ of a set A is the set of points in $\mathbb{R}^n$ that are neither Lebesgue density points of A nor of its complement.

If A is a set of finite perimeter, then the variation (scalar) measure $|D\mathbf{1}_A|$ can be written as a restriction of the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ in the form

$$|D\mathbf{1}_A| = \mathcal{H}^{n-1}\llcorner(\partial^* A), \tag{1.3}$$

[2, (3.63)], and the perimeter $P(A) = |D\mathbf{1}_A|(\mathbb{R}^n)$ equals $\mathcal{H}^{n-1}(\partial^* A)$. In the case where A has Lipschitz boundary, we have $\partial A = \partial^* A$ and $P(A)$ coincides with the usual surface area of A.

For a general set A with finite perimeter, the distributional derivative $D\mathbf{1}_A$ can be decomposed as

$$D\mathbf{1}_A = \Delta_{\mathbf{1}_A}\,|D\mathbf{1}_A|;$$

see (2.4), below. The density $\Delta_{\mathbf{1}_A}$ is an S^{n-1}-valued function defined $\mathcal{H}^{n-1}$-almost everywhere on $\partial^* A$ and can be interpreted as a generalized inner unit normal vector field to A. (In fact there exists a subset of $\partial^* A$ of full $\mathcal{H}^{n-1}$ measure called *reduced boundary* and a representative ν_A of $-\Delta_{1_A}$ defined there such that the half-space $\{y:\ y\cdot\nu_A(a)\le 0\}$ coincides with the approximate tangent cone of A at a for any a from the reduced boundary, see [2, §3.5].) Thus, it is natural to define the *generalized surface area measure* of a set A with finite perimeter as

$$S^*_{n-1}(A;\,\cdot\,) = \mathcal{H}^{n-1}\{a\in\partial^* A:\ -\Delta_{\mathbf{1}_A}(a)\in\,\cdot\,\}. \tag{1.4}$$

Clearly, $S^*_{n-1}(A;\,\cdot\,)$ coincides with $S_{n-1}(A;\,\cdot\,)$ if A has Lipschitz boundary.

Sets with finite perimeter have already appeared in the context of stochastic geometry. Villa [23] considered the (outer) Minkowski content and the spherical contact distribution function of inhomogeneous Boolean models with grains that have finite perimeter. The second author considered in [19] random sets of finite perimeter and established, among other things, a Crofton formula for these. Galerne and Lachièze-Rey [11] developed a theory of random measurable (not necessarily closed) sets and discussed the covariogram realizability problem in this framework. Their paper is based on an earlier one by Galerne [10], who showed an extension of the formula (1.1) for sets with finite volume and finite perimeter, namely

$$\lim_{r\to 0+}\frac{C(A,ru)-C(A,0)}{r} = -\frac{1}{2}\int_{S^{n-1}}|u\cdot v|\,S^*_{n-1}(A;dv), \tag{1.5}$$

and applied it to random sets.

Our main result is an analogous extension of (1.2) for the case of finite sets Q:

Theorem 1.1. *Assume that $A\subset\mathbb{R}^n$ has finite perimeter. If $\emptyset\ne Q\subset\mathbb{R}^n$ is finite then*

$$\lim_{r\to 0+}\frac{\lambda_n((A\oplus rQ)\setminus A)}{r} = \int h(Q,v)^+\,S^*_{n-1}(A;dv). \tag{1.6}$$

If, in addition, A has bounded volume then also

$$\lim_{r\to 0+}\frac{\lambda_n(A\oplus rQ)-\lambda_n(A)}{r} = \int h(Q,v)\,S^*_{n-1}(A;dv). \tag{1.7}$$

We show in Example 4.3 that the result is no longer true if we allow Q to be infinite, even if Q is countable and compact.

The case when Q is an n-dimensional convex body was considered by Chambolle et al. [4]. They showed that (1.7) is true whenever it holds for $Q = B(0,1)$ (which, however, need not be true). They also proved the convergence in (1.7) in a weaker sense (Γ-convergence) for any n-dimensional convex body Q. Related results for special sets A can be found in [14].

Extending or complementing corresponding results in [23] and [10], we conclude with an application of Theorem 1.1 for the contact distribution of stationary random sets. Recall that for a stationary random *closed* set $Z \subset \mathbb{R}^n$ (in the sense of Matheron; see, e.g. [22]) with volume fraction $\overline{p} = \Pr(0 \in Z)$, the contact distribution function of Z with compact structuring element $Q \subset \mathbb{R}^n$ is defined by

$$H_Q(r) = \Pr(Z \cap rQ \neq \emptyset \mid 0 \notin Z), \quad r \geq 0.$$

We will derive a formula for the one-sided derivative of H_Q at zero when Q is finite. The framework of sets with finite perimeter seems to be particularly well-suited for this problem, as the result does not require any of the usual integrability assumptions. In addition, it even holds for the more general class of random measurable sets (RAMS) introduced in [11].

A RAMS is a random element from the space of Lebesgue measurable subsets of $\mathbb{R}^n$ modulo differences of Lebesgue measure zero, with topology induced by the L^1_{loc} convergence of the indicator functions. This setting includes random closed sets in the sense of Matheron as a special case. The definitions of the volume fraction $\overline{p}$ and the contact distribution function H_Q can be extended to stationary RAMS $Z \subset \mathbb{R}^n$; see Section 5. We use the notion of *specific perimeter* $\overline{P}(Z)$ of Z given as the (constant) density of the variation measure $|D\mathbf{1}_Z|$ with respect to λ_n (cf. [10] where the notion 'specific variation' is used, or [19]), and oriented rose of directions $\mathcal{R}^*$ given as the distribution of the outer normal $-\Delta_{\mathbf{1}_Z}(z)$ at a typical point $z \in \partial^* Z$ in case $\overline{P}(Z) < \infty$; see Section 5 for exact definitions.

Theorem 1.2. *Let $Q \neq \emptyset$ be finite. If Z is a stationary RAMS, then the right sided derivative $H'_Q(0+)$ of H_Q at 0 satisfies*

$$(1 - \overline{p})H'_Q(0+) = \overline{P}(Z) \int_{S^{n-1}} h(-Q, v)^+ \mathcal{R}^*(dv) \tag{1.8}$$

when $\overline{P}(Z) < \infty$. If $\overline{P}(Z)$ is infinite, and $\operatorname{conv}(Q \cup \{0\})$ has interior points,

$$(1 - \overline{p})H'_Q(0+) = \infty.$$

If Z is stationary and isotropic, and $\overline{P}(Z) \in [0, \infty]$, then

$$(1 - \overline{p})H'_Q(0+) = \tfrac{1}{2} b(\operatorname{conv}(Q \cup \{0\}))\overline{P}(Z) \tag{1.9}$$

where $b(\,\cdot\,)$ is the mean width.

We would like to stress that the methods of proofs are different from the classical approaches in stochastic geometry when dealing with sets with finite perimeter. Namely, we use typically approximations of characteristic functions by smooth functions of bounded variation, show related formulas for them, and apply continuity arguments to obtain the desired results. This means that we have to define functionals to be dealt with not only for sets but also for functions.

The paper is organized as follows. In Section 2 we recall the usual and directional variation of a function f, discuss basic properties, and define sets of finite perimeter. The notion of the variation $V^Q(f)$ of f with respect to a compact set Q is introduced and discussed in Section 3. This is a special case of anisotropic variation with respect to a Finsler metric, see [1]. In particular, $V^{-Q}(\mathbf{1}_A)$ coincides with the right hand side of (1.7) when $0 \in Q$. Section 4 is devoted to the proof of the main result, Theorem 1.1. While one equality (Proposition 4.2) is obtained by standard methods (similarly as the same inequality for n-dimensinal convex bodies in [4]), the other inequality (Corollary 4.6) is more difficult. The above mentioned application to random sets and the proof of Theorem 1.2 is described in Section 5.

2 Preliminaries

We present here some definitions and properties of functions of bounded variation and sets with finite perimeter. As reference we use mostly the book [2].

Let Ω be a nonempty open subset of $\mathbb{R}^n$ and $0 \neq u \in \mathbb{R}^n$. We write $L^1_{\text{loc}}(\Omega)$ for the space of all functions on Ω that are locally Lebesgue-integrable. The *distributional directional derivative* of a function $f \in L^1_{\text{loc}}(\Omega)$ in direction u is the linear functional

$$D_u f : \phi \mapsto -\int_\Omega \frac{\partial \phi}{\partial u}(x) f(x)\, dx, \quad \phi \in C_c^\infty(\Omega). \tag{2.1}$$

Here $\frac{\partial \phi}{\partial u}(x)$ is the classical directional derivative of a smooth function, dx denotes the integration w.r.t. Lebesgue measure and $C_c^\infty(\Omega)$ stands for the space of infinitely differentiable functions on Ω with compact support. We define the *directional variation* of $f \in L^1_{\text{loc}}(\Omega)$ in the direction $u \in S^{n-1}$ as

$$V_u(f, \Omega) := \sup \left\{ D_u f(\phi) :\ \phi \in C_c^\infty(\Omega),\ \|\phi\|_\infty \le 1 \right\}.$$

If the last expression is finite and $f \in L^1(\Omega)$, we say that f has *finite directional variation* (in Ω and) in direction u. We denote by $\mathrm{BV}_u(\Omega)$ the space of all such functions. Note that, by the Riesz representation theorem, $f \in \mathrm{BV}_u(\Omega)$ if and only if the distributional directional derivative $D_u f$ can be represented as a finite Radon measure on Ω. In this case we have $V_u(f, \Omega) = |D_u f|(\Omega)$, where $|\mu|$ denotes the variation measure of the (real- or vector-valued) Radon measure μ given by

$$|\mu|(A) = \sup \Big\{ \sum_{h=1}^{\infty} |\mu(E_h)| : (E_1, E_2, \ldots) \text{ forms a Borel partition of } A \Big\}$$

for any Borel set $A \subset \Omega$.

