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FUNCTIONAL ANALYSIS IX

Proceedings of the Postgraduate School and Conference Held at the Inter-University Centre, Dubrovnik, Croatia, 15–23 June 2005

Editors: G. Muić, J. Hoffmann-Jørgensen

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Foreword

This volume contains notes and papers related to the lectures given at the Postgraduate School and Conference on Functional Analysis held at the Inter-University Centre of Postgraduate Studies, Dubrovnik, Croatia, 15–23 June 2005. The lectures were devoted to various parts of functional analysis (such as operator algebras, theory of representations) and its applications (in probability).

There were single 45–minutes lectures and series of lectures (the number of 45–minute lectures being given in parentheses):

- Dražen Adamović:
  Realization of certain admissible $A_1^{(1)}$*–modules*
- Jeffrey D. Adams:
  Shimura correspondence for split real groups
- Ljiljana Arambašić:
  Representations of Hilbert C^**–modules*
- Ioan Badulescu:
  Jacquet-Langlands transfert for some unitay representations
- Damir Bakić:
  Semiorthogonal Parseval wavelets
- Dubravka Ban:
  On Arthur's R–group
- Ivana Baranović:
  Principal subspace basis for modules of type D_4 *affine Lie algebra and Capparelli-Lepowsky-Milas' method*
- Dan Barbasch :
  Unitary dual for Hecke algebras with unequal parameters
- Corrine Blondel:
  Tadić's philosophy and propagation of types
- Janko Bračič:
  Synthesis with respect to Banach modules
- Matej Brešar:
  Derivations on Banach algebras
- Zhen-Qing Chen:
  On the Robin Problem in Fractal Domains
- Zhen-Qing Chen:
  Exit System and Lévy System of Time Changed Processes
- Pavle Goldstein:
  Crossed product of a quantum group by canonical endomorphism and Pimsner algebras
- Neven Grbac:
  On the residual spectrum of the hermitian quaternionic group of split rank 2
- Marcela Hanzer:
  Unitary dual of the non-split inner form of $Sp(8, F)$
- Guy Henniart:
  Representations of $GL(2)$*, old and new (3)*
- Dijana Ilišević:
  Bicircular projections on C^**–algebras*

- Niels Jacob:
  Potential theory for L^p*–sub-Markovian semigroups (3)*
- Miroslav Jerković:
  Recurrence relations for characters of type A_2 *affine Lie algebra*
- Andreas Kyprianou:
  Classic exit problems for spectrally one sided Levy processes: a modern slice of theory (3)
- Franz Luef:
  Hilbert C^**–modules in Gabor Analysis*
- Bojan Magajna:
  On cogenerators among operator modules
- Thomas Mikosch:
  Tales on regular variation (3)
- Dragan Miličić:
  Is "hard duality" really hard?
- Harry I. Miller:
  More new results on the A*–statistical convergence of double sequences*
- Allen Moy:
  The Bernstein center and orbital integrals (3)
- Goran Muić:
  Construction of generalized Steinberg representations for reductive p*–adic groups*
- Pavle Pandžić:
  A simple proof of Bernstein-Lunts equivalence
- Gilles Pisier:
  Similarity problems, amenability and completely bounded maps (3)
- Alexander Potrykus:
  Approximating a Feller semigroup by using the Yosida approximation of the symbol of its generator
- Mirko Primc:
  Combinatorial bases of Feigin-Stoyanovsky type subspaces and intertwining operators
- Rajna Rajić:
  Some generalizations of the numerical range in Hilbert C^**–modules*
- David Renard:
  Towards the unitary dual of $GL(n, D)$
- Gordan Savin:
  A series of representations for universal central extensions of $SO(p, q)$
- Gordan Savin:
  On the structure of internal modules with an application to minimal representations
- Tomislav Šikić:
  Partition of number n*, Weyl-Kac Character Formula (case* $A_n^{(1)}$*) and* q*–Series Identities*
- Boris Širola:
  On certain pairs of Lie algebras
- Renming Song:
  Estimates on the density of Brownian motion with singular drift
- András Telcs:
  Telcs Heat kernel estimates and the space-time scaling
- András Telcs:
  Parabolic mean value and Harnack inequalities for weakly homogeneous graphs

- Peter Trapa:
  Unipotent representations of $Sp(p,q)$ and $O^*(2n)$ (2)

- Goran Trupčević:
  Principal subspace basis for modules of type A_l affine Lie algebra and Capparelli-Lepowsky-Milas' method

- Aleksej Turnšek:
  On maps that preserve orthogonality in normed spaces

- David Vogan:
  Branching laws for representations of real reductive groups (4)

- Peter Wood:
  Hilbert C-modules and wavelets (3)*

- Darko Žubrinić:
  Fractal analysis of spiral trajectories of some planar vector fields, I

- Vesna Županović:
  Fractal analysis of spiral trajectories of some planar vector fields, II

We would like to thank all participants of the School and Conference, the University of Dubrovnik, and the employees of the Inter-University Centre in Dubrovnik for all they have done to make the present meeting yet another splendid event. All of this, however, would not be possible without financial support from the Croatian Ministry of Science and Technology and the Inter-University Centre, which we gratefully acknowledge. In addition, we would like to express our thanks to the Department of Mathematical Sciences, University of Aarhus, for publishing the present proceedings.

G.M. J.H.J.

Announcement. The next conference, Functional Analysis X, will take place at the Inter-University Centre in Dubrovnik from June 29 to July 6 June 2008. More information on this event can be found at the conference web-page:

`http://www.math.hr/~congress/Dubrovnik06`

List of articles

ON REALIZATION OF CERTAIN ADMISSIBLE $A_1^{(1)}$–MODULES

DRAŽEN ADAMOVIĆ

Abstract. We discuss our recent results on a realization of certain irreducible modules for the affine Lie algebra $A_1^{(1)}$. These modules are constructed by using the theory of generalized vertex algebras associated to rational lattices. In this paper we will show that our construction gives also a natural framework for studying simple current extensions of the simple vertex operator algebra $L(-\frac{4}{3}\Lambda_0)$.

1. Introduction

The explicit vertex operator construction of integrable modules for affine Kac–Moody Lie algebras gave important examples of vertex operator algebras (cf. [DL], [FLM], [F], [FFR]). Admissible modules are a broad class of highest weight modules which contains integrable modules as a subclass. These representations have many applications in the representation theory and in conformal field theory (cf. [KW], [Wak]). The admissible representations can be also investigated by using the framework of vertex operator algebras (cf. [A1], [A2], [AM], [DLM2], [G], [LMRS], [P]). But only in the case of symplectic affine Lie algebra $C_n^{(1)}$ of level $-\frac{1}{2}$, it was known an explicit realization of admissible representations (cf. [FF], [We]). Note that in the case $n = 1$ this construction gives the realization of four admissible $A_1^{(1)}$–modules.

In this paper we review our new realization of certain admissible $A_1^{(1)}$–modules from [A4] of level $-\frac{4}{3}$ which uses the generalized vertex algebra V_L. This realization was motivated by the representation theory of the Virasoro Lie algebra. In fact, we proved in [A4] that the vertex operator algebra $L(-\frac{4}{3}\Lambda_0)$ contains certain vertex subalgebras associated to (1,p) models for the Virasoro Lie algebra investigated in [A3].

The vertex operator algebra $L(-\frac{4}{3}\Lambda_0)$ is not rational. It contains a large class of weak modules outside the category $\mathcal{O}$. In [A4] we identify some weak $L(-\frac{4}{3}\Lambda_0)$–modules which are obtained by applying spectral flow automorphisms. In this paper we shall also prove a new result saying that certain simple current extensions of the vertex operator algebra $L(-\frac{4}{3}\Lambda_0)$ can be realized as subalgebras of V_L.

2. Vertex operator algebra $L(k\Lambda_0)$

In this section we recall some basic facts about vertex operator algebras associated to affine Lie algebras (cf. [FZ], [K2], [Li1], [MP]).

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and let $(\cdot,\cdot)$ be a nondegenerate symmetric bilinear form on $\mathfrak{g}$. The affine Lie algebra $\hat{\mathfrak{g}}$ associated with $\mathfrak{g}$ is defined as $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c\oplus\mathbb{C}d$, where c is the canonical central element [K1] and the Lie algebra structure is given by

$$[x\otimes t^n, y\otimes t^m] = [x,y]\otimes t^{n+m} + n(x,y)\delta_{n+m,0}c,$$

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$$[d, x \otimes t^n] = nx \otimes t^n$$

for $x, y \in \mathfrak{g}$. We write $x(n)$ for $x \otimes t^n$ and identify $\mathfrak{g}$ with $\mathfrak{g} \otimes t^0$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ and let $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the Cartan subalgebra of $\hat{\mathfrak{g}}$.

For arbitrary $\lambda \in \hat{\mathfrak{h}}^*$, let $L(\lambda)$ be the irreducible highest weight $\hat{\mathfrak{g}}$–module with highest weight λ.

For every $k \in \mathbb{C}$, $k \neq -h^\vee$, the irreducible $\hat{\mathfrak{g}}$–module $L(k\Lambda_0)$ carries the structure of a simple vertex operator algebra. Let ω be the Virasoro vector obtained by using the Sugawara construction (cf. [FZ], [MP]) and let $L(z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$ be the associated Virasoro field.

Admissible modules for affine Lie algebras are irreducible highest weight modules whose highest weights are admissible weights.

We call a weight $\lambda \in \hat{\mathfrak{h}}^*$ admissible if it satisfies the following two conditions:

(1) $(\lambda + \hat{\rho})(\alpha^\vee) \notin \{0, -1, -2, \cdots\}$ for all real positive coroots $\alpha^\vee$,

(2) $\mathbb{Q}R^\lambda = \mathbb{Q}\Pi^\vee$ where $\hat{\rho}$ is the sum of all fundamental weights, $\Pi^\vee$ is the set of simple coroots and $R^\lambda = \{\alpha^\vee : \text{a positive real coroot} \mid (\lambda + \hat{\rho})(\alpha^\vee) \in \mathbb{Z}\}$.

Let now $\mathfrak{g} = sl_2(\mathbb{C})$ with generators e, f, h and relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Let Λ_0, Λ_1 be the fundamental weights for $\hat{\mathfrak{g}}$.

We fix the normalized Killing form $(\cdot, \cdot)$ on $\mathfrak{g}$ such that $(h, h) = 2$. Let Λ_0, Λ_1 be the fundamental weights for $\hat{\mathfrak{g}}$.

A rational number $k = \frac{t}{u}$ is called admissible if t, u are coprime integers such that $u \geq 1$ and $2u + t - 2 \geq 0$. Let $k = t/u \in \mathbb{Q}$ be admissible. Define:

$$\lambda_{k,l,n} = (k - n + l(k+2))\Lambda_0 + (n - l(k+2))\Lambda_1$$

$$P^k = \{\lambda_{k,l,n}, \quad l, n \in \mathbb{Z}_{\geq 0},\ n \leq 2u + t - 2,\ l \leq u - 1\}.$$

The modules $L(\lambda)$, $\lambda \in P^k$, are all admissible $\hat{\mathfrak{g}}$–modules of level k (cf. [KW]). It was proved by that the admissible $\hat{\mathfrak{g}}$–modules of level k provides all irreducible $L(k\Lambda_0)$–modules from the category $\mathcal{O}$ (cf. [AM]).

In order to construct and investigate modules which don't belong to the category $\mathcal{O}$, one can consider the spectral flow automorphisms π_s of $U(\hat{\mathfrak{g}})$, which are uniquely determined by

$$\pi_s(e(n)) = e(n - s), \tag{2.1}$$
$$\pi_s(f(n)) = f(n + s), \tag{2.2}$$
$$\pi_s(h(n)) = h(n) - sk\delta_{n,0}. \tag{2.3}$$

One can also show that π_s can be extended to an automorphism of the vertex operator algebra $L(k\Lambda_0)$ and that

$$\pi_s(L(n)) = L(n) - \tfrac{s}{2}h(n) + \tfrac{1}{4}ks^2\delta_{n,0}. \tag{2.4}$$

For $h \in \mathfrak{h}$ we define

$$\Delta(h, z) = z^{h(0)} \exp\left(\sum_{n=1}^{\infty} \frac{h(n)}{-n}(-z)^{-n}\right).$$

Let $\alpha = -\frac{h}{2}$. Assume that M is a $L(k\Lambda_0)$–module. Let $s \in \mathbb{Z}$. The weak $L(k\Lambda_0)$–module $\pi_s(M)$ associated to the automorphism π_s is realized as (see [A4], [Li2]):

$$\pi_s(M) := \mathbb{C}e^{s\alpha} \otimes M \quad \text{as a vector space},$$

where $\mathbb{C}e^{s\alpha}$ is a one dimensional vector space with a distinguished basis element $e^{s\alpha}$, with action

$$Y(v,z)(e^{s\alpha}\otimes w)=e^{s\alpha}\otimes Y(\Delta(s\alpha,z)v,z)w \quad \text{for } v\in L(k\Lambda_0),\ w\in M.$$

Moreover, M is a weak $L(k\Lambda_0)$–module if and only if $\pi_s(M)$ is a weak $L(k\Lambda_0)$–module. It is important to notice that

$$\pi_{-1}(L(k\Lambda_0))=L(k\Lambda_1).$$

In the case when $k\in\mathbb{Z}_{>0}$ one has

$$\pi_{2i+1}(L(k\Lambda_0))=L(k\Lambda_1),\quad \pi_{2i}(L(k\Lambda_0))=L(k\Lambda_0)\quad (i\in\mathbb{Z}).$$

On the other hand when $k\notin\mathbb{Z}_{\geq 0}$, then the set $\{\pi_s(L(k\Lambda_0))\mid s\in\mathbb{Z}\}$ contains infinitely many inequivalent irreducible modules. Therefore in this case the vertex operator algebra $L(k\Lambda_0)$ has infinitely many irreducible weak modules outside the category $\mathcal{O}$. Relation (2.4) implies that in general these modules are not $\mathbb{Z}_{\geq 0}$–graded, so they can be consider only as weak modules.

Remark 2.1. *In the terminology of* [DLM1] *and* [Li2], *modules $\pi_s(L(k\Lambda_0)$ belong to an important family of (weak) $L(k\Lambda_0)$ modules called simple currents. In what follows we shall see that $e^{s\alpha}$ has nice interpretation in the context of (generalized) vertex algebras.*

3. Lattice construction of the vertex operator algebra $L(-\frac{4}{3}\Lambda_0)$

In this section we shall recall the realization of the vertex operator algebra $L(-\frac{4}{3}\Lambda_0)$ and their modules from [A4].

Let us first say that in the case $k=-\frac{4}{3}$ the set

$$\{L(-\tfrac{4}{3}\Lambda_0),\ L(-\tfrac{2}{3}\Lambda_0-\tfrac{2}{3}\Lambda_1),\ L(-\tfrac{4}{3}\Lambda_1)\}$$

provides all irreducible $L(-\frac{4}{3}\Lambda_0)$–modules from the category $\mathcal{O}$.

Define the following lattice

$$\widetilde{L}=\mathbb{Z}\gamma+\mathbb{Z}\delta,\qquad \langle\gamma,\gamma\rangle=-\langle\delta,\delta\rangle=\frac{1}{6},\ \langle\gamma,\delta\rangle=0.$$

Let $\mathfrak{h}=\mathbb{C}\otimes_{\mathbb{Z}}\widetilde{L}$. Extend the form $\langle\cdot,\cdot\rangle$ on $\widetilde{L}$ to $\mathfrak{h}$. Let $\hat{\mathfrak{h}}=\mathfrak{h}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c$ be the affinization of $\mathfrak{h}$.

For every $\lambda\in\mathfrak{h}$ denote by $M(1,\lambda)$ the irreducible highest weight $\hat{\mathfrak{h}}$–module generated by the highest weight vector v_λ satisfying

$$cv_\lambda=v_\lambda,\quad h(n)v_\lambda=\delta_{n,0}\langle\lambda,h\rangle v_\lambda\ \ (n\geq 0).$$

For $\alpha\in\mathfrak{h}$ and $n\in\mathbb{Z}$ write $\alpha(n)=\alpha\otimes t^n$. Set $\alpha(z)=\sum_{n\in\mathbb{Z}}\alpha(n)z^{-n-1}$.

Then $M(1):=M(1,0)$ is a vertex algebra which is generated by the fields $\alpha(z)$, $\alpha\in\mathfrak{h}$, and $M(1,\lambda)$, for $\lambda\in\mathfrak{h}$, are irreducible modules for $M(1)$.

Now we shall study the generalized vertex algebra $V_{\widetilde{L}}$ associated to the rational lattice $\widetilde{L}$.

Let $\kappa=e^{\frac{\pi i}{12}}$. Define the 2-cocycle $\varepsilon:\widetilde{L}\times\widetilde{L}\to\mathbb{C}^\times$ by

$$\varepsilon(A\gamma+B\delta,C\gamma+D\delta)=\kappa^{(A+B)(C-D)}$$

where $A, B, C, D \in \mathbb{Z}$. Let $\mathbb{C}\{\widetilde{L}\}$ be the twisted group algebra with basis $\{e^\alpha,\ \alpha \in \widetilde{L}\}$ and multiplication $e^\alpha e^\beta = \varepsilon(\alpha,\beta)e^{\alpha+\beta}$. On the vector space

$$V_{\widetilde{L}} = M(1) \otimes \mathbb{C}\{\widetilde{L}\}$$

exists a natural structure of a generalized vertex algebra.

We shall describe the vertex operator map

$$Y : V_{\widetilde{L}} \to (\mathrm{End} V_{\widetilde{L}})\{z\}.$$

Let $v = \alpha_1(-n_1)\alpha_2(-n_2)\cdots\alpha_k(-n_k) \otimes e^\beta$ be an element of $V_{\widetilde{L}}$.

$$\begin{aligned} Y(v,z) := \quad & {\circ \atop \circ} \left(\tfrac{1}{(n_1-1)!}\left(\tfrac{\partial}{\partial z}\right)^{n_1-1}\alpha_1(z)\right)\cdots \\ & \cdot\left(\tfrac{1}{(n_k-1)!}\left(\tfrac{\partial}{\partial z}\right)^{n_k-1}\alpha_k(z)\right) \\ & E^-(-\beta,z)E^+(-\beta,z)e^\beta z^\beta {\circ \atop \circ}, \end{aligned}$$

where

$$E^{\pm}(\beta,z) := \exp\left(\sum_{n=1}^{\infty}\frac{\beta(\pm n)}{\pm n}z^{\mp n}\right).$$

The action of $\alpha(n)$, e^β, z^β on $u \otimes e^\lambda \in V_{\widetilde{L}}$ are

$$\begin{aligned} \alpha(n).(u\otimes e^\lambda) &= \begin{cases} (\alpha(n)u)\otimes e^\lambda & \text{if } n<0 \\ (n\frac{\partial}{\partial\alpha(-n)}u)\otimes e^\lambda & \text{if } n>0 \\ \langle\alpha,\lambda\rangle u\otimes e^\lambda & \text{if } n=0 \end{cases} \\ e^\beta.(u\otimes e^\lambda) &:= \varepsilon(\beta,\lambda)(u\otimes e^{\beta+\lambda}), \\ z^\beta.(u\otimes e^\lambda) &:= u\otimes e^\lambda z^{\langle\beta,\lambda\rangle}. \end{aligned}$$

Now we define the screening operators:

$$\begin{aligned} Q &= \mathrm{Res}_z Y(e^{-6\gamma},z) = e_0^{-6\gamma}, \\ \widetilde{Q} &= \mathrm{Res}_z Y(e^{2\gamma},z) = e_0^{2\gamma}, \end{aligned}$$

and the following Virasoro element

$$\omega = 3\gamma(-1)^2 - 2\gamma(-2) - 3\delta(-1)^2.$$

Then ω generates the Virasoro vertex operator algebra $L^{Vir}(-6,0)$. Set

$$L(z) = Y(\omega,z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}.$$

Define the following vectors in $V_{\widetilde{L}}$:

$$\begin{aligned} e &= e^{3(\gamma-\delta)}, \\ h &= 4\delta(-1), \\ f &= -\tfrac{2}{9}Qe^{3(\gamma+\delta)} \\ &= -(4\gamma(-1)^2 - \tfrac{2}{3}\gamma(-2))e^{-3(\gamma-\delta)}. \end{aligned}$$

Then e, h, f are primary vectors of conformal weight 1 for the Virasoro algebra.

Define

$$L = \mathbb{Z}(\gamma + \delta) + \mathbb{Z}(\gamma - \delta).$$

Clearly, V_L is a subalgebra of $V_{\widetilde{L}}$ which contains vectors e, f and h.

Theorem 3.1. [A4] *The vectors e, f and h span a subalgebra of the generalized vertex algebra V_L isomorphic to the vertex operator algebra $L(-\frac{4}{3}\Lambda_0)$. Moreover, V_L is a weak module for the vertex operator algebra $L(-\frac{4}{3}\Lambda_0)$. In particular, the linear map $\rho : \hat{\mathfrak{g}} \to End\, V_L$ determined by*

$$\begin{aligned}
&\rho : c \mapsto -\tfrac{4}{3} Id \\
&\rho : d \mapsto -L(0) \\
&\rho : e(n) \mapsto e_n^{3(\gamma-\delta)} \\
&\rho : h(n) \mapsto 4\delta(n) \\
&\rho : f(n) \mapsto -\tfrac{2}{9}(Q e_n^{3(\gamma-\delta)} - e_n^{3(\gamma-\delta)} Q)
\end{aligned}$$

for $n \in \mathbb{Z}$ is a representation of $\hat{\mathfrak{g}}$ on V_L.

Theorem 3.1 gives that V_L carries the structure of a $U(\hat{\mathfrak{g}})$–module. This module is not completely reducible. On the other hand it contains an infinite family of irreducible $U(\hat{\mathfrak{g}})$–submodules. We have the following result.

Theorem 3.2. [A4] *For every $s \in \mathbb{Z}$ we have:*

(1)

$$U(\hat{\mathfrak{g}}).e^{2s\delta} \cong \pi_{-s}(L(-\tfrac{4}{3}\Lambda_0)).$$

(2)

$$U(\hat{\mathfrak{g}}).e^{-\gamma+(2s+1)\delta} \cong \pi_{-s}(L(-\tfrac{2}{3}\Lambda_0 - \tfrac{2}{3}\Lambda_1)).$$

4. Simple current extensions of $L(-\frac{4}{3}\Lambda_0)$

In this section we will use the explicit realization of $L(-\frac{4}{3}\Lambda_0)$–modules from [A4], and show that on the generalized vertex algebra V_L we can realize some extensions of $L(-\frac{4}{3}\Lambda_0)$.

For every $s \in \mathbb{Z}$, we define the following linear map

$$\Phi^{(s)} = e^{-2s\delta}\varepsilon(-2s\delta, \cdot) : V_L \to V_L.$$

It is clear that $\Phi^{(s)}$ is a linear bijection.

Lemma 4.1. *We have*

$$x(n)\Phi^{(s)} = \Phi^{(s)}\pi_s(x(n))$$

for every $s, n \in \mathbb{Z}$, $x \in \mathfrak{g}$.

Proof. First we notice that $h(n)\Phi^{(s)} = \Phi^{(s)}\pi_s(h(n))$. It remains to prove that

$$\Phi^{(s)}Y(e,z) = z^s Y(e,z)\Phi^{(s)}, \tag{4.5}$$

$$\Phi^{(s)}Y(f,z) = z^{-s} Y(f,z)\Phi^{(s)}. \tag{4.6}$$

We have

$$
\begin{aligned}
& \Phi^{(s)}Y(e,z) \\
= \; & e^{-2s\delta}\varepsilon(-2s\delta, 3\gamma-3\delta)\cdot \\
& E^-(-3\gamma+3\delta,z)E^+(-3\gamma+3\delta,z)e^{3\gamma-3\delta}z^{3\gamma-3\delta}\varepsilon(-2s\delta,\cdot) \\
= \; & \varepsilon(-2s\delta, 3\gamma-3\delta)^2\cdot \\
& E^-(-3\gamma+3\delta,z)E^+(-3\gamma+3\delta,z)e^{3\gamma-3\delta}e^{-2s\delta}z^{3\gamma-3\delta}\varepsilon(-2s\delta,\cdot) \\
= \; & E^-(-3\gamma+3\delta,z)E^+(-3\gamma+3\delta,z)e^{3\gamma-3\delta}z^{3\gamma-3\delta}e^{-2s\delta}z^{s}\varepsilon(-2s\delta,\cdot) \\
= \; & z^sY(e,z)\Phi^{(s)},
\end{aligned}
$$

and therefore (4.5) holds. Next we notice that $\gamma(n)e^{-2s\delta} = e^{-2s\delta}\gamma(n)$. Then the proof of relation (4.6) is similar to that of (4.5). □

By using Lemma 4.1 we see that

$$
\begin{aligned}
& \Phi^{(r)}\pi_s(L(-\tfrac{4}{3}\Lambda_0)) = \Phi^{(r)}U(\hat{\mathfrak{g}}).e^{-2s\delta} = \\
= \; & U(\hat{\mathfrak{g}}).e^{-(2r+2s)\delta} = \pi_{r+s}(L(-\tfrac{4}{3}\Lambda_0)). \qquad (4.7)
\end{aligned}
$$

Define now:

$$\widetilde{V} = \bigoplus_{s\in\mathbb{Z}} \Phi^{(s)}L(-\tfrac{4}{3}\Lambda_0) = \bigoplus_{s\in\mathbb{Z}} \pi_s(L(-\tfrac{4}{3}\Lambda_0)), \qquad (4.8)$$

$$\overline{V} = \bigoplus_{s\in\mathbb{Z}} \Phi^{(3s)}L(-\tfrac{4}{3}\Lambda_0) = \bigoplus_{s\in\mathbb{Z}} \pi_{3s}(L(-\tfrac{4}{3}\Lambda_0)). \qquad (4.9)$$

Theorem 4.1. *We have*

(1) *The vector space* $\widetilde{V}$ *carries the structure of a generalized vertex algebra.*

(2) *The vector space* $\overline{V}$ *carries the structure of a vertex algebra.*

Proof. Let $a \in \pi_r(L(-\frac{4}{3}\Lambda_0))$, $b \in \pi_s(L(-\frac{4}{3}\Lambda_0))$. By using Lemma 4.1, relation (4.7) and the vertex operator formula, one has:

$$a_n b \in \pi_{r+s}(L(-\tfrac{4}{3}\Lambda_0)) \quad (n \in \mathbb{Z}).$$

This implies that $\widetilde{V}$ and $\overline{V}$ are subalgebras of the generalized vertex algebra V_L, and therefore they are also generalized vertex algebras. Next we notice that

$$\overline{V} \subset V_{L_0} \subset V_L$$

where $L_0 = \mathbb{Z}(3\gamma - 3\delta) + \mathbb{Z}(3\gamma + 3\delta)$. Since L_0 is an even lattice, we have that V_{L_0} is a vertex algebra (see also Remark 4.2 of [A4]). Therefore $\overline{V}$ has the vertex algebra structure. □

Remark 4.1. *The vertex algebra* $\overline{V}$ *is the extension of the vertex operator algebra* $L(-\frac{4}{3}\Lambda_0)$ *by an infinite family of simple current* $L(-\frac{4}{3}\Lambda_0)$*–modules. Our proof that* $\overline{V}$ *is a vertex algebra is based on the embedding of* $\overline{V}$ *into the lattice vertex algebra* V_{L_0}*. On the other hand, we have the extension theory of vertex operator algebras of affine type developed by C. Dong, H.Li and G. Mason in* [DLM1], [Li3]*. This theory is based on the construction of simple currents by using semisimple primary elements.*

In our case simple currents are constructed from the elements of the lattice $D = 6\mathbb{Z}\delta \subset \mathfrak{h}$*. By using results and constructions from Section 3 of* [Li3] *one can also see that the vector space*

$$\mathbb{C}[D] \otimes L(-\tfrac{4}{3}\Lambda_0) = \bigoplus_{\beta \in D} e^{\beta} \otimes L(-\tfrac{4}{3}\Lambda_0)$$

has the vertex algebra structure, and that this vertex algebra is isomorphic to our vertex algebra $\overline{V}$*. So we get an explicit realization of the extended vertex algebra* $\mathbb{C}[D] \otimes L(-\frac{4}{3}\Lambda_0)$ *on the lattice vertex algebra* V_{L_0}*.*

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ZELEVINSKY INVOLUTION AND MOEGLIN-WALDSPURGER ALGORITHM FOR $\mathrm{GL}_n(D)$

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Abstract. In this short note, we remark that the algorithm of Moeglin and Waldspurger for computing the dual (as defined by Zelevinsky) of an irreducible representation of GL_n still works for the inner forms of GL_n, the proof being basically the same.

1. Segments, multisegments and the involution

A **multiset** is a finite set with finite repetitions $(a, a, b, c, d, d, d, e, a, ...)$. A **segment** Δ is the void set or a set of consecutive integers $\{b, b+1, ..., e\}$, $b, e \in \mathbb{Z}$, $b \leq e$. We then call e the **ending** of Δ and the integer $e-b+1$ the **length** of Δ. By convention, the length of the void segment is 0. Let $\Delta = \{b, b+1, ..., e\}$ and $\Delta' = \{b', b'+1, ..., e'\}$ be two segments. We say Δ **precede** Δ' if $b < b'$, $e < e'$ and $b' \leq e+1$. We also write $\Delta \geq \Delta'$ if $b > b'$ or $b = b'$ and $e \geq e'$. This is a total order on the set of segments.

A **multisegment** is a multiset of segments. We identify multisegments obtained from each other by dropping or adding void segments. The **full extended length** of a multisegment is the sum of the lengths of all its elements and is 0 if the multisegment is void. The **support** of a multisegment m is the multiset of integers obtained by taking the union (with repetitions) of the segments in m. A multisegment $(\Delta_1, \Delta_2, ..., \Delta_k)$ is said to be ordered if $(\Delta_1 \geq \Delta_2 \geq ... \geq \Delta_t)$. The lexicographic order induces a total order on ordered multisegments : if $m = (\Delta_1, \Delta_2, ..., \Delta_t)$ and $m' = (\Delta'_1, \Delta'_2, ..., \Delta'_{t'})$ are multisegments, then $m \geq m'$ if $\Delta_1 > \Delta'_1$, or $\Delta_1 = \Delta'_1$ and $\Delta_2 > \Delta'_2$, and so on, or $t \geq t'$ and $\Delta_i = \Delta'_i$ for all $i \in \{1, 2, ..., t'\}$.

If $\Delta = \{b, b+1, ..., e\}$ is a non void segment, we set $\Delta^- = \{b, b+1, ..., e-1\}$ with the convention that Δ^- is void if $b = e$.

Let m be a non void multisegment. We associate to m a multisegment $m^\#$ in the following way : let d be the biggest ending of a segment in m. Then chose a segment Δ_{i_0} in m containing d and maximal for this property. Then we define the integers $i_1, i_2, ..., i_r$ inductively : Δ_{i_s} is a segment of m preceding $\Delta_{i_{s-1}}$ with ending $d-s$, maximal with these properties, and r is such that there's no possibility to find such a i_{r+1}. Set $m^- = (\Delta'_1, \Delta'_2, ..., \Delta'_t)$, where $\Delta'_i = \Delta_i$ if $i \notin \{i_0, i_1, ..., i_r\}$, and $\Delta'_i = \Delta_i^-$ if $i \notin \{i_0, i_1, ..., i_r\}$. Then $\{d-r, d-r+1, ..., d\}$ is the first segment of $m^\#$. Starting from the beginning with m^- what we have done with m, we find the second segment of $m^\#$, and so on (so at the end we have that $m^\#$ is the multiset union of $\{\{d-s, d-s+1, ..., d\}\}$ and $(m^-)^\#$). This multisegment $m^\#$ is independent of the choices made for the construction. The map $m \mapsto m^\#$ is an involution of the set of non void multisegments. It preserves the support. See [MW].

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2. Representations of G_n

2.1. **Generalities.** Let F be an non-Archimedean local field of any characteristic with norm $|\ |_F$. For all $n \in \mathbb{N}^*$ let G_n be the group $GL_n(F)$, $\mathcal{A}_n$ be the set of equivalence classes of smooth finite length representations of G_n and $\mathcal{R}_n$ be the Grothendieck group of smooth finite length representations of G_n. As usual, we will slightly abuse notation by identifying representations and their equivalence classes, and sometimes, representations with their image in the Grothendieck group $\mathcal{R}_n$.

The set B_n of classes of smooth irreducible representations of G_n is a basis of $\mathcal{R}_n$. If $\pi_1 \in B_{n_1}$ and $\pi_2 \in B_{n_2}$, then $\pi_1 \otimes \pi_2$ is a representation of $G_{n_1} \times G_{n_2}$. This group may be seen as the subgroup L of matrices diagonal by two blocks of size n_1 and n_2 of $G_{n_1+n_2}$. We set

$$\pi_1 \times \pi_2 = \mathrm{ind}_P^{G_{n_1+n_2}}(\pi_1 \otimes \pi_2)$$

where "ind" is the normalized parabolic induction functor and P is the parabolic subgroup of $G_{n_1+n_2}$ containing L and the group of upper triangular matrices. We generalize this notation in an obvious way to any finite number of elements $\pi_i \in B_{n_i}$, $i \in \{1, 2, ..., k\}$.

Let $\mathcal{C}_n$ be the set of cuspidal representations of G_n and $\mathcal{D}_n$ the set of essentially square integrable representations of G_n (we assume irreducibility in the definition of cuspidal and essentially square integrable representations).

If χ is a smooth character of G_n and $\pi \in \mathcal{A}_n$, then $\chi\pi$ will denote the tensor product representation $\chi \otimes \pi$. Let ν_n be the character $g \mapsto |det(g)|_F$ of G_n. We will drop the index n when no confusion may occur.

2.2. **Irreducible representations.** Let $k \in \mathbb{N}^*$ and n_i, $i \in \{1, 2, ..., k\}$ be positive integers. For each i let $\sigma_i \in \mathcal{D}_{n_i}$. The representations σ_i being essential square integrable, for all $i \in \{1, 2, ..., k\}$ there exists a unique real number a_i such that $\nu^{a_i}\sigma_i$ is unitary. If the σ_i are ordered such that the sequence (a_i) is increasing, then $S = \sigma_1 \times \sigma_2 \times ... \times \sigma_k$ is called a **standard representation** and has a unique irreducible quotient $\theta(S)$. The representation S doesn't depend on the order of the σ_i as long as the condition that the sequence (a_i) is increasing is fullfiled. So S and $\theta(S)$ depend only on the multiset $(\sigma_1, \sigma_2, ..., \sigma_k)$. We call this multiset the **esi-support** of S or of $\theta(S)$ ("esi" : essentially square integrable).

2.3. **Standard elements.** The image in $\mathcal{R}_n$ of a standard representation is called a **standard element** of $\mathcal{R}_n$. The set H_n of standard elements of $\mathcal{R}_n$ is a basis of $\mathcal{R}_n$. The map $W_n : S \mapsto \theta(S)$ is a bijection from H_n to B_n (see [DKV]).

2.4. **The involution.** On $\mathcal{R}_n$, we consider the involution I_n from [Au], which transforms irreducible representations to irreducible representations up to a sign. The involution commutes with induction ([Au]), i.e. if $\pi_1 \in B_{n_1}$ and $\pi_2 \in B_{n_2}$, then $I_{n_1+n_2}(\pi_1 \times \pi_2) = I_{n_1}(\pi_1) \times I_{n_2}(\pi_2)$. Forgetting signs, the involution in [Au] gives rise to a permutation $|I_n|$ of B_n (which is the involution defined in [Ze]). We will call $|I_n|(\pi)$ the **dual** of π. See [Au] and [Ze].

The algorithm of Moeglin and Waldspurger ([MW]) computes the esi-support of the dual of a smooth irreducible representation π from the esi-support of π.

2.5. **Essentially square integrable representations.** Following [Ze], if k is a positive integer such that $k|n$, if we set $p = n/k$ and chose $\rho \in \mathcal{C}_p$, then $\rho \times \nu\rho \times \nu^2\rho\times, ..., \times\nu^{k-1}\rho$

has a unique irreducible quotient $Z(k,\rho)$ which is an essentially square integrable representation of G_n. Any element σ of $\mathcal{D}_n$ is obtained in this way and σ determines k and ρ such that $\sigma = Z(k,\rho)$. If $\rho \in \mathcal{C}_p$ for some p, given a segment $\Delta = \{b, b+1, ..., e\}$, we set

$$< \Delta >_\rho = Z(\nu^b\rho, e-b+1) \in \mathcal{D}_{p(e-b+1)}.$$

2.6. **Rigid representations.** If $\rho \in \mathcal{C}_p$ for some p we call the set $\{\nu^k\rho\}_{k\in\mathbb{Z}}$ **the ρ-line.** If $\pi \in B_n$ we say π is **ρ-rigid** if the cuspidal support of π is included in the ρ-line (of course, it is the $\nu\rho$-line too). An irreducible representation is called **rigid** if it is ρ-rigid for some ρ. If $\pi_1 \in B_{n_1}$ and $\pi_2 \in B_{n_2}$ are such that the cuspidal supports of π_1 and π_2 are disjoint, then $\pi_1 \times \pi_2$ is irreducible. So any $\pi \in B_n$ is a product of rigid representations π_i. Then we know ([Ze]) that the esi-support of π is the reunion with multiplicities of the esi-supports of the π_i. As I_n commutes with induction, to compute the esi-support of duals of irreducible representations, we need only to compute the esi-support of duals of rigid representations.

2.7. **Multisegments and representations.** If $m = (\Delta_1, \Delta_2, ..., \Delta_k)$ is an ordered multisegment of full length q and $\rho \in \mathcal{C}_p$, then m and ρ define a standard element $\pi_\rho(m)$ of $\mathcal{R}_{pq}$, precisely

$$\pi_\rho(m) = < \Delta_1 >_\rho \times < \Delta_2 >_\rho \times, ..., \times < \Delta_k >_\rho \in H_{pq},$$

and an irreducible representation

$$< m >_\rho = W_n(\pi_\rho(m)) \in B_{pq}.$$

The map $m \mapsto < m >_\rho$ realizes a bijection between the set of multisegments of full length q and the set $B_{n,\rho}$ of ρ-rigid irreducible representations of G_{pq}.

2.8. **The algorithm for G_n.** The result of Moeglin and Waldspurger in [MW] is : the dual of $< m >_\rho$ is $< m^\# >_\rho$.

2.9. **The proof.** We recall here their argument:

Let (p,ρ) be a couple such that p is a positive integer and $\rho \in \mathcal{C}_p$. Fix a multiset s with integer entries, and let S be the (finite) set of all the multisegments m having support s. They all have the same full length, let's call it k. Set $n = pk$. Let $B_\rho = \{< m >_\rho, m \in S\}$ and $H_\rho = \{\pi_\rho(m), m \in S\}$. Let $\mathcal{R}_\rho$ be the (finite dimensional) submodule of $\mathcal{R}_n$ generated by B_ρ. Then B_ρ and H_ρ are basis of the space $\mathcal{R}_\rho$. On B_ρ and H_ρ consider the decreasing order induced by the order on multisegments in S. Then we know that for this order the matrix M of H_ρ in the basis B_ρ is upper triangular and unipotent ([Ze] or [DKV]). The space $\mathcal{R}_\rho$ is stable under I_n. It is important to notice here that the involution $(-1)^{n-k}I_n$ of $\mathcal{R}_\rho$ transforms every irreducible representation in an irreducible one, since all the elements here have the same cuspidal support, of full length k (see [Au]). In other words, the restriction of $|I_n|$ to B_ρ is $(-1)^{n-k}I_n$.

Let T_1 (resp. T_2) be the matrix of the involution $(-1)^{n-k}I_n$ of $\mathcal{R}_\rho$ in the basis B_ρ (resp. H_ρ). *Then the matrix T_1 doesn't depend on the couple (p,ρ).* The argument, attributed in [MW] to Oesterlé, is the following:

We have already seen that T_1 is a permutation matrix ([Au]). Then as M is an upper triangular unipotent matrix, the relation $T_2 = M^{-1}T_1M$ is a Bruhat decomposition for T_2 and this implies that T_1 is determined by T_2.

Now, T_2 itself doesn't depend on the couple (p,ρ) because:

(c1) if $m=(\Delta_1,\Delta_2,...,\Delta_t)$ with Δ_i of length n_i/p, then

$$I_n(\pi_\rho(m))=I_{n_1}(<\Delta_1>_\rho)\times I_{n_2}(<\Delta_2>_\rho)\times...\times I_{n_t}(<\Delta_t>_\rho),$$

(c2) if $\Delta=\{b,b+1,...,e\}$, then

$$I_{(e+1-b)p}(<\Delta>_\rho)=(-1)^{(e+1-b)(p-1)}<m_\Delta>_\rho,$$

where $m_\Delta=(\{b\},\{b+1\},...,\{e\})$,

(c3) one has $<m_\Delta>_\rho=\sum_{m'\leq m_\Delta}(-1)^{d(m')+e-b+1}\pi_\rho(m')$, where $d(m')$ is the cardinality of m' (as a multiset of segments) ([Ze]).

So it is enough to show that the dual of $<m>_\rho$ is $<m^\#>_\rho$ for a particular ρ. The authors conclude their proof by showing this relation holds for a clever choice of the cuspidal representation ρ.

3. Representations of G'_n

Let D be a central division algebra of dimension d^2 over F (with $d\in\mathbb{N}^*$) and let G'_n be the group $GL_n(D)$. We use the notation for objects relative to G_n, but with a prime, for objects relative to G'_n : $\mathcal{A}'_n$, $\mathcal{C}'_n$, $\mathcal{D}'_n$, $\mathcal{R}'_n$, B'_n... The involution I'_n ([Au]) on $\mathcal{R}'_n$, has the same properties as I_n : it transforms irreducible representations into irreducible representations, up to a sign, and commutes with induction.

If $g'\in G'_n$, one can define the characteristic polynomial $P_{g'}\in F[X]$ of g', and $P_{g'}$ is monic of degree nd ([Pi]). If $g'\in G'_n$, the determinant $det(g')$ of g' is the constant term of its characteristic polynomial. We write ν'_n for the character $g'\mapsto|det(g')|_F$ of G'_n, and we drop the index n when no confusion may occur.

For a given n, if $g\in G_{nd}$ and $g'\in G'_n$ we write $g\leftrightarrow g'$ if the characteristic polynomial of g is separable (i.e. has distinct roots in an algebraic closure of F) and is equal to the characteristic polynomial of g'. If $\pi\in\mathcal{R}_{nd}$ or $\pi\in\mathcal{R}'_n$, we denote by χ_π the character of π. It is well defined on the set of elements with separable characteristic polynomial even if the characteristic of F is not zero. The Jacquet-Langlands correspondence is the following result :

Theorem 3.1. *There exists a unique bijection* $\mathbf{C}:\mathcal{D}_{nd}\to\mathcal{D}'_n$ *such that for all* $\pi\in\mathcal{D}_{nd}$ *one has*

$$\chi_\pi(g)=(-1)^{nd-n}\chi_{\mathbf{C}(\pi)}(g')$$

for all $g\leftrightarrow g'$.

This well known result of [DKV] is also true in non-zero characteristic ([Ba1]).

One can extend the Jacquet-Langlands correspondence to a linear map between Grothendieck groups ([Ba2]) :

Proposition 3.2. *a) There exists a unique group morphism* $\mathbf{LJ}:\mathcal{R}_{nd}\to\mathcal{R}'_n$ *such that for all* $\pi\in\mathcal{R}_{nd}$ *one has*

$$\chi_\pi(g)=(-1)^{nd-n}\chi_{\mathbf{LJ}(\pi)}(g')$$

for all $g\leftrightarrow g'$.

The morphism $\mathbf{LJ}$ *is defined on the basis* H_{nd} *: if* $S=\sigma_1\times\sigma_2\times...\times\sigma_k$, *with* $\sigma_i\in\mathcal{D}_{n_i}$, *then*

- if for all $i\in\{1,2,...,k\}$, $d|n_i$,

$$\mathbf{LJ}(S)=\mathbf{C}(\sigma_1)\times\mathbf{C}(\sigma_2)\times...\times\mathbf{C}(\sigma_k),$$

- if not, $\mathbf{LJ}(S)=0$.

b) For all $\pi \in \mathcal{R}_{nd}$, $\mathbf{LJ}(I_{nd}(\pi)) = (-1)^{nd-n} I'_n(\mathbf{LJ}(\pi))$.

The classification of irreducible representations is similar to the one for G_n, and we can define the esi-support of an irreducible representation, the standard elements H'_n and the bijection $W'_n : H'_n \to B'_n$. Knowing the esi-support of $\pi' \in B'_n$, one would like to compute the esi-support of $|I'_n|(\pi')$.

The classification of essentially square integrable representations on G'_n differs slightly from that on G_n (it is more general, since $G'_n = G_n$ when $D = F$). If $\rho' \in \mathcal{D}'_n$, then $\mathbf{C}^{-1}(\rho') \in \mathcal{D}_{nd}$. Following [Ta], if $\mathbf{C}^{-1}(\rho') = Z(k, \rho)$, we set $s(\rho') = k$, and $\nu_{\rho'} = (\nu')^{s(\rho')}$. Given a positive integer k such that $k|n$ and a $\rho' \in \mathcal{C}'_p$ where $p = n/k$, the representation $\rho' \times \nu_{\rho'}\rho \times \nu^2_{\rho'}\rho' \times ... \times \nu^{k-1}_{\rho'}\rho'$ has a unique irreducible quotient σ' which is an essentially square integrable representation of G'_n. We set then $\sigma' = T(k, \rho')$. Any $\sigma' \in \mathcal{D}'_n$ is obtained in this way and σ' determines k and ρ' such that $\sigma' = T(k, \rho')$. See [Ta] for details.

If $\rho' \in \mathcal{C}_p$ for some p, given a segment $\Delta = \{b, b+1, ..., e\}$, we set

$$< \Delta >_{\rho'} = T(\nu^b_{\rho'}\rho', e - b + 1) \in \mathcal{D}'_{p(e-b+1)}.$$

A line in this setting is a set of the form $\{\nu^k_{\rho'}\rho'\}_{k\in\mathbb{Z}}$ where ρ' is a cuspidal representation. The definition of ρ'-rigid and rigid representations and their properties are similar to the ones for G_n ([Ta]), and as for G_n, one needs only to compute the esi-support of the duals for rigid representations.

If $m = (\Delta_1, \Delta_2, ..., \Delta_k)$ is an ordered multisegment and $\rho' \in \mathcal{C}'_p$, then m and ρ' define a standard element of some $\mathcal{R}'_n$, more precisely

$$\pi'_{\rho'}(m) = < \Delta_1 >_{\rho'} \times < \Delta_2 >_{\rho'} \times, ..., \times < \Delta_k >_{\rho'},$$

and an irreducible representation

$$< m >_{\rho'} = W'_n(\pi'_{\rho'}(m)).$$

The map $\pi'_{\rho'}$ realizes a bijection between the set of multisegments of full length k and the set of ρ'-rigid representations of G'_{pk}. Now, we claim that the algorithm for G'_n is the same as for G_n, namely :

Theorem 3.3. *The dual of the representation* $< m >_{\rho'}$ *is* $< m^\# >_{\rho'}$.

For the proof, we follow the argument in [MW] :

Let (p, ρ') be a couple such that p is a positive integer and $\rho \in \mathcal{C}'_p$, let k be a positive integer and set $n = pk$. Let $B'_{\rho'} = \{< m >_{\rho'}, m \in S\}$ and $H'_{\rho'} = \{\pi'_{\rho'}(m), m \in S\}$ (S has already been defined in the section 2.9). Let $\mathcal{R}'_{\rho'}$ be the finite dimensional submodule of $\mathcal{R}'_n$ generated by $B'_{\rho'}$. Then $B'_{\rho'}$ and $H'_{\rho'}$ are bases of $\mathcal{R}'_{\rho'}$. On $B'_{\rho'}$ and $H'_{\rho'}$ consider the decreasing order induced by the order on multisegments in S. Then the matrix M' of $H'_{\rho'}$ in the basis $B'_{\rho'}$ is upper triangular and unipotent ([DKV] and [Ta]). The involution $(-1)^{n-k} I'_n$ induces an involution of $\mathcal{R}'_{\rho'}$ which carries irreducible representations to irreducible representations. Let T'_1 (resp. T'_2) be the matrix of this involution in the basis $B'_{\rho'}$ (resp. $H'_{\rho'}$).

As for G_n, the matrix T'_1 doesn't depend on (p, ρ'), because Oesterlé's argument works again. First of all (see [Au]), T'_1 is a permutation matrix so the relation $T'_2 = M'^{-1} T'_1 M'$ is a Bruhat decomposition for T'_2 and this implies that T'_1 is determined by T'_2.

As for G_n, T'_2 itself doesn't depend on (p, ρ') because, as we will explain shortly afterwards, we have :

(c'1) If $m = (\Delta_1, \Delta_2, ..., \Delta_t)$ with Δ_i of length n_i/p, then

$$I'_n(\pi'_{\rho'}(m)) = I'_{n_1}(< \Delta_1 >_{\rho'}) \times I'_{n_2}(< \Delta_2 >_{\rho'}) \times ... \times I'_{n_t}(< \Delta_t >_{\rho'}).$$

(c'2) If $\Delta = \{b, b+1, ..., e\}$, then

$$I'_{(e+1-b)p}(< \Delta >_{\rho'}) = (-1)^{(e+1-b)(p-1)} < m_\Delta >_{\rho'},$$

where $m_\Delta = (\{b\}, \{b+1\}, ..., \{e\})$.

(c'3) One has $< m_\Delta >_{\rho'} = \sum_{m' \leq m_\Delta} (-1)^{d(m')+e-b+1} \pi'_{\rho'}(m')$, where $d(m')$ is the cardinality of m' (as a multiset of segments).

The relation (c'1) is clear since the involution commutes with induction ([Au]).

(c'2) is true too: from the formula for I'_n to be found in [Au], and the computation in [DKV] of all normalized parabolic restrictions of essentially square integrable representations of G'_n, one may see $I'_{(e+1-b)p}(< \Delta >_{\rho'})$ is an alternate sum of representations $\pi'_{\rho'}(m_i)$, where m_i runs over the set of multisegments with same support as Δ. It is obvious that the maximal one is $\pi'_{\rho'}(m_\Delta)$. It appears in the sum with coefficient $(-1)^{(e+1-b)(p-1)}$, and so $W'_n(\pi'_{\rho'}(m_\Delta)) =< m_\Delta >_{\rho'}$, has to appear with coefficient $(-1)^{(e+1-b)(p-1)}$ in the final result. As we know a priori that this result is plus or minus an irreducible representation, (c'2) follows.

(c'3) is the combinatorial inversion formula ([Ze]), which is still true here since for all $m' \leq m_\Delta$ one has $\pi'_{\rho'}(m') = \sum_{m'' \leq m'} < m'' >_{\rho'}$.

So it is enough to show that the dual of $< m >_{\rho'}$ is $< m^\# >_{\rho'}$ for a particular ρ'. Or, equivalently, to show that for some ρ' we have $T'_2 = T_2$. Let $\rho \in \mathcal{C}_d$ and set $\rho' = \mathbf{C}(\rho)$. Then $\rho' \in \mathcal{C}'_1$ and $s(\rho') = 1$. The map **LJ** induces a bijection from H_ρ to $H'_{\rho'}$ commuting with the bijections from S onto these sets. The point b) of the proposition 3.2 implies then $T'_2 = T_2$.

Note: After submitting this paper, we learned from Alberto Minguez that he proved the same result in his PhD thesis.

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Covers and propagation in symplectic groups

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The purpose of these notes is to give a simplified overview of the theory of types and covers, due to Bushnell and Kutzko, and explain what is the counterpart, in the study of parabolically induced complex representations of reductive p-adic groups, of Tadić's philosophy in terms of covers. As an illustration, we describe a setting in which we can 'propagate' covers in symplectic groups and detail an example in $Sp(12, F)$.

1 Tadić's philosophy and covers

Let F be a local non archimedean field, with ring of integers $\mathfrak{o}_F$, maximal ideal $\mathfrak{p}_F$, residue field k_F of cardinality q_F, and residual characteristic $p = \operatorname{char} k_F$. We will occasionnally use a uniformizing element ϖ_F of $\mathfrak{p}_F$ and a character $\psi : F \to \mathbb{C}^\times$ trivial on $\mathfrak{p}_F$ and non trivial on $\mathfrak{o}_F$. Analogous notations will be used for field extensions of F.

Let $G = \mathbf{G}(F)$ be the group of F-rational points of a connected reductive algebraic group $\mathbf{G}$ defined over F. Let $\mathcal{R}(G)$ be the category of smooth complex representations of G. (Recall that a vector in a representation of G is smooth if its fixator is an open subgroup of G; a representation is smooth if all vectors in its space are smooth.)

There are two basic tools in the study of those representations: the functor of parabolic induction and Jacquet's restriction functor. Let P be a parabolic subgroup of G, let N be its unipotent radical and let M be a Levi factor of P. We need the modular character $\Delta_P : P \longrightarrow \mathbb{R}^{\times +}$ defined by: $\Delta_P(a) = \dfrac{\operatorname{vol}(aKa^{-1})}{\operatorname{vol}(K)}$, $a \in P$, where vol denotes the volume of an open compact subgroup of P with respect to some right Haar measure on P.

The (normalized) **parabolic induction functor** $\operatorname{Ind}_P^G : \mathcal{R}(M) \longrightarrow \mathcal{R}(G)$ is defined as follows. Let $(\sigma, V) \in \mathcal{R}(M)$. Then $\operatorname{Ind}_P^G(\sigma)$ is the representation of G by right translations in the space of functions

$$\operatorname{Ind}_P^G(V) = \{f\colon G \to V \ / \text{ for } m \in M, n \in N, g \in G : f(mng) = \Delta_P(m)^{1/2}\sigma(m)f(g), \text{ and } f \text{ is a smooth vector for the action of } G\}.$$

The **Jacquet restriction functor** $r_N : \mathcal{R}(G) \longrightarrow \mathcal{R}(M)$ is defined as follows. Let $(\pi, V) \in \mathcal{R}(G)$ and let $V(N) = \operatorname{Span}_{\mathbb{C}}\{\pi(n)v - v \ / \ v \in V, n \in N\}$. Then $r_N(\pi)$ is the natural quotient action of M in the space $r_N(V) = V_N = V/V(N)$. The normalized restriction functor r_P^G, defined as the twist of r_N by the character $\Delta_P^{-1/2}$, is left-adjoint to Ind_P^G.

We are using normalized induction here because it is common use; however, from the point of view of types, unnormalized and normalized induction or restriction make no difference: they differ by an unramified character.

The **supercuspidal** representations of G are those smooth representations (π, V) of G which satisfy, for any proper parabolic subgroup $P = MN$ of G: $r_N(\pi) = 0$ (or equivalently, for complex representations, $\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\sigma)) = 0$ for any $\sigma \in \mathcal{R}(M)$).

Let σ be an irreducible supercuspidal representation of M. The **inertial class** $[M, \sigma]_G$ is the equivalence class of the pair (M, σ) under the equivalence relation defined by conjugacy in G and twisting of σ by unramified characters of M – that is, one-dimensional representations of M that are trivial on every compact subgroup. We denote by $\mathcal{R}^{[M,\sigma]}(G)$ the subcategory of smooth representations π of G with the following property: any irreducible subquotient of π is a subquotient of $\mathrm{Ind}_P^G(\sigma \otimes \chi)$ for some unramified character χ of M. Bernstein's decomposition of $\mathcal{R}(G)$ is the decomposition of $\mathcal{R}(G)$ as the direct sum, over inertial classes in G, of the subcategories $\mathcal{R}^{[M,\sigma]}(G)$. We will come back to this later: it is indeed the goal of the theory of types to describe each piece in this decomposition with a type.

The study of $\mathcal{R}(G)$ has for long followed two distinct paths: the study of supercuspidal representations of G, and the study of the parabolically induced representations $\mathrm{Ind}_P^G(\sigma)$ as above. As for the second problem, worked upon by many, Tadić's approach has been to study such induced representations using all possible functors $r_{N'} : \mathcal{R}(G) \longrightarrow \mathcal{R}(M')$, where $M'N'$ is a parabolic subgroup of G with unipotent radical N'. The philosophy of this approach is that *having more parabolic subgroups gives more possibilities to compare informations coming from the Jacquet modules of various parabolic subgroups* ([9], Introduction). The method is indeed very efficient and has produced a lot of results, by Tadić and others.

We will here explain a completely different approach that belongs to a line of thought initialized more than thirty years ago: via compact open subgroups. Let us start with supercuspidal representations; it has been an open question, for at least that amount of time, whether supercuspidal representations were (compactly) induced, that is:

1.1. *Given an irreducible supercuspidal representation π of G, does there exist an open subgroup K of G, compact modulo the center of G, and an irreducible smooth representation κ of K, such that $\pi \simeq \text{c-Ind}_K^G \kappa$?*

Here, if V is the space of κ, $\text{c-Ind}_K^G \kappa$ is the representation of G by right translations in the space

$$\text{c-Ind}_K^G(V) = \{f\colon G \to V \;/\; \text{for } k \in K, g \in G : f(kg) = \kappa(k) f(g), \text{ and } \\ f \text{ is compactly supported mod the center of } G\}.$$

It is a basic fact that if the representation $\text{c-Ind}_K^G \kappa$ is irreducible, then it is supercuspidal (supercuspidal representations are also characterized by the fact that their coefficients are compactly supported mod the center). Furthermore its irreducibility is equivalent to the fact that the **intertwining** of κ, that is, the set of $g \in G$ such that $\mathrm{Hom}_{K \cap K^g}(\kappa, \kappa^g) \neq \{0\}$, is equal to K.

There have been too many works on question 1.1 to even try to cite them all: thirty years of efforts led eventually to a positive answer for $\mathbf{G} = GL(N)$ ([5], 1993) and many other reductive groups have followed. We only wish here to give an idea of types, starting with types for supercuspidal representations of $GL(N, F)$, and of covers, following Bushnell and Kutzko's formalism ([6]).

1.1 Types for supercuspidal representations of $GL_N(F)$

We first define the notion of *maximal simple type* $(J(\beta, \mathfrak{A}_0), \lambda(\beta, \mathfrak{A}_0))$ in $G = GL_N(F)$, following [5]. We will skip the technical definitions, to be found in *loc. cit.*, but will give an easy example instead.

We start with a principal $\mathfrak{o}_F$-order $\mathfrak{A}_0$ in $M_N(F)$ (e.g. $M_N(\mathfrak{o}_F)$) and an element β in $M_N(F)$ generating a field extension $E = F[\beta]$ of F. We let $\hat{E}$ be the commutant algebra of E in $M_N(F)$. We assume that:

- $E^\times$ normalises $\mathfrak{A}_0$;
- $n_0 = -\text{val}_E(\beta)$ satisfies $n_0 > 0$;
- $\mathfrak{B}_0 = \mathfrak{A}_0 \cap \hat{E}$ is a *maximal* $\mathfrak{o}_E$-order in $\hat{E}$;
- $[E : F]$ is minimal among the degrees of field extensions possibly generated by elements of $\beta + \mathfrak{A}_0$.

Then we call $[\mathfrak{A}_0, n_0, 0, \beta]$ a *maximal simple stratum* in $M_N(F)$. The simplest object attached to this stratum is a function ψ_β on G defined by $\psi_\beta(g) = \psi \circ \text{tr}\,(\beta(g-1))$, $g \in G$. This function restricts to a character on suitable subgroups.

With these data $\mathfrak{o}_F$-lattices in $M_N(F)$ are built:

$$\mathfrak{H}^1(\beta, \mathfrak{A}_0) \subset \mathfrak{J}^1(\beta, \mathfrak{A}_0) \subset \mathfrak{J}^0(\beta, \mathfrak{A}_0),$$

and compact open subgroups of $GL_N(F)$:

$$H^1(\beta, \mathfrak{A}_0) \subset J^1(\beta, \mathfrak{A}_0) \subset J(\beta, \mathfrak{A}_0),$$

with $H^1 = 1 + \mathfrak{H}^1$, $J^1 = 1 + \mathfrak{J}^1$, $J = \mathfrak{J}^{0\times}$.

A simple type will be constructed by stages up this tower of subgroups. First, the crucial notion of simple character: it is a difficult generalization of an easy construction given in the example below. A *simple character* is a rather special character of $H^1(\beta, \mathfrak{A}_0)$ (one-dimensional representation), among its properties are the following:

- the restriction of a simple character to a suitable subgroup of H^1 coincides with ψ_β;
- the intertwining of a simple character is $J\hat{E}^\times J$.

Let θ_0 be a simple character. Next, Heisenberg construction provides η_0, unique irreducible representation of $J^1(\beta, \mathfrak{A}_0)$ containing θ_0; as such, η_0 has the same intertwining as θ_0.

The third step is the β-*extension* step: extending the representation η_0 to $J(\beta, \mathfrak{A}_0)$ without shrinking its intertwining. It is possible, if difficult. We thus pick κ_0, a β-*extension* of η_0 to $J(\beta, \mathfrak{A}_0)$; the intertwining of κ_0 is the full intertwining of θ_0.

Let $f = N/[E:F]$. The last ingredient in our simple type is σ_0, a cuspidal representation of $GL(f, k_E)$. We inflate it to $J(\beta, \mathfrak{A}_0)$ through:

$$J(\beta, \mathfrak{A}_0)/J^1(\beta, \mathfrak{A}_0) \simeq GL(f, k_E).$$

Our maximal simple type is now defined by $\lambda(\beta, \mathfrak{A}_0) = \kappa_0 \otimes \sigma_0$.

For a complete account of maximal simple types, we also must allow the so-called 'level 0 case': where $\mathfrak{A}_0$ is a maximal order and we take $\beta = 0$, $E = F$, $\lambda(\beta, \mathfrak{A}_0) = \sigma_0$.

Theorem 1.2 (Bushnell-Kutzko [5]). *For any extension $\lambda^\star$ of $\lambda(\beta, \mathfrak{A}_0)$ to $J^\star = E^\times J(\beta, \mathfrak{A}_0)$, the induced representation* c-$\mathrm{Ind}_{J^\star}^{GL_N(F)} \lambda^\star$ *is irreducible and supercuspidal.*

Every supercuspidal representation of $GL_N(F)$ has this form for a suitable choice of β and $\mathfrak{A}_0$.

1.2 An example in $GL(4, F)$

Assume the residual characteristic p is not 2. Let $\nu \in \mathfrak{o}_F^\times$ be an element that is not a square and let E_0 be the unramified quadratic extension of F, viewed as $E_0 = F[\left(\begin{smallmatrix} 0 & \nu \\ 1 & 0 \end{smallmatrix}\right)]$ in $M_2(F)$. Let $a \geq 1$ be an integer and let $u \in \mathfrak{o}_{E_0}^\times$ satisfy $u^2 \notin \mathfrak{o}_F^\times(1 + \mathfrak{p}_{E_0})$.

Define $\beta = \left(\begin{smallmatrix} 0 & u\varpi_F^{-a} \\ u\varpi_F^{1-a} & 0 \end{smallmatrix}\right)$ in $M_2(E_0) \subset M_4(F)$. Then $E = F[\beta]$ is a ramified quadratic extension of E_0, viewed inside $M_4(F)$, in which β has odd valuation $-n_0 = 1 - 2a$.

Now $\mathfrak{A}_0 = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \end{pmatrix}$ is a principal $\mathfrak{o}_F$-order with radical $\mathfrak{P}_0 = \begin{pmatrix} \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix}$, normalized by $E^\times$, and $[\mathfrak{A}_0, n_0, 0, \beta]$ is a maximal simple stratum in $M_4(F)$.

This is deliberately an elementary situation, in which definitions in [5] immediately give:

$$H^1(\beta, \mathfrak{A}_0) = J^1(\beta, \mathfrak{A}_0) = (1 + \mathfrak{p}_E)\ (1 + \mathfrak{P}_0^{\frac{n_0+1}{2}})\ ; \quad J(\beta, \mathfrak{A}_0) = \mathfrak{o}_E^\times\ (1 + \mathfrak{P}_0^{\frac{n_0+1}{2}}).$$

Simple characters are obtained as follows. The function ψ_β is a character on $1 + \mathfrak{P}_0^{\frac{n_0+1}{2}}$. Let θ_0 be some character of $(1+\mathfrak{p}_E)$ agreeing with ψ_β on the intersection $(1+\mathfrak{p}_E) \cap (1+\mathfrak{P}_0^{\frac{n_0+1}{2}})$. Then $\theta_0\psi_\beta$ is a simple character of $H^1(\beta, \mathfrak{A}_0)$, still denoted by θ_0.

Our maximal simple type is here $\lambda(\beta, \mathfrak{A}_0) = \theta^\star \psi_\beta$ for some character $\theta^\star$ of $\mathfrak{o}_E^\times$ extending θ_0.

1.3 Types for non-supercuspidal representations

We come back to the general setting of the beginning of this paragraph and give the definition of a type for a subcategory $\mathcal{R}^{[M,\sigma]}(G)$, or equivalently for an inertial class $[M, \sigma]_G$:

Definition 1.3 ([6]). *A pair (J, λ), J an open compact subgroup of G, λ a smooth irreducible representation of J, is a type for $[M, \sigma]_G$ if, for any smooth irreducible representation π of G, the following conditions are equivalent:*

- $\pi|_J$ *contains* λ*;*

- π *is a subquotient of* $\mathrm{Ind}_P^G \sigma \otimes \chi$ *for some unramified character* χ *of* M*.*

(In the situation of Theorem 1.2, the pair $(J(\beta, \mathfrak{A}_0), \lambda(\beta, \mathfrak{A}_0))$ is a type for the inertial class of $\text{c-Ind}_{J^\star}^{GL_N(F)} \lambda^\star$ in ${}_{GL_N(F)}$.)

To explain the interest of this definition, we immediately need to define the **Hecke algebra** of a pair (J, λ) as above; it is the intertwining algebra of the representation $\text{c-Ind}_J^G \lambda$, which can also be described as the following convolution algebra, where W_λ denotes the space of λ (and a Haar measure is fixed on G):

$$\mathcal{H}(G, \lambda) = \{f : G \longrightarrow \mathrm{End}_{\mathbb{C}}(W_\lambda) \ / \ f \ \text{ compactly supported} \\ \text{and } \ f(ugv) = \lambda(u) f(g) \lambda(v) \ \text{ for } \ u, v \in J, \ g \in G\}.$$

Note that the support of an element of $\mathcal{H}(G, \lambda)$ is a finite union of J-double cosets.

Now the whole point of the definition is to replace the study of induced representations of the form $\mathrm{Ind}_P^G \sigma \otimes \chi$ as above, by the study of corresponding right modules over the algebra $\mathcal{H}(G, \lambda)$, via the following functor, which is an **equivalence of categories** whenever (J, λ) is a type for $[M, \sigma]_G$ (see [6]):

$$\begin{array}{rcl} \mathcal{R}^{[M,\sigma]}(G) & \overset{\mathcal{M}_\lambda}{\underset{\sim}{\longrightarrow}} & \text{Mod-}\mathcal{H}(G, \lambda) \\ \pi & \longmapsto & \mathrm{Hom}_J(\lambda, \pi) \end{array}$$

The right action of $f \in \mathcal{H}(G, \lambda)$ on $\phi \in \mathrm{Hom}_J(\lambda, \pi)$ is given by

$$\phi.f(w) = \int_G \pi(g^{-1}) \, \phi(f(g) w) \, dg \quad (w \in W_\lambda).$$

1.4 Covers

Types are not known yet to exist in all cases; for supercuspidal representations, existence is very closely related to question 1.1. Anyhow, the best way to construct types in the non-supercuspidal case, say $[M, \sigma]_G$ with M a proper Levi subgroup, is to try to start with a type (J_M, λ_M) for σ in M and to build a G-**cover** or *induced type* of (J_M, λ_M) in G: the shape of type best adapted to parabolic induction.

Let as above $P = MN$ be a parabolic subgroup of G of Levi M and unipotent radical N and let $P^- = MN^-$ be the parabolic subgroup opposite to P relative to M.

Definition 1.4 ([6])**.** *A pair* (J, λ)*,* J *an open compact subgroup of* G*,* λ *a smooth irreducible representation of* J*, is a* G-**cover** *of an analogous pair* (J_M, λ_M) *in* M *if*

- $J = (J \cap N^-) \ (J \cap M) \ (J \cap N)$ *and* $J \cap M = J_M$*,*

- λ *is trivial on* $J \cap N^-$ *and* $J \cap N$ *and* $\lambda_{|J \cap M} = \lambda_M$*;*

- *for any smooth irreducible representation* (π, V) *of* G*, the Jacquet functor* r_N *is injective on the isotypic component of* $\pi_{|J}$ *of type* λ*:*

$$V^\lambda \overset{r_N}{\underset{\sim}{\longrightarrow}} V_N^{\lambda_M}.$$

The two first conditions, dealing with the Iwahori decomposition of the group **and** the representation, are relatively easy to fulfill; pairs (J, λ) satisfying those two conditions are said to be **decomposed** above (J_M, λ_M) with respect to P – note that the representation λ is then entirely determined by λ_M. Given the first two, the third condition is a very strong one and it has indeed very strong consequences:

Theorem 1.5 (Bushnell-Kutzko [6]). *A G-cover of a type is a type. With the notation above, if (J, λ) is a G-cover of (J_M, λ_M) and (J_M, λ_M) is a type for $[L, \sigma]_M$ in M, then (J, λ) is a type for $[L, \sigma]_G$ in G.*

Yet there is more to covers than this theorem. Indeed, if we start as before with a decomposed pair (J, λ) above (J_M, λ_M), we can define an injective homomorphism of vector spaces $T : \mathcal{H}(M, \lambda_M) \hookrightarrow \mathcal{H}(G, \lambda)$, by :

$$f \in \mathcal{H}(M, \lambda_M) \text{ and } \operatorname{Supp} f = J_M m J_M \quad \Rightarrow \quad \operatorname{Supp} T(f) = JmJ \text{ and } T(f)(m) = f(m).$$

Further, there is a subalgebra $\mathcal{H}(M, \lambda_M)^+$ of $\mathcal{H}(M, \lambda_M)$ on which T restricts to an homomorphism of *algebras*

$$T^+ : \mathcal{H}(M, \lambda_M)^+ \hookrightarrow \mathcal{H}(G, \lambda). \tag{1.6}$$

It is the subalgebra of functions supported on (P, J)-*positive* elements of M, namely those $m \in M$ satisfying $m(J \cap N)m^{-1} \subset J \cap N$ and $m(J \cap N^-)m^{-1} \supset J \cap N^-$.

We twist our morphism by the character $\Delta_P^{1/2}$, which is trivial on compact subgroups of M, letting $T_P^+(f) = T^+(\Delta_P^{1/2} f)$, $f \in \mathcal{H}(M, \lambda_M)^+$. The remarkable feature of covers is the following:

Theorem 1.7 (Bushnell-Kutzko [6]). *Let (J, λ) be a decomposed pair above (J_M, λ_M) with respect to P. Then (J, λ) is a G-cover of (J_M, λ_M) if and only if the morphism T_P^+ extends to a homomorphism of algebras*

$$t_P : \mathcal{H}(M, \lambda_M) \hookrightarrow \mathcal{H}(G, \lambda).$$

This extension is unique, injective, and gives rise to the commutative diagram :

$$\begin{array}{ccc} \mathcal{R}^{[L,\sigma]}(G) & \xrightarrow[\sim]{\mathcal{M}_\lambda} & \text{Mod-}\mathcal{H}(G, \lambda) \\ \uparrow {\scriptstyle \operatorname{Ind}_{P^-}^G} & & \uparrow {\scriptstyle (t_P)_*} \\ \mathcal{R}^{[L,\sigma]}(M) & \xrightarrow[\sim]{\mathcal{M}_{\lambda_M}} & \text{Mod-}\mathcal{H}(M, \lambda_M) \end{array} \tag{1.8}$$

Here $(t_P)_*(Y) = \operatorname{Hom}_{\mathcal{H}(M,\lambda_M)}(\mathcal{H}(G, \lambda), Y)$, the structure of right $\mathcal{H}(M, \lambda_M)$-module on $\mathcal{H}(G, \lambda)$ being given by t_P.

In this context, a counterpart to Tadić's philosophy is given by the crucial homomorphism of algebras t_P, plus the transitivity of covers ([6] again):

1.9. *Let $P' = M'N'$ be another parabolic subgroup of G of Levi M' and unipotent radical N'. Assume $P \subset P'$ and $M \subset M'$. Then (J, λ) is a G-cover of $(J \cap M, \lambda_{|J \cap M})$ if and only if (J, λ) is a G-cover of $(J \cap M', \lambda_{|J \cap M'})$ and $(J \cap M', \lambda_{|J \cap M'})$ is a M'-cover of $(J \cap M, \lambda_{|J \cap M})$.*

Indeed, the point of types is to replace the study of parabolic induction by the study of modules over the relevant Hecke algebras. The point of covers is to do so using the commutative diagram 1.8. You do need information on the Hecke algebra $\mathcal{H}(G,\lambda)$ and a crucial tool is the homomorphism of algebras defined in Theorem 1.7. Now, thanks to transitivity, each intermediate Levi subgroup will give rise to such an algebra homomorphism so *the more intermediate parabolic subgroups we have, the more information we get on the Hecke algebra*. We describe in the next paragraph an instance of this technique; more details on the use of these homomorphisms will be found in section 2.5.

2 Propagation of types in symplectic groups

¿From now on we assume the residual characteristic p is different from 2. We look at the following situation: we fix integers $N \geq 1$ and $k \geq 0$ and let an integer $t \geq 1$ vary. We put

$$G_t = Sp\,(2tN + 2k, F)$$

and consider:

- the standard Levi subgroup $M_t = GL_N(F) \times \cdots \times GL_N(F) \times Sp_{2k}(F)$ of G_t
- the standard parabolic subgroup P_t of block-upper-triangular matrices with Levi component M_t and unipotent radical U_t.

We pick an irreducible supercuspidal representation π of $GL_N(F)$ and an irreducible supercuspidal representation ρ of $Sp_{2k}(F)$ and form, for complex numbers $a_1, \ldots, a_t$, the representation of G_t:

$$I(\pi,t,\rho) = \mathrm{Ind}_{P_t}^{G_t}\ \pi|\det|^{a_1} \otimes \cdots \otimes \pi|\det|^{a_t} \otimes \rho.$$

Tadić has proved in [12] that the knowledge of reducibility points (that is, the values of $a_1, \ldots, a_t$ such that the representation is reducible) and composition series of $I(\pi,1,\rho)$ implies the knowledge of reducibility points and composition series of $I(\pi,t,\rho)$ for $t \geq 1$. This is of course in accordance with the philosophy stated in the previous paragraph: the case of maximal parabolic subgroup is the most difficult, if t is greater than 1 there are more parabolics to work with and one can obtain results through the use of Jacquet functors.

What could be the counterpart of this result in terms of types and covers ? To express it, we need first to pick (Γ,γ), a $[GL_N(F),\pi]$- type in $GL_N(F)$, and (Δ,δ), an $[Sp_{2k}(F),\rho]$- type in $Sp_{2k}(F)$. Knowledge of reducibilities for $I(\pi,1,\rho)$ can be replaced by the knowledge of a G_1-cover (Ω_1,ω_1) of $(\Gamma\times\Delta,\gamma\otimes\delta)$ and of its Hecke algebra $\mathcal{H}(G_1,\omega_1)$, through the equivalence of categories:

$$\mathcal{R}^{[M_1,\ \pi\otimes\rho]}(G_1) \xrightarrow[\sim]{\mathcal{M}_{\omega_1}} \text{Mod-}\mathcal{H}(G_1,\omega_1).$$

In this setting, a result similar to Tadić's would be the following:

There exists a G_t-cover (Ω_t, ω_t) of $(\Gamma^{\times t} \times \Delta, \gamma^{\otimes t} \otimes \delta)$ with Hecke algebra $\mathcal{H}(G_t, \omega_t)$ admitting a presentation by generators and relations ***entirely determined*** *by t and $\mathcal{H}(G_1, \omega_1)$.*

Indeed, the reducibilities for $I(\pi, t, \rho)$ would then be known through the equivalence of categories:

$$\mathcal{R}^{[M_t,\ \pi^{\otimes t}\otimes\rho]}(G_t) \overset{\mathcal{M}_{\omega_t}}{\underset{\sim}{\longrightarrow}} \text{Mod-}\mathcal{H}(G_t, \omega_t).$$

This is the question we will examine in this section, in the case where the representation π is **self-dual**, that is, equivalent to the contragredient representation $\check{\pi}$ defined as the dual action of $GL_N(F)$ in the space of smooth vectors in the dual of the space of π. This is the most interesting case: if no representation $\pi|\det|^a$ is self-dual, then it follows from the work of Tadić that reducibilities for $I(\pi, t, \rho)$ all come from reducibilities for the parabolically induced representation of $\pi|\det|^{a_1} \cdots \otimes \pi|\det|^{a_t}$ in $GL_{tN}(F)$, which are well known.

2.1 The principle of propagation

We now need to get more technical and fix some more notations. We let w_i be the anti-diagonal matrix $w_i = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ and define $Sp\,(2i, F)$ as the symplectic group of F^{2i} with respect to the symplectic form of matrix $\begin{pmatrix} 0 & w_i \\ -w_i & 0 \end{pmatrix}$. For any matrix x we write ${}^{\tau}x$ for the transpose of x with respect to the anti-diagonal; in particular the identification of $GL_N(F) \times Sp_{2k}(F)$ with M_1 reads:

$$(x, g) \longmapsto \begin{pmatrix} x & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & {}^{\tau}x^{-1} \end{pmatrix}.$$

We start with a G_1-cover (Ω_1, ω_1) of $(\Gamma \times \Delta, \gamma \otimes \delta)$ and think of it in terms of blocks: from the definition, the pair (Ω_1, ω_1) is in particular decomposed above $(\Gamma \times \Delta, \gamma \otimes \delta)$ so is the product of its intersections with U_1^-, M_1 and U_1. We visualize as follows:

$$\Omega_1 = \begin{pmatrix} \Gamma & \mathcal{B}_{12} & \mathcal{B}_{13} \\ \mathcal{B}_{21} & \Delta & \mathcal{B}_{23} \\ \mathcal{B}_{31} & \mathcal{B}_{32} & {}^{\tau}\Gamma \end{pmatrix},$$

this schematization meaning of course that the off-diagonal blocks $\mathcal{B}_{ij}$ are suitable lattices in the relevant matrix space and that Ω_1 is the intersection with $Sp\,(2N + 2k, F)$ of the set of matrices whose block entries belong to the corresponding subgroup or lattice.

We want to produce a G_t-cover (Ω_t, ω_t) of $(\Gamma^{\times t} \times \Delta, \gamma^{\otimes t} \otimes \delta)$; with the same convention of visualization, this would look like

$$\Omega_t = \begin{pmatrix} \Gamma & \mathcal{C}_{12} & \cdots & \mathcal{C}_{1,t+1} & & \cdots & \mathcal{C}_{1,2t+1} \\ \mathcal{C}_{21} & \Gamma & & \mathcal{C}_{2,t+1} & & & \vdots \\ & \vdots & \ddots & \vdots & & & \\ \mathcal{C}_{t+1,1} & \mathcal{C}_{t+1,2} & \cdots & \Delta & \cdots & \cdots & \mathcal{C}_{t+1,2t+1} \\ & & & \vdots & \ddots & & \\ \vdots & & & \mathcal{C}_{2t,t+1} & & \mathrm{T} & \mathcal{C}_{2t,2t+1} \\ \mathcal{C}_{2t+1,1} & \cdots & & \mathcal{C}_{2t+1,t+1} & & \cdots & \mathrm{T} \end{pmatrix} = \begin{pmatrix} \Gamma_t & \mathcal{D}_{12} & \mathcal{D}_{13} \\ \mathcal{D}_{21} & \Delta & \mathcal{D}_{23} \\ \mathcal{D}_{31} & \mathcal{D}_{32} & \mathrm{T}_t \end{pmatrix}.$$

The first shape shows what we mean by propagation: we would expect a relationship between the blocks $\mathcal{C}_{i,t+1}$ or $\mathcal{C}_{t+1,i}$ and the blocks $\mathcal{B}_{j2}$ or $\mathcal{B}_{2j}$ in Ω_1, at best equality for instance, and a relationship between the off-$Sp\,(2k,F)$-part $\begin{pmatrix} \Gamma & \mathcal{B}_{13} \\ \mathcal{B}_{31} & \mathrm{T} \end{pmatrix}$ and what is obtained from Ω_t by removing the $t+1$-th row and column, namely $\begin{pmatrix} \Gamma_t & \mathcal{D}_{13} \\ \mathcal{D}_{31} & \mathrm{T}_t \end{pmatrix}$ in the second shape.

It seems hopeless to try such a 'propagation' without any specific knowledge of the cover (Ω_1, ω_1). Besides, the best hint we have is the following consequence of the transitivity of covers: the block Γ_t should hold a $GL_{tN}(F)$-cover of $(\Gamma^{\times t}, \gamma^{\otimes t})$. We will thus rely strongly on Bushnell-Kutzko types in $GL_{iN}(F)$.

2.2 A family of $GL_{2tN}(F)$-covers of $(\Gamma \times \cdots \times \Gamma, \gamma \otimes \cdots \otimes \gamma)$

We now pick (Γ, γ), our $[GL_N(F), \pi]$-type in $GL_N(F)$, as a Bushnell-Kutzko maximal simple type $(\Gamma, \gamma) = (J(\beta, \mathfrak{A}_0),\ \lambda(\beta, \mathfrak{A}_0))$. We use the notations of section 1.1 and will shorten $\mathfrak{H}^1 = \mathfrak{H}^1(\beta, \mathfrak{A}_0)$, $\mathfrak{J}^0 = \mathfrak{J}^0(\beta, \mathfrak{A}_0)$... Recall $E = F[\beta]$.

In the book [5], Bushnell and Kutzko do produce a $GL_{2tN}(F)$-cover of $(\Gamma^{\times 2t}, \gamma^{\otimes 2t})$. Starting with this cover, and with some additional work, one obtains a family $(\Gamma(2t,r), \gamma(2t,r))$ of $GL_{2tN}(F)$-covers of $(\Gamma^{\times 2t}, \gamma^{\otimes 2t})$, indexed by an integer r, $t \leq r \leq 2t$; the cover given in [5] corresponds to $r = t$ while the cover given by $r = 2t$ is the best suited to propagation. In block-matrix form, we have:

$$\Gamma(2t,r)=\begin{array}{c}\longleftarrow \quad r \quad \longrightarrow \\ \begin{pmatrix} \Gamma & \mathfrak{J}^0 & \dots & \mathfrak{J}^0 & \varpi_E^{-1}\mathfrak{H}^1 & \dots & \varpi_E^{-1}\mathfrak{H}^1 \\ \mathfrak{H}^1 & \Gamma & \ddots & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & \ddots & \varpi_E^{-1}\mathfrak{H}^1 \\ \mathfrak{H}^1 & & & & & & \mathfrak{J}^0 \\ \varpi_E\mathfrak{J}^0 & \ddots & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & \Gamma & \mathfrak{J}^0 \\ \varpi_E\mathfrak{J}^0 & \dots & \varpi_E\mathfrak{J}^0 & \mathfrak{H}^1 & \dots & \mathfrak{H}^1 & \Gamma \end{pmatrix} \\ \longleftarrow \quad 2t-r \quad \longrightarrow \end{array}$$

and the representation $\gamma(2t,r)$ is trivial on upper or lower triangular block matrices and equal to $\gamma\otimes\cdots\otimes\gamma$ on block diagonal matrices. (Actually we have such a family of covers $(\Gamma(i,r),\gamma(i,r))$, $[\frac{i+1}{2}]\leq r\leq i$, for any integer $i\geq 1$.)

It is convenient to think of this subgroup as a 2×2 block-matrix subgroup and to introduce a notation for the lattices in $M_{tN}(F)$ corresponding to the upper and lower unipotent parts; we write:

$$\Gamma(2t,r)=\begin{pmatrix}\Gamma\ (t,t) & \Gamma^+(t,r)\\ \Gamma^-(t,r) & \Gamma\ (t,t)\end{pmatrix},\quad t\leq r\leq 2t.$$

Note that $\Gamma(t,t)$ is the most obvious 'propagated type' from $\Gamma(2,2)=\begin{pmatrix}\Gamma & \mathfrak{J}^0\\ \mathfrak{H}^1 & \Gamma\end{pmatrix}$ and that $\mathfrak{J}^0$ and $\mathfrak{H}^1$ are both rings. Subdiagonals in $\Gamma^+(t,r)$ have either all entries in $\mathfrak{J}^0$ or all entries in $\varpi_E^{-1}\mathfrak{H}^1$ and accordingly for $\Gamma^-(t,r)$, with $\varpi_E\mathfrak{J}^0$ and $\mathfrak{H}^1$.

We are almost ready to produce propagated types. The last ingredient we need is the specific properties that the type of a self-dual supercuspidal representation can be assumed to satisfy, useful to obtain decomposed pairs in symplectic groups from the ones we have in linear groups.

2.3 Type of a self-dual supercuspidal representation

Heuristically, we want to produce decomposed pairs – hopefully covers – in $Sp(2tN,F)$ with the subgroups $\Gamma(2t,r)$ in $GL_{2tN}(F)$. Intersecting with $Sp(2tN,F)$ will give interesting subgroups only if $\Gamma(2t,r)$ is stable under the involution defining the symplectic group. Up to some conjugacy, we can achieve this condition when π is self-dual, provided that we find a type for π that is itself self-dual in some sense. This is the motivation of the following theorem:

Theorem 2.1 (Blondel [2]). *Let π be a self-dual irreducible supercuspidal representation of $GL_N(F)$. One can choose a maximal simple type $(\Gamma,\gamma) = (J(\beta,\mathfrak{A}_0),\ \lambda(\beta,\mathfrak{A}_0))$ for π satisfying the following properties:*

1. *$\mathfrak{A}_0$ is a τ-stable principal $\mathfrak{o}_F$-order in $M_N(F)$.*
2. *If $\beta\neq 0$, $E=F[\beta]$ is a quadratic extension of $F[\beta^2]$. We let $x\mapsto\bar{x}$ denote the non trivial element of* $\mathrm{Gal}(F[\beta]/F[\beta^2])$.
3. *There is an element σ in $U(\mathfrak{A}_0)$ such that $\sigma^{-1}x\sigma = {}^\tau\bar{x}$ for all $x\in E$.*
4. *$\mathfrak{H}^1=\mathfrak{H}^1(\beta,\mathfrak{A}_0)$ and $\mathfrak{J}^0=\mathfrak{J}^0(\beta,\mathfrak{A}_0)$ are stable under $x\mapsto\sigma\,{}^\tau x\sigma^{-1}$.*
5. *The pairs (H^1,θ_0), (J^1,η_0) and (Γ,γ) are stable under $x\mapsto\sigma\,{}^\tau x^{-1}\sigma^{-1}$ (the group is stable and the representation transformed into an equivalent representation).*

Example. In our previous example in $GL_4(F)$ (1.2), the extension E is τ-stable and conditions (i) to (iv) hold with $\sigma=\begin{pmatrix}-I_2 & 0\\ 0 & I_2\end{pmatrix}$. The fifth condition holds if and only if we choose a character $\theta^\star$ such that $\theta^\star(x\bar{x})=1$ for all $x\in\mathfrak{o}_E^\times$.

2.4 The propagation theorem

We fix a maximal simple type (Γ,γ) for our self-dual representation π having the properties in Theorem 2.1 and use the corresponding notations. It follows that the conjugate of $\Gamma(2t,r)$ by $\Sigma=\ \mathrm{diag}(I_N,\cdots,I_N,\sigma,\cdots,\sigma)$ (t blocks I_N, t blocks σ), namely

$$\Gamma(2t,r)^\Sigma=\begin{pmatrix}\Gamma\ (t,t) & \Gamma^+(t,r)\sigma\\ \sigma^{-1}\Gamma^-(t,r) & {}^\tau\Gamma\ (t,t)\end{pmatrix},\ t\le r\le 2t,$$

is stable by the involution defining the symplectic group. We also fix a type (Δ,δ) for the supercuspidal representation ρ of $Sp(2k,F)$ such that $\rho=\text{c-Ind}_\Delta^{Sp(2k,F)}\delta$; the existence of such a type follows from recent work of Stevens ([11]).

We first state the hypotheses needed on the G_1-cover of $(\Gamma\times\Delta,\gamma\otimes\delta)$. The main point is that we actually need two such covers, related to the twin groups $\Gamma(2,1)$ and $\Gamma(2,2)$ in $GL_{2N}(F)$, or rather a little less: two decomposed pairs that are 'almost' covers. (As a consequence of the theorem they will turn out to be covers.)

The first hypothesis thus concerns the existence of two decomposed pairs of a special shape: there must exist $\mathfrak{o}_F$-lattices $\Delta^+\subset M_{N,2k}(F)$ and $\Delta^-\subset M_{2k,N}(F)$ such that

$$\Omega(1,1)=\begin{pmatrix}\Gamma & \Delta^+ & \varpi_E^{-1}\mathfrak{H}^1\sigma\\ \Delta^- & \Delta & \alpha\,{}^\tau\Delta^+\\ \sigma^{-1}\mathfrak{J}^0\varpi_E & {}^\tau\Delta^-\alpha^{-1} & {}^\tau\Gamma\end{pmatrix}\ \text{(intersected with } G_1)$$

$$\text{and}\ \ \Omega(1,2)=\begin{pmatrix}\Gamma & \Delta^+ & \mathfrak{J}^0\sigma\\ \Delta^- & \Delta & \alpha\,{}^\tau\Delta^+\\ \sigma^{-1}\mathfrak{H}^1 & {}^\tau\Delta^-\alpha^{-1} & {}^\tau\Gamma\end{pmatrix}\ \text{(intersected with } G_1),\ \ (\alpha=\begin{pmatrix}-I_k & 0\\ 0 & I_k\end{pmatrix}),$$

are subgroups of G_1 holding representations $\omega(1,1)$ and $\omega(1,2)$ such that $(\Omega(1,1),\omega(1,1))$ and $(\Omega(1,2),\omega(1,2))$ are decomposed pairs above $(\Gamma\times\Delta,\gamma\otimes\delta)$.

To explain the second hypothesis, we have to come back to the definition of covers: the third condition in definition 1.4 is equivalent to the inversibility, in the Hecke algebra, of an element supported on a special double coset; the exact statement is rather technical, hence omitted. Suffice to say that in practice, showing that a given decomposed pair is a cover is very much related to finding invertible generators for the Hecke algebra, that make it a convolution algebra on an affine (or extended affine) Weyl group (possibly twisted).

Now the elements s and q defined below are the generators of the affine Weyl group of type $\tilde{C}_2$ adapted to the situation in the maximal case. The assumptions below imply that the elements e_s and e_q are invertible (b_q and b_s are non zero); if they did belong to the same Hecke algebra, both their invertibilities would imply that the corresponding decomposed pair is a cover. Heuristically, 2.2 and 2.3 say that the two pairs are 'halfway' to being covers, and 'complementarily' so.

We thus let $s=\begin{pmatrix}0&0&\sigma\\0&I_{2k}&0\\-{}^\tau\sigma^{-1}&0&0\end{pmatrix}$, $q=\begin{pmatrix}0&0&-{}^\tau\sigma\,{}^\tau\varpi_E^{-1}\\0&I_{2k}&0\\\sigma^{-1}\varpi_E&0&0\end{pmatrix}$, and assume we have elements

- $e_q\in\mathcal{H}(G_1,\omega(1,1))$, supported on $\Omega(1,1)\ q\ \Omega(1,1)$, such that:

$$e_q^2=a_q\,e_q+b_q\,\mathcal{I}\qquad(a_q\in\mathbb{C},b_q\in\mathbb{C}^\times);\tag{2.2}$$

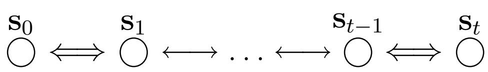
- $e_s\in\mathcal{H}(G_1,\omega(1,2))$, supported on $\Omega(1,2)\ s\ \Omega(1,2)$, such that:

$$e_s^2=a_s\,e_s+b_s\,\mathcal{I}\qquad(a_s\in\mathbb{C},b_s\in\mathbb{C}^\times).\tag{2.3}$$

With those assumptions, propagation holds. We can 'propagate' the subgroups $\Omega(1,1)$ and $\Omega(1,2)$ into the following family in G_t:

$$\Omega(t,r)=\begin{pmatrix}\Gamma(t,t)&M_{t,1}(\Delta^+)&\Gamma^+(t,r)\sigma\\M_{1,t}(\Delta^-)&\Delta&M_{1,t}(\alpha\,{}^\tau\Delta^+)\\\sigma^{-1}\Gamma^-(t,r)&M_{t,1}({}^\tau\Delta^-\alpha^{-1})&{}^\tau\Gamma(t,t)\end{pmatrix}\cap G_t\quad(t\geq 1\ \text{ and } t\leq r\leq 2t).$$

Theorem 2.4 (Blondel [3])**.** *For $t\geq 1$ and $t\leq r\leq 2t$, $\Omega(t,r)$ is a subgroup of G_t and holds a (unique) representation $\omega(t,r)$ such that $(\Omega(t,r),\omega(t,r))$ is a G_t-cover of $(\Gamma^{\times t}\times\Delta,\gamma^{\otimes t}\otimes\delta)$. The Hecke algebra $\mathcal{H}(G_t,\omega(t,r))$ is a convolution algebra on an affine Weyl group of type $\tilde{C}_t$, with parameters:*



$$(a_{\mathbf{s}_0},b_{\mathbf{s}_0})=(a_s,b_s),$$
$$(a_{\mathbf{s}_i},b_{\mathbf{s}_i})=(q_E^f-1,q_E^f)\quad \textit{for } 1\leq i\leq t-1,$$
$$(a_{\mathbf{s}_t},b_{\mathbf{s}_t})=(a_q,b_q).$$

(In the next subsection we give precise information on the generators of this Hecke algebra.)

Given the assumptions, this theorem fulfills the requirements in the beginning of section 2: the groups $\Omega(t,r)$ are entirely built from the groups $\Omega(1,1)$ and $\Omega(1,2)$, using propagation and the structure of $GL_{2tN}(F)$-covers; the Hecke algebra $\mathcal{H}(G_t,\omega(t,r))$ is completely determined by the $t=1$ case. Note that the parameters (q_E^f-1, q_E^f) only depend on the (inertial class of the) representation π: they are parameters of the Hecke algebra $\mathcal{H}(GL_{tN}(F),\gamma(t,t))$ (see [5]).

At this point of course, we are left with two difficulties: first, producing covers with the required properties for $t=1$, second, compute the corresponding parameters. So far, the cases in which this theorem applies – that is, the assumptions are fulfilled – are: the level zero case, due to Morris [8]; the $k=0$ case [2]; the case of $Sp(4,F)$ $(2N+2k=4)$ [1]; and a special case with $k=N$ [3] of which we detail an example for $N=4$ below.

2.5 Some elements in the proof

Past the preliminary checking (technical and boring) that $(\Omega(t,r),\omega(t,r))$, $t\le r\le 2t$, are indeed decomposed pairs, the proof that they are covers proceeds along two main lines.

One is the use of a *family* of decomposed pairs and not just one. Technically, the proof works alternatively with $\Omega(t,t)$ and $\Omega(t,2t)$: we work at each step with the most convenient of the two. A crucial intermediate result ([3], Proposition 5) states that the Hecke algebras $\mathcal{H}(G_t,\omega(t,r))$ are all isomorphic in a very precise way. To simplify matters we now shorten: $(\Omega,\omega)=(\Omega(t,r),\omega(t,r))$, $\mathcal{H}=\mathcal{H}(G_t,\omega(t,r))$.

The other is the most related to Tadić's philosophy: it is the one we will roughly explain here. Indeed, transitivity of covers (1.9) and a similar property of transitivity for decomposed pairs allows to move back and forth from one Levi subgroup to the full group and then down to another Levi subgroup.

First, to show that (Ω,ω) is a cover, we produce specific invertible elements in $\mathcal{H}$ as the images, by homomorphisms of the form $T^+:\ \mathcal{H}(M,\omega_M)^+\hookrightarrow\mathcal{H}(G_t,\omega)$ (see 1.6) for well chosen Levi subgroups M, of invertible elements in the subalgebra $\mathcal{H}(M,\omega_M)^+$ itself.

Next, to obtain a full description of the Hecke algebra $\mathcal{H}$ by generators and relations, we use the fact that our covers are covers of their intersection with any intermediate Levi subgroup: the images of the corresponding Hecke algebras by the homomorphisms as defined in Theorem 1.7 will span the algebra $\mathcal{H}$. Precisely, we define elements s_0, s_i $(1\le i\le t-1)$, s_t of G_t, together with Levi subgroups particularly relevant to those elements.

$$s_0=\begin{pmatrix} I_{(t-1)N} &&&& \\ & 0 & 0 & \sigma & \\ & 0 & I_{2k} & 0 & \\ & -{}^{\tau}\sigma^{-1} & 0 & 0 & \\ &&&& I_{(t-1)N}\end{pmatrix},\quad M^0=\begin{pmatrix}\ddots &&&& \\ & GL_N(F) &&& \\ && G_1 && \\ &&& GL_N(F) & \\ &&&& \ddots\end{pmatrix};$$

$$s_i=\begin{pmatrix} I_{(t-i-1)N} &&&&&&& \\ & 0 & I_N &&&&& \\ & I_N & 0 &&&&& \\ &&& I_{2k+2(i-1)N} &&&& \\ &&&& 0 & I_N & \\ &&&& I_N & 0 & \\ &&&&&& I_{(t-i-1)N}\end{pmatrix},\quad M^i=\begin{pmatrix} GL_{tN}(F) && \\ & Sp_{2k}(F) & \\ && GL_{tN}(F)\end{pmatrix};$$

$$s_t = \begin{pmatrix} 0 & 0 & -{}^\tau\sigma\,{}^\tau\varpi_E^{-1} \\ 0 & I_{2(t-1)N+2k} & 0 \\ \sigma^{-1}\varpi_E & 0 & 0 \end{pmatrix}, \quad M^t = \begin{pmatrix} \star & 0 & \star & 0 & \star \\ 0 & GL_{(t-1)N}(F) & 0 & 0 & 0 \\ \star & 0 & \star & 0 & \star \\ 0 & 0 & 0 & GL_{(t-1)N}(F) & 0 \\ \star & 0 & \star & 0 & \star \end{pmatrix} \simeq GL_{(t-1)N}(F) \times G_1.$$

The generator e_{s_j} of $\mathcal{H} = \mathcal{H}(G_t, \omega(t,r))$ attached to the Coxeter generator $\mathbf{s}_j$, $0 \le j \le t$, in Theorem 2.4, is an element of $\mathcal{H}$ with support $\Omega\ s_j\ \Omega$; indeed, easy representation-theoretic considerations imply that the double coset $\Omega\ s_j\ \Omega$ supports a one-dimensional subspace of $\mathcal{H}$. We need to produce such an element e_{s_j} that satisfies the quadratic relation $e_{s_j}^2 = a_{\mathbf{s}_j} e_{s_j} + b_{\mathbf{s}_j}\mathcal{I}$. Now, with a good choice of r ($r = t$ for s_t, $r = 2t$ for s_i, $i \le t-1$), the element s_j is Ω-positive in M^j (1.6) and there is an element ϵ_j in $\mathcal{H}(M^j, \omega_{|\Omega \cap M^j})$, with support $(\Omega \cap M^j) s_j (\Omega \cap M^j)$, that satisfies this quadratic relation. Indeed:

- If $1 \le j \le t-1$: $\mathcal{H}(M^j, \omega_{|\Omega\cap M^j}) \simeq \mathcal{H}(GL_{tN}(F), \Gamma(t,t)) \otimes \mathcal{H}(Sp_{2k}(F), \delta)$; there is an element $u_j \in \mathcal{H}(GL_{tN}(F), \Gamma(t,t))$, with support $\Gamma(t,t) \begin{pmatrix} I_{(t-j-1)N} & & & \\ & 0 & I_N & \\ & I_N & 0 & \\ & & & I_{(j-1)N} \end{pmatrix} \Gamma(t,t)$, such that $u_j^2 = (q_E^f - 1)u_j + q_E^f$ ([5]).
- If $j = 0$: $\mathcal{H}(M^0, \omega_{|\Omega\cap M^0}) \simeq \mathcal{H}(GL_N(F), \Gamma)^{t-1} \otimes \mathcal{H}(G_1, \omega(1,2))$; we use assumption 2.3.
- If $j = t$: $\mathcal{H}(M^t, \omega_{|\Omega\cap M^t}) \simeq \mathcal{H}(GL_{(t-1)N}(F), \Gamma(t-1,t-1)) \otimes \mathcal{H}(G_1, \omega(1,1))$; we use assumption 2.2.

Using the homomorphism $T^+ : \mathcal{H}(M^j, \omega_{|\Omega\cap M^j}) \hookrightarrow \mathcal{H}$ of 1.6 gives the result for $e_{s_j} = T^+(\epsilon_j)$.

Finally, to prove that the family $(e_{s_j})_{0 \le j \le t}$ generates $\mathcal{H}$, we use the 'Bernstein presentation' given by Bushnell and Kutzko in [6] and [7].

3 An example in $Sp(12, F)$

We come back to our example in section 1.2 and notice that E/F is a biquadratic extension – that is, its Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and its norm subgroup is $N_{E/F}(E^\times) = F^{\times 2}$. We have a maximal simple stratum $[\mathfrak{A}_0, n_0, 0, \beta]$ in $M_4(F)$ and a simple character θ_0 of $H^1(\beta, \mathfrak{A}_0)$. Skipping to the notations of §2.3, we let $\sigma = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}$ and make the additional assumption that $\theta_0(x\bar{x}) = 1$ for all $x \in 1 + \mathfrak{p}_E$.

The involution defining the symplectic group is actually $x \mapsto \sigma\,{}^\tau x \sigma^{-1}$. Our stratum is thus a *skew* stratum and our simple character θ_0 is a *skew* simple character ([10]): the groups $H^1(\beta, \mathfrak{A}_0) = J^1(\beta, \mathfrak{A}_0)$ and $J(\beta, \mathfrak{A}_0)$ are stable under this involution, as is the character θ_0.

We let $G = GL(4, F)$, $\bar{G} = Sp(4, F)$, and for any subgroup K of G we let $\bar{K} = K \cap \bar{G}$. According to [10], Theorem 5.2, the restriction to $\bar{H}^1 = \bar{J}^1 = \left((1 + \mathfrak{p}_E) \cap \bar{G}\right)\left((1 + \mathfrak{P}_0^{\frac{n_0+1}{2}}) \cap \bar{G}\right)$ of the skew simple character θ_0 underlies supercuspidal representations of $Sp(4, F)$: for any extension δ of θ_0 to $\Delta = \bar{J}(\beta, \mathfrak{A}_0)$, the representation $\rho = \text{c-Ind}_\Delta^{Sp(4,F)}\delta$ is irreducible supercuspidal and the pair (Δ, δ) satisfies the assumptions in Theorem 2.4. We have here $\Delta = \bar{\mathfrak{o}}_E^\times \bar{H}^1$ so there are two such extensions δ.

On the other hand the same skew simple character θ_0 underlies self-dual supercuspidal representations of G – but it turns out that the character we must use here is not θ_0 but its square θ_0^2, attached to the skew simple stratum $[\mathfrak{A}_0, n_0, 0, 2\beta]$ (see [2], Lemma 4.3.1). Anyway, θ_0^2 has two extensions to a self-dual character γ of $\Gamma = J(\beta, \mathfrak{A}_0) = \mathfrak{o}_E^\times H^1$, i.e. satisfying $\gamma(x\bar{x}) = 1$ for all $x \in \mathfrak{o}_E^\times$; note that γ is either trivial or quadratic on $\mathfrak{o}_{E_0^\times}$. Now (Γ, γ) is a maximal simple type contained in a self-dual supercuspidal representation π of G and satisfying the properties in Theorem 2.1.

By arguments similar to [2] §3.3, one can show that

$$\Omega(1,1) = \begin{pmatrix} \Gamma & \mathfrak{J}^0 & \varpi_E^{-1}\mathfrak{H}^1\sigma \\ \mathfrak{H}^1 & \Delta & \mathfrak{J}^0\sigma \\ \sigma^{-1}\mathfrak{J}^0\varpi_E & \sigma^{-1}\mathfrak{H}^1 & {}^\tau\Gamma \end{pmatrix} \cap Sp(12, F)$$

$$\text{and} \quad \Omega(1,2) = \begin{pmatrix} \Gamma & \mathfrak{J}^0 & \mathfrak{J}^0\sigma \\ \mathfrak{H}^1 & \Delta & \mathfrak{J}^0\sigma \\ \sigma^{-1}\mathfrak{H}^1 & \sigma^{-1}\mathfrak{H}^1 & {}^\tau\Gamma \end{pmatrix} \cap Sp(12, F)$$

are subgroups holding decomposed pairs above $(\Gamma\times\Delta, \gamma\otimes\delta)$ with respect to P_1 (remember that $\Delta = \bar{\Gamma}$) and that assumptions 2.2 and 2.3 hold. The propagation theorem 2.4 thus applies, giving covers in $Sp\,(4(2t+1), F)$ attached to the inertial class $[GL(4, F)^{\times t} \times Sp(4, F), \pi^{\otimes t} \otimes \rho]_{Sp(4(2t+1),\, F)}$ and the structure of their Hecke algebra, provided we know the parameters of the $t = 1$ case.

In the present situation, we can compute the parameters (a_s, b_s) and (a_q, b_q), following the recipe in [1], §1.d; in particular we use the Haar measure on $Sp(12, F)$ giving $\Omega(1, 1)$ and $\Omega(1, 2)$ volume 1 and normalise e_s and e_q by $e_s(s) = 1$, $e_q(q) = 1$. Let L be the quadratic character of $\mathfrak{o}_{E_0}^\times$ and let G be the Gauss sum

$$G(x) = \sum_{z \in \mathfrak{o}_{E_0}^\times / 1+\mathfrak{p}_{E_0}} L(z)\ \psi \circ \mathrm{tr}_{E_0/F}(zx), \quad x \in \mathfrak{o}_{E_0}^\times.$$

The residual field of E_0 has cardinality q_F^2 so -1 is a square in $k_{E_0}^\times$ and $G(x)^2 = q_F^2$; we write $G(x) = \epsilon(x)q_F$ with $\epsilon(x) = \pm 1$. We find:

$$b_s = q_F^2, \quad a_s = \delta(-1)(q_F^2 - 1), \quad b_q = q_F^8, \quad a_q = \begin{cases} 0 & \text{if } \gamma \text{ is trivial on } \mathfrak{o}_{E_0^\times}, \\ (-1)^{\frac{q+1}{2}}\,\epsilon(u)\, q_F^3\,(q_F^2 - 1) & \text{otherwise.} \end{cases} \tag{3.1}$$

The interest of this example lies in the fact that, according to the class of u mod $(\mathfrak{o}_{E_0}^\times)^2$, the representation π is either generic or non generic ([4]). When it is not, reducibilities for $I(\pi, t, \rho)$ are not attained by the usual methods; the construction of a cover above plus the computation of the Hecke algebra do, however, give those reducibilities. One can notice that, in the case studied above, reducibilities will be the same whether the representation is generic or not.

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THE PRIME SPECTRUM OF A MODULE

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ABSTRACT. We study prime submodules in the realm of modules over not necessarily commutative rings.

1. Introduction

Prime submodules were introduced by J. Dauns in [2, 3]. Various authors have studied related questions however the most of work was done in the realm of modules over commutative rings, see, for instance, [4, 6, 7, 8, 9, 10, 11, 12], and the references cited therein. In our research we are relying on the known facts about prime submodules in commutative case and about prime ideals in (not necessarily commutative) rings. In section 2 we give a characterisation of prime submodules and show, for instance, that every maximal submodule is prime. In Section 3 we study three topologies on the spectrum of a module which generalize the Zariski topology.

Throughout the paper the letter R always stands for a ring with identity. All modules are unital. The set of all prime ideals in R is denoted by $\mathrm{Spec}(R)$ and called the *prime spectrum* of R. The *Zariski topology* on $\mathrm{Spec}(R)$ is given by the family of sets $\omega(\mathfrak{a}) = \mathrm{Spec}(R) \setminus \mathbf{h}(\mathfrak{a})$, where $\mathbf{h}(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(R);\ \mathfrak{a} \subseteq \mathfrak{p}\}$ is the *hull* of an ideal $\mathfrak{a} \subseteq R$.

2. Prime submodules

Let R be a ring and M be a left R-module. The *quotient* of a submodule $N \subseteq M$ is defined by $(N : M) = ann(M/N)$. A submodule $N \subseteq M$ is a *co-ideal* if there exists a left ideal $\mathfrak{a} \subseteq R$ such that $N = \mathfrak{a} \cdot M = \{\sum_{i=1}^{n} a_i \cdot x_i;\ a_i \in \mathfrak{a}, x_i \in M, i = 1, \ldots, n, n \in \mathbb{N}\}$. For a left ideal $\mathfrak{a} \subseteq R$, we shall say that $\mathfrak{a} \cdot M$ is its co-ideal. A module M is a *multiplication module* (see [5]) if each submodule in M is a co-ideal. The following proposition lists several useful properties of quotients. An easy proof is omitted.

Proposition 2.1. *Let R be a ring and M a left R-module.*
(i) For submodules $N \subseteq L \subseteq M$, we have $(N : M) \subseteq (L : M)$.
(ii) If $\{N_i\}_{i\in\mathbb{I}}$ is an arbitrary family of submodules in M, then

$$(\cap_{i\in\mathbb{I}} N_i : M) = \cap_{i\in\mathbb{I}}(N_i : M).$$

Let R be a ring and M be a left R-module. In [2] Dauns has defined prime submodules as follows. A proper submodule $P \subset M$ is *prime* if, for $a \in R$ and $x \in M$, the inclusion $(aR) \cdot x \subseteq P$ implies $a \in (P : M)$ or $x \in P$.

The set of all prime submodules in a left R-module M will be denoted by $\mathrm{Spec}(M)$ and called the *prime spectrum* of M. A module M is said to be *primeless* ([11]) if $\mathrm{Spec}(M)$ is empty. ¿From now on we shall always assume that a module under consideration is not primeless.

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Theorem 2.2. *Let R be a ring and M be a left R-module. For a proper submodule P in M, the following assertions are equivalent.*

(a) For a two-sided ideal $\mathfrak{a} \subseteq R$ and a submodule $N \subseteq M$, the inclusion $\mathfrak{a} \cdot N \subseteq P$ implies $\mathfrak{a} \subseteq (P : M)$ or $N \subseteq P$.

(b) P is prime.

(c) For a left (or right) ideal $\mathfrak{a} \subseteq R$ and a submodule $N \subseteq M$, the inclusion $\mathfrak{a} \cdot N \subseteq P$ implies $\mathfrak{a} \subseteq (P : M)$ or $N \subseteq P$.

Proof. $(a) \Rightarrow (b)$ Let $a \in R$ and $x \in M$ be such that $a \cdot R \cdot x \subseteq P$. It follows, since P is a submodule, that $(RaR) \cdot (R \cdot x) \subseteq P$. Now, by (a), we get $RaR \subseteq (P : M)$ or $R \cdot x \subseteq P$. If the former is true, then $a \in (P : M)$, and the latter inclusion gives $x \in P$.

$(b) \Rightarrow (c)$ Assume that $\mathfrak{a} \cdot N \subseteq P$, for a left (right) ideal $\mathfrak{a} \subseteq R$ and a submodule $N \subseteq M$. If $N \not\subseteq P$, then there exists $y \in N \setminus P$. For every $a \in \mathfrak{a}$, we have $(aR) \cdot y = a \cdot (R \cdot y) \subseteq \mathfrak{a} \cdot N \subseteq P$ which gives, by (b), $a \in (P : M)$.

$(c) \Rightarrow (a)$ Obvious. □

Corollary 2.3. *If R is a commutative ring and M is a left R-module, then a proper submodule $P \subset M$ is a prime submodule if and only if, for arbitrary $a \in R$ and $x \in M$, we have $a \in (P : M)$ or $x \in P$ whenever $a \cdot x \in P$.*

Proof. This is clear by the equivalence

$$a \cdot x \in P \qquad \text{if and only if} \qquad Ra \cdot x \subseteq P.$$ □

Consider R as a left R-module over itself. It is straightforward that $\mathfrak{a} = (\mathfrak{a} : R)$ if $\mathfrak{a} \subseteq R$ is a two-sided ideal. Thus, if we replace in Theorem 2.2 and Corollary 2.3 the phrase *a proper submodule P* with the phrase *a proper two-sided ideal $\mathfrak{p}$*, we get two well-known assertions. It follows, for instance, that every prime ideal in R is a prime submodule when R is considered as a left module over itself. If R is commutative all submodules are of this type.

In the following theorem, which should be compared with Lemma 1.1 in [11], we use a simple fact that M/N is a left $R/(N : M)$-module with multiplication $(a + (N : M)) \cdot (x + N) = a \cdot x + N$, where $a + (N : M) \in R/(N : M)$ and $x + N \in M/N)$ are arbitrary, whenever N is a submodule in a left R-module M. Note that any R-module X is an $R/ann(X)$-module as well.

Theorem 2.4. *Let R be a ring and M be a left R-module. A proper submodule $P \subseteq M$ is prime if and only if the quotient $(P : M)$ is a prime ideal in R and (0) is a prime submodule in the $R/(P : M)$-module M/P.*

Proof. Let P be a prime submodule in M. Since P is proper, the identity of R is not in the quotient of P, which means that $(P : M)$ is a proper ideal in R. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two-sided ideals in R such that $\mathfrak{a}\mathfrak{b} \subseteq (P : M)$. Assume that $\mathfrak{b} \not\subseteq (P : M)$. Choose $b \in \mathfrak{b}$ which is not in $(P : M)$. Then there exists $x \in M$ such that $y = b \cdot x$ is not in P. It is clear that x is not in P. Let $a \in \mathfrak{a}$ be arbitrary. It follows from the inclusion $\mathfrak{a}\mathfrak{b} \subseteq (P : M)$ that $aRb \subseteq (P : M)$ and consequently $(aR) \cdot y = (aRb) \cdot x \subseteq P$, which gives $(RaR) \cdot (R \cdot y) \subseteq P$. By the assumption, P is a prime submodule, which means that we have $RaR \subseteq (P : M)$ or $R \cdot y \subseteq P$. However, the latter case is impossible because of $y \notin P$. Hence $a \in RaR \subseteq (P : M)$, which gives $\mathfrak{a} \subseteq (P : M)$.

Assume that $a + (P : M) \in R/(P : M)$ and $x + P \in M/P$ are such that $(a + (P : M))R/(P : M) \cdot (x + P) \subseteq (0)$. Then we have $aR \cdot x \subseteq P$ and consequently $a \in (P : M)$

or $x \in P$, by Theorem 2.2. Now we conclude that $a + (P : M) \in ((0) : M/P)$ or $x + P \in (0)$, which means, by Theorem 2.2, that (0) is a prime submodule in a left $R/(P : M)$-module M/P. (Note that $((0) : M/P)$ is the quotient of the trivial submodule in the left $R/(P : M)$-module M/P and that it is a trivial ideal in $R/(P : M)$.)

In order to prove the opposite implication assume that a two-sided ideal $\mathfrak{a} \subseteq R$ and a submodule $N \subseteq M$ are such that $\mathfrak{a} \cdot N \subseteq P$. It is easily seen that $\mathfrak{a}(N : M) \subseteq (P : M)$. Since, by the assumption, $(P : M)$ is a prime ideal, we conclude that $\mathfrak{a} \subseteq (P : M)$ or $(N : M) \subseteq (P : M)$. If the former is true, we have done. Assume therefore that $\mathfrak{a} \not\subseteq (P : M)$ and let $a \in \mathfrak{a} \setminus (P : M)$. Then, for an arbitrary $y \in N$, the inclusion $(a + (P : M))R/(P : M) \cdot (y + P) \subseteq (0)$ gives $a + (P : M) \in ((0) : M/P)$ or $y + P \in (0)$, by the assumption. However, the former is not the case because of $a \notin (P : M)$. Thus, $y \in P$. $\square$

Corollary 2.5. *Let R be a ring and M be a left R-module. There is a well defined mapping $\nu : \operatorname{Spec}(M) \to \operatorname{Spec}(R)$ which is given by*

$$\nu(P) = (P : M).$$

Following [4] we call ν the *natural mapping.* For a prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$, let $\operatorname{Spec}_{\mathfrak{p}}(M) = \nu^{-1}(\{\mathfrak{p}\})$. By Proposition 2.1 (i), it is clear that $\operatorname{Spec}_{\mathfrak{p}}(M)$ is empty if $ann(M) \not\subseteq \mathfrak{p}$. Also, if $\mathfrak{p} \cdot M = M$, for $\mathfrak{p} \in \operatorname{Spec}(R)$, then $\operatorname{Spec}_{\mathfrak{p}}(M) = \emptyset$. On the other hand, if $\mathfrak{m} \in \operatorname{Spec}(R)$ is maximal, then $\operatorname{Spec}_{\mathfrak{m}}(M)$ is non-empty if and only if $\mathfrak{m} \cdot M \neq M$.

In the rest of this section we study maximal submodules.

Definition 2.6. *A submodule N in a left R-module M is* co-cyclic *if M/N is a cyclic R-module. An element $u \in M$ is* co-cyclic *for N if $u + N$ is cyclic for M/N.*

It is obvious that a proper submodule does not contain co-cyclic elements. In the following proposition we list some properties of co-cyclic and maximal submodules.

Proposition 2.7. *Let R be a ring and M a left R-module.*

(i) If $L \subseteq M$ is a submodule such that it contains a co-cyclic submodule $N \subseteq M$, then L is co-cyclic as well. Moreover, if $u \in M$ is co-cyclic for N, then it is co-cyclic for L.

(ii) If $K \subseteq X$ is a maximal submodule, then M/K is simple. Thus, K is co-cyclic.

(iii) If K is a maximal submodule in M and $x \in M$ is not in K, then $R \cdot x + K = M$.

(iv) Every proper co-cyclic submodule is included in a maximal submodule.

(v) Every maximal submodule in M is prime.

Proof. The claims (i), (ii), and (iii) are easily proven.

(iv) Let N be a proper co-cyclic submodule with a co-cyclic element u. Denote by $\mathcal{F}$ the family of all the submodules in M that contain N and do not contain u. This family is partially ordered. The union of all submodules in a given linearly ordered subfamily of $\mathcal{F}$ is the upper bound of this subfamily. Thus, by Zorn's lemma, the family $\mathcal{F}$ has the maximal element, say K. If K were not a maximal submodule, there would be a maximal submodule K' such that $N \subseteq K \subset K'$ and $u \in K'$. However, by (i), this is impossible since u has to be co-cyclic for K'.

(v) Let K be a maximal submodule. Assume that $a \in R$ and $x \in M$ are such that $aR \cdot x \subseteq K$. Suppose that $x \notin K$. Then $x + K$ is a non-zero element in M/K, which means that $x + K$ is cyclic for this module. Hence, for every $y \in M$, there exists $b \in R$ such that $y + K = b \cdot (x + K)$. It follows that $y - b \cdot x \in K$ and therefore $a \cdot y - ab \cdot x \in K$. However, by the assumption, $ab \cdot x \in K$ and we conclude $a \cdot y \in K$ and consequently $a \in (K : M)$. $\square$

It follows, by the statement (v) of the previous proposition and by Theorem 2.4, that the quotient of a maximal submodule is a prime ideal. However, it is not necessarily that the quotient of a maximal submodule is a maximal ideal.

3. Topologies on the prime spectrum

Let R be a ring and M be a left R-module. For a given non-empty subset $S \subseteq \operatorname{Spec}(M)$, the intersection of all submodules in S is denoted by $\mathbf{k}(S)$ and we set $\mathbf{k}(\emptyset) = M$.

Definition 3.1. *The* outer hull *of a submodule $N \subseteq M$ is*

$$\mathbf{v}(N) = \{P \in \operatorname{Spec}(M);\ (N : M) \subseteq (P : M)\}$$

and the inner hull *of a submodule $N \subseteq M$ is*

$$\mathbf{h}(N) = \{P \in \operatorname{Spec}(M);\ N \subseteq P\}.$$

The assertions in the following lemma are straightforward to prove.

Lemma 3.2. *(i) For every submodule $N \subseteq M$, we have $\mathbf{h}(N) \subseteq \mathbf{v}(N)$.*

(ii) $\mathbf{v}((0)) = \mathbf{h}((0)) = \operatorname{Spec}(M)$ and $\mathbf{v}(M) = \mathbf{h}(M) = \emptyset$.

(iii) If $S_1 \subseteq S_2$, then $\mathbf{k}(S_1) \supseteq \mathbf{k}(S_2)$.

(iv) $S \subseteq \mathbf{h}(\mathbf{k}(S)) \subseteq \mathbf{v}(\mathbf{k}(S))$, for every $S \subseteq \operatorname{Spec}(M)$.

(v) If $N \subseteq L \subseteq M$, then $\mathbf{v}(N) \supseteq \mathbf{v}(L)$ and $\mathbf{h}(N) \supseteq \mathbf{h}(L)$.

(vi) For $S \subseteq \operatorname{Spec}(M)$, we have $\mathbf{k}(\mathbf{v}(\mathbf{k}(S))) \subseteq \mathbf{k}(\mathbf{h}(\mathbf{k}(S))) = \mathbf{k}(S)$; in general the inclusion might be proper.

(vii) Equalities $\mathbf{v}(N) = \mathbf{v}(\mathbf{k}(\mathbf{v}(N)))$ and $\mathbf{h}(N) = \mathbf{h}(\mathbf{k}(\mathbf{h}(N)))$ hold for any submodule $N \subseteq M$.

(viii) For an arbitrary family $\{N_i\}_{i\in\mathbb{I}}$ of submodules in M, we have

$$\cap_{i\in\mathbb{I}}\mathbf{h}(N_i) = \mathbf{h}\big(\sum_{i\in\mathbb{I}} N_i\big).$$

(ix) If N and L are submodules in M, then $\mathbf{h}(N) \cup \mathbf{h}(L) \subseteq \mathbf{h}(N \cap L)$.

Let us show that the family of outer hulls is closed under arbitrary intersections and finite unions.

Proposition 3.3. *Let R be a ring and M a left R-module.*

(i) For an arbitrary family $\{N_i\}_{i\in\mathbb{I}}$ of submodules in M, we have

$$\cap_{i\in\mathbb{I}}\mathbf{v}(N_i) = \mathbf{v}\big(\sum_{i\in\mathbb{I}}(N_i : M) \cdot M\big).$$

(ii) If N and L are submodules in M, then $\mathbf{v}(N) \cup \mathbf{v}(L) = \mathbf{v}(N \cap L)$.

Proof. (i) If $P \in \cap_{i\in\mathbb{I}}\mathbf{v}(N_i)$, then $(N_i : M) \subseteq (P : M)$, for all $i \in \mathbb{I}$. It follows $(N_i : M) \cdot M \subseteq (P : M) \cdot M$, for all $i \in \mathbb{I}$, and consequently

$$\sum_{i\in\mathbb{I}}(N_i : M) \cdot M \subseteq (P : M) \cdot M.$$

Since $(P : M) \cdot M \subseteq P$, we conclude

$$\big(\sum_{i\in\mathbb{I}}(N_i : M) \cdot M : M\big) \subseteq (P : M).$$

On the other hand, if P is in the outer hull of $\sum_{i\in\mathbb{I}}(N_i : M)\cdot M$, then

$$(N_j : M) \subseteq ((N_j : M)\cdot M : M) \subseteq (\sum_{i\in\mathbb{I}}(N_i : M)\cdot M : M) \subseteq (P : M)$$

gives $P \in \mathbf{v}(N_j)$, for all $j \in \mathbb{I}$.

(ii) It is easily seen that $(N : M)(L : M) \subseteq (N \cap L : M)$, for any pair of submodules $N,\ L \subseteq M$. Thus, if $P \in \mathbf{v}(N \cap L)$, then

$$(N : M)(L : M) \subseteq (N \cap L : M) \subseteq (P : M)$$

gives $(N : M) \subseteq (P : M)$ or $(N : M) \subseteq (P : M)$, by primeness of $(P : M)$ (see Theorem 2.4). The opposite inclusion is easily seen. □

Let R be a ring and M be a left R-module. For each submodule $N \subseteq M$, let us denote $\omega(N) = \mathrm{Spec}(M) \setminus \mathbf{v}(N)$.

Theorem 3.4 (cf. [4])**.** *The family*

$$\tau = \{\omega(N);\ N \textit{ is a submodule in } M\}$$

is a topology on Spec(M).

Proof. Since $\omega((0)) = \emptyset$ and $\omega(M) = \mathrm{Spec}(M)$ we have $\emptyset \in \tau$ and $\mathrm{Spec}(M) \in \tau$. Using the claim (i) of Proposition 3.3 we get

$$\cup_{i\in\mathbb{I}}\omega(N_i) = \omega\Big(\sum_{i\in\mathbb{I}}(N_i : M)\cdot M\Big),$$

for an arbitrary family $\{N_i\}_{i\in\mathbb{I}}$ of submodules in M. This shows that τ is closed under arbitrary unions. Similarly, by the statement (ii) of Proposition 3.3, we have $\omega(N) \cap \omega(L) = \omega(N \cap L)$, for all submodules N and L in M, which shows that τ is closed under finite intersections. □

Let $\tau' = \{\omega(\mathfrak{a}\cdot M);\ \mathfrak{a} \text{ is a left ideal in } R\}$. If R is a commutative ring and M is a left module over it, then τ' is a topology on Spec(M) (see [8]). We are going to prove that this is true in general.

Lemma 3.5 (cf. Lemma 3.1 in [11])**.** *Let R be a ring and M be a left R-module. For a left ideal $\mathfrak{a} \subseteq R$ and a submodule $N \subseteq M$, we have*

$$\mathbf{v}(\mathfrak{a}\cdot M) \cup \mathbf{v}(N) = \mathbf{v}(\mathfrak{a}\cdot N) = \mathbf{v}(\mathfrak{a}\cdot M \cap N).$$

Proof. We shall follow the proof of Lemma 3.1 in [11] modifying the arguments when needed. The inclusions

$$\mathbf{v}(\mathfrak{a}\cdot M) \cup \mathbf{v}(N) \subseteq \mathbf{v}(\mathfrak{a}\cdot M \cap N) \subseteq \mathbf{v}(\mathfrak{a}\cdot N)$$

are obvious. Suppose that $P \in \mathbf{v}(\mathfrak{a}\cdot N)$. Then $(\mathfrak{a}\cdot N : M) \subseteq (P : M)$. It is easily seen that $\mathfrak{a}(N : M) \subseteq (\mathfrak{a}\cdot N : M)$. Since, by Theorem 2.4, the quotient $(P : M)$ is a prime ideal the inclusion $\mathfrak{a}(N : M) \subseteq (P : M)$ gives $\mathfrak{a} \subseteq (P : M)$ or $(N : M) \subseteq (P : M)$. If the former is true, then we have the inclusion $\mathfrak{a}\cdot M \subseteq P$ and consequently $(\mathfrak{a}\cdot M : M) \subseteq (P : M)$, which proves the assertion. Obviously, if $(N : M) \subseteq (P : M)$ we have done as well. □

Corollary 3.6. *For arbitrary left ideals $\mathfrak{a}$ and $\mathfrak{b}$ in R, we have*

$$\mathbf{v}(\mathfrak{a}\cdot M) \cup \mathbf{v}(\mathfrak{b}\cdot M) = \mathbf{v}(\mathfrak{a}\mathfrak{b}\cdot M) = \mathbf{v}(\mathfrak{a}\cdot M \cap \mathfrak{b}\cdot M).$$

Theorem 3.7. *Let R be a ring and M be a left R-module. The family τ' is a topology on* $\mathrm{Spec}(M)$.

Proof. Use the claim (i) of Proposition 3.3 and Corollary 3.6. □

It is obvious that $\tau' \subseteq \tau$. Note however that the topology τ is also quite weak. For instance, the τ-closure of a singleton $\{P\} \subseteq \mathrm{Spec}(M)$ is $\mathrm{Spec}_{(P:M)}(M)$, a set that might be quite large (see [1]).

For a submodule N in a left R-module M, set $\omega^*(N) = \mathrm{Spec}(M) \setminus \mathbf{h}(N)$ and let $\tau^* = \{\omega^*(N);\ N \text{ is a submodule in } M\}$. It is well-known that τ^* is not always a topology on $\mathrm{Spec}(M)$. If it is, the module is said to be a *top* module (see [11]).

Theorem 3.8. *Every multiplication module is a top module.*

Proof. The proof of Lemma 3.1 in [11] works also in the case when the involved ring is not commutative. Thus, we may use Corollary 3.2 in [11], which says $\mathbf{h}(\mathfrak{a} \cdot M) \cup \mathbf{h}(\mathfrak{b} \cdot M) = \mathbf{h}(\mathfrak{a}\mathfrak{b} \cdot M)$. Since the module under consideration is a multiplication module the last equality shows that τ^* is closed under finite intersections. That τ^* is closed under arbitrary unions follows from Lemma 3.2(viii). □

At the end let us show that the natural mapping is continuous.

Theorem 3.9. *Let R be a ring and M be a left R-module. The natural mapping $\nu :$ $\mathrm{Spec}(M) \to \mathrm{Spec}(R)$ is continuous if $\mathrm{Spec}(M)$ is endowed with the topology τ' and $\mathrm{Spec}(R)$ has the Zariski topology. More precisely, for every left ideal $\mathfrak{a} \subseteq R$, we have $\nu^{-1}(\mathbf{h}(\mathfrak{a})) = \mathbf{v}(\mathfrak{a} \cdot M)$.*

Proof. It is enough to prove the last equality. If $P \in \mathrm{Spec}(M)$ is in $\nu^{-1}(\mathbf{h}(\mathfrak{a}))$, then $\mathfrak{a} \subseteq (P : M)$ and consequently $\mathfrak{a} \cdot M \subseteq (P : M) \cdot M \subseteq P$, which gives $(\mathfrak{a} \cdot M : M) \subseteq (P : M)$. On the other hand, if $P \in \mathrm{Spec}(M)$ is in $\mathbf{v}(\mathfrak{a} \cdot M)$, then $\mathfrak{a} \subseteq (\mathfrak{a} \cdot M : M) \subseteq (P : M)$ and therefore $(P : M) \in \nu^{-1}(\mathbf{h}(\mathfrak{a}))$. □

Corollary 3.10 (cf. [4])**.** *The natural mapping is continuous if* $\mathrm{Spec}(M)$ *is endowed with the topology τ.*

Corollary 3.11. *Let M be a multiplication module. Then the natural mapping is continuous if* $\mathrm{Spec}(M)$ *is endowed with the topology τ^*.*

Proof. Since $\mathbf{h}(\mathfrak{a} \cdot M) \subseteq \mathbf{v}(\mathfrak{a} \cdot M)$ we have $\mathbf{h}(\mathfrak{a} \cdot M) \subseteq \nu^{-1}(\mathbf{h}(\mathfrak{a}))$, by Theorem 3.9. The opposite inclusion is proved in the first part of the proof of Theorem 3.9. Hence $\nu^{-1}(\mathbf{h}(\mathfrak{a})) = \mathbf{h}(\mathfrak{a} \cdot M)$. □

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On Feynman-Kac Perturbation of Symmetric Markov Processes

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Let X be an m-symmetric right process on Luzin space E and $(\mathcal{E}, \mathcal{F})$ be its associated quasi-regular Dirichlet form. Let μ be a signed smooth measure of X and A^μ be the continuous additive functional (CAF in abbreviation) of X with signed Revuz measure μ. It defines a symmetric Feynman-Kac semigroup

$$T_t f(x) := \mathbf{E}_x \left[\exp(A_t^\mu) f(X_t) \right] \qquad \text{for } t > 0 \text{ and } f \geq 0. \tag{1}$$

Define a symmetric quadratic form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ by

$$\begin{aligned} \mathcal{D}(\mathcal{E}^\mu) &:= \left\{ u \in \mathcal{F} : u \in L^2(E, |\mu|) \right\}, \\ \mathcal{E}^\mu(u, v) &= \mathcal{E}(u, v) - \int_E u(x) v(x) \mu(dx) \qquad \text{for } u, v \in \mathcal{D}(\mathcal{C}). \end{aligned}$$

We say that the form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is bounded from below if there is some $\alpha_0 \geq 0$ such that

$$\mathcal{E}^\mu_{\alpha_0}(u, u) := \mathcal{E}^\mu(u, u) + \alpha_0 (u, u) \geq 0 \qquad \text{for every } u \in \mathcal{D}(\mathcal{E}^\mu).$$

For a non-negative smooth measure ν, we say it is in the Kato class of X if

$$\lim_{t \to 0} \sup_{x \in E} [A_t^\nu] = 0.$$

It is known (see, e.g., [1, Proposition 2.1(i)] and [6, Theorem 3.1]) that if ν is in the Kato class of X, then for every $\varepsilon > 0$, there is some constant $A_\varepsilon > 0$ such that

$$\int_E u(x)^2 \nu(dx) \leq \varepsilon\, \mathcal{E}(u, u) + A_\varepsilon \int_E u(x)^2 m(dx) \qquad \text{for every } u \in \mathcal{F}.$$

This implies that for a signed smooth measure μ with $|\mu|$ in the Kato class of X, $\mathcal{F}^\mu = \mathcal{F}$ and the quadratic form $(\mathcal{E}^\mu, \mathcal{F})$ is bounded from below. Moreover, it is know (see, e.g.,

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[1, Proposition 3.1(ii)]) that when $|\mu|$ is in the Kato class of X, the semigroup $\{T_t, t \geq 1\}$ defined by (1) is the strongly continuous semigroup in $L^2(E,m)$ that is associated with the quadratic form $(\mathcal{E}^\mu, \mathcal{F}^\mu)$.

The purpose of this note is to give an alternative, direct proof of the following result due to S. Albeverio and Z.-M. Ma [1]. For definitions and properties of Dirichlet form, smooth measure, additive functional, $\mathcal{E}$-nest, we refer the reader to [5].

Theorem 1 *The semigroup $\{T_t, t \geq 0\}$ defined by (1) is strongly L^2-continuous in $L^2(E,m)$ if and only if the form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is bounded from below. In this case, the closed quadratic form $(\mathcal{C}, \mathcal{D}(\mathcal{C}))$ associated with $\{T_t, t \geq 0\}$ is the smallest closed form that is form bigger than $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ in the following sense:*

(i) *$\mathcal{D}(\mathcal{E}^\mu) \subset \mathcal{D}(\mathcal{C})$ and $\mathcal{E}^\mu(f,f) \geq \mathcal{C}(f,f)$ for every $f \in \mathcal{D}(\mathcal{E}^\mu)$;*

(ii) *Suppose $(\mathcal{S}, \mathcal{D}(\mathcal{S}))$ is a closed symmetric form that has the property in (i) with $(\mathcal{S}, \mathcal{D}(\mathcal{S}))$ in place of $(\mathcal{C}, \mathcal{D}(\mathcal{C}))$. Then $\mathcal{D}(\mathcal{C}) \subset \mathcal{D}\mathcal{S})$ and $\mathcal{C}(f,f) \leq \mathcal{S}(f,f)$ for every $f \in \mathcal{D}(\mathcal{C})$.*

To prove the above theorem, Albeverio and Ma [1] used $\mu_k := (\mathbf{1}_{F_k}\mu^+) - \mu^-$ to approximate μ, where $\{F_k, k \geq 1\}$ is an $\mathcal{E}$-nest so that $\mathbf{1}_{F_k}\mu^+$ is in Kato class of X for every $k \geq 1$. In this paper we will use $\mathbf{1}_{F_k}\mu$ to approximate μ instead, where $\{F_k, k \geq 1\}$ is an $\mathcal{E}$-nest so that $\mathbf{1}_{F_k}|\mu|$ is in Kato class of X for every $k \geq 1$. The advantage of our approximation is that it is more intrinsic since it does not depend on the Jordan decomposition $\mu^+ - \mu^-$ of μ. The idea of our approach is applicable to study similar problems for more general type for perturbations for symmetric Markov processes such as those investigated in [3].

We first establish the following. Let $\{F_k, k \geq 1\}$ be an $\mathcal{E}$-nest so that $\mathbf{1}_{F_k}|\mu|$ is in Kato class of X for every $k \geq 1$. Such an $\mathcal{E}$-nest always exists since $|\mu|$ is a smooth measure of X (see, e.g., [2, Theorem 2.4] or[4, Proposition 2.2]). Observe that the CAF associated with $\mathbf{1}_{F_k}\mu$ is $t \mapsto \int_0^t \mathbf{1}_{F_k}(X_s) dA_s^\mu$.

Theorem 2 *The semigroup $\{T_t, t \geq 0\}$ is strongly L^2-continuous in $L^2(E,m)$ if and only if the form $\left(\mathcal{E}^\mu, \bigcup_{k\geq 1} \mathcal{F}_{F_k}\right)$ is bounded from below. In this case, the closed quadratic form $(\mathcal{C}, \mathcal{D}(\mathcal{C}))$ associated with $\{T_t, t \geq 0\}$ is the smallest closed form that is form bigger than $(\mathcal{E}^\mu, \bigcup_{k\geq 1} \mathcal{F}_{F_k})$ in the following sense:*

(i) *$\bigcup \mathcal{F}_{F_k} \subset \mathcal{D}(\mathcal{C})$ and $\mathcal{E}^\mu(f,f) \geq \mathcal{C}(f,f)$ for every $f \in \bigcup_{k\geq 1} \mathcal{F}_{F_k}$;*

(ii) *Suppose $(\mathcal{S}, \mathcal{D}(\mathcal{S}))$ is a closed symmetric form that has the property in (i) with $(\mathcal{S}, \mathcal{D}(\mathcal{S}))$ in place of $(\mathcal{C}, \mathcal{D}(\mathcal{C}))$. Then $\mathcal{D}(\mathcal{C}) \subset \mathcal{D}\mathcal{S})$ and $\mathcal{C}(f,f) \leq \mathcal{S}(f,f)$ for every $f \in \mathcal{D}(\mathcal{C})$.*

Proof. Let $\{F_k, k \geq 1\}$ be an $\mathcal{E}$-nest so that $\mathbf{1}_{F_k}|\mu|$ is in the Kato class of X for every $k \geq 1$. In particular, $\mathbf{1}_{F_k}|\mu|$ is in the Kato class of the subprocess X^{F_k} of X killed upon leaving F_k. Define

$$\tau_{F_k} := \inf\{t > 0 : X_t \notin F_k\}.$$

It is well-known that the CAF of X^{F_k} having Revuz measure $\mathbf{1}_{F_k}\mu$ is $\{A^\mu_{t\wedge\tau_{F_k}}, t \geq 0\}$. Thus the semigroup $\left\{T_t^{(k)}, t \geq 0\right\}$ defined by

$$T_t^{(k)} f(x) = \mathbf{E}_x \left[\exp(A_t^\mu) f(X_t); t < \tau_{F_k}\right]$$

is a strongly L^2-continuous semigroup on $L^2(F_k, m) \subset L^2(E, m)$ whose associated quadratic form is $(\mathcal{E}^\mu, \mathcal{F}_{F_k})$, where

$$\mathcal{F}_{F_k} := \{u \in \mathcal{F} : u = 0 \text{ q.e. on } F_k^c\}.$$

Note that $\cup_{k\geq 1}\mathcal{F}_{F_k} \subset \mathcal{D}(\mathcal{E}^\mu)$.

(1) Assume that $\left(\mathcal{E}^\mu, \bigcup_{k\geq 1} \mathcal{F}_{F_k}\right)$ is bounded from below. Then there is some $\alpha_0 \geq 0$ such that $\mathcal{E}^\mu_{\alpha_0}(u, u) \geq 0$ for every $u \in \bigcup_{k\geq 1} \mathcal{F}_{F_k}$. This in particular implies that for every $k \geq 1$,

$$\mathcal{E}^\mu_{\alpha_0}(u, u) \geq 0 \qquad \text{for every } u \in \mathcal{F}_{F_k}.$$

This yields

$$\|T_t^{(k)}\|_{2,2} \leq e^{\alpha_0 t} \qquad \text{for every } t \geq 0.$$

For every $f \in L^2(E, m)$, we have by Fatou's theorem,

$$\|T_t f\|_2 \leq \|T_t|f|\|_2 \leq \varliminf_{k\to\infty} \|T_t^{(k)}|f|\| \leq e^{\alpha_0 t}\|f\|_2.$$

Thus we have

$$\|T_t\|_{2,2} \leq e^{\alpha_0 t} \qquad \text{for every } t \geq 0.$$

Note that for $f \in L^2_+(E, m)$, by dominated convergence theorem, $T_t^{(k)} f$ converges in $L^2(E, m)$ to $T_t f$.

Let $\{G_\alpha, \alpha > \alpha_0\}$ be the resolvent of $\{T_t, t \geq 0\}$ and define

$$\mathcal{D}(\mathcal{C}) = \left\{u \in L^2(E, m) : \sup_{t>0} \frac{1}{t}(u - T_t u, u) < \infty\right\} \tag{2}$$

Note that as $\lim_{t\to 0}(u - T_t u, u) = 0$ for every $u \in \mathcal{D}(\mathcal{C})$, we have

$$\lim_{t\to 0} \|T_t u - u\|_2^2 = \lim_{t\to\infty} \left((T_{2t}u, u) - 2(T_t u, u) + (u, u)\right) = 0 \qquad \text{for } u \in D(\mathcal{C}). \tag{3}$$

Since for $f \in \mathcal{F}^+_{F_k}$, we have $T_t^{(k)} f \leq T_t f$ and so $f \in \mathcal{D}(\mathcal{C})$. This implies that $\mathcal{F}_{F_k} \subset \mathcal{D}(\mathcal{C})$ and so $\bigcup_{k\geq 1} \mathcal{F}_{F_k} \subset \mathcal{D}(\mathcal{C})$. Since $\bigcup_{k\geq 1} \mathcal{F}_{F_k}$ is dense in $L^2(E, m)$ and T_t is a bounded symmetric operator on $L^2(E, m)$, we conclude from (3) that

$$\lim_{t\to 0} \|T_t u - u\|_2 = 0 \qquad \text{for every } u \in L^2(E, m).$$

In other words $\{T_t, t \geq 0\}$ is a strongly continuous semigroup in $L^2(E, m)$. Let $(\mathcal{C}, \mathcal{D}(\mathcal{C}))$ be the Dirichlet form of $\{T_t, t \geq 0\}$, that is, $\mathcal{D}(\mathcal{C})$ is given by (2) and

$$\mathcal{C}(u, u) = \lim_{t\to\infty} \frac{1}{t}(u - T_t u, u) = \sup_{t>0} \frac{1}{t}(u - T_t u, u) \qquad \text{for } u \in \mathcal{D}(\mathcal{C}).$$

We claim that for $f \in \bigcup_{k\geq 1} \mathcal{F}_{F_k}$, $\mathcal{E}^\mu(f,f) \geq \mathcal{C}(f,f)$. This is because for $f \in \mathcal{F}_{F_k}$, by [5, Lemma 1.3.4],

$$\begin{aligned}
\mathcal{C}_{\alpha_0}(f,f) &= \sup_{t>0} \frac{1}{t}(f - e^{-\alpha_0 t} T_t f, f) = \sup_{t>0} \lim_{n\to\infty} \frac{1}{t}(f - e^{-\alpha_0 t} T_t^{(n)} f, f) \\
&\leq \lim_{n\to\infty} \sup_{t>0} \frac{1}{t}(f - e^{-\alpha_0 t} T_t^{(n)} f, f) = \lim_{n\to\infty} \mathcal{E}^\mu_{\alpha_0}(f,f) = \mathcal{E}^\mu_{\alpha_0}(f,f).
\end{aligned}$$

Now suppose $(\mathcal{S}, \mathcal{D}(\mathcal{S}))$ is another closed form that is greater than $(\mathcal{C}, \cup_{k\geq 1}\mathcal{F}_{F_k})$. Take $\alpha > \alpha_0$. Let $f = G_\alpha u$ for some $u \in L^2(E,m)$. Then we know $f_n := G_\alpha^{(n)} u \in \mathcal{F}_{F_n}$ converges in L^2 to f. It follows then

$$\sup_{n\geq 1} \mathcal{S}_\alpha(f_n, f_n) \leq \sup_{n\geq 1} \mathcal{C}_\alpha(f_n, f_n) = \sup_{n\geq 1} (u, G_\alpha^{(n)} u) < \infty.$$

This implies that $f \in \mathcal{D}(\mathcal{S})$ with

$$\begin{aligned}
\mathcal{S}_\alpha(f,f) &\leq \varliminf_{n\to\infty} \mathcal{S}_\alpha(f_n, f_n) \leq \varliminf_{n\to\infty} \mathcal{E}^\mu_\alpha(f_n, f_n) = \lim_{n\to\infty} (u, G_\alpha^{(n)} u) \\
&= (u, G_\alpha u) = \mathcal{C}_\alpha(G_\alpha u, G_\alpha u) = \mathcal{C}_\alpha(f,f).
\end{aligned}$$

Since $G_\alpha(L^2(E,m))$ is $\mathcal{C}_1$-dense in $\mathcal{D}(\mathcal{C})$, it follows that $\mathcal{D}(\mathcal{C})) \subset \mathcal{D}(\mathcal{S})$ and $\mathcal{S}(f,f) \leq \mathcal{C}(f,f)$ for every $f \in \mathcal{D}(\mathcal{C})$. In other words, $(\mathcal{S}, \mathcal{D}(\mathcal{S}))$ is form bigger than $(\mathcal{C}, \mathcal{D}(C))$. Hence we conclude that $(\mathcal{C}, \mathcal{D}(\mathcal{C}))$ is *the* smallest closed symmetric form that is form bigger than $(\mathcal{C}, \cup_{k\geq 1}\mathcal{F}_{F_k})$.

(2) Now suppose that $\{T_t, t \geq 0\}$ defined by (1) is a strongly L^2-continuous semigroup on $L^2(E;m)$. We first show that there is some $\alpha_0 \geq 0$ such that $\|T_t\|_{2,2} \leq e^{\alpha_0 t}$ for every $t \geq 0$. Let $\alpha_0 = 0 \vee \log \|T_1\|$ and define $\widehat{T}_t = e^{-\alpha_0 t} T_t$. Then $\{\widehat{T}_t, t \geq 0\}$ is a strongly continuous symmetric semigroup with $\|\widehat{T}_1\| \leq 1$. Thus for every $f \in L^2$ with $\|f\|_2 = 1$,

$$\|\widehat{T}_{1/2} f\|_2^2 = (f, T_1 f) \leq 1.$$

This proves $\|\widehat{T}_{1/2}\| \leq 1$. Iterating, we have $\|\widehat{T}_{1/2^n}\| \leq 1$ for every integer $n \geq 1$. Using semigroup property, we have $\|\widehat{T}_{k/2^n}\| \leq 1$ for integers $k, n \geq 1$. By the strong continuity of $\{\widehat{T}_t, t \geq 0\}$, we conclude that $\|\widehat{T}_t\| \leq 1$ and so $\|T_t\| \leq e^{\alpha_0 t}$ for every $t \geq 0$.

Next let $\{F_k\}$ be an $\mathcal{E}$-nest so that $\mathbf{1}_{F_k}|\mu|$ is in Kato class of X for every $k \geq 1$. As we observed earlier, for every $k \geq 1$, the semigroup $\{T_t^{(k)}, t \geq 0\}$ defined by

$$T_t^{(k)} f := \mathbf{E}_x \left[\exp(A_t^\mu) f(X_t); t < \tau_{F_k} \right]$$

is the strongly continuous semigroup associated with the closed quadratic form $(\mathcal{E}^\mu, \mathcal{F}_{F_k})$. For every $f \in L^2(F_k, m)$ and $t \geq 0$,

$$\|T_t^{(k)} f\|_2 \leq \|T_t^{(k)} |f|\|_2 \leq \|T_t |f|\|_2 \leq e^{\alpha_0 t} \|f\|_2.$$

Thus we have $\|T_t^{(k)}\|_{2,2} \leq e^{\alpha_0 t}$. It follows then for $u \in \mathcal{F}_{F_k}$,

$$\mathcal{E}^\mu_{\alpha_0}(u,u) = \lim_{t\to\infty} \frac{1}{t}(u - e^{\alpha_0 t} T_t^{(k)} u, u) \geq 0.$$

This proves the form $\mathcal{E}^\mu$ bounded from below on $\bigcup_{k\geq 0}\mathcal{F}_{F_k}$. □

Theorem 2 is in fact equivalent to Theorem 1, due to the following fact.

Lemma 3 *Let $\{F_k, k\geq 1\}$ be an $\mathcal{E}$-nest so that $\mathbf{1}_{F_k}|\mu|$ is in Kato class of X for every $k\geq 1$. Then $\bigcup_{k\geq 1}\mathcal{F}_{F_k}$ is dense in $\mathcal{D}(\mathcal{E}^\mu)$ with respect to the $\mathcal{E}_1^{|\mu|}$-norm. Here $\mathcal{E}_1^{|\mu|}(u,u) := \mathcal{E}_1(u,u)+\int_E u(x)^2|\mu|(dx)$ for $u\in\mathcal{D}(\mathcal{E}^\mu)$.*

Proof. As we already noted, $\bigcup_{k\geq 1}\mathcal{F}_{F_k}\subset\mathcal{D}(\mathcal{E}^\mu)$. For $u\in\mathcal{D}(\mathcal{E}^\mu)^+$, let

$$u_k(x) := u(x) - \mathbf{E}_x\left[e^{-\sigma_{F_k^c}}\,u\big(X_{\sigma_{F_k^c}}\big)\right]$$

be the $\mathcal{E}_1$-orthogonal projection of u onto $\mathcal{F}_{F_k}$. Clearly, u_k is $\mathcal{E}_1$-convergent to u as $k\to\infty$. By [5, Theorem 1.4.2(v)], $|u_k|$ converges to u in $\mathcal{E}_1$-norm as $k\to\infty$. Taking a subsequence if necessary, we may assume without loss of generality that $|u_k|$ converges to u q.e. on E. Define $f_k := |u_k|\wedge u$. Clearly f_k converges to u in $L^2(E,|\mu|)$, $f_k\in\mathcal{F}$ and $f_k\in\mathcal{F}_{F_k}$ as $u_k=0$ q.e. on F_k^c. As

$$\sup_{k\geq 1}\mathcal{E}(f_k,f_k)\leq\sup_{k\geq 1}\mathcal{E}(|u_k|,|u_k|)+\mathcal{E}(u,u)<\infty,$$

there is a subsequence $\{n_k, k\geq 1\}$ so that $g_j := \frac{1}{j}\sum_{k=1}^j f_{n_k}$ converges to u in $\mathcal{E}_1$-norm as $j\to\infty$. Note that $g_j\in\bigcup_{k\geq 1}\mathcal{F}_{F_k}$ and g_j converges to u in $L^2(E,|\mu|)$. Thus we have proved that every $u\in\mathcal{D}(\mathcal{C})^+$ can be approximated in $\mathcal{E}_1^{|\mu|}$-norm by members of $\bigcup_{k\geq 1}\mathcal{F}_{F_k}$. It follows then $\bigcup_{k\geq 1}\mathcal{F}_{F_k}$ is $\mathcal{E}_1^{|\mu|}$-dense in $\mathcal{D}(\mathcal{E}^\mu)$. □

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CORRESPONDENCE BETWEEN THE RESIDUAL SPECTRA OF RANK TWO SPLIT CLASSICAL GROUPS AND THEIR INNER FORMS

NEVEN GRBAC

Introduction

In this paper the correspondence between the residual spectra of the classical rank two split groups SO_4 and Sp_4 and their inner forms G_1' and H_1' is obtained. As algebraic groups over an algebraic number field, hermitian quaternionic groups G_1' and H_1' are defined in Section 1. They are non–quasi split. The problem of comparing the residual spectra of split groups and their inner forms is still open even for the general linear group as mentioned in [3] and Section 25 of [2]. In this paper, the correspondence is obtained for the low rank case by the explicit decomposition of the residual spectra of all the groups involved.

For quasi–split groups the residual spectrum has been considered by several authors. Among others Mœglin and Walspurger in [21], Mœglin in [18], [19] and [20], Kim in [11], [12] and [15], Žampera in [29], Kon–No in [16]. All those papers use Langland–Shahidi method described in [25] and [26] for normalization of standard intertwining operators.

For the split group Sp_4 the residual spectrum is decomposed into irreducible constituents by Kim in [11] and in this paper we use his result recalled in Theorem 3.4. Decomposition of the residual spectrum for the split group SO_4 is obtained by the same method in Theorem 3.3 of this paper.

However, groups G_1' and H_1' are out of scope of Langlands–Shahidi method since they are non–quasi split. For non–quasi split groups the residual spectrum was considered in author's papers [6] and [7]. In those papers we developed a new technique for normalization of standard intertwining operators which is applied in this paper. It is based on Jacquet–Langlands correspondence explained in [5] and comparison of Plancherel measures as in [24].

Although, in principle, the residual spectra of G_1' and H_1' could be decomposed using Arthur's trace formula explained in [2], we use more direct approach of Langlands spectral theory explained in [17] and [22]. Decompositions of the residual spectra for groups G_1' and H_1' are obtained in Theorems 3.1 and 3.2. Already for these low rank cases, the results show certain ambiguities of hermitian quaternionic groups such as condition on non–triviality of local components at all non–split places in Theorem 3.2.

Finally, as a consequence of decomposition Theorems, we obtain correspondences between the residual spectra of G_1' and SO_4 in Corollary 3.5 and between the residual spectra of H_1' and Sp_4 in Corollary 3.6.

The paper is divided into three Sections. In the first Section we introduce notation and recall basic structural facts for the groups involved. In the second Section we collect global normalizing factors required in calculation and obtained in [6] and [7]. Finally, in the third Section we decompose the residual spectra and give the correspondences.

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1. Group Structure and Notation

In this Section we define groups considered in this paper, review their structure and introduce notation. Let k be an algebraic number field, k_v its completion at place v and $\mathbb{A}$ its ring of adeles. Let D be a quaternion algebra central over k and τ the usual involution fixing the center of D. Then D splits at all but finitely many places v of k, i.e. at those places the completion $D \otimes_k k_v$ is isomorphic to the additive group $M(2, k_v)$ of 2×2 matrices with coefficients in k_v. At finitely many places v of k where D is non–split the completion $D \otimes_k k_v$ is isomorphic to the quaternion algebra D_v central over k_v. The finite set of places of k where D is non–split is denoted by S. The cardinality of S, denoted by $|S|$, is even for every D.

The algebraic group over k of invertible elements of D is denoted GL'_1. At a split place $v \notin S$ it is isomorphic to $GL'_1(k_v) \cong GL_2(k_v)$, where GL_2 is a split group over k of invertible 2×2 matrices. At a non–split place $v \in S$ it is isomorphic to $GL'_1(k_v) \cong D_v^\times$.

Let $\det'$ denote the reduced norm of the simple algebra $D \otimes_k \mathbb{A}$ and $\det'_v$ the corresponding reduced norm at place v. If $v \notin S$ is split, then $\det'_v = \det_v$ is just the determinant for 2×2 matrices, while if $v \in S$ is non–split, then $\det'_v$ is the reduced norm of the quaternion algebra D_v. The absolute value of the reduced norm $\det'$ and $\det'_v$ is denoted by ν.

Let V be a hyperbolic plane over D, i.e. a two–dimensional right vector space over D equipped with a hermitian form $(\cdot, \cdot)$ defined in the basis $\{e_1, e_2\}$ of V by

$$(e_1 x_1 + e_2 x_2, e_1 x'_1 + e_2 x'_2) = \tau(x_1) x'_2 + \epsilon \tau(x_2) x'_1$$

for all $x_1, x_2, x'_1, x'_2 \in D$, where $\epsilon \in \{\pm 1\}$. The group of isometries of the form $(\cdot, \cdot)$ regarded as a reductive algebraic group defined over k will be denoted by G'_1 if $\epsilon = -1$ and by H'_1 if $\epsilon = 1$. Then, G'_1 is an inner form of the split group SO_4, while H'_1 is an inner form of the split group Sp_4. Hence $G'_1(k_v) \cong SO_4(k_v)$ and $H'_1(k_v) \cong Sp_4(k_v)$ for every place $v \notin S$.

Both, G'_1 and H'_1, have only one standard proper parabolic subgroup defined over k. Abusing the notation, we let P'_0 denote that subgroup for both groups. The Levi factor M'_0 of P'_0 is isomorphic to GL'_1. For the split groups SO_4 and Sp_4, let P_0 denote the standard parabolic subgroup with Levi factor $M_0 \cong GL_2$ such that P'_0 is an inner form of P_0. The maximal split torus isomorphic to $GL_1 \times GL_1$ for those split groups is denoted by T.

The Weyl groups for G'_1 and H'_1 with respect to the maximal split torus are $W' = \{1, w\}$, where w is the unique nontrivial element. For the corresponding split case Levi factor $M_0 \cong GL_2$ in SO_4 or Sp_4 let $W(M_0)$ denote the subgroup of the Weyl group W consisting of elements fixing the Levi factor M_0. Then $W' \cong W(M_0)$ and again by w we denote the nontrivial element of $W(M_0)$.

Let Δ be the set of simple roots of split group SO_4 or Sp_4. If e_i is the character of T defined by $e_i(t_1, t_2) = t_i$ for $(t_1, t_2) \in T \cong GL_1 \times GL_1$, then

$$\Delta = \begin{cases} \{e_1 - e_2, e_1 + e_2\}, & \text{for the group } SO_4, \\ \{e_1 - e_2, 2e_2\}, & \text{for the group } Sp_4. \end{cases}$$

Let w_1 in both cases be the simple reflection with respect to the root $e_1 - e_2$ and w_2 the simple reflection with respect to $e_1 + e_2$ for SO_4 and $2e_2$ for Sp_4. Then the Weyl group equals

$$W = \begin{cases} \{1, w_1, w_2, w_1w_2\}, & \text{for the group } SO_4, \\ \{1, w_1, w_2, w_1w_2, w_2w_1, w_1w_2w_1, w_2w_1w_2, w_1w_2w_1w_2\}, & \text{for the group } Sp_4. \end{cases}$$

Let $\mathfrak{a}^*_{M_0,\mathbb{C}} \cong X(M_0') \otimes_{\mathbb{Z}} \mathbb{C}$ denote the complexification of the $\mathbb{Z}$–module $X(M_0')$ of k–rational characters of M_0'. It is one–dimensional and we identify it with $\mathbb{C}$ taking for the basis the reduced norm on GL_1'. In the split case of M_0 in SO_4 and Sp_4 the complexification of the $\mathbb{Z}$–module $X(M_0)$ of k–rational characters of M_0 is again $\mathfrak{a}^*_{M_0,\mathbb{C}}$. For the torus T in SO_4 and Sp_4 the complexification of the $\mathbb{Z}$–module $X(T)$ of k–rational characters of T is denoted by $\mathfrak{a}^*_{T,\mathbb{C}}$. It is two–dimensional and we identify it with $\mathbb{C}^2$ taking for the basis reduced norms on every copy of GL_1 inside T.

We should remark that in this paper the usual parabolic induction from standard parabolic subgroup P of G with the Levi factor M will be denoted by Ind_M^G instead of Ind_P^G. This will not cause any confusion since all the parabolic subgroups appearing in the paper are standard.

2. Normalizing factors for intertwining operators

In this Section we recall results on normalization of intertwining operators obtained in [6] and [7]. The global normalizing factors are given for standard global intertwining operators in all maximal parabolic subgroup cases needed in the sequel. These factors are scalar valued meromorphic functions with the property that the normalized intertwining operator becomes holomorphic and non–vanishing in the closure of the positive Weyl chamber, except at the origin in some cases. We do not repeat the proof of holomorphy and non–vanishing in the cases needed in this paper since it can be found in [6] and [7]. The strategy of the proof is local. At split places the normalization is obtained using Langlands–Shahidi method explained in [25] and [26]. Non–split places for non–quasi split groups are out of scope of Langlands–Shahidi method and the normalization is obtained using a new technique developed in [6] and [7] which is based on transfer of Plancherel measures between split groups and their inner forms as in [24].

The normalizing factors are defined using Jacquet–Langlands correspondence explained in Section 8 of [5]. More precisely, let $\pi' \cong \otimes_v \pi_v'$ be a higher–dimensional cuspidal automorphic representation of $GL_1'(\mathbb{A})$. Then, at non–split places $v \in S$, by the local Jacquet–Langlands correspondence of Theorem (8.1) in [5], π_v' corresponds to a square–integrable representation π_v of $GL_2(k_v)$. At the split places $v \notin S$ we have $GL_1'(k_v) \cong GL_2(k_v)$ and we take $\pi_v \cong \pi_v'$. Then, by the global Jacquet–Langlands correspondence, π' corresponds to a representation $\pi \cong \otimes_v \pi_v$ of $GL_2(\mathbb{A})$. By Theorem (8.3) of [5], π is isomorphic to a cuspidal automorphic representation of $GL_2(\mathbb{A})$.

Otherwise, a cuspidal automorphic representation π' of $GL'_1(\mathbb{A})$ is one–dimensional. Then, there exists a unitary character χ of $\mathbb{A}^\times/k^\times$ such that $\pi' \cong \chi \circ \det'$.

Now, we are ready to give global normalizing factors for standard intertwining operators in maximal proper parabolic cases needed in the sequel. Those cases have Levi factors $GL'_1 \subset G'_1$, $GL'_1 \subset H'_1$ for non–quasi split groups and $GL_2 \subset SO_4$, $GL_2 \subset Sp_4$, $GL_1 \times GL_1 \subset GL_2$ and $GL_1 \subset Sp_2 \cong SL_2$ for split groups. Furthermore, for non–quasi split groups the normalizing factors depend on dimension of a cuspidal automorphic representation of $GL'_1(\mathbb{A})$, i.e. factors are different for higher–dimensional and one–dimensional representations. The normalizing factor in general proper parabolic case is just a product of normalizing factors in maximal cases appearing in decomposition of standard intertwining operator given in Section 2.1 of [25].

For a cuspidal automorphic representation σ of one of the Levi factors $M(\mathbb{A})$, $\underline{s} \in \mathfrak{a}^*_{M,\mathbb{C}}$ and the unique nontrivial Weyl group element w_0 corresponding to M, the standard intertwining operator, denoted by $A(\underline{s}, \sigma, w_0)$, acts on the induced representation $I(\underline{s}, \sigma)$ where I denotes normalized induction for the corresponding maximal proper parabolic case. Observe that $\underline{s} = s \in \mathbb{C}$ in all cases except $GL_1 \times GL_1 \subset GL_2$ when $\underline{s} = (s_1, s_2) \in \mathbb{C}^2$. The normalizing factor for the standard intertwining operator $A(\underline{s}, \sigma, w_0)$ is denoted by $r(\underline{s}, \sigma, w_0)$. Then, the normalized intertwining operator $N(\underline{s}, \sigma, w_0)$ is given by

$$A(\underline{s}, \sigma, w_0) = r(\underline{s}, \sigma, w_0) N(\underline{s}, \sigma, w_0).$$

It is proved in Section 1 of [6] and Section 1 of [7] that the normalized operators obtained in this way are holomorphic and non–vanishing in the region required in this paper. Thus, calculation of poles of standard intertwining operators is reduced to poles of normalizing factors.

When defining global normalizing factors in [6] and [7], the first step is to define local normalizing factors and local normalized intertwining operators at all places. In this paper we use local normalized intertwining operators when studying images of global ones. However, for precise definition one should consult [6] and [7].

In the case $GL'_1 \subset G'_1$, the normalizing factor for the intertwining operator $A(s, \pi', w)$ acting on the induced representation

$$\mathrm{Ind}_{GL'_1(\mathbb{A})}^{G'_1(\mathbb{A})} \pi' \nu^s,$$

where π' is a higher–dimensional cuspidal automorphic representation of $GL'_1(\mathbb{A})$, is given by

$$r(s, \pi', w) = \frac{L(2s, \omega_\pi)}{L(1+2s, \omega_\pi)\varepsilon(2s, \omega_\pi)}, \tag{1}$$

where L–function and ε–factor are the ones of Hecke for the central character ω_π of a cuspidal automorphic representation π of $GL_2(\mathbb{A})$ corresponding to π' by the global Jacquet–Langlands correspondence.

In the case $GL'_1 \subset G'_1$, the normalizing factor for the intertwining operator $A(s, \chi \circ \det', w)$ acting on the induced representation

$$\mathrm{Ind}_{GL'_1(\mathbb{A})}^{G'_1(\mathbb{A})} (\chi \circ \det') \nu^s,$$

where $\chi \circ \det'$ is a one–dimensional cuspidal automorphic representation of $GL'_1(\mathbb{A})$, is given by

$$r(s, \chi \circ \det', w) = \frac{L(2s, \chi^2)}{L(1+2s, \chi^2)\varepsilon(2s, \chi^2)}, \tag{2}$$

where L–function and ε–factor are the ones of Hecke.

In the case $GL_1' \subset H_1'$, the normalizing factor for the intertwining operator $A(s,\pi',w)$ acting on the induced representation

$$\mathrm{Ind}_{GL_1'(\mathbb{A})}^{H_1'(\mathbb{A})}\pi'\nu^s,$$

where π' is a higher–dimensional cuspidal automorphic representation of $GL_1'(\mathbb{A})$, is given by

$$r(s,\pi',w) = \frac{L(s,\pi)}{L(1+s,\pi)\varepsilon(s,\pi)} \cdot \frac{L(2s,\omega_\pi)}{L(1+2s,\omega_\pi)\varepsilon(2s,\omega_\pi)}, \tag{3}$$

where L–functions and ε–factors are principal Jacquet L–functions and ε–factors for π and the ones of Hecke for the central character ω_π of π. Here π is a cuspidal automorphic representation of $GL_2(\mathbb{A})$ corresponding to π' by the global Jacquet–Langlands correspondence.

In the case $GL_1' \subset H_1'$, the normalizing factor for the intertwining operator $A(s,\chi\circ\det',w)$ acting on the induced representation

$$\mathrm{Ind}_{GL_1'(\mathbb{A})}^{H_1'(\mathbb{A})}(\chi\circ\det')\nu^s,$$

where $\chi\circ\det'$ is a one–dimensional cuspidal automorphic representation of $GL_1'(\mathbb{A})$, is given by

$$r(s,\chi\circ\det',w) =$$

$$= \frac{L(2s,\chi^2)}{L(1+2s,\chi^2)\varepsilon(2s,\chi^2)} \cdot \frac{L(s-1/2,\chi)}{L(s+3/2,\chi)\varepsilon(s+1/2,\chi)\varepsilon(s-1/2,\chi)} \prod_{v\in S} \frac{L(s+1/2,\chi_v)}{L(1/2-s,\chi_v^{-1})}, \tag{4}$$

where L–functions and ε–factors are the global and the local ones of Hecke.

In the case $GL_2 \subset SO_4$, the normalizing factor for the intertwining operator $A(s,\pi,w)$ acting on the induced representation

$$\mathrm{Ind}_{GL_2(\mathbb{A})}^{SO_4(\mathbb{A})}\pi\nu^s,$$

where π is a cuspidal automorphic representation of $GL_2(\mathbb{A})$, is given by

$$r(s,\pi,w) = \frac{L(2s,\omega_\pi)}{L(1+2s,\omega_\pi)\varepsilon(2s,\omega_\pi)}, \tag{5}$$

where L–function and ε–factor are the ones of Hecke for the central character ω_π of π.

In the case $GL_2 \subset Sp_4$, the normalizing factor for the intertwining operator $A(s,\pi,w)$ acting on the induced representation

$$\mathrm{Ind}_{GL_2(\mathbb{A})}^{Sp_4(\mathbb{A})}\pi\nu^s,$$

where π is a cuspidal automorphic representation of $GL_2(\mathbb{A})$, is given by

$$r(s,\pi,w) = \frac{L(s,\pi)}{L(1+s,\pi)\varepsilon(s,\pi)} \cdot \frac{L(2s,\omega_\pi)}{L(1+2s,\omega_\pi)\varepsilon(2s,\omega_\pi)}, \tag{6}$$

where L–functions and ε–factors are principal Jacquet L–functions and ε–factors for π and the ones of Hecke for the central character ω_π of π.

In the case $GL_1 \times GL_1 \subset GL_2$, the normalizing factor for the standard intertwining operator $A((s_1,s_2),\chi_1\otimes\chi_2,w_1)$ acting on the induced representation

$$\mathrm{Ind}_{GL_1(\mathbb{A})\times GL_1(\mathbb{A})}^{GL_2(\mathbb{A})}\left(\chi_1|\cdot|^{s_1}\otimes\chi_2|\cdot|^{s_2}\right),$$

where χ_1 and χ_2 are unitary characters of $\mathbb{A}^\times/k^\times$, is given by

$$r((s_1,s_2),\chi_1\otimes\chi_2,w_1)=\frac{L(s_1-s_2,\chi_1\chi_2^{-1})}{L(1+s_1-s_2,\chi_1\chi_2^{-1})\varepsilon(s_1-s_2,\chi_1\chi_2^{-1})}, \tag{7}$$

where L–functions and ε–factors are the ones of Hecke.

In the case $GL_1\subset SL_2$, the normalizing factor for the intertwining operator $A(s,\chi,w_2)$ acting on the induced representation

$$\mathrm{Ind}_{GL_1(\mathbb{A})}^{SL_2(\mathbb{A})}\chi|\cdot|^s,$$

where χ is a unitary character of $\mathbb{A}^\times/k^\times$, is given by

$$r(s,\chi,w_2)=\frac{L(s,\chi)}{L(1+s,\chi)\varepsilon(s,\chi)}, \tag{8}$$

where L–functions and ε–factors are the ones of Hecke.

Observe that for SO_4 the intertwining operator $A((s_1,s_2),\chi_1\otimes\chi_2,w_2)$, where χ_1 and χ_2 are unitary characters of $\mathbb{A}^\times/k^\times$, is in fact intertwining operator for $GL_1\times GL_1\subset GL_2$ case since the Levi factor of the parabolic subgroup corresponding to the simple root e_1+e_2 is isomorphic to GL_2. However,

$$r((s_1,s_2),\chi_1\otimes\chi_2,w_2)=r((s_1,-s_2),\chi_1\otimes\chi_2^{-1},w_1)=\frac{L(s_1+s_2,\chi_1\chi_2)}{L(1+s_1+s_2,\chi_1\chi_2)\varepsilon(s_1+s_2,\chi_1\chi_2)}. \tag{9}$$

Finally, we recall well-known analytic properties of local and global L–functions appearing in the normalizing factors above. The proofs for Hecke L–functions can be found in [28] and for principal Jacquet L–functions for GL_2 in [10]. Observe that the global Hecke L–function $L(s,\mathbf{1})$ for the trivial character $\mathbf{1}$ of $\mathbb{A}^\times/k^\times$ is nothing else than the complete ζ–function of algebraic number field k.

Lemma 2.1. *The global principal Jacquet L–function $L(s,\sigma)$ of a cuspidal automorphic representation σ of $GL_2(\mathbb{A})$ is entire. It has no zeroes for $Re(s)\geqslant 1$.*

The global Hecke L–function $L(s,\mu)$ of a unitary character μ of $\mathbb{A}^\times/k^\times$ has simple poles at $s=0$ and $s=1$ if μ is trivial and it is entire otherwise. It has no zeroes for $Re(s)\geqslant 1$. The local Hecke L–function $L(s,\mu_v)$ of a unitary character μ_v of $k_v^\times$ has a simple real pole at $s=0$ if μ_v is trivial and it is entire otherwise. It has no zeroes.

3. Calculation of the residual spectra

In this Section we decompose the residual spectra of $G_1'(\mathbb{A})$ and $H_1'(\mathbb{A})$ and parts of the residual spectra of $SO_4(\mathbb{A})$ and $Sp_4(\mathbb{A})$ required for defining the correspondence. The strategy of decomposition is Langlands spectral theory explained in [17] and [22]. Constituents of the residual spectrum are spaces of automorphic forms obtained as iterated residues inside the positive Weyl chamber of Eisenstein series attached to cuspidal automorphic representations of Levi factors of standard proper parabolic subgroups which are square–integrable. The calculation of poles reduces to the poles of the constant term of the Eisenstein series since it inherits all analytic properties from the Eisenstein series. Furthermore, the constant term equals the sum of standard intertwining operators and their poles are given by the normalizing factors of the previous Section. Square integrability of the iterated residues is checked using Langlands square integrability criterion from page 104 of [17].

During calculation of poles of Eisenstein series we always assume that they are real. There is no loss in generality because that can be achieved just by twisting a cuspidal automorphic representation of a Levi factor by the appropriate imaginary power of the absolute value of the reduced norm of the determinant. Hence, this assumption is just a convenient choice of coordinates.

Let $L^2_{res}(G'_1)$ and $L^2_{res}(H'_1)$ be the residual spectra of $G'_1(\mathbb{A})$ and $H'_1(\mathbb{A})$, respectively. Since those groups have only one proper parabolic subgroup P'_0 with the Levi factor $M'_0 \cong GL'_1$, constituents of the residual spectrum are iterated residues of the Eisenstein series attached to cuspidal automorphic representations of $M'_0(\mathbb{A})$.

Let $L^2_{res}(SO_4)$ and $L^2_{res}(Sp_4)$ be the residual spectra of $SO_4(\mathbb{A})$ and $Sp_4(\mathbb{A})$, respectively. For every standard proper parabolic subgroup P of SO_4 or Sp_4 with the Levi factor M let $L^2_{res,M}(SO_4)$ or $L^2_{res,M}(Sp_4)$ be the part of the residual spectrum which is obtained as iterated residues of the Eisenstein series attached to cuspidal automorphic representations of $M(\mathbb{A})$. Accordingly, the residual spectrum decomposes into the direct sum

$$L^2_{res}(SO_4) \cong \oplus_M L^2_{res,M}(SO_4) \quad \text{and} \quad L^2_{res}(Sp_4) \cong \oplus_M L^2_{res,M}(Sp_4).$$

However, only parts attached to Levi factors $M_0 \cong GL_2$ and $T \cong GL_1 \times GL_1$ are involved in the correspondence between the residual spectra of $G'_1(\mathbb{A})$ and $H'_1(\mathbb{A})$ and the residual spectra of $SO_4(\mathbb{A})$ and $Sp_4(\mathbb{A})$. Thus, in this paper we decompose only those parts of the residual spectra.

Theorem 3.1. *The residual spectrum $L^2_{res}(G'_1)$ of the group $G'_1(\mathbb{A})$ decomposes into the direct sum*

$$L^2_{res}(G'_1) \cong \left(\oplus_{\pi'} \mathcal{A}_{G'_1}(\pi')\right) \oplus \left(\oplus_{\chi} \mathcal{A}_{G'_1}(\chi)\right).$$

The former sum is over all higher–dimensional cuspidal automorphic representations π' of $M'_0(\mathbb{A}) \cong GL'_1(\mathbb{A})$ having trivial central character. The latter sum is over all quadratic characters χ of $\mathbb{A}^\times/k^\times$.

The irreducible space of automorphic forms $\mathcal{A}_{G'_1}(\pi')$ is spanned by the residue at $s = 1/2$ of the Eisenstein series attached to π' which is by the constant term map isomorphic to the image of the normalized intertwining operator $N(1/2, \pi', w)$.

The irreducible space of automorphic forms $\mathcal{A}_{G'_1}(\chi)$ is spanned by the residue at $s = 1/2$ of the Eisenstein series attached to the one–dimensional cuspidal automorphic representation $\chi \circ \det'$ of $M'_0(\mathbb{A}) \cong GL'_1(\mathbb{A})$ which is by the constant term map isomorphic to the image of the normalized intertwining operator $N(1/2, \chi \circ \det', w)$.

Proof. The constant term of the Eisenstein series attached to both π' and $\chi \circ \det'$ is the sum over W' of standard intertwining operators. Since w is the only nontrivial element of W', poles of the Eisenstein series inside the positive Weyl chamber $s > 0$ are those of the standard intertwining operator corresponding to w. Moreover, by the previous Section, the normalized intertwining operator is holomorphic and non–vanishing for $s > 0$ and hence poles of the standard intertwining operator for $s > 0$ coincide with poles of the normalizing factor.

For higher–dimensional π' the normalizing factor is given by (1) and by Lemma 2.1 it has a pole if and only if ω_π is trivial, where π corresponds to π' by Jacquet–Langlands correspondence. Then the simple pole is at $s = 1/2$ and its residue, up to a non–zero constant, equals $N(1/2, \pi', w)$. Observe that $\omega_\pi = \omega_{\pi'}$ because central characters are invariant for Jacquet–Langlands correspondence.

For one–dimensional $\chi \circ \det'$ the normalizing factor is given by (2) and by Lemma 2.1 it has a pole if and only if χ^2 is trivial. Then the simple pole is at $s = 1/2$ and its residue, up to a non–zero constant, equals $N(1/2, \chi \circ \det', w)$.

Langlands square–integrability criterion is obviously satisfied in both cases. It remains to prove that the images of $N(1/2, \pi', w)$ and $N(1/2, \chi \circ \det', w)$ are irreducible. That is done locally for every place v of k. If π'_v or $\chi_v \circ \det'_v$ is tempered, the image is irreducible by Langlands classification since $1/2$ is in the positive Weyl chamber. Observe that this is the case for all $v \in S$.

Let $v \notin S$ and $\pi_v \cong \pi'_v$ a non–tempered representation of $GL_2(k_v)$. Then, as a local component of a cuspidal automorphic representation of $GL_2(\mathbb{A})$, it is unitary and generic. Thus π_v is a complementary series, i.e. a fully induced representation

$$\pi_v \cong \mathrm{Ind}_{GL_1(k_v)\times GL_1(k_v)}^{GL_2(k_v)} \left(\mu_v|\cdot|^r \otimes \mu_v|\cdot|^{-r}\right),$$

where $0 < r < 1/2$ and μ_v is a unitary character of $k_v^\times$. Then, the image of $N(1/2, \pi'_v, w)$ is the same as the image of $N((1/2+r, 1/2-r), \mu_v \otimes \mu_v, w_1 w_2)$ which is irreducible by the Langlands classification since $(1/2+r, 1/2-r)$ is in the positive Weyl chamber for $T \subset SO_4$ and $w_1 w_2$ the longest Weyl group element.

Similarly, $\chi_v \circ \det_v$ at $v \notin S$ is the Langlands quotient of the standard module

$$\mathrm{Ind}_{GL_1(k_v)\times GL_1(k_v)}^{GL_2(k_v)} \left(\chi_v|\cdot|^{1/2} \otimes \chi_v|\cdot|^{-1/2}\right).$$

In other words it is the image of the normalized intertwining operator $N((1/2, -1/2), \chi_v \otimes \chi_v, w_1)$. Hence, the image of $N(1/2, \chi_v \circ \det'_v, w)$ is the same as the image of $N((1, 0), \chi_v \otimes \chi_v, w_1 w_2)$ which is irreducible by Langlands classification as above. □

Before decomposing $L^2_{res}(H'_1)$ we introduce some notation. Let $\chi \circ \det' = \otimes_v(\chi_v \circ \det'_v)$ be a one–dimensional cuspidal automorphic representation of the Levi factor $GL'_1(\mathbb{A})$ in $H'_1(\mathbb{A})$ where χ is a quadratic character of $\mathbb{A}^\times/k^\times$. For a non–split place $v \in S$ let Π'_v denote the image of the local normalized intertwining operator $N(1/2, \chi_v \circ \det'_v, w)$. Since $\chi_v \circ \det'_v$ is supercuspidal and $1/2$ is in the open positive Weyl chamber, that image is irreducible by the Langlands classification.

For a split place $v \notin S$ consider the image of the normalized operator $N(1/2, \chi_v \circ \det'_v, w)$. It is isomorphic to the image of the normalized intertwining operator $N((1, 0), \chi_v \otimes \chi_v, w_1 w_2 w_1 w_2)$ because $w = w_2 w_1 w_2$ and $\chi_v \circ \det'$ is the image of the $GL_2(k_v)$ normalized intertwining operator $N((1/2, -1/2), \chi_v \otimes \chi_v, w_1)$. By decomposition property of Section 2.1 in [25], normalized operator

$$N((1, 0), \chi_v \otimes \chi_v, w_1 w_2 w_1 w_2) = N((1, 0), \chi_v \otimes \chi_v, w_1 w_2 w_1) N((1, 0), \chi_v \otimes \chi_v, w_2).$$

By inducing in stages we have

$$\mathrm{Ind}_{T(k_v)}^{Sp_4(k_v)} \left(\chi_v|\cdot| \otimes \chi_v\right) \cong \mathrm{Ind}_{GL_1(k_v)\times SL_2(k_v)}^{Sp_4(k_v)} \left(\chi_v|\cdot| \otimes \mathrm{Ind}_{GL_1(k_v)}^{SL_2(k_v)} \chi_v\right).$$

Observe that $N((1, 0), \chi_v \otimes \chi_v, w_2)$ intertwines the $SL_2(k_v)$ induced representation with itself. Furthermore,

$$\mathrm{Ind}_{GL_1(k_v)}^{SL_2(k_v)} \chi_v \cong \tau_v^+ \oplus \tau_v^-$$

where $\tau_v^\pm$ are irreducible tempered representations of $SL_2(k_v)$ and the sign in the exponent denotes the sign of action of $SL_2(k_v)$ normalized intertwining operator $N((1, 0), \chi_v \otimes \chi_v, w_2)$. Moreover, $\tau_v^\pm$ are both nontrivial unless χ_v is trivial and then τ_v^- is trivial. If χ_v is unramified, then the unramified component of the induced representation is τ_v^+ since, by definition, the normalized operator acts as identity on the unramified component. Now, $w_1w_2w_1$ is the longest Weyl group element for the standard proper parabolic subgroup of Sp_4 with Levi factor isomorphic to $GL_1 \times SL_2$, 1 is in the positive Weyl chamber and $\chi_v \otimes \tau_v^{\pm}$ is tempered. Therefore, the image of the operator $N((1,0), \chi_v \otimes \chi_v, w_1w_2w_1)$ acting on

$$\mathrm{Ind}_{GL_1(k_v)\times SL_2(k_v)}^{Sp_4(k_v)} \left(\chi_v|\cdot| \otimes \tau_v^{\pm}\right)$$

is irreducible by Langlands classification and we denote it by $\Pi_v'^{\pm}$. Then the image of the normalized intertwining operator $N(1/2, \chi_v \circ \det_v', w)$ decomposes into the direct sum

$$\Pi_v'^{+} \oplus \Pi_v'^{-},$$

where $\Pi_v'^{-}$ is trivial if χ_v is trivial. Observe that representations Π_v' at $v \in S$ and $\Pi_v'^{\pm}$ at $v \notin S$ depend on the quadratic character χ_v although it is not explicit in our notation. Moreover, since τ_v^{+} is the unramified component for unramified χ_v, $\Pi_v'^{-}$ is never unramified.

Theorem 3.2. *The residual spectrum $L^2_{res}(H_1')$ of the group $H_1'(\mathbb{A})$ decomposes into the direct sum*

$$L^2_{res}(H_1') \cong \left(\oplus_{\pi'} \mathcal{A}_{H_1'}(\pi')\right) \oplus \left(\oplus_{\chi} \mathcal{A}_{H_1'}(\chi)\right) \oplus \mathcal{A}_{H_1'}(\mathbf{1}).$$

The former sum is over all higher–dimensional cuspidal automorphic representations π' of $M_0'(\mathbb{A}) \cong GL_1'(\mathbb{A})$ which have trivial central character and $L(1/2, \pi) \neq 0$ for the global principal Jacquet L–function of a cuspidal automorphic representation π corresponding to π' by the Jacquet–Langlands correspondence. The latter sum is over all quadratic characters χ of $\mathbb{A}^\times/k^\times$ such that χ_v is nontrivial at all non–split places $v \in S$. Finally, $\mathbf{1}$ *denotes the trivial character of $\mathbb{A}^\times/k^\times$.*

The irreducible space of automorphic forms $\mathcal{A}_{H_1'}(\pi')$ is spanned by the residue at $s = 1/2$ of the Eisenstein series attached to π' which is by the constant term map isomorphic to the image of the normalized intertwining operator $N(1/2, \pi', w)$.

The space of automorphic forms $\mathcal{A}_{H_1'}(\chi)$, for nontrivial χ, is spanned by the residue at $s = 1/2$ of the Eisenstein series attached to the one–dimensional cuspidal automorphic representation $\chi \circ \det'$ of $M_0'(\mathbb{A}) \cong GL_1'(\mathbb{A})$ which is by the constant term map isomorphic to the image of the normalized intertwining operator $N(1/2, \chi \circ \det', w)$. It decomposes into the sum of irreducible constituents which are by the constant term map isomorphic to irreducible representations of $H_1'(\mathbb{A})$ of the form $\otimes_v \Pi_v'$, where Π_v' at a non–split place $v \in S$ is defined above, while at a split place $v \notin S$ it is one of representations $\Pi_v'^{\pm}$ defined above and it is $\Pi_v'^{+}$ at almost all places.

The irreducible space of automorphic forms $\mathcal{A}_{H_1'}(\mathbf{1})$ is spanned by the iterated residue at $s = 3/2$ of the Eisenstein series attached to $\mathbf{1} \circ \det'$ which is by the constant term map isomorphic to the image of the normalized intertwining operator $N(3/2, \mathbf{1} \circ \det', w)$.

Proof. The proof goes along the same lines as the proof of the previous Theorem 3.1. Poles of the Eisenstein series inside the positive Weyl chamber $s > 0$ coincide with poles of the normalizing factors for the standard intertwining operator corresponding to w.

For higher–dimensional π' the normalizing factor is given by (3) and by Lemma 2.1 it has a pole if and only if ω_π is trivial and $L(1/2, \pi) \neq 0$, where π corresponds to π' by Jacquet–Langlands correspondence. Then the simple pole is at $s = 1/2$ and its residue, up to a non–zero constant, equals $N(1/2, \pi', w)$. Observe that $\omega_\pi = \omega_{\pi'}$ because central characters are invariant for Jacquet–Langlands correspondence.

For one–dimensional $\chi \circ \det'$ the normalizing factor is given by (4) and by Lemma 2.1 its possible poles are at $s = 1/2$ and $s = 3/2$. The pole at $s = 1/2$ occurs if and only if

χ^2 is trivial but χ_v is nontrivial at all non–split places $v \in S$. It is simple and the residue, up to a non–zero constant, equals $N(1/2, \chi \circ \det', w)$. The local condition on χ_v appears since otherwise the pole of local L–functions $L(1/2-s, \chi_v^{-1})$ in denominator of (4) would cancel the pole of numerator. The pole at $s = 3/2$ occurs if and only if χ is trivial. It is simple and the residue, up to a nonzero constant, equals $N(3/2, \mathbf{1} \circ \det', w)$.

Langlands square–integrability criterion is obviously satisfied in both cases. Irreducibility of images of $N(1/2, \pi', w)$ and $N(3/2, \mathbf{1} \circ \det', w)$ follows as in the proof of the previous Theorem 3.1. Moreover, the same argument shows that the image of the local normalized intertwining operator $N(1/2, \chi_v \circ \det_v, w)$ at a split place $v \in S$ is the same as the image of $N((1,0), \chi_v \otimes \chi_v, w_1 w_2 w_1 w_2)$. By the discussion preceding the statement of this Theorem, that image is the sum of $\Pi_v'^{+}$ and $\Pi_v'^{-}$. At a non–split place $v \in S$ the image of the local normalized operator $N(1/2, \chi_v \circ \det_v', w)$ is irreducible by Langlands classification and denoted by Π_v' above. From images of the local normalized intertwining operators at all places we obtain decomposition of the image of the global normalized intertwining operator $N(1/2, \chi \circ \det', w)$. At almost all places Π_v' is $\Pi_v'^{+}$ because $\Pi_v'^{-}$ is never an unramified representation of $Sp_4(k_v)$ as explained above. □

Theorem 3.3. *The part $L^2_{res,M_0}(SO_4)$ of the residual spectrum $L^2_{res}(SO_4)$ of the group $SO_4(\mathbb{A})$ decomposes into the direct sum*

$$L^2_{res,M_0}(SO_4) \cong \oplus_\pi \mathcal{A}^{M_0}_{SO_4}(\pi),$$

where the sum is over all cuspidal automorphic representations π of $M_0(\mathbb{A}) \cong GL_2(\mathbb{A})$ having trivial central character. The irreducible space of automorphic forms $\mathcal{A}^{M_0}_{SO_4}(\pi)$ is spanned by the iterated residue at $s = 1/2$ of the Eisenstein series attached to π which is by the constant term map isomorphic to the image of the normalized intertwining operator $N(1/2, \pi, w)$.

The part $L^2_{res,T}(SO_4)$ of the residual spectrum $L^2_{res}(SO_4)$ of the group $SO_4(\mathbb{A})$ decomposes into the direct sum

$$L^2_{res,T}(SO_4) \cong \oplus_\chi \mathcal{A}^{T}_{SO_4}(\chi),$$

where the sum is over all quadratic characters χ of $\mathbb{A}^\times/k^\times$. The irreducible space of automorphic forms $\mathcal{A}^{T}_{SO_4}(\chi)$ is spanned by the iterated residue at $(s_1, s_2) = (1, 0)$ of the Eisenstein series attached to cuspidal automorphic representation $\chi \otimes \chi$ of $T(\mathbb{A}) \cong GL_1(\mathbb{A}) \times GL_1(\mathbb{A})$ which is by the constant term map isomorphic to the image of the normalized intertwining operator $N((1,0), \chi \otimes \chi, w_1 w_2)$.

Proof. First we decompose the part $L^2_{res,M_0}(SO_4)$. Since the only nontrivial element of $W(M_0)$ is w, poles inside the positive Weyl chamber $s > 0$ of the Eisenstein series attached to a cuspidal automorphic representation π of $M_0(\mathbb{A})$ coincide with poles of the normalizing factor for the standard intertwining operator corresponding to w. It is given by (5) and by Lemma 2.1 the pole occurs if and only if the central character ω_π is trivial. Then, the pole is at $s = 1/2$, it is simple and its residue, up to a non–zero constant, equals $N(1/2, \pi, w)$. The Langlands square–integrability criterion is obviously satisfied and the image of that operator is irreducible by the same argument as in the proof of Theorem 3.1.

For the part $L^2_{res,T}(SO_4)$, the constant term of the Eisenstein series attached to a cuspidal automorphic representation $\chi_1 \otimes \chi_2$ of $T(\mathbb{A})$ is the sum over $W = \{1, w_1, w_2, w_1 w_2\}$ of the standard intertwining operators. Normalizing factors corresponding to w_1 and w_2 are given by (7) and (9) and the normalizing factor corresponding to $w_1 w_2$ is just the product of those two. Hence, by Lemma 2.1, possible poles inside the positive Weyl chamber $s_1 > |s_2|$ of the normalizing factors are along hyperplanes $s_1 - s_2 = 1$ and $s_1 + s_2 = 1$. The hyperplane $s_1 - s_2 = 1$ is singular if and only if $\chi_1 = \chi_2$ and hyperplane $s_1 + s_2 = 1$ is singular if and only if $\chi_1 = \chi_2^{-1}$. Then the pole along both is simple. Therefore, the only possible iterated pole is at their intersection, i.e. for $(s_1, s_2) = (1, 0)$ and the Weyl group element $w_1 w_2$. It occurs if and only if $\chi = \chi_1 = \chi_2$ is a quadratic character and its residue, up to a non–zero constant, equals $N((1,0), \chi \otimes \chi, w_1 w_2)$. That image is irreducible by the Langlands classification because $(1, 0)$ is in the positive Weyl chamber and $w_1 w_2$ is the longest Weyl group element. The Langlands square integrability criterion is satisfied since $w_1 w_2 (1, 0) = (-1, 0)$. □

The residual spectrum for the remaining case of the split group $Sp_4(\mathbb{A})$ is decomposed by H. Kim in [11]. For convenience, we state his Theorem 3.3 and Theorem 5.4 in our notation in the following Theorem 3.4 below. Recall that before the statement of Theorem 3.2 we decomposed the image of the local normalized intertwining operator $N((1,0), \chi_v \otimes \chi_v, w_1 w_2 w_1 w_2)$ into the sum

$$\Pi_v^+ \oplus \Pi_v^-,$$

where Π_v^- is trivial if χ_v is trivial.

Theorem 3.4 (Kim, [11]). *The part $L^2_{res,M_0}(Sp_4)$ of the residual spectrum $L^2_{res}(Sp_4)$ of the group $Sp_4(\mathbb{A})$ decomposes into the direct sum*

$$L^2_{res,M_0}(Sp_4) \cong \oplus_\pi \mathcal{A}^{M_0}_{Sp_4}(\pi),$$

where the sum is over all cuspidal automorphic representations π of $M_0(\mathbb{A}) \cong GL_2(\mathbb{A})$ having trivial central character and such that $L(1/2, \pi) \neq 0$. The irreducible space of automorphic forms $\mathcal{A}^{M_0}_{Sp_4}(\pi)$ is spanned by the iterated residue at $s = 1/2$ of the Eisenstein series attached to π which is by the constant term map isomorphic to the image of the normalized intertwining operator $N(1/2, \pi, w)$.

The part $L^2_{res,T}(Sp_4)$ of the residual spectrum $L^2_{res}(Sp_4)$ of the group $Sp_4(\mathbb{A})$ decomposes into the direct sum

$$L^2_{res,T}(Sp_4) \cong \left(\oplus_\chi \mathcal{A}^T_{Sp_4}(\chi)\right) \oplus \mathcal{A}^T_{Sp_4}(\mathbf{1}),$$

where the sum is over all nontrivial quadratic characters χ of $\mathbb{A}^\times / k^\times$ and $\mathbf{1}$ is the trivial character of $\mathbb{A}^\times / k^\times$.

The space of automorphic forms $\mathcal{A}^T_{Sp_4}(\chi)$, for nontrivial χ, is spanned by the iterated residue at $(s_1, s_2) = (1, 0)$ of the Eisenstein series attached to cuspidal automorphic representation $\chi \otimes \chi$ of $T(\mathbb{A}) \cong GL_1(\mathbb{A}) \times GL_1(\mathbb{A})$. It decomposes into the sum of irreducible constituents which are by the constant term map isomorphic to irreducible representations of $Sp_4(\mathbb{A})$ of the form $\otimes_v \Pi_v$, where Π_v is one of representations $\Pi_v^\pm$, it is Π_v^+ at almost all places and the product of the signs over all places equals 1.

The irreducible space of automorphic forms $\mathcal{A}^T_{Sp_4}(\mathbf{1})$ is spanned by the iterated residue at $(s_1, s_2) = (2, 1)$ of the Eisenstein series attached to the trivial cuspidal automorphic representation $\mathbf{1} \otimes \mathbf{1}$ of $T(\mathbb{A}) \cong GL_1(\mathbb{A}) \times GL_1(\mathbb{A})$ which is by the constant term map isomorphic to the image of the normalized intertwining operator $N((2,1), \mathbf{1} \otimes \mathbf{1}, w_1 w_2 w_1 w_2)$.

Finally, we obtain correspondence between the residual spectra of $G_1'(\mathbb{A})$ and $SO_4(\mathbb{A})$ and between the residual spectra of $H_1'(\mathbb{A})$ and $Sp_4(\mathbb{A})$.

Corollary 3.5. *Mapping $\imath$ defined, in notation of Theorem 3.1 and Theorem 3.3, by*

$$\begin{aligned}\imath\left(\mathcal{A}_{G_1'}(\pi')\right) &= \mathcal{A}_{SO_4}^{M_0}(\pi),\\ \imath\left(\mathcal{A}_{G_1'}(\chi)\right) &= \mathcal{A}_{SO_4}^{T}(\chi),\end{aligned}$$

where π corresponds to π' by the Jacquet–Langlands correspondence, is an injective map from irreducible constituents of $L^2_{res}(G_1')$ to irreducible constituents of $L^2_{res,M_0}(SO_4) \oplus L^2_{res,T}(SO_4)$. The image of the map $\imath$ consists of

(a) *all irreducible constituents $\mathcal{A}_{SO_4}^{M_0}(\pi)$ of $L^2_{res,M_0}(SO_4)$ such that π_v is square–integrable at every place $v \in S$, and*
(b) *all irreducible constituents $\mathcal{A}_{SO_4}^{T}(\chi)$ of $L^2_{res,T}(SO_4)$.*

Proof. Corollary is a direct consequence of decompositions in Theorem 3.1 and Theorem 3.3. Local square integrability condition in part (a) of the image of $\imath$ comes from the global Jacquet–Langlands correspondence. Namely, by Theorem (8.3) of [5], it is a bijection between all higher–dimensional cuspidal automorphic representations of $GL_1'(\mathbb{A})$ and all cuspidal automorphic representations of $GL_2(\mathbb{A})$ such that the local component at every place $v \in S$ is square–integrable. □

Corollary 3.6. *Mapping $\jmath$ is defined, in notation of Theorem 3.2 and Theorem 3.4, by*

$$\begin{aligned}\jmath\left(\mathcal{A}_{H_1'}(\pi')\right) &= \mathcal{A}_{Sp_4}^{M_0}(\pi),\\ \jmath\left(\otimes_v \Pi_v'\right) &= \oplus\left(\otimes_v \Pi_v\right),\\ \jmath\left(\mathcal{A}_{H_1'}(\mathbf{1})\right) &= \mathcal{A}_{Sp_4}^{T}(\mathbf{1}),\end{aligned}$$

where π corresponds to π' by the Jacquet–Langlands correspondence. In the second row $\otimes_v \Pi_v'$ is an irreducible constituent of $\mathcal{A}_{H_1'}(\chi)$ and the sum is over $2^{|S|-1}$ irreducible constituents $\otimes_v \Pi_v$ of $\mathcal{A}_{Sp_4}^{T}(\chi)$ such that $\Pi_v \cong \Pi_v'$ for $v \notin S$, Π_v is one of $\Pi_v^{\pm}$ for $v \in S$ and the product of all signs equals 1. Then $\jmath$ is an injective map from irreducible constituents of $L^2_{res}(H_1')$ to not necessarily irreducible constituents of $L^2_{res,M_0}(Sp_4) \oplus L^2_{res,T}(Sp_4)$. The image of the map $\jmath$ consists of

(a) *all irreducible constituents $\mathcal{A}_{Sp_4}^{M_0}(\pi)$ of $L^2_{res,M_0}(Sp_4)$ such that π_v is square–integrable at every place $v \in S$,*
(b) *all constituents of the form $\oplus(\otimes_v \Pi_v)$ as above of $\mathcal{A}_{Sp_4}^{T}(\chi)$ in $L^2_{res,T}(Sp_4)$ such that χ_v is nontrivial at all $v \in S$, and*
(c) *the irreducible constituent $\mathcal{A}_{Sp_4}^{T}(\mathbf{1})$ of $L^2_{res,T}(Sp_4)$.*

Observe that the sum of all the constituents in (b) *gives the whole space $\mathcal{A}_{Sp_4}^{T}(\chi)$.*

Proof. Corollary is a direct consequence of decompositions in Theorem 3.2 and Theorem 3.4. Local square integrability condition in part (a) of the image of $\jmath$ comes from the global Jacquet–Langlands correspondence as in the previous Corollary 3.5. Local non–triviality condition in part (b) comes from Theorem 3.2. The reason lies in different form of local normalizing factors for standard intertwining operators at split and non–split places resulting with local Hecke L–functions appearing in the global normalizing factor (4).

Observe that, for nontrivial χ, in decomposition of spaces $\mathcal{A}_{H_1'}(\chi)$ and $\mathcal{A}_{Sp_4}^{T}(\chi)$ given in Theorem 3.2 and Theorem 3.4 into irreducible constituents, the choice of local components at a split place is exactly the same. At a non–split place there is one choice for the local component of an irreducible constituent of $\mathcal{A}_{H_1'}(\chi)$ and two choices for the local component of an irreducible constituent of $\mathcal{A}^T_{Sp_4}(\chi)$. However, the parity condition for irreducible constituents of $\mathcal{A}^T_{Sp_4}(\chi)$ reduces the freedom of choice. Therefore, $\jmath$ sends irreducible constituents of $\mathcal{A}_{H'_1}(\chi)$ to the sum of $2^{|S|-1}$ irreducible components of $\mathcal{A}^T_{Sp_4}(\chi)$ as defined in the Corollary. □

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The comparison of the local and the global method for proving unitarizability

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1 Introduction

In this paper we compare two methods which we use for the calculation of the unitary dual of the non-split forms of the groups $Sp(8, F)$ and $SO(8, F)$, where F is a non-archimedean local field. These non-split forms are the hermitian quaternionic groups which we will denote by $G_2(D; 1)$ and $G_2(D, -1)$; we will explain the notation shortly. The complete classification of the non-cuspidal part of the unitary dual for these groups is obtained in [5] and [4].

The global method includes the calculation of the residual spectrum; the calculation of the complete residual spectrum of the non-split inner form of $SO(8, F)$ was done elsewhere ([2]). The main idea of finding a representation as a local component of an automorphic representation belonging to the residual spectrum was first used by Speh (archimedean case) and Tadić (non-archimedean case), and since then, by a number of others. Of course, to be able to use this kind of idea to prove unitarizability it is not always necessary to find (i.e. to decompose) the whole residual spectrum, maybe only some parts which are more accessible. Another problem sometimes appears in this situation: the appearance of the global representation with the required local component is sometimes conditional, i.e. it depends on non–vanishing of a certain global L–function. This situation will appear with the group $G_2(D; 1)$, we will give some details later.

The local method is based on the idea which is developed further elsewhere ([3]): in short, to prove unitarizability of the certain representation of the group $G_2(D, \varepsilon)$ we form a parabolically induced representation of a certain larger group; our representation serves as a tensor factor of a representation of the Levi subgroup; the other tensor factors will be unitarizable. First, we have to prove that the induced representation is irreducible, and then we have to prove that it is unitarizable. From that, it will follow that the original representation of the Levi subgroup is unitarizable, too. To prove these claims, we will mainly use the results of Tadić about the Jacquet functors ([14]). Although these results are formulated in terms of the split groups $Sp(n, F)$ and $SO(2n + 1, F)$, we can still use the majority of the results cited, because of the similarity between so–called structure formula for these split groups ([13]) and for the hermitian quaternionic groups ([5]).

We will illustrate this situation by an example of a representation which is a subquotient of a principal series representation.

The calculation we present here is somewhat technical, but we believe that it can still be applied further, even with (seemingly) less technical difficulties. We believe that the local method presented here can be further generalized to the case of the quasi-split

classical groups of the larger rank and that it can be used for the determination of the part of the unitary dual of these groups. We hope that we can proceed in that direction using the following developments: the classification of the discrete series of such groups ([7]) and the knowledge of the structure of the generalized principal series ([8]) (something we do not yet know for the hermitian quaternionic groups of larger rank; so in this example we had to do more direct calculations with Jacquet modules).
In this way, using this local method, we hope we can by–pass some already mentioned problems involving residual spectrum: not only it can be rather difficult constructing representations in the residual spectrum, but also their actual appearance depends on non-vanishing of some global L–function; a condition which is rather difficult to check for a representation which we construct, but without the explicit knowledge of most of its local components and behaviour.

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2 Notation

Let F be a non-archimedean local field of characteristic zero, and let D be a quaternion algebra, central over F and let τ be an involution, fixing the center of D (involution of the first kind). The division algebra D defines a reductive group G over F as follows. Let

$$V_n = e_1D \oplus \cdots \oplus e_nD \oplus e_{n+1}D \oplus \cdots \oplus e_{2n}D$$

be a right vector space over D. Fix $\varepsilon \in \{1, -1\}$. The relations $(e_i, e_{2n-j+1}) = \delta_{ij}$ for $i = 1, 2, \ldots, n$ define a hermitian form on V_n:

$$(v, v') = \varepsilon\tau((v', v)), \quad \forall v, v' \in V_n,$$
$$(vx, v'x') = \tau(x)(v, v')x', \quad \forall x, x' \in D.$$

Let $G_n(D, \varepsilon)$ be the group of isometries of the form $(\cdot, \cdot)$. The group $G_n(D, 1)$ is a non-split inner form of the group $Sp(4n, F)$, and $G_n(D, -1)$ is a non-split inner form of the group $SO(8, F)$.

We will fix a maximal F- split torus A_0 of the group $G_n(D, \varepsilon)$:

$$A_0(F) = \left\{ \begin{pmatrix} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_n & & & & \\ & & & & \lambda_n^{-1} & & & \\ & & & & & \ddots & & \\ & & & & & & \lambda_2^{-1} & \\ & & & & & & & \lambda_1^{-1} \end{pmatrix} : \lambda_i \in F^* \right\}.$$

A set of positive roots with respect to the above torus is fixed in such a way that the minimal parabolic subgroup defined by those roots consists of upper–triangular matrices.

Then, the Levi subgroup of standard F–parabolic subgroup is of the following form:

$$M(F)=\left\{\begin{pmatrix} A_1 & & & & & & \\ & \ddots & & & & & \\ & & A_k & & & & \\ & & & g & & & \\ & & & & J{A_k}^{-*}J & & \\ & & & & & \ddots & \\ & & & & & & JA_1^{-*}J \end{pmatrix} : \begin{array}{c} A_i\in GL(n_i,D), \\ g\in G_r(D,\varepsilon) \end{array}\right\},$$

for some positive integers $n_1,\ldots,n_k$ and a non-negative integer r such that $\sum n_i+r=n$. The matrices J are of the appropriate size having $1'$s on the second diagonal and $0'$s elsewhere and A^{-*} denotes the involution (induced by the involution on the algebra D) on the space of n_i by n_i matrices applied to the matrix A^{-1} ([5], [4]).

In this situation, let $\tau_1\otimes\tau_2\otimes\cdots\otimes\tau_k\otimes\sigma$ denote an admissible irreducible representation of the group $M(F)$. In future, we always by M, G, P consider the groups of their F–rational points, i.e. $M(F)$, $G(F)$, $P(F)$, so we drop F from the future notation (in this situation). Then the parabolically induced representation $\mathrm{Ind}_P^{G_n(D,\varepsilon)}\tau_1\otimes\tau_2\otimes\cdots\otimes\tau_k\otimes\sigma$ is denoted $\tau_1\times\tau_2\times\cdots\times\tau_k\rtimes\sigma$ ([14]). For every segment $[\tau\nu^{l_1},\tau\nu^{l_2}]$ of cuspidal representations ([12]) (where l_1, $l_2\in\mathbb{R}$ and l_2-l_1 is a positive integer) of $GL(n,D)$–groups the induced representation $\tau\nu^{l_2}\times\cdots\times\tau\nu^{l_1}$ contains a unique irreducible subrepresentation denoted by $\delta([\tau\nu^{l_1},\tau\nu^{l_2}])$. It is essentially square–integrable. The Jacquet module of the representation π of the group G with respect to the Levi subgroup M will be denoted by $r_{M,G}(\pi)$.

Before proceeding with an example which illustrates a local technique we use, we introduce some notation, mainly following Zelevinsky and Tadić.
For an induced representation of a hermitian quaternionic group, for which $r=0$ (as we previously explained our notation) we use 1 instead of σ.
A segment of cuspidal representations of the group $GL(n,D)$ (which is a set of cuspidal representations of the form $\{\rho,\nu\rho,\ldots,\nu^k\rho\}$ ([16],[12]) we usually denote by $[\rho,\nu^k\rho]$ or by Δ. Here ν denotes the norm on the division algebra D composed with the absolute value on F. Then the representation $\rho\times\nu\rho\times\cdots\times\nu^k\rho$ (with $\Delta=\{\rho,\nu\rho,\ldots,\nu^k\rho\}$) has a unique subrepresentation, denoted by $<\Delta>$. By Zelevinsky's theory, for a set of segments $\Delta_1,\ldots,\Delta_r$ such that for each pair of indices i, j with $i<j$, Δ_i does not precede Δ_j ([16], Section 4), the representation $<\Delta_1>\times\cdots\times<\Delta_k>$ has a unique irreducible submodule; we denote it by $Z(\Delta_1,\ldots,\Delta_r)$. ([16], Section 6). Although the corresponding results from ([16]) are formulated for $GL(n,F)$ groups, by the work of Tadić in ([12]), we see that they are valid in the analogous form for $GL(n,D)$ groups.
When we work in the Grothendieck group of the finite length admissible representations of the hermitian quaternionic groups, then the relation $\pi\leq\tau$, where π and τ are finite length admissible representations of $G_r(D,\varepsilon)$, means that each irreducible subquotient of π appears in τ with the greater multiplicity (i.e. $s.s.(\pi)\leq s.s.(\tau)$ in the appropriate Grothendieck group; 's.s.' stands for the semisimplification).

Let $\delta_1,\ldots,\delta_k$ denote square integrable representations of the general linear groups over the division algebra D, let τ be a tempered representation of $G_r(D,\varepsilon)$ and let $s_1,\ldots,s_k$

be real numbers such that $s_1 \geq s_2 \geq \cdots \geq s_k > 0$. Then the representation $\delta_1\nu^{s_1} \times \delta_2\nu^{s_2} \times \cdots \times \delta_k\nu^{s_k} \rtimes \tau$ has a unique quotient (Langlands' quotient). We denote it by $L(\delta_1\nu^{s_1}, \delta_2\nu^{s_2}, \ldots, \delta_k\nu^{s_k}; \tau)$.
Every maximal Levi subgroup of the group $G_r(D, \varepsilon)$ is isomorphic to $GL(k, D) \times G_r(D, \varepsilon)$. Having that in mind, for an admissible, finite length representation π of $G_k(D, \varepsilon)$ we define

$$\mu^*(\pi) = \sum_{r=0}^{k} s.s r_{GL(r,D)\times G_{k-r}(D,\varepsilon)}(\pi).$$

The object on the right–hand side belongs to the tensor product $R(G) \otimes R(G)$, where $R(G)$ denotes the direct sum of the Grotehndieck groups of the finite length smooth representations of $G_l(D, \varepsilon)$ over all l's; for $l = 0$ we get a trivial representation of the trivial group. In this way, $\mu^*(\pi)$ contains information about the Jacquet modules of the representation π with respect to all the maximal parabolic subgroups, and, as such, is very valuable when studying the representation π itself (the length, square–integrability etc.) Without explicitly mentioning it, we use the following expression (the 'structure formula' ([13],[5]))when calculating Jacquet modules of the induced representation

$$\mu^*(\sigma \rtimes \pi) = M^*(\sigma) \rtimes \mu^*(\pi).$$

M^* denotes a certain homomorphism acting on the Grothendieck group of the finite length representations of the general linear group over D; all the details can be found in the above papers.

In the following sections, we study the case $n = 2$ in some detail. We concentrate to the following situation: Let τ be an irreducible, self-dual representation of D^*, of dimension greater than one, with the trivial central character. Here D^* denotes the multiplicative group of D. We consider the following principal series representation of the group $G_2(D, \varepsilon)$:

$$\tau\nu^{\frac{3}{2}} \times \tau\nu^{\frac{1}{2}} \rtimes 1.$$

Lemma 2.1. *Let τ be an irreducible, self-dual representation of D^*, of dimension greater than one, with the trivial central character. Then, for $s \geq 0$, the representation $\tau\nu^s \rtimes 1$ reduces only for $s = \frac{1}{2}$; this is the case for both $G_1(D, 1)$ and $G_1(D, -1)$.*

Proof. By the Jacquet–Langlands correspondence, τ corresponds to a cuspidal representation τ' of the group $GL(2, F)$, and the reducibility of the representation $\tau'\nu^s \rtimes 1$ is completely governed by the zeroes and the poles of the Plancherel measure $\mu(s, \tau')$ for the group $Sp(4, F)$ or the group $SO(4, F)$ (we use completely analogous notation for the hermitian quaternionic groups and their split inner forms). But for both of these groups, and the same cuspidal τ', $\mu(s, \tau')$ has the same zeroes (poles) ([11],[10]). By the results of Muić and Savin ([9]) about the transfer of the Plancherel measure in the Siegel case ($\mu(s, \tau) = \mu(s, \tau')$), the claim follows. □

The length of the representation $\tau\nu^{\frac{1}{2}} \rtimes 1$ equals two, and one of the subquotients is a square–integrable representation which we denote by $\delta[\tau\nu^{\frac{1}{2}}; 1]$. Because of the previous Lemma, the representation $\tau\nu^{\frac{3}{2}} \times \tau\nu^{\frac{1}{2}} \rtimes 1$ has the same length and an analogous structure for both $G_2(D, 1)$ and $G_2(D, -1)$. In an appropriate Grothendieck group, we have ([5])

$$\tau\nu^{\frac{3}{2}} \times \tau\nu^{\frac{1}{2}} \rtimes 1 = L(\delta(\tau\nu^{\frac{1}{2}}, \tau\nu^{\frac{3}{2}}); 1) + L(\tau\nu^{\frac{3}{2}}; \delta[\tau\nu^{\frac{1}{2}}; 1]) + L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1) + \pi,$$

where π is a square integrable representation.

The goal of this paper is to compare the local and the global method of proving unitarizability on the example of representation $L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)$.

3 The global calculation

The global method amounts to constructing an automorphic representation belonging to the residual spectrum of the appropriate group, which has, as a local component, the representation $L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)$.

In [2], Grbac calculated the residual spectrum of the group $G_2(\mathbf{D}, -1)$. Here $\mathbf{D}$ denotes a quaternion algebra over an algebraic number field k, such that, for certain finite set of places S of k, and $v \in S$, we have $k_v \cong F$, and $G_2(\mathbf{D}, -1)_v \cong G_2(D, -1)$ and for $v \notin S$, we have $G_2(\mathbf{D}, -1)_v \cong SO(8, k_v)$. He proved that there is an automorphic representation in the residual spectrum of the group $G_2(\mathbf{D}, -1)$ which has a local component isomorphic to $L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)$, thus proving that this representation is unitarizable. In [4], we proved the analogous fact for the group $G_2(\mathbf{D}, 1)$, but, it is interesting to note that in this case, the appearance of this global representation in the residual spectrum is conditional; namely there is a (global) condition on non vanishing of certain global L–function involved; a feature which does not occur for $G_2(\mathbf{D}, -1)$.

4 The local calculation

In order to prove unitarizability of $L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)$ using local techniques we study first the representation $\tau\nu^{\frac{1}{2}} \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)$. We divide the proof of unitarizability into three steps:
Firstly, we study the composition series of the representation
$\tau\nu^{\frac{1}{2}} \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)$ (Proposition 4.1).
Secondly, we prove that all of its irreducible subquotients are unitarizable (Proposition 4.2).
Thirdly, using the Jantzen filtration, we prove that the representation
$\tau \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)$ is unitarizable (Proposition 4.3).

Then, from the irreducibility of the representation $\tau \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)$ and the fact that $L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)$ is hermitian, it follows that it must be unitarizable.

The representation $\delta([\tau\nu^{-\frac{1}{2}}, \tau\nu^{\frac{1}{2}}]) \rtimes 1$ is reducible; this easily follows from the fact that $\tau\nu^{\frac{1}{2}} \rtimes 1$ reduces. We introduce tempered representations T_i, $i = 1, 2$ such that $\delta([\tau\nu^{-\frac{1}{2}}, \tau\nu^{\frac{1}{2}}]) \rtimes 1 = T_1 \oplus T_2$; let T_2 be the one which appears as a subquotient in $\tau\nu^{\frac{1}{2}} \rtimes L(\tau\nu^{\frac{1}{2}}; 1)$.

Proposition 4.1. *In an appropriate Grothendieck group, we have*

$$\tau\nu^{\frac{1}{2}} \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1) = L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}, \tau\nu^{\frac{1}{2}}; 1) + L(\tau\nu^{\frac{3}{2}}; T_2).$$

Proof. There are epimorphisms

$$\tau\nu^{\frac{3}{2}} \times \tau\nu^{\frac{1}{2}} \times \tau\nu^{\frac{1}{2}} \rtimes 1 \twoheadrightarrow L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}) \times \tau\nu^{\frac{1}{2}} \rtimes 1 \cong \tau\nu^{\frac{1}{2}} \times L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}) \rtimes 1 \twoheadrightarrow \tau\nu^{\frac{1}{2}} \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1),$$

and, because the representation $\tau\nu^{\frac{3}{2}} \times \tau\nu^{\frac{1}{2}} \times \tau\nu^{\frac{1}{2}} \rtimes 1$ has a unique irreducible quotient, it follows that $L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}, \tau\nu^{\frac{1}{2}}; 1)$ is a quotient of the representation $\tau\nu^{\frac{1}{2}} \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)$.

Modifying some ideas from Section 8 of [14], we proceed as follows: We have an epimorphism

$$\tau\nu^{\frac{3}{2}} \times \tau\nu^{-\frac{1}{2}} \times \tau\nu^{\frac{1}{2}} \rtimes 1 \cong \tau\nu^{-\frac{1}{2}} \times \tau\nu^{\frac{3}{2}} \times \tau\nu^{\frac{1}{2}} \rtimes 1 \twoheadrightarrow \tau\nu^{-\frac{1}{2}} \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1).$$

If we assume that the restriction of this epimorphism to the subrepresentation $\tau\nu^{\frac{3}{2}} \times L(\tau\nu^{\frac{1}{2}}, \tau\nu^{-\frac{1}{2}}) \rtimes 1$ is still an epimorphism, it would follow that

$$\mu^*(\tau\nu^{\frac{3}{2}} \times L(\tau\nu^{\frac{1}{2}}, \tau\nu^{-\frac{1}{2}}) \rtimes 1) \geq \mu^*(\tau\nu^{-\frac{1}{2}} \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)).$$

Since $L(\tau\nu^{-\frac{1}{2}}, \tau\nu^{-\frac{3}{2}}) \otimes 1 \leq \mu^*(L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1))$, we get that

$$\tau\nu^{\frac{1}{2}} \times L(\tau\nu^{-\frac{1}{2}}, \tau\nu^{-\frac{3}{2}}) \otimes 1 \leq \mu^*(\tau\nu^{-\frac{1}{2}} \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)) \leq \mu^*(\tau\nu^{\frac{3}{2}} \times L(\tau\nu^{\frac{1}{2}}, \tau\nu^{-\frac{1}{2}}) \rtimes 1).$$

There is a similar situation in the case of general linear groups over F and over division algebra D: the representation $\tau\nu^{\frac{1}{2}} \times L(\tau\nu^{-\frac{1}{2}}, \tau\nu^{-\frac{3}{2}})$ has a unique subrepresentation, denoted by $Z(\{\tau\nu^{\frac{1}{2}}\}, \{\tau\nu^{-\frac{1}{2}}, \tau\nu^{-\frac{3}{2}}\})$(using Zelevinsky's notation). But when we examine closer $\mu^*(\tau\nu^{\frac{3}{2}} \times L(\tau\nu^{\frac{1}{2}}, \tau\nu^{-\frac{1}{2}}) \rtimes 1)$ (using the structure formula ([13],[5]), we see that in the appropriate Jacquet module $r_{GL(3,D),G_3(D,\varepsilon)}(\tau\nu^{\frac{3}{2}} \times L(\tau\nu^{\frac{1}{2}}, \tau\nu^{-\frac{1}{2}}) \rtimes 1)$ there is no subquotient with $\tau\nu^{-\frac{1}{2}}$, $\tau\nu^{\frac{1}{2}}$ and $\tau\nu^{-\frac{3}{2}}$ in the cuspidal support. This is a contradiction and we conclude that restriction of the epimorphism above to the subrepresentation $\tau\nu^{\frac{3}{2}} \times L(\tau\nu^{\frac{1}{2}}, \tau\nu^{-\frac{1}{2}}) \rtimes 1$ is not surjective. So, there is an intertwining from $\tau\nu^{\frac{3}{2}} \times \delta([\tau\nu^{-\frac{1}{2}}, \tau\nu^{\frac{1}{2}}]) \rtimes 1$ onto a quotient of $\tau\nu^{-\frac{1}{2}} \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)$.

We recall the definition of the tempered representations T_1 and T_2:
$\delta([\tau\nu^{-\frac{1}{2}}, \tau\nu^{\frac{1}{2}}]) \rtimes 1 = T_1 \oplus T_2$; and T_2 is a subquotient of $\tau\nu^{\frac{1}{2}} \rtimes L(\tau\nu^{\frac{1}{2}}; 1)$. Using the structure formula, we obtain:

$$r_{GL(2,D),G_2(D,\varepsilon)}(T_1 + T_2) = 2\delta([\nu^{-\frac{1}{2}}\tau, \nu^{\frac{1}{2}}\tau]) \otimes 1 + \nu^{\frac{1}{2}}\tau \times \nu^{\frac{1}{2}}\tau \otimes 1.$$

So, T_1 is a tempered subquotient of $\delta([\tau\nu^{-\frac{1}{2}}, \tau\nu^{\frac{1}{2}}]) \rtimes 1$ for which $r_{GL(2,D),G_2(D,\varepsilon)}(T_1) = \nu^{\frac{1}{2}}\tau \times \nu^{\frac{1}{2}}\tau \otimes 1 + \delta([\nu^{-\frac{1}{2}}\tau, \nu^{\frac{1}{2}}\tau]) \otimes 1$, and $r_{GL(2,D),G_2(D,\varepsilon)}(T_2) = \delta([\nu^{-\frac{1}{2}}\tau, \nu^{\frac{1}{2}}\tau]) \otimes 1$ (this agrees with our previous remark on T_2).

If we assume that there is a non-zero intertwining form $\tau\nu^{\frac{3}{2}} \rtimes T_1$ onto a quotient of $\tau\nu^{-\frac{1}{2}} \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)$, it would follow that $L(\tau\nu^{\frac{3}{2}}; T_1) \leq \tau\nu^{-\frac{1}{2}} \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)$.

But, since $\tau\nu^{-\frac{3}{2}} \otimes T_1 \leq \mu^*(\tau\nu^{-\frac{3}{2}} \rtimes T_1)$ and $\tau\nu^{\frac{1}{2}} \times \tau\nu^{\frac{1}{2}} \leq \mu^*(T_1)$, it would follow that

$$\tau\nu^{-\frac{3}{2}} \otimes \tau\nu^{\frac{1}{2}} \times \tau\nu^{\frac{1}{2}} \otimes 1 \leq r_{D^* \times GL(2,D),G_3(D,\varepsilon)}(\tau\nu^{-\frac{1}{2}} \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)).$$

But since

$$\mu^*(L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}};1)) = 1\otimes L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}};1)+\tau\nu^{-\frac{3}{2}}\otimes L(\tau\nu^{\frac{1}{2}};1)+L(\tau\nu^{-\frac{1}{2}},\tau\nu^{-\frac{3}{2}})\otimes 1,$$

we easily get that $\tau\nu^{-\frac{3}{2}}\otimes\tau\nu^{\frac{1}{2}}\times\tau\nu^{\frac{1}{2}}\otimes 1$ cannot appear as a subquotient of the appropriate Jacquet module of $\tau\nu^{-\frac{1}{2}}\rtimes L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}};1)$.

We conclude that there is a non-zero intertwining from $\tau\nu^{\frac{3}{2}}\rtimes T_2$ onto a quotient of $\tau\nu^{-\frac{1}{2}}\rtimes L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}};1)$, so $L(\tau\nu^{\frac{3}{2}};T_2)\leq\tau\nu^{\frac{1}{2}}\rtimes L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}};1)$.

It remains to prove that the length of the representation $\tau\nu^{\frac{1}{2}}\rtimes L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}};1)$ equals two. We examine $r_{GL(3,D),G_3(D\varepsilon)}(\tau\nu^{\frac{1}{2}}\rtimes L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}};1))$, which, in Grothendieck group equals $\tau\nu^{-\frac{1}{2}}\times L(\tau\nu^{-\frac{1}{2}},\tau\nu^{-\frac{3}{2}})\otimes 1+\tau\nu^{\frac{1}{2}}\times L(\tau\nu^{-\frac{1}{2}},\tau\nu^{-\frac{3}{2}})\otimes 1$. So, the maximum length of the representation $\tau\nu^{\frac{1}{2}}\rtimes L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}};1)$ equals three. If we assume that it is three, there is a subquotient, call it π, of our representation, such that $r_{GL(3,D),G_3(D,\varepsilon)}(\pi) = L(\tau\nu^{\frac{1}{2}},\tau\nu^{-\frac{1}{2}},\tau\nu^{-\frac{3}{2}})\otimes 1$, so $r_{D^*\times D^*\times D^*,G_3(D,\varepsilon)}(\pi)=\tau\nu^{-\frac{3}{2}}\otimes\tau\nu^{-\frac{1}{2}}\otimes\tau\nu^{\frac{1}{2}}$. When we compare this with the Jacquet module $r_{D^*\times G_2(D,\varepsilon),G_3(D,\varepsilon)}(\tau\nu^{\frac{1}{2}}\rtimes L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}};1))$ we get that then $r_{D^*\times G_2(D,\varepsilon),G_3(D,\varepsilon)}(\pi)=\tau\nu^{-\frac{3}{2}}\otimes L(\tau\nu^{\frac{1}{2}},\tau\nu^{\frac{1}{2}};1)$, but this leads to contradiction, since

$$r_{D^*\times D^*,G_2(D,\varepsilon)}(L(\tau\nu^{\frac{1}{2}},\tau\nu^{\frac{1}{2}};1))=\tau\nu^{-\frac{1}{2}}\otimes\tau\nu^{\frac{1}{2}}\otimes 1+2\tau\nu^{-\frac{1}{2}}\otimes\tau\nu^{-\frac{1}{2}}\otimes 1.$$

□

If we assume that the representation $L(\tau\nu^{\frac{3}{2}},T_2)$ can also be a quotient, we get a contradiction, because

$$\mathrm{Hom}(\tau\nu^{-\frac{1}{2}}\underline{\rtimes}L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}};1),L(\tau\nu^{\frac{3}{2}};T_2)) = \mathrm{Hom}(\tau\nu^{-\frac{1}{2}}\otimes L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}};1),r_{D^*\times G_2(D,\varepsilon),G_3(D,\varepsilon)}L(\tau\nu^{\frac{3}{2}};T_2))=0,$$

and $\tau\nu^{-\frac{1}{2}}\otimes L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}};1)$ does not appear in the Jacquet module $r_{D^*\times G_2(D,\varepsilon),G_3(D,\varepsilon)}(\tau\nu^{\frac{3}{2}}\rtimes T_2)$.

We claim that the representations $L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}},\tau\nu^{\frac{1}{2}};1)$ and $L(\tau\nu^{\frac{3}{2}};T_2)$ are unitarizable. To prove that, we will study certain complementary series representations, and then observe that these representations appear at the end of the complementary series, and as such, are unitarizable.

Proposition 4.2. *The representations* $L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}},\tau\nu^{\frac{1}{2}};1)$ *and* $L(\tau\nu^{\frac{3}{2}};T_2)$ *are unitarizable.*

Proof. First we prove unitarizability of $L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}},\tau\nu^{\frac{1}{2}};1)$. The first step is showing that it is a subquotient of $\nu^{\frac{1}{2}}L(\tau\nu,\tau,\tau\nu^{-1})\rtimes 1$. In the Grothendieck group,we have

$$\tau\nu^{\frac{3}{2}}\times\tau\nu^{\frac{1}{2}}\times\tau\nu^{-\frac{1}{2}}\rtimes 1 = L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}})\times\tau\nu^{-\frac{1}{2}}\rtimes 1+\nu\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\times\tau\nu^{-\frac{1}{2}}\rtimes 1 = \\ = L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}},\tau\nu^{-\frac{1}{2}})\rtimes 1+Z(\Delta_1,\Delta_2)\rtimes 1+\nu\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\times\tau\nu^{-\frac{1}{2}}\rtimes 1.$$

Here $\Delta_1=\left\{\tau\nu^{\frac{1}{2}},\tau\nu^{\frac{3}{2}}\right\}$, and $\Delta_2=\left\{\tau\nu^{-\frac{1}{2}}\right\}$ are segments of cuspidal representations, and $Z(\Delta_1,\Delta_2)$ denotes a unique subrepresentation of $L(\tau\nu^{\frac{3}{2}},\tau\nu^{\frac{1}{2}})\times\tau\nu^{-\frac{1}{2}}$, in Zelevinsky's notation.

We now show that the representation $L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}, \tau\nu^{\frac{1}{2}}; 1)$ is not a subquotient of the second and of the third term of the above sum. The representation $L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}, \tau\nu^{\frac{1}{2}}; 1)$ occurs with multiplicity one in $\tau\nu^{\frac{3}{2}} \times \tau\nu^{\frac{1}{2}} \times \tau\nu^{-\frac{1}{2}} \rtimes 1$, and, after calculating Jacquet module

$$r_{D^*\times G_2(D,\varepsilon),G_3(D,\varepsilon)}(\nu\delta([\tau\nu^{-\frac{1}{2}}, \tau\nu^{\frac{1}{2}}]) \times \tau\nu^{-\frac{1}{2}} \rtimes 1)$$

we see that there is no subquotient of this Jacquet module which is of the form $\tau\nu^{-\frac{3}{2}} \otimes \pi$, for some representation π, and $r_{D^*\times D^*\times D^*,G_3(D,\varepsilon)}(L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}, \tau\nu^{\frac{1}{2}}; 1))$ has to have $\tau\nu^{-\frac{3}{2}} \otimes \tau\nu^{-\frac{1}{2}} \otimes \tau\nu^{-\frac{1}{2}}$ as a subquotient.
We conclude that

$$L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}, \tau\nu^{\frac{1}{2}}; 1) \leq \nu L(\tau\nu^{\frac{1}{2}}, \tau\nu^{-\frac{1}{2}}) \times \tau\nu^{-\frac{1}{2}} \rtimes 1.$$

On the other hand, we further calculate

$$\begin{aligned} r_{D^*\times G_2(D,\varepsilon),G_3(D;\varepsilon)}(Z(\Delta_1, \Delta_2) \rtimes 1) &= \tau\nu^{\frac{1}{2}} \otimes \tau\nu^{\frac{3}{2}} \times \tau\nu^{-\frac{1}{2}} \rtimes 1 \\ &+ \tau\nu^{-\frac{3}{2}} \otimes T_1 + \tau\nu^{-\frac{3}{2}} \otimes T_2 + \tau\nu^{\frac{1}{2}} \otimes \nu L(\tau\nu^{\frac{1}{2}}, \tau\nu^{-\frac{1}{2}}) \rtimes 1. \end{aligned}$$

So, we can not have $\tau\nu^{-\frac{3}{2}} \otimes \tau\nu^{-\frac{1}{2}} \otimes \tau\nu^{-\frac{1}{2}}$ as a subquotient of $r_{D^*\times D^*\times D^*,G_3(D;\varepsilon)}(Z(\Delta_1, \Delta_2) \rtimes 1)$, and $L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}, \tau\nu^{\frac{1}{2}}; 1) \not\leq Z(\Delta_1, \Delta_2) \rtimes 1$, so $L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}, \tau\nu^{\frac{1}{2}}; 1) \leq \nu^{\frac{1}{2}} L(\tau\nu, \tau, \tau\nu^{-1}) \rtimes 1$, as claimed.

We now consider family of representations $\nu^s L(\tau\nu, \tau, \tau\nu^{-1}) \rtimes 1$, $s \geq 0$. By [14], Theorem 9.1(ii), the first point of reducibility of this representation is $s = \frac{1}{2}$. But the representation $L(\tau\nu, \tau, \tau\nu^{-1})$ is unitarizable, so the representation $L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}, \tau\nu^{\frac{1}{2}}; 1)$ is unitarizable as a subquotient of the representation at the end of the complementary series.

Now we prove unitarizability of $\tau\nu^{\frac{3}{2}} \rtimes T_2$ by examining the family of representations $\tau\nu^s \rtimes T_2$ for $s \geq 0$. It is not difficult to see that for $s = 0$ we get an irreducible representation; it immediately follows from the fact that $\tau \rtimes 1$ is irreducible. Similarly, from the factorization of the long intertwining operator, it follows that $\tau\nu^s \rtimes T_2$ is irreducible for $s > 0$, $s \notin \frac{1}{2} + \mathbb{Z}$. Now we prove that the representation $\tau\nu^{\frac{1}{2}} \rtimes T_2$ is irreducible. To do that, we prove that the representation

$$\tau\nu^{\frac{1}{2}} \times \delta([\tau\nu^{-\frac{1}{2}}, \tau\nu^{\frac{1}{2}}]) \rtimes 1$$

is of length three. The following holds:

$$\begin{aligned} \tau\nu^{\frac{1}{2}} \times \delta([\tau\nu^{-\frac{1}{2}}, \tau\nu^{\frac{1}{2}}]) \rtimes 1 &= \tau\nu^{\frac{1}{2}} \rtimes T_1 + \tau\nu^{\frac{1}{2}} \rtimes T_2 = \\ &\delta([\tau\nu^{-\frac{1}{2}}, \tau\nu^{\frac{1}{2}}]) \rtimes \delta[\tau\nu^{\frac{1}{2}}; 1] + \delta([\tau\nu^{-\frac{1}{2}}, \tau\nu^{\frac{1}{2}}]) \rtimes L(\tau\nu^{\frac{1}{2}}; 1). \end{aligned}$$

We now want to prove that the representation $\delta([\tau\nu^{-\frac{1}{2}}, \tau\nu^{\frac{1}{2}}]) \rtimes \delta[\tau\nu^{\frac{1}{2}}; 1]$ is irreducible. Using the structure formula, we obtain

$$\begin{aligned} r_{D^*\times G_2(D,\varepsilon),G_3(D,\varepsilon)}(\delta([\tau\nu^{-\frac{1}{2}}, \tau\nu^{\frac{1}{2}}]) \rtimes \delta[\tau\nu^{\frac{1}{2}}; 1]) &= 2\tau\nu^{\frac{1}{2}} \otimes L(\tau\nu^{\frac{1}{2}}; \delta[\tau\nu^{\frac{1}{2}}; 1]) + \\ &3\tau\nu^{\frac{1}{2}} \otimes T_1 + \tau\nu^{\frac{1}{2}} \otimes T_2, \end{aligned}$$

$$r_{GL(3,D),G_3(D,\varepsilon)}(\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\rtimes\delta[\tau\nu^{\frac{1}{2}};1]) = 2\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\times\tau\nu^{\frac{1}{2}}\otimes 1 + \tau\nu^{\frac{1}{2}}\times\tau\nu^{\frac{1}{2}}\times\tau\nu^{\frac{1}{2}}\otimes 1.$$

If we assume that $\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\rtimes\delta[\tau\nu^{\frac{1}{2}};1])$ reduces, then there exist two non-isomorphic tempered representations T_1' and T_2' such that

$$\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\rtimes\delta[\tau\nu^{\frac{1}{2}};1]) = T_1'\oplus T_2'.$$

So, let, for example, T_2' be such that $r_{GL(3,D),G_3(D,\varepsilon)}(T_2')\geq\tau\nu^{\frac{1}{2}}\times\tau\nu^{\frac{1}{2}}\times\tau\nu^{\frac{1}{2}}\otimes 1$. Then, since in the cuspidal support of $\tau\nu^{\frac{1}{2}}\times\tau\nu^{\frac{1}{2}}\times\tau\nu^{\frac{1}{2}}$ we have just multiples of $\tau\nu^{\frac{1}{2}}\otimes\tau\nu^{\frac{1}{2}}\otimes\tau\nu^{\frac{1}{2}}$, we must have $r_{D^*\times G_2(D,\varepsilon),G_3(D,\varepsilon)}(T_2')\geq 3\tau\nu^{\frac{1}{2}}\otimes T_1$. But then $3\tau\nu^{\frac{1}{2}}\otimes\tau\nu^{\frac{1}{2}}\otimes\tau\nu^{-\frac{1}{2}}\leq r_{D^*\times D^*\times D^*,G_3(D,\varepsilon)}(T_2')$. Now we compare this with $r_{GL(3,D),G_3(D,\varepsilon)}(\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\rtimes\delta[\tau\nu^{\frac{1}{2}};1])$. Since $r_{D^*\times D^*\times D^*,GL(3,D)}(\tau\nu^{\frac{1}{2}}\times\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}]) = 2\tau\nu^{\frac{1}{2}}\otimes\tau\nu^{\frac{1}{2}}\otimes\tau\nu^{-\frac{1}{2}}+\tau\nu^{\frac{1}{2}}\otimes\tau\nu^{-\frac{1}{2}}\otimes\tau\nu^{\frac{1}{2}}$, we must have $2\tau\nu^{\frac{1}{2}}\times\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\leq r_{GL(3,D),G_3(D,\varepsilon)}(T_2')$, but then $T_2' = \delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\rtimes\delta[\tau\nu^{\frac{1}{2}};1])$, and we conclude that $\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\rtimes\delta[\tau\nu^{\frac{1}{2}};1])$ is irreducible.

Now we prove that $\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\rtimes L(\tau\nu^{\frac{1}{2}};1)$ is of length two.

Examining Langlands' parameters and cuspidal support of the representations $\tau\nu^{\frac{1}{2}}\rtimes T_i,\quad i=1,2$, we see that these representations cannot have other non-tempered subquotients but their Langlands quotients. So, besides $L(\tau\nu^{\frac{1}{2}};T_i),\quad i=1,2$ the representation $\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\rtimes L(\tau\nu^{\frac{1}{2}};1)$ has only tempered subquotients.

We have

$$r_{D^*\times G_2(D,\varepsilon),G_3(D,\varepsilon)}(\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\rtimes L(\tau\nu^{\frac{1}{2}};1)) = 2\tau\nu^{\frac{1}{2}}\otimes L(\tau\nu^{\frac{1}{2}},\tau\nu^{\frac{1}{2}};1) + 2\tau\nu^{\frac{1}{2}}\otimes T_2+\tau\nu^{-\frac{1}{2}}\otimes T_1+\tau\nu^{-\frac{1}{2}}\otimes T_2,$$

so the only irreducible subquotients above coming from these tempered subquotients are $2\tau\nu^{\frac{1}{2}}\otimes T_2$ (so the rest of the subquotients in the above Jacquet module comes from $L(\tau\nu^{\frac{1}{2}};T_1)$ and $L(\tau\nu^{\frac{1}{2}};T_2)$).

But, since

$$r_{GL(3,D),G_3(D,\varepsilon)}(\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\rtimes L(\tau\nu^{\frac{1}{2}};1)) = 2\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\times\tau\nu^{-\frac{1}{2}}\otimes 1 + \delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\times\tau\nu^{\frac{1}{2}}\otimes 1 + L(\tau\nu^{\frac{1}{2}},\tau\nu^{-\frac{1}{2}})\times\tau\nu^{\frac{1}{2}}\otimes 1$$

no irreducible subquotient from above is compatible with our requirement on the Jacquet module $r_{D^*\times G_2(D,\varepsilon),G_3(D,\varepsilon)}$.

We conclude that

$$\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\rtimes L(\tau\nu^{\frac{1}{2}};1)) = L(\tau\nu^{\frac{1}{2}};T_1)+L(\tau\nu^{\frac{1}{2}};T_2).$$

Hence, exactly one of the representations $\tau\nu^{\frac{1}{2}}\times T_i,\ i=1,2$ reduces. If we assume that $\tau\nu^{\frac{1}{2}}\rtimes T_2$ reduces, then, it would follow that

$$\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\rtimes\delta[\tau\nu^{\frac{1}{2}};1]\leq\tau\nu^{\frac{1}{2}}\rtimes T_2.$$

We noted above that $\tau\nu^{\frac{1}{2}}\times\tau\nu^{\frac{1}{2}}\times\tau\nu^{\frac{1}{2}}\otimes 1\leq r_{GL(3,D),G_3(D,\varepsilon)}(\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\rtimes\delta[\tau\nu^{\frac{1}{2}};1])$, on the other hand, the following holds

$$r_{GL(3,D),G_3(D,\varepsilon)}(\tau\nu^{\frac{1}{2}}\rtimes T_2) = \tau\nu^{\frac{1}{2}}\times\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\otimes 1+\tau\nu^{-\frac{1}{2}}\times\delta([\tau\nu^{-\frac{1}{2}},\tau\nu^{\frac{1}{2}}])\otimes 1.$$

The conclusion is that $\tau\nu^{\frac{1}{2}} \rtimes T_2$ is irreducible. So, the first (possible) point of reducibility for the family of hermitian representations $\tau\nu^s \rtimes T_2,\ s \geq 0$ is $s = \frac{3}{2}$, hence $L(\tau\nu^{\frac{3}{2}}; T_2)$ is a unitarizable representation as a subquotient of a representation at the end of a complementary series.

□

Proposition 4.3. *The representation* $L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)$ *is unitarizable.*

Proof. Let X denote the compact picture of the representation $\tau\nu^s \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)$, and let

$$A(s) : \tau\nu^s \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1) \to \tau\nu^{-s} \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)$$

be the usual intertwining operator (given as a meromorphic continuation of the integral intertwining operator converging for $s >> 0$). Also we note that

$$A(s)A(-s) = (s - \frac{1}{2})h(s), \tag{1}$$

where $h(s)$ in holomorphic and nonzero in the neighborhood of $\frac{1}{2}$. This follows from the simplicity of the pole of the Plancherel measure at $s = \frac{1}{2}$ by the result of Heiremann ([6]). Namely, the representation $\tau\nu^{\frac{3}{2}} \times \tau\nu^{\frac{1}{2}} \times \tau\nu^{\frac{1}{2}} \rtimes 1$ has a square–integrable subquotient, and, in that situation, this result gives the order of the pole of the Plancherel measure with respect to the long intertwining operator. But, similarly as in [1], from that, we can calculate the order of the pole of the Plancherel measure in question. By the definition of the Jantzen filtration ([15]), we study hermitian forms on X, indexed by $s \in [0, \frac{1}{2}]$ and generated by the intertwining operators $(\cdot,\cdot)_s = (A(s)\cdot,\cdot)$, where $(\cdot,\cdot)$ denotes the usual hermitian form on X given by the integration over a maximal compact subgroup K of G (this form exists because the inducing representation is hermitian). Then, by ([15]), a sequence of G-invariant spaces is given

$$X^0_{\frac{1}{2}} = X \supset X^1_{\frac{1}{2}} = L(\tau\nu^{\frac{3}{2}}; T_2) \supset X^2_{\frac{1}{2}} \supset \ldots \supset \{0\}.$$

The space $X^i_{\frac{1}{2}}$ is given as the radical of the hermitian form $(\cdot,\cdot)^{i-1}_{\frac{1}{2}}$ defined on $X^{i-1}_{\frac{1}{2}}$ and given by

$$\lim_{s \to \frac{1}{2}} \frac{1}{(s - \frac{1}{2})^{i-1}} (A(s)\cdot,\cdot).$$

First, we want to show that $X^2_{\frac{1}{2}}$ is zero, i.e. that the hermitian form $(\cdot,\cdot)^1_{\frac{1}{2}}$ defined on $L(\tau\nu^{\frac{3}{2}}; T_2)$ is non-degenerate. We can always focus our attention to a certain K–type $m_\delta V_\delta$ (where δ is an irreducible representation of K on the space V_δ and m_δ is the multiplicity of that representation in X) such that $m_\delta V_\delta \cap L(\tau\nu^{\frac{3}{2}}; T_2) \neq \{0\}$. Then, on this K-type (which is a finite–dimensional subspace) we have the following expansion:

$$A(s) = A(\frac{1}{2}) + (s - \frac{1}{2})A'(\frac{1}{2}) + \cdots .$$

The operator $A(s)$ is holomorphic in the neighborhood of $s = \frac{1}{2}$ (we get this from the factorization of the intertwining operator on the appropriate larger space). For

$f \in m_\delta V_\delta \cap L(\tau\nu^{\frac{3}{2}}; T_2)$ we have $\lim_{s\to\frac{1}{2}} \frac{1}{s-\frac{1}{2}} A(s)f = A'(\frac{1}{2})f$, so $A(-\frac{1}{2})A'(\frac{1}{2})f = h(\frac{1}{2})f \neq 0$. Then, because $A'(\frac{1}{2})f \notin \mathrm{Ker} A(-\frac{1}{2})$ we can choose appropriate $v' \in L(\tau\nu^{\frac{3}{2}}; T_2)|_K$ such that $(A'(\frac{1}{2})f, v') = (f, v')_{\frac{1}{2}} \neq 0$. So, if we denote the signature on the space $X/L(\tau\nu^{\frac{3}{2}}; T_2) \cong L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}, \tau\nu^{\frac{1}{2}}; 1)$ by $(p_0, 0)$, then $(p_1, q_1) = (0, q_1)$ is a signature on the space $L(\tau\nu^{\frac{3}{2}}; T_2)$. It follows ([15]) that the representation $\tau\nu^s \rtimes L(\tau\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1)$ has a signature equal to $(p_0 + q_1, 0)$, i.e. is unitarizable for $s \in [0, \frac{1}{2})$.

□

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Some Remarks on the Potential Theory of L^p-Sub-Markovian Semigroups

Niels Jacob

This article summarises results on L^p-potential theory and function spaces associated with sub-Markovian semigroups presented during the meeting in Dubrovnik. It is based on several joint papers written with W. Farkas, W. Hoh and R. Schilling.

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1 L^p-Sub-Markovian Semigroups and their Generators

We start with

Definition 1.1. *A family* $(T_t)_{t\geq 0}$ *of linear operator* $T_t : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, *is called an* L^p-sub-Markovian semigroup *if*

$$T_t \circ T_s = T_{t+s} \quad \text{and} \quad T_0 = \mathrm{id} \qquad \textit{(semigroup property)} \tag{1.1}$$

$$\lim_{t\to 0} \|T_t u - u\|_{L^p} = 0 \qquad \textit{(strong continuity)} \tag{1.2}$$

$$\|T_t u\|_{L^p} \leq \|u\|_{L^p} \qquad \textit{(contraction property)} \tag{1.3}$$

$$0 \leq u \leq 1 \ \textit{a.e. implies}\ 0 \leq T_t u \leq 1 \ \textit{a.e.} \qquad \textit{(sub-Markovian property)} \tag{1.4}$$

Note that if necessary to emphasize p we will write $T_t^{(p)}$ instead of T_t. A strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ has a *generator* $\big(A, D(A)\big)$ where

$$D(A) := \{u \in L^p(\mathbb{R}^n)\,;\, \lim_{t\to 0} \frac{T_t u - u}{t} \text{ exists as strong limit in } L^p(\mathbb{R}^n)\} \tag{1.5}$$

and

$$Au = \lim_{t\to 0} \frac{T_t u - u}{t}\,, u \in D(A), \qquad \text{(strong limit).} \tag{1.6}$$

The *resolvent* $(R_\lambda)_{\lambda>0}$ associated with $(T_t)_{t\geq 0}$ is defined on $L^p(\mathbb{R}^n)$ by

$$R_\lambda u = \int_0^\infty e^{-\lambda t} T_t u \,\mathrm{d}t. \tag{1.7}$$

For an L^p-sub-Markovian semigroup we know that the generator $\big(A, D(A)\big)$ is densely defined, closed and it is a *Dirichlet operator* in the sense that

$$\int_{\mathbb{R}^n} (Au)\big((u-1)^+\big)^{p-1}\,\mathrm{d}x \le 0 \tag{1.8}$$

holds for all $u \in D(A)$. Dirichlet operators are dissipative, i.e.

$$\|\lambda u - Au\|_{L^p} \ge \lambda \|u\|_{L^p} \quad \text{for } u \in D(A), \lambda > 0. \tag{1.9}$$

In addition they satisfy

$$\int_{\mathbb{R}^n} (Au)(u^+)^{p-1}\,\mathrm{d}x \le 0 \quad \text{and} \quad \int_{\mathbb{R}^n} (Au)(u^-)^{p-1}\,\mathrm{d}x \ge 0, \tag{1.10}$$

and they are negative definite in the sense that

$$\int_{\mathbb{R}^n} (Au)(\operatorname{sign} u)|u|^{p-1}\,\mathrm{d}x \le 0. \tag{1.11}$$

Note that (1.11) implies (1.9).

The resolvent $(R_\lambda)_{\lambda>0}$ is a strongly continuous contraction resolvent which is sub-Markovian, i.e. $R_\lambda : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ and it holds

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu \quad \text{(resolvent equation)} \tag{1.12}$$

$$\lim_{\lambda\to\infty} \|\lambda R_\lambda u - u\|_{L^p} = 0 \quad \text{(strong continuity)} \tag{1.13}$$

$$\|\lambda R_\lambda u\|_{L^p} \le \|u\|_{L^p} \quad \text{(contraction property)} \tag{1.14}$$

$$0 \le u \le 1 \text{ a.e. implies } 0 \le \lambda R_\lambda u \le 1 \text{ a.e.} \text{(sub-Markovian property)}. \tag{1.15}$$

Moreover for $u \in L^p(\mathbb{R}^n)$ we have

$$(\lambda - A)^{-1} u = R_\lambda u = \int_0^\infty e^{-\lambda t} T_t u\,\mathrm{d}t \tag{1.16}$$

As bounded linear operator each T_t, $(T_t)_{t\ge0}$ being an L^p-sub-Markovian semigroup, has an adjoint operator $T_t^* : L^{p'}(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{p'} = 1$, $1 < p < \infty$. Using the reflexivity of $L^p(\mathbb{R}^n)$, $1 < p < \infty$, we can prove that $(T_t^*)_{t\ge0}$ is a strongly continuous contraction semigroup on $L^{p'}(\mathbb{R}^n)$ with generator being the conjugate to $\big(A, D(A)\big)$, i.e.

$$\Big\{u \in L^{p'}(\mathbb{R}^n)\,;\, \lim_{t\to0} \frac{T_t^* u - u}{t} \text{ exists as strong } L^{p'}\text{-limit}\Big\} = D(A^*) \tag{1.17}$$

and

$$\lim_{t\to0} \frac{T_t^* u - u}{t} = A^* u \qquad \text{(strong } L^{p'}\text{-limit)}. \tag{1.18}$$

However it is important to note that $(T_t^*)_{t\ge0}$ is in general *not* sub-Markovian.

It is easy to see that every sub-Markovian operator is *positivity preserving* in the sense that

$$u \ge 0 \quad \text{a.e. implies } T_t u \ge 0 \text{ a.e.} \tag{1.19}$$

Let $(T_t)_{t\ge0}$ be a strongly continuous contraction semigroup on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, which is positivity preserving and let $\big(A, D(A)\big)$ be its generator.

Theorem 1.2. *The strongly continuous contraction semigroup is positivity preserving if and only if*

$$\int_{\mathbb{R}^n} (Au)(u^+)^{p-1} \mathrm{d}x \leq 0 \tag{1.20}$$

for all $u \in D(A)$. Moreover, for $1 < p < \infty$, the adjoint of a positivity preserving strongly continuous contraction semigroup on $L^p(\mathbb{R}^n)$ is a positivity preserving semigroup on $L^{p'}(\mathbb{R}^n)$.

Our aim is to learn more about the structure of operators satisfying (1.8) or (1.20), respectively. We start with

Definition 1.3. *Let $\big(A, C_0^\infty(\mathbb{R}^n)\big)$ be a linear operator with range in $C(\mathbb{R}^n)$. We say that A satisfies the* positive maximum principle *if for $u \in C_0^\infty(\mathbb{R}^n)$*

$$u(x_0) = \sup_{x \in \mathbb{R}^n} u(x) \geq 0 \quad \textit{implies } Au(x_0) \leq 0. \tag{1.21}$$

The following result is due to R. Schilling [17]:

Theorem 1.4. *Let A be a linear operator on $C_0^\infty(\mathbb{R}^n)$ mapping $C_0^\infty(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, $p > 1$. If A satisfies the positive maximum principle then the following assertions are equivalent*

$$\int_{\mathbb{R}^n} (Au)(u^+)^{p-1} \mathrm{d}x \leq 0 \quad , u \in C_0^\infty(\mathbb{R}^n), \tag{1.22}$$

and

$$\int_{\mathbb{R}^n} (Au)\big((u-1)^+\big)^{p-1} \mathrm{d}x \leq 0, u \in C_0^\infty(\mathbb{R}^n). \tag{1.23}$$

If these assertions hold, then $\big(A, C_0^\infty(\mathbb{R}^n)\big)$ is closable as an operator in $L^p(\mathbb{R}^n)$ and (1.22) as well as (1.23) holds also for the closure.

We want to understand more the structure of operators satisfying the positive maximum principle.

Definition 1.5. *Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a function. We call ψ a* negative definite function *if*

$$\psi(0) \geq 0 \tag{1.24}$$

and

$$\xi \mapsto e^{-t\psi(\xi)} \quad \textit{is positive definite for all } t > 0. \tag{1.25}$$

We are only interested in continuous negative definite functions and they are characterised by the *Lévy-Khinchin formula*:

Theorem 1.6. *A continuous function $\psi : \mathbb{R}^n \to \mathbb{C}$ is negative definite if and only if there exists a constant $C \geq 0$, a vector $d = (d_1, \dots, d_n) \in \mathbb{R}^n$, a positive semidefinite matrix $(a_{kl})_{k,l=1,\dots,n}$, i.e. $a_{kl} = a_{lk}$ and $\sum_{k,l=1}^n a_{kl}\xi_k\xi_l \geq 0$ for all $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, and a measure ν on $\mathbb{R}^n \backslash \{0\}$ satisfying $\int_{\mathbb{R}^n\backslash\{0\}} (1 \wedge |y|^2)\nu(\mathrm{d}y) < \infty$, such that*

$$\begin{aligned}\psi(\xi) =& c + i\sum_{j=1}^n d_j\xi_j + \sum_{k,l=1}^n a_{kl}\xi_k\xi_l \\ &+ \int_{\mathbb{R}^n\backslash\{0\}} \left(1 - e^{-iy\cdot\xi} - \frac{iy\xi}{1+|y|^2}\right)\nu(\mathrm{d}y).\end{aligned} \tag{1.26}$$

Here are some examples of continuous negative definite functions:

$$\xi \mapsto |\xi|^{2\alpha}, 0 < \alpha \leq 1; \tag{1.27}$$

$$\xi \mapsto 1 - e^{-s\xi}, s > 0, n = 1; \tag{1.28}$$

$$\xi \mapsto (i\xi)^\alpha, 0 < \alpha \leq 1, n = 1; \tag{1.29}$$

$$\xi \mapsto -\ln\left(\frac{1}{K_1(1)}\frac{K_1(\sqrt{1+|\xi|^2})}{\sqrt{1+|\xi|^2}}\right), K_1 \text{ Bessel function;} \tag{1.30}$$

$$\xi \mapsto \ln(1+\xi^2) + i\arctan\xi, n = 1. \tag{1.31}$$

We refer to [11] and [13] for more examples.

An important result due to Ph. Courrège [3] links continuous negative definite functions and operators satisfying the positive maximum principle.

Theorem 1.7. *Let $A : C_0^\infty(\mathbb{R}^n) \to C(\mathbb{R}^n)$ satisfy the positive maximum principle. Then there exists a function $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ which is measurable and locally bounded such that $\xi \mapsto q(x, \xi)$ is for every $x \in \mathbb{R}^n$ a continuous negative definite function and it holds*

$$Au(x) = -q(x, D)u(x) := -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}q(x,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi \tag{1.32}$$

where $\widehat{u}(\xi) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)\,\mathrm{d}x$ denotes the Fourier transform of u.

Using the Lévy-Khinchin formula we can rewrite (1.32) as

$$\begin{aligned}Au(x) =& -c(x)u(x) + \sum_{j=1}^n d_j(x)\frac{\partial u(x)}{\partial x_j} + \sum_{k,l=1}^n a_{kl}(x)\frac{\partial^2 u(x)}{\partial x_k \partial x_l} \\ &+ \int_{\mathbb{R}^n\backslash\{0\}} \left(u(x+y) - u(x) + \sum_{j=1}^n \frac{y_j}{1+|y|^2}\frac{\partial u(x)}{\partial x_j}\right)\nu(x, \mathrm{d}y)\end{aligned} \tag{1.33}$$

with continuous function $c : \mathbb{R}^n \to \mathbb{R}$, $c(x) \geq 0$, $d_j : \mathbb{R}^n \to \mathbb{R}$, $a_{kl} : \mathbb{R}^n \to \mathbb{R}$, $a_{kl}(x) = a_{lk}(x)$ and $\sum_{k,l=1}^n a_{kl}(x)\xi_k\xi_l \geq 0$, and a kernel $\nu(x, \mathrm{d}y)$ integrating for each $x \in \mathbb{R}^n$ the function $y \mapsto 1 \wedge |y|^2$.

Whereas it is possible to prove that a generator of a *Feller semigroup* $(T_t)_{t\geq 0}$, i.e. a positivity preserving strongly continuous contraction semigroup on $C_\infty(\mathbb{R}^n)$ must have (under mild assumptions) a generator of type (1.32) or (1.33), such a result does not hold, or is at least unknown, for generators of L^p-sub-Markovian semigroups. However, there are strong indications that many operator of type (1.32) or (1.33) extend to generators of L^p-sub-Markovian semigroups and hence we will often decide to work with such operators also in the L^p-context. One central open question in the theory is: Given a continuous *negative definite symbol*, i.e. a function $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ which is continuous and for every $x \in \mathbb{R}^n$ the mapping $\xi \mapsto q(x, \xi)$ is a continuous negative definite function, find conditions such that $-q(x, D)$ with domain $C_0^\infty(\mathbb{R}^n)$ (or $\mathcal{S}(\mathbb{R}^n)$) extends to a generator of an L^p-sub-Markovian semigroup.

A related problem is whether we can determine $D(A)$, the domain of the closure (if existing) of $\big(-q(x, D), C_0^\infty(\mathbb{R}^n)\big)$ (or $(-q(x, D), \mathcal{S}(\mathbb{R}^n))$) in $L^p(\mathbb{R}^n)$. In case of uniformly elliptic second order differential operators with smooth coefficients the domain is of course known and it is the Sobolev space $W^{2,p}(\mathbb{R}^n) = H^{2,p}(\mathbb{R}^n)$. Note that $W^{2,p}(\mathbb{R}^n)$ or $H^{2,p}(\mathbb{R}^n)$ are defined using the Laplace operator and for a general second order uniformly elliptic operator $L(x, D)$ the result is obtained by using estimates of type

$$\|L(x, D)u\|_{L^p} + \|u\|_{L^p} \sim \|\Delta u\|_{L^p} + \|u\|_{L^p}. \tag{1.34}$$

Thus a first idea to get some progress in the study of operator such as (1.32) is to introduce suitable constant coefficient operators and to start to study these operators. Of course, the reasonable choice is to study operators of type

$$\psi(D)u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)\widehat{u}(\xi)\,\mathrm{d}\xi, \tag{1.35}$$

where $\psi : \mathbb{R}^n \to \mathbb{R}$ is a real-valued continuous negative definite function. The semigroup generated by $-\psi(D)$ has on $\mathcal{S}(\mathbb{R}^n)$ the representation

$$T_t u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t\psi(\xi)}\widehat{u}(\xi)\,\mathrm{d}\xi. \tag{1.36}$$

Using the fact that every continuous negative definite function corresponds to a unique convolution semigroup $(\mu_t)_{t\geq 0}$ of (sub-)probability measures μ_t on $\mathbb{R}^n$ by the relation

$$\widehat{\mu}_t(\xi) = (2\pi)^{-\frac{n}{2}} e^{-t\psi(\xi)}, \tag{1.37}$$

the convolution theorem yields

$$T_t u(x) = \int_{\mathbb{R}^n} u(x - y)\mu_t(\mathrm{d}y) \tag{1.38}$$

for all $u \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.

2 Function Spaces Associated with $\psi(D)$. Part I

As seen in the last section we need to construct and to investigate function spaces associated with continuous negative definite functions. We will start with such an investigation in this section which is based on our joint work [4] and [5] with W. Farkas and R. Schilling, compare also [12].

In case $p = 2$ and the operator $-\psi(D)$ originally defined on $\mathcal{S}(\mathbb{R}^n)$ by

$$Au(x) = -\psi(D)u(x) = -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)\widehat{u}(\xi)\mathrm{d}\xi \tag{2.1}$$

with a real-valued continuous negative definite function as symbol it is easy to determine the domain of its L^2-closure. An application of Plancherel's theorem gives for the graph norm of A

$$\|u\|_{L^2}^2 + \|Au\|_{L^2}^2 = \int_{\mathbb{R}^n} \big(1+\psi(\xi)^2\big)|\widehat{u}(\xi)|^2\mathrm{d}\xi, \tag{2.2}$$

and since

$$1+\psi(\xi)^2 \sim \big(1+\psi(\xi)\big)^2 \tag{2.3}$$

we find with

$$\|u\|_{H^{\psi,2}}^2 := \int_{\mathbb{R}^n} \big(1+\psi(\xi)\big)^2|\widehat{u}(\xi)|^2\,\mathrm{d}\xi \tag{2.4}$$

that

$$\|u\|_{L^2}^2 + \|Au\|_{L^2}^2 \sim \|u\|_{H^{\psi,2}}^2. \tag{2.5}$$

Introducing the scale

$$H^{\psi,s}(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n)\,;\ \big(1+\psi(\cdot)\big)^{\frac{s}{2}}\widehat{u} \in L^2(\mathbb{R}^n)\} \tag{2.6}$$

and the norms

$$\|u\|_{H^{\psi,s}} = \big\|\big(1+\psi(\cdot)\big)^{\frac{s}{2}}\widehat{u}(\cdot)\big\|_{L^2} = \|F^{-1}\big((1+\psi(\cdot)\big)^{\frac{s}{2}}\widehat{u}\|_{L^2} \tag{2.7}$$

we can prove that the domain of the L^2-closure of $-\psi(D)$ is given by $H^{\psi,2}(\mathbb{R}^n)$.

The obvious and correct generalisation to an L^p-setting is to introduce the norms

$$\|u\|_{H_p^{\psi,s}} = \|F^{-1}\big((1+\psi(\cdot)\big)^{\frac{s}{2}}\widehat{u})\|_{L^p}. \tag{2.8}$$

Note that the norms

$$\|u\|_{\psi,s,p} := \|(1+\psi(\cdot))^{\frac{s}{2}}\widehat{u}\|_{L^p} \tag{2.9}$$

also give rise to a scale of function spaces, the spaces $B^s_{\psi,p}(\mathbb{R}^n)$, which were investigated in [11], Section 3.10. These spaces are analogue to the Hörmander spaces $B_{k,p}(\mathbb{R}^n)$, have their merits for regularity investigations but are not suited for existence results or to determine domains of closed extensions. The exception if of course the case $p = 2$ where due to Plancherel's theorem (2.9) and (2.8) coincide.

Since continuous negative definite functions have at most quadratic growth, (2.8) is well-defined on $\mathcal{S}(\mathbb{R}^n)$. Extending the expression (2.8) to certain non-smooth $u \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, is by no means trivial: $\psi : \mathbb{R}^n \to \mathbb{R}$ need not be differentiable, multiplying $\widehat{u}$ for $u \in L^p(\mathbb{R}^n)$ with ψ is hence not straightforward, and even if possible the inverse Fourier transform of the product need not belong to $L^p(\mathbb{R}^n)$ – at least this is not obvious. As worked out in detail in [5] we can use the Lévy-Khinchin representation

$$\psi(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos y \cdot \xi)\nu(\mathrm{d}y) \tag{2.10}$$

in order to give a meaning to (2.9) for $u \notin \mathcal{S}(\mathbb{R}^n)$. The essential idea is that for $R > 0$

$$\psi_R(\xi) := \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos y \cdot \xi)\chi_{B_R(0)}(y)\nu(\mathrm{d}y) \tag{2.11}$$

is a smooth function and

$$\widetilde{\psi}_R(\xi) := \psi(\xi) - \psi_R(\xi) \tag{2.12}$$

is a bounded function. Hence we have for $u \in L^p(\mathbb{R}^n)$ the possibility to define (in the sense of tempered distributions)

$$\psi(D)u := F^{-1}(\psi_R \widehat{u}) + \int_{B^c_R(0)} \big(u(\cdot + y) - u(\cdot)\big)\nu(\mathrm{d}y). \tag{2.13}$$

Now we may consider

$$\|u\|_{H^{\psi,R,p}} := \|(\mathrm{id} + \psi_R(D))u\|_{L^p}. \tag{2.14}$$

For different values of R we will obtain equivalent norms and for $u \in \mathcal{S}(\mathbb{R}^n)$ it holds

$$\|(\mathrm{id} + \psi(D))u\|_{L^p} \sim \|u\|_{\psi,R,p} \tag{2.15}$$

as well as by the definition of $\mathrm{id} + \psi(D)$

$$\|F^{-1}((1 + \psi(\cdot))\widehat{u})\|_{L^p} \sim \|u\|_{\psi,R,p}. \tag{2.16}$$

Omitting longer considerations, compare [5], it is possible to prove that for the L^p-extension $A^{(p)}$ of $\big(-\psi(D), \mathcal{S}(\mathbb{R}^n)\big)$ we have

$$D(\mathrm{id} - A^{(p)}) = H^{\psi,2}_p(\mathbb{R}^n) \tag{2.17}$$

where $H^{\psi,2}_p(\mathbb{R}^n)$ is defined as the space of all $u \in L^p(\mathbb{R}^n)$ for which (2.14) is finite. Sometimes we will use the not quite correct expression $\|F^{-1}((1 + \psi(\cdot))\widehat{u}(\cdot))\|_{L^p}$ for all elements in $H^{\psi,2}_p(\mathbb{R}^n)$. This allows us to give a short definition of the scale $H^{\psi,s}_p(\mathbb{R}^n)$ by introducing the norms

$$\|u\|_{H^{\psi,s}_p} := \|F^{-1}((1 + \psi(\cdot))^{\frac{s}{2}}\widehat{u}(\cdot))\|_{L^p}. \tag{2.18}$$

The spaces $H_p^{\psi,s}(\mathbb{R}^n)$ are Banach spaces. We may try to apply the general machinery of the theory of function spaces to this scale. However, the non-smoothness and the an-isotropy of ψ cause a larger problem. Here are some results. For a given continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ we have the continuous embeddings

$$H_p^{\psi,s+t}(\mathbb{R}^n) \hookrightarrow H_p^{\psi,s}(\mathbb{R}^n \hookrightarrow L^p(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \tag{2.19}$$

where $s, t \geq 0$. More general embedding results follow from a Fourier multiplier theorem. By definition $m \in \mathcal{S}'(\mathbb{R}^n)$ is a *Fourier multiplier of type* (p, q), $m \in M_{p,q}$, if

$$\sup \left\{ \frac{\|F^{-1}(m\widehat{\varphi})\|_{L^q}}{\|\varphi\|_{L^p}} \, ; \, 0 \neq \varphi \in \mathcal{S}(\mathbb{R}^n) \right\} < \infty. \tag{2.20}$$

It is known that $M_{2,2} = L^\infty(\mathbb{R}^n)$ and $F(L^1) \subset M_{p,p}$, $1 < p < \infty$, where $F(L^1)$ denotes the image of $L^1(\mathbb{R}^n)$ under the Fourier transform. In [5] we proved

Theorem 2.1. *The embedding*

$$H_p^{\psi,s}(\mathbb{R}^n) \hookrightarrow C_\infty(\mathbb{R}^n) \text{ and } \|u\|_\infty \leq c\|u\|_{H_p^{\psi,s}} \tag{2.21}$$

holds if and only if $(1+\psi)^{-\frac{s}{2}} \in M_{p,\infty}$.

A sufficient criterium for (2.21) to hold is the estimate

$$1 + \psi(\xi) \geq c_0(1 + |\xi|^2)^{r_0}, r_0 > 0, c_0 > 0, \tag{2.22}$$

together with the condition

$$s > \frac{n}{r_0} \frac{p+1}{p}. \tag{2.23}$$

Note that in [6] W. Farkas and H.-G. Leopold considered function spaces of generalised smoothness which cover some of our spaces.

So far we have summarised the "typical" results expected from the theory of function spaces. The positive maximum principle satisfied by $-\psi(D)$, or equivalently the fact that the associated semigroup is sub-Markovian allows us to introduce into our considerations capacities. We will first discuss the general setting, Section 3, and return to the spaces $H_p^{\psi,s}(\mathbb{R}^n)$ in Section 4.

3 On Capacities Associated with L^p-Sub-Markovian Semigroups

Let $(T_t^{(p)})_{t\geq 0}$, $1 < p < \infty$, be an L^p-sub-Markovian semigroup on $L^p(\mathbb{R}^n)$ with generator $\big(A^{(p)}, D(A^{(p)})\big)$. For $r \geq 0$ we define the Γ-transform of $(T_t^{(p)})_{t\geq 0}$ by

$$V_r^{(p)}u := \frac{1}{\Gamma(\frac{r}{2})} \int_0^\infty t^{\frac{r}{2}-1} e^{-t} T_t^{(p)} u dt, \; u \in L^p(\mathbb{R}^n). \tag{3.1}$$

Note that $(V_r^{(p)})_{r\geq 0}$ is a subordinate semigroup, compare [12] for details. In particular $(V_r^{(p)})_{r\geq 0}$ is itself an L^p-sub-Markovian semigroup. With $(V_r^{(p)})_{r\geq 0}$ we can associate a generalised Bessel potential space by

$$\mathcal{F}_{r,p} := V_r^{(p)}\big(L^p(\mathbb{R}^n)\big) \tag{3.2}$$

with norm

$$\|u\|_{\mathcal{F}_{r,p}} = \|f\|_{L^p} \quad \text{where } u = V_r^{(p)} f. \tag{3.3}$$

The space $\big(\mathcal{F}_{r,p}, \|.\|_{\mathcal{F}_{r,p}}\big)$ is a Banach space and it holds

$$\mathcal{F}_{r,p} = D\big((\mathrm{id} - A^{(p)})^{\frac{r}{2}}\big), \tag{3.4}$$

for the latter compare [4] or [5]. Moreover we have

$$V_r^{(p)} = (\mathrm{id} - A^{(p)})^{-\frac{r}{2}}, \tag{3.5}$$

and it holds

$$\|u\|_{\mathcal{F}_{r,p}} = \|(\mathrm{id} - A^{(p)})^{-\frac{r}{2}} f\|_{L^p},\ u = V_r^{(p)} f. \tag{3.6}$$

One can show that $(\mathrm{id} - A^{(p)})^{\frac{r}{2}} : \mathcal{F}_{r,p} \to L^p(\mathbb{R}^n)$ is a bijective continuous operator with a continuous inverse given by $V_r^{(p)}$. It holds

$$\mathcal{F}_{r,p} = \{u \in L^p(\mathbb{R}^n)\ ;\ \|(\mathrm{id} - A^{(p)})^{\frac{r}{2}} u\|_{L^p} < \infty\} \tag{3.7}$$

as well as

$$\|u\|_{\mathcal{F}_{r,p}} = \|(\mathrm{id} - A^{(p)})^{\frac{r}{2}} u\|_{L^p},\ u \in \mathcal{F}_{r,p}. \tag{3.8}$$

For an open set $G \subset \mathbb{R}^n$ we define

$$C_{r,p}(G) := \{u \in \mathcal{F}_{r,p}; u \geq 1 \text{ a.e. on } G\} \tag{3.9}$$

and

$$\mathrm{cap}_{r,p}(G) := \inf\{\|u\|^p_{\mathcal{F}_{r,p}}; u \in C_{r,p}(G)\}. \tag{3.10}$$

If we define for an arbitrary set $A \subset \mathbb{R}^n$

$$\mathrm{cap}_{r,p}(A) := \inf\{\mathrm{cap}_{r,p}(G)\ ;\ A \subset G, G \text{ open}\} \tag{3.11}$$

then $\mathrm{cap}_{r,p}$ becomes a Choquet capacity.

Theorem 3.1. *For $2 < p < \infty$, $G \subset \mathbb{R}^n$ open, $\mathrm{cap}_{r,p}(G) < \infty$, there exists a unique $u_G \in \mathcal{F}_{r,p}$ such that $u_G \geq 1$ a.e. on G, and*

$$\mathrm{cap}_{r,p}(G) = \|u_G\|_{\mathcal{F}^p_{r,p}}. \tag{3.12}$$

Moreover, there exists $f \in L^p(\mathbb{R}^n)$, $f \geq 0$, a.e. such that

$$u_G = V_r^{(p)} f. \tag{3.13}$$

Definition 3.2. *The unique function u_G in Theorem 3.1 is called the* (r,p)-equilibrium potential *of the set G (with respect to* $(T_t^{(p)})_{t\geq 0}$.

Our aim is to characterise equilibrium potentials. Here we have a kind of dual approach as in the paper [16] of M. Rao and Z. Vondraček.

We introduce on $\mathcal{F}_{r,p}$ the L^p-energy

$$E_{r,p}(u) := \frac{1}{p}\|u\|^p_{\mathcal{F}_{r,p}} = \frac{1}{p}\int_{\mathbb{R}^n} \left|(\mathrm{id} - A^{(p)})^{\frac{r}{2}} u\right|^p \mathrm{d}x. \tag{3.14}$$

The Gâteaux derivative of $E_{r,p}$ is given by the operator

$$\mathcal{A}_r^{(p)} : \mathcal{F}_{r,p} \to \mathcal{F}^*_{r,p} \tag{3.15}$$

with

$$\mathcal{A}_r^* u = (\mathrm{id} - A^{(p)*})^{\frac{r}{2}} \left(|(\mathrm{id} - A^{(p)})^{\frac{r}{2}} u|^{p-2} (\mathrm{id} - A^{(p)})^{\frac{r}{2}} u\right). \tag{3.16}$$

Recall that $A^{(p)}$ could be a pseudo-differential operator with negative definite symbol. The following results are taken from our joint paper with W. Hoh [10].

Theorem 3.3. *The equilibrium potential u_G is the unique solution to the variational inequality*

$$\left\langle p\mathcal{A}_r^{(p)} u_G, \varphi - u_G\right\rangle \geq 0 \quad \textit{for all } \varphi \in C_{r,p}(G). \tag{3.17}$$

Our aim is to bring the L^p-theory as close to the theory of Dirichlet forms, compare M. Fukushima et al. [8], as possible. For this we introduce the *mutual energy* of two elements in $\mathcal{F}_{r,p}$ as

$$\mathcal{E}_r^{(p)} : \mathcal{F}_{r,p} \times \mathcal{F}_{r,p} \to \mathbb{R} \tag{3.18}$$

with

$$\mathcal{E}_r^{(p)}(u,v) := \int_{\mathbb{R}^n} \left|(\mathrm{id} - A^{(p)})^{\frac{r}{2}} u\right|^{p-2} (\mathrm{id} - A^{(p)})^{\frac{r}{2}} u (\mathrm{id} - A^{(p)})^{\frac{r}{2}} v \mathrm{d}x. \tag{3.19}$$

Note that $p = 2$ and $r = 1$ gives

$$\mathcal{E}_1^{(2)}(u,v) = \left((\mathrm{id} - A^{(2)})^{\frac{1}{2}} u, (\mathrm{id} - A^{(2)})^{\frac{1}{2}} v\right)_{L^2} \tag{3.20}$$

and for $u = v \in \mathcal{F}_{1,2}$ it follows

$$\mathcal{E}_1^{(2)}(u,u) \approx \left((-A^{(2)})^{\frac{1}{2}} u, (-A^{(2)})^{\frac{1}{2}} u\right)_{L^2} + (u,u)_{L^2}. \tag{3.21}$$

Combining (3.19) with (3.17) we find

Corollary 3.4. *The equilibrium potential u_G is the unique solution to*

$$\mathcal{E}_r^{(p)}(u_G, \varphi - u_G) \geq 0 \quad \textit{for all } \varphi \in C_{r,p}(G). \tag{3.22}$$

Finally we want to achieve $u_G = 1$ a.e. on G. This however can not hold in general, we need a further assumption on $\mathcal{F}_{r,p}$.

Definition 3.5. *We say that the* truncation property *holds in* $\mathcal{F}_{r,p}$ *if for every Lipschitz mapping* $T : \mathbb{R} \to \mathbb{R}$ *with Lipschitz constant* 1*, i.e.*

$$|Tx - Ty| \leq |x - y|,$$

the following two conditions are fulfilled.

$$u \in \mathcal{F}_{r,p} \text{ implies } T \circ u \in \mathcal{F}_{r,p} \tag{3.23}$$

and

$$\|T \circ u\|_{\mathcal{F}_{r,p}} \leq \|u\|_{\mathcal{F}_{r,p}}. \tag{3.24}$$

Note that the condition of Definition 3.5 is often described as "the normal contraction operates on $\mathcal{F}_{r,p}$".

From classical Sobolev space theory, i.e. the Bessel potential spaces associated with the Brownian semigroup, we know that the truncation property does in general not hold. For example it does not hold in $H^k(\mathbb{R}^n) = \mathcal{F}_{k,2}$ for $k \geq 2$.

Theorem 3.6. *Suppose that* $\mathcal{F}_{r,p}$ *satisfies the truncation property. Then the equilibrium potential* u_G *is the unique element in* $\mathcal{F}_{r,p}$ *satisfying* $u_G = 1$ *a.e. on* G *and it holds*

$$\mathcal{E}_r^{(p)}(u_G, v) \geq 0 \quad \text{for all } v \in \mathcal{F}_{r,p} \text{ and } v \geq 0 \text{ a.e. on } G. \tag{3.25}$$

Moreover we have

$$\mathcal{E}_r^{(p)}(u_G, v) = \mathrm{cap}_{r,p}(G) \tag{3.26}$$

for all $v \in \mathcal{F}_{r,p}$ *with* $v = 1$ *a.e. on* G*.*

This theorem gives a characterisation of equilibrium potentials completely analogous to that in the theory of Dirichlet forms.

Using capacities we can refine the notion of Lebesgue exceptional sets, i.e. sets of (Lebesgue) measure zero. We need a few definitions.

Definition 3.7. *Let* $(T_t^{(p)})_{t\geq 0}$ *be an* L^p*-sub-Markovian semigroup.*
A. We call a set $N \subset \mathbb{R}^n$ *an* (r,p)-exceptional set *(with respect to* $(T_t^{(p)})_{t\geq 0}$*) if* $\mathrm{cap}_{r,p}(N) = 0$*.*
B. A statement is said to hold (r,p)-quasi everywhere *(q.e.) if there exists an* (r,p)*-exceptional set* N *such that the statement holds in* N^c*.*
C. Let $u : \mathbb{R}^n \to \mathbb{C}$ *be a function. We call* u (r,p)-quasi-continuous *if for every* $\varepsilon > 0$ *there exists an open set* $G_\varepsilon \subset \mathbb{R}^n$ *such that* $\mathrm{cap}_{r,p}(G_\varepsilon) < \varepsilon$ *and* $u|_{G_\varepsilon^c}$ *is continuous.*
D. Let $u \in \mathcal{F}_{r,p}$*. We call* $\widetilde{u} \in \mathcal{F}_{r,p}$ *an* (r,p)-quasi continuous modification *of* u *if* $\widetilde{u}$ *is* (r,p)*-quasi continuous and* $\widetilde{u} = u$ *a.e.*

Theorem 3.8. *If* $\mathcal{F}_{r,p} \cap C_0(\mathbb{R}^n)$ *is dense in* $\mathcal{F}_{r,p}$ *then every* $u \in \mathcal{F}_{r,p}$ *admits a quasi continuous modification.*

Note that by Theorem 3.8 it follows that the statement "$\mathrm{cap}_{r,p}(A) = 0$ implies $A = \emptyset$" has as consequence that every element in $\mathcal{F}_{r,p}$ has a (unique) continuous representative.

We will now apply these results to the spaces $H_p^{\psi,r}(\mathbb{R}^n)$.

4 Function Spaces Associated with $\psi(D)$. Part II.

Let us now return to the space $H_p^{\psi,r}(\mathbb{R}^n)$ already introduced in Section 2. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a real-valued continuous negative definite function. The operator $-\psi(D)$ originally defined on $\mathcal{S}(\mathbb{R}^n)$ extends to a generator of an L^p-sub-Markovian semigroup $(T_t^{\psi,p})_{t\geq 0}$ and on $\mathcal{S}(\mathbb{R}^n)$ this semigroup is given by

$$T_t^{\psi,p}u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t\psi(\xi)} \widehat{u}(\xi)\,\mathrm{d}\xi \tag{4.1}$$

compare (1.31). Now, as proved in [5] the Bessel potential space $\mathcal{F}_{r,p}^{\psi}$ associated with $(T_t^{\psi,p})_{t\geq 0}$ is the space $H_p^{\psi,r}(\mathbb{R}^n)$,

$$\mathcal{F}_{r,p}^{\psi} = H_p^{\psi,r}(\mathbb{R}^n), \tag{4.2}$$

and for smooth functions we find of course

$$\|u\|_{\mathcal{F}_{r,p}} = \|(\mathrm{id} + \psi(D))^{\frac{r}{2}} u\|_{L^p} = \|F^{-1}\big((1+\psi(\cdot))^{\frac{r}{2}}\big)\widehat{u}(\cdot)\|_{L^p} \tag{4.3}$$

Thus we can transfer the notions of the capacity $\mathrm{cap}_{r,p}$, (r,p)-exceptional sets or (r,p)-equilibrium potentials to the spaces $H_p^{\psi,r}(\mathbb{R}^n)$. In particular every $u \in H_p^{\psi,r}(\mathbb{R}^n)$ has an (r,p)-quasi continuous modification $\widetilde{u}$. Let us first state some interesting results in the context of $H_p^{\psi,r}$-spaces which hold more generally, compare M. Fukushima [7].

Theorem 4.1. *A. Each $u \in H_p^{\psi,r}(\mathbb{R}^n)$ admits an (r,p)-quasi continuous modification $\widetilde{u}$ which is unique up to (r,p)-quasi everywhere equality. Moreover, for $\rho > 0$ the following Chebyshev-type inequality*

$$\mathrm{cap}_{r,p}(\{|\widetilde{u}| > \rho\}) \leq \frac{2}{\rho^p}\|u\|^p_{H^{\psi,r}} \tag{4.4}$$

holds.
B. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $H_p^{\psi,r}(\mathbb{R}^n)$ converging to $u \in H_p^{\psi,r}(\mathbb{R}^n)$. Then there exists a subsequence $(u_{k_l})_{l\in\mathbb{N}}$ such that

$$\lim_{l\to\infty} u_{kl}(x) = \widetilde{u}(x) \qquad (r,p)\text{-quasi everywhere.} \tag{4.5}$$

We have seen already (and more will come below) that the truncation property is essential in order to get improved properties of equilibrium potentials. It turns out that a general necessary and sufficient result for the truncation property to hold in $H_p^{\psi,r}(\mathbb{R}^n)$ is difficult to establish. Just to indicate one of the problems consider the two continuous negative definite functions $\psi_1(\xi) = |\xi|^2$ and $\psi_2(\xi) = |\xi|$. For simplicity take $p = 2$, i.e. we may use Plancherel's theorem. Clearly it holds $H^{\psi_1,s}(\mathbb{R}^n) = H^{\psi_2,2s}(\mathbb{R}^n)$. In the first case we are tempted to take $s \leq 1$ in order to prove the truncation property, but in the second case we then should choose $s \leq 2$ in order to prove the truncation property in $H^{\psi_2,s}$. (But see also below for a surprise). This problem extends obviously to any pairing ψ and ψ^{α}, $0 < \alpha < 1$. In his work on function spaces related to Dirichlet forms, F. Hirsch, compare [9] obtained some results which combined with observations of D. Bakry [2] leads to

Theorem 4.2. *For any continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$, $0 < s < 1$, and $p \geq 2$ the Lipschitz functions operate on the space $H_p^{\psi,s}(\mathbb{R}^n)$.*

The fact that the truncation property holds in $H_p^s(\mathbb{R}^n)$ for $1 < p < \infty$ and $0 \leq s \leq 1$ is not too difficult to see, at least in the case $p = 2$. An interesting result due to H. Triebel [18], see also [19], tells us that s can be larger than 1! The precise result is proved for Triebel-Lizorkin spaces $F_{p,q}^s$, note that $H_p^s = F_{p,2}^s$, and reads as follows:

A. The truncation property holds in $F_{p,q}^s$ for $0 < s < 1$, $1 < p$, $q < \infty$

B. The Lipschitz functions $x \mapsto |x|$ and $x \mapsto \frac{1}{2}\big(|x| - x\big)$ (and some others) operate on $F_{p,q}^s$ and $B_{p,q}^s$ for $0 < s < 1 + \frac{1}{p}$. (Here $B_{p,q}^s$ denotes of course the Besov spaces.)

Note that the proof of Theorem 3.6 uses the Lipschitz function $t \mapsto (0 \vee t) \wedge 1$. At least when working in bounded domains Triebel's result will be of some use for our purposes.

5 Further Capacities

L^p-potential theory owes much to pioneering work of V. G. Maz'ya and V. P. Havin, see [15], as well as to the efforts of D. Adams and the late L. I. Hedberg, see [1]. In their considerations integral representations of potentials (including equilibrium potentials) are central and these representations may lead to different notions of capcities. The kernels under considerations are of course kernels representing the operators $V_r^{(p)}$. In [14] R. Schilling and the author discussed the existence of such presentations in more detail.

Define for an open set $G \subset \mathbb{R}^n$

$$\mathrm{kap}_{r,p}(G) := \inf\left\{\|u\|_{L^p}^p\,;\ u \in L^p(\mathbb{R}^n), u \geq 0 \text{ and } V_r^{(p)}u \geq \chi_G \text{ a.e.}\right\} \tag{5.1}$$

where χ_G denotes the characteristic function of the set G. We may extend $\mathrm{kap}_{r,p}$ to a Choquet capacity by setting

$$\mathrm{kap}_{r,p}(A) := \inf\left\{\mathrm{kap}_{r,p}(G)\,;\ G \supset A, G \text{ open}\right\}. \tag{5.2}$$

One can prove that

$$\mathrm{cap}_{r,p}(A) = \mathrm{kap}_{r,p}(A). \tag{5.3}$$

Next, let $K \subset \mathbb{R}^n$ be a compact set and put

$$L_{r,p}(K) := \{f \in \mathcal{F}_{r,p} \cap C_b\,;\ f = 1 \text{ in a neighbourhood of } K\} \tag{5.4}$$

and

$$N_{r,p}(K) := \inf\left\{\|f\|_{\mathcal{F}_{r,p}}^p\,;\ f \in L_{r,p}(K)\right\} \tag{5.5}$$

Extend $N_{r,p}$ to an outer capacity by defining for an open set G

$$N_{r,p}(G) := \sup\{N_{r,p}(K)\,;\ K \subset G, K \text{ compact}\}, \tag{5.6}$$

and finally for an arbitrary set $A \subset \mathbb{R}^n$ we get a Choquet capacity by

$$N_{r,p}(A) := \inf\{N_{r,p}(G) \, ; \, G \supset A, G \text{ open}\}. \tag{5.7}$$

It is easy to see that

$$\text{kap}_{r,p}(K) = \text{cap}_{r,p}(K) \leq N_{r,p}(K). \tag{5.8}$$

However, in general $\text{kap}_{r,p}/\text{cap}_{r,p}$ are not equivalent to $N_{r,p}$. Again the truncation property is needed. Adapting the proof of Corollary 3.3.4 in D. Adams and L. F. Hedberg [1] in [14] we proved

Proposition 5.1. *Suppose that the Lipschitz functions operate on $\mathcal{F}_{r,p}$. Then the capacities $\text{cap}_{r,p}$ and $N_{r,p}$ are equivalent, i.e.*

$$\text{cap}_{r,p}(A) \leq N_{r,p}(A) \leq c \, \text{cap}_{r,p}(A) \tag{5.9}$$

for all $A \subset \mathbb{R}^n$. If the normal contraction operators on $\mathcal{F}_{r,p}$ i.e. the truncation property holds, then we have in (5.9) *equality.*

In [14] these considerations are now extended to measures using a duality theory which is in some sense complementary to that used by M. Rao and Z. Vondraček in [16].

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DYADIC PFW'S AND W_o-BASES

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The rich structure of the set of dyadic PFW's has been presented in [7]. As mentioned in [7, Remark 3.7], the subclass of the so called non-W_0-frames with various levels of W_0-linear independence (denoted by $\mathcal{P}_{0,+}$ in [7]) needs extra attention. The recent observation in [5] on W_0-Schauder bases enables us to properly restructure this subclass ($\mathcal{P}_{0,+}$) and to show that it contains a wealth of examples. This is the purpose of our article. Let us, however, begin by defining some of the notions mentioned above.

We denote by $\mathcal{P}$ the set of all *Parseval frame wavelets* (PFW's, for short), i.e., the set of functions $\psi \in L^2(\mathbb{R})$ such that the system

$$\{\psi_{jk}(x)\} := \{2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$$

forms a normalized (frame bounds 1) tight frame for $L^2(\mathbb{R})$. This is equivalent to having the reproducing property, i.e., for every $f \in L^2(\mathbb{R})$

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{jk}\rangle \psi_{jk} \tag{1}$$

unconditionally in $L^2(\mathbb{R})$. Interestingly enough, the property of being a PFW (on the entire $L^2(\mathbb{R})$) still allows a very varied behaviour on the main (or zeroeth) resolution level

$$W_0 := \overline{span}\{\psi_{0k} : k \in \mathbb{Z}\}. \tag{2}$$

If the family $\{\psi_{0k} : k \in \mathbb{Z}\}$ satisfies the property "*abc*" within W_0, we shall say that ψ is a "$W_0 - abc$" (for example, if it is a Riesz basis within W_0, we say that ψ is a W_0-Riesz basis). A very useful tool to study such properties is the periodization function $p_\psi : \mathbb{R} \to [0, +\infty)$, defined by

$$p_\psi(\xi) := \sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 \qquad , \xi \in \mathbb{R}\,,$$

where the Fourier transform is chosen so that

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}dx, \text{ for } f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

Recall that the Hilbert space $W_0 \subset L^2(\mathbb{R})$ is isometrically isomorphic to the space $L^2(\mathbb{T}; p_\psi)$ (the L^2 space on the torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ with the measure $p_\psi(\xi)\frac{d\xi}{2\pi}$; observe that p_ψ is 2π-periodic) via isomorphism

$$\mathcal{I}_\psi : L^2(\mathbb{T}; p_\psi) \to W_0\ ,$$

given by,

$$\mathcal{I}_\psi(t) := (t\hat{\psi})^\vee\ . \tag{3}$$

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If $\psi \in \mathcal{P}$, then (see [6])

$$p_\psi \leq 1 \qquad \text{a.e.} \tag{4}$$

Let us consider also an operator

$$\tilde{\mathcal{I}}_\psi : L^2(\mathbb{T}; \frac{d\xi}{2\pi}) \to L^2(\mathbb{T}; p_\psi) \ ,$$

given by $\tilde{\mathcal{I}}_\psi(t) = t$. It follows by (4) that $\tilde{\mathcal{I}}_\psi$ is a bounded linear operator.

Let us now introduce the main subclasses of $\mathcal{P}$ which are going to be of interest for us here (for a complete picture see [7]). In the following we shall assume that the reader is familiar with the notions of frame, normalized tight frame, biorthogonal sequence, Schauder basis, Riesz basis and orthonormal basis (see [3] and [8] for basic results and some further references).

It is shown in [7] that the class $\mathcal{P}_{\cdot,+}$, defined by

$$\mathcal{P}_{\cdot,+} := \Big\{\psi \in \mathcal{P} : ker(\tilde{\mathcal{I}}_\psi) = \{0\}\Big\} \ , \tag{5}$$

is the disjoint union

$$\mathcal{P}_{\cdot,+} = \mathcal{P}_{0,+} \cup \mathcal{P}_{f,+} \cup \mathcal{P}_{tf,+} \ , \tag{6}$$

where

$$\mathcal{P}_{0,+} = \{\psi \in \mathcal{P}_{\cdot,+} : \psi \text{ is not a } W_0\text{-frame}\}$$
$$\mathcal{P}_{f,+} = \{\psi \in \mathcal{P}_{\cdot,+} \backslash \mathcal{P}_{0,+} : \psi \text{ is not a } W_0\text{-normalized tight frame}\}$$
$$\mathcal{P}_{tf,+} = \{\psi \in \mathcal{P}_{\cdot,+} : \psi \text{ is a } W_0\text{-normalized tight frame}\}$$

Recall (see also Corollary 2) that $\mathcal{P}_{tf,+}$ is actually the set of orthonormal wavelets, while $\mathcal{P}_{f,+}$ is the set of non-semiorthogonal W_0-Risez basis. However, the class $\mathcal{P}_{0,+}$ has been touched only briefly in [7], where it was shown that $\mathcal{P}_{0,+}$ is non-empty. Following the most recent advances in [5], we are able to show that $\mathcal{P}_{0,+}$ has a much richer structure with various levels of linear independence (or basis) type conditions which naturally fit with other subclasses of $\mathcal{P}_{\cdot,+}$.

Remark. Parseval frames are orthogonal projections of orthonormal bases, and as such, are not necessarily linearly independent. However, in the special case of PFW's, there is more structure. First of all, for every $\psi \in \mathcal{P}$ the family $\{\psi_{0k} : k \in \mathbb{Z}\}$ is linearly independent. Suppose contrary that the family $\{e^{i\xi k} : k \in \mathbb{Z}\}$ (here we are using (3)) is linearly dependent in $L^2(\mathbb{T}; p_\psi)$. Since $\|\psi\|_2 > 0$, we have that the Lebesgue measure $|\{\xi \in \mathbb{T} : p_\psi(\xi) > 0\}| > 0$. Hence, without loss of generality, we would have that a constant function 1 is a.e. on a set of positive Lebesgue measure equal to a finite linear combination of non-constant exponentials $\{e^{i\xi k}\}$, which is not possible. Additionally, every frame (even Bessel sequence) can be partitioned into linearly independent sets [1]. Combining these two results, it is natural to conjecture that PFW's must already be linearly independent. This is also true, and will appear in future work. ∎

We can not expect more than linear independence of $\{\psi_{0k} : k \in \mathbb{Z}\}$ outside of $\mathcal{P}_{\cdot,+}$. However, inside $\mathcal{P}_{\cdot,+}$ we have a rich hierarchy of subclasses. Let us remind the reader (see [8] for more details) that a sequence $\{x_n : n \in \mathbb{N}\}$ in a Banach space is said to be *ℓ^2-linearly independent* if

$$\Big(\{\alpha_n\} \in \ell^2 \ , \ \sum_{n=1}^{\infty} \alpha_n x_n = 0 \ \Rightarrow \ \alpha_n = 0 \ , \forall n \in \mathbb{N}\Big) \ . \tag{7}$$

Theorem 1. *Supose that $\psi \in \mathcal{P}$. Then, the following are equivalent:*

(a) $\psi \in \mathcal{P}_{\cdot,+}$;
(b) $p_\psi > 0$ *a.e.* ;
(c) $\{\psi_{0k} : k \in \mathbb{Z}\}$ *is l^2-linearly independent in W_0.*

Observe that in (7), the tacit assumption is that $\lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n x_n$ exists in the Banach space, so the particular ordering of vectors matters. This will be the case in several other statements in this article. Here (Theorem 1.(c)), as throughout the article, we assume that $\mathbb{Z}$ is ordered as

$$\{0, 1, -1, 2, -2, ...\} \ . \tag{8}$$

Proof. The equivalence $(a) \Leftrightarrow (b)$ is more or less obvious (and is given already in [7]). Let us prove $(b) \Leftrightarrow (c)$.

Suppose first that $p_\psi > 0$ a.e., i.e., that $\ker(\tilde{\mathcal{I}}_\psi) = \{0\}$. Let us denote by $\{x_n : n \in \mathbb{N}\}$ the family $\{\psi_{0k} : k \in \mathbb{Z}\}$ ordered by (8). Suppose that $\{\alpha_n\} \in \ell^2$ and $\Sigma_{n=1}^{\infty}\alpha_n x_n = 0$. Let us denote by $\{e_n : n \in \mathbb{N}\}$ the family of exponentials $\{e^{i\xi k} : k \in \mathbb{Z}\}$ ordered by (8). Since $\tilde{\mathcal{I}}_\psi$ is a bounded linear operator, and $\tilde{\mathcal{I}}_\psi(e_n) = x_n$, we get $\tilde{\mathcal{I}}_\psi(\sum_{n=1}^{\infty} \alpha_n e_n) = 0$. Since the kernel of $\tilde{\mathcal{I}}_\psi$ is trivial, we get $\sum_{n=1}^{\infty} \alpha_n e_n = 0$ in $L^2(\mathbb{T}; \frac{d\xi}{2\pi})$. However, $\{e_n : n \in \mathbb{N}\}$ forms an orthonormal basis in $L^2(\mathbb{T}; \frac{d\xi}{2\pi})$, which implies $\alpha_n = 0\,, \forall n \in \mathbb{N}$.

Suppose now that the set $\{\xi : p_\psi(\xi) = 0\}$ has a positive Lebesgue measure. Observe that $h := 1_{\{p_\psi = 0\}}$ is a non-zero function in $L^2(\mathbb{T}; \frac{d\xi}{2\pi})$, but is a zero function in $L^2(\mathbb{T}; p_\psi)$. Hence, the Fourier coefficients of h are not trivial, i.e., there exists an ℓ^2-sequence $\{\alpha_n\}$, whose elements are not all equal to zero, such that $h = \sum_{n=1}^{\infty} \alpha_n e_n$ in $L^2(\mathbb{T}; \frac{d\xi}{2\pi})$. Since $\tilde{\mathcal{I}}_\psi$ is a bounded linear operator, we get that

$$0 = h = \tilde{\mathcal{I}}_\psi(h) = \sum_{n=1}^{\infty} \alpha_n x_n$$

in $L^2(\mathbb{T}; p_\psi)$. Hence, $\{x_n\}$ is not ℓ^2-linearly independent. Q.E.D.

If we combine this result with the recent note [5], we get a nice description of the structure of $\mathcal{P}_{\cdot,+}$. Let us remind the reader that a measurable, 2π-periodic function $w : \mathbb{R} \to (0, \infty)$ is an $\mathcal{A}_2$-weight (see [4] for more details) if there exists a constant $M > 0$ such that for every interval $I \subseteq \mathbb{R}$,

$$\left(\frac{1}{|I|}\int_I w(\xi)\, d\xi\right)\left(\frac{1}{|I|}\int_I \frac{1}{w(\xi)}\, d\xi\right) \le M\,,$$

where $|I|$ is the Lebesgue measure of I.

Corollary 2. *Let $\psi \in \mathcal{P}$. Then*

(a) $\psi \in \mathcal{P}_{\cdot,+}$ *if and only if* $\{\psi_{0k} : k \in \mathbb{Z}\}$ *is ℓ^2-linearly independent;*
(b) $\psi \in \mathcal{P}_{\cdot,+}$ *if and only if* $p_\psi > 0$ *a.e.* ;
(c) $\{\psi_{0k} : k \in \mathbb{Z}\}$ *belongs to a biorthogonal sequence in W_0 if and only if* $\frac{1}{p_\psi} \in L^1(\mathbb{T}; \frac{d\xi}{2\pi})$
(d) ψ *is a W_0-Schauder basis if and only if p_ψ is an $\mathcal{A}_2(\mathbb{T})$-weight;*
(e) ψ *is a W_0-Riesz basis if and only if $\frac{1}{p_\psi}$ is bounded;*
(f) ψ *is a W_0-orthonormal basis (and, therefore, an orthonormal wavelet) if and only if $p_\psi \equiv 1$ a.e.*

Furthermore, any ψ from either (b),(c), or (d), which is also a W_0-frame, belongs to (e). For any W_0-Schauder basis, which is not a W_0-frame, the Schauder basis $\{\psi_{0k}\}$ in W_0 is a conditional basis.

Following [7] we can expect that some of these classes are not easy to understand; partially for the lack of examples. It is not a priori clear that any of those subclasses is non-empty. However, as we shall see here, the approach via the function p_ψ is not just suitable for the characterization of these subclasses, but is also helpful for the construction of examples. We shall construct a continuum of examples of PFW's whose periodization function can be arbitrarily chosen on an interval, so as to satisfy any of the desired conditions from Corollary 2. Hence, every subclass is far from being empty.

We shall construct sets with the properties that $[-\pi, \pi)$ will be divided into three disjoint sets, say A, B, C (each of them is a union of intervals). The periodization function will be chosen arbitrarily on A (and, hence, can be adjusted to satisfy any of the requirements given in Corollary 2), will be equal to 1 on B (and, hence, will not affect any of the properties from Corollary 2). On C the periodization function will be of the form $\frac{1}{2}a + \frac{1}{2}b$, where a and b will be bounded from below. Observe that in this fashion, we can construct examples that satisfy any condition from Corollary 2, parts (a) – (e). The only condition which is not included is Corollary 2, part (f) , i.e., $p_\psi \equiv 1$. But this is the case of orthonormal wavelets, which has been studied thoroughly and many examples are known.

Let us denote by τ the translation projection on $\mathbb{R}$, defined by $\tau(\xi) = \eta$, where $\eta \in [-\pi, \pi)$ is such that $\xi - \eta = 2\pi k$ for some $k \in \mathbb{Z}$. The dilation projection d is defined on $\mathbb{R}\backslash\{0\}$ by $d(\xi) = \eta$, where η belongs to $[-2\pi, -\pi) \cup [\pi, 2\pi)$ and $\eta/\xi = 2^k$ for some $k \in \mathbb{Z}$. We recall the following theorem, which plays a crucial role in our construction (see [2] and [9]).

Theorem 3. [2]. *Let $E \subseteq [-\pi, \pi)$ and $F \subseteq [-2\pi, -\pi) \cup [\pi, 2\pi)$ be measurable sets such that $0 \in E^\circ$ and $E^\circ \neq \emptyset$. Then, there exists measurable $G \subseteq R$ such that $\tau|_G$ and $d|_G$ are injective functions with $\tau(G) = E$ and $d(G) = F$.*

Example 4. Let I be a small interval around $\frac{9\pi}{4}$ so that $d(I) \cap d(I + \pi) = \emptyset$, $\tau(I) \cap \tau(I + \pi) = \emptyset$, and $\tau(I) \cap \tau(2^j I + 2k\pi) = \emptyset$ for $j = 1, 2$ and $k = 1, 2$. Moreover, we require I to be small enough to ensure that E, defined by $E := [-\pi, \pi)\backslash\big(\tau(I) \cup \tau(I + \pi) \cup \tau(2I) \cup \tau(4I)\big)$, satisfies the requirements of Theorem 3. It is easy to see that such a choice of I is possible. We define F by $F := \big[[-2\pi, -\pi) \cup [\pi, 2\pi)\big]\backslash\big[d(I) \cup d(I + \pi)\big]$. Using Theorem 3, we obtain G such that $\tau|_G$ and $d|_G$ are $1 - 1$ and $\tau(G) = E$, $d(G) = F$.

Take any function $f : I \to \mathbb{C}$, such that $0 < |f(\xi)| < 1$, for every $\xi \in I$. We claim that ψ is a PFW, when defined by

$$\hat{\psi}(\xi) := \begin{cases} 1 & \xi \in G \\ f(\xi) & \xi \in I \\ f(\xi - \pi) & \xi \in I + \pi \\ (1/\sqrt{2})\sqrt{1 - |f(\xi/2)|^2} & \xi \in 2I \\ -(1/\sqrt{2})\sqrt{1 - |f\big((\xi - 2\pi)/2\big)|^2} & \xi \in 2I + 2\pi \\ (1/\sqrt{2})\sqrt{1 - |f(\xi/4)|^2} & \xi \in 4I \\ (1/\sqrt{2})\sqrt{1 - |f\big((\xi - 4\pi)/4\big)|^2} & \xi \in 4I + 4\pi \\ 0 & \text{otherwise} . \end{cases}$$

We first need to show that $\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 = 1$ a.e. Clearly, it is enough to check this for $\xi \in [-2\pi, -\pi) \cup [\pi, 2\pi)$. This means that ξ is either in G, in I or in $(I + \pi)$ (modulo dilations by 2). If $\xi \in G$, then $\hat{\psi}(2^j\xi) = 0$, for every $j \in \mathbb{Z} \backslash \{0\}$. Hence,

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = |\hat{\psi}(\xi)|^2 = 1 .$$

If $\xi \in I$, then $\hat{\psi}(2^j\xi) = 0$, for every $j \in \mathbb{Z}\backslash\{0, 1, 2\}$. Hence

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 |\hat{\psi}(\xi)|^2 + |\hat{\psi}(2\xi)|^2 + |\hat{\psi}(4\xi)|^2 =$$

$$|f(\xi)|^2 + \frac{1}{2}(1 - |f(\xi)|^2) + \frac{1}{2}(1 - |f(\xi)|^2) = 1 .$$

The case $\xi \in (I + \pi)$ goes along the same line.

Secondly, we need to show that for every odd integer q and for almost every ξ we have

$$t_q(\xi) := \sum_{j \geq 0} \hat{\psi}(2^j\xi)\overline{\hat{\psi}\big(2^j(\xi + 2q\pi)\big)} = 0 .$$

Let us observe terms of the form

$$\hat{\psi}(2^j\xi) \cdot \overline{\hat{\psi}\big(2^j(\xi + 2q\pi)\big)} .$$

Such a product is zero except in the following cases

$$\begin{array}{ll} \xi \in 2I, & j = 0,\ q = 1 \\ \xi \in 2I + 2\pi, & j = 0,\ q = -1 \\ \xi \in 2I, & j = 1,\ q = 1 \\ \xi \in 2I + 2\pi, & j = 1,\ q = -1. \end{array}$$

In the first case ($\xi \in 2I, j = 0, q = 1$), we get

$$\begin{aligned} t_q(\xi) =& \hat{\psi}(\xi)\overline{\hat{\psi}(\xi + 2\pi)} + \hat{\psi}(2\xi)\overline{\hat{\psi}(2\xi + 4\pi)} = \\ =& -\frac{1}{2}\sqrt{1 - |f(\xi/2)|^2}\sqrt{1 - |f(\xi/2)|^2} + \\ &+\frac{1}{2}\sqrt{1 - |f(\xi/2)|^2}\sqrt{1 - |f(\xi/2)|^2} \ = \ 0 . \end{aligned}$$

Other cases are checked in a similar way.

Observe that we have a PFW ψ and the function p_ψ has the following properties on $[-\pi, \pi)$ (since p_ψ is 2π-periodic this is all we need)

$$p_\psi(\xi) = \begin{cases} 1 & \xi \in G \\ |f(\xi)|^2 & \xi \in I \\ |f(\xi - \pi)|^2 & \xi \in I + \pi \\ \frac{1}{2}\left[(1 - |f(\frac{\xi}{2})|^2) + (1 - |f(\frac{\xi}{2})|^2)\right] & \xi \in 2I \\ \frac{1}{2}\left[(1 - |f(\frac{\xi}{4})|^2) + (1 - |f(\frac{\xi}{4})|^2)\right] & \xi \in 4I \ . \end{cases}$$

In particular, $p_\psi > 0$ a.e. Hence, for each choice of f, $f : I \to \mathbb{C}$, $0 < |f(\xi)| < 1$, we get a PFW ψ which is in $\mathcal{P}_{\cdot,+}$. Observe that if we require $|f|$ to be bounded away from 1, then $p_{\psi|_{G \cup 2I \cup 4I}}$ is bounded below and away from zero. Since we can adjust p_ψ freely on I (through the choice of f) it is clear that we get a continuum of examples for any subclass described in Corollary 2. (except the subclass described in Corollary 2.(f), i.e., orthonormal wavelets). ⧫

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AN EXERCISE ON UNITARY REPRESENTATIONS IN THE CASE OF COMPLEX CLASSICAL GROUPS

MARKO TADIĆ

Introduction

Generic irreducible unitary representations of classical groups have been classified in [LMT]. The purpose of this paper is to present that classification in a very special case, in the case of classical complex groups. In this case the theorem and the proofs are technically very simple, but they still contain main ideas used in the general case.

Further, the same ideas show up in the proof of the exhaustion in the external approach to the classification of spherical unitary duals of classical groups.

Now we shall describe the result that we shall prove here (as we already mentioned, this is a special case of [LMT]). We shall denote by $(\mathbb{C}^\times)\hat{}$ the group of all unitary characters of $\mathbb{C}^\times$. The character $z \mapsto z\bar{z} = |z|^2$ of $\mathbb{C}^\times$ is denoted by ν. We fix a series of symplectic or the series of odd-orthogonal groups. Denote by S_q the group of rank q from that series. The minimal parabolic subgroup in S_q is denoted by P_{min} (see the second section for more details regarding notation). Then we have the following classification theorems. Before we state them, let us note that these results can be stated uniformly, as one theorem (see Theorem 5.2 of the paper). Moreover, they are special case of even a more general theorem, which addresses all the classical groups and all the local fields in the same time (see [LMT]).

Theorem $(SO(2n+1,\mathbb{C}))$**.** *(i) Take real numbers* $0 < a_1, \dots, a_m, b_1, \dots, b_l < 1/2$*, and characters* $\varphi_1, \dots, \varphi_l, \chi_1, \dots, \chi_r \in (\mathbb{C}^\times)\hat{}$ *such that* $\varphi_1, \dots, \varphi_l$ *are all non-trivial (possibilities* $m = 0$ *or* $l = 0$ *or* $r = 0$ *are not excluded). Denote* $n = m + l + r$*. Then*

$$\pi = Ind_{P_{min}}^{SO(2n+1,\mathbb{C})}(\nu^{a_1} \otimes \cdots \otimes \nu^{a_m} \otimes \nu^{b_1}\varphi_1 \otimes \nu^{b_1}\bar{\varphi}_1 \otimes \cdots \otimes \nu^{b_l}\varphi_l \otimes \nu^{b_l}\bar{\varphi}_l \otimes \chi_1 \otimes \cdots \otimes \chi_r)$$

is irreducible (and generic) representation of $SO(2n+1,\mathbb{C})$*. This representation is unitarizable.*

(ii) Each irreducible principal series (resp. generic) representation of $SO(2n+1,\mathbb{C})$ *which is unitarizable, is equivalent to some representation* π *from (i).*

Theorem $(Sp(2n,\mathbb{C}))$**.** *(i) Take* $0 < \alpha_1 \le \cdots \le \alpha_k \le 1/2 < \beta_1 < \cdots < \beta_\ell < 1$*,* $0 < b_1, \dots, b_l < 1/2$ *and characters* $\varphi_1, \dots, \varphi_l, \chi_1, \dots, \chi_r \in (\mathbb{C}^\times)\hat{}$ *such that* $\varphi_1, \dots, \varphi_l$ *are all non-trivial (possibilities* $k = 0$ *or* $\ell = 0$ *or* $l = 0$ *or* $r = 0$ *are not excluded). Denote* $n = k + \ell + l + r$*. Suppose that holds:*

(a) $\alpha_i + \beta_j \neq 1$ *for* $1 \le i \le k$*,* $1 \le j \le \ell$ *and* $\alpha_{k-1} + \alpha_k < 1$ *if* $k \ge 2$*;*
(b) $card\{i \in \{1, 2, \dots, k\}; 1 - \alpha_i < \beta_1\}$ *is even.*
(c) $card\{i \in \{1, 2, \dots, k\}; \beta_j < 1 - \alpha_i < \beta_{j+1}\}$ *is odd if* $j \in \{1, 2, \dots, \ell - 1\}$*.*

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Then the representation π *defined as*

$$Ind_{P_{min}}^{Sp(2n,\mathbb{C})}(\nu^{\alpha_1}\otimes\cdots\otimes\nu^{\alpha_k}\otimes\nu^{\beta_1}\otimes\cdots\otimes\nu^{\beta_\ell}\otimes\nu^{b_1}\varphi_1\otimes\nu^{b_1}\bar{\varphi}_1\otimes\cdots\otimes\nu^{b_l}\varphi_l\otimes\nu^{b_l}\bar{\varphi}_l\otimes\chi_1\otimes\cdots\otimes\chi_r)$$

is irreducible (and generic) representation of $Sp(2n,\mathbb{C})$*. This representation is unitarizable.*

(ii) Each irreducible principal series (resp. generic) representation of $Sp(2n,\mathbb{C})$ *which is unitarizable, is equivalent to some representation* π *from (i).*

In the paper all equivalences among representations π in the theorem are explained.

Following G. Muić's suggestion, we prepared this paper, which is based on an older manuscript. We thank him for the suggestion. The referee has found a number of typos in the previous version of the paper. He also gave a number of useful suggestions, which helped a lot to improve the readability and the style of the paper . We are thankful to him for that.

At the University of Minnesota in 2003, we gave a series of three talks explaining (among others things) the material presented here. We are thankful to D. Jiang and University of Minnesota for the hospitality.

In the first section we recall the basic simple constructions of irreducible unitary representations. The notation that we shall use in this paper from representation theory of general linear groups, is introduced in the second section, while the third sections does the same for classical groups. The fourth section recalls the very old, simple and well known representation theory of complex rank one groups (essentially $SL(2,\mathbb{C})$). The fifth section recalls the classification theorem from [LMT] in the complex case. A lemma giving upper bounds for complementary series is in the sixth section. The (very short) seventh section gives the proof of the classification theorem in the case of odd-orthogonal groups, while the eighth section brings the proof in the symplectic case.

1. Simple constructions of irreducible unitary representations

In this paper we shall deal with representations of (connected) classical complex groups. For such a group G we shall fix a maximal compact subgroup K of G. The complexified Lie algebra of G, viewed as a real Lie group, will be denoted by $\mathfrak{g}$. A $(\mathfrak{g},K)$-module will be called simply a representation of G in this paper. Such a representation is called unitarizable (resp. Hermitian) if on the representation space there exists a positive definite (resp. non-degenerate) K-invariant Hermitian form which is skew-symmetric for the action of $\mathfrak{g}$. Contragredient (resp. Hermitian contragredient) of a representation π will be denoted by $\tilde{\pi}$ (resp. π^+). Complex conjugate will be denoted by $\bar{\pi}$.

We shall denote by $\tilde{G}$ the set of all equivalence classes of irreducible representations of G, and by $\hat{G}$ the subset of unitarizable classes. Then $\hat{G}$ is in a natural bijection with the unitary dual of G, i.e. with the set of equivalence classes of (topologically) irreducible unitary representations of G. The set $\tilde{G}$ is called nonunitary (or admissible) dual of G.

We shall list here a simple and well known constructions of irreducible unitary representations of reductive groups. Let $P = MN$ be a parabolic subgroup of G and σ a representation of M.

(UI) Unitary parabolic induction: If σ unitarizable, then parabolically induced representation $\mathrm{Ind}_P^G(\sigma)$ is unitarizable.

(UR) Unitary parabolic reduction: If σ is a Hermitian representation, such that parabolically induced representation $\mathrm{Ind}_P^G(\sigma)$ is irreducible and unitarizable, then σ is (irreducible) unitarizable representation.

(D) Deformation (or complementary series): Suppose that X is a connected set of characters of M. Suppose that each representation $\mathrm{Ind}_P^G(\chi\sigma)$ is Hermitian and irreducible for $\chi \in X$. If there exists $\chi_0 \in X$ such that $\mathrm{Ind}_P^G(\chi_0\sigma)$ is unitarizable, then all $\mathrm{Ind}_P^G(\chi\sigma)$, $\chi \in X$ are unitarizable.

(ED) Ends of deformations: Let Y be a set of characters of M and X a dense subset of Y. Suppose that X satisfies the conditions of (D). Then each irreducible subquotient of each $\mathrm{Ind}_P^G(\chi\sigma)$, $\chi \in Y$ is unitarizable.

2. $GL(n,\mathbb{C})$

The standard absolute value on $\mathbb{C}$ is denoted by $|\ \ |$. We shall denote by $|\ \ |_{\mathbb{C}}$ the square of the standard absolute value on $\mathbb{C}$, i.e. $|z|_{\mathbb{C}} = z\bar{z}$ (observe that the standard absolute value is without index $\mathbb{C}$) . We denote

$$\nu : \mathbb{C}^\times \to \mathbb{R}^\times, \quad z \mapsto |z|_{\mathbb{C}}.$$

In each $GL(n,\mathbb{C})$ we fix the maximal compact subgroup $K = U(n,\mathbb{C})$ consisting of all unitary matrices. Further, we fix the minimal parabolic subgroup of $\mathrm{GL}(n,\mathbb{C})$ consisting of all upper triangular matrices in $\mathrm{GL}(n,\mathbb{C})$. For representations π_i of $\mathrm{GL}(n_i,\mathbb{C})$, $i = 1,2$, denote by $\pi_1 \times \pi_2$ the representation of $\mathrm{GL}(n_1+n_2,\mathbb{C})$ parabolically induced by $\pi_1 \otimes \pi_2$ from the standard parabolic subgroup having Levi factor naturally isomorphic to $\mathrm{GL}(n_1,\mathbb{C}) \times \mathrm{GL}(n_2,\mathbb{C})$. If π_i's have finite length, then $\pi_1 \times \pi_2$ is also of finite length. Then

$$\pi_1 \times \pi_2 \text{ and } \pi_2 \times \pi_1 \text{ have the same composition series} \tag{2-1}$$

(this follows from the fact concerning parabolic induction from associate parabolic subgroups and representations). In particular:

$$\text{if } \pi_1 \times \pi_2 \text{ is irreducible, then } \pi_1 \times \pi_2 \cong \pi_2 \times \pi_1. \tag{2-2}$$

A consequence of a general simple fact about induction in stages is

$$\pi_1 \times (\pi_2 \times \pi_3) \cong (\pi_1 \times \pi_2) \times \pi_3. \tag{2-3}$$

Using the determinant homomorphism, we identify characters of $GL(n,\mathbb{C})$ and $\mathbb{C}^\times$. Then for a character χ of $\mathbb{C}^\times$ holds

$$\chi(\pi_1 \times \pi_2) \cong (\chi\pi_1) \times (\chi\pi_2). \tag{2-4}$$

Clearly,

$$(\pi_1 \times \pi_2)\tilde{}\cong \tilde{\pi}_1 \times \tilde{\pi}_2. \tag{2-5}$$

3. Complex classical groups

Denote

$$J_n = \begin{bmatrix} 00 & \dots & 01 \\ 00 & \dots & 10 \\ \vdots & & \\ 10 & \dots & 0 \end{bmatrix} \in GL(n,\mathbb{C}).$$

Further, the identity matrix in $GL(n,\mathbb{C})$ is denoted by I_n. For $g \in GL(2n,\mathbb{C})$ let

$$ {}^{\times}g = \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} {}^{t}g \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix}, $$

where ${}^{t}g$ denotes the transposed matrix of g. Then ${}^{\times}(g_1 g_2) = {}^{\times}g_2 \, {}^{\times}g_1$. Symplectic group is defined as

$$ Sp(2n,\mathbb{C}) = \{g \in GL(2n,\mathbb{C});\ {}^{\times}g\, g = I_{2n}\}. $$

We take $\mathrm{Sp}(0,\mathbb{C})$ to be the trivial group. We take formally that the (trivial) element of this group is 0×0 matrix.

By ${}^{\tau}g$ we shall denote the transposed matrix of $g \in GL(n,\mathbb{C})$ with respect to the second diagonal. Then we define odd (special) orthogonal group as

$$ SO(2n+1,\mathbb{C}) = \{g \in SL(2n+1,\mathbb{C});\ {}^{\tau}g\, g = I_{2n+1}\}. $$

We shall fix one series of classical groups, either symplectic or odd orthogonal. The group of rank n will be denoted by S_n. We fix the minimal the parabolic subgroup $P_{\min}$ in S_n consisting of all upper triangular matrices in S_n. Fix maximal compact subgroup in S_n consisting of unitary matrices in S_n. Sometimes we shall write $\rtimes_{Sp}$ or $\rtimes_{SO}$ to indicate with which series of groups we are working.

Let τ be a representation of S_n and let π be a representation of $\mathrm{GL}(m,\mathbb{C})$. We denote by $M_{(n)}$ the Levi subgroup in S_{n+m} consisting of matrices

$$ \begin{bmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & {}^{\tau}g^{-1} \end{bmatrix} $$

where $g \in GL(m,\mathbb{C})$ and $h \in S_n$. Denote by $\pi \rtimes \sigma$ the representation of S_{n+m} parabolically induced from $M_{(n)}P_{min}$ by $\pi \otimes \sigma$. Here $\pi \otimes \sigma$ maps the above matrix into $\pi(g) \otimes \sigma(h)$.

Suppose that π, π_1 and π_2 are admissible representations of $\mathrm{GL}(m,\mathbb{C})$, $\mathrm{GL}(m_1,\mathbb{C})$ and $\mathrm{GL}(m_2,\mathbb{C})$ respectively. Let σ be a representation of S_n. Then

$$ \pi_1 \rtimes (\pi_2 \rtimes \sigma) \cong (\pi_1 \times \pi_2) \rtimes \sigma \qquad (3\text{-}1) $$

and

$$ (\pi \rtimes \sigma)^{\sim} \cong \tilde{\pi} \rtimes \tilde{\sigma} \qquad (3\text{-}2) $$

(this corresponds to (2-3) and (2-5) respectively). Further

$$ \pi \rtimes \sigma \text{ and } \tilde{\pi} \rtimes \sigma \text{ have the same composition series.} \qquad (3\text{-}3) $$

In particular,

$$ \text{if } \pi \rtimes \sigma \text{ is irreducible, then } \pi \rtimes \sigma \cong \tilde{\pi} \rtimes \sigma \qquad (3\text{-}4) $$

(this corresponds to (2-1) and (2-2) respectively).

4. Rank one groups

The trivial (one dimensional) representation of a group G will be denoted by 1_G. In the case of trivial group, this representation will be also denoted simply by 1. Recall that $|\ \ |$ denotes the standard absolute value on $\mathbb{C}$, while $|\ \ |_{\mathbb{C}}$ denotes the square of the standard absolute value, i.e. $|\ \ |_{\mathbb{C}} = |\ \ |^2$.

4.1. **Characters of $\mathbb{C}^\times$.** Observe that $(\mathbb{C}^\times)\hat{}$ is just the set of unitary characters of $\mathbb{C}^\times$, and $(\mathbb{C}^\times)\tilde{}$ is the set of all characters of $\mathbb{C}^\times$. Let χ be a character of $\mathbb{C}^\times$. Then we can find unique $\chi^u \in (\mathbb{C}^\times)\hat{}$ and $e(\chi) \in \mathbb{R}$ such that

$$\chi(z) = \nu^{e(\chi)}\chi^u(z) = |z|_{\mathbb{C}}^{e(\chi)}\chi^u(z).$$

This defines χ^u and $e(\chi)$.

For $x, y \in \mathbb{C}$ satisfying $x - y \in \mathbb{Z}$ and $z \in \mathbb{C}^\times$ set

$$\gamma(x,y)(z) = (z/|z|)^{x-y}|z|^{x+y}.$$

Then $\gamma(x,y)$ is a character of $\mathbb{C}^\times$ and $e(\gamma(x,y)) = 1/2\,\mathrm{Re}(x+y)$. Observe $\gamma(x,y)\gamma(x',y') = \gamma(x+x', y+y')$. Further

$$\gamma(x,y)\tilde{} = \gamma(-x,-y), \quad \gamma(x,y)^- = \gamma(\bar{y},\bar{x}), \quad \gamma(x,y)^+ = \gamma(-\bar{y},-\bar{x}).$$

Also $\gamma(x,x)(z) = |z|_{\mathbb{C}}^x$.

4.2. **Representations of $GL(2,\mathbb{C})$.** The representation $\gamma(x_1,y_1) \times \gamma(x_2,y_2)$ reduces if and only if

$$x_1 - x_2 \in \mathbb{Z} \quad \text{and} \quad (x_1 - x_2)(y_1 - y_2) > 0.$$

From this follows that if $\gamma(x_1,y_1) \times \gamma(x_2,y_2)$ reduces, then $(x_1+y_1)/2 - (x_2+y_2)/2 \in (1/2)\mathbb{Z}\backslash\{0, \pm 1/2\}$. For the same reason, if $x_1 - y_1 \neq x_2 - y_2$, then reducibility of $\gamma(x_1,y_1) \times \gamma(x_2,y_2)$ implies $(x_1+y_1)/2 - (x_2+y_2)/2 \in (1/2)\mathbb{Z}\backslash\{0, \pm 1/2, \pm 1\}$.

Therefore, if $\chi_1 \times \chi_2$ reduces ($\chi_1,\ \chi_2 \in (\mathbb{C}^\times)\tilde{}$), then $e(\chi_1) - e(\chi_2) \in (1/2)\mathbb{Z}\backslash\{0, \pm 1/2\}$. If additionally $\chi_1^u \neq \chi_2^u$, then $e(\chi_1) - e(\chi_2) \in (1/2)\mathbb{Z}\backslash\{0, \pm 1/2, \pm 1\}$.

Observe that if $\nu^x \times \nu^y = \gamma(x,x) \times \gamma(y,y)$ reduces for some $x, y \in \mathbb{R}$, then $x - y \in \mathbb{Z}$.

If we have reducibility of $\gamma(x_1,y_1) \times \gamma(x_2,y_2)$, then the composition series of this representation consists of

$$L(\gamma(x_1,y_1), \gamma(x_2,y_2)) \quad \text{and} \quad \gamma(x_1,y_2) \times \gamma(x_2,y_1).$$

Unitary dual of $GL(2,\mathbb{C})$ consists of the trivial representation, the unitary principal series are $\chi_1 \times \chi_2$, $\chi_1,\ \chi_2 \in (\mathbb{C}^\times)\hat{}$, and complementary series $\nu^\alpha\chi \times \nu^{-\alpha}\chi$, where $\chi \in (\mathbb{C}^\times)\hat{}$ and $0 < \alpha < 1/2$.

4.3. **Representations of $SL(2,\mathbb{C})$.** Restricting $\gamma(x,y) \times \gamma(0,0)$ to $SL(2,\mathbb{C})$ we get $\gamma(x,y) \rtimes_{Sp} 1$. Therefore $\gamma(x,y) \rtimes_{Sp} 1$ reduces if and only if

$$x \in \mathbb{Z} \quad \text{and} \quad xy > 0.$$

Further, if $\gamma(x,y) \rtimes_{Sp} 1$ reduces, then $(x+y)/2 \in (1/2)\mathbb{Z}\backslash\{0 \pm 1/2\}$.

In other words, if $\chi \rtimes_{Sp} 1$ reduces ($\chi \in (\mathbb{C}^\times)\tilde{}$), then $e(\chi) \in (1/2)\mathbb{Z}\backslash\{0, \pm 1/2\}$. Further, if $\nu^x \rtimes_{Sp} 1 = \gamma(x,x) \rtimes_{Sp} 1$ reduces for some $x \in \mathbb{R}$, then $x \in \mathbb{Z}$.

In the case of reducibility of $\gamma(x,y) \rtimes_{Sp} 1$, composition series consists of

$$L(\gamma(x,y), 1) \quad \text{and} \quad \gamma(x,-y) \rtimes_{Sp} 1.$$

Unitary dual of $SL(2,\mathbb{C})$ consists of the trivial representation, the unitary principal series and complementary series $\nu^\alpha 1_{\mathbb{C}^\times} \rtimes_{Sp} 1$, where $0 < \alpha < 1$.

4.4. **Representations of $SO(3,\mathbb{C})$.** Consider the epimorphism $SL(2,\mathbb{C})@>>> SO(3,\mathbb{C})$ which comes from the adjoint action on the Lie algebra. Using this epimorphism, the representation $\gamma(x,y)\rtimes_{SO} 1$ pulls back to $\gamma(2x,2y)\rtimes_{Sp} 1$. Therefore, $\gamma(x,y)\rtimes_{SO} 1$ reduces if and only if

$$x\in(1/2)\mathbb{Z} \quad \text{and} \quad xy>0.$$

From this follows: if $\gamma(x,y)\rtimes_{SO} 1$ reduces, then $x+y\in(1/2)\mathbb{Z}\backslash\{0,\pm1/2\}$.

Thus, if $\chi\rtimes_{SO}1$ reduces ($\chi\in(\mathbb{C}^\times)\tilde{}$), then $2e(\chi)\in(1/2)\mathbb{Z}\backslash\{0,\pm1/2\}$. Also, if $\nu^x\rtimes_{SO}1=\gamma(x,x)\rtimes_{SO}1$ reduces for some $x\in\mathbb{R}$, then $x\in(1/2)\mathbb{Z}$.

If we have reducibility of $\gamma(x,y)\rtimes_{SO} 1$, composition series again consists of

$$L(\gamma(x,y),1) \quad \text{and} \quad \gamma(x,-y)\rtimes_{SO} 1.$$

Unitary dual of $SO(3,\mathbb{C})$ consists of the trivial representation, the unitary principal series and complementary series $\nu^\alpha 1_{\mathbb{C}^\times}\rtimes_{SO} 1$, where $0<\alpha<1/2$.

4.5. Observe that for $\alpha\in\mathbb{Z}$ if $\nu^\alpha 1_{\mathbb{C}^\times}\rtimes 1_{S_0}$ reduces, then it has a tempered subquotient. For $\alpha=0$, we have irreducibility, but the whole induced representation is tempered.

5. Unitarizable irreducible principal series representations

All irreducible tempered representations of semisimple complex groups are fully induced by unitary characters. We fix a series S_n of classical groups, symplectic or odd orthogonal.

The set of equivalence classes of irreducible tempered representations of groups S_n. $n\in\mathbb{Z}_{\geq 0}$, is denoted by T. Observe that T consists of all classes $\chi_1\times\cdots\times\chi_n\rtimes 1$, $\chi_i\in(\mathbb{C}^\times)\hat{}$, $n\in\mathbb{Z}_{\geq0}$. If we replace some of χ_i by χ_i^{-1}, or change the order of $\chi_1,\dots,\chi_n$ in $\chi_1\times\cdots\times\chi_n\rtimes 1$, we get the same class. These are the only equivalence among representations $\chi_1\times\ldots\times\chi_n\rtimes 1$.

Denote $(\mathbb{C}^\times)\tilde{}_+=\{\chi\in(\mathbb{C}^\times)\tilde{};e(\chi)>0\}$. A finite multiset in $(\mathbb{C}^\times)\tilde{}_+$ is defined to be an unordered n-tuple of characters in $(\mathbb{C}^\times)\tilde{}_+$, $n\in\mathbb{Z}_{\geq0}$. The set of all finite multisets in $(\mathbb{C}^\times)\tilde{}_+$ will be denoted by $M((\mathbb{C}^\times)\tilde{}_+)$. For $t=(d,\tau)=((\chi_1,\dots,\chi_n),\tau)\in M((\mathbb{C}^\times)\tilde{}_+)\times T$ take a permutation p of order n such that $e(\chi_{p(1)})\geq e(\chi_{p(2)})\geq\cdots\geq e(\chi_{p(n)})$. Denote

$$\lambda(t)=\chi_{p(1)}\times\chi_{p(2)}\times\ldots\times\chi_{p(n)}\rtimes\tau.$$

Then $\lambda(t)$ has a unique irreducible quotient, which will be denoted by $L(t)$. In this way we get parameterization of admissible duals of all groups S_n by the set $M((\mathbb{C}^\times)\tilde{}_+)\times T$. This is Langlands classification (of admissible duals of these groups). The representation $\lambda(t)$ is called standard module. The formula for Hermitian contragredient is simply

$$L(((\chi_1,\dots,\chi_n),\tau))^+\cong L(((\bar\chi_1,\dots,\bar\chi_n),\tau)) \tag{5-1}$$

By Vogan's result [V], representation $L(t)$ is generic if and only if $\lambda(t)$ is irreducible, i.e. if and only if $\lambda(t)=L(t)$.

As we already mentioned in the introduction, we shall present here the proof of classification of irreducible unitarizable generic representations of classical groups S_n. By Vogan's result, it is the same as classifying unitarizable representations among irreducible principal series representations, since by Vogan's result, irreducible generic representations of a classical group S_n are exactly principal series representations $\chi_1\times\cdots\times\chi_n\rtimes 1$, $\chi_i\in(\mathbb{C}^\times)\tilde{}$, which are irreducible (use (2-2), (2-3), (3-1), (3-4) and the following proposition). Because of this, for us is important the following

Proposition 5.1. *Let $\chi_1,\dots,\chi_k \in (\mathbb{C}^\times)\tilde{}$. Then $\chi_1\times\chi_2\times\cdots\times\chi_k \rtimes 1$ is irreducible if and only if all the representations $\chi_i\times\chi_j,\ \chi_i\times\tilde{\chi}_j,\ 1\le i<j\le k$, and $\chi_i\rtimes 1,\ 1\le i\le k$, are irreducible.*

Proof. If some of the representations $\chi_i\times\chi_j,\ \chi_i\times\tilde{\chi}_j$ or $\chi_i\rtimes 1$ is reducible, then (3-1), (2-3), (2-1) (3-3) and (2-5) imply that $\chi_1\times\chi_2\times\cdots\times\chi_k\rtimes 1$ is reducible.

For the other implication, suppose that these representations are irreducible. To prove irreducibility of $\chi_1\times\chi_2\times\cdots\times\chi_k\rtimes 1$, using (3-1), (2-3), (2-2) and (3-4) we can easily reduce the proof to the case $e(\chi_1)\ge\cdots\ge e(\chi_k)\ge 0$. Recall that by Langlands classification, the space of intertwining operators $\chi_1\times\chi_2\times\cdots\times\chi_k\rtimes 1\to\chi_1^{-1}\times\chi_2^{-1}\times\cdots\times\chi_k^{-1}\rtimes 1$ is one dimensional, and the image of non-zero intertwining is (irreducible) Langlands quotient. To be consistent with Langlands classification as we have described it above, take minimal $1\le i\le k$ such that $e(\chi_i)>0$ if such i exists, and take $i=0$ otherwise. Denote $\tau=\chi_{i+1}\times\cdots\times\chi_k\rtimes 1$. Then the space of intertwining operators $\chi_1\times\cdots\times\chi_i\rtimes\tau\to\chi_1^{-1}\times\cdots\times\chi_i^{-1}\rtimes\tau$ is one dimensional. But $\chi_1\times\cdots\times\chi_i\rtimes\tau\cong\chi_1\times\chi_2\times\cdots\times\chi_k\rtimes 1$ and $\chi_1^{-1}\times\cdots\times\chi_i^{-1}\rtimes\tau\cong\chi_1^{-1}\times\chi_2^{-1}\times\cdots\times\chi_k^{-1}\rtimes 1$ by (2-2), (2-3), (3-1) and (3-4). Observe that (3-1), (2-3), (2-2) and (3-4) imply that $\chi_1\times\chi_2\times\cdots\times\chi_k\rtimes 1\cong\chi_1^{-1}\times\chi_2^{-1}\times\cdots\times\chi_k^{-1}\rtimes 1$. This implies the irreducibility of $\chi_1\times\chi_2\times\cdots\times\chi_k\rtimes 1$. □

Now we have a special case of classification theorem of [LMT][1] (one can find in the introduction of the paper formulation of the theorem separately for symplectic and odd-orthogonal groups):

Theorem 5.2. *(i) Take $\varphi_1,\dots,\varphi_l\in(\mathbb{C}^\times)\hat{}\backslash\{1_{\mathbb{C}^\times}\}$, $0<a_1,\dots,a_m<1$, $0<b_1,\dots,b_l<1/2$ and $\tau\in T$. Suppose that holds:*

If $\nu^{1/2}\rtimes 1$ reduces, then all $a_i<1/2$.

If $\nu^{1/2}\rtimes 1$ does not reduce, write the numbers $a_1,\dots,a_m$ as a non-decreasing sequence $\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_\ell$ where $\alpha_k\le 1/2<\beta_1$ (possibilities $k=0$ or $\ell=0$ are not excluded). Assume

$$\beta_1<\beta_2<\cdots<\beta_\ell. \tag{5-2}$$

Further assume that hold

(a) *$\alpha_i+\beta_j\ne 1$ for $1\le i\le k$, $1\le j\le\ell$ and $\alpha_{k-1}+\alpha_k<1$ if $k\ge 2$;*

(b) *$card\big\{i\in\{1,2,\dots,k\};1-\alpha_i<\beta_1\big\}$ is even.*

(c) *$card\big\{i\in\{1,2,\dots,k\};\beta_j<1-\alpha_i<\beta_{j+1}\big\}$ is odd if $j\in\{1,2,\dots,\ell-1\}$.*

Then the representation

$$\pi=\nu^{a_1}\times\cdots\times\nu^{a_m}\times(\nu^{b_1}\varphi_1\times\nu^{b_1}\bar{\varphi}_1)\times\cdots\times(\nu^{b_l}\varphi_l\times\nu^{b_l}\bar{\varphi}_l)\rtimes\tau \tag{5-3}$$

is irreducible (and generic) representation of some S_n. This representation is unitarizable.

(ii) Each irreducible principal series (resp. generic) representation of S_q which is unitarizable, is equivalent to some representation π from (i).

Recall that τ is equivalent to $\chi_1\times\cdots\times\chi_n\rtimes 1$ for some $\chi_i\in(\mathbb{C}^\times)\hat{}$.

[1]The classification theorem in [LMT] holds over all locally compact non-discrete fields. In that classification theorem shows up also reducibility at 0, which does not show up in the complex case

6. Lemma

Observe that for $\chi \in (\mathbb{C}^\times)\hat{}$, $\chi = \tilde{\chi}$ if and only if $\chi = 1_{\mathbb{C}^\times}$.

Lemma 6.1. *Suppose that* $\pi = \mu_1 \times \cdots \times \mu_s \rtimes 1$, $\mu_i \in (\mathbb{C}^\times)\tilde{}$, *is an irreducible unitarizable representation. Let* $1 \leq i \leq s$. *Then:*

(1) *If* $\mu_i^u \neq 1_{\mathbb{C}^\times}$, *then* $|e(\mu_i)|_{\mathbb{C}} < 1/2$.
(2) *If* $\mu_i^u = 1_{\mathbb{C}^\times}$, *then* $\mu_i \rtimes 1$ *is unitarizable.*

Proof. Observe that by Proposition 5.1, all $\mu_i \times \mu_j$, $\mu_i \times \tilde{\mu}_j$ and $\mu_i \rtimes 1$ are irreducible.

Using relations of the second and third section, we reduce the lemma to the case when all $e(\mu_j) \geq 0$. Further, we need to consider only the case $e(\mu_i) > 0$.

(1) Suppose $\mu_i^u \neq \tilde{\mu}_i^u$ (i.e $\mu_i^u \neq 1_{\mathbb{C}^\times}$). Relations of the second and third section imply that after renumeration we can assume that $i = 1$. Since $\pi = \mu_1 \times \cdots \times \mu_s \rtimes 1$ is Hermitian, by (5-1) there exists $j \neq 1$ such that $\mu_j = \bar{\mu}_1$. Now using relations of sections two and three, we can take $j = 2$. This implies $\tilde{\mu}_2 = \mu_1^+$. Now the relations of the second and the third section imply

$$\pi \cong \mu_1 \times \tilde{\mu}_2 \times \mu_3 \times \cdots \times \mu_s \rtimes 1 = (\mu_1 \times \mu_1^+) \times \mu_3 \times \cdots \times \mu_s \rtimes 1$$

Since $(\mu_1 \times \mu_1^+) \otimes (\mu_3 \times \cdots \times \mu_s \rtimes 1)$ is Hermitian (and irreducible), unitary parabolic reduction (UR) implies that $(\mu_1 \times \mu_1^+) \otimes (\mu_3 \times \cdots \times \mu_s \rtimes 1)$ is unitarizable. From this directly follows that $\mu_1 \times \mu_1^+ = \nu^{e(\mu_1)}\mu_1^u \times \nu^{-e(\mu_1)}\mu_1^u$ is unitarizable. Now the description of the unitary dual of $GL(2,\mathbb{C})$ in 4.2 implies $e(\mu_1) < 1/2$. This ends the proof of (1).

(2) Suppose now $\mu_i^u = 1_{\mathbb{C}^\times}$. If $e(\mu_i) < 1/2$, then from 4.3 and 4.4 we know that $\mu_i \rtimes 1$ is unitarizable, and thus (2) holds. Therefore, it remains to consider the case $e(\mu_i) \geq 1/2$, and we shall assume this in the sequel. Relations of the second and third section imply that after renumeration we can assume that $i = 1$. Further, using reduction as in the proof of (1), we can suppose that $\mu_j^u = \tilde{\mu}_j^u$ for all j (i.e. $\mu_j^u = 1_{\mathbb{C}^\times}$). We need only to consider the case $s \geq 2$.

Using deformation (D) and the fact that the reducibility happens on a closed set (see Proposition 5.1 and 4.2–4.4), twisting μ_j, $j \geq 2$, by ν^{ε_j} for small enough (by real absolute value) real numbers ε_j, we can assume that $e(\mu_u) \pm e(\mu_v) \notin \mathbb{Q}$ for all $u \neq v$, and $e(\mu_u) \notin (1/2)\mathbb{Z}$ for all $u \geq 2$

We can write $\mu_1 = \gamma(a,a)$, where $a = e(\mu_1)$. Since $a \geq 1/2$, we could deform a to the case $a > 1/2$ (and not in $\mathbb{Q}$) in a way that $\mu_1 \times \cdots \times \mu_s \rtimes 1$ stays irreducible and unitarizable. Take $k \in \mathbb{Z}_{>0}$ such that

$$|a - k| < 1/2.$$

Therefore, the representation $\gamma(a-k, a-k) \times \gamma(-(a-k), -(a-k))$ is an irreducible unitarizable (complementary series) representation of $GL(2,\mathbb{C})$. Therefore

$$\gamma(a-k, a-k) \times \gamma(-(a-k), -(a-k)) \times \mu_1 \times \cdots \times \mu_s \rtimes 1 \tag{6-1}$$

is unitarizable. By 4.2 (and relations of the second and the third section), one subquotient of the above representation is

$$\gamma(a-k, a) \times \gamma(a, a-k) \times \gamma(-(a-k), -(a-k)) \times \mu_2 \times \cdots \times \mu_s \rtimes 1 \tag{6-2}$$

Therefore, this subquotient is unitarizable. Note that by the relations of the second and the third section, the representation (6-2) is isomorphic to

$$\big[\gamma(a-k,a)\times\gamma(-\bar{a},-\overline{(a-k)})\big]\times\big[\gamma(-(a-k),-(a-k))\times\mu_2\times\cdots\times\mu_s\rtimes 1\big].$$

Using parabolic reduction we conclude that both representations

$$\gamma(a-k,a)\times\gamma(-\bar{a},-\overline{(a-k)}) \text{ and } \gamma(-(a-k),-(a-k))\times\mu_2\times\cdots\times\mu_s\rtimes 1 \qquad (6\text{-}3)$$

are unitarizable.

From the unitary dual of $GL(2,\mathbb{C})$ we know that $|e(\gamma(a-k,a))| = |a-k/2| < 1/2$.

Suppose that $e(\mu_1)\geq 1$. First, we can deform it to $e(\mu_1) = a > 1$ so that π stays irreducible and unitarizable. Now from the inequalities $|a-k|, |a-k/2| < 1/2$ we get $k/2 = |(k-a)+(a-k/2)| < 1$. This implies $k = 1$. From this and $|a-k/2| < 1/2$ we get $a - 1/2 < 1/2$, which implies $a < 1$. This contradiction shows that $|e(\mu_1)| < 1$. This implies that (2) holds for symplectic groups, since complementary series for $Sp(2,\mathbb{C}) = SL(2,\mathbb{C})$ end at 1 (see (4.3)).

It remains to prove (2) for odd-orthogonal groups. Recall that we have started the proof with assumption $e(\mu_1)\geq 1/2$ and got that $\gamma(-(a-k),-(a-k))\times\mu_2\times\cdots\times\mu_s\rtimes 1$ is unitarizable (see (6-3)), where $|a-k| < 1/2$. Repeating this reductions and using relations of the second and third section, we can suppose that that $\mu_1\times\cdots\times\mu_s\rtimes 1$ is unitarizable, where $e(\mu_1) > 1/2$ and $0 < e(\mu_j) < 1/2$ for $j\geq 2$.

Denote $b = |a-k|$. Suppose $e(\mu_2) < b$. Then $\nu^\alpha\times\mu_1\times\mu_3\times\ldots\times\mu_s\rtimes 1$, $0\leq\alpha\leq e(\mu_2)$ is a continuous family of irreducible Hermitian representations (use Proposition 5.1 and 4.2 - 4.4 to see this), and for $\alpha = e(\mu_2)$ we have unitarizability. Therefore, we have unitarizability for $\alpha = 0$. Now using unitary parabolic reduction (UR) we get that $\mu_1\times\mu_3\times\cdots\times\mu_s\rtimes 1$ is unitarizable.

Consider now the case $e(\mu_2) > b$. Then $\mu_1\times\mu_3\times\cdots\times\mu_s\times\nu^\alpha\rtimes 1$, $e(\mu_2)\leq\alpha < 1/2$ is a continuous family of irreducible Hermitian representations (again use Proposition 5.1, 4.2 and 4.4 to see this), and for $\alpha = e(\mu_2)$ we have unitarizability. So, all the representations in the family are unitarizable. For $\alpha = 1/2$, one irreducible subquotient is $\mu_1\times\mu_3\times\cdots\times\mu_s\times\mu\rtimes 1$ for some unitary character μ of $\mathbb{C}^\times$ (to be precise, for $\mu = \gamma(1/2,-1/2)$). Using (ED) we conclude that the last representation is unitarizable. Now using using the relations of the second and third section we get $\mu_1\times\mu_3\times\cdots\times\mu_s\times\mu\rtimes 1\cong\mu\times\mu_1\times\mu_3\times\cdots\times\mu_s\rtimes 1$. Using parabolic reduction (R) we get that $\mu_1\times\mu_3\times\cdots\times\mu_s\rtimes 1$ is unitarizable.

Applying above reductions $s-1$ times, we get (2) for odd-orthogonal groups. □

7. Proof of Theorem 5.2 for odd-orthogonal groups

Observe that in this case $\nu^{1/2}\rtimes 1$ reduces.

Proof. Irreducibility follows directly from Proposition 5.1 and rank one reducibility 4-2–4.4.

To prove the unitarizability of π, consider the family

$$\nu^{x_1}\times\cdots\times\nu^{x_m}\times(\nu^{y_1}\varphi_1\times\nu^{y_1}\bar{\varphi}_1)\times\cdots\times(\nu^{y_l}\varphi_l\times\nu^{y_l}\bar{\varphi}_l)\rtimes\tau$$

where $0\leq x_i\leq a_i$ and $0\leq y_j\leq b_j$. Now Proposition 5.1, 4.2 and 4.4 imply directly that this is a continuous family of irreducible Hermitian representations, which contains π. Since this family contains unitarizable representation (for all $x_i = y_j = 0$), by (D) all these representations are unitarizable. Thus, π is unitarizable.

Recall that by Vogan's result [V], irreducible generic representations of a classical group S_n are exactly principal series representations $\chi_1 \times \cdots \times \chi_n \rtimes 1$, $\chi_i \in (\mathbb{C}^\times)\tilde{}$, which are irreducible. Using this, exhaustion follows now directly from the last lemma. □

8. Proof of Theorem 5.2 for symplectic groups

Proof. Irreducibility of the representations π from the theorem follows from Proposition 5.1, conditions on exponents a_i and b_j in (i) ($0 < a_i < 1$, $0 < b_j < 1/2$ and (a)), and rank one reducibility facts 4.2, 4.3.

Now we shall prove unitarizability of the representations π in the theorem by induction on $m + 2l$. If $m + 2l = 1$, then $m = 1$ and $l = 0$. In this case we obviously have unitarizability (see 4.3). Therefore, it remains to consider the case $m + 2l \geq 2$.

First suppose that $l \geq 1$. Then $\pi \cong (\nu^{b_1}\varphi_1 \times \nu^{b_1}\bar{\varphi}_1) \rtimes \pi' \cong (\nu^{b_1}\varphi_1 \times \nu^{-b_1}\varphi_1) \rtimes \pi'$, where π' satisfy conditions of (i) of the theorem. Since π' is unitarizable by the inductive assumption, and $0 < b_1 < 1/2$, 4.2 implies that π is unitarizable.

Therefore, we need to consider the case when $l = 0$ and $m = k + \ell \geq 2$. Suppose first that $\ell = 0$. Then we get unitarizability of π in the same way as in the case of odd-orthