K-THEORY

JOHAN DUPONT

LECTURE NOTES SERIES NO. 11

1968

REVISED NOVEMBER 1976
REVISED NOVEMBER 2009

DEPARTMENT OF MATHEMATICAL SCIENCES
FACULTY OF SCIENCE, AARHUS UNIVERSITY

Contents

Introduction

These notes are based on a series of lectures given at Matematisk Institut, Aarhus Universitet, in the fall af 1968.

They are intended as an introduction to the subject presupposing only some general topology, linear algebra and basic analysis.

The main line is that of M. F. Atiyah [7], only the definition of a vector bundle is changed half the way, suggested by D. W Anderson [4] and M. Karoubi [21]. Our *projection bundle* is a *vector bundle* in the sense of Anderson.

The proof of the periodicity is that of Atiyah and Bott [11] with the modification suggested by Atiyah [6]. Only the basic complex K-theory is developed. Neither real, symplectic nor equivariant K-theory is touched upon. Also the important representation of relative K-theory by complexes of vector bundles is skipped.

Therefore, literature for further study is included among the references.

Århus, December 1968 Johan Dupont

Special thanks to Toki Schnoor Kruse for retyping the original manuscript in LaTeX, and to Lars Madsen for copy-editing it.

Århus, November 2009 Johan Dupont

Chapter 1

Vector Bundles

1.1 Basic definitions

Unless otherwise specified, all vector spaces considered are finite dimensional *complex* vector spaces. We leave it to the reader to carry through all constructions with *real* or *quaternionic* vector spaces when possible.

Definition 1.1. Let X be a topological space. A *family of vector spaces over* X is a topological space E, together with:

(1) a continuous map $p\colon E \to X$

(2) a finite dimensional vector space structure on each

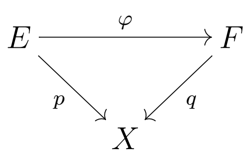
$$E_x = p^{-1}(x) \quad \text{for} \quad x \in X$$

compatible with the topology on E_x induced from E.

The map p is called the *projection map*, the space E is called the *total space* and X is called the *base space* of the family, and if $x \in X$, E_x is called the *fibre over* x.

Definition 1.2. Let $p\colon E \to X$ and $q\colon F \to X$. A homomorphism φ from (E, p) to (F, q) is a continuous map $\varphi\colon E \to F$ such that:

(1) $q \circ \varphi = p$



(2) for each $x \in X$ $\varphi_x\colon E_x \to F_x$ is a linear map of vector spaces.

So we can talk about the category of vector space families over X. An isomorphism is a homomorphism with an an inverse homomorphism (which is continuous by definition).

Example 1.3. Let V be a vector space, and let $E = X \times V$, $p\colon E \to X$ be the projection on to the first factor. E is called the *product family* with fibre V. A family which is isomorphic to a product family is called a *trivial family*, and the isomorphism is called a *trivialization*.

If Y is a subspace of X, and if (E, p) is a family of vector spaces over X, $p\colon p^{-1}(Y) \to Y$ is clearly a family over Y. This is called the *restriction* of E to Y and denoted by $E_{|Y}$.

Now if X and Y are two spaces and $f\colon Y \to X$ is a continuous map we define the *induced* family $(f^*(E), f^*(p))$ by:

$$f^*(E) \subseteq Y \times E$$

is the subspace of points (y, e) such that $f(y) = p(e)$. $f^*(p)$ is the restriction to $f^*(E)$ of the projection $Y \times E \to Y$.

We also call $f^*(E)$ the *pull-back* of E under f.

Exercise 1.4.

(1) f^* is a covariant functor from the category of families over X to the category of families over Y.

(2) If $f\colon Y \to X$, $g\colon Z \to Y$ then the functors $g^* \circ f^*$ and $(fg)^*$ are naturally equivalent.

(3) $Y \subseteq X$, $i\colon Y \to X$, the injection, then i^* is naturally equivalent to the restriction functor: $E \to E_{|Y}$.

Remark 1.5. For the concepts *Category*, *Functor* and *natural equivalence* we refer to S. MacLane [22], ch. I § 7–8.

Until now our families of vector spaces are not supposed to have much structure which relates the fibres over neighbouring points. If V is a vector space $\mathrm{Hom}(V, V)$ has a natural topology defined by the norm of the operators. $\mathcal{P}(V) \subseteq \mathrm{Hom}(V, V)$ is the subspace of projection operators in V, i.e. $\pi \in \mathcal{P}(V)$ iff $\pi^2 = \pi$.

Definition 1.6. Let X be a topological space. A *projection bundle* over X is a continuous map $\pi\colon X \to \mathcal{P}(V)$, where V is some finite dimensional vector space.

A projection bundle defines a family of vector spaces over X in the following way: $E \subseteq X \times V$ is the subspace of points (x, v) such that $\pi_x v = v$. $p\colon E \to X$ is the restriction of the projection onto X. This family is called the *embedded vector bundle* belonging to (V, π).

Example 1.7.

(1) In $\mathbb{R}^n$ consider

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

$$\tau\colon S^{n-1} \to \mathcal{P}(\mathbb{R}^n), \qquad \tau(x)v = v - (x, v)x.$$

This defines a real embedded vector bundle called the tangent bundle of S^{n-1}.

(2) Let V be a finite dimensional complex vector space with a hermitian structure.

$$P(V) = \{V \setminus \{0\}\} \,/\, \{v \sim \lambda v \mid v \in V,\ \lambda \in \mathbb{C} \setminus \{0\}\}$$

is called the "*complex projective space* of V". $v \in V$ $v \neq 0$ determines a unique point $\{v\} \in P(V)$.

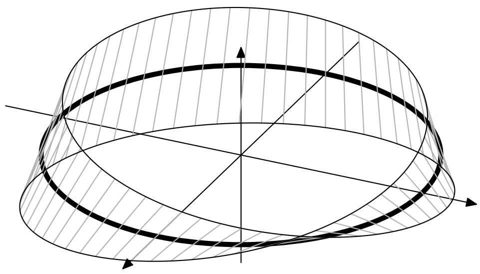
$$\pi\colon P(V) \to \mathcal{P}(V), \qquad \pi(v)w = (w,v)\frac{v}{(v,v)}$$

The corresponding embedded vector bundle is called the *dual Hopf bundle* and is denoted H^*.

(3) The analogous construction in the real case gives for $V = \mathbb{R}^2$ a bundle over $S^1 \cong P(\mathbb{R}^2)$ (the homeomorphism $S^1 \to P(\mathbb{R}^2)$ is defined by

$$(\cos\theta, \sin\theta) \to \left\{\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right\}, \quad -\pi \leq \theta \leq \pi).$$

This bundle is isomorphic to the *Möbius bundle*: The fibre over $[\cos\theta, \sin\theta]$ is the line in $\mathbb{R}^3$ spanned by $(\cos\frac{\theta}{2}\cos\theta, \cos\frac{\theta}{2}\sin\theta, \sin\frac{\theta}{2})$.



In view of Example (3) it is natural to consider the vector bundles independently from the defining projection bundle.

Definition 1.8. A vector *vector bundle* (E,p) over X is a family of vector spaces over X isomorphic to an embedded vector bundle.

Proposition 1.9. *Let $f\colon Y \to X$ be a continuous map and (E,p) a vector bundle over X, then*

$$(f^*(E), f^*(p))$$

is a vector bundle over Y.

Proof. Let $\pi\colon X \to \mathcal{P}(V)$ define an embedded bundle $E' \subseteq X \times V$ and let

$$\varphi\colon E \to E', \qquad \varphi^{-1}\colon E' \to E.$$

Consider the projection $\pi_2\colon X \times V \to V$.

$\pi \circ f \colon Y \to \mathcal{P}(V)$ defines an embedded bundle $E'' \subseteq Y \times V$. Now define $\psi \colon f^*(E) \to E''$ by

$$\psi(y, e) = (y, \pi_2 \circ \varphi(e))$$

and $\psi' \colon E'' \to f^*(E)$ by

$$\psi'(y, v) = (y, \varphi^{-1}(f(y), v)).$$

ψ and ψ' are clearly homomorphisms and each others inverse. □

For convenience we will denote a vector bundle (E, p) by the total space E.

A vector bundle has the important property af being *locally trivial*; we state this as

Proposition 1.10. *Let E be a vector bundle over X, $x \in X$. There exists a neighbourhood U of x such that $E_{|U}$ is trivial.*

Obviously we only have to show this for an embedded vector bundle, and in that case the statement follows from the following lemma.

Lemma 1.11. *Let V be a finite dimensional vector space. π_1, π_2 projections in V with $V_i = \pi_i V$ where $i = 1, 2$. If $\|\pi_1 - \pi_2\| < 1$ then $\pi_1 \colon V_2 \to V_1$ is an isomorphism.*

Proof. Put $A = \pi_1 - \pi_2$ then $I + A$ has an inverse B,

$$B = \sum_{i=0}^{\infty} (-1)^i A^i$$

$(I - A)(\pi_2 v) = (I + \pi_1 - \pi_2)(\pi_2 v) = \pi_1 \pi_2 v$, so $I + A_{|V_2} = \pi_{1|V_2}$ and therefore

$$\pi_1 \colon V_2 \to V_1$$

is injective and consequently $\dim V_2 \leq \dim V_1$. Interchanging π_1 and π_2 we get equality, and therefore π_1 is also onto. □

Let us now return to the proof of Proposition 1.10.

Proof of Proposition 1.10 Consider a projection bundle $\pi \colon X \to \mathcal{P}(V)$ and a point $x_0 \in X$. Let U be a neighourhood of x_0 contained in

$$\pi^{-1}\{\pi' \mid \|\pi' - \pi(x_0)\| < 1\}.$$

Let $\overline{I + A}$, $\overline{B} \colon U \times V \to U \times V$ be defined by

$$\overline{I + A}(x, v) = (x, (I + \pi_{x_0} - \pi_x)v)$$
$$\overline{B}(x, v) = (x, (I + \pi_{x_0} - \pi_x)^{-1}v)$$

$\overline{I + A}$ is clearly continuous and $\overline{B}$ is because the map $\mathrm{GL}(V) \to \mathrm{GL}(V)$, which takes an operator to its inverse, is continuous. $\overline{I + A}$ and $\overline{B}$ induce isomorphisms between the embedded bundle defined by $\pi_{|U}$ and the product bundle $U \times \pi_{x_0} V$. □

Actually the property of being locally trivial characterizes the vector bundles over a compact Hausdorff space. We do not need this fact, but we show the importance of local triviality by the following study of homomorphisms.

Proposition 1.12. *Let $\varphi\colon E \to F$ be a homomorphism of vector bundles over X such that $\varphi_x\colon E_x \to F_x$ is an isomorphism of vector spaces. Then φ^{-1} is continuous, i.e. φ is an isomorphism.*

Proof. Clearly φ is an bijection, so we need only to consider φ locally. So assume without loss of generality that $E = X \times V$ and $F = X \times W$, where V and W are vector spaces of the same dimension.

φ determines a map $\Phi\colon X \to \mathrm{Hom}(V, W)$ by

$$\varphi(x, v) = (x, \Phi(x)(v)).$$

$\mathrm{Hom}(V, W)$ has the topology determined by the operator norm. This is the same as the topology induced from the hermitian norm defined in the following way:

Choose bases $\{e_i\}$ and $\{f_j\}$ for V and W respectively; then $\Phi \in \mathrm{Hom}(V, W)$ has coordinates Φ_{ij} determined by

$$\Phi e_i = \sum_j \Phi_{ij} f_j.$$

So if φ is continuous, then $x \mapsto \Phi(x)_{ij}$ is clearly continuous and therefore $x \mapsto \Phi(x)$ is continuous.

Now $\mathrm{Iso}(V, W)$ is open in $\mathrm{Hom}(V, W)$ because

$$\mathrm{Iso}(V, W) = \det^{-1}(\mathbb{C} \setminus \{0\})$$

and $\Phi \to \Phi^{-1}$ defines a continuous map in $\mathrm{Iso}(V, W)$. Therefore $x \mapsto \Phi(x)^{-1}$ is continuous and

$$\varphi^{-1}(x, w) = (x, \Phi(x)^{-1} w)$$

defines a continuous map. □

Proposition 1.13. *Let $\varphi\colon E \to F$ be a homomorphism of vector bundles over X. The set U of points $x \in X$ such that $\varphi_x\colon E_x \to F_x$ is an isomorphism, is open in X*

Proof. Clearly we only need to consider a neighbourhood of a point $x_0 \in U$. So again without loss of generality we have

$$E = X \times V, \qquad F = X \times W$$

$$\varphi(x, v) = (x, \Phi(x)(v)), \qquad (x, v) \in X \times V$$

where $\Phi\colon X \to \mathrm{Hom}(V, W)$ is continuous. This gives us that $U = \Phi^{-1}(\mathrm{Iso}(V, W))$ is open because $\mathrm{Iso}(V, W)$ is open in $\mathrm{Hom}(V, W)$. □

In the examples of projection bundles mentioned in this section all projections were orthogonal. In general every embedded vector bundle is defined by orthogonal projections.

In fact if $\pi\colon X \to \mathcal{P}(V)$, and V has a hermitian structure, let $\pi_x^\perp$ be the orthogonal projection onto $\pi_x V$; $x \mapsto \pi_x^\perp$ is a projection bundle defining the same embedded vector bundle. This is a consequence of

Lemma 1.14. *For $\pi \in \mathcal{P}(V)$ let $\pi^\perp$ be the orthogonal projection onto πV. The map $\perp$ sending π into $\pi^\perp$ is continuous.*

Proof. (Gram-Schmidt) Induction on $\dim \pi V$:

(1) $\dim \pi V = 0$ is trivial.

(2) $\dim \pi_0 V > 0$.
Consider $e_0 \in \pi_0 V$ with $\|e_0\| = 1$. The set

$$U = \{\pi | \|\pi - \pi_0\| < 1\}$$

is open in $\mathcal{P}(V)$. For $\pi \in U$ is $\dim \pi V = \dim \pi_0 V$. Hence we only need to consider $\perp$ restricted to U. $\pi e_0 \neq 0$. Define Q_π by

$$Q_\pi v = \pi v - (\pi v, \pi e_0)\frac{\pi e_0}{\|\pi e_0\|^2}$$

$Q_\pi \in \mathcal{P}(V)$, $Q_\pi V \subseteq \pi V$, $\pi e_0 \perp Q_\pi V$ and clearly

$$\pi V = Q_\pi V + \{\pi e_0\}.$$

Hence denoting the orthogonal projection onto $\{\pi e_0\}$ by $P_{\pi_{e_0}}$ we get

$$\pi^\perp = Q_\pi^\perp + P_{\pi_{e_0}}.$$

By induction, the map $Q_\pi \mapsto Q_\pi^\perp$ is continuous, and the maps $\pi \mapsto Q_\pi$ and $\pi \mapsto P_{\pi_{e_0}}$ are clearly continuous. □

This lemma also has another consequence:

Definition 1.15. Let V be a finite dimensional vector space and let

$$G_n(V) = \{W \subseteq V \mid W \text{ subspace of dimension } n\}.$$

If V is given a hermitian structure, we can identify $W \subseteq V$ with the orthogonal projection onto W. The norm topology in $\mathcal{P}(V)$ then defines the topology in $G_n(V)$. Lemma 1.14 shows that the topology does not depend on the choice of hermitian structure.

$G_n(V)$ is called the *"Grassmann manifold"* of n-planes in V.

$$E_n(V) = \{(W, v) \in G_n(V) \times V \mid v \in W\}$$

constitute clearly an embedded vector bundle, called the *universal* bundle over $G_n(V)$. For $n = 1$ $G_1(V) = P(V)$ and $E_1(V) = H^*$.

Proposition 1.16. *Every bundle over a connected space is isomorphic to a bundle induced from a universal bundle. Specifically if* $\pi\colon X \to \mathcal{P}(V)$ *is a projection bundle* $n = \dim \pi_x V$ *and* $f(x) = \pi_x V \in G_n(V)$*; then (if we identify* $x \in X$ *with* $(x, f(x)) \in X \times G_n(V)$*) the corresponding vector bundle is*

$$f^* E_n(V).$$

Note that n is well defined because $\dim E_x$ is locally constant in x by Proposition 1.10 and therefore constant on the whole of X. In general if $\dim E_x = n$ is constant, we say that E is an n-dimensional bundle.

We conclude this section with a remark concerning hermitian structures on a vector bundle. First define

$$E \oplus E = \bigcup_x E_x \times E_x$$

with the topology induced from $E \times E$. A hermitian structure is a continuous map

$$h\colon E \oplus E \to \mathbb{C}$$

such that h restricted to $E_x \times E_x$ is a hermitian structure.

If E is an embedded vector bundle in $V \times V$ and we give V a hermitian structure, this also defines a hermitian structure on E. As a consequence every vector bundle can be given a hermitian structure.

Exercise 1.17. If E is a vector bundle with a hermitian structure, then E is locally isomorphic to a product bundle in such a way that, if $\varphi\colon E_{|U} \to U \times V$ is a trivialization, then φ_x is unitary for all $x \in X$.

1.2 Operations on vector bundles

We start with some linear algebra. In the category of finite dimensional vector spaces and linear maps there are some important functors:

$V \oplus W$	*direct sum*	(1.1)
$V \otimes W$	*tensor product*	(1.2)
$\mathrm{Hom}(V, W)$		(1.3)
V^*	*dual space*	(1.4)
$\lambda^i(V)$	i^{th} *exterior power.*	(1.5)

(1.1) : $V \oplus W$ is the vector space consisting of pairs (v, w) where $v \in V$ and $w \in W$.

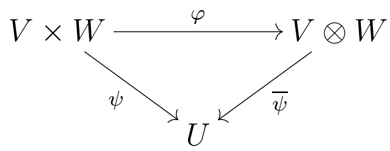
(1.2) :
$$V \otimes W = \coprod_{(v,w) \in V \times W} \mathbb{C}(v, w) \Big/ R$$

$R =$ the subspace spanned by all elements of the form

$$\lambda(v,w) - (\lambda v, w), \quad \lambda(v,w) - (v, \lambda w)$$
$$((v+v^{'}),w) - (v,w) - (v^{'},w), \quad (v,(w+w^{'})) - (v,w) - (v,w^{'})$$

where $\lambda \in \mathbb{C}$, $v, v^{'} \in V$ and $w, w^{'} \in W$. The image of (v,w) in $V \otimes W$ is denoted $v \otimes w$.

The tensor product has the following universal property: Let $\varphi \colon V \times W \to V \otimes W$ be the bilinear map $\varphi(v,w) = v \otimes w$. If $\psi \colon V \times W \to U$, where U is some vector space such that ψ is bilinear, then there exist a unique linear map $\overline{\psi} \colon V \otimes W \to U$ such that the following diagram commutes:



The tensor product is a functor which is covariant in both variables. The universal property determines the tensor product uniquely up to isomorphism; but we have given the explicit definition in order to secure the functoriality.

(1.3) and (1.4) need not any comments except that $\mathrm{Hom}(-,-)$ is contravariant in the first and covariant in the last variable.

(1.4): Define

$$\mathcal{T}(V) = \coprod_{i=0}^{\infty} \underbrace{V \otimes \cdots \otimes V}_{i \text{ times}} = \coprod_{i=0}^{\infty} V^{\otimes i}$$

The 0-th term is $\mathbb{C}$. $\mathcal{T}$ has a product:

$$(v_1 \otimes \cdots \otimes v_i)(w_1 \otimes \cdots \otimes w_j) = v_1 \otimes \cdots \otimes v_i \otimes w_1 \otimes \cdots \otimes w_j.$$

In $\mathcal{T}(V)$ we consider the ideal $\mathcal{J}$ generated by elements of the form $v \otimes v$.

$$\Lambda^*(V) = \mathcal{T}(V)/\mathcal{J}.$$

The image of $V^{\otimes i}$ in $\Lambda^*(V)$ is denoted $\lambda^i(V)$.

$\mathcal{T}(V)$ is called the *tensor algebra* of V.
$\Lambda^*(V)$ is called the *exterior algebra* of V.
$\lambda^i(V)$ is called the *i-th exterior power* of V.

The image of $v_1 \otimes \cdots \otimes v_i$ in $\lambda^i(V)$ is denoted $v_1 \wedge \cdots \wedge v_i$. λ^i is clearly a functor.

Proposition 1.18.

(1) *If* $\{e_1, \ldots, e_n\}$ *and* $\{f_1, \ldots, f_m\}$ *are bases of* V *and* W*, then*

$$\{e_i \otimes f_j\}_{i,j}$$

is a basis of $V \otimes W$.

(2) (a) $V \otimes W \cong W \otimes V$.
(b) $V \otimes (U \otimes W) \cong (V \otimes U) \otimes W$.
(c) $V \otimes (U \oplus W) \cong V \otimes U \oplus V \otimes W$.
All isomorphisms are natural.

(3) $\mathrm{Hom}(V, W) \cong V^* \otimes W$ *naturally.*

(4) (a) $\lambda^0(V) = \mathbb{C}$, $\lambda^1(V) = V$, $\lambda^i(V) = 0$ *for* $i > \dim V$.
(b) *If* V *is* n*-dimensional then* $\dim \lambda^n(V) = 1$. *If* $A\colon V \to V$ *is linear, then*

$$\lambda^n(A)\colon \lambda^n(V) \to \lambda^n(V)$$

is multiplication by $\det(A)$. *In general, the coefficients of the characteristic polynomial* $\det(tA - I)$ *is* $\mathrm{tr}(\lambda^i(A))$.
(c) *If* $\{e_1, \dots, e_n\}$ *is a basis of* V, *then*

$$\{e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_i}\}_{1 \le j_1 < j_2 < \cdots < j_i \le n}$$

is a basis of $\lambda^i(V)$.

(5)

$$\lambda^k(V \oplus W) \cong \coprod_{i+j=k} \lambda^i(V) \otimes \lambda^j(W) \text{ naturally.}$$

Proof. (1) It is easy to see that the bilinear map $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ given by $(x, y) \mapsto xy$ induces an isomorphism $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$. The statement is a consequence of this together with (c).

(2) Is proved by general nonsense using universal properties etc.

(3) Define $\varphi\colon V^* \otimes W \to \mathrm{Hom}(V, W)$ by $\varphi(f \otimes w)(v) = f(v)w$. Let us show that φ is onto: $\{e_1, \dots, e_n\}$ is a basis of V. Define $k_i \in V^*$ by $k_i(\sum_j a_j e_j) = a_i$. If $\psi \in \mathrm{Hom}(V, W)$, then $\psi = \varphi(\sum_i k_i \otimes \psi(e_i))$, and counting dimensions gives the result.

(4) (a) By linearity and anticommutativity we see that every class in $\lambda^i(V)$ is a linear combination of the classes given in part (c). In particular, we get that $\lambda^i(V) = 0$ if i exceeds the dimension of V.

(b) It follows from (a) that $\lambda^n(V)$ is generated by $e_1 \wedge e_2 \wedge \dots e_n$. So unless this vectorspace is trivial, it has dimension 1. We can as well assume that V equals $\mathbb{C}^n$. Then, the determinant gives a multilinear map $V \times V \times \cdots \times V \to \mathbb{C}$. This gives a linear map $V^{\otimes n} \to \mathbb{C}$. Since this determinant vanishes if two of the vectors are equal, the determinant factors over $\lambda^n(V)$. So this vector space is non trivial, and the single class $e_1 \wedge e_2 \wedge \cdots \wedge e_n$ spans it.

(c) The classes in question span $\lambda^i(V)$, so we have to check that they are linearly indedent. Let $\{e_1, \dots, e_n\}$ be a basis of V and put

$$V_k = \mathrm{span}\{e_1, \dots, e_k\}$$

We have seen that $\lambda^k(V_k)$ is 1-dimensional. We want to show that

$$\lambda^i(V_{n-1}) \cap \lambda^{i-1}(V_{n-1}) \wedge e_n = 0.$$

Assume

$$A = B \wedge e_n \qquad A \in \lambda^i(V_{n-1}),\ B \in \lambda^{i-1}(V_{n-1})$$

Write

$$B = \sum a_{i_1 \dots i_{i-1}} e_{i_1} \wedge \dots \wedge e_{i_{i-1}}$$

The summation is over all $i_1 \dots i_{i-1}$ satisfying $1 \le i_1 < \dots < i_{i-1} < n$. For fixed $i_1 \dots i_{i-1}$ let $j_1 \dots j_{n-i} < n$ such that

$$e_{i_1} \wedge \dots \wedge e_{i_{i-1}} \wedge \dots \wedge e_{j_{n-i}} = \pm e_1 \wedge \dots \wedge e_{n-1}$$

Then

$$A \wedge e_1 \wedge \dots \wedge e_{n-1} \in \lambda^n(V_{n-1}) = 0.$$

Hence

$$\pm a_{i_1 \dots i_{i-1}} e_1 \wedge \dots \wedge e_n = 0$$

and therefore is $B = 0$. We have actually shown

$$\lambda^i(V_n) = \lambda^i(V_{n-1}) \oplus \lambda^{i-1}(V_{n-1}) \wedge e_n$$

or

$$\lambda^i(V_n) \cong \lambda^i(V_{n-1}) \oplus \lambda^{i-1}(V_{n-1}).$$

By the triangle of Pascal is $\dim \lambda^i(V_n) = \binom{n}{i}$.