The *variation* of a function $f \in L^1_{\text{loc}}(\Omega)$ is defined as

$$V(f, \Omega) := \sup \left\{ \int_\Omega f(x) \operatorname{div} \varphi(x)\, dx :\ \varphi \in C_c^\infty(\Omega, \mathbb{R}^n),\ \||\varphi|\|_\infty \le 1 \right\}.$$

Here, $C_c^\infty(\Omega, \mathbb{R}^n)$ is the vector space of $\mathbb{R}^n$-valued infinitely differentiable functions on Ω with compact support, and $\| |\varphi| \|_\infty$ is the L^∞-norm of the Euclidean norm $|\varphi| = \sqrt{\varphi_1^2 + \cdots + \varphi_n^2}$ of $\varphi = (\varphi_1, \ldots, \varphi_n)$. If $V(f, \Omega)$ is finite and $f \in L^1(\Omega)$, we say that f has *bounded variation* in Ω. The vector space of all functions of bounded variation is denoted by $\mathrm{BV}(\Omega)$. Functions $f \in L^1_{\mathrm{loc}}(\Omega)$ with bounded variation in any relatively compact open subset of Ω are said to have *locally* bounded variation in Ω. We have $f \in \mathrm{BV}(\Omega)$ if and only if $f \in \mathrm{BV}_u(\Omega)$ for all $u \in S^{n-1}$ and then,

$$V(f, \Omega) = (2\kappa_{n-1})^{-1} \int_{S^{n-1}} V_u(f, \Omega)\, \mathcal{H}^{n-1}(du), \tag{2.2}$$

cf. [10]. Here and in the following $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure in $\mathbb{R}^n$, and κ_k is the k-dimensional volume of the Euclidean unit ball in $\mathbb{R}^k$.

If $f \in \mathrm{BV}(\Omega)$ then there exists a finite $\mathbb{R}^n$-valued Radon measure Df on Ω such that $Df(A) \cdot u = D_u f(A)$ for all Borel-sets $A \subset \Omega$, and $u \neq 0$; Df represents the *distributional derivative* of f, cf. [2, §3.1]. The variation of f is the total variation of Df:

$$V(f, \Omega) = |Df|(\Omega). \tag{2.3}$$

Let

$$Df = \Delta_f\, |Df| \tag{2.4}$$

be the *polar decomposition* of Df, i.e., $\Delta_f \in L^1(\Omega, |Df|)$ taking values in S^{n-1} is the Radon-Nikodým density of Df w.r.t. $|Df|$ (cf. [2, Corollary 1.29]). Note that if $f \in \mathrm{BV}(\Omega)$ and $u \neq 0$ then $V_u(f, \Omega)$ can be written in the form

$$V_u(f, \Omega) = \int_\Omega |u \cdot \Delta_f(x)|\, |Df|(dx).$$

Note also that if $f \in C^1(\Omega) \cap \mathrm{BV}(\Omega)$ then $Df(dx) = \nabla f(x)\, dx$, $|Df|(dx) = |\nabla f(x)|\, dx$, and

$$\Delta_f(x) := \begin{cases} \frac{\nabla f(x)}{|\nabla f(x)|}, & \nabla f(x) \neq 0, \\ 0, & \text{otherwise}, \end{cases}$$

is a version of Δ_f, where $\nabla f(x)$ denotes the gradient of f at x.

Let (f_j) be a sequence of functions in $\mathrm{BV}(\Omega)$ and let $f \in \mathrm{BV}(\Omega)$. Following [2, 3.14], we say that (f_j) *converges strictly* to f if $f_j \to f$ in $L^1(\Omega)$ and, additionally, $V(f_j, \Omega) \to V(f, \Omega)$. As a basic example, consider any function $f \in \mathrm{BV}(\Omega)$ and a sequence of C^∞ mollifiers ρ_j (i.e., $\rho_j(y) = j^n \rho(jy)$ with a nonnegative function $\rho \in C_c^\infty$ fulfilling $\int_{\mathbb{R}^n} \rho dx = 1$). Then, the convolutions $f * \rho_j$ (*mollifications of* f) belong to $C^\infty(\Omega')$ and $f * \rho_j \to f$ strictly in a slightly "shrunk" open set $\Omega' = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$ (cf. [2, §2.1, 3.1]). That the set Ω has to be replaced by a smaller one can be avoided by mollifying $f\varphi_h$, where (φ_h) is a smooth partition of unity in Ω relative to a locally finite covering (Ω_h) with open, relative compact sets. The corresponding result can be found in [24, Theorem 5.3.3] and implies the third statement in the following collection of well-known basic properties of the variation.

Proposition 2.1 (Basic properties of the variation)**.**

(a) *For* $f \in \mathrm{BV}(\Omega) \cap C^1(\Omega)$,

$$V(f,\Omega) = \int_\Omega |\nabla f| dx.$$

(b) *If* $f_j \to f$ *in* $L^1(\Omega)$ *then* $V(f,\Omega) \le \liminf_{j\to\infty} V(f_j,\Omega)$.

(c) *For* $f \in \mathrm{BV}(\Omega)$, *there is a sequence of functions* (f_j) *in* $C^\infty(\Omega) \cap \mathrm{BV}(\Omega)$ *converging strictly to* f.

The following lemma states that the positive and negative parts of $D_u f$, $u \neq 0$, have the same total mass when $f \in \mathrm{BV}(\Omega)$ and $\Omega = \mathbb{R}^n$. This is not necessarily true when $\Omega \neq \mathbb{R}^n$. For instance, $f(x) = x$ on $\Omega = (0,1)$ satisfies $(D_1 f)^+(\Omega) = 1$, but $(D_1 f)^-(\Omega) = 0$.

Lemma 2.2. *For* $f \in \mathrm{BV}$ *we have* $Df(\mathbb{R}^n) = 0$. *In particular,*

$$D_u f(\mathbb{R}^n) = \int_{\mathbb{R}^n} (u \cdot \Delta_f(x))\, |Df|(dx) = 0 \tag{2.5}$$

for all $u \neq 0$.

Proof. Fix $f \in \mathrm{BV}$ and put $\phi_m = \mathbf{1}_{B(0,m)} * \rho$, where $0 \le \rho \in C^\infty$ is a mollifier with support in $B(0,1)$. Clearly, $\nabla\phi_m$ is zero outside the annulus $R_m = B(0,m+1) \setminus B(0,m-1)$, and $\|\frac{\partial\phi_m}{\partial x_i}\|_\infty < \kappa_n \||\nabla\rho|\|_\infty$, so

$$\left|\int_{\mathbb{R}^n} \phi_m (Df)_i(dx)\right| = \left|-\int_{\mathbb{R}^n} \frac{\partial\phi_m}{\partial x_i} f(x) dx\right| \le \kappa_n \||\nabla\rho|\|_\infty \int_{R_m} |f| dx.$$

for all $i \in \{1,\dots,n\}$. As $f \in L^1$, the right hand side converges to 0. The left hand side converges to $\left|\int_{\mathbb{R}^n} (Df)_i(dx)\right|$, as ϕ_m is an increasing sequence with pointwise limit 1, and $(Df)_i$ is a finite Radon measure. We conclude $Df(\mathbb{R}^n) = 0$ and

$$\int_{\mathbb{R}^n} (u \cdot \Delta_f(x))\, |Df|(dx) = u \cdot \int_{\mathbb{R}^n} Df(dx) = 0,$$

as claimed. □

We shall work with the following generalization of directional variations. Let L be a linear subspace of $\mathbb{R}^n$ of dimension $k \in \{1,\dots,n\}$. If $C_c^\infty(\Omega, L)$ denotes the vector space of all functions in $C_c^\infty(\Omega,\mathbb{R}^n)$ with values in L, we may define the *L-variation* in Ω of $f \in L^1_{\mathrm{loc}}(\Omega)$ as

$$V_L(f,\Omega) := \sup\left\{ \int_\Omega f(x) \sum_{i=1}^k \frac{\partial(\varphi \cdot u_i)}{\partial u_i}(x)\, dx : \varphi \in C_c^\infty(\Omega, L),\ \||\varphi|\|_\infty \le 1 \right\},$$

where $\{u_1,\dots,u_k\}$ is an orthonormal basis of L. This definition does not depend on the choice of the orthonormal basis.

Clearly, when $f \in L^1(\Omega)$, $V_L(f,\Omega) < \infty$ if and only if $V_u(f,\Omega) < \infty$ for all unit vectors $u \in L$, and in this case, we say that f has *finite directional variation in L*, writing $f \in \mathrm{BV}_L(\Omega)$. We have $V_{\mathbb{R}^n}(f,\Omega) = V(f,\Omega)$, and $V_{\mathrm{span}\{u\}}(f,\Omega) = V_u(f,\Omega)$ when $u \in S^{n-1}$. If $L \subseteq L'$ are two subspaces then $V_L(f,\Omega) \le V_{L'}(f,\Omega)$.

Proposition 2.3 (Basic properties of the directional variation)**.** *The following assertions hold for a linear subspace $\{0\} \neq L \subset \mathbb{R}^n$.*

(a) *For $f \in \mathrm{BV}(\Omega)$ we have*

$$V_L(f,\Omega) = |p_L(Df)|(\Omega) = \int_\Omega |p_L \Delta_f(x)|\,|Df|(dx), \tag{2.6}$$

where p_L denotes the orthogonal projection on L. If, in addition, $f \in C^1(\Omega)$,

$$V_L(f,\Omega) = \int_\Omega |p_L \nabla f| dx.$$

(b) *If $f_j \to f$ in $L^1(\Omega)$ then $V_L(f,\Omega) \leq \liminf_{j\to\infty} V_L(f_j,\Omega)$.*

(c) *If (f_j) is a sequence converging strictly to $f \in \mathrm{BV}(\Omega)$, then $V_L(f_j,\Omega) \to V_L(f,\Omega)$ as $j \to \infty$.*

Proof. The first two statements generalize slightly [2, Proposition 3.6] and we can skip the proof as it is quite obvious. To show (c) let (f_j) be a sequence converging strictly to $f \in \mathrm{BV}(\Omega)$. By [2, Proposition 3.13] the measures Df_j converge weakly to Df in Ω and their total variations converge to $|Df|(\Omega)$. The claim now follows from a special case of the Reshetnyak continuity theorem, Lemma 2.4, below, which is quoted here from the literature for easy reference. □

Lemma 2.4 ([2, Proposition 2.39])**.** *Let $\mu_0, \mu_1, \dots$ be finite vector-valued Radon measures on an open set $\Omega \subset \mathbb{R}^n$, such that μ_j converges weakly to μ_0 in Ω and $|\mu_j|(\Omega) \to |\mu_0|(\Omega)$ as $j \to \infty$. Then*

$$\int_\Omega h(g_j(x))\, d|\mu_j|(x) \to \int_\Omega h(g_0(x))\, d|\mu_0|(x), \quad j \to \infty,$$

for every continuous and bounded function $h : \Omega \to \mathbb{R}$, where g_j is the Radon-Nikodým density of μ_j with respect to $|\mu_j|$.

The *perimeter* of a measurable set $A \subseteq \mathbb{R}^n$ in an open set Ω is defined as

$$P(A,\Omega) = V(\mathbf{1}_A, \Omega).$$

If the last quantity is finite, we say that A has *finite perimeter in* Ω. Sets A with $P(A,\mathbb{R}^n) < \infty$ are simply called sets of *finite perimeter*. This class is closed under set complement operation: a Borel set A has finite perimeter if and only if its complement has finite perimeter. In all the above notions, we skip from now on the argument Ω if $\Omega = \mathbb{R}^n$. If A has finite volume, $\mathbf{1}_A$ is in $L^1(\mathbb{R}^n)$ and thus A has finite perimeter if and only if $\mathbf{1}_A \in \mathrm{BV}$.

If the perimeter of a set A is finite, it is the variation of $D(\mathbf{1}_A)$ on Ω. This variation measure can in turn be expressed by means of the $(n-1)$-dimensional Hausdorff measure. To do so, let the *reduced boundary* $\mathcal{F}A$ be the set of all points $x \in \Omega$ in the support of $|D(\mathbf{1}_A)|$ such that the limit

$$\nu_A(x) = -\lim_{\rho\to 0_+} \frac{D(\mathbf{1}_A)(B(0,\rho))}{|D(\mathbf{1}_A)|(B(0,\rho))}$$

exists in $\mathbb{R}^n$ and is a unit vector. Here and in the following, $B(x,\rho)$ denotes the Euclidean ball with radius $\rho \geq 0$ centered at $x \in \mathbb{R}^n$. The negative sign in this definition is included here, so that the function $\nu_A : \mathcal{F}A \to S^{n-1}$ can be interpreted as *generalized outer normal to* A. By the Besicovitch derivation theorem [2, Theorem 2.22], $|D(\mathbf{1}_A)|$ is concentrated on $\mathcal{F}A$, and $D(\mathbf{1}_A) = -\nu_A|D(\mathbf{1}_A)|$. A comparison with the polar decomposition (2.4) yields $-\nu_A(x) = \Delta_{\mathbf{1}_A}(x)$ for $|D(\mathbf{1}_A)|$-almost every x. De Giorgi has shown that $\mathcal{F}A$ is countably $(n-1)$-rectifiable and $|D(\mathbf{1}_A)| = \mathcal{H}^{n-1} \llcorner \mathcal{F}A$, see, for instance [2, Theorem 3.59].

If $A \subset \mathbb{R}^n$ has finite perimeter and $u \in \mathbb{R}^n$, let $\mathcal{F}_{u+}A$, $\mathcal{F}_{u-}A$ denote the set of all points $x \in \mathcal{F}A$ such that $\nu_A(x) \cdot u$ is positive or negative, respectively. When $u \neq 0$, these sets are connected to the positive and negative parts of the measure $D_u\mathbf{1}_A$ as follows:

$$\begin{aligned}(D_u\mathbf{1}_A)^+(B) &= \int_{B\cap\mathcal{F}_{u-}A} |u \cdot \nu_A(x)|\, \mathcal{H}^{n-1}(dx),\\ (D_u\mathbf{1}_A)^-(B) &= \int_{B\cap\mathcal{F}_{u+}A} |u \cdot \nu_A(x)|\, \mathcal{H}^{n-1}(dx),\end{aligned} \tag{2.7}$$

where B is any bounded Borel subset of $\mathbb{R}^n$.

It is sometimes convenient to replace $\mathcal{F}A$ with larger sets, that are easier to handle. Let $\partial^*A = \mathbb{R}^n \setminus (A^0 \cup A^1)$ be the *essential boundary* of A, where

$$A^t := \left\{ x \in \mathbb{R}^n : \lim_{r\to 0+} \frac{\lambda_n(A \cap B(x,r))}{\lambda_n(B(x,r))} = t \right\} \tag{2.8}$$

is the set of all points with *Lebesgue density* $t \in [0,1]$. Then we have $\mathcal{F}A \subset \partial^*A$ ([2, Theorem 3.61]). If A is a set of finite perimeter in Ω, it can be shown that $\mathcal{H}^{n-1}(\Omega \cap \partial^*A \setminus \mathcal{F}A) = 0$, see [2, Theorem 3.61], and thus we have

$$|D(\mathbf{1}_A)| = \mathcal{H}^{n-1} \llcorner \mathcal{F}A = \mathcal{H}^{n-1} \llcorner \partial^*A \tag{2.9}$$

on Ω, and, in particular,

$$P(A,\Omega) = \mathcal{H}^{n-1}(\mathcal{F}A \cap \Omega) = \mathcal{H}^{n-1}(\partial^*A \cap \Omega). \tag{2.10}$$

When $\Omega = \mathbb{R}^n$, the *generalized surface area measure* of A, as defined in the introduction, can therefore also be written as

$$S^*_{n-1}(A;\,\cdot\,) = \mathcal{H}^{n-1}\left(\{a \in \mathcal{F}A : \nu_A(a) \in \,\cdot\,\}\right). \tag{2.11}$$

Remark 2.5. As functions with bounded variation, sets with finite perimeter are considered not as individual sets in $\mathbb{R}^n$, but as equivalence classes $\mathbf{1}_A \in L^1$. Thus, two sets with finite perimeter are considered as identical if the Lebesgue measure of their symmetric difference vanishes.

3 The variation with respect to a compact set

The *support function* $h(Q,\,\cdot\,)$ of a non-empty compact set Q in $\mathbb{R}^n$ is defined as the (usual) support function of its convex hull $\operatorname{conv} Q$. Explicitly, we have

$$h(Q,u) = \max\{u \cdot x : x \in Q\}, \qquad u \in S^{n-1}.$$

If $x^+ = \max\{x, 0\}$ denotes the positive part of $x \in \mathbb{R}$, $h(Q \cup \{0\}, \cdot\,) = h(Q, \cdot\,)^+$. Properties and applications of the support function of convex sets can be found in [21]. We only mention here that the *mean width* $b(K)$ of a non-empty compact convex set $K \subset \mathbb{R}^n$ can be defined using its support function:

$$b(K) = \frac{2}{n\kappa_n} \int_{S^{n-1}} h(K, u) du.$$

For an open set $\Omega \subset \mathbb{R}^n$ and $f \in \mathrm{BV}(\Omega)$ with polar decomposition (2.4), we define a functional

$$V^Q(f, \Omega) = \int_\Omega h(Q, \Delta_f(x))^+ \, |Df|(dx)$$

and call it the *variation of f with respect to Q in* Ω. As $V^Q(f, \Omega) = V^{\mathrm{conv}(Q \cup \{0\})}(f, \Omega)$, this variation depends on Q only through the convex hull of $Q \cup \{0\}$. We follow our usual convention and write $V^Q(f) = V^Q(f, \mathbb{R}^n)$. If this definition is applied to the indicator function of a set $A \subset \mathbb{R}^n$ of finite perimeter with $\Omega = \mathbb{R}^n$, (2.9) and (2.11) give

$$V^Q(\mathbf{1}_A) = \int_{S^{n-1}} h(-Q \cup \{0\}, u) S^*_{n-1}(A; du). \tag{3.1}$$

If A is a convex body, $V^Q(\mathbf{1}_A) = nV(-Q \cup \{0\}, A, \ldots, A)$ is a mixed volume, so $V^Q(\mathbf{1}_A)$ generalizes certain mixed volumes to sets of finite perimeter. If $\mathrm{conv}\, Q$ is symmetric w.r.t. the origin then $V^Q(f)$ is a special case of the generalized (anisotropic) variation defined in [1]. Indeed, we have $V^Q(f) = |Df|_\phi(\mathbb{R}^n)$ with Finsler metric $\phi(x, v) = h_{Q \cup \{0\}}(v)$, $x \in \mathbb{R}^n$, $v \in \mathbb{R}^n \setminus \{0\}$, in the sense of [1, Definition 3.1].

Let $B_L = B(0, 1) \cap L$ be the unit ball in L. One motivation to call $V^Q(f)$ a "variation" comes from the fact that

$$V^{B_L}(f, \Omega) = V_L(f, \Omega), \tag{3.2}$$

which follows directly from the definitions as $h(B_L, \cdot\,) = |p_L|$. In particular, we have $V^{\{-u,u\}}(f, \Omega) = V_u(f, \Omega)$ whenever $u \in S^{n-1}$. Another motivation is that averaging Q-variations gives the usual variation, that is,

$$\int_{\mathrm{SO}(n)} V^{\vartheta Q}(f, \Omega) \, d\vartheta = c_Q \, V(f, \Omega).$$

where $c_Q = (1/2) b\,(\mathrm{conv}(Q \cup \{0\}))$. This follows directly from the definitions and an application of Fubini's theorem. We now summarize connections and basic inequalities between the variation with respect to Q and the L-variations when $\Omega = \mathbb{R}^n$.

Lemma 3.1 (Ordinary variation and variation with respect to Q)**.** *Let* $f \in \mathrm{BV}$ *and a non-empty compact set* $Q \subset \mathbb{R}^n$ *be given. Then the following statements hold.*

(a) *For* $u \in S^{n-1}$ *we have*

$$2V^{\{u\}}(f) = V_u(f).$$

(b) $V^Q(f) \geq sV_L(f)$, *where* $L = \mathrm{span}\, Q$ *and* s *is the relative inradius of* $\mathrm{conv}(Q \cup \{0\})$ *in* L *(i.e.,* s *is the maximum radius of a ball in* L *contained in* Q*).*

(c) $V^Q(f) \leq R\,V_L(f) \leq R\,V(f)$, *where* R *denotes the circumradius of* $\operatorname{conv}(Q \cup \{0\})$, *that is the radius of the unique smallest ball containing this set.*

Proof. The claim in (a) follows from the definitions of $V_u(f)$ and $V^Q(f)$, in combination with (2.5). To verify (b), let $B_L(y,s)$ be a ball in L included in $K := \operatorname{conv}(\{0\} \cup Q)$. From the basic properties of support functions we get for $u \in S^{n-1}$

$$\begin{aligned}(h(Q,u))^+ &= h(K,u) = h(K,p_L u)\\ &\geq h(B(y,s),p_L u) = y \cdot p_L u + s|p_L(u)| = y \cdot u + s|p_L(u)|.\end{aligned}$$

Setting $u = \Delta_f(x)$ and integrating w.r.t. $|Df|$, equations (2.5) and (2.6) imply

$$V^Q(f) \geq sV_L(f),$$

as required. The proof of assertion (c) is analogous. ☐

For a non-empty compact set $Q \subset \mathbb{R}^n$ we define the *Q-variation measure* $|\mu|_Q$ of the $\mathbb{R}^n$-valued Radon measure μ on the open set $\Omega \subset \mathbb{R}^k$ by

$$|\mu|_Q(A) = \sup\Big\{\sum_{h=0}^{\infty} h(Q,\mu(E_h))^+ : (E_1,E_2,\ldots) \text{ forms a partition of } A\Big\}$$

for any Borel set $A \subset \Omega$. Using the subadditivity of the support function, it is easy to show that $|\mu|_Q$ is a positive Radon measure; one can for instance adapt the proof of [2, Theorem 1.6] and observe that $Q \subset B(0,r)$ implies $|\mu|_Q \leq |\mu|_{B(0,r)} = r|\mu|$ to prove finiteness on compact sets. The identity (3.2) shows that the following Proposition contains Proposition 2.3 as special case.

