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# RIEMANNIAN GEOMETRY: A METRIC ENTRANCE

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## Contents



## 0. Introduction

This is intended as a rapid introduction to basic Riemannian geometry with minimal prerequisites. In contrast to all other books on the topic, the basic theory of differentiable manifolds is not a prerequisite nor a topic which is treated first. Rather, the theory of smooth manifolds emerges naturally halfway through the text, in the passage from local to global considerations, and is only treated then with this as motivation.

The idea behind our approach is to begin with *local Riemannian geometry* from the point of view of special metric spaces. The only prerequisites needed for this are basic linear algebra, analysis and metric spaces. The obvious *length metric* on say the graph in $\mathbb{R}^n \times \mathbb{R}$ of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ serves as the motivation for the general concept of a *Riemannian patch* (which is nothing but a Riemannian metric on an open subset of $\mathbb{R}^n$). In this context, *geodesics* defined as locally shortest curves (parametrized by arclength) are easily seen to exist. - The first *variation of arclength formula* for smooth curves motivates the concept of *connections* and the *exponential map.* Via the*Gauss-lemma* one then sees that the metrically defined geodesics are actually smooth curves satisfying the second order *geodesic equation* used to construct the exponential map. Likewise it follows that *isometries*, i.e. distance preserving maps, between Riemannian patches are smooth, a key observation in the passage from local to *global Riemannian spaces.* Indeed, a metric space which is locally isometric to a Riemannian patch, is in particular a *smooth manifold.* This allows us to carry over immediately all considerations for Riemannian patches to global Riemannian spaces, i.e. *Riemannian manifolds* . In particular, the essential notions of *curvature* (introduced as a measurement for the deviation of the exponential map form being an isometry), *Jacobi fields* and *parallel transport* carry over to global spaces since they are isometry invariants.

Once the global notions are in place, the equivalence between metric and geodesic *completeness* is proved. This so-called *Hopf-Rinow theorem* is the germ of most if not all global results in *metric differential geometry.* Two of the most classical results concerning relations between geometry and topology, the *Hadamard-Cartan* and the *Bonnet-Myers theorems*, are easily obtained at this relatively early stage in our treatment.

A general pedestrian (but possibly terse) treatment of *bundles*, *forms* and *tensors* in general is given primarily for the purpose of *submanifold theory* and *relative curvature* (second fundamental form). The *Theorema Egregium* is used to provide explicit *models* for *spaces of constant curvature.* The *dual* notion of *Riemannian submersions* and the corresponding *Gray-O'Niell formula* is treated with the immediate purpose of discussing *homogeneous spaces.*

We have chosen to terminate our treatment at this point, where Riemannian geometry bifurcates into many different directions. One of these, now referred to as *comparison theory* has been the driving force behind our point of view in this geometric introduction to Riemannian geometry.

As an introduction to the subject it is important to stress that the problems form an integral part of the text. In an attempt to strive for clarity in the exposition, details of proofs are often deferred to the problems. - The notes are based on one semester courses taught at the University of Maryland and at the University of Aarhus. It is a pleasure to thank the students at both universities for their important input. The final draft of the notes was written while on sabbatical at the University of Aarhus during 1996/1997. I am grateful for the support an excellent research atmosphere provided by the mathematics department.

## 1. Riemannian length and distance

Recall that the *euclidean distance* between points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ is given by

$$\rho(x,y) = \|x-y\| = \langle x-y, x-y\rangle^{\frac{1}{2}} = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{\frac{1}{2}}. \tag{1.1}$$

In terms of this, the euclidean length of a (continuous) curve $c : [a,b] \to \mathbb{R}^n$ is defined as

$$L_0(c) = \sup L_0(P) = \sup \sum_{i=1}^{k} \rho(c(t_{i-1}), c(t_i)), \tag{1.2}$$

where the supremum is taken over all *partitions* $P = \{a = t_0 < t_1 < \cdots < t_k = b\}$ of $[a, b]$. Clearly any parameter change of $c$ yields a curve with the same length. Moreover

$$L_0(c) = L_0(c_{|[a,t]}) + L_0(c_{|[t,b]}), \tag{1.3}$$

for any $t \in [a, b]$, and

$$L_0(c) \geq L_0(c_{|[a_1,b_1]}), \tag{1.4}$$

whenever $[a_1, b_1] \subset [a, b]$. In particular, $c_{|[a_1,b_1]}$ has finite length if $c$ does. Such curves are called *rectifiable*. Any $C^1$-curve, $c$ is rectifiable, in fact

$$L_0(c) = \int_a^b \|c'(t)\| dt, \tag{1.5}$$

where $c' : [a, b] \to \mathbb{R}^n$ is the derivative, $c'(t) = \frac{d}{dt}c(t)$ of $c$. Clearly

$$\rho(c(a), c(b)) \leq L_0(c) \tag{1.6}$$

for any curve $c$, with equality if and only if $c$ is the line segment from $c(a)$ to $c(b)$.

Now let $U \subset \mathbb{R}^n$ be a connected open subset in $\mathbb{R}^n$. If confined to $U$, the induced euclidean distance is not a reasonable measure for the distance between points in $U$, unless it is convex. Instead define

$$\mathrm{dist}_0(x, y) = \inf L_0(c), \tag{1.7}$$

where $c : [a, b] \to U$ joins $x$ and $y$, i.e. $c(a) = x$ and $c(b) = y$. Here it is sufficient to take the infimum over curves $c$ that are *piecewise regular*. A continuous curve $c : [a, b] \to U \subset \mathbb{R}^n$ is piecewise regular provided $[a, b]$ admits a partition $a = t_0 < t_1 < \cdots < t_k = b$ such that the restriction $c_| : [t_{i-1}, t_i] \to U$, is $C^1$ with $c'_| \neq 0$, $i = 1, \ldots, k$. Clearly

$$\mathrm{dist}_0 \geq \rho \tag{1.8}$$

also defines a metric on $U$ with the same topology.

More generally consider a function $f : U \to \mathbb{R}$ of class $C^k, k \geq 1$. Any curve $c$ on the graph of $f$ in $U \times \mathbb{R} \subset \mathbb{R}^{n+1}$ is of the form $c(t) = (c_1(t), f \circ c_1(t))$ for some curve $c_1$ in $U$. Moreover, if $c_1$ is $C^1$

$$L_0(c) = \int_a^b (\|c_1'(t)\|^2 + (f \circ c_1)'(t)^2)^{\frac{1}{2}} dt = \int_a^b (g_{c(t)}(c_1'(t), c_1'(t))^{\frac{1}{2}} dt, \tag{1.9}$$

where for each $x \in U, g_x$ is the inner product on $\mathbb{R}^n$ defined in terms of the euclidean inner product $\langle , \rangle$ by $g_x = \langle , \rangle \circ D(id, f)_x \times D(id, f)_x$. This motivates the following

**Definition 1.10.** A *riemannian $C^k$-structure* on an open set $U \subset \mathbb{R}^n$ is a $C^k$-map $g$, which to each $x \in U$ assigns an inner product on $\mathbb{R}^n$. The pair $(U, g)$ is called a *riemannian patch.*

If $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$ , the inner products

$$g_{ij} = g(e_i, e_j), \qquad i, j = 1, \ldots, n \tag{1.11}$$

determine $g$ completely. These coordinate functions are all $C^r$. Let $\mathrm{G}_x : \mathbb{R}^n \to \mathbb{R}^n$ be the symmetric linear map, whose matrix with respect to $e_1, \ldots, e_n$ is $\{g_{ij}(x)\}$. With this notation

$$g_x(u, v) = \langle u, G_x v\rangle \tag{1.12}$$

for all $x \in U, u, v \in \mathbb{R}^n$.

Although less will often do, we assume for convenience that all maps considered (including *riemannian metrics*, $g$) are smooth, i.e. $C^\infty$, unless otherwise explicitely stated.

The *(riemannian) length* of a piecewise $C^1$-curve $c : [a, b] \to U$ is defined as

$$L(c) = \int_a^b \left(g_{c(t)}(c'(t), c'(t))\right)^{\frac{1}{2}} dt. \tag{1.13}$$

We will only use subscripts as e.g. $L_g$, if we need to be specific about the riemannian structure. As in (1.7), if $U$ is connected, we define the (riemannian) distance between $x, y \in U$ by

$$\mathrm{dist}(x, y) = \inf L(c), \tag{1.14}$$

where the infimum is taken over all piecewise regular curves joining $x$ and $y$ in $U$. Comparing $g$ with a (constant) euclidean inner product shows that

$$\mathrm{dist} : U \times U \to \mathbb{R} \tag{1.15}$$

is a metric whose topology coincides with the original one on $U$.

**Example 1.16.** Let $(U, g)$ be any riemannian patch and $f$ a positive function on $U$. Then $(U, f \cdot g)$ is another riemannian patch. Two such patches are said to be *conformally related*. An important special case that can be described that way is the socalled *hyperbolic space*. Here $U = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ is equipped with the riemannian structure

$$g_x = \frac{4}{(1 - \|x\|^2)^2}\langle , \rangle,$$

also called the *Poincaré metric*.

We will see that the hyperbolic space, together with the euclidean space and the sphere are the simply connected *model spaces*, i.e. spaces of constant curvature.

**Problem 1.17.** Prove (1.3), (1.4), and give an example of a non-rectifiable curve.

**Problem 1.18.** Prove (1.6), (1.15) and the statements following them.

**Problem 1.19.** Prove the claim following (1.7).

**Problem 1.20.** Show that any $((a, b), \mathrm{dist}_g)$ is locally isometric to $((a, b), \rho_1)$. Is it true globally?

**Problem 1.21.** Extend the situation in (1.9) to the case where $f : U \to \mathbb{R}$ is replaced by a $C^k$ map $F :\to \mathbb{R}^m$ (It follows from Nash's embedding theorem that any riemannian path can be obtained this way).

## 2. Geodesics

In analogy with (1.2) we can define the length of any continuous curve $c : [a, b] \to (U, \mathrm{dist})$ by

$$L(c) = \sup \sum_{i=1}^{k} \mathrm{dist}(c(t_{i-1}), c(t_i)), \tag{2.1}$$

where the supremum is taken over all partititions $a = t_0 < \cdots < t_k = b$ of $[a, b]$. Again taking $\inf L(c)$ over all curves with fixed endpoints defines a metric which in general is larger than the original one. In our case, however

$$\mathrm{dist}(x, y) = \inf L(c), \tag{2.2}$$

where the infimum is taken over all continuous curves $c$ from $x$ to $y$ in $U$. A metric space with this property is called a *length space* (or inner metric space).

A *normal geodesic* in $(U, \mathrm{dist})$ is a curve $c$ which is locally distance preserving i.e. for any $t$ in the domain of $c$

$$\mathrm{dist}(c(t_1), c(t_2)) = |t_1 - t_2| \tag{2.3}$$

whenever $t_1$ and $t_2$ are sufficiently close to $t$. A *geodesic* is a curve which up to an affine change of parameter is a normal geodesic. A geodesic $c : [a, b] \to U$ is called *minimal* if $\mathrm{dist}(c(a), c(b)) = L(c)$. Since $U$ is locally compact, the following yields the existence of "short" minimal geodesics.

**Lemma 2.4.** *For $x \in U$ choose $\varepsilon > 0$ so that the ball $B(x, \varepsilon) = \{y \in U \mid \mathrm{dist}(x, y) < \varepsilon\}$ has compact closure. Then any $y \in B(x, \varepsilon)$ can be joined to $x$ by a minimal geodesic.*

**Proof:** Select a sequence of curves $c_n : [0, 1] \to U$ parametrized proportional to arc length, such that $c_n(0) = x, c_n(1) = y$ and $L(c_n) \to \mathrm{dist}(x, y)$. Then $\{c_n\}$ is an equicontinuous family. Moreover, for $n$ sufficiently large $c_n([0, 1]) \subset \overline{B(x, \varepsilon)}$. By Ascoli's theorem a subsequence $\{c_{n_k}\}$ of $\{c_n\}$ converges uniformly to a continuous curve $c : [0, 1] \to \overline{B(x, \varepsilon)}$ from $x$ to $y$. Moreover, $L(c) = \mathrm{dist}(x, y)$ and $c$ is parametrized proportional to arc length. □

The key point in the above proof is that all $c_n$ map into a compact set for $n$ large. In particular, if the closure of any ball in $(U, \mathrm{dist})$ is compact, any two points can be joined by a minimal geodesic.

The following is a generalization of the Heine-Borel theorem for $\mathbb{R}^n$.

**Theorem 2.5.** *Suppose the riemannian patch $(U, g)$ is a complete metric space. Then every closed and bounded set is compact. Moreover, any two points in $U$ may be joined by a minimal geodesic.*

**Proof:** We need to show that every closed ball

$$\overline{B(x, r)} = \{y \in U \mid \mathrm{dist}(x, y) \le r\} \tag{2.6}$$

is compact. Since $U$ is locally compact, it suffices to show that $\overline{B(x,R)}$ is compact provided $\overline{B(x,r)}$ is compact for all $r < R$. Now let $\{x_n\}$ be any sequence in $\overline{B(x,R)}$. Since

$$\text{dist}\,\overline{(B(x,r_1)}, \overline{B(y,r_2))} \leq \text{dist}(x,y) - r_1 - r_2 \tag{2.7}$$

for any $x, y \in U$ and $r_1, r_2$ positive, we see that $\overline{B(x,R-\varepsilon)} \cap \overline{B(x_n, 2\varepsilon)} \neq \phi$ for every $0 < \varepsilon < R$. Let $\{\varepsilon_p\}$ be a decreasing sequence of positive numbers such that $\varepsilon_p \to 0$. For each $p$, pick $y_n^p$ so that

$$y_n^p \in \overline{B(x, R-\varepsilon_p)} \text{ and } \text{dist}(x_n, y_n^p) \leq 2\varepsilon_p.$$

By assumption $\{y_n^p\}$ has a convergent subsequence for each $p$. Hence the diagonal proceedure yields a subsequence $\{n_k\}$ such that $\{y_{n_k}^p\}$ is convergent for all $p$. The sequence $\{x_{n_k}\}$ being the uniform limit of $\{y_{n_k}^p\}$ is a Cauchy-sequence. Thus $\{x_{n_k}\}$ is convergent and consequently $\overline{B(x,R)}$ is compact.

Since closed balls are compact the existence of minimal geodesics has already been proved above. □

In the next few sections we will show that geodesics are smooth curves which are solutions to a second order differential equation.

**Problem 2.8.** Prove (2.2) and give an example of a metric space where it is false.

**Problem 2.9.** Prove that (2.6) holds for all length spaces, and show that it is wrong in general.

**Problem 2.10.** Prove that (2.7) holds in all length spaces, but not in general.

**Problem 2.11.** Prove that 2.4 and 2.5 hold for all locally compact length spaces.

## 3. First variation of arc length

A *one parameter variation* of a curve $c : [a,b] \to U$ is a map

$$V : [a,b] \times (-\varepsilon, \varepsilon) \to U$$

such that $V(t,0) = c(t)$ for all $t \in [a,b]$. We assume that $V$ is piecewise $C^k, k \geq 1$, i.e. there is a partition $a = t_0 < \cdots < t_m = b$ of $[a,b]$ so that $V_{|[t_{i-1},t_i]\times(-\varepsilon,\varepsilon)}$ is $C^k, i = 1, \ldots, m$. In particular, for each $s \in (-\varepsilon, \varepsilon)$, the curve $c_s = V( , s)$ is piecewise $C^k$, and for each $t \in [a,b]$ the curve $\sigma_t = V(t, \cdot)$ is $C^k$. The curve

$$X(t) = \frac{\partial}{\partial s} V(t,s)_{|s=0} = \sigma_t'(0)$$

is called the *variation field* of $V$ along $c$.

We are interested in the variation of $L(c_s)$. The general case can easily be analyzed after having dealt with the case $m = 1$, i.e. $V$ is $C^k$. Assume moreover that $c = c_0$ is parametrized proportionally to arc length i.e. $g_{c(t)}(c'(t), c'(t)) = l^2$ is constant.

Then $L(c_s)$ is a $C^k$ function of $s$ near zero, and

$$\begin{aligned}\frac{d}{ds}L(c_s) =& \frac{d}{ds}\int_a^b g_{c_s(t)}(c_s'(t), c_s'(t))^{\frac{1}{2}}dt\\ =& \int_a^b \frac{\partial}{\partial s}\langle c_s'(t), G_{c_s(t)}c_s'(t)\rangle^{\frac{1}{2}}dt\\ =& \int_a^b \frac{1}{2}g_{c_s(t)}(c_s'(t), c_s'(t))^{-\frac{1}{2}}\frac{\partial}{\partial s}\langle c_s'(t), G_{c_s(t)}c_s'(t)\rangle dt\end{aligned}$$

Now

$$\begin{aligned}\frac{\partial}{\partial s}\langle c_s'(t), G_{c_s(t)}c_s'(t)\rangle =& \langle\frac{\partial}{\partial s}\frac{\partial}{\partial t}V(t,s), G_{c_s(t)}c_s'(t)\rangle +\\ & \langle c_s'(t), \frac{\partial}{\partial s}[G_{c_s(t)}c_s'(t)]\rangle\\ =& \langle\frac{\partial}{\partial t}\frac{\partial}{\partial s}V(t,s), G_{c_s(t)}c_s'(t)\rangle +\\ & \langle c_s'(t), DG_{c_s(t)}(\frac{\partial}{\partial s}V(t,s))c_s'(t) + G_{c_s(t)}\frac{\partial}{\partial s}\frac{\partial}{\partial t}V(t,s)\rangle\\ =& 2\langle\frac{\partial}{\partial t}\frac{\partial}{\partial s}V(t,s), G_{c_s(t)}c_s'(t)\rangle +\\ & \langle c_s'(t), G_{c_s(t)}G^{-1}_{c_s(t)}DG_{c_s(t)}(\frac{\partial}{\partial s}V(t,s))c_s'(t)\rangle.\end{aligned}$$

Thus the evaluation at $s = 0$ yields

$$\frac{d}{ds}L(c_s)_{|s=0} = l^{-1}\int_a^b g_{c(t)}(\nabla_c^1 X(t), c'(t))dt \tag{3.1}$$

where

$$\nabla_c^1 X(t) = \frac{d}{dt}X(t) + {}^1\Gamma_{c(t)}(c'(t), X(t)) \tag{3.2}$$

and

$${}^1\Gamma_x(u,v) = \frac{1}{2}G_x^{-1}DG_x(v)u \tag{3.3}$$

for all $x \in U, u, v \in \mathbb{R}^n$. Note, that for each $x \in U, {}^1\Gamma_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is bilinear, and

$$\nabla_c^1(X+Y) = \nabla_c^1 X + \nabla_c^1 Y, \tag{3.4}$$

$$\nabla_c^1(fX) = f'X + f\nabla_c^1 X \tag{3.5}$$

holds for any fields $X, Y$ along $c$, and any function $f : [a,b] \to \mathbb{R}$. Moreover, for any fields $X, Y$ along $c$ we have

$$g_c(X,Y)' = g_c(\nabla_c^2 X, Y) + g_c(X, \nabla_c^2 Y) \tag{3.6}$$

where

$$\nabla_c^2 X(t) = \frac{d}{dt}X(t) + {}^2\Gamma_{c(t)}(c'(t), X(t)) \tag{3.7}$$

and ${}^2\Gamma_x(u,v) = {}^1\Gamma_x(v,u)$ for all $x \in U, u, v \in \mathbb{R}^n$.

Given any smooth map $\Gamma$ that assigns to any $x \in U$ a bilinear map $\Gamma_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ the expression

$$\nabla_c X(t) = \frac{d}{dt} X(t) + \Gamma_{c(t)}(c'(t), X(t)) \tag{3.8}$$

is called the *covariant derivative* of $X$ along $c$ with *Christoffel symbol* $\Gamma$. Clearly (3.4) and (3.5) hold for any $\Gamma$. If moreover $\Gamma$ is *symmetric*, i.e. $\Gamma_x$ is symmetric for each $x \in U$, then

$$\nabla_{c_s} \frac{\partial}{\partial s} V(t,s) = \nabla_{\sigma_t} \frac{\partial}{\partial t} V(t,s) \tag{3.9}$$

holds for any $C^k$ variation $V$.

Note, that if ${}^1\Gamma$ is symmetric, then ${}^1\Gamma = {}^2\Gamma$ and the formulas (3.4), (3.5), (3.6) and (3.7) would all be valid for $\nabla$ with $\Gamma = {}^1\Gamma = {}^2\Gamma$. In this case a separate direct calculation shows that

$$\frac{d}{ds} L(c_s)_{|s=0} = l^{-1} \int_a^b g_{c(t)}(\nabla_c X(t), c'(t)) dt \tag{3.10}$$

or equivalently

$$\frac{d}{ds} L(c_s)_{|s=0} = l^{-1} \left\{ g_c(X, c') \Big|_a^b - \int_a^b g_c(X, \nabla_c c') \right\} \tag{3.11}$$

Observe that any $X$ is the variation field for a variation of $c$ (take e.g. $V(t,s) = c(t) + sX(t)$). In particular, if the smooth curve $c$ is a minimal geodesic, then $\frac{d}{ds} L(c_s)_{|s=0} = 0$ for all $X$ with $X(a) = 0, X(b) = 0$. We conclude, that any smooth geodesic $c$, will satisfy the differential equation.

$$\nabla_c c' = c''(t) + \Gamma_{c(t)}(c'(t), c'(t)) = 0. \tag{3.12}$$

Given any $\Gamma$, the equation (3.12) is called the *geodesic equation* for the covariant derivative, or connection $\nabla$. This equation is equivalent to the system

$$c' = d \quad ; \quad d' = -\Gamma_c(d,d) \tag{3.13}$$

of first order equations. From the existence and uniqueness theorem for first order ordinary differential equations, we get immediately

**Theorem 3.14.** *For every $(x_0, y_0) \in U \times \mathbb{R}^n$ there is an interval $J(x_0, v_0) \ni 0$ and a unique maximal solution $c_{(x_0,v_0)} : J(x_0, v_0) \to U$ to (3.12) satisfying $c_{(x_0,v_0)}(0) = x_0$, and ${c_{(x_0,v_0)}}'(0) = v_0$. Moreover, the set $W = \{(x,v,t) \mid (x,v) \in U \times \mathbb{R}^n, t \in J(x,v)\}$ is open in $U \times \mathbb{R}^n \times \mathbb{R}$, and the map $W \to U, (x,v,t) \to c_{(x,v)}(t)$ is smooth.*

Note, that if $c$ is a $\nabla$-*geodesic*, i.e. $c$ is a solution to (3.12), and $c(0) = x, c'(0) = v$, then by homogeneity $\bar{c}(t) = c(at)$ (a constant) is a $\nabla$-geodesic with $\bar{c}(0) = x$ and $\bar{c}'(0) = a \cdot v$.

In the next section we will see, that for any riemannian structure $g$ on $U$, there is a unique symmetric $\Gamma$ such that (3.6) holds. A first variation argument which we will carry out in section 5 then shows that the corresponding $\nabla$-geodesics are exactly the geodesics as defined in section 2.

**Problem 3.15.** Show that any regular curve $c$ can be parametrized so that $c'$ has a constant $g$-norm.

**Problem 3.16.** Prove (3.4), (3.5), and (3.6).

**Problem 3.17.** Prove (3.9) and (3.10).

**Problem 3.18.** Prove the homogeneity property of $\nabla$-geodesics stated above.

**Problem 3.19.** Show that in general ${}^1\Gamma \neq {}^2\Gamma$.

**Problem 3.20.** If $g$ is constant in $U$, show that ${}^1\Gamma = {}^2\Gamma = 0$. What are the $\nabla$-geodesics in this case?

## 4. The Levi Civita connection

A pair $(x, v) \in U \times \mathbb{R}^n$, as considered in the previous section, is called a *tangent vector* at $x \in U$. The set $\{x\} \times \mathbb{R}^n$ of tangent vectors at $x$ clearly form a vector space isomorphic to $\mathbb{R}^n$ via $I_x : \mathbb{R}^n \to \{x\} \times \mathbb{R}^n, v \to (x, v)$, called the *tangent space* to $U$ at $x$. We will also use the notation $v_x$ for the tangent vector $(x, v), T_xU$ for the tangent space $\{x\} \times \mathbb{R}^n$, and $TU$ for the union of tangent spaces $\bigcup_x T_xU = U \times \mathbb{R}^n$.

If $c : [a, b] \to U$ is a differentiable curve, the tangent vector $(c(t), c'(t))$ is called the *velocity vector* of $c$ at $c(t) \in U$ (or more precisely at $t \in [a, b]$). This is simply denoted by $\dot{c}(t)$. Note, that any tangent vector $v_x$ is the velocity vector of a curve, e.g. $v_x = \dot{c}(0)$, where $c(t) = x + tv$. Curves $c_1, c_2$ are *tangent* at $x$ if $c_1(t_1) = c_2(t_2) = x$ and $\dot{c}_1(t_1) = \dot{c}_2(t_2)$. This is clearly an equivalence relation, and the set of equivalence classes of curves through $x \in U$ is naturally isomorphic to the tangent space $T_xU$. This view point is extremely useful.

We can now interpret a riemannian structure $g$ on $U \subset \mathbb{R}^n$ as a map that assigns to any $x \in U$ an inner product, $g_x = \langle, \rangle_x$ on the tangent space $T_xU$. Moreover, any map $Y : U \to \mathbb{R}^n$ can be viewed as a map, that assigns to any $x \in U$ a tangent vector $(x, Y(x)) \in T_xU$ at $x$. With this interpretation, $Y$ is called a *vector field* on $U$. Similarly, if $c : [a, b] \to U$, a map $Y : [a, b] \to \mathbb{R}^n$ may be viewed also as a map that assigns to any $t \in [a, b]$ a tangent vector at $c(t)$. This is why such a map is then called a *vector field along* $c$.

Now fix a map $\Gamma : U \to L^2(\mathbb{R}^n, \mathbb{R}^n; \mathbb{R}^n)$ as in section 3. If $Y : U \to \mathbb{R}^n$ is a vector field on $U$ and $c : [a, b] \to U$ is a curve, then $Y \circ c$ is a vector field along $c$. By (3.8)

$$\nabla_c(Y \circ c)(t) = DY_{c(t)}(c'(t)) + \Gamma_{c(t)}(c'(t), Y(c(t)))$$

only depends on $\dot{c}(t)$ and $Y$ near $c(t)$. Therefore if $v_x = (x, v)$ is a tangent vector at $x$ we define

$$\nabla_{v_x} Y = v_x[Y] + \Gamma_x(v, Y(x)), \tag{4.1}$$

where $v_x[Y] = DY_x(v)$ is the *directional derivative* of $Y$ in direction $v_x$. $\nabla_{v_x} Y$ is called the covariant derivative of $Y$ in direction $v_x$. This again we may view as a tangent vector at $x$ if we please. Clearly

$$\nabla_{av_x + bu_x} Y = a\nabla_{v_x} Y + b\nabla_{u_x} Y \tag{4.2}$$

for all $a, b \in \mathbb{R}$ and $v_x, u_x \in T_xU$. Moreover, as in (3.4) and (3.5)

$$\nabla_{v_x}(Y_1 + Y_2) = \nabla_{v_x} Y_1 + \nabla_{v_x} Y_2 \tag{4.3}$$

$$\nabla_{v_x}(fY) = v_x[f] \cdot Y + f \cdot \nabla_{v_x} Y, \tag{4.4}$$

hold for all vector fields $Y_1, Y_2$ on $U$ and functions $f : U \to \mathbb{R}$. Here again $v_x[f] = Df_x(v)$ is the directional derivative of $f$ in direction $v_x$.