(5) Define a map from right to left and show that it is an epimorphism. The result again follows by counting dimensions. □

We now turn to the main problem of this section: We want to extend these functors to the category of vector bundles over a space X. We restrict to functors of one variable only, and leave the obvious generalizations for functors of several variables to the reader.

Let T be a covariant functor in the category of vector spaces end linear maps. We say that T is continuous if

$$T\colon \operatorname{Hom}(V, W) \to \operatorname{Hom}(TV, TW)$$

is continuous.

If $\pi \in \mathcal{P}(V)$, denote by i the inclusion $\pi V \to V$ then $\pi = i \circ \overline{\pi}$ where $\overline{\pi}\colon V \to \pi V$. Now $T(\pi)$ is a projection in $T(V)$ and $T(i)$ is injective because

$$T(\overline{\pi}) \circ T(i) = T(\overline{\pi} \circ i) = \mathrm{Id}_{\pi V}\,.$$

$T(\pi)T(V) = T(i)T(\pi V)$ so in the following we will not distinguish between $T(\pi)T(V)$ and $T(\pi V)$.

Let E be a vector bundle over X i.e

$$E = \bigcup_{x \in X} E_x$$

and put

$$T(E) = \bigcup_{x \in X} T(E_x)$$

The problem is to give $T(E)$ a reasonable topology. If $\varphi\colon E \to E^{'}$ is an isomorphism and $E^{'}$ is an embedded vector bundle defined by $\pi\colon X \to \mathcal{P}(V)$, then

$$T(\varphi) = \bigcup_{x \in X} T\varphi_x\colon T(E) \to T(E^{'}) \subseteq X \times T(V)$$

is a bijection. $T(E^{'})$ has a natural topology induced from $X \times T(V)$ and is actually an embedded vector bundle defined by

$$T \circ \pi\colon X \to \mathcal{P}(T(V))$$

This analysis forces $T(E)$ to have the topology arising from requiring $T(\varphi)$ to be a homeomorphism. We have to show that this topology is independent of the choice of $E^{'}$ and ψ:

Lemma 1.19. *Let*

$$\pi_1\colon X \to \mathcal{P}(V)$$
$$\pi_2\colon X \to \mathcal{P}(W)$$

be a projection bundles defining the embedded vector bundles E_1 and E_2 respectively. Let

$$\varphi\colon E_1 \to E_2$$

be a bundle homomorphism and T a continuous functor. Then

$$T\varphi = \bigcup_{x \in X} T\varphi_x$$

defines a bundle homomorphism from

$$TE_1 = \bigcup_{x \in X} TE_{1x} \quad \text{to} \quad TE_2 = \bigcup_{x \in X} TE_{2x}.$$

Proof. Let $i_2\colon E_2 \to X \times W$ be the inclusion.

$$\overline{\varphi}\colon X \times V \to X \times W$$

defined by

$$\overline{\varphi}(x, v) = i_2\varphi(x, \pi_{1x}v)$$

is clearly continuous.

$$\overline{\varphi}_{|E_{1x}} = \varphi_x\colon E_{1x} \to E_{2x}.$$

$\overline{\varphi}$ defines a continuous map

$$\Phi\colon X \to \mathrm{Hom}(V, W), \qquad T \circ \Phi\colon X \to \mathrm{Hom}(TV, TW)$$

is continuous, and this corresponds to

$$T\overline{\varphi}\colon X \times TV \to X \times TW; \quad T\overline{\varphi}_{|TE_1} = T\varphi,$$

which is therefore continuous. □

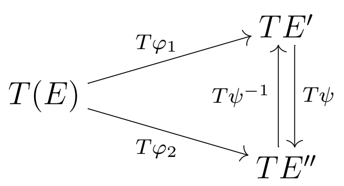
The topology of $T(E)$ is now easily shown independent of E': Let $\varphi_1\colon E \to E'$ and $\varphi_2\colon E \to E''$ be two isomorphisms, where E' and E'' are embedded vector bundles.

$$\psi = \varphi_2\varphi_1^{-1}, \qquad \psi^{-1} = \varphi_1\varphi_2^{-1}$$

are homomorphisms, for which we can apply Lemma 1.19. So

$$T\psi = \bigcup_{x \in X} T\varphi_{2x}T\varphi_{1x}^{-1}, \qquad T\psi^{-1} = \bigcup_{x \in X} T\varphi_{1x}T\varphi_{2x}^{-1}$$

are both continuous and each others inverse.



Also by Lemma 1.19 it is clear that T is well defined on homomorphisms. We have actually shown:

Theorem 1.20. *If T is a continuous functor in the category of vector spaces and linear maps, there exist a unique functor T in the category of vector bundles over a space X such that*

(1) $TE = \bigcup_{x \in X} TE_x$, *and if $\varphi\colon E \to E'$ is a homomorphism, then*

$$T\varphi = \bigcup_{x \in X} T\varphi_x.$$

(2) $$T(X \times V) = X \times TV.$$

Furthermore if $f\colon Y \to X$ is a continuous map, then the functors

$$f^* \circ T \qquad T \circ f^*$$

are naturally equivalent.

Exercise 1.21. If T and $T^{'}$ are continuous functors in the category of vector spaces and

$$\eta\colon T \to T^{'}$$

a natural transformation, then η extends to a natural transformation between the corresponding functors in the category of vector bundles over X. Especially if T and $T^{'}$ are naturally equivalent then also the extended functors are naturally equivalent.

As a consequence of Theorem 1.20 we have the following functors on vector bundles over a space X:

$$E \oplus F \tag{1.6}$$
$$E \otimes F \tag{1.7}$$
$$\mathrm{Hom}(E, F) \tag{1.8}$$
$$E^* \tag{1.9}$$
$$\lambda^i(E) \tag{1.10}$$

From Exercise 1.21 and Proposition 1.18 we get the natural equivalences:

(1) $E \otimes F \cong F \otimes E$.
(2) $E \otimes (F \otimes G) \cong (E \otimes F) \otimes G$.
(3) $E \otimes (F \oplus G) \cong E \otimes F \oplus E \otimes G$.
(4) $\mathrm{Hom}(E, F) \cong E \otimes F$.
(5) (a) $\lambda^0(E) \cong X \times \mathbb{C}$.
    (b) $\lambda^1(E) \cong E$.
    (c) $\lambda^i(E) = 0$ for $i > n$, if E has dimension n.
(6) $\lambda^k(E \oplus F) \cong \coprod_{i+j=k} \lambda^i(E) \otimes \lambda^j(F)$.

Exercise 1.22.
(1) Let E be a vector bundle over X and F a vector bundle over Y. Then $E \times F$ is a vector bundle over $X \times Y$ which is isomorphic to $p_1^* E \oplus p_2^* F$, where $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$ are the projections.
(2) If $\Delta\colon X \to X \times X$ is the diagonal, then for any bundles E and F over X we have

$$E \oplus F \cong \Delta^*(E \times F).$$

Remark 1.23. $E \oplus F$ is often called the *Whitney sum* of E and F.

Definition 1.24. A *section* s of a vector bundle E is a continuous function $s\colon X \to E$ such that $p \circ s = \mathrm{Id}_X$.

Exercise 1.25.

(1) The set of all sections of E is a vector space denoted $\Gamma(E)$.

(2) There is a 1–1 correspondence between the section of $\mathrm{Hom}(E, F)$ and homomorphisms from E to F.

We conclude with some lemmas for technical purpose later on.

Proposition 1.26. *Let*

$$\pi_1\colon X \to \mathcal{P}(V)$$
$$\pi_2\colon X \to \mathcal{P}(W)$$

define embedded vector bundles E_1 and E_2 and let $\varphi\colon E_1 \to E_2$ be an isomorphism. Consider E_1 and E_2 as subspaces of $X \times (V \oplus W)$. Then there exists an isomorphism

$$\psi\colon X \times (V \oplus W) \to X \times (V \oplus W)$$

such that $\psi_{|E_1} = \varphi$.

Proof. As in the proof of Lemma 1.19 φ defines

$$\Phi_1\colon X \to \mathrm{Hom}(V, W)$$

and φ^{-1} defines

$$\Phi_{-1}\colon X \to \mathrm{Hom}(W, V)$$

Define ψ by the map $X \to \mathrm{Hom}(V \oplus W, V \oplus W)$ defined by the matrix

$$\begin{pmatrix} 1 - \pi_1 & \Phi_{-1} \\ \Phi_1 & 1 - \pi_2 \end{pmatrix}$$

$\psi^2 = \mathrm{Id}$ so is clearly an isomorphism. □

The next lemma uses the

Tietze Extension Theorem. Let X be a normal space, $Y \subseteq X$ a closed subspace, V a vector space, and $f\colon Y \to V$ a continuous map. Then there exists a continuous map $g\colon X \to V$ such that

$$g(y) = f(y) \quad \text{for} \quad y \in Y.$$

Lemma 1.27. *Let X be a normal space, $Y \subseteq X$ a closed subset, and E a vector bundle over X. Then any section $s\colon Y \to E$ can be extended to X.*

Proof. Without loss of generality E is embedded defined by $\pi\colon X \to \mathcal{P}(V)$. So actually

$$s(y) = (y, \overline{s}y) \quad \text{for} \quad y \in Y,\ \overline{s}\colon Y \to V.$$

We extend $\overline{s}$ to $\overline{t}\colon X \to V$ and put

$$t(x) = (x, \pi_x \overline{t}(x))$$

Then t is an extension of s. □

Using this lemma on the vector bundle $\mathrm{Hom}(E, F)$ we get from Proposition 1.13:

Corollary 1.28. *Let X be a normal space, $Y \subseteq X$ a closed subspace and E,F two vector bundles. If $\psi\colon E_{|Y} \to F_{|Y}$ is an isomorphism, then there exists an open set $U \supseteq Y$ and an extension $\varphi\colon E_{|U} \to F_{|U}$, which is an isomorphism.*

1.3 Sub-bundles and quotient bundles

Definition 1.29. Let (E, p) be a vector bundle over X; $F \subseteq E$ is a *sub-bundle* of E if (F, p) itself is a vector bundle such that F_x is a subspace of E_x for all x

Definition 1.30. $\varphi\colon F \to E$ is a *monomorphism* (respectively *epimorphism*) if for all x $\varphi_x\colon F_x \to E_x$ is monomorphic (respectively epimorphic).

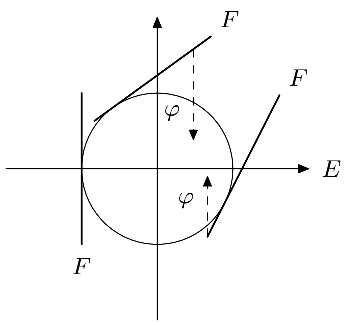
Definition 1.31. $\varphi\colon F \to E$ is said to be a *strict homomorphism* if $\dim\ker\varphi_x$ (and so $\dim\mathrm{Im}\,\varphi_x$) is locally constant.

In fact not every homomophism is strict:

Example 1.32.

$$\begin{aligned} X &= S^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}, \\ F &= \text{tangent bundle of } S^1, \\ E &= S^1 \times \{x \in \mathbb{R}^2 \mid x_2 = 0\}, \end{aligned}$$

φ is the projection in $\mathbb{R}^2$ along $\{x \mid x_1 = 0\}$.



Proposition 1.33. *Let $\varphi\colon X \times V \to X \times W$ be a strict homomorphism; then the families*

$$\bigcup_{x \in X} \ker \varphi_x \quad \textit{and} \quad \bigcup_{x \in X} \operatorname{Im} \varphi_x$$

are embedded vector bundles.

Before proving this we draw some immediate consequences.

Corollary 1.34. *Let $\varphi\colon E \to X \times W$ be a monomorphism; then*

$$\bigcup_{x \in X} \operatorname{Im} \varphi_x$$

is an embedded vector bundle. Especially every subbundle of a trivial bundle is actually embedded.

Proof. Without loss of generality E is an embedded vector bundle defined by $\pi\colon X \to \mathcal{P}(V)$. Apply proposition 1.33 to the homomorphism

$$\overline{\varphi}\colon X \times V \to X \times W$$

defined by

$$\overline{\varphi}(x, v) = \varphi(x, \pi_x v).$$ □

Corollary 1.35. *Let $\varphi\colon F \to E$ be a strict homomorphism; then*

(1) $\bigcup_{x \in X} \ker \varphi_x$ *is a sub-bundle of F,*
(2) $\bigcup_{x \in X} \operatorname{Im} \varphi_x$ *is a sub-bundle of E.*

In particular this is the case for φ a monomorphism or an epimorphism.

Proof. Without loss of generality F is embedded in $X \times V$ and E is embedded in $X \times W$ and E is embedded in $X \times W$, and let $\pi\colon X \to \mathcal{P}(V)$ be the projection bundle defining F. Again define $\overline{\varphi}\colon X \times V \to X \times W$ by

$$\overline{\varphi}(x, v) = \varphi(x, \pi_x v).$$

$\operatorname{Im} \varphi = \operatorname{Im} \overline{\varphi}$ is a sub-bundle of $X \times W$ by Proposition 1.33 and hence a sub-bundle of E. This proves (2).

$F' = \ker \overline{\varphi}$ is embedded in $X \times V$ by 1.33 and $\overline{\pi}\colon X \times V \to X \times V$ defined by

$$\overline{\pi}(x, v) = (x, \pi_x v)$$

defines a homomorphism

$$\overline{\pi}\colon F' \to F.$$

$\operatorname{Im} \overline{\pi} = \ker \varphi$, so $\overline{\pi}$ is strict. Hence by (2) we have proved (1). □

Corollary 1.36. *$F \subseteq E$ is a sub-bundle, then*

$$E/F = \bigcup_{x \in X} E_x/F_x$$

with the quotient topology is a vector bundle.

Proof. Without loss of generality E is embedded defined by an orthogonal projection bundle $\pi_1\colon X \to \mathcal{P}(V)$. By Corollary 1.34 F is embedded defined by $\pi_2\colon X \to \mathcal{P}(V)$, π_{2x} is the orthogonal projection onto F_x.

$\pi_1 - \pi_2$ is a projection bundle defining an embedded bundle isomorphic to E/F, the isomorphism simply given by

$$e_x \mapsto (\pi_1 - \pi_2)e_x.$$

Actually we have given E a hermitian structure and shown that $E/F \cong F^\perp = \bigcup_{x \in X} F_x^\perp$ is a vector bundle. Hence

$$E \cong F \oplus E/F. \qquad \square$$

Proof of Proposition 1.33 If a family $E = \bigcup_x E_x \subseteq X \times V$, and we give V a hermitian structure, then E is an embedded vector bundle iff the orthogonal projection onto E_x is continuous in x. This is a local property; hence using Lemma 1.14, we only need to show that E locally is defined by some, not necessarily orthogonal, projection bundle. Now consider a homomorphism

$$\varphi\colon X \times V \to X \times W$$

and fix a point $x_0 \in X$.
Let V_0 be a complementary subspace of $\ker \varphi_{x_0}$ in V and W_0 a complementary subspace of $\operatorname{Im} \varphi_{x_0}$ in W. Consider the maps:

$$\psi_1\colon\ X \times (V_0 \oplus W_0) \xrightarrow{\ \ i\ \ } X \times (V \oplus W) \xrightarrow{\varphi\oplus 1} X \times (W \oplus W) \xrightarrow{\ \ \nabla\ \ } X \times W$$

$$\psi_2\colon\ X \times V \xrightarrow{\ \ \Delta\ \ } X \times (V \oplus V) \xrightarrow{\varphi\oplus 1} X \times (W \oplus V) \xrightarrow{\ \ j\ \ } X \times (W/W_0 \oplus V/V_0)$$

Here i is the injection, ∇ the sum map, Δ the diagonal and j the projection.

$\psi_{1_{x_0}}$ is clearly an isomorphism, so by Proposition 1.13 ψ_{1x} is an isomorphism in a neighbourhood U of x_0.

$$\psi_{1x}(V_0) \subseteq \varphi_x V \quad \text{for} \quad x \in X$$

and equality sign holds for $x \in X$ provided φ is strict. Taking π_x to be the projection onto $\varphi_x V$ along $W_0(x \in U)$,

$$\pi_x = \psi_{1x} \pi_0 \psi_{1x}^{-1}$$

where π_0 is the projection in $V_0 \oplus W_0$ onto V_0. π_x is therefore continuous in x, and hence

$$\bigcup_{x \in X} \operatorname{Im} \pi_x$$

is an embedded vector bundle. The *dual* analysis of ψ_2 gives the result concerning $\bigcup_{x \in X} \ker \varphi_x$. □

Remark 1.37. We have actually proved that if $\varphi\colon F \to E$ is a homomorphism, then

$$\operatorname{rank} \varphi_x \geq \operatorname{rank} \varphi_{x_0}$$

for all x in a neighbourhood of x_0.

1.4 Homotopy properties of vector bundles

Definition 1.38. $f, g\colon X \to Y$ are *homotopic*, denoted $f \sim g$ if there is a continuous map

$$F\colon X \times I \to Y$$

such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$ for all $x \in X$.

Exercise 1.39.

(1) $\sim$ is an equivalence relation in the set of maps $X \to Y$. The equivalence class containing $f\colon X \to Y$ is called the *homotopy class* of f. The set of all homotopy classes of maps from X into Y is denoted $[X, Y]$.

(2) Let $f_1, f_2\colon X \to Y$, $g_1, g_2\colon Y \to Z$ be continuous maps such that $f_1 \sim f_2$ and $g_1 \sim g_2$; then $g_1 f_1 \sim g_2 f_2$.

We can, therefore, talk about the category of spaces, with morphisms homotopy classes of maps. This gives rise to notions such as *homotopy inverse* and *homotopy equivalence*. A space which is homotopy equivalent to a point is said to be *contractible*. Note that a homeomorphism is a homotopy equivalence.

Definition 1.40. Let $Y \subseteq X$ and denote the inclusion map i.

(1) $r\colon X \to Y$ is called a *retraction* if $r \circ i = \mathrm{Id}$ and Y is called a *retract* of X.

(2) A retraction is called a *deformation retraction* and Y a *deformation retract* of X if $i \circ r \sim \mathrm{Id}$, that is if there is a map $F\colon X \times I \to X$ such that $F(x, 0) = x$ and $F(x, 1) = r(x) \in Y$. If $y \in Y$ then $F(y, 1) = r(y) = y$.

(3) If $F(y, t) = y$ for all $y \in Y$ and $t \in I$, then Y is called a *strong deformation retract of X*.

Exercise 1.41.

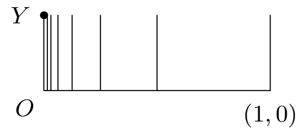
(1) Let X consists of two circles with exactly one point in common, and Y is one of them. Then Y is a retract of X. (Y is not a deformation retract, but this is not easy to show).

(2) Let $P \neq Q$ be two points and $X = P \cup Q$, $Y = P$. Then Y is a retract, but not a deformation retract of X.

(3) (The comb space) Let X consists of the points $(x, y) \in \mathbb{R}^2$ satisfying either $0 \leq x \leq 1$, $y = 0$ or $x = 0, 0 \leq y \leq 1$ or

$$x = \tfrac{1}{n}, \quad 0 \leq y \leq 1, \quad n = 1, 2, 3, \ldots$$

Let Y consists of the point $(0, 1)$. Then Y is a deformation retract, but not a strong deformation retract of X.



(4) The central circle of a solid hole of a solid torus is a strong deformation retract.

(5) Make a small hole in the surface of a torus. Find two circles, which are a strong deformation retract.

Proposition 1.42. *Let Y be a compact Hausdorff space and $Z \subseteq Y$ a closed subset. Let $f_0, f_1 \colon Y \to X$ be homotopic through maps f_t such that for $t \in I$*

$$f_t(z) = f_0(z) \quad \textit{for all} \quad z \in Z$$

Let E be a vector bundle over X. Then there exists an isomorphism

$$\varphi\colon f_0^*(E) \to f_1^*(E)$$

such that φ restricted to Z is the identity.

Proof. Let I be the unit interval; $f\colon Y \times I \to Y$ denote the homotopy, so that $f(y, t) = f_t(y)$, and let $\pi\colon Y \times I \to Y$ denote the projection. Fix $t_0 \in I$ and consider the bundles $f^*(E)$ and $(f_{t_0} \circ \pi)^*(E)$. Over $A = Y \times \{t_0\} \cup Z \times I$ there is an obvious isomorphism

$$\varphi\colon f^*(E) \to (f_{t_0} \circ \pi)^*(E).$$

(Actually φ is the identity because f and $f_{t_0} \circ \pi$ agree on A). According to Corollary 1.28 φ has an extension over a neigbourhood U of A in $Y \times I$.

By compactness of Y we can find an open interval $\delta_{t_0} \subseteq I$ containing t_0 such that $Y \times \delta_{t_0} \subseteq U$. Now consider an arbitrary $t \in \delta_{t_0}$ and let $i_t\colon Y \to Y \times I$ be the map sending y to (y, t). $f \circ i_t = f_t$ and $f_{t_0} \circ \pi \circ i_t = f_{t_0}$. Hence

$$f_t^*(E) \cong i_t^* f^*(E) \xrightarrow[i_t^*\varphi]{\cong} i_t^*(f_{t_0} \circ \pi)^*(E) \cong f_{t_0}^*(E).$$

By inspection it is easily seen that this isomorphism $f_t^*(E) \to f_{t_0}^*(E)$ is the identity over Z.

Thus for every $t_0 \in I$ there exists a δ_{t_0} such that $f_t^*(E) \cong f_{t_0}^*(E)$ for $t \in \delta_{t_0}$, the isomorphism being the identity over Z. From the compactness of I the proposition follows. □

Remark 1.43. The proposition holds more generally over paracompact spaces; for a proof see for example D. Husemoller [20]. In the following we will assume all base spaces to be compact Hausdorff spaces.

Proposition 1.44. *Let $X \subseteq Y$ be a strong deformation retract of Y, i.e.*

$$F\colon Y \times I \to Y$$

such that $F(y,1) \in X$, $F(y,0) = y$, $F(x,t) = x$, for $x \in X$. Let $\pi\colon Y \times I \to Y$ be the projection and E a vector bundle over Y. Then there is an isomorphism

$$\Phi\colon \pi^*(E) \to F^*(E)$$

which is the identity over $Y \times 0 \cup X \times I$.

Proof. Use Proposition 1.42 on the homotopy

$$G_s\colon Y \times I \to Y$$

defined by

$$G_s(y,t) = F(y,st).$$

□

Definition 1.45. Two isomorphisms $\alpha_0, \alpha_1\colon E \to F$ are called homotopic ($\alpha_0 \sim \alpha_1$), if there is an isomorphism

$$\Phi\colon \pi^*E \to \pi^*F \qquad (\pi\colon X \times I \to X)$$

such that the composites

$$E \longrightarrow \pi^*E_{|0} \xrightarrow{\Phi} \pi^*F_{|0} \longrightarrow F$$

and

$$E \longrightarrow \pi^*E_{|1} \xrightarrow{\Phi} \pi^*F_{|1} \longrightarrow F$$

are α_0 and α_1 respectively.

Remark 1.46. if $E = X \times V$ and $F = X \times W$ then α_0 and α_1 correspond to

$$\bar{\alpha}_0, \bar{\alpha}_1\colon X \to \mathrm{Iso}(V, W)$$

$\alpha_0 \sim \alpha_1$ as isomorphisms of bundles iff $\bar{\alpha}_0 \sim \bar{\alpha}_1$ as continuous maps. We leave it to the reader to verify that $\sim$ is an equivalence relation.

Corollary 1.47. *$X \subseteq Y$ a strong deformation retract, E a bundle over Y and $\varphi\colon E \to E$ an isomorphism such that $\varphi_{|X} = \mathrm{Id}$. Then $\varphi \sim \mathrm{Id}$, and the homotopy is the identity over $X \times I$.*

Proof. Let Φ be the isomorphism according to Proposition 1.44. Then $\Psi = \Phi^{-1}F^*(\varphi)\Phi$ is a homotopy between φ and Id. □

Suppose now Y is closed in X, E is a vector bundle over X and $\alpha\colon E_{|Y} \to Y \times W$ is a trivialization. Let $\pi\colon Y \times W \to W$ denote the projection and define an equivalence relation on E by

$$e \sim e' \iff \begin{cases} \text{either } e = e' \text{ or} \\ (1)\ pe \in Y \text{ and } pe' \in Y \\ (2)\ \pi\alpha(e) = \pi\alpha(e'). \end{cases}$$

The quotient space of E is denoted E/α. It has a natural structure of a family of vector spaces over X/Y, and is in fact a vector bundle.

Proposition 1.48. *Let $Y \subseteq X$ be a closed subset and $\alpha\colon E_{|Y} \to Y \times W$ a trivialization. Then E/α is a vector bundle over X/Y, and its isomorphism class depends only on the homotopy class of α. Futhermore, if $j\colon X \to X/Y$ then $j^*E/\alpha \cong E$.*

Proof. Without loss of generality E is embedded defined by $\pi'\colon X \to \mathcal{P}(V)$. Extend α to a neighbourhood U of Y in X. According to Urysohn there is a function $f\colon X \to [0,1]$ such that

$$f(Y) = 1, \qquad f(X \setminus U') = 0$$

for some neighbourhood U' of Y satisfying $Y \subseteq U' \subseteq \bar{U}' \subseteq U$. Considering $E \subseteq X \times V$, we define

$$\varphi\colon E \to X \times (V \oplus W) \quad \text{by} \quad \varphi(x,v) = (x, (1 - f(x)) \cdot v \oplus f(x)\alpha_x v).$$

φ is clearly continuous, and if $e \sim e'$, then φe and $\varphi e'$ has the same second component, hence there is a continuous map $\bar{\varphi}$ such that

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & X \times (V \oplus W) \\ \downarrow & & \downarrow \\ E/\alpha & \xrightarrow{\bar{\varphi}} & X/Y \times (V \oplus W) \end{array}$$

commutes. According to Corollary 1.34 $\operatorname{Im}\varphi$ is embedded, hence

(1) φ is a homeomorphism onto its image and therefore $\bar{\varphi}$ is, and

(2) if π_x is the projection onto $\operatorname{Im}\varphi_x$, then $x \mapsto \pi_x$ is continuous which shows that $\operatorname{Im}\bar{\varphi}$ is embedded. This proves the first part.

If $\Phi\colon \pi^*E_{|Y} \to Y \times I \times W$ is a homotopy connecting two trivializations α_0 and α_1, consider π^*E/Φ, a bundle over $X \times I/Y \times I$. Let $g\colon X/Y \times I \to X \times I/Y \times I$ and consider

$$g^*(\pi^*E/\Phi) \quad \text{over} \quad (X/Y) \times I.$$

Clearly this bundle restricted to $(X/Y) \times 0$ and $(X/Y) \times 1$ is E/α_0 and E/α_1 respectively. The injections

$$(X/Y) \longrightarrow (X/Y) \times 1 \longrightarrow (X/Y) \times I$$

and

$$(X/Y) \longrightarrow (X/Y) \times 0 \longrightarrow (X/Y) \times I$$

are homotopic, so the second statement follows from Proposition 1.42. The proof of the last statement is left as an exercise. □

Definition 1.49. $\mathrm{Vect}(X)$ denotes the set of isomorphism classes of vector bundles over X, and $\mathrm{Vect}_n(X)$ denotes the subset of n-dimensional bundles.

$\mathrm{Vect}(X)$ is an abelian semigroup under the Whitney sum operation $\oplus$. For example if $X =$ a point $*$ then $\mathrm{Vect}(*) \cong \mathbb{Z}_+$. Vect is actually a functor taking a map $f\colon X \to Y$ into

$$f^*\colon\ \mathrm{Vect}(Y) \to \mathrm{Vect}(X).$$

We state this as

Theorem 1.50.

(1) Vect *is a contravariant functor from the category of compact spaces with morphisms homotopy classes of maps into the abelian semigroups.*

(2) *Especially if $f\colon X \to Y$ is a homotopy equivalence, then f^* is an isomorphism.*

Remark 1.51. It follows that every bundle over a contractible space is trivial, so Proposition 1.48 is applicable if Y is contractible.

Corollary 1.52. *Let $Y \subseteq X$ be a closed contractible subspace. Then $f\colon X \to X/Y$ induces an isomorphism.*

$$f^*\colon\ \mathrm{Vect}(X/Y) \to \mathrm{Vect}(X).$$

Proof. We construct an inverse map: Let E be a vector bundle over X. $E_{|Y}$ is trivial, so we choose a trivialization $\alpha\colon E_{|Y} \to Y \times W$. E/α defines then an element in $\mathrm{Vect}(X/Y)$. We have to prove that this element is independent of the choice of α: So let α_1, α_2 be two trivializations

$$\alpha_1\colon E_{|Y} \to Y \times V, \qquad \alpha_2\colon E_{|Y} \to Y \times W$$

$\beta = \alpha_2\alpha_1^{-1}\colon Y \times V \to Y \times W$ is an isomorphism. Y is contractible, so

$$\beta \sim \mathrm{Id} \times \beta_0, \qquad \beta_0 \colon V \to W.$$

Put $\gamma = (\mathrm{Id} \times \beta_0) \circ \alpha_1$. By Proposition 1.48 is $E/\alpha_2 \cong E/\gamma$. Let

$$\pi_1 \colon Y \times V \to V, \qquad \pi_2 \colon Y \times W \to W.$$

Then

$$\begin{aligned} e \sim e' \quad \text{in } E/\gamma &\iff \pi_2\gamma e = \pi_2\gamma e' \\ &\iff \beta_0\pi_1\alpha_1 e = \beta_0\pi_1\alpha_1 e' \\ &\iff \pi_1\alpha_1 e = \pi_1\alpha_1 e' \\ &\iff e \sim e' \quad \text{in } E/\alpha_1. \end{aligned}$$

So actually $E/\alpha_1 = E/\gamma$. □

We will now describe another construction on vector bundles similar to E/α. We consider a compact space X and closed subsets X_1 and X_2 satisfying $X_1 \cup X_2 = X$. Put $A = X_1 \cap X_2$. Assume E_1, E_2 are vector bundles over X_1 and X_2 respectively and suppose there is given an isomorphism $\varphi\colon E_{1|A} \to E_{2|A}$. We then define the family $E_1 \cup_\varphi E_2$ over X as follows. The total space is the quotient of the disjoint sum $E_1 + E_2$ by the equivalence relation which identifies $e_1 \in E_{1|A}$ with $\varphi(e_1) \in E_{2|A}$. Identifying X with the corresponding quotient of $X_1 + X_2$ we obtain a natural projection

$$p\colon E_1 \cup_\varphi E_2 \to X,$$

and this defines a family of vector spaces.

Proposition 1.53.

(1) *$E_1 \cup_\varphi E_2$ is a vector bundle.*

(2) *If E is a bundle over X and $E_i = E_{|X_i}$, then the identity defines an isomorphism $I_A\colon E_{1|A} \to E_{2|A}$, and $E_1 \cup_{I_A} E_2 \cong E$.*

(3) *If $\beta_i\colon E_i \to E_i'$ are isomorphisms on X_i and $\varphi'\beta_1 = \beta_2\varphi$, then $E_1 \cup_\varphi E_2 \cong E_1' \cup_{\varphi'} E_2'$.*

(4) $(E_1 \cup_\varphi E_2) \oplus (E_1' \cup_{\varphi'} E_1') \cong E_1 \oplus E_1' \cup_{\varphi\oplus\varphi'} E_2 \oplus E_2'$.
$(E_1 \cup_\varphi E_2) \otimes (E_1' \cup_{\varphi'} E_1') \cong E_1 \otimes E_1' \cup_{\varphi\otimes\varphi'} E_2 \otimes E_2'$.
$(E_1 \cup_\varphi E_2)^* \cong E_1^* \cup_{(\varphi^*)^{-1}} E_2^*$.

(5) *The isomorphism class of $E_1 \cup_\varphi E_2$ depends only on the homotopy class of the isomorphism $\varphi\colon E_{1|A} \to E_{2|A}$.*

Proof. We leave the proof of (2)–(5) as an exercise for the reader and concentrate on (1).

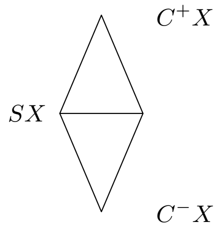
Without loss of generality E_1, E_2 are embedded bundles defined by $\pi_i\colon X_i \to \mathcal{P}(V)$, and according to Proposition 1.26 we can asumme φ to be the restriction of an isomorphism $\Phi\colon A \times V \to A \times V$. Extend Φ to a neighbourhood U of A in X and choose an Urysohn function $f\colon X \to [0,1]$ such that $f(A) = 1$, $f(X \setminus U') = 0$ for some neighbourhood U' satisfying $A \subseteq U' \subseteq \bar{U}' \subseteq U$. Considering $E_1 + E_2$ as embedded in $(X_1 + X_2) \times V$, we define $\psi\colon E_1 + E_2 \to (X_1 + X_2) \times (V \oplus V)$ by

$$\psi(x_2, v) = (x_2, 0 \oplus v), \quad x_2 \in X_2,$$
$$\psi(x_1, w) = (x_1, (1 - f(x_1))w \oplus f(x_1)\Phi_{x_1} w), \quad x_1 \in X_1.$$

The rest of the proof proceeds as in the proof of Proposition 1.48. □

Remark 1.54. E_1, E_2 are called *clutching data* and φ is called a *clutching function.*

Definition 1.55. Let X be a topological space and $I = [-1, 1]$. The *suspension* SX of X is $X \times I$ with $X \times (-1)$ identified to one point and $X \times (+1)$ identified to one point.



Theorem 1.56. *There is for any compact space X a natural isomorphism*

$$\mathrm{Vect}_n(SX) \cong [X, \mathrm{GL}(n, \mathbb{C})]\,.$$

Proof. We define a map from the right to the left in the following way:

$$\bar{\varphi}\colon X \to \mathrm{GL}(n, \mathbb{C})$$

corresponds to an isomorphism

$$\varphi\colon X \times \mathbb{C}^n \to X \times \mathbb{C}^n.$$

$$C^+X = X \times [0,1]\,/X \times (1),$$
$$C^-X = X \times [-1,0]\,/X \times (-1).$$
$$C^+X \cup C^-X = SX \quad \text{and} \quad C^+X \cap C^-X = X \times (0).$$

Put $E = (C^+X \times \mathbb{C}^n) \cup_\varphi (C^-X \times \mathbb{C}^n)$. By Proposition 1.53 E is a vector bundle over SX whose isomorphism class only depends on the homotopy class of $\bar{\varphi}$.

We now want to construct an inverse map, so let E be an arbitrary bundle of dimension n over SX. C^+X and C^-X are contractible spaces, so according to Theorem 1.50 the restriction of E to these spaces gives trivial bundles. Choose trivializations

$$\alpha^+\colon E_{|C^+X} \to C^+X \times \mathbb{C}^n$$
$$\alpha^-\colon E_{|C^-X} \to C^-X \times \mathbb{C}^n$$

and put $\varphi = \alpha^-_{|X} \circ (\alpha^+_{|X})^{-1}$. Let $\bar{\varphi}\colon X \to \mathrm{GL}(n,\mathbb{C})$ be the corresponding map. By Proposition 1.53 we have

$$E = E_{|C^+X} \cup_{I_X} E_{|C^-X} \cong (C^+X \times \mathbb{C}^n) \cup_\varphi (C^-X \times \mathbb{C}^n),$$

so we have certainly finished if only we can show that the homotopy class of $\bar{\varphi}$ is independent of the above choices. It is sufficient to show that the homotopy class of for example α^+ is well defined. Let α_1^+ and α_2^+ be two trivializations. Then $\alpha = \alpha_2^+(\alpha_1^+)^{-1}$ is an isomorphim

$$\alpha\colon C^+X \times \mathbb{C}^n \to C^+X \times \mathbb{C}^n,$$

whose homotopy class corresponds to the homotopy class of

$$\bar{\alpha}\colon C^+X \to \mathrm{GL}(n,\mathbb{C}).$$

C^+X is contractible so without loss of generality $\bar{\alpha}$ is assumed to be the constant. Now $\mathrm{GL}(n,\mathbb{C})$ is pathwise connected, so $\bar{\alpha}$ is homotopic to the map which is constantly the unit of $\mathrm{GL}(n,\mathbb{C})$, This means that α is homotopic to the identity, and hence $\alpha_2^+ \sim \alpha_1^+$. □

Appendix. $\mathrm{GL}(n,\mathbb{C})$ is pathwise connected.

Proof. Let $T \in \mathrm{GL}(n,\mathbb{C})$; we will show that T can be connected by a path to I. Because every complex linear map has an eigenvector it is easy to show by induction on n that T is equivalent to a matrix with zeros under the diagonal. Hence $T = A^{-1}(D+N)A$, where D is a diagonal matrix and N is nilpotent.

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

$\det T = \det D = d_1 \cdots d_n \neq 0$, hence $d_i \neq 0$ for all i. d_i can be connected to 1 by a path $d_i^{(t)}$ contained in $\mathbb{C} \setminus \{0\}$. Put

$$D^{(t)} = \begin{pmatrix} d_1^{(t)} & & 0 \\ & \ddots & \\ 0 & & d_n^{(t)} \end{pmatrix}$$

$T^{(t)} = A^{-1}(D^{(t)} + N)A$. $\det T^{(t)} = \det D^{(t)} \neq 0$, so T is connected to $A^{-1}(I+N)A$. Now $I + sN \in \mathrm{GL}(n,\mathbb{C})$ for all $s \in [0,1]$ because N is nilpotent. Hence T is connnected to $A^{-1}IA = I$. □

Remark 1.57. Taking $X = S^0$ in Definition 1.55 we get that every bundle over S^1 is trivial.

Definition 1.58. Let A be *directed* set, i.e. a partially ordered set such that if $\alpha, \beta \in A$ then there exists $\gamma \in A$ satisfying $\gamma \geq \alpha$ and $\gamma \geq \beta$. A system $\{X_\alpha\}_{\alpha \in A}$ of sets with maps $f_{\beta\alpha}\colon X_\alpha \to X_\beta$ for every pair of $(\alpha, \beta) \in A \times A$ satisfying $\beta \geq \alpha$, is said to be a *direct system* of sets, provided

1. $f_{\alpha\alpha} = \mathrm{Id}_{X_\alpha}$
2. $f_{\gamma\beta} \circ f_{\beta\alpha} = f_{\gamma\alpha}$.

The *direct limit* $\varinjlim_{\alpha \in A} X_\alpha$ is defined as the disjoint union $\bigcup_{\alpha \in A} X_\alpha$ with the identifications

$$x \sim y \quad \text{for} \quad x \in X_\alpha, y \in X_\beta$$

and there is a $\gamma \geq \alpha, \beta$ such that $f_{\gamma\beta}(y) = f_{\gamma\alpha}(x)$.

Exercise 1.59.

(1) The above defined relation is an equivalence relation, so $\varinjlim_{\alpha \in A} X_\alpha$ is well defined.

(2) A map $h\colon \{X_\alpha\} \to \{Y_\alpha\}$ of directed systems indexed by A is a collection of maps $h_\alpha\colon X_\alpha \to Y_\alpha$ making the diagram

$$\begin{array}{ccc} X_\alpha & \xrightarrow{h_\alpha} & Y_\alpha \\ {\scriptstyle f_{\beta\alpha}}\downarrow & & \downarrow{\scriptstyle g_{\beta\alpha}} \\ X_\beta & \xrightarrow{h} & Y_\beta \end{array}$$

commutative for any $\beta \geq \alpha$. Show that $\varinjlim_A$ is a functor from the category of directed systems of sets indexed by A to the category of sets.

(3) If X_α is an abelian group for every $\alpha \in A$, and $f_{\beta\alpha}$ is a homomorphism for any $\beta \geq \alpha$, then $\varinjlim_A X_\alpha$ is again an abelian group and analogous for homomorphisms.

(4) Let B be a directed set and A_β a directed set for every $\beta \in B$ then $C = \bigcup_{\beta \in B} A_\beta$ is a natural way a directed set. Let $\{X_{\alpha_\beta}\}$ be a directed system, then

$$\left\{\varinjlim_{A_\beta} X_{\alpha_\beta}\right\}_{\beta \in B}$$

is a directed system, and

$$\varinjlim_{C} X_{\alpha_\beta} \cong \varinjlim_{B} \varinjlim_{A} X_{\alpha_\beta}$$

(5) Let $\{G_\alpha\}_{\alpha \in A}$ and $\{F_\alpha\}_{\alpha \in A}$ be two directed systems of abelian groups satisfying $F_\alpha \subseteq G_\alpha$, and such that the inclusions define a map of directed systems. Then $\{G_\alpha / F_\alpha\}_{\alpha \in A}$ is also a directed sytem, $\varinjlim_A F_\alpha$ is a subgroup of $\varinjlim_A G_\alpha$ and

$$\varinjlim_A G_\alpha \,/\, F_\alpha = \varinjlim_A G_\alpha \,/\, \varinjlim_A F_\alpha.$$

We now want to prove another classification theorem for vector bundles. First note that the injection $\mathbb{C}^{m_1} \to \mathbb{C}^{m_1} \oplus \mathbb{C}^{m_2} \cong \mathbb{C}^{m_1+m_2}$ induces an injection of Grassmannians

$$i_{m_1+m_2,m_1} \colon G_n(\mathbb{C}^{m_1}) \to G_n(\mathbb{C}^{m_1+m_2}),$$

and note that

$$i^*_{m_1+m_2,m_1} E_n(\mathbb{C}^{m_1+m_2}) \cong E_n(\mathbb{C}^{m_1}).$$

Theorem 1.60. *The natural map*

$$\varinjlim_m \, [X, G_n(\mathbb{C}^m)] \to \mathrm{Vect}_n(X)$$

induced by sending $f \colon X \to G_n(\mathbb{C}^m)$ into $f^*E_n(\mathbb{C}^m)$, is a bijection for all compact Hausdorff spaces X.

Proof. We shall construct an inverse map. As in Proposition 1.16 every n-dimensional bundle is isomorphic to $f^*E_n(\mathbb{C}^m)$ for m and some $f \colon X \to G_n(\mathbb{C}^m)$ defined by

$$f(x) = \pi_x \mathbb{C}^m, \quad \text{where} \quad \pi \colon X \to \mathcal{P}(\mathbb{C}^m)$$

is a projection bundle. The only problem is to show that the homotopy class of f is uniquely determined by the isomorphism class of E. So let $\varphi_0 \colon E \to X \times \mathbb{C}^{m_0}$ and $\varphi_1 \colon E \to X \times \mathbb{C}^{m_1}$ be isomorphisms onto embedded bundles. Let $I = [0, 1]$ and $E \times I = \pi^*E$, $\pi \colon X \times I \to X$. Define

$$\Phi \colon E \times I \to X \times I \times (\mathbb{C}^{m_0} \oplus \mathbb{C}^{m_1})$$

by

$$\Phi(e, t) = (p(e), t, (1 - t)\varphi_0(e) \oplus t\varphi_1(e)).$$

According to Corollary 1.34 $\mathrm{Im}\,\Phi$ is an embedded bundle over $X \times I$ which is $\mathrm{Im}\, j_0 \circ \varphi_0$ over $X \times 0$ and $\mathrm{Im}\, j_1 \circ \varphi_1$ over $X \times 1$, where

$$j_0 \colon \mathbb{C}^{m_0} \to \mathbb{C}^{m_0} \oplus \mathbb{C}^{m_1} \quad \text{and} \quad j_1 \colon \mathbb{C}^{m_1} \to \mathbb{C}^{m_0} \oplus \mathbb{C}^{m_1}.$$

Hence there is a homotopy

$$\pi_t \colon X \to \mathcal{P}(\mathbb{C}^{m_0} \oplus \mathbb{C}^{m_1})$$

such that π_0 and π_1 define $\mathrm{Im}\, j_0 \circ \varphi_0$ and $\mathrm{Im}\, j_1 \circ \varphi_1$ respectively. So if

$$f_0 \colon X \to G_n(\mathbb{C}^{m_0}) \quad \text{and} \quad f_1 \colon X \to G_n(\mathbb{C}^{m_1})$$

are defined by means of $\mathrm{Im}\,\varphi_0$ and $\mathrm{Im}\,\varphi_1$ respectively, then $j_0 f_0$ and $j_1 f_1$ are homotopic, where

$$j_0 \colon G_n(\mathbb{C}^{m_0}) \to G_n(\mathbb{C}^{m_0} \oplus \mathbb{C}^{m_1}),$$
$$j_1 \colon G_n(\mathbb{C}^{m_1}) \to G_n(\mathbb{C}^{m_0} \oplus \mathbb{C}^{m_1}).$$

Now identifying $\mathbb{C}^{m_0} \oplus \mathbb{C}^{m_1}$ with $\mathbb{C}^{m_0+m_1}$, $j_0 = i_{m_0+m_1,m_0}$ whereas $j_1 = \bar{T} \circ i_{m_0+m_1,m_1}$, where $\bar{T}$ is induced by

$$(z_1, \ldots, z_{m_1}, z_{m_1+1}, \ldots z_{m_0+m_1}) \to (z_{m_1+1}, \ldots z_{m_0+m_1}, z_1, \ldots z_{m_1}).$$

That is

$$T \colon \mathbb{C}^{m_0+m_1} \to \mathbb{C}^{m_0+m_1}$$

and if $V \in G_n(\mathbb{C}^{m_0+m_1})$ then $\bar{T}V \in G_n(\mathbb{C}^{m_0+m_1})$ is the image TV of T restricted to V. Now $T \in U(m_0 + m_1)$ and this is an arc-connected space. Hence there is an arc $T_t,\ t \in [0,1]$, such that $T_t \in U(m_0+m_1)$. If we can show that the corresponding family $\bar{T}_t$ is continuous, then $\bar{T} \sim \mathrm{Id}$ and thus

$$i_{m_0+m_1,m_0} \circ f_0 = j_0 \circ f_0 \sim j_1 \circ f_1 = \bar{T} \circ i_{m_0+m_1,m_1} \circ f_1 \sim i_{m_0+m_1,m_1} \circ f_1$$

Hence f_0 and f_1 become homotopic in the limit and our inverse map is well defined.

Continuity of the map

$$G_n(\mathbb{C}^{m_0+m_1}) \times I \to G_n(\mathbb{C}^{m_0+m_1})$$

sending (V,t) to $\bar{T}_t(V) = T_t(V)$: Put $m = m_0 + m_1$ and identify $G_n(\mathbb{C}^m)$ with the space $\mathcal{P}_n^{\perp}(\mathbb{C}^m)$ of orthogonal projections with n-dimensional image. $\bar{T}$ is under this identification the map

$$\bar{T} \colon \mathcal{P}_n^{\perp}(\mathbb{C}^m) \to \mathcal{P}_n^{\perp}(\mathbb{C}^m) \quad \text{defined by} \quad \bar{T}(\pi) = T\pi T^{-1}$$

The map $\mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^m) \times U(m) \to \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^m)$ taking (A,T) to TAT^{-1} is clearly continuous. Hence the map

$$\mathcal{P}_n^{\perp}(\mathbb{C}^m) \times T \to \mathcal{P}_n^{\perp}(\mathbb{C}^m) \times U(m) \to \mathcal{P}_n^{\perp}(\mathbb{C}^m)$$

taking (π, t) first to (π, T_t) and then to $T_t \pi T_t^{-1}$, is clearly continuous. □

Chapter 2

K-theory

2.1 The Grothendieck Construction

Vect(X) is a commutative semigroup under Whitney sum for every space X.

In this section we shall associate in a universal fashion a commutative group to any such semigroup S. The result is called the *Grothendieck group* of S and is denoted $\mathcal{G}(S)$.

More precisely $\mathcal{G}$ is a functor from the category of commutative semigroups to the category of Abelian groups, such that for any S there is a natural isomorphism

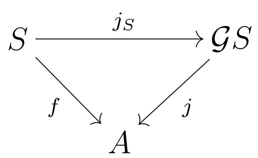
$$j_S \colon S \to \mathcal{G}(S)$$

satisfying the following universal property:

If $f : S \to A$ is a homomorphism into an Abelian group A, then there exists a unique homomorphism

$$j \colon \mathcal{G}(S) \to A$$

such that the diagram



commutes.

The actual construction of $\mathcal{G}(S)$ is similar to the construction of the integers out of the natural numbers:

In $S \times S$ we define the equivalence relation:

$$\begin{aligned}(s, s') \sim (t, t') \iff & \text{there exists } \hat{s} \text{ and } \hat{t} \text{ such that} \\ & (s, s') + (\hat{s}, \hat{s}) = (t, t') + (\hat{t}, \hat{t}) \\ & \text{i.e.} \quad s + \hat{s} = t + \hat{t} \\ & \text{and} \quad s' + \hat{s} = t' + \hat{t}.\end{aligned}$$

$\sim$ is clearly an equivalence relation in $S \times S$, and the quotient $\mathcal{G}(S)$ is clearly an Abelian semigroup. $(\hat{s}, \hat{s})$ where $\hat{s}$ is arbitrary, represents the zero element in $\mathcal{G}(S)$, and the inverse of $(s, s^{'})$ is $(s^{'}, s)$, so $\mathcal{G}(S)$ is actually a group.

If $s \in S$ we define $j_S(s)$ as the element represented by $(s + \hat{s}, \hat{s})$ for $\hat{s}$ arbitrary. $j_S(s)$ is also denoted $[s]$. Clearly $(s, s^{'})$ represents $[s] - \left[s^{'}\right]$ in $\mathcal{G}(S)$.

If S has a product which is distributive over the addition, then $\mathcal{G}(S)$ has the structure of a ring. This is seen either by constructing the multiplication directly, or by using the universal property of $\mathcal{G}(S)$.

Definition 2.1. If X is any compact space we define

$$K(X) = \mathcal{G}(\mathrm{Vect}(X)).$$

$K(X)$ is a commutative ring, where the product is inherited from the tensor product of vector bundles.

Every element in $K(X)$ is of the form $[E] - [F]$, where $E, F \in \mathrm{Vect}\, X$.

The vector bundle F is isomorphic to an embedded vector bundle, so according to the remark following Corollary 1.36, there exists a vector bundle $F^{\perp}$ such that $F \oplus F^{\perp}$ is trivial. Denoting the trivial bundle of dimension n by $\underline{n}$ we have $F \oplus F^{\perp} = \underline{n}$ for some n and

$$[E] - [F] = \left[E \oplus F^{\perp}\right] - \left[F \oplus F^{\perp}\right] = \left[E \oplus F^{\perp}\right] - [\underline{n}] .$$

Thus every element of $K(X)$ is of the form $[G] - [\underline{n}]$ where $G \in \mathrm{Vect}(X)$. Two bundles E and F are said to be *stably equivalent* if there are natural numbers n and m such that

$$E \oplus \underline{n} \cong F \oplus \underline{m}.$$

This implies that $[E] - [F] = [\underline{m}] - [\underline{n}]$. On the other hand assume that $[E] - [F]$ is an integer mutiple of $[1]$. Without loss of generality

$$[E] = [F] + [\underline{m}] = [F \oplus \underline{m}] .$$

Then there is a bundle G such that

$$E \oplus G \cong F \oplus \underline{m} \oplus G$$
$$E \oplus G \oplus G^{\perp} \cong F \oplus \underline{m} \oplus G \oplus G^{\perp}$$

where $G \oplus G^{\perp} \cong \underline{n}$. Thus $E \oplus \underline{n} \cong F \oplus \underline{m} \oplus \underline{n} = F \oplus \underline{m+n}$. Especially two bundles E and F of the same dimension are stable equivalent iff $[E] = [F]$.