Proposition 3.2 (Basic properties of the variation with respect to Q)**.**
Let $Q \subset \mathbb{R}^n$ *be non-empty and compact.*

(a) *For* $f \in \mathrm{BV}(\Omega)$ *we have* $V^Q(f,\Omega) = |Df|_Q(\Omega)$. *If, in addition,* $f \in C^1(\Omega)$, *then*
$$V^Q(f,\Omega) = \int_\Omega h\left(Q,\nabla f(x)\right)^+ dx. \tag{3.3}$$

(b) *Assume that* $\Omega = \mathbb{R}^n$ *or that the origin is a relative interior point of* $\operatorname{conv} Q$. *If* $f_j \to f$ *in* $L^1(\Omega)$ *then* $V^Q(f,\Omega) \leq \liminf_{j\to\infty} V^Q(f_j,\Omega)$.

(c) *If* (f_j) *is a sequence converging strictly to* $f \in \mathrm{BV}(\Omega)$, *then* $V^Q(f_j,\Omega) \to V^Q(f,\Omega)$ *as* $j \to \infty$.

Proof. In order to prove that $V^Q(f,\Omega) = |Df|_Q(\Omega)$ in (a), it is enough to show that if an $\mathbb{R}^n$-valued finite measure μ has density g with respect to a positive measure ν, then $|\mu|_Q$ has density $h(Q,g(\,\cdot\,))^+$ with respect to ν, and apply this to $\mu = Df$, $\nu = |Df|$. With this notation, and exploiting that we may assume $0 \in Q$, we have to prove

$$|\mu|_Q(B) = \int_B h(Q,g(x))\,\nu(dx) \tag{3.4}$$

for all measurable sets B. The inequality $|\mu|_Q(B) \le \int_B h(Q,g)d\nu$ follows from the convexity, positive homogeneity and continuity of $h(Q,\,\cdot\,)$. To show the reverse inequality let $\varepsilon > 0$ and choose a dense sequence (z_h) in $\operatorname{conv} Q$. Define

$$\sigma(x) = \min\{h \in \mathbb{N} : z_h \cdot g(x) \ge (1-\varepsilon)h(Q, g(x))\},$$

and the level sets $B_h = \sigma^{-1}(h) \cap B$, that form a partition of B. Then

$$\begin{aligned}(1-\varepsilon)\int_B h(Q,g)d\nu &= \sum_h \int_{B_h} (1-\varepsilon)h(Q,g)d\nu \le \sum_h \int_{B_h} z_h \cdot g(x)d\nu \\ &= \sum_h z_h \cdot \mu(B_h) \le \sum_h h(Q, \mu(B_h)) \le |\mu|_Q(B),\end{aligned}$$

yielding (3.4). If f is also in $C^1(\Omega)$, Df has Lebesgue-density ∇f and (3.4) with $\mu = Df$, $g = \nabla f$ and Lebesgue measure ν yields the second claim in (a).

Let us show (b). We may assume that $\liminf_j V^Q(f_j, \Omega) < \infty$, and pass to a subsequence (again denoted by (f_j)) for which $\lim_{j\to\infty} V^Q(f_j,\Omega) < \infty$ exists. Set $L := \operatorname{span} Q$. Except the trivial case $Q = \{0\}$, we always have $\dim L \ge 1$. If $\Omega = \mathbb{R}^n$ let $s > 0$ be the inradius of $\operatorname{conv}(\{0\} \cup Q)$ in L. Then,

$$V_L(f_j, \Omega) \le \frac{1}{s} V^Q(f_j, \Omega) \tag{3.5}$$

due to Lemma 3.1.(b). If the origin is a relative interior point of $\operatorname{conv} Q$, there is $s > 0$ such that $sB_L \subset \operatorname{conv} Q$ and hence $V^Q(f_j, \Omega) \ge V^{sB_L}(f_j, \Omega) = sV^{B_L}(f_j, \Omega) = sV_L(f_j, \Omega)$, implying again (3.5). In either case, the sequence $V_L(f_j, \Omega)$ is bounded. Hence, by Proposition 2.3.(a), $\mu_j := p_L(Df_j)$, $j = 1, 2, 3, \ldots$, are L-valued finite vector measures. We can show exactly as in the proof of [2, Proposition 3.13] that $\mu_j \to \mu = p_L(Df)$ weakly* (we use the relative weak* compactness of (μ_j) and verify that any cumulative point of (μ_j) must agree with $p_L(Df)$). Note that the measures μ_j, and μ have polar decompositions (2.4)

$$d\mu_j = \frac{p_L \Delta_{f_j}}{|p_L \Delta_{f_j}|}\, d|\mu_j|, \quad \mu = \frac{p_L \Delta_f}{|p_L \Delta_f|}\, d|\mu|,$$

and we can write

$$V^Q(f, \Omega) = \int_\Omega h\left(\operatorname{conv}(\{0\} \cup Q), \frac{p_L \Delta_f}{|p_L \Delta_f|}\right) d|\mu|,$$

and analogously with f_j and μ_j. Since the support function $h(\operatorname{conv}(\{0\} \cup Q),\,\cdot\,)$ is continuous and positively 1-homogeneous, we may apply the Reshetnyak lower semi-continuity theorem [2, Theorem 2.38] and we obtain $V^Q(f, \Omega) \le \liminf_j V^Q(f_j, \Omega)$, as requested.

Assertion (c) follows directly from Lemma 2.4 with $h = h(\operatorname{conv}(\{0\} \cup Q),\,\cdot\,)$. □

Remark 3.3. If $\operatorname{conv} Q$ is symmetric w.r.t. the origin then assertion (b) of Proposition 3.2 follows from [1, Theorem 5.1].

4 Dilation volumes

Let $A \oplus Q = \{a + q : a \in A, q \in Q\}$ be the *Minkowski sum* of the sets A and Q in $\mathbb{R}^n$. For measurable A and compact $Q \neq \emptyset$ we are interested in the volume

$$\lambda_n((A \oplus Q) \setminus A) = \int_{\mathbb{R}^n} \big(\max_{u \in Q} \mathbf{1}_{A+u}(x) - \mathbf{1}_A(x)\big)^+ dx,$$

and therefore define more generally the functional

$$G(Q, f) = \int_{\mathbb{R}^n} \big(\sup_{u \in Q} f(x - u) - f(x)\big)^+ dx \tag{4.1}$$

for any measurable function f on $\mathbb{R}^n$. Note that the family $\{f(\,\cdot\, - u) : u \in Q\}$ is a permissible class, and thus, $\sup_{u \in Q} f(\,\cdot\, - u)$ is Lebesgue-measurable; see e.g. [17, Appendix C] for a short summary or [8, Section III] for details. By definition,

$$G(Q, \mathbf{1}_A) = \lambda_n((A \oplus Q) \setminus A). \tag{4.2}$$

Note that the mapping $f \mapsto G(Q, f)$ may depend in general on the particular representation f and, hence, cannot be considered as a mapping on L^1. When Q is at most countable, independence of the representative is straightforward.

Lemma 4.1 (Properties of $G(Q, \cdot)$ for countable Q)**.** *If the compact set $Q \neq \emptyset$ is at most countable then the mapping $G(Q, \cdot\,)$ is well-defined and lower semi-continuous on L^1. Moreover, if $f_j = f * \rho_j$ is a mollification of $f \in L^1$ with non-negative ρ, then*

$$G(Q, f_j) \leq G(Q, f) \tag{4.3}$$

and thus $G(Q, f_j) \to G(Q, f)$, as $j \to \infty$.

Proof. For integrable f, let $f^Q(x) = \sup_{u \in Q} f(x - u)$. If g is another representative of the L^1-equivalence class of f, then $f = g$ outside a set N of Lebesgue measure zero. Then, $f^Q = g^Q$ outside the set $N \oplus Q$, the latter being a Lebesgue-null set as Q is at most countable. Hence $G(Q, \cdot\,)$ is well-defined on L^1.

To show the semi-continuity, let (f_j) be a sequence that converges to f in L^1. This implies that (f_j) converges in measure and if we consider a subsequence of (f_j) such that the limit inferior (of $(G(Q, f_j))$) becomes an ordinary limit, there is a sub-subsequence $(f_{j'})$ that converges outside a Lebesgue-null set N. As Q is at most countable, $M = N \oplus (Q \cup \{0\})$ is a Lebesgue-null set, and we have that $\lim_{j \to \infty} f_{j'}(x - u) = f(x - u)$ for all $u \in Q \cup \{0\}$ and $x \notin M$. Fatou's lemma and the lower semi-continuity of the supremum operation now yield

$$\begin{aligned}
\liminf_{j \to \infty} G(Q, f_j) &\geq \int_{\mathbb{R}^n} \liminf_{j \to \infty} \sup_{u \in Q \cup \{0\}} (f_{j'}(x - u) - f_{j'}(x))\, dx \\
&\geq \int_{\mathbb{R}^n \setminus M} \sup_{u \in Q \cup \{0\}} \liminf_{j \to \infty} (f_{j'}(x - u) - f_{j'}(x))\, dx \\
&= \int_{\mathbb{R}^n \setminus M} \sup_{u \in Q \cup \{0\}} (f(x - u) - f(x))\, dx \\
&= G(Q, f).
\end{aligned}$$

It remains to prove (4.3). We may assume without loss of generality that $0 \in Q$, as $G(Q, f) = G(Q \cup \{0\}, f)$. Then, the positive part can be dropped in the definition of $G(Q, f)$. We have

$$\begin{aligned} G(Q, f_j) &= \| \sup_{u \in Q} (f_j(\,\cdot\, - u) - f_j) \|_1 \\ &= \| \sup_{u \in Q} [(f(\,\cdot\, - u) - f) * \rho_j] \, \|_1 \\ &\le \left\| \left[\sup_{u \in Q} (f(\,\cdot\, - u) - f) \right] * \rho_j \right\|_1 \\ &= \| \sup_{u \in Q} (f(\,\cdot\, - u) - f) \|_1 \\ &= G(Q, f). \end{aligned}$$