If now also $X : U \to \mathbb{R}^n$ is a vector field on $U$ we define the covariant derivative $\nabla_X Y$ of $Y$ in direction $X$, as the vector field

$$(\nabla_X Y)(x) = \nabla_{X_x} Y. \tag{4.5}$$

Then by (4.2), (4.2) and (4.4)

$$\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y \tag{4.6}$$

$$\nabla_{fX} Y = f \nabla_X Y \tag{4.7}$$

$$\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2 \tag{4.8}$$

$$\nabla_X (fY) = X[f]Y + f \nabla_X Y, \tag{4.9}$$

where $X[f] : U \to \mathbb{R}$ is defined by $X[f](x) = X_x[f] = Df_x(X(x))$.

A map $\nabla$ that assigns to any pair of vector fields $X, Y$ a vector field $\nabla_X Y$ such that (4.6)-(4.9) hold is also called a *connection*. Using these properties it is easy to see that a connection $\nabla$ is completely determined by

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \Gamma_{ij}^k e_k, \tag{4.10}$$

where $e_i : U \to \mathbb{R}^n, i = 1, \ldots, n$ are the *standard coordinate vector fields* on $U$. Now $\Gamma_{ij}^k : U \to \mathbb{R}, i, j, k = 1, \ldots, n$ define a map $\Gamma : U \to L^2(\mathbb{R}^n, \mathbb{R}^n; \mathbb{R}^n)$ by $\Gamma(x)(\sum u_i e_i, \sum v_j e_j) = \sum u_i v_j \Gamma_{ij}^k e_k$, and the connection defined by this $\Gamma$ coincides with the connection we started out with. In other words there is a one to one correspondence between connections $\nabla$ and Christoffel symbols $\Gamma$.

Observe that for vector fields in $U$ that

$$\nabla_X Y - \nabla_Y X = X[Y] - Y[X] \tag{4.11}$$

if $\Gamma$ is symmetric. The vector field $[X, Y](x) = X_x[Y] - Y_x[X]$ is called the *Lie bracket* of $X$ and $Y$. In general

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{4.12}$$

at each $x \in U$ only depends on $X_x, Y_x \in T_x U$. Thus for each $x \in U, T_x : T_x U \times T_x U \to T_x U$ is bilinear map called the *torsion tensor* of $\nabla$. Clearly $\Gamma$ is symmetric if and only if $\nabla$ is torsion free, i.e. $T = 0$.

By definition $[X, Y](x)$ depends on $X$ and $Y$ near $x \in U$, and the directional derivative is computed as

$$[X, Y]_x[f] = X_x[Y[f]] - Y_x[X[f]] \tag{4.13}$$

We say that $X$ and $Y$ *commute*, provided $[X, Y] = 0$. The coordinate vector fields clearly commute.

The advantage of (4.13) is that it allows an invariant definition of the Lie bracket. The point is that tangent vectors can also be viewed abstractly as "directional derivatives". Given a tangent vector $v_x \in T_x U$, clearly

$$v_x[f + h] = v_x[f] + v_x[h], \tag{4.14}$$

$$v_x[f \cdot h] = v_x[f] \cdot h(x) + f(x) v_x[h] \tag{4.15}$$

for all smooth functions $f, h$ defined near $x$.

The tangent vector $v_x$ thus induces what is called a *derivation* on the set of functions near $x$. The map that assigns to each tangent vector the corresponding directional derivative is a linear isomorphism between the tangent space at $x$ and the vector space of derivations on functions defined near $x$ (cf. Problems 4.28-4.34).

Similarly we can view vector fields on $U$ as derivations on the ring of smooth functions on $U$. (cf. Problem 4.35).

In section 3 we have seen the use of a connection $\nabla$ which is torsion free and *metric*, i.e.

$$X[g(Y,Z)] = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{4.16}$$

for all vector fields $X, Y$ and $Z$ on $U$, or equivalently

$$g(X,Y)' = g(\nabla_c X, Y) + g(X, \nabla_c Y) \tag{4.17}$$

for all curves $c$ in $U$ and vector fields $X, Y$ along $c$. The following is sometimes referred to as *the fundamental lemma of riemannian geometry.*

**Theorem 4.18.** *For every riemannian structure $g$ on an open set $U \subset \mathbb{R}^n$ there is one and only one torsion free metric connection $\nabla$.*

**Proof:** Uniqueness: If $\nabla$ is a torsion free metric connection, then

$$\frac{\partial}{\partial x_i} g_{jk} = e_i[\langle e_j, e_k \rangle] = \langle \nabla_{e_i} e_j, e_k \rangle + \langle e_j, \nabla_{e_i} e_k \rangle \tag{4.19}$$

by (4.16). Permuting the indices cyclicly and using (4.11) with $[e_i, e_j] = 0$ then gives

$$\langle \nabla_{e_i} e_j, e_k \rangle = \frac{1}{2} \left( \frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ik} - \frac{\partial}{\partial x_k} g_{ij} \right). \tag{4.20}$$

On the other hand

$$\langle \nabla_{e_i} e_j, e_k \rangle = \sum_{l=1}^{n} \Gamma_{ij}^l g_{lk} \tag{4.21}$$

Thus if $\{g^{kl}\}$ is the matrix of $G^{-1}$ we get

$$\Gamma_{ij}^l = \frac{1}{2} \sum_{k=1}^{n} \left( \frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ik} - \frac{\partial}{\partial x_k} g_{ij} \right) g^{kl}, \tag{4.22}$$

which expresses $\Gamma$ in terms of $g$ alone.

Existence: Define $\nabla$ using (4.22). Then $\Gamma$ is clearly symmetric i.e. $\nabla$ is torsion free. Moreover (4.20) and then clearly (4.19) hold for this connection. However, it is straight forward to see that (4.16) follows from (4.19). Hence $\nabla$ defined by (4.22) is a torsion free metric connection. □

The connection given by (4.22) is called the *Levi Civita Connection* of $g$. Unless otherwise explicitly stated, this is the connection that we will use from now on.

**Problem 4.23.** Show that the relation "tangency" among curves is an equivalence relation. Give an explicit bijective map between the set of such equivalence classes and tangent vectors.

**Problem 4.24.** Show that there is a one-to-one correspondence between Christoffel symbols $\Gamma$ and connections $\nabla$.

**Problem 4.25.** Find the coordinates of $[X, Y]$ as defined in (4.11), and prove (4.13).

**Problem 4.26.** Show that

$$[fX, hY] = f \cdot h[X, Y] + (f \cdot X[h])Y - (h \cdot Y[f])X$$

for all functions $f, h$ and vector fields $X, Y$ on $U$.

**Problem 4.27.** User (4.26) to show that $T$ as defined in (4.12) is bilinear with respect to functions on $U$.

**Problem 4.28.** Let $x \in U$. Suppose $U_1 \subset U, U_2 \subset U$ are open neighbourhoods of $x$, and $f : U_1 \to \mathbb{R}$, $h : U_2 \to \mathbb{R}$ are smooth functions. Define $f + h, f \cdot h$ as the functions defined on $U_1 \cap U_2$ by pointwise operations. The set of locally defined smooth functions with these operations is denoted by $\mathcal{F}_x$. A *derivation* on $\mathcal{F}_x$. is a map $D : \mathcal{F}_x \to \mathbb{R}$ such that

$$D(af + bh) = aD(f) + bD(h) \tag{4.29}$$

$$D(f \cdot h) = D(f) \cdot h(x) + f(x) \cdot D(h). \tag{4.30}$$

Show that the set of derivations, $Der_x$ on $\mathcal{F}_x$ is a vector space, and that the map $T_xU \to Der_x$ that assigns to any tangent vector $v_x$ the corresponding directional derivative is a linear map.

**Problem 4.31.** Show that $D(1) = 0$ for any $D \in Der_x$ where 1 is the constant function 1 defined in a neighbourhood of $x$. Use this to show that $D(f) = D(h)$ if $f = h$ in a neighbourhood of $x$.

**Problem 4.32.** Let $V \subset \mathbb{R}^n$ be an open convex set. Prove that for any smooth function $f : V \to \mathbb{R}$, and any fixed $x \in V$

$$f(y) - f(x) = \sum_{i=1}^{n} (y_i - x_i) f_i(y), \quad y \in Y,$$

where $f_i : V \to \mathbb{R}$ are smooth functions with $f_i(x) = \frac{\partial f}{\partial x_i}(x), i = 1, \ldots, n$.

Hint:$f(y) - f(x) = \int_0^1 \frac{d}{dt} f(x + t(y - x))dt$.

**Problem 4.33.** Using (4.31) and (4.32) on any function $f$ defined near $x \in U$, show that

$$D(f) = \sum_{i=1}^{n} D(h_i) \frac{\partial f}{\partial x_i}(x) = \sum_{i=1}^{n} D(h_i) e_{i_x}[f],$$

for any derivation $D \in Der_x$, where $h_i(y) = y_i - x_i, i = 1, \ldots, n$.

**Problem 4.34.** Show that the map $T_xU \to Der_x$ defined in (4.28) is a linear isomorphism.

**Problem 4.35.** Show that the vector space of smooth vector fields on $U$ is isomorphic to the vector space of derivations on $C^\infty(U, \mathbb{R})$.

**Problem 4.36.** Interpret $[X, Y]$ as a derivation.

**Problem 4.37.** Show that $X, Y \to [X, Y]$ is bilinear, anti symmetric, and satisfies the *Jacobi identity.*

$$[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]]=0. \tag{4.38}$$

By definition of a *Lie algebra*, this makes the vector space of vector fields on $U$ with the product $[,]$ into a Lie algebra.

**Problem 4.39.** Show that (4.16) is equivalent to (4.17).

**Problem 4.40.** Fill in the missing details in the proof of (4.18)

**Problem 4.41.** What is the Levi Civita connection if $g$ is constant? What are the $\nabla$-geodesics in this case?

**Problem 4.42.** Let $U = H = \{x \in \mathbb{R}^n \mid x_n > 0\}$ be the upper half space with the riemannian structure $g_x = \frac{1}{x_n^2}\langle , \rangle$. Find the Christoffel symbols $\Gamma_{ij}^k$ for the Levi Civita connection of $g$.

## 5. The exponential map

In this section we will show that the $\nabla$-geodesics for the Levi Civita connection are exactly the geodesics as defined in section 2.

From (3.14) and the homogeneity of (3.12), it follows that the set

$$O = \{(x, v) \in U \times \mathbb{R}^n \mid 1 \in J(x, v)\} \tag{5.1}$$

is an open neighbourhood of $U \times \{0\}$ in $U \times \mathbb{R}^n$. Moreover, $O_x = O \cap T_xU$ is starshaped around $0_x \in T_xU$. The *exponential* map, $\exp : O \to U$ is defined by

$$\exp(v_x) = c_{(x,v)}(1) \tag{5.2}$$

for all tangent vectors $v_x = (x, v) \in T_xU$ , $x \in U$. By (3.14), the exponential map is clearly smooth and $\exp(0_x) = x$ for all $x \in U$. Consider the smooth map

$$(\pi, \exp) : O \to U \times U \quad , (x, v) \to (x, \exp(v_x)) \tag{5.3}$$

and observe that

$$\begin{array}{ccc} & \begin{array}{cc} n & n \end{array} & \\ D(\pi, \exp)_{(x,0)} = & \begin{pmatrix} id & 0 \\ id & id \end{pmatrix} & \begin{array}{c} n \\ n \end{array} \end{array} \tag{5.4}$$

for all $x \in U$. By the inverse function theorem and $(\pi, \exp)_{|U\times\{0\}}$ injective, it follows that

**Theorem 5.5.** *There are open neighbourhoods $\mathcal{D}$ of $U \times \{0\}$ in $U \times \mathbb{R}^n$ and $\mathcal{V}$ of $\triangle(U)$ in $U \times U$, such that $(\pi, \exp) : \mathcal{D} \to \mathcal{V}$ is a* diffeomorphism*, i.e. it is a smooth bijective map with smooth inverse.*

If $\exp : O_x \to U$ is the restriction of exp to $O_x$ we get immediately

**Corollary 5.6.** *For each $x \in U$ there are open neighbourhoods $\mathcal{D}_x$ of $0_x \in T_xU$ and $\mathcal{V}_x$ of $x \in U$ such that $\exp_x : \mathcal{D}_x \to \mathcal{V}_x$ is a diffeomorphism.*

Since for all $v_x \in O$, $\exp(tv_x) = c_{v_x}(t)$ for $t \in [0,1]$ (cf. 5.4) and $L(c_{v_x|[0,1]}) = \|v_x\| = \langle v_x, v_x\rangle^{\frac{1}{2}}$, we see that $\exp_x$ maps the line segment from $0_x$ to $v_x$ in $T_xU$ onto the $\nabla$-geodesic segment in direction $v_x$ of length $\|v_x\|$. The comparison of euclidean geometry of $T_xU$ near $0_x$ with riemannian geometry of $U$ near $x$ via $\exp_x$ is crucial for the understanding of (local) riemannian geometry.

In the first step of this comparison, it is convenient to view the differential of a map as a map between tangent spaces. To be precise if $f : U \to \mathbb{R}^m$ is differentiable at $x \in U$ with differential $Df_x : \mathbb{R}^n \to \mathbb{R}^m$, the map

$$f_{*_x} : T_xU \to T_{f(x)}\mathbb{R}^m \tag{5.7}$$

defined by $f_{*_x}(x, v) = (f(x), Df_x(v))$ is called the *tangent map*, or the *induced map* of $f$ at $x \in U$. If there is no confusion, it may also simply be referred to as the differential of the map.

If $f$ is differentiable at all points $x \in U$, the induced map $f_* : TU = U \times \mathbb{R}^n \to T\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m$ is simply given by $f_* |_{T_xU} = f_{*_x}$. With this formalism the chain rule takes the pleasant form

$$(f \circ h)_* = f_* \circ h_* \tag{5.8}$$

when $h : U \to V$, $f : V \to \mathbb{R}^k$ are differentiable maps defined on open subsets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$. With this notation we have.

**Theorem 5.9.** *For all $x \in U$, $\exp_x : O_x \to U$ is a radial isometry, i.e.*

$$\langle \exp_*(v_v), \exp_*(u_v)\rangle_{\exp(v)} = \langle v_v, u_v\rangle_x$$

*for all $v \in O_x$ and tangent vectors $u_v \in T_vO_x$.*

**Proof:** Consider the variation

$$V(t,s) = \exp_x(x, t(v + su))$$

for $t \in [0,1]$ and $s$ near zero. Note that each $c_s$ is a $\nabla$-geodesic and the first variation formula (cf. 3.11) gives

$$\frac{d}{ds}L(c_s)_{|s=0} = \|v\|_x^{-1}\langle \exp_{*_v}(u_v), \exp_{*_v}(v_v)\rangle_{\exp(v)}.$$

On the other hand, $L(c_s) = \|v+su\|_x$ and so $\frac{d}{ds}L(c_s)_{|s=0} = \|v\|_x^{-1}\langle u, v\rangle_x = \|v\|_x^{-1}\langle u_v, v_v\rangle_x$. $\square$

This result is often referred to as the *Gauss-lemma*. Using it we will now show that $\nabla$-geodesics are locally length minimizing, i.e. that $\nabla$-geodesics are geodesics in the sense of section 2. To do this, let $\sigma : [0,1] \to O_x$ be any piecewise $C^1$-curve with $\sigma(0) = 0_x$ and $\sigma(1) = v_x$. Then

$$L(\exp \circ \sigma) \geq L(c_{v_x|[0,1]}) = \|v_x\|_x, \tag{5.10}$$

and strict inequality holds if there is a $t_0 \in (0,1]$ such that the component of $\dot\sigma(t_0)$ orthogonal to $\sigma(t_0)_{\sigma(t_0)}$ is not annihilated by $\exp_{*_{\sigma(t_0)}}$.

In proving (5.10) we may assume that $v_x \neq 0$ and $\sigma(t) \neq 0_x$ for all $t \in (0,1]$. Let $a(t)$ be the unit radial vector field along $\sigma$ in $T_xU$. If $\dot\sigma(t)$ is not proportional to $a(t)$ set

$$b(t) = \frac{\dot\sigma(t) - \langle \dot\sigma(t), a(t)\rangle_x a(t)}{\|\dot\sigma(t) - \langle \dot\sigma(t), a(t)\rangle_x a(t)\|_x}$$

For such $t$, $\dot\sigma = \langle a(t), \dot\sigma(t)\rangle_x a(t) + \langle b(t), \dot\sigma(t)\rangle_x b(t)$ and hence by the Gauss-lemma (5.9)

$$\|(\dot{\exp\circ\sigma})(t)\|^2_{\exp(\sigma(t))} = \langle a(t), \dot\sigma(t)\rangle_x^2 + \langle b(t), \dot\sigma(t)\rangle_x^2 \|\exp_* b(t)\|^2_{\exp(\sigma(t))}$$

In particular, $\|(\dot{\exp\circ\sigma})(t)\|_{\exp(\sigma(t))} \geq | \langle a(t), \dot\sigma(t)\rangle_x |$ for all $t \in (0,1]$, and strict inequality follows if there is a $t_0$ as described under (5.10). On the other hand, $\frac{d}{dt}\|\sigma\|_x = \langle \dot\sigma, a\rangle_x$ and hence

$$\begin{aligned} L(\exp\circ\sigma) &= \int_0^1 \|(\dot{\exp\circ\sigma})(t)\|_{\exp(\sigma(t))} dt \\ &\geq \int_0^1 | \langle a(t), \dot\sigma(t)\rangle_x | \, dt \\ &\geq \int_0^1 \frac{d}{dt}\|\sigma(t)\|_x dt \\ &= \|v\|_x \\ &= L(c_{v_x|[0,1]}) \end{aligned}$$

This proves (5.10) and the equality discussion.

Now choose $\delta > 0$ so that $\exp_x$ is a diffeomorphism from $D_x(\delta) = \{(x,v) \in O_x \mid \|v\|_x < \delta\}$ onto $\mathcal{V}_x(\delta)$. From (5.10) it follows that for each $y \in \mathcal{V}_x(\delta)$ the geodesic, $c(t) = \exp_x(t \exp_x^{-1}(y))$ is a curve from $x$ to $y$ whose length is shorter than the length of any other piecewise $C^1$-curve in $U$ from $x$ to $y$. In particular, $\mathcal{V}_x(\delta) = B(x,\delta)$ is the metric $\delta$-ball in $U$ centered at $x$.

**Problem 5.11.** Show that $O_x$ is open and starshaped.

**Problem 5.12.** Verify (5.4).

**Problem 5.13.** Show that $B(x,\delta) = \{y \in U \mid \operatorname{dist}(x,y) < \delta\} = \exp_x(D_x(\delta))$ for $\delta$ sufficiently small.

**Problem 5.14.** Show that $\operatorname{dist}_{g_1} = \operatorname{dist}_{g_2}$ if and only if $g_1 = g_2$.

**Problem 5.15.** Consider the metric space $(\mathbb{R}^n, \operatorname{dist})$, where $\operatorname{dist}(x,y) = \max|x_i - y_i|$. Show that $\operatorname{dist} \neq \operatorname{dist}_g$ for any riemannian structure $g$ on $\mathbb{R}^n$.

## 6. Isometries

Let $(U_1, g_1), (U_2, g_2)$ be two riemannian patches, and $F : (U_1, \operatorname{dist}_{g_1}) \to (U_2, \operatorname{dist}_{g_2})$ an *isometry*, i.e. $F$ is a distance preserving map with $F(U_1) = U_2$. The inverse $F^{-1} : U_2 \to U_1$ is also an isometry. In particular, $F$ is a homeomorphism and $\dim U_1 = \dim U_2 = n$. The following is crucial in our treatment of riemannian geometry.

**Theorem 6.1.** *An isometry $F$ between riemannian patches $(U_1, g_1), (U_2, g_2)$ is a diffeomorphism. Moreover, $F_{*_x} : T_xU_1 \to T_{F(x)}U_2$ is a linear isometry for all $x \in U_1$, and $F \circ \exp_x = \exp_{F(x)} \circ F_{*_x}$.*

**Proof:** Fix $x \in U_1$ and choose $\delta > 0$ such that $\exp_y : D_y(2\delta) \to B(y, 2\delta)$ and $\exp_{F(y)} : D_{F(y)}(2\delta) \to B(F(y), 2\delta)$ are diffeomorphisms for all $y \in B(x, 2\delta)$ (cf. 5.5) as in the last paragraph of section 5.

Since $F$ is an isometry, it maps for each $y \in B(x, \delta)$ the unique minimal geodesic from $x$ to $y$, to the unique minimal geodesic from $F(x)$ to $F(y)$ (cf. 6.2). Since $\exp_x : D_x(\delta) \to B(x, \delta)$ and $\exp_{F(x)} : D_{F(x)}(\delta) \to B(F(x), \delta)$ are diffeomorphisms, we only need to show that $\exp_{F(x)}^{-1} \circ F \circ \exp_x : D_x(\delta) \to D_{F(x)}(\delta)$ is smooth. In fact, we claim that it is the restriction of a linear isometry $T_xU_1 \to T_{F(x)}U_2$. We have already seen, that it preserves the norm of vectors. To show that it preserves angles, consider vectors $u_x, v_x \in D_x(\delta)$, and let $V : [0,1] \times [0,1] \to B(x, 2\delta)$ be the variation

$$V(t, s) = \exp_{\exp(v_x)}(t \cdot \exp_{\exp(v_x)}^{-1}(\exp_x(s \cdot u_x))).$$

For each $s$, this is the unique minimal geodesic from $\exp(v_x)$ to $\exp(su_x)$. The first variation formula (3.11) then yields

$$\frac{d}{ds} L(c_s)_{|s=0} = \|v_x\|_x^{-1} \langle u_x, -v_x \rangle_x = -\|u\|_x \cos \vartheta,$$

where $\vartheta$ is the angle between $u_x$ and $v_x$. Again, however, since $F$ is an isometry it maps the variation $V$ to the corresponding variation for the vectors $\exp_{F(x)}^{-1} \circ F \circ \exp_x(u_x)$, and $\exp_{F(x)}^{-1} \circ F \circ \exp_x(v_x)$. Repeating the argument above, then shows that the angle between these vectors is also $\vartheta$. The argument also shows that $F_{*_x} = \exp_{F(x)}^{-1} \circ F \circ \exp_x$, which completes the proof. □

**Problem 6.2.** Show that isometries map geodesics to geodesics in any length-space.

**Problem 6.3.** Show that isometries between riemannian patches preserve the covariant derivative of the corresponding Levi-Civita connections. Is this true for the connection defined in (3.3) and (3.7)?

**Problem 6.4.** What are the geodesics on $(U, g)$ corresponding to the hemisphere, graph $(f)$; $f : U = \{x \in \mathbb{R}^n \mid \|x\| < 1\} \to \mathbb{R}$, $x \to \sqrt{1 - \|x\|^2}$ (cf. (1.9) and the paragraph below it).

Hint: It is possible to argue using local uniqueness of geodesics together with (6.2).

**Problem 6.5.** Let $F : U_1 \to U_2$ be a smooth bijective map between riemannian patches $(U_1, g_1), (U_2, g_2)$ such that

$$g_2(F_*(v_*), F_*(u_*)) = g_1(v_x, u_x)$$

for all $x \in U_1$ and tangent vectors $u_x, v_x \in T_xU_1$. Show that $F : (U_1, \operatorname{dist} g_1) \to (U_2, \operatorname{dist} g_2)$ is an isometry.

**Problem 6.6.** Let $(U_{-1}, g_{-1})$ be the 2-dimensional Poincaré disc (cf. 1.16) and $(H, g)$ the upper half plane (cf. 4.42). Identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and show that $z \to i\frac{z-i}{z+i}$ defines an isometry $F : H \to U_{-1}$. What are the geodesics on $H, U_{-1}$?

## 7. Jacobi fields and curvature

In the last section we saw that a bijective map of riemannian patches is an isometry if and only if it is smooth and its induced map is an isometry on all tangent spaces. To continue our comparison between local euclidian and local riemannian geometry we now proceed to investigate the map induced by the exponential map.