The exterior power operations λ^i give also rise to operations in $K(X)$. This is seen as follows:

Let $K(X)\,[[t]]$ denote the ring of power series in t with coefficients in $K(X)$. If E is a vector bundle over X, we define $\lambda_t(E) \in K(X)[[t]]$ to be the power series

$$\sum_{i=0}^{\infty} \left[\lambda^i(E)\right] t^i.$$

According to Section 1.2 there is an isomorphism

$$\lambda^k(E \oplus F) \cong \coprod_{i+j=k} \lambda^i(E) \otimes \lambda^j(F)$$

for any vector bundles E, F over X. This can be expressed as a relation between power series:

$$\lambda_t(E \oplus F) = \lambda_t(E)\lambda_t(F).$$

The power series $\lambda_t(E)$ is invertible in $K(X)\,[[t]]$, because it has constant leading term 1. Thus we have a homomorphism

$$\lambda_t\colon \operatorname{Vect}(X) \to 1 + tK(X)\,[[t]]^+$$

of the additive semigroup $\operatorname{Vect}(X)$ into the multiplicative group of power series over $K(X)$ with constant term 1. By the universal property of $K(X)$ this extends uniquely to a homomorphism

$$\lambda^i\colon K(X) \to 1 + tK(X)\,[[t]]^+\,.$$

Taking the coefficient of t^i we have

$$\lambda^i\colon K(X) \to K(X).$$

For any $x, y \in K(X)$ we have

$$\lambda^k(x+y) = \sum_{i+j=k} \lambda^i(x)\lambda^j(y)$$

and λ_t is explicitly defined by

$$\lambda^k([E]-[F]) = \lambda_t(E)(\lambda_t(F))^{-1}.$$

At last we remark that if $f\colon X \to Y$ is a continuous function, then $f^*\colon \operatorname{Vect}(Y) \to \operatorname{Vect}(X)$ extend in a unique way to $f^*\colon K(Y) \to K(X)$. f^* is a ring homomorphism and commutes with the operations λ^i. By Theorem 1.50 this homomorphism depends only on the homotopy class of f.

2.2 Elementary Algebra of Modules

Let R be a commutative ring with a unit 1. An *R-module* is an Abelian group A, with a product

$$R \times A \to A$$

satisfying

$$r(a+a') = ra + ra', \qquad (r+r')a = ra + r'a,$$
$$r(r'a) = (rr')a, \qquad 1a = a,$$

where $r, r' \in R$ and $a, a' \in A$.

A map $f\colon A \to B$, where A and B are R-modules is said to be an R-homomorphism if

$$f(a + a') = f(a) + f(a') \quad \text{and} \quad f(ra) = rf(a)$$

for all $a, a' \in A$ and $r \in R$.

The kernel

$$\ker f = f^{-1}(0)$$

is a module. If C is a submodule of A, then A/C is again a module. The image $\operatorname{Im} f$ of f is a submodule of B and

$$\operatorname{Coker} f = B/\operatorname{Im} f$$

is called the cokernel of f. As usual we have a natural isomorphism

$$\operatorname{Im} f \cong A/\ker f.$$

Note that if $f\colon A \to B$ is a module homomorphism, $C \subseteq A$, $D \subseteq B$ submodules satisfying $f(C) \subseteq D$, then there is a homomorphism $\bar{f}\colon A/C \to B/D$, such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A/C & \xrightarrow{\bar{f}} & B/D \end{array}$$

commutes.

Let us consider a sequence of modules and homomorphisms (not necessarily infinite)

$$\cdots \longrightarrow A_{q+1} \xrightarrow{f_{q+1}} A_q \xrightarrow{f_q} A_{q-1} \longrightarrow \cdots$$

The sequence is said to be *exact* at A_q when $\operatorname{Im} f_{q+1} = \ker f_q$. The sequence is said to be exact if it is exact at every of its modules.

Example 2.2.

$0 \longrightarrow A \xrightarrow{i} B$	is exact $\iff$	i is monomorphic.
$A \xrightarrow{j} B \longrightarrow 0$	is exact $\iff$	j is epimorphic.
$0 \longrightarrow A \longrightarrow 0$	is exact $\iff$	$A = 0$.
$0 \longrightarrow A \xrightarrow{i} B \longrightarrow 0$	is exact $\iff$	i is an isomorphism.
$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{f} C$	is exact $\iff$	i defines an isomorphism of A onto $\ker f$.
$B \xrightarrow{f} C \xrightarrow{j} D \longrightarrow 0$	is exact $\iff$	j defines an isomorphism of $\operatorname{Coker} f$ onto D.

$$0 \longrightarrow A \xrightarrow{\;i\;} B \xrightarrow{\;j\;} C \longrightarrow 0 \quad \text{is exact} \iff$$ i is an ismorphism onto $\ker j$ and j induces an isomorphism $B/iA \cong C$

B is called an extension of A by C, and the sequence is said to be *short exact.*

Lemma 2.3. *If a module homomorphism $p_2 \colon B \to A_2$ has a right inverse, i.e. a homomorphism $i_2 \colon A_2 \to B$ such that $p_2 i_2 = \mathrm{Id}_{A_2}$, then*

$$B = A_1 \oplus i_2 A_2,$$

where $A_1 = \ker p_2$.

i_2 is called a *splitting* of p_2

Proposition 2.4. *The following properties of a short exact sequence*

$$0 \longrightarrow A_1 \xrightarrow{\;i_1\;} B \xrightarrow{\;p_2\;} A_2 \longrightarrow 0$$

are equivalent:

1. *p_2 has a right inverse $i_2 \colon A_2 \to B$ with $p_2 \circ i_2 = 1$.*
2. *i_1 has a left inverse $p_1 \colon B \to A_1$ with $p_1 \circ i_1 = 1$.*
3. *There is a commutative diagram:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{\;i_1\;} & B & \xrightarrow{\;p_2\;} & A_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\cong} & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & A_1 \oplus A_2 & \longrightarrow & A_2 & \longrightarrow & 0
\end{array}$$

A short exact sequence with one of these properties is said to be *split exact.*

Lemma 2.5 (Five lemma). *Consider the commutative diagram*

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
\downarrow{f_1} & & \downarrow{f_2} & & \downarrow{f_3} & & \downarrow{f_4} & & \downarrow{f_5} \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}$$

with exact rows. If f_1, f_2, f_4, f_5 are isomorphisms, then f_3 is also an isomorphism.

Lemma 2.6 (Nine lemma). *Consider the commutative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_{11} & \longrightarrow & B_{12} & \longrightarrow & B_{13} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_{21} & \longrightarrow & B_{22} & \longrightarrow & B_{23} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_{31} & \longrightarrow & B_{32} & \longrightarrow & B_{33} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact columns and exact middle row. The top row is exact iff the bottom row is exact.

If $\{A_i\}_{i\in I}$ is a collection of R-modules, then the product $\prod_{i\in I} A_i$ is again an R-module.

$$\coprod_{i\in I} A_i$$

is the submodule constisting of sequences in which all but a finite number of elements are 0.

We also have the tensor product $A \otimes_R B$ of R-modules. It is constructed as the tensor product of vector spaces and satisfies a similar universal property:

$$\psi\colon A\times B \to A\otimes B \quad \text{defined by} \quad \psi(a,b) = a\otimes b$$

is bilinear, and for every bilinear $\varphi\colon A\times B\to C$, C an R-module, there exists a unique $\bar{\varphi}\colon A\otimes B\to C$ such that the diagram

$$\begin{array}{ccc}
A\times B & \longrightarrow & A\otimes B \\
 & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\bar{\varphi}} & \\
 & C &
\end{array}$$

commutes.

Lemma 2.7.

1. *If $\left\{A_i\right\}_{i\in I}$ is a collection of R-modules, then*

$$\Big(\coprod_{i\in I} A_i\Big)\otimes B \cong \coprod_{i\in I}(A_i\otimes B)$$

2. $R\otimes B\cong B$

3. If F is a free R-module, i.e.

$$F = \coprod_{i \in I} R a_i,$$

and $A \to B \to C$ is an exact sequence, then

$$F \otimes A \longrightarrow F \otimes B \longrightarrow F \otimes C$$

is exact.

Remark 2.8. Tensoring with a module is not always exact, for example let $R = \mathbb{Z}$ and $\mathbb{Z}_2 \to \mathbb{Z}_4$ the natural inclusion. $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2$ but the image of $i \otimes 1$ in $\mathbb{Z}_4 \otimes \mathbb{Z}_2$ is zero.

Nevertheless tensoring with an arbitrary module is right exact.

Lemma 2.9. *If $A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ is exact, then for an arbitrary module M:*

$$A \otimes M \xrightarrow{i \otimes 1} B \otimes M \xrightarrow{j \otimes 1} C \otimes M \longrightarrow 0$$

is exact.

Proof. Let L be the cokernel of $i \otimes 1$ and $l\colon B \otimes M \to L$ the natural map. $(j \otimes 1) \circ (i \otimes 1) = 0$, so there exist $f\colon L \to C \otimes M$ such that $f \circ l = j \otimes 1$. The map $g\colon B \times M \to B \otimes M \to L$ contains $iA \times M$ in its kernel. Hence there exists a map $\bar{g}\colon C \times M \to L$ such that $\bar{g}(jb, m) = l(b \otimes m)$, and thus this extends to $h\colon C \otimes M \to L$. f and h are easily seen to be each others inverse. □

2.3 Cohomology Theory Properties of K

Let $\mathcal{C}$ denote the category of compact spaces and $\mathcal{C}^+$ the category of compact spaces with a distinguished point, i.e. an object of $\mathcal{C}^+$ consists of a compact space X together with a point $x_0 \in X$, and a map

$$f\colon (X, x_0) \to (Y, y_0)$$

is a continuous function $f\colon X \to Y$ taking x_0 to y_0. In $\mathcal{C}^+$ we define homotopy of maps in an obvious way and the set of homotopy classes of base point preserving maps from X to Y is also denoted $[X, Y]$. Occasionally all the base points are identified to a single one denoted $+$.

$\mathcal{C}^2$ denotes the category of pairs of compact spaces, i.e. an object is a pair (X, A) with X and A compact and satisfying $A \subseteq X$. A map

$$f\colon (X, A) \to (Y, B)$$

is a continuous function $f\colon X \to Y$ satisfying $f(A) \subseteq B$. Again we can define homotopy in this category.

We include $\mathcal{C}$ in $\mathcal{C}^2$ by sending X into $(X, \emptyset)$. there is a functor

$$/\colon \mathcal{C}^2 \to \mathcal{C}^+$$

sending (X, A) into X/A, the base point being the point A. Here $X/\emptyset$ is understood to be the disjoint union of X with a point $+$. We also denote $X/\emptyset$ by X^+.

Remark 2.10. The functor $+\colon \mathcal{C} \to \mathcal{C}^+$ can be extended over the category of *locally compact* spaces and *proper* maps. (A map $f\colon X \to Y$ of locally compact spaces is proper if $f^{-1}(C)$ is compact for any compact $C \subseteq Y$).

For any locally compact X (not necessarily non-compact) X^+ denotes the one-point compactification of X with the base point $+ = \infty$.

Note that $X/A = (X - A)^+$ for any $(X, A) \in \mathcal{C}^2$. Actually there is an obvious map from the right to the left, which is clearly continuous and therefore a homeomorphism.

In $\mathcal{C}^+$ we have the *"smash product"*: If $X, Y \in \mathcal{C}^+$ we put

$$X \wedge Y = X \times Y/X \vee Y$$

where $X \vee Y = X \times (+) \cup (+) \times Y$.

Exercise 2.11. For any three spaces $X, Y, Z \in \mathcal{C}^+$ we have the natural homeomorphisms

$$X \wedge (Y \wedge Z) \cong X \wedge Y \wedge Z \cong (X \wedge Y) \wedge Z.$$

Here $X \wedge Y \wedge Z$ is $X \times Y \times Z$ with (x, y, z) identified with the base point if either x or y or z is the base point.

In $\mathcal{C}^2$ we have the product

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$$

and clearly this corresponds to $\wedge$ under the functor $/$:

$$X/A \wedge Y/B \cong X \times Y/X \times B \cup A \times Y.$$

In fact the the natural map

$$X \times Y \longrightarrow X/A \times Y/B \longrightarrow X/A \wedge Y/B$$

induces the homeomorphisms.

If especially $A = B = \emptyset$ this equation reads

$$X^+ \wedge Y^+ \cong (X \times Y)^+.$$

Exercise 2.12. If X and Y are locally compact, then $X \times Y$ is locally compact and

$$X^+ \wedge Y^+ \cong (X \times Y)^+$$

In the Euclidean n-space $\mathbb{R}^n$ we consider the subspaces

$$\begin{aligned} B^n &= \{\underline{x} = (x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_n^2 \leq 1\} \\ S^{n-1} &= \{\underline{x} = (x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_n^2 = 1\}. \end{aligned}$$

It is well known that

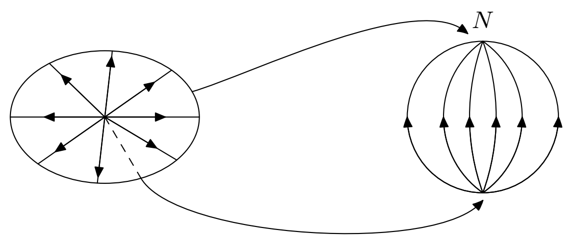
$$B^n / S^{n-1} \cong (\mathbb{R}^n)^+ \cong S^n.$$

Explicitly define $(B^n, S^{n-1}) \to ((\mathbb{R}^n)^+, \infty)$ by sending x to

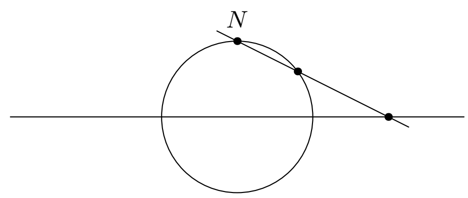
$$\frac{x}{\sqrt{1 - |x|^2}} \in \mathbb{R}^n \cup \{\infty\}.$$

Also define (denoting the north pole N) a map $(B^n, S^{n-1}) \to (S^n, N)$, by sending x to

$$(2x\sqrt{1 - |x|^2}) \oplus (2|x|^2 - 1) \in \mathbb{R}^n \oplus \mathbb{R}.$$



The identification $(\mathbb{R}^n)^+ \cong S^n$ is called *stereographic projection.*



Taking the north pole as the base point in S^n we have

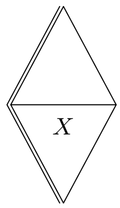
$$S^n \wedge S^m \cong (\mathbb{R}^n)^+ \wedge (\mathbb{R}^m)^+ \cong (\mathbb{R}^n \times \mathbb{R}^m)^+ \cong (\mathbb{R}^{n+m})^+ \cong S^{n+m}.$$

Especially

$$S^n \cong \underbrace{S^1 \wedge S^1 \wedge \cdots \wedge S^1}_{n \text{ factors}}.$$

For $X \in \mathcal{C}^+$ the space $X \wedge S^1$ is called the *reduced suspension* of X. From the above considerations it follows that the n-th iterated suspension $SS \dots SX$(n times) is naturally homeomorphic to $X \wedge S^n$. It is briefly written $S^n X$.

Note that SX for $X \in \mathcal{C}^+$ differs from the suspension defined by Definition 1.55. Actually the reduced suspension is the suspension with $(+) \times I$ identified to a point.



Because I is contractible K agrees on the two suspensions according to Corollary 1.52, so the use of SX for both of them leads to no problems.

Remark 2.13. For any locally compact space X (including the compact ones)

$$S^n(X^+) \cong (X \times \mathbb{R}^n)^+.$$

For a compact pair (X, A)

$$S^n(X/A) \cong X \times B^n / X \times S^{n-1} \cup A \times B^n.$$

Definition 2.14. If $X \in \mathcal{C}^+$, $i\colon + \to X$ the inclusion of the base point, define

$$\tilde{K}(X) = \ker[\, K(X) \xrightarrow{i^*} K(+)\,].$$

If X is locally compact define

$$K(X) = \tilde{K}(X^+).$$

If $(X, A) \in \mathcal{C}^2$ define $K(X, A) = \tilde{K}(X/A)$.

For any $X \in \mathcal{C}^+$ the collapsing map $c\colon X \to +$ induces a splitting of i^*, so clearly

$$K(X) = \tilde{K}(X) \oplus K(+) = \tilde{K}(X) \oplus \mathbb{Z}.$$

K is clearly a functor on $\mathcal{C}^+$. Also the definition $K(X) = \tilde{K}(X^+)$ agrees on compact spaces with the original $K(X)$, and K is a functor on the category of locally compact spaces and proper maps. Finally note that for $X \in \mathcal{C}$ $K(X) = K(X, \emptyset)$ and $K(-,-)$ is a functor on $\mathcal{C}^2$.

Definition 2.15. For $n \geq 0$

$$\begin{aligned}
\tilde{K}^{-n}(X) &= \tilde{K}(S^n X) && \text{for } X \in \mathcal{C}^+,\\
K^{-n}(X, A) &= \tilde{K}^{-n}(X/A) \\
&\cong K(X \times B^n, X \times S^{n-1} \cup A \times B^n) && \text{for } (X, A) \in \mathcal{C}^2,\\
K^{-n}(X) &= \tilde{K}^{-n}(X^+) \cong K(X \times \mathbb{R}^n) && \text{for } X \text{ locally compact.}
\end{aligned}$$

Clearly these definitions agree on their common domains. They all give functors on the appropriate categories. Note that $\tilde{K}(X)$ for $X \in \mathcal{C}^+$ is the set of equivalence classes of stably equivalent vector bundles over X.

Lemma 2.16. *For $(X, A) \in \mathcal{C}^2$ we have the exact sequence*

$$K(X, A) \xrightarrow{j^*} K(X) \xrightarrow{i^*} K(A),$$

where $i\colon A \to X$ and $j\colon (X, \emptyset) \to (X, A)$ are inclusions. Moreover, if A is contractible then

$$j^*\colon K(X, A) \to \tilde{K}(X)$$

is an isomorphism.

Proof. i^*j^* is induced by $j \circ i\colon (A, \emptyset) \to (X, A)$ and so factors through the zero group $K(A, A)$. Thus $i^*j^* = 0$.

Suppose now that $x \in K(X)$ is in the kernel of i^*. $x = [E] - [\underline{n}]$, where E is a vector bundle over X. From $i^*x = 0$ it follows that $\left[E_{|A}\right] = [\underline{n}]$ in $K(A)$, so for some integer m we have

$$(E \oplus m)_{|A} \cong \underline{n} \oplus \underline{m},$$

i.e. we have a trivalization of $E \oplus \underline{m}_{|A}$. This defines a bundle $E \oplus \underline{m}/\alpha$ on X/A. Put $y = [E \oplus \underline{m}/\alpha] - [\underline{n} \oplus \underline{m}] \in \tilde{K}(X/A)$. Then $j^*y = [E \oplus \underline{m}] - [\underline{n} \oplus \underline{m}] = x$. The last statement follows immediately from Corollary 1.52. ☐

Corollary 2.17. *If $(X, A) \in \mathcal{C}^2$ and $A \in \mathcal{C}^+$ (and so $X \in \mathcal{C}^+$ with the same base point) then the sequence*

$$K(X, A) \longrightarrow \tilde{K}(X) \longrightarrow \tilde{K}(A)$$

is exact.

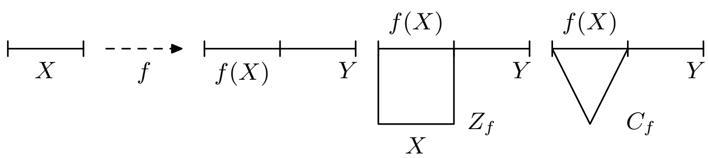
Definition 2.18.

(1) For $X \in \mathcal{C}^+$ is $CX = X \times B^1/X \times (-1) \cup (+) \times B^1$ called the cone on X

(2) If $f\colon X \to Y$ is a map in $\mathcal{C}^+$ then Z_f is the disjoint union $(X \times B^1) + Y$ with the identifications:
$(x, 1) \in B^1$ is identified with $f(x) \in Y$, and $(+) \times B^1$ is identified to the base point.

(3) If $f\colon X \to Y$ is the map in $\mathcal{C}^+$ then

$$C_f = Z_f/X \times (-1).$$



Corollary 2.19. *If $a(f)$ denotes the inclusion of Y in C_f, then the sequence*

$$\tilde{K}(C_f) \xrightarrow{a(f)^*} \tilde{K}(Y) \xrightarrow{f^*} \tilde{K}(X)$$

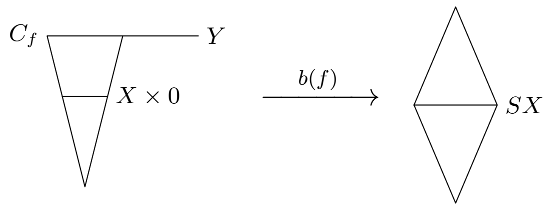
is exact.

Proof. Consider the pair $(Z_f, X \times (-1))$ and the inclusion $v\colon Y \to Z_f$. Using Corollary 2.17 we get the commutative diagram with exact row:

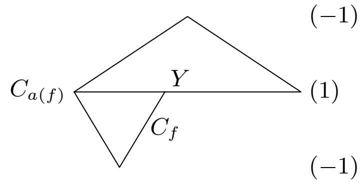
$$\begin{array}{ccccc} & & \tilde{K}(Y) & & \\ & \swarrow^{f^*} & \uparrow{\scriptstyle v^*} & \nwarrow^{a(f)^*} & \\ \tilde{K}(X) & \longleftarrow & \tilde{K}(Z_f) & \longleftarrow & \tilde{K}(C_f) \end{array}$$

Now Y is easily seen to be a deformation retract of Z_f (by pressing the cylinder onto $f(X)$) so v^* is an isomorphism. $\square$

We now want to use this on the map $a(f)\colon Y \to C_f$. First define the map $b(f)\colon C_f \to SX$ by collapsing Y.



The cone $C_{a(f)}$ is viewed as the disjoint union $CX + CY$ with the identifications: $(x, 1) \in CX$ is identified with $(f(x), 1) \in CY$.



Collapsing CY in $C_{a(f)}$ we get a map

$$r\colon C_{a(f)} \to SX.$$

We obtain the diagram

$$\begin{array}{ccc} C_f & & \\ {\scriptstyle b(f)}\downarrow & \searrow^{a(a(f))} & \\ SX & \xleftarrow{\ r\ } & C_{a(f)} \\ {\scriptstyle Sf}\downarrow & & \downarrow{\scriptstyle b(a(f))} \\ SY & \xleftarrow{\ l\ } & SY \end{array} \tag{2.1}$$

Here $l\colon SY \to SY$ is the homeomorphism induced by sending $(x, t) \mapsto (x, -t)$.

The upper triangle is clearly commutative.

Lemma 2.20. *The lower square in* (2.1) *is commutative up to homotopy.*

Proof. Define a homotopy $h_s\colon C_{a(f)} \to SY$ by the map $CX + CY \to SY$ sending

$$(x,t) \in CX \quad \text{to} \quad (f(x), (1-s)(t+1)-1)$$

and

$$(y,t) \in CY \quad \text{to} \quad (y, 1 - s(t+1)).$$

When $(x,t) \sim (f(x), 1)$ in $C_{a(f)}$ we have

$$(1-s)2 - 1 = 1 - 2s$$

so h_s is well defined.

$$h_0 = S(f) \circ r \quad \text{and} \quad h_1 = l \circ b(a(f))$$ □

Corollary 2.21. *Let $f\colon X \to Y$ be a map in $\mathcal{C}^+$. Then the sequence*

$$\tilde{K}(SY) \xrightarrow{Sf^*} \tilde{K}(SX) \xrightarrow{b(f)^*} \tilde{K}(C_f) \xrightarrow{a(f)^*} \tilde{K}(Y) \xrightarrow{f^*} \tilde{K}(X)$$

is exact.

Proof. By Corollary 2.19 this sequence is exact at $\tilde{K}(Y)$. Using Corollary 2.19 we get from the diagram (2.1) that

$$\begin{array}{ccccc} \tilde{K}(SX) & \xrightarrow{b(f)^*} & \tilde{K}(C_f) & \xrightarrow{a(f)^*} & \tilde{K}(Y) \\ \downarrow{\scriptstyle r^*} & \nearrow{\scriptstyle a(a(f))^*} & & & \\ \tilde{K}(C_{a(f)}) & & & & \end{array}$$

is commutative and the lower sequence is exact. r^* is also an isomorphism because CY is contractible, so the upper sequence is exact. Using this sequence on $a(f)$ and the lower square of (2.1) we get

$$\begin{array}{ccccc} \tilde{K}(SY) & \xrightarrow{(Sf)^*} & \tilde{K}(SX) & \xrightarrow{b(f)^*} & \tilde{K}(C_f) \\ {\scriptstyle l^*}\downarrow & & {\scriptstyle r^*}\downarrow & \nearrow{\scriptstyle a(a(f))^*} & \\ \tilde{K}(SY) & \xrightarrow[b(a(f))^*]{} & \tilde{K}(C_{a(f)}) & & \end{array}$$

with exact lower sequence and vertical isomorphisms. □

Remark 2.22. Corollary 2.21 is a formal consequence of Lemma 2.16, and thus this is true for any functor from the category of spaces with morphism homotopy classes of maps to Abelian groups, if only it satisfies Lemma 2.16. Such a functor is called a *half exact* homotopy functor.

Exercise 2.23.

(1) C_f is a functor in f, i.e. for any commutative diagram in $\mathcal{C}^+$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

there is a map $C_f \to C_{f'}$ such that there is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{a(f)} & C_f & \xrightarrow{b(f)} & SX \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[a(f')]{} & C_{f'} & \xrightarrow[b(f')]{} & SX' \end{array}$$

(2) For $f\colon X \to Y$ a map in $\mathcal{C}^+$ there is a natural homeomorphism

$$\eta_f\colon C_{Sf} \to SC_f$$

such that if $T\colon S^1 \wedge S^1 \to S^1 \wedge S^1$ permutes the factors, then there is a commutative diagram

$$\begin{array}{ccccc} SY & \xrightarrow{a(Sf)} & C_{Sf} & \xrightarrow{b(Sf)} & S^2X \\ \downarrow & & \downarrow{\scriptstyle \eta_f} & & \downarrow{\scriptstyle 1\wedge T} \\ SY & \xrightarrow[S(a(f))]{} & SC_f & \xrightarrow[s(b(f))]{} & S^2X \end{array}$$

η_f is natural in the sense that for any commutative diagram as in (1), there is a commutative diagram

$$\begin{array}{ccc} C_{Sf} & \xrightarrow{\eta_f} & SC_f \\ \downarrow & & \downarrow \\ C_{Sf'} & \xrightarrow{\eta_{f'}} & SC_{f'} \end{array}$$

Note that Corollary 2.21 can be used on Sf, S^2f etc., and hence we can extend the sequence infinitely to the left and in this way obtain the *Puppe sequence* of f and K.

Returning to a compact pair (X, A) with $A \in \mathcal{C}^+$, the cone on $i\colon A \to X$ contains the contractible subspaces CA. Hence if $q\colon C_i \to X/A$ is the collapsing map,

$$q^*\colon \tilde{K}(X/A) \to \tilde{K}(C_i)$$

is an isomorphism and we define $\delta = (q^*)^{-1} \circ b(i)^*\colon \tilde{K}(SA) \to \tilde{K}(X/A)$.

In this way Corollary 2.21 gives the exact sequence

$$\tilde{K}^{-1}(X) \xrightarrow{i^*} \tilde{K}^{-1}(A) \xrightarrow{\delta} K(X, A) \xrightarrow{j^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(A).$$

Substituting (S^nX, S^nA) in this sequence we get

Corollary 2.24. *For $(X, A) \in \mathcal{C}^2$ with $A \in \mathcal{C}^+$ there is a natural exact sequence (infinite to the left):*

$$\cdots \tilde{K}^{-2}(A) \xrightarrow{\delta} K^{-1}(X, A) \xrightarrow{j^*} \tilde{K}^{-1}(X) \xrightarrow{i^*} \tilde{K}^{-1}(A) \xrightarrow{\delta} K^0(X, A)$$

$$\xrightarrow{j^*} \tilde{K}^0(X) \xrightarrow{i^*} \tilde{K}^0(A)$$

The naturality is with respect to base points preserving maps of pairs. If $(X, A) \in \mathcal{C}^2$ the sequence for (X^+, A^+) is isomorphic to the sequence

$$\cdots K^{-2}(A) \xrightarrow{\delta} K^{-1}(X, A) \xrightarrow{j^*} K^{-1}(X) \xrightarrow{i^*} K^{-1}(A) \xrightarrow{\delta} K^0(X, A)$$

$$\xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(A)$$

This is natural in (X, A).

Corollary 2.25. *Let $X, Y \in \mathcal{C}^+$, then*

$$\tilde{K}^{-n}(X \vee Y) = \tilde{K}^{-n}(X) \oplus \tilde{K}^{-n}(Y) \quad n \geq 0.$$

Proof. Consider the exact sequence for the pair $(X \vee Y, Y)$ and the splittings induced by $X \vee Y \to Y$ and $X \to X \vee Y$:

$$\begin{array}{ccccc} K^{-n}(X \vee Y, Y) & \longrightarrow & \tilde{K}^{-n}(X \vee Y) & \rightleftarrows & \tilde{K}^{-n}(Y) \\ \downarrow \cong & \nearrow\!\!\swarrow & & & \\ \tilde{K}^{-n}(X) & & & & \end{array}$$

□

We conclude this section with reformulations of Theorem 1.56 and Theorem 1.60:

Theorem 2.26. *For any map $f\colon X \to \mathrm{GL}(n, \mathbb{C})$ let E_f denote the corresponding vector bundle over SX. Then $f \mapsto [E_f] - [\underline{n}]$ induces a group isomorphism F:*

$$\varinjlim_n [X, \mathrm{GL}(n, \mathbb{C})] \to \tilde{K}(SX) \cong K^{-1}(X)$$

where the group structure on the left is induced from that of $\mathrm{GL}(n, \mathbb{C})$.

Proof. The inclusions $\mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(n+1, \mathbb{C})$ defined by sending a matrix A into the matrix

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

make the sets $[X, \mathrm{GL}(n, \mathbb{C})]$ into a direct system. If $f\colon X \to \mathrm{GL}(n, \mathbb{C})$ and $f \oplus 1$ denotes the composite $X \to \mathrm{GL}(n+1, \mathbb{C})$, then by Propositon 1.53 and the construction in Theorem 1.56 $E_{f\oplus 1} \cong E_f \oplus \underline{1}$. Thus

$$[E_{f\oplus 1}] - [\underline{n+1}] = [E_f] - [\underline{n}]$$

and the map F is well defined. F is clearly onto by Theorem 1.56. Now we want to show that F is one to one: Let $f\colon X \to \mathrm{GL}(n)$ and $g\colon X \to \mathrm{GL}(m)$ satisfy

$$[E_f] - [\underline{n}] = [E_g] - [\underline{m}]$$
$$[E_f \oplus \underline{m}] = [E_g \oplus \underline{n}] \,.$$

Then there is a natural number s such that

$$E_f \oplus \underline{m} \oplus \underline{s} \cong E_g \oplus \underline{n} \oplus \underline{s} \quad \text{or} \quad E_{f\oplus(m+s)} \cong E_{g\oplus(n+s)}.$$

According to Proposition 1.56 $f \oplus (\underline{m} + \underline{s}) \sim g \oplus (\underline{n} + \underline{s})$ and thus f and g become homotopic in the limit.

Finally F is a homomorphism: Let $f, g\colon X \to \mathrm{GL}(n)$. $f \cdot g\colon X \to \mathrm{GL}(n)$ is the composite

$$X \xrightarrow{f\times g} \mathrm{GL}(n) \times \mathrm{GL}(n) \xrightarrow{m} \mathrm{GL}(n)$$

where $m(A, B) = AB$. On the other hand we have according to Proposition 1.53:

$$[E_f] - [\underline{n}] + [E_g] - [\underline{n}] = [E_{f\oplus g}] - [2\underline{n}]$$

where $f \oplus g$ is the composite

$$X \xrightarrow{f\times g} \mathrm{GL}(n) \times \mathrm{GL}(n) \xrightarrow{\rho_0} \mathrm{GL}(2n)$$

and ρ_0 is given by

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

The fact that F is a homomorphism thus follows from the homotopy connecting ρ_0 and the map ρ_1

$$(A, B) \mapsto \begin{pmatrix} AB & 0 \\ 0 & 1 \end{pmatrix}.$$

Both are maps $\mathrm{GL}(n) \times \mathrm{GL}(n) \to \mathrm{GL}(2n)$. This homotopy is given explicitly by

$$\rho_{2t/\pi}(A \times B) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

where $0 \le t \le \frac{\pi}{2}$. This gives the first isomorphism.

$$K^{-1}(X) = \tilde{K}(S(X^+)) \cong \tilde{K}(SX \vee S^1) \cong \tilde{K}(SX) \oplus \tilde{K}(S^1) = \tilde{K}(SX). \qquad \square$$

Remark 2.27. The isomorphisms in Theorem 2.26 are natural for maps in $\mathcal{C}$. If $X \in \mathcal{C}^+$ we get a similar theorem taking base point preserving homotopies.

Corollary 2.28. *Let $l\colon SX \to SX$ be given by $(x,t) \mapsto (x,-t)$. Then $l^*x = -x$ for $x \in \tilde{K}(SX)$.*

Proof. If $x \in \tilde{K}(SX)$ then $x = [E_f] - [\underline{n}]$, $f\colon X \to \mathrm{GL}(n,\mathbb{C})$. $-x = [E_{f^{-1}}] - [\underline{n}]$ according to Theorem 2.26. But clearly $E_{f^{-1}} = l^*E_f$ □

Corollary 2.29. *Let*

$$T_\sigma = S^nX \to S^nX$$

be the map induced by a permutation σ of the the n factors in

$$S^n = S^1 \wedge \cdots \wedge S^1.$$

Then $(T_\sigma)^*x = \mathrm{sgn}(\sigma)x$ for $x \in \tilde{K}(S^nX)$.

Proof. $T\colon S^nX \to S^nX$ corresponds under the homeomorphism $S^nX \cong (X \times \mathbb{R}^n)^+$ to the map which computes the coordinates in $\mathbb{R}^n$ by means of σ. Let $\bar{T}_\sigma \in \mathrm{GL}(n,\mathbb{R})$ be the corresponding permutation matrix.

If $\mathrm{sgn}\,\sigma = 1$ then $\bar{T}_\sigma$ can be joined by a curve in $\mathrm{GL}(n,\mathbb{R})$ to the identity and so $\bar{T}_\sigma$ is homotopic to the identity.

If $\mathrm{sgn}\,\sigma = -1$ then $\bar{T}_\sigma$ can be joined by a curve in $\mathrm{GL}(n,\mathbb{R})$ to the matrix

$$\begin{pmatrix} 1 & & \\ & \ddots & & \Large 0 \\ \Large 0 & & 1 & \\ & & & -1 \end{pmatrix}$$

In this case $\bar{T}_\sigma$ is homotopic to $l\colon S(S^{n-1}X) \to S(S^{n-1}X)$, and thus $T_\sigma^* = -\,\mathrm{Id}$. □

Theorem 2.30. *If $X \in \mathcal{C}^+$ is connected then the natural map*

$$\varinjlim_{n,m}\,[X, G_n(\mathbb{C}^m)] \to \tilde{K}(X)$$

induced by sending $f\colon X \to G_n(\mathbb{C}^m)$ into

$$[f^*E_n(\mathbb{C}^m)] - [\underline{n}]$$

is an isomorphism.

Here $\{G_n(\mathbb{C}^m)\}_{(n,m)\in\mathbb{Z}_+\times\mathbb{Z}_+}$ is a direct system in the following way: $\mathbb{Z}_+ \times \mathbb{Z}_+$ has the partial order

$$(n,m) \le (n',m'), \text{if} \quad n \le n' \quad \text{and} \quad m + n' \le m' + n.$$

If $(n,m) \leq (n',m')$ then $G_n(\mathbb{C}^m) \to G_{n'}(\mathbb{C}^{m'})$ is the map

$$G_n(\mathbb{C}^m) \to G_{n+(n'-n)}(\mathbb{C}^{n'-n} \oplus \mathbb{C}^m \oplus \mathbb{C}^{m'-(n'-n+m)})$$

sending $V \in G_n(\mathbb{C}^m)$ into $\mathbb{C}^{n'-n} \oplus V \oplus 0$. None of the sets $[X, G_n(\mathbb{C}^m)]$ are groups, but the direct limit is, the addition being induced by the map

$$G_{n_1}(\mathbb{C}^{m_1}) \times G_{n_2}(\mathbb{C}^{m_2}) \to G_{n_1+n_2}(\mathbb{C}^{m_1+m_2})$$

sending (V,W) into $V \oplus W \subseteq \mathbb{C}^{m_1} \oplus \mathbb{C}^{m_2} = \mathbb{C}^{m_1+m_2}$.

2.4 External multiplication

Definition 2.31. For $(X,A) \in \mathcal{C}^2$ put

$$K^{\#}(X,A) = \prod_{n=0}^{\infty} K^{-n}(X,A).$$

In this section we will make $K^{\#}(X)$ into a graded ring and $K^{\#}(X,A)$ into a graded module over $K^{\#}(X)$.

Lemma 2.32. *Let $X, Y \in \mathcal{C}^+$. Then*

$$\tilde{K}^{-n}(X \times Y) \cong \tilde{K}^{-n}(X \wedge Y) \oplus \tilde{K}^{-n}(X) \oplus \tilde{K}^{-n}(Y).$$

Proof. Let

$$\begin{aligned} i_1 &\colon X \to X \times (+) \subseteq X \times Y \\ i_2 &\colon Y \to (+) \times Y \subseteq X \times Y \\ p_1 &\colon X \times Y \to Y \\ p_2 &\colon X \times Y \to Y \\ j &\colon X \times Y \to X \wedge Y. \end{aligned}$$

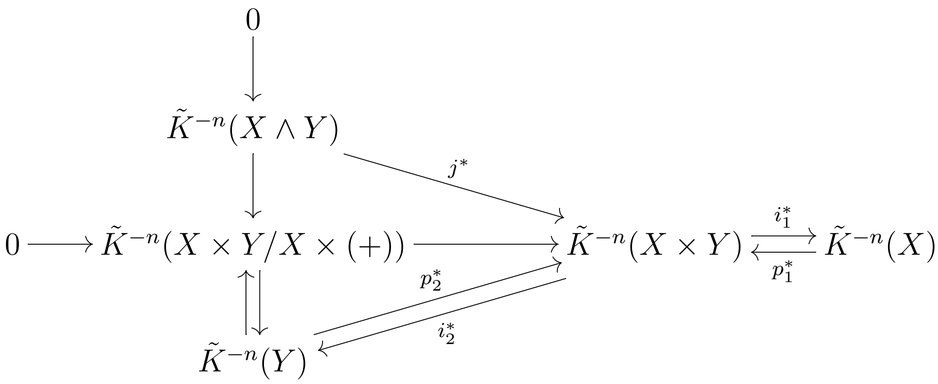
We have the exact sequence

$$\xrightarrow{\delta} \tilde{K}^{-n}(X \times Y / X \times (+)) \longrightarrow \tilde{K}^{-n}(X \times Y) \underset{p_1^*}{\overset{i_1^*}{\rightleftarrows}} \tilde{K}^{-n}(X)\,, \quad n \geq 0.$$

p_1^* is a splitting and i_1^* so i_1^* is epi, i.e. $\delta = 0$. Thus we have actually a short split exact sequence. In the same way we have the split exact sequence

$$0 \longrightarrow \tilde{K}^{-n}(X \wedge Y) \longrightarrow \tilde{K}^{-n}(X \times Y / X \times (+)) \longrightarrow \tilde{K}^{-n}(Y) \longrightarrow 0$$

Hence we have the commutative diagram



Hence $\tilde{K}^{-n}(X \times Y) = p_1^* \tilde{K}(X) \oplus p_2^* \tilde{K}^{-n}(Y) \oplus j^* \tilde{K}^{-n}(X \wedge Y)$. Note that $\tilde{K}^{-n}(X \wedge Y)$ is in the kernel of

$$i_1^* \oplus i_2^* \colon \tilde{K}^{-n}(X \times Y) \to \tilde{K}^{-n}(X) \oplus \tilde{K}^{-n}(Y). \qquad \square$$

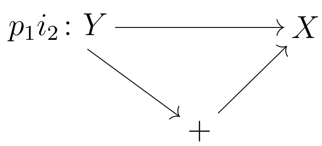
$K(X \times Y)$ is a ring, so we have a pairing

$$K(X) \otimes K(Y) \xrightarrow{p_1^* \cdot p_2^*} K(X \times Y) \tag{2.2}$$

Now, if $x \in \tilde{K}(X)$ and $y \in K(Y)$, then

$$i_2^*(p_1^* x p_2^* y) = ((p_1 i_2)^* x) y = 0$$

because



is commutative.

Hence, if $x \in \tilde{K}(X)$ and $y \in \tilde{K}(Y)$, then there is a unique element $x \cdot y \in \tilde{K}(X \wedge Y)$ such that $j^* x \cdot y = (p_1^* x)(p_2^* y)$. Thus the above pairing (2.2) induces a pairing

$$\tilde{K}(X) \otimes \tilde{K}(Y) \longrightarrow \tilde{K}(X \wedge Y)\,. \tag{2.3}$$

Using the map

$$S^n X \wedge S^m Y = X \wedge S^n \wedge Y \wedge S^m \to X \wedge Y \wedge S^n \wedge S^m$$

the above pairing (2.3) induces a pairing

$$\tilde{K}^{-n}(X) \otimes \tilde{K}^{-m}(Y) \longrightarrow \tilde{K}^{-(n+m)}(X \wedge Y)\,. \tag{2.4}$$

Substituting X/A and Y/B for (X, A), $(Y, B) \in \mathcal{C}^2$, we have

$$K^{-n}(X, A) \otimes K^{-m}(Y, B) \longrightarrow K^{-(n+m)}((X, A) \times (Y, B)) \,. \tag{2.5}$$

For X and Y locally compact we have

$$K^{-n}(X) \otimes K^{-m}(Y) \longrightarrow K^{-(n+m)}(X \times Y) \,.$$

Let $X \in \mathcal{C}$ and A and B be subspaces of X. Then the diagonal $\Delta\colon X \to X \times X$ induces

$$\Delta\colon (X, A \cup B) \to (X, A) \times (X, B).$$

Thus Δ^* induces a pairing

$$K^{-n}(X, A) \otimes K^{-m}(X, B) \longrightarrow K^{-(n+m)}((X, A) \times (X, B)) \xrightarrow{\Delta^*} K^{-n-m}(X, A \cup B). \tag{2.6}$$

Especially for $A = B = \emptyset$, $K^{\#}(X)$ is a ring and for $B = \emptyset$, $K^{\#}(X, A)$ is a right module over $K^{\#}(X)$.

Note that for X locally compact $\Delta\colon X \to X \times X$ is a proper map and thus induces a ring structure on $K^{\#}$.

Exercise 2.33.

(1) The multiplication in $K^{\#}$ for $X \in \mathcal{C}$ agrees with the original multiplication in $K^0(X)$.

(2) The pairing (2.5) is natural with respect to pairs of maps $(X, A) \to (X', A')$ and $(Y, B) \to (Y', B')$.

(3) Let $p_1\colon X \times X \to X$ and $p_2\colon X \times Y \to Y$. If $x \in K^{\#}(X)$ and $y \in K^{\#}(Y)$ then

$$x \cdot y = p_1^*(x) p_2^*(y) \in K^{\#}(X \times Y),$$

where the left hand side uses (2.5) and the right hand side uses (2.6).

(4) The pairing (2.6) is natural with respect to maps $X \to X'$ carrying A to A' and B to B'.

(5) For $X \in \mathcal{C}$ the original multiplication in $K^0(SX)$ is identically 0. (Hint: Use (4) on the identity map of SX carrying $+$ to C^+X and $+$ to C^-X).

(6) For X locally compact and $x \in K^{-n}(X)$ and $y \in K^{-m}(X)$,

$$x \cdot y = (-1)^{n \cdot m} y \cdot x \quad \text{in} \quad K^{-m}(X).$$

(Hint: Use Corollary 2.29).

In the next section we will see that for example

$$\tilde{K}^{-1}(S^1) \otimes \tilde{K}^{-1}(S^1) \to \tilde{K}^{-2}(S^2)$$

is different from zero, so Exercise 2.33. (5) shows that external multiplication

$$\tilde{K}^{-1}(X) \otimes \tilde{K}^{-1} \to \tilde{K}^{-2}(X)$$

is not to be mixed up with the internal multiplication in $\tilde{K}(SX)$.

We now want to show that the maps in the exact sequence for a pair (X, A) are right module maps over $K^{\#}(X)$. Here $K^{\#}(A)$ is a module over $K^{\#}(X)$ by

$$K^{-n}(A) \otimes K^{-m}(X) \longrightarrow K^{-n}(A) \otimes K^{-m}(A) \longrightarrow K^{-(n+m)}(A)\,.$$

Clearly $i^*\colon K^{\#} \to K^{\#}$ and $j^*\colon K^{\#}(X,A) \to K^{\#}(X)$ are $K^{\#}$-module maps by Exercise 2.33.(4). So it remains to prove it for

$$\delta\colon K^{-n-1}(A) \to K^{-n}(X,A).$$

By defintion $\delta = (q^*)^{-1}b(i)^*$ where

$$A \xrightarrow{\;i\;} X \longrightarrow C_i \xrightarrow{\;b_i\;} SA$$

and $q\colon C_i \to X/A$. More generally let us consider a map $f\colon X \to Y$ of spaces $X, Y \in \mathcal{C}^+$. It is not hard to see that if

$$f \wedge 1_Z\colon X \wedge Z \to Y \wedge Z$$

then $C_{f\wedge 1_Z} \cong C_f \wedge Z$. As a special case we get $C_{S^n f} \cong S^n C_f$. Putting $Z = Y$, there is a natural map

$$C_f \to C_{f\wedge 1_Y}$$

such that there is a commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{\;f\;} & Y & \longrightarrow & C_f & \xrightarrow{\;b(f)\;} & SX \\
\downarrow{\scriptstyle 1\wedge f} & & \downarrow{\scriptstyle \Delta} & & \downarrow & & \downarrow{\scriptstyle S(1\wedge f)} \\
X \wedge Y & \longrightarrow & Y \wedge Y & \longrightarrow & C_{f\wedge 1_Y} & \xrightarrow{\;b(f\wedge 1)\;} & S(X\wedge Y) \\
 & & & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle T} \\
 & & & \searrow{\scriptstyle a_{f\wedge 1}} & C_f \wedge Y & \xrightarrow{\;b(f)\wedge 1\;} & SX \wedge Y
\end{array} \tag{2.7}$$

Call the composite map $\bar{\Delta}\colon C_f \to C_f \wedge Y$. Then there is a pairing

$$\tilde{K}(C_f) \otimes \tilde{K}^{-m}(Y) \longrightarrow \tilde{K}^{-m}(C_f \wedge Y) \xrightarrow{\;\bar{\Delta}^*\;} \tilde{K}^{-m}(C_f) \tag{2.8}$$

Also there is a pairing

$$\tilde{K}(X) \otimes \tilde{K}^{-m}(Y) \longrightarrow \tilde{K}^{-m}(X) \otimes \tilde{K}^{-m}(X) \longrightarrow \tilde{K}^{-m}(X) \tag{2.9}$$

which is the same as

$$\tilde{K}(X) \otimes \tilde{K}^{-m}(Y) \longrightarrow \tilde{K}^{-m}(X \wedge Y) \xrightarrow{\;(1\wedge f)^*\;} \tilde{K}^{-m}(X)$$

by Exercise 2.33.(2).

Lemma 2.34. *For $x \in \tilde{K}^{-n-1}(X)$ and $y \in \tilde{K}^{-m}(Y)$*

$$b(f)^*(x \cdot y) = (b(f)^* x) \cdot y$$

in $\tilde{K}^{-n-m}(C_f)$.

Proof. This follows immediately from taking $\tilde{K}^{-n-m}$ on the diagram (2.7). □

Corollary 2.35. *If $(X, A) \in \mathcal{C}^2$ and X and A have a common base point, then*

$$\delta\colon \tilde{K}^{\#}(A) \to K^{\#}(X, A)$$

is a $\tilde{K}^{\#}(X)$-module map.

Proof. If $i\colon A \to X$ is the inclusion then the module structure of $K^{\#}(A)$ is clearly the pairing (2.9). From the commutative diagram

$$\begin{array}{ccc} C_i & \xrightarrow{\quad q \quad} & X/A \\ \Big\downarrow{\scriptstyle \bar{\Delta}} & & \Big\downarrow{\scriptstyle \Delta} \\ C_i \wedge X & \xrightarrow{q \wedge 1} & X/A \wedge X \end{array}$$

it follows that the pairing (2.6) corresponds to the pairing (2.8) and thus the statement follows directly from Lemma 2.34 because

$$\delta = (q^*)^{-1} b(i)^*.$$ □

Corollary 2.36. *If $(X, A) \in \mathcal{C}^2$ then*

$$\delta\colon K^{\#}(A) \to K^{\#}(X, A)$$

is a $K^{\#}(X)$-module map.

2.5 The periodicity theorem

Until now we have not proved the existence of any non-trivial bundle, so a priori $K(X)$ may be identically 0 and thus the whole theory would be quite uninteresting. The purpose of this section is to prove a general theorem which enables us to calculate for example $\tilde{K}(S^{2n})$ which is infinite cyclic.

We start with the important case S^2.