We have used the inequality

$$\sup_{u \in Q} [g_u * h] \le [\sup_{u \in Q} g_u] * h$$

valid for any integrable functions $h \ge 0$ and $g_u, u \in Q$. □

Proposition 4.2. *If $f \in C^1 \cap BV$, $Q \subset \mathbb{R}^n$ is non-empty and compact, and $r > 0$ then*

$$\liminf_{r \to 0+} \tfrac{1}{r} G(rQ, f) \ge V^{-Q}(f). \tag{4.4}$$

If Q is in addition at most countable, (4.4) *holds for any $f \in \mathrm{BV}$.*

Proof. Assume first that $f \in C^1 \cap \mathrm{BV}$. Using the function

$$g_x(r) = \max_{u \in Q} f(x - ru) - f(x), \quad r \ge 0,$$

we may write

$$\tfrac{1}{r} G(rQ, f) = \int_{\mathbb{R}^n} \left(\tfrac{1}{r} g_x(r) \right)^+ dx. \tag{4.5}$$

Fix $x \in \mathbb{R}^n$. As f is Lipschitz in a neighborhood of x with Lipschitz constant M_x, say, g_x is Lipschitz in a neighborhood V of 0 with constant bounded by $M_x \max_{u \in Q} |u|$. Hence g_x is differentiable almost everywhere in V, this derivative is essentially bounded uniformly in V, and

$$g_x(r) = g_x(0) + \int_0^r g_x'(s) ds = r \int_0^1 g_x'(rs) ds.$$

Inserting this into (4.5), and using the fact that $g_x'(rs)$ coincides almost everywhere with the right sided derivative $g_x'(rs+)$, this gives

$$\tfrac{1}{r} G(rQ, f) = \int_{\mathbb{R}^n} \left(\int_0^1 g_x'(rs+) ds \right)^+ dx. \tag{4.6}$$

To determine the limit inferior we first fix $x \in \mathbb{R}^n$. For every $r > 0$ there is some $v_r \in Q$ with

$$g_x(r) = f(x - r v_r) - f(x).$$

Thus, for all $u \in Q$, $f(x - ru) - f(x) \le f(x - rv_r) - f(x)$ and division with r and taking the limit $r \to 0_+$ yields

$$(-\nabla f(x)) \cdot u \le (-\nabla f(x)) \cdot v,$$

for all $u \in Q$, where $v \in Q$ is any accumulation point of a subsequence of (v_r). Hence,

$$h(Q, -\nabla f(x)) = (-\nabla f(x)) \cdot v. \tag{4.7}$$

A lower bound for $g_x'(r+)$ is now obtained from

$$\begin{aligned} g_x'(r+) &= \lim_{s\to 0+} \tfrac{1}{s}(g_x(r+s) - g_x(r)) \\ &\ge \tfrac{1}{s} \lim_{s\to 0+} \big((f(x-(r+s)v_r) - f(x)) - (f(x - rv_r) - f(x))\big) \\ &\ge (-\nabla f(x - rv_r)) \cdot v_r. \end{aligned}$$

Considering a subsequence of (v_r) such that the limit inferior becomes a limit and is converging to some $v \in Q$, we can take the limit and get from (4.7) that

$$\liminf_{r\to 0+} g_x'(r+) \ge (-\nabla f(x)) \cdot v = h(Q, -\nabla f(x)).$$

As $g_x'(r+)$ is essentially bounded by $M_x \max_{u\in Q} |u|$ in V, dominated convergence implies

$$\liminf_{r\to 0+} \int_0^1 g_x'(rs+)\, ds \ge h(Q, -\nabla f(x)).$$

This can be used in (4.6), after applying Fatou's lemma, to obtain

$$\liminf_{r\to 0+} \tfrac{1}{r} G(rQ, f) \ge V^{-Q}(f).$$

This yields the assertion for continuously differentiable f.

Let now f be a general function of bounded variation, and let $f_j = f * \rho_j$ be smooth mollifications of f with mollifiers $\rho_j \ge 0$ (cf. Section 2). Let Q be non-empty and at most countable. Then inequality (4.4) holds for all f_j and by Lemma 4.1 and Proposition 3.2.(b) also for f. This completes the proof. $\square$

The arguments in the proof of [10, Proposition 11] show that

$$\lim_{r\to 0_+} \tfrac{1}{r} G(rQ, f) = V^{-Q}(f)$$

when $Q = \{u\}, u \ne 0$, which is a version of (1.5) for BV functions f. One might thus expect that the limit inferior in (4.4) is indeed an ordinary limit, and equality holds, for at most countable sets Q. However, when Q is infinite, this need not be true. In the following we give a counterexample where f is the indicator function of a compact set of finite perimeter. This example is adapted from the known example of a set of positive reach with infinite outer Minkowski content, see e.g. [2, pp. 109f].

Example 4.3. Let $n \geq 2$. For every $m \in \mathbb{N}$ define the open annulus

$$R_m = \operatorname{int}\left(B\left(0, \frac{1}{m}\right) \setminus B\left(0, \frac{1}{m+1}\right)\right),$$

and choose a finite set $A_m \subset R_m$ with

$$R_m \subset A_m \oplus B(0, (2^m m)^{-1}).$$

Let (r_m) be a sequence of positive numbers and set $A = \{0\} \cup \bigcup_{m=1}^{\infty} (A_m \oplus B(0, r_m))$. The sequence (r_m) can be chosen in such a way that

$$\mathcal{H}^{n-1}(\partial A) \leq \sum_{m=1}^{\infty} (\# A_m) \mathcal{H}^{n-1}(\partial B(0,1)) r_m^{n-1} < \infty,$$

(here we use the assumption $n \geq 2$), $A_m \oplus B(0, r_m) \subset R_m$, and

$$\lambda_n(A_m) \leq \sum_{m=1}^{\infty} (\# A_m) \lambda(B(0,1)) r_m^n < \frac{\lambda_n(R_m)}{2} \tag{4.8}$$

for all $m \in \mathbb{N}$. In particular, A is a compact set of finite perimeter. In a similar way, choose finite sets $Q_m \subset B(0, 1/m)$ with $Q_m \oplus B(0, r_m) \supset B(0, 1/m)$ for all $m \in \mathbb{N}$, and set $Q = \{0\} \cup \bigcup_{m=1}^{\infty} Q_m$. Then Q is a compact countable subset of the unit ball. For $0 < r < 1/2$ let m be such that $2^{-m} < r \leq 2^{-m+1}$. Then

$$\begin{aligned}
(A \oplus rQ) \setminus A &\supset [(A \oplus rQ) \setminus A] \cap R_m \\
&\supset \left[\left(A_m \oplus B(0, r_m) \oplus rQ\right) \cap R_m\right] \setminus A_m \\
&\supset \left[\left(A_m \oplus r(Q \oplus B(0, r_m))\right) \cap R_m\right] \setminus A_m \\
&\supset \left[\left(A_m \oplus B(0, (2^m m)^{-1})\right) \cap R_m\right] \setminus A_m \\
&= R_m \setminus A_m.
\end{aligned}$$

It follows from (4.2) and (4.8) that there is a constant $c > 0$ with

$$\tfrac{1}{r} G(rQ, \mathbf{1}_A) \geq \frac{\lambda_n(R_m)}{2r} \geq c \frac{(\log_2 \frac{1}{r})^{-(n+1)}}{r} \to \infty,$$

as $r \to 0_+$. In particular, $\frac{1}{r} G(rQ, \mathbf{1}_A)$ does not converge to $V^{-Q}(\mathbf{1}_A) \leq \mathcal{H}^{n-1}(\partial A) < \infty$ (we use here Lemma 3.1(c)).

We will show now that the desired convergence result is true when f is the indicator of a set of finite perimeter and Q is finite. This requires some auxiliary lemmas. We recall the notation $\mathcal{F}_{u+}A$, $\mathcal{F}_{u-}A$ introduced in Section 2.

Lemma 4.4. *Let $0 \neq u \in \mathbb{R}^n$ and $r > 0$ be given.*

(i) *If $f \in \mathrm{BV}$ and $U \subset \mathbb{R}^n$ is open then*

$$\int_U (f(x) - f(x + ru))^+ \, dx \leq r V^{\{-u\}}(f, U \oplus (0, ru)).$$

(ii) *If* $A \subset \mathbb{R}^n$ *has finite perimeter then*

$$\lambda_n\left(\{x \in A:\, x + ru \notin A,\, [x, x+ru] \cap \mathcal{F}_{u+}A = \emptyset\}\right) = 0.$$

(iii) *If* A *is as in (ii) and* $0 < s < 1$ *then*

$$\lambda_n\left(\{x \in A:\, x + sru \notin A,\, x + ru \in A\}\right) = o(r), \quad r \to 0.$$

Proof. In fact, (i) is a local and signed version of [10, Proposition 11] and we proceed with a similar proof. If f belongs to $C^1(U \oplus (0, ru)) \cap \mathrm{BV}(U \oplus (0, ru))$ then

$$f(x) - f(x+ru) = \int_0^1 r\left(-\frac{\partial f}{\partial u}(x + tru)\right) dt \le \int_0^1 r\left(\frac{\partial f}{\partial u}(x+tru)\right)^- dt$$

for all $x \in U$, and, applying Fubini's theorem and (3.3), we get

$$\begin{aligned}&\int_U (f(x) - f(x-ru))^+\, dx \\ &\qquad \le \int_0^1 r \int_U \left(\frac{\partial f}{\partial u}(x+tru)\right)^- dx\, dt \le rV^{\{-u\}}(f, U \oplus (0, ru)).\end{aligned}$$

The case $f \in \mathrm{BV}$ can be shown by strict approximation: By Proposition 2.1.(c) there is a sequence f_j in $C^\infty(U \oplus (0, ru)) \cap \mathrm{BV}(U \oplus (0, ru))$ converging strictly to f on $U \oplus (0, ru)$. Now, (i) holds with f replaced by f_j, and taking the limit $j \to \infty$ it also holds for f due to Proposition 3.2.(c) and since $f_j \xrightarrow{L^1} f$ implies $(f_j(\,\cdot\,) - f_j(\,\cdot\, + ru))^+ \xrightarrow{L^1} (f(\,\cdot\,) - f(\,\cdot\, + ru))^+$.