Fix $x \in U$ and $v \in O_x$. We have already seen a glimpse of how to describe $(\exp_x)_{*v} : T_v O_x \to T_{\exp(v)} U$. Namely, any tangent vector to $O_x$ at $v$ is represented by a curve of the form $\sigma(s) = v + s \cdot u$ for some $u \in T_x U$. Therefore, for this tangent vector $u_v \in T_v O_x$, $\exp_*(u_v)$ is represented by the curve $\exp(v + s \cdot u)$. To get a better description of this consider as in the proof of 5.9 the variation

$$V(t, s) = \exp_x(t(v + su))$$

for $t \in [0, 1]$ and $s$ near zero. Using the notation from section 3, $\sigma_1 = \exp \circ \sigma$, and for $s = 0$, the variation field $X = X_0$ along $c$ is given by $X(t) = \dot{\sigma}_t(0) = \exp_*(tu_{t \cdot v})$. Since each $c_s$ is a geodesic and hence satisfies a second order equation we expect each $X_s$ to do the same. Now $\nabla_{c_s} \dot{c}_s = 0$ for all $t$ and all $s$ gives

$$\begin{aligned} 0 =& \nabla_{\sigma_t} \nabla_{c_s} \dot{c}_s \\ =& \nabla_{\sigma_t} \nabla_{c_s} \dot{c}_s - \nabla_{c_s} \nabla_{\sigma_t} \dot{c}_s + \nabla_{c_s} \nabla_{\sigma_t} \dot{c}_s \\ =& \nabla_{\sigma_t} \nabla_{c_s} \dot{c}_s - \nabla_{c_s} \nabla_{\sigma_t} \dot{c}_s + \nabla_{c_s} \nabla_{c_s} \dot{\sigma}_t \end{aligned} \tag{7.1}$$

by (4.11) (cf. also 3.9). For fixed $s$, the last term is $\nabla_{c_s} \nabla_{c_s} X_s$. Moreover, if $Z$ is any vector field along $V$, then

$$\begin{aligned} \nabla_{\sigma_t} \nabla_{c_s} Z(t, s) =& \nabla_{\sigma_t} \left( \frac{\partial}{\partial t} Z + \Gamma_{c_s(t)}(c_s', Z) \right) \\ =& \frac{\partial}{\partial s} \frac{\partial}{\partial t} Z + \frac{\partial}{\partial s} [\Gamma_{V(t,s)}(c_s', Z)] + \\ & \Gamma_{V(t,s)}(\sigma_t', \frac{\partial}{\partial t} Z + \Gamma_{V(t,s)}(c_s', Z)) \\ =& \frac{\partial}{\partial s} \frac{\partial}{\partial t} Z + D\Gamma_{V(t,s)}(\frac{\partial}{\partial s} V)(\frac{\partial}{\partial t} V, Z) + \\ & \Gamma_{V(t,s)}(\frac{\partial}{\partial t} V, \frac{\partial}{\partial s} Z) + \Gamma_{V(t,s)}(\frac{\partial}{\partial s} \frac{\partial}{\partial t} V, Z) \\ & + \Gamma_{V(t,s)}(\frac{\partial}{\partial s} V, \frac{\partial}{\partial t} Z) + \Gamma_{V(t,s)}(\frac{\partial}{\partial s} V, \Gamma_{V(t,s)}(\frac{\partial}{\partial t} V, Z)) \end{aligned}$$

and similarly

$$\begin{aligned} \nabla_{c_s} \nabla_{\sigma_t} Z(t, s) =& \frac{\partial}{\partial t} \frac{\partial}{\partial s} Z + D\Gamma_{V(t,s)}(\frac{\partial}{\partial t} V)(\frac{\partial}{\partial s} V, Z) + \\ & \Gamma_{V(t,s)}(\frac{\partial}{\partial s} V, \frac{\partial}{\partial t} Z) + \Gamma_{V(t,s)}(\frac{\partial}{\partial t} \frac{\partial}{\partial s} V, Z) + \\ & \Gamma_{V(t,s)}(\frac{\partial}{\partial t} V, \frac{\partial}{\partial s} Z) + \Gamma_{V(t,s)}(\frac{\partial}{\partial t} V, \Gamma_{V(t,s)}(\frac{\partial}{\partial s} V, Z)). \end{aligned}$$

Hence

$$\begin{aligned} \nabla_{\sigma_t} \nabla_{c_s} Z - \nabla_{c_s} \nabla_{\sigma_t} Z =& D\Gamma_{V(t,s)}(\sigma_t')(c_s', Z) - D\Gamma_{V(t,s)}(c_s')(\sigma_t', Z) + \\ & \Gamma_{V(t,s)}(\sigma_t', \Gamma(c_s', Z)) - \Gamma_{V(t,s)}(c_s', \Gamma(\sigma_t', Z)) \end{aligned} \tag{7.2}$$

which clearly depends on $Z, \sigma_t'$ and $c_s'$ at $(t, s)$. Inserting (7.2) into (7.1) at $s = 0$ yields

$$\nabla_c \nabla_c X + R(X, \dot{c})\dot{c} = 0 \tag{7.3}$$

for the variation field $X = X_0$ along $c = c_0$. Here for each $x \in U$, $R_x : T_xU \times T_xU \times T_xU \to T_xU$ is the 3-linear map defined by the right hand side of (7.2), i.e.

$$\begin{aligned} R_x(u_x, v_x)z_x =& D\Gamma_x(u)(v,z) - D\Gamma_x(v)(u,z) + \\ & \Gamma_x(u, \Gamma_x(v,z)) - \Gamma_x(v, \Gamma_x(u,z)) \end{aligned} \tag{7.4}$$

for all $u, v, z \in \mathbb{R}^n$. Observe, that for any $u_x, v_x, z_x \in T_xU$ there is a variation $V$ and a vector field $Z$ such that $V(0,0) = x$, $\frac{\partial}{\partial s}V(0,0) = u$, $\frac{\partial}{\partial t}V(0,0) = v$ and $Z(0,0) = z$ (take e.g. $V(t,s) = x + t \cdot v + s \cdot u$ and $Z(t,s) = z$ for all $t, s$ near zero).

In general for vector fields $X, Y$, and $Z$ on $U$ one finds

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{7.5}$$

One way to see this is to note that the right hand side is linear with respect to functions in all 3 variables. It then follows, that each $x \in U$ it only depends on $X, Y$ and $Z$ at $x$. Choosing $X, Y$, and $Z$ as above together with (7.2) and (7.4), then proves (7.5). This expression makes sense for any connection $\nabla$. It is called the *curvature tensor* of $\nabla$.

The equation (7.3) is called the *Jacobi equation*, and any vector field $X$ along a geodesic $c$, which satisfies this equation is called a *Jacobi field* along $c$. The Jacobi equation is clearly a second order linear differential equation. In particular

**Theorem 7.6.** *For any (maximal) geodesic $c : J \to U$ and tangent vectors $u, v \in T_{c(0)}U$ there is a unique Jacobi field $X : J \to TU$, along $c$ with $X(0) = u$, and $\nabla_c X(0) = v$.*

Using the above, we now see that for $v \in O_x$ and $u \in T_xU$, $X(t) = \exp_*(tu_{tv})$ is the unique Jacobi field along $c_v : J_v \to U$ with initial conditions

$$X(0) = 0_x, \quad \nabla_{c_v} X(0) = u. \tag{7.7}$$

Because of the Gauss-lemma (5.9) we are particularly interested in $\|X(t)\|_{\exp(tv)}$ when $u$ is perpendular to $v$. Rather than $\|X(t)\|$, let us consider $f(t) = \|X(t)\|^2 = \langle X(t), X(t)\rangle_{\exp(tv)}$. Abbreviating $\nabla_c X$ simply by $X'$ we have

$$\begin{aligned} f(0) =& \langle X(t), X(t)\rangle(0) = 0 \\ f'(0) =& 2\langle X'(t), X(t)\rangle(0) = 0 \\ f''(0) =& 2\{\langle X''(t), X(t)\rangle + \langle X'(t), X'(t)\rangle\}(0) = 2\|u\|^2 \\ f'''(0) =& 2\{\langle X'''(t), X(t)\rangle + 3\langle X''(t), X'(t)\rangle\}(0) \\ =& 2\{\langle X'''(0), X(0)\rangle - 3\langle R(X(0), v)v, u\rangle\} = 0 \\ f''''(0) =& 2\{\langle X''''(t), X(t)\rangle + 4\langle X'''(t), X'(t)\rangle + 3\langle X''(t), X''(t)\rangle\}(0) \\ =& 8\langle X'''(0), u\rangle. \end{aligned}$$

The Taylor expansion for $\|X(t)\|^2$ therefore looks like

$$\|X(t)\|^2 \sim \|u\|^2 t^2 + \frac{1}{3}\langle X'''(0), u\rangle t^4 + \dots$$

Here the first term only depends on $g$ at $x$. For the second term, however

$$\begin{aligned} X''' =& -(R(X, \dot{c})\dot{c})' \\ =& -R'(X, \dot{c})\dot{c} - R(X', \dot{c})\dot{c} - R(X, \dot{c}')\dot{c} - R(X, \dot{c})\dot{c}' \end{aligned}$$

by (7.3) and (7.13), i.e. $X'''(0) = -R(u,v)v$. Assuming without loss of generality, that $\|u\| = \|v\| = 1$, we get

$$\|\exp_*(tu)_{tv}\|^2 = t^2 - \frac{1}{3}sec(p)t^4 + 0(t^5), \tag{7.8}$$

where $sec(p) = \langle R(u,v)v, u\rangle$ is called the *sectional curvature* of the plane $p$ spanned by $u$ and $v$ in $T_xU$. It is easy to see that for linearly independent $u$ and $v$,

$$sec(p) = \frac{\langle R(u,v)v, u\rangle}{\|u\|^2\|v\|^2 - \langle u, v\rangle^2} \tag{7.9}$$

only depends on the 2-plane $p$ spanned by $u, v$. From (7.8) it is evident that it is the sectional curvature that determines whether exp is expanding or contracting.

**Problem 7.10.** Prove formula 7.5.

**Problem 7.11.** Prove Theorem 7.6.

**Problem 7.12.** Let $X$ be a Jacobi field along the geodesic $c$. Write $X = X^\perp + X^\top$, where $X^\top$ is proportional to $\dot{c}$ and $X^\perp$ is perpendicular to $\dot{c}$. Show that $X^\perp$ and $X^\top$ are Jacobi fields along $c$. Show that $X^\top(t) = (a+bt)\dot{c}(t)$ for constants $a, b \in \mathbb{R}$.

**Problem 7.13.** Show that for each vector field $X$,

$$\nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y,Z)\nabla_X W$$

is linear with respect to functions, in all three variables $Y, Z$ and $W$. Therefore, it defines for each $x \in U$ a 3-linear map $(\nabla_X R)_x : T_xU \times T_xU \times T_xU \to T_xU$ called the covariant derivative of $R$ in direction $X$.

**Problem 7.14.** In the spirit of 7.13, show that $\nabla_X g = 0$ for all vector fields $X$.

**Problem 7.15.** Show the *Bianchi identity*

$$(\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(X,Y)W = 0$$

for all vector fields $X, Y, Z$ and $W$ (cf. 7.13).

**Problem 7.16.** Show that (7.9) defines a function on two-dimensional subspaces of $T_xU, x \in U$.

## 8. Curvature identities

In this section we will show that knowing the curvature tensor $R$ or the sectional curvature function $sec$ amounts to the same thing.

**Theorem 8.1.** *The curvature tensor $R$ of the Levi Civita connection $\nabla$ for a riemannian structure $\langle,\rangle$ on $U$ satisfies the identities*

(i) $R(X,Y)Z = -R(Y,X)Z$
(ii) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$
(iii) $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$
(iv) $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$

*for all vector fields $X, Y, Z$ and $W$ on $U$.*

**Proof:** The first identity is obvious by 7.5. To prove the remaining identities first note that all expression are *tensorial*, i.e. for each $x \in U$ they depend only on $X, Y, Z$, and $W$ at $x$. To prove them we are therefore free to choose $X, Y, Z$, and $W$ so that all Lie brackets are zero (e.g. extend $X_x, Y_x, Z_x$, and $W_x$ to constant vector fields on $U$).

Then by (4.11) and (7.5)

$$\begin{aligned}\nabla_X Y &= \nabla_Y X\\ R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z\end{aligned}$$

and similarly for all other combinations of $X, Y, Z$, and $W$. The identity (ii) is then a straight forward computation. Moreover, (iii) is equivalent to

(iii)' $\langle R(X,Y)Z, Z\rangle = 0$

for all $X, Y$, and $Z$. Now

$$\begin{aligned}\langle R(X,Y)Z,Z\rangle =&\langle \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z, Z\rangle\\ =&X[\langle\nabla_Y Z, Z\rangle] - \langle\nabla_Y Z, \nabla_X Z\rangle\\ &- Y[\langle\nabla_X Z, Z\rangle] + \langle\nabla_X Z, \nabla_Y Z\rangle\\ =&X[\frac{1}{2}Y[\langle Z,Z\rangle]] - Y[\frac{1}{2}X[\langle Z,Z\rangle]]\\ =&\frac{1}{2}[X,Y][\langle Z,Z\rangle]\\ =&0,\end{aligned}$$

where we have used (4.16) and (4.13). The last identity is a purely formal consequence of the first three. From (ii) we have

(iv)' $\langle R(X,Y)Z,W\rangle + \langle R(Y,Z)X,W\rangle + \langle R(Z,X)Y,W\rangle = 0$

and interchanging $W$ in turn with $X, Y$, and $Z$ yields

(iv)" $\langle R(W,Y)Z,X\rangle + \langle R(Y,Z)W,X\rangle + \langle R(Z,W)Y,X\rangle = 0$
(iv)"' $\langle R(X,W)Z,Y\rangle + \langle R(W,Z)X,Y\rangle + \langle R(Z,X)W,Y\rangle = 0$
(iv)"" $\langle R(X,Y)W,Z\rangle + \langle R(Y,W)X,Z\rangle + \langle R(W,X)Y,Z\rangle = 0$

By adding (iv)'-(iv)"" and using (i) and (iii) we get

$$\langle R(X,W)Z + R(W,Z)X, Y\rangle + \langle R(W,Y)Z,X\rangle = 0.$$

Hence (ii) implies

$$-\langle R(Z,X)W,Y\rangle + \langle R(W,Y)Z,X\rangle = 0$$

which up to a change of letters is (iv).

We are now ready to show that the map

$$(X,Y,Z,W) \to \langle R(X,Y)Z,W\rangle$$

is completely determined by the "bi quadratic" form

$$k(X,Y) = \langle R(X,Y)Y,X\rangle \tag{8.2}$$

for all $X, Y$. First we want to rewrite $R(X,Y)Z$ :

$$\begin{aligned}R(X,Y+Z)(Y+Z) =&R(X,Y)Y + R(X,Y)Z + R(X,Z)Y + R(X,Z)Z\\ R(X+Z,Y)(X+Z) =&R(X,Y)X + R(X,Y)Z + R(Z,Y)X + R(Z,Y)Z\\ 0 =&R(X,Y)Z + R(Y,X)Z\end{aligned}$$

Adding these three equations and applying (i) and (ii) from 8.1 gives

$$\begin{aligned} R(X,Y+Z)(Y+Z)-R(Y,X+Z)(X+Z) =&3R(X,Y)Z+R(X,Y)Y\\ &-R(Y,X)X+R(X,Z)Z-R(Y,Z)Z \end{aligned}$$

and hence

$$\begin{aligned} R(X,Y)Z =&\frac{1}{3}\{R(X,Y+Z)(Y+Z)-R(Y,X+Z)(X+Z)\\ &-R(X,Y)Y+R(Y,X)X-R(X,Z)Z+R(Y,Z)Z\} \end{aligned} \tag{8.3}$$

Now fix a vector field $B$ on $U$ and consider the map $\omega_B$ defined by

$$\omega_B(X,Y)=\langle R(X,B)B,Y\rangle$$

for all vector fields $X,Y$ on $U$. From (i), (iii) and (iv) of 8.1 we see that $\omega_B$ is symmetric. In particular

$$\begin{aligned} 2\langle R(X,B)B,Y\rangle =&2\omega_B(X,Y)\\ =&\omega_B(X+Y,X+Y)-\omega_B(X,X)-\omega_B(Y,Y)\\ =&k(X+Y,B)-k(X,B)-k(Y,B). \end{aligned} \tag{8.4}$$

Combining (8.3) and (8.4) results in

$$\begin{aligned} \langle R(X,Y)Z,W\rangle =&\frac{1}{6}\{k(X+W,Y+Z)-k(Y+W,X+Z)\\ &-k(X+W,Y)-k(X+W,Z)+k(Y+W,X)\\ &+k(Y+W,Z)-k(X,Y+Z)+k(Y,X+Z)\\ &+k(W,X+Z)-k(W,Y+Z)+k(X,Z)\\ &-k(Y,Z)-k(W,X)+k(W,Y)\}. \end{aligned} \tag{8.5}$$

Since $k(u_x,v_x)=\langle R(u_x,v_x)v_x,u_x\rangle = sec(span\ (u_x,v_x))$ for linearly independent $u_x,v_x\in T_xU$ and $k(u_x,v_x)=0$ for linearly dependent $u_x,v_x$, it is clear that the sectional curvature determines the curvature tensor $R$. In particular, $R=0$ if and only if $sec=0$.

The *Ricci-tensor*, $c_1R$ is defined for each $x\in U$ as the bilinear map

$$c_1R(u_x,v_x)=trace\ (w_x\to R(w_x,u_x)v_x) \tag{8.6}$$

for all $u_x,v_x\in T_xU$. Clearly Ricci $=\frac{c_1R(u_x,u_x)}{\|u_x\|^2}$, $u_x\neq 0$ only depends on the line spanned by $u_x$. This is called the *Ricci curvature*. Clearly

$$Ric(l)=\sum_{i=1}^{n-1}sec(\sigma_i), \tag{8.7}$$

for $l=span\{X\}$ and $\sigma_i=span\{X,u_i\}$, where $X,u_1,\dots,u_{n-1}$ is an orthonormal basis for $T_xU$. Averaging once more defines the *scalar curvature* $Scal:U\to\mathbb{R}$, i.e.

$$Scal(x)=\sum_{i=1}^{n}c_1R(u_i,u_i)=2\sum_{1\le i<j\le n}sec(\sigma_{ij}), \tag{8.8}$$

where $u_1,\dots,u_n$ is an orthonormal basis for $T_xU$ and $\sigma_{ij}=span\{u_i,u_j\}$. Note, that it is only for $n>2$ that these curvatures are essentially different.

**Problem 8.9.** Show that $R_1$ defined by

$$R_1(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y$$

is tensorial, and satisfies (i)-(iv) of 8.1. This will be shown to be the curvature tensor of a space with constant (sectional) curvature 1. Note that the corresponding bi quadratic form $k_1$ is given by

$$k_1(X,Y) = \|X\|^2\|Y\|^2 - \langle X,Y\rangle^2,$$

and hence $sec(span(u_x,v_x)) = \frac{k(u_x,v_x)}{k_1(u_x,v_x)}$, for $u_x, v_x$ linearly independent (equivalently $k_1(u_x,v_x) > 0$).

**Problem 8.10.** Prove (8.7) and the second equality in (8.8).

## 9. Second variation of arc length and convexity

We have seen how curvature controlles the local behaviour of geodesics. To see that it also affects the length of curves near a geodesic, we will compute the second variation of arc length.

A *two parameter variation* of a curve $c : [a,b] \to U$ is a continuous map

$$W : [a,b] \times (-\varepsilon_1,\varepsilon_1) \times (-\varepsilon_2,\varepsilon_2) \to U$$

such that $W(t,0,0) = c(t)$ for $t \in [a,b]$. We assume $W$ is piecewise $C^k,\ k \geq 2$, i.e. there is a partion $a = t_0 < \cdots < t_m = b$ of $[a,b]$ for which $W_{|[t_{i-1},t_i]\times(-\varepsilon_1,\varepsilon_1)\times(-\varepsilon_2,\varepsilon_2)}$ is $C^k,\ i = 1,\dots,m$. As in the case of a one parameter variation we let $c_{s_1,s_2}$ be the curve $W(\cdot,s_1,s_2)$ , and we are interested in $L(c_{s_1,s_2})$ for $s_1, s_2$ near zero when $c_{0,0} = c$ is a geodesic. Again it is sufficient to understand the case $m = 1$, i.e. $W$ is of class $C^k$. If $e_0, e_1, e_2$ are the standard coordinate vector fields on $[a,b]\times(-\varepsilon_1,\varepsilon_1)\times(-\varepsilon_2,\varepsilon_2)$ we set

$$T = W_* \circ e_0,\ X_1 = W_* \circ e_1,\ X_2 = W_* \circ e_2$$

and call $X_1, X_2$ the variation vector fields along $W$. Assume $\|\dot c\| = l \neq 0$, then $L(c_{s_1,s_2})$ is of class $C^2$ near $(0,0)$ and

$$\text{(9.1)} \qquad \frac{\partial_2}{\partial s_1 \partial s_2} L(c_{s_1,s_2})(0,0) = l^{-1}\{\langle \nabla_{X_1} X_2, T\rangle \mid_a^b + I(X_1^\perp, X_2^\perp)\}$$

where

$$\text{(9.2)} \qquad I(Y_1,Y_2) = \int_a^b \langle \nabla_c Y_1, \nabla_c Y_2\rangle - \langle R(Y_1,\dot c)\dot c, Y_2\rangle$$

is the socalled *index form* on the vector space of vector fields along $c$. The formula (9.1) is referred to as the *second variation of arc length.*

To prove (9.1) we proceed as follows (cf. 3.10)

$$\begin{aligned}\frac{\partial}{\partial s_2} L(c_{s_1,s_2}) &= \int_a^b \frac{\partial}{\partial s_2}\langle T,T\rangle^{\frac{1}{2}} dt\\ &= \int_a^b \|T\|^{-1}\langle \nabla_{X_2} T, T\rangle dt\\ &= \int_a^b \|T\|^{-1}\langle \nabla_T X_2, T\rangle dt,\end{aligned}$$

and therefore

$$\begin{aligned}\frac{\partial^2}{\partial s_1 \partial s_2} L(c_{s_1,s_2}) &= \int_a^b (-1)\|T\|^{-3}\langle \nabla_{X_1} T, T\rangle\langle \nabla_T X_2, T\rangle dt \\ &+ \int_a^b \|T\|^{-1}\{\langle \nabla_{X_1}\nabla_T X_2, T\rangle + \langle \nabla_T X_2, \nabla_{X_1} T\rangle\} dt \\ &= -\int_a^b \|T\|^{-3}\langle \nabla_T X_1, T\rangle\langle \nabla_T X_2, T\rangle dt \\ &+ \int_a^b \|T\|^{-1}\{\langle R(X_1,T)X_2, T\rangle + \langle \nabla_T \nabla_{X_1} X_2, T\rangle \\ &\quad + \langle \nabla_T X_2, \nabla_T X_1\rangle\} dt.\end{aligned}$$

At $(s_1, s_2) = (0,0)$ we have

$$\begin{aligned}\frac{\partial^2}{\partial s_1 \partial s_2} L(c_{s_1,s_2}) =& l^{-1}\{\langle \nabla_{X_1} X_2, T\rangle \mid_a^b - \int_a^b \langle R(X_1,T)T, X_2\rangle \\ &+ \int_a^b \langle \nabla_T X_1, \nabla_T X_2\rangle - \langle \nabla_T X_1, \frac{T}{\|T\|}\rangle\langle \nabla_T X_2, \frac{T}{\|T\|}\rangle\}\end{aligned} \tag{9.3}$$

Now $X_i = X_i^\perp + \langle X_i, \frac{T}{\|T\|}\rangle \frac{T}{\|T\|}$ and hence $\nabla_T X_i = \nabla_T X_i^\perp + \langle \nabla_T X_i, \frac{T}{\|T\|}\rangle \frac{T}{\|T\|}$ since $c$ is a geodesic. Since $\langle X_i^\perp, T\rangle = 0$ also $\langle \nabla_T X_i^\perp, T\rangle = 0$ and (9.1) follows from (9.3).

If in particular $W$ is a variation with fixed end points, i.e. $W(a, s_1, s_2) = c(a)$ and $W(b, s_1, s_2) = c(b)$ for all $s_1, s_2$, then $\frac{\partial^2}{\partial s_1 \partial s_2} L(c_{s_1,s_2})(0,0) = l^{-1} \cdot I(X_1^\perp, X_2^\perp)$.

Observe that if $I(X,Y) = 0$ for all vector fields $Y$ along $c$ which vanishes at the end points, then

$$\nabla_c \nabla_c X + R(X, \dot c)\dot c = 0$$

i.e. $X$ is a Jacobi field along $c$.

We will now use (9.1) to prove

**Theorem 9.4.** *For every $x \in U$ there is an $\varepsilon > 0$ such that the ball $B(x,\varepsilon)$ is strictly convex, i.e. for any $y, z \in B(x,\varepsilon)$ there is a unique minimal geodesic $c$ in $U$ from $y$ to $z$, and $c$ is contained in $B(x,\varepsilon)$.*

**Proof:** Fix $x \in U$ and $\delta > 0$ such that $\exp_y : D(2\delta) \to B(y, 2\delta)$ is a diffeomorphism for all $y \in B(x,\delta)$. For $y, z \in B(x,\delta)$ let $c : [0, \operatorname{dist}(y,z)] \to B(x, 2\delta)$ be the unique minimal geodesic from $y$ to $z$ and consider the 2-parameter variation

$$W(t, s_1, s_2) = \exp_x(t \cdot \exp_x^{-1} c(s_1 + s_2))$$

for $t \in [0,1]$. For fixed $s_1, s_2$ we have that $L(c_{s_1,s_2}) = \operatorname{dist}(x, c(s_1+s_2))$ and therefore $\frac{\partial^2}{\partial s_1 \partial s_2} L(c_{s_1,s_2})$ at $s = s_1 + s_2$ is the same as $\frac{d^2}{ds^2}\operatorname{dist}(x, c(s))$, assuming $c(s) \neq x$. Thus using (9.1) we get

$$\frac{d^2}{ds^2}\operatorname{dist}(x, c(s)) = \operatorname{dist}(x, c(s))^{-1} I(X^\perp, X^\perp), \tag{9.5}$$

where $X_1 = X_2 = X$ is the Jacobi field along the minimal geodesic $c_s$ from $x$ to $c(s)$ with $X(0) = 0_x$ and $X(1) = \dot c(s)$. Since $X^\perp$ is a Jacobi field, we obtain by (9.2), that

$$\begin{aligned}\frac{d^2}{ds^2}\operatorname{dist}(x,c(s)) &= \operatorname{dist}(x,c(s))^{-1}\langle\nabla_{c_s}X^\perp(1),X^\perp(1)\rangle\\ &= \frac{1}{2}\operatorname{dist}(x,c(s))^{-1}\langle X^\perp,X^\perp\rangle'(1)\end{aligned}$$

Now suppose $c(s)\notin B(x,\delta)$ for some $s\in(0,1)$. Then $\operatorname{dist}(x,c(s))$ has a local maximum at say $s_0\in(0,1)$. For the corresponding $c_{s_0}$ we have $X=X^\perp$ and $\frac{d^2}{ds^2}\operatorname{dist}(x,c(s_0))\le 0$. For $\delta$ sufficiently small, however, this is impossible according to the following lemma.