Lemma 2.37. *If L is a line bundle over X then $L \otimes L^*$ is trivial. Hence $[L]^{-1} = [L^*]$ in $K(X)$.*

Proof. The map sending $(e_X \otimes \varphi_X)$ into $\varphi_X(e_X) \in \mathbb{C}$ defines a trivialization. □

In Example 1.7.(2) of Section 1.1 we considered the line bundle H^* over $P(\mathbb{C}^2)$ the fibre of which over a point $\{z_1, z_2\} \in P(\mathbb{C}^2)$ is the complex line in $\mathbb{C}^2$ spanned by (z_1, z_2).

Put $H = (H^*)^*$. The map $P(\mathbb{C}^2) \to (\mathbb{C})^+$ taking

$$\{z_1, z_2\} \to \frac{z_1}{z_2} \quad (\text{and } \{1, 0\} \text{ to } \infty)$$

defines a homeomorphism.

$$(\mathbb{C})^+ \approx S^2 \approx S(S^1).$$

Consider S^1 as the unit circle in $\mathbb{C}$ and put

$$B_0 = \{z \in \mathbb{C} \mid |x| \leq 1\} \quad B_\infty = \{z \in (\mathbb{C})^+ \mid |x| \geq 1\}.$$

Under the homeomorphism to $S(S^1)$, B_0 corresponds to $C^-(S^1)$ and B_∞ to $C^+(S^1)$. Let z denote the map $z\colon S^1 \to \mathrm{GL}(1, \mathbb{C})$ which takes $\lambda \in S^1 \subseteq \mathbb{C}$ into the linear map which is multiplication with λ. $z^{-1}\colon S^1 \subseteq \mathrm{GL}(1, \mathbb{C})$ is the map which takes λ to multiplication with λ^{-1}.

Lemma 2.38. *Under the isomorphism from Theorem 2.26:*

$$\tilde{K}(P(\mathbb{C}^2)) \cong \tilde{K}(S(S^1)) \cong \lim \left[S^1, \mathrm{GL}(n, \mathbb{C})\right]$$

corresponds $[H] - [1]$ *to* z *and* $[H^*] - [1]$ *to* z^{-1}. *Futhermore* $([H] - [1])^2 = 0$ *in* $K(P(\mathbb{C}^2))$.

Proof. $H^*_{|B_0}$ has the trivialization

$$(z_1, z_2) \in H^*_{\{z_1, z_2\}} \quad \text{goes to} \quad (z_1/z_2, z_2) \in B_0 \times \mathbb{C}.$$

Analogously $H^*_{|B_\infty}$ has the trivialization

$$(z_1, z_2) \in H^*_{\{z_1, z_2\}} \quad \text{goes to} \quad (z_1/z_2, z_1) \in B_\infty \times \mathbb{C}.$$

Over $S^1 = B_0 \cap B_\infty$ the map

$$B_\infty \times \mathbb{C}_{|S^1} \to H^*_{S^1} \to B_0 \times \mathbb{C}_{|S^1}$$

sends $(z_1/z_2, z_1)$ to $(z_1/z_2, z_2)$. Thus putting $\lambda = z_1/z_2 \in S_1$

$$(\lambda, z_1) \quad \text{goes to} \quad (\lambda, \lambda^{-1} z_1).$$

Hence H^* has clutching function z^{-1}. According to Proposition 1.53 $H = (H^*)^*$ has clutching function z. This shows the first part of the lemma.

Now Lemma 2.37 shows that $[H]^{-1} - [1]$ corresponds to z^{-1} which according to Theorem 2.26 corresponds to $-([H] - [1])$. Hence $-([H] - [1]) = [H]^{-1} - [1]$ and hence $([H] - [1])^2 = 0$. □

Note that the last part of this lemma also follows from Exercise 2.33.(5) in Section 2.4.

Put

$$b = [H] - [1] \quad \text{in} \quad \tilde{K}((\mathbb{C})^+) = K(\mathbb{R}^2).$$

External (right-)multiplication by b defines a natural homomorphism

$$\beta\colon K(X) \to K^{-2}(X)$$

called the *Bott homomorphism.*

Theorem 2.39 (Periodicity Theorem). *For any compact (or locally compact) X*

$$\beta\colon K(X) \to K^{-2}(X)$$

is an isomorphism.

To prove the theorem we will construct a map $\alpha\colon K^{-2}(X) \to K(X)$ for all compact spaces (we will sometimes write α_X). This map satisfies the following axioms

(A1) α is a natural transformation

(A2) α is a (left) $K(X)$-module homomorphism

(A3) $\alpha(b) = 1, \alpha\colon K^{-2}(+) \to K(+)$.

We first show that such an α actually is an inverse of β, and then we will explicitly construct α.

Lemma 2.40. *Let α satisfy Axioms* (A1), (A2), (A3)*. Then α can be extended to a natural homomorphism*

$$\alpha\colon K^{-q-2}(X) \to K^{-q}(X)$$

which commutes with left multiplication by elements of $K^{-p}(X)$.

Proof. We first extend α to a locally compact X. From the exact sequence for $(X^+, +)$ and Axiom (A1) we get the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & K^{-2}(X) & \longrightarrow & K^{-2}(X^+) & \longrightarrow & K^{-2}(+) \\
& \downarrow & & \downarrow \alpha & & \downarrow \alpha \\
0 \longrightarrow & K(X) & \longrightarrow & K(X^+) & \longrightarrow & K(+)
\end{array}$$

Thus α induces a map $\alpha\colon K^{-2}(X) \to K(X)$. Replacing X by $X \times \mathbb{R}^q$ we then get a map

$$\alpha\colon K^{-q-2} \to K^{-q}(X)$$

which is clearly a natural transformation. For any compact X and Y we have the commutative diagram

$$\begin{array}{ccc} K(X)\otimes K^{-2}(Y) & \xrightarrow{\varphi} & K^{-2}(X\times Y) \\ \downarrow{\scriptstyle 1\otimes\alpha_Y} & & \downarrow{\scriptstyle \alpha_{X\times Y}} \\ K(X)\otimes K(Y) & \xrightarrow{\psi} & K(X\times Y) \end{array}$$

where φ and ψ are external multiplications. To see this consider

$$p_1\colon X\times Y\to X \quad p_2\colon X\times Y\to Y$$

From Axiom (A1)

$$\begin{aligned} p_2^*\alpha_Y(v) &= \alpha_{X\times Y}(p_2^*v) && v\in K^{-2}(Y)\\ \psi(u\otimes\alpha_Y(v)) &= p_1^*u\cdot p_2^*\alpha_Y(v) = p_1^*u\cdot\alpha_{X\times Y}(p_2^*v)\\ &= \alpha_{X\times Y}(p_1^*u\cdot p_2^*v) = \alpha_{X\times Y}\circ\varphi(u\otimes v), && u\in K(X) \end{aligned}$$

according to Axiom (A2) and Exercise 2.33.(3) in Section 2.4.

The commutativity of the corresponding diagram for locally compact X, Y follows now by passage to X^+, Y^+. Replacing X and Y by $X\times\mathbb{R}^p$ and $X\times\mathbb{R}^q$ and using the diagonal map we get a commutative diagram

$$\begin{array}{ccc} K^{-p}(X)\otimes K^{-q-2}(X) & \longrightarrow & K^{-p-q-2}(X) \\ \downarrow{\scriptstyle 1\otimes\alpha_X} & & \downarrow{\scriptstyle \alpha_X} \\ K^{-p}(X)\otimes K^{-q}(X) & \longrightarrow & K^{-p-q}(X) \end{array}$$

which proves that α commutes with left multiplication. □

Proposition 2.41. *Suppose there exist an α satisfying Axioms* (A1), (A2), (A3). *Then Theorem 2.39 holds and α is the inverse of β.*

Proof. It is enough to prove Theorem 2.39 for compact X. Then it clearly follows for locally compact X. Now Axioms (A1), (A2), (A3) imply

$$\alpha\circ\beta(x) = \alpha(x\cdot b) = x\alpha(b) = x \qquad \text{for } x\in K(X)$$

and

$$\beta\circ\alpha(x) = \alpha(x)\cdot b = b\alpha(x) = \alpha(bx) = \alpha(xb) = x\alpha(b) = x \quad \text{for } x\in K^{-2}(X).$$

Here we have used Exercise 2.33.(6) of Section 2.4. □

For the definition of α we will construct a map

$$I_X\colon K(X\times S^2)\to K(X)$$

which is natural and a left $K(X)$-module map. Futhermore I satisfies

$$I([H]) = 1 \quad \text{and} \quad I([1]) = 0.$$

Then α is the composite

$$K^{-2}(X) \longrightarrow K(X \times S^2) \xrightarrow{I_X} K(X)$$

In the sequel we shall integrate certain function on the unit circle with values in a complex vector space V of finite dimension. Thus if $f\colon S^1 \to V$ and $e_1, \ldots, e_n$ is a basis for V then

$$f(z) = \sum_i f_i(z) e_1 \quad \text{and} \quad \int_{S^1} f(z)dz = \sum_i \Big(\int_{S^1} f_i(z)dz \Big) e_i.$$

This integral is clearly independent of the choice of basis for V. We leave the proof of the following lemma as an exercise in analysis

Lemma 2.42. *Let V be a complex vector space of finite dimension and X a compact space. Then for any continuous map $f\colon X \times S^1 \to V$,*

$$F(x) = \int_{S^1} f(x,z)dz$$

is a continuous function of X into V.

Let X, V, f be as in Lemma 2.42. Put

$$\begin{aligned} a_n(x) &= \frac{1}{2\pi i} \int_{S^1} f(x,z) \frac{dz}{z^{k-1}}, \quad -\infty < k < \infty, \\ s_n(x,z) &= \sum_{k=-n}^{n} a_x(x) z^k, \\ f_n(x,z) &= \frac{1}{n} \sum_{i=0}^{n} s_i(x,z). \end{aligned}$$

The usual proof of Fejér's theorem extends to

Lemma 2.43. *Let $f\colon X \times S^1 \to V$ be a continuous map, X and V as in Lemma 2.42. Then the sequence $\{f_n\}$ converges to f uniformly on $X \times S^1$ for any chosen norm on V.*

Now let E be a vector bundle over $X \times S^2$. Let $\pi\colon X \times S^1 \to X$, $\pi_0\colon X \times B_0 \to X$ and $\pi_\infty\colon X \times B_\infty \to X$ be the projections. $X \times 0$ is a strong deformation retract of $X \times B_0$; thus π_0 is a homotopy equivalence and according to Proposition 1.42 there is an isomorphism

$$\alpha_0\colon E_{|X \times B_0} \to \pi_0^* E^0, \quad E^0 = E_{|X \times 0},$$

and α_0 is the identity over $X \times 0$. Analogously there is an isomorphism

$$\alpha_\infty \colon E_{|X\times B_\infty} \to \pi_\infty^* E^\infty, \quad E^\infty = E_{|X\times\infty},$$

and α_∞ is the identity over $X \times \infty$. Putting

$$f = \alpha_\infty \circ \alpha_0^{-1} \colon \pi^* E^0_{|X\times S^1} \to \pi^* E^\infty_{|X\times S^1}$$

we have by Proposition 1.53 $E \cong \pi_0^* E^0 \cup_f \pi_\infty^* E^\infty$. We denote for short this bundle by (E^0, f, E^∞).

Lemma 2.44. *The homotopy class of*

$$f \colon \pi^* E^0 \to \pi^* E^\infty$$

is uniquely determined by the isomorphism class of E.

Proof. We only need to show that the homotopy class of the isomorphism

$$\alpha_0 \colon E_{|X\times B_0} \to \pi_0^* E^0$$

is uniquely determined by E. Let β_0 be another such isomorphism. Then $\varphi = \beta_0^{-1} \circ \alpha_0$ is the identity over $X \times 0$ and by Corollary 1.47 $\varphi \sim \mathrm{Id}$. Hence $\beta_0 \sim \alpha_0$. □

Remark 2.45. By Lemma 2.38 $H \cong (1, z^{-1}, 1)$ (we have interchanged B_0 and B_∞). According to Proposition 1.53 $H^k \cong (1, z^{-k}, 1)$ for $-\infty \leq k \leq \infty$.

Definition 2.46. A *Laurent clutching function* is an isomorphism $f \colon \pi^* E^0 \to \pi^* E^\infty$ of the form

$$f_{x,z} = \sum_{k=-n}^{n} (a_k)_x z^k$$

where a_k is a homomorphism $a_k \colon E^0 \to E^\infty$ over X.

Lemma 2.47. *If f is any clutching function*

$$f \colon \pi^* E^0 \to \pi^* E^\infty.$$

Then there exist homomorphisms $a_k \colon E^0 \to E^\infty$ over X such that for fixed $x \in X$

$$(a_k)_x = \frac{1}{2\pi i} \int_{S^1} f_{(x,z)} \frac{dz}{z^{k-1}}$$

Proof. For fixed $x \in X$, $f_{(x,z)}$ is a continuous map of S^1 into $\mathrm{Hom}(E^0_x, E^\infty_x)$, which is a complex vector space of finite dimension. Thus we only have to show that a_k defines a continuous homomorphism. According to Proposition 1.26 we can assume E^0 and E^∞ to be embedded in a trivial bundle $X \times V$ and f is the restriction of an isomorphism

$$g \colon X \times S^1 \times V \to X \times S^1 \times V.$$

g defines

$$\bar{g}\colon X \times S^1 \to \mathrm{Hom}(V, V).$$

Hence according to Lemma 2.42

$$A_x = \frac{1}{2\pi i} \int_{S^1} \bar{g} \frac{dz}{z^{k-1}}$$

defines a homomorphism $X \times V \to X \times V$. If π_{1x} and π_{2x} are the projections onto E^0_x and E^∞_x respectively, then

$$(A_k)_x \pi_{1x} = \frac{1}{2\pi i} \int_{S^1} (g_{(x,z)} \circ \pi_{1x}) \frac{dz}{z^{k-1}} = \pi_{2x}(A_k)_x.$$

Hence A_k maps E^0 to E^∞ and the restrictions to E^0_x is clearly $(a_k)_x$. □

Again defining the Laurent series

$$s_n = \sum_{-n}^{n} a_k z^k \quad f_n = \frac{1}{n} \sum_{0}^{n} s_k$$

of homomorphisms $\pi^* E^0 \to \pi^* E^\infty$ we have

Lemma 2.48. *If f is any clutching function $\pi^* E^0 \to \pi^* E^\infty$ and $\{f_n\}$ is the sequence of Cesaro means of the Fourier series of f. Then for n sufficiently large f_n is an isomorphism homotopic to f. Thus $(E^0, f, E^\infty) \cong (E^0, f_n, E^\infty)$.*

Proof. According to the proof of Lemma 2.47 we can assume E^0 and E^∞ embedded in $X \times V$, f is the restriction of an isomorphism g and the Fourier coefficients a_k of f are restrictions of the Fourier coefficients A_k of g. Let $\{g_k\}$ denote the series of Cesaro means of g. Thus g_k restricted to E^0 is f_k. The corresponding sequence

$$\bar{g}_k\colon X \times S^1 \to \mathrm{Hom}(V, V)$$

converges uniformly to $\bar{g}$ according to Lemma 2.43. (Here we have chosen a norm on V). Now $\mathrm{Iso}(V, V)$ is open in $\mathrm{Hom}(V, V)$, hence by the compactness of X there exists an $\varepsilon > 0$ such that $|h(x, z) - \bar{g}(x, z)| < \varepsilon \ \forall (x, z) \in X \times S^1$ imply that $h(x) \in \mathrm{Iso}(V, V)$. For large n we have

$$|\bar{g_n}(x, z) - \bar{g}(x, z)| < \varepsilon \quad \forall (x, z) \in X \times S^1.$$

Hence

$$g_{nt} = tg + (1 - t) g_n.$$

defines a homotopy of isomorphisms. Restricting to E^0 we get a homotopy of clutching functions.

$$f_{nt} = tf + (1 - t) f_n.$$

Hence we only need to consider Laurent clutching functions. We start with a *linear* clutching function: Thus let $p\colon \pi^*E^0 \to \pi^*E^\infty$ be an isomorphism of the form

$$p = az + b \quad a, b \text{ homomorphism } E^0 \to E^\infty.$$

Assume that E^0 and E^∞ are embedded in $X \times V$ and that a and b are restrictions of homomorphisms in $X \times V$ (denoted by A and B) such that $P = Az + B$ is an isomorphism of $X \times V$ □

Define

$$Q = \frac{1}{2\pi i}\int_{S^1}(Az+B)^{-1}Adz.$$

Lemma 2.49. *Q_x is a projection in V such that*

$$Q_{x|E^0} = \frac{1}{2\pi i}\int_{S^1}(a_x z + b_x)^{-1}a_x dz$$

and thus $\operatorname{Im} Q_{|E^0}$ *is a subbundle of* E^0.

Proof. We will show that

$$Q_x = \frac{1}{2\pi i}\int_{S^1}(A_x z + B_x)^{-1}A_x dz$$

is a projection in V and leave the rest as an exercise for the reader. We leave out the sub-index x:

If $z \neq w$ we have

$$\begin{aligned}\frac{(Az+B)^{-1}}{w-z} + \frac{(Aw+B)^{-1}}{z-w} &= (Aw+B)^{-1}\frac{Aw+B}{w-z}(Az+B)^{-1}\\ &\quad + (Aw+B)^{-1}\frac{Az+B}{z-w}(Az+B)^{-1}\\ &= (Aw+B)A(Az+B)^{-1}.\end{aligned}$$

Now $Az+B$ is an isomorphism for $|z|=1$ and hence also for $1-\varepsilon < |z| < 1+\varepsilon$ for sufficiently small $\varepsilon > 0$. Thus choosing r_1 and r_2 such that $1-\varepsilon < r_1 < r_2 < 1+\varepsilon$ we have

$$Q = \frac{1}{(2\pi i)}\int_{|z|=r_1}(Az+B)^{-1}Adz = \frac{1}{2\pi i}\int_{|w|=r_2}(Aw+B)^{-1}Adw$$

and

$$\begin{aligned}Q^2 &= \frac{1}{(2\pi i)^2}\int_{|w|=r_2}\int_{|z|=r_1}(Aw+B)^{-1}A(Az+B)^{-1}Adzdw\\ &= \frac{1}{(2\pi i)^2}\int_{|w|=r_2}\int_{|z|=r_1}\frac{(Az+B)^{-1}}{w-z}A + \frac{(Aw+B)^{-1}}{z-w}Adzdw\\ &= \frac{1}{(2\pi i)^2}\int_{|z|=r_1}\int_{|w|=r_2}\frac{(Az+B)^{-1}}{w-z}Adwdz\\ &= \frac{1}{2\pi i}\int_{|z|=r_1}(Az+B)^{-1}Adz = Q.\end{aligned}$$

□

Definition 2.50. For (E^0, p, E^∞) with p linear define a vector bundle over X by

$$M_1(E^0, p, E^\infty) = \operatorname{Im} Q_{|E^0}.$$

Definition 2.51. For

$$p = \sum_{k=0}^{n} a_k z^k$$

a polynomial clutching function define a linear clutching function

$$L^n(p)\colon \pi^*(\underbrace{E^0 \oplus \cdots \oplus E^0}_{n+1 \text{ copies}}) \to \pi^*(E^\infty \oplus \underbrace{E^0 \oplus \cdots \oplus E^0}_{n \text{ copies}})$$

by the matrix

$$L^n(p) = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ -z & 1 & 0 & \dots & 0 & 0 \\ 0 & -z & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -z & 1 \end{pmatrix}.$$

We thus get clutching data

$$((n+1)E^0, L^n(p), E^\infty \oplus nE^0),$$

which defines a bundle

$$L^n(E^0, p, E^\infty).$$

If E^0 and E^∞ are embedded in V and p is the restriction of a corresponding polynomial isomorphism, then we can define

Definition 2.52. $M_n(E^0, p, E^\infty) = M_1((n+1)E^0, L^n(p), E^\infty \oplus nE^0)$.

Proposition 2.53. *Let p be a polynomial clutching function of degree n for E^0 and E^∞. Then*

(1) $L^{n+1}(E^0, p, E^\infty) \cong L^n(E^0, p, E^\infty) \oplus (E^0, 1, E^0)$,
(2) $L^{n+1}(E^0, zp, E^\infty) \cong L^n(E^0, p, E^\infty) \oplus (E^0, z, E^0)$.

Proof. (1)

$$L^{n+1}(p) = \begin{pmatrix} & & & & 0 \\ & L^n(p) & & & 0 \\ & & & & 0 \\ 0 & 0 & \dots & -z & 1 \end{pmatrix}$$

Multiplying the z on the bottom row by t gives us a homotopy between $L^{n+1}(p)$ and $L^n(p) \oplus 1$.

(2)

$$L^{n+1}(zp) \begin{pmatrix} 0 & a_0 & a_1 & \dots & a_{n-1} & a_n \\ -z & 1 & 0 & \dots & 0 & 0 \\ 0 & -z & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -z & 1 \end{pmatrix}$$

We multiply the 1 on the second row by t and obtain a homotopy between

$$L^{n+1}(zp) \quad \text{and} \quad L^n(p) \oplus (-z).$$

Clearly

$$(E^0, -z, E^0) \cong (E^0, z, E^0).$$

□

Corollary 2.54. *Let p be as in proposition 2.53. Then*

(1) $M_{n+1}(E^0, p, E^\infty) \cong M_n(E^0, p, E^\infty)$
(2) $M_{n+1}(E^0, zp, E^\infty) \cong M_n(E^0, p, E^\infty) \oplus E^0$.

Definition 2.55. Let E be a bundle over $X \times S^2$. Let $E \cong (E^0, f_n, E^\infty)$, f_n a Laurent clutching function approximating f. Put

$$I(E) = \left[nE^0\right] - \left[M_{2n}(E^0, z^n f_n, E^\infty)\right] \in K(X).$$

Proposition 2.56. *$I(E)$ is independent of the choices made.*

Proof. First note that if p_{nt} is a homotopy through polynomial clutching functions of degree n, then

$$M_n(E^0, p_{n0}, E^\infty) \cong M_n(E^0, p_{n1}, E^\infty).$$

Indeed, substituting X by $X \times I$ in the construction of M_n we obtain a bundle over $X \times I$, whose restrictions to $X \times 0$ and $X \times 1$ are the two bundles respectively.

Now if n is sufficiently large we have $f_n \sim f_{n+1}$ by the homotopy $tf_n + (1-t)f_{n+1}$; thus

$$z^{n+1} f_{n+1} \sim z(z^n f_n)$$

through polynomial clutching functions of degree $\leq 2(n+1)$. Hence

$$\begin{aligned} M_{2n+2}(E^0, z^{n+1} f_{n+1}, E^\infty) &\cong M_{2n+2}(E^0, z(z^n f_n), E^\infty) \\ &\cong M_{2n+1}(E^0, z(z^n f_n), E^\infty) \\ &\cong M_{2n}(E^0, z^n f_n, E^\infty) \oplus E^0 \end{aligned}$$

by Corollary 2.54. Thus $I(E)$ is independent of n for n sufficiently large.

Finally $I(E)$ does not change under the homotopy of f. Explicitly let f_t be a homotopy of clutching functions. Substituting X by $X \times I$ in Lemma 2.48 gives a homotopy f_{nt} of Laurent clutching functions, and thus we have finished by the first part of this proof. □

Theorem 2.57. *I induces a map $K(X \times S^2) \to K(X)$ such that if α denotes the composite*

$$K^{-2}(X) \to K(X \times S^2) \to K(X)$$

then α satisfies Axioms (A1), (A2), (A3).

Proof. Clearly $I(E \oplus F) = I(E) + I(F)$ so I induces a homomorphism $K(X \times S^2) \to K(X)$. Also I is natural so α obviously satisfies (A1).

Let $p_1 \colon X \times S^2 \to X$ be the projection. For any bundle F over X and any $E = (E^0, f, E^\infty)$ over $X \times S^2$ we have

$$(p_1^* F) \otimes E \cong (F \otimes E^0, 1 \otimes f, F \otimes E^\infty).$$

Hence from the construction of I it is easy to see that $I((p_1^* F) \otimes E) = [F] \cdot I(E)$, and hence α satisfies (A2).

Clearly $\underline{1} \cong (\underline{1}, 1, \underline{1})$ and $M_0(\underline{1}, 1, \underline{1}) = 0$. Hence $I(\underline{1}) = 0$. $H = (\underline{1}_1, z^{-1}, \underline{1})$. $z \cdot z^{-1} = 1$ and

$$L^2(1) = \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence $M_2(\underline{2}, L^2(1), \underline{2}) = 0$ and $I(H) = [\underline{1}] - 0 = [\underline{1}]$.

This ends the proof of the periodicity theorem. □

Corollary 2.58.

$$\tilde{K}(S^n) = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

The generator of $\tilde{K}(S^{2n})$ is b^n (external multiplication).

Corollary 2.59. *For any compact X, the map*

$$K(X)\,[t]\,/(t-1)^2 \to K(X \times S^2)$$

taking t to $[H]$, is an isomorphism

Proof. The pair $(X \times S^2, X \times (+))$ gives the split exact sequence

$$0 \longrightarrow K^{-2}(X) \xrightarrow{j^*} K(X \times S^2) \underset{p_1^*}{\overset{i^*}{\rightleftarrows}} K(X) \longrightarrow 0$$

Here $j^*(1 \cdot b) = [p_2^* H] - [1]$. Thus every element in $K(X \times S^2)$ can be written in a unique way as a linear combination of $[p_2^* H] - [1]$ and $[1]$ with coefficients from $K(X)$. Hence also $p_2^*[H] = 1 \cdot [H]$ and $[1]$ constitute a basis as a $K(X)$-module in $K(X \times S^2)$. □

Definition 2.60. For $n \geq 0$ put

$$\begin{aligned} K^n(X, A) &= K^{-n}(X, A) \quad (X, A) \in \mathcal{C}^2 \\ K^n(X) &= K^{-n}(X) \qquad \text{for } X \text{ locally compact.} \end{aligned}$$

Also put

$$K^*(X, A) = \coprod_{-\infty < n < \infty} K^n(X, A).$$

Theorem 2.61. *$K^*(X)$ is a graded ring and $K^*(X, A)$ is a graded right-module over $K^*(X)$. For any compact pair we have an exact triangle*

$$\begin{array}{ccc} & K^*(X) & \\ {}^{j^*}\nearrow & & \searrow^{i^*} \\ K^*(X, A) & \xleftarrow[\delta]{} & K^*(A) \end{array}$$

where i^*, j^* pereserve the grading and δ lifts the grading by 1. Futhermore i^*, j^* and δ are $K^*(X)$-module homomorphisms.

Proof. From Corollary 2.36 it follows that

$$\delta_{-n} \colon K^{-n}(A) \to K^{-n+1}(X, A) \quad n \geq 1$$

commutes with the periodicity isomorphisms β. Extend β by stipulating

$$\begin{aligned} &\beta = \mathrm{Id} \colon K^1(X, A) \to K^{-1}(X, A) \\ &\beta \colon K^n(X, A) \to K^{n-2}(X, A) \quad \text{is} \\ &\beta^{-1} \colon K^{-n}(X, A) \to K^{-(n-2)}(X, A) \quad n \geq 2 \end{aligned}$$

Define for $n \geq 0$, $\delta_n \colon K^n(A) \to K^{n+1}(X, A)$ by $\delta_n = \beta^{-i}\delta_{n-2i}\beta^i$ where $n - 2i < 0$. Actually δ_n is independent of i and is the map

$$\begin{array}{ccccccc} K^n(A) & & & & & & K^{n+1}(X, A) \\ \downarrow = & & & & & & \downarrow = \\ K^{-n}(A) & \xrightarrow{\beta^{i-n}} & K^{n-2i}(A) & \xrightarrow{\delta_{n-2i}} & K^{n-2i+1}(X, A) & \xrightarrow{\beta^{-(i-n-1)}} & K^{-n-1}(X, A) \end{array} \quad \square$$

Clearly δ commutes with the extended periodicity isomorphism and hence we get the exact sequence from Corollary 2.24.

Analogously we extend the external product by means of the periodicity isomorphims. We leave the details to the reader.

Chapter 3

Computations and applications of K

3.1 Thom isomorphims and the splitting principle

If E is a vector bundle over a compact Hausdorff space X, then the total space E is locally compact. For example, consider $E = X \times \mathbb{C}^n$. It is a direct consequence of the Periodicity Theorem that multiplication with $b^n \in K(\mathbb{C}^n)$ yields an isomorphism

$$K(X) \to K(X \times \mathbb{C}^n)$$

In this section we want to generalize this to general bundles.

Definition 3.1. Let (E, p) be a vector bundle over X. Include X in E by means of the 0-section and put

$$P(E) = (E \setminus X)/e \sim \lambda e \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

$P(E)$ is called the *projective bundle* belonging to E. The map $p\colon E \to X$ induces a map $p\colon P(E) \to X$ such that

$$p^{-1}(x) = P(E_x).$$

Define the *"dual Hopf bundle"* H^* as the family of vector spaces contained in p^*E, the fibre of which over $\{e_x\} \in P(E_x)$ is the line in E_x through e_x.

Lemma 3.2. *H^* is a subbundle of p^*E.*

Proof. Without loss of generality E is embedded in $X \times V$. Then $P(E)$ is a subspace of $X \times P(V)$. Hence p^* is embedded in $P(E) \times V$. If

$$q\colon X \times P(V) \to P(V)$$

is the projection and H_V^* is the dual Hopf bundle over $P(V)$, then the restriction of $q^*H_V^*$ to $P(E)$ is H^*. Thus this is embedded in $P(V) \times V$ and the lemma follows from Corollary 1.35. □

As usual we define $H = (H^*)^*$, which is called the Hopf bundle over $P(E)$.

Definition 3.3. The *Thom complex* of E is E^+.

Lemma 3.4. *$E^+ \cong P(E \oplus 1)/P(E)$.*

If E is given a hermitian metric and

$$B(E) = \{e \in E \mid |e| \leq 1\} \quad S(E) = \{e \in E \mid |e| = 1\}$$

then $E^+ \cong B(E)/S(E)$.

Proof. $P(E) \subseteq P(E \oplus 1)$ by $\{e\}$ included as $\{e \oplus 0\}$. Define

$$E \to P(E \oplus 1) \quad \text{by} \quad e \mapsto \{e \oplus 1\}.$$

Finally sending $\infty \in E^+$ to $P(E)$ gives the homeomorphism. We leave the second part as an exercise for the reader. □

Note that especially

$$S^{2n} \cong (\mathbb{C}^n) \cong P(\mathbb{C}^{n+1})/P(\mathbb{C}^n).$$

Definition 3.5. X is a *cell complex of dimension* $2n$ *with even dimensional cells* if there is a sequence

$$\emptyset \subseteq X_0 \subseteq X_2 \subseteq \cdots \subseteq X_{2n} = X$$

such that X_{2i}/X_{2i-2} is homemorphic to a finite wedge of $2i$-dimensional spheres.

Example 3.6. $X = P(\mathbb{C}^{n+1})$ is a cell complex of dimension $2n$. The filtration is $\emptyset \subseteq P(\mathbb{C}) \subseteq P(\mathbb{C}^2) \subseteq \cdots \subseteq P(\mathbb{C}^{n+1})$.

Lemma 3.7. *Let X be a cell complex, as in Definition 3.5*

(1) *$K^0(X)$ is a free Abelian group of rank the number of cells in X. $K^{-1}(X) = 0$.*
(2) *For Y an arbitrary space, the product defines an isomorphism*

$$K^*(Y) \otimes K(X) \to K^*(Y \times X).$$

Proof. (1) Use induction on the dimension of X. From the exact sequence for (X_{2n}, X_{2n-2}) we have the exact sequences

$$0 \longrightarrow \tilde{K}(\textstyle\bigvee_i S_i^{2n}) \longrightarrow K(X_{2n}) \longrightarrow K(X_{2n-2}) \longrightarrow 0$$

and

$$0 \longrightarrow K^{-1}(X_{2n}) \longrightarrow 0$$

Hence $K^{-1}(X_{2n}) = 0$, and $K(X_{2n})$ is an extension of a free Abelian group by another free Abelian group, and thus itself a free Abelian group.

(2) It is clearly sufficient to prove it for Y compact. In fact if Y is locally compact

$$0 \longrightarrow K^*(Y) \longrightarrow K^*(Y^+) \longrightarrow K^*(+) \longrightarrow 0$$

is split exact. Hence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^*(Y) \otimes K(X) & \longrightarrow & K^*(Y^+) \otimes K(X) & \longrightarrow & K^*(+) \otimes K(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K^*(X \times Y) & \longrightarrow & K^*(Y^+ \times X) & \longrightarrow & K^*(X) & \longrightarrow & 0 \end{array}$$

is commutative with exact rows.

For compact Y we prove the lemma by induction on the number of cells in X. Thus we have a filtration of cell complexes

$$\emptyset \subseteq X^0 \subseteq \cdots \subseteq X^{m-1} \subseteq X^m = X$$

such that X^{i+1}/X^i is homeomorphic to a sphere of dimension $2k_i$, and we will prove the lemma by induction on i. By means of (1) we have the split exact sequence ($K(X^i)$ is free)

$$0 \longrightarrow K(X^{(i+1)}, X^{(i)}) \longrightarrow K(X^{(i+1)}) \longrightarrow K(X^{(i)}) \longrightarrow 0$$

and thus for any l we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^l(Y) \otimes K(X^{(i+1)}, X^{(i)}) & \longrightarrow & K^l(Y) \otimes K(X^{(i+1)}) & \rightarrow & K^l(Y) \otimes K(X^{(i)}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_l} & & \downarrow & & \downarrow{\scriptstyle \varphi_l} & & \\ \cdots & \rightarrow & K^l((Y,\emptyset) \times (X^{(i+1)}, X^{(i)})) & \underset{v_l^*}{\longrightarrow} & K^l(Y \times X^{(i+1)}) & \underset{u_l^*}{\longrightarrow} & K^l(Y \times X^{(i)}) & \longrightarrow & \cdots \end{array}$$

φ_l is an isomorphism by induction hypothesis. It follows that u_l^* is epi. Hence by exactness v_{l+1}^* is injective. Substituting l by $l-1$ we have v_l^* injective.

$$X^{(i+1)}/X^{(i)} \cong S^{2k_i}$$

so ψ_l is actually the iterated periodicity map β^{k_i} which is an isomorphism. Hence the conclusion follows from the five-lemma. □

Lemma 3.8. *Let $p\colon B \to X$ be a map of compact spaces, and let $a_1, \ldots, a_n$ be elements of $K^0(B)$. Let M be the free Abelian group generated by $a_1, \ldots, a_n$.*

Suppose that every point $x \in X$ has a neighbourhood U such that for all $V \subseteq U$ the natural map

$$K^*(V) \otimes M \longrightarrow K^*(p^{-1}V) \otimes K^0(B) \longrightarrow K^*(p^{-1}V)$$

is an isomorphism. Then for any $Y \subseteq X$, the map

$$K^*(X, Y) \otimes M \to K^*(B, p^{-1}(Y))$$

is an isomorphism.

Proof. Notice that if $V_1 \supseteq V_2$, and if

$$K^*(V_i) \otimes M \to K^*(p^{-1}(V_i))$$

is an isomorphism for $i = 1, 2$, then from the commutative diagram

$$\begin{array}{ccccccccc}
\cdots K^*(V_2)\otimes M & \longrightarrow & K^*(V_1,V_2)\otimes M & \longrightarrow & K^*(V_1)\otimes M & \to & K^*(V_2)\otimes M & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots K^*(p^{-1}(V_2)) & \to & K^*(p^{-1}(V_1),p^{-1}(V_2)) & \to & K^*(p^{-1}(V_1)) & \to & K^*(p^{-1}(V_2)) & \longrightarrow & 0
\end{array}$$

and the five-lemma we obtain an isomorphism

$$K^*(V_1, V_2) \otimes M \cong K^*(p^{-1}(V_1), p^{-1}(V_2)).$$

Suppose now U_1 and U_2 are any two subspaces of X such that $V_1 \subseteq U_1$, $V_2 \subseteq U_2$ implies that

$$K^*(U_i, V_i) \otimes M \cong K^*((p^{-1}U_i), p^{-1}(V_i)) \quad i = 1, 2$$

Then if $U = U_1 \cup U_2$, $V \subseteq U$, put $W = U_1 \cup V$, $V_1 = V \cap U_1$ and $V_2 = W \cup U_2$. Thus $V \subseteq W \subseteq U$, and considering $(U/V, W/V)$ we have

$$\begin{array}{ccccccc}
\cdots \longrightarrow & K^*(W,V)\otimes M & \longrightarrow & K^*(U,W)\otimes M & \longrightarrow & K^*(U,V)\otimes M & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & K^*(p^{-1}(W),p^{-1}(V)) & \to & K^*(p^{-1}(U),p^{-1}(W)) & \to & K^*(p^{-1}(U),p^{-1}(V)) & \to \cdots
\end{array}$$

Now $W/V = U_1/V_1$ and $U/W = U_2/V_2$. Hence again the five-lemma gives isomorphism

$$K^*(U, V) \otimes M \cong K^*(p^{-1}(U), p^{-1}(V)).$$ □

We now proceed by induction on the number of sets in an open cover of X.

Corollary 3.9. *Let L be a line-bundle over a compact space X. Then $K^*(P(L \oplus 1))$ is a free $K^*(X)$-module on generators* $[1]$ *and* $[H]$*.* $[H]$ *satisfies the equation*

$$([H]\,[L] - [1])([H] - [1]) = 0.$$

Proof. $K^*(P(L \oplus 1))$ is a $K^*(X)$-module by the pairing

$$\begin{array}{ccc}
K^*(P(L\oplus 1))\otimes K^*(X) & \longrightarrow & K^*(P(L\oplus 1))\otimes K^*(P(L\oplus 1)) \\
& & \downarrow \\
& & K^*(P(L\oplus 1))
\end{array}$$

L is locally trivial and clearly the restriction of H to $P(L \oplus 1_{|U}) \cong U \times P(\mathbb{C}^2)$ is the pull-back of the Hopf-bundle over $P(\mathbb{C}^2)$. Hence the first assertion follows from Corollary 2.59 and Lemma 3.8. The equation is a special case of the next proposition. □

Proposition 3.10. *Let E be a n-dimensional bundle over a compact space. Then if H denotes the Hopf bundle over $P(E)$*

$$\sum_{i=0}^{n}(-1)^i \left[\lambda^i(E)\right][H]^i = 0 \quad \text{in} \quad K^0(P(E)).$$

Proof. Again $\lambda^i(E)$ is an abbreviation for $p^*\lambda^i E = \lambda(p^*E)$. Now H^* is a subbundle of P^*E and according to Corollary 1.36 there is a bundle F such that $p^*E = F \oplus H^*$, where $p\colon P(E) \to X$ is the map induced by the projection $E \to X$. Using the operation λ_t (see Section 2.1). We have the equation of polynomials

$$\lambda_t(p^*E) = \lambda_t(F) \cdot \lambda_t(H^*) = \lambda_t(F) \cdot ([1] + [H^*]\, t).$$

substituting $t = -[H]$ and using Lemma 2.37 we have $\lambda_{-H}(p^*E) = 0$. Especially for $E = L \oplus 1$, L a line bundle, we have

$$\lambda^1(L \oplus 1) = L \oplus 1 \quad \text{and} \quad \lambda^2(L \oplus 1) \cong L \otimes 1 \cong L,$$

and the above equation reads

$$0 = [1] - ([L] + [1])\,[H] + [L]\,[H]^2 = ([L]\,[H] - [1])([H] - [1]). \qquad \square$$

Proposition 3.11. *Let H_n denote the Hopf bundle over $P(\mathbb{C}^n)$. Then $K(P(\mathbb{C}^n))$ is a free Abelian group on generators $1, H_n, \ldots, H_n^{n-1}$. Futhermore $(H_n - 1)^n = 0$.*

Proof. The relation $(H_n - 1)^n = 0$ is an immediate consequence of Proposition 3.10. We will prove the first part by induction on n.

$n = 1$ is trivial and $n = 2$ is a special case of Corollary 2.59. Now assume the proposition shown for n.

$$\begin{array}{ccc} P(H_n^* \oplus 1) & \xrightarrow{\ q\ } & P(\mathbb{C}^{n+1}) \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle p} & & \\ P(\mathbb{C}^n) & & \end{array}$$

where p is induced by the projection, s is defined by $s(x) = \{0 \oplus 1_x\}$ and q is defined in the following way: If $L_x \subseteq (H_n^* \oplus 1)_x \subseteq \mathbb{C}^n \oplus \mathbb{C}$ is a line, it is also a line in $\mathbb{C}^n \oplus \mathbb{C} = \mathbb{C}^{n+1}$. Clearly q induces a homeomorphism

$$P(H_n^* \oplus 1)/s(P(\mathbb{C}^n)) \xrightarrow{q} P(\mathbb{C}^{n+1}).$$

Let G denote the Hopf bundle over $P(H_n^* \oplus 1)$. Then $q^*H_{n+1} \cong G$ and $s^*G \cong 1$.

By Corollary 3.9 $K(P(H_n^* \oplus 1))$ is a free $K(P(\mathbb{C}^n))$-module on generators $[1]$ and $[G]$.

From the split exact sequence for the pair $(P(H_n^* \oplus 1), s(P(\mathbb{C}^n)))$ every element of $(K(P(H_n^* \oplus 1), s(P(\mathbb{C}))))$ can be written in a unique way as

$$a \cdot ([G] - [1]),$$

where $a \in K(P(\mathbb{C}^n))$ has a unique representation as a linear combination of $1, H_n$, $\dots, H_n^{n-1}$. By Corollary 3.9

$$([G]\,[H_n^*] - [1])([G] - [1]) = 0$$

or

$$[H_n]\,([G] - [1]) = [G]\,([G] - [1]).$$

Hence

$$1 \cdot ([G] - [1]), [G]\,([G] - [1]), \dots, [G]^{n-1}\,([G] - [1]),$$

is a basis for

$$(K(P(H_n^* \oplus 1)), s(P(\mathbb{C}^n)))$$

or

$$[H_{n+1}] - [1]\,, [H_{n+1}]\,([H_{n+1}] - [1]), \dots, [H_{n+1}]^{n-1}\,([H_{n+1}] - [1]),$$

is a basis for $\tilde{K}(P(\mathbb{C}^{n+1}))$. From this it clearly follows that

$$[1]\,, [H_{n+1}]\,, \dots, [H_{n+1}]^n\,,$$

is a basis for $K(P(\mathbb{C}^{n+1}))$. □

Theorem 3.12. *Let E be an n-dimensional bundle over a compact space X, and H the Hopf bundle over $P(E)$. Then $K^*(P(E))$ is a free $K^*(X)$-module on generators $[1]\,, [H]\,, \dots, [H]^{n-1}$. $[H]$ satisfies the single relation*

$$\sum_{i=0}^{n} (-1)^i \left[\lambda^i(E)\right] [H]^i = 0 \quad \text{in} \quad K^0(P(E)).$$

Proof. For E a trivial bundle the assertion is clear from Proposition 3.11 and Lemma 3.7. For E an arbitrary bundle the theorem is an immediate application of Lemma 3.8 and the fact that E is locally trivial. □

Corollary 3.13 (The Splitting Principle). *Let E be a bundle over a compact space X. Then there exists a space $F(E)$ and a map $f\colon F(E) \to X$ such that*

(1) *f^*E is a sum of line bundles.*
(2) *$f^*\colon K^*X \to K^*(F(E))$ is injective and maps onto a direct summand.*

Proof. By induction on the dimension of E. If E is a line bundle there is nothing to prove. If not, consider $p\colon P(E) \to X$ and note that there is a bundle E' over $P(E)$ such that $E' \oplus H^* \cong p^*E$. By the induction hypothesis there is a space $F(E')$ and a map $f'\colon F(E') \to P(E)$ such that $(f')^*E'$ is a sum of line bundle, and furthermore $(f')^*$ is injective and maps onto a direct sum in $K^*(F(E'))$. Put $F(E) = F(E')$ and $f = p \circ f'$. □

We now return to Thom complexes. Notice that if $f\colon Y \to X$ is a continuous function of compact spaces and E is a vector bundle over X then there is an obvious map

$$\tilde{f}\colon f^*(E) \to E$$

which is clearly a proper map of locally compact spaces. Hence we have an induced map

$$K^*(E) \to K^*(f^*(E)).$$

Now let F be a bundle over X and E a bundle over Y. Then $F \times E$ is a bundle over $X \times Y$ and we have a pairing

$$K^*(F) \otimes K^*(E) \to K^*(X \times Y).$$

If $X = Y$ and $\Delta\colon X \to X \times Y$ is the diagonal $\Delta^*(F \times E) \cong F \oplus E$ and we have a pairing

$$K^*(F) \otimes K^*(E) \longrightarrow K^*(F \times E) \xrightarrow{\tilde{\Delta}^*} K^*(F \oplus E)\ .$$

Especially $F = 0$ gives a module structure

$$K^*(X) \otimes K^*(E) \to K^*(E).$$

Exercise 3.14.

(1) According to Lemma 3.4 $E^+ \approx B(E)/S(E)$. Under this homeomorphism the above pairing corresponds to

$$\begin{array}{ccc} K^*(X) \otimes K^*(B(E), S(E)) & \longrightarrow & K^*(B(E)) \otimes K^*(B(E), S(E)) \\ & & \downarrow \\ & & K^*(B(E), S(E)). \end{array}$$

(2) Under the homeomorphism $E^+ \approx P(E \oplus 1)/P(E)$ the above pairing corresponds to

$$\begin{array}{ccc} K^*(X) \otimes K^*(P(E \oplus 1), P(E)) & \longrightarrow & K^*(P(E \oplus 1)) \otimes K^*(P(E \oplus 1), P(E)) \\ & & \downarrow \\ & & K^*(P(E \oplus 1), P(E)). \end{array}$$

Consider the exact sequence for the pair $(P(E \oplus 1), P(E))$:

$$\cdots \longrightarrow K^*(P(E \oplus 1), P(E)) \xrightarrow{j^*} K^*(P(E \oplus 1)) \xrightarrow{i^*} K^*(P(E)) \longrightarrow \cdots$$

Let H denote the Hopf bundle over $P(E \oplus 1)$. Then i^*H is the Hopf bundle over $P(E)$. $1, [i^*H], \ldots, [i^*H]^{n-1}$ constitute a basis for $K^*(P(E))$ as $K^*(X)$-module and hence i^* is onto. Hence j^* is injective.

$$\lambda_{-H}(E) = \sum_{i=0}^{n} (-1)^i \left[\lambda^i(E)\right] [H]^i$$

is in the kernel of i^*. Hence there is a unique element (the Thom class) denoted by

$$\lambda_E \in K(P(E \oplus 1), P(E))$$

such that $j^*\lambda_E = \lambda_{-H}(E)$. The coefficient to H^n in $\lambda_{-H}(E)$ is $(-1)^n [\lambda^n(E)]$ which is invertible because $\lambda^n(E)$ is a line bundle. Hence

$$1, [H], \ldots, [H]^{n-1}, \lambda_{-H}(E),$$

constitute a basis for $K^*(P(E \oplus 1))$ as $K^*(X)$-module. The kernel of i^* is thus a free $K^*(X)$-module on the single generator $\lambda_{-H}(E)$. We have thus proved:

Theorem 3.15 (The Thom Isomorphism Theorem). *Let $\lambda_E \in K(E)$ denote the unique element such that the image in $K(P(E \oplus 1))$ is*

$$\sum_{i=0}^{n} (-1)^i \left[\lambda^i(E)\right] [H]^i .$$

Multiplication with λ_E induces an isomorphism

$$K^*(X) \xrightarrow{\lambda_E} K^*(E)$$

Remark 3.16. If $E = X \times \mathbb{C}$ then $\lambda_E = -b$ and thus the Thom isomorphism is $-\beta\colon K(X) \to K^{-2}(X)$.

We conclude this section by proving that if E and F are bundles over X and Y respectively then

$$\lambda_{E \times F} = \lambda_E \cdot \lambda_F \in K(E \times F).$$

Lemma 3.17. *If E' is a bundle such that $E' \oplus H^* = p^*(E \oplus 1)$ over $P(E \oplus 1)$ then $\lambda_{-H}(E) = \lambda_{-1}(E')$.*

Proof. Here $\lambda_{-1}(E') = \sum_{i=0}^{n} (-1)^i \lambda^i(E')$. Notice that

$$0 = \lambda_{-H}(E \oplus 1) = \lambda_{-H}(E)\lambda_{-H}(1) = \lambda_{-H}(E) \cdot (1 - H),$$

i.e.

$$H \cdot \lambda_{-H}(E) = \lambda_{-H}(E).$$

And so

$$\lambda_{-H}(E) = H^{-n}\lambda_{-H}(E) = \sum_{i=0}^{n} (-1)^i \lambda^i(E)(H^*)^{n-i}.$$

Let t denote an indeterminate and consider the equation $E' - 1 = E - H^*$. Thus

$$\lambda_t(E') \cdot \frac{1}{1+t} = \lambda_t(E) \cdot \frac{1}{1+H^*t}.$$

Hence

$$\lambda_{-H}(E) = \left[\lambda_{-t}(E)\frac{1}{1-H^*t}\right]_n = \left[\lambda_{-t}(E')\frac{1}{1-t}\right]_n = \sum_{i=0}^{n}(-1)^i\lambda^i(E').$$

Here $[p(t)]_n$ denotes the n-th coefficient of the power series $p(t)$. □

Lemma 3.18. *Let $E_n^\perp$ denote the orthogonal complement to the universal bundle E_n over $G_n(\mathbb{C}^m)$ (i.e. $E_n^\perp \oplus E_n = G_n(\mathbb{C}^m) \times \mathbb{C}^m$). Then*

$$G_{n+1}(\mathbb{C}^{m+1})/G_{n+1}(\mathbb{C}^m) \approx (E_n^\perp)^+$$

Proof. Define $q\colon P(E_n^\perp \oplus 1) \to G_{n+1}(\mathbb{C}^{m+1})$ by:

(u_V, λ) where $V \in G_n(\mathbb{C}^m)$, $u_V \perp V$ and $\lambda \in \mathbb{C}$ is sent into the $n+1$-dimensional subspace

$$V \oplus \{u_V, \lambda\} \subseteq \mathbb{C}^{m+1}.$$

Clearly $P(E_n)$ goes to $G_{n+1}(\mathbb{C}^m)$ under this map, and thus q induces a homeomorphism

$$P(E_n^\perp \oplus 1)/P(E_n) \to G_{n+1}(\mathbb{C}^{m+1})/G_{n+1}(\mathbb{C}^m).$$

Notice that $q^*E_{n+1} \cong E_n \oplus H^*$ where H is the Hopf bundle over $P(E_n^\perp \oplus 1)$. □

Remark 3.19. This lemma is also of general interest. It can be used to compute $K^*(G_n(\mathbb{C}^m))$ by induction by means of the exact sequence and the Thom isomorphism. In fact

$$K^{-1}(G_n(\mathbb{C}^m)) = 0$$

and $K(G_n(\mathbb{C}^m))$ is a free Abelian group of rank $\binom{m}{n}$.