We will show (ii) by contradiction, i.e., assume that $\lambda_n(Z) > 0$, where

$$Z := (A \setminus (A - ru)) \setminus (\mathcal{F}_{u+}A \oplus [0, -ru]).$$

Note that, in particular, $(D_u\mathbf{1}_A)^-(Z \oplus [0, ru]) = 0$ (cf. (2.7)). Since the measure $(D_u\mathbf{1}_A)^-$ is outer regular, we can find an open set $V \supset Z \oplus [0, ru]$ such that $(D_u\mathbf{1}_A)^-(V) < r^{-1}\lambda_n(Z)$. Let, further, $U \supset Z$ be an open set such that $U \oplus [0, ru] \subset V$ (we can set $U = V \ominus [0, ru]$, where $\ominus$ is the Minkowski subtraction, and use [21, (3.15)]). Then, applying (i) with $f = \mathbf{1}_A$, we obtain

$$\begin{aligned}\lambda_n(Z) \le \lambda_n((A \setminus (A - ru)) \cap U) &= \int_U (\mathbf{1}_A(x) - \mathbf{1}_A(x+ru))^+\, dx \\ &\le rV^{\{-u\}}(\mathbf{1}_A, U \oplus [0, ru]) \\ &= r(D_u\mathbf{1}_A)^-(U \oplus [0, -ru]) \\ &\le r(D_u\mathbf{1}_A)^-(V) < \lambda_n(Z),\end{aligned}$$

a contradiction completing the proof of (ii).

In order to prove (iii), we apply (ii) and get

$$\lambda_n\left(\{x \in A:\, x + sru \notin A,\, x + ru \in A\}\right) \le \lambda_n(\{z:\, \#([x, x+ru] \cap \mathcal{F}A) \ge 2\}).$$

The last measure is of order $o(r)$ since $\mathcal{F}A$ is $\mathcal{H}^{n-1}$-rectifiable (see, e.g., [18, Lemma 1]), and the proof is finished. □

Let now a set $A \subset \mathbb{R}^n$ of finite perimeter and a finite set $Q = \{u_0 = 0, u_1, \dots, u_k\} \subset \mathbb{R}^n$ be given. To any $x \in \mathcal{F}A$ we assign the (unique) smallest number $0 \le i(x) \le k$ for which $\nu_A(x) \cdot u_{i(x)} = \max_j \nu_A(x) \cdot u_j$, and we consider the partition

$$\mathcal{F}A = \bigcup_i \partial_i A$$

with $\partial_i A := \{x \in \mathcal{F}A : i(x) = i\}$, $i = 0, \dots, k$. Note that $\partial_i A \subset \mathcal{F}_{u_i+}A$, $i = 1, \dots, k$, by definition. Denoting further

$$A_{Q,r} := \bigcup_{i=0}^{k} (\partial_i A \oplus [0, ru_i]),$$

we have, using Fubini's theorem and the area formula for the orthogonal projection of A_i onto $u_i^\perp$ (see [2, Theorem 2.91]),

$$\begin{aligned}\lambda_n(A_{Q,r}) &\le \sum_{i=0}^{k} r\mathcal{H}^{n-1}(p_{u_i^\perp}(\partial_i A))|u_i| = \sum_{i=0}^{k} r \int_{\partial_i A} |u_i \cdot \nu_A(x)| \, \mathcal{H}^{n-1}(dx) \\ &= r \int_{\mathcal{F}A} \max_i (u_i \cdot \nu_A(x)) \, \mathcal{H}^{n-1}(dx) \\ &= rV^{-Q}(\mathbf{1}_A). \end{aligned} \tag{4.9}$$

Lemma 4.5. *Let $A \subset \mathbb{R}^n$ have finite perimeter and let $0 \in Q \subset \mathbb{R}^n$ be finite. Then we have*

$$\lambda_n\Big(\big((A \oplus rQ) \setminus A\big) \setminus A_{Q,r}\Big) = o(r), \qquad r \to 0.$$

Proof. First, we shall show that it is sufficient to consider sets $Q = \{0, u_1, \dots, u_k\}$ such that for all $1 \le i < j \le k$, the vectors u_i, u_j are either linearly independent, or linearly dependent, but pointing in opposite directions. To see this, consider a larger set $Q' = Q \cup \{su_k\}$ with some $0 < s < 1$. We have clearly $A_{Q',r} = A_{Q,r}$, $r > 0$, and

$$\begin{aligned}\lambda_n\big((A \oplus rQ') \setminus (A \oplus rQ)\big) &\le \lambda_n\big((A + sru_k) \setminus (A \oplus \{0, ru_k\})\big) \\ &\le \lambda_n(\{z : z \notin A, z + sru_k \in A, z + ru_k \notin A\}),\end{aligned}$$

and the last expression is of order $o(r)$ by Lemma 4.4.(iii) applied to the complement of A.

Any point $z \in ((A \oplus rQ) \setminus A) \setminus A_{Q,r}$ has the following properties: $z \notin A$, $z - ru_i \in A$ for some $1 \le i \le k$ and $[z - ru_j, z] \cap \partial_j A = \emptyset$ for all $1 \le j \le k$. By Lemma 4.4.(ii), λ_n-almost all such points z have the additional property that there exists a point $x \in [z - ru_i, z] \cap \mathcal{F}_{u_i+}A$ and, clearly, this x must belong to $\partial_j A$ for some $j \ne i$, $j \ge 1$. Hence,

$$\lambda_n\Big(\big((A \oplus rQ) \setminus A\big) \setminus A_{Q,r}\Big) \le \sum_{j \ne i} \lambda_n(V_{ij}^r)$$

with

$$V_{ij}^r := \{z : [z - ru_i, z] \cap F_{ij} \ne \emptyset, [z - ru_j, z] \cap F_{ij} = \emptyset\},$$

where

$$F_{ij} := \partial_j A \cap \mathcal{F}_{u_i+}A.$$

It is thus enough to show that $\lambda_n(V_{ij}^r) = o(r)$ for any $1 \le i \ne j \le k$.

Note that if $u_j = -su_i$ for some $s > 0$ then $F_{ij} = \emptyset$ (indeed, in this case $u_i \cdot \nu_A(x) > 0$ implies $u_j \cdot \nu_A(x) < 0 < u_i \cdot \nu_A(x)$). Thus, we can assume in the sequel that u_i, u_j are linearly independent.

Applying the Fubini's theorem and the generalized area formula [9, §3.2.22] with the orthogonal projection $p_{u_i^\perp}|F_{ij}$ (note that $F_{ij} \subset \mathcal{F}A$ is countably $(n-1)$-rectifiable and its Jacobian $J_{n-1}(p_{u_i^\perp}|F_{ij})$ is at most 1), we get

$$\begin{aligned}
\lambda_n(V_{ij}^r) &= \lambda_n\big(V_{ij}^r \cap (F_{ij} \oplus [0, ru_i])\big) \\
&= \int_{u_i^\perp} \lambda_1\Big(V_{ij}^r \cap (F_{ij} \oplus [0, ru_i]) \cap \big(y + \operatorname{span}(u_i)\big)\Big)\, \lambda_{n-1}(dy) \\
&\le \int_{u_i^\perp} \sum_{x \in F_{ij} \cap (y + \operatorname{span}(u_i))} \lambda_1(V_{ij}^r \cap [x, ru_i])\, \lambda_{n-1}(dy) \\
&= \int_{F_{ij}} J_{n-1}(p_{u_i^\perp} \mid F_{ij})(x)\, \lambda_1(V_{ij}^r \cap [x, x + ru_i])\, \mathcal{H}^{n-1}(dx) \\
&\le \int_{F_{ij}} \lambda_1(V_{ij}^r \cap [x, x + ru_i])\, \mathcal{H}^{n-1}(dx).
\end{aligned}$$

Hence we have

$$r^{-1}\lambda_n(V_{ij}^r) \le \int_{F_{ij}} \varphi^r(x)\, \mathcal{H}^{n-1}(dx),$$

where

$$\varphi^r(x) := r^{-1}\lambda_1(V_{ij}^r \cap [x, x + ru_i]), \quad x \in F_{ij}.$$

We will show that

$$\lim_{r \to 0} \varphi^r = 0 \quad \mathcal{H}^{n-1} - \text{a.e. on } F_{ij}. \tag{4.10}$$

Applying then the Lebesgue dominated convergence theorem (note that $|\varphi^r(x)| \le |u_i|$ for any x) we obtain $\lambda_n(V_{ij}^r) = o(r)$, proving the lemma.

We will verify (4.10). Since F_{ij} is countably $(n-1)$-rectifiable, the approximate tangent cone $\operatorname{Tan}^{n-1}(F_{ij}, x)$ is a hyperplane at $\mathcal{H}^{n-1}$-a.a. $x \in F_{ij}$ by [9, §3.2.19], and we thus get $\operatorname{Tan}^{n-1}(F_{ij}, x) = \nu_A(x)^\perp$ at $\mathcal{H}^{n-1}$-a.a. $x \in F_{ij}$ by [2, Theorem 3.59]. (Concerning rectifiability, we use the terminology from [2] which is slightly different from [9].)

Denote $L := \operatorname{span}(u_i, u_j)$. We apply the generalized co-area formula [9, §3.2.22] to the mapping $f := p_{L^\perp}|F_{ij} : F_{ij} \to L^\perp$. We get that $f^{-1}\{z\} = F_{ij} \cap (z + L)$ is countably 1-rectifiable for $\mathcal{H}^{n-2}$-a.a. $z \in L^\perp$ and, thus, for $\mathcal{H}^1$-a.a. $x \in F_{ij} \cap (z+L) = F_{ij} \cap (x + L)$, the one-dimensional Lebesgue density $\Theta^1(F_{ij} \cap (x + L), x) = 1$ (cf. [9, §3.2.19]) and

$$\operatorname{Tan}^1(F_{ij} \cap (x + L), x) = \nu_A(x)^\perp \cap L. \tag{4.11}$$

Let N denote the set of all $x \in F_{ij}$ for which (4.11) is not true. We have $\mathcal{H}^1(N \cap f^{-1}\{z\}) = 0$ for $\mathcal{H}^{n-2}$-a.a. $z \in L^\perp$, hence, again by the co-area formula,

$$\int_N J_{n-2}f(x)\, \mathcal{H}^{n-1}(dx) = \int_{L^\perp} \mathcal{H}^1(N \cap f^{-1}\{z\})\, \mathcal{H}^{n-2}(dz) = 0.$$

As $J_{n-2}f(x) \neq 0$ for $x \in F_{ij}$ (recall that both $\nu_A(x) \cdot u_i$ and $\nu_A(x) \cdot u_j$ are positive if $x \in F_{ij}$), we have $\mathcal{H}^{n-1}(N) = 0$, hence, (4.11) is true for $\mathcal{H}^{n-1}$-a.a. $x \in F_{ij}$.

