Differential Geometry

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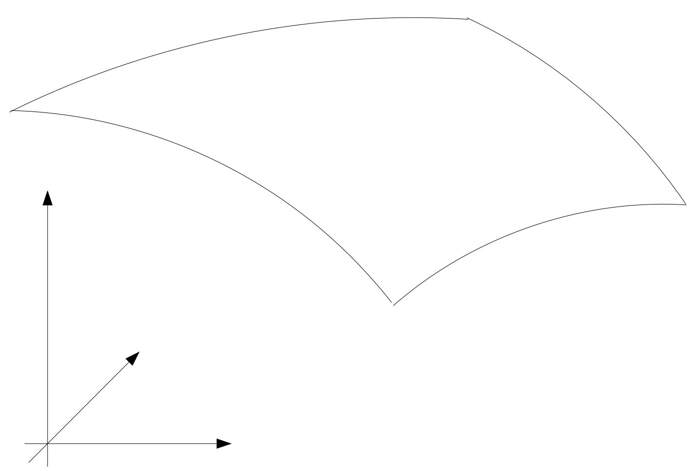
Chapter 0 INTRODUCTION

Differential Geometry concerns geometric concepts studied by means of differential and integral calculus. Its origin goes far back in the history of Mathematics, and an important step in the development was the investigation by C.F. Gauss in 1827 of surfaces in Euclidean 3-space. However, the real break-through in the theory was made by B. Riemann in 1854 in his famous inaugural lecture “Über die Hypothesen, welche der Geometrie zu Grunde liegen”: “On the basic hypotheses underlying Geometry”.

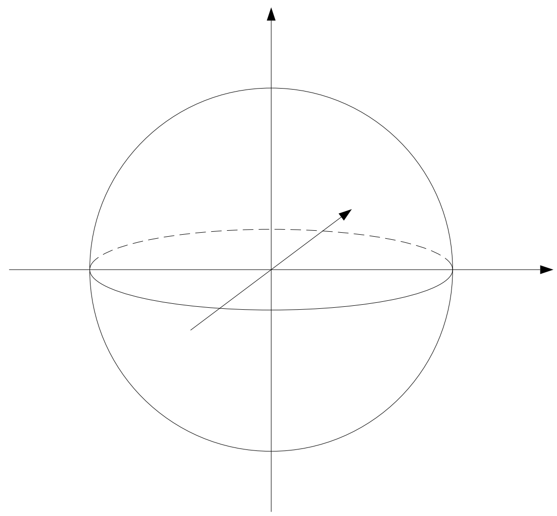
When you read this lecture you will be surprised to find that he here not only made the foundations of Differential Geometry, but also that of other branches of Mathematics, which were not existing at the time: Set-Theory and Topology. In his theory Riemann wanted to create a framework, which on one hand included both Euclidean and the non Euclidean geometries (which were new at the time) and on the other hand would generalize the Gaussian theory of surfaces to higher dimensions. Furthermore, he expected that his theory would be useful in formulating various parts of mathematical physics, a subject he was also studying at about the same time.

As a starting point Riemann took the basic concepts of Euclidean Geometry: *“points”*, *“lines”* and *“distance”*. First of all the *“points”* should constitute the domain of his geometry and this should locally be described by n real parameters, (n the dimension), so for this purpose he introduced the concept of an $n-$*dimensional manifold.*

As an example we can think of a submanifold in Euclidean space, e.g. a surface in 3-space



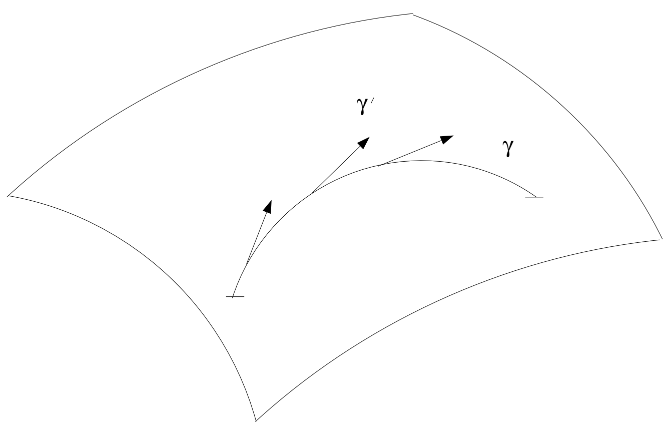
or even more concretely the $n-$sphere in $(n+1)$-space



But the main point made by Riemann was that a manifold is an *intrinsically defined object* disregarded from the surrounding Euclidean space.

The other two concepts “lines” and “distance” are closely related since in Euclid′s “Elements” lines are characterized as the set of points lying “straight” between two given points, i.e. it is a curve, which realizes the shortest distance between two points.

So how do we measure “distances” or rather “arc length” for curves in a manifold? Let us look at our example with a submanifold M in Euclidean space $\mathbb{R}^N$



and let $\gamma : [a,b] \to M$ be a differentiable curve. Now the arc length in $\mathbb{R}^N$ is well-known and is given by the formula

$$L(\gamma) = \int_a^b \parallel \gamma'(t) \parallel dt$$

where $\gamma'(t)$ is the field of tangents along γ and $\|v\|^2 =< v, v >$ defines the norm of the vector $v \in \mathbb{R}^N$. Now again Riemann′s point is that this formula should be *intrinsically* defined in M, that is, the *norm of tangent vectors* to M should be part of the structure; furthermore this norm should come from an inner product in each tangent space. Thus we are led to the following definition.

Definition 0.1

A *Riemannian manifold* is a smooth (C^∞) manifold M together with a *Riemannian metric*, that is, for each point $p \in M$ there is given an *inner product* in the tangent space T_pM, i.e. a *symmetric positive definite bilinear* form.

$$g_p(\cdot, \cdot) : T_pM \times T_pM \to \mathbb{R}.$$

Furthermore g is *smooth* in the sense that for any two smooth vector fields X, Y in M the function $p \to g_p(X_p, Y_p)$ is a smooth function.

Remark

For $u, v \in T_pM$ we shall often write $g_p(u, v) = \langle u, v\rangle_p = \langle u, v\rangle$ depending on the context.

Exercise 0.2

Let $M \subseteq \mathbb{R}^N$ be a submanifold of $\mathbb{R}^N$.Then for each $p \in M$ the tangent space T_pM is naturally identified with a subspace of $\mathbb{R}^N$ and thus inherits the Euclidean inner product from $\mathbb{R}^N$. Show that this defines a Riemannian metric on M in the sense of Definition 0.1.

Return to M a general manifold with Riemannian metric g. The following definition now makes good sense:

Definition 0.3

i) Let $\gamma : [a, b] \to M$ be a smooth curve. Then the *arc length* of γ is defined by

$$L(\gamma) = L_a^b(\gamma) = \int_a^b \| \gamma'(t) \| \, dt \quad \text{where } \|v\|^2 = g_p(v, v) \quad \text{for } v \in T_pM.$$

ii) Let $\gamma : [a, b] \to M$ be *piecewise smooth,* i.e. γ is continuous and there is a subdivision $a = t_0 < t_1 < \ldots < t_k = b$ such that $\gamma|[t_{i-1}, t_i]$ is smooth for $i = 1, \ldots, k$. Then define

$$L(\gamma) = L_a^b(\gamma) = \sum_{i=1}^{k} L_{t_{i-1}}^{t_i}(\gamma)$$

iii) Assume M is *connected* and let $p, q \in M$. Then the *distance* between p and q is defined

$$d(p,q) = \inf \{L(\gamma) \mid \gamma \text{ piecewise smooth, } \gamma : [a,b] \to M, \gamma(a) = p, \gamma(b) = q\}$$

Remark

We shall see later that d defines a metric in M in the usual sense and that this metric defines the same topology as the given one (M being a manifold is a Hausdorff space to begin with!)

Coming back to the geometric concept of a "line" as defined in Euclid′s book, we would again like to define a line in a Riemannian manifold as a set of points such that the arc length between any two points on the line equals the distance in the sense of Definition 0.3. However, this raises the following two questions:

Question 1

Given two points $p, q \in M$; does there exist an arc, called a *geodesic curve*, connecting the two points and whose arc length equals the distance?

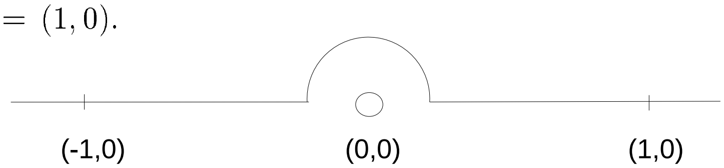
Question 2

Is the curve in Question 1, if it exists, uniquely determined?

Unfortunately the answers to both questions are "No":

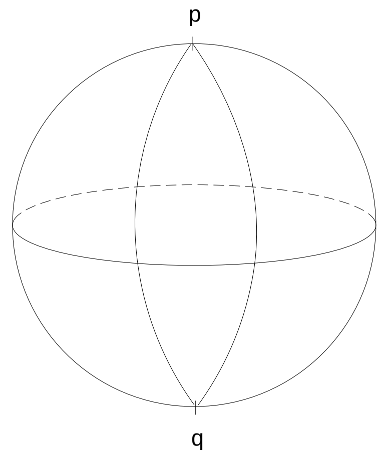
Examples 0.4

i) Let $M = \mathbb{R}^2 - \{(0,0)\}$ with the Riemannian metric inherited from $\mathbb{R}^2$. Let $p = (-1,0)$, $q = (1,0)$.



Clearly, as seen on the figure, $d(p,q) = 2$. However, a geodesic in M would also be a geodesic in $\mathbb{R}^2$, and since that is unique it would contain the point $(0,0)$, which is not allowed, so there is no geodesic curve connecting p and q inside M.

ii) Let $M = S^2 \subseteq \mathbb{R}^3$, $p =$ North pole, $q =$ South pole



The shortest length of an arc from p to q is realized by any half great circle through p and q. Hence there are infinitely many geodesic curves from p to q.

In spite of these examples it turns out that geodesic curves do exist “locally”, that is, if the endpoints p and q are not too far apart, and also in this case the geodesic curves are unique. To show these facts will be our first task in the development of Riemannian Geometry. However, before we get that far, we need some preparations.

Chapter 1 DIFFERENTIABLE MANIFOLDS

In this chapter we review the basic notions of differentiable manifolds

Definition 1.1

Let M be a Hausdorff space with a countable basis for the topology. Let $n \in \mathbb{N}$ (or zero).

i) A *chart* or *coordinate system* on M is a pair $(U, \mathbf{x})$, $U \subseteq M$ open , $\mathbf{x} : U \to U' \subseteq \mathbb{R}^n$ a homeomorphism onto an open set.

ii) Two charts $(U, \mathbf{x})$ and $(V, \mathbf{y})$ are said to have *smooth overlap* if

$$\mathbf{y} \circ \mathbf{x}^{-1} : \mathbf{x}(U \cap V) \to \mathbf{y}(U \cap V)$$

and

$$\mathbf{x} \circ \mathbf{y}^{-1} : \mathbf{y}(U \cap V) \to \mathbf{x}(U \cap V)$$

are smooth mappings (hence diffeomorphisms).

iii) An *atlas* for a *differentiable structure* on M is a set $\mathcal{A} = \{U_\alpha, \mathbf{x}_\alpha\}_{\alpha \in I}$ of charts with pairwise smooth overlap, such that $\{U_\alpha\}$ covers $M,$ i.e. $M = \cup_{\alpha \in I} U_\alpha$. Two atlasses $\mathcal{A}$ and $\mathcal{A}'$ define the same *differentiable structure* if $\mathcal{A} \cup \mathcal{A}'$ is again an atlas.

iv) A *differentiable manifold* of dimension n is a Hausdorff space M as above with a differentiable structure, i.e. with an atlas $\mathcal{A}$. A chart $(U, \mathbf{x})$ belongs to the differentiable structure for $M,$ if it belongs to an atlas for this, i.e. if it has smooth overlap with any chart in $\mathcal{A}$. We shall often write M or M^n to denote the differentiable manifold.

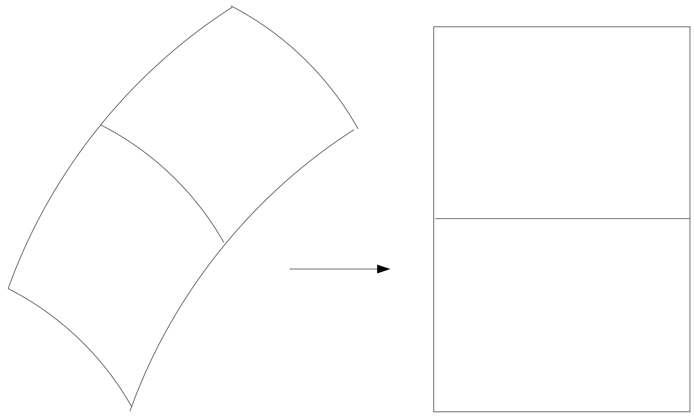
Notation

If $(U, \mathbf{x})$ is a chart we can write $\mathbf{x} = \big(x^1, \ldots, x^n\big)$. Then $x^i : U \to \mathbb{R}$ are called the *local coordinates* for the manifold.

Definition 1.2

Let N^n be a differentiable manifold. A *submanifold* $M^m \subseteq N^n$ is a set such that for each $p \in M$ there exist charts $(U, \mathbf{x})$ around p in N such that

$$M \cap U = \big\{q \in U \bigm| x^{m+1}(q) = x^{m+2}(q) = \ldots = x^n(q) = 0\big\}$$



Exercise 1.3

a) Show that for $M \subseteq N$ a submanifold, the set of charts $(U \cap M,\ \mathbf{x}|U \cap M)$ where $(U, \mathbf{x})$ are charts in N as in definition 1.2, makes M into a manifold.

b) Show that $S^n \subseteq \mathbb{R}^{n+1}$, $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ is a submanifold of $\mathbb{R}^{n+1}$.

c) Let $M \subseteq \mathbb{R}^2$ be the subset

$$M = \{(x_1, x_2) \mid |x_1| + |x_2| = 1\},$$

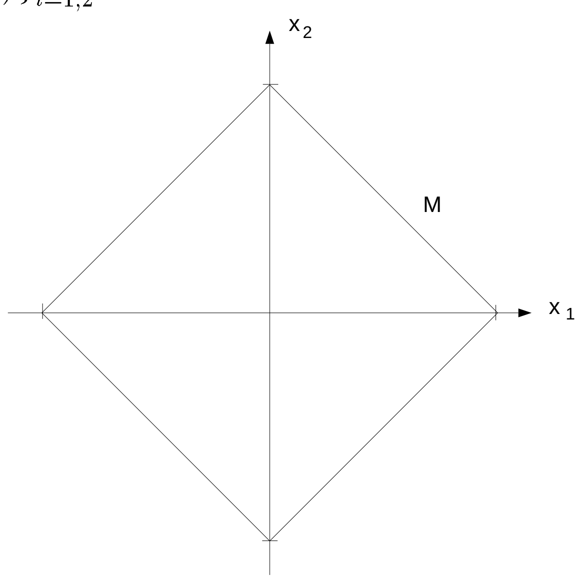
cover M by the sets

$$U^i_{\pm} = \{(x_1, x_2) \in M \mid \pm x_i > 0\}\ ,\ i = 1, 2,$$

and let $\mathbf{x}^i_{\pm} : U^i_{\pm} \to \mathbb{R}$ be defined by

$$\mathbf{x}^1_{\pm}(x_1, x_2) = x_2\ ,\ \mathbf{x}^2_{\pm}(x_1, x_2) = x_1.$$

Show that $\{(U^i_{\pm}\ ,\ \mathbf{x}^i_{\pm})\}_{i=1,2}$ is an atlas for a differentiable structure on M; but M



is *not* a submanifold.

Definition 1.4

A continuous mapping $f: M^m \to N^n$ between two manifolds is called *smooth* or *differentiable* if for every chart $(U, \mathbf{x})$ on M and $(V, \mathbf{y})$ on N

$$(*) \qquad \begin{array}{rcl} \mathbf{y}\circ f\circ \mathbf{x}^{-1}: & \mathbf{x}\big(f^{-1}(V)\cap U\big) & \to \mathbf{y}(V) \\ & \cap| & \quad \cap| \\ & \mathbb{R}^m & \quad \mathbb{R}^n \end{array}$$

is smooth.

Notation

The set of smooth functions $f: M \to \mathbb{R}$ is denoted $C^\infty(M)$.

The Tangent Space

Let M be an $n-$dimensional differentiable manifold and let $p \in M$.

Definition 1.5

i) A tangent vector at p is an equivalence class of pairs $(\mathbf{x}, v)$ where $(U, \mathbf{x})$ is a chart around p and $v \in \mathbb{R}^n$ a vector.

$$\begin{array}{ll} & (\mathbf{x}, v) \underset{p}{\sim} (\mathbf{y}, w) \\ \text{if} & D\big(\mathbf{x}\circ\mathbf{y}^{-1}\big)_{\mathbf{y}(p)}(w) = v \qquad D(f) = \text{ differential of } f. \end{array}$$

$[\mathbf{x}, v]_p$ denotes an equivalence class, $v = (v_1, \ldots, v_n)$ are called the *coordinates* for the tangent vector in the coordinate system $(U, \mathbf{x})$.

ii) The set of tangent vectors in p is denoted T_pM and is called the *tangent space.*

The following is an easy consequence of the definition

Proposition 1.6

α) *T_pM is an $n-$dimensional vector space over $\mathbb{R}$.*

β) *If $(U,\mathbf{x})$ is a chart around p, then there is a natural isomorphism $\mathbf{x}_*: T_pM \to \mathbb{R}^n$ given by*

$$\mathbf{x}_*[\mathbf{x}, v]_p = v$$

γ) *If* $(\mathbf{y}, V)$ *is some other chart around* p *we have a commutative diagram*

$$\begin{array}{ccc} & T_pM & \\ \mathbf{y}_*\swarrow & & \searrow \mathbf{x}_* \\ \mathbb{R}^n & \overset{D(\mathbf{x}\circ\mathbf{y})_{\mathbf{y}\,(p)}}{\longrightarrow} & \mathbb{R}^n \end{array}$$

We can now talk about tangent vectors to arcs in M :

Definition 1.7

Let $\gamma : (a,b) \to M$, $(a,b) \subseteq \mathbb{R}$ be a differentiable curve and let $t_0 \in (a,b)$ with $\gamma(t_0) = p$. The tangent $\gamma'(t_0) \in T_pM$ is defined by $\frac{d\gamma}{dt}(t_0) = \gamma'(t_0) = [\mathbf{x}, v]_p$ where $v = \frac{d}{dt}(\mathbf{x} \circ \gamma)\Big|_{t=t_0}$ for $(U,\mathbf{x})$ any chart around p.

(*Note:* this is well-defined!)

Remark

For $M = U \subseteq \mathbb{R}^n$ an open set and $\mathbf{x} = \mathrm{id}$ we get a natural isomorphism $\mathrm{id}_* : T_pU \overset{\cong}{\to} \mathbb{R}^n$. We shall usually identify T_pU with $\mathbb{R}^n$ via this isomorphism. More generally for $M \subseteq \mathbb{R}^n$ a submanifold (see definition 1.2) the tangent space T_pM for $p \in M$ naturally identifies with a linear subspace of $\mathbb{R}^n$, namely the subspace of tangent vectors to curves through p lying entirely in M.

Another way of characterizing tangent vectors is in terms of the associated *directional derivation:*

Definition 1.8

Let $X_p = [\mathbf{x}, v]_p \in T_pM$ and let $f \in C^\infty(M)$. Then the *directional derivative* of f with respect to X_p is

$$X_p(f) = v_{\mathbf{x}(p)}\big(f \circ \mathbf{x}^{-1}\big) = D\big(f \circ \mathbf{x}^{-1}\big)_{\mathbf{x}(p)}(v)$$

(*Note:* again this is well-defined.)

Notation

Let $(U,\mathbf{x})$ be a chart around p and let $e_1 \ldots e_n \in \mathbb{R}^n$ be the canonical basis vectors, that is, $e_i = (0, \ldots, 0, 1_i, \ldots, 0)$. Then we will denote the corresponding tangent vectors in T_pM

$$\frac{\partial}{\partial x^1}\bigg|_p \;, \ldots, \;\frac{\partial}{\partial x^n}\bigg|_p \;, \quad \text{that is,} \quad \frac{\partial}{\partial x^i}\bigg|_p = [\mathbf{x}, e_i]_p$$

Remark

If $f \in C^\infty(M)$ then the derivation corresponding to $\frac{\partial}{\partial x^i}\Big|_p$ is given by

$$\frac{\partial}{\partial x^i}\Bigg|_p (f) = D\big(f \circ \mathbf{x}^{-1}\big)_{\mathbf{x}(p)}(e_i) = \frac{\partial\big(f \circ \mathbf{x}^{-1}\big)}{\partial x^i}\Bigg|_{\mathbf{x}(p)} .$$

The directional derivative $f \to X_p(f) : C^\infty(M) \to \mathbb{R}$ is a derivation in the following sense:

Definition 1.9

$l : C^\infty(M) \to \mathbb{R}$ is a *derivation at* p if l is $\mathbb{R}-$linear and

$$l(f \cdot g) = l(f) \cdot g(p) + f(p)\, l(g)$$

for all $f, g \in C^\infty(M)$.

Theorem 1.10

If $l : C^\infty(M) \to \mathbb{R}$ is a derivation at p then there is a unique $X_p \in T_pM$ such that $l = X_p(\cdot)$.

For the proof we need a few lemmas.

Lemma 1.11

Let $f \in C^\infty(U)$, U open neighbourhood of $p \in M$. Then there is $\tilde{f} \in C^\infty(M)$ such that $\tilde{f}$ and f agree in a neighbourhood $V \subseteq U$ and $\tilde{f} \equiv 0$ outside a bigger neighbourhood W , $V \subseteq W \subseteq \overline{W} \subseteq U$.

Proof

By taking U smaller we can suppose U is the domain of a chart $\mathbf{x} : U \to U' \subseteqq \mathbb{R}^n$. Now choose a "bump-function" $\phi : U' \to \mathbb{R}$, that is, ϕ is smooth, $\phi \equiv 1$ on some $V' \subseteqq U'$ with $\mathbf{x}(p) \in V'$ and $\phi \equiv 0$ outside $W' \supseteq V'$, $\overline{W'} \subseteq U'$. Then put

$$\tilde{f}(q) = \begin{cases} 0 & \text{outside } W = \mathbf{x}^{-1}(W') \\ (\phi \circ \mathbf{x}(q)) \cdot f(q) & q \in W \end{cases}$$

Lemma 1.12

Let $f \in C^\infty(U)$, where $(U, \mathbf{x})$ is a chart around p. Assume $\mathbf{x}(p) = 0$ and $\mathbf{x}(U) \subseteq \mathbb{R}^n$ is an open ball with centre 0. Then there exist $g_i \in C^\infty(U)$, $i = 1, \ldots, n$, such that

1) $f(q) = f(p) + \sum\limits_{i=1}^{n} x^i(q) \cdot g_i(q) \qquad \forall q \in U$

2) $g_i(p) = \frac{\partial f}{\partial x^i}(p)$

Proof

It suffices to take $U \subseteq \mathbb{R}^n$an open ball, $p = 0$ and $\left(x^1, \ldots, x^n\right)$ the identity chart. Then we must show that for f defined in $U \subseteq \mathbb{R}^n$ we have

$$(*) \qquad f\left(x^1, \ldots, x^n\right) = f(0) + \sum_{i=1}^{n} x^i g_i\left(x^1, \ldots, x^n\right)$$

for some $g_i : U \to \mathbb{R}$ with $g_i(0) = \frac{\partial f}{\partial x^i}\ (0)$.

To show (*) consider $f\left(tx^1, \ldots, tx^n\right), \quad t \in [0, 1]$,

$$\begin{aligned} f\left(x^1, \ldots, x^n\right) - f(0) &= \int_0^1 \frac{\partial}{\partial t}\, f\left(tx^1, \ldots, tx^n\right) dt \\ &= \sum_{i=1}^{n} \int_0^1 x^i\ (D_i f)\left(tx^1, \ldots, tx^n\right) dt \\ &= \sum_{i=1}^{n} x^i g_i\left(x^1, \ldots, x^n\right) \end{aligned}$$

with

$$g_i\left(x^1, \ldots, x^n\right) = \int_0^1 D_i f\left(tx^1, \ldots, tx^n\right) dt, \qquad g(0) = \int_0^1 D_i f(0) dt = \frac{\partial f}{\partial x^i}\ (0).$$

Proof of theorem 1.10

Let $l : C^\infty(M) \to \mathbb{R}$ be a derivation at p.

I. $l(1) = l(1 \cdot 1) = l(1) \cdot 1 + 1 \cdot\ l(1)$. Hence $l(1) = 0$. By linearity $l(k) = 0$ for k any constant.

II. If f is zero in a neighbourhood of p then $l(f) = 0$. In fact as in the proof of lemma 1.11 we can find $h \in C^\infty(M)$, such that $h \equiv 1$ near p and $h \equiv 0$ outside the neighbourhood where f vanishes. Hence $f \cdot h \equiv 0$ so that

$$0 = l(f \cdot h) = l(f) \cdot 1 + f(p) \cdot l(h) = l(f)$$

We conclude that if $f, g \in C^\infty(M)$ and $f \equiv g$ in a neighbourhood of p then $l(f) = l(g)$.

III. By step II we can extend l to all C^∞ functions, which are just defined in some neighbourhood of p, simply by setting $l(f) = l\left(\tilde{f}\right)$ where f is defined near p , $\tilde{f} \in C^\infty(M)$ and f and $\tilde{f}$ agree in a neighbourhood of p. Given $f, \tilde{f}$ exists by lemma 1.11. Extended in this way l is still a derivation on the set of all such locally defined functions.

IV. Now let $(U,\mathbf{x})$ be a chart around p as in lemma 1.12. Then

(**) $$l = \sum_{i=1}^{n} l\big(x^i\big) \frac{\partial}{\partial x^i}\Big|_p$$

In fact by that lemma, $f = f(p) + \sum_{i=1}^{n} x^i g_i$, and since $\mathbf{x}(p) = 0$

$$\begin{aligned} l(f) &= 0 + \sum_{i=1}^{n} l\big(x^i\big) \cdot g_i(p) + 0 \\ &= \sum_{i=1}^{n} l\big(x^i\big)\ \frac{\partial f}{\partial x^i}(p). \end{aligned}$$

Now the right hand side of (**) is the derivation by a tangent vector, which shows the theorem.

Definition 1.13

Let $f : M \to N$ be a differentiable mapping between differentiable manifolds. The *differential* of f or the *tangent mapping* $f_*(= Tf = df)$ at the point $p \in M$ is defined by

$$f_*[\mathbf{x}, v]_p = \Big[\mathbf{y},\ D\big(\mathbf{y} \circ f \circ \mathbf{x}^{-1}\big)_{\mathbf{x}(p)}(v)\Big]_{f(p)}$$

where $(U,\mathbf{x})$ and $(V,\mathbf{y})$ are charts around p and $f(p)$ respectively.

(*Note:* This is well-defined.)

Remark

For $M = U \subseteq \mathbb{R}^m$, $N = V \subseteq \mathbb{R}^n$ open sets, the differential of $f : U \to V$ at $p \in U$ identifies with the usual differential $Df_p : \mathbb{R}^m \to \mathbb{R}^n$ via the natural isomorphisms $T_pU = \mathbb{R}^m$, $T_{f(p)}V = \mathbb{R}^n$.

The following is straight forward:

Proposition 1.14

i) *$f_* : T_pM \to T_{f(p)}N$ is linear.*

ii) *Let $X_p \in T_pM$. Then f_*X_p corresponds to the derivation*

$$(f_*X_p)(u) = X_p(u \circ f) \qquad u \in C^\infty(N)$$

iii) *The identity* $\mathrm{id} : M \to M$ has $\mathrm{id}_* = \mathrm{id} : T_pM \to T_pM$, *and if* $f : M \to N$ *and* $g : N \to L$ *are differentiable mappings we have the "chain-rule"*

$$(g \circ f)_* = g_* \circ f_* \ , \quad \textit{that is, } (g \circ f)_{*p} = g_{*f(p)} \circ f_{*p} :$$

$$\begin{tikzcd} T_pM \arrow[rr, "f_*"] \arrow[dr, "(g\circ f)_*"'] & & T_{f(p)}N \arrow[dl, "g_*"] \\ & T_{(g\circ f)(p)}L & \end{tikzcd}$$

Smooth Vector Fields

Definition 1.15

A family of tangent vectors $\{X_p\}_{p\in M}$, $X_p \in T_pM$, is called a *smooth vector field* if for every chart $(U, \mathbf{x})$ the mapping $U \to \mathbb{R}^n$ given by $p \mapsto \mathbf{x}_* X_p$ is differentiable.

Remark

For $M = U \subseteq \mathbb{R}^n$ we can identify a smooth vector field X with a smooth map $X : U \to \mathbb{R}^n$ using the natural identification $T_pU \cong \mathbb{R}^n$. More generally for $M \subseteq \mathbb{R}^n$ a submanifold we shall identify a smooth vector field X on M with a smooth map $X : M \to \mathbb{R}^n$ such that $X(p) \in T_pM \subseteq \mathbb{R}^n$, for all $p \in M$ (cf. remark following definition 1.7).

Example 1.16

For $(U, \mathbf{x})$ a chart and $i = 1, \ldots, n$, $\left\{ \frac{\partial}{\partial x^i} \Big|_p \right\}_{p\in U}$ is a smooth vector field in U, which we of course denote $\frac{\partial}{\partial x^i}$.

If now X is any vector field in U, then by definition

$$X = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}$$

where $v^i : U \to \mathbb{R}$ are smooth iff X is smooth.

Exercise 1.17

A family of tangent vectors $X_p \in T_pM$, $p \in M$, defines a smooth vector field iff for every $f \in C^\infty(M)$ the function $X(f)$ defined by

$$X(f)(p) = X_p(f) \qquad p \in M$$

is smooth.

Next let us define the *Lie bracket* of two smooth vector fields X and Y on M : For each $p \in M$ define

$$[X,Y]_p(f) = X_p(Y(f)) - Y_p(X(f)) \qquad f \in C^\infty(M)$$

Proposition 1.18

i) *The mapping $[X,Y]_p(\cdot) : C^\infty(M) \to \mathbb{R}$ is a derivation, hence defines a tangent vector at p.*

ii) *The family $[X,Y]_p$, $p \in M$, defines a smooth vector field.*

Proof

Let $f, g \in C^\infty(M)$. Then

$$Y(f \cdot g) = Y(f) \cdot g + f \cdot Y(g)$$

Hence

$$X_p(Y(f \cdot g)) = X_p(Y(f)) \cdot g(p) + Y_p(f) \cdot X_p(g) + X_p(f) \cdot Y_p(g) + f(p) \cdot X_p(Y(g))$$

Similarly

$$Y_p(X(f \cdot g)) = Y_p(X(f)) \cdot g(p) + X_p(f) \cdot Y_p(g) + Y_p(f) \cdot X_p(g) + f(p) Y_p(X(g))$$

Subtracting we get

$$[X,Y]_p\,(f \cdot g) = [X,Y]_p(f) \cdot g(p) + f(p)[X,Y]_p(g)$$

which shows i).

ii) is obvious in view of exercise 1.17.

Definition 1.19

For X, Y smooth vector fields the vector field $[X,Y]$ given by $[X,Y]_p$ above is called the *Lie bracket* of X and Y.

Notation

Sometimes we shall write

$$L_X(Y) = [X,Y].$$

Remark

If $(U, \mathbf{x})$ is any chart then the vector fields $\frac{\partial}{\partial x^i}$ satisfy

$$\left[\frac{\partial}{\partial x^i}\,,\;\frac{\partial}{\partial x^j}\right] = 0 \qquad \forall\, i, j.$$

The following proposition is straight forward

Proposition 1.20

The Lie bracket satisfies

i) $[X, Y]$ *is* $\mathbb{R}-$*linear in both* X *and* Y,

ii) $L_X(f \cdot Y) = X(f) \cdot Y + f L_X(Y)\;,\quad f \in C^\infty(M)$,

iii) *(Jacobi identity)*
$[X, [Y, Z]\,] + [Y, [Z, X]\,] + [Z, [X, Y]\,] = 0$,

iv) *equivalently*
$L_X([Y, Z]) = [L_X(Y), Z] + [Y, L_X(Z)]$,

v) $[X, Y] = \;-[Y, X]$.

Now let us return to the notion of a Riemannian metric:

Recall that a Riemannian metric on M is a collection of inner products

$$g_p(\cdot, \cdot)\;\; : T_pM \times T_pM \to \mathbb{R}$$

such that for X, Y smooth vector fields on M the function $p \mapsto g_p(X_p, Y_p)$ is smooth. Now suppose (U,$\mathbf{x}$) is a chart; then in particular the functions $g_{ij} = g\big(\frac{\partial}{\partial x^i}\,,\;\frac{\partial}{\partial x^j}\big)$ are smooth and for each $p \in U$ the symmetric matrix $\{g_{ij}(p)\}$ determines the inner product g_p in T_pM. In fact, if $v, w \in T_pM$ have coordinates

$$v = \sum_i v^i \frac{\partial}{\partial x^i}\bigg|_p \qquad\qquad w = \sum_j w^j \frac{\partial}{\partial x^j}\bigg|_p$$

Then

$$\begin{aligned} g_p(v, w) &= g_p\left(\sum_i v^i \frac{\partial}{\partial x^i}\bigg|_p\,,\;\sum_j w^j \frac{\partial}{\partial x^j}\bigg|_p\right) \\ &= \sum_{i,j} v^i\, w^j\; g_p\left(\frac{\partial}{\partial x^i}\bigg|_p\,,\;\frac{\partial}{\partial x^j}\bigg|_p\right) \\ &= \sum_{i,j} v^i\, w^j\; g_{ij}(p) \end{aligned}$$

That is, g_p is the bilinear form given by the matrix $\{g_{ij}(p)\}$ with respect to the basis $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \right\}$. Notice that since g_p is symmetric and positive definite, the matrix $\{g_{ij}(p)\}$ is a positive symmetric matrix.

Remark

Notice that the Euclidean metric in $U \subset \mathbb{R}^n$ with coordinates $\left(x^1, \ldots, x^n\right)$ corresponds to the matrix

(***) $$g_p = \begin{Bmatrix} 1 & & \bigcirc \\ & \ddots & \\ \bigcirc & & 1 \end{Bmatrix} \quad \forall\, p.$$

Therefore if (U,**x**) is a chart in a manifold M with metric g such that (***) holds, then we shall say that the metric with respect to $(U, \mathbf{x})$ is Euclidean or *flat*. We shall see later that given a metric we can *not* always find a local chart, which makes the metric Euclidean.

Theorem 1.21

Every differentiable manifold has at least one Riemannian metric.

Proof

Let M be an arbitrary manifold. To begin with let (U,**x**) be a chart and observe that at least in U we can find a Riemannian metric simply by choosing the Euclidean one with respect to **x**, i.e. such that (***) holds. Hence if we cover M by coordinate charts $\{(U_\alpha\ ,\ \mathbf{x}_\alpha)\}_{\alpha \in I}$ then in each U_α we can find a metric

$$g^{(\alpha)}(p)(\cdot,\cdot)\ : T_pM \times T_pM \to \mathbb{R}, \quad p \in U_\alpha.$$

Now we can choose a *partition of unity* subordinate $\{U_\alpha\}_{\alpha \in I}$, that is, a family $\{\phi_\alpha\}_{\alpha \in I}$ of smooth functions $\phi_\alpha : M \to \mathbb{R}$, such that

i) $0 \le \phi_\alpha \le 1$

ii) $\mathrm{Supp}\ \phi_\alpha \subseteq U_\alpha$

iii) For each $p \in M$ there is a neighbourhood V, which intersects $\mathrm{Supp}\ \phi_\alpha$ for only finitely many α and $\sum_\alpha \phi_\alpha = 1$.

Then we can define a metric g by

$$g_p(v,w) = \sum_\alpha \phi_\alpha(p) \cdot g_p^{(\alpha)}(v,w).$$

In fact for fixed p, g_p is a *finite* convex combination of positive symmetric bilinear functions and hence is a positive definite symmetric bilinear function. Furthermore g is *smooth:*

Given X and Y smooth vector fields the function

$$p \mapsto g_p(X_p, Y_p) = \sum_\alpha \phi_\alpha(p) \cdot g_p^{(\alpha)}(X_p, Y_p)$$

is *locally* a *finite* sum of smooth functions. This proves the theorem.

Appendix A PARTITION OF UNITY

Definition A1

Let X be a topological space and $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ a covering of X.

1) A covering $\mathcal{V} = \{V_\beta\}_{\beta \in J}$ is called a refinement of $\mathcal{U}$ if, for every $\beta \in J$, there is an $\alpha \in I$ such that $V_\beta \subseteq U_\alpha$.

2) A set of subsets $\{A_\alpha\}_{\alpha \in I}$ of X is called *locally finite* if every point of X has a neighbourhood U such that $U \cap A_\alpha \neq \emptyset$ for at most finitely many $\alpha \in I$.

Definition A2.

Let M be a smooth manifold and $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ an open covering of M. A *partition of unity subordinate* $\mathcal{U}$ is a family $\{\phi_\alpha\}_{\alpha \in I}$ of C^∞ functions on M such that the following holds:

i) $0 \leq \phi_\alpha \leq 1$ and $\operatorname{Supp} \phi_\alpha \subseteq U_\alpha$, for all $\alpha \in I$,

ii) $\{\operatorname{Supp} \phi_\alpha \mid \alpha \in I\}$ is locally finite,

iii) for all $p \in M$ we have

$$\sum_{\alpha \in I} \phi_\alpha(p) = 1$$

Theorem A3

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of a smooth manifold M. Then there exists a partition of unity subordinate $\mathcal{U}$.

For the proof we need a few preparations:

Notation

For $r > 0$ put

$$K(r) = \left\{ x \in \mathbb{R}^m \;\middle|\; |x| < r \right\}.$$

We shall use without proof the existence of *"bump functions"* (see e.g. Warner [ ]):

Lemma A4

There exists a non-negative smooth function ψ on $\mathbb{R}^m$ such that

i) $\psi | \overline{K(1)} = 1$

ii) $\operatorname{Supp} \psi \subseteq K(2)$.

Next we prove:

Lemma A5

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of an $m-$dimensional smooth manifold M Then there exists an atlas

$$\mathcal{A} = \left\{\mathbf{x}_i : V_i \to V_i' \subseteq \mathbb{R}^m\right\}, \ i \in J,$$

with J countable, such that

a) $\{V_i\}_{i \in J}$ *is a locally finite refinement of $\mathcal{U}$,*

b) $V_i' = K(3)$,

c) *If we put $W_i = \mathbf{x}_i^{-1}(K(1))$, then $\mathcal{W} = \{W_i\}_{i \in J}$ is a covering of M.*

Proof

1. First we construct a sequence of compact subsets A_n , $n \in \mathbb{N}$, such that $A_n \subseteq \mathring{A}_{n+1}$ and $\cup_{n=1}^{\infty} \mathring{A}_n = M$. In fact, since M is locally compact with a countable basis, we can cover M by countably many open sets $\mathcal{O} = \{O_1, \ldots, O_n, \ldots\}$ such that $\overline{O}_n$ is compact for every n. Then we define A_n by induction: For $n = 1$ put $A_1 = \overline{O}_1$. For higher n suppose that A_n is found such that $O_1 \cup \ldots \cup O_n \subseteq A_n$. Then since A_n is compact and $\mathcal{O}$ is a covering we can find an $N \geq n + 1$ such that

$$A_n \subseteq O_1 \cup \ldots \cup O_N.$$

Then we put $A_{n+1} = \overline{O}_1 \cup \ldots \cup \overline{O}_N$.

2. Now fix $n \in \mathbb{N}$. Then we have (putting $A_0 = A_{-1} = \emptyset$)

$$C = A_{n+1} \backslash \mathring{A}_n \subseteq \mathring{A}_{n+2} \backslash A_{n-1} = B$$

where C is compact and B is open. Around every point of C we can now choose a chart $\mathbf{x} : V \to V' = K(3)$ such that for some $\alpha \in I$ we have $V \subseteq B \cap U_\alpha$. Let $W = \mathbf{x}^{-1}(K(1))$. Since C can be covered by finitely many such W's we obtain a finite set of charts

$$(V_{n,\alpha_1}, \ \mathbf{x}_{n,\alpha_1}), \ldots, \left(V_{n,\alpha_{k_n}}, \ \mathbf{x}_{n,\alpha_{k_n}}\right) \tag{A.6}$$

such that

$$A_{n+1} \backslash \mathring{A}_n \subseteq W_{n,\alpha_1} \cup \ldots \cup W_{n,\alpha_{k_n}}.$$

Let $\mathcal{A}$ be the atlas consisting of all the charts $\{(V_i, \mathbf{x}_i)\}_{i \in J}$ in (A.6) for all $n \in \mathbb{N}$. Then clearly $\mathcal{V} = \{V_i\}_{i \in J}$ is a refinement of $\mathcal{U}$.

By construction there are at most finitely many $V_i{}'$s inside $\mathring{A}_n$, hence $\mathcal{V}$ is locally finite. also since

$$M = \bigcup_{n=1}^{\infty} A_n \backslash A_{n-1} \subseteq \bigcup_{n=1}^{\infty} A_{n+1} \backslash \mathring{A}_{n-1}$$

it is clear that the $W_i{}'$s cover M. This proves the lemma since J is clearly countable.

Proof of theorem A3.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of M and choose an atlas $\mathcal{A}$ as in lemma A5. Choosing a "bump function" as in lemma A4 we define $\psi_i : M \to \mathbb{R}$, $i \in J$, by

$$\psi_i(p) = \begin{cases} \psi \circ \mathbf{x}_i(p) \quad , & p \in V_i, \\ 0, & p \notin V_i. \end{cases}$$

Then clearly ψ_i is smooth and satisfies

i) $0 \leq \psi_i \leq 1$,

ii) $\operatorname{Supp} \psi_i \subseteq \mathbf{x}_i^{-1}(K(2))$,

iii) $\psi | W_i \equiv 1$.

Note that $S : M \to \mathbb{R}$ defined by

$$S(p) = \sum_{i \in J} \psi_i(p)$$

is well-defined and smooth since $\mathcal{V}$ is locally finite. Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, we can choose a function $\gamma : J \to I$ such that $V_i \subseteq U_{\gamma(i)}$ for all $i \in J$. Now we define for each $\alpha \in I$

$$\phi_\alpha(p) = \sum_{\gamma(i)=\alpha} \psi_i(p)/S(p) \ , \qquad p \in M$$

(and $\phi_\alpha = 0$ if $\gamma(i) \neq \alpha$ for all $i \in J$). Note that since the $W_i{}'$s cover M we have $S(p) \geq 1$ for all $p \in M$, and also ϕ_α is well-defined and smooth on M. Clearly

$$0 \leq \sum_{\gamma(i)=\alpha} \psi_i \leq \sum_{i \in J} \psi_i = S$$

so that $0 \leq \phi_\alpha \leq 1$. Also

$$\operatorname{Supp} \phi_\alpha \subseteq \bigcup_{\gamma(i)=\alpha} \operatorname{Supp} \psi_i \subseteq \bigcup_{\gamma(i)=\alpha} V_i \subseteq U_\alpha$$

and $\{\operatorname{Supp} \phi_\alpha\}_{\alpha \in I}$ is locally finite. In fact, suppose U is a neighbourhood of p such that only $V_{i_1}, \ldots, V_{i_k}$ has non-empty intersection with U. Then $\operatorname{Supp} \phi_\alpha \cap U \neq \emptyset$ only if $\alpha \in \{\gamma(i_1), \ldots, \gamma(i_k)\}$ which is a finite set. Finally we clearly have

$$\sum_{\alpha \in I} \phi_\alpha(p) = \sum_{\alpha \in I} \sum_{\gamma(i)=\alpha} \psi_i(p)/S(p)$$
$$= \sum_{i \in J} \psi_i(p)/S(p) = 1.$$

Hence $\{\phi_\alpha\}_{\alpha \in J}$ is a partition of unity subordinate $\mathcal{U}$.

Chapter 2 CONNECTIONS

In the introduction we mentioned the problem of finding the curve between two given points on a Riemannian manifold such that the arc length is minimal. This is actually a classical problem in Variational Calculus and for a solution curve there is a necessary condition known as "the Euler differential equation". It turns out that this is most conveniently expressible in terms of a first order differential operator called a "linear connection" associated with the Riemannian metric.

We therefore start by studying the formal properties of such a linear connection.

Definition 2.1

Let M be a smooth manifold. A *linear connection* in M is an operator $\bigtriangledown$ ("dell"), which to any two smooth vector fields X and Y on M associates a third $\bigtriangledown_X(Y)$ such that

1) $\bigtriangledown_X(Y)$ is smooth.

2) $\bigtriangledown_{X_1+X_2}(Y) = \bigtriangledown_{X_1}(Y) + \bigtriangledown_{X_2}(Y)$

3) $\bigtriangledown_X(Y_1 + Y_2) = \bigtriangledown_X(Y_1) + \bigtriangledown_X(Y_2)$

4) $\bigtriangledown_{fX}(Y) = f\bigtriangledown_X(Y) \qquad f \in C^\infty(M)$

5) $\bigtriangledown_X(fY) = f\bigtriangledown_X(Y) + X(f)\cdot Y \qquad f \in C^\infty(M)$.

Proposition 2.2

$\bigtriangledown_X(Y)(p)$ *depends only on* X_p *(and* Y*), that is, if* $X_p = X'_p$ *then* $\bigtriangledown_X(Y)(p) = \bigtriangledown_{X'}(Y)(p)$.

Proof

First assume that X and X' agree in some neighbourhood of p, and choose a "bump function" f on M, which is 1 near p and 0 outside a neighbourhood in which X and X' agree. Then $fX = fX'$ so that by 4)

$$f\bigtriangledown_X(Y) = \bigtriangledown_{fX}(Y) = \bigtriangledown_{fX'}(Y) = f\bigtriangledown_{X'}(Y).$$

Evaluating at p yields

$$\bigtriangledown_X(Y)(p) = \bigtriangledown_{X'}(Y)(p).$$

Hence $\bigtriangledown_X(Y)(p)$ makes sense just X is defined in a neighbourhood of p. Now choose a local chart $(U, \mathbf{x})$ and write $X = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}$ in U. Then by 2) and 4)

$$\bigtriangledown_X(Y) = \bigtriangledown_{\sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}}(Y) = \sum_{i=1}^{n} v_i \bigtriangledown_{\frac{\partial}{\partial x^i}}(Y)$$

Hence

$$\nabla_X(Y)(p) = \sum_{i=1}^{n} v^i(p)\ \nabla_{\frac{\partial}{\partial x^i}}(Y)(p)$$

where the right hand side only involves $v^i(p)$, $i = 1, \ldots, n$, that is, the coordinates of X_p.

Notation

We shall write $\nabla_{X_p}(Y) = \nabla_X(Y)(p)$

Remark

The Lie-derivation $L_X(Y)$ does *not* define a connection since it does *not* satisfy 4) in the definition 2.1.

Exercise 2.3

Let $M^n \subseteq \mathbb{R}^N$ be a submanifold, and as usual identify T_pM , $p \in M$, with a subspace in $\mathbb{R}^N$. (See remark following definition 1.7.) For $v \in \mathbb{R}^N$ let ${}^{\mathrm{T}}v = {}^{\mathrm{T}}v_p$ denote the *orthogonal projection* of v onto $T_p(M)$. Now define for $X_p \in T_pM$, ∇_{X_p} as follows:

Identify Y, a smooth vector field on M, with a map $Y : M \to \mathbb{R}^N$ and define

$$\nabla_{X_p}(Y) = {}^{\mathrm{T}}\big(D_{X_p}(Y)\big)_p$$

where $D_{X_p}(Y) = D(Y)(X_p) = \big(X_p\big(Y^1\big), \ldots, X_p\big(Y^N\big)\big)$is the directional derivative of Y in the direction X_p. Show that $\nabla_{X_p}(Y)$ defines a connection in M in the sense of definition 2.1.

Notation

For $M^n = U$ an open set in $\mathbb{R}^n$ the connection in Exercise 2.3 is called the *Euclidean* or *flat* connection.

Next let us express a connection in local coordinates:

Let $\big(U, u^1, \ldots, u^n\big)$ be a local coordinate system on M and let us put $\partial_k = \frac{\partial}{\partial u^k}, k = 1, \ldots, n,$ for short. Then ∇ (restricted to U) is determined by n^3 smooth functions Γ^k_{ij} given by

$$\nabla_{\partial_i}(\partial_j) = \sum_k \Gamma^k_{ij}\ \partial_k \tag{2.4}$$

In fact let $X = \sum\limits_i x^i \ \partial_i$, $Y = \sum\limits_j y^j \partial_j$ be smooth vector fields on U then

$$\begin{aligned}
\nabla_X(Y) &= \nabla_{\sum\limits_i x^i\partial_i}(Y) = \sum_i x^i \nabla_{\partial_i}\left(\sum_j y^j \partial_j\right) \\
&= \sum_i x^i \left(\sum_j \big(\partial_i y^j\big)\partial_j + y^j \nabla_{\partial_i}(\partial_j)\right) \\
&= \sum_i \sum_j x^i \big(\partial_i y^j\big)\partial_j + \sum_i \sum_j x^i y^j \left(\sum_k \Gamma_{ij}^k \partial_k\right) \\
&= \sum_k \left[\sum_i x^i \Big(\partial_i y^k\Big) + \sum_i x^i \left(\sum_j \Gamma_{ij}^k y^j\right)\right]\partial_k
\end{aligned}$$

That is

$$\begin{aligned}
\nabla_X(Y) &= \sum_k \left(\sum_i x^i y^k_{,i}\right)\partial_k \\
y^k_{,i} &= \partial_i y^k + \sum_j \Gamma_{ij}^k y^j
\end{aligned} \tag{2.5}$$

We summarize in

Proposition 2.6

Let ∇ be a connection in M and $\big(U, u^1, \ldots, u^n\big)$ a local chart in M. Then $\nabla \,\big|\, U$, is determined by the n^3 smooth functions Γ_{ij}^k in (2.4). Furthermore given n^3 such functions we can define ∇ in U by the formulas (2.5) and this defines a connection in U.

Proof

It remains to prove that ∇ given by (2.5) satisfies definition 2.1. The only non-trivial point is equation 5). So let $f \in C^\infty(U)$. Then

$$\nabla_X(fY) = \sum_k \left(\sum_i x^i \Big(fy^k\Big)_{,i}\right)\partial_k$$

where

$$\Big(fy^k\Big)_{,i} = \partial_i\Big(fy^k\Big) + \sum_j \Gamma_{ij}^k f y^j = \partial_i(f)\cdot y^k + f \cdot y^k{}_{,i}$$

Hence

$$\nabla_X(fY) = \sum_k \left(\sum_i x^i \partial_i(f)\cdot y^k\right)\partial_k + \sum_k \left(\sum_i x^i f y^k{}_{,i}\right)\partial_k = X(f)Y + f\cdot \nabla_X(Y).$$

From this we conclude

Corollary 2.7

Every smooth manifold has a connection.

Proof

The proof is similar to the proof of the existence of a Riemannian metric: Cover M by local coordinate charts. In each U_α choose any connection, say put $\Gamma_{ij}^k \equiv 0,$ which defines a connection ∇^{U_α} in U_α. Now choose a partition of unity $\{\phi_\alpha\}_{\alpha\in I}$ subordinate $\{U_\alpha\}_{\alpha\in I}$, and put

$$\nabla = \sum_\alpha \phi_\alpha \nabla^{U_\alpha}$$

that is

$$\nabla_X(Y) = \sum_\alpha \phi_\alpha \, \nabla_X^{U_\alpha} \, (Y)$$

One checks easily that this defines a connection in M.

Parallel Transport

Now consider a differentiable manifold with given connection $\bigtriangledown$. The reason why $\bigtriangledown$ is called a "connection" is that it provides a way of "connecting" the tangent space at one point p by the tangent space at another point q. We cannot expect to find a canonical isomorphism $T_pM \overset{\sim}{=} T_qM$ for any two points unless M is parallellizable (which not all manifolds are). However, given a smooth curve

$$\gamma : [a,b] \to M, \quad \gamma(a) = p, \; \gamma(b) = q,$$

we can *parallel transport* a tangent vector $v \in T_pM$ *along* γ to a vector in $T_q(M)$.