**Lemma 9.6.** *For any $x\in U$ there is an $\varepsilon>0$ such that for every $y\in B(x,\varepsilon)$ and every unit vector $u\in T_yU$ perpendicular to the unique minimal geodesic $c_{xy}:[0,1]\to B(x,\varepsilon)\subset U$ from $x$ to $y$ there is a unique Jacobi field $X$ along $c_{xy}$ with $X(0)=0$ and $X(1)=u$. Moreover $\langle X,X\rangle'(1)>0$ for all such $X$.*

**Proof:** The first follows from section 7 when $\varepsilon>0$ is chosen so that $\exp_x : D_x(\varepsilon)\to B(x,\varepsilon)$ is a diffeomorphism. By the construction of $X$ in section 7 we get

$$\|X(t)\|\le\|\exp_{*_{tv}}\|\cdot\|X'(0)\|, \tag{9.7}$$

where $v=\exp_x^{-1}(y)\in D_x(\varepsilon)$, and $\|\exp_{*_{tv}}\|$ denotes the operator norm of $\exp_{*_{tv}}: T_{tv}O_x\to T_{c_{xy}(t)}U$. In particular

$$\langle X,X\rangle'(0)=0,\quad \langle X,X\rangle''(0)\ge\|\exp_{*_v}\|^{-1}. \tag{9.8}$$

Since $X$ is a Jacobi field, then

$$\begin{aligned}\langle X,X\rangle'(1) &= \int_0^1 2\{\langle X'(t),X'(t)\rangle-\langle R(X(t),\dot c_{xy}(t))\dot c_{xy}(t),X(t)\rangle dt\\ &\ge 2\int_0^1\|X'(t)\|^2dt-2\cdot E^2\cdot\|X'(0)\|^2\cdot\varepsilon^2\cdot C,\end{aligned} \tag{9.9}$$

where $E$ is an upper bound for $\|\exp_*\|$ on $D_x(\varepsilon)$ and $C$ is an upper bound for $sec^+$ the non-negative of the sectional curvature on $B(x,\varepsilon)$. Now

$$\begin{aligned}\big|\|X'(t)\|^2-\|X'(0)\|^2\big| &= \left|\int_0^t -2\langle R(X,\dot c_{xy})\dot c_{xy},X'\rangle\right|\\ &\le 2\cdot\varepsilon^2\cdot E\cdot\|X'(0)\|\cdot\max\|X'\|\cdot r\end{aligned} \tag{9.10}$$

where $r$ is an upper bound for $\|R\|$ on $B(x,\varepsilon)$. Using (9.10) for $\|X'(t)\|=\max\|X'\|$ gives

$$\max\|X'\|\le\|X'(0)\|\{\sqrt{1+\varepsilon^4E^2r^2}+\varepsilon^2Er\}\le\|X'(0)\|\{1+2\varepsilon^2E\cdot r\} \tag{9.11}$$

Combining (9.9)-(9.11) clearly shows that $\langle X,X\rangle'(1)>0$ for sufficiently small $\varepsilon$ independent of $X$.

**Problem 9.12.** Let $(U,g)$ be a riemannian patch and $x\in U$. Show that the metric on $B(x,\varepsilon)$ induced from $\operatorname{dist}_g$ is the same as $\operatorname{dist}_{g_|}$ when $B(x,\varepsilon)$ is strictly convex. Here $g_|$ is the restriciton of $g$ to $B(x,\varepsilon)$.

**Problem 9.13.** Let $(U,g)$ be riemannian patch with non-positive curvature, i.e. $Sec\le 0$. Show that any geodesic $c:[a,b]\to U$ is shorter than any other curve from $c(a)$ to $c(b)$ near $c$.

## 10. Parallel transport

There is yet another geometric aspect associated with riemannian patches $(U, g)$. Given any connection $\nabla$ on $U$ we say that a vector field $X$ along a curve $c : [a, b] \to U$ is *parallel* if

$$\nabla_c X(t) = \frac{d}{dt} X(t) + \Gamma_{c(t)}(X(t), c'(t)) = 0 \tag{10.1}$$

for all $t \in [a, b]$. Since this is a first order linear differential equation in $X$ we get immediately

**Theorem 10.2.** *Let $\nabla$ be a connection on $U$ and $c : J \to U$ a $C^1$-curve in $U$. For $u \in T_{c(t_0)}U$ there is a unique parallel field $X$ along $c$ with $X_{c(t_0)} = u$. Moreover, the map $T_{c(t_0)}U \to T_{c(t_1)}U$ defined by $X(t_0) \to X(t_1)$, $X$ parallel along $c$ is a linear isomorphism.*

The linear isomorphism $\mathcal{T}_c^{t_0 t_1} : T_{c(t_0)}U \to T_{c(t_1)}U$ of 10.2 is called *parallel transport* along $c$ from $c(t_0)$ to $c(t_1)$. In terms of parallel transport we can express the covariant derivative of a vector field $X$ along $c$ as

$$\nabla_c X(t_0) = \frac{d}{dt}(t \to \mathcal{T}_c^{tt_0}(X(t)))_{|t=t_0} \tag{10.3}$$

To see this, simply let $X_1, \ldots, X_n$ be a basis of parallel fields along $c$ and write $X = \sum_{i=1}^n x_i \cdot X_i$.

As we will see later on, the importance of parallel transport is partly due to the fact that it allows us to compare vector fields along say geodesics in different patches (manifolds).

Let us now see how parallel transport relates to curvature.

Fix $x \in U$, $u_x, v_x, z_x \in T_x U$. To describe $R(u_x, v_x)z_x$ we let $V : J_1 \times J_2 \to U$ be a map so that $V(0, 0) = x$, and $V_{*(0,0)}(e_1) = u_x$, $V_{*(0,0)}(e_2) = v_x$. Here $J_i, i = 1, 2$ are intervals around $0 \in \mathbb{R}$ and $e_1, e_2$ are the coordinate vector fields in $\mathbb{R}^2$. (Take e.g. $V(t, s) = x + tu + s \cdot v$).

Define a vector field $z$ along $V$ as follows

$$z(t, s) = \mathcal{T}_{c_s}^{0t} \mathcal{T}_{\sigma_0}^{0s}(z_x).$$

From the existence and uniqueness theorem for differential equations we get that $z$ is smooth. By (7.2) and (7.4)

$$\begin{aligned} R(u_x, v_x)z_x =& (\nabla_{c_s}\nabla_{\sigma_t} z - \nabla_{\sigma_t}\nabla_{c_s} z)(0, 0) \\ =& \nabla_{c_s}\nabla_{\sigma_t} z(0, 0) \end{aligned}$$

since $z$ is a parallel along any $c_s$ curve. Now using (10.3) we have

$$R(u_x, v_x)z_x = \lim_{t \to 0} \frac{1}{t}(\mathcal{T}_{c_0}^{t0}(\nabla_{\sigma_t} z(t, 0)) - \nabla_{\sigma_t} z(0, 0))$$

and similarly

$$\begin{aligned}\nabla_{\sigma_t} z(t,0) &= \lim_{s\to 0} \frac{1}{s}(\mathcal{T}^{s0}_{\sigma_t} z(t,s) - z(t,0))\\ \nabla_{\sigma_0} z(0,0) &= \lim_{s\to 0} \frac{1}{s}(\mathcal{T}^{s0}_{\sigma_0} z(0,s) - z(0,0))\\ &= 0,\end{aligned}$$

where the last equality is by construction of $z$. Thus

$$R(u_x, v_x) z_x = \frac{\partial^2}{\partial t \partial s} Z(0,0), \tag{10.4}$$

where $Z(t,s) \in T_x U$ is defined in terms of parallel translation along "coordinate loops", by $Z(t,s) = \mathcal{T}^{t0}_{c_0} \mathcal{T}^{s0}_{\sigma_t} \mathcal{T}^{0t}_{c_s} \mathcal{T}^{0s}_{\sigma_0} z_x$.

This formula gives a geometric interpretation of the curvature tensor in terms of parallel transport.

**Problem 10.5.** Prove 10.2.

**Problem 10.6.** Show that $R \equiv 0$ if and only if parallel transport is locally path independent.

**Problem 10.7.** Fill out the details in the proof of (10.4).

**Problem 10.8.** Let $(U, g)$ be a riemannian patch with sectional curvature $Sec \geq 1$. Show that any normal minimal geodesic $c : [0, l] \to U$ has length $L(c) = l \leq \pi$.

Hint: Let $X$ be a parallel field along $c$ with $X \perp \dot{c}$, and show that $I(Y, Y) < 0$ for $Y(t) = \sin(t \cdot \frac{\pi}{l}) X(t)$.

**Problem 10.9.** Let $(U_1, g_1)$, $(U_2, g_2)$ be riemannian patches. The patch $(U, g)$ with $U = U_1 \times U_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ and

$$G = \left(\begin{array}{c|c} G_1 & 0 \\ \hline 0 & G_2 \end{array}\right)$$

is called the riemannian product of $U_1$ and $U_2$. Find the Levi Civita connection $\nabla$ of $g$ in terms of the Levi Civita connections $\nabla_1, \nabla_2$ for $g_1, g_2$.

What are the geodesics on $U$?

Express $\operatorname{dist}_g$ in terms of $\operatorname{dist} g_1$ and $\operatorname{dist} g_2$.

Let $F_1 : U_1 \to U_1$, $F_2 : U_2 \to U_2$ be isometries. Show that $F = F_1 \times F_2 : U_1 \times U_2 \to U_1 \times U_2$ is an isometry.

Find the curvature tensor $R$ on $U$ in terms of $R_1$ and $R_2$.

Show that all "mixed sectional curvatures" (curvature of two planes spanned by a vector tangent to $U_1$ and a vector tangent to $U_2$) are zero.

**Problem 10.10.** Let $(U, g)$ be a riemannian patch. Show that the sectional curvature $Sec \equiv 0$ if and only if $\exp_x$ is an isometry near $0_x$ for all $x \in U$.

## 11. Manifolds and maps

From section 1, 2 we know that the graph of a function $f : U \to \mathbb{R}$ in a natural way is a length space which is isometric to a riemannian patch. Since many spaces may be described locally as the graph of a function, we make the following general

**Definition 11.1.** A *riemannian n-space* is a length space, $(M^n, \mathrm{dist})$ which is locally isometric to a riemannian $n$-patch, i.e., for any $p \in M$ there is an open neighbourhood $p \in V_\alpha$, a riemannian patch $(U_\alpha, g_\alpha)$ and an isometry $\varphi_\alpha : (V_\alpha, \mathrm{dist}) \to (U_\alpha, \mathrm{dist}_{g_\alpha})$.

From (6.1), (9.4) and (9.12) it follows that all the *coordinate changes* $\varphi_{\beta|} \circ \varphi_{\alpha|}^{-1} : \varphi_\alpha(V_\alpha \cap V_\beta) \to \varphi_\beta(V_\alpha \cap V_\beta)$ above are diffeomorphisms between open sets in $\mathbb{R}^n$. By definition therefore, the collection of homeomorphisms $\varphi_\alpha : V_\alpha \to U_\alpha \subset \mathbb{R}^n$ form an *atlas* for a *differentiable structure* on $M^n$. Two atlases $\{(V_\alpha, \varphi_\alpha)\}$ and $\{(W_\beta, \psi_\beta)\}$ on $M^n$ are said to be equivalent if they together form an atlas on $M$, i.e., if all possible coordinate changes are smooth. An equivalence class of atlases is called a differentiable structure, and $M$ together with such a structure is called a *differentiable manifold.* In particular we have

**Theorem 11.2.** *A riemannian n-space* $(M, \mathrm{dist})$ *has the structure of an n-dimensional differentiable manifold.*

Clearly $\{(\mathbb{R}^n, id_{\mathbb{R}^n})\}$ form an atlas on $\mathbb{R}^n$, and this way $\mathbb{R}^n$ is given a structure of a differentiable manifold. This particular structure is called the *standard differentiable structure* on $\mathbb{R}^n$. Also any open set $V \subset M$ of a differentiable manifold carries an induced differentiable structure.

We say that a continuous map $f : M^n \to N^m$ between differentiable manifolds is $C^k$ if and only if $\psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(f^{-1}(W_\beta)) \to \psi_\beta(W_\beta)$ is of class $C^k$ for all *charts* $(V_\alpha, \varphi_\alpha)$ on $M^n$ and all charts $(W_\beta, \psi_\beta)$ on $N$.

Now let $p \in M$ and consider all differentiable curves $c : J \to M$ through $p$ i.e. $0 \in J$ and $c(0) = p$. We say that $c_1$ is tangent to $c_2$ if for one and hence all charts $\varphi_\alpha : V_\alpha \to U_\alpha \subset \mathbb{R}^n$ around $p$, that $(\varphi_\alpha \circ c_1)'(0) = (\varphi_\alpha \circ c_2)'(0)$. The set of equivalence classes is the *tangent space*, $T_pM$, of $M$ at $p$. The bijection $T_pM \to T_{\varphi_\alpha(p)}U_\alpha$ yields a well defined linear structure on $T_pM$. The collection $TM = \cup_{p\in M} T_pM$ of all tangent spaces to $M$ is called the *tangent bundle* of $M$. As in section 4 we can also view $T_pM$ as the vector space of derivations of the set of smooth functions $f \in \mathcal{F}_p$ defined near $p$.

**Theorem 11.3.** *Let $M^n$ be an n-dimensional differentiable manifold. Then $TM$ has the structure of a $2n$-dimensional differentiable manifold, and the projection $\pi : TM \to M$, $\pi(T_pM) = \{p\}$, is smooth.*

**Proof:** Let $\{(V_\alpha, \varphi_\alpha)\}$ be an atlas on $M$. Let $\frac{\partial}{\partial x_\alpha^1}(p), \ldots, \frac{\partial}{\partial x_\alpha^n}(p)$ be the basis for $T_pM$ corresponding to the canonical basis $e_1(\varphi_\alpha(p)), \ldots, e_n(\varphi_\alpha(p))$ for $T_{\varphi_\alpha(p)}U_\alpha \cong \mathbb{R}^n$ via the isomorphism $T_pM \to T_{\varphi_\alpha(p)}U_\alpha, p \in V_\alpha$. Then any tangent vector $u_p \in T_pM$ may be written uniquely as $u_p = \sum_{i=1}^n u_i \frac{\partial}{\partial x_\alpha^i}(p)$. With this presentation define $\Phi_\alpha : \pi(V_\alpha) \to U_\alpha \times \mathbb{R}^n$ by

$$\Phi_\alpha\left(\sum_{i=1}^n u_i \frac{\partial}{\partial x_\alpha^i}(p)\right) = (p, u_1, \ldots, u_n)$$

for all $u_p \in T_pM$. Then $\Phi_\beta \circ \Phi_\alpha^{-1}(p, u) = (\varphi_\beta \circ \varphi_\alpha^{-1}(p), D(\varphi_\beta \circ \varphi_\alpha^{-1})_p u)$ and hence $(\pi^{-1}(V_\alpha), \Phi_\alpha)$ defines an atlas for $TM$ as desired (a topology of $TM$ is uniquely determined by the requirements, $\pi^{-1}(V_\alpha)$ is open and $\Phi_\alpha$ is a homeomorphism for all $\alpha$). To see that $\pi : TM \to M$ is smooth it is enough to observe that

$$\varphi_\alpha \circ \pi \circ \Phi_\alpha^{-1} : U_\alpha \times \mathbb{R}^n \to U_\alpha, \quad (x, u) \to x$$

is smooth for all $\alpha$. $\square$

As in section 5, any smooth map $f : M^n \to N^m$ gives rise to an *induced map* (or tangent map) $f_* : TM \to TN$, defined by $f_*([c]) = [f \circ c]$ for any tangent vector $v_p = [c]$ represented by the curve $c$. With the differentiable structures on $TM$, and $TN$ given above, we get immediately

**Theorem 11.4.** *The induced map $f_* : TM \to TN$ of a smooth map $f : M \to N$ is smooth, and satisfies $\pi_N \circ f_* = f \circ \pi_M$.*

A *vector field* $X$ on $M$ is a map $X : M \to TM$ such that $\pi \circ X = id_M$, i.e., $X(p) \in T_pM$ for all $p \in M$. Unless otherwise stated we will only consider smooth vector fields $X : M \to TM$. The *coordinate vector fields* $\frac{\partial}{\partial x_\alpha^1}, \dots, \frac{\partial}{\partial x_\alpha^n}$ on $V_\alpha \subset M$ are clearly smooth, and $\frac{\partial}{\partial_\alpha^i} = (\varphi_\alpha^{-1})_* \circ e_i$, where $e_i$, $i = 1, \dots, n$, are the constant basis vector fields in $U_\alpha \subset \mathbb{R}^n$.

Viewing vector fields $X$ as derivations on the algebra $C^\infty(M, \mathbb{R})$, of smooth functions $f : M \to \mathbb{R}$, the *Lie bracket* $[X, Y]$ of vector fields $X$ and $Y$ on $M$ is defined by

$$[X, Y][f] = X[Y[f]] - Y[X[f]]$$

for all $f \in C^\infty(M, \mathbb{R})$ (cf. section 4). If $[X, Y] = 0$ we say that $X$ and $Y$ *commute*. Clearly the coordinate vector fields $\frac{\partial}{\partial x_\alpha^i}, \dots, \frac{\partial}{\partial x_\alpha^n}$ on $V_\alpha$ commute.

**Problem 11.5.** Give an example of a metric space $(S, \mathrm{dist})$ which is locally isometric to a riemannian patch, but not a length space. Show, however, that the associated length space $(S, \mathrm{dist}_l)$ is a riemannian $n$-space.

**Problem 11.6.** Show that $\{(\mathbb{R}, \varphi)\}$ with $\varphi(t) = t^3$ for all $t \in \mathbb{R}$ is an atlas for a differentiable structure on $\mathbb{R}$. Is this the same as the standard structure on $\mathbb{R}$? Exhibit a diffeomorphism between them.

**Problem 11.7.** Let $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ be the standard *$n$-sphere* in $\mathbb{R}^{n+1}$. Exhibit an atlas for $S^n$ using orthogonal projections onto coordinate hyper planes in $\mathbb{R}^{n+1}$. Exhibit an atlas for $S^n$ using stereographic projection. Show that these atlases define the same differentiable structure on $S^n$. What is the least number of charts needed for a differentiable structure on $S^n$?

Show that the length-space structure on $S^n$ induced from the euclidean distance in $\mathbb{R}^{n+1}$ makes $S^n$ into a riemannian $n$-space.

Show that the orthogonal group $O(n+1)$ is the group of isometries on $S^n$.

What are the geodesics on $S^n$?

Show that $S^n$ has constant curvature.

**Problem 11.8.** Let $B^n$ and $F^k$ be differentiable manifolds. Show that $B^n \times F^k$ has a differentiable structure with atlas $\{(V_\alpha \times W_\beta, \varphi_\alpha \times \psi_\beta)\}$ where $\{(V_\alpha, \varphi_\alpha)\}$ and $\{(W_\beta, \psi_\beta)\}$ are atlases for $B$ and $F$ respectively. What is the dimension of $B \times F$?

**Problem 11.9.** Let $M^{n+k}, B^n$, and $F^k$ be differentiable manifolds and $\pi : M \to B$ be a smooth map. The triple $(M, \pi, B)$ is called a *fiber bundle* with *fiber $F$* if each $p \in B$ has an open neighbourhood $U$ such that $\pi^{-1}(U) \subset M$ is diffeomorphic to $U \times F$ via a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times F$ and the diagram

(11.10)
$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi_{|}"'] & & U \times F \arrow[dl, "P_1"] \\ & U & \end{tikzcd}$$

is commutative, i.e. $P_1 \circ \Phi = \pi_{|}$. Here $P_1 : U \times F \to U$ is the projection onto the first factor.

If $B$ can be chosen as $U$ we say that the bundle is *trivial* (or a product). The property (11.10) is referred to as the bundle $\pi : M \to B$ being *locally trivial*.

Show that the tangent bundle $\pi : TM \to M$ is a fiber bundle with fiber $\mathbb{R}^n$. In fact, show that the *local trivializations* $\Phi$ may be chosen so that $\Phi_p = \Phi_{|\pi^{-1}(p)} : \pi^{-1}(p) \to \{p\} \times \mathbb{R}^n$ is a linear isomorphism for all $p \in U$. By definition therefore $\pi : TM \to M$ is called an $n$-dimensional *vector bundle* over $M$.

**Problem 11.11.** Let $\pi : E^{n+k} \to M^n$ be a $k$-dimensional vector bundle over $M^n$. A map $s : M \to E$ such that $\pi \circ s = id_M$ is called a *section* of $\pi$. Sections $s_1, \ldots, s_m$ are said to be *linearly independent* if $s_1(p), \ldots, s_m(p) \in \pi^{-1}(p) = E_p$ are linearly independent vectors in the vector space $E_p$ for all $p \in M$.

Show that $\pi : E \to M$ is trivial if and only if there are $k$ linearly independent smooth sections $s_1, \ldots, s_k$ of $\pi$.

**Problem 11.12.** Let $F : M \to N$ be a smooth map between differentiable manifolds. For each $v \in T_pM$, show that $F_*(v) \in T_{F(p)}N$ as a derivation on functions, $f$ defined near $F(p)$ is given by $F_*(v)[f] = v[f \circ F]$.

**Problem 11.13.** A point in the real projective space $\mathbb{R}P^n$ is by definition a pair of antipodal points $(x, -x)$ on the $n$-sphere $S^n$. Show that $\mathbb{R}P^n$ has a differentiable structure such that $\pi : S^n \to \mathbb{R}P^n, x \to (x, -x)$ is a fiber bundle with fiber $F = \{1, -1\}$. A fiber bundle with discrete fiber is called a *covering space*.

**Problem 11.14.** Consider the trivial vector bundle $\tilde{\pi} : S^n \times \mathbb{R} \to S^n$. Show that by identifying antipodal points $(x, t) \sim (-x, -t)$ in $S^n \times \mathbb{R}$ and $x \sim -x$ in $S^n$ as in (11.13), $\tilde{\pi}$ induces a map $\pi$ from $E = S^n \times \mathbb{R}/_\sim$ to $\mathbb{R}P^n = S^n/_\sim$ such that $(E, \pi, \mathbb{R}P^n)$ is a one dimensional vector bundle over $\mathbb{R}P^n$. This is called the *canonical line bundle* over $\mathbb{R}P^n$. For $n = 1$, this is the "infinite" Möbius band over $\mathbb{R}P^1 = S^1$.

View $\mathbb{R}P^n \subset E$ as the image of the zero section of $\pi : E \to \mathbb{R}P^n$. Show that $E - \mathbb{R}P^n$ is path connected and conclude that $\pi$ is a non-trivial bundle.

**Problem 11.15.** Show that a differentiable manifold $M$ is connected if and only if it is path connected.

**Problem 11.16.** Let $\{V_\alpha\}$ be a cover of $M$, i.e. $M = \cup_\alpha V_\alpha$, $V_\alpha \subset M$. Suppose $\varphi_\alpha : V_\alpha \to \mathbb{R}^n$ are injective maps, such that $\varphi_\alpha(V_\alpha) = U_\alpha$ and $\varphi_\alpha(V_\alpha \cap V_\beta) = U_{\alpha\beta}$ are open subsets in $\mathbb{R}^n$, and $\varphi_\beta \circ \varphi_\alpha^{-1} : U_{\alpha\beta} \to U_{\beta\alpha}$ are all continuous.

Show that there is a unique topology on $M$ such that each $V_\alpha \subset M$ is open and $\varphi_\alpha : V_\alpha \to U_\alpha$ is a homeomorphism. With this topology, $M$ is a *topological $n$-manifold* (a topological space locally homeormorphic to $\mathbb{R}^n$).

## 12. Completeness

From section 2 we know in particular, that if $M^n$ is a complete riemannian $n$-space, then any two points of $M$ can be joined by a minimal geodesic. We will begin this section by showing that the same conclusion holds provided $M$ is *geodesically complete*, i.e. any geodesic defined on all of $\mathbb{R}$.

First however observe, that if $X, Y$ are smooth vector fields on $M$, then the Levi Civita connection $\nabla^\alpha$ on the $U_\alpha$'s define uniquely $\nabla_X Y$ as the vector field

$$\nabla_X Y_{|V_\alpha} = (\varphi_\alpha^{-1})_* \circ \nabla^\alpha_{X_\alpha} Y_\alpha \circ \varphi_\alpha, \tag{12.1}$$

where $X_\alpha$ is the vector field $(\varphi_\alpha)_* \circ X \circ \varphi_\alpha^{-1}$, and similarly for $Y_\alpha$ (cf. 6.3). The map $\nabla$ clearly has the properties

$$\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y \tag{12.2}$$

$$\nabla_{fX} Y = f \cdot \nabla_X Y \tag{12.3}$$

$$\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2 \tag{12.4}$$

$$\nabla_X (f \cdot Y) = X[f] \cdot Y + f \cdot \nabla_X Y \tag{12.5}$$

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0 \tag{12.6}$$

$$Z[g(X, Y)] = g(\nabla_Z X, Y) + g(X, \nabla_Z Y), \tag{12.7}$$

for all smooth vector fields $X_i, Y_i, Z$ and functions $f$ on $M$. Here $g$ is the map that to each $p \in M$ assigns the inner product $g_p$ on the tangent space $T_pM$ induced from $g^\alpha$ on $T_{\varphi_\alpha(p)}U_\alpha$ by

$$g_p(u, v) = g^\alpha_{\varphi_\alpha(p)}\left((\varphi_\alpha)_*(u), (\varphi_\alpha)_*(v)\right) \tag{12.8}$$

for all $u, v \in T_pM$. Clearly the function $g(X, Y) : M \to \mathbb{R}$ defined by $p \to g_p(X_p, Y_p)$ is smooth whenever $X, Y$ are smooth vector fields on $M$. A differentiable $n$-manifold $M$ together with such a *riemannian structure* $g$ is called a riemannian $n$-manifold. Conversely, a connected riemannian $n$-manifold $(M, g)$ with dist : $M \times M \to \mathbb{R}$ defined in analogy with (1.14) via (1.13) is a riemannian $n$-space.

Because of (12.2)-(12.5) we say that $\nabla$ is a connection on $M$ (or more precisely on the tangent bundle $TM \to M$), and $\nabla_X Y$ is the covariant derivative of $Y$ in direction $X$. The left hand side, $T(X, Y)$ of (12.6) is bilinear with respect to functions on $M$; it is called the torsion tensor of $\nabla$. As in section 4 we say that $\nabla$ is torsion free (or symmetric) if (12.6) holds, and metric if (12.7) holds. These properties uniquely determine $\nabla$, which we refer to as the *Levi Civita connection* on $(M, g)$. In fact proceeding as in the proof of Theorem 4.18 with general vector fields $X, Y$, and $Z$ instead of $e_i, e_j$, and $e_k$ and using (12.6) and (12.7) one gets

$$\begin{aligned} \langle \nabla_X Y, Z \rangle = \frac{1}{2}\{ &X[\langle Y, Z\rangle] - Z[\langle X, Y\rangle] + Y[\langle Z, X\rangle] \\ &- \langle X, [Y, Z]\rangle + \langle Z, [X, Y]\rangle + \langle Y, [Z, X]\rangle\}. \end{aligned} \tag{12.9}$$

For fixed $X, Y$, the right hand side is linear in $Z$ with respect to functions on $M$. Therefore (12.9) is a coordinate free description of $\nabla$ equivalent to (4.22).

In terms of the Levi Civita connection $\nabla$ on a riemannian $n$-manifold (space), $M^n$, the geodesics on $M$ are the smooth curves $c$ on $M$ which satisfy the geodesic equation $\nabla_c \dot{c} = 0$. In particular, for each tangent vector $v \in TM$, there is a unique maximal geodesic $c_v : J_v \to M$ such that $\dot{c}_v(0) = v$. Moreover, if $W = \{(v, t) \in TM \times \mathbb{R} \mid t \in J_v\}$ then $W \subset TM \times \mathbb{R}$ is open and the map $W \to M$, $(v, t) \to c_v(t)$ is smooth (cf. 3.14).

As in section 5 we let $O = \{v \in TM \mid 1 \in J_v\}$. Then $O \subset TM$ is an open neighbourhood of the zero section $M \subset TM$, and

$$\exp : O \to M, \quad v \to c_v(1)$$

is a smooth map. Clearly $M$ is geodesically complete if and only if $J_v = \mathbb{R}$ for all $v \in TM$, or equivalently $O = TM$.