Fix now a point $x \in F_{ij}$ for which (4.11) holds, set

$$q := \frac{\nu_A(x) \cdot u_i}{\nu_A(x) \cdot u_j} \in (0, 1],$$
$$w := u_i - qu_j \in \nu_A(x)^\perp \cap L,$$

and choose an $\varepsilon > 0$. Note that small positive multiples of the vector w lie in the open triangle

$$C := \{tu_i - su_j : 0 < \frac{s}{q+\varepsilon} < t < 1\}$$

and, consequently, also

$$\Theta^1(F_{ij} \cap (x + rC), x) = \tfrac{1}{2}$$

for any $r > 0$. If π denotes the projection from $x + L$ onto $x + \mathrm{span}(u_i)$ along u_j, we get as a consequence that

$$\Theta^1(\pi(F_{ij} \cap (x + rC)), x) = \tfrac{1}{2}.$$

On the other hand, if $z = x + tu_i \in V_{ij}^r$ for some $0 < t < \frac{r}{q+\varepsilon}$ then $z \notin \pi(F_{ij} \cap (x + rC))$ and, consequently,

$$\begin{aligned}\lambda_1(V_{ij}^r \cap [x, x + ru_i]) &\le \left(r - \frac{r}{q+\varepsilon}\right)^+ + \lambda_1\left([x, x + ru_i] \setminus \pi(F_{ij} \cap (x + rC))\right) \\ &\le \varepsilon r + o(r).\end{aligned}$$

Since $\varepsilon > 0$ can be arbitrarily small, we obtain (4.10) and the proof is finished. □

Corollary 4.6. *Let $A \subset \mathbb{R}^n$ have finite perimeter and let $\emptyset \neq Q \subset \mathbb{R}^n$ be finite. Then*

$$\limsup_{r\to 0_+} \tfrac{1}{r} G(rQ, \mathbf{1}_A) \le V^{-Q}(\mathbf{1}_A).$$

Proof. As both sides of the stated equality remain unchanged when Q is replaced by $Q \cup \{0\}$ we may assume that $0 \in Q$. The claim then follows from (4.9) and Lemma 4.5. □

Proposition 4.2 and Corollary 4.6 yield already our main result:

Proposition 4.7. *Assume that $A \subset \mathbb{R}^n$ has finite perimeter. If $\emptyset \neq Q \subset \mathbb{R}^n$ is finite then*

$$\lim_{r\to 0+} r^{-1} G(rQ, \mathbf{1}_A) = V^{-Q}(\mathbf{1}_A).$$

Proof of Theorem 1.1. The first statement, (1.6), follows directly from Proposition 4.7 in combination with (4.2) and (3.1). If $0 \in Q$ and $\lambda_n(A) < \infty$ then (1.7) holds as it then coincides with (1.6). It is thus enough to show that the two sides of (1.7) do not change, when Q is replaced by a translation $Q - x$ with $x \in Q$. This is trivial for the left hand side and follows, using (2.5), also for the right hand side. Hence (1.7) also holds without the additional restriction $0 \in Q$. □

5 An application: Contact distributions of stationary random sets

In this section we apply the geometric results to random sets; see the book [22] for details on random closed sets, and [11] for random measurable sets in $\mathbb{R}^n$. Galerne and Lachièze-Rey [10, 11] define the mean covariogram of a random measurable set and discusses its properties. With the results of the previous section, similar relations for the *mean generalized dilation volume* with a finite structuring element could be established. We will not do so here, but instead present an approximation of the contact distribution function of a random set at zero, as the contact distribution function is an important summary statistics in applications.

We recall the notion of a *random measurable set* in $\mathbb{R}^n$. Let $\mathcal{M}$ denote the space of all Lebesgue measurable subsets of $\mathbb{R}^n$ modulo set differences of Lebesgue measure zero, equiped with the topology of L^1_{loc} convergence of the indicator functions. If $\mathcal{B}(\mathcal{M})$ denotes the corresponding Borel σ-algebra, $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ is a standard Borel space, and a random measurable set (RAMS) is a measurable mapping

$$Z : (\Omega, \Sigma, \Pr) \to (\mathcal{M}, \mathcal{B}(\mathcal{M}))$$

from a probability space Ω. (As remarked in [11, Remark 1], the random sets of finite perimeter from [19] are just random measurable sets with finite specific perimeter.) We restrict attention to *stationary* random measurable sets Z in $\mathbb{R}^n$ (that is, random measurable sets with translation-invariant distribution).

If Z is a stationary random *closed* set with volume fraction $\overline{p} = \Pr[0 \in Z] < 1$, its *contact distribution function* (sometimes called hit distribution function) with a compact structuring element $Q \subset \mathbb{R}^n$ is defined by

$$H_Q(r) = \Pr(Z \cap rQ \neq \emptyset \mid 0 \notin Z), \quad r \geq 0. \tag{5.1}$$

If $\overline{p} = 1$, we set $H_Q(r) = 1$. For convex Q with $0 \in Q$ and $\overline{p} < 1$, $H_Q(\,\cdot\,)$ coincides with the function

$$\tilde{H}_Q(r) = \Pr(d_Q(Z) \leq r \mid 0 \notin Z),$$

where $d_Q(Z) = \min\{t \geq 0 : Z \cap tQ \neq \emptyset\}$. In general we have

$$\tilde{H}_Q(r) = H_{\operatorname{star} Q}(r),$$

where $\operatorname{star} Q = \bigcup_{y \in Q} [0, y]$ is the star-hull of Q with respect to 0.

Notice that (5.1) does not give sense if Z is a stationary RAMS since $[0 \in Z]$ or $[Z \cap rQ \neq \emptyset]$ are not events (measurable subsets of Ω) any more. (Indeed, one cannot determine whether 0 belogs to $Z(\omega)$ since $Z(\omega)$ is given only up to measure zero.) Nevertheless, under stationarity, and for finite Q, we can give a meaning to (5.1) as follows. We consider the *shift randomization* $\widetilde{Z}$ of Z defined on the larger probability space $\widetilde{\Omega} := \Omega \times [0, 1]^n$ with $\widetilde{\Pr} := \Pr \otimes (\lambda_n|_{[0,1]^n})$ and $\widetilde{\Sigma}$ being the completion of the product σ-algebra $\Sigma \otimes \mathcal{B}(\mathcal{R}^n)$ as follows:

$$\widetilde{Z}(\omega, x) := Z(\omega) - x, \qquad (\omega, x) \in \widetilde{\Omega}.$$

By stationarity, we get the equality in distribution, $\widetilde{Z} \stackrel{d}{=} Z$. In Lemma 5.1 below, we show that $[0 \in \widetilde{Z}]$ and $[\widetilde{Z} \cap rQ \neq \emptyset]$ are random events, and we can define the volume fraction of Z as $\overline{p} := \widetilde{\Pr}[0 \in \widetilde{Z}]$ and the contact distribution function $H_Q(r)$ of Z using (5.1), where $\widetilde{\Pr}, \widetilde{Z}$ are used instead of $\Pr, Z$. This contact distribution function satisfies

$$H_Q(r) = 1 - \frac{1 - \mathbb{E}\lambda_n \left((Z \oplus (-rQ \cup \{0\})) \cap [0,1]^n \right)}{1 - \mathbb{E}\lambda_n (Z \cap [0,1]^n)},$$

$r \geq 0$, which is a known representation of H_Q when Z is a RACS; cf. [22, p. 44].

Lemma 5.1. *Let Z be a stationary RAMS in $\mathbb{R}^n$ and $\widetilde{Z}$ its shift randomization. Then $[x \in \widetilde{Z}]$ is a random event (i.e., a measurable subset of $\widetilde{\Omega}$) for any $x \in \mathbb{R}^n$. If $Q \subset \mathbb{R}^n$ is at most countable then $[\widetilde{Z} \cap Q \neq \emptyset]$ is also a random event.*

Proof. According to [11, Proposition 1], Z admits a measurable graph representative, i.e., a subset $Y \subset \Omega \times \mathbb{R}^n$ measurable w.r.t. $\Sigma \otimes \mathcal{B}(\mathbb{R}^n)$ such that for a.a. $\omega \in \Omega$, $\lambda_n(Z(\omega) \Delta Y_\omega) = 0$, where $Y_\omega := \{x \in \mathbb{R}^n : (\omega, x) \in Y\}$. Then we have by Fubini's theorem

$$\widetilde{\Pr}\Big([0 \in \widetilde{Z}] \Delta \big(Y \cap (\Omega \times [0,1]^n) \big) \Big) = \int_\Omega \lambda_n \left((Z(\omega) \Delta Y_\omega) \cap [0,1]^n \right) \Pr(d\omega) = 0.$$

Since Y is product-measurable and $\widetilde{\Sigma}$ is complete, also $[0 \in \widetilde{Z}]$ is in $\widetilde{\Sigma}$. When $x \in \mathbb{R}^n$ is given, $Z - x$ is a RAMS, and thus $[x \in \tilde{Z}] = [0 \in \tilde{Z} - x] = [0 \in \widetilde{Z - x}]$ is measurable. The second assertion now follows from this and the fact that

$$[\tilde{Z} \cap Q = \emptyset] = \bigcap_{u \in Q} [u \notin \tilde{Z}],$$

and the proof is finished. □

Let Z be a stationary RAMS. If Z has a.s. locally finite perimeter (i.e. $P(Z, \Omega) < \infty$ almost surely for all bounded open sets Ω), its derivative, the random $\mathbb{R}^n$-valued Radon measure $D\mathbf{1}_Z$ exists, and inherits stationarity from Z. Hence, $|D\mathbf{1}_Z|$ is a stationary nonnegative Radon measure, and there is $\overline{P}(Z) \in [0, \infty]$ such that $\mathbb{E}|D\mathbf{1}_Z| = \overline{P}(Z)\lambda_n$. The constant $\overline{P}(Z)$ is called the *specific perimeter* of Z (see [10, 19]) and we extend it by $\overline{P}(Z) := \infty$ to those Z which do not almost surely have locally bounded variation. By definition, for any open $\Omega \subset \mathbb{R}^n$ the random variable $P(Z, \Omega)$ is an unbiased estimator of $\overline{P}(Z)\lambda_n(\Omega)$. The specific perimeter can also be obtained as usual by an averaging process over increasing windows.

Lemma 5.2. *Let Z be a stationary RAMS. Then*

$$\overline{P}(Z) = \lim_{r \to \infty} \frac{\mathbb{E}P(Z \cap rW)}{\lambda_n(rW)}, \tag{5.2}$$

where $W \subset \mathbb{R}^n$ is a compact convex set with positive volume.

Proof. Due to stationarity, we may assume $0 \in \operatorname{int} W$. For $\Omega = r(\operatorname{int} W)$, we have

$$(\partial^* Z) \cap \Omega \subset \partial^*(Z \cap rW) \subset [(\partial^* Z) \cap \Omega] \cup r\partial W.$$

Applying the $(n-1)$st Hausdorff-measure, and taking expectations, yields

$$\mathbb{E}\mathcal{H}^{n-1}(\partial^* Z \cap \Omega) \le \mathbb{E}P(Z \cap rW) \le \mathbb{E}\mathcal{H}^{n-1}(\partial^* Z \cap \Omega) + r^{n-1}\mathcal{H}^{n-1}(\partial W). \tag{5.3}$$

If Z has a.s. locally finite perimeter, a comparison with the definition of $\overline{P}(Z)$ yields (5.2). Otherwise, there is some open bounded set $\tilde{\Omega}$ such that $\mathcal{H}^{n-1}(\partial^* Z \cap \tilde{\Omega}) = \infty$ with positive probability. Then the expectation on the left hand side of (5.3) is infinite for all sufficiently large r, and the limit in (5.2) equals infinity, as required. □

If Z is a stationary RAMS with $\overline{P}(Z) < \infty$, then, for almost all realizations of Z, the generalized inner normal $\Delta_{\mathbf{1}_Z}(z)$ is defined for $\mathcal{H}^{n-1}$-almost all $z \in \partial^* Z$. Consider the random measure on $\mathbb{R}^n \times S^{n-1}$ given by

$$\Psi(B \times U) = \mathcal{H}^{n-1}\{z \in \partial^* Z \cap B : -\Delta_{\mathbf{1}_Z}(z) \in U\},$$

$B \times U \in \mathcal{B}(\mathbb{R}^n \times S^{n-1})$; cf. [19, Proposition 4.2]. Since Ψ is stationary in the first component and with finite intensity, its intensity measure can be disintegrated as

$$\mathbb{E}\Psi(B \times U) = \overline{P}(Z)\lambda_n(B)\mathcal{R}^*(U)$$

with a Borel probability measure $\mathcal{R}^*$ on S^{n-1}. If $\overline{P}(Z) > 0$ then $\mathcal{R}^*$ is uniquely determined and it is called *oriented rose of directions* of Z (cf. [19]). Note that this notion is in general different from the usual *oriented rose of directions* $\mathcal{R}$, which is defined under regularity conditions on Z such that there is an outer normal at $\mathcal{H}^{n-1}$-almost all points in ∂Z. Both notions coincide if $\mathcal{H}^{n-1}(\partial Z \setminus \partial^* Z) = 0$, for instance when Z is a topologically regular element of the extended convex ring, like in the case of a Boolean model Z of full-dimensional convex particles.

We are now ready to prove our second main result.

Proof of Theorem 1.2. If $\overline{p} = 1$ then $Z = \mathbb{R}^n$ almost surely, $\overline{P}(Z) = 0$, and (1.8) holds. For $\overline{p} < 1$ observe that

$$(1-\overline{p})H_Q'(0+) = \lim_{t\to\infty}(t^n\kappa_n)^{-1}\lim_{r\to 0+} r^{-1}, \mathbb{E}\lambda_n(M_{r,t})$$

with the set $M_{r,t} = [(Z \oplus (-rQ)) \setminus Z] \cap B(0,t)$. We may assume $Q \subset B(0,1)$, and abbreviate $Z_s = Z \cap B(0,s)$, $s \ge 0$. For $t > 1$, $r \in (0,1)$ and $R_{r,t}$ being the annulus $B(0,t-1+r) \setminus B(0,t-1)$, we have

$$[Z_{t-1} \oplus (-rQ)] \setminus Z_{t-1} \subset M_{r,t} \cup R_{r,t}$$

and

$$M_{r,t} \subset [Z_{t+1} \oplus (-rQ)] \setminus Z_{t+1}$$

Due to

$$\lim_{t\to\infty}(t^n\kappa_n)^{-1}\lim_{r\to 0+} r^{-1}\lambda_n(R_{r,t}) = 0,$$

$\lim_{t\to\infty} t^n/(t\pm 1)^n = 1$, and $\lambda_n([Z_t \oplus (-rQ)] \setminus Z_t) = G(-rQ, \mathbf{1}_{Z_t})$ we have

$$(1-\overline{p})H'_Q(0+) = \lim_{t\to\infty}(t^n\kappa_n)^{-1} \lim_{r\to 0+} r^{-1}\mathbb{E}G(-rQ, \mathbf{1}_{Z_t}). \tag{5.4}$$

Assume that $\overline{P}(Z) < \infty$. Then (4.1), Lemma 4.4.(i) and Lemma 3.1.(c) imply

$$r^{-1}G(-rQ, \mathbf{1}_{Z_t}) \le \sum_{0\ne u\in Q} V^{\{u\}}(\mathbf{1}_{Z_t}) \le (\#Q)V(\mathbf{1}_{Z_t}),$$

which gives the uniformly integrable upper bound $(\#Q)P(Z\cap B(0,t))$. This allows us to use Lebesgue's dominated convergence theorem for the limit $r \to 0+$ when t is fixed. Hence, Proposition 4.7 gives

$$\lim_{r\to 0+} r^{-1}\mathbb{E}G(-rQ, \mathbf{1}_{Z_t}) = \mathbb{E}V^Q(\mathbf{1}_{Z_t}). \tag{5.5}$$

As

$$(\partial^* Z)\cap \operatorname{int} B(0,t) \subset \partial^* Z_t \subset [(\partial^* Z)\cap \operatorname{int} B(0,t)] \cup tS^{n-1},$$

$\lim_{t\to\infty}(t^n\kappa_n)^{-1}\mathcal{H}^{n-1}(tS^{n-1}) = 0$, $0 \le h(-Q,\,\cdot\,)^+ \le 1$, the definition of $V^Q(\,\cdot\,)$ and (2.9) yield

$$\begin{aligned}\lim_{t\to\infty}\,&(t^n\kappa_n)^{-1}\mathbb{E}V^Q(\mathbf{1}_{Z_t})\\ &= \lim_{t\to\infty}(t^n\kappa_n)^{-1}\mathbb{E}\int_{(\partial^* Z)\cap \operatorname{int} B(0,t)} h(Q, \Delta_{\mathbf{1}_{Z_t}}(x))^+\,\mathcal{H}^{n-1}(dx).\end{aligned}$$

As Δ_{1_Z} is locally defined according to [2, p. 154], we have $\Delta_{\mathbf{1}_{Z_t}}(x) = \Delta_{\mathbf{1}_Z}(x)$ for $\mathcal{H}^{n-1}$-almost every $x \in (\partial^* Z)\cap \operatorname{int} B(0,t)$, so

$$\begin{aligned}\lim_{t\to\infty}(t^n\kappa_n)^{-1}\mathbb{E}V^Q(\mathbf{1}_{Z_t}) &= \lim_{t\to\infty}(t^n\kappa_n)^{-1}\mathbb{E}\int_{\operatorname{int} B(0,t)\times S^{n-1}} h(-Q,v)^+\,\Psi(d(x,v))\\ &= \overline{P}(Z)\int_{S^{n-1}} h(-Q,v)^+\mathcal{R}^*(dv).\end{aligned}$$

The combination of this with (5.5) and (5.4) completes the proof in the case $\overline{P}(Z) < \infty$.

Consider the case where $\overline{P}(Z) = \infty$. Approximating $\mathbf{1}_{Z_t}$ by mollifications $f_j \in C_c^1$ with non-negative ρ, inequality (4.3), Proposition 4.2 and Lemma 3.1.(b) give

$$\liminf_{r\to 0+} r^{-1}G(-rQ, \mathbf{1}_{Z_t}) \ge \liminf_{r\to 0+} r^{-1}G(-rQ, f_j) \ge V^Q(f_j) \ge sV(f_j),$$

where $s > 0$ is the inradius of $\operatorname{conv}(Q\cup\{0\})$; note that the latter set has interior points by assumption. Proposition 2.3.(c) now implies

$$\liminf_{r\to 0+} r^{-1}G(-rQ, \mathbf{1}_{Z_t}) \ge sV(\mathbf{1}_{Z_t}),$$

and insertion into (5.4) and using Lebesgue's dominated convergence theorem gives

$$(1-\overline{p})H'_Q(0+) \ge \lim_{t\to\infty} s\frac{\mathbb{E}P(Z\cap B(0,t))}{t^n\kappa_n} = s\overline{P}(Z) = \infty, \tag{5.6}$$

due to Lemma 5.2.

Now let Z be isotropic. If $\overline{P}(Z) = 0$, the claim is trivial. If $0 < \overline{P}(Z) < \infty$, the measure $\mathcal{R}^*$ is the uniform distribution on S^{n-1} and the definition of the mean width gives the required relation. If $\overline{P}(Z) = \infty$, equation (1.9) holds for $Q = \{0\}$, so we may assume that there is an $u_0 \in S^{n-1}$ and a number $s > 0$ such that $su_0 \in Q$. Then (5.4), $G(-rQ, \cdot\,) \geq sG(r\{-u_0\}, \cdot\,)$, Proposition 4.7 and Lemma 3.2.(a) yield

$$(1-\overline{p})H_Q'(0+) \geq \frac{s}{2} \lim_{t\to\infty} (t^n\kappa_n)^{-1}\mathbb{E}V_{u_0}(\mathbf{1}_{Z_t}).$$

As Z is isotropic, $\mathbb{E}V_{u_0}(\mathbf{1}_{Z_t}) = \mathbb{E}V_u(\mathbf{1}_{Z_t})$ for all $u \in S^{n-1}$, and (2.2) gives

$$(1-\overline{p})H_Q'(0+) \geq \lim_{t\to\infty} s(2n\kappa_n)^{-1} \int_{S^{n-1}} \frac{\mathbb{E}V_u(\mathbf{1}_{Z_{t-1}})}{t^n\kappa_n} \mathcal{H}^{n-1}(du) = \frac{s\kappa_{n-1}}{n\kappa_n}\overline{P}(Z) = \infty.$$

Thus, assertion (1.9) is shown and the proof is complete. □

Note that the only assumption on the random set Z in Theorem 1.2 is stationarity. The use of the bounded variation concept allows us to avoid any kind of integrability condition, which is usually present in similar results. For instance, (1.8) was shown in [13] for "gentle" random sets and compact Q. A variant of (1.8) for non-stationary Z, where $H_Q(\,\cdot\,)$ also depends on the position of (the compact, convex set) Q and on the outer normal of the contact point, was shown in [12] for a grain model with compact convex grains. A related result is given in [23, Theorem 4.1], where the derivative of the spherical contact distribution function of certain non-stationary Boolean models Z is determined for $0 \leq r \leq R$, where R is the reach of the typical grain of Z. Under appropriate assumptions, even the (right sided) second derivative at zero is given there. All three named papers rely on the (local) finiteness of certain measures associated to Z. The price to pay for the generality of Theorem 1.2 are the severe restrictions on the structuring element Q. However, (1.8) cannot hold for general compact Q, as the example of a stationary hyperplane process together with $Q = B(0, 1)$ shows.

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