Let $\gamma : [a,b] \to M$ be a smooth curve. By a *smooth vector field along* γ we mean a family $\{V_t\}_{t\in[a,b]}$ of tangent vectors $V_t \in T_{\gamma(t)}M$ which is *smooth* in the following sense:

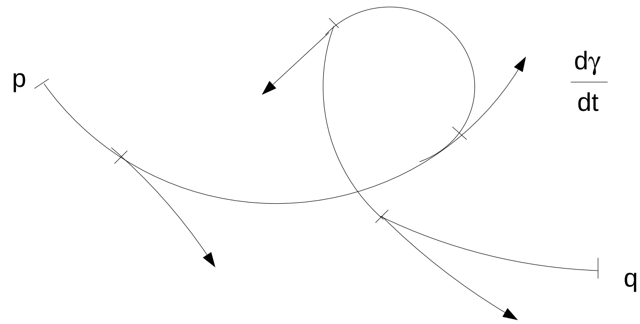
Suppose $\left(U, u^1, \ldots, u^n\right)$ are local coordinates near $\gamma(t_0)$ and

$$V_t = \sum_{i=1}^n v^i(t) \frac{\partial}{\partial u^i} \Big|_{\gamma(t)}$$

for t in an interval around t_0; then we require the functions $v^i(t)$ to be smooth (note that this is independent of choice of chart).

Example.

The velocity vector field $\frac{d\gamma}{dt}$ is a smooth vector field along γ.



Now our connection makes it possible to make the *covariant derivative* of a vector field along γ:

Lemma 2.8

There is a uniquely determined operator in the set of vector fields along γ

$$V \mapsto \frac{DV}{dt} = V' = \nabla_{\frac{d\gamma}{dt}}(V)$$

called covariant derivation along γ *such that*

a) $\frac{D(V+W)}{dt} = \frac{DV}{dt} + \frac{DW}{dt}$

b) For $f \in C^\infty(a,b)$: $\frac{D(fV)}{dt} = \frac{df}{dt}V + f\frac{DV}{dt}$

c) If $V_t = Y_{\gamma(t)}$, Y a vector field on M, then

$$\frac{DV}{dt}(t) = \nabla_{\frac{d\gamma}{dt}}(Y)\,(\gamma(t))$$

Proof

It is clearly enough to prove this locally, so we can assume that there are local coordinates $\left(U, u^1, \ldots, u^n\right)$, such that $\gamma[a,b] \subseteq U$. Put $\partial_i = \frac{\partial}{\partial u^i}$, and write $V = \Sigma_i\, v^i \partial_i$, $v^i : [a,b] \to \mathbb{R}$ smooth, and $\gamma^j = u^j \circ \gamma,\ i,j = 1,\ldots,n.$

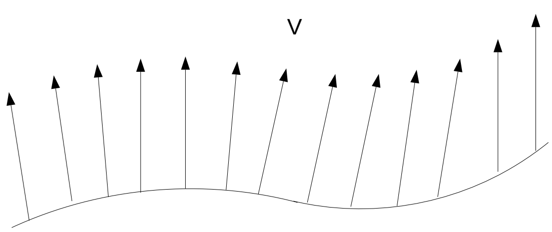
Uniqueness: Using a) – c) we obtain

$$\begin{aligned}\frac{DV}{dt} &= \sum_{i=1}^{n} \frac{D}{dt}\big(v^i \partial_i\big) = \sum_i \left(\frac{dv^i}{dt} \cdot \partial_i + v^i \frac{D}{dt} \partial_i \right) \\ &= \sum_i \left(\frac{dv^i}{dt}\, \partial_i + v^i \nabla_{\sum_j \frac{d\gamma^j}{dt} \partial_j} (\partial_i) \right) \\ &= \sum_i \left(\frac{dv^i}{dt}\, \partial_i + v^i \sum_j \frac{d\gamma^j}{dt}\, \nabla_{\partial_j} \partial_i \right) \\ &= \sum_i \left(\frac{dv^i}{dt}\, \partial_i + v^i \sum_{j,k} \frac{d\gamma^j}{dt}\, \Gamma^k_{ji}\, \partial_k \right) \\ &= \sum_k \left(\frac{dv^k}{dt} + \sum_{ij} \frac{d\gamma^j}{dt}\, \Gamma^k_{ji}\, v^i \right) \partial_k\end{aligned}$$

That is given ∇ and γ, $\frac{DV}{dt}$ depends only on V, which proves uniqueness.

Existence simply follows by defining $\frac{DV}{dt}$ by the above formula.

Definition 2.9
A vector field V along γ is called *parallel* if $\frac{DV}{dt} \equiv 0$



Proposition 2.10
Given a smooth curve $\gamma : [a, b] \to M$ and a vector $V_a \in T_{\gamma(a)}M$. Then there is a unique parallel vector field V_t along γ extending V_a.

Proof
By subdivision it is clearly enough to do this for γ lying inside a coordinate chart $\big(U, u^1, \ldots, u^n\big)$. Looking at the proof of lemma 2.8 we see that we must solve the differential equations

$$\frac{dv^k}{dt} + \sum_{ij} \frac{d\gamma^j}{dt}\, \Gamma^k_{ji}\, v^i = 0 \qquad k = 1, \ldots, n$$

for $t \in [a,b]$ and initial condition $v^i(a) = v^i_a,\ i = 1,\dots,n$.

Now this is a differential equation of the form

$$v' = g \cdot v \qquad , \quad v : [a,b] \to \mathbb{R}^n$$

for $g(t)$ an $n \times n$ matrix-valued function. From the general theorem for existence and uniqueness of differential equations we know that such solutions exist at least in a small interval. However, since the equation is *linear* the solution exists over all of $[a,b]$. In fact suppose $b_0 < b$ is the supremum of end points of intervals $[a,\beta)$ on which solutions do exist. Now in a small interval around b_0 we can find solutions $v_1,\dots,v_n$ such that for each t in the interval $v_1(t),\dots,v_n(t)$ is a basis for $\mathbb{R}^n$. Now given $v(a)$ choose $b^* < b_0$ in this interval so that we have a solution v defined on $[a,\beta)$, $b^* < \beta < b_0$. Then $v(b^*) = x_1v_1(b^*) + \ldots + x_nv_n(b^*),\ x_i \in \mathbb{R}$ and $x_1v_1 + \ldots + x_nv_n$ is a solution in a small interval around b_0 extending v. This is a contradiction so $b_0 = b$.

Furthermore the solution is unique which proves the proposition.

Definition 2.11

$\tau_\gamma : T_pM \to T_qM$ given by $\tau_\gamma(V_a) = V_b$ for V_t a parallel field along γ is called the *parallel transport* along γ.

Remark

$\tau_\gamma : T_pM \to T_qM$ is a linear isomorphism.

Example 2.13

$M = \mathbb{R}^n$ with the Euclidean connection. Then $\tau : T_p\mathbb{R}^n \to T_q\mathbb{R}^n$ is the usual parallel translation.

Exercise 2.12

a) Let $S^2 \subseteq \mathbb{R}^3$ be the unit sphere with connection induced from $\mathbb{R}^3$ as in exercise 2.3 Let γ be the horizontal circle parametrized by

$$\gamma(t) = (\alpha\cos t\ ,\ \alpha\sin t\ ,\ \beta), \qquad 0 < \alpha < 1\ ,\ -1 < \beta < 1$$
$$0 \le t \le 2\pi\ , \qquad \alpha^2 + \beta^2 = 1$$

Let $X(t) = (-\sin t\ ,\ \cos t\ ,\ 0) = \frac{1}{\alpha}\ \frac{d\gamma}{dt}$ be the normalized velocity field and $Y(t) = (-\beta\cos t\ ,\ -\beta\sin t, \alpha)$ be the orthogonal field pointing to the north pole. Show that for any angle $\theta_0 \in [0,2\pi]$ the field Z given by

$$Z(t) = \cos(\theta_0 - \beta t)X(t) + \sin(\theta_0 - \beta t)Y(t) \qquad t \in [0,2\pi]$$

is parallel along γ.

b) Let $C \subseteq \mathbb{R}^3$ be the cone

$$C = \left\{ (x, y, z) \;\middle|\; \alpha^2 \big(x^2 + y^2\big) = z^2 \beta^2 \ , \ z > 0 \right\}$$

where $0 < \alpha, \beta < 1$ are fixed and satisfy $\alpha^2 + \beta^2 = 1$. Again let C have the induced connection from $\mathbb{R}^3$, and let $L \subseteq C$ be the ray through the point $(\beta, 0, \alpha)$. Finally let $\mathbb{R}^2$ have the Euclidean connection and let $U \subseteq \mathbb{R}^2$ be the sector given by polar coordinates

$$U = \left\{ (r, \theta) \;\middle|\; 0 < \theta < 2\pi\beta \right\}$$

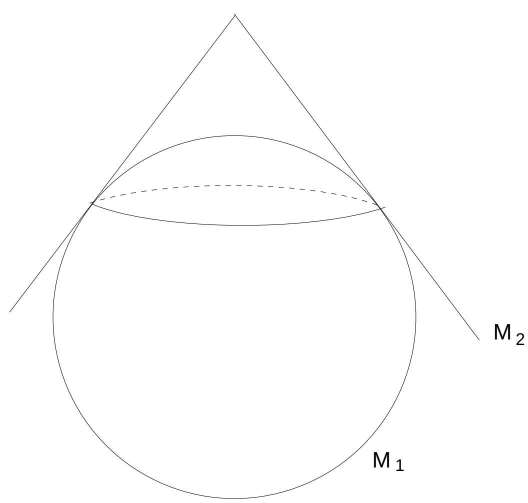
where $(x, y) = (r\cos\theta \ , \ r\sin\theta)$.
Show that there is a diffeomorphism $\phi : U \to C - L$ given by

$$(r, \theta) \to \left(r\beta\cos\frac{\theta}{\beta} \ , \ r\beta\sin\frac{\theta}{\beta} \ , \ r\alpha \right)$$

and that it preserves the connections.
(*Hint:* Express the Euclidean connection in $\mathbb{R}^2$ by polar coordinates, i.e., find Γ_{ij}^k).

c) Let $M_1, M_2 \subseteq \mathbb{R}^N$ be two submanifolds which "touch" along a curve $\gamma : [a, b] \to M_1 \cap M_2$, that is, $T_{\gamma(t)}M_1 = T_{\gamma(t)}M_2$ for all $t \in [a, b]$. Show that in the induced connections from $\mathbb{R}^N$ parallel transport along γ in M_1 is the same as parallel transport along γ in M_2.

d) Give a "geometric proof of a) using c) with $M_1 = S^2$ and M_2 a cone in $\mathbb{R}^3$ touching S^2 along the circle γ.



We now again turn to a manifold with a Riemannian metric, and we want to show that there is a canonical connection associated to it, the so-called "Riemannian connection" or the "Levi-Civita connection".

Proposition 2.13

Let M be a Riemannian manifold with metric $< \cdot \, , \, \cdot >$. For ∇ a connection in M the following are equivalent:

i) *For all smooth vector fields X, Y, Y'*

$$X\big\langle Y, Y'\big\rangle \;=\; \big\langle \nabla_X(Y), Y'\big\rangle \;+\; \big\langle Y, \nabla_X Y'\big\rangle$$

ii) *For every smooth curve $\gamma : [a,b] \to M$, the parallel transport $\tau_\gamma : T_{\gamma(a)}M \to T_{\gamma(b)}M$ is a linear isometry.*

Proof

First let us show that i) is equivalent to

i′) For $\gamma : [a,b] \to M$ any smooth curve and V, W vector fields along γ

$$\frac{d}{dt}\left\langle V, W\right\rangle = \left\langle \frac{DV}{dt}, W\right\rangle + \left\langle V, \frac{DW}{dt}\right\rangle.$$

In fact clearly i′) implies i). On the other hand, if i′) is true for V, W, and $f : [a,b] \to \mathbb{R}$ is smooth, then it is easy to see that i′) is true for V and W replaced by fV and W or by V and fW. Now if i) is true for M then, it is clearly true locally, hence in particular it follows for any X and $Y, Y' = \frac{\partial}{\partial u^i} \; , \; \frac{\partial}{\partial u^j}$ in some local coordinate system. Now since $\nabla_X(Y)(p)$ only depends on $X(p)$ we thus have i′) valid for $V, W = \frac{\partial}{\partial u^i} \; , \; \frac{\partial}{\partial u^j}$ restricted to γ. And since any vector field along γ is a linear combination of these where the coefficients are smooth functions, i′) follows by our previous remark. Thus i) is equivalent to i′).

Now $\mathrm{i}) \;\Rightarrow\; \mathrm{ii})$. In fact by i′) for V and W *parallel* along γ we have since

$$\frac{DV}{dt} \equiv 0 \; , \; \frac{DW}{dt} \equiv 0$$

that $\frac{d}{dt}\langle V, W\rangle = 0$, that is, $\langle V, W\rangle$ is constant along γ and in particular

$$\langle V_a, W_a\rangle = \langle V_b, W_b\rangle$$

which proves ii).

To prove $\mathrm{ii}) \;\Rightarrow\; \mathrm{i})$ or rather $\mathrm{ii}) \;\Rightarrow\; \mathrm{i}')$ first choose parallel fields along γ , $P_1, \ldots, P_n$, such that $P_1(a), \ldots, P_n(a)$ is an orthonormal basis in $T_{\gamma(a)}M$. Then by assumption $P_1(t), \ldots, P_n(t)$ is again an orthonormal basis for $T_{\gamma(t)}M$ for each $t \in [a,b]$. Now for $V(t) = \Sigma_i \; v^i(t)P_i(t)$, $W(t) = \Sigma_j w^j(t)P_j(t)$ any two vector fields along γ we then have

$$\langle V, W\rangle = \sum_i v^i w^i,$$

hence

$$\frac{d}{dt}\langle V, W\rangle = \sum_i \frac{dv^i}{dt}\cdot w^i + \sum_i v^i \frac{dw^i}{dt}.$$

But since

$$\frac{DV}{dt} = \sum_i \frac{dv^i}{dt} P_i\,, \qquad \frac{DW}{dt} = \sum_j \frac{dw^j}{dt} P_j$$

(because the $P_i{}'$s are parallel) this equation is just i$'$) which proves the proposition.

Definition 2.14

A connection $\bigtriangledown$ is called *symmetric* or *torsion free* if for all smooth vector fields X and Y

$$\bigtriangledown_X(Y) - \bigtriangledown_Y(X) = [X, Y]$$

Exercise 2.15

Define for X, Y as above the *torsion*

$$T(X,Y) = \bigtriangledown_X(Y) - \bigtriangledown_Y(X) - [X, Y]$$

a) Show that T is a tensor, that is, $T(X,Y)(p)$ depends only on X_p and Y_p.
(*Hint*: Show that for $f \in C^\infty(M)$, $T(fX, Y) = T(X, fY) = fT(X,Y)$ and compare the proof of proposition 2.2)

b) If $\left(U, u^1, \ldots, u^n\right)$ is a local coordinate system and Γ^k_{ij} are defined by (2.4), then $T = 0$ iff $\Gamma^k_{ij} = \Gamma^k_{ji}$ $\forall i, j$.

Theorem 2.16

Let M be a Riemannian manifold. Then there is precisely one connection $\bigtriangledown$ such that

a) $\bigtriangledown$ *is symmetric.*

b) *Either* i) *or* ii) *in proposition* 2.13 *is fulfilled.*

Notation

The connection given by theorem 2.16 is called the *Riemannian connection* or the *Levi-Civita connection* for the Riemannian metric.

Proof of theorem 2.16.

Uniqueness. It is enough to do it locally. Thus let $(U, u^1, \ldots, u^n)$ be a coordinate chart and as before let $\partial_i = \frac{\partial}{\partial u^i}$, $i = 1, \ldots, n$, and $g_{ij} = \langle \partial_i \ , \ \partial_j \rangle$, where $\langle \cdot \ , \ \cdot \rangle$ is the Riemannian metric. Now by assumption using proposition 2.13 i)

(2.17) $$\partial_i \big(g_{jk} \big) = \partial_i \langle \partial_j \ , \ \partial_k \rangle = \langle \nabla_{\partial_i}(\partial_j) \ , \ \partial_k \rangle + \langle \partial_j \ , \ \nabla_{\partial_i}(\partial_k) \rangle$$

and as in (2.4) let

$$\nabla_{\partial_i}(\partial_j) = \sum_l \Gamma^l_{ij} \ \partial_{l.}$$

Furthermore put

(2.18) $$[ij, k] = \langle \nabla_{\partial_i}(\partial_j) \ , \ \partial_k \rangle = \sum_l \Gamma^l_{ij} \ \ g_{lk}$$

Then (2.17) reads

(2.19)$'$ $$\partial_i \ g_{jk} = [ij, k] + [ik, j]$$

and by cyclic permutations

(2.19)$''$ $$\partial_j \ g_{ki} = [jk, i] + [ji, k]$$

(2.19)$'''$ $$\partial_k \ g_{ij} = [ki, j] + [kj, i]$$

Now since the connection is symmetric $[ij, k] = [ji, k]$ and we obtain from $(2.19)^{\prime - \prime\prime\prime}$:

(2.20) $$[ij, k] = \frac{1}{2}\big(\partial_i \ g_{jk} + \partial_j \ g_{ki} - \partial_k \ g_{ij} \big) = \sum_l \Gamma^l_{ij} \ g_{lk} \qquad i, j, k = 1, \ldots, n$$

Now introducing the inverse matrix

$$\{g^{ij}\} = \{g_{ij}\}^{-1}$$

we can write (2.20) in the form

(2.21) $$\Gamma^l_{ij} = \sum_k \ [ij, k] \ g^{kl} \qquad i, j, l = 1, \ldots, n$$

so that Γ^l_{ij} are uniquely determined by $\{g_{ij}\}$ and the derivatives of $\{g_{ij}\}$. This clearly shows uniqueness.

Existence now follows locally by simply defining in a chart a connection corresponding to the functions Γ^l_{ij} given by (2.21). On overlapping charts the corresponding connections will agree by the uniqueness. This ends the proof.

Notation

The functions $[ij,k]$ and Γ_{ij}^k are called *Christoffel-symbols* (of the first respectively the second kind) for the connection, and the equations (2.20) and (2.21) are called the *Christoffel identities*.

Exercise 2.22

Let $M \subset \mathbb{R}^N$ be a submanifold and let $\bigtriangledown$ be the connection in M induced from $\mathbb{R}^N$ as in exercise 2.3.

i) Show that $\bigtriangledown$ is symmetric.

ii) Show that $\bigtriangledown$ is the Riemannian connection associated to the Riemannian metric induced from $\mathbb{R}^N$ (cf. exercise 0.2).

Chapter 3 GEODESICS AND THE EXPONENTIAL MAP

Let M be a manifold with a linear connection $\bigtriangledown$.

Definition 3.1

A smooth curve $\gamma : [a,b] \to M$ is called a *geodesic* if the velocity vector field $\frac{d\gamma}{dt}$ is parallel along γ, that is if

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) \equiv 0$$

Let us express this condition in local coordinates: let $\left(U, u^1, \ldots, u^n\right)$ be the local coordinates and let Γ_{ij}^k be the Christoffel symbols. Let $\gamma^i(t) = u^i(\gamma(t)) \quad i = 1, \ldots, n$ so that $\frac{d\gamma}{dt} = \Sigma_i \ \frac{d\gamma^i}{dt}\,\partial_i.$ Now in the proof of proposition 2.10 we found that a vector field along γ given in local coordinates by $V_t = \Sigma_i \ v^i \partial_i$ is parallel iff it satisfies

$$\frac{dv^k}{dt} + \sum_{ij} \frac{d\gamma^j}{dt} \Gamma_{ji}^k \ v^i = 0 \qquad k = 1, \ldots, n.$$

Hence γ is a geodesic iff it satisfies

$$\frac{d^2\gamma^k}{dt^2} + \sum_{ij} \Gamma_{ij}^k \frac{d\gamma^i}{dt}\ \frac{d\gamma^j}{dt} = 0 \qquad k = 1, \ldots, n. \tag{3.2}$$

This is a system of 2nd order differential equations of the following form: Let $\mathbf{u} = \left(u^1, \ldots, u^n\right) \in \mathbb{R}^n$, $U \subseteq \mathbb{R}^{2n}$ an open set with coordinates $(\mathbf{u}, \mathbf{v})$, $F : U \to \mathbb{R}^n$ is smooth, and the equation is

$$\frac{d^2\mathbf{u}}{dt^2} = F\left(\mathbf{u}, \frac{d\mathbf{u}}{dt}\right)$$

In this situation we need the following

Theorem 3.3

Given $(\mathbf{u}_1, \mathbf{v}_1) \in U$ there exists a neighbourhood W of $(\mathbf{u}_1, \mathbf{v}_1)$ and $\epsilon > 0$ such that for each $(\mathbf{u}_0, \mathbf{v}_0) \in W$ the differential equation

$$\frac{d^2\mathbf{u}}{dt^2} = F\left(\mathbf{u}, \frac{d\mathbf{u}}{dt}\right)$$

has a unique solution $t \ \mapsto \ \mathbf{u}(t)$ defined for $|t| < \epsilon$ and satisfying

$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \left.\frac{d\mathbf{u}}{dt}\right|_{t=0} = \mathbf{v}_0.$$

Furthermore the solution depends smoothly on the initial data $(\mathbf{u}_0, \mathbf{v}_0)$.

“Proof”

This follows from the corresponding existence and uniqueness theorem for the 1st order equations

$$\frac{d\mathbf{u}}{dt} = \mathbf{v} \qquad \frac{d\mathbf{v}}{dt} = F(\mathbf{u}, \mathbf{v})$$

with initial condition $\mathbf{u}(0) = \mathbf{u}_0$, $\qquad \mathbf{v}(0) = \mathbf{v}_0$. See e.g. Lang [4].

We shall now “translate” this theorem into a statement concerning geodesics in a manifold. This involves the *tangent bundle* of the manifold considered as a manifold. Thus

$$TM = \bigsqcup_{p \in M} T_pM$$

is given the structure of a smooth manifold determined by the local charts given as follows: Let as usual $\left(U, u^1, \ldots, u^n\right)$ be local coordinates in M and $\partial_i = \frac{\partial}{\partial u^i}$ the vector fields in U. Then in $TU = \bigsqcup\limits_{p \in U} T_pM$ every tangent vector is uniquely expressible in the form

$$v_p = \sum_{i=1}^{n} v^i \partial_i \Big|_p$$

so

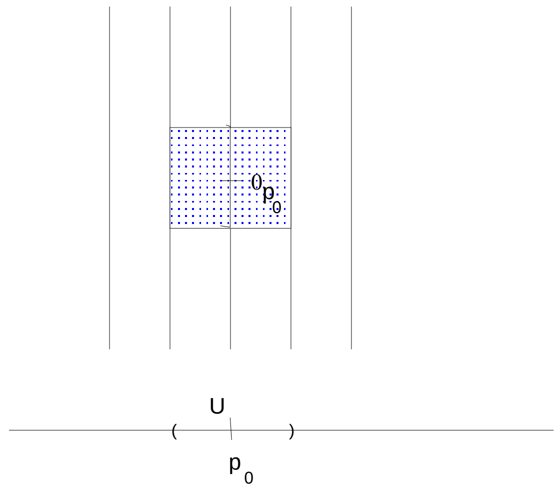
$$v_p \ \mapsto \ \left(u^1(p), \ldots, u^n(p),\ v^1, \ldots, v^n\right) \in \mathbb{R}^{2n}$$

gives a chart in TM. For details we refer to appendix B (theorem B.4).

Now suppose M is given a Riemannian metric (always possible by theorem 1.21). Then a neighbourhood of the 0 vector in $T_{p_0}M$, 0_{p_0}, contains a neighbourhood of the form

$$\left\{ v_p \in TM \;\middle|\; p \in U \ , \ \| v_p \| < \epsilon \right\}$$

for some $U \subseteq M$ neighbourhood of p_0 and some $\epsilon > 0$. Again we refer to appendix B for a proof (see proposition B.5).



Again let $\bigtriangledown$ be a connection in M (not necessarily the Riemannian one). Then we can reformulate theorem 3.3 applied to the equations (3.2):

Corollary 3.4

For every point $p_0 \in M$ *there is a neighbourhood* U *of* p_0 *and real numbers* $\epsilon, \delta > 0$ *such that: for each* $p \in U$ *and* $v \in T_pM$ *with* $\| v \| < \epsilon$ *there is a unique geodesic*

$$\gamma_v : (-2\delta \ , \ 2\delta) \to M$$

such that

$$\gamma_v(0) = p \ , \ \frac{d\gamma_v}{dt}(0) = v.$$

Furthermore $\gamma_v(t)$ *depends smoothly on* t *and* $v_p \in \left\{ v \in TU \ \middle| \ \| v \| < \epsilon \right\}$.

Remarks

1. In this corollary δ can be taken to be 1 by replacing ϵ by $\epsilon \cdot \delta$. In fact if $\gamma_v : (-2\delta \ , \ 2\delta) \to M$ is a geodesic then $\gamma_{\delta v}(t) = \gamma_v(\delta t)$, $t \in (-2, 2)$, is also a geodesic.

2. By the usual arguments for solutions to differential equations there is a unique *maximal* geodesic $\gamma_v(t)$ (i.e. defined on the largest possible interval $(-a, b)$, $a, b > 0$).

Definition 3.5

Suppose $\gamma_v(t)$ is defined for $t = 1$ then put

$$\exp_p(v) = \gamma_v(1)$$

Remarks

1. By Remark 1 above $\exp_p(v)$ is defined for $v \in T_pM$ of length $< \epsilon$, p in a neighbourhood of p_0. Furthermore the map $v \mapsto \exp_p(v)$ is smooth on this set. We shall prove a more global statement in appendix C.

2. Note also that $\gamma_v(t) = \exp_p(tv)$ by the same argument.

Example 3.6

Consider $S^1 \subseteq \mathbb{R}^2 = \mathbb{C}$ with the induced connection. Then $\exp_1 : T_1(S^1) \to S^1$ is given by $\exp\left(t\frac{d}{dt}\Big|_1\right) = \exp(it)$. In fact let $\gamma(t) = \exp(it)$; then $\gamma'(t) = i\exp(it)$ and $\frac{D\gamma'}{dt}(\gamma(t))$ is the orthogonal projection onto $\operatorname{span}_{\mathbb{R}}\{\gamma'(t)\}$ of $\gamma''(t) = -\exp(it)$ which is clearly 0, so that γ is a geodesic.

Next let us study the local properties of geodesics. Again suppose M has a metric and for $p \in M$ let, as in Corollary 3.4, U be a neighbourhood of p and let $\epsilon > 0$ be chosen such that $\exp_q(v)$ exists for $q \in U, \| v \| < \epsilon$, and so that it depends smoothly on q, v. Let $V \subseteq TU \subset TM$ be the set of $v \in T_qM$, $q \in U$ with $\| v \| < \epsilon$ and define

$$F : V \to M \times M$$

by

$$F(v_q) = \big(q \ , \ \exp_q v\big).$$

Then clearly F is smooth, and $F(0_p) = (p, p)$.

Proposition 3.7

F_* is non-singular at 0_p.

Proof

Assume $(U, u^1, \ldots, u^n)$ are local coordinates around p and as usual let $\partial_i = \frac{\partial}{\partial u^i}$. Then any $v = v_q \in V$ is of the form $v_q = \Sigma_i \, v^i \partial_i \big|_q$ and we have a local coordinate system on V given by

$$v_q \ \mapsto \ \big(u^1(q), \ldots, u^n(q) \ , \ v^1, \ldots, v^n\big).$$

Now $M \times M$ has a local coordinate system $\big(U \times U \ , \ u_1^1, \ldots, u_1^n, u_2^1, \ldots, u_2^n\big)$ around (p, p). The differential F_* at the point 0_p is given by

$$F_*\left(\frac{\partial}{\partial u^i}\Bigg|_{0_p}\right) = \frac{\partial}{\partial u_1^i}\Bigg|_p + \frac{\partial}{\partial u_2^i}\Bigg|_p$$
$$F_*\left(\frac{\partial}{\partial v^i}\Bigg|_{0_p}\right) = \frac{\partial}{\partial u_2^i}\Bigg|_p .$$

So F_* has matrix $\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$. Q.E.D.

Hence F is a diffeomorphism in a neighbourhood of 0_p. Again suppose this neighbourhood is of the form

$$V = \big\{ v_q \ \big| \ q \in V \quad \| v_q \| < \epsilon \big\}$$

Choose W neighbourhood of p such that

$$F(V) \supseteq W \times W$$

Then we have actually proved:

Theorem 3.8

Let M be a Riemannian manifold and $\bigtriangledown$ *a connection in M (not necessarily the Riemannian connection). For each* $p \in M$ *there is a neighbourhood* W *and a real number* $\epsilon > 0$ *such that*

1) $\forall q, q' \in W$ $\exists$ *unique* $v \in T_q(M)$ *with* $\| v \| < \epsilon$ *such that* $t \mapsto \exp_q(tv)$ *is a geodesic from q to* q'.

2) *The map* $W \times W \to TM$ *given by* $(q, q') \mapsto v$ *is smooth.*

3) *For each* $q \in W$, $\exp_q : \{v \in T_qM \mid \| v \| < \epsilon\} \to M$ *is a diffeomorphism onto an open set* $U_q \supseteq W$.

Notation

A neighbourhood of the form U_q as in 3) above is called a *normal* neighbourhood of q.

Remarks

1. Notice that W in theorem 3.8 can be chosen as a normal neighbourhood of p.

2. One can actually prove that for small ϵ a normal neighbourhood W of radius ϵ is actually *convex*, that is, the geodesic between any two points in W stays inside W. (See e.g. Helgason, [2, Chapter I § 6].)

Geodesics for Riemannian connections.

So now assume that $\bigtriangledown$ is the Riemannian connection. Recall that for $\gamma : [a, b] \to M$ the length L is given by

$$L(\gamma) = L_a^b(\gamma) = \int_a^b \left\| \frac{d\gamma}{dt} \right\| dt$$

and consider the *arc length function*

$$s(t) = L_a^t(\gamma) = \int_a^t \left\| \frac{d\gamma}{du} \right\| du.$$

Now if γ is a geodesic then

$$\frac{d}{dt} \left\| \frac{d\gamma}{dt} \right\|^2 = 2 \left\langle \frac{D}{dt} \frac{d\gamma}{dt} \, , \, \frac{d\gamma}{dt} \right\rangle = 0$$

so that $\left\|\frac{d\gamma}{dt}\right\|$ is constant, in fact if $\gamma = \gamma_v,\ v \in T_pM$, then $\left\|\frac{d\gamma}{dt}\right\| \equiv \|v\|$ so that $s(t) = \|v\| \cdot t$ for $\gamma_v : (-2\delta\ ,\ 2\delta) \to M$. Hence we have shown that a geodesic is parametrized proportionally to arc length.

Notation

If $\gamma(s) = \exp sv$ with $\| v \| = 1$ then γ is called a *normalized geodesic* and s is just the arc length.

Remark

If $\exp_q(v) = q'$, then $\| v \|$ is the length of the geodesic $\exp_q(tv)$ from q to q', hence 1) in theorem 3.8 can be expressed by saying that up to parametrization by a constant there is a unique geodesic from q to q' of length $< \epsilon$.

We shall now show that locally geodesics are the curves of shortest length:

Theorem 3.9

Let M be a Riemannian manifold and $\bigtriangledown$ the Riemannian connection. Let W and ϵ be as in theorem 3.8 and $\gamma : [0,1] \to M$ a geodesic of length $L(\gamma) < \epsilon$ joining two points $\gamma(0) = q$, $\gamma(1) = q'$, $q, q' \in W$.

Then for any path ω in M joining q and q' we have $L(\gamma) \leq L(\omega)$.

Furthermore = holds iff ω and γ agree after reparametrization.

For the proof we need several lemmas. First a small technical one.

Lemma 3.10

Let $\alpha : \Omega \to M$, $\Omega \subseteq \mathbb{R}^2$ with parameters (s,t), be a smooth map. Then at every point of Ω

$$\frac{D}{dt}\frac{\partial\alpha}{\partial s} = \frac{D}{ds}\frac{\partial\alpha}{\partial t}$$

Proof

Clearly it is enough to prove this locally, so assume $\alpha(\Omega) \subseteq U$, $\left(U, u^1, \ldots, u^n\right)$ is a local chart. As usual let $\left\{\Gamma^k_{ij}\right\}$ be the Christoffel symbols for the connection. Recall from lemma 2.8 that $\frac{D}{dt}$ is given by

$$\frac{DV}{dt} = \sum_k \left(\frac{dv^k}{dt} + \sum_{ij}\Gamma^k_{ij}\frac{\partial\alpha^j}{dt}v^i\right)\partial_k \qquad V = \sum_i v^i\partial_i$$

where $\alpha^j = u^j \circ \alpha$, and V a vector field along $t \mapsto \alpha(s,t), s$ fixed. In particular for $V = \frac{\partial \alpha}{\partial s} = \sum\limits_i \frac{\partial \alpha^i}{\partial s} \partial_i$

$$\frac{D}{dt} \frac{\partial \alpha}{\partial s} = \sum_k \left(\frac{\partial^2 \alpha^k}{\partial t \partial s} + \sum_{ij} \Gamma^k_{ji} \frac{\partial \alpha^j}{\partial t} \frac{\partial \alpha^i}{\partial s} \right) \partial_k .$$

Now since $\bigtriangledown$ is symmetric, i.e. $\Gamma^k_{ij} = \Gamma^k_{ji}$, the above expression is completely symmetric in s and t, which proves the statement.

Next we have the famous

Lemma 3.11 (Gauss).
Let $U_q \subseteq M$ be a normal neighbourhood of q. Then the geodesics through q are orthogonal trajectories to the hypersurfaces

$$S_q(c) = \left\{ \exp_q(v) \;\middle|\; v \in T_qM \,,\, \| v \| = c \right\} \,,\; c < \epsilon .$$

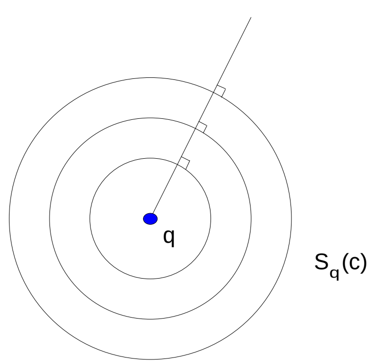
Proof

We shall show that tangents to a curve in $S_q(c)$ are orthogonal to the *radial* geodesics, i.e. the geodesics emanating from q. So let $v(t) \in T_qM, t \in [a,b]$, be a curve with $\| v(t) \| = 1$. We shall show that

$$\frac{d}{dt} \exp_q(cv(t)) \Big|_{t=t_0}$$

is orthogonal to

$$\frac{d}{dr} \exp_q(rv(t_0)) \Big|_{r=c} .$$



So consider $\alpha : (-\epsilon, \epsilon) \times [a,b] \to M$ given by

$$\alpha(r,t) = \exp_q(rv(t))$$

and let us prove

$$\left\langle \frac{\partial\alpha}{\partial r}\,,\;\frac{\partial\alpha}{\partial t}\right\rangle = 0 \qquad \forall\; r,t \in (-\epsilon,\epsilon)\times[a,b].$$

Now

$$\begin{aligned}\frac{\partial}{\partial r}\left\langle \frac{\partial\alpha}{\partial r}\,,\;\frac{\partial\alpha}{\partial t}\right\rangle &= \left\langle \frac{D}{\partial r}\,\frac{\partial\alpha}{\partial r}\,,\;\frac{\partial\alpha}{\partial t}\right\rangle + \left\langle \frac{\partial\alpha}{\partial r}\,,\;\frac{D}{\partial r}\,\frac{\partial\alpha}{\partial t}\right\rangle \\ &= \qquad 0 \qquad + \left\langle \frac{\partial\alpha}{\partial r}\,,\;\frac{D}{\partial t}\,\frac{\partial\alpha}{\partial r}\right\rangle\end{aligned}$$

since $r \mapsto \alpha(r,t)$ is a geodesic and in view of the previous lemma. Furthermore

$$\left\|\frac{\partial\alpha}{\partial r}\right\|^2 = \|v\|^2 = 1$$

so

$$0 = \frac{d}{dt}\left\langle \frac{\partial\alpha}{\partial r}\,,\;\frac{\partial\alpha}{\partial r}\right\rangle = \left\langle \frac{D}{dt}\,\frac{\partial\alpha}{\partial r}\,,\;\frac{\partial\alpha}{\partial r}\right\rangle + \left\langle \frac{\partial\alpha}{\partial r}\,,\;\frac{D}{\partial t}\,\frac{\partial\alpha}{\partial r}\right\rangle = 2\cdot\langle\ ,\ \rangle.$$

It follows that

$$\frac{\partial}{\partial r}\left\langle \frac{\partial\alpha}{\partial r}\,,\;\frac{\partial a}{\partial t}\right\rangle = 0$$

so that $\left\langle \frac{\partial\alpha}{\partial r}\,,\,\frac{\partial\alpha}{\partial t}\right\rangle$ is constant in r. However, for $r = 0$, we have $\alpha(0,t) = q$, for all $t \in [a,b]$ so that $\frac{\partial\alpha}{\partial t}(0,t) \equiv 0$; hence $\left\langle \frac{\partial\alpha}{\partial r}\,,\,\frac{\partial\alpha}{\partial t}\right\rangle = 0$, for all r and t.

In the next lemma again U_q is a normal neighbourhood of radius $\epsilon > 0$ around $q \in M$:

Lemma 3.12

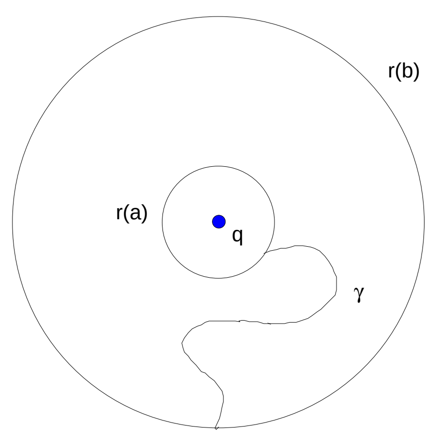
Let $\gamma : [a,b] \to U_q - \{q\}$ *be a piecewise smooth curve and write*

$$\gamma(t) = \exp_q\left(r(t)\cdot v(t)\right)\,, \qquad \|v\| \equiv 1\,,\; 0 < r(t) < \epsilon.$$

Then

$$L_a^b(\gamma) \geq |r(b) - r(a)|$$

and = holds iff v *is constant and* r *monotone.*



Proof

Again consider

$$\alpha(r,t) = \exp_q(rv(t)) \, , \qquad 0 < r < \epsilon \, , \; t \in [a,b].$$

Then by the chain rule

$$\frac{d\gamma}{dt} = r'(t)\frac{\partial \alpha}{\partial r} + \frac{\partial \alpha}{\partial t}$$

and by Gauss's lemma $\frac{\partial\alpha}{\partial r} \perp \frac{\partial\alpha}{\partial t}$; hence

$$\left\|\frac{d\gamma}{dt}\right\|^2 = r'(t)^2\left\|\frac{\partial\alpha}{\partial r}\right\|^2 + \left\|\frac{\partial\alpha}{\partial t}\right\|^2 \geq r'(t)^2$$

and = holds iff $\frac{\partial\alpha}{\partial t} = 0$ (where this makes sense).

It follows that

$$L_a^b(\gamma) = \int_a^b \left\|\frac{d\gamma}{dt}\right\| \geq \int_a^b |r'| = \operatorname{var}(r) \geq |r(b) - r(a)|.$$

Furthermore = holds iff

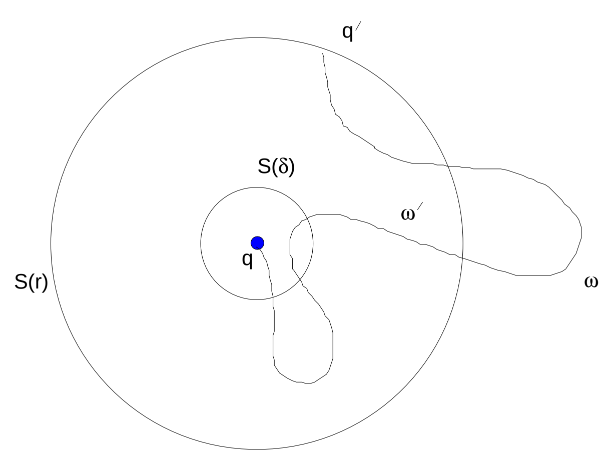
1) $\operatorname{var}(r) = |r(b) - r(a)|$, i.e. r is monotone, and

2) $\frac{\partial\alpha}{\partial t} = 0$ "almost everywhere", i.e., since $\exp_q$ is a diffeomorphism, $v(t)$ is constant.

This proves the lemma.

Proof of theorem 3.9

Again let U_q be the normal neighbourhood of q of radius $\epsilon > 0$ and $q' = \exp_q(rv)$ with $\|v\| = 1$, $0 < r < \epsilon$. We shall show that if ω is any piecewise smooth curve from q to q' then $L(\omega) \geq r$. Now let $0 < \delta < r$ and consider the two spheres in $U_q, S(\delta)$ and $S(r)$ of radius δ and r respectively.



Then there is some segment ω' of ω connecting $S(\delta)$ and $S(r)$ and lying totally in between. In fact ω must cross $S(\delta)$ sometime by continuity and hence there is a "last" point $a' \in [a,b]$ with $\omega(a') \in S(\delta)$ and a "first" point $b' \in [a,b]$ with $\omega(b') \in S(r)$. Clearly $L(\omega) \geq L(\omega')$, and since $\omega' : [a',b'] \to U_q - \{q\}$ we conclude from the previous lemma that

$$L(\omega) \geq L\big(\omega'\big) \geq r - \delta \qquad \text{for all } \delta > 0.$$

Hence $L(\omega) \geq r$. Now suppose $L(\omega) = r$. Then again for $\epsilon > \delta > 0$ we conclude from the previous lemma that ω contains a ray connecting $S(\delta)$ and $S(r)$. In fact the segment ω' from $S(\delta)$ to $S(r)$ has $L(\omega') \geq r - \delta;$ but also $r = L(\omega) \geq L(\omega_1) + L(\omega') + L(\omega_2)$ where $\omega = \omega_1 * \omega' * \omega_2$ and $L(\omega_1) \geq \delta$ by the first part of the lemma, so that $L(\omega') \leq r - \delta,$ i.e. $L(\omega') = r - \delta$. For different choices of δ these rays either have the same direction or are disjoint, so for small δ they must have the same direction, and thus, if we reparametrize ω by arc length, it contains the ray $t \mapsto \exp(tv),\ 0 \leq t \leq r,\ \|v\| = 1$ where v is the common direction of the rays. Therefore ω coincides after reparametrization with this geodesic.

Now recall the definition of distance

$$d(p,q) = \inf \{L(\gamma) |\ \gamma \text{ a piecewise smooth curve from } p \text{ to } q\}.$$

Corollary 3.13

i) d *is a metric.*

ii) *The topology of* M *agrees with the metric topology given by* d.

Proof

First notice that for ϵ small the normal neighbourhood U_q around q of radius ϵ is just the set

$$U_q = \big\{q' \in M \ \big|\ d\big(q, q'\big) < \epsilon\big\}.$$

In fact clearly $\subseteq$ holds since $t \to \exp_q(tv)\ ,\ t \in [0,1]$ is a geodesic from q to $q' = \exp_q(v)$ of length $\|v\| < \epsilon$. And on the other hand if $q' \notin U_q$ then a curve ω from q to q' must cut the sphere $S(r)$ of any radius $r < \epsilon$ and so by theorem 3.9 has $L(\omega) \geq r \quad \forall\ r < \epsilon,$ hence $L(\omega) \geq \epsilon$, that is, $d(q,q') \geq \epsilon$.

To see that d is a metric the only non-trivial statement is that $d(q,q') = 0 \Rightarrow q = q'$. By the above remark $q' \in U_q$ so that $q' = \exp_q(rv), \qquad \|v\| = 1,\ 0 \leq r < \epsilon$. Now if $r > 0$ then any curve from q to q' has $L(\omega) \geq r$ so that $d(q,q') \geq r > 0,$ which contradicts the assumption. Hence $r = 0$, that is, $q = q'$. It is straight forward to prove the triangle inequality:

$$d(p,q) \leq d\big(p,q'\big) + d\big(q',q\big) \ \ \forall\, p,q,q'.$$

For this choose $\epsilon > 0$ and a curve ω joining p and q' of length $L(\omega) \le d(p,q') + \epsilon$ and a curve ω' joining q' and q of length $L(\omega') \le d(q',q) + \epsilon$. Then ω followed by ω' gives a curve from p to q of length $\le d(p,q') + d(q',q) + 2\epsilon$, so that

$$d(p,q) \le d\big(p,q'\big) + d\big(q',q\big) + 2\epsilon, \qquad \forall\ \epsilon.$$

Hence the result.

For the second part we just observe that the topology of M has a basis consisting of the neighbourhoods $U_q(\epsilon)$ for ϵ small. However, this is exactly the topology defined by the metric by our first statement.

We can now prove a global version of the last part of theorem 3.9:

Corollary 3.14

Let $\omega : [0,l] \to M$ be a piecewise smooth curve parametrized by arc length and suppose that ω has length less than or equal to the length of any other curve from $\omega(0)$ to $\omega(l)$. Then ω is a geodesic (and in particular a smooth curve).

Proof

It is clearly enough to show locally that ω is a geodesic curve. So consider a segment $\omega|[a,b]$ contained in an open set W as in theorem 3.8. Then by theorem 3.9 $\omega|[a,b]$ must agree with a geodesic ($\omega|[a,b]$ clearly has smaller length than any other curve from $\omega(a)$ to $\omega(b)$).

Definition 3.15

A geodesic realizing the distance between two points is called a *minimal* geodesic.

Remark

Thus corollary 3.14 says that any curve realizing the distance between its endpoints is (after reparametrization) a minimal geodesic. Also by theorem 3.9 small segments of a geodesic are minimal.

We shall now find conditions which ensure the existence of minimal geodesics (although they are *not unique*).

Definition 3.16

A connection $\bigtriangledown$ on a manifold M is called *geodesically complete* if any geodesic can be extended infinitely in both directions, i.e., if every *maximal* geodesic is defined on $\mathbb{R}$.

Theorem 3.17 (Hopf-Rinow)

Let M be a Riemannian manifold. Then the following are equivalent:

a) *The Riemannian connection is geodesically complete.*

b) (M, d) *is a complete metric space (i.e., every Cauchy sequence converges to some point).*

If case a) *and* b) *holds we have furthermore*

c) *Any two points of M can be joined by a minimal geodesic.*

Proof

b) $\Rightarrow$ a) : Let $\gamma : (a, b) \to M$ be a maximal geodesic and suppose $b < \infty$. Let $\{t_n\}$ be an increasing sequence $t_n \to b$. Since γ is parametrized proportionally to arc length let us assume without loss of generality that t is the arc length. Then

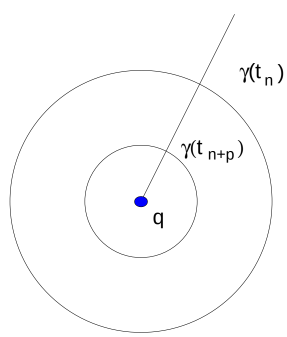
$$d(\gamma(t_{n+p}), \gamma(t_n)) \le |t_{n+p} - t_n| \qquad p, n \in \mathbb{N}.$$

Hence, since $t_n \to b$, $\{\gamma(t_n)\}$ is a Cauchy sequence. Let by assumption $q = \lim_{n\to\infty} \gamma(t_n)$ and choose a neighbourhood W around q and an $\epsilon > 0$ as in theorem 3.8.

For large n we have $\gamma(t_n) \in W$ and $\gamma(t)$ is the minimal geodesic from $\gamma(t_n)$ to $\gamma(t_{n+p})$ so that

$$d(\gamma(t_{n+p}) \ , \ \gamma(t_n)) = t_{n+p} - t_n .$$

Letting $p \to \infty$, $d(q \ , \ \gamma(t_n)) = b - t_n$. Therefore $\gamma(t)$, $t \in [t_n \ , \ t_{n+p}]$ is a curve from the sphere of radius $b - t_n$ to the sphere of radius $b - t_{n+p}$ centered at q, so that



$\gamma \mid [t_n \ , \ t_{n+p}]$ is just a radial geodesic by lemma 3.12, that is

$$\gamma(t) = \exp_q((b - t)v) \ , \quad b - \epsilon < t < b$$

for some $v \in T_qM$, $\|v\| = 1$. But then we can simply extend γ by

$$\gamma(t) = \exp_q((b - t)v) \ , \quad b \le t < b + \epsilon.$$

This contradicts the maximality of γ so that $b = \infty$. Similarly $a = -\infty$.

In order to prove $\mathrm{a}) \Rightarrow \mathrm{b})$ we shall prove

I. $\mathrm{a}) + \mathrm{c}) \Rightarrow \mathrm{b})$ II. $\mathrm{a}) \Rightarrow \mathrm{c})$

I. To see that any Cauchy sequence converges it is enough to see that any *bounded set* X is contained in a compact set. (Since a Cauchy sequence is bounded it then has a converging subsequence, which in turn implies that the original sequence converges.)

To see this choose $q \in X$ and suppose $d(q, q') < K \ \ \forall\, q' \in X$. But by c)

$$q' = \exp_q(v) \ , \ v \in T_q M$$

and by a) $\exp_q : T_q M \to M$ is always defined and *continuous* (cf. appendix C), so in particular

$$q' \in \mathrm{Image}\left(\exp_q : \left\{ v \ \middle|\ \|v\| \leq K \right\} \to M \right\}\right)$$

which is compact. It thus remains to prove

II a) $\Rightarrow$ c). Consider $p, q \in M \ , \ d(p, q) = r$. Choose a normal $\epsilon-$ball U_p around p and let $S(\delta) \subseteq U_p$ be the shell of radius $\delta < \epsilon$. Since $S(\delta)$ is compact there exists $p_0 \in S(\delta)$ with $d(p_0, q)$ minimal. Put $p_0 = \exp_p(\delta v) \ , \ \|v\| = 1 \ , \ v \in T_p M$.

Claim: $q = \exp_p(rv)$.

We shall prove this in the following way. We put $\gamma(t) = \exp_p(tv) \ , \ t \in \mathbb{R}$ (which is well-defined by a)), and we shall show

(*) $$d(\gamma(t), q) = r - t \qquad \forall\, t \in [\delta, r].$$

In particular for $t = r$ (*) is the claim.

First notice that we always have

$$d(\gamma(t), q) \geq r - t$$

in fact; $r = d(p, q) \leq d(p, \gamma(t)) + d(\gamma(t), q) \leq t + d(\gamma(t), q)$. Next observe that (*) holds for $t = \delta$. In fact

$$r = \inf \left\{ L(\gamma) \ \middle|\ \gamma \text{ arc from } p \text{ to } q \right\}$$

$$\geq \min_{s \in S(\delta)} \left(d(p, s) + d(s, q)\right) = \delta + d(p_0, q);$$

hence $d(p_0, q) \leq r - \delta$, which proves (*) for $t = \delta$. Also notice that if (*) holds for t then it also holds for all *smaller* $t' \geq \delta$ since then

$$d\big(\gamma\big(t'\big), q\big) \leq d\big(\gamma\big(t'\big), \gamma(t)\big) + d(\gamma(t), q) \leq t - t' + r - t = r - t'.$$

By continuity we can now find a maximal t_0 such that (*) holds for $t \leq t_0$. Suppose $t_0 < r$ and we want to get a contradiction:

Do as above: Take a small shell $S(\delta')$ of radius δ' around $\gamma(t_0)$. Let p_0' again be of minimal distance from q. Then as above

$$d\big(p_0', q\big) = d(\gamma(t_0), q) - \delta' = r - t_0 - \delta'.$$

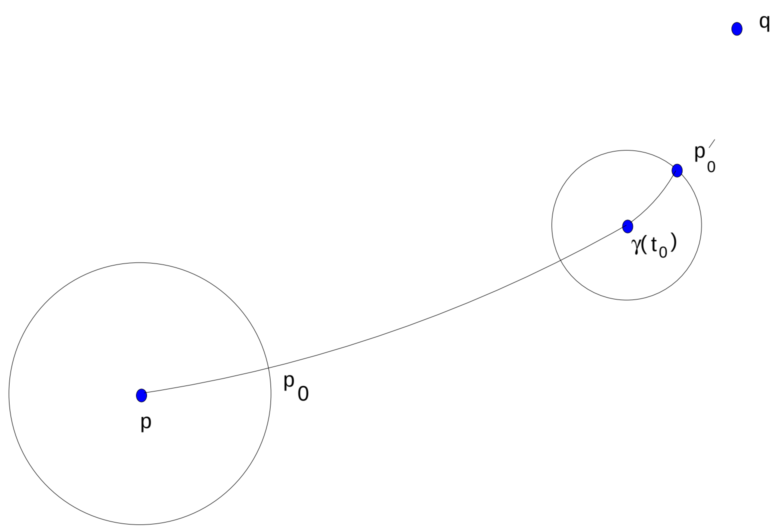
Then

$$\begin{aligned} d\big(p, p_0'\big) \geq d(p, q) - d\big(p_0', q\big) &= r - \big(r - t_0 - \delta'\big) \\ &= t_0 + \delta'. \end{aligned}$$

But the curve γ from p to $\gamma(t_0)$ followed by the geodesic ray from $\gamma(t_0)$ to p_0' is a curve of length $t_0 + \delta'$ and hence must be the (unbroken) geodesic from p to p_0'. Since it agrees with γ on the first part, it must be equal to γ, that is, $p_0' = \gamma(t_0 + \delta')$ and so

$$d\big(\gamma\big(t_0 + \delta'\big), q\big) = d\big(p_0', q\big) = r - \big(t_0 + \delta'\big).$$

Hence (*) holds for $t = t_0 + \delta'$ contradicting the maximality of t_0.