**Lemma 12.10.** *Suppose $O_p = O \cap T_pM = T_pM$ for some $p \in M$. Then any $q \in M$ can be joined to $p$ by a minimal geodesic.*

**Proof:** Choose $\delta > 0$ so that $\exp_p : D_p(2\delta) \to B(p, 2\delta)$ is a diffeomorphism. In particular, for any $q \in \overline{B(p,\delta)} = \{x \in M \mid \mathrm{dist}(p,x) \le \delta\}$ there is a unique minimal geodesic from $p$ to $q$ (cf. section 5). Now suppose $\mathrm{dist}(p,q) > \delta$. Since $\partial B(p,\delta) = \exp_p(S(p,\delta))$, $S(p,\delta) = \{v \in T_pM \mid \|v\| = \delta\}$, is compact, there is a $q' \in \partial B(p,\delta)$ such that $\mathrm{dist}(q, \partial B(p,\delta)) = \mathrm{dist}(q,q')$. In particular

$$\mathrm{dist}(p,q) = \delta + \mathrm{dist}(q',q).$$

Now let $c : \mathbb{R} \to M$ be the unique maximal normal ($\|\dot{c}\| = 1$) geodesic such that $c_{|[0,\delta]}$ is the minimal geodesic from $p$ to $q'$. We claim that $c(\mathrm{dist}(p,q)) = q$. To see this, consider the set

$$A = \{t \in [0, \mathrm{dist}(p,q)] \mid \mathrm{dist}(c(t),q) = \mathrm{dist}(p,q) - t\}.$$

Clearly $A$ is closed and $[0,\delta] \subset A$. For any $t \in \mathbb{R}$

$$\mathrm{dist}(p,q) \le \mathrm{dist}(p,c(t)) + \mathrm{dist}(c(t),q) \le t + \mathrm{dist}(c(t),q).$$

Thus if $t \in A$, we see that $c_{|[0,t]}$ and hence any subsegment $c_{|[0,t']}$, $t' \in [0,t]$, is minimal. It follows that

$$\begin{aligned}\mathrm{dist}(p,q) =& t' + \mathrm{dist}(c(t'),c(t)) + \mathrm{dist}(c(t),q)\\ =& t' + \mathrm{dist}(c(t'),q)\end{aligned}$$

by the triangle inequality, and therefore $t' \in A$. The set $A$ is therefore a closed interval $[0,t_0]$. Assume $t_0 < \mathrm{dist}(p,q)$ and choose $0 < \delta_0 < \min\{t_0, \mathrm{dist}(p,q) - t_0\}$ so small that $\overline{B(c(t_0),\delta_0)}$ is a ball like $\overline{B(p,\delta)}$ above. Like there, choose $q'' \in \partial B(c(t_0),\delta_0)$ with

$$\mathrm{dist}(c(t_0),q) = \delta_0 + \mathrm{dist}(q'',q),$$

and let $c' : [t_0, t_0+\delta_0] \to M$ be the unique minimal geodesic from $c(t_0)$ to $q''$. Now

$$\mathrm{dist}(p,q) = t_0 + \delta_0 + \mathrm{dist}(q'',q)$$

whence, by the triangle inequality

$$\mathrm{dist}(p,q'') = t_0 + \delta_0.$$

But then $\mathrm{dist}(p,q'') = L(c'')$, where $c''$ is the piecewise smooth curve $c''_{|[0,t_0]} = c_{|[0,t_0]}$, $c''_{|[t_0,t_0+\delta_0]} = c'$. Since $c''$ is a minimal curve, it is a geodesic. In particular $c'' = c_{|[0,t_0+\delta_0]}$ and $q'' = c(t_0+\delta_0)$. Thus

$$\mathrm{dist}(p,q) = (t_0+\delta_0) + \mathrm{dist}(c(t_0+\delta_0),q),$$

i.e. $t_0 + \delta_0 \in A$, contradicting the definition of $t_0$. Therefore $t_0 = \mathrm{dist}(p,q)$, and $c_{|[0,\mathrm{dist}(p,q)]}$ is a minimal geodesic from $p$ to $q$. $\square$

It follows directly from this lemma, that if $M$ is geodesically complete, then any pair of points $p, q \in M$ can be joined by a minimal geodesic. This point in direction of the following result called the *Hopf-Rinov Theorem*:

**Theorem 12.11.** *A riemannian $n$-space is complete if and only if it is geodesically complete.*

**Proof:** Assume $M^n$ is geodesically complete, and let $\{q_k\}$ be a Cauchy-sequence in $M$. Fix $p \in M$ and choose according to 12.10 minimal geodesics $c_k : [0, \operatorname{dist}(p, q_k)] \to M$ from $p$ to $q_k$ for each $k$. Clearly $\{\operatorname{dist}(p, q_k)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete we have $\lim \operatorname{dist}(p, q_k) = t$. Moreover $\{\dot{c}_k(0)\}$ is a sequence of unit vectors in $T_pM$. By compactness of $S_p(1) \subset T_pM$ this sequence has a convergent subsequence, $\lim \dot{c}_{k_m}(0) = v$. In particular $\operatorname{dist}(p, q_{k_m}) \cdot \dot{c}_{k_m}(0) \to t \cdot v$, and by continuity of $\exp_p : T_pM \to M$ we get $q_{k_m} = \exp_p(t \cdot v)$. It follows that the Cauchy sequence $\{q_k\}$ itself converges to $q$, i.e. $M^n$ is complete.

Now suppose $M^n$ is complete, and let $p \in M$ be an arbitrary point in $M$. For $v \in T_pM$ we must show that $J_v = \mathbb{R}$. if $J_v = (a, b)$ with $b < \infty$ let $\{t_k\}$ be an increasing sequence in $J_v$ with $t_k \to b$. Clearly $\{c_v(t_k)\}$ is a Cauchy-sequence in $M$. Let $q = \lim c_v(t_k)$ and define $c : [0, b] \to M$ by $c(t) = c_v(t)$ for $t < b$ and $c(b) = q$. Then $c$ is a continuous path in $M$. Now choose $\delta > 0$ according to (5.4) so that $(\pi, \exp)$ is a diffeomorphism when restricted to the set

$$\{v \in TM \mid \|v\| < 2\delta,\ \pi(v) \in B(q, \delta)\}.$$

Then any two points in $B(q, \delta)$ are joined by a unique minimal geodesic of length $< 2\delta$. Moreover, the corresponding maximal geodesic is defined on an interval containing $(-2\delta, 2\delta)$. Pick $t < s < b$ so that $c_v[t, b] \subset B(q, \delta)$ and $c_{|[t,s]}$ is the unique minimal geodesic from $c_v(t)$ to $c_v(s)$ and $\operatorname{dist}(c_v(t), c_v(s)) < \delta$. By construction, however, this minimal geodesic and hence $c_v$ can be extended through and beyond $q$. This contradicts the definition of $b$ when $b < \infty$. Similarly we see that $a = -\infty$, i.e. $J_v = \mathbb{R}$ and $M$ is geodesically complete.

**Problem 12.12.** Show that $\nabla_X Y$ is well defined by (12.1).

**Problem 12.13.** Show that $g_p$ is well defined by (12.8).

**Problem 12.14.** Show that $g(\frac{\partial}{\partial x_i^\alpha}, \frac{\partial}{\partial x_j^\alpha}) = g_{ij}^\alpha \circ \varphi_\alpha$ and conclude that $g(X, Y)$ is smooth for smooth $X, Y$.

**Problem 12.15.** Prove (12.9) and show that it uniquely defines the Levi Civita connection.

**Problem 12.16.** Show that for each $v \in TM$ there is a unique maximal geodesic $c_v : J_v \to M$. Show that $W = \{(v, t) \mid t \in J_v\} \subset TM \times \mathbb{R}$ is open and $W \to M$, $(v, t) \to c_v(t)$ is smooth. Show that $O$ is open, $\exp : O \to M$ is smooth and $O_p = O \cap T_pM$ is starshaped around $O_p \in T_pM$.

**Problem 12.17.** Show that a local isometry between riemannian $n$-manifolds is a local diffeomorphism. Let $M$ be a complete riemannian $n$-space and $F : M \to M$ an isometry. Show that $F$ is completely determined by $F_{*_p}$ for any $p \in M$. (Hint: show that $\exp_{F(p)} \circ F_{*_p} = F \circ \exp_p$).

## 13. Global effects of curvature

In sections 7 and 9 we investigated local and semi-global effects of curvature. In this section we will prove two classical theorems about curvature and topology of Riemannian spaces.

Let $M^n$ be a Riemannian $n$-space with Riemannian structure $\langle\ ,\ \rangle$. Since isometries between Riemannian patches preserve the curvature tensor, it is clear that for $p \in M$, $R_p\colon T_pM \times T_pM \times T_pM \to T_pM$ given by

$$R_p(u,v)w = (\phi_\alpha^{-1})_{*\phi_\alpha(p)} R^\alpha_{\phi_\alpha(p)}(\phi_{\alpha*}(u), \phi_{\alpha_*}(v))\phi_{\alpha*}w, \tag{13.1}$$

is well defined. For vector fields $X$, $Y$ and $Z$ on $M$ the *curvature tensor* $R$ of (13.1) is also given in terms of the Levi–Civita connection $\nabla$ of section 12 by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{13.2}$$

The *sectional curvature* , *Ricci curvature* , and *scalar curvature* are all defined from $R$ and $\langle\ ,\ \rangle$ as in section 8.

The following result is called the *Hadamard–Cartan theorem.*

**Theorem 13.3.** *A simply connected complete Riemannian n-space of non-positive curvature is diffeomorphic to* $\mathbb{R}^n$*.*

**Proof:** Let $M^n$ be any complete Riemannian $n$-space with $\sec M \le 0$. We will prove the theorem by showing that in fact $\exp_p\colon T_pM \to M$ is a covering map for any $p \in M$ (cf. 12.11).

First we claim that for any $v \in T_pM$, $(\exp_p)_{*v}\colon T_v(T_pM) \to T_{\exp(v)}M$ is a linear isomorphism and hence by the inverse function theorem $\exp_p$ is a local diffeomorphism. If not, there is a $u \perp v$ such that $(\exp_p)_*(u_v) = 0 \in T_{\exp(v)}M$. The vector field $X(t) = (\exp_p)_*(tu_{tv})$ along the geodesic $c_v\colon \mathbb{R} \to M$ is a *Jacobi field* i.e.

$$\nabla_{c_v}\nabla_{c_v}X + R(X,\dot c_v)\dot c_v = 0 \tag{13.4}$$

with $X \perp \dot c_v$, $X(0) = 0 \in T_pM$ and $X(1) = (\exp_p)_*(u_v) = 0 \in T_{c_v(1)}M$ (cf. section 5 and 7). By assumption on the curvature and (13.4), $\langle X'', X\rangle \ge 0$ and therefore also $\langle X, X\rangle'' = 2\{\langle X'', X\rangle + \langle X', X'\rangle\} \ge 0$, i.e. $\|X\|^2\colon [0,1] \to \mathbb{R}$ is convex. Since $\|X(0)\| = \|X(1)\| = 0$, we conclude that $X \equiv 0$, contradicting the fact that $\exp_p$ is a diffeomorphism near $0 \in T_pM$. Now the local diffeomorphism $\exp_p\colon T_pM \to M$ induces a Riemannian structure on $T_pM$ such that for the corresponding metric space, $\exp_p$ is a local isometry. Moreover, since the straight lines through $0 \in T_pM$ are also geodesics in this new metric space, it follows from 12.11 that $T_pM$ is a complete Riemannian $n$-space.

For $q \in M$ choose $\epsilon > 0$ so that $B(q,\epsilon)$ is strictly convex (cf. section 9). Then by completeness of $T_pM$, the balls $B(v,\epsilon) \subset T_pM$, $\exp_p(v) = q$ are mutually disjoint. Otherwise, there would be a geodesic loop at $q$ of length less that $2\epsilon$. By the same reasoning, in fact $\exp_{p|}\colon B(v,\epsilon) \to B(q,\epsilon)$ is a diffeomorphism for each $v \in \exp_p^{-1}(q)$, in particular $\exp_p\colon T_pM \to M$ is a covering map (cf. 11.13).

Since $T_pM \simeq \mathbb{R}^n$ is simply connected, $\exp_p\colon T_pM \to M$ is a diffeomorphism if $M$ is simply connected.

At the other extreme we have the so called *Bonnet–Myers theorem.*

**Theorem 13.5.** *Let $M^n$ be a complete Riemannian n-space with Ricci curvature* $\operatorname{Ric} M \geq (n-1)k$, $k > 0$. *Then $M$ is compact, and the fundamental group $\pi_1(M)$ is finite.*

**Proof:** We will show that $\operatorname{dist}(p,q) \leq \pi/\sqrt{k}$ for any pair of points $p, q \in M$. Then by completeness of $M$, it is compact and $\operatorname{diam}(M) \leq \pi/\sqrt{k}$.

Let $c\colon [0,l] \to M$, $c(0) = p$, $c(l) = q$ be a normal minimal geodesic, and suppose $l > \pi/\sqrt{k}$. Let $X_1, \ldots, X_{n-1}, X_n$ be *parallel fields* along $c$, i.e.

$$\nabla_c X_i = 0, \quad i = 1, \ldots, n-1 \tag{13.6}$$

with $\{X_i(t)\}$ an orthonormal basis, and $X_n = \dot{c}$. For $Y_i(t) = \sin(t \cdot \pi/l) X_i(t)$, $i = 1, \ldots, n-1$ we have

$$I(Y_i, Y_i) = \int_0^l \left\{ \left(\frac{\pi}{l}\right)^2 \cos^2\left(t \cdot \frac{\pi}{l}\right) - \sin^2\left(t \cdot \frac{\pi}{l}\right) sec_i(t) \right\} dt,$$

where $I$ is the index form (cf. 9.2) and $sec_i(t)$ is the sectional curvature of the 2-plane spanned by $X_i(t)$ and $\dot{c}(t)$. By (8.7) therefore

$$\begin{aligned}
\sum_{i=1}^{n-1} I(Y_i, Y_i) &= \int_0^l \left\{ (n-1)\left(\frac{\pi}{l}\right)^2 \cos^2\left(t \cdot \frac{\pi}{l}\right) - \sin^2\left(t \cdot \frac{\pi}{l}\right) Ric(\dot{c}) \right\} dt \\
&\leq (n-1) \int_0^l \left\{ \left(\frac{\pi}{l}\right)^2 \cos^2\left(t \cdot \frac{\pi}{l}\right) - k \sin^2\left(t \cdot \frac{\pi}{l}\right) \right\} dt \\
&= (n-1) \int_0^\pi \left(\frac{l}{\pi}\right) \left\{ \left(\frac{\pi}{l}\right)^2 \cos^2 u - k \sin^2 u \right\} du \\
&= \left(\frac{l}{\pi}\right) (n-1) \left\{ \left(\frac{\pi}{l}\right)^2 \cdot \frac{\pi}{2} - k \cdot \frac{\pi}{2} \right\} \\
&< 0.
\end{aligned}$$

In particular $I(Y_i, Y_i) < 0$ for some $i$, and hence $c$ is not locally the shortest curve according to the second variation formula (9.1). Thus $l \leq \pi/\sqrt{k}$.

To show the last part of the theorem, consider the universal cover $\pi\colon \tilde{M} \to M$, with fiber $\pi_1(M)$. As in the proof of (13.3), $\pi$ induces a Riemannian structure on $\tilde{M}$ so that $\pi$ is a local isometry. In particular, $Ric\,\tilde{M} \geq (n-1)k$ and also $\tilde{M}$ is complete. Thus it follows from what we have already seen, that $\tilde{M}$ is compact, and therefor $\pi_1(M)$ is finite.

**Problem 13.7.** Show that the curvature tensor $R_p$ is well defined by (13.1).

**Problem 13.8.** Show that the right hand side of (13.2) is linear with respect to functions on $M$ in each variable $X$, $Y$ and $Z$. Show (13.2).

**Problem 13.9.** Define Jacobi fields, and parallel fields locally and then prove (13.4) and (13.6).

**Problem 13.10.** Show that the first and second variation formulas (3.11) and (9.1) hold globally in a Riemannian space $M$.

**Problem 13.11.** Let $c\colon [0,l] \to M$ be a geodesic in a Riemannian space $M$, and $u, v \in T_{c(0)}M$. Show that there is a vector field $Y$ along $c_u(s)$ s.t. $Y'(0) = v$. Define the variation $V(t,s) = \exp_{c_u(s)} t \cdot Y(s)$, and show that the variation field $X$ along $c$ is the unique Jacobi field along $c$ with $X(0) = u$, $X'(0) = v$.

## 14. Vector bundles and tensors

We have already seen examples of vector bundles and tensors. Here we will discuss these topics in general.

A (smooth) *k-dimensional vector bundle* is a triple $(E, \pi, M)$, where $E$, $M$ are differentiable manifolds, and $\pi\colon E \to M$ is a smooth map with the following properties:

For every $p \in M$ there is an open neighborhood $U \subset M$ of $p$ and a diffeomorphism $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that

(14.1)
$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi_|"'] & & U \times \mathbb{R}^k \arrow[dl, "P_1"] \\ & U & \end{tikzcd}$$

is a commutative diagram. Moreover, $\pi^{-1}(p)$ is a $k$-dimensional vector space, for all $p \in M$, such that each map $\Phi_q\colon \pi^{-1}(q) \to \mathbb{R}^k$ given by $\Phi_q = p_2 \circ \Phi_{|\pi^{-1}(q)}$ is a linear isomorphism. $E$ is called the *total space* , $M$ the *base space* , and $\pi$ the *projection* of the bundle. The vector space $E_q = \pi^{-1}(q)$ is called the *fiber* over $q \in M$. The map $\Phi$ in (14.1) is referred to as a *trivialization* of $(E, \pi, M)$ over $U$, it is also called a local trivialization, and the bundle is said to be locally trivial. If $U = M$, the bundle is called (globally) *trivial* .

A (smooth) map $s\colon M \to E$ such that $\pi \circ s = \mathrm{id}_M$ is called a *section* of the bundle. With this terminology, a vector field $X$ on a manifold $M$ is a section of the tangent bundle $(TM, \pi, M)$. We say that sections $s_1, \dots, s_m$ are *linearly independent* if and only if $s_1(p), \dots, s_m(p)$ are linearly independent in $E_p$. According to (11.11) a $k$-dimensional vector bundle $(E, \pi, M)$ is trivial if and only if it has $k$ linearly independent sections. In particular, this is always true locally.

The space of smooth sections of $(E, \pi, M)$ will be denoted by $\mathcal{S}^\infty(E)$.

A *bundle map* between two vector bundles $(E_1, \pi_1, M_1)$ and $(E_2, \pi_2, M_2)$ is a pair $(F, f)$, where $F\colon E_1 \to E_2$ and $f\colon M_1 \to M_2$ are smooth, and

(14.2)
$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"'] \\ M_1 \arrow[r, "f"'] & M_2 \end{tikzcd}$$

is a commutative diagram. Moreover, for every $p_1 \in M_1$, the map $F_{p_1} = F_{|\pi_1^{-1}(p_1)}\colon E_{1p_1} \to E_{2f(p_1)}$ is linear. If $M = M_1 = M_2$ and $f = \mathrm{id}_M$ the pair $(F, \mathrm{id}_M)$ is called a *bundle isomorphism* provided $F$ is a diffeomorphism.

**Example 14.3.** (pull back) Let $(E, \pi, M)$ be a $k$-dimensional vector bundle and $f\colon N \to M$ a smooth map between differentiable manifolds. In $N \times E$ consider the set $f^*(E) = \{(p, e) \mid f(p) = \pi(e)\}$ and define $\tilde\pi\colon f^*(E) \to N$, $F\colon f^*(E) \to E$ as the projections $P_1\colon N \times E \to N$, $P_2\colon N \times E \to E$ restricted to $f^*(E) \subset N \times E$. There is a unique differentiable structure on $f^*(E)$ such that $(f^*(E), \tilde\pi, N)$ is a $k$-dimensional vector bundle and $(F, f)$ a bundle map. This is called the *pull back* of $(E, \pi, M)$ by $f$. It is also determined by the requirement that a section $s^*\colon U \to f^*(E)$, $\tilde\pi \circ s^* = \mathrm{id}_U$ is smooth if and only if the corresponding *section along* $f$, $s_f = F \circ s^*\colon U \to E$, $\pi \circ s_f = f$, is smooth. Note that $s^* = (\mathrm{id}, s_f)$.

In the next two examples we consider two vector bundles $(E_i, \pi_i, M)$, $i = 1, 2$ over $M$.

**Example 14.4.** (Sum) Consider the vector bundle $(E_1 \times E_2, \pi_1 \times \pi_2, M \times M)$ over $M \times M$ and the diagonal map $\Delta\colon M \to M \times M$. The pull back $(\Delta^*(E_1 \times E_2), \widetilde{\pi_1 \times \pi_2}, M)$ is called the *direct sum* of $(E_1, \pi_1, M)$ and $(E_2, \pi_2, M_2)$. Clearly the fiber $\Delta^*(E_1 \times E_2)(p) = \widetilde{\pi_1 \times \pi_2}^{-1}(p) = \{p\} \times E_1(p) \times E_2(p) \cong E_1(p) \oplus E_2(p)$, $p \in M$ and we use the notation $(E_1 \oplus E_2, \pi, M)$.

**Example 14.5.** (Hom) Consider the set $L(E_1; E_2) = \bigcup_{p\in M} L(E_1(p); E_2(p))$, the collection of all linear maps from fibers $E_1(p) = \pi_1^{-1}(p)$ in $E_1$ to corresponding fibers $E_2(p) = \pi_2^{-1}(p)$ in $E_2$. Moreover define $\pi\colon L(E_1; E_2) \to M$ by $\pi(L(E_1(p); E_2(p))) = p$, $p \in M$. Then $L(E_1; E_2)$ carries a unique differentiable structure such that $(L(E_1; E_2), \pi, M)$ is a vector bundle in which $s\colon U \to L(E_1; E_2)$, $\pi \circ s = \mathrm{id}_U$ is a smooth section if and only if the pair $(S, \mathrm{id}_U)\colon (E_{1|U}, \pi_{1|}, U) \to (E_{2|U}, \pi_{2|}, U)$ given by $S(v_1) = s(\pi_1(v_1))(v_1)$ is a bundle map, $U \subset M$ open.

In view of the above examples, note that a general bundle map $(F, f)\colon (E_1, \pi_1, M_1) \to (E_2, \pi_2, M_2)$ may be viewed as a smooth section in $(L(E_1; f^*(E_2)), \pi, M_1)$ and vise versa. This also illustrates the thesis: "Knowing a bundle is knowing its sections".

**Example 14.6.** (Dual bundle) For a bundle $(E, \pi, M)$ consider its *dual bundle* $(E^*, \pi, M)$ where $E^* = \bigcup_{p\in M} E(p)^* \simeq L(E; M \times \mathbb{R})$ as in 14.5. The dual bundle $(T^*M, \pi, M)$ of the tangent bundle $(TM, \pi, M)$ of $M$ is also called the *cotangent bundle* of $M$.

Smooth sections of $(T^*M, \pi, M)$ are also called 1-*forms* or differential forms of degree 1. If $f\colon M \to \mathbb{R}$ is a smooth function the *differential* $df$ of $f$ given by $df_p\colon T_pM \to \mathbb{R}$, $v \mapsto v[f]$ clearly is a 1-form on $M$. If $x^1_\alpha, \dots, x^n_\alpha$ are the coordinate functions of a coordinate patch $\phi_\alpha\colon V_\alpha \to U_\alpha \subset \mathbb{R}^n$, $V_\alpha \subset M$, the sections $dx^1_\alpha, \dots, dx^n_\alpha$ are linearly independent. In fact $dx^1_\alpha(p), \dots, dx^n_\alpha(p) \subset T_pM^*$ is the dual basis of $\frac{\partial}{\partial x^1_\alpha}(p), \dots, \frac{\partial}{\partial x^n_\alpha}(p) \in T_pM$ In particular any 1-form on $V_\alpha$ is written uniquely as $\omega = \sum_{i=1}^n f_i dx^i_\alpha$, where $f_i\colon V_\alpha \to \mathbb{R}$, $i = 1, \dots, n$ are smooth functions.

Given vector bundles $(E_i, \pi_i, M)$, $i = 1, \dots, k$ and $(E, \pi, M)$ we obtain by iterating 14.5 a vector bundle $(L(E_1; L(E_2, \dots, L(E_k; E) \cdots)), \pi, M)$ of iterated linear maps. This of course is canonically isomorphic to the bundle $(L(E_1, \dots, E_k; E), \pi, M)$ of $k$-*linear maps* from $E_1(p) \times \cdots \times E_k(p)$ to $E(p)$, $p \in M$. This way we can interpret the torsion tensor, $T$, of a connection $\nabla$ on $M$ as a smooth section in the bundle $(L(TM, TM; TM), \pi, M) = (L^2(TM; TM), \pi, M)$ of bilinear maps $T_pM \times T_pM \to T_pM$, $p \in M$. The curvature tensor, $R$, is a smooth section in $(L^3(TM; TM), \pi, M)$, and a Riemannian structure $g$ is a smooth section in $(L^2(TM; \mathbb{R}), \pi, M)$.

**Example 14.7.** (Tensor product) If $(E_i, \pi_i, M)$, $i = 1, 2$ are vector bundles over $M$, the bundle $(L(E_1, E_2; \mathbb{R}), \pi, M)$ of bilinear maps $E_1(p) \times E_2(p) \to \mathbb{R}$, $p \in M$ is also denoted by $(E_1^* \otimes E_2^*, \pi, M)$ and called the *tensor product* of $(E_1^*, \pi_1, M)$ and $(E_2^*, \pi_2, M)$. If $e_1, \dots, e_k \in E_1(p)$, $f_1, \dots, f_l \in E_2(p)$ are bases with corresponding dual bases $e_1^*, \dots, e_k^* \in E_1(p)^*$, $f_1^*, \dots, f_l^* \in E_2(p)^*$, $\{e_i^* \otimes e_j^*\}$, $i = 1, \dots, k$, $j = 1, \dots, l$ is a basis for $E_1^*(p) \otimes E_2^*(p)$. An isomorphism $E_1^*(p) \otimes E_2^*(p) \cong L(E_1(p), E_2(p); \mathbb{R})$ is then given where $e_i^* \otimes e_j^*\colon E_1(p) \times E_2(p) \to \mathbb{R}$ is the map $(v_1, v_2) \to e_i^*(v_1) \cdot e_j^*(v_2)$. With this notation we can write any bilinear map $B\colon E_1(p) \times E_2(p) \to \mathbb{R}$ uniquely as $B = \sum b_{ij} e_i^* \otimes e_j^*$. In the case $E_i = TM$

observe that a Riemannian structure (metric tensor) on $M$ is given locally i.e. in charts $\phi_\alpha\colon V_\alpha \to U_\alpha \subset \mathbb{R}^n$ by $g_| = \sum g_{ij}^\alpha dx_i^\alpha \otimes dx_j^\alpha$.