Also some special cases are well-known:

$$n = 0 \quad \text{gives} \quad P(\mathbb{C}^{m+1}) \,/\, P(\mathbb{C}^m) \cong (\mathbb{C}^m)^+$$

and

$$n = m-1 \quad \text{gives} \quad P(\mathbb{C}^{m+1}) \cong (H_m^*)^+,$$

Where H_m^* is the dual Hopf bundle over $P(\mathbb{C}^m)$. (Compare the proof of Proposition 3.11).

Proposition 3.20. *Let E and F be bundles over X and Y respectively. Then*

$$\lambda_{E\times F} = \lambda_E \cdot \lambda_F.$$

Proof. Notice that if $f\colon Y \to X$ and E a bundle over X then $\lambda_{f^*E} = \tilde{f} * \lambda_E$ where $\tilde{f}\colon (f^*E)^+ \to E^+$ is the obvious map. Now every bundle is the pull-back of a universal bundle over a Grassman-manifold; hence we only need to prove the formula for E and F universal bundles over Grassmanians. The map

$$\nu\colon G_{m-n}(\mathbb{C}^m) \to G_n(\mathbb{C}^m)$$

taking a space V to $V^\perp$, the orthogonal complement of V in $\mathbb{C}^m$ satisfies $\nu^* E_n^\perp \cong E_{m-n}$. Hence it is enough to consider the following cases:

$$E = E_n^\perp \text{ over } G_n(\mathbb{C}^m) \quad \text{and} \quad F = E_{n'}^\perp \text{ over } G_{n'}(\mathbb{C}^{m'}).$$

Using the notation of the proof of Lemma 3.18

$$q^*E_{n+1} \cong E_n \oplus H^* \quad \text{or} \quad q^*E_{n+1}^\perp \oplus H^* = 1 \oplus E_n^\perp.$$

According to Lemma 3.17 $\lambda_{-H}(E_n^\perp) = q^*\lambda_{-1}(E_{n+1}^\perp)$. The Proposition now follows from the multiplicative property of λ_t and the following diagram which commutes up to homotopy:

$$\begin{array}{ccc}
G_{n+1}(\mathbb{C}^{m+1}) \times G_{n'}(\mathbb{C}^{m'+1}) & \xleftarrow{\;q\times q'\;} & P(E_n^\perp \oplus 1) \times P(E^{\perp_{n'}} \oplus 1) \\
\Big\downarrow & & \Big\downarrow \mu \\
 & & P(E_n^\perp \times E_{n'}^\perp \oplus 1)/P(E_n^\perp \times E_{n'}^\perp) \\
 & & \Big\downarrow \nabla \\
G_{n+n'+2}(\mathbb{C}^{m+m'+2})/G_{n+n'+2}(\mathbb{C}^{m+m'+1}) & \xleftarrow[\;\tilde{q}\;]{} & P(E_{n+n'+1}^\perp \oplus 1)/P(E_{n+n'+1}^\perp)
\end{array}$$

Here $\mu((u,s)\times(u',s')) = ((s'u \times su'), ss')$ induces the map

$$(E_n^\perp)^+ \times (E_{n'}^\perp)^+ \to (E_n^\perp \times E_{n'}^\perp)^+,$$

and ∇ is induced by the map

$$G_n(\mathbb{C}^m) \times G_{n'}(\mathbb{C}^{m'}) \to G_{n+n'}(\mathbb{C}^{m+m'}) \to G_{n+n'+1}(\mathbb{C}^{m+m'+1}).$$ □

Corollary 3.21.

$$\lambda_{\mathbb{C}^n} = (-b)^n \in K(\mathbb{C}^n).$$

This means that the Thom isomorphism $K(X) \to K(X \times \mathbb{C}^n)$ is $(-1)^n\beta^n$. Note also that

$$j^*\lambda_{\mathbb{C}^n} = (1-H)^n$$

where $j\colon P(\mathbb{C}^{n+1}) \to (\mathbb{C}^n)^+$ and H is the Hopf bundle over $P(\mathbb{C}^{n+1})$. Thus $j^*b^n = (H-1)^n$ where $b = H - 1 \in \tilde{K}(S^2)$.

Corollary 3.22. *Let E and F be bundles over X then multiplication by λ_F gives an isomorphism*

$$K(E) \xrightarrow{\cdot\lambda_F} K(E \oplus F).$$

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} K(X) & & \\ {\scriptstyle \cdot\lambda_E}\downarrow & \searrow^{\cdot\lambda_{E\oplus F}} & \\ K(E) & \xrightarrow[\cdot\lambda_F]{} & K(E \oplus F) \end{array}$$

□

3.2 The Adams operations

Definition 3.23. An operation $\varphi\colon K \to K$ is a natural transformation

$$\varphi_X\colon K(X) \to K(X)$$

commuting with induced maps.

Example 3.24.

(1) We have already used the operation λ^k several times

(2) The map $E \to E^*$ induces clearly an operation. If E is given a Hermitian metric $E^* \cong \bar{E}$ where $\bar{E}$ is the vector bundle over X the total space of which is E, but where the scalar multiplication by λ in $\bar{E}_x$ is multiplication by $\bar{\lambda}$ in E; i.e. the identity induces an anti-linear homomorphism $E \to \bar{E}$.

This operation is denoted by $\bar{\mathrm{Id}}$.

Lemma 3.25. *Let φ and $\varphi^{'}\colon K \to K$ be two operations. If*

$$\varphi([E] - n) = \varphi^{'}([E] - n)$$

for E a sum of line bundles, then $\varphi = \varphi^{'}$.

Proof. This follows immediately from Corollary 2.49 □

Especially if both φ and $\varphi^{'}$ are additive we only have to check that they agree on line bundles.

We now define some special operations.

Definition 3.26. In the ring $K(X)\,[[t]]$ define

$$\psi_t(x) = -t\frac{d}{dt}(\log \lambda_{-t}(x)).$$

The coefficient of t^k is denoted $\psi^k(x)$ for $k \geq 1$.

Remark 3.27. This looks a little artificial; but it is to be interpreted as

$$\psi_t \lambda_{-t} = -t\frac{d}{dt}\lambda_{-t} \quad \text{or} \quad \psi^k + \sum_{j=1}^{k-1}(-1)^j \psi^{k-j}\lambda^j = (-1)^{k+1}k\lambda^k.$$

Thus ψ^k is defined inductively by this *Newton identity.* The operations ψ^k are called *Adams operations.*

Proposition 3.28. *For any $k > 0$ we have*

(1) $\psi^k(x+y) = \psi^k(x) + \psi^k(y) \quad x, y \in K(X).$

(2) $\psi^k(L) = [L]^k \quad$ *L a line bundle.*

(3) *ψ^k is uniquely detemined by* (1) *and* (2).

Proof. (1) Clearly $\psi_t(x+y) = \psi_t(x) + \psi_t(y)$, so that

$$\psi^k(x+y) = \psi^k(x) + \psi^k(y).$$

(2) $\lambda_{-t}(L) = 1 - Lt$, so by Remark 3.27

$$\psi_t(L) = \frac{1}{1-Lt} \cdot (-t(-L)) = \sum_{i=1}^{\infty} L^i t^i.$$

(3) Is immediate from Lemma 3.25. □

Proposition 3.29.

(1) $\psi^k(xy) = \psi^k(x)\psi^k(y) \quad$ *for* $\quad x, y \in K(X) \quad k \geq 1.$

(2) $\psi^k(\psi^l(x)) = \psi^{kl}(x) \quad$ *for* $\quad x \in K(X) \quad k, l \geq 1.$

(3) *If p is a prime,*

$$\psi^p(x) \equiv x^p \mod p.$$

(4) *If $u \in \tilde{K}(S^{2n})$ then*

$$\psi^k(u) = k^n u \quad \textit{for} \quad k \geq 1.$$

Proof. (1) and (2) follows from Proposition 3.28 and the splitting principle.

(3) If $x = x_1 + \cdots + x_n$, where x_i is a line bundle, then

$$\psi^p(x) = x_1^p + \cdots + x_n^p.$$

and

$$(x_1 + \cdots + x_n)^p = \sum_{i_1+\cdots+i_n=p} \binom{p}{i_1 \ldots i_n} x_i^{i_1} \ldots x_n^{i_n},$$

where

$$\binom{p}{i_1 \dots i_n} = p!/(i_1! \dots i_n!)$$

is divisible by p unless one of the i_j is p. Hence $(x_1 + \cdots + x_n)^p = x_1^p + \cdots + x_n^p$ mod p.

Thus by the splitting principle for x arbitrary $\psi^p(x) - x^p$ is divisible by p in a group in which $K(X)$ is a direct summand and hence divisible by p in $K(X)$.

(4) Let $X = S^2 \times \cdots \times S^2$ n times, $p_i \colon X \to S^2$ is the projection onto the i-th factor and $j \colon X \to S^{2n}$ is the collapsing map.

$$\tilde{K}(S^{2n}) = \mathbb{Z}(b^n) \quad \text{where} \quad j^* b^n = \prod_{i=1}^{n} p_i^* b.$$

and

$$\begin{aligned} \psi^k(b) &= \psi^k(H-1) \\ &= H^k - 1 \\ &= ((H-1)+1)^k - 1 \\ &= 1 + k(H-1) - 1 \\ &= k(H-1) \qquad \text{(because } (H-1)^2 = 0). \end{aligned}$$

Hence $j^* \psi^k(b^n) = k^n \prod_{i+1}^{n} p_i^* b = j^*(k^n b^n)$ and hence $\psi^k(b^n) = k^n b^n$. □

Remark 3.30. It is possible to prove the properties of ψ^k without using the splitting principle and thus the periodicity theorem. See for example M.F. Atiyah [9].

3.3 Almost complex structures on $\boldsymbol{S^{2n}}$

The classical application of K-theory is to prove the nonexistence of elements of Hopf invariant one, and to show that this implies that $\mathbb{R}^n$ is a division algebar only $n = 1, 2, 4, 8$. For this we refer to D. Husemoller [20], ch. 14. Another consequence is that S^{2n} does not have an almost complex structure except for $n = 1, 3$. In this section we will give a direct proof of this fact.

Until now we have only considered complex vector bundles. If E is a complex vector bundle it has an underlying real vector bundle. On the other hand if F is a real vector bundle of even dimension does there exist a complex vector bundle whose underlyimasng real bundle is F?

Definition 3.31. A real vector bundle F has a *complex structure* if there exists a complex vector bundle E whose underlying real bundle is F.

If $F = \mathfrak{T}(M^{2n})$ is the tangent bundle of a $2n$-dimensional manifold and F has a complex structure, then M^{2n} is said to have an *almost complex structure.*

Example 3.32. $S^2 \subseteq P(\mathbb{C}^2)$ is a complex manifold and therefore has a complex structure on the tangent bundle.

Alternatively we can represent the tangent bundle $\mathfrak{T}$ of S^2 as the bundle for which the fibre at $a \in S^2$ is the plane in $\mathbb{R}^3$ orthogonal to a. We can directly define multiplication by i by the map

$$J_a \colon \mathfrak{T}_a \to \mathfrak{T}_a, \quad \text{where} \quad J_a(y) = a \times y,$$

the vector product of a and y. J_a is linear an $J_a^2 = -\operatorname{Id}$.

In the same way it is possible to define an almost linear structure on S^6. In fact there is a vector product in $\mathbb{R}^7$ induced from the Cayley multiplication in the same way as the vector product in $\mathbb{R}^3$ is induced from the multiplication of quarternions.

It is still unsolved whether S^6 can be given the structure of a complex manifold.

Theorem 3.33. *S^{2n} does not have an almost complex structure for $n \neq 1, 3$.*

We need some preperations. First note that that if F is a real bundle then again we can define the Thom complex of F as F^+.

If we give F a metric then we can define $B(F)$ and $S(F)$ as usual. Letting ε denote the trivial real bundle of dimension one $S(F \oplus \varepsilon)$ has the section $x \mapsto (0, 1_x)$.

Lemma 3.34. *For any real bundle F we have*

$$F^+ \approx B(F)/S(F) \cong S(F \oplus \varepsilon)/s(X).$$

Proof. There is a map

$$\Theta \colon (B(F), S(F)) \to (S(F \oplus \varepsilon), s(X))$$

defined by sending v to

$$(2v\sqrt{1 - |v|^2},\ 2|v|^2 - 1).$$

This map induces the homeomorphism. Notice that if $t \colon X \to B(F)$ is the 0-section then $\Theta \circ t \colon X \to S(F \oplus \varepsilon)$ is the map sending x to $(0, -1_x)$. □

Lemma 3.35. *If $\mathfrak{T}$ is the tangent bundle of S^n and Δ is the diagonal in $S^n \times S^n$ then*

$$\mathfrak{T}^+ \approx S^n \times S^n/\Delta.$$

Proof. $\mathfrak{T}_x \subseteq S^n \times \mathbb{R}^{n+1}$ is orthogonal to $x \in \mathbb{R}^{n+1}$. Thus letting $\varepsilon_x = \{x\} \subseteq \mathbb{R}^{n+1}$ we have $\mathfrak{T} \oplus \varepsilon = S^n \times \mathbb{R}^{n+1}$. Hence $S(\mathfrak{T} \oplus \varepsilon) = S^n \times S^n$ and clearly the section s corresponds to the diagonal. □

Remark 3.36. We have to distinguish between the two S^n, so

$$S(\mathfrak{T} \oplus \varepsilon) = S_1^n \times S_2^n \quad \text{where} \quad p_1 \colon S_1^n \times S_2^n \to S_1^n$$

is the projection in $S(\mathfrak{T} \oplus \varepsilon)$. Notice that the section $t \colon S^n \to S(\mathfrak{T} \oplus \varepsilon)$ is the map sending x to $(x, -x)$.

Proof of Theorem 3.33 If $\mathfrak{T}$ is the underlying real bundle of a complex vector bundle E, then there is by Theorem 3.15 an element $\lambda_E \in K(\mathfrak{T})$ such that the map

$$K(S^{2n}) \xrightarrow{\lambda \cdot E} K(\mathfrak{T})$$

is an isomorphism. Using Lemma 3.35 we thus have

$$\lambda_E = \Theta^* c \quad c \in K(S_1^{2n} \times S_2^{2n}, \Delta)$$

such that the map

$$K(S^{2n}) \xrightarrow{\pi_1^*} K(S_1^{2n} \times S_2^{2n}) \xrightarrow{c\cdot} K(S_1^{2n} \times S_2^{2n}, \Delta)$$

is an isomorphism

Now $\tilde{K}(S^{2n}) = \mathbb{Z}(h)$ where $h = b^n$. Hence

$$K(S^{2n}) = \mathbb{Z}(1) \oplus \mathbb{Z}(h).$$

According to Lemma 3.7 $\tilde{K}(S_1^{2n} \times S_2^{2n})$ has a basis $\{h_1, h_2, h_1 \cdot h_2\}$ where $h_i = p_i^* h$.

$$p_1 \colon S_1^{2n} \times S_2^{2n} \to S_i^{2n}.$$

The exact sequence for $(S_1^{2n} \times S_2^{2n},\ \Delta)$ reads

$$0 \longrightarrow \tilde{K}(S_1^{2n} \times S_2^{2n}/\Delta) \xrightarrow{j^*} \tilde{K}(S_1^{2n} \times S_2^{2n}) \xrightarrow{\Delta^*} K(S^{2n}) \longrightarrow 0$$

Here $\Delta^* h_i = (p_1 \circ \Delta)^* h = h$ is clearly onto, and the kernel is generated by $h_1 \cdot h_2$ and $h_1 - h_2$ let a and b in $K(S_1^{2n} \times S_2^{2n}/\Delta)$ be generators such that $j^* a = h_1 \cdot h_2$ and $j^* b = h_1 - h_2$. $c = \alpha a + \beta b$ and by the Thom isomorphism $b = (\alpha_0 + \alpha_1 h_1) \cdot c$, where $\alpha, \beta, \alpha_0, a \in \mathbb{Z}$. I.e.

$$\begin{aligned} h_1 - h_2 &= (\alpha_0 + \alpha_1 h_1)(\alpha h_1 h_2 + \beta(h_1 - h_2)) \\ &= \alpha_0 \beta (h_1 - h_2) + (\dots) h_1 h_2. \end{aligned}$$

Hence $\alpha_0 \beta = 1$, $\beta = \pm 1$ and $c = \alpha a \pm b$.

Now in general for a complex bundle E over X, if $t \colon X \to B(E)$ is the 0-section and $j_1 \colon B(E) \to B(E)/S(E)$, then it is clear that

$$t^* j_1^* \lambda_E = \sum_{i=0}^{n} (-1)^i \lambda^i(E) \in K(X).$$

Put $U = t^* j_1^* \lambda_E$. From Remark 3.36 we have the commutative diagram

$$\begin{array}{ccccc} \mathfrak{T}^+ & \xleftarrow{j_1} & B(\mathfrak{T}) & \nwarrow^{t} & \\ \downarrow \Theta & & \downarrow \Theta & & \\ S_1^{2n} \times S_2^{2n}/\Delta & \xleftarrow[j]{} & S_1^{2n} \times S_2^{2n} & \xleftarrow[t']{} & S^{2n}. \end{array}$$

where t' is the map sending x to $(x,-x)$. Hence $U = t'^* j^* c = t'(\alpha h_1 \cdot h_2 \pm (h_1 - h_2))$. $t'^* h_1 = (p_1 t')^* h = h$; but the antipodal map in S^{2n} induces multiplication by -1 (compare the proof of Corollary 2.29), hence $t'^* h_2 = -h$. Hence $U = \pm 2h \in \tilde{K}(S^{2n})$.

On the other hand $[E] \in K(S^{2n})$ and $[E] - [n] \in \tilde{K}(S^{2n})$. From Remark 3.27 and from Exercise 2.33.(6) in Section 2.4 it follows that

$$\psi^k(x) = (-1)^{k+1} k \lambda^k(x) \quad \text{for} \quad x \in \tilde{K}(S^{2n})$$

or

$$\lambda^k(x) = (-1)^{k+1} \tfrac{1}{k} \psi^k(x) \quad \text{for} \quad x \in \tilde{K}(S^{2n}).$$

Hence

$$\begin{aligned} \lambda^k(E) &= \lambda^k((E-n)+n) \\ &= \sum_{i+1}^{n} \binom{n}{k-i} (-1)^{i+1} \frac{1}{i} \psi^i(E-n) + \binom{n}{k} \\ &= \binom{n}{k} + \Big(\sum_{i=1}^{n} \binom{n}{k-i} (-1)^{i+1} \; i^{n-1} \Big)(E-n). \end{aligned}$$

Here we have used Proposition 3.29.(4). From this follows

$$U = \sum_{k=0}^{n} (-1)^k \lambda^k(E) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} + \Big(\sum_{k=1}^{n} \sum_{i=1}^{k} \binom{n}{k-i} (-1)^{k+i+1} \; i^{n-1} \Big)(E-n)$$

$$\sum_{k=1}^{n} \sum_{i=1}^{k} \binom{n}{k-i} (-1)^{k+i+1} \; i^{n-1} = \sum_{i=1}^{n} \sum_{k=i}^{n} \binom{n}{k-i} (-1)^{k+i+1} \; i^{n-1}.$$

According to W. Feller: *An introduction to probability theory*, Vol 1, p.61 (12.7)

$$\sum_{k=0}^{n-1} (-1)^{k+1} \binom{n}{k} = (-1)^{n-i+1} \binom{n-1}{n-i}$$

and using p. 63 (12.17)

$$\sum_{i=1}^{n} (-1)^{n-i+1} \binom{n-1}{i-1} \; i^{n-1} = -\frac{n!}{n} = -(n-1)!.$$

Hence

$$U = \Big(\sum_{i=1}^{n} (-1)^{n-i+1} \binom{n-1}{n-i} \; i^{i-1} \Big)(E-n) = -(n-1)!(E-n).$$

We thus have

$$(n-1)!(E-n) = \pm 2h \tag{3.1}$$

where h is the generator of $\tilde{K}(S^{2n})$, and thus 2 is divisible by $(n-1)!$. Hence $n \leq 3$.

It remains to consider the case $n = 2$:

The above equation (3.1) gives $E - 2 = \pm 2b^2$. Now there exist natural transformations

$$\varepsilon_U \colon KR \to K$$
$$\varepsilon_R \colon K \to KR$$

where KR is the K-theory on real bundles. ε_R is induced from the map which to each complex bundle assigns the underlying real bundle. ε_U is induced from the map which to each real bundle F assigns the complex bundle $F \otimes_R \mathbb{C}$. It is easy to see that

$$\varepsilon_U \circ \varepsilon_R = \mathrm{Id} + \bar{\mathrm{Id}}.$$

By definition $\varepsilon_R(E) = \mathfrak{T}$, which is stably trivial, and hence

$$[\mathfrak{T}] - [4\varepsilon] = 0 \quad \text{in} \quad KR(S^4).$$

and

$$\begin{aligned} 0 &= \varepsilon_U([\mathfrak{T}] - [4\varepsilon]) \\ &= \varepsilon_U \varepsilon_R([E] - [2]) \\ &= \pm(\mathrm{Id} + \bar{\mathrm{Id}})(b^2). \end{aligned}$$

Now $\bar{b} = \left[\bar{H}\right] - 1 = [H^*] - 1 = [H]^{-1} - 1 = 1 - [H]$ in $K(S^2)$. Hence $\bar{\mathrm{Id}}(b^2) = (-b)^2 = b^2$ and $0 = \pm 2(2b^2) = \pm 4b^2$ which is a contradiction. □

Remark 3.37. The above proof gives one further information concerning the case $n = 3$. If $\mathfrak{T}$ has a complex structure E then

$$[E] = [3] \pm b^3 \quad \text{in} \quad K(S^6)$$

and thus is uniquely determined as an element of $K(S^6)$. Because $\pi_5(U(3))$ is in the stable range this shows that an almost complex structure on S^6 is uniquely determined up to isomorphism

Warning. In differential geometry an almost complex structure is a particular automorphism J of the tangent bundle satisfying $J^2 = -\mathrm{Id}$. If φ is an arbitrary automorphism then $\varphi J \varphi^{-1}$ may be another almost complex structure in this sense, though the two complex bundles clearly are isomorphic (under φ). It can be shown that the particular almost complex structure on S^6 defined in the beginning of this section, cannot be lifted to a complex structure, but it is not known what happens, when it is changed by an automorphism.

Bibliography

[1] J. F. Adams. *Vector fields on spheres.* Ann. of Math. 75 (1962), 603–632.

[2] J. F. Adams. *On the groups $J(X)$ I–IV.* Topology 2, 181–195; 3 137–171; 3, 193–222 and 5, 31–71.

[3] D. W. Anderson. *A new cohomology theory.* Thesis, Berkeley 1964.

[4] D. W. Anderson. *K-theory.* Lecture notes. Nordic Summer School in Mathematics, Aarhus 1968.

[5] J. F. Adams and M. F Atiyah. *K-theory and the Hopf invariant.* Quart. J. Math., Oxford (2), 17 (1966), 31–8.

[6] M. F. Atiyah. *Bott periodicity and the index of elliptic operators.* Quart. J. Math., Oxford (2), 19 (1968), 113–40.

[7] M. F. Atiyah. *K-theory.* Benjamin 1967.

[8] M. F. Atiyah. *K-theory and the reality.* Quart. J. Math., Oxford (2), 17 (1966).

[9] M.F Atiyah. *Power operations in K-theory.* Quart. J. Math., Oxford (2), 17 (1966), 165–93.

[10] M. F. Atiyah. *Thom complexes.* Proc. London Math Soc., 11, 1961, 291–310.

[11] M. F. Atiyah and R. Bott *On the periodicity theorem for complex vector bundles.* Acta Math. 112 (1964), 229-47.

[12] M. F. Atiyah, R. Bott and A. Shapiro. *Clifford modules.* Topology 3 (Supp. 1) (1964), 3–38.

[13] M. F. Atiyah and F. Hirzebruch. *Vector bundles on homogeneous spaces.* Proc. Symp. in Pure Math., 3, A.M.S(1961), 7–38.

[14] M. F. Atiyah and G.B Segal. *Equivariant K-theory.* Lecture notes (Oxford), (1965).

[15] M. F. Atiyah and I. M. Singer. *The index of elliptic operators, I.* Ann. of Math., 87 (1968), 484–530.

[16] A. Dold. *Halbexakte Homotopifunktoren.* Lecture Notes in Mathematics, 12 (Springer 1966)

[17] J. L. Dupont. *Symplectic bundles and KR-theory.* Math. Scand. 23 (1968).

[18] B. Eckmann. *Cohomologie et classes caracteristiques.* Centro Internazionale Matematico Estivo, III Ciclo, L'Aquilla, 2–10, Sep. 1966, Edizione Cremonese (Roma 1967)

[19] F. Hirzebruch. *Topological methods in algebraic geometry, 3rd edition.* Springer, 1966.

[20] D. Husemoller. *Fibre bundles.* McGraw-Hill, 1966.

[21] M. Karoubi. *Algebra de Clifford et K-theorie.* Ec. Norm. Sup. (4) 1 (1968), 161–270.

[22] S. MacLane. *Homology.* Springer, 1963.