Examples of geodesics

Example 3.18

i) $\mathbb{R}^n$ with the Euclidean metric has as Riemannian connection the Euclidean connection (exercise 2.22) which has $\Gamma_{ij}^k \equiv 0$. Hence the differential equation for a geodesic is just

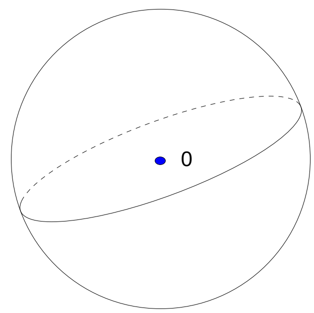
$$\frac{d^2\gamma}{dt^2} \equiv 0\,, \qquad \gamma : [a, b] \to \mathbb{R}^n.$$

The solutions are of course just the straight lines in $\mathbb{R}^n$. Notice that $\mathbb{R}^n$ is complete.

ii) $M = U \subseteq \mathbb{R}^n$ an open subset with the Euclidean metric has of course again straight lines as geodesics. However, unless $U = \mathbb{R}^n, U$ will never be complete. Notice, however, that if U is *convex* any two points can be joined by a minimal geodesic so that c) in theorem 3.17 is *not* equivalent to completeness.

Example 3.19

The *sphere* $S^n \subseteq \mathbb{R}^{n+1}$ with the induced metric has as geodesics the *great circles*, that is, the intersections of S^n with the 2-planes through 0.



To see this let $V \subseteq \mathbb{R}^{n+1}$ be such a 2-plane and let $I : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the isometry reflecting in V, i.e.,

$$I(x) = \begin{cases} x & x \in V \\ -x & x \in V^{\perp} \end{cases}$$

Then clearly $I : S^n \to S^n$ has $C = V \cap S^n$ as fixed point set. We shall show that C is a geodesic. Clearly it suffices to show that if $x, y \in C$ are *close*, then the smaller segment of C joining them is the connecting minimal geodesic (which exists by theorem 3.9). So let γ be the minimal geodesic joining x and y. Then $I(\gamma)$ is also a minimal geodesic joining x and y. Hence if γ is parametrized by arc length we get $I(\gamma) = \gamma$, that is, $\gamma \subseteq C$.

Example 3.20. The Poincaré upper halfplane

Let

$$\mathcal{H}^2 = \left\{ z = x + iy \in \mathbb{C} \;\middle|\; y > 0 \right\}$$

be given the metric

$$(g_{ij}(z)) = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$$

Proposition 3.21

i) *The group*

$$\mathrm{Gl}(2, \mathbb{R})^{+} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad - bc > 0 \right\}$$

acts on $\mathcal{H}^2$ as a group of isometries by

$$z \mapsto (az+b)\,/\,(cz+d)$$

ii) *The geodesics in $\mathcal{H}^2$ are the half circles and half lines perpendicular to the line $y = 0$. In particular $\mathcal{H}^2$ is complete.*

Proof

i) Identifying a tangent vector in $\mathcal{H}^2$ by a complex number we first notice that the inner product at a point $z \in \mathcal{H}^2$ is given by

$$\langle v, w\rangle_z = \frac{1}{(\mathrm{Im}\ z)^2}\,\mathrm{Re}\,(v\overline{w}), \qquad v, w \in \mathbb{C}.$$

Now for a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ consider the map $g : \mathcal{H}^2 \to \mathcal{H}^2$ given by $g(z) = (az+b)/(cz+d)$. The differential at a point z is given by multiplication by the complex derivative

$$g'(z) = \frac{(cz+d)a - (az+b)c}{(cz+d)^2} = \frac{\Delta}{(cz+d)^2}$$

where $\Delta = ad - cb$ is the determinant.

Also we compute

$$\begin{aligned}\mathrm{Im}\,(g(z)) = \frac{1}{2i}\left(\frac{az+b}{cz+d} - \frac{a\overline{z}+b}{c\overline{z}+d}\right) &= \frac{1}{2i}\frac{(az+b)(c\overline{z}+d) - (a\overline{z}+b)(cz+d)}{|cz+d|^2} \\ &= \frac{\Delta\ \mathrm{Im}\ z}{|cz+d|^2}.\end{aligned}$$

Therefore

$$\begin{aligned}\langle g_*v, g_*w\rangle_{g(z)} &= \frac{1}{(\mathrm{Im}\ g(z))^2}\ \mathrm{Re}\Big(g'(z)v\overline{g'(z)}\ \overline{w}\Big) \\ &= \frac{|g'(z)|^2}{(\mathrm{Im}\ g(z))^2}\ \mathrm{Re}\,(v\overline{w}) = \frac{1}{(\mathrm{Im}\ z)^2}\ \mathrm{Re}\,(v\overline{w}) = \langle v, w\rangle_z\end{aligned}$$

which proves that y acts as an isometry.

ii) First let us show that the half $y-$axis,

$$l = \Big\{iy \ \Big|\ y > 0\Big\}$$

is a geodesic. For this we use the reflection $r : \mathcal{H}^2 \to \mathcal{H}^2$ given by $r(x+iy) = -x+iy$, which is easily seen to be an isometry (not covered by i)) with l as fixed point set. The argument of example 3.19 then shows that l is a geodesic. If we write $l(t) = iy(t)$ and parametrize by arc length then y must satisfy

$$\left|\frac{y'(t)}{y(t)}\right| = 1$$

that is $y'(t) = \pm y(t)$.Therefore considered as a geodesic through the point i, l is given by

$$l(t) = ie^t, \quad -\infty < t < \infty,$$

in particular it is defined for all t. In order to get geodesics in all directions through the point $i,$ let g in i) be the isometry given by the rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \theta \in \mathbb{R}$$

Then clearly $y(i) = i$ and the differential is given by multiplication by

$$g'(i) = \frac{1}{(i\sin\theta + \cos\theta)^2} = e^{-2i\theta}.$$

Hence $g(l)$ is the geodesic through i pointing in the direction determined by the angle $\frac{\pi}{2} - 2\theta$. In order to get geodesics through any other point just notice that $\mathrm{Gl}(2,\mathbb{R})^+$ acts transitively on $\mathcal{H}^2$. In fact if $z = ai + b$, $a > 0$ then $z = g(i)$ for g given by the matrix

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \qquad \Delta = a > 0.$$

In all cases the geodesics are the images of l under some Möbius transformation as in i). These always take circles (or lines) to circles (or lines) and also preserve angles. Therefore, and since the $x-$axis is mapped onto itself, the images will always be either a circle perpendicular to the $x-$axis or a line perpendicular to the $x-$axis. (One might think that it is enough that only "one end" of the geodesic is perpendicular to the $x-$axis, but by first rotating l by 180° we see that also the other end must be perpendicular to the $x-$axis).

Exercise 3.22 The disc model for the hyperbolic plane.

Let $D \subseteq \mathbb{C}$ be the unit disc $D = \left\{z \in \mathbb{C} \;\middle|\; |z| < 1\right\}$ with the Riemannian metric given by

$$\langle v, w\rangle_z = \frac{4 \text{ Re } (v\overline{w})}{\left(1 - |z|^2\right)^2}.$$

i) Show that the *Cayley transform* $c : D \to \mathcal{H}^2$ given by

$$c(z) = -i\,\frac{z+i}{z-i}$$

is an isometry.

ii) Show that for $a, b \in \mathbb{C}$ with $|a|^2 - |b|^2 = 1$, the transformation

$$z \mapsto \frac{az + b}{\bar{b}z + \bar{a}}$$

is an isometry of D onto itself.

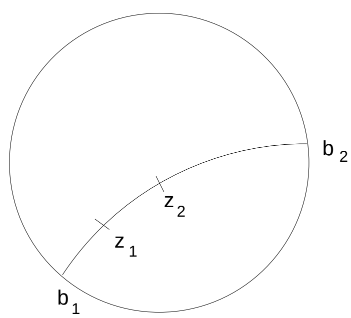
iii) Show that the geodesics through 0 are the Euclidean straight lines and that the distance from 0 to any other point is given by

$$d(0, z) = \log \frac{1 + |z|}{1 - |z|}.$$

iv) Show that all other geodesics are circular arcs perpendicular to the boundary circle $|z| = 1$ and that for any two points $z_1, z_2 \in D$ the distance is given by

$$d(z_1, z_2) = \log \left(\frac{z_1 - b_2}{z_1 - b_1} \Big/ \frac{z_2 - b_2}{z_2 - b_1} \right)$$

where b_1 and b_2 are the points of intersection of the joining geodesic circular arc and the boundary (cf. the figure).



Example 3.23. The hyperbolic *n*-space
In $\mathbb{R}^{n+1}$ let F be the bilinear form

$$F(x, y) = -x^0 y^0 + x^1 y^1 + \ldots + x^n y^n$$

and consider

$$H^n = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; F(x, x) = -1 \right\}$$

Proposition 3.24

i) *H^n is an n-dimensional submanifold of $\mathbb{R}^{n+1}$ with two connected components, $H^n = H^n_+ \cup H^n_-$, where $e_0 = (1, 0, 0, \ldots, 0) \in H^n_+$.*

ii) *For $a \in H^n$ the tangent space $T_a H^n$ is naturally identified with*

$$T_a H^n = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; F(x,a) = 0 \right\}$$

and F restricted to $T_a H^n$ is positive definite, so that H^n gets a natural Riemannian metric defined by $\langle v, w \rangle_a = F(v,w)$, $v, w \in T_a H^n$.

iii) *Let $\mathrm{O}(1,n) \subseteq \mathrm{Gl}(n+1, \mathbb{R})$ be the subgroup of linear maps A satisfying*

$$F(Ax, Ay) = F(x,y) \qquad \forall\, x, y \in \mathbb{R}^{n+1}$$

and let $\mathrm{O}(1,n)^+ \subseteq \mathrm{O}(1,n)$ consist of those satisfying further

$$F(Ae_0, e_0) < 0.$$

Then $\mathrm{O}(1,n)^+$ is a subgroup acting transitively on H^n_+ as a group of isometries. Furthermore the subgroup of $\mathrm{O}(1,n)^+$ fixing e_0 is $\mathrm{O}(n)$ acting on $\mathrm{span}\,\{e_1, \ldots, e_n\}$.

iv) *The geodesics in H_+ are all curves of the form $H^n_+ \cap E$, $E \subseteq \mathbb{R}^{n+1}$ is a 2-plane through 0 such that $F|E$ is non-degenerate of type (1,1).*

v) *H^n_+ is complete.*

vi) *There is an isometry of D in exercise 3.22 onto H^2_+ given by*

$$z \;\mapsto\; \frac{\left(1 + |z|^2 \;,\; 2\,\mathrm{Re}\, z \;,\; 2\mathrm{Im}\, z\right)}{1 - |z|^2}$$

Proof

i) That H^n is a submanifold follows easily from the Implicit Function Theorem. To see that it has at least 2 components observe that $x^0 \neq 0$, $\quad \forall\; x \in H^n$, so it suffices to see that, say H_+, is connected. But if $x = \left(x^0, \ldots, x^n\right) \in H_+$ then

$$\left(x^1\right)^2 + \ldots + (x^n)^2 = \left(x^0\right)^2 - 1 \geq 0$$

and = holds only for $x = e_0$. Hence $\left(x^1, \ldots, x^n\right)$ lies on a sphere and thus for $n > 1$ it can be connected to the point $\left(\sqrt{\left(x^0\right)^2 - 1}\;,\; 0, \ldots, 0\right)$. Hence we can suppose that $x^2 = x^3 = \ldots = x^n = 0$. Thus we are reduced to the case $n = 1$ where H^n consists of two hyperbolas.

ii) Let $a = \left(a^0, \ldots, a^n\right) \in H^n$. By differentiation of $F(x,x) = -1$ it follows easily that

$$T_a H^n = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; F(x,a) = 0 \right\}$$

which is clearly a $n-$dimensional vector space. To see that $F|T_aH^n$ is positive definite let us assume $a \neq e_0$ (otherwise $T_aH^n = \mathrm{span}\,\{e_1, \ldots, e_n\}$ which is a trivial case). Then for $x = \left(x^0, \ldots, x^n\right) \in T_aH^n,\ x^1a^1 + \ldots + x^na^n = x^0a^0$ so that

$$\begin{aligned} \left|x^0a^0\right| = \left|x^1a^1 + \ldots + x^na^n\right| &\leq \sqrt{\sum_1^n \left(x^i\right)^2}\ \sqrt{\sum_1^n \left(a^i\right)^2} \\ &= \sqrt{F(x,x) + \left(x^0\right)^2}\ \sqrt{\left(a^0\right)^2 - 1}. \end{aligned}$$

Hence

$$\left|x^0\right| \leq \sqrt{F(x,x) + \left(x^0\right)^2}\ \frac{\sqrt{\left(a^0\right)^2 - 1}}{\left|a^0\right|}$$

and since $\sqrt{a_0^2 - 1}/\left|a^0\right| < 1$ this is only possible if either $x_0 = 0$ or $F(x,x) > 0$. But if $x_0 = 0$ and $F(x,x) = 0$ then clearly $x = 0$. Hence $F|T_aH^n$ is positive definite.

iii) To prove that $\mathrm{O}(1,n)^+$ is a subgroup first notice that if $a, b \in H_+^n$ then $F(a,b) < 0$. In fact $a = \left(a^0, \ldots, a^n\right)$, $b = \left(b^0, \ldots, b^n\right)$, with $a^0, b^0 > 0$ and $F(a,a) = F(b,b) = -1$. Then

$$F(a,b) = -a^0b^0 + \sum_1^n a^ib^i < 0$$

since

$$\left|\sum_1^n a^ib^i\right| \leq \sqrt{\sum_1^n \left(a^i\right)^2}\sqrt{\sum_1^n \left(b^i\right)^2} = \sqrt{\left(a^0\right)^2 - 1}\ \sqrt{\left(b^0\right)^2 - 1} < \left|a^0b^0\right|.$$

Therefore, for $A, B \in \mathrm{O}(1,n)^+$, $a = Ae_0$, $b = Be_0$ satisfies

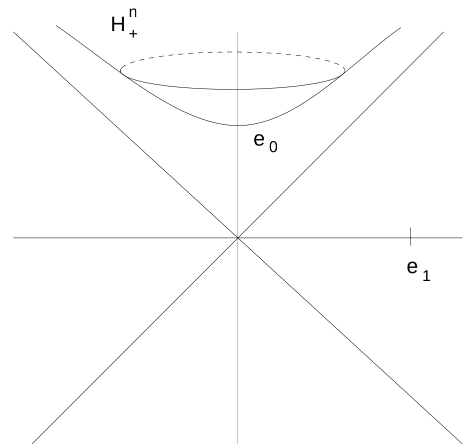
$$F\left(B^{-1}Ae_0, e_0\right) = F(Ae_0\ ,\ Be_0) < 0$$

that is, $B^{-1}A \in \mathrm{O}(1,n)^+$. To see that $\mathrm{O}(1,n)^+$ acts transitively on H_+^n is now easily proved using the usual Gram-Schmidt procedure to extend $a \in H_+^n$ to an orthonormal basis for $\mathbb{R}^{n+1}$ (orthonormal with respect to F).

The remaining statements we leave as

Exercise 3.25

Prove all unproven statements in proposition 3.24.


H^n_+
e_0
e_1

Appendix B THE TANGENT BUNDLE

Let M be a differentiable manifold. We will show how to make the disjoint union

$$TM = \bigsqcup_{p \in M} T_pM$$

into a manifold. This manifold is called the *tangent bundle* for M. The first step is to make TM into a topological space.

More generally consider a set X together with a covering by subsets $\{U_\alpha\}_{\alpha \in I}$, that is, $X = \cup_{\alpha \in I} U_\alpha$. Suppose further that for each $\alpha \in I$ there is associated a topological space U'_α and a bijection $h_\alpha : U_\alpha \to U'_\alpha$. In this situation we have:

Lemma B.1

Assume that the following holds for each $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$:

1. $h_\alpha\big(U_\alpha \cap U_\beta\big)$ *is open in* U'_α,
2. $h_\beta \circ h_\alpha^{-1} : h_\alpha\big(U_\alpha \cap U_\beta\big) \to U'_\beta$ *is continuous.*

Then there is a unique topology on X such that for all $\alpha \in I$ the following holds:

1. U_α *is open in* X
2. $h_\alpha : U_\alpha \to U'_\alpha$ *is a homeomorphism.*

Proof

First notice that it follows from i) and ii) that furthermore $h_\beta\big(U_\alpha \cap U_\beta\big)$ is open in U'_β and

(B.2)
$$h_\beta \circ h_\alpha^{-1} : h_\alpha\big(U_\alpha \cap U_\beta\big) \to h_\beta\big(U_\alpha \cap U_\beta\big)$$

is a homeomorphism. Now define a topology on X by stipulating $U \subseteq X$ is open iff for all $\alpha \in I$, $h_\alpha(U \cap U_\alpha)$ is open in U'_α. This is easily seen to define a topology on X and clearly 1) is fulfilled. To show 2) we fix $\alpha \in I$ and consider $V \subseteq U_\alpha$. If V is open in X then clearly $h_\alpha(V)$ is open in U'_α by the definition of the topology. We must show the converse, i.e., that if $h_\alpha(V)$ is open in U'_α then V is open, that is, for every $\beta \in I$, $h_\beta\big(V \cap U_\beta\big)$ is open in U'_β. for this notice that $V \cap U_\beta \subseteq U_\alpha \cap U_\beta$ and since the map in (B.2) is a homeomorphism, clearly $h_\beta(V \cap U_\beta)$ is open in $h_\beta\big(U_\alpha \cap U_\beta\big)$ iff $h_\alpha\big(V \cap U_\beta\big)$ is open in $h_\alpha\big(U_\alpha \cap U_\beta\big) \subseteq U'_\alpha$. But this obviously follows from the assumption that $h_\alpha(V)$ is open in U'_α, which proves the lemma.

Now return to M an $n-$dimensional manifold and cover it by coordinate charts $(U_\alpha, \mathbf{u}_\alpha), \alpha \in I$, $\mathbf{u}_\alpha = \big(u^1_\alpha, \ldots, u^n_\alpha\big) : U_\alpha \to U'_\alpha \subseteq \mathbb{R}^n$. Then TM is covered by the sets

$$TU_\alpha = \bigsqcup_{p \in U_\alpha} T_p M$$

and for each $\alpha \in I$ we have a bijection

$$h_\alpha : TU_\alpha \to TU'_\alpha = U'_\alpha \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$$

given by

(B.3) $$h_\alpha(v_p) = \left(u^1_\alpha(p), \ldots, u^n_\alpha(p), v^1, \ldots, v^n\right)$$

where

$$v_p = \sum_{i=1}^n v^i \left.\frac{\partial}{\partial u^i_\alpha}\right|_p \in T_p M \ , \ p \in U_\alpha.$$

It is easily checked that $X = TM$ together with the covering $\{TU_\alpha\}_{\alpha \in I}$ and the bijections h_α , $\alpha \in I,$ satisfy lemma B.1, so that TM becomes a topological space. Furthermore the following is straightforward.

Theorem B.4

TM is a differentiable manifold of dimension $2n$ with local coordinate charts given by (TU_α, h_α) as defined by (B.3).

Now let M be equipped with a Riemannian metric $\langle \cdot \ , \ \cdot \rangle,$ and let $\| \cdot \|$ be the associated norm in the tangent spaces. The following is used in connection with corollary 3.4:

Proposition B.5

Let $0_{p_0} \in T_{p_0}M$ be the $0-$vector and let $W \subseteq TM$ be an open neighbourhood of 0_{p_0} in TM. Then there is an open neighbourhood $V \subseteq W$ of the form

$$V = \left\{ v_p \in TM \ \middle| \ p \in U \ , \ \|v_p\| < \epsilon \right\}$$

for a suitable open neighbourhood U of p_0 in M and $\epsilon > 0$.

Proof

Let $\left(U_1, u^1, \ldots, u^n\right)$ be a coordinate chart around p_0 and let the metric be given by the matrix $\{g_{ij}\},$ i.e.

$$g_{ij}(p) = \left\langle \partial_i \big|_p \ , \ \partial_j \big|_p \right\rangle \ , \ p \in U_1.$$

That is if $v_p = \Sigma_i \ v^i \partial_i \ \big|_p$ then

$$\|v_p\|^2 = \sum_{i,j} \ g_{ij}(p) v^i v^j.$$

Since this is clearly a differentiable expression as a function in the coordinates of v, it follows that a set V as defined in the proposition is an open set. Let $h : TU_1 \to \mathbb{R}^{2n}$ be the chart

$$h(v_p) = \big(u^1(p), \ldots, u^n(p), v^1, \ldots, v^n\big),$$

where $v_p = \sum_i v^i \, \partial_i \, \big|_p$. Since h is a homeomorphism onto its image we can clearly choose a neighbourhood $V_1 \subseteq W$ of 0_{p_0} of the form

$$V_1 = \big\{ v_p \in TM \;\big|\; p \in U_2 \;,\; |v_p| < \epsilon_1 \big\}$$

where $U_2 \subseteq U_1$ is a neighbourhood of p_0 and

$$|v| = \sqrt{\left(v^1\right)^2 + \ldots + \left(v^n\right)^2}$$

is the usual Euclidean norm. Now let $B \subseteq U_2$ be a compact neighbourhood of p_0 of the form

$$B = \big\{ p \in U_2 \;\big|\; |\mathbf{u}(p) - \mathbf{u}(p_0)| \leq \delta \big\}$$

for some $\delta > 0$. We claim that there is an $\epsilon_2 > 0$ such that

$$\|v_p\| \geq \epsilon_2 |v_p|$$

for all v_p satisfying $p \in B$. In that case

$$V = \Big\{ v_p \in TM \;\Big|\; |\mathbf{u}(p) - \mathbf{u}(p_0)| < \delta \;,\; \|v_p\| < \epsilon \Big\},$$

where $\epsilon = \epsilon_1 \epsilon_2$, is a neighbourhood of 0_{p_0} of the required form and $V_1 \subseteq W$. That ϵ_2 exists follows from the continuity of the function $v_p \;\mapsto\; \|v_p\|$ restricted to the compact set

$$S = \big\{ v_p \in TM \;\big|\; p \in B \;,\; |v_p| = 1 \big\}.$$

This ends the proof of the proposition.

Appendix C DIFFERENTIABILITY OF THE EXPONENTIAL MAP

In the proof of Hopf-Rinow′s theorem (3.17) it is used that for a geodesically complete Riemannian manifold M the exponential map $\exp : TM \to M$ is smooth.

In this appendix we shall prove this statement using the corresponding local statement in corollary 3.4. Thus let M be an $n-$dimensional manifold with connection $\bigtriangledown$ and metric $\langle \cdot \, , \, \cdot \rangle$.

Theorem C.1

Let $V \subseteq TM$ be the domain for the exponential map, that is, $v = v_q \in V$ iff there exists a geodesic curve $\gamma_v : (-\epsilon, a) \to M$ such that $\gamma_v(0) = q$, $\frac{d\gamma}{dt}(0) = v$, $\epsilon > 0$ and $a > 1$. Then $V \subseteq TM$ is an open set and $\exp : V \to M$ is a smooth map.

For this we need the following:

Lemma C.2

Let N be a manifold and let $\gamma : [a,b] \times N \to M$ be a smooth map. Suppose there is a coordinate chart $\left(U, \; u^1, \ldots, u^n\right)$ such that $\gamma(a \times N) \subseteq U$. For $x \in N$ let

$$\tau_{\gamma_x} : T_{\gamma(a,x)}M \to T_{\gamma(b,x)}M$$

be the parallel transport along $\gamma(t,x)$, $t \in [a,b]$. Then the mapping $\Theta : \mathbb{R}^n \times N \to TM$ given by

$$\Theta(v,x) = \tau_{\gamma_x}\Big(\Sigma_i \, v^i \partial_i \, \big|_{\gamma(a,x)}\Big) \, , \; v = \left(v^1, \ldots, v^n\right), \;\; x \in N,$$

is a smooth map.

Proof

It clearly suffices to prove this locally; hence we can assume $\gamma([a,b] \times N) \subseteq U$. Now for $x \in N$ let $V_{(t,x)}$ be a parallel field along $\gamma(t,x)$, $t \in [a,b]$ and write

$$V_{(t,x)} = \sum_i v^i(t,x) \frac{\partial}{\partial u^i} \Bigg|_{\gamma(t,x)} .$$

That is, $\left(v^1, \ldots, v^n\right)$ are solutions to the linear system of differential equations

$$\frac{dv^k}{dt} \, + \, \sum_{ij} \Gamma^k_{ij} \, \frac{\partial \gamma^i}{\partial t} \, v^j = 0 \; , \quad k = 1, \ldots, n.$$

Since these solutions depend smoothly on the initial values $\left(v^1(a,x), \ldots, v^n(a,x)\right)$ and since

$$\Theta(v,x) = \sum_i \; v^i(b,x)\partial_i \quad \text{for } v = \sum_i \; v^i(a,x)\partial_i,$$

the lemma clearly follows.

Proof of theorem C.1

Suppose $v_0 \in T_pM$ is in the domain for $\exp_p$ and put $\omega(t) = \exp_p(tv_0)$, $t \in [0,1]$. We must show that exp is defined and smooth on an open subset of TM containing the line segment $\{tv_0 \mid t \in [0,1]\}$. By compactness of the interval we can find a subdivision

$$0 = t_0 < t_1 < \ldots < t_N = 1$$

and for each j a neighbourhood W_j of $\omega(t_j)$ and an $\epsilon_j > 0$ as in theorem 3.8 such that $\omega([0,1]) \subseteq W_0 \cup \ldots \cup W_N$. By possibly subdividing further we can assume that both $\omega(t_j)$ and $\omega(t_{j+1})$ lie in W_j , $j = 0, \ldots, N-1$. By induction we shall now show that exp is defined and is smooth on an open set containing $\{tv_0 \mid t \in [0,t_j]\}$: The case $j = 0$ is obvious by theorem 3.8. So assume the statement holds for j and we shall prove it for $j+1$. First we choose a coordinate chart $(U, u^1, \ldots, u^n)$ around p and notice that (by possibly making U smaller) we can assume that $\exp_q(w)$ is defined and smooth for $q \in U$ and $w = \Sigma_i\, w^i\, \partial_i|_q$, where $(w^1, \ldots, w^n) \in \Omega$ for some open set $\Omega \subseteq \mathbb{R}^n$ containing the segment

$$\left\{ t\big(v_0^1, \ldots, v_0^n\big) \;\middle|\; t \in [0, t_j] \right\}, \quad \text{where } v_0 = \sum_i \, v_0^i \, \partial_i \,\Big|_p .$$

Also, we can assume that $\exp_q(w) \in W_0 \cup \ldots \cup W_j$. Now put $N = U \times \Omega$ and let $\gamma : [0,1] \times N \to M$ be defined by

$$\gamma(s, q, \underline{w}) = \exp_q(sw) , \qquad s \in [0,1],$$

where $\underline{w} = (w^1, \ldots, w^n)$ and $w = \Sigma_i\, w^i \partial_i|_q$. Now let $\Theta : \mathbb{R}^n \times N \to TM$ be defined as in lemma C.2. Then clearly

$$\exp_q(sw) = \exp_{\exp_q(w)}\big((s-1)\Theta(\underline{w}, q, \underline{w})\big) \tag{C.3}$$

is defined for $s \geq 1$ as long as

$$\begin{gathered} (q, \underline{w}) \in N, \text{ and} \\ \|(s-1)\Theta(\underline{w}, q, \underline{w})\| < \epsilon_k \,, \quad \text{for } \exp_q(w) \in W_k. \end{gathered} \tag{C.4}$$

Furthermore the expression in (C.3) is smooth as a function of $q, \underline{w}$, and s. The set $V' \subseteq TM$ of vectors $v = sw$ satisfying (C.4) is clearly an open subset, and thus exp is smooth on this set by (C.3). Since $\omega(t_{j+1}) \in W_j$, we have that $\|(t-t_j)\Theta(\underline{v}_0, p, \underline{v}_0)\| < \epsilon_j$ for $t \in [t_j, t_{j+1}]$ and

$$\omega(t) = \exp_{\omega(t_j)}\big((t-t_j)\Theta(\underline{v}_0, p, \underline{v}_0)\big).$$

Hence $\{t\, v_0 | t \in [t_j, t_{j+1}]\} \subseteq V'$. Since already $\{t\, v_0 | t \in [0, t_j]\} \subseteq V'$ we have completed the induction.

Chapter 4 THE CURVATURE TENSOR AND THE STRUCTURAL EQUATIONS

In this section we are concerned with the problem of determining when two Riemannian manifolds are *isometric*. In particular when a Riemannian manifold is isometric to the Euclidean space $\mathbb{R}^n$ with the usual Euclidean metric. We shall see that already *locally* this presents a non-trivial problem. First a few elementary facts about isometries:

Definition 4.1

A diffeomorphism $\Phi : M \to N$ of Riemannian manifolds M and N is called an *isometry* if the differential

$$\Phi_* : T_qM \to T_{\Phi(q)}N$$

is a linear isometry for each $q \in M$, i.e. if

$$\langle \Phi_* v,\ \Phi_* w\rangle = \langle v, w\rangle \qquad \forall\ v, w \in T_qM.$$

We have already seen examples of isometries in the previous chapter §§ 3.19 - 3.24, where we have implicitly used the following

Exercise 4.2

Let M and N be Riemannian manifolds and let $\Phi : M \to N$ be a diffeomorphism.

a) If Φ is an isometry then Φ preserves arc length and distances, and maps geodesics to geodesics.

b) If Φ is distance preserving, i.e., if $d(\Phi(p),\ \Phi(q)) = d(p, q)\ \ \forall\ p, q,$ then Φ is an isometry.
(*Hint:* first show that geodesics are mapped to geodesics).

Let us return to the problem of determining when two Riemannian manifolds M and N are locally isometric. More precisely choose points $p \in M$, $p' \in N$ and suppose we have given a *linear isometry*

$$\phi : T_pM \to T_{p'}N$$

and we ask for an isometry Φ of a neighbourhood of p onto a neighbourhood of p' such that

$$\Phi(p) = p'\ ,\ \Phi_{*p} = \phi.$$

With these data there is at *most one choice* for Φ :

To see this choose $\epsilon > 0$ so that the balls $B_p \subseteq T_pM$ and $B_{p'} \subseteq T_{p'}N$ of radius ϵ are mapped diffeomorphically under the exponential maps $\exp_p$ and $\exp_{p'}$ onto open sets U_p and $U_{p'}$ respectively. Suppose now that $\Phi : U_p \to U_{p'}$ is an isometry and $\Phi_{*p} = \phi$. Then for $v \in B_p$ the curve

$$\gamma(t) = \Phi\big(\exp_p(tv)\big) \qquad |t| \leq 1$$

is clearly a geodesic with

$$\gamma(0) = p' \quad , \quad \frac{d\gamma}{dt}(0) = \phi(v)$$

so that $\gamma(t) = \exp_{p'}(t\phi(v))$ and hence

$$\Phi\big(\exp_p(v)\big) = \exp_{p'}(\phi(v)).$$

Hence we have proved

Proposition 4.3
The isometry $\Phi : U_p \to U_{p'}$ is uniquely determined by

$$\Phi = \exp_{p'} \circ \phi \circ \exp_p^{-1} : \; U_p \to U_{p'}.$$

However, Φ defined by this formula is not always an isometry, and we shall give rather complicated looking necessary and sufficient conditions for this.

As usual the first step is to reduce the problem to a question about connections. So now let M and N have linear connections (denoted by $\bigtriangledown$ and $\bigtriangledown'$ respectively) and suppose $\Phi : M \to N$ is a diffeomorphism.

Notation:
Given a vector field X on M we have the $\Phi-$*transformed* vector field X^Φ on N defined by

$$X^\Phi_{q'} = \Phi_*\big(X_{\Phi^{-1}(q')}\big)$$

Definition 4.4
$\Phi : M \to N$ is called an *affine transformation* if for all C^∞ vector fields X, Y on M

$$\bigtriangledown'_{X^\Phi}\Big(Y^\Phi\Big) = \left(\bigtriangledown_X(Y)\right)^\Phi.$$

The following exercise was implicitly used in exercise 2.12:

Exercise 4.5

Let $\Phi : M \to N$ be a diffeomorphism.

a) For $\left(U',\ u'^1, \ldots, u'^n\right)$ a coordinate chart for N, let $\left(U,\ u^1, \ldots, u^n\right)$ be the corresponding coordinate chart $U = \Phi^{-1}U'$, $u^i = u'^i \circ \Phi$ on M. Show that $\Phi : U \to U'$ is an affine transformation iff $\Gamma^k_{ij} = \Gamma'^k_{ij} \circ \Phi$

b) Show that affine transformations preserve parallel transport and geodesics.

Proposition 4.6

Let $\Phi : M \to N$ be a diffeomorphism of connected Riemannian manifolds. Suppose that at some point $p \in M$, $(\Phi_*)_p$ is an isometry. Then Φ is an isometry iff Φ is an affine transformation with respect to the Riemannian connections.

Proof

$\Rightarrow$ Let $\triangledown'$ be the Riemannian connection on N. Then the formula

$$\triangledown_X(Y) = \left[\triangledown'_{X^\Phi}\left(Y^\Phi\right)\right]^{\Phi^{-1}} , \quad X, Y \text{ vector fields on } M,$$

is easily seen to define a connection on M, and also it is easily seen that it is symmetric and Riemannian. Hence $\triangledown$ is *the* Riemannian connection on M, hence Φ is an affine transformation.

$\Leftarrow$ Let $q \in M$ and choose a piecewise smooth curve γ from p to q. Let $\tau : T_pM \to T_qM$ be the parallel transport along γ. Since Φ is affine it preserves parallel transport and so

$$(\Phi_*)_q = \tau' \circ \Phi_{*p} \circ \tau^{-1}$$

where τ' is parallel transport along $\Phi \circ \gamma$. Now since τ' and τ are both isometries, we conclude that also $(\Phi_*)_q$ is an isometry.

Now let M be a manifold with connection $\triangledown$. We have already defined the *torsion tensor field* T for $\triangledown$ (see exercise 2.15) by the formula

(4.7) $$T(X, Y) = \triangledown_X(Y) - \triangledown_Y(X) - [X, Y]$$

where X and Y are smooth vector fields on M. Similarly we define the *curvature tensor field* R, which to any 3 smooth vector fields X, Y, Z associates a 4th by the formula

(4.8) $$R(X, Y)Z = \triangledown_X\triangledown_Y(Z) - \triangledown_Y\triangledown_X(Z) - \triangledown_{[X,Y]}(Z).$$

Exercise 4.9

Show that (4.7) and (4.8) define *tensors*, that is, their values at a point $q \in M$ depend only on X_q, Y_q, Z_q.

We now want to show that in some sense R and T determine the connection. For this purpose we shall use a *local moving frame*, that is, an open neighbourhood U and a set of smooth vector fields $X_1, \ldots, X_n$ on U, such that for every point $q \in U$, the set $\{X_1(q), \ldots, X_n(q)\}$ is a basis for $T_q M$.

Example 4.10

If $\left(U, u^1, \ldots, u^n\right)$ is a local coordinate system then $\left(U, \frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n}\right)$ is a local moving frame.

Notice that the connection restricted to U is determined by the n^3 functions $\left\{\Gamma_{ij}^k\right\}$ given by

$$\nabla_{X_i}(X_j) = \sum_k \Gamma_{ij}^k X_k. \tag{4.11}$$

Similarly the tensors T and R are determined by the functions $\left\{T_{ij}^k\right\}$ and $\left\{R_{lij}^k\right\}$ respectively given by

$$T(X_i, X_j) = \sum_k T_{ij}^k X_k \tag{4.12}$$

$$R(X_i, X_j)X_l = \sum_k R_{lij}^k X_k. \tag{4.13}$$

The so-called *structural equations* are really just a reformulation of these equations in terms of differential forms.

First let us recall the basic facts of the calculus of differential forms (see e.g. Spivak [6, I, chapter 7] or Warner [7, chapter II]):

Differential forms

A differential form ω of degree k associates to k smooth vector fields $X_1, \ldots, X_k$ a real valued smooth function $\omega(X_1, \ldots, X_k)$ such that it has the “tensor property” (see exercise 4.9 above), and such that it is *multilinear* and *alternating* in $X_1, \ldots, X_k$.

For ω_1 an $l-$form and ω_2 a $k-$form, the product $\omega_1 \wedge \omega_2$ is the $(k+l)-$form given by

$$\begin{aligned}
&\omega_1 \wedge \omega_2(X_1, \ldots, X_{k+l}) \\
&= \frac{1}{(k+l)!} \sum_\sigma \operatorname{sign}(\sigma)\, \omega_1\big(X_{\sigma(1)}, \ldots, X_{\sigma(l)}\big)\, w_2\big(X_{\sigma(l+1)}, \ldots, X_{\sigma(l+k)}\big)
\end{aligned}$$

where σ runs through all permutations of $1, \ldots, k+l$. This product is *associative* and *graded* commutative, i.e. $\omega_1 \wedge \omega_2 = (-1)^{kl} \omega_2 \wedge \omega_1$. Furthermore there is an *exterior differential d*, which to any $k-$form ω associates a $(k+1)-$form $d\omega$ given by

$$(d\omega)(X_1, \ldots, X_{k+1}) = \frac{1}{k+1} \Bigg[\sum_{i=1}^{k+1} (-1)^{i+1} X_i \Big(\omega \Big(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1} \Big) \Big) + \\ + \sum_{i<j} (-1)^{i+j} \omega \Big([X_i, X_j],\ X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1} \Big) \Bigg]$$

where the “hat” means that the term is left out.

d has the following properties:

i) d is linear over $\mathbb{R}$,

ii) $dd = 0$,

iii) $d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$
for ω_1 a $k-$form,

iv) For f a smooth function df is the 1–form given by
$(df)(X) = X(f)$, for X a smooth vector field,

v) d is *local*, that is, for any open set $U \subseteq M$,

$$\omega | U = 0 \quad \Rightarrow \quad d\omega | U = 0$$

(i.e. $d\omega|U$ depends only on $\omega|U$).

In a local coordinate system $\big(U, u^1, \ldots, u^n\big)$ any $k-$form ω has a unique presentation

$$\omega = \sum_{1 \leq i_1 < \ldots < i_k \leq n} a_{i_1 \ldots i_k} du^{i_1} \wedge \ldots \wedge du^{i_k}$$

where $a_{i_1 \ldots i_k}$ are smooth functions on U.

Finally if $F : M \to N$ is a smooth map of manifolds M and N, and if ω is a $k-$form on N then there is a unique *induced* $k-$form $F^*\omega$ on M such that for any k vector fields $X_1, \ldots, X_k$ on M

$$F^*(\omega)(X_1, \ldots, X_k)(q) = \omega_{F(q)} \big(F_* X_{1q}, \ldots, F_* X_{kq} \big) \qquad \forall\, q \in M.$$

F^* preserves $\wedge$ and commutes with d.

The structural equations

Now return to our manifold M with connection ∇ and consider a local moving frame $(U, X_1, \ldots, X_n)$ which will be kept fixed in the following. In U we have the dual 1–forms $\{\theta^1, \ldots, \theta^n\}$ defined by

$$\theta^i(X_j) = \delta^i{}_j$$

where $\delta^i{}_j$ is the Kronecker δ, that is, $\delta^i{}_j = 0$ for $i \neq j$ and $\delta^i{}_i = 1$. Dual to equation (4.11) we consider the 1–forms

$$\omega^i{}_j = \sum_k \Gamma^i_{kj}\, \theta^k \tag{4.14}$$

so that $\{\theta^i\}$ and $\{\omega^i{}_j\}$ together determine Γ^i_{kj} and hence the connection.

Theorem 4.15 (The structural equations)

i) $d\theta^i = -\sum\limits_p \omega^i{}_p \wedge \theta^p + \frac{1}{2} \sum\limits_{jk} T^i{}_{jk}\, \theta^j \wedge \theta^k$

ii) $d\omega^i{}_l = -\sum\limits_p \omega^i{}_p \wedge \omega^p{}_l + \frac{1}{2} \sum\limits_{jk} R^i{}_{ljk}\, \theta^j \wedge \theta^k$

Proof

We shall prove only i) since ii) is completely analogous. First define the functions $c^i{}_{rs}$ by

$$[X_r, X_s] = \sum_i c^i{}_{rs}\, X_i.$$

Then we shall check i) by evaluating both sides on X_r, X_s :

$$\begin{aligned}\big(d\theta^i\big)(X_r, X_s) &= \frac{1}{2}\big(X_r\big(\theta^i(X_s)\big) - X_s\big(\theta^i(X_r)\big) - \theta^i([X_r, X_s])\big)\\ &= -\frac{1}{2}\, c^i{}_{rs}.\end{aligned}$$

On the other hand

$$\begin{aligned}\left(-\sum_p \omega^i{}_p \wedge \theta^p\right)(X_r, X_s) &= -\frac{1}{2}\sum_p \big(\omega^i{}_p(X_r)\theta^p(X_s) - \omega^i{}_p(X_s)\theta^p(X_r)\big)\\ &= -\frac{1}{2}\big(\omega^i{}_s(X_r) - \omega^i{}_r(X_s)\big) = -\frac{1}{2}\big(\Gamma^i{}_{rs} - \Gamma^i{}_{sr}\big)\end{aligned}$$

and

$$\left(\frac{1}{2}\sum_{jk} T^i_{jk}\theta^j \wedge \theta^k\right)(X_r, X_s) = \frac{1}{2}\, T^i_{rs}.$$

But since

$$\sum_i T^i{}_{rs} X_i = T(X_r, X_s) = \nabla_{X_r}(X_s) - \nabla_{X_s}(X_r) - [X_r, X_s]$$
$$= \sum_i \left(\Gamma^i{}_{rs} - \Gamma^i{}_{sr} - c^i{}_{rs}\right) X_i$$

we have $T^i{}_{rs} = \Gamma^i{}_{rs} - \Gamma^i{}_{sr} - c^i{}_{rs}$. The equation i) now clearly follows from this.

Theorem 4.15 becomes particularly useful in the following special moving frame: Consider a point $p \in M$ and let U be a "normal neighbourhood" around p defined as follows. Let $X_1, \dots, X_n \in T_pM$ be a basis and write any vector $v \in T_pM$ in the form $v = \sum_i v^i X_i$. Choose $\epsilon > 0$ so that the $\epsilon-$ball

$$B_p = \left\{ v \in T_pM \;\middle|\; \left(\sum_i \left(v^i\right)^2\right)^{\frac{1}{2}} < \epsilon \right\} \subseteq T_pM$$

maps diffeomorphically under $\exp_p$ onto an open neighbourhood U (this is possible by theorem 3.8 by choosing any metric on M such that $X_1, \dots, X_n$ is orthonormal at p). Now we get a moving frame $X_1^*, \dots, X_n^*$ on U by parallel transporting $X_1, \dots, X_n$ along the radial geodesics, and we consider the structural equations with respect to this moving frame:

Consider $V \subseteq \mathbb{R} \times T_pM$ given by $V = \left\{(t, v) \mid tv \in B_p\right\}$. V is an open set with coordinates $(t, v^1, \dots, v^n)$. Let $\Psi : V \to U$ be the map $\Psi(t, v) = \exp_p(tv)$.
Then we have

Theorem 4.16 (The structural equations in polar coordinates)

i) $\Psi^*\theta^i = v^i dt + \overline{\theta}^i$, $\Psi^*\omega^i{}_j = \overline{\omega}^i{}_j$
where $\overline{\theta}^i$ *and* $\overline{\omega}^i{}_j$ *do* ***not*** *contain* dt.

ii) *Furthermore* $\overline{\theta}^i$ *and* $\overline{\omega}^i{}_l$ *satisfy the differential equations:*

a) $\frac{\partial \overline{\theta}^i}{\partial t} = dv^i + \sum_k v^k \overline{\omega}^i{}_k + \sum_{jk} T^i{}_{jk} v^j \overline{\theta}^k$

b) $\frac{\partial \overline{\omega}^i{}_l}{\partial t} = \sum_{jk} R^i{}_{ljk}\, v^j\, \overline{\theta}^k$

with initial conditions $\overline{\theta}^i \,\big|_{t=0} = 0, \qquad \overline{\omega}^i{}_l \,\big|_{t=0} = 0.$

Proof

i) Since $\Psi_*\big(\frac{d}{dt}\big) = \frac{d}{dt}\ \exp_p(tv) = \sum\limits_i v^i X_i^*$ and since θ^i gives the ith coordinate with respect to $\{X_i^*\}$ the first equation is obvious. Similarly

$$\Psi^*\big(\omega^i{}_j\big)\left(\frac{d}{dt}\right) = \Psi^*\left(\sum_k \Gamma^i_{kj}\theta^k\right)\left(\frac{d}{dt}\right) = \sum_k \Gamma^i_{kj}\ v^k.$$

However, the fields X_j^* are parallel along the curve $\gamma_v(t) = \exp_p(tv)$ with $\frac{d\gamma_v}{dt} = v^* = \sum\limits_i v_i X_i^*$. It follows that for each j , $\nabla_{v^*}\Big(X_j^*\Big)$ vanishes along γ_v, i.e.

$$0 = \nabla_{v^*}\big(X_j^*\big) = \sum_k v^k \sum_i \Gamma^i_{kj} X_i^* \quad \text{along } \gamma_v$$

or equivalently

$$0 = \sum_k \Gamma^i_{kj} v^k \quad \text{ along } \gamma_v \text{ for all } i, j.$$

This proves i).

ii) We prove only equation a); the other one is similar:

By i) we have

$$d\big(\Psi^*\theta^i\big) = dv^i \wedge dt + dt \wedge \frac{\partial}{\partial t}\,\overline{\theta}^i + \text{ terms not involving } dt.$$

Similarly using theorem 4.15 we have

$$\Psi^*\big(d\theta^i\big) = -\sum_p \overline{\omega}^i_p \wedge (v^p dt) + \frac{1}{2}\sum_{jk} T^i_{jk}\Big(v^j dt \wedge \overline{\theta}^k + v^k \overline{\theta}^j \wedge dt\Big) + \text{ terms not involving } dt.$$

Using $T^i_{jk} = -T^i_{kj}$ and comparing the coefficients of dt in the two equations we obtain

$$-dv^i + \frac{\partial}{\partial t}\overline{\theta}^i = \sum_p v^p\, \overline{\omega}^i_p + \sum_{jk} T^i_{jk} v^j\, \overline{\theta}^k$$

which is just a). The initial conditions follows trivially from the fact that $\Psi(0, v)$ is constantly equal to p.

Corollary 4.17

A Riemannian manifold M is locally isometric to $\mathbb{R}^n$ with the Euclidean metric iff the curvature tensor R is constantly equal to 0*.*

Notation

A Riemannian manifold satisfying $R \equiv 0$ is called *flat*.

Proof of 4.17

$\Rightarrow$ is trivial.

$\Leftarrow$ For a Riemannian connection $T \equiv 0$ so $\overline{\omega}^i{}_l \equiv 0$ and $\overline{\theta}^i = t dv^i$ are the solutions of a) and b) of theorem 4.16. In particular for $\Psi_1 = \exp_p$, $\Psi_1 : B_p \to U$ we have

$$\Psi_1^* \theta^i = dv^i \text{ and } \Psi_1^* \omega^i{}_j = 0.$$

Thus, if T_pM is given the usual Euclidean metric in the coordinates $\left(v^1, \ldots, v^n\right)$ then clearly Ψ_1 is an affine transformation. Hence if $X_1, \ldots, X_n$ are chosen orthonormal then Ψ_1 is an isometry by proposition 4.6.

More generally the uniqueness of the solutions to the equations a) and b) in theorem 4.16 is expressed as follows:

Corollary 4.18

Let $(M, \bigtriangledown)$ *and* $(N, \bigtriangledown')$ *be manifolds with connections and let* T, R *and* T', R' *be the torsion and curvature tensor fields for* $\bigtriangledown$ *and* $\bigtriangledown'$ *respectively. Let* $p \in M, p' \in N$ *and let* $\phi : T_pM \to T_{p'}N$ *be a linear isomorphism. Let* U *and* U' *be normal neighbourhoods around* p *and* p' *of the same radius (with respect to a fixed basis of* T_pM *respectively its image in* $T_{p'}N$*). Furthermore for each* $q \in U$, $q = \exp_p(v)$, *let* $q' = \exp_{p'}(\phi v) \in U'$ *and let* τ_q *and* $\tau_q{}'$ *be the parallel transports along* $\exp_p(tv)$ *respectively* $\exp_{p'}(t\phi v)$.

Then ϕ *extends to an affine transformation* $\Phi : U \to U'$ *iff the following is satisfied:* $\forall\, q \in U$ *the linear isomorphism*

$$\tilde{\phi}_q = \tau_{q'} \circ \phi \circ \tau_q^{-1} : T_qM \to T_{q'}N$$

satisfies

a. $T'\Big(\tilde{\phi}_q(v)\ ,\ \tilde{\phi}_q(w)\Big) = \tilde{\phi}_q T(v, w) \qquad \forall\, v, w \in T_qM$

b. $R'\Big(\tilde{\phi}_q(v)\ ,\ \tilde{\phi}_q(w)\Big)\tilde{\phi}_q(z) = \tilde{\phi}_q R(v, w) z \qquad \forall\, v, w, z \in T_qM$

Furthermore Φ *is unique and is given by*

$$\Phi = \exp_{p'} \circ \phi \circ \exp_p^{-1}$$

Remark

In particular for Riemannian manifolds M and N and a linear isometry $\phi : T_pM \to T_{p'}N$, the above map is an isometry iff b) above is satisfied.

Proof of 4.18

Let $X_1, \ldots, X_n \in T_pM$ be a basis and $X_1^*, \ldots, X_n^*$ the associated moving frame in U. Let $\left\{\theta^i\right\}$ and $\left\{\omega^i{}_j\right\}$ be the corresponding frame and connection forms. Put

$X'_1 = \phi X_1, \ldots, X'_n = \phi X_n$ and similarly consider the frame $X'^*_1, \ldots, X'^*_n$ in U' and the corresponding forms $\{\theta'^i\}$ and $\left\{\omega'^i{}_j\right\}$. Also let $V \subseteq \mathbb{R} \times T_pM$ and $V' \subseteq \mathbb{R} \times T_{p'}N$ be as in theorem 4.16 and let $\Psi : V \to U$ and $\Psi' : V' \to U'$ be the maps defined by

$$\Psi(t, v) = \exp_p(tv), \qquad \Psi'(t', v') = \exp_{p'}\left(t'v'\right).$$

Again we define

$$\Phi = \exp_{p'} \circ \phi \circ \exp^{-1}_{p'}$$

and observe that

$$\Phi \circ \Psi = \Psi' \circ \phi,$$

where $\phi : V \to V'$ is the map $(t, v) \mapsto (t, \phi(v))$. With respect to the above frames the torsion and curvature tensors have components $\left\{T^i_{jk}\right\}$, $\left\{R^i_{ljk}\right\}$ and $\left\{T'^i_{jk}\right\}$, $\left\{R'^i_{ljk}\right\}$ which by our assumptions satisfy

$$T^i_{jk} = T'^i_{jl} \circ \phi \text{ and } R^i_{ljk} = R'^i_{ljk} \circ \phi$$

in V. Hence, in the notation of theorem 4.16, both

$$\left\{\overline{\theta}^i\,,\ \overline{\omega}^i_j\right\} \text{ and } \left\{\phi^*\overline{\theta}'^i\,,\ \phi^*\overline{\omega}'^i{}_j\right\}$$

satisfy the structural equations in polar coordinates. Since they also have the same initial values they must be the same set of 1–forms. Hence we conclude that

$$\Psi^*\theta^i = \phi^*\left(\Psi'^*\phi'^i\right) = \Psi^*\Phi^*\theta'^i$$

and

$$\Psi^*\omega^i_j = \phi^*\left(\Psi'^*\omega'^i{}_j\right) = \Psi^*\Phi^*\omega'^i{}_j.$$

In particular, since $\Psi_1 = \exp_p$ is a diffeomorphism, we obtain the equations

$$\theta^i = \Phi^*\theta'^i \text{ and } \omega^i_j = \Phi^*\omega'^i{}_j. \tag{4.19}$$

By the first equation in (4.19) we get

$$\left(X^*_i\right)^\Phi = X'^*_i, \qquad i = 1, \ldots, n.$$

Writing

$$\nabla_{X^*_i}\left(X^*_j\right) = \sum_k \Gamma^k_{ij}\ X^*_k \quad \text{and} \quad \nabla'_{X'^*_i}\left(X'^*_j\right) = \sum_k \Gamma'^k_{ij}\ X'^*_k$$

we conclude from the second equation in (4.19) that $\Gamma^k_{ij} = \Gamma'^k_{ij} \circ \Phi$. Hence

$$\nabla_{\left(X^*_i\right)^\Phi}\left(\left(X^*_j\right)^\Phi\right) = \nabla'_{X'^*_i}\left(X'^*_j\right) = \sum_k \Gamma'^k_{ij}\ X'^*_k = \left(\nabla_{X^*_i}\left(X^*_j\right)\right)^\Phi.$$

That is, Φ is an affine transformation.