Again the construction (notation) in 14.7 can be iterated to yield $L(E_1, \ldots, E_k; \mathbb{R}) \cong E_1^* \otimes \cdots \otimes E_k^*$ for vector bundles $(E_i, \pi_i, M)$, $i = 1, \ldots, k$. Since canonically $(E^{**}, \pi, M) \cong (E, \pi, M)$ we have the *tensor product bundle* $(E_1 \otimes \cdots \otimes E_r, \pi, M) \cong (L(E_1^*, \ldots, E_k^*; \mathbb{R}), \pi, M)$. With this notation, $g$ is a section in $T^*M \otimes T^*M$, $T$ is a section in $T^*M \otimes T^*M \otimes TM$, and $R$ is a section in $T^*M \otimes T^*M \otimes T^*M \otimes TM$. A *tensor of type* $(r, s)$ on a manifold $M$ is by definition a section in the bundle $(T_s^r(TM), \pi, M)$, where in $T_s^r(TM) = TM \otimes \cdots \otimes TM \otimes T^*M \otimes \cdots \otimes T^*M$ there are $r$ factors of $TM$ and $s$ factors of $T^*M$.

We will now give a different interpretation of tensors. For purposes of exposition let us return to the case $(E_1^* \otimes E_2^*, \pi, M) \cong (L(E_1, E_2; \mathbb{R}, \pi, M)$. Clearly any smooth section $B$ of this bundle induces a map

$$B\colon \mathcal{S}^\infty(E_1) \times \mathcal{S}^\infty(E_2) \to C^\infty(M) \tag{14.8}$$

that assigns to section $s_i \in \mathcal{S}^\infty(E_i)$ of $(E_i, \pi_i, M)$ the map $p \to B(p)(s_1(p), s_2(p))$, $p \in M$. This map is bilinear with respect to $C^\infty(M)$. In order to see that the converse is also true, i.e., a bilinear map (14.8) defines a section in $(E_1^* \otimes E_2^*, \pi, M)$, we need to use so-called *localization functions* .

Consider the function $h\colon \mathbb{R} \to \mathbb{R}$ defined by

$$h(t) = \begin{cases} e^{-t^{-2}}, & t > 0 \\ 0, & t \le 0 \end{cases} \tag{14.9}$$

Then $h \ge 0$ is smooth, and $h(t) > 0$ for $t > 0$. Therefore, if $0 < r < R$, the function $\phi\colon \mathbb{R}^n \to \mathbb{R}$ given by

$$\phi(x) = \frac{h(R^2 - \|x\|^2)}{h(R^2 - \|x\|^2) + h(\|x\|^2 - r^2)} \tag{14.10}$$

is smooth and $\phi(x) = 1$ if $||x|| \le r$ and $\phi(x) = 0$ if $||x|| \ge R$. Clearly if $p \in M$ and $U \subset M$ is an open neighborhood of $p$, there is a smooth function $\phi\colon M \to [0, 1]$ s.t. $\phi \equiv 1$ near $p$ and $\operatorname{supp} \phi = \overline{\{q \in M \mid \phi(q) \neq 0\}} \subset U$ (provided $M$ is Hausdorff). Such a function is called a localization function at $p \in M$.

Now suppose $B\colon \mathcal{S}^\infty(E_1) \times \mathcal{S}^\infty(E_2) \to C^\infty(M)$ is bilinear with respect to $C^\infty(M)$. Let $s_1, \bar{s}_1$ be sections in $(E_1, \pi, M)$ with $s_1(p) = \bar{s}_1(p)$. In a local basis of sections $e_1, \ldots, e_k$ defined on $U \ni p$ we have $s_{1|U} = \sum t_i e_i$, $\bar{s}_{1|U} = \sum \bar{t}_i e_i$, and $t_i(p) = \bar{t}_i(p)$. Let $\phi$ be a localization function at $p$ with $\operatorname{supp} \phi \subset U$. Then we get globally defined sections $E_i$, and functions $T_i$, $i = 1, \ldots k$ that are zero outside $U$ and $\phi \cdot e_i$, resp.

$\phi \cdot t_i$ inside $U$. Using this we get

$$\begin{aligned} B(s_1, s_2)(p) &= \phi^2(p)B(s_1, s_2)(p) \\ &= B(\phi^2 s_1, s_2)(p) \\ &= B\left(\sum T_i E_i, s_2\right)(p) \\ &= \left(\sum T_i B(E_i, s_2)\right)(p) \\ &= \sum t_i(p)B(E_i, s_2)(p) \\ &= \sum \bar{t}_i(p)B(E_i, s_2)(p) \\ &= B(\bar{s}_1, s_2)(p) \end{aligned}$$

and similarly $B(s_1, s_2)(p) = B(s_1, \bar{s}_2)(p)$ if $s_2(p) = \bar{s}_2(p)$. Moreover if $v_1 \in E_1(p)$, $v_2 \in E_2(p)$ we may construct sections $s_1 \in S^\infty(E_1)$, $s_2 \in S^\infty(E_2)$ such that $s_i(p) = v_i$, $i = 1, 2$. This again uses a localization function $\phi$ and local trivializations of $(E_i, \pi_i, M)$ near $p$.

In view of the above we may also view a tensor of type $(r, s)$ on $M$ as a map

$$(14.11)\quad T\colon \underbrace{\mathcal{S}^\infty(T^*M) \times \cdots \times \mathcal{S}^\infty(T^*M)}_{r} \times \underbrace{\mathcal{S}^\infty(TM) \times \cdots \times \mathcal{S}^\infty(TM)}_{s} \to C^\infty(M)$$

which is $(r + s)$-linear with respect to $C^\infty(M)$.

**Problem 14.12.** Complete the proof of the claims in 14.3 and 14.5.

**Problem 14.13.** Show that $df$ as defined on page 36 is a smooth section of the cotangent bundle.

**Problem 14.14.** Fill in the missing details in 14.7.

**Problem 14.15.** Prove the existence of localization functions (cf. 14.9, 14.10, ...). Show that it is necessary that $M$ is Hausdorff.

**Problem 14.16.** On a Riemannian manifold $(M, \langle\ ,\ \rangle)$ exhibit a canonical bundle isomorphism $(m, \mathrm{id}_m)$, $m\colon TM \to T^*M$. Using this the *gradient* of a smooth $f\colon M \to \mathbb{R}$ is defined by $m \circ \operatorname{grad} f = df$.

What is $\operatorname{grad} f$ in local coordinates?

**Problem 14.17.** Let $\nabla$ be the Levi–Civita connection on a Riemannian manifold $(M, \langle\ ,\ \rangle)$. The *divergence* of a vector field $X$ on $M$ is defined as the function $(\operatorname{div} X)(p) = trace(v \to \nabla_v X)$, $v \in T_pM$.

What is $\operatorname{div} X$ in local coordinates?

The *Laplacian* of a function $f\colon M \to \mathbb{R}$ is defined by $\Delta f = \operatorname{div} \operatorname{grad} f$.

Find the local expression of $\Delta f$.

**Problem 14.18.** Show that a connection $\nabla$ on a manifold $M$ can be defined as a linear map

$$\nabla\colon S^\infty(TM) \to S^\infty(T^*M \otimes TM)$$

so that $\nabla(f \cdot X) = df \otimes X + f\nabla X$, $f \in C^\infty(M)$, $X \in S^\infty(TM)$.

## 15. Connections and differential forms

A connection in a vector bundle $(E, \pi, M)$ is a map $\nabla$ which assigns to any vector field $X$ on $M$, and section $s \in S^\infty(E)$ a section $\nabla_X s \in S^\infty(E)$ such that (12.2)–(12.5) hold with obvious modifications. The first two of these expresses that $\nabla_X s$

is tensorial in the $X$-variable. Using the notation from the previous section we can therefore also say that a connection $\nabla$ in $(E, \pi, M)$ is a linear map

$$\nabla\colon S^\infty(E) \to S^\infty(T^*M \otimes E) \tag{15.1}$$

which satisfies

$$\nabla(fs) = df \otimes s + f\nabla s \tag{15.2}$$

for all $f \in C^\infty(M)$, and $s \in S^\infty(E)$. Using localization maps it is easy to see that

$$(\nabla_X s_1)_{|U} = (\nabla_X s_2)_{|U} \tag{15.3}$$

whenever $s_{1|U} = s_{2|U}$, $U \subset M$ open. This also allows to define $\nabla_X s$ when $X$ and $s$ are defined only on $U \subset M$. If $X_1, \dots, X_n$ and $s_1, \dots, s_k$ define local trivializations of $TM_{|U}$ and $E_{|U}$ respectively we set

$$\nabla_{X_i} s_j = \sum_{l=1}^{k} \Gamma_{ij}^l s_l, \tag{15.4}$$

and call $\Gamma_{ij}^l$ the components of $\nabla$ relative to these trivializations.

The curvature tensor $R$ is defined by

$$R(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \tag{15.5}$$

for all vector field $X$, $Y$ and sections $s$.

**Example 15.6.** Let $\nabla$ be a connection on $(E, \pi, M)$, and $f\colon N \to M$ a smooth map. There is a unique connection $\nabla^*$ on the pull back bundle $(f^*(E), \tilde{\pi}, N)$ such that $\nabla^*_X(\mathrm{id}, s \circ f) = (\mathrm{id}, \nabla_{f_* X} s)$ for vector fields $X$ on $N$ and sections $s$ in $(E, \pi, M)$.

In particular if $c\colon J \to M$ is a smooth curve in $M$, we get an induced connection $\nabla^*$ on $(c^*(E), \tilde{\pi}, J)$. This allows us to differentiate sections $s_c$ along $c$ in direction $\frac{\partial}{\partial t} \in C^\infty(TJ)$, $\nabla^*_{\frac{\partial}{\partial t}} s_c$. Earlier we have used the notation $\nabla_c s$ for the case $E = TM$. A section $s$ along $c$ is called *parallel* if and only if $\nabla_c s = 0$. Also in this generality this leads to the notion of *parallel transport* along $c$. This in turn gives a geometric interpretation of the curvature tensor as in section 10.

**Example 15.7.** Given connections $\nabla^i$ in $(E_i, \pi_i, M)$, $i = 1, 2$. There is a unique connection $\nabla$ in $(L(E_1; E_2), \pi, M)$ such that

$$(\nabla_X T)s_1 = \nabla_X(Ts_1) - T\nabla_X s_1$$

for all $T \in \mathcal{S}^\infty(L(E_1; E_2), \pi, M)$, $X \in \mathcal{S}^\infty(TM)$ and $s_1 \in \mathcal{S}^\infty(E_1)$.

Clearly now using the above example iteratively one constructs canonical connections on tensor bundles (cf. also section 7).

We can use the notion of parallel transport to define orientability of a vector bundle $(E, \pi, M)$. First recall that an *orientation* of $\mathbb{R}^k$ is an equivalence class of bases, where $\{e_1, \dots, e_k\}$ is equivalent to $\{\bar{e}_1, \dots, \bar{e}_k\}$ (*determine the same orientation*) if and only if the linear map $T\colon \mathbb{R}^k \to \mathbb{R}^k$ that maps $\{e_1, \dots, e_k\}$ into $\{\bar{e}_1, \dots, \bar{e}_k\}$ has positive determinant. The *holonomy group* $\Phi_p$, of the connection $\nabla$ at $p \in M$ is by definition the subgroup of $GL(E(p))$ consisting of parallel transports along loops at $p$. Note that $\Phi_p \cong \Phi_q$ for any $p, q \in M$. The bundle $(E, \pi, M)$ is said to be orientable if and only if $\det(\Phi_p) \subset \mathbb{R}_+$. An *orientation* of each fiber $E(q)$ is then obtained from an orientation at a fixed fiber $E(p)$ by means of parallel transport along paths from $p$ to $q$.

The concept of orientability is independent of connections, however. This will follow from a description in terms of exterior powers given below.

**Example 15.8.** For a fixed $k$-dimensional vector bundle $(E, \pi, M)$ consider the bundle $(L^r(E;\mathbb{R}), \pi, M)$ of $r$-linear maps from the fibers of $E$ to $\mathbb{R}$. For fixed $p \in M$ consider the subspace $A^r(E(p);\mathbb{R}) \subset L^r(E(p);\mathbb{R})$ consisting of alternating $r$-linear maps $\omega$ i.e. $\omega(v_1, \ldots, v_r) = \operatorname{sign}\sigma\, \omega(v_{\sigma(i)}, \ldots, v_{\sigma(r)})$ for every permutation $\sigma$ of $\{1, \ldots, r\}$, $r \leq k$. Clearly there is a unique vector bundle $(A^r(E;\mathbb{R}), \pi, M)$ where a section $\omega\colon U \to A^r(E_{|U};\mathbb{R})$ is smooth if and only if $\omega(s_1, \ldots, s_r)$ is a smooth function on $U$ whenever $s_1 \ldots, s_r$ are smooth sections of $E_{|U} \to U$. If $e_1, \ldots, e_k$ is a basis for $E(p)$ with dual basis $e_1^*, \ldots, e_k^*$, then $\{e_{i_1}^* \wedge \ldots \wedge e_{i_r}^*\}$, $1 \leq i_1 < \cdots < i_r \leq k$ form a basis for a $\binom{k}{r}$-dimensional vector space $\Lambda^r(E(p)^*) \cong A^r(E(p);\mathbb{R})$ where the isomorphism is established so that $e_{i_1}^* \wedge \ldots \wedge e_{i_r}^* \colon E(p) \times \cdots \times E(p) \to \mathbb{R}$ is the map $(v_1, \ldots, v_r) \to \det\{e_{i_j}^*(v_l)\} = \sum_\sigma \operatorname{sign}\sigma\, e_{i_1}^*(v_{\sigma(1)}) \cdot \cdots \cdot e_{i_r}^*(v_{\sigma(r)})$. Using this we say that $(E, \pi, M)$ is *orientable* if and only if the one-dimensional bundle $(A^k(E,\mathbb{R}), \pi, M) \cong (\Lambda^k(E^*), \pi, M)$ is trivial. A choice of a non-trivial section determines an *orientation* on every fiber $E(p)$, $p \in M$. If $(E, \pi, M)$ is equipped with a connection $\nabla$, the two notions of orientability are naturally the same.

Consider now the special case $E = TM$. A smooth section in $(A^r(TM;\mathbb{R}), \pi, M) \cong (\Lambda^r(T^*M), \pi, M)$ is called an *$r$-form* on $M$, or a differential form of degree $r$. Locally, in a coordinate system $(x_1, \ldots, x_n)$ an $r$-form $\omega$ is written uniquely as $\omega_| = \sum_{1 \leq i_1 < \cdots < i_r \leq n} u_{i_1 \cdots i_r} dx_{i_1} \wedge \ldots \wedge dx_{i_r}$. The obvious product $\Lambda^r(T_pM^*) \times \Lambda^s(T_pM^*) \xrightarrow{\wedge} \lambda^{r+s}(T_pM^*)$ called the wedge product takes the form

$$\text{(15.9)}\quad \alpha \wedge \beta(v_1, \ldots, v_{r+s}) = \frac{1}{r!}\frac{1}{s!}\, sign\, \sigma \sum_\sigma \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) \cdot \beta(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)})$$

when viewed as a map $A^r(T_pM;\mathbb{R}) \times A^s(T_pM;\mathbb{R}) \to A^{r+s}(T_pM;\mathbb{R})$. If we denote the space of $r$-forms on $M$ by $\mathcal{D}^r(M)$, then clearly $\mathcal{D}(M) = \bigoplus_{r=0}^n \mathcal{D}^r(M)$ becomes a graded algebra. Here we set $\mathcal{D}^0(M) = C^\infty(M)$.

**Theorem 15.10.** *There is a unique linear extension $d\colon \mathcal{D}(M) \to \mathcal{D}(M)$ of the differential on $\mathcal{D}^0(M)$ so that*

(i) $d\mathcal{D}^r \subset \mathcal{D}^{r+1}(M)$ *($d$ has degree 1)*
(ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^s \omega \wedge d\eta$, $\omega \in \mathcal{D}^s(M)$, $\eta \in \mathcal{D}(M)$
(iii) $d^2 = d \circ d = 0$.

**Proof:** *Uniqueness:* First observe that by using localization functions we conclude $d\omega_{|U} = d\eta_{|U}$ if $\omega_{|U} = \eta_{|U}$. If $\omega_| = \sum u_{i_1 \ldots i_r} dx_{i_1} \wedge \ldots \wedge dx_{i_r}$ in a chart $U \subset M$ we get (again via localization functions) that $d\omega_| = \sum du_{i_1 \ldots i_r} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r}$, which shows uniqueness. *Existence:* Given $\omega \in \mathcal{D}^r(M)$ and suppose $\omega_{|U} = \sum u_{i_1 \ldots i_r} dx_{i_1} \wedge \ldots \wedge dx_{i_r}$. Defining $d_U \colon \mathcal{D}(U) \to \mathcal{D}(U)$ so that $d_U \sum u_{i_1 \ldots i_r} dx_{i_1} \wedge \ldots \wedge dx_{i_r} = \sum du_{i_1 \ldots i_r} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r}$ one checks that $d$ satisfies the conditions in the theorem. Hence $d\colon \mathcal{D}(M) \to \mathcal{D}(M)$ given as $d\omega_{|U} = d_U \omega_{|U}$ is well defined by the uniqueness part. Moreover, this $d$ clearly satisfies the conditions as stated in the theorem.

The map $d\colon \mathcal{D}(M) \to \mathcal{D}(M)$ is called the *exterior derivative* . Since $d^2 = 0$, by definition

$$\text{(15.11)}\qquad \mathcal{D}^0(M) \to \cdots \to \mathcal{D}^r(M) \to \mathcal{D}^{r+1}(M) \to \cdots \to \mathcal{D}^n(M) \to 0$$

is a *chain complex* whose cohomology groups

$$H^r(M) = \frac{\mathrm{Ker}\{d\colon \mathcal{D}^r \to \mathcal{D}^{r+1}\}}{\mathrm{Im}\{d\colon \mathcal{D}^{r-1} \to \mathcal{D}^r\}} \tag{15.12}$$

are the so called *de Rham cohomology groups* (vector spaces) of $M$. It is a surprizing fact that $H^r(M)$ are homotopy invariant of $M$.

Differential forms constitute a powerfull tool in geometry and topology, not only because they form a differential graded algebra $(\mathcal{D}(M), d)$, but also because they behave nicely with respect to maps $f\colon M \to N$. In fact, for every $\omega \in \mathcal{D}^r(N)$ we have $f^*(\omega) \in \mathcal{D}^r(M)$ defined by

$$f^*\omega_p(X_1(p), \dots, X_r(p)) = \omega_{f(p)}(f_{*p}X_1(p), \dots, f_{*p}X_r(p)), \quad p \in M \tag{15.13}$$

whenever $X_1, \dots X_r$ are smooth vector fields on $M$. Then $f^*\colon \mathcal{D}(N) \to \mathcal{D}(M)$ is a homomorphism of commutative graded differential algebras. Here commutative refer to the property $\omega \wedge \eta = (-1)^{r+s}\eta \wedge \omega$, $\omega \in \mathcal{D}^r(M)$, $\eta \in \mathcal{D}^s(M)$.

**Problem 15.14.** Prove (15.3) and the statements in (15.6) and (15.7).

**Problem 15.15.** Fill in the missing details in (15.8), (15.9), and (15.10).

**Problem 15.16.** Show that $d\omega(X, Y) = X[\omega(Y)] - Y[\omega(X)] - \omega([X, Y])$ for any 1-form $\omega$.

**Problem 15.17.** Show that $f^*$ is well defined by (15.13) and prove the stated properties of it. If $M$ and $N$ are orientable, what whould it mean that $f\colon M \to N$ is *orientation preserving* ?

**Problem 15.18.** Let $\Gamma^l_{ij}$ be the components of a connection $\nabla$ on $(E, \pi, M)$ relative to trivializations of $TM$, $E$ over $U$ determined by $X_1, \dots, X_n$ and $s_1, \dots, s_k$ (cf. 15.4). For fixed $l$ and $j$ consider the 1-form $\omega^l_j$ on $U$ defined by $\omega^l_j(X_i) = \Gamma^l_{ij}$. Show that $\nabla s_j = \sum_l \omega^l_j \otimes s_l$, and that any $k \times k$-matrix of 1-forms $\{\omega^l_j\}$ in this way determine a connection on $(E_{|U}, \pi, U)$. These 1-forms are called *connection forms* .

**Problem 15.19.** Let $\nabla\colon S^\infty(E) \to S^\infty(TM^* \otimes E)$ be a connection in $(E, \pi, M)$. Show that there is a unique linear map

$$\hat{\nabla}\colon S^\infty(TM^* \otimes E) \to S^\infty(\Lambda^2(TM^*) \otimes E)$$

such that

$$\hat{\nabla}(\theta \otimes s) = d\theta \otimes s - \theta \wedge \nabla(s)$$

for every 1-form $\theta$ on $M$ and section $s \in S^\infty(E)$. Show that moreover

$$\hat{\nabla}(f(\theta \otimes s)) = df \wedge (\theta \otimes s) + f\hat{\nabla}(\theta \otimes s).$$

Hint: Consider local sections $\theta_1, \dots, \theta_n$ and $s_1, \dots, s_k$.

**Problem 15.20.** Show that the composed map $R = \hat{\nabla} \circ \nabla\colon S^\infty(E) \to S^\infty(\Lambda^2(TM^*) \otimes E)$ is linear with respect to $C^\infty(M)$. Thus $R$ can be viewed as a section in $L(E; \Lambda^2(TM^*) \otimes E) \cong \Lambda^2(TM^*) \otimes L(E; E) \cong A^2(TM; L(E, E))$. In a trivialization of $E$ by sections $s_1, \dots, s_k$ show that

$$R(s_j) = \hat{\nabla}(\sum \omega^l_j \otimes s_l) = \sum \Omega^l_j \otimes s_l,$$

where $\Omega^l_j$ is the 2-form given by

$$\Omega^l_j = d\omega^l_j - \sum_\alpha \omega^\alpha_j \wedge \omega^l_\alpha.$$

These 2-forms are called the curvature forms of the connection $\nabla$.

**Problem 15.21.** Find the expression for $\Omega^l_j$ (cf. 15.10) in terms of the $\Gamma^l_{ij}$'s, and show that $R$ defined in 15.20 is in fact the curvature tensor of $\nabla$.

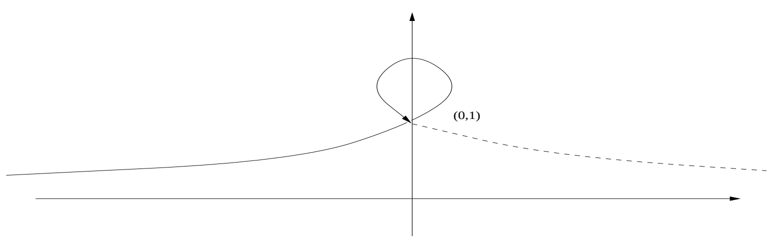
The form interpretation of curvature is important in the development of characteristic classes for bundles.

## 16. Submanifolds

Let $M^n$ be an $n$-dimensional manifold and $L \subset M$ a subset. We say that $L$ is a $k$-dimensional (embedded) *submanifold* of $M$ if for every $p \in L$ there is a chart $\phi\colon V \to U \subset \mathbb{R}^n$ for $M$ such that $\phi(p) = 0$ and $\phi(V \cap L) = U \cap \mathbb{R}^k$. We will refer to such charts as *submanifold charts* . It is clear that the collection $\{\phi_{|L}\}$ of submanifold charts for $L$ form an atlas for $L$, making it into a $k$-dimensional smooth manifold. Moreover the manifold topology of $L$ obtained this way coincides with the induced topology from $M$ (cf. 16.7).

More generally we say that a map $f\colon L^k \to M^n$ between smooth manifolds is an *immersion* provided $f_{*p}\colon T_pL \to T_{f(p)}M$ is injective for every $p \in L$. From the inverse function theorem it follows that there are charts $\psi_\alpha\colon V_\alpha \to U_\alpha \subset \mathbb{R}^k$ around $p$ and $\phi_\beta\colon V_\beta \to U_\beta \subset \mathbb{R}^n$ around $f(p)$, such that $\phi_\beta \circ f \circ \psi_\alpha^{-1}(x) = (x, 0)$ for all $x \in U_\alpha$ (cf. 16.8). Observe that $f(V_\alpha) \subset M$ is a $k$-dimensional submanifold of $M$. If in addition $f\colon L \to M$ is globally injective, we say that the subset $f(L) \subset M$ is an *immersed submanifold.* Finally, an injective immersion $f\colon L \to M$ is called an *embedding* if $f\colon L \to f(L)$ is a homeomorphism when $f(L)$ is given the induced topology from $M$. Clearly, the image $f(L)$ of an imbedding $f$ is a submanifold. Conversely, if $L \subset M$ is a submanifold, the inclusion map $i\colon L \hookrightarrow M$ is an embedding.

**Example 16.1.** An immersion from an interval is simply a regular curve, $c\colon J \to M$, i.e. $\dot{c}(t) \neq 0$, $t \in J$. The picture below represents such a regular curve $c\colon \mathbb{R} \to \mathbb{R}^2$




with say $c(-1) = c(1) = (0,1)$ and $c_|\colon \mathbb{R} - \{-1,1\} \to \mathbb{R}^2$ is injective. Clearly $c_|\colon (-\infty, 1) \to \mathbb{R}^2$ is 1-1 immersion, but not an embedding.

Another such example is given by $c\colon \mathbb{R} \to S^1 \times S^1$ where $c(t) = (e^{2\pi it}, e^{2\pi i\alpha t})$, $\alpha \in \mathbb{R} - \mathbb{Q}$. Here $c(\mathbb{R})$ is dense in $T^2 = S^1 \times S^1$.

Locally, of course any submanifold is the solution set of a system of equations. The following global version of this is very useful: Let $f\colon M \to N$ be a smooth map, and $q \in N$. Then the solution set $f^{-1}(q) = L \subset M$ is a submanifold of $M$ provided $f_{*p}\colon T_pM \to T_qN$ is surjective for all $p \in f^{-1}(q)$. The existence of submanifold charts around $p \in L$ is again guaranteed by the inverse function theorem (cf. 16.10). More generally, if $f\colon M \to N$ and $L \subset N$ is a submanifold, one says that $f$ is *transversal* to $L$ if $f_{*p}(T_pM) + T_{f(p)}L = T_{f(p)}N$ for all $p \in f^{-1}(L)$. Also in this case

$f^{-1}(L) \subset M$ is a submanifold (16.11). A map $f\colon M \to N$ is called a *submersion* if $f_{*p}\colon T_pM \to T_{f(p)}N$ is surjective for all $p \in M$. Clearly a submersion is transversal to every submanifold $L \subset N$.

**Example 16.2.** Let $(E, \pi, M)$ be a vector bundle over $M$ and $f\colon N \to M$ a smooth map. Then $\Delta = \{(p,p) \in M \times M \mid p \in M\}$ is a submanifold of $M \times M$ and $f \times \pi\colon N \times E \to M \times M$ is transversal to $\Delta$. In particular the total space $f^*(E) = (f \times \pi)^{-1}(\Delta)$ of the bundle induced by $f$ is a submanifold of $N \times E$ (cf. 14.3).

**Example 16.3.** If $(E^k, \pi, M)$ is a vector bundle with Riemannian structure $g$, i.e. $g$ is a smooth section in $(L^2(E;\mathbb{R}), \pi, M)$ which is symmetric and positive definite at each $p \in M$. The map $\| \|^2\colon E \to \mathbb{R}$ is smooth and a submersion when restricted to $E - M$. Here $M$ is identified with the image of the zero section in $(E, \pi, M)$. In particular the unit sphere bundle $(S(E), \pi, M)$, $S(E) = \| \|^{-1}(1)$ is a fiber bundle with fiber $S^{k-1}$.

**Example 16.4.** The image of any section $s\colon M \to E$ of a vector bundle $(E, \pi, M)$ is a submanifold of $E$.

The following observation is often used

**Lemma 16.5.** *Let $M, N$ and $L$ be smooth manifolds and $f\colon M \to N$, $g\colon N \to L$ maps. Then*

(i) *If $g$ is an imbedding, $f$ is smooth if and only if $g \circ f$ is smooth.*
(ii) *If $f$ is a submersion onto $N$, then $g\colon N \to L$ is smooth if and only if $g \circ f$ is smooth.*

**Example 16.6.** Let $L(n,n)$ denote the $n^2$-dimensional vector space of all $n \times n$ matrices, and $GL(n) \subset L(n,n)$ the invertible ones. Clearly $\det\colon L(n,n) \to \mathbb{R}$ is smooth and in particular, $GL(n) = \det^{-1}(\mathbb{R}\backslash\{0\})$ is an open subset of $L(n,n)$. Moreover the group operations $GL(n) \times GL(n) \to GL(n)$, $(A,B) \mapsto AB$ and $GL(n) \to GL(n)$, $A \mapsto A^{-1}$ are smooth. By definition, a *Lie Group* is a smooth manifold $G$ which at the same time is a group and the group operations are smooth. In particular, the *general linear group* $GL(n)$ is a Lie group. This group has many important subgroups, that are also Lie groups. Here let us point out only two such subgroups:
(i) $SL(n) = \det^{-1}(1) \subset GL(n)$ is called the *special linear group* . Clearly, $\det\colon GL(n) \to \mathbb{R} - \{0\}$ is a submersion, and in particular $SL(n)$ is a submanifold of $GL(n)$. By using (16.5)(i) it is easy to see that the group operations on $SL(n)$ are smooth. Thus, $SL(n)$ is a Lie group.
(ii) $O(n) = \{A \in GL(n) \mid AA^t = \mathrm{id}\}$ is the *orthogonal group* . Let $\mathrm{Sym}(n)$ denote the vector space of symmetric $n \times n$ matrices. Then $f\colon GL(n) \to \mathrm{Sym}(n)$ defined by $f(A) = AA^t$ is smooth and $f_{*A}\colon T_AGL(n) \to T_{f(A)}\mathrm{Sym}(n)$ is surjective for all $A \in f^{-1}(\mathrm{id}) = O(n)$. By the above arguments therefore also $O(n)$ is a Lie group. Note that $O(n)$ has two connected components. The identity component $SO(n) = O(n) \cap SL(n)$ is called the *special orthogonal group*.

**Problem 16.7.** Show that the manifold topology of a submanifold $L \subset M$ is the same as the induced topology on $L$ from $M$.

**Problem 16.8.** Let $f\colon L^k \to M^n$ be a smooth map and suppose $f_{*p}\colon T_pL \to T_{f(p)}M$ is 1-1 for some $p \in L$. Show that $k \le n$, and use the inverse function theorem to construct charts $\psi_\alpha\colon V_\alpha \to U_\alpha \subset \mathbb{R}^k$ around $p$ and $\phi_\beta\colon V_\beta \to U_\beta \subset \mathbb{R}^n$ around $f(p)$ so that $\phi_\beta(f(\psi_\alpha^{-1}(x))) = (x, 0)$ for all $x \in U_\alpha$.

**Problem 16.9.** Show that the image of $c\colon \mathbb{R} \to S^1 \times S^1$ in (16.1) is dense in $S^1 \times S^1$.

**Problem 16.10.** Let $f\colon M^n \to N^k$ be a smooth map and $p \in M$ a point so that $f_{*p}\colon T_pM \to T_{f(p)}N$ is surjective. Show that $n \geq k$, and use the inverse function theorem to construct charts $\phi_\alpha\colon V_\alpha \to U_\alpha \subset \mathbb{R}^n$ around $p$ and $\psi_\beta\colon V_\beta \to U_\beta \subset \mathbb{R}^k$ around $f(p)$ so that $\psi_\beta \circ f \circ \phi_\alpha^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_k)$ for all $(x_1, \dots, x_n) \in U_\alpha$.

**Problem 16.11.** Let $f\colon M^n \to N^k$ be a smooth map, and $L \subset N$ a submanifold of codimension $l$, i.e. $\dim N - \dim L = l$. Show that if $f$ is transversal to $L$, then $f^{-1}(L)$ is a submanifold of $M$ with codimension $l$.
Hint: For $p \in f^{-1}(L)$ choose a submanifold chart around $f(p) \in L \subset N$, and reduce to the case where $L$ is a point.

**Problem 16.12.** Prove the statements in 16.3.

**Problem 16.13.** Prove the statements in 16.4.

**Problem 16.14.** Prove Lemma 16.5.

**Problem 16.15.** Prove the statements in in 16.6.

**Problem 16.16.** Let $G$ be an $n$-dimensional Lie group. For each $a \in G$ the maps $L_a, R_a\colon G \to G$, $g \mapsto a \cdot g, g \cdot a$ are called *left translation* , resp. *right translation* in $G$ by $a$.
(i) Show that all left and right translations are diffeomorphisms of $G$

A vector field $X$ on $G$ is called *left invariant* (*right invariant* ) if $X_{a \cdot b} = (L_a)_{*b}(X_b)$ (resp. $(R_b)_{*a}(X_a)$) for all $a, b \in G$.
(ii) For any $v \in T_eG$ let $X_a \in T_aG$ be the vector $X_a = (L_a)_{*e}(v)$. Show that $X$ defined this way is a smooth left invariant vector field on $G$.
(iii) Define a vector space isomorphism between $T_eG$ and the space of left invariant vector fields on $G$. Here $e \in G$ is the neutral element.
(iv) Let $X$ and $Y$ be left invariant vector fields on $G$. Show that $[X, Y]$ is left invariant.

This gives the vector space $\mathfrak{g}$ of left invariant vector fields on $G$ the structure of an $n$-dimensional Lie algebra $\mathfrak{g} \cong T_eG$.

Define a connection $\nabla$ on G such that $\nabla X = 0$ , i.e. $X$ is parallel for all left invariant vector fields $X \in \mathfrak{g}$.
(v) Show that $T(X, Y) = -[X, Y]$, and $R \equiv 0$.

Let $\langle\ ,\ \rangle$ be an inner product for $T_eG$. Define $\langle\ ,\ \rangle_a$ by $\langle L_{a_*}u, L_{a_*}v\rangle_a = \langle u, v\rangle$ for all $u, v \in T_eG$.
(vi) Show that $\langle\ ,\ \rangle$ is a Riemannian structure on $G$.
(vii) Show that all left translations $L_a$ are isometries in this metric.

## 17. Relative curvature

Consider a submanifold $M^n \subset N^{n+k}$ of a Riemannian manifold $(N, \langle\ ,\ \rangle)$. Then for each $p \in M$ the tangent space $T_pN$ admits an orthogonal splitting

$$T_pN = T_pM \oplus T_pM^\perp \tag{17.1}$$

where $T_pM^\perp$ is called the *normal space* to $M$ at $p$. Let $TM^\perp = \bigcup_{p \in M} T_pM^\perp$ and define $\pi\colon TM^\perp \to M$ by $\pi(T_pM^\perp) = p$ for all $p \in M$. Then

**Theorem 17.2.** *$TM^\perp$ admit a unique differentiable structure such that $(TM^\perp, \pi, M)$ is a $k$-dimensional sub vector bundle of $(TN_{|M}, \pi, M)$.*

Naturally $(TM^\perp, \pi, M)$ is called the *normal bundle* to $M$ in $N$. Clearly the Riemannian metric $\langle\ ,\ \rangle$ on $TN$ restrict to smooth inner products on $TM$ and $TM^\perp$ respectively. In particular $M$ is a Riemannian manifold. We will use $\langle\ ,\ \rangle$ to denote its restrictions also.

Any smooth vector field $Z\colon M \to TN$ along $M$ decomposes uniquely as $Z = Z^\top + Z^\perp$, where $Z^\top$ is a smooth vector field on $M$ and $Z^\perp$ is a smooth normal field to $M$. If we let $\bar\nabla$ denote also the restriction of the Levi–Civita connection on $N$ to $(TN_{|M}, \pi, M)$ (cf. 15.6) we get

$$\bar\nabla_X Y = \nabla_X Y + \alpha(X,Y), \tag{17.3}$$

where

$$\nabla_X Y = (\bar\nabla_X Y)^\top, \qquad \alpha(X,Y) = (\bar\nabla_X Y)^\perp \tag{17.4}$$

for all smooth vector fields $X, Y$ on $M$. Moreover

$$\bar\nabla_X \eta = S(\eta, X) + \nabla_X \eta, \tag{17.5}$$

where

$$S(\eta, X) = (\bar\nabla_X \eta)^\top, \qquad \nabla_X \eta = (\bar\nabla_X \eta)^\perp \tag{17.6}$$

for all smooth vector (resp. normal) fields $X, \eta$ on $M$.

**Theorem 17.7.** (1) $\nabla$ *defined in (17.4) is the Levi–Civita connection on $M$.*

(2) $\nabla$ *defined in (17.5) is a metric connection for the normal bundle of $M$.*

(3) $\alpha$ *and $S$ defined in (17.4) and (17.5) are tensorial. Moreover $\langle \alpha(X,Y), \eta\rangle = -\langle S(\eta, X), Y\rangle$ for all $X$ ,$Y$ and $\eta$, and $\alpha$ is symmetric.*

**Proof:** It is easy to see that $\nabla$ defined in (17.4) and (17.5) are metric connections (cf. 12.2–12.5, 12.7). Since the Lie bracket of vector fields $X$, $Y$ tangent to a submanifold is again tangent to the submanifold (cf. 17.16), this completes the proof of (1) and (2).

To prove (3) first note that $\alpha$ clearly is bilinear and linear with respect to $C^\infty(M)$ in the $X$-variable. Again, using that $[X,Y]$ is tangent to $M$, and $\bar\nabla$ is symmetric, we get that $\alpha(X,Y) = \alpha(Y,X)$ and thus in particular also tensorial in the $Y$-variable. To see that $S$ is linear with respect to $C^\infty(M)$ in the $\eta$-variable we proceed by $S(f\eta, X) = (\bar\nabla_X f\eta)^\top = (X[f]\cdot\eta)^\top + (f\cdot\bar\nabla_X\eta)^\top = fS(\eta, X)$. Finally

$$\begin{aligned}\langle \alpha(X,Y), \eta\rangle &= \langle \bar\nabla_X Y, \eta\rangle \\ &= X\langle Y, \eta\rangle - \langle Y, \bar\nabla_X \eta\rangle \\ &= -\langle Y, S(\eta, X)\rangle\end{aligned}$$

since $\bar\nabla$ is metric and $Y$ is tangential, and $\eta$ is normal to $M$.

For $p \in M$, the symmetric bilinear map

$$\alpha_p\colon T_pM \times T_pM \to T_pM^\perp \tag{17.8}$$

is called the *second fundamental tensor* for $M$ in $N$ at $p$. If $\eta_p$ is a unit normal vector to $M$ at $p$, then $l_{n_p} = \langle \alpha_p(\cdot,\cdot), \eta_p\rangle$ is called the second fundamental form in direction $\eta_p$. The trace of $\alpha_p$,

$$H_p = -\frac{1}{\dim M}\sum_{i=1}^{n} \alpha_p(e_i, e_i), \tag{17.9}$$

$e_1, \ldots, e_n$ and orthonormal basis for $T_pM$, is called the *mean curvature vector* at $p$ (cf. 17.7). $M$ is said to be a *minimal submanifold* if and only if its mean curvature vector field $H$ is identically zero. Even stronger, if $\alpha \equiv 0$, we say that $M$ is *totally geodesic* . A minimal submanifold can be characterized geometrically, like geodesics, as being stationary for the volume function under variations. The geometric significance of $M$ being totally geodesic is, that geodesics in $M$ are also geodesics in $N$ (cf. 17.18). According to (17.7) it is equivalent to consider for $p \in M$ the bilinear map

$$S_p \colon T_pM^\perp \times T_pM \to T_pM. \tag{17.10}$$

For fixed unit normal vector $\eta_p$ the symmetric linear map

$$S_{\eta_p} = S_p(\eta_p, \cdot) \colon T_pM \to T_pM \tag{17.11}$$

is called the *shape operator* (or *Weingarten map* ) to $M$ in direction $\eta_p$. Its eigenvalues are called the *principal curvatures* of $M$ relative to $\eta_p$. The corresponding eigenvectors are called principal curvature directions. In analogy with the above definition,

$$H_{\eta_p} = \frac{1}{\dim M} \operatorname{trace} S_{\eta_p} \tag{17.12}$$

is called the *mean curvature* with respect to $\eta_p$, and moreover

$$G_{\eta_p} = \det S_{\eta_p} \tag{17.13}$$

is called the *Gauss–Kronecker* curvature with respect to $\eta_p$.

All such curvatures are called *relative* , or *extrinsic curvatures* of $M$ in $N$. The situation is of course particularly simple if codim $M = 1$.

Let us now give another geometric interpretation of the second fundamental tensor, or equivalently, shape operator.

Consider a variation

$$V \colon M \times (-\epsilon, \epsilon) \to N$$

of $M$ in $N$, and assume for simplicity, that each $V_s \colon M \to N$ is an embedding, and that the variation field $\eta_p = V_{*(p,0)}\left(\frac{\partial}{\partial s}\right)$ is normal to $M$ in $N$. For each $s \in (-\epsilon, \epsilon)$ we get an induced Riemannian metric $\langle\ ,\ \rangle_s$ on $M$. Then

**Proposition 17.14.** *For any vector fields $X, Y$ on $M$,*

$$\frac{d}{ds}\langle X, Y\rangle_{s|s=0} = 2\langle S_\eta X, Y\rangle.$$

**Proof:** Observe that $\langle X, Y\rangle_s = \langle V_{s*}X, V_{s*}Y\rangle$ and thus

$$\begin{aligned}
\frac{d}{ds}\langle X, Y\rangle_{s|s=0} &= \eta[\langle V_{s*}X, V_{s*}Y\rangle] \\
&= \langle \nabla_\eta V_{s*}X, V_{s*}Y\rangle_{|s=0} + \langle V_{s*}X, \nabla_\eta V_{s*}Y\rangle_{|s=0} \\
&= \langle \nabla_X \eta, Y\rangle + \langle X, \nabla_Y \eta\rangle \\
&= \langle S_\eta(X), Y\rangle + \langle X, S_\eta(Y)\rangle \\
&= 2\langle S_\eta(X), Y\rangle
\end{aligned}$$

where we have used that $[X, \frac{\partial}{\partial s}] = 0$ on $M \times (-\epsilon, \epsilon)$ (cf. 17.16) and $S_\eta$ symmetric.

We can interpret (17.14) by saying that the second fundamental tensor $(-2\alpha)$ is the "gradient" of the Riemannian metric restricted to $M$. Finally we want to establish a relation between extrinsic and intrinsic curvatures of $M$. For surfaces in $\mathbb{R}^3$ this is the famous *Theorema egregium* of Gauss.

**Theorem 17.15.** *Let $N^{n+k}$ be a Riemannian manifold and $M^n \subset N$ a submanifold with the induced Riemannian structure, and $n \geq 2$. For $p \in M$, $\eta_1, \ldots, \eta_k \in T_pM^\perp$ and orthonormal basis and any $u, v, w, z \in T_pM$, we have*

(i) $$R(u,v)w = (\bar{R}(u,v)w)^\top - \sum_{i=1}^{k} \{l_i(v,w)S_i(u) - l_i(u,w)S_i(v)\}$$

(ii) $$\langle R(u,v)w, z\rangle = \langle \bar{R}(u,v)w, z\rangle + \sum_{i=1}^{k} \{l_i(v,w)l_i(u,z) - l_i(u,w)l_i(v,z)\}$$

(iii) $$k(u,v) = \bar{k}(u,v) + \sum_{i=1}^{k} \det \begin{pmatrix} l_i(u,u) & l_i(u,v) \\ l_i(v,u) & l_i(v,v) \end{pmatrix},$$

*where $R$, $\bar{R}$ are the curvature tensors of $M$, $N$, $l_i = l_{\eta_i}$, $S_i = S_{\eta_i}$, $i = 1, \ldots, k$ and $k$, $\bar{k}$ the biquadratic forms of $R$, $\bar{R}$ (cf. 8.2).*

**Proof:** Since (ii) and (iii) are straight forward consequences of (i) we only prove this. First extend $u$, $v$, $w$, and $z$ to smooth vector fields near $p$, and $\eta_1, \ldots, \eta_k$ to smooth normal fields near $p$. Then from

$$\bar{\nabla}_v w = \nabla_v w + \sum_{i=1}^{k} \langle \bar{\nabla}_v w, \eta_i\rangle \eta_i$$

we get

$$\bar{\nabla}_u \bar{\nabla}_v w = \bar{\nabla}_u \nabla_v w + \sum_{i=1}^{k} (u[\langle \bar{\nabla}_v w, \eta_i\rangle]\eta_i + \langle \bar{\nabla}_v w, \eta_i\rangle \bar{\nabla}_u \eta_i)$$

and hence by definition of $S_i$, $l_i$ and 17.7

$$(\bar{\nabla}_u \bar{\nabla}_v w)^\top = \nabla_u \nabla_v w + \sum_{i=1}^{k} l_i(v,w)S_i(u).$$

Interchanging $u$ and $v$ and using $(\bar{\nabla}_{[u,v]}w)^\top = \nabla_{[u,v]}w$ we obtain (i) by definition of $R$, $\bar{R}$.

Observe that in the special case where $N = \mathbb{R}^3$ with its usual flat metric, and $M$ is a 2-dimensional submanifold of $\mathbb{R}^3$, then the sectional curvature of $M$ is the same as the Gauss–(Kronecker) curvature of $M$. In general, according to a deep theorem of J. Nash any Riemannian manifold $M^n$ can be isometrically embedded in some Euclidean space $\mathbb{R}^{n+k}$.

**Problem 17.16.** Let $X$ be a smooth vector field on a manifold $M$. Use the existence and uniqueness theorem for ordinary differential equations to show:
(1) For every $p \in M$ there is an open neighborhood $U \ni p$, and $\epsilon > 0$ and a smooth

map $\Phi\colon U \times (-\epsilon, \epsilon) \to M$ s.t.

$$\begin{gathered}\Phi_{*(q,t)}(\frac{\partial}{\partial t}) = X(\Phi(q,t)), \quad (q,t) \in U \\ \Phi(q, t+s) = \Phi(\Phi(q,t), s) \quad \text{when defined} \\ \Phi(q,0) = q, \quad q \in U\end{gathered}$$

(2) Show that $\phi_t = \Phi(\cdot\ , t)\colon U \to M$ is a diffeomorphism onto its image.
Now let $Y$ be another vector field on $M$. Show that

$$(*) \qquad [X,Y]_p = \lim_{t\to 0} \frac{1}{t}[Y_p - ((\phi_t)_* Y)_p], \quad p \in M,$$

where $(\phi_* Y)_p = \phi_{*\phi^{-1}(p)} Y_{\varphi^{-1}(p)}$, $\phi$ a diffeomorphism.
Hint: Let $f$ be a function defined near $p$ and consider $f(q,t) = f(\phi_t(q)) - f(q)$. Define $g(q,t) = \int_0^1 f'(q,ts)\, ds$ and show

$$f \circ \phi_t = f + t \cdot g_t, \quad g_0 = X[f]$$

where $g_t = g(\ ,t)$. Now set $p(t) = \phi_t^{-1}(p)$ and show ($t$ fixed)

$$((\phi_t)_* Y)_p[f] = Y_{p(t)}[f \circ \phi_t] = Y_{p(t)}[f] + tY_{p(t)}[g_t]$$

and hence

$$\begin{aligned}\lim_{t\to 0} \frac{1}{t}[Y - (\phi_t)_* Y]_p[f] &= \lim_{t\to 0}\frac{1}{t}[Y[f](p) - Y[f](p(t))] - \lim_{t\to 0} Y[g_t](p(t)) \\ &= X_p[Y[f]] - Y_p[g_0].\end{aligned}$$

The right hand side of $(*)$ is also denoted by $L_X(Y)$ and called the *Lie derivative* of $Y$ in direction $X$.
(3) Show that $L_X$ extends to a type preserving derivation on all tensor fields, and $L_X f = X[f]$ for every function $f$.

**Problem 17.17.** Show that the definition of $H_p$ in (17.9) is independent of the orthonormal basis $e_1, \ldots, e_n$.

**Problem 17.18.** Show that $M \subset N$ is totally geodesic i.e. $\alpha \equiv 0$ if and only if all geodesics in $M$ are geodesics in $N$.

The remaining problems all deal with *surfaces of revolution* in $\mathbb{R}^3$. Let $c = (c_1, c_2)\colon J \to \mathbb{R}^2$ be a regular curve with $c_1 > 0$ on $J$. The surface of revolution gotten by rotating $c$ around the second axis in 3-space is the image of the immersion $f\colon J \times R \to \mathbb{R}^3$ defined by

$$f(t,s) = (c_1(t)\cos(s), c_1(t)\sin(s), c_2(t))$$

for all $(t,s) \in J \times \mathbb{R}$.

**Problem 17.19.** Show that

$$Z(t,s) = \frac{c_2'(t)\cos s e_1 + c_2'(t)\sin s e_2 - c_1'(t) e_3}{\|\dot c(t)\|}$$

is a normal field along $f$ ($e_1$, $e_2$, $e_3$ standard coordinate fields in $\mathbb{R}^3$). Show that

$$S_Z f_*(e_1) = \frac{c_1' c_2'' - c_2' c_1''}{((c_1')^2 + (c_2')^2)^{3/2}} f_* e_1$$
$$S_Z f_*(e_2) = \frac{c_2'}{c_1((c_1')^2 + (c_2')^2)^{1/2}} f_* e_2$$

(here $e_1$, $e_2$ are the coordinate fields in $J \times \mathbb{R}$). What are the principal curvatures?

**Problem 17.20.** Let $c_1(t) = R + r\cos t$, $c_2(t) = r \sin t$, $t \in \mathbb{R}$ and $0 < r < R$. The corresponding surface of revolution is $T^2 \subset \mathbb{R}^3$. Show that the principal curvatures of this torus are given by

$$\lambda_1 = \frac{1}{r} \quad \text{and} \quad \lambda_2 = \frac{\cos t}{R + r\cos t}$$

whence

$$H = \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{R + 2r\cos t}{2r(R + r\cos t)}$$

is the mean curvature, and

$$sec = G = \lambda_1 \lambda_2 = \frac{\cos t}{r(R + r\cos t)}$$

is the Gauss and sectional curvature. Where is $sec > 0, = 0, < 0$?.

**Problem 17.21.** Consider the *Catenoid* , i.e. the surface of revolution determined by $c(t) = (\cosh t, t)$, $t \in \mathbb{R}$. Show that the principal curvatures are given by

$$\lambda_1 = -\frac{1}{\cosh^2 t}, \qquad \lambda_2 = \frac{1}{\cosh^2 t}.$$

In particular, the Catenoid is a minimal surface ($H \equiv 0$) and $sec = -\cosh^{-4} t < 0$ everywhere.

## 18. Space forms

We already know that $\mathbb{R}^n$ with its usual Riemannian structure has constant curvature $sec = 0$. We also know that a sphere $S^n(r) = \{x \in \mathbb{R}^{n+1} \mid \|x\| = r\}$ has constant positive curvature (cf. 11.7, 13.3). Here we will show that it has constant curvature $sec = \frac{1}{r^2}$.

First let us consider in general a Riemannian manifold $N^{n+1}$ and a smooth function $f : N \to \mathbb{R}$. Let $M = f^{-1}(a)$, $a \in \mathbb{R}$ and suppose $df \neq 0$ on $M$. Then $M^n$ is a codimension 1 submanifold of $N$ (cf. 16.10). We will use the results in the previous section to compute the curvature of $M$ with its induced Riemannian structure.

By definition $\langle \operatorname{grad} f, v \rangle = df(v)$ and we see that $\operatorname{grad} f$ is a nowhere vanishing normal field to $M$. Thus $\eta = \operatorname{grad} f / \| \operatorname{grad} f \|$ is a unit normal field of $M$. The shape operator of $M$ is therefore given by

$$S(v) = (\nabla_v \eta)^\top = \frac{(\nabla_v \operatorname{grad} f)^\top}{\| \operatorname{grad} f \|} \tag{18.1}$$

for tangent vectors $v$ to $M$.

By definition, the *Hessian tensor* of $f$ is the $(1,1)$ tensor given by $H_f(X) = \nabla_X \operatorname{grad} f$, and the hessian is the $(2,0)$ tensor $h_f(X,Y) = \langle H_f(X), Y \rangle$. If $p \in N$

is a critical point for $f$, i.e. $df_p = 0$, and $(x_1, \dots, x_{n+1})$ are local coordinates near $p$ then $h_f(p)$ is given by the matrix

$$\left\{ \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right\} \tag{18.2}$$

with respect to the basis $\frac{\partial}{\partial x_1}(p), \dots, \frac{\partial}{\partial x_{n+1}}(p)$, for $T_pN$.

With this terminology we get the following expression for the second fundamental form $l = l_\eta$ of $M$:

$$l(u,v) = -\frac{h_f(u,v)}{\|\operatorname{grad} f\|} \tag{18.3}$$

for all $u, v \in T_pM$, $p \in M$. Thus from (17.15) we obtain

$$\langle R(u,v)w, z\rangle = \langle \bar{R}(u,v)w, z\rangle + \frac{h_f(v,w)h_f(u,z) - h_f(u,w)h_f(v,z)}{\|\operatorname{grad} f\|^2}. \tag{18.4}$$

In the special case $N = \mathbb{R}^{n+1}$ with its standard flat metric, and $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$, $x \mapsto \|x\|^2 = \sum x_i^2$ we get for $M = f^{-1}(r^2)$, that

$$\langle R(u,v)w, z\rangle = \|\operatorname{grad} f\|^{-2}\{h_f(v,w)h_f(u,z) - h_f(u,w)h_f(v,z)\} \tag{18.5}$$

and so it only remains to compute $\operatorname{grad} f$, and $h_f$. Now clearly

$$\operatorname{grad} f = 2\rho, \quad \rho(x) = \sum_{i=1}^{n+1} x_i e_i \tag{18.6}$$

and therefore

$$H_f(u) = 2\nabla_u \sum x_i e_i = 2u, \qquad h_f(u,v) = 2\langle u, v\rangle. \tag{18.7}$$

Since $\|\operatorname{grad} f\|^2 = 4r^2$ on $M = S^2(r)$ we get by substituting into (18.5) that indeed $sec \equiv r^{-2}$.

It is possible to construct examples of manifolds with constant negative curvature in a similar way. Rather than using $\mathbb{R}^{n+1}$ with its standard flat Riemannian metric, one has to replace this by its standard flat *Lorenz metric*. Since we have not developed this concept here, we will proceed by appealing to another useful tool: conformal change of metric.

Two Riemannian metrics $g$ and $\tilde{g}$ on a smooth manifold $M$ are said to be *conformally equivalent* if there is a function $f \in C^\infty(M)$ such that $\tilde{g} = e^f \cdot g$. Similarly, a diffeomorphism $F\colon M \to \tilde{M}$ between Riemannian manifolds is said to be *conformal* if $F^*\tilde{g} = e^f \cdot g$. This is the case if and only if $F_*$ preserves angles.

Now suppose $\tilde{g} = e^f \cdot g$, and let $\tilde{\nabla}$, $\nabla$ be the Levi–Civita connections of $\tilde{g}$ and $g$ respectively.

**Theorem 18.8.** *For all smooth vector field $X$, $Y$, and $Z$ on $M$,*

(i) $$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{X[f]Y + Y[f]X - g(X,Y)\operatorname{grad} f\}$$

(ii)
$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z &\\ &+ \frac{1}{2}\{h_f(X,Z)Y - h_f(Y,Z)X + g(X,Z)H_f(Y) - g(Y,Z)H_f(X)\}\\ &+ \frac{1}{4}\{(Y[f]Z[f] - g(Y,Z)\|\operatorname{grad} f\|^2)X\\ &\qquad - (X[f]Z[f] - g(X,Z)\|\operatorname{grad} f\|^2)Y\\ &\qquad + (X[f]g(Y,Z) - Y[f]g(X,Z))\operatorname{grad} f\}\end{aligned}$$

*and in particular,*

(iii)
$$e^{f(p)}\tilde{sec}(\sigma) = sec(\sigma) - \frac{1}{2}\{h_f(v,v) + h_f(u,u) + \frac{1}{2}(\|\operatorname{grad} f\|^2 - v[f]^2 - u[f]^2)\}$$

*for any two plane $\sigma = \operatorname{span}(u,v)$, where $u, v \in T_pM$ are orthonormal with respect to $g_p$. All the right hand sides above are expressed entirely in terms of $g$.*

**Proof:** (ii) and (iii) follow from (i) by a simple, but rather lengthy computation. To prove (i) we use the characterization (12.9) of the Levi–Civita connection (cf. also 4.20, 4.22).

Since the difference of connections is a tensor (cf. 18.13) one only needs to check (i) on coordinate vector fields.

We will now use 18.8 when $g$ is the usual flat Riemannian metric on $\mathbb{R}^n$. For $k \in \mathbb{R}$, let

$$U_k = \begin{cases} \mathbb{R}^n, & k \geq 0 \\ D^n\left(\frac{1}{\sqrt{|k|}}\right) = \left\{x \in \mathbb{R}^n \;\middle|\; \|x\|^2 < \frac{1}{|k|}\right\}, & k < 0 \end{cases}$$

be equipped with the metric $g_k = \phi_k \cdot g$, where

$$\phi_k(x) = \frac{4}{(1 + k\|x\|^2)^2}, \quad x \in U_k.$$

Thus $g_k$ is conformally equivalent to the flat metric $g$ and $f_k = \log 4 - 2\log(1 + k\|x\|^2)$.

**Theorem 18.9.** *The Riemannian manifold $(U_k, g_k)$ has constant curvature $sec \equiv k$. Moreover, for $k \leq 0$ it is complete.*

**Proof:** From our computation of $\operatorname{grad}\|\cdot\|^2$ in the last section we obtain

$$\operatorname{grad} f_k = -\frac{4k}{1 + k\|\cdot\|^2}\rho$$

and hence

$$v_x[f_k] = \langle \operatorname{grad} f_k, v\rangle = \frac{-4k\langle \rho, v\rangle}{1+k\|x\|^2}$$
$$\|\operatorname{grad} f_k(x)\|^2 = \frac{16k^2\|x\|^2}{(1+k\|x\|^2)^2}$$
$$H_{f_k}(v) = \nabla_v \operatorname{grad} f_k = \frac{8k^2\langle \rho, v\rangle}{(1+k\|x\|^2)^2}\rho(x) - \frac{4k}{1+k\|x\|^2}v.$$

In particular

$$h_{f_k}(v,v) = \frac{8k^2\langle \rho, v\rangle^2}{(1+k\|x\|^2)^2} - \frac{4k}{1+k\|x\|^2}$$

for unit vectors $v \in T_x\mathbb{R}^n$. From (18.8)(iii) therefore

$$\phi_k(x)sec(\sigma) = 0 - \frac{1}{2}\left(-\frac{8k}{1+k\|x\|^2} + \frac{8k^2\|x\|^2}{(1+k\|x\|^2)^2}\right)$$
$$= \phi_k(x)\cdot k$$

i.e. $\sec(\sigma) = k$ for every two plane $\sigma$.

The completeness is obvious for $k = 0$. Now suppose $k < 0$. Since the metric is obviously radially symmetric, it follows that Euclidean rays emanating from the origin are geodesics (up to parametrization). In order to show completeness, therefor according to Hopf–Rinow, we only have to show that these curves are infinitely long. Now

$$\int_0^{\frac{1}{\sqrt{|k|}}} \frac{2}{1+kt^2}\,dt$$

diverges. The manifold $(U_{-1}, g_{-1})$ is called the *Poincaré model* for the hyperbolic $n$-space. Another model, also due to Poincaré, is the *upper half space* $H = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ with the metric $\tilde{g}(x) = \frac{1}{x_n^2}g(x)$.

**Problem 18.10.** Prove 18.2 and conclude that $h_f$ is independent of the Riemannian metric at a critical point.

**Problem 18.11.** Prove 18.6 and 18.7.

**Problem 18.12.** Prove that a diffeomorphism is conformal if and only if it preserves angles.

**Problem 18.13.** If $\nabla^1$ and $\nabla^2$ are connections in a vector bundle $(E, \pi, M)$, show that

$$T(X,s) = \nabla^1_X s - \nabla^2_X s$$

is tensorial in $X$ as well as in $s$.

**Problem 18.14.** Complete the proof of 18.8

## 19. Riemannian submersions

Numerous manifolds arize naturally as solution sets to equations, i.e. as submanifolds of other manifolds. We have seen how to compute curvature in such cases. Another important class of manifolds are more naturally described as quotient spaces. This is the case for example for orbit spaces of group actions.

The dual concept to isometric immersions, is that of Riemannian submersions. A submersian $\pi\colon E \to M$ between Riemannian manifolds $(E, \bar{g})$ and $(M, g)$ is called a *Riemannian submersion* if

$$g(\pi_* u, \pi_* v) = \bar{g}(u, v) \tag{19.1}$$

for all $u, v \in (\operatorname{Ker} \pi_{*p})^{\perp}$, $p \in E$. Since $\pi(E)$ is open we assume w.l.o.g. that $\pi\colon E \to M$ is surjective.

Rather than using $\bar{g}$, $g$ we will use the notation $\langle\ ,\ \rangle$ for both. The tangent space $T_pE$ splits orthogonally as

$$T_pE = E_p^{\top} \oplus E_p^{\perp}$$

where $E_p^{\top} = \operatorname{Ker} \pi_{*p}$ is the tangent space to the fiber $\pi^{-1}(\pi(p))$. $E_p^{\top}$ is called the *vertical space* at $p$ and $E_p^{\perp}$ the *horizontal space* at $p$. Every vector field $Z$ on $E$ splits accordingly

$$Z = Z^{\top} + Z^{\perp}, \quad Z_p^{\top} \in E_p^{\top},\ Z_p^{\perp} \in E_p^{\perp}$$

and $Z$ is smooth if and only if $Z^{\top}$ and $Z^{\perp}$ are smooth. For every smooth vector field $X$ on $M$ there is a unique smooth horizontal vector field $\bar{X}$ on $E$, i.e. $\bar{X}^{\top} = 0$, such that $\pi_*(\bar{X}_p) = X(\pi(p))$ for all $p \in E$. $\bar{X}$ is called the *horizontal lift* of $X$ (cf. also 19.7).

**Theorem 19.2.** *The Levi–Civita connections $\bar{\nabla}$, $\nabla$ on $E$, and $M$ respectively are related by*

$$(\bar{\nabla}_{\bar{X}}\bar{Y})^{\perp} = (\overline{\nabla_X Y})$$

*for all smooth vector fields $X$, $Y$ on $M$.*

**Proof:** Again using (12.9) and the properties of $\pi$ (cf. 19.7) we get immediately:

$$\begin{aligned}
2\langle \bar{\nabla}_{\bar{X}}\bar{Y}, \bar{Z}\rangle &= \bar{X}\langle \bar{Y}, \bar{Z}\rangle + \bar{Y}\langle \bar{Z}, \bar{X}\rangle - \bar{Z}\langle \bar{X}, \bar{Y}\rangle \\
&\qquad + \langle \bar{Z}, [\bar{X}, \bar{Y}]\rangle - \langle \bar{X}, [\bar{Y}, \bar{Z}]\rangle + \langle \bar{Y}, [\bar{Z}, \bar{X}]\rangle \\
&= X\langle Y, Z\rangle \circ \pi + Y\langle Z, X\rangle \circ \pi - Z\langle X, Y\rangle \circ \pi \\
&\qquad + \langle Z, [X, Y]\rangle \circ \pi - \langle X, [Y, Z]\rangle \circ \pi + \langle Y, [Z, X]\rangle \circ \pi \\
&= 2\langle \nabla_X Y, Z\rangle \circ \pi \\
&= 2\langle \overline{\nabla_X Y}, \bar{Z}\rangle
\end{aligned}$$

for all vector fields $X$, $Y$ and $Z$ on $M$.

Now note that if $T$ is a vertical field on $E$, i.e. $\pi_* T = 0$, then $T$ is $\pi$-related to the zero field on $M$ (cf. 19.7), so for any lift $\bar{X}$ of a vector field $X$ on $M$ we have $\pi_*[T, \bar{X}] = [0, X] = 0$ (cf. 19.7) and therefore

$$[T, \bar{X}]^{\perp} = 0 \tag{19.3}$$

for any vertical $T$.

**Theorem 19.4.** *For any smooth vector fields $X$, $Y$ on $M$,*

$$(\bar{\nabla}_{\bar{X}}\bar{Y})^{\top} = \frac{1}{2}[\bar{X}, \bar{Y}]^{\top} = A(X, Y)$$

*and $A$ defined this way is a tensor.*

**Proof:** By (12.9), (19.3) and $T\langle \bar{X}, \bar{Y}\rangle = 0$ we get $2\langle \bar{\nabla}_{\bar{X}}\bar{Y}, T\rangle = \langle [\bar{X}, \bar{Y}], T\rangle$, for all $X$, $Y$ on $M$ and vertical $T$ on $E$.

In particular $\bar\nabla_{\bar X}\bar X$ is horizontal. This shows, that geodesics in $M$ lift to horizontal geodesics in $E$. Conversely a geodesic in $E$ which is horizontal at one point is horizontal at all points, and in fact a horizontal lift of a geodesic in $M$. In short

$$\pi\circ\exp^E_{|E_p^\perp}=\exp^M\circ\pi_{*|E_p^\perp}. \tag{19.5}$$

We are now ready to compare curvature tensors.

**Theorem 19.6.** *For $X, Y, Z, U$ smooth vector fields on $M$ we have*

(i) $$\langle R(X,Y)Z,U\rangle=\langle\bar R(\bar X,\bar Y)\bar Z,\bar U\rangle+\frac14\langle[\bar Y,\bar Z]^\top,[\bar X,\bar U]^\top\rangle -\frac14\langle[\bar X,\bar Z]^\top,[\bar Y,\bar U]^\top\rangle-\frac12\langle[\bar X,\bar Y]^\top,[\bar Z,\bar U]^\top\rangle,$$

(ii) $$sec(\mathrm{span}(X,Y))=s\bar ec(\mathrm{span}(\bar X,\bar Y))+\frac34\|[\bar X,\bar Y]^\top\|^2,$$

*when $X$, $Y$ locally form an orthonormal 2-frame field on $M$.*

**Proof:** (ii) follows directly from (i) which is proved as follows

$$\begin{aligned}
\langle R(X,Y)Z,U\rangle\circ\pi &= (\langle\nabla_X\nabla_YZ,U\rangle-\langle\nabla_Y\nabla_XZ,U\rangle-\langle\nabla_{[X,Y]}Z,U\rangle)\circ\pi\\
&=\langle\bar\nabla_{\bar X}\overline{\nabla_YZ},\bar U\rangle-\langle\bar\nabla_{\bar Y}\overline{\nabla_XZ},\bar U\rangle-\langle\nabla_{\overline{[X,Y]}}\bar Z,\bar U\rangle\\
&=\langle\bar\nabla_{\bar X}(\bar\nabla_{\bar Y}\bar Z-(\bar\nabla_{\bar Y}\bar Z)^\top),\bar U\rangle-\langle\bar\nabla_{\bar Y}(\bar\nabla_{\bar X}\bar Z-(\bar\nabla_{\bar X}\bar Z)^\top),\bar U\rangle\\
&\quad-\langle\bar\nabla_{[\bar X,\bar Y]-[\bar X,\bar Y]^\top}\bar Z,\bar U\rangle\\
&=\langle\bar R(\bar X,\bar Y)\bar Z,\bar U\rangle-\langle\bar\nabla_{\bar X}(\bar\nabla_{\bar Y}\bar Z)^\top,\bar U\rangle\\
&\quad+\langle\bar\nabla_{\bar Y}(\bar\nabla_{\bar X}\bar Z)^\top,\bar U\rangle+\langle\bar\nabla_{[\bar X,\bar Y]^\top}\bar Z,\bar U\rangle\\
&=\langle\bar R(\bar X,\bar Y)\bar Z,\bar U\rangle+\langle(\bar\nabla_{\bar Y}\bar Z)^\top,\bar\nabla_{\bar X}\bar U\rangle-\langle(\bar\nabla_{\bar X}\bar Z)^\top,\bar\nabla_{\bar Y}\bar U\rangle\\
&\quad+\langle\bar\nabla_{\bar Z}[\bar X,\bar Y]^\top,\bar U\rangle\\
&=\langle\bar R(\bar X,\bar Y)\bar Z,\bar U\rangle+\frac14\langle[\bar Y,\bar Z]^\top,[\bar X,\bar U]^\top\rangle-\frac14\langle[\bar X,\bar Z]^\top,[\bar Y,\bar U]^\top\rangle\\
&\quad-\langle[\bar X,\bar Y]^\top,\bar\nabla_{\bar Z}\bar U\rangle
\end{aligned}$$

where we have used (19.2), (19.3), (19.4) and of course that $\bar\nabla$ is the Levi–Civita connection.

**Problem 19.7.** Let $f\colon M\to N$ be a smooth map. Vector fields $\bar X$ on $M$ and $X$ on $N$ are said to be $f$-related provided $f_{*p}(\bar X_p)=X_{f(p)}$ for all $p\in M$. Show that if $\bar X$ and $X$ are $f$-related, $\bar Y$ and $Y$ are $f$-related, then $[\bar X,\bar Y]$ and $[X,Y]$ are $f$-related.

**Problem 19.8.** Let $\pi\colon E\to M$ be a submersion of a smooth manifold $E$ onto a Riemannian manifold $(M,g)$. Show that (provided $E$ admits any Riemannian metric then) $E$ admits a Riemannian metric $\bar g$ such that $\pi$ is a Riemannian submersion.

**Problem 19.9.** Let $\pi\colon E\to M$ be a Riemannian submersion. Show that if $E$ is complete, so is $M$. In this case, show that all fibers are diffeomorphic, and in fact $\pi\colon E\to M$ is a locally trivial fiber bundle.

**Problem 19.10.** Let $\pi\colon E\to M$ be a vector bundle with connection $\nabla$. For $U\in E$ let $E_u^\perp\subset T_uE$ be the subset of all tangent vectors at $u\in E$ represented by parallel fields through $u$. Show that $E_u^\perp$ is a subspace and $\pi_{*|}\colon E_u^\perp\to T_{\pi(u)}M$ is a linear isomorphism. Define a Riemannian metric on $E$ so that $\pi\colon E \to M$ is a Riemannian submersion.

## 20. Lie groups and homogeneous spaces

Suppose $G$ is a Lie group (cf. 16.6 and 16.6) with *bi-invariant Riemannian metric* $\langle\ ,\ \rangle$, i.e. left as well as right translations are isometries (e.g. any compact Lie group admits such a metric, cf. 20.8 and 20.9).

**Theorem 20.1.** *Inversion $\iota\colon G \to G$, $g \mapsto g^{-1}$ is an isometry. Moreover $G$ is a symmetric space.*

**Proof:** Representing tangent vectors by curves we see that

$$\mu_{*(e,e)}(u,v) = u + v, \quad u, v \in T_eG \tag{20.2}$$

where $\mu\colon G \times G \to G$ is the multiplication. In particular

$$\iota_{*e}(u) = -u, \quad u \in T_eG \tag{20.3}$$

and so $\iota_{*e}$ is an isometry. In general $\iota = L_{g^{-1}} \circ \iota \circ R_{g^{-1}}$ for any $g \in G$ so by (20.3) $\iota_{*g}\colon T_gG \to T_{g^{-1}}G$ is an isometry.

Now let $I_g = L_g \circ \iota \circ L_{g^{-1}}$, $g \in G$. Then each $I_g$ is an isometry, $I_e = \iota$ and

$$I_g(g) = g, \qquad (I_g)_{*g} = -\mathrm{id}_{T_gG}. \tag{20.4}$$

By definition therefore, $G$ is a *symmetric space* , since at each point there is an isometry which reverses geodesics through the point.

Now let $c\colon J \to G$ be a geodesic with $c(0) = g$, $c(t_1) = g_1$. The isometry $I_{g_1} \circ I_g$ maps $c(t)$ to $c(2t_1 + t)$ whenever $t, t_1, 2t_1, 2t_1 + t \in J$. This shows that $c$ can be extended to $\mathbb{R}$, i.e. $G$ is complete.

A *one-parameter subgroup* of Lie group $G$, is a smooth homomorphism $\varphi : (\mathbb{R}, +) \to G$.

**Theorem 20.5.** *The geodesics through $e \in G$ coincide with the one-parameter subgroups of $G$. The Levi–Civita connection $\nabla$ is given by*

$$\text{(i)} \qquad \nabla_X Y = \frac{1}{2}[X,Y], \quad X, Y \in \mathfrak{g}$$

$$\text{(ii)} \qquad R(X,Y)Z = -\frac{1}{4}\big[[X,Y],Z\big], \quad X,Y,Z \in \mathfrak{g}$$

$$\text{(iii)} \qquad \langle R(X,Y)Z, U\rangle = -\frac{1}{4}\langle [X,Y],[Z,U]\rangle, \quad X,Y,Z,U \in \mathfrak{g}.$$

*In particular $G$ has nonnegative sectional curvature.*

**Proof:** Let $c\colon \mathbb{R} \to G$ be a geodesic with $c(0) = e$. Using the reflexion $I_{c(t_1)}$ we obtain $c(t_1 - t) = I_{c(t_1)}c(t_1 + t) = c(t_1)c(t_1 + t)^{-1}c(t_1)$ and by induction $c(mt_1) = c(t_1)^m$ for any $t_1 \in \mathbb{R}$ and $m \in \mathbb{Z}$. But then

$$c\Big(\frac{p}{q} + \frac{r}{s}\Big) = c\Big((ps + qr)\frac{1}{qs}\Big) = c\Big(\frac{1}{qs}\Big)^{ps+qr} = c\Big(\frac{p}{q}\Big)c\Big(\frac{r}{s}\Big)$$

for all $p, q, r, s \in \mathbb{Z}$ and hence $c(t + s) = c(t)c(s)$ for all $t, s \in \mathbb{R}$. It also follows, that the left translates of the one-parameter subgroup, $c$ are exactly all the integral curves of the left invariant vector field $X \in \mathfrak{g}$ determined by $X(e) = \dot{c}(0)$. Since these are all geodesics we get $\nabla_X X = 0$ for any $X \in \mathfrak{g}$. We get (i) by polarization and symmetry of $\nabla$. (ii) and (iii) are now straightforward.

Before we discuss homogenous spaces, let us consider in general a smooth isometric action $M \times H \to M$ of a Lie group $H$ on a Riemannian manifold $M$, i.e.

$$M \times H \to M, \quad p(h_1 h_2) = (ph_1)h_2$$

for all $p \in M$, $h_1, h_2 \in H$. Moreover $pe = p$ for all $p \in M$, and $h\colon M \to M$, $p \mapsto p \cdot h$ is an isometry for all $h \in H$. The action is *free* if all *isotropy subgroups* $H_p = \{h \in H \mid h(p) = p\}$ are trivial.

**Theorem 20.6.** *If $M \times H \to M$ is a free, proper isometric action, then the orbit space $N = M/H$ admits a structure of a smooth Riemannian manifold such that $\pi\colon M \to M/H$ is a Riemannian submersion.*

**Proof:** Fix $p \in M$, consider the map $p\colon H \to M$, $h \mapsto ph$. This is clearly 1-1 and smooth. Moreover for $X_h \in T_h H$ let $X \in \mathfrak{h}$ be the corresponding left invariant vector field on $H$. If $c$ is the integral curve with $c(0) = e$, then $h \cdot c$ represents $X_h$ and hence $phc$ represents $p_*(X_h)$. Thus if $p_*(X_h) = 0$, then $(ph)c(t)$ represents the zero vector for all $t \in \mathbb{R}$. This, however, would only be possible if $phc$ is constant, which is impossible since the action is free. This shows that $p\colon H \to M$ is a 1-1 immersion, and hence an embedded submanifold by the assumption on properness. Now consider the restriction of $\exp\colon \widetilde{TM} \to M$ to the normal bundle $T(p \cdot H)^\perp$ of $p \cdot H \subset M$. Clearly (for any submanifold) exp has maximal rank along the zero section of this normal bundle. Consequently it is a diffeomorphism when restricted to a suitable neighborhood. In our case, since $H$ acts by isometries we can choose an $\epsilon > 0$ such that $\exp\colon \{\eta \in T(pH)^\perp \mid \|\eta\| < \epsilon\} \to M$ is an equivariant (i.e. commutes with action of $H$) diffeomorphism onto $\{q \in M \mid \mathrm{dist}(q, p \cdot H) < \epsilon\}$. Since the action is free, $T(pH)^\perp \to pH$ is a trivial bundle, and hence each orbit in this neighborhood is represented by a unique point in a fixed normal $\epsilon$-disc at $p$. It follows that $M/H$ is a smooth manifold, and in fact $\pi\colon M \to M/H$ a locally trivial fiber bundle with fiber $H$. Moreover the equivariant metric in the normal bundle to each orbit exhibits a Riemannian metric on $M/H$ so that $\pi\colon M \to M/H$ is a Riemannian submersion.

Now let us consider the special case where $G$ is a Lie group with biinvariant metric, and $H$ a closed Lie subgroup. $N = G/H$ will denote the space of left cosets of $H$ in $G$ with the quotient topology of $\pi\colon G \to G/H$. Letting $H$ act on the right of $G$ we obviously get an isometric, proper action whose orbits coincide with the left coset of $H$. According to 20.6, therefore $G/H$ admits a structure of a smooth Riemannian manifold such that $\pi\colon G \to G/H$ is a Riemannian submersion. The homogeneous space $G/H$ with this metric is called a *normal homogeneous space* . From (20.5) and (19.6) we compute the sectional curvature of $G/H$ by

$$sec(\sigma) = \frac{1}{4}\|[\bar{X}, \bar{Y}]^\perp\|^2 + \|[\bar{X}, \bar{Y}]^\top\|^2, \tag{20.7}$$

where $\sigma = \mathrm{span}(X, Y)$, $X$, $Y$ orthonormal in $G/H$ and $\bar{X}$, $\bar{Y}$ horizontal lifts to $G$. Note that we only have to compute the curvatures at one point.

**Problem 20.8.** If $G$ is a compact Lie group there is a biinvariant measure on it with finite volume. Show by averaging that $G$ has a biinvariant metric.

**Problem 20.9.** Let $G$ be a Lie group and $X \in \mathfrak{g} \simeq T_e G$. The $\mathrm{ad}\, X\colon G \to \mathfrak{g}$ is the map $Y \to [X, Y]$ and the *Killing–Cartan form* on $\mathfrak{g}$ is defined by

$$\phi(X, Y) = \mathrm{trace}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y).$$

Show that $\phi$ is bilinear, symmetric and biinvariant. When $G$ is semisimple and compact, then it is negative definite. ($G$ semisimple $\Leftrightarrow$ $\phi$ nondegenerate. $G$ compact $\Leftrightarrow$ $\phi$ negative semidefinite.)

**Problem 20.10.** Show that the Grassman manifold $G_{n,k}$ of all $k$-dimensional subspaces in $\mathbb{R}^{n+k}$ can be viewed as a normal homogeneous space.

**Problem 20.11.** Same problem for complex Grassman manifold of complex $k$-dimensional subspaces in $\mathbb{C}^{n+k}$. For $k = 1$, this is the complex projective space $\mathbb{C}P^n$.

**Problem 20.12.** View $\mathbb{C}P^n$ as an orbit space $S^{2n+1}/S^1$ and conclude that $\mathbb{C}P^n$ as positive curvature.

**Problem 20.13.** Prove (20.2), (20.3) and (20.4).

**Problem 20.14.** Complete the proof of 20.5.

**Problem 20.15.** Fill in the missing details in the proof of 20.6.

**Problem 20.16.** Prove 20.7.

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# Index