Remark

It follows from the proof that $\left(X^*_i\right)^\Phi = X'^*_i$, $i = 1, \ldots, n$. Hence, for all $q \in U$,

$$\tilde{\phi}_q = \tau_{q'} \circ \phi \circ \tau^{-1}_q = \Phi_{*q}.$$

Chapter 5 THE SECTIONAL CURVATURE

In the previous section we saw that a Riemannian metric is locally determined by its curvature tensor field. This does not sound as a great advance since we need to determine the n^4 functions $\left\{R^k_{lij}\right\}$ instead of the n^2 functions $\{g_{ij}\}$! However, the curvature tensor field satisfies certain symmetry relations which enable us to cut down on the number of invariants needed to determine the metric up to isometry.

If M is a Riemannian manifold with metric $g(\cdot\,,\,\cdot) = \langle \cdot\,,\,\cdot \rangle$ and curvature tensor field R we shall also consider the *curvature tensor* in 4 vector field variables $\langle R(X,Y)Z, W\rangle$.

Proposition 5.1

The curvature tensor of a Riemannian manifold with metric g satisfies:

i) $R(X,Y)Z + R(Y,X)Z = 0$

ii) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \qquad (Bianchi's\ identity)$

iii) $\langle R(X,Y)Z, W\rangle + \langle R(X,Y)W, Z\rangle = 0$

iv) $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$

Proof

Recall the definition

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The skew-symmetry relation i) is obvious from this. ii) Since R is a tensor it suffices to prove ii) for $X = \frac{\partial}{\partial u^i}, Y = \frac{\partial}{\partial u^j},\ Z = \frac{\partial}{\partial u^k}$, where $\left(U, u^1, \ldots, u^n\right)$ is a local chart in M. In this case all Lie brackets $[X,Y]$, $[X,Z]$, $[Y,Z]$ are zero, so we must show the identity

(5.2) $$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X + \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y = 0$$

However, the symmetry of the connection implies

$$\nabla_Y Z - \nabla_Z Y = [X,Z] = 0\ ,\ \nabla_Z X - \nabla_X Z = 0\ ,\ \nabla_X Y - \nabla_Y X = 0$$

from which (5.2) clearly follows.

To prove iii) it clearly suffices to prove for all X, Y, Z :

$$\langle R(X,Y)Z, Z\rangle = 0.$$

Again we may assume $[X, Y] = 0$ so that we must show

$$
\begin{aligned}
\langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \ , \ Z \rangle = 0 \\
\langle \nabla_X \nabla_Y Z, Z \rangle = \langle \nabla_Y \nabla_X Z, Z \rangle
\end{aligned}
\tag{5.3}
$$

Now since $\bigtriangledown$ is Riemannian we have

$$X\langle Z, Z\rangle = 2\langle \nabla_X Z, Z\rangle$$

and

$$YX\langle Z, Z\rangle = 2Y\langle \nabla_X Z, Z\rangle = 2\langle \nabla_Y \nabla_X Z, Z\rangle + 2\langle \nabla_X Z, \nabla_Y Z\rangle$$

so that

$$\langle \nabla_Y \nabla_X Z, Z\rangle = \frac{1}{2} YX\langle Z, Z\rangle - \langle \nabla_X Z, \nabla_Y Z\rangle.$$

but since

$$
\begin{aligned}
0 = [X, Y] = XY - YX \text{ we get} \\
\langle \nabla_Y \nabla_X Z, Z\rangle &= \frac{1}{2} XY\langle Z, Z\rangle - \langle \nabla_Y Z \ , \ \nabla_X Z\rangle \\
&= \langle \nabla_X \nabla_Y Z, Z\rangle
\end{aligned}
$$

by symmetry in X and Y. This proves (5.3) and thus iii).

iv) now follows completely algebraically from i)-iii):

In fact by i) and ii) we obtain

$$\langle R(X,Y)Z, W\rangle = -\langle R(Y,X)Z, W\rangle = \langle R(X,Z)Y, W\rangle + \langle R(Z,Y)X, W\rangle \tag{5.4}$$

and by iii) and ii)

$$\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle = \langle R(Y,W)X, Z\rangle + \langle R(W,X)Y, Z\rangle. \tag{5.5}$$

Adding (5.4) and (5.5) now gives

$$
\begin{aligned}
&2\langle R(X,Y)Z, W\rangle \\
=&\langle R(X,Z)Y, W\rangle + \langle R(Z,Y)X, W\rangle + \langle R(Y,W)X, Z\rangle + \langle R(W,X)Y, Z\rangle.
\end{aligned}
\tag{5.6}
$$

In this interchange X and Z respectively Y and W to obtain

$$
\begin{aligned}
&2\langle R(Z,W)X, Y\rangle \\
=&\langle R(Z,X)W, Y\rangle + \langle R(X,W)Z, Y\rangle + \langle R(W,Y)Z, X\rangle + \langle R(Y,Z)W, X\rangle \\
=&\langle R(X,Z)Y, W\rangle + \langle R(W,X)Y, Z\rangle + \langle R(Y,W)X, Z\rangle + \langle R(Z,Y)X, W\rangle
\end{aligned}
\tag{5.7}
$$

where we have used i) and iii) in each term. Since the right hand sides of (5.6) and (5.7) are the same we have proved iv).

With these identities proved we can now show that the curvature tensor field R in some sense is determined by the so-called *sectional curvature:*

Definition 5.8

Let M be a Riemannian manifold of $\dim \geq 2$ with metric g and let R be the corresponding curvature tensor field. For $p \in M$ and $S \subseteq T_pM$ a 2–plane choose y, z spanning S and define the *sectional curvature* by

$$K_p(S) = -\frac{\langle R(y,z)y, z\rangle}{|y \wedge z|^2}$$

where $|y \wedge z|^2 = \langle y, y\rangle\langle z, z\rangle - \langle y, z\rangle^2$ is the square of the area of the parallelogram spanned by y and z.

Proposition 5.9

$K_p(S)$ does not depend on the choice of y and z.

Proof

Let

$$\begin{aligned} y_1 &= \alpha_1 y + \beta_1 z \\ z_1 &= \alpha_2 y + \beta_2 z. \end{aligned}$$

Then clearly

$$|y_1 \wedge z_1|^2 = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}^2 |y \wedge z|^2.$$

Now by proposition 5.1 the curvature tensor $\langle R(X,Y)Z, W\rangle$ is alternating in X, Y as well as Z, W, hence

$$\begin{aligned} \langle R(y_1, z_1)y_1, z_1\rangle &= \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \langle R(y,z)y_1, z_1\rangle \\ &= \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}^2 \langle R(y,z)y, z\rangle \end{aligned}$$

from which the proposition clearly follows.

We can now reformulate corollary 4.18 in the Riemannian case in terms of the sectional curvature:

Theorem 5.10

Let M and N be Riemannian manifolds with metrics $\langle \cdot\,,\,\cdot \rangle$ and $\langle \cdot\,,\,\cdot \rangle'$ respectively and let K, K' be the associated sectional curvatures. Let $p \in M$, $p' \in N$ and $\phi : T_pM \to T_{p'}N$ be a linear isometry. Let U and U' be normal neighbourhoods around p and p' of the same radius. Furthermore for each $q \in U$, $q =$

$\exp_p(v)$, *let* $q' = \exp_{p'}(\phi v) \in U'$ *and let* τ_q , $\tau_{q'}$ *be the parallel transports along* $\exp_p(tv)$ *respectively* $\exp_{p'}(t\phi(v))$.

Then ϕ *extends to an isometry* $\Phi : U \to U'$ *iff the following is satisfied:*

$\forall\, q \in U$ *the linear isometry*

$$\tilde{\phi}_q = \tau_{q'} \circ \phi \circ \tau_q^{-1} : T_qM \to T_{q'}N$$

satisfies

$$K'_{q'}\left(\tilde{\phi}S\right) = K_q(S) \qquad \forall\, S \subset T_qM \ a\, 2-plane.$$

Furthermore Φ *is unique and given by*

$$\Phi = \exp_{p'} \circ \phi \circ \exp_p^{-1}$$

Proof

By corollary 4.18 it suffices to show for all $q \in U$ that

$$R'\left(\tilde{\phi}_q(x), \tilde{\phi}_q(y)\right)\tilde{\phi}_q z = \tilde{\phi}_q(R(x,y)z) \qquad \forall\, x, y, z \in T_qM. \tag{5.11}$$

In the following let us fix $q \in U$ and write $\tilde{\phi} = \tilde{\phi}_q$, $K = K_q$ etc. Then (5.11) is equivalent to

$$\left\langle R'\left(\tilde{\phi}x, \tilde{\phi}y\right)\tilde{\phi}z, \tilde{\phi}w\right\rangle' = \left\langle \tilde{\phi}(R(x,y)z), \tilde{\phi}w\right\rangle' \tag{5.12}$$

or, since $\tilde{\phi}$ is an isometry

$$\left\langle R'\left(\tilde{\phi}x, \tilde{\phi}y\right)\tilde{\phi}z, \tilde{\phi}w\right\rangle' = \langle R(x,y)z, w\rangle \qquad \forall\, x, y, z, w \in T_qM. \tag{5.13}$$

Again since $\tilde{\phi}$ is an isometry clearly

$$\left|\tilde{\phi}y \wedge \tilde{\phi}z\right|^2 = |y \wedge z|^2 \qquad \forall\, y, z \in T_qM$$

so our assumption $K'\left(\tilde{\phi}S\right) = K(S)$, $\forall\, S \subseteq T_qM$, is equivalent to

$$\left\langle R'\left(\tilde{\phi}y, \tilde{\phi}z\right)\tilde{\phi}y, \tilde{\phi}z\right\rangle' = \langle R(y,z)y, z\rangle \qquad \forall\, y, z \in T_qM. \tag{5.14}$$

Thus we want to conclude (5.13) from (5.14). For this we write

$$B(x,y,z,w) = \left\langle R'\left(\tilde{\phi}x, \tilde{\phi}(y)\right)\tilde{\phi}z, \tilde{\phi}w\right\rangle' - \langle R(x,y)z, w\rangle, \qquad x, y, z, w \in T_qM.$$

and we notice that

$$B(y,z,y,z) = 0 \qquad \forall\, y, z \in T_qM. \tag{5.15}$$

Now by proposition 5.1 we have

$$B(x,y,z,w) = B(z,w,x,y) \qquad \forall\, x,y,z,w \in T_qM$$

so in particular $B(x,y,x,w)$ is symmetric in y and w. By (5.15) and the "parallelogram identity" we obtain

$$B(x,y,x,w) = 0 \qquad \forall\, x,y,w \in T_qM$$

or

(5.16) $$B(x,y,z,w) = -B(z,y,x,w) \qquad \forall\, x,y,z,w \in T_qM.$$

Now using ii) in proposition 5.1 we obtain

(5.17) $$B(x,y,z,w) + B(y,z,x,w) + B(z,x,y,w) = 0.$$

But

$$B(y,z,x,w) = -B(z,y,x,w) = B(x,y,z,w)$$

and

$$B(z,x,y,w) = -B(y,x,z,w) = B(x,y,z,w)$$

so that by (5.17) we obtain $3B(x,y,z,w) = 0$ for all $x,y,z,w \in T_qM$. This proves the theorem.

Remark

In case of M a *surface* i.e. $\dim M = 2$ there is only one plane at each point $p \in M$ so the sectional curvature in this case is just a function $K : M \to \mathbb{R}$.

Next let us see how to calculate the sectional curvature: First a trivial lemma:

Lemma 5.18

Let M be a manifold with connection $\bigtriangledown$ and curvature tensor R and let $\left(U, u^1, \ldots, u^n\right)$ be a local coordinate system. Also with $\partial_i = \frac{\partial}{\partial u^i}$, $i = 1, \ldots, n$, write

$$\nabla_{\partial_i}(\partial_j) = \sum_k \Gamma^k_{ij}\partial_k \ and \ R(\partial_i, \partial_j)\partial_l = \sum_k R^k_{lij}\partial_k.$$

Then R^k_{lij} are given by

$$R^k_{lij} = \frac{\partial \Gamma^k_{jl}}{\partial u^i} - \frac{\partial \Gamma^k_{il}}{\partial u^j} + \sum_p \left(\Gamma^p_{jl}\Gamma^k_{ip} - \Gamma^p_{il}\Gamma^k_{jp}\right).$$

Proof

Since $(U, u^1, \ldots, u^n)$ is a coordinate chart we have $[\partial_i, \partial_j] = 0$ for all i, j. Hence

$$\begin{aligned}
R(\partial_i, \partial_j)\partial_l &= \nabla_{\partial_i}\nabla_{\partial_j}(\partial_l) - \nabla_{\partial_j}\nabla_{\partial_i}(\partial_l) \\
&= \nabla_{\partial_i}\left(\sum_p \Gamma^p_{jl}\partial_p\right) - \nabla_{\partial_j}\left(\sum_p \Gamma^p_{il}\partial_p\right) \\
&= \sum_p \partial_i\Gamma^p_{jl}\partial_p + \sum_p \Gamma^p_{jl}\sum_k \Gamma^k_{ip}\partial_k \\
&- \sum_p \partial_j\Gamma^p_{il}\partial_p - \sum_p \Gamma^p_{il}\sum_k \Gamma^k_{jp}\partial_k \\
&= \sum_k \left(\partial_i\Gamma^k_{jl} - \partial_j\Gamma^k_{il}\right)\partial_k + \sum_k\sum_p \left(\Gamma^p_{jl}\Gamma^k_{ip} - \Gamma^p_{il}\Gamma^k_{jp}\right)\partial_k
\end{aligned}$$

which proves the lemma.

Now if M is a Riemannian manifold with metric $\langle \cdot\, , \,\cdot \rangle$ and we again look at the curvature tensor $\langle R(X, Y)Z, W\rangle$ in the local coordinate system $(U, u^1, \ldots, u^n)$, then it is determined by the functions

$$R_{klij} = \langle R(\partial_i, \partial_j)\partial_l, \partial_k\rangle = \sum_\mu g_{k\mu} R^\mu{}_{lij}. \tag{5.19}$$

In general this formula gives too messy calculations for practical use even for surfaces. However, for $M = M^2$ a *surface* one can choose the coordinates in a convenient way to simplify the calculations: Thus consider $p \in M$ and choose an orthonormal basis $\{e_1, e_2\}$ for T_pM. Then in a normal neighbourhood $U \subseteq M$ centered at p of radius $\rho > 0$ we have the *geodesic polar coordinates* (r, θ) , $0 < r < \rho$, θ in an interval of length $< 2\pi$, such that

$$q = \exp_p\left(r(q)(\cos\theta(q)e_1 + \sin\theta(q)e_2)\right) \qquad q \in U - \{p\}.$$

Notice that by the Gauss lemma (3.11) the vector fields $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial\theta}$ are perpendicular, and since

$$r \to \exp_p\left(r(\cos\theta e_1 + \sin\theta e_2)\right)$$

is a normalized geodesic, $\frac{\partial}{\partial r}$ has length 1. Therefore the metric g is given in the coordinates (r, θ) by the matrix $\{g_{ij}\}$, $i, j = 1, 2$:

$$g_{11} = 1\ ,\ g_{12} = g_{21} = 0\ ,\ g_{22} = G, \tag{5.20}$$

where $G = \left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right\rangle > 0$. The inverse matrix $\{g^{ij}\}$ is then

$$g^{11} = 1\ ,\ g^{12} = g^{21} = 0\ ,\ g^{22} = 1/G, \tag{5.21}$$

and the Christoffel symbols (2.20) and (2.21) are easily calculated:

$$(5.22)\qquad \begin{aligned}&[22,1] = -\frac{1}{2}\frac{\partial G}{\partial r}\,, \qquad [11,1] = [12,1] = [21,1] = 0\\ &[22,2] = \frac{1}{2}\frac{\partial G}{\partial \theta}\,, \qquad [12,2] = [21,2] = \frac{1}{2}\frac{\partial G}{\partial r}\,, \qquad [11,2] = 0\end{aligned}$$

$$(5.23)\qquad \begin{aligned}&\Gamma^1_{22} = -\frac{1}{2}\frac{\partial G}{\partial r}\,, \qquad \Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{21} = 0\\ &\Gamma^2_{22} = \frac{1}{2G}\frac{\partial G}{\partial \theta}\,, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2G}\frac{\partial G}{\partial r}\,, \qquad \Gamma^2_{11} = 0.\end{aligned}$$

Exercise 5.24
Verify (5.22) and (5.23).

We now have

Proposition 5.25 (Gauss)
Let M be a surface with geodesic polar coordinates (r,θ) and metric given by (5.20)*. Then the sectional curvature is the function K given by*

$$K = -\frac{1}{\sqrt{G}}\,\frac{\partial^2\left(\sqrt{G}\right)}{\partial r^2}.$$

Proof
Since at every point $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ spans the tangent space, the sectional curvature is given by

$$K = \frac{-\left\langle R\left(\frac{\partial}{\partial r},\frac{\partial}{\partial \theta}\right)\frac{\partial}{\partial r},\frac{\partial}{\partial \theta}\right\rangle}{\begin{vmatrix}1 & 0\\ 0 & G\end{vmatrix}} = \frac{\left\langle R\left(\frac{\partial}{\partial r},\frac{\partial}{\partial \theta}\right)\frac{\partial}{\partial \theta},\frac{\partial}{\partial r}\right\rangle}{G} = \frac{R_{1212}}{G}$$

where by (5.19) and lemma 5.18

$$\begin{aligned}R_{1212} = g_{11}{R^1}_{212} = {R^1}_{212} &= \frac{\partial \Gamma^1_{22}}{\partial r} - \frac{\partial \Gamma^1_{12}}{\partial \theta} + \sum_p \left(\Gamma^p_{22}\Gamma^1_{1p} - \Gamma^p_{12}\,\Gamma^1_{2p}\right)\\ &= -\frac{1}{2}\frac{\partial^2 G}{\partial r^2} - \frac{1}{2G}\frac{\partial G}{\partial r}\left(-\frac{1}{2}\frac{\partial G}{\partial r}\right) = -\left(\frac{1}{2}\frac{\partial^2 G}{\partial r^2} - \frac{1}{4G}\left(\frac{\partial G}{\partial r}\right)^2\right)\\ &= -\sqrt{G}\,\frac{\partial^2}{\partial r^2}\left(\sqrt{G}\right).\end{aligned}$$

This proves the proposition.

Example 5.26

Let us calculate the curvature of the hyperbolic plane using the disc model D in exercise 3.22. By (ii) in that exercise the geodesic polar coordinates at the point 0 is given by (r,θ) where $e^{i\theta(z)} = z/|z|$ and $e^{r(z)} = (1+|z|)/(1-|z|)$ or

$$z = \frac{e^r - 1}{e^r + 1}\, e^{i\theta}.$$

Then

$$G = g\left(\frac{\partial z}{\partial \theta}, \frac{\partial z}{\partial \theta}\right) = 4\,\frac{\left(\frac{e^r-1}{e^r+1}\right)^2}{\left(1-|z|^2\right)^2} \;=\; 4\frac{(e^r-1)^2(e^r+1)^2}{\left((e^r+1)^2-(e^r-1)^2\right)^2}$$
$$= 4\,\frac{\left(e^{2r}-1\right)^2}{\left(4e^r\right)^2} = \sinh^2 r.$$

Therefore by proposition 5.25 the sectional curvature is given by

$$K = \,-\frac{1}{\sinh r}\,\frac{\partial^2}{\partial r^2}(\sinh r) = -1.$$

Notice that the fact that K is constant follows without any calculations just from the fact that the isometry group acts transitively on the hyperbolic plane.

Exercise 5.27

For the 2–sphere $S^2 \subseteq \mathbb{R}^3$ of radius 1 show that the metric in geodesic polar coordinates is given by

$$g_{11} = 1\ ,\ g_{12} = g_{21} = 0\ ,\ g_{22} = \sin^2 r$$

and show that the sectional curvature is $K = 1$.

Now return to a Riemannian manifold of dimension $n > 2$ and let us show that the computation of sectional curvatures can be reduced to surfaces:

Proposition 5.28

Let M be a Riemannian manifold, let $p \in M$ and let $S \subseteq T_pM$ be a 2-plane. Choose $B \subseteq T_pM$ a ball such that $\exp_p : B \to U \subseteq M$ is a diffeomorphism onto a normal neighbourhood, and consider the surface $N = \exp_p(B \cap S)$. Let N be given the metric induced from M. Then the sectional curvatures K^N and K^M of N and M respectively satisfy:

$$K_p^N(S) = K_p^M(S)$$

Proof

It is convenient to use what is called a *Riemannian normal coordinate system* around p defined as follows: Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for T_pM; then, since $\exp_p : B \to U$ is a diffeomorphism, we have a local coordinate system $\big(U, u^1, \ldots, u^n\big)$ determined by

$$q = \exp_p\left(\sum_{i=1}^{n} u^i(q) e_i\right) \ , \ \forall\, q \in U.$$

In this coordinate system the Christoffel symbols satisfy:

$$\Gamma^k_{ij}(p) = 0 \qquad \forall\, i, j, k. \tag{5.29}$$

In fact for $v = \big(v^1, \ldots, v^n\big) \in \mathbb{R}^n$ the curve γ_v

$$t \to \exp_p\big(t\big(v^1 e_1 + \ldots + v^n e_1\big)\big) \qquad , \ |t| \text{ small,}$$

is a geodesic through p with coordinates $\gamma^i = u^i \circ \gamma_v(t) = tv^i$, so by the differential equation (3.2) we have for $t = 0$:

$$\sum_{ij} \Gamma^k_{ij}(p) v^i v^j = 0 \qquad k = 1, 2 \ldots n. \tag{5.30}$$

For fixed k this is a quadratic form (since the connection is symmetric) which is constantly zero, hence the coefficients are all zero, which proves (5.29). The components ${R^k}_{lij}$ for the curvature tensor field in this coordinate system are now given by lemma 5.18 which at the point p reduces to

$${R^k}_{lij}(p) = \left.\frac{\partial \Gamma^k_{jl}}{\partial u^i}\right|_p - \left.\frac{\partial \Gamma^k_{il}}{\partial u^j}\right|_p \tag{4.31}$$

and hence by (5.19)

$$\begin{aligned} R_{klij}(p) &= \langle R(e_i, e_j)e_l, e_k\rangle = \sum_\mu g_{k\mu}(p) {R^\mu}_{lij}(p) \\ &= \sum_\mu g_{k\mu}(p) \left.\frac{\partial \Gamma^\mu_{jl}}{\partial u^i}\right|_p - \sum_\mu g_{k\mu}(p) \left.\frac{\partial \Gamma^\mu_{il}}{\partial u^j}\right|_p \\ &= \left.\frac{\partial}{\partial u^i}\left(\sum_\mu g_{k\mu}\Gamma^\mu_{jl}\right)\right|_p - \left.\frac{\partial}{\partial u^j}\left(\sum_\mu g_{k\mu}\Gamma^\mu_{il}\right)\right|_p \\ &= \frac{\partial}{\partial u^i}[jl, k](p) - \frac{\partial}{\partial u^j}[il, k](p). \end{aligned} \tag{5.32}$$

Now choose the basis $\{e_1, \ldots, e_n\}$ in T_pM such that $S = \operatorname{span}\{e_1, e_2\}$. Clearly for $v \in S$ the curve $t \to \exp_p(tv)$ is a geodesic in N so that (N, u^1, u^2) is a Riemannian normal coordinate system for N. It follows that the metric in N with respect to this coordinate system is just given by the matrix$\{g_{ij}\}, i,j \leq 2,$ where $\{g_{ij}\}$ is the matrix defining the metric in M in the coordinate system $\left(U, u^1, \ldots, u^n\right)$

Now clearly by (5.32) and (2.20)

$$K_p(S) = -R_{2112}(p) = R_{1212}(p) = \frac{\partial}{\partial u^1}\,[22,1](p) - \frac{\partial}{\partial u^2}\,[12,1](p)$$

is the same for M and N which proves the proposition.

As an application let us calculate the sectional curvatures for the $n-$sphere and hyperbolic $n-$space (see example 3.23):

Proposition 5.33

i. *For $S^n \subseteq \mathbb{R}^{n+1}$ the sphere of radius* 1 *the sectional curvature is constantly equal* 1, *that is,* $K_p(S) = 1, \qquad \forall\, p \in S^n\ ,\ S \subseteq T_p(S^n).$

ii. *For H^n_+ the hyperbolic $n-$space the sectional curvature is constantly equal to* –1, *that is,* $K_p(S) = -1, \quad \forall\, p \in H^n_+\ ,\ S \subseteq T_p\big(H^n_+\big).$

Proof

Let us prove ii), the proof of i) is entirely similar. Since the isometry group $\mathrm{O}(1,n)^+$ acts transitively on H^n_+ (proposition 3.24 iii)) it suffices to calculate $K_p(S)$ for $p = e_0 = (1, 0, \ldots, 0)$, and since the isometry group fixing e_0 is the orthogonal group we get the same sectional curvature in all plane directions. Thus at least $K_p(S)$ is constant and we just need to calculate $K_{e_0}(S)$ for $S = \operatorname{span}\{e_1, e_2\} \subseteq T_{e_0}\big(H^n_+\big)$. Now clearly by proposition 3.24 iv) we have $\exp_{e_0}(S) = H^2_+ \subseteq \operatorname{span}\{e_0, e_1, e_2\}$ so by proposition 5.28 $K_{e_0}(S)$ can be calculated in H^2_+. Hence by example 5.26 $K_{e_0}(S) = -1$ which proves ii).

Exercise 5.34

Prove proposition 5.33 i).

Exercise 5.35

i. Let $r > 0$ be a positive real number and let $\Phi : M \to N$ be a diffeomorphism of Riemannian manifolds with metrics $\langle\cdot\ ,\ \cdot\rangle$ and $\langle\cdot\ ,\ \cdot\rangle'$ respectively such that

$$\langle\Phi_* x, \Phi_* y\rangle' = r^2\langle x, y\rangle \qquad \forall\, x, y \in T_qM\ ,\ \forall\, q \in M.$$

Show that Φ is an affine transformation and deduce that the sectional curvatures K and K' for M and N respectively are related by

$$K_{\Phi(q)}(\Phi_* S) = \frac{1}{r^2}K_q(S) \qquad \text{for all 2-planes } S \subseteq T_qM,\ \forall\, q \in M.$$

ii. For $r > 0$ let $S_r^n \subseteq \mathbb{R}^{n+1}$ be the sphere of radius r, i.e.,

$$S_r^n = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \left(x^1\right)^2 + \ldots + \left(x^{n+1}\right)^2 = r^2 \right\}$$

with the usual induced Riemannian metric. Show that the sectional curvature K of S_r^n is constant and $K \equiv 1/r^2$.

iii. For $r > 0$ let $H_{r+}^n \subseteq \mathbb{R}^{n+1}$ be the hyperbolic space of "radius ir", i.e., $H_{r+}^n \subseteq \mathbb{R}^{n+1}$ is a connected component of the submanifold

$$H_r^n = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; -\left(x^0\right)^2 + \left(x^1\right)^2 + \ldots + \left(x^n\right)^2 = -r^2 \right\}$$

with the metric induced from the bilinear form F as in example 3.23. Show that the sectional curvature K of H_{r+}^n is constant and that $K \equiv -\frac{1}{r^2}$.

Corollary 5.36

Let M be a Riemannian manifold of constant sectional curvature K

i. *If $K > 0$ then M is locally isometric to S_r^n with $r = 1/\sqrt{K}$.*

ii. *If $K = 0$ then M is locally isometric to $\mathbb{R}^n$.*

iii. *If $K < 0$ then M is locally isometric to H_{r+}^n with $r = 1/\sqrt{-K}$.*

Proof

This is immediate from theorem 5.10.

Chapter 6 CURVATURE FOR SUBMANIFOLDS OF EUCLIDEAN SPACE

In this section we shall study the curvature for an $n-$dimensional submanifold $M \subseteq \mathbb{R}^k$ with the Riemannian metric in M induced from the Euclidean metric in $\mathbb{R}^k$ which we shall denote by $\langle \cdot\ ,\ \cdot \rangle$. As usual we shall identify the tangent space $T_pM\ ,\ p \in M$ with the corresponding subspace of $\mathbb{R}^k$, so that a vector field X on M is identified with a function $X : M \to \mathbb{R}^k$. In $\mathbb{R}^k$ the Riemannian connection is just given by the directional derivative $D_{X_p}Y$ for Y a vectorfield on $\mathbb{R}^k$ and $X_p \in \mathbb{R}^k$ any (tangent) vector. We shall often consider *vector fields along M,* i.e. functions $Y : M \to \mathbb{R}^k$ and for these the directional derivative $D_{X_p}Y$ is defined for $X_p \in T_pM\ ,\ p \in M$. In particular, as we have seen in exercises 2.3 and 2.22, the Riemannian connection in M is given for Y a tangent field to M and $X_p \in T_pM$ by

(6.1)
$$\nabla_{X_p}(Y) = {}^{\mathrm{T}}\big(D_{X_p}(Y)\big)_p$$

where ${}^{\mathrm{T}}v$ for $v \in \mathbb{R}^k$ is the orthogonal projection of v onto T_pM. Now we shall also need the *normal space* $T_pM^{\perp}$ of vectors perpendicular to T_pM , and for $v \in T_p\mathbb{R}^k$ we denote $v^{\perp} = v - {}^{\mathrm{T}}v$ the projection of v onto the normal space at p. Thus as in (6.1) let Y be a tangent field to M and $X_p \in T_pM$ and consider

(6.2)
$$S(X_p, Y) = \big(D_{X_p}(Y)\big)_p^{\perp}$$

Then we have

Proposition 6.3

1. $S(X_p, Y)$ *does only depend on* Y_p, *that is,* S *defines a bilinear map*

$$S : T_pM \times T_pM \to T_pM^{\perp}$$

such that for X *and* Y *smooth tangent fields on* M *the map* $p \mapsto S(X_p, Y_p)$ *is smooth.*

2. S *is symmetric, i.e.*

$$S(x,y) = S(y,x) \quad \forall\, x, y \in T_pM\ ,\ p \in M.$$

3. *For* Y *a tangent field on* M *and* $X_p \in T_pM$

$$D_{X_p}(Y) = \nabla_{X_p}(Y) + S(X_p, Y_p)$$

Proof

i) We shall see that $S(X_p, Y)$ satisfies

$$S(X_p, fY) = f(p)S(X_p, Y) \text{ for } f \in C^\infty(M).$$

In fact

$$\begin{aligned} S(X_p, fY) = D_{X_p}(fY)^\perp &= \big(X_p(f)Y + f(p)D_{X_p}(Y)\big)^\perp \\ &= f(p)\big(D_{X_p}(Y)\big)^\perp = f(p)S(X_p, Y). \end{aligned}$$

This proves the first statement. The smoothness of $p \mapsto S(X_p, Y_p)$ is obvious from the definition.

ii) By i) is enough to prove this for $X = \frac{\partial}{\partial u^i} =\big|_p$, $Y = \frac{\partial}{\partial u^j}\big|_p$ for $\big(U, u^1, \ldots, u^n\big)$ a local coordinate chart in M around p. In this case we have

$$D_X Y = D_Y X. \tag{6.4}$$

In fact if $f : U' \to U \subseteq M \subseteq \mathbb{R}^k$ is the inverse of the coordinate chart and we consider this as a function from $U' \subseteq \mathbb{R}^n$ into $\mathbb{R}^k$ then (6.4) is just the formula

$$\frac{\partial^2 f}{\partial u^i \partial u^j} = \frac{\partial^2 f}{\partial u^j \partial u^i} \,. \tag{6.5}$$

Now ii) just follows from (6.4) by taking normal components.

iii) is obvious from the definition.

We can now calculate the curvature tensor in terms of the above bilinear map S:

Theorem 6.6 (Gauss$'$ equations)

For X, Y, Z, W tangent fields on M we have

$$\langle R(X,Y)Z, W\rangle = \langle S(Y,Z)\ ,\ S(X,W)\rangle - \langle S(X,Z)\ ,\ S(Y,W)\rangle.$$

Proof

Since both side of the equation are multilinear with respect to functions it suffices to prove this for $X = \frac{\partial}{\partial u^i}$, $Y = \frac{\partial}{\partial u^j}$, $Z = \frac{\partial}{\partial u^l}$, where $\big(U, u^1, \ldots, u^n\big)$ is a local coordinate system. Then we have

$$D_X(D_Y Z) = D_Y(D_X Z) \tag{6.7}$$

similar to (6.4) above, and also

$$R(X,Y)Z = \nabla_X \nabla_Y(Z) - \nabla_Y \nabla_X(Z). \tag{6.8}$$

Now

$$\begin{aligned} D_X(D_Y Z) &= D_X(\nabla_Y Z) + D_X(S(Y,Z)) \\ &= \nabla_X \nabla_Y Z + S(X, \nabla_Y Z) + D_X(S(Y,Z)) \end{aligned} \tag{6.9}$$

and similarly

$$D_Y(D_X Z) = \nabla_Y \nabla_X Z + S(Y, \nabla_X Z) + D_Y(S(X, Z)) \tag{6.10}$$

subtracting (6.10) from (6.9) we then obtain

$$0 = R(X,Y)Z + S(X, \nabla_Y Z) - S(Y, \nabla_X Z) + D_X(S(Y,Z)) - D_Y(S(X,Z)) \tag{6.11}$$

or

$$R(X,Y)Z = -S(X, \nabla_Y Z) + S(Y, \nabla_X Z) - D_X(S(Y,Z)) + D_Y(S(X,Z)). \tag{6.12}$$

Since W is a tangent field and $S(X, \nabla_Y Z)$ and $S(Y, \nabla_X Z)$ are normal fields we therefore get from (6.12):

$$\langle R(X,Y)Z, W\rangle = -\langle D_X S(Y,Z), W\rangle + \langle D_Y S(X,Z), W\rangle. \tag{6.13}$$

But again $\langle S(Y,Z), W\rangle = 0$ so that

$$\begin{aligned} 0 = X\langle S(Y,Z), W\rangle &= \langle D_X S(Y,Z), W\rangle + \langle S(Y,Z), D_X W\rangle \\ &= \langle D_X S(Y,Z), W\rangle + \langle S(Y,Z), S(X,W)\rangle \end{aligned} \tag{6.14}$$

and similarly

$$0 = \langle D_Y S(X,Z), W\rangle + \langle S(X,Z), S(Y,W)\rangle. \tag{6.15}$$

Substituting (6.14) and (6.15) in (6.13) now yields the desired equation.

In particular let us consider the case of a *hypersurface,* i.e. $M^n \subseteq \mathbb{R}^{n+1}$ and let us suppose that we have given a *unit normal* vector field N, that is, a smooth vector field along M^n such that $T_pM^\perp = \operatorname{span}\{N_p\}$ for all $p \in M$ and such that the length of N_p is one. Notice that at least locally such a field always exists and it is unique up to a sign ± 1. In this situation the bilinear function S defines the “second fundamental form” II by the equation

$$S(X,Y) = \mathrm{II}(X,Y)N \tag{6.16}$$

for X and Y vector fields on M. We then clearly obtain:

Corollary 6.17 (Gauss′ Theorema Egregium)

1. *For X, Y, Z, W tangent fields on a hypersurface M in $\mathbb{R}^{n+1}$, we have*

$$\langle R(X,Y)Z, W\rangle = \mathrm{II}\,(Y,Z)\mathrm{II}\,(X,W) - \mathrm{II}\,(X,Z)\mathrm{II}\,(Y,W).$$

2. *In particular the sectional curvature for $S = \operatorname{span}\{y,z\} \subseteq T_pM$ is given by*

$$K_p(S) = \frac{\mathrm{II}(y,y)\mathrm{II}(z,z) - \mathrm{II}(y,z)^2}{\langle y,y\rangle\langle z,z\rangle - \langle y,z\rangle^2}.$$

Remark

Classically the metric $\langle \cdot\ ,\ \cdot \rangle$ is called the "first fundamental form" and is denoted I, so that the above formula can be memorized as "the second fundamental form divided by the first". For M a surface in $\mathbb{R}^3$ this is called the *Gaussian curvature.*

The second fundamental form can also be expressed in terms of the derivative of N:

Proposition 6.18

Let $M \subseteq \mathbb{R}^{n+1}$ be a hypersurface with unit normal field N.

1. *For all $X_p \in T_pM\ \ wehave\ D_{X_p}N \in T_pM.$*
2. *For Y a tangent field in M and $X_p \in T_pM$ we have the **Weingarten equations***

$$\big\langle D_{X_p}N, Y_p \big\rangle = -\big\langle N, D_{X_p}Y \big\rangle = -\mathrm{II}(X_p, Y_p).$$

3. *In particular for $X_p, Y_p \in T_pM$*

$$\big\langle D_{X_p}N, Y_p \big\rangle = \big\langle D_{Y_p}N, X_p \big\rangle$$

Proof

i) Since $\langle N, N \rangle \equiv 1$ we have

$$0 = X_p\langle N, N \rangle = 2\big\langle D_{X_p}N, N_p \big\rangle$$

this proves that $D_{X_p}N$ is perpendicular to N_p hence is a tangent vector.

ii) Since $\langle N, Y \rangle \equiv 0$,

$$0 = X_p\langle N, Y \rangle = \big\langle D_{X_p}N, Y_p \big\rangle + \big\langle N_p, D_{X_p}Y_p \big\rangle$$

which proves ii).

iii) Clearly follows from ii) and the symmetry of II (i.e. the symmetry of S).

The normal field N considered as a map $N : M \to \mathbb{R}^{n+1}$ is clearly a smooth map into the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ called the *Gauss map* $N : M \to S^n$. Then as noticed above $T_{N(p)}S^n = T_pM$ so that the differential

$$N_* : T_pM \to T_{N(p)}S^n = T_pM$$

maps T_pM to itself. In these terms ii) above becomes

$$\langle N_* x, y \rangle = -\mathrm{II}(x, y) \qquad \forall\, x, y \in T_pM. \tag{6.19}$$

For $n = 2$ this gives another characterization of the curvature:

Corollary 6.20

Let $M \subseteq \mathbb{R}^3$ be a surface with normal field N. Then the Gaussian curvature K_p of M at p is given by $K_p = \det N_*$ where $N_* : T_pM \to T_pM$ is the differential of the Gauss map.

Proof

Let$\{e_1, e_2\}$ be an orthonormal basis for T_pM, and let $\{n_{ij}\}$ be the matrix for N_* with respect to $\{e_1, e_2\}$, that is, by (6.19)

$$n_{ij} = \langle N_* e_i, e_j \rangle = -\mathrm{II}(e_i, e_j).$$

Then by corollary 6.17 the Gaussian curvature is given by

$$\begin{aligned} K_p &= \mathrm{II}(e_1, e_1)\mathrm{II}(e_2, e_2) - \mathrm{II}(e_1, e_2)^2 = n_{11}n_{22} - n_{12}^2 \\ &= \begin{vmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{vmatrix} \end{aligned}$$

which is the determinant of N_*.

Exercise 6.21

a. Let $H(u, v) = \sum_{ij} h_{ij} u^i v^j$, $u, v \in \mathbb{R}^n$ be a symmetric bilinear form. Let $M \subseteq \mathbb{R}^{n+1}$ be the hypersurface

$$M = \left\{ \left(u, \frac{1}{2} H(u, u) \right) \;\middle|\; u \in \mathbb{R}^n \right\}$$

so that the tangent space at $0 \in M$ is naturally identified with $\mathbb{R}^n$ given by the first n coordinates in $\mathbb{R}^{n+1}$. Show that the second fundamental form for M at 0 is $\mathrm{II}_0 = H$.

b. For n=2 describe the 3 different cases $K_0 \overset{>}{\underset{<}{=}} 0$ for the Gaussian curvature K_0 in terms of the bilinear form H, and draw pictures of the standard forms of M for each case.

Exercise 6.22

Let $M \subseteq \mathbb{R}^3$ be a surface with normal field N and second fundamental form II. Let $\gamma : [a, b] \to M$ be a smooth curve without self-intersection . Also suppose $\mathrm{II}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right) \neq 0$ and $K_{\gamma(t)} \neq 0$ for all t in $[a, b]$.

a. Show that there is a smooth vector field Y along γ tangent to M such that for all $t \in [a, b]$, $\mathrm{II}\left(Y(t)\,,\, \frac{d\gamma}{dt}\right) = 0$ and also $Y(t)$ and $\frac{d\gamma}{dt}$ are linearly independent.

b. Now define $f : \mathbb{R} \times [a, b] \to \mathbb{R}^3$ by $f(s, t) = sY(t) + \gamma(t)$ and show that for some $\epsilon > 0$ the set

$$M_\epsilon = \left\{ y = f(s, t) \;\middle|\; s \in (-\epsilon, \epsilon)\,,\; t \in [a, b] \right\}$$

is a submanifold of $\mathbb{R}^3$ which touches M along γ in the sense of exercise 2.12 c).

c. Show that the Gaussian curvature of M_ϵ is identically zero and conclude that M_ϵ is locally isometric to $\mathbb{R}^2$ (M_ϵ is called an "osculating developable").

Chapter 7 THE GAUSS-BONNET THEOREM

In this section we shall prove the classical Gauss-Bonnet theorem stating that the integral of the curvature function over a compact surface M is 2π times the Euler characteristic $\chi(M)$. First a few remarks about integration:

Let M be a Riemannian manifold with Riemannian metric $g = g(\cdot\,,\,\cdot)$. Then for a suitable class of "integrable functions" we shall define the integral as follows: For convenience we assume the functions have compact support.

First suppose that the support of the function f is contained in a neighbourhood such that $(U, \mathbf{u})$, $\mathbf{u} : U \to U' \subseteq \mathbb{R}^n$ is a coordinate chart. Let $\{g_{ij}\}$ be the matrix defining the metric in this coordinate chart and let $G = \det\{g_{ij}\}$. Then we define f to be integrable if $\left(f \circ \mathbf{u}^{-1}\right)\sqrt{G \circ \mathbf{u}^{-1}}$ is (Lebegues or Riemann) integrable in $U' \subseteq \mathbb{R}^n$ and

$$
\begin{aligned}
\int_M f &= \int_{U'} \left(f \circ \mathbf{u}^{-1}\right) \cdot \sqrt{G \circ \mathbf{u}^{-1}} \\
&= \int_{\mathbb{R}^n} f\left(u^1, \dots, u^n\right)\sqrt{G(u^1, \dots, u^n)}\, du^1 \,\dots\, du^n
\end{aligned}
\tag{7.1}
$$

Notice that if $(V, \mathbf{v})$, $\mathbf{v} : V \to V' \subseteq \mathbb{R}^n$ is another coordinate chart and if $\{\tilde{g}_{ij}\}$ is the matrix defining g in this chart then

$$
\begin{aligned}
\tilde{G} = \det\{\tilde{g}_{ij}\} &= \det\left\{ g\left(\frac{\partial}{\partial v^i}\,,\,\frac{\partial}{\partial v^j}\right)\right\} \\
&= \det\left\{ g\left(\sum_k \frac{\partial u^k}{\partial v^i}\cdot\frac{\partial}{\partial u^k}\,,\,\sum_l \frac{\partial u^l}{\partial v^j}\frac{\partial}{\partial u^l}\right)\right\} \\
&= \det\left\{\frac{\partial u^i}{\partial v^j}\right\}^2 \det\left\{ g\left(\frac{\partial}{\partial u^k}\,,\,\frac{\partial}{\partial u^l}\right)\right\} \\
&= \det\left\{\frac{\partial u^i}{\partial v^j}\right\}^2 G \\
&= \det\left(D(\mathbf{u}\circ\mathbf{v})\right)^2 G
\end{aligned}
$$

Therefore if $\operatorname{Supp} f \subseteq U \cap V$ we get from the transformation formula for integrals that

$$
\begin{aligned}
\int_{V'} \left(f \circ \mathbf{v}^{-1}\right)\sqrt{\tilde{G} \circ \mathbf{v}^{-1}} &= \int_{V'} \left(f \circ \mathbf{v}^{-1}\right)\left|\det\left(D\left(\mathbf{u}\circ\mathbf{v}^{-1}\right)\right)\right|\sqrt{G \circ \mathbf{v}^{-1}} \\
&= \int_{U'} \left(f \circ \mathbf{u}^{-1}\right)\sqrt{G \circ \mathbf{u}^{-1}}
\end{aligned}
\tag{7.2}
$$

where one integrand is integrable iff the other one is integrable. Hence the integral in (7.1) does not depend on the choice of coordinate chart.

In the general case we cover the support of the function f by coordinate neighbourhoods $\{U_\alpha\}_{\alpha\in I}$ and since we have assumed supp f to be compact we can take the covering to be finite. Now choose a partition of unity $\{\phi_\alpha\}_{\alpha\in I}$ subordinate to $\{U_\alpha\}$. Then f is defined to be integrable iff $f\cdot\phi_\alpha$ is integrable for all $\alpha\in I$ and we define

$$\int_M f \quad = \sum_{\alpha\in I} \int_M f\cdot\phi_\alpha \tag{7.3}$$

where the right hand side is defined by (7.1) since $f\cdot\phi_\alpha$ has support in U_α. This definition is independent of choice of covering and partition of unity. In fact if $\left\{V_\beta\right\}_{\beta\in J}$ is another finite covering of supp$f$ and $\left\{\psi_\beta\right\}_{\beta\in J}$ is a subordinate partition of unity then clearly for $\alpha\in I,\ f\cdot\phi_\alpha$ is integrable iff $f\cdot\phi_\alpha\cdot\psi_\beta$ is integrable for all $\beta\in J$ and

$$\int_M f\phi_\alpha = \sum_\beta \int_M f\phi_\alpha\psi_\beta.$$

It thus follows that $f\cdot\phi_\alpha$ is integrable $\forall\,\alpha\in I$ iff $f\cdot\phi_\alpha\psi_\beta$ is integrable $\forall\,\alpha\in I\ ,\ \beta\in J$ and by symmetry iff $f\cdot\psi_\beta$ is integrable $\forall\beta\in J$.

Similarly in this case

$$\sum_\alpha \int_M f\phi_\alpha = \sum_{\alpha,\beta}\int_M f\phi_\alpha\psi_\beta = \sum_\beta\int_M f\psi_\beta$$

which shows that (7.3) is well-defined.

Now let us return to a 2-dimensional Riemannian manifold M and let us define some nice domains of integration in M:

Definition 7.4

A *polygonal domain* $P\subseteq M$ is a compact connected subset such that the boundary consists of finitely many piecewise geodesic curves with no double points.

We shall decompose such a polygonal domain into some particularly simple ones namely *small geodesic triangles* defined as follows: Let $W\subseteq M$ and $\epsilon>0$ be as in theorem 3.8, that is, for $q,q'\in W$ there is a unique minimal geodesic of length $<\epsilon$ joining q and q', and furthermore for all $q\in W$, there is a normal neighbourhood around q of radius ϵ. Then we will define

Definition 7.5

Let W be as above and $\Sigma\subseteq W$ any subset.

i) Σ is *geodesically convex* if whenever $q,q'\in\Sigma$ also the minimal geodesic arc connecting q,q' is contained in Σ.

ii) A subset $\Sigma\subseteq W$ is called a *small geodesic triangle* with *vertices* p_1,p_2,p_3 if

a) Σ is geodesically convex and contains p_1,p_2,p_3,

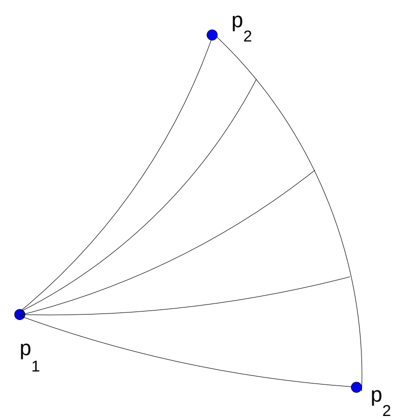
b) whenever $\Sigma' \subseteq W$ is geodesically convex and contains p_1, p_2, p_3 then $\Sigma \subseteq \Sigma'$,

c) p_1, p_2, p_3 do not lie on one geodesic line.

We shall show that there are plenty of small geodesic triangles: For this let $p_1 \in W$ be any point and let $p_2, p_3 \in W$ be chosen in such a way that

α) the minimal geodesic arc γ_1 connecting p_2 and p_3 is contained in $W - \{p_1\}$ and is *not* part of a radial geodesic,

β) the radial geodesics emanating from p_1 and joining a point on γ_1 are also contained in W.



Notice that if just p_2 and p_3 are chosen sufficiently close to p_1 then α) and β) are fulfilled if just p_1, p_2, p_3 are not on the same geodesic line. In fact take $U_{p_1} \subseteq W$ a normal neighbourhood of radius $\delta < \epsilon$ and let p_2, p_3 have distance less than $\frac{\delta}{4}$ from p_1. Then $d(p_2, p_3) < \frac{\delta}{2}$ and any point on the minimal geodesic γ_1 has distance less than $\frac{3}{4}\delta$ from p_1 hence lies in U_{p_1}.

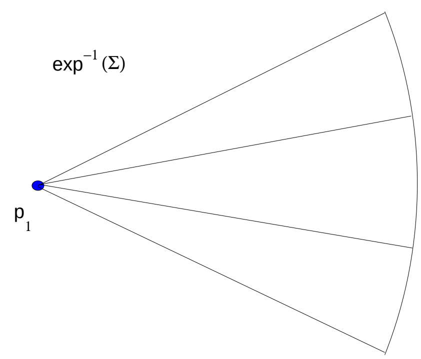
Proposition 7.7

Let W and $\epsilon > 0$ be as above and let $p_1, p_2, p_3 \in W$ satisfy (7.6) α) *and* β) *and let $\Sigma \subseteq W$ be the union of geodesic rays from p_1 to the minimal geodesic arc γ_1 joining p_2 and p_3. Then Σ is a small geodesic triangle with vertices p_1, p_2, p_3. Conversely every small geodesic triangle contained in W is the union of the geodesic rays joining one vertex to its opposite side. Furthermore Σ is a polygonal domain bounded by the 3 minimal geodesics joining the vertices.*

Proof

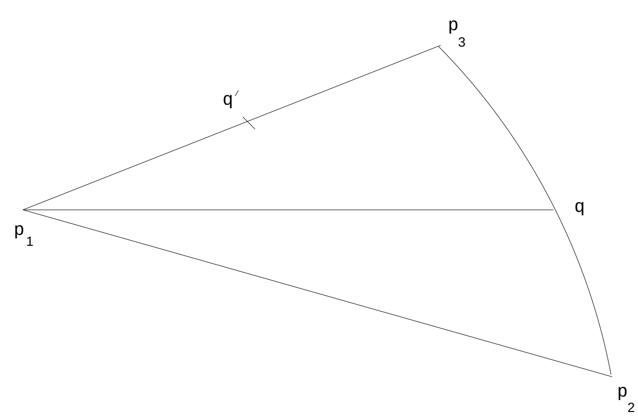
Let $\Sigma \subseteq W$ be constructed as in (7.6); then we shall show that Σ is a small geodesic triangle with vertices p_1, p_2, p_3..

First notice that Σ is a polygonal domain bounded by the minimal geodesics $\gamma_1, \gamma_2, \gamma_3$ joining p_2 and p_3 , p_1 and p_3, respectively p_1 and p_2. In fact clearly every geodesic ray emanating from p_1 intersects γ_1 precisely once so that the inverse image of Σ by the exponential map $\exp_{p_1}$ is a compact set in $T_{p_1}M$ bounded by two line segments and a curve.

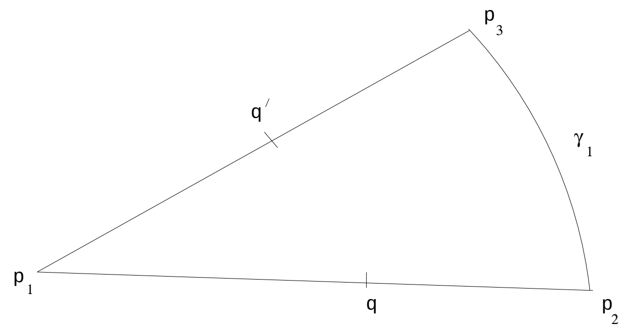


Next let us show that Σ is geodesically convex. For this let q and q' be two points in Σ and let ω be the minimal geodesic joining them and suppose ω does not stay inside Σ. Then clearly some segment of ω will join two boundary points and otherwise stay entirely outside Σ, so we can assume that q, q' lie on either γ_1, γ_2 or γ_3 and ω lies entirely outside Σ except for the endpoints.

Case 1. $q, q' \in \gamma_1$ then ω is clearly part of γ_1, which is a contradiction.



Case 2. $q \in \gamma_1, q' \in \gamma_2$. Then by replacing p_2 by q we can suppose we are in case 3 below.



Case 3. $q \in \gamma_3$, $q' \in \gamma_2$.

It suffices to show that ω stays in the same angular component as γ_1. In fact it can never intersect γ_1, since in that case either ω would be part of γ_2 or γ_3, or we would get new intersection points on γ_1 as in case 1.

Now ω has only two possible angular components to stay in, and for q' close to p_3 and q close to p_2 it must be the same component as for γ_1. Hence by continuity the component must be the same as the one containing γ_1 for all q and q'. This shows that Σ is convex. That Σ satisfies b) in definition 7.5 ii) is obvious from the definition.

Now conversely given $\Sigma \subseteq W$ a small geodesic triangle with vertices p_1, p_2, p_3. Then clearly by convexity the sides $\gamma_1, \gamma_2, \gamma_3$ are also contained in Σ. Furthermore the set Σ' consisting of all geodesic rays joining p_1 to γ_1 is then also contained in $\Sigma \subseteq W$ and clearly satisfies (7.6). On the other hand we have just shown then that Σ' is geodesically convex. Hence by 7.5 ii) b) $\Sigma' \supseteq \Sigma,$ that is, $\Sigma' = \Sigma$. This ends the proof.

Proposition 7.8

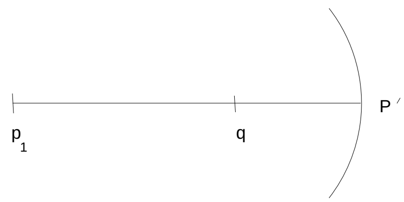
Every polygonal domain $P \subseteq M$ can be triangulated into small geodesic triangles. That is, there exist finitely many small geodesic triangles $\Sigma_1, \ldots, \Sigma_k,$ such that

1. $P = \bigcup_{i=1}^{k} \Sigma_i.$

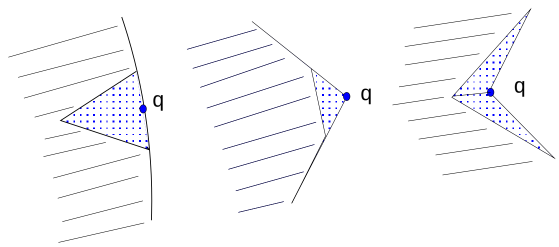
2. *Any two triangles intersect in at most one common vertex or one common side.*

Proof

First observe that by proposition 7.7 any interior point q in P has a neighbourhood which is the interior of a geodesic triangle. In fact choose $W \subseteq P$ a neighbourhood of q as in 7.5 above, let $p_1 \in W \quad p_1 \neq q$ be any point, and let $p' \in W$ be any point opposite p_1 with respect to q. Then for γ any geodesic through p' different from the line p, p' and for any choice of p_2, p_3 on γ sufficiently close to p' and on opposite sides, the points p_1, p_2, p_3 will determine a geodesic triangle containing q in the interior.



Next for each point q on the boundary we have a neighbourhood as in either of the figures



It follows by compactness that we can cover P by finitely many geodesic triangles such that i) the interiors cover the interior of P, ii) the boundaries cover the boundary of P.

Now if two triangles Σ_1 and Σ_2 intersect then we subdivide the union $\Sigma_1 \cup \Sigma_2$ into smaller triangles using all necessary new vertices, in order to get a triangulation. Adding one triangle at a time we eventually obtain the desired triangulation.

With these preliminaries out of the way we can now prove the Gauss-Bonnet theorem for a small triangle:

Theorem 7.9 (Gauss)

Let Σ be a small geodesic triangle with vertices A, B, C and opposite sides α, β, γ respectively. Let $\angle A, \angle B, \angle C$ be the angles between the geodesics $(\gamma, \beta), (\gamma, \alpha)$ and (α, β) respectively. Let K be the curvature function. Then

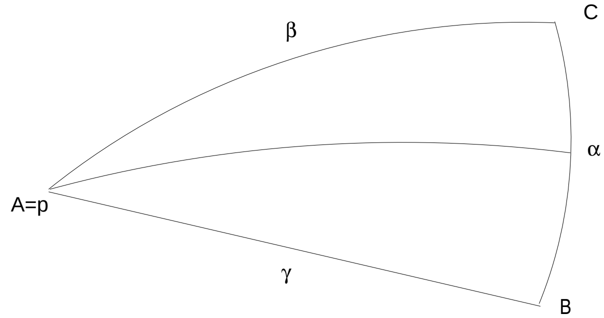
$$\int_{\Sigma} K = \angle A + \angle B + \angle C - \pi.$$

Note 1

Here $\int\limits_{\Sigma} K = \int\limits_{M} 1_{\Sigma} \cdot K$ where 1_{Σ} is the characteristic function for Σ, that is, $1_{\Sigma}(q) = 0$ for $q \notin \Sigma$, $1_{\Sigma}(q) = 1$ for $q \in \Sigma$.

Note 2

The angle between two geodesics is just the angle between their tangents in the tangent space at the point of intersection.



For the proof of theorem 7.9 we shall consider polar coordinates (r, θ) with respect to the point $p = A$. Notice that since Σ is "small" it lies inside a normal neighbourhood of p and is the union of all geodesic rays joining p to the opposite side α.

Recall that the metric is given by

$$g_{11} = 1 \ , \ g_{12} = g_{21} = 0 \ , \ g_{22} = G$$

where $G = g\left(\frac{\partial}{\partial\theta} \ , \ \frac{\partial}{\partial\theta}\right)$ is a positive function. A priori G is not defined for $r = 0$. However, we have

Lemma 7.10

1. $\lim_{r \to 0+} \sqrt{G(r, \theta)} = 0$ *uniformly in* θ.

2. $\lim_{r \to 0+} \frac{\partial\sqrt{G}}{\partial r}(r, \theta) \to 1$ *uniformly in* θ.

3. $\lim_{r \to 0+} \frac{\partial^2\sqrt{G}}{\partial r^2}(r, \theta) \to 0$ *uniformly in* θ.

Proof

Let $v = r\cos\theta \ e_1 + r\sin\theta \ e_2 \in T_pM$, then at (r, θ), $\sqrt{G}(r, \theta)$ is the length of the vector

$$\left(\exp_p\right)_{*v}\left(\frac{\partial}{\partial\theta}\right) \in T_{\exp_p(v)}(M)$$

where now (r, θ) are the polar coordinates in T_pM. Now since $\left(\exp_p\right)_{*v}$ depends continuously on $v \in T_pM$ and since $\left(\exp_p\right)_{*0} = \mathrm{id} : T_pM \to T_pM$ we get for given $\epsilon > 0$ that

$$\left| \left\| \left(\exp_p\right)_{*v}(u) \right\| - \|u\| \right| \le \epsilon \ \|u\| \qquad \forall\, u \in T_pM, \tag{7.11}$$

v close to 0.

Now for

$$v = r\cos\theta\, e_1 + r\sin\theta\, e_2$$
$$\frac{\partial}{\partial\theta} = -r\sin\theta\, e_1 + r\cos\theta\, e_2$$

so that $\left\|\frac{\partial}{\partial\theta}\right\| = r$. Hence we conclude from (7.11) for $u = \frac{\partial}{\partial\theta}$:

(7.12) $$\left|\sqrt{G}\,(r,\theta) - r\right| \leq \epsilon\cdot r \qquad \text{for } r \text{ close to } 0.$$

Clearly i) follows from (7.12). Next recall from proposition 5.25 that

(7.13) $$\frac{\partial^2\sqrt{G}}{\partial r^2} = -K\sqrt{G}$$

and since K is clearly continuous at zero (converging to K_p) iii) follows.

Now for $\delta_1 > \delta_2$ we get from (7.13)

(7.14) $$\frac{\partial\sqrt{G}}{\partial r}(\delta_1,\theta) - \frac{\partial\sqrt{G}}{\partial r}(\delta_2,\theta) = -\int_{\delta_2}^{\delta_1} K\sqrt{G}\ dr,$$

hence letting $\delta_2 \to 0$ it follows for fixed θ that

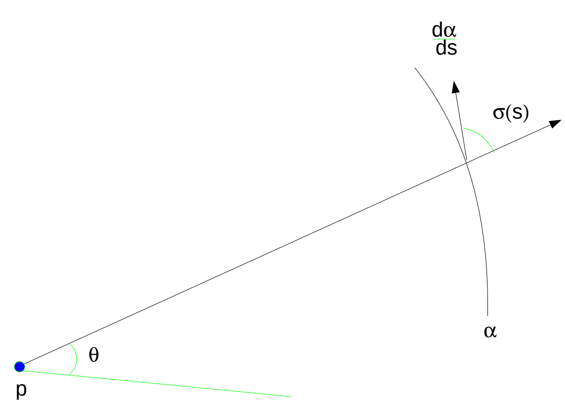
$$\lim_{r\to 0}\frac{\partial\sqrt{G}}{\partial r}(r,\theta)$$

exists and by (7.12) the limit must be 1. Hence by (7.14) we conclude for δ_1 small and θ arbitrary

(7.15) $$\frac{\partial\sqrt{G}}{\partial r}(\delta_1,\theta) = 1 - \int_0^{\delta_1} K\sqrt{G}\ dr,$$

which clearly converges uniformly to 1 as $\delta_1 \to 0$. This proves the lemma.

Next let us investigate $\sqrt{G}$ along the geodesic α : For this we parametrize α by arc length s and we let $\sigma(s)$ denote the angle between α and the radial geodesic, i.e. the angle between the tangent vectors $\frac{d\alpha}{ds}$ and $\frac{\partial}{\partial r}$ at $\alpha(s)$



Notice that since a radial geodesic intersects the geodesic α only once the polar coordinate θ is increasing in s, so we can express the angle σ as a function in θ along α. Then we have:

Lemma 7.15

Along the geodesic α we have

$$\frac{\partial\sqrt{G}}{\partial r}(\alpha(s)) = -\frac{d\sigma}{ds} \Big/ \frac{d\theta}{ds} = -\frac{d\sigma}{d\theta} \, .$$

Proof

Write $\alpha(s)$ in polar coordinates (r, θ), that is,

$$\alpha(s) = \exp_p\left(r(s)\cos\theta(s)e_1 + r(s)\sin\theta(s)e_2\right).$$

Then the equations (3.2) for the geodesic α give with $\alpha^1(s) = r(s)$, $\alpha^2(s) = \theta(s)$:

$$\frac{d^2\alpha^1}{ds^2} + \sum_{ij}\Gamma^1_{ij}\frac{d\alpha^i}{ds}\,\frac{d\alpha^j}{ds} = 0 \tag{7.16}$$

or by the identities (5.23):

$$\frac{d^2 r}{ds^2} = \frac{1}{2}\,\frac{\partial G}{\partial r}\,\left(\frac{d\theta}{ds}\right)^2 \tag{7.17}$$

The angle $\sigma(s)$ is determined by

$$\cos\sigma(s) = g\left(\frac{d\alpha}{ds}\, ,\, \frac{\partial}{\partial r}\bigg|_{\alpha(s)}\right) = \frac{dr}{ds} \tag{7.18}$$

since both vectors have unit length. Differentiating this we obtain using (7.17):

$$-\sin\sigma(s)\cdot\frac{d\sigma}{ds} = \frac{d^2 r}{ds^2} = \frac{1}{2}\,\frac{\partial G}{\partial r}\,\left(\frac{d\theta}{ds}\right)^2, \tag{7.19}$$

but since α is parametrized by arc length

$$\left(\frac{dr}{ds}\right)^2 + \left(\frac{d\theta}{ds}\right)^2 G = 1$$

so that by (7.18)

$$\sin^2\sigma(s) = 1 - \cos^2\sigma(s) = 1 - \left(\frac{dr}{ds}\right)^2 = \left(\frac{d\theta}{ds}\right)^2 G. \tag{7.20}$$

Hence we obtain from (7.19)

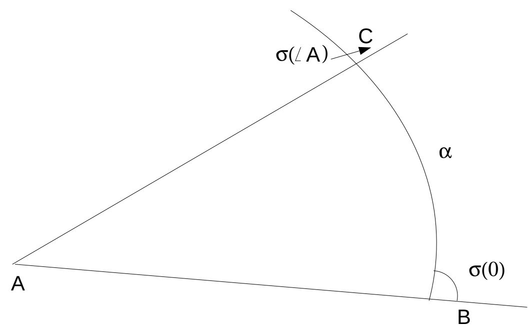
$$-\frac{d\theta}{ds}\cdot\sqrt{G}\cdot\frac{d\sigma}{ds} = \frac{1}{2}\,\frac{\partial G}{\partial r}\,\left(\frac{d\theta}{ds}\right)^2$$

or

$$\frac{d\sigma}{ds} = -\frac{1}{2}\frac{\frac{\partial G}{\partial r}}{\sqrt{G}}\frac{d\theta}{ds} = -\frac{\partial\sqrt{G}}{\partial r}\frac{d\theta}{ds}. \tag{7.21}$$

This gives the lemma.

We now prove the theorem:



Proof of theorem 7.9

As above let Σ be parametrized by polar coordinates (r,θ) at the point $p = A$ and choose the basis $\{e_1, e_2\} \in T_A(M)$ such that e_1 is the tangent vector of γ. Then for points in Σ, θ varies between 0 and $\angle A$, and given θ, r varies between 0 and $r_\alpha(\theta)$ where $(r_\alpha(\theta),\theta)$ are the polar coordinates of the unique point on the geodesic α. Now by (7.1) and proposition 5.25 we have

$$\begin{aligned}
\int_\Sigma K &= \int_0^{\angle A} d\theta \int_0^{r_\alpha(\theta)} K\sqrt{G}\,dr = -\int_0^{\angle A} d\theta \int_0^{r_\alpha(\theta)} \frac{\partial^2\sqrt{G}}{\partial r^2}\,dr \\
&= -\int_0^{\angle A}\left(\frac{\partial\sqrt{G}}{\partial r}(r_\alpha(\theta),\theta) - 1\right)d\theta \text{ by lemma 7.10} \\
&= \int_0^{\angle A}\left(\frac{d\sigma}{d\theta} + 1\right)d\theta \qquad\qquad \text{by lemma 7.15} \\
&= \angle A + \sigma(\angle A) - \sigma(0) \\
&= \angle A + \angle C - (\pi - \angle B) \\
&= \angle A + \angle C + \angle B - \pi
\end{aligned}$$

which was to be proven.

Next we want to extend this theorem to more general polygonal domains. So let $P \subseteq M$ be a polygonal domain. The boundary ∂P by definition consists of finitely many closed curves, which are broken geodesics. Each of the finitely many non-smooth points of the boundary $p_1, \ldots, p_l$ we call a *vertex* and to each of these we can attach a well-defined *interior angle* $\beta_1, \ldots, \beta_l$ where $0 < \beta_i < 2\pi$, $i = 1, \ldots, l$. The quantities

$$\alpha_i = \pi - \beta_i \,, \qquad -\pi < \alpha_i < \pi \,, \qquad i = 1, \ldots, l \tag{7.22}$$

are called the *exterior angles.*

We shall also need the notion of the *Euler characteristic* of P :

Definition 7.23

Let $P \subseteq M$ be a polygonal domain triangulated into finitely many triangles as in proposition 7.8. Let V be the number of vertices, E the number of edges, T the number of triangles in this triangulation. Then the Euler characteristic $\chi(P)$ is

$$\chi(P) = V - E + T.$$

Note

From Topology it is known that $\chi(P)$ is a topological invariant, independent of choice of triangulation. In particular for $P = M$ an *oriented* 2-manifold

$$\chi(M) = 2(1 - g)$$

where g is the *genus* of M. This in turn is the number of handles, which should be attached to a 2-sphere in order to obtain a manifold diffeomorphic to M (by the classification of surfaces this is always possible).

Theorem 7.24 (Gauss-Bonnet)

Let M be a 2-dimensional Riemannian manifold with curvature function K. Let $P \subseteq M$ be a polygonal domain and let $\alpha_1, \ldots, \alpha_l$ be the exterior angles at the vertices of P. Then

$$\int_P K = 2\pi \,\chi\,(P) - \sum_{i=1}^{l} \alpha_i.$$

For the proof we need the following trivial lemma:

Lemma 7.25

Let $P \subseteq M$ be a polygonal domain triangulated into small geodesic triangles $\Sigma_1, \ldots, \Sigma_k$. Then for any integrable function f on P

$$\int_P f = \sum_{i=1}^{k} \int_{\Sigma_i} f.$$

Proof

It is easily seen from the definition (7.3) of the integral $\int\limits_M$ that it is a linear functional on the vector space of integrable functions. Now the function

$$1_P \cdot f \, - \, \sum_{i=1}^{k} 1_{\Sigma_i} f \tag{7.26}$$

is zero outside the boundaries of the triangles Σ_i. In a coordinate neighbourhood the boundaries of triangles are clearly sets of measure zero, so the integral of the function (7.26) is clearly zero by the definition. Hence

$$\int\limits_P f = \int\limits_M 1_P \cdot f = \sum_{i=1}^{k} \int\limits_M 1_{\Sigma_i} f = \sum_{i=1}^{k} \int\limits_{\Sigma_i} f.$$

Proof of Theorem 7.24

Choose a triangulation of P as in proposition 7.8. Let $\Sigma_1, \ldots, \Sigma_T$ be the small geodesic triangles in this and $q_1, \ldots, q_V$ all the vertices. Let $q_1, \ldots, q_m$, $m \le V$ be all the vertices on the boundary ∂P. This includes the vertices $p_1, \ldots, p_l$ of ∂P (i.e. the non-smooth points), but there may be more. For each Σ_j , $j = 1, \ldots, T$ we name the 3 vertices A_j, B_j, C_j considered as vertices of Σ_j. Then by lemma 7.25 and theorem 7.9 we have

$$\int\limits_P K = \sum_{j=1}^{T} \int\limits_{\Sigma_j} K = \sum_{j=1}^{T} (\angle A_j + \angle B_j + \angle C_j) - \pi T. \tag{7.27}$$

In this sum each vertex q_i , $i = 1, \ldots, V$ may occur several times as a vertex in different triangles.

For each interior vertex the different angles clearly add up to 2π whereas for $q_i \in \partial P$ they add up to the interior angle β_i at q_i. Hence

$$\begin{aligned} \sum_{j=1}^{T} (\angle A_i + \angle B_j + \angle C_j) &= \sum_{i=1}^{m} \beta_i + (V - m)2\pi \\ &= 2\pi V - \sum_{i=1}^{m} (\pi - \beta_i) - m\pi \end{aligned} \tag{7.28}$$

Notice that since the boundary consists of closed curves the number m is also the number of edges in the boundary for the triangulation of P. Then a simple combinatorial argument gives

$$3T = 2E - m. \tag{7.29}$$

In fact counting the 3 vertices of all triangles gives a doubling of a vertex every time two triangles have a common edge, i.e. at all edges, which are not part of the boundary.

Since there are $E-m$ of these the formula (7.29) is proved. Inserting (7.29) in (7.28) gives

$$\sum_{j=1}^{T}(\angle A_i+\angle B_j+\angle C_j)=2\pi V+3T\pi-2E\pi-\sum_{i=1}^{m}\alpha_i \tag{7.30}$$

with $\alpha_i=\pi-\beta_i$ for each q_i , $i=1,\dots,m.$

Hence (7.27) reduces to

$$\int_P K=2\pi V+2\pi T-2\pi E-\sum_{i=1}^{m}\alpha_i=2\pi\chi(P)-\sum_{i=1}^{m}\alpha_i. \tag{7.31}$$

Notice that if q_i is a smooth point of the boundary then the exterior angle is 0 so (7.31) is just the required formula in theorem 7.24.

Remark

Notice that it follows from theorem 7.24 that the Euler characteristic $\chi(P)$ is independent of choice of triangulation for P.

Let us make a number of corollaries.

Corollary 7.32

For M a compact Riemannian 2-manifold with curvature function K

$$\int_M K=2\pi\ \chi(M).$$

In particular for M oriented,

$$\int_M K=4\pi(1-g)$$

where g is the genus of M.

Definition 7.33

A *Polygon* in M is a polygonal domain homeomorphic to a disk.

Corollary 7.34

Suppose M has constant curvature K and let $P\subseteq M$ be a polygon with interior angles $\beta_1,\dots,\beta_l$ at the vertices $p_1,\dots,p_l\in\partial P$. Let $A(P)=\int_P 1$ be the area of P. Then

$$\sum_{j=1}^{l}\beta_j=(l-2)\pi+A(P)\cdot K.$$

In particular for P a triangle with vertices A, B, C

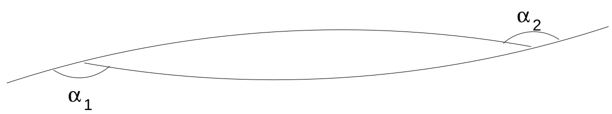
$$\angle A + \angle B + \angle C = \pi + A(P) \cdot K.$$

Proof

Immediate from theorem 7.24 and the fact that for a polygon P, $\chi(P) = 1$

Corollary 7.35

Suppose M has curvature function K satisfying $K \leq 0$; then there cannot exist a polygon with 2 vertices.



Proof

In fact if P was a polygon with 2 vertices with exterior angles α_1, α_2 then $\chi(P) = 1$ and so

$$\int_P K = 2\pi - (\alpha_1 + \alpha_2) > 0$$

contradicting $K \leq 0$.

Exercise 7.36

Let $F : M \to L$ be a smooth map of $n-$dimensional *oriented* manifolds. Suppose M is compact and L is connected. Then the *degree* of F is defined by

$$\deg(F) = \sum_{x \in F^{-1}(y)} \epsilon_x(F)$$

where $y \in L$ is *regular* value (i.e. F_{*x} is non-singular for all $x \in F^{-1}(y)$) and $\epsilon_x(F) = \pm 1$ depending on F_{*x} being orientation preserving or reversing.

1. Suppose M and L have Riemannian metrics g_M and g_L respectively, and define for $x \in M$

$$\det(F)(x) = \det\left\{g_L\left(F_* e_i, e'_j\right)\right\}$$

where $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ are *positively oriented* orthonormal bases for $T_x M$ and $T_{F(x)}(L)$ respectively. Show that for any integrable function f on L

$$\int_M (f \circ F) \det(F) = \deg(F) \int_L f.$$

2. Suppose $M \subseteq \mathbb{R}^3$ is a compact oriented submanifold with normal field N. Show that the Gauss-map $N : M \to S^2$ satisfies

$$\mathrm{deg} N = \frac{1}{2}\chi(M).$$

$\frac{d\alpha}{ds}$ and $\frac{\partial}{\partial r}$ at $\alpha(s)$

Chapter 8 LOCALLY AND GLOBALLY SYMMETRIC SPACES

We have seen that in some sense a Riemannian manifold is determined locally by its curvature. In particular manifolds of the same *constant* curvature are locally isometric. In this section we shall show that this is even true globally provided the manifolds are simply connected. This we shall show in the more general context of *symmetric spaces*.

Definition 8.1

1. A manifold M with connection $\bigtriangledown$ and associated torsion and curvature vector fields T and R, is called *locally affine symmetric* if 1) $T = 0$, 2) R is *parallel along geodesics,* i.e. if R satisfies:

   For every pair of points $p, q \in M$ let $\tau : T_pM \to T_qM$ denote the parallel transport along a geodesic from p to q. Then

   $$R(\tau v, \tau w)\tau z = \tau(R(v,w)z) \qquad \forall\, v, w, z \in T_pM.$$

2. A Riemannian manifold M is called a *Riemannian locally symmetric* space if M with the Riemannian connection is locally affine symmetric.

Remark

One can show that if R is parallel along geodesics then R is parallel along any curve (see Helgason [chapter I §7]).

Exercise 8.2

a. Show that a manifold with affine connection is an affine locally symmetric space just 1) and 2) above are satisfied locally.

b. Show that a Riemannian manifold of constant sectional curvature is a locally symmetric space.

The phrase "locally symmetric" is justified by the following:

Proposition 8.3

A manifold M with connection $\bigtriangledown$ is locally affine symmetric iff for all points $p \in M$ there is a normal neighbourhood U_p such that the map $s_p : U_p \to U_p$ taking $\exp_p(v)$ to $\exp_p(-v)$ is an affine transformation.

In particular a Riemannian manifold is Riemannian locally symmetric iff $s_p : U_p \to U_p$ is an isometry for all p.

Proof

Clearly the second statement follows from the first and proposition 4.6 since $(s_p)_* = -\mathrm{id} : T_pM \to T_pM$ is an isometry.

Let us prove the first statement:

$\Rightarrow$ Consider $\phi = -\mathrm{id} : T_pM \to T_pM$. Then clearly for $T = 0$ and R parallel a) and b) of Corollary 4.18 are satisfied. Therefore s_p is an affine transformation.

For $\Leftarrow$ we shall use the following lemma:

Lemma 8.4

Let M be a manifold with connection $\bigtriangledown$ and let $\gamma : (-\epsilon, \epsilon) \to M$ be a smooth curve. For V_t a smooth vector field along γ the covariant derivative along γ at $t = 0$ is given by

$$\frac{DV}{dt}\Big|_{t=0} = \lim_{s\to 0} \frac{1}{s}\left(\tau_s^{-1}(V_s) - V_0\right)$$

where $\tau_s : T_{\gamma(0)}(M) \to T_{\gamma(s)}(M)$ is the parallel transport along γ.

Proof

For given $s > 0$ let Z_t , $0 \le t \le s$ be a parallel vector field along γ such that $Z_0 = \tau_s^{-1}V_s$. Choose local coordinates $\left(u^1, \dots, u^n\right)$ around $\gamma(0)$ and put $\partial_i = \frac{\partial}{\partial u^i}$ as usual. Then

$$Z_t = \sum_i z^i(t)\partial_i \ , \ V_t = \sum_j v^j(t)\partial_j \ , \quad \text{for } t \le s.$$

By the mean value theorem

$$z^k(s) = z^k(0) + s\frac{dz^k}{dt}(t^*) \text{ for some } t^* \in [0, s].$$

Therefore the $k-$th component (with respect to the frame $\{\partial_i\}$) of $\frac{1}{s}\left(\tau_s^{-1}(V_s) - V_0\right)$ is

$$\frac{1}{s}\Big(z^k(0) - v^k(0)\Big) = \frac{1}{s}\left(z^k(s) - s\frac{dz^k}{dt}(t^*) - v^k(0)\right).$$

Since Z is parallel along γ we have

$$\frac{dz^k}{dt} + \sum_{ij} \Gamma_{ij}^k \frac{d\gamma^i}{dt} z^j = 0 \text{ for all } t \le s,$$

where $\gamma^i = u^i \circ \gamma$. Therefore, for $s \to 0$, $\frac{1}{s}\big(z^k(0) - v^k(0)\big)$ converges to

$$\sum_{ij} \Gamma_{ij}^k \frac{d\gamma^i}{dt}(0)\, v^j(0) + \frac{dv^k}{dt}(0)$$

which is exactly the expression in local coordinates of $\frac{DV}{dt}\big|_{t=0}$ (see proof of lemma 2.8).

We can now finish the proof of proposition 8.3:

To show that R is parallel amounts to show that for γ a geodesic and X, Y, Z parallel fields along γ the vector field $V = R(X,Y)Z$ is also parallel along γ. Now if m is any point of γ we can assume $m = \gamma(0)$ and we just have to prove

$$\frac{DV}{dt}\bigg|_{t=0} = 0. \tag{8.5}$$

Now choose a normal neighbourhood U_m around m such that $s_m : U_m \to U_m$ is an affine transformation. By lemma 8.4

$$\begin{aligned}\frac{DV}{dt}\bigg|_{t=0} &= \frac{1}{2}\lim_{s\to 0+}\left[\frac{1}{s}\big(\tau_s^{-1}(V_s) - V_0\big) + \frac{1}{-s}\big(\tau_{-s}^{-1}(V_{-s}) - V_0\big)\right]\\ &= \frac{1}{2}\lim_{s\to 0+}\frac{1}{s}\big[\tau_s^{-1}V_s - \tau_{-s}^{-1}(V_{-s})\big],\end{aligned}$$

so in order to prove (8.5) it is enough to show that for small $s > 0$ the parallel transport τ from $p = \gamma(-s)$ to $q = \gamma(s)$ along γ takes V_{-s} to V_s (because in that case $\tau_s^{-1}(V_s) = \tau_s^{-1}(\tau(V_{-s})) = \tau_{-s}^{-1}(V_{-s})$). Equivalently we must show

$$R(\tau x, \tau y)\tau z = \tau R(x,y)z \qquad \forall\, x, y, z \in T_pM. \tag{8.6}$$

Now since s_m is an affine transformation we have

a) $$T(s_{m_*}v\ ,\ s_{m_*}w) = s_{m_*}T(v,w)$$

b) $$R(s_{m_*}v\ ,\ s_{m_*}w)s_{m_*}z = s_{m_*}R(v,w)z$$

for all $v, w, z \in T_pM$. For $p = q = m$ a) immediately gives $T(-v,-w) = -T(v,w)$, hence $T = 0$. Therefore in order to finish the proof we shall just prove that b) implies (8.6) above. This obviously follows if we show

$$s_{m_*} = -\tau : T_pM \to T_qM. \tag{8.7}$$

For this let L_t be a parallel vector field along $\gamma(t)$, $t \in [-s,s]$. Then since s_m is an affine transformation, the field $s_{m_*}(L_t)$ is a parallel field along the curve $t \to \gamma(-t)$, or equivalently $-s_{m_*}(L_{-t})$ is parallel along $\gamma(t)$, $t \in [-s,s]$. But

$$s_{m_*} = \ -\mathrm{id} : T_mM \to T_mM$$

so

$$-s_{m_*}(L_{-t})\big|_{t=0} = \ -s_{m_*}L_0 = L_t\big|_{t=0}\ .$$

Hence by uniqueness of parallel transport

$$-s_{m_*}(L_{-t}) = L_t \qquad \text{for all } t \in [-s,s].$$

In particular $t = s$ gives (8.7), which finishes the proof of proposition 8.3.

In order to study global properties of symmetric spaces we introduce the following notion:

Definition 8.8

A connected and *complete* Riemannian manifold M is called a *Riemannian globally symmetric* space if at each point p there is an isometry $s_p : M \to M$ such that

$$s_p\big(\exp_p(v)\big) = \exp_p(-v) \text{ for all } v \in T_pM.$$

Exercise 8.9

Show that the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ and the hyperbolic $n-$space H_+^n (example 3.23) are Riemannian globally symmetric spaces.

Example 8.10. Siegel′s upper half space

This is a generalization of example 3.20: Let Let $M_n(\mathbb{C})$ denote the set of complex $n \times n-$matrices. For $Z \in M_n(\mathbb{C})$ let Z^t denote the transpose of Z and tr$Z$ is the trace. Let $\mathfrak{S}_n \subseteq M_n(\mathbb{C})$ be the set of *symmetric* matrices $Z = X + iY$ where X and Y are real and Y is a *positive* definite matrix. $\mathfrak{S}_n$ is clearly an open subset of set $S_n(\mathbb{C})$ of all symmetric matrices, which is naturally identified with $\mathbb{C}^{\frac{n(n+1)}{2}}$. The tangent space of $\mathfrak{S}_n$ at any point is thus naturally identified with $S_n(\mathbb{C})$ and we define the Riemannian metric at a point $Z = X + iY \in \mathfrak{S}_n$ by

$$g_Z(V, W) = \mathrm{Re}\left[\mathrm{tr}\big(Y^{-1}VY^{-1}\overline{W}\big)\right], \qquad V, W \in S_n(\mathbb{C}). \tag{8.11}$$

Exercise 8.12

1. Show that (8.11) defines a Riemannian metric in $\mathfrak{S}_n$.

2. For $Q + iP \in \mathfrak{S}_n$ define $h : \mathfrak{S}_n \to \mathfrak{S}_n$ by $h(Z) = P^{\frac{1}{2}}ZP^{\frac{1}{2}} + Q$, $\qquad Z \in \mathfrak{S}_n$. Show that h defines an isometry and observe that $h(iI) = Q + iP$. Conclude that the group of isometries of $\mathfrak{S}_n$ acts transitively on $\mathfrak{S}_n$.

3. Show that $\mathfrak{S}_n$ is complete. (*Hint:* this is a direct consequence of ii).)

4. Show that the map $s : \mathfrak{S}_n \to \mathfrak{S}_n$ defined by $s(Z) = -Z^{-1}$ is an isometry keeping the point iI fixed and with $s_* =$—id at that point.

5. Conclude that $\mathfrak{S}_n$ is a Riemannian symmetric space.

6. Let $\mathfrak{D}_n \subseteq \mathfrak{S}_n$ be the set of matrices D of the form

$$D = \begin{pmatrix} id_1 & & \bigcirc \\ & \ddots & \\ \bigcirc & & id_n \end{pmatrix} \qquad d_1 > 0, \ldots, d_n > 0.$$

   Show that $\mathfrak{D}_n$ in the induced metric is isometric to Euclidean $n-$space via the isometry $\mathbb{R}^n \to \mathfrak{D}_n$ given by

$$(t_1, \ldots, t_n) \ \mapsto \ \begin{pmatrix} ie^{t_1} & & \bigcirc \\ & \ddots & \\ \bigcirc & & ie^{t_n} \end{pmatrix}$$

7. Let $\mathcal{Z} \subseteq \mathfrak{S}_n$ be the set of matrices of the form zI where $z = x + iy$, $y > 0$. Show that $\mathcal{Z}$ in the induced metric is isometric to the hyperbolic plane of radius $i\sqrt{n}$ (see Exercise 5.35).

8. One can show that the submanifolds $\mathfrak{D}_n$ and $\mathcal{Z}$ are *geodesic*, that is, whenever two points are in the submanifold, the whole geodesic line joining them lies in the submanifold as well. Given this, show that the sectional curvature of $\mathfrak{S}_n$ at the point iI is *not* constant provided $n > 1$.

Remark

There is the following analogue of proposition 3.21 1) for $\mathfrak{S}_n$, which we shall state without proof:

The real *symplectic* group $\mathrm{Sp}(2n, \mathbb{R})$ is the subgroup of $\mathrm{Gl}(2n, \mathbb{R})$ consisting of matrices g satisfying

$$g^t J g = J$$

where J is the $2n \times 2n-$matrix

$$J = \begin{pmatrix} \bigcirc & I \\ -I & \bigcirc \end{pmatrix}.$$

$\mathrm{Sp}(2n, \mathbb{R})$ acts as a group of isometries on $\mathfrak{S}_n$ by

$$g(Z) = (AZ + B)(CZ + D)^{-1}\,, \quad Z \in \mathfrak{S}_n \text{ where } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Notice that the isometries considered in exercise 8.11 ii) and iii) are of this form.

A Riemannian globally symmetric space is obviously locally symmetric by proposition 8.3. For the other direction we shall prove the following:

Theorem 8.13

Let M be a connected and complete Riemannian locally symmetric space. If furthermore M is simply connected then M is globally symmetric.

Before proving this theorem let us recall a few facts from the theory of *covering spaces* (see e.g. Greenberg-Harper [1]).

Let X be a connected topological space (we do *not* assume X to be a Hausdorff space!). A mapping $\pi : \tilde{X} \to X$ is called a *covering* of X if each point in X has an open neighbourhood U such that $\pi^{-1}U$ is a union of open sets of $\tilde{X}$ each mapping homeomorphically onto U under π. Also recall that for a path connected space X, the *fundamental group* $\pi_1(X) = \pi_1(X,\ x_0)$, $x_0 \in X$ a base point, is the set of *homotopy classes* of loops $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$. X is called *simply connected* or 1-*connected* if $\pi_1(X) = 1$.

The basic theorem about covering spaces is, that for any connected, locally connected and "semi-locally 1–connected" space X, there is a *universal* covering space $\pi : \tilde{X} \to X$, that is, a covering with $\tilde{X}$ simply connected. Furthermore $\tilde{X}$ is unique in the following sense: Let $x_0 \in X$ be a base point and for each covering $\pi : \tilde{X} \to X$ let $\tilde{x}_0$ be a point over x_0. Given a different universal covering $\rho : \tilde{Y} \to X$ with $\tilde{y}_0 \in \tilde{Y}$ lying over x_0 there is a unique homeomorphism $h : \tilde{X} \to \tilde{Y}$ such that $h(\tilde{x}_0) = \tilde{y}_0$ and such that the diagram

$$\begin{tikzcd} \tilde{X} \arrow[rr, "h"] \arrow[dr, "\pi"'] & & \tilde{Y} \arrow[dl, "\rho"] \\ & X & \end{tikzcd}$$

commutes. In particular there is a 1–1 correspondence between the points of $\pi^{-1}(x_0)$ and the group of homeomorphisms $h : \tilde{X} \to \tilde{X}$ covering the identity. This group is called the group of *deck-transformations* of π and is, in a natural way, isomorphic to $\pi_1(X, x_0)$. $\pi : \tilde{X} \to X$ is clearly an open mapping, so we can identify X with the quotient space $\pi_1(X)\backslash\tilde{X}$. On the other hand suppose Γ is a group of homeomorphisms of some simply connected (and locally connected etc. ...) space $\tilde{X}$ such that Γ acts *properly discontinuously* on $\tilde{X}$, that is, for each $x \in \tilde{X}$ there is a neighbourhood U of x such that $U \cap h(U) = \emptyset$ for all $h \neq \mathrm{id}$, $h \in \Gamma$. Then the quotient space $\Gamma\backslash\tilde{X}$ has fundamental group isomorphic to Γ and the projection $\tilde{X} \to \Gamma\backslash\tilde{X}$ is the universal covering.

Now suppose M is a connected Riemannian manifold and let $\pi : N \to M$ be a covering. Then clearly N can be given a C^∞ structure such that π is a local diffeomorphism, and the Riemannian metric pulls back to a Riemannian metric on N.

Proposition 8.14

a. *A curve $\gamma : [a,b] \to N$ is a geodesic iff $\pi \circ \gamma$ is a geodesic in M,*

b. *N is complete iff M is complete,*

c. *N is locally symmetric iff M is locally symmetric.*

d. *If $\pi : N \to M$ is the universal covering then the deck transformations act as isometrics, and the action is properly discontinuous.*

Exercise 8.15

Prove proposition 8.14.

Remark

In particular the universal covering N of a complete Riemannian locally symmetric space is also a complete Riemannian locally symmetric space, hence by theorem 8.13 a Riemannian globally symmetric space.

On the other hand, suppose we have given a group Γ of isometries of a connected Riemannian manifold N. The action is called *free* if $\gamma \in \Gamma,\ \gamma \neq$id, implies that γ has no fixed point, and it is called *discontinuous* if no orbit has an accumulation point.

Proposition 8.16

The action of Γ on N is properly discontinuous iff it is free and discontinuous.

Proof

If Γ acts properly discontinuously on N then clearly the action is free. Suppose it is not discontinuous. Then $\exists x_0 \in N$ and a sequence $\{\gamma_n\}$ of *different* elements of Γ such that $\gamma_n x_0 \to y_0$. That is, for any open neighbourhood U of y_0 we can find 2 different (actually infinitely many) $p, q \in \mathbb{N}$ such that $\gamma_p x_0$, $\gamma_q x_0 \in U$. Hence $g = \gamma_p \gamma_q^{-1} \neq \mathrm{id}$ and $g(\gamma_q x_0) = \gamma_p x_0 \in U$ so that $gU \cap U \neq \emptyset$, which contradicts the proper discontinuity.

Conversely suppose the action is free and discontinuous and let $x \in N$ be arbitrary. Then

$$r = \inf\{d(\gamma x, x) | \gamma \in \Gamma\ ,\ \gamma \neq \mathrm{id}\} > 0$$

and we consider a normal neighourhood U of x of radius $< r/2$. Suppose there is a $\gamma \in \Gamma$ such that $U \cap \gamma U \neq \emptyset$, that is, there is a $y \in U$ such that $\gamma y \in U$. Then

$$d(x, \gamma x) \leq d(x, \gamma y) + d(\gamma y, \gamma x) = d(x, \gamma y) + d(y, x) < r,$$

hence $\gamma = \mathrm{id}$ by the definition of r, which shows that the action is properly discontinuous.

It is now straightforward to prove the following:

Proposition 8.17

a. *Suppose N is a simply connected Riemannian manifold and Γ a group of isometries of N, such that the action is free and discontinuous. Then $M = \Gamma\backslash N$ is in a natural way a Riemannian manifold such that the projection $\pi : N \to M$ is the universal covering with Γ as the group of deck transformations.*

b. *If N is the universal covering of a Riemannian manifold M_0 and Γ is the group of deck transformations then M_0 is isometric to $\Gamma\backslash N$.*

Exercise 8.18

Prove proposition 8.17. (*Warning:* Remember to show that $M = \Gamma\backslash N$ is a Hausdorff space.).

Once we have proved theorem 8.13 we conclude from propositions 8.14 and 8.17:

Corollary 8.19

Every connected and complete Riemannian locally symmetric space M is isometric to a quotient $M = \Gamma\backslash N$ of a simply connected Riemannian globally symmetric space N by a group of isometries Γ acting properly discontinuously on N.

Remark

Thus if one wants to classify all complete locally symmetric spaces, this should be done in two steps:

I. Find all simply connected Riemannian globally symmetric spaces.

II. Given a globally symmetric space find all isometry groups acting properly discontinuously.

The first problem was completely solved by Élie Cartan. For details see e.g. Helgason [2] or Wolf [8]. The second problem is quite hard and in general unsolved (see e.g. Wolf, op.cit.). For spaces of constant curvature this problem is called the "Clifford-Klein space form" problem.

We now turn to the proof of theorem 8.13, which follows from the more general

Theorem 8.20

Let M and N be connected and simply connected complete Riemannian locally symmetric spaces. Let $p \in M$, $p' \in N$ and $\phi : T_pM \to T_{p'}N$ a linear isometry. Then there is an isometry $\Phi : M \to N$ with $\Phi_{*p} = \phi$ iff

$$R'(\phi v, \phi w)\phi z = \phi(R(v,w)z) \qquad \text{for all } v, w, z \in T_pM,$$

where R and R' are the curvature tensor fields.

Furthermore Φ is uniquely determined and is given by

$$\Phi\big(\exp_p(v)\big) = \exp_{p'}(\phi(v)) \qquad \forall\, v \in T_pM.$$

Remark 1

Clearly theorem 8.13 follows from theorem 8.20 by taking $M = N$, $p = p'$ and $\phi = \ -\mathrm{id}$.

Remark 2

Let $r > 0$ and suppose that p and p' has normal neighbourhoods U_p and $U_{p'}$ of radius r (i.e. the exponential mappings are diffeomorphisms onto U_p and $U_p{}'$). Then theorem 8.20 is true for $M = U_p$ and $N = U_p{}'$ by corollary 4.18.

The second remark is essential for the proof of theorem 8.20, which will be divided into a few lemmas. First a trivial one:

Lemma 8.21

Let M and N be connected Riemannian manifolds and $\Phi, \Psi : M \to N$ two isometries. Suppose that at some point $p \in M$

(*) $$\Phi(p) = \Psi(p) \text{ and } \Phi_{*p} = \Psi_{*p}.$$

Then $\Phi = \Psi$.

Proof

It suffices to show that the set $A \subseteq M$ of points p, for which (*) holds, is both open and closed. By continuity it is clearly closed, and if (*) holds for p then by proposition 4.3, Φ and Ψ agree in a normal neighbourhood around p. Hence the set A is also open which proves the lemma.

Now let M and N be connected Riemannian manifolds and let Φ be an isometry of an open neighbourhood U of a point $p \in M$ onto an open set in N. Let $p' = \Phi(p)$. We now want to extend Φ to all of M by extending Φ along curves. So let $\gamma : [0,1] \to M$ be a piecewise smooth curve with $\gamma(0) = p$. We say that Φ is *extendable* along γ if there is a family $\Phi_t(t \in [0,1])$ of isometries of open neighbourhoods U_t of $\gamma(t)$ onto open sets of N, such that if $|t - t'|$ is sufficiently small then $U_t \cap U_{t'} \neq \emptyset$ and

$$\Phi_t\big|U_t \cap U_t' = \Phi_{t'}\big|U_t \cap U_{t'} \ ,$$

and such that $U_0 = U$, $\Phi_0 = \Phi$. $(\Phi_t \ , \ U_t)$ is called a *continuation* of Φ *along* γ. If (Ψ_t, V_t) is another continuation of Φ along γ then, by the argument of lemma 8.20 above the set of $t \in [0,1]$ for which

$$\Phi_t(\gamma(t)) = \Psi_t(\gamma(t)) \text{ and } (\Phi_t)_{*\gamma(t)} = (\Psi_t)_{*\gamma(t)}$$

is all of $[0, 1]$. Hence for all $t \in [0, 1]$, Ψ_t and Φ_t agree on the connected component of $U_t \cap V_t$ containing $\gamma(t)$. So if continuations exist they are in some sense unique.

Now let M and N be complete locally symmetric spaces and let Φ be an isometry of a normal neighbourhood U_p of a point $p \in M$ onto a normal neighbourhood ${U_p}'$, $p' = \Phi(p)$.

Lemma 8.22

Let $\gamma : [0, 1] \to M$, $\gamma(0) = p$, be a piecewise smooth curve. Then Φ is extendable along γ. Furthermore there is a real number $r > 0$ such that for all t we can choose U_t to be the normal neighbourhood around $\gamma(t)$ of radius r.

Proof

Let $l =$ length (γ). The sets $\{q|\ d(q, p) \leq 2l\} \subseteq M$ (resp. $\{q'|\ d(q', p') \leq 2l\} \subseteq N)$ are compact. Hence by theorem 3.8 these sets can be covered by a finite number of open sets W (resp. W') such that for som $\rho > 0$ the normal neighbourhoods U_q of radius 2ρ exist and contain W for all $q \in W$ (similarly for W'). Let $r > 0$ be smaller than any of these ρ. We shall now extend Φ along γ such that $U_t, t \in [0, 1]$ is the normal neighbourhood of radius r and center $\gamma(t)$.

Let s^* be the supremum of the $s \in [0, 1]$ such that Φ is extendable along $\gamma \mid [0, s]$. Put $p_t = \gamma(t)$, $t \in [0, 1]$ and $q_t = \Phi_t(p_t)$, $t < s^*$. By completeness there is a point $q^* \in N$ such that

$$\Phi_t(p_t) \to q^* \ , \quad \text{for } t \to s^*.$$

Then clearly $d(q^*, p') < 2l$ so we can choose neighbourhoods W and W' as above around p_{s^*} and q^* respectively. Now choose $s_0 < s^*$ such that

$$\left.\begin{array}{cc} p_t \in W \ , & q_t \in W' \\ d(p_t \ , \ p_{s^*}) < r \ , & d(q_t \ , \ q^*) < r \end{array}\right\} \text{for all } t \ , \ s_0 \leq t < s^*.$$

By remark 2 following theorem 8.20, Φ_{s_0} has an extension to all of the normal neighbourhood V of radius $2r$ around p_{s_0}. But if U_t is the normal neighbourhood around p_t of radius r, then clearly $U_t \subseteq V$ for $s_0 \leq t \leq s^*$ and the extension of Φ_{s_0} agrees with Φ_t on U_t , $s_0 \leq t < s^*$. Hence Φ is extendable along $\gamma|[0, s^*]$ and if $s^* < 1$ Φ is extendable along γ beyond s^* by the same argument. This shows that $s^* = 1$ and ends the proof of the lemma.

Now let $\gamma, \delta : [0, 1] \to M$ be two piecewise smooth curves such that

$$\gamma(0) = \delta(0) = p \text{ and } \gamma(1) = \delta(1) = q$$

and let Φ_t and Ψ_t be the continuations of Φ along γ and δ respectively.

Lemma 8.23

If γ and δ are homotopic then Φ_1 and Ψ_1 agree in a neighbourhood of q.

Proof

That γ and δ are homotopic means that there is a family $\alpha^s (s \in [0,1])$ of piecewise smooth curves with $\alpha^s(0) = p$, $\alpha^s(1) = q$, $\alpha^0 = \gamma$, $\alpha^1 = \delta$, such that the map $(s,t) \to \alpha^s(t)$ is continuous. Let Φ_t^s be a continuation of ϕ along α^s so that

$$\Phi_t = \Phi_t^0 \ , \ \Psi_t = \Phi_t^1.$$

We shall show that the set of $s \in [0,1]$ such that

(*) Φ_1^s and Φ_1 agree in a neighbourhood of q,

is open and closed. By the argument of lemma 8.21, this is clearly closed. So now suppose (*) holds for $s = \sigma$ and we shall find $\epsilon > 0$ such that (*) holds for $|s - \sigma| < \epsilon$. By lemma 8.22 we can find $r > 0$ such that we can suppose that Φ_t^σ is defined on the normal neighbourhood V_t of $\alpha^\sigma(t)$ of radius $2r$. By uniform continuity we can find $\epsilon > 0$ such that

$$d(\alpha^\sigma(t) \ , \ \alpha^s(t)) < r \qquad \text{for } 0 \leq t \leq 1 \text{ and } \quad |\sigma - s| < \epsilon.$$

For such s the normal neighbourhoods U_t^s of $\alpha^s(t)$ of radius r is contained in V_t so $(\Phi_t^\sigma \ , \ V_t^s)$ is a continuation of Φ along α^s. By uniqueness of continuations Φ_t^σ and Φ_t^s agree in a neighbourhood of $\alpha^s(t)$. In particular Φ_1^s and Φ_1^s agree near q.

This proves the lemma.

Proof of theorem 8.20

As above choose normal neighbourhoods U and U' around p and p' and let $\Phi_0 : U \to U'$ be the isometry with $(\Phi_0)_{*p} = \phi$ according to the remark 2 following the statement of the theorem. For any point $q \in M$ choose a piecewise smooth curve $\gamma : [0,1] \to M$ with $\gamma(0) = p$, $\gamma(1) = q$, and let Φ_t be the continuation of Φ along γ. Define

$$\Phi(q) = \Phi_1(q).$$

Since M is simply connected this is well-defined by lemma 8.23. Clearly $\Phi|U = \Phi_0$ and in general if (Φ_t, U_t) is a continuation of Φ_0 along γ then $\Phi|U_1 = \Phi_1$. It follows that Φ is locally an isometry.

Similarly using $\phi^{-1} : T_{p'}N \to T_pM$ we get a map $\Psi : N \to M$ extending Φ_0^{-1} and also Ψ is a local isometry. Now clearly $\Psi \circ \Phi$ is defined by extending the identity of U along curves in M, so by uniqueness $\Psi \circ \Phi = \mathrm{id}$. Similarly $\Phi \circ \Psi = \mathrm{id}$, which ends the proof of the theorem 8.20 and hence also theorem 8.13. Notice that we have implicitly used the following:

Exercise 8.24

Let $\gamma, \delta : [0,1] \to M$ be piecewise smooth curves in a Riemannian manifold and suppose $\gamma(0) = \delta(0)$ and $\gamma(1) = \delta(1)$. Show that if γ and δ are homotopic through continuous curves then they are homotopic through piecewise smooth curves.

Corollary 8.25

Let $M^n, n > 1,$ be a complete connected Riemannian manifold of constant curvature K. Then the universal covering is isometric to

a. *The sphere* $S^n_{1/\sqrt{K}}$ *of radius* $1/\sqrt{K}$ *if* $K > 0$.

b. *Euclidean space* $\mathbb{R}^n$ *if* $K = 0$.

c. *The hyperbolic space* $H^n_{1/\sqrt{-K}+}$ *if* $K < 0$.

Proof

Let M_K denote the above mentioned standard model for a complete manifold of constant curvature K. Notice that M_K is simply connected. Also by proposition 8.14 we can assume M simply connected. Now choose any points $p \in M$, $p' \in M_K$ and normal neighbourhoods $U_p, U_{p'}$ of the same radius. By corollary 5.36 there is an isometry $\Phi : U_p \to U_{p'}$ and in particular the condition of theorem 8.20 is fulfilled for $\phi = \Phi_{*p}$. Hence Φ extends to an isometry of M to M_K.

Exercise 8.26

a. Show that lemma 8.22 remains valid even if M is not complete.

b. Now let M be a locally symmetric space and N a simply connected globally symmetric space and let $\pi : \tilde{M} \to M$ be the universal covering. Choose basepoints $p \in M$, $\tilde{p} \in \tilde{M}$ with $\pi\tilde{p} = p$ and $p' \in N$ and suppose a linear isometry $\phi : T_pM \to T_{p'}N$ satisfies $R(\phi v, \phi w)\phi z = \phi R(v,w)z$, for all $v, w, z \in T_pM$. Show that there is a unique local isometry $\Phi : \tilde{M} \to N$ with $\Phi(\tilde{p}) = p'$, $\Phi_{*\tilde{p}} = \phi \circ \pi_{*\tilde{p}}$.

c. Show that Φ in b) is an isometry iff M is complete.

(*Note:* Φ in b) is called the *developing map* for M in N).

Exercise 8.27

Let M be a complete locally symmetric space.

a. Show that if M is simply connected then the group of isometries acts transitively on M.

b. Show that if for some point $p \in M$ every linear isometry $\phi : T_pM \to T_pM$ satisfies $R(\phi v, \phi w)\phi z = \phi R(v, w)z$ for all $v, w, z \in T_pM,$ then M has constant curvature

Exercise 8.28

For K a real number and $n \geq 2$ let $M_K = M_K^n$ be the standard model for the $n-$dimensional simply connected manifold of constant curvature K (see corollary 8.25). Let G denote the group of isometries given in the 3 cases by:

a. $G = \mathrm{O}(n + 1).$

b. $G = \mathrm{E}(n),$ the group of Euclidean motions of $\mathbb{R}^n,$ that is, for $v \in \mathbb{R}^n$ and $A \in \mathrm{O}(n)$, $g \in \mathrm{E}(n)$ is given by $g(x) = v + Ax$, $x \in \mathbb{R}^n.$

c. $G = \mathrm{O}(1, n)^+$ (see proposition 3.24).

1. Show that in all 3 cases G is the full group of isometries of $M_K.$

2. Show that G has a natural topology as a subset of some Euclidean space such that
   - $\alpha)$ G is a *topological group,* i.e., the map $G \times G \to G$ given by $(g, h) \to gh^{-1}$ is continuous
   - $\beta)$ The action of G on M_K is continuous, i.e., the map $G \times M \to M$ given by $(g, x) \to gx$ is continuous.

3. Show that if $\Gamma \subseteq G$ is a subgroup acting without fixed points on M_K then the action is discontinuous iff Γ is a *discrete* subset of G.

4. Conclude that the complete Riemannian manifolds of constant curvature K are, up to isometry, all manifolds M of the form $M = \Gamma\backslash M_K$, where $\Gamma \subseteq G$ is a *discrete* subgroup acting without fixed points on $M_K.$ In particular for $K > 0$, $\Gamma \subseteq \mathrm{O(n+1)}$ is a *finite* group acting without fixed points on $S^n_{1/\sqrt{K}}.$

Chapter 9 LIE GROUPS AND LIE ALGEBRAS

For the further study of symmetric spaces we need the basic properties of Lie groups and Lie algebras.

Definition 9.1

A C^∞ manifold G with a multiplication $G \times G \to G$ is called a *Lie group* if

a. G is a group,

b. The map $G \times G \to G$ given by $(g, h) \to gh^{-1}$ is a C^∞ map.

Examples 9.2

a. $(\mathbb{R}^n, +)$ is a Lie group.

b. The quotient group $\mathbb{R}/\mathbb{Z}$ is a Lie group.

c. Let $M(n, \mathbb{R})$ be the set of $n \times n$ real matrices and let $\mathrm{Gl}(n, \mathbb{R}) \subseteq M(n, \mathbb{R})$ be the open subset of non-singular matrices with the usual matrix multiplication. Then $\mathrm{Gl}(n, \mathbb{R})$, the *general linear group* of *order* n over $\mathbb{R}$, is a Lie group.

c′) Similarly the general linear groups $\mathrm{Gl}(n, \mathbb{C})$ over $\mathbb{C}$ and $\mathrm{Gl}(n, \mathbb{H})$ over $\mathbb{H}$ are Lie groups ($\mathbb{C}$ and $\mathbb{H}$ being the complex numbers and the quarternions respectively).

d) $\mathrm{O}(n) \subseteq \mathrm{Gl}(n, \mathbb{R})$, the subgroup of orthogonal matrices, i.e., matrices A satisfying $A^t A = AA^t = I$, makes a Lie group, the *orthogonal group.*

d′) $\mathrm{U}(n) \subseteq \mathrm{Gl}(n, \mathbb{C})$, the subgroup of unitary matrices, i.e. matrices A satisfying $A^* = \overline{A}^t = A^{-1}$, makes a Lie group, the *unitary group.*

e) $\mathrm{Sl}(n, \mathbb{R}) \subseteq \mathrm{Gl}(n, \mathbb{R})$ or $\mathrm{Sl}(n, \mathbb{C}) \subseteq \mathrm{Gl}(n, \mathbb{C})$, the *special linear* groups of matrices A with det $A = 1$, are Lie groups.

e′) $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{Sl}(n, \mathbb{R})$, $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{Sl}(n, \mathbb{C})$ are Lie groups.

Exercise 9.3

Show that examples a) - e′) are Lie groups.

Definition 9.4

Two Lie groups G and G' are called *isomorphic* if there is an *isomorphism* $\phi : G \to G'$, i.e., a group isomorphism, which is also a diffeomorphism.

Exercise 9.5

Show that there are natural isomorphisms of Lie groups: $\mathbb{R}/\mathbb{Z} \simeq \mathrm{U}(1) \simeq \mathrm{SO}(2)$.

Exercise 9.6

Show that if G is a Lie group and if a subgroup $\Gamma \subseteq G$ satisfies

1. Γ lies in the center of G, that is, $\gamma g = g\gamma \; \forall \, g \in G \; , \; \forall \, \gamma \in \Gamma$,
2. Γ is discrete, that is, every $\gamma \in \Gamma$ has a neighbourhood not meeting $\Gamma - \{\gamma\}$;

Then the quotient group G/Γ is in a natural way a Lie group.

Exercise 9.7

a. Show that if G_1 and G_2 are Lie groups then also $G_1 \times G_2$ is a Lie group.

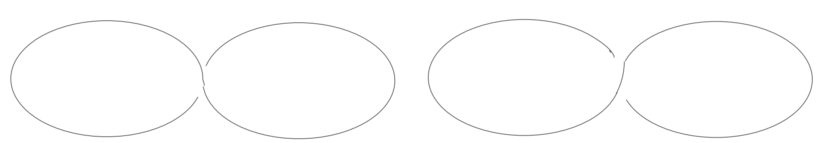
b. Show that $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \simeq \mathbb{R}^2/\mathbb{Z}^2$.

Definition 9.8

Let G be a Lie group. A subgroup $H \subseteq G$ is called a *Lie subgroup* if H has a smooth structure, such that i) H is a Lie group and ii) The inclusion $i : H \hookrightarrow G$ is an *immersion* of manifolds.

Warning

If a subset $N \subseteq M$ of an $n-$dimensional manifold has a smooth structure of dimension $k \leq n$, such that the inclusion $i : N \hookrightarrow M$ is an immersion, then N is not necessarily a submanifold, but what is called an *immersed submanifold*, i.e., every point in N has a coordinate neighbourhood $(U, \mathbf{x})$ in M and a neighbourhood V in N such that i) $x_{k+1}(p) = \ldots = x_n(p) = 0$ for $p \in V$, and ii) $\mathbf{x} : V \to \mathbb{R}^k \times 0 \subseteq \mathbb{R}^n$ is a coordinate chart on N. We do *not* know that V can be chosen as $U \cap N$. Furthermore a subset $N \subseteq M$ may have different differentiable structures such that the inclusions are submanifolds as the following drawing shows:



Example 9.9

Let $G = \mathbb{R}^2/\mathbb{Z}^2$ and $H \subseteq G$ the image of a line in $\mathbb{R}^2$ with *irrational* slope. Then $H \simeq \mathbb{R}$ is a Lie subgroup, but H is *not* a submanifold in G. In fact H is dense in G.

Proposition 9.10

Let G be a Lie group. Let G_0 denote the connected component containing the identity e. Then G_0 is an open submanifold and is a Lie subgroup.

Proof

G_0 is closed since components are always closed. G_0 is open because every point of $g \in G_0$ has an open neighbourhood diffeomorphic with an open ball in a Euclidean space, i.e., a connected open neighbourhood in G. Hence G_0 is an open submanifold. G_0 is also a subgroup: In fact, fix $a \in G_0$; then $a^{-1}G_0$ is connected since $x \mapsto a^{-1}x$ is a diffeomorphism. Since $a^{-1}a = e \in G_0 \cap a^{-1}G_0$ we obtain $a^{-1}G_0 = G_0$.

Exercise 9.11

a. Let G be a Lie group and G_0 the identity component. Show that G_0 is normal in G.

b. For $G = \mathrm{O}(n)$ show that $G_0 = \mathrm{SO}(n)$ and $G/G_0 \simeq \mathbb{Z}/2$.

c. For $G = \mathrm{Gl}(n, \mathbb{R})$ find G_0 and G/G_0.

d. For $G = \mathrm{O}(1, n)$ find G_0 and G/G_0.

Notation

For $a \in G$ we shall use

$$L_a : G \to G\ ,\ L_a(x) = ax$$
$$R_a : G \to G\ ,\ R_a(x) = xa.$$

Notice that L_a and R_a are diffeomorphisms of G.

Definition 9.12

A vector field X on G is called *left invariant* (respectively *right invariant*) if $L_{a*}X_b = X_{ab}$ (respectively $R_{a*}X_b = X_{ba}$) for all $a, b \in G$.

Proposition 9.13

There is a natural 1–1 *correspondence between T_eG and the set of left invariant smooth vector fields on G given by: X a left invariant vector field corresponds to $X_e \in T_eG$.*

Proof

Since a left invariant vector field X satisfies $X_a = X_{ae} = L_{a*}(X_e)$, it is clearly determined by X_e. It suffices to show that every vector $X_e \in T_eG$ extends to a smooth vector field X on G by

$$X_a = L_{a*}(X_e).$$

For this it is enough to show that X thus defined is smooth in a neighbourhood U of e, since for $b \in G$, bU is a neighbourhood of b and

$$X_{ba} = L_{b*}(X_a)\ , \qquad a \in U.$$

Choose a coordinate system $(U, \mathbf{x})$ around e. Then it is enough to consider $X_e = \frac{\partial}{\partial x^j}\Big|_e$, $j = 1, \ldots, n$. Now also let $V \subseteq U$ such that $a, b \in V$ implies $ab \in U$. We shall show that $X(x^i)$ is smooth in V for any $i = 1, \ldots, n$:

$$X\big(x^i\big)(a) = (L_{a*}X_e)\big(x^i\big) = X_e\big(x^i \circ L_a\big) = \frac{\partial}{\partial x^j}\big(x^i \circ L_a\big).$$

Here $x^i \circ L_a(b) = x^i(a \cdot b)$ and since

$$V \times V \to U$$

given by $(a, b) \mapsto a \cdot b$ is smooth and since x^i is smooth also $\frac{\partial}{\partial x_2^j}\big(x^i(a \cdot b)\big)$ is smooth where $x_2^1 \ldots x_2^n$ are the coordinates $x^1 \ldots x^n$ used on the second copy of V in $V \times V$. This ends the proof.

Proposition 9.14

Let X_1 and X_2 be left invariant vector fields on G. Then also the Lie bracket $[X_1, X_2]$ is left invariant.

This follows from the following more general situation: Let $\Phi : M \to N$ be a smooth map between two manifolds. Two vector fields X on M and Y on N are called $\Phi-$*related* if $X(f \circ \Phi) = Y(f) \circ \Phi$ for all $f \in C^\infty(N)$.

Lemma 9.15

Let $\Phi : M \to N$ as above and let X_1, X_2 be vector fields on M, Y_1, Y_2 vector fields on N. Suppose X_i and Y_i are $\Phi-$related, $i = 1, 2$. Then also $[X_1, X_2]$ and $[Y_1, Y_2]$ are $\Phi-$related.

Proof

$\beta\alpha$

$$\begin{aligned}
[X_1, X_2](f \circ \Phi) &= X_1(X_2(f \circ \Phi)) - X_2(X_1(f \circ \Phi)) \\
&= X_1(Y_2(f) \circ \Phi) - X_2(Y_1(f) \circ \Phi) \\
&= Y_1(Y_2(f)) \circ \Phi - Y_2(Y_1(f)) \circ \Phi \\
&= [Y_1, Y_2](f) \circ \Phi.
\end{aligned}$$

Proof of proposition 9.14.

Take $G = M = N$ and $\Phi = L_a$. A vector field X on G is left invariant iff $\forall\, a \in G$,

$$X(f \circ L_a) = X(f) \circ L_a \qquad \forall\, f \in C^\infty(G),$$

i.e. iff X is L_a-related to itself. Therefore if X_1 and X_2 are left invariant vector fields, X_i is L_a-related to itself, $i = 1, 2$, and so by the lemma also $[X_1, X_2]$ is L_a-related to itself, i.e. left invariant.

Definition 9.16

The *Lie algebra* of G is the vector space $\mathcal{L}(G) = T_eG$ with the Lie product

$$\mathcal{L}(G) \times \mathcal{L}(G) \to \mathcal{L}(G)$$

given as follows: For $X, Y \in \mathcal{L}(G)$ let $\tilde{X}$ and $\tilde{Y}$ be the corresponding left invariant vector fields. Then $[X, Y]$ is the unique vector in T_eG such that the corresponding left invariant vector field is $\left[\tilde{X}, \tilde{Y}\right]$. That is, $[X, Y]^\sim = \left[\tilde{X}, \tilde{Y}\right]$.

Definition 9.17

A *Lie algebra* (over $\mathbb{R}$) is a finite dimensional real vector space $\mathfrak{g}$ with a *bilinear* operation

$$[-\,,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying

1. $[X, X] = 0 \qquad\qquad \forall\, X \in \mathfrak{g}$,

2. (Jacobi identity)
   $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \qquad \forall\, X, Y, Z \in \mathfrak{g}$.

Proposition 9.18

For G a Lie group, $\mathcal{L}$(G) is a Lie algebra as in definition 9.17

Proof

This is immediate from proposition 1.20.

Notation

We shall often use the notation $\mathfrak{g} = \mathcal{L}(G)$.

Let us now determine $\mathcal{L}(G)$ in the examples above. First notice that if $U \subseteq \mathbb{R}^n$ is an open set and if we identify a vector field X on U with a function $X : U \to \mathbb{R}^n$ (cf. remark following definition 1.15), then given two smooth vector fields X and Y the Lie bracket is given by

$$[X, Y] = D_X(Y) - D_Y(X) \tag{9.19}$$

where D_{X_p} denotes the directional derivative, that is, the differential evaluated at X_p. (9.19) follows by direct calculation and is just the fact that the torsion T in Euclidean space is zero (exercise 2.22).

Examples 9.20

a. Consider $G = (\mathbb{R}^n, +)$ and let us calculate the Lie Bracket in the Lie algebra $T_0(G) = \mathbb{R}^n$: For $x \in \mathbb{R}^n$, $L_x(y) = x + y$ so $L_{x*} = \mathrm{id}$. Hence a left invariant vector field is just a constant function $X : \mathbb{R}^n \to \mathbb{R}^n$. Hence by (9.19)

$$[X, Y] = D_X(Y) - D_Y(X) = 0.$$

That is, $\mathcal{L}(G) = \mathbb{R}^n$ with $[X, Y] = 0 \qquad \forall\ X, Y$. This Lie algebra is called *commutative.*

b. Consider $G = \mathrm{Gl}(n, \mathbb{R}) \subseteq M(n, \mathbb{R})$. G is an open set in $M(n, \mathbb{R}) \simeq \mathbb{R}^{n^2}$, so we shall identify the tangent space of G at every point with $M(n, \mathbb{R})$. In particular $\mathcal{L}(G) = T_eG = M(n, \mathbb{R})$ and $X \in M(n, \mathbb{R})$ corresponds to the left invariant vector field

$$\tilde{X}_P = PX.$$

(Note that $L_P(X) = PX$ so $L_{P*}X = PX$.) Given $X, Y \in M(n, \mathbb{R})$ we must compute by (9.19)

$$\left[\tilde{X}, \tilde{Y}\right] = D_{\tilde{X}}\left(\tilde{Y}\right) - D_{\tilde{Y}}\left(\tilde{X}\right).$$

Here the differential of $\tilde{Y}$ is given by

$$d\tilde{Y} = d(PY) = (dP) \cdot Y \qquad \text{(for } Y \text{ fixed)}$$

so

$$D_{\tilde{X}_P}\left(\tilde{Y}\right) = \tilde{X}_P \cdot Y = P \cdot X \cdot Y$$

Similarly

$$D_{\tilde{Y}_P}\left(\tilde{X}\right) = P \cdot Y \cdot X.$$

Hence

$$\begin{aligned} \left[\tilde{X}, \tilde{Y}\right]_P &= P \cdot X \cdot Y - P \cdot Y \cdot X = P(XY - YX) \\ &= (XY - YX)^{\sim}_P. \end{aligned}$$

It follows that the Lie bracket is just given by

(9.21) $$[X, Y] = XY - YX$$

using the usual matrix multiplication.

Notation
$\mathcal{L}(\mathrm{Gl}(n,\mathbb{R})) = \mathfrak{gl}_n(\mathbb{R}) = M(n,\mathbb{R})$ with the Lie product (9.21).

Next let us notice the functorial properties of $\mathcal{L}$:

Definition 9.22
A homomorphism of Lie groups is a smooth map $\phi : H \to G$ of the Lie groups H and G, such that ϕ is a group homomorphism.

Proposition 9.23

a. *A homomorphism of Lie groups $\phi : H \to G$ induces a* ***Lie algebra homomorphism*** *$\phi_* : \mathcal{L}(H) \to \mathcal{L}(G)$ given by the differential of ϕ at e. That is,*

$$\phi_*[X_1 X_2] = [\phi_* X_1\ ,\ \phi_* X_2] \qquad \forall\, X_1, X_2 \in \mathcal{L}(H).$$

b. *In particular if $H \subseteq G$ is a* ***Lie subgroup*** *then $\mathcal{L}(H) \subseteq \mathcal{L}(G)$ is a Lie subalgebra, i.e., if $X_1, X_2 \in \mathcal{L}(H)$ then also $[X_1, X_2] \in \mathcal{L}(H)$.*

Proof
$X \in \mathcal{L}(H)$ and $Y = \phi_* X \in \mathcal{L}(G)$ and let $\tilde{X},$ respectively $\tilde{Y}$, be the corresponding left invariant vector fields on H, respectively G. Then $\tilde{X}$ and $\tilde{Y}$ are $\phi-$related, in fact for $f \in C^\infty(G)$ and $a \in H$

$$\begin{aligned}\tilde{X}_a(f\circ\phi) &= (L_{a*}X)(f\circ\phi) = \phi_*(L_{a*}X)(f) \\ &= \big(L_{\phi(a)*}\phi_* X\big)(f) = \tilde{Y}_{\phi(a)}(f)\end{aligned}$$

where we have used that $\phi_* \circ L_{a*} = L_{\phi(a)} \circ \phi_*$ since $\phi \circ L_a = L_{\phi(a)} \circ \phi$.

Now let $X_1, X_2 \in \mathcal{L}(H)$ and $Y_1 = \phi_* X_1,\ \ Y_2 = \phi_* X_2,$ and let $\tilde{X}_i, \tilde{Y}_i$ be the corresponding left invariant vector fields. Then since $\tilde{X}_i$ and $\tilde{Y}_i$ are $\phi-$related it follows from lemma 9.15 that $\left[\tilde{X}_1, \tilde{X}_2\right]$ and $\left[\tilde{Y}_1, \tilde{Y}_2\right]$ are $\phi-$related. In particular for $f \in C^\infty(G)$

$$\begin{aligned}\phi_*[X_1, X_2](f) &= [X_1, X_2](f\circ\phi) = [Y_1, Y_2](f) \\ &= [\phi_* X_1, \phi_* X_2](f)\end{aligned}$$

so indeed

$$\phi_*[X_1, X_2] = [\phi_* X_1, \phi_* X_2].$$

b) is immediate from a).

The basic theorem about the interplay between Lie groups and Lie algebras is the following:

Theorem 9.24

Let G be a Lie group, $\mathfrak{g} = \mathcal{L}(G)$ and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a subalgebra. Then there is a unique connected Lie subgroup $H \subseteq G$ such that $\mathcal{L}(H) = \mathfrak{h}$.

For the proof we need the Frobenius theorem about integrable distributions:

In general let M be a manifold of dimension n. A *distribution* Δ of dimension k on M is a collection of $k-$dimensional subspaces $\Delta_p \subseteq T_pM$, one for each $p \in M$. Furthermore Δ is assumed to be *smooth* in the following sense: Locally we can find smooth vector fields $X_1, \ldots, X_k$ such that $X_1(p), \ldots, X_k(p)$ span Δ_p for each p in a neighbourhood.

The distribution is called *integrable* if whenever X and Y are smooth vector fields belonging to Δ (i.e. takes values in Δ) then also $[X, Y]$ belongs to Δ. The following is straightforward:

Lemma 9.25

a) Δ *is integrable iff* Δ *is locally integrable, i.e., iff every point has a neighbourhood* U *such that* $\Delta|U$ *is integrable.*

b) Δ *is locally integrable iff every point has a neighbourhood* U *with smooth vector fields* $X_1, \ldots, X_k$ *satisfying*

1. $X_1(p), \ldots, X_k(p)$ *span* Δ_p *for all* $p \in U$

2. *there are smooth function* $c_{ij}^{\alpha} \in C^{\infty}(U)$, $\alpha, i, j = 1, \ldots, k,$ *such that*

$$[X_i, X_j] = \sum_{\alpha=1}^{k} c_{ij}^{\alpha} X_{\alpha}.$$

The notion of integrability is relevant for finding *integral manifolds* for Δ. By this we mean a connected $k-$dimensional *immersed* submanifold $N \subseteq M$ (that is, N has a differentiable structure such that the inclusion $i : N \hookrightarrow M$ is an immersion cf. the remarks following 9.8 above) such that $T_pN = \Delta_p$ for all $p \in N$. We state the following theorem without proof (see e.g. Spivak [6 Vol. I chap. 6] or Warner [7 §§ 1.60 – 1.64]):

Theorem 9.26 (Frobenius)

Let Δ be a $k-$dimensional distribution in M and suppose Δ is integrable.

Then there is a ***unique foliation*** *$\mathcal{F}$ of M with all* ***leaves*** *as integral manifolds for Δ. That is, $M = \bigcup_{\alpha \in I} \mathcal{F}_{\alpha}$ a disjoint union of* ***leaves*** *$\mathcal{F}_{\alpha}$, each of which is a maximal integral manifold for Δ. Furthermore the leaves $\mathcal{F}_{\alpha}$ are unique. Also, there are coordinate charts $(U, \mathbf{x})$ for M such that*

$$\mathbf{x}(U) = (-\epsilon, \epsilon) \times \ldots \times (-\epsilon, \epsilon) \subseteq \mathbb{R}^n , \qquad \epsilon > 0,$$

and such that each $\mathcal{F}_\alpha \cap U$ is a union of sets of the form

$$U_a = \left\{ q \in U \mid x^{k+1}(q) = a^{k+1}, \ldots, x^n(q) = a^n \right\}$$

with $a^i, i = k+1, \ldots, n$, satisfying $\left|a^i\right| < \epsilon$.

Remark

Notice that we obtain coordinate charts for $\mathcal{F}_\alpha$ of the form $\left(U_a, \left(x^1, \ldots, x^k\right)\middle| U_a\right)$, where $(U, \mathbf{x})$ is a coordinate chart for M as above and $U_a \subseteq \mathcal{F}_\alpha \cap U$ one of the sets defined there. This determines the differential structure of $\mathcal{F}_\alpha$ uniquely.

Proof of theorem 9.24

Let Δ be the distribution in G given by $\Delta_a = L_{a*}\mathfrak{h} \subseteq T_aG$. Clearly if $X_1, \ldots, X_k$ span $\mathfrak{h}$ and $\tilde{X}_1, \ldots, \tilde{X}_k$ are the corresponding left invariant vector fields then $\tilde{X}_{1^a}, \ldots, \tilde{X}_{k^a}$ span $\Delta_{^a}$. Hence Δ is smooth and it is also integrable by lemma 9.25 since

$$[X_i, X_j] = \sum_{\alpha=1}^{k} c_{ij}^{\alpha}\, X_\alpha$$

for some constants c_{ij}^{α}, so that also

$$\left[\tilde{X}_i, \tilde{X}_j\right] = \sum_{\alpha=1}^{k} c_{ij}^{\alpha}\, \tilde{X}_\alpha.$$

Now let $\mathcal{F}$ be the corresponding foliation and let H be the leaf of $\mathcal{F}$ through e. For any $b \in G$ we clearly have

$$(L_b)_* \Delta_a = L_{b*} L_{a*} \mathfrak{h} = \Delta_{ba}$$

so by the uniqueness of the foliation it follows that L_b permutes the leaves of $\mathcal{F}$. In particular for $b \in H$, $L_{b^{-1}}H$ is a leaf containing $b^{-1}b = e$, hence $L_{b^{-1}}H = H$. It follows that H is a subgroup. It remains to show that the multiplication in H is C^∞.

For this first notice that $L_b : H \to H$, for $b \in H$, is smooth. Also choose a chart $(U, \mathbf{x})$ around e in G as in theorem 9.26 with $\mathbf{x}(e) = 0$ and let $V \subseteq U$ be chosen such that $a^{-1}b \in U$ for all $a, b \in V$. Put

$$U_0 = \left\{ q \in U \mid x^{k+1}(q) = \ldots = x^n(q) = 0 \right\}, \qquad V_0 = U_0 \cap V.$$

so that $V_0 \subset U_0$ are neighbourhoods of e in H. If we choose U and V such that $\mathbf{x}(U)$ and $\mathbf{x}(V)$ are $n-$balls in $\mathbb{R}^n$ then $\mathbf{x}(U_0)$ are $k-$ balls in $\mathbb{R}^k \times 0$ so in particular they are connected. Now given $a \in V_0$ we have $L_{a^{-1}}V_0 \subseteq U \cap H$ and $L_{a^{-1}}V_0$ contains $e = a^{-1}a$ so that $L_{a^{-1}}V_0 \subseteq U_0$. That is, if $a, b \in V_0$ then $a^{-1}b \in U_0$. Since $(U_0\ ,\ \mathbf{x}|U_0)$ is a chart for H it follows that the map $(a, b) \to a^{-1}b$ is smooth in $V_0 \times V_0$. Hence the theorem follows from the following:

Lemma 9.27

Let G be a group with a C^∞ structure such that left translation is C^∞ and G is connected. Suppose V is a neighbourhood of e such that the map $V \times V \to G$ given by $(a, b) \to a^{-1}b$ is smooth. Then G is a Lie group.

Exercise 9.28

a. Prove lemma 9.27.

b. Prove the uniqueness of H in theorem 9.24.

Theorem 9.29

Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Let $\Phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then there is a neighbourhood U of $e \in G$ and a C^∞ map $\phi : U \to H$ such that

1. $\phi(ab) = \phi(a)\phi(b) \qquad \forall\, a, b \in U$ *with* $ab \in U$,

2. $\phi_{*e} = \Phi.$

Moreover if $\phi, \psi : G \to H$ are Lie group homomorphisms with $\phi_{*e} = \psi_{*e} : \mathfrak{g} \to \mathfrak{h}$ then $\phi|_{G_0} = \psi|_{G_0}$ where G_0 is the connected component containing e.

Proof

First notice that the Lie group $G \times H$ has Lie algebra $\mathfrak{g} \oplus \mathfrak{h} = \mathfrak{g} \times \mathfrak{h}$ with Lie product

$$[X_1 \oplus X_2\ ,\ Y_1 \oplus Y_2] = [X_1\ ,\ Y_1] \oplus [X_2, Y_2].$$

Let $\mathfrak{k} \subseteq \mathfrak{g} \oplus \mathfrak{h}$ be the set

$$\mathfrak{k} = \left\{ X \oplus \Phi(X) \;\middle|\; X \in \mathfrak{g} \right\}$$

and notice that $\mathfrak{k}$ is a subalgebra: Indeed, if $X, Y \in \mathfrak{g}$ then

$$[X \oplus \Phi X\ ,\ Y \oplus \Phi Y] = [X, Y] \oplus [\Phi X\ ,\ \Phi Y] = [X, Y] \oplus \Phi[X, Y].$$

Hence by theorem 9.24 there is a unique connected Lie subgroup $K \subseteq G \times H$ with Lie algebra $\mathfrak{k}$. Let $\pi_1 : G \times H \to G$ be the projection. Then clearly

$$\omega = \pi_1|_K : K \to G$$

is a Lie group homomorphism and clearly $\omega_* : \mathfrak{k} \to \mathfrak{g}$ is given by

$$\omega_*(X \oplus \Phi X) = X \qquad \forall\, X \in \mathfrak{g}. \tag{9.30}$$

Hence $\omega_* : T_eK \to T_eG$ is an isomorphism so there is an open neighbourhood V of $(e, e) \in K$ such that $\omega : V \to U \subseteq G$ is a diffeomorphism onto the open neighbourhood U of $e \in G$. Now let $\pi_2 : G \times H \to H$ be the projection onto H and put

$$\phi = \pi_2 \circ \omega^{-1} : U \to H.$$

Then since ω is a homomorphism clearly

$$\phi(ab) = \phi(a)\phi(b) \text{ for } a, b \in U \text{ with } ab \in U.$$

Also $\phi_{*e} = \pi_{2*e} \circ \omega_{e*}^{-1}$ so by (9.30)

$$\phi_{*e}(X) = \pi_{2*e}(X \oplus \Phi X) = \Phi X \ , \ \forall\, X \in \mathfrak{g}.$$

This proves the first part.

As for the second part, let $\phi, \psi : G \to H$ with $\phi_{*e} = \psi_{*e} = \Phi : \mathfrak{g} \to \mathfrak{h}$ and suppose G is connected. Let

$$\begin{aligned} \theta_\phi &: G \to G \times H & \qquad \theta_\phi(g) &= (g, \phi(g)) \\ \theta_\psi &: G \to G \times H & \qquad \theta_\psi(g) &= (g, \psi(g)) \end{aligned}$$

Then clearly θ_ϕ and θ_ψ are immersions and

$$\theta_{\phi*}(X) = (X, \Phi X) = \theta_{\psi*}(X).$$

Hence $\theta_\phi = \theta_\psi$ by theorem 9.24 and so $\phi = \psi$.

Corollary 9.31

If G and H are Lie groups with isomorphic Lie algebras, then they are locally isomorphic. I.e., there are neighbourhoods $U \subseteq G$, $V \subseteq H$ and a diffeomorphism $\phi : U \to V$ with

$$\phi(ab) = \phi(a)\phi(b) \qquad \forall\, a, b \in U \ \text{with} \ ab \in U.$$

Proof

Obvious from theorem 9.29.

As another application of theorem 9.29 one can show (for a proof see Spivak [6 chap. 10 problem 8] or Warner [7 theorem 3.27]):

Theorem 9.32

a. *Let G be a connected and* ***simply connected*** *Lie group and let H be another Lie group. Suppose $\Phi : \mathcal{L}(G) \to \mathcal{L}(H)$ is a Lie algebra homomorphism, then there is a unique Lie group homomorphism $\phi : G \to H$ with $\phi_* = \Phi$.*

b. *In particular if G and H are simply connected Lie groups and $\mathcal{L}(G)$ and $\mathcal{L}(H)$ are isomorphic, then G and H are isomorphic.*

With this the following proposition is straightforward.

Proposition 9.33

a. *The universal covering G of a connected Lie group G is in a canonical way a Lie group.*

b. *Two connected Lie groups G and H have isomorphic Lie algebras iff the universal covering groups $\tilde{G}$ and $\tilde{H}$ are isomorphic.*

Exercise 9.34

Prove proposition 9.33.

We shall prove theorem 9.32 in the following special case:

Corollary 9.35

Let G be a Lie group and $X \in \mathcal{L}(G)$. Then there is a unique Lie group homomorphism $\phi : \mathbb{R} \to G$ such that $\phi_{*0}\left(\frac{d}{dt}\right) = X$.

Proof

Let us identify $\mathcal{L}(\mathbb{R})$ with $\mathbb{R}$ and let $\Phi : \mathbb{R} \to \mathcal{L}(G)$ be given by $\Phi(\alpha) = \alpha X$. Then by theorem 9.29 there is a real number $\epsilon > 0$ and a differentiable map $\phi : (-\epsilon, \epsilon) \to G$ with

1. $\phi(s+t) = \phi(s)\phi(t)$ whenever $|s|, |t|\ ,\ |s+t| < \epsilon$,

2. $\frac{d\phi}{dt}\ \big|_{t=0} = \phi_{*0}\left(\frac{d}{dt}\right) = X$.

In order to extend ϕ to $\mathbb{R}$ we write every $t \in \mathbb{R}$ in the form

$$t = k\left(\frac{\epsilon}{2}\right) + r\ , \qquad k \in \mathbb{Z}\ ,\ |r| < \frac{\epsilon}{2},$$

and we define

$$\phi(t) = \begin{cases} \phi(\epsilon/2)^k \cdot \phi(r) & \text{if } k \geq 0 \\ \phi(-\epsilon/2)^{-k} \cdot \phi(r) & \text{if } k < 0. \end{cases}$$

The uniqueness of ϕ is contained in theorem 9.29.

Notation

The homomorphism ϕ above is called a *one-parameter group* with *infinitesimal generator* X. We also define

$$\exp(X) = \phi(1). \tag{9.36}$$

Clearly $\phi(t) = \exp(tX)$, for all $t \in \mathbb{R}$. The map

$$\exp : \mathcal{L}(G) \to G$$

is called the *exponential map* for G.

Proposition 9.37

For G a Lie group there is a left invariant connection $\bigtriangledown$ on G (that is, $\nabla^{L_g} = \nabla$, $\forall\, g \in G$) such that the geodesics through e are precisely the one-parameter groups. In particular $\bigtriangledown$ is complete.

Proof

Let $X_1, \ldots, X_n$ be a basis for $\mathcal{L}(G)$ and let $\tilde{X}_1, \ldots, \tilde{X}_n$ be the corresponding moving frame on G of left invariant vector fields. Let $\bigtriangledown$ be the connection given by the equation (4.11) with $\Gamma^k_{ij} = 0$, that is, $\nabla_{\tilde{X}}\left(\tilde{Y}\right) = 0$ for $\tilde{X}, \tilde{Y}$ any left invariant vector fields. $\bigtriangledown$ is obviously left invariant. Let us show that a one-parameter group $\phi : \mathbb{R} \to G$ is a geodesic: Let $X = \phi_{*0}\left(\frac{d}{dt}\right)$ and let $\tilde{X}$ be the associated left invariant vector field. Since $\phi(s+t) = \phi(s) \cdot \phi(t)$ we get

$$\frac{d\phi}{dt}(s) = L_{\phi(s)} \frac{d\phi}{dt}(0) = L_{\phi(s)} X = \tilde{X}_{\phi(s)}$$

so clearly

$$\frac{D}{dt}\, \frac{d\phi}{dt}\bigg|_s = \nabla_{\tilde{X}_{\phi(s)}}\left(\tilde{X}\right) = 0,$$

which shows that $\phi(t)$, $t \in \mathbb{R}$, is a geodesic.

Remark

If G is compact there is a left and right invariant Riemannian metric (that is, L_g and R_g are isometries for all $g \in G$, see theorem 9.50 below). One can then show that the one-parameter groups are geodesics with respect to the corresponding Riemannian connection. But for G non-compact one *cannot* in general find a Riemannian metric with this property. In fact for $G = \mathrm{Sl}(2, \mathbb{R})$ the exponential map is not surjective in contrast to Hopf-Rinow's theorem for a Riemannian connection (theorem 3.17), cf. exercise 9.39 below.

Example 9.38

Consider $G = \mathrm{Gl}(n, \mathbb{R})$, $\mathcal{L}(G) = \mathfrak{gl}(n, \mathbb{R})$ = the set of $n \times n$ matrices. The exponential map is in this case given by

$$\mathrm{Exp}\,(A) = I + \frac{A}{1!} + \frac{A^2}{2!} + \ldots + \frac{A^n}{n!} + \ldots \;, \quad A \in \mathfrak{gl}(n, \mathbb{R}),$$

where this series is absolutely convergent in the operator norm. To see that this is the exponential map consider for fixed A

$$\operatorname{Exp}(tA) = I + \frac{A}{1!} + \frac{A^2}{2!}t^2 + \ldots + \frac{A^n}{n!}t^n + \ldots , \quad t \in \mathbb{R},$$

which is an absolutely convergent power series in t. Hence

$$\begin{aligned}\frac{d}{dt}\operatorname{Exp}(tA) &= \frac{A}{1!} + \frac{2}{2!}A^2 t + \ldots + \frac{A^n}{n!} n\, t^{n-1} + \ldots \\ &= A\operatorname{Exp}(tA)\end{aligned}$$

so $E(t) = \operatorname{Exp}(tA)$ is the unique solution to the linear differential equation

$$\frac{d}{dt}E(t) = AE(t) , \qquad E(0) = I.$$

Now given $s \in \mathbb{R}$

$$\frac{d}{dt}E(t+s) = AE(t+s) , \qquad E(0+s) = E(s)$$

and also

$$\begin{aligned}\frac{d}{dt}(E(t)\cdot E(s)) = \left(\frac{d}{dt}E(t)\right)\cdot E(s) =& A\,E(t)E(s), \\ & E(0)E(s) = E(s).\end{aligned}$$

Hence by uniqueness of the solutions we obtain

$$E(t+s) = E(t)E(s)$$

so that indeed $E : \mathbb{R} \to \mathrm{Gl}(n, \mathbb{R})$ is a one-parameter group.

Exercise 9.39

Let

$$\mathrm{Sl}(2,\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \det g = ad - bc = 1 \right\}$$

a. Show that the Lie algebra is

$$\mathfrak{sl}(2,\mathbb{R}) = \left\{ X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

b. Show that $\operatorname{Exp} : \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{Sl}(2,\mathbb{R})$ is given by

$$\operatorname{Exp} X = \begin{cases} \left(\cosh\sqrt{-\det X}\right) I + \frac{\sinh\sqrt{-\det X}}{\sqrt{-\det X}}\; X\ , & \text{if } \det X < 0, \\ I + X, & \text{if } \det X = 0, \\ \left(\cos\sqrt{\det X}\right) I \quad + \frac{\sin\sqrt{\det X}}{\sqrt{\det X}}\; X\ , & \text{if } \det X > 0. \end{cases}$$

c. Let us consider 1-parameter groups the same if they have proprotional infinitesimal generators.
Show that

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad \lambda \neq 1,$$

lies in exactly one 1-parameter subgroup if $\lambda > 0$, in infinitely many if $\lambda = -1$ and in *no* 1-parameter subgroup if $\lambda < 0$, $\lambda \neq -1$.

Let us list a few properties of the exponential map:

Proposition 9.40

a. $\exp : \mathcal{L}(G) \to G$ *is smooth and maps a neigbourhood of* 0 *diffeomorphically onto a neighbourhood of* e.

b. *If* $\phi : H \to G$ *is a homomorphism of Lie groups then* $\exp \circ \phi_* = \phi \circ \exp$, *i.e., the diagram below commutes*

$$\begin{array}{ccc} \mathcal{L}(H) & \xrightarrow{\phi_*} & \mathcal{L}(G) \\ \downarrow \exp & & \downarrow \exp \\ H & \xrightarrow{\phi} & G. \end{array}$$

c. *In particular for* $H \subseteq G$ *a Lie subgroup*

$$\exp_G \big| \mathcal{L}(H) = \exp_H : \mathcal{L}(H) \to H.$$

Proof

a) follows from theorem 3.8, theorem C.1 in appendix C and proposition 9.37.

b) if $\gamma : \mathbb{R} \to H$ is a 1-parameter group with $\gamma_{*0}\left(\frac{d}{dt}\right) = X$, then clearly $\phi \circ \gamma : \mathbb{R} \to G$ is a 1-parameter group with infinitesimal generator

$$(\phi \circ \gamma)_{*0}\left(\frac{d}{dt}\right) = \phi_{*e} \circ \left(\gamma_{*0}\left(\frac{d}{dt}\right)\right) = \phi_* X,$$

so $\exp(\phi_* X) = \phi \circ \gamma(1) = \phi(\exp X)$,

c) clearly follows from b).

Remark

It follows in particular that the differentiable structure on a Lie subgroup $H \subseteq G$ is uniquely determined by the requirement that

$$\exp_G : \mathcal{L}(H) \to H$$

is a diffeomorphism in a neighbourhood of 0.

Exercise 9.41

For G a Lie group let $\bigtriangledown$ be the left invariant connection given by $\nabla_{\tilde{X}}(\tilde{Y}) = 0$ for $\tilde{X}, \tilde{Y}$ any left invariant vector fields (cf. proposition 9.37).

a. Show that the torsion tensor field T is left invariant and is given by $T(X,Y) = -[X,Y],\ X,Y \in \mathcal{L}(G)$. Show also that the curvature tensor field is zero.

b. Show that if G is connected then a vector field is left invariant iff it is parallel along all 1-parameter groups and their left cosets.

c. For G and H two connected Lie groups and $\phi : G \to H$ a diffeomorphism with $\phi(e) = e$, show that ϕ is an affine transformation iff it is a Lie group isomorphism. (*Hint:* First observe that if ϕ is an affine transformation then it preserves left invariance of vector fields.)

As an application of the exponential map we shall study *homogeneous spaces.* Let G be a Lie group and $H \subseteq G$ a *closed* Lie subgroup (in particular H is an embedded submanifold). Consider the set G/H of cosets gH with the quotient topology.

Exercise 9.42

Show that G/H is a Hausdorff space and the map $G \times G/H \to G/H$ defined by $(g, g'H) \to gg'H$ is continuous.

Theorem 9.43

For G a Lie group and $H \subseteq G$ a closed Lie subgroup, the quotient space G/H is in a natural way a smooth manifold and the map $G \times G/H \to G/H$ defined by $(g, g'H) \to gg'H$ is C^∞.

Proof

Let $\mathfrak{g} \supseteq \mathfrak{h}$ be the Lie algebras of G and H respectively and let $\mathfrak{m} \subseteq \mathfrak{g}$ be a complementary subspace, i.e. $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. Let $\pi : G \to G/H$ be the canonical projection. Since the map $\Phi : \mathfrak{g} \to G$ defined by

$$\Phi(A,B) = \exp(A) \cdot \exp(B)\ , \qquad A \in \mathfrak{m}\ ,\ B \in \mathfrak{h},$$

has differential equal to the identity at e, we can find neighbourhoods $U_\mathfrak{m} \subseteq \mathfrak{m}\ ,\ U_\mathfrak{h} \subseteq \mathfrak{h}$ of 0 in $\mathfrak{m}$ and $\mathfrak{h}$, such that

$$\Phi : U_\mathfrak{m} \times U_\mathfrak{h} \to G$$

is a diffeomorphism onto an open neighbourhood of e in G. In particular $\exp U_\mathfrak{h} \subseteq H$ is an open subset, so we can choose V a neighbourhood of e in G such that

$$V \cap H = \exp U_\mathfrak{h}.$$

By continuity we can find a compact neighbourhood $U \subseteq U_{\mathfrak{m}}$ of 0 such that

$$\exp\left(-X'\right) \cdot \exp\left(X''\right) \in V \qquad \forall\, X', X'' \in U.$$

Claim. $\pi : \exp U \to G/H$ is a homeomorphism. (The set $\exp U \subseteq G$ is called a *local cross section*).

Clearly $\exp : U \to \exp U \subseteq G$ is continuous and one-to-one, hence a homeomorphism, so we shall just see that π is one-to-one on $\exp U$: Suppose

$$\pi \exp\left(X'\right) = \pi \exp\left(X''\right), \qquad X', X'' \in U.$$

Then there exists $h \in H$ such that

$$h = \exp\left(-X'\right) \cdot \exp\left(X''\right) \in V,$$

hence $h = \exp Z$, $Z \in U_{\mathfrak{h}}$. It follows that

$$\exp\left(X'\right)\ \exp\left(Z\right) = \exp\left(X''\right).$$

But since Φ is one-to-one, $Z = 0$ and $X' = X''$. This proves the claim above.

Now put $\psi = \pi \circ \exp :\ U \to G/H$. Then we shall use $(\psi U, \psi^{-1})$ as a coordinate chart around $H \in G/H$, and in general

$$\psi^{-1} \circ L_{g^{-1}} :\ g\,\psi(U) \to U$$

shall be a coordinate chart around $gH \in G/H$. It is now straightforward to check that this gives a smooth structure on G/H and that the map $G \times G/H \to G/H$ given by $(g, g'H) \to gg'H$ is smooth.

Remark

Notice that the above map $G \times G/H \to G/H$ is a *left action* of G on G/H, so that G *acts smoothly* on G/H.

Examples 9.44

a. In $\mathrm{Gl}(n, \mathbb{R})$ consider for given $k < n$ the subgroup $A(k, n-k) \subseteq \mathrm{Gl}(n, \mathbb{R})$ of matrices of the form

$$g = \begin{pmatrix} I & A \\ 0 & g_2 \end{pmatrix}$$

where I is the $k \times k$ identity matrix, $A \in M(k, n-k)$ an arbitrary $k \times (n-k)$-matrix, and $g_2 \in \mathrm{Gl}(n-k, \mathbb{R})$. Then $\mathrm{Gl}(n, \mathbb{R})/A(k, n-k)$ is in a natural way diffeomorphic to the open set $W_{n,k} \subseteq M(n, k) = \mathbb{R}^{n \cdot k}$ of k linearly independent column-vctors in $\mathbb{R}^n$, and the left action of $\mathrm{Gl}(n, \mathbb{R})$ corresponds to the natural left matrix multiplication on such columns. In particular $\mathrm{Gl}(n, \mathbb{R})/A(1, n-1) \approx \mathbb{R}^n - \{0\}$.

b. Let $P(k, n-k) \subseteq \mathrm{Gl}(n, \mathbb{R})$ be the subgroup of matrices

$$g = \begin{pmatrix} g_1 & A \\ 0 & g_2 \end{pmatrix}$$

where $g_1 \in \mathrm{Gl}(k, \mathbb{R})$, $g_2 \in \mathrm{Gl}(n-k, \mathbb{R})$, $A \in M(k, n-k)$. Then by theorem 9.43 we get a manifold structure on $G_k(\mathbb{R}^n) = \mathrm{Gl}(n,\ \mathbb{R})/P(k, n-k)$. Notice that $P(k, n-k) = A(k, n-k) \cdot \mathrm{Gl}(k, \mathbb{R})$ and hence by a) we have a bijection

$$G_k(\mathbb{R}^n) \simeq W_{n,k}/\mathrm{Gl}(k, \mathbb{R})$$

where on the right we have used the usual right matrix multiplication of $\mathrm{Gl}(k, \mathbb{R})$ on $W_{n,k} \subseteq M(n, k)$. Now the set $W_{n,k}/\mathrm{Gl}(k, \mathbb{R})$ is in a natural way identified with the set of k-dimensional linear subspaces of $\mathbb{R}^n$. Therefore the manifold $G_k(\mathbb{R}^n)$ is usually identified with this set and is called the *Grassmann manifold* of k-planes in $\mathbb{R}^n$. For $k = 1$, $G_1(\mathbb{R}^n) = P(\mathbb{R}^n)$ is called the $n-1$-dimensional (real) *projective space*.

Exercise 9.45

a. Let $1 \times \mathrm{O}\,(n-k) \subseteq \mathrm{O}\,(n)$ be the subgroup of orthogonal matrices of the form

$$g = \begin{pmatrix} I & 0 \\ 0 & g_2 \end{pmatrix}, \qquad g_2 \in \mathrm{O}(n-k).$$

Show that $\mathrm{O}(n)/1 \times \mathrm{O}(n-k)$ is diffeomorphic to the submanifold $V_{n,k} \subseteq M(n, k) = \mathbb{R}^{n \cdot k}$ of matrices X satisfying

$$X^t X = I$$

that is, the set of k orthonormal vectors in $\mathbb{R}^n$. $V_{n,k}$ is called the *Stiefel manifold* of *orthogonal k-frames* in $\mathbb{R}^n$. So in particular

$$\mathrm{O}(n)/1 \times \mathrm{O}(n-1) \approx V_{n,1} = S^{n-1}.$$

b. Show that

$$G_k(\mathbb{R}^n) \approx \mathrm{O}(n)/\mathrm{O}(k) \times \mathrm{O}(n-k) \simeq V_{n,k}/\mathrm{O}(k)$$

where $\mathrm{O}(k) \times \mathrm{O}(n-k) \subseteq \mathrm{O}(n)$ is the subgroup of matrices of the form

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}, \qquad g_1 \in \mathrm{O}(k)\ ,\ g_2 \in \mathrm{O}(m-k).$$

In particular $P(\mathbb{R}^n)$ is identified with $S^{n-1}/\mathrm{O}(1)$ where $\mathrm{O}(1) = \{1, -1\}$ acts by $v(\pm 1) = \pm v$.

We end this chapter with a few remarks on the *adjoint representation* of a Lie group G with Lie algebra $\mathfrak{g} = \mathcal{L}(G)$.

Definition 9.46

For $g \in G$ define $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ to be the differential at e of the inner automorphism $x \mapsto gxg^{-1}$.

Proposition 9.47

a. $\mathrm{Ad}(g)$ *is invertible and*

$$\mathrm{Ad} : G \to \mathrm{Gl}(\mathfrak{g})$$

is a Lie group homomorphism.

b. *The differential of* Ad *is given by*

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$

where $\mathrm{ad}(X)(Y) = [X, Y]$, $X, Y \in \mathfrak{g}$.

c. $\mathrm{Ad}(\exp(X)) = \mathrm{Exp}(\mathrm{ad}X)$, $X \in \mathfrak{g}$, *where* $\mathrm{Exp} : \mathrm{End}(\mathfrak{g}) \to \mathrm{Gl}(\mathfrak{g})$ *is given in example* 9.38.

d. $\exp(\mathrm{Ad}(g)X) = g\exp(X)g^{-1}$, $\forall\, X \in \mathfrak{g}$, $g \in G$.

Proof

a) For $g \in G$ let $\sigma_g : G \to G$ be given by

$$\sigma_g(x) = gxg^{-1} , \qquad x \in G,$$

so that

$$\mathrm{Ad}(g) = (\sigma_g)_{*e} : \mathfrak{g} \to \mathfrak{g}.$$

Clearly $\sigma_g\sigma_{g^{-1}} = \sigma_e = \mathrm{id}$, so $\mathrm{Ad}(g)\mathrm{Ad}\big(g^{-1}\big) = \mathrm{id}$. Similarly

$$\mathrm{Ad}\big(gg'\big) = \mathrm{Ad}(g) \circ \mathrm{Ad}\big(g'\big) , \qquad g, g' \in G.$$

Also the map $\phi : G \times G \to G$ given by $\phi(g, x) = gxg^{-1}$ is clearly differentiable so that $\phi_* : T(G \times G) \to TG$ is differentiable. Now

$$\mathrm{Ad}(g)(X) = \phi_*\Big((0 \oplus X)_{(g,e)}\Big) \in T_e(G) , \qquad X \in T_eG , \ g \in G,$$

so the map $G \times \mathfrak{g} \to \mathfrak{g}$ given by $(g, X) \mapsto \mathrm{Ad}(g)(X)$ is differentiable from which a) easily follows.

b) For $X \in \mathfrak{g}$ put $g_t = \exp(tX)$, $t \in \mathbb{R}$, and notice that

$$\mathrm{Ad}_*(X) = \frac{d}{dt}\,\mathrm{Ad}(g_t)\,\Big|_{t=0}$$

so that

$$\mathrm{Ad}_*(X)(Y) = \frac{d}{dt}\,\frac{d}{ds}\,g_t(\exp(sY))g_t^{-1}\,\Big|_{s=0,t=0} .$$

That is, for $f \in C^\infty(G)$

$$\mathrm{Ad}_*(X)(Y)(f) = \frac{\partial^2}{\partial t \partial s}\,f(g_t(\exp sY)g_{-t})\,\Big|_{s=0,t=0} . \tag{9.48}$$

Now, for fixed $s \in \mathbb{R}$

$$\begin{aligned} \frac{\partial}{\partial t}\,f(g_t(\exp sY)g_{-t})\,\Big|_{t=0} &= \frac{\partial}{\partial t}\,f(g_t \exp sY)\,\Big|_{t=0} \\ &\quad - \frac{\partial}{\partial t}\,f((\exp sY)g_t)\,\Big|_{t=0} \end{aligned} \tag{9.49}$$

On the other hand the left invariant vector field $\tilde{Y}$ corresponding to Y clearly satisfies

$$\begin{aligned} \tilde{Y}_g(f) = Y(f \circ L_g) &= \frac{d}{ds}\,f \circ L_g(\exp sY)\,\Big|_{s=0} \\ &= \frac{d}{ds}\,f(g \exp sY)\,\Big|_{s=0} , \qquad \forall\, g \in G, \end{aligned}$$

so that

$$\frac{\partial^2}{\partial t \partial s}\,f(g_t(\exp sY))\,\Big|_{t=0,s=0} = \frac{d}{dt}\,\tilde{Y}_{g_t}(f)\,\Big|_{t=0} = X\Big(\tilde{Y}(f)\Big).$$

Similarly

$$\frac{\partial^2}{\partial s \partial t}\,f((\exp sY)(\exp tX))\,\Big|_{t=0,s=0} = Y\Big(\tilde{X}(f)\Big).$$

Hence by (9.48)

$$\mathrm{Ad}_*(X)(Y)[f] = X\Big(\tilde{Y}(f)\Big) - Y\Big(\tilde{X}(f)\Big) = [X,Y](f)$$

so that indeed

$$\mathrm{Ad}_*(X)(Y) = [X,Y] = \mathrm{ad}(X)(Y).$$

c) now follows from b) and proposition 9.40 applied to $\mathrm{Ad} : G \to \mathrm{Gl}(\mathfrak{g})$.

d) is obvious from the definitions. In fact, the one parameter group $t \mapsto g\exp(tX)g^{-1}$, $t \in \mathbb{R}$, has infinitesimal generator

$$\frac{d}{dt}\big(g \exp(tX)g^{-1}\big)\,\Big|_{t=0} = \mathrm{Ad}(g)(X).$$

As an application of the adjoint representation let us prove:

Theorem 9.50

If K is a compact Lie group then it has a bi-invariant Riemannian metric, that is, L_g and R_g , $g \in K$, are isometries.

Proof

First choose an inner product $\langle \cdot \ , \ \cdot \rangle$ in T_eK and extend it to a Riemannian metric on K by

$$\langle x, y \rangle_g = \left\langle L_{g^{-1}{}_*} x \ , \ L_{g^{-1}{}_*} y \right\rangle \ , \ \forall \, x, y \in T_gK \ , \ g \in K.$$

As in chapter 7 this enables us to integrate functions on K. By construction $L_g : K \to K$ is an isometry for every $g \in G$. Hence clearly

$$\int_K f = \int_K f \circ L_g \qquad \forall \, g \in K \text{ and } f \text{ integrable on } K.$$

We claim that also

$$\int_K f = \int_K f \circ R_g \qquad \forall \, g \in K \text{ and } f \text{ integrable on } K. \tag{9.51}$$

For this notice that (cf. exercise 7.36)

$$\int_K f = \int_K (f \circ R_g) \cdot |\det R_g|$$

where $|\det R_g|$ is the function $K \to \mathbb{R}$ whose value at $k \in K$ is $\pm$ the determinant of $R_{g*} : T_kK \to T_{kg}K$ with respect to an orthonormal basis. Since left translation is an isometry $(R_g)_{*k}$ has the same determinant up to sign) for all k and

$$\begin{aligned} \det R_{g*} &= \det \left(R_{g*} : T_eK \to T_gK \right) \\ &= \ \pm \det \left(L_{g^{-1}{}_*} \circ R_{g*} : T_eK \to T_eK \right) \\ &= \ \pm \det \ \mathrm{Ad}\big(g^{-1}\big) \end{aligned}$$

However, $\det \ \circ \mathrm{Ad} \ : \ K \ \to \ \mathbb{R}^* \ = \ \mathrm{Gl}(1, \mathbb{R})$ defines a Lie group homomorphism, and since K is compact the image in $\mathbb{R}^*$ is a compact subgroup of $\mathbb{R}^*$ and hence is ± 1. It follows that $|\det R_g| = 1$ and hence (9.51) is proved.

Now define another inner product in T_eK by

$$\langle\langle x, y \rangle\rangle = \int_K \langle \mathrm{Ad}(k)x \ , \ \mathrm{Ad}(k)y \rangle dk \ , \ x, y \in T_eK.$$

Then

$$\langle\langle \mathrm{Ad}(g)x \ , \ \mathrm{Ad}(g)y \rangle\rangle = \langle\langle x, y \rangle\rangle \ , \qquad \forall \, g \in K, \ x, y \in T_eK. \tag{9.52}$$

In fact

$$\langle\langle \mathrm{Ad}(g)x, \mathrm{Ad}(g)y\rangle\rangle = \int_K \langle \mathrm{Ad}(k\cdot g)x,\ \mathrm{Ad}(k\cdot g)y\rangle dk$$
$$= \int_K \langle \mathrm{Ad}(k)x, \mathrm{Ad}(k)y\rangle dk \quad = \langle\langle x,y\rangle\rangle$$

follows using (9.51). It is now easily checked that the metric defined by

$$\langle\langle x,y\rangle\rangle_g = \left\langle\left\langle L_{g^{-1}*}x\ ,\ L_{g^{-1}*}y\right\rangle\right\rangle\ ,\ x,y\in T_eK,$$

is bi-invariant: As before it is left invariant and also

$$\begin{aligned}\langle\langle R_{k*}x\ ,\ R_{k*}y\rangle\rangle_{gk} &= \left\langle\left\langle L_{(gk)^{-1}*}R_{k*}x\ ,\ L_{(gk)^{-1}*}R_{k*}y\right\rangle\right\rangle \\ &= \left\langle\left\langle R_{k*}L_{k^{-1}*}L_{g^{-1}*}x\ ,\ R_{k*}L_{k^{-1}*}L_{g^{-1}*}y\right\rangle\right\rangle \\ &= \left\langle\left\langle \mathrm{Ad}\big(k^{-1}\big)L_{g^{-1}*}x\ ,\ \mathrm{Ad}\big(k^{-1}\big)L_{g^{-1}*}y\right\rangle\right\rangle = \left\langle\left\langle L_{g^{-1}*}x\ ,\ L_{g^{-1}*}y\right\rangle\right\rangle \\ &= \langle\langle x,y\rangle\rangle_g\end{aligned}$$

by (9.52). This proves the theorem.

Exercise 9.53

Let G be a Lie group and $H\subseteq G$ a closed Lie subgroup. Let $\mathfrak{g}\supseteq\mathfrak{h}$ be the Lie algebras of G and H respectively.

a. Show that the projection $\pi: G\to G/H$ gives rise to an exact sequence

$$0\to\mathfrak{h}\to\mathfrak{g}\overset{\pi_*}{\to}T_e(G/H)\to 0$$

and that for $h\in H$ there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{h} & \to & \mathfrak{g} & \to & T_e(G/H) & \to & 0 \\ & & \downarrow \mathrm{Ad}(h) & & \downarrow \mathrm{Ad}(h) & & \downarrow L_{h*} & & \\ 0 & \to & \mathfrak{h} & \to & \mathfrak{g} & \to & T_e(G/H) & \to & 0 \end{array}$$

b. Show that there is a one-one correspondence between left invariant Riemannian metrics on G/H and inner products $\langle\cdot\ ,\ \cdot\rangle$ in $\mathfrak{g}/\mathfrak{h}$ satisfying

$$\langle \mathrm{Ad}(h)x\ ,\ \mathrm{Ad}(h)y\rangle = \langle x,y\rangle \qquad \forall\, x,y\in\mathfrak{g}/\mathfrak{h}\ ,\ h\in H.$$

c. Show that if $K\subseteq G$, with K compact, then there exist left invariant metrics on G/K.

Remark

Riemannian manifolds of the form considered in b) above are called *homogeneous* Riemannian manifolds. In the next section we shall see that every simply connected Riemannian symmetric space is a homegenous Riemannian manifold with compact isotropy subgroup.

Chapter 10 THE ISOMETRY GROUP OF A SYMMETRIC SPACE

From now on we shall call a connected Riemannian globally symmetric space just a *symmetric space*. In this section we shall study the group of isometries of a symmetric space, and we shall show that it is in a natural way a Lie group. As a first step we make it into a *topological group*. For convenience we assume M simply connected.

Definition 10.1

For M a connected Riemannian manifold let $I(M)$ denote the group of isometries of M. $I(M)$ is given the *compact open* topology, that is, a basis for the topology is given by the finite intersections of sets of the form

$$W(C,U) = \left\{ g \in I(M) \;\middle|\; g(C) \subseteq U \right\}$$

where $C \subseteq M$ is compact and $U \subseteq M$ is open.

Proposition 10.2

$I(M)$ is a Hausdorff space with a countable basis for the topology.

Proof

Since M is a connected manifold it has a countable basis $\{\mathcal{O}_n\}$ and since M is locally compact we can assume that $\overline{\mathcal{O}}_n$ is compact for all n. Then it is easily seen that the set of finite intersections of the sets $W\left(\overline{\mathcal{O}}_n, \mathcal{O}_m\right)$ constitutes a countable basis for the topology of $I(M)$.

Remark

The usefulness of this proposition lies in the fact that if $f : X \to Y$ is a function and X has a countable basis then f is continuous iff f maps convergent sequences to convergent sequences.

Proposition 10.3

Let $\{\phi_n\} \subseteq I(M)$ and $f \in I(M)$. Then the following are equivalent:

a) $\phi_n \to f$ *pointwise on* M.

b) $\phi_n \to f$ *uniformly on compact sets.*

c) $\phi_n \to f$ *in the compact open topology.*

Proof

b) $\Rightarrow$ c) $\Rightarrow$ a) is easy. a) $\Rightarrow$ b): Given C compact and $\epsilon > 0$. Find finitely many points $p_i \in C$ such that any point $p \in C$ has distance less than ϵ from some point p_i. Let N be so large that for all i

$$d(\phi_n(p_i)\ ,\ f(p_i)) < \epsilon \qquad \text{for all } n > N.$$

Then if $p \in C$ has distance less than ϵ from p_i we have

$$\begin{aligned} d(\phi_n(p)\ ,\ f(p)) &\leq d(\phi_n(p)\ ,\ \phi_n(p_i)) + d(\phi_n(p_i)\ ,\ f(p_i)) + \\ &\quad + d(f(p_i)\ ,\ f(p)) \\ &= d(p, p_i) + d(\phi_n(p_i)\ ,\ f(p_i)) + d(p_i, p) < 3\epsilon. \end{aligned}$$

for $n > N$. This ends the proof.

More generally let $U \subseteq M$ be an open set and let $I(U, M)$ denote the set of isometries of U onto an open set of M. Again $I(U, M)$ is given the compact open topology.

Proposition 10.4

Let M be a simply connected symmetric space. For any open set $U \subseteq M$ is the restriction map $\rho : I(M) \to I(U, M)$ a homeomorphism.

Proof

By lemma 8.21 ρ is injective and by theorem 8.20 it is onto. Clearly ρ is continuous. In order to see that ρ^{-1} is continuous let $\{\phi_n\} \in I(M)$ be a sequence and $f \in I(M)$ such that $\phi_n \to f$ on U. In view of proposition 10.3 it is enough to prove that $\phi_n(q) \to f(q) \qquad \forall\, q \in M$.

For this we shall need normal neighbourhoods V, which are *regular* in the following sense: For every $q, q' \in V$ there is a unique $v \in T_q(M)$ (of small length) such that $q' = \exp_q(v)$, and the mapping $V \times V \to TM$ sending (q, q') to v is a diffeomorphism onto an open neighbourhood. By theorem 3.8 every point has a regular normal neighbourhood. Furthermore notice that for any two points p and p' there is an isometry of M taking p to p' (in fact the symmetry in the midpoint of a geodesic from p to p' does this). Therefore, we can find an $r > 0$ such that any point has a regular normal neighbourhood of radius r. With this in mind we shall prove

Lemma 10.5

Let $M, \{\phi_n\}$ and f be as above and suppose M has regular normal neighbourhoods of radius $2r$. Suppose $\phi_n(q) \to f(q)$ for q in a neighbourhood of $p \in M$. Then

$$\phi_n(q) \to f(q) \qquad \textit{for} \quad d(p, q) < r.$$

Given this lemma we can end the proof of the proposition as follows: Suppose $\phi_n \to f$

on U and choose $p_0 \in U$. Let $q_0 \in M$ be arbitrary and choose a curve $\gamma : [0,1] \to M$ from p_0 to q_0. Then by an argument similar to the proof of lemma 8.22 we have for all $t \in [0,1]$ that $\phi_n(q) \to f(q)$ for q in a neighbourhood of $\gamma(t)$. In particular $\phi_n(q_0) \to f(q_0)$. It remains to prove lemma 10.5:

Proof of lemma 10.5

Suppose $\phi_n(q) \to f(q)$ for q in a neighbourhood of p.

Case 1. $\phi_n(p) = f(p) \quad \forall\, n$. Since $\phi_n\big(\exp_p(v)\big) = \exp_p(\phi_{n*}(v))$ for small $v \in T_p(M)$ and since $\exp_p$ is a homeomorphism on a small neighbourhood we conclude that $\phi_{n*}(v) \to f_*(v)$ for small v and hence by linearity for all $v \in T_p(M)$. Hence $\phi_n\big(\exp_p(v)\big)$ converges to $f\big(\exp_p(v)\big)$ for all v, that is, $\phi_n(q) \to f(q) \;\; \forall\, q \in M$.

General case. Clearly it is enough to consider $f = \mathrm{id}$. So $\phi_n(q) \to q$ in a neighbourhood of p. Let p_n be the midpoint of the geodesic from p to $\phi_n(p)$ and let s_n be the symmetry in p_n. Hence $s_n(p) = \phi_n(p)$ or equivalently

$$s_n \circ \phi_n(p) = p.$$

also let s_0 be the symmetry in p. We claim that it is enough to show that $s_n \to s_0$ on the normal neighbourhood V of radius r. Because then by proposition 10.3 this convergence is uniform on compact subsets of V and it follows easily that $\phi_n \to \mathrm{id}$ on V iff $s_n \circ \phi_n \to s_0$ on V which reduces the general case to case 1.

It therefore remains to prove that if $p_n \to p$ then the corresponding symmetries s_n converges to s_0 in the neighbourhood V. But by assumption the normal neighbourhood of radius $2r$ is regular. Therefore the mapping

$$\big(q, q'\big) \;\mapsto\; \exp_q(-v) = s_q\big(q'\big) \qquad \big(q' = \exp_q(v)\big)$$

is differentiable on $V \times V$ and is in particular continuous as a function in q. This ends the proof of the lemma and hence of proposition 10.4.

As a corollary we get:

Corollary 10.6

For a simply connected symmetric space the mapping $M \to I(M)$ given by sending q to the symmetry s_q is continuous.

Proof

As seen in the proof of lemma 10.5 the mapping $q \mapsto s_q$ of V into $I(V, M)$ (for V a regular normal neighbourhood of a point p) is continuous. The statement therefore follows from proposition 10.4.

Corollary 10.7

Let M be a simply connected symmetric space and choose $p \in M$. Let $K \subseteq I(M)$ be the isotropy subgroup at p, that is,

$$K = \left\{ k \in I(M) \;\middle|\; k(p) = p \right\}.$$

Then the homomorphism $k \mapsto k_{*p}$ is a homeomorphism of K onto a compact subgroup of the group of linear isometries of $T_p(M)$.

Proof

Put $V = T_p(M)$ and let $O(V)$ be the group of isometries of V. On $O(V)$ all possible topologies agree: the *matrix topology* induced from a choice of a basis of V, the *norm topology* with respect to the inner product, the compact open topology and the topology of pointwise convergence. From case 1 in the proof of lemma 10.5 it follows that a sequence $k_n \in K$ converges in a neighbourhood of p iff $(k_n)_{*p}$ converges in $O(V)$. It therefore follows from proposition 10.4. that the topology on K induced from $I(M)$ is the same as the one induced from $O(V)$. Now by theorem 8.20 the image of K in $O(V)$ is the set of linear isometries $\phi : T_p(M) \to T_p(M)$ satisfying $R(\phi v, \phi w)\phi z = \phi R(v, w)z$, where R is the curvature tensor field at p. This is clearly a closed subset of the compact group $O(V)$.

Corollary 10.8

$I(M)$ is locally compact.

Proof

Let us find a compact neighbourhood of $\mathrm{id} \in I(M)$. Again for C and U subsets of M we put $W(C, U) = \{g \in I(M) |\; g(C) \subseteq U\}$. Choose $p \in M$ and an open neighbourhood U such that $\overline{U}$ is compact and is contained in a normal neighbourhood of p. Then

$$\mathrm{id} \in W(p, U) \subseteq W\big(p, \overline{U}\big)$$

and we claim that $W\big(p, \overline{U}\big)$ is compact. So let $\phi_n \in W\big(p, \overline{U}\big)$. Replacing $\{\phi_n\}$ with a subsequence we can suppose that $\phi_n(p)$ converges to some point $q \in \overline{U}$. Let s_0 be the symmetry in the midpoint between p and q and also let s_n be the symmetry in the midpoint between p and $\phi_n(p)$. By corollary 10.6 $s_n \to s_0$. Also $s_n \circ \phi_n(p) = p$ so since K is compact we can find a convergent subsequence of $\{s_n \circ \phi_n\}$ and hence a convergent subsequence of $\{\phi_n\}$. This ends the proof.

We shall now study the action of $I(M)$ on M.

Proposition 10.9

Let M be a simply connected symmetric space.

a. *$I(M)$ is a topological group.*

b. *The evaluation map $I(M) \times M \to M$ sending (g,p) to $g(p)$ is continuous.*

c. *Given $p \in M$ let K be the isotropy group at p. Then the mapping $\pi : I(M) \to M$ sending g to $g(p)$ induces a homeomorphism $\overline{\pi} : I(M)/K \xrightarrow{\cong} M$, where the left hand side is the set of cosets $\{g\, K\}$ with the quotient topology.*

Proof

a) and b) are trivial.

c) Consider the diagram of maps

$$\begin{array}{ccc} I(M)/K & \overset{\overline{\pi}}{\to} & M \\ \uparrow \rho & \nearrow \pi & \\ I(M) & & \end{array}$$

As remarked above $I(M)$ acts transitively on M hence $\overline{\pi}$ is clearly bijective. Also $\overline{\pi}$ is obviously continuous. We shall prove that $\overline{\pi}$ is open. It is clearly enough to find an open neighbourhood W of id $\in I(M)$ such that $W = \rho^{-1}(\rho W)$ and such that $\overline{\pi}|\rho W$ is open. For this choose p and U as in the proof of corollary 10.8, and put $W = W(p,U)$. Clearly $W = \pi^{-1}(U) = \rho^{-1}(\rho W)$. Also $W\big(p,\overline{U}\big)$ is compact. Since $\overline{\pi}$ is continuous $I(M)/K$ is Hausdorff hence $\rho\big(W\big(p,\overline{U}\big)\big)$ is compact and $\overline{\pi}|\rho\big(W\big(p,\overline{U}\big)\big)$ is a homeomorphism onto $\overline{U}$. Hence $\overline{\pi}|\rho(W)$ is a homeomorphism onto U.

In order to make $I(M)$ into a Lie group we first consider the isotropy subgroup K at $p \in M$:

Proposition 10.10

The isotropy group K at $p \in M$ is a compact Lie group.

Proof

As in the proof of corollary 10.7 we put $V = T_pM$ and we identify K with the subgroup of the orthogonal group $O(V)$ given by the set of linear isometries $g : V \to V$ satisfying $R(gv, gw)gz = g\, R(v,w)z$, for all $v, w, z \in V$, where R is the curvature tensor field at p. Putting $B(x,y,z,w) = \langle R(x,y)z, w\rangle$ we can identify K with the subgroup

$$K = \big\{ g \in O(V) \;\big|\; B(gx, gy, gz, gw) = B(x,y,z,w)\;,\; \forall\, x,y,z,w \in V \big\},$$

and we shall show that this is a Lie subgroup of the Lie group $O(V)$. For this let W denote the vector space of multilinear functions $V \times V \times V \times V \;\longrightarrow\; \mathbb{R}$ and let $F : O(V) \to W$ be the differentiable map given by

$$F(g) = B \circ (g \times g \times g \times g).$$

Also let $\mathfrak{o}(V) \subseteq \mathrm{End}(V)$ denote the Lie algebra of $O(V)$, i.e. the set of skew-adjoint linear endomorphisms of V. Then by differentiation it is easily checked that

$$\mathfrak{k} = \mathrm{Ker}\; F_{*e} \subseteq \mathfrak{o}(V)$$

can be identified with the subspace of skew-adjoint linear maps A satisfying

$$B(Ax, y, z, w) + B(x, Ay, z, w) + B(x, y, Az, w) + B(x, y, z, Aw) = 0,$$
$$\forall\, x, y, z, w \in V.$$

Now by the Implicit Function Theorem there is a neighbourhood U of e in $O(V)$ and an (embedded) submanifold $L \subseteq U$ such that $K \cap U \subseteq L$ and $T_eL = \mathfrak{k}$. On the other hand it is easily checked that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{o}(V)$ so by theorem 9.24 there is a corresponding connected Lie subgroup $K_0 \subseteq O(V)$ and we claim that $K_0 \subseteq K$. In fact for $A \in \mathfrak{k}$ it is easy to check by differentiation that $F(\mathrm{Exp}(tA))$ is constant in t so that $\mathrm{Exp}(A) \in K$; that is, $\mathrm{Exp}(\mathfrak{k}) \in K$. It follows that

$$K_0 \cap U \subseteq K \cap U \subseteq L$$

and since K_0 and L have the same tangent space at e we get, by possibly making U smaller that $K_0 \cap U = K \cap U = L$. Hence K is a submanifold of $O(V)$ in a neighbourhood of e. Since for $k \in K$ left-multiplication by k is a diffeomorphism of $O(V)$ it follows easily that $K \subseteq O(V)$ is a submanifold and hence a Lie subgroup. We have already seen that K is compact.

Remark

Notice that it follows from the proof that K is actually an embedded submanifold of $O(T_pM)$. In particular the manifold structure is uniquely determined.

We shall now prove:

Theorem 10.11

a. *$I(M)$ has the structure of a Lie group such that the action*

$$I(M) \times M \to M$$

is C^∞.

b. *For $p_0 \in M$ and K the isotropy group at p the homeomorphism $\overline{\pi} : I(M)/K \cong M$ is a diffeomorphism.*

The idea of the proof is simply to construct the local cross section as in theorem 9.43 and use the differentiable structure on M together with the differentiable structure on K. Recall that in theorem 9.43 the local cross section is given by local 1-parameter subgroups with infinitesimal generator in the complement of the Lie algebra of the isotropy group. So we first construct these local 1-parameter subgroups of isometries.

Definition 10.12

Let γ be a geodesic of M. A *transvection* along γ is an isometry T of M, which preserves γ and induces parallel translation along γ on the tangent spaces at points of γ. If γ goes through p then T is said to be a transvection at p.

Example 10.13

In Euclidean space with the flat metric the transvections are of course the parallel translations.

Exercise 10.14

Describe the transvections on a sphere.

Lemma 10.15

a. *Given γ a geodesic through p. Then there is a unique family T_t , $t \in \mathbb{R}$, of transvections along γ such that $T_t(p) = \gamma(t)$.*

b. $T_{t+s} = T_t \circ T_s \qquad \forall\, t, s \in \mathbb{R}$.

c. *Let N be a normal neighbourhood of p. Then the map $\psi : N \to I(M)$ given by associating to $\exp_p(v)$ the transvection T_1 along $\exp_p(tv)$ is continuous.*

Proof

a) is obvious from theorem 8.20, explicitly $T_t = s_{\gamma(t/2)} \circ s_p$ as shown in the proof of proposition 8.3.

b) is obvious from the uniqueness of a) and the fact that $T_s(\gamma(t)) = \gamma(s+t)$, which is easily proved.

c) By a) ψ is given explicitly by

$$\psi\big(\exp_p(v)\big) = s_{\exp_p\left(\frac{1}{2}v\right)} \circ s_p$$

so c) follows from corollary 10.6

Proof of theorem 10.11

Let $B = \psi(N)$ as in Lemma 10.15 above. Then clearly

$$\psi : B \to N \subseteq M$$

is a homeomorphism, so choosing local coordinates on N we get a C^∞ structure on B. Clearly the map $(x, k) \mapsto \psi(x) \cdot k$,

$$N \times K \to B \cdot K \subseteq I(M)$$

is 1–1 onto an open set, and since K is a Lie group it follows that $B \cdot K$ has a C^∞ structure. Let

$$\phi : B \cdot K \to N \times K$$

be the inverse map. For any $g \in I(M)$ we give $g \cdot B \cdot K \subseteq I(M)$ the C^∞ structure induced by

$$\phi_g = \phi \circ L_{g^{-1}} \ : \ \ g \cdot B \cdot K \to N \times K,$$

where as usual L_g denotes left-multiplication by g. We must show that these structures are compatible on their overlaps. It is enough to show that if $gBK \cap BK \neq \emptyset$ then $\phi_e \circ \phi_g^{-1}$ is C^∞. Now if $g\, b_2 k_2 = bk,\ b_2,\ b \in B,\ k_2,\ k \in K,$ then $g\, b_1 p = bp$ and it follows that $g(p)$ is near to p so if we replace B above by some smaller B' then g can be written in the form $g = b_1 k_1$ for $b_1 \in B,\ k_1 \in K$. So it is clearly enough to show

Claim

If $b_1 b_2 \in B$, $k_1 k_2 \in K$ runs through all elements such that

$$b_1 k_1 b_2 k_2 = bk \ , \qquad b \in B \ , \ k \in K,$$

then b and k are C^∞ functions of b_1, b_2, k_1, k_2.

This will prove that $I(M)$ is a C^∞ manifold. But at the same time it will prove that $I(M)$ is a Lie-group. In fact suppose we have proved the claim and put $V = B \cdot K \subseteq I(M)$. Then V is a neighbourhood of e and by the claim the multiplication restricted to V is C^∞. Also since K is a Lie group and since clearly $b \mapsto b^{-1}$ is C^∞ on B (via ψ this map is equivalent to s_p) it follows easily that $g \mapsto g^{-1}$ is C^∞ on V. Hence given the claim $I(M)$ is a Lie group due to the following lemma, the proof of which is left to the reader:

Lemma 10.16

Let G be a topological group with a C^∞ structure such that left translation is C^∞. Suppose V is a neighbourhood of e such that the multiplication $V \times V \to G$ and the map $g \mapsto g^{-1}$ of V into G are C^∞. Then G is a Lie group.

Proof of claim

Suppose

$$b_1 k_1 b_2 k_2 = bk,\ b_1 b_1, b_2 \in B \ , \ k_1 k_1, k_2 \in K.$$

First notice that $k_1 b_2 k_1^{-1} = b^*$ is a transvection at p. In fact if b_2 is the transvection from p to $b_2(p) = \exp_p(v)$ then clearly $k_1 b_2 k_1^{-1}$ is the transvection from p to $\exp_p((k_1)_* v) = k_1(b_2 p)$. Since K acts C^∞ on $T_p(M)$ (by definition of the C^∞ structure on K) via the mapping $k_1 \mapsto (k_1)_*$ and since $\exp_p$ is a diffeomorphism onto the neighbourhood N

it follows that $b^*p = k_1(b_2p)$ is a C^∞ function of b_2 and k_1, so b^* is a C^∞ function of b_2 and k_1.

Now $b_1k_1b_2k_2 = b_1b^*k_1k_2 = bk$ where b is the transvection from p to b_1b^*p. So we must show that b_1b^*p depends C^∞ on b_1p and b^*p. Here b_1b^*p is constructed as follows from b_1p and b^*p: Take $X \in T_p(M)$ such that $\exp_p(X) = b^*p$; parallel translate X along the geodesic to b_1p and apply $\exp_{b_1p}$. Each of these are C^∞ operations as functions of b^*p and b_1p. Similarly the matrix coordinates of the differential at p of the map

$$b^{-1}\, b_1\, b^* = k\,(k_1\, k_2)^{-1}$$

depends C^∞ on b_1p and b^*p so the coordinates of $k(k_1\, k_2)^{-1}$ depends C^∞ on b_1, b_2, k_1, k_2. Hence since K is a Lie group k depends C^∞ on b_1, b_2, k_1, k_2. This proves the claim and hence that $I(M)$ is a Lie group.

Notice that since $\pi : B \to N$ is a diffeomorphism we have also proved that the action $((bk), q) \mapsto bk \cdot q$ is C^∞ as long as $bk \cdot q \in N$. It is easy from this to prove the remaining statements of theorem 10.11.

Chapter 11 SYMMETRIC SPACES AND ORTHOGONAL INVOLUTIVE ALGEBRAS

In chapter 9 we associated to a Lie group its Lie algebra. Similarly we shall construct to a symmetric space an associated algebraic object called an *orthogonal involutive algebra.*

First a few remarks on the adjoint representation af a Lie Group G. Recall that for $g \in G$ the mapping $G \to G$ defined by $x \mapsto gxg^{-1}$ has differential $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ and the mapping $\mathrm{Ad} : G \to \mathrm{Gl}(\mathfrak{g})$ is called the *adjoint* representation of G. The differential of this is the map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ defined by

$$\mathrm{ad}(X)(Y) = [X, Y] \ , \quad X, Y \in \mathfrak{g}.$$

For an arbitrary Lie algebra $\mathfrak{g}$ we still have $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ defined as above and the image $\mathrm{ad}(\mathfrak{g}) \subseteq \mathrm{End}(\mathfrak{g})$ is a Lie subalgebra of $\mathrm{End}(\mathfrak{g})$ which is the Lie algebra of $\mathrm{Gl}(\mathfrak{g})$. Let $\mathrm{Int}(\mathfrak{g}) \subseteq \mathrm{Gl}(\mathfrak{g})$ be the connected Lie subgroup corresponding to $\mathrm{ad}(\mathfrak{g})$. If $\mathfrak{g}$ is the Lie algebra of a Lie group G then it follows from 9.47 that $\mathrm{Ad} : G \to \mathrm{Gl}(\mathfrak{g})$ is a Lie group homomorphism onto $\mathrm{Int}(\mathfrak{g})$. Notice that the topology on $\mathrm{Int}(\mathfrak{g})$ need not be induced from the topology on $\mathrm{Gl}(\mathfrak{g})$.

Now let $\mathfrak{k} \subseteq \mathfrak{g}$ be a Lie subalgebra. Then $\mathrm{ad}(\mathfrak{k}) \subseteq \mathrm{ad}(\mathfrak{g})$ is a Lie subalgebra and hence corresponds to a Lie subgroup $K^* \subseteq \mathrm{Int}(\mathfrak{g}) \subseteq \mathrm{Gl}(\mathfrak{g})$. We now have:

Proposition 11.1

The following are equivalent

1. K^* *is a compact Lie group.*

2. *The set* K^* *is compact in the induced topology from* $\mathrm{Gl}(\mathfrak{g})$.

3. K^* *is closed in* $\mathrm{Gl}(\mathfrak{g})$ *and* $\mathfrak{g}$ *has a* K^* *invariant positive definite symmetric bilinear form.*

4. K^* *is closed in* $\mathrm{Gl}(\mathfrak{g})$ *and* $\mathfrak{g}$ *has an* $\mathrm{ad}(\mathfrak{k})-$*invariant positive definite symmetric bilinear form* Q*, i.e.*

$$Q([X, Y], Z) + Q(Y, [X, Z]) = 0 \qquad \forall\, X \in \mathfrak{k},\ Y, Z \in \mathfrak{g}.$$

Proof

a) $\Rightarrow$ b) is trivial. On the other hand if $K^* \subseteq \mathrm{Gl}(\mathfrak{g})$ is compact then by proposition 7.6 it is a closed Lie subgroup in a canonical way which proves b) $\Rightarrow$ a).

b) $\Rightarrow$ c) by integrating an arbitrary positive definite symmetric bilinear form over K^* (cf. theorem 9.50).

c) $\Rightarrow$ b) is obvious since the orthogonal group is compact.

c) $\Rightarrow$ d) by differentiating the equation

$$Q(\mathrm{Exp}\,(t\,\mathrm{ad}X)Y\ ,\ \mathrm{Exp}\,(t\,\mathrm{ad}X)\ Z) = Q(Y,Z), \qquad X \in \mathfrak{k}\ ,\ Y, Z \in \mathfrak{g},$$

with respect to t.

d) $\Rightarrow a)$. Let $O(\mathfrak{g}) \subseteq \mathrm{Gl}(\mathfrak{g})$ be the orthogonal group corresponding to the positive definite bilinear form Q. It follows from d) that $\mathrm{ad}(\mathfrak{k})$ is contained in the Lie algebra of $O(\mathfrak{g})$ hence $K^* \subseteq O(\mathfrak{g})$ which is compact.

Definition 11.2

If either of a) - d) above holds we say that $\mathfrak{k}$ is *compactly embedded* in $\mathfrak{g}$.

If $\mathfrak{k}$ is compactly embedded in $\mathfrak{k}$ then $\mathfrak{k}$ is called a *compact* Lie-algebra. I.e. $\mathfrak{k}$ is compact iff $\mathrm{Int}(\mathfrak{k})$ is a compact group.

Notice that K compact clearly implies that the corresponding Lie algebra $\mathfrak{k}$ is compact. However, the other direction is in general not true (e.g. for Abelian Lie algebras).

Now return to M a simply connected symmetric space and let $p_0 \in M$ with symmetry s_0. Let $G_0 = I_0(M)$ be the connected component of the isometry group $I(M)$ of M. Furthermore let $K_0 = I_0(M) \cap K$ where K is the isotropy group for p_0. Finally let $\pi : G_0 \to M$ be the mapping $\pi(g) = gp_0$. With this notation we now have:

Theorem 11.3

1. *K_0 is connected and the mapping $\overline{\pi} : G_0/K_0 \to M$ induced by π is a diffeomorphism.*

2. *The mapping $\sigma : G \to G$ given by $g \mapsto s_0 \circ g \circ s_0$ is an involutive automorphism (that is $\sigma^2 = \mathrm{id}$) such that K_0 is the connected component of the group K_σ of fixed points for σ.*

3. *Let $\mathfrak{g}$ be the Lie algebra of G_0. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where*

$$\mathfrak{k} = \{X \in \mathfrak{g} |\ \sigma_* X = X\}\ , \quad \mathfrak{p} = \{X \in \mathfrak{g} |\ \sigma_* X = -X\}$$

and $\mathfrak{k}$ is the Lie algebra of K_0. Furthermore $\pi_*(\mathfrak{k}) = 0$ and $\pi_* : \mathfrak{p} \to T_{p_0}(M)$ is an isomorphism.

4. *If $X \in \mathfrak{p}$ $then$ $t \mapsto \exp(tX) \cdot p_0$ is the geodesic with tangent vector $\pi_* X$ and $\exp(tX)$ is the family of transvections along that curve.*

5. *For* $k \in K_0$ *we have* $\mathrm{Ad}(k)(\mathfrak{p}) \subseteq \mathfrak{p}$ *and*

$$\pi_* \mathrm{Ad}(k)(X) = k_* \pi_*(X) \qquad \forall\, X \in \mathfrak{p}.$$

Proof

a) For every $p \in M$ there is a transvection taking p_0 to p; hence G_0 acts transitively on M, so clearly $G_0/K_0 \;\cong\; M$. Now let us show that $K_0 = K \cap G_0$ is connected. Consider $\pi : G_0 \to M$ and notice that around any point of M we can find a neighbourhood U such that $\pi^{-1}(U)$ is homeomorphic to $U \times K_0$ (compare the proof of theorem 10.11. Using this it is easy to see that if $\gamma : [0,1] \to G$ is a curve and $\overline{\alpha}^s : [0,1] \to M\ ,\ s \in [0,1]$, is a homotopy of $\pi \circ \gamma$ keeping the endpoints fixed (that is, $\overline{\alpha}^0 = \pi \circ \gamma$ and $\overline{\alpha}^s(0) = \overline{\alpha}^0(0)\ ,\ \overline{\alpha}^s(1) = \overline{\alpha}^0(1) \qquad \forall\, s \in [0,1]$), then we can find a lifted homotopy (keeping the endpoints fixed) $\alpha^s : [0,1] \to M\ ,\ s \in [0,1]$, such that $\pi \circ \alpha^s = \overline{\alpha}^s$ and $\alpha^0 = \gamma$. (To show this subdivide $[0,1] \times [0,1]$ into small squares).

Now let $k \in K \cap G_0$ and join k by a curve $\gamma : [0,1] \to G_0$ to e. Since M is simply connected $\pi \circ \gamma$ is homotopic to the constant loop and if α^s is a lift of the homotopy as above, then clearly $\alpha^1 : [0,1] \to G_0$ is a curve lying inside K_0 and joining k to e. Hence K_0 is connected.

b) K_σ is clearly a closed subgroup of G with Lie algebra $\mathfrak{k} = \{X \in \mathfrak{g} |\ \sigma_* X = X\}$. It is therefore enough to prove that $\mathfrak{k}$ is actually the Lie algebra of K_0 or equivalently that for $X \in \mathfrak{k}\quad \exp(X) \in K$. But for $X \in \mathfrak{k}$ we have

$$\exp(tX)p_0 = \sigma(\exp(tX))p_0 = s_0((\exp tX)p_0) \qquad \forall\, t \in \mathbb{R},$$

so since p_0 is the only fixed point in a neighbourhood of p_0, it follows that $\exp(tK)p_0 = p_0 \qquad \forall\, t \in \mathbb{R}\ ,$ hence $\exp(tX) \in K \quad \forall\, t$.

c) Clearly $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and we have just proved that $\mathfrak{k}$ is the Lie algebra of K_0. Obviously $\pi_* \mathfrak{k} = 0$. Now let $v \in T_{p_0}(M)$ and let T_t be the family of transvections along $\exp_{p_0}(tv) = T_t(p_0)$. Now T_t is a one-parameter group so that $T_t = \exp(tX)$ for some $X \in \mathfrak{g}$. Then

$$\pi_*(X) = \frac{d}{dt}\, T_t(p_0)\ \Big|_{t=0} = \frac{d}{dt}\ \exp_{p_0}(tv)\ \Big|_{t=0} = v$$

so $\pi_* : \mathfrak{g} \to T_{p_0}(M)$ is onto. Hence by dimension reasons $\pi_* : \mathfrak{p} \to T_{p_0}(M)$ is an isomorphism.

d) Notice that X in the above argument lies in $\mathfrak{p}$. In fact T_{-t} is the one parameter group of transvections along the geodesic

$$\begin{aligned}\exp_{p_0}(-tv) &= s_0 \exp_{p_0}(tv) = (s_0\ \exp(tX) s_0)p_0 = \\ &= \sigma(\exp(tX))p_0 = \exp(t\,\sigma_*\,X)p_0,\end{aligned}$$

so that $\sigma_*(X) = -X$, that is, $X \in \mathfrak{p}$. Since $\pi_* : \mathfrak{p} \to T_{p_0}(M)$ is an isomorphism an arbitrary $X \in \mathfrak{p}$ occurs as the infinitesimal generator of the one parameter group of transvections along the geodesic $\exp_{p_0}(tv)$ where $v = \pi_* X$.

e) For $k \in K_0$ and $X \in \mathfrak{p}$ we have

$$\sigma\big(k(\exp\, tX)k^{-1}\big) = k(\exp\,(-tX))k^{-1} \qquad , \ \forall\, t \in \mathbb{R},$$

or equivalently

$$\exp\,(\sigma_* \mathrm{Ad}(k)(tX)) = \exp\,(\mathrm{Ad}(k)(-tX)) \qquad , \ \forall\, t \in \mathbb{R}.$$

Hence $\sigma_* \mathrm{Ad}(k)(X) = \ -\mathrm{Ad}(k)(X)$ so that $\mathrm{Ad}(k)(X) \in \mathfrak{p}$. Also

$$\begin{aligned}(\exp\, t\, \mathrm{Ad}(k)(X))\, p_0 \ &= \ \big(k\, \exp\, (tX)k^{-1}\big)p_0 = k\big(\exp_{p_0}(\pi_* tX)\big) \\ &= \ \exp_{p_0}(t\, k_* \pi_*(X))\ , \qquad\qquad \forall\, t \in \mathbb{R},\end{aligned}$$

so clearly

$$\pi_* \mathrm{Ad}(k)(X) = k_* \pi_*(X).$$

This ends the proof.

To the symmetric space M we can associate the Lie algebra $\mathfrak{g}$ of G_0 together with the involutive automorphism $\sigma_* : \mathfrak{g} \to \mathfrak{g}$. In general consider a Lie algebra $\mathfrak{g}$ with an involution $s : \mathfrak{g} \to \mathfrak{g}$. Then again $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where

$$\mathfrak{k} = \big\{X \in \mathfrak{g} \,\big|\, sX = X\big\}\ , \quad \mathfrak{p} = \big\{X \in \mathfrak{g} \,\big|\, sX \ = \ -X\big\},$$

and it is easy to see that

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}\ , \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}\ , \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}.$$

In the following whenever $\mathfrak{g}$ is a Lie algebra with involution s, we shall denote the $+1$ and -1 eigenspaces by $\mathfrak{k}$ and $\mathfrak{p}$ respectively.

Definition 11.4

An *orthogonal involutive Lie algebra* is a triple $(\mathfrak{g}\ ,\ s\ ,\ Q)$ where $\mathfrak{g}$ is a Lie algebra with involution s and Q is a positive definite symmetric bilinear form on $\mathfrak{p}$ which is $\mathrm{ad}(\mathfrak{k})-$invariant, that is,

$$Q([X, Y], Z)\ +\ Q(Y, [X, Z]) = 0 \qquad \forall\, X \in \mathfrak{k};\ Y, Z \in \mathfrak{p}.$$

Furthermore we require that $\mathfrak{k} \subseteq \mathfrak{g}$ is *compactly embedded* and that $\mathfrak{k}$ *does not* contain any non-zero ideal of $\mathfrak{g}$.

Remark

Since $\mathfrak{k}$ is compactly embedded we can extend Q to a positive definite symmetric form on $\mathfrak{g}$, which is s-invariant and ad($\mathfrak{k}$)-invariant. However, only $Q|\mathfrak{p}$ is part of the structure.

Proposition 11.5

For a simply connected symmetric space M let $\mathfrak{g}$ be the associated Lie algebra with involution σ_* as in theorem 11.3, and let Q be the bilinear form on $\mathfrak{p}$ induced from the Riemannian metric on $T_{p_0}(M)$ via the isomorphism $\pi_* : \mathfrak{p} \overset{\to}{\simeq} T_{p_0}(M)$.

Then $(\mathfrak{g},\ \sigma_*,\ Q)$ is an orthogonal involutive Lie algebra.

Proof

Most of this is already contained in theorem 11.3. Since K_0 is compact clearly $\mathrm{Ad}(K_0) \subseteq \mathrm{Gl}(\mathfrak{g})$ is compact so $\mathfrak{k}$ is compactly embedded. It remains to see that $\mathfrak{k}$ does not contain an ideal of $\mathfrak{g}$. So suppose $\mathfrak{a} \subseteq \mathfrak{k}$ is an ideal of $\mathfrak{g}$. Then $[\mathfrak{a}, \mathfrak{p}] \subseteq \mathfrak{a} \cap \mathfrak{p} = 0$ which implies $\mathfrak{a} = 0$. In fact suppose $Y \in \mathfrak{k}$ and $[Y, \mathfrak{p}] = 0$ then by theorem 11.3 e) we have

$$\begin{aligned}(\exp\ t\,Y)_*\,\pi_*(X) &= \pi_*\,\mathrm{Ad}\,(\exp\ t\,Y)(X)\\ &= \pi_*\,\mathrm{Exp}\,(\mathrm{ad}\,(t\,Y)\,(X)) = 0 \qquad \forall\, X \in \mathfrak{p}\ ,\ t \in \mathbb{R},\end{aligned}$$

so since K_0 acts effectively on $T_{p_0}(M)$ we have $\exp\ t\,Y = \mathrm{id} \quad \forall\, t \in \mathbb{R}$, hence $Y = 0$, which ends the proof.

We will now prove that the orthogonal involutive Lie algebra determines the symmetric space. In fact the curvature tensor field of M is determined by the Lie-product of $\mathfrak{g}$:

Theorem 11.6

Let M be a symmetric space as above and $(\mathfrak{g},\ s,\ Q)$ the associated orthogonal involutive Lie algebra. Then the curvature tensor field of M at the point p_0 is given by

$$R_{p_0}(\pi_*\,X,\ \pi_*\,Y)\,\pi_*\,Z = \ -\pi_*[[X, Y], Z]\ , \quad X, Y, Z \in \mathfrak{p}.$$

Proof

An element $X \in \mathfrak{g}$ determines a 1-parameter group of isometries of M namely $\{\exp\ t\,X\ ,\ t \in \mathbb{R}\}$ and hence a vector field X^* on M, that is,

$$X^*_p = \frac{d}{dt}\,((\exp\ tX)p)\,\Big|_{t=0}$$

Now let $\tilde{X}$ be the *right* invariant vector field on G_0 corresponding to X. Then $\tilde{X}$ and X^* are $\pi-$*related*, that is ,

$$\pi_*\left(\tilde{X}_g\right) = X^*_{\pi(g)} \qquad \forall\, g \in G.$$

In fact both sides are equal to $\frac{d}{dt}((\exp tX)gp_0)\,\big|_{t=0}$. It follows that for $X, Y \in \mathfrak{g}$ the vector fields $\left[\tilde{X}, \tilde{Y}\right]$ and $[X^*, Y^*]$ are also $\pi-$related. Now it is easy to see that the

Lie multiplication on $\mathfrak{g}$ defined by *right-invariant* vector fields and the one defined by *left*-invariant vector fields differ by a – sign. It follows that for $X, Y \in \mathfrak{g}$

$$\pi_*[X,Y] = -[X^*,Y^*]_{p_0}$$

so we can identify the Lie-algebra $\mathfrak{g}$ (except for a minus) with the Lie algebra of vector fields on M of the form X^* as above. For $X \in \mathfrak{g}$ we call X^* an *infinitesimal isometry.* Similarly for $X \in \mathfrak{g}(X \in \mathfrak{k})$, X^* is called an *infinitesimal transvection (rotation).*

In order to prove the theorem we must therefore prove

(*) $$(R(X^*,Y^*)Z^*)_{p_0} = -[[X^*,Y^*]\ ,\ Z^*]_{p_0} \qquad \forall\, X,Y,Z \in \mathfrak{p}.$$

Notice that the left hand side only depends on $X^*_{p_0}$, $Y^*_{p_0}$ and $Z^*_{p_0}$. Also $[X^*,Y^*]_{p_0} = 0$ since $[X,Y] \in \mathfrak{k}$, so the right hand side does not change if we replace Z^* with another vector field with the same value at p_0. Instead of (*) we shall therefore prove

(**) $$\big(R(X^*,Y^*)Z'\big)_{p_0} = -\big[[X^*,Y^*]\ ,\ Z'\big]_{p_0}$$

where $Z'_p = Z^*_{p_0}$ and Z' is parallel along the geodesics through p_0.

In the following we write $L_X(Y) = [X,Y]$ for X and Y vector fields on M.

By definition

$$R(X^*,Y^*)Z' = \nabla_{X^*}\nabla_{Y^*}Z' - \nabla_{Y^*}\nabla_{X^*}Z' - \nabla_{[X^*,Y^*]}Z'.$$

Since $[X^*,Y^*]_{p_0} = 0$ the last term vanishes at p_0. Notice also that $(\nabla_{Y^*}(Z'))_{p_0} = 0$ since Z' is parallel in any direction at p_0. Since the torsion is zero we therefore have

$$\nabla_{X^*}\big(\nabla_{Y^*}Z'\big)_{p_0} = \big(L_{X^*}\nabla_{Y^*}Z'\big)_{p_0}$$

Now any isometry g is an affine transformation, i.e. for arbitrary vector fields Y and Z on M with $g-$transformed vector fields Y^g and Z^g we have

$$\nabla_{Y^g}(Z^g) = (\nabla_Y(Z))^g.$$

Letting $g = \exp tX$, $X \in \mathfrak{g}$, $t \in \mathbb{R}$, and differentiating with respect to t, we therefore have

$$L_{X^*}\nabla_Y(Z) = \nabla_{[X^*,Y]}(Z) + \nabla_Y(L_{X^*}(Z))$$

for arbitrary vector fields Y and Z.

It follows that

$$\begin{aligned}\nabla_{X^*}\big(\nabla_{Y^*}Z'\big)_{p_0} &= \nabla_{[X^*,Y^*]}\big(Z'\big)_{p_0} + \big(\nabla_{Y^*}L_{X^*}\big(Z'\big)\big)_{p_0}\\ &= \big(\nabla_{Y^*}L_{X^*}\big(Z'\big)\big)_{p_0}\end{aligned}$$

since again $[X^*,Y^*]_{p_0} = 0$.

Finally $L_{X^*}(Z')_{p_0} = 0$ since Z' is parallel and $\exp(tX)$ preserves parallel translation. Hence again

$$\left(\nabla_{Y^*} L_{X^*}\left(Z'\right)\right)_{p_0} = \left(L_{Y^*} L_{X^*}\left(Z'\right)\right)_{p_0}$$

so we conclude that

$$\nabla_{X^*}\left(\nabla_{Y^*} Z'\right)_{p_0} = \left(L_{Y^*} L_{X^*}\left(Z'\right)\right)_{p_0}.$$

It follows that

$$\begin{aligned}\left(R(X^*, Y^*)Z'\right)_{p_0} &= \left(L_{Y^*} L_{X^*}\left(Z'\right)\right)_{p_0} - \left(L_{X^*} L_{Y^*}\left(Z'\right)\right)_{p_0} \\ &= \left(L_{[Y^*, X^*]}\left(Z'\right)\right)_{p_0} = -\left[[X^*, Y^*]\ ,\ Z'\right]_{p_0}\end{aligned}$$

by the Jacobi identity. This proves (**) and ends the proof of the theorem.

Corollary 11.7

Let M and N be simply connected symmetric spaces such that the associated orthogonal involutive Lie algebras are isomorphic. Then M and N are isometric.

Proof

Obvious from theorem 11.6 and theorem 8.20.

Example 11.8

Euclidean space $\mathbb{R}^n$ with the usual metric is a symmetric space. The group of isometries $E(n)$ is the *Euclidean group* generated by the orthogonal group $\mathrm{O}(n)$ and the translations $x \mapsto x + v$ for $v \in \mathbb{R}^n$. The Lie algebra of $\mathrm{O}(n)$ is denoted $\mathfrak{o}(n)$ and is the Lie subalgebra of $\mathrm{End}(\mathbb{R}^n)$ of skew-symmetric endomorphisms. The orthogonal involutive Lie-algebra associated to $\mathbb{R}^n$ is then $\left(\mathfrak{o}(n) \oplus \mathbb{R}^n\ ,\ s\ ,\ Q\right)$, where $[\ ,\ ] \bigm| \mathbb{R}^n = 0$ and $[\ ,\ ] : \mathfrak{o}(n) \times \mathbb{R}^n \to \mathbb{R}^n$ is given by the usual action of $\mathfrak{o}(n) \subseteq \mathrm{End}(\mathbb{R}^n)$ on $\mathbb{R}^n$. The involution s is given by $s|\mathfrak{o}(n) = \mathrm{id}$ and $s|\mathbf{R}^n = -\mathrm{id}$. Finally Q is the usual inner product on $\mathbb{R}^n$.

Notice that if $\mathfrak{k} \subseteq \mathfrak{o}(n)$ is a Lie subalgebra, then also $(\mathfrak{k} \oplus \mathbb{R}^n\ ,\ s\ ,\ Q)$ is an orthogonal involutive Lie algebra.

Definition 11.9

$(\mathfrak{g}_0,\ s_0,\ Q_0)$ is an orthogonal involutive Lie *subalgebra* of $(\mathfrak{g},\ s,\ Q)$ if $\mathfrak{g}_0 \subseteq \mathfrak{g}\ ,\ s \bigm| \mathfrak{g}_0 = s_0\ ,\quad \mathfrak{p}_0 = \mathfrak{p}$ and $Q_0 = Q$.

Definition 11.10

An orthogonal involutive Lie subalgebra of $(\mathfrak{o}(n) \oplus \mathbb{R}^n,\ s,\ Q)$ in the above example 11.8 is called a *Euclidean* orthogonal involutive Lie algebra.

Equivalently $(\mathfrak{g},\ s,\ Q)$ is Euclidean if $[\mathfrak{p}, \mathfrak{p}] = 0$.

Now let $(\mathfrak{g},\ s,\ Q)$ be any orthogonal involutive Lie algebra. We want to construct an associated symmetric space. For this we need the following proposition which will be proved in section 13 (see remark following theorem 13.5).

Proposition 11.11

Let $(\mathfrak{g},\ s,\ Q)$ be an orthogonal involutive Lie algebra. Then there is a Lie group $\tilde{G}$ with Lie algebra $\mathfrak{g}$.

Taking the universal covering we can assume $\tilde{G}$ to be simply connected. Then there is also an involutive automorphism σ of $\tilde{G}$ with differential $\sigma_* = s$ and clearly the fixed point set of σ is a closed subgroup. The identity component we denote by $\tilde{K}$. Clearly the Lie algebra of $\tilde{K}$ is $\mathfrak{k}$, the fixed points of the involution s. Now let $M = \tilde{G}/\tilde{K}$ and let $\pi : \tilde{G} \to M$ be the natural projection. Since $\tilde{K}$ is closed M is a manifold and

$$\pi_* : \mathfrak{p} \to T_o(M)\ , \qquad o = \left\{\tilde{K}\right\} \in M,$$

is an isomorphism. We give M a Riemannian metric as follows: $\tilde{G}$ clearly acts on M by left translations and $\tilde{K}$ fixes the point o. Also notice that the adjoint action of $\tilde{K}$ on the Lie algebra $\mathfrak{g}$ keeps the subspace $\mathfrak{p}$ invariant. In fact

$$\begin{aligned}\sigma_*(\mathrm{Ad}(\exp X)(Y)) = \mathrm{Ad}(\exp s_* x)(\sigma_* Y) = &-\mathrm{Ad}(\exp X)(Y)\\ &\text{for } X \in \mathfrak{k}\ ,\ Y \in \mathfrak{p}.\end{aligned}$$

Since the bilinear form Q on $\mathfrak{p}$ is $\mathrm{ad}(\mathfrak{k})-$invariant it is also $\mathrm{Ad}\left(\tilde{K}\right)-$ invariant. We now identify $T_o(M)$ with $\mathfrak{p}$ via the map π_*, and it is easy to see that under this identification the adjoint action of $\tilde{K}$ on $\mathfrak{p}$ corresponds to the induced action by $\tilde{K}$ on $T_o(M)$. We can therefore extend the bilinear form to a Riemannian metric on all of M by the formula

$$Q(X,Y) = Q\left(\left(L_{g^{-1}}\right)_* X,\ \left(L_{g^{-1}}\right)_* Y\right), \qquad X, Y \in T_{go}(M)$$

where L_g denotes left-multiplication by $g \in \tilde{G}$. By definition Q is $\tilde{G}-$invariant.

Theorem 11.12

The Riemannian manifold M constructed above is a simply connected symmetric space and the associated orthogonal involutive Lie algebra contains $(\mathfrak{g},\ s,\ Q)$ as an orthogonal involutive Lie subalgebra.

Proof

M is simply connected since $\tilde{G}$ is simply connected and $\tilde{K}$ is connected. In fact suppose $\gamma : [0,1] \to M$ is a curve with $\gamma(0) = \gamma(1) = o$. Using local cross sections of $\pi : \tilde{G} \to \tilde{G}/\tilde{K}$ one can lift γ to a curve $\tilde{\gamma} : [0,1] \to \tilde{G}$ such that $\tilde{\gamma}(0) = e\ ,\ \pi\,\tilde{\gamma}(1) = o\ ,$ that is, $\tilde{\gamma}(1) \in \tilde{K}$. Since $\tilde{K}$ is connected we can join $\tilde{\gamma}(1)$ to e by a curve inside $\tilde{K}$. Since γ is clearly homotopic to a curve which is constant near 1 we can therefore assume that $\tilde{\gamma}(1) = e$. Now since $\tilde{\gamma}$ is homotopic to the constant loop the same is true for $\gamma = \pi \circ \tilde{\gamma}$.

Now let us show that M is symmetric. In fact the symmetry s_0 at o is induced by $\sigma : \tilde{G} \to \tilde{G}$, that is,

$$s_0\Big(g\,\tilde{K}\Big) = \sigma(g)\,\tilde{K}\ , \qquad g \in \tilde{G}.$$

Clearly

$$s_0(gp) = L_{\sigma(g)} \circ s_0(p)\ , \qquad g \in \tilde{G}\ ,\ p \in M,$$

hence

$$s_0 \circ L_g = L_{\sigma(g)} \circ s_0\ , \qquad \forall\, g \in \tilde{G}.$$

It follows easily from this and the definition of the metric that $s_0 : M \to M$ is an isometry. To see that s_0 is the geodesic symmetry at o it is enough to observe that $s_{0*} = -\mathrm{id} : T_o(M) \to T_o(M)$ which is obvious since by definition $s_0\pi = \pi\sigma$ and since $\sigma_*\big|\mathfrak{p} = -\mathrm{id}$. The symmetry at $p = gK$ is clearly $s_p = L_g \circ s_0 \circ L_{g^{-1}}$. It follows that M is at least locally symmetric and it remains to see that M is *complete*. However, this follows simply because the symmetries s_p are globally defined. In fact suppose $\gamma : (a,b) \to M$ is a maximal geodesic and choose $t_0 \in (a,b)$ near b. Then with $p = \gamma(t_0)$ the geodesic

$$s_p\left(\gamma(2t_0 - t)\right)\ , \qquad 2t_0 - b < t < 2t_0 - a,$$

agrees with the geodesic γ on the interval $|t - t_0| < b - t_0$. Therefore γ can be prolonged to the interval $(a,\, 2t_0 - a)$ which is larger than (a,b) for $t_0 > \frac{a+b}{2}$ contrary to the maximality. This proves that M is complete, and hence is a globally symmetric space.

Now let G_0 as before denote the connected component of the isometry group of M. The involution σ_0 of G_0 is given by

$$\sigma_0(h) = s_0 \circ h \circ s_0\ , \qquad h \in G_0.$$

Let $\mathfrak{g}_0$ be the corresponding Lie algebra with involution σ_0*. clearly there is a natural map $L : \tilde{G} \to G_0$ sending g to L_g, and as shown above

$$L_{\sigma(g)} = s_0 \circ L_g \circ s_0 = \sigma_0(L_g)\ , \qquad g \in G$$

so L and hence $L_* : \mathfrak{g} \to \mathfrak{g}_0$ preserve the involutions. Also if $\pi_0 : G_0 \to M$ is evaluation at o clearly $\pi = \pi_0 \circ L$ and since $\pi_*\big|\mathfrak{p}$ and $\pi_{0*}\big|\mathfrak{p}_0$ are both isomorphism onto $T_o(M)$ it follows that $L_* : \mathfrak{p} \to \mathfrak{p}_0$ is an isomorphism.

Now let $\mathfrak{a} \subseteq \mathfrak{g}$ be the kernel of L_*. Then $\mathfrak{a}$ is an $s-$invariant ideal of $\mathfrak{g}$, hence $\mathfrak{a} = \mathfrak{a} \cap \mathfrak{k} \oplus \mathfrak{a} \cap \mathfrak{p}$. But since $L_* : \mathfrak{p} \to \mathfrak{p}_0$ is an isomorphism $\mathfrak{a} \cap \mathfrak{p} = 0$ so $\mathfrak{a} \subseteq \mathfrak{k}$ and hence $\mathfrak{a} = 0$ by assumption. Hence L_* is injective, which proves the theorem.

Remark 1

It follows that the kernel $Z \subseteq \tilde{G}$ of L is a discrete invariant subgroup of $\tilde{G}$ contained in $\tilde{K}$, so if we put $G = \tilde{G}/Z$, $K = \tilde{K}/Z$, then $M = G/K$ and here the map $L : G \to I_0(M)$ is injective.

Remark 2

In general we cannot expect L to be onto. In fact any Euclidean orthogonal involutive Lie algebra gives $M = \mathbb{R}^n$ as the associated symmetric space and here $I_0(M)$ is the whole Euclidean group.

Definition 11.13.

An orthogonal involutive Lie algebra $(\mathfrak{g},\ s,\ Q)$ is called *maximal* if it is *not* a subalgebra (in the sense of definition 11.9) of any bigger one.

Corollary 11.14

There is a 1–1 *correspondence between simply connected symmetric spaces and maximal orthogonal involutive Lie algebras.*

Proof

Suppose $(\mathfrak{g},\ s,\ Q)$ is maximal then by definition $L_* : \mathfrak{g} \to \mathfrak{g}_0$ in the proof of theorem 11.11 is onto.

Now suppose M is a simply connected symmetric space. We must show that the associated orthogonal involutive Lie algebra $(\mathfrak{g},\ s,\ Q)$ is maximal. But suppose $(\mathfrak{g}',\ s',\ Q')$ is any bigger one and let M' be the associated symmetric space. Then since $\mathfrak{p} = \mathfrak{p}'$ the spaces M and M' are isometric by theorem 11.6 and theorem 8.20 and hence by theorem 11.12, $\mathfrak{g}' \subseteq \mathfrak{g}$.

Chapter 12 SEMI-SIMPLE LIE ALGEBRAS AND LIE GROUPS

For the study of orthogonal involutive Lie algebras we need a few facts from the theory of *semi-simple* Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra. The *Killing form* B on $\mathfrak{g}$ is the symmetric bilinear form defined by

$$B(X,Y) = \operatorname{trace}\left((\operatorname{ad} X) \circ (\operatorname{ad} Y)\right) \qquad X, Y \in \mathfrak{g}.$$

Clearly B is invariant under automorphism of $\mathfrak{g}$, that is, if $\sigma : \mathfrak{g} \to \mathfrak{g}$ is an automorphism then $B(\sigma X, \sigma Y) = B(X,Y)$, $X, Y \in \mathfrak{g}$. Also B is $\operatorname{ad}(\mathfrak{g})-$invariant, that is,

$$B([Z,X],Y) + B(X,[Z,Y]) = 0 \ , \qquad X, Y, Z \in \mathfrak{g}.$$

Definition 12.1

$\mathfrak{g}$ is called *semi-simple* if B is *non-degenerate*, i.e., if

$$B(X,Z) = 0 \qquad \forall\, X \in \mathfrak{g} \Rightarrow Z = 0.$$

$\mathfrak{g}$ is called *simple* if it is semi-simple and has no proper ideals.

A Lie group is called semi-simple (simple) if its Lie algebra is semi-simple (simple).

Remark

If $\mathfrak{a} \subseteq \mathfrak{g}$ is an *ideal* then it is easy to see that the Killing form of $\mathfrak{a}$ is the restriction of the Killing form of $\mathfrak{g}$.

Lemma 12.2

If $\mathfrak{g}$ is semi-simple then it has no non-zero Abelian ideal.

Proof

Suppose $\mathfrak{a} \subseteq \mathfrak{g}$ is an Abelian ideal, that is, $[\mathfrak{a}, \mathfrak{a}] = 0$. Then for $Z \in \mathfrak{a}$ and $X \in \mathfrak{g}$,

$$\operatorname{ad}(Z) \operatorname{ad}(X)(\mathfrak{a}) = 0 \ , \qquad \operatorname{ad}(Z) \operatorname{ad}(X)(\mathfrak{g}) \subseteq \mathfrak{a};$$

hence $B(Z,X) = \operatorname{trace}\left(\operatorname{ad}(Z) \operatorname{ad}(X)\right) = 0$. So since $\mathfrak{g}$ is semi-simple $\mathfrak{a} = 0$.

Lemma 12.3

Let $\mathfrak{g}$ be semi-simple and $\mathfrak{a} \subseteq \mathfrak{g}$ an ideal. Let

$$\mathfrak{a}^{\perp} = \left\{ X \in \mathfrak{g} \;\middle|\; B(X,Z) = 0 \quad \forall\, Z \in \mathfrak{a} \right\}.$$

Then $\mathfrak{a}^\perp$ is an ideal and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ as Lie algebras. Furthermore $\mathfrak{a}$ and $\mathfrak{a}^\perp$ are semi-simple.

Proof

Since

$$B([Y,X],Z) = -B(X,[Y,Z]) = 0\ , \qquad \begin{array}{r} X \in \mathfrak{a}^\perp,\ Y \in \mathfrak{g} \\ Z \in \mathfrak{a}, \end{array}$$

$\mathfrak{a}^\perp$ is an ideal. Hence also $\mathfrak{a} \cap \mathfrak{a}^\perp$ is an ideal. Also

$$B([X,Y],Z) = -B(Y,[X,Z]) = 0\ , \qquad Z \in \mathfrak{g}\ ,\ Y \in \mathfrak{a}^\perp\ ,\ X \in \mathfrak{a},$$

so $\mathfrak{a} \cap \mathfrak{a}^\perp$ is Abelian; hence $\mathfrak{a} \cap \mathfrak{a}^\perp = 0$ by lemma 12.2. This proves $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ as Lie algebras. It follows that $B\big|\mathfrak{a}$ (or $B\big|\mathfrak{a}^\perp$) is non-degenerate, hence by the remark above $\mathfrak{a}$ is semi-simple.

Corollary 12.4

If $\mathfrak{g}$ is semi-simple then

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \ldots \oplus \mathfrak{g}_r$$

as Lie algebras, where $\mathfrak{g}_0, \ldots, \mathfrak{g}_r$ are all the simple ideals of $\mathfrak{g}$.

Proof

Clearly by lemma 12.3 we can decompose $\mathfrak{g}$ into a direct sum of simple ideals. Now suppose $\mathfrak{a} \subseteq \mathfrak{g}$ is a simple ideal different from $\mathfrak{g}_i\ ,\ i = 0\ ,\ \ldots r$. Then $\mathfrak{a} \cap \mathfrak{g}_i = 0 \quad \forall\, i\,;$ but then $[\mathfrak{a},\mathfrak{a}] \subseteq [\mathfrak{g},\mathfrak{a}] = 0$ so $\mathfrak{a} = 0$.

For a Lie algebra $\mathfrak{g}$ the *center* $\mathfrak{z}$ is defined as

$$\mathfrak{z} = \big\{X \in \mathfrak{g} \,\big|\, [X,Y] = 0 \quad \forall\, Y \in \mathfrak{g}\big\}.$$

We then obviously have

Corollary 12.5

Let $\mathfrak{g}$ be semi-simple. Then

a) *The center of $\mathfrak{g}$ is zero.*

b) $\mathfrak{g} \cong \mathrm{ad}(\mathfrak{g})$. *In particular $\mathfrak{g}$ is isomorphic to the Lie algebra of the Lie group* $\mathrm{Int}\,(\mathfrak{g}) \subseteq \mathrm{Gl}(\mathfrak{g})$.

Recall that in general $\mathrm{Int}\,(\mathfrak{g})$ *does not* have the topology induced from $\mathrm{Gl}\,(\mathfrak{g})$. However, for $\mathfrak{g}$ semi-simple $\mathrm{Int}\,(\mathfrak{g})$ is a closed subgroup as we shall now see:

Let $\mathrm{Aut}\,(\mathfrak{g}) \subseteq \mathrm{Gl}(\mathfrak{g})$ be the group of automorphisms of $\mathfrak{g}$. This is clearly a closed subgroup, and so is a Lie group. The Lie algebra is easily seen to be the set of *derivations*, $\mathrm{Der}\,(\mathfrak{g}) \subseteq \mathrm{End}\,(\mathfrak{g})$. That is, D is a derivation if

$$D[X,Y] = [D\,X,Y] + [X,\ DY]\ , \quad X,Y \in \mathfrak{g}.$$

By the Jacobi identity $\mathrm{ad}\,(\mathfrak{g}) \subseteq \mathrm{Der}\,(\mathfrak{g})$.

Proposition 12.6

If $\mathfrak{g}$ is semi-simple then $\mathrm{ad}\,(\mathfrak{g}) = \mathrm{Der}\,(\mathfrak{g})$. In particular $\mathrm{Int}\,(\mathfrak{g})$ is the identity component of $\mathrm{Aut}\,(\mathfrak{g})$.

Proof

The second statement clearly follows from the first. Now

$$[D,\ \mathrm{ad}\,X] = D \circ \mathrm{ad}\,(X) - \mathrm{ad}\,(X) \circ D = \mathrm{ad}\,(DX),$$

hence $\mathfrak{a} = \mathrm{ad}\,(\mathfrak{g})$ is an ideal of $\mathrm{Der}\,(\mathfrak{g})$. Since $\mathfrak{a} \cong \mathfrak{g}$ the Killing form is non-degenerate on $\mathfrak{a}$. Let $\mathfrak{a}^{\perp} \subseteq \mathrm{Der}\,(\mathfrak{g})$ be the ideal of $\mathrm{Der}\,(\mathfrak{g})$ "orthogonal" to $\mathfrak{a}$ under B. Then clearly $\mathfrak{a} \cap \mathfrak{a}^{\perp} = 0$. It follows that

$$[D,\ \mathrm{ad}\,X] = \mathrm{ad}\,(DX) = 0 \qquad \forall\, X \in \mathfrak{g}\ ,\ D \in \mathfrak{a}^{\perp}.$$

Hence $DX = 0 \quad \forall\, X \in \mathfrak{g}$, so $\mathfrak{a}^{\perp} = 0$ which proves the proposition.

Proposition 12.7

Let $\mathfrak{g}$ be a Lie algebra with center $\mathfrak{z}$ and let $\mathfrak{k} \subseteq \mathfrak{g}$ be a subalgebra such that $\mathfrak{z} \cap \mathfrak{k} = 0$. If $\mathfrak{k}$ is compactly embedded then $B\big|\mathfrak{k}$ is negative definite.

In particular a semi-simple Lie algebra is compact iff B is negative definite.

Proof

By assumption we have a positive definite symmetric bilinear form Q on $\mathfrak{g}$ invariant under $\mathrm{ad}(\mathfrak{k})$. For $T \in \mathfrak{k}$, $\mathrm{ad}\,(T)$ is therefore given by a skew-symmetric matrix with respect to an orthogonal basis for Q. Hence $\mathrm{ad}\,T$ has imaginary eigenvalues $i\lambda_1, \ldots, i\lambda_k$ and it follows that

$$B\,(T,T) = \mathrm{trace}\,((\mathrm{ad}\,T) \circ (\mathrm{ad}\,T)) = -\Sigma_i\, \lambda_i^2 < 0$$

unless $\mathrm{ad}\,T = 0$, that is, unless $T \in \mathfrak{k} \cap \mathfrak{z} = 0$.

Theorem 12.8

Let G be a semi-simple Lie group with Lie algebra $\mathfrak{g}$. Then G is compact iff $\mathfrak{g}$ is compact.

Sketch proof

As remarked before $\Rightarrow$ is trivial.

$\Leftarrow$. It is clearly enough to consider G simply connected. Then the adjoint homomorphism

$$\mathrm{Ad} : G \to \mathrm{Int}\,(\mathfrak{g})$$

is the universal covering and the kernel is the center Z of G. So we must prove that if G is semi-simple and G/Z is compact then Z is *finite*. We can clearly give G/Z a bi-invariant metric, and it follows that the induced metric is bi-invariant, that G/Z and hence G is complete and that the geodesics of G through e are exactly the one-parameter subgroups. Now suppose Z is infinite and choose $z_n \in Z$ such that

$$d(z_n, e) \to \infty \,, \qquad n \to \infty.$$

Let $X_n \in \mathfrak{g}$ such that $\|X_n\| = 1$ and

$$\exp t_n X_n = z_n \qquad \text{for some } t_n \in \mathbb{R}.$$

By going to a subsequence we can assume that $\{X_n\}$ converges, i.e. let $X = \lim\limits_{n\to\infty} X_n$. One can then prove that $\mathrm{span}\,\{X\} \subseteq \mathfrak{g}$ is in the center of $\mathfrak{g}$ contradicting the semi-simplicity of $\mathfrak{g}$. We refer to Helgason [chapter II, § 6] for details.

Until now we have studied simple and semi-simple Lie algebras over $\mathbb{R}$. These notions, of course, also make sense over $\mathbb{C}$. Now let $\mathfrak{g}$ be a real Lie algebra and let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification, i.e. as a real vector space $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ and

$$[iX, iY] = -[X, Y] \,,\ [iX, Y] = [X, iY] = i[X, Y] \,, \qquad X, Y \in \mathfrak{g}.$$

Proposition 12.9

Let $\mathfrak{g}$ be a real Lie algebra.

a) *$\mathfrak{g}$ is semi-simple iff $\mathfrak{g}_{\mathbb{C}}$ is semi-simple.*

b) *Suppose $\mathfrak{g}$ is simple. Then $\mathfrak{g}_{\mathbb{C}}$ is not simple iff $\mathfrak{g}$ is the underlying real Lie algebra of a complex Lie algebra.*

Proof

a) is straightforward.

b) First assume that $\mathfrak{g}_{\mathbb{C}}$ is not simple and write

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_0 \oplus \ldots \oplus \mathfrak{g}_r \,, \qquad r \geq 1,$$

as in corollary 12.4, where $\mathfrak{g}_0, \ldots, \mathfrak{g}_r$ are complex simple ideals in $\mathfrak{g}_{\mathbb{C}}$. For each $i = 0, 1, \ldots, r$, we consider the homomorphism of real Lie algebras $\phi_i : \mathfrak{g} \to \mathfrak{g}_i$ defined as the composition of the inclusion $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ and the projection onto the i-th factor. Clearly ϕ_i is non-zero and since $\mathfrak{g}$ is simple ϕ_i is injective. In particular, for each i,

$$\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} = \dim_{\mathbb{R}} \mathfrak{g} \leq \dim_{\mathbb{R}} \mathfrak{g}_i = 2 \dim_{\mathbb{C}} \mathfrak{g}_i .$$

It follows that $r = 1$ and $\dim_{\mathbb{C}} \mathfrak{g}_i = \frac{1}{2} \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}$. Hence $\dim_{\mathbb{R}} \mathfrak{g}_0 = \dim_{\mathbb{R}} \mathfrak{g}$ and $\phi_0 : \mathfrak{g} \to \mathfrak{g}_0$ is an isomorphism over the reals. Thus $\mathfrak{g}$ is isomorphic to the real Lie algebra underlying $\mathfrak{g}_0$.

On the other hand let $\mathfrak{h}$ be a complex Lie algebra and let $\mathfrak{g}$ be the underlying real Lie algebra. Also let $j : \mathfrak{g} \to \mathfrak{g}$ be the $\mathbb{R}$-linear map given by multiplication by $\sqrt{-1}$ on $\mathfrak{h}$.

Then the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ has a non-trivial ideal consisting of all elements of the form $v + ijv$, $v \in \mathfrak{h}$, so indeed $\mathfrak{g}_{\mathbb{C}}$ is not simple.

Definition 12.10

Let $\mathfrak{g}$ be a *complex* Lie algebra. A *real form* of $\mathfrak{g}$ is a subalgebra $\mathfrak{g}_0$ of the underlying real Lie algebra of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$.

Equivalently a real form $\mathfrak{g}_0$ is the set of fixed points for an *anti-linear* involution σ of the underlying real Lie algebra of $\mathfrak{g}$. Here $\sigma : \mathfrak{g} \to \mathfrak{g}$ is *anti-linear* if

$$\sigma(\lambda X) = \overline{\lambda}\, \sigma(X) \ , \qquad X \in \mathfrak{g} \ , \ \lambda \in \mathbb{C},$$

and $\overline{\lambda}$ is the complex conjugate of λ. σ is called the *conjugation* of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. We state without proof (see Helgason [chapter 3 § 6]) the following:

Theorem 12.11

Every complex semi-simple Lie algebra has a compact real form.

We prove that the compact real form is unique in some sense. More generally we have:

Proposition 12.12

Let $\mathfrak{g}_0$ be a semi-simple real Lie algebra and let $\mathfrak{g}$ be the complexification. Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$ and let σ and τ be the conjugations of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ and $\mathfrak{u}$ respectively. Then there is $\phi \in \mathrm{Int}\,(\mathfrak{g})$ such that the compact real form $\phi\mathfrak{u}$ is invariant under σ.

Proof

Notice that the conjugation with respect to $\phi\mathfrak{u}$ is $\tau_1 = \phi\tau\phi^{-1}$ and that $\phi\mathfrak{u}$ is $\sigma-$invariant iff $\sigma\tau_1 = \tau_1\sigma$.

Since $\mathfrak{u}$ is compact we can define a Hermitian inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathfrak{g}$ by

$$\langle X, Y\rangle = -B(X, \tau Y)\,, \qquad X, Y \in \mathfrak{g}.$$

Now let $N = \sigma\tau$ and it is easy to see that

$$\langle X, NY\rangle = \langle NX,\, Y\rangle\,, \qquad X, Y \in \mathfrak{g}.$$

Observe also that $\tau N \tau = N^{-1}$. Clearly the self-adjoint operator $P = N^2$ has positive eigenvalues so that the one-parameter group of automorphisms

$$P^t = \operatorname{Exp} t \log P\,, \qquad t \in \mathbb{R},$$

is well-defined. It follows that $P^t \in \operatorname{Int}(\mathfrak{g})$. Again $\tau P^t \tau = P^{-t}$.

We want to use $P^{\frac{1}{4}}$ as ϕ. So let $\tau_1 = P^t \tau P^{-t}$. Then

$$\begin{aligned}\sigma\tau_1 &= \sigma P^t \tau P^{-t} = \sigma\tau P^{-2t} = NP^{-2t}\\ \tau_1\sigma &= P^t\tau P^{-t}\sigma = P^{2t}N^{-1} = N^{-1}P^{2t} = NP^{2t-1}\end{aligned}$$

Hence for $t = \frac{1}{4}$, $\sigma\tau_1 = \tau_1\sigma$ and the proposition is proved.

Corollary 12.13

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. If $\mathfrak{u}_0$ and $\mathfrak{u}_1$ are compact real forms then there is an automorphism $\phi \in \operatorname{Int}(\mathfrak{g})$ such that $\phi\mathfrak{u}_0 = \mathfrak{u}_1$.

Proof

Let τ_0 and τ_1 be the conjugations with respect to $\mathfrak{u}_0$ and $\mathfrak{u}_1$ respectively. By the proposition we can assume that $\mathfrak{u}_1$ is invariant under τ_0 and so

$$\mathfrak{u}_1 = \mathfrak{u}_0 \cap \mathfrak{u}_1 \oplus (i\mathfrak{u}_0) \cap \mathfrak{u}_1.$$

But since B is negative definite on $\mathfrak{u}_1$ and positive definite on $i\mathfrak{u}_0$ we have $i\mathfrak{u}_0 \cap \mathfrak{u}_1 = 0$ so $\mathfrak{u}_0 = \mathfrak{u}_1$.

Remark

It follows from this and the theorem 12.8 and 12.11 that the classification of compact semi-simple simply connected Lie groups is equivalent to the classification of complex semi-simple Lie algebras.

We now turn to the study of *non-compact* semi-simple Lie algebras and show that this is closely related to the study of orthogonal involutive Lie algebras.

Let $\mathfrak{g} = \mathfrak{g}_{0\mathbb{C}}$ be a real semi-simple Lie algebra, let $\mathfrak{g} = \mathfrak{g}_0$ be the complexification and let τ be the conjugation with respect to $\mathfrak{g}_0$. Now let $\mathfrak{u}$ be a compact real form with corresponding conjugation σ of $\mathfrak{g}$. By proposition 12.12 we can assume that σ and τ commute or equivalently that $\mathfrak{g}_0$ is $\sigma-$invariant. Then

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0\,, \qquad \mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{u}\,, \qquad \mathfrak{p}_0 = \mathfrak{g}_0 \cap i\mathfrak{u}$$

is the eigenspace decomposition of $\mathfrak{g}_0$ with respect to σ. This is called a *Cartan decomposition of* $\mathfrak{g}_0$. Notice that $B|\mathfrak{p}_0$ is positive definite. It follows that the bilinear form Q defined on $\mathfrak{g}_0$ by

$$Q|\mathfrak{k}_0 = -B\,, \quad Q|\mathfrak{p}_0 = B\,, \quad Q(\mathfrak{k}_0, \mathfrak{p}_0) = 0$$

is positive definite and $\mathrm{ad}(\mathfrak{k}_0)-$invariant. Therefore $\mathfrak{k}_0$ is compactly embedded in $\mathfrak{g}_0$ if just $\mathrm{Int}(\mathfrak{k}_0)$ is closed in $\mathrm{Gl}(\mathfrak{g}_0)$. But this is obvious since $\mathrm{Int}(\mathfrak{k}_0)$ is the identity component of the subgroup of $\mathrm{Int}(\mathfrak{g}_0)$ fixed by the involution $g \mapsto \sigma \circ g \circ \sigma$.

Therefore the triple $(\mathfrak{g}_0,\ \sigma|\mathfrak{g}_0,\ B|\mathfrak{p}_0)$ has all the properties of an orthogonal involutive Lie algebra *except* that $\mathfrak{k}_0$ may contain a non-zero ideal of $\mathfrak{g}_0$. This may very well happen if $\mathfrak{g}_0$ in the decomposition of corollary 12.4 contains a compact ideal. We therefore restrict to *simple* Lie-algebras:

Corollary 12.14

If $\mathfrak{g}_0$ is a non-compact simple real Lie algebra then there is a corresponding orthogonal involutive Lie algebra $(\mathfrak{g}_0,\ \sigma,\ Q)$ such that $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{g}_0$. The bilinear form Q is the restriction to $\mathfrak{p}_0$ of the Killing form B.

Exercise 12.15

Show that a Cartan decomposition is unique up to isomorphism with an element $\phi \in \mathrm{Int}(\mathfrak{g}_0)$.

Example 12.16

Suppose $\mathfrak{g}$ is a complex semi-simple Lie algebra and let $\mathfrak{k} \subseteq \mathfrak{g}$ be a compact real form. Then it is easily seen that the underlying real Lie algebra of $\mathfrak{g}$ is also semi-simple and that $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ is a Cartan decomposition ($\mathfrak{p} = i\mathfrak{k}$). Notice also that if $\mathfrak{g}$ is simple then the underlying real algebra is also simple so this gives an example of corollary 12.14. (In fact suppose $\mathfrak{a} \subseteq \mathfrak{g}$ is a real ideal, then $\mathfrak{a} \cap i\mathfrak{a}$ and $\mathfrak{a} + i\mathfrak{a}$ are complex ideals so by simplicity $\mathfrak{g} = \mathfrak{a} \oplus i\mathfrak{a}$. But then $[i\mathfrak{a},\ \mathfrak{a}] \subseteq i\mathfrak{a} \cap \mathfrak{a} = 0$ so also $[\mathfrak{a}, \mathfrak{a}] = 0;$ hence $[\mathfrak{g}, \mathfrak{g}] = 0$ which contradicts the semi-simplicity of $\mathfrak{g}$.)

Exercise 12.17

Describe the Lie algebra and its Cartan decomposition for the following non-compact simple groups:

a) $\mathrm{Sl}(n, \mathbb{R})$: The group of $n \times n$ matrices of determinant one.

b) SO(p, q): The group of matrices of $\mathrm{Sl}(p+q, \mathbb{R})$ which leaves invariant the symmetric bilinear form:

$$-x_1 y_1 - x_2 y_2 - \ldots - x_p y_p + x_{p+1} y_{p+1} + \ldots + x_{p+q} y_{p+q}.$$

c) $\mathrm{Sp}(n, \mathbb{R})$: The group of matrices of $\mathrm{Gl}(2n, \mathbb{R})$ leaving invariant the alternating bilinear form

$$x_1 y_{n+1} - x_{n+1} y_1 + x_2 y_{n+2} - x_{n+2} y_2 + \ldots + x_n y_{2n} - x_{2n} y_n.$$

Chapter 13 THE STRUCTURE OF ORTHOGONAL INVOLUTIVE LIE ALGEBRAS

In this chapter we shall reduce the classification of orthogonal involutive Lie algebras (and equivalently of simply connected symmetric spaces) to that of compact simple Lie algebras and their involutions. In particular we shall establish a *duality* between symmetric spaces of *compact* and *non-compact type* generalizing the classical "duality" between spherical and hyberbolic geometry.

Recall that an orthogonal involutive Lie algebra is a triple $(\mathfrak{g},\ s,\ Q)$ where s is an involution of the Lie algebra $\mathfrak{g}$, and if $\mathfrak{k}$ and $\mathfrak{p}$ denote the $+1$ and -1 eigenspaces of s, then Q is a positive definite symmetric bilinear form on $\mathfrak{p}$. Furthermore Q is $\mathrm{ad}(\mathfrak{k})$-invariant, $\mathfrak{k}$ is compactly embedded in $\mathfrak{g}$ and finally no non-zero ideal of $\mathfrak{g}$ is contained in $\mathfrak{k}$. Also B denote the Killing form on $\mathfrak{g}$.

We need the following 3 lemmas:

Lemma 13.1

a. $B(\mathfrak{k},\mathfrak{p}) = 0.$

b. $B|\mathfrak{k}$ *is negative definite.*

c. *If* $\mathfrak{u},\ \mathfrak{w} \subseteq \mathfrak{p}$ *are* $\mathrm{ad}(\mathfrak{k})-$*invariant subspaces such that* $B(\mathfrak{u},\mathfrak{w}) = 0$ *then* $[\mathfrak{u},\mathfrak{w}] = 0.$

Proof

a) Clearly B is $s-$invariant so $B(X,Y) = -B(X,Y)$ for $X \in \mathfrak{k},\ Y \in \mathfrak{p}$.

b) follows from proposition 12.7 provided $\mathfrak{k}$ does not contain any element in the center of $\mathfrak{g}$. But suppose X is such an element then $\mathrm{span}\,\{X\}$ is a one-dimensional ideal of $\mathfrak{g}$ contained in $\mathfrak{k}$, which is a contradiction.

c) Let $X \in \mathfrak{u}$, $Y \in \mathfrak{w}$, $Z = [X,Y] \in \mathfrak{k}$. Then $W = [Y,Z] \in \mathfrak{w}$. Hence

$$B(Z,Z) = B([X,Y],Z) = B(X,W) = 0$$

and hence $Z = 0$ by b).

Lemma 13.2

Let $\mathfrak{z} = \{Z \in \mathfrak{g}|\ [Z,\mathfrak{p}] = 0\}$.

Then $\mathfrak{z}$ *is an Abelian ideal contained in* $\mathfrak{p}$. *Furthermore* $\mathfrak{g}$ *is semi-simple iff* $\mathfrak{z} = 0$.

Proof

$\mathfrak{z}$ is an ideal: In fact $[\mathfrak{p}, \mathfrak{z}] = 0$ by definition, and if $X \in \mathfrak{k}$ and $Z \in \mathfrak{z}$ then

$$[[X, Z], Y] = -[[Z, Y], X] - [[Y, X], Z] = 0, \text{ for } Y \in \mathfrak{p}$$

so $[X, Z] \in \mathfrak{z}$.

Also $\mathfrak{z} \cap \mathfrak{k}$ is an ideal so $\mathfrak{z} \cap \mathfrak{k} = 0$. On the other hand $\mathfrak{z}$ is $s-$invariant so $\mathfrak{z} = \mathfrak{z} \cap \mathfrak{p} \subseteq \mathfrak{p}$. Hence $\mathfrak{z}$ is Abelian.

It follows that if $\mathfrak{g}$ is semi-simple then $\mathfrak{z} = 0$ by lemma 12.2. On the other hand suppose $\mathfrak{g}$ is not semi-simple. Then

$$\mathfrak{n} = \{X \in \mathfrak{g} |\ B(X, Y) = 0 \ \ \forall\, Y \in \mathfrak{g}\}$$

is an $s-$invariant ideal and by lemma 13.1, b) $\mathfrak{n} \cap \mathfrak{k} = 0$ so $\mathfrak{n} \subseteq \mathfrak{p}$. Hence by lemma 13.1, c), $[\mathfrak{n}, \mathfrak{p}] = 0$ so $\mathfrak{n} \subseteq \mathfrak{z}$. Therefore $\mathfrak{n}$ non-zero gives $\mathfrak{z} \neq 0$.

Lemma 13.3

Suppose $\mathfrak{g}$ is semi-simple. Then $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$ and $(\mathfrak{g},\ s,\ Q)$ is a maximal orthogonal involutive Lie algebra.

Proof

Notice that if $(\mathfrak{g},\ s,\ Q) \subseteq \left(\tilde{\mathfrak{g}},\ \tilde{s},\ \tilde{Q}\right)$ and $\mathfrak{g}$ is semi-simple then also $\tilde{\mathfrak{g}}$ is semi-simple by lemma 13.2 because $\tilde{\mathfrak{p}} = \mathfrak{p}$. It is therefore enough to prove $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$, since then also $\tilde{\mathfrak{k}} = \left[\tilde{\mathfrak{p}}, \tilde{\mathfrak{p}}\right] = [\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$.

So let $\mathfrak{h} = [\mathfrak{p}, \mathfrak{p}] + \mathfrak{p} \subseteq \mathfrak{g}$. By the Jacobi identity $[\mathfrak{k}, \mathfrak{h}] \subseteq \mathfrak{h}$ and clearly $[\mathfrak{p}, \mathfrak{h}] \subseteq \mathfrak{h}$ so $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Let $\mathfrak{h}^{\perp}$ be the orthogonal complement with respect to B. So $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ and $\left[\mathfrak{h}, \mathfrak{h}^{\perp}\right] = 0$. In particular $\mathfrak{h}^{\perp} \subseteq \mathfrak{z} = 0$. Hence $\mathfrak{h} = \mathfrak{g}$.

Definition 13.4

$(\mathfrak{g},\ s,\ Q)$ is called *irreducible* if it is not Euclidean and if $\mathfrak{p}$ does not contain a proper $\mathrm{ad}(\mathfrak{k})-$invariant subspace.

We can now prove:

Theorem 13.5

Let $(\mathfrak{g},\ s,\ Q)$ be an orthogonal involutive Lie algebra. Then there is a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_t$$

into a direct sum of $s-$invariant ideals, such that there are corresponding decompositions

$$\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_t$$

and

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_t \qquad (Q(\mathfrak{p}_i, \mathfrak{p}_j) = 0,\ i \neq j)$$

of $\mathfrak{k}$ and $\mathfrak{p}$, and such that

1. $(\mathfrak{g}_0,\ s|\mathfrak{g}_0,\ Q|\mathfrak{p}_0)$ *is Euclidean,*

2. $(\mathfrak{g}_i,\ s|\mathfrak{g}_i,\ Q|\mathfrak{p}_i)\ ,\ i \geq 1,$ *are irreducible and* $\mathfrak{g}_i,\ i \geq 1,$ *are semi-simple.*

Furthermore the decomposition is unique up to a permutation of the irreducible factors.

Proof

Define $\alpha : \mathfrak{p} \to \mathfrak{p}$ by

$$B(X, Y) = Q(\alpha X, Y)\ , \qquad X, Y \in \mathfrak{p}.$$

Then clearly α has real eigenvalues $\lambda_0 = 0, \lambda_1, \ldots, \lambda_r,\ r \geq 0$. Let $\mathfrak{p}_0$ be the eigenspace belonging to $\lambda_0 = 0$ and let $\mathfrak{p}_i'$ be the eigenspaces belonging to $\lambda_i \neq 0, \quad i > 0$. Clearly $\mathfrak{p}_0$ and $\mathfrak{p}_i'\ ,\ i = 1, \ldots, r,$ are mutually orthogonal (with respect to Q). Each $\mathfrak{p}_i'$ we split into an orthogonal sum of irreducible $\mathrm{ad}(\mathfrak{k})-$invariant subspaces. So let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t\ ,\ t \geq r\ ,$ be all these $\mathrm{ad}(\mathfrak{k})-$invariant subspaces and let $\lambda_1, \ldots, \lambda_t$ be the corresponding eigenvalues for α. By construction

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_t$$

is an orthogonal direct sum with respect to Q and since $B|\mathfrak{p}_i'$ is a multiple of Q we have

$$B(\mathfrak{p}_i\ ,\ \mathfrak{p}_j) = 0, \qquad i \neq j, \quad i, j \geq 0.$$

It follows by lemma 13.1, c) that

$$[\mathfrak{p}_i\ ,\ \mathfrak{p}_j] = 0 \qquad i \neq j, \quad i, j \geq 0.$$

Now define

$$\mathfrak{g}_i = [\mathfrak{p}_i\ ,\ \mathfrak{p}_i] + \mathfrak{p}_i\ , \qquad i \geq 1.$$

By construction and by the Jacobi identity $\mathfrak{g}_i$ are $\mathrm{ad}(\mathfrak{k})-$invariant. Similarly

$$[\mathfrak{g}_i\ ,\ \mathfrak{p}_j] = 0\ , \qquad i \neq j\ , \quad i \geq 1\ ,\ j \geq 0.$$

Hence $\mathfrak{g}_i,\ \ i = 1, \ldots, t,$ are ideals and

$$[\mathfrak{g}_i\ ,\ \mathfrak{g}_j] = 0 \qquad i \neq j\ ,\ i, j \geq 1.$$

For $i \geq 1\ ,\ B|\mathfrak{p}_i = \lambda_i\, Q$ with $\lambda_i \neq 0$ and $B|[\mathfrak{p}_i\ ,\ \mathfrak{p}_i]$ is negative definite. Hence $B|\mathfrak{g}_i\ ,\ i \geq 1,$ is non-degenerate, i.e., $\mathfrak{g}_i$ is semi-simple. Let

$$\mathfrak{g}' = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_t.$$

Then clearly $\mathfrak{g}'$ is a semi-simple ideal so if $\mathfrak{g}_0 = \{X \in \mathfrak{g}|\ B(X, \mathfrak{g}') = 0\}$ then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}'$.

Again $\mathfrak{g}_0$ is an $s-$invariant ideal and $\mathfrak{p}\cap\mathfrak{p}_0 = \mathfrak{p}_0$ by definition. Also let $\mathfrak{k}_0 = \mathfrak{k}\cap\mathfrak{g}_0$. Then $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ and since $B(\mathfrak{p}_0\ ,\ \mathfrak{p}) = 0$ we conclude from lemma 13.1, c) that $[\mathfrak{p}_0\ ,\ \mathfrak{p}] = 0$ so in particular $(\mathfrak{g}_0\ , s|\mathfrak{g}_0\ ,\ Q|\mathfrak{p}_0)$ is Euclidean. This proves the existence of a decomposition.

To prove the uniqueness first notice that $\mathfrak{p}_0\ ,\ \mathfrak{p}_1, \ldots, \mathfrak{p}_t$ must necessarily be eigenspaces for the operator α, so in particular the decomposition

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}'\ , \qquad \text{where } \mathfrak{p}' = \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_t\ ,$$

is unique. It follows that the decomposition of $(\mathfrak{g},\ s,\ Q)$ into a Euclidean and a semi-simple part is unique. We can therefore assume $\mathfrak{g}$ semi-simple. Notice also that $(\mathfrak{g},\ s,\ Q)$ is irreducible iff there is no decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ into $s-$invariant ideals. It follows using corollary 12.4 that if $(\mathfrak{g},\ s,\ Q)$ is irreducible and $\mathfrak{g}$ is semi-simple then either $\mathfrak{g}$ is simple, or $\mathfrak{g}\ \cong\ \mathfrak{h} \oplus \mathfrak{h}$, with $\mathfrak{h}$ a simple ideal of $\mathfrak{g}$ and s interchanges the factors. Therefore if $\mathfrak{g}$ is semi-simple with involution s it follows easily from corollary 12.4 that $\mathfrak{g}$ has a unique decomposition into $s-$invariant ideals (up to a permutation) which proves the uniqueness of the decomposition in the theorem.

Remark

It follows from theorem 13.5 that if $(\mathfrak{g},\ s,\ Q)$ is an orthogonal involutive Lie algebra then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}'$ with $\mathfrak{g}'$ semi-simple and $\mathfrak{g}_0$ a subalgebra of the Lie algebra of the Euclidean group. Hence by corollary 12.5 $\mathfrak{g}$ is the Lie algebra of some Lie group. This proves proposition 11.11.

Using the 1–1 correspondence between orthogonal involutive Lie algebras and simply connected symmetric spaces we can also give a geometric formulation of theorem 13.5: We say that a simply connected symmetric space is *irreducible* if the corresponding orthogonal involutive Lie algebra is irreducible.

Theorem 13.6

A simply connected symmetric space M has a unique decomposition

$$M = M_0 \times M_1 \times \ldots \times M_t\ , \qquad t \geq 0,$$

such that M_0 is some Euclidean space and $M_i\ ,\ i > 0\ ,$ is irreducible.

In particular M is irreducible iff M is not Euclidean and does not factorize into a product of symmetric spaces.

We shall now investigate the irreducible orthogonal involutive Lie algebras. For this we introduce the notion of "duality":

Suppose $(\mathfrak{g},\ s,\ Q)$ is an orthogonal involutive Lie algebra. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$, and let s also denote the extension of the involution to $\mathfrak{g}_{\mathbb{C}}$. As usual $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and consider the real vector space

$$\mathfrak{g}^* = \mathfrak{k} \oplus i\,\mathfrak{p} \subseteq \mathfrak{g}_{\mathbb{C}}.$$

$\mathfrak{g}^*$ is clearly an $s-$invariant subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Define Q^* on $i\,\mathfrak{p}$ by

$$Q^*(X,Y) = Q(i\,X\ ,\ i\,Y)\ , \qquad X, Y \in \mathfrak{p},$$

and let $s^* = s|\mathfrak{p}^*$. Then it is straightforward to see that

$$(\mathfrak{g},\ s,\ Q)^* = (\mathfrak{g}^*,\ s^*,\ Q^*)$$

is again an orthogonal involutive Lie algebra. $(\mathfrak{g}^*,\ s^*,\ Q^*)$ is called the *dual* of $(\mathfrak{g},\ s,\ Q)$. Obviously

$$(\mathfrak{g},\ s,\ Q)^{**} = (\mathfrak{g},\ s,\ Q)$$

Lemma 13.7

1. *If*

$$(\mathfrak{g},\ s,\ Q) = \bigoplus_i\ (\mathfrak{g}_i,\ s_i,\ Q_i)$$

as in theorem 13.5*, then*

$$(\mathfrak{g}^*,\ s^*,\ Q^*) = \bigoplus_i\ (\mathfrak{g}_{\mathfrak{i}}^*,\ s_i^*,\ Q_i^*)$$

2. $(\mathfrak{g},\ s,\ Q)^* \ \cong\ (\mathfrak{g},\ s,\ Q)$ *iff* $(\mathfrak{g},\ s,\ Q)$ *is Euclidean.*

3. $(\mathfrak{g},\ s,\ Q)$ *is irreducible iff* $(\mathfrak{g}^*,\ s^*,\ Q^*)$ *is irreducible.*

4. *If* $(\mathfrak{g},\ s,\ Q)$ *is irreducible then precisely one of* $\mathfrak{g}$ *or* $\mathfrak{g}^*$ *is compact.*

Proof

1) and 3) are obvious.

4) By the proof of theorem 13.5, $B|\mathfrak{p} = \lambda\,Q\ ,\quad \lambda \neq 0$. Hence $B|i\,\mathfrak{p} = -\lambda\,Q\ ,$ so either $B|\mathfrak{p}$ or $B|i\,\mathfrak{p}$ is negative definite (and not both are). Now by lemma 13.1 $B|\mathfrak{k}$ is negative definite so 4) follows from proposition 12.7.

2) If $(\mathfrak{g},\ s,\ Q)$ is isomorphic to its dual, then it cannot have any irreducible factors by 4).

Definition 13.8

If $\mathfrak{g}$ is compact then $(\mathfrak{g},\ s,\ Q)$ and the associated symmetric space M is said to be of the *compact type*. If $\mathfrak{g}$ is not compact then $(\mathfrak{g},\ s,\ Q)$ and M are said to be of the *non-compact type.*

Remark

By theorem 12.8 a symmetric space of the compact type is in fact a compact space.

The compact and non-compact types are also distinguished by the sectional curvature:

Proposition 13.9

Suppose M is an irreducible symmetric space with associated orthogonal involutive Lie algebra $(\mathfrak{g},\ s,\ Q)$. Let $\mathfrak{p}$ be identified with the tangent space T_pM at the point p. Then the sectional curvature for a two plane $S \subseteq \mathfrak{p}$ is given by

$$K_p(S) = \lambda\, B([X,Y]\,,\ [X,Y]),$$

where $\{X,Y\}$ is an orthonormal basis for S and $\lambda \neq 0$ is given by $\lambda\, B|\mathfrak{p} = Q$.

In particular $K_p(S) \geq 0$ or ≤ 0 according to whether M is of the compact or non-compact type.

Proof

As noticed above there is $\lambda \neq 0$ such that $\lambda\, B|\mathfrak{p} = Q$. Then

$$\begin{aligned} K_p(S) = \ -Q(R(X,Y)X,Y) &= \lambda\, B([[X,Y],X],Y) \\ &= \lambda\, B([X,Y],[X,Y]). \end{aligned}$$

Now $[X,Y] \in \mathfrak{k}$ and $B|\mathfrak{k}$ is negative definite. Hence $K_p(S)$ and λ have opposite sign and since M is of the compact type iff $\lambda < 0$ this proves the proposition.

Remark

One can prove that if M is simply connected and has non-positive sectional curvature then $\exp_p : T_p(M) \to M$ is a diffeomorphism (see e.g. Helgason [chapter I, § 13] or Milnor, [theorem 19.2]). In particular a simply connected symmetric space of the non-compact type is diffeomorphic to Euclidean space.

We now have the following classification of the irreducible orthogonal involutive Lie algebras:

Theorem 13.10

The irreducible orthogonal involutive Lie algebras $(\mathfrak{g},\ s,\ Q)$ fall into the following 4 disjoint classes:

I. $\mathfrak{g}$ *is a compact simple Lie algebra,* s *is an automorphism of order* 2, $Q = \lambda B|\mathfrak{p}$, $\lambda < 0$.

II. $\mathfrak{g} = \mathcal{L} \oplus \mathcal{L}$, $\mathcal{L}$ *is a compact simple Lie algebra,* $s(X,Y) = (Y,X)$, $X,Y \in \mathcal{L}$ *and* $Q = \lambda\, B|\mathfrak{p}$, $\lambda < 0$.

III. $\mathfrak{g}$ *is a non-compact simple Lie algebra and* $\mathfrak{g}_{\mathbb{C}}$ *is simple.* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *is a Cartan decomposition and* $Q = \lambda B|\mathfrak{p}\ ,\quad \lambda > 0.$

IV. $\mathfrak{g}$ *is the underlying real Lie algebra of a complex simple Lie algebra, and* $\mathfrak{k}$ *is a compact real form. Again* $\mathfrak{g} = \mathfrak{k} \oplus i\,\mathfrak{k}$ *is a Cartan decomposition with* $\mathfrak{p} = i\,\mathfrak{k}$ *and* $Q = \lambda B|\mathfrak{p}\ ,\quad \lambda > 0.$

Classes I *and* II *are the ones of compact type and classes* III *and* IV *are the ones of non-compact type. Furthermore duality interchanges class* I *and* III *and interchanges class* II *and* IV.

Proof

Since in all these classes $\mathfrak{g}$ cannot be decomposed into smaller $s-$invariant ideals, $(\mathfrak{g},\ s,\ Q)$ is irreducible in all cases. Also it is clear from the above that $Q = \lambda B|\mathfrak{p}$ with $\lambda < 0$ in the compact cases and $\lambda > 0$ in the non-compact cases.

Now suppose $(\mathfrak{g},\ s,\ Q)$ is irreducible and $\mathfrak{g}$ is compact semi-simple. Then as remarked in the proof of the uniqueness of the decomposition in theorem 13.5, either $\mathfrak{g}$ is simple in which case we are in class I, or $\mathfrak{g} = \mathcal{L} \oplus \mathcal{L}$ where $\mathcal{L}$ is a simple ideal and s interchanges the factors. Since $\mathfrak{g}$ is compact also $\mathcal{L}$ is compact and we are in class II.

Also it follows using proposition 12.9 that

$$\mathrm{I}^* \subseteq \mathrm{III} \text{ and } \mathrm{II}^* \subseteq \mathrm{IV},$$

and that III and IV are disjoint.

By duality every irreducible orthogonal involutive Lie algebra with $\mathfrak{g}$ non-compact is either in I^* or II^* hence either in III or IV and it follows that I^* = III and II^* = IV. This ends the proof.

An irreducible symmetric space is of course said to be of class I, II, III or IV according to the class of the associated orthogonal involutive Lie algebra. Thus the symmetric spaces of class II are exactly the simply connected simple compact Lie groups with a bi-invariant metric. By duality the classification of these is equivalent to the classification of complex simple Lie algebras, which is given by the so-called Dynkin-diagrams (see e.g. Humphreys [3]). For a complete classification of the symmetric spaces of class I (or by duality of class III) see e.g. Helgason [chapter 10, § 6] or Wolf [chapter 8, § 11]. Notice that this also gives a classification of the non-compact simple Lie-algebras by corollary 13.13 and the exercise 12.15.

Exercise 13.11

Each Cartan decomposition of the Lie algebras considered in exercise 12.17 gives rise to an irreducible symmetric space of type III. Describe in each case the corresponding dual symmetric space of the compact type I.

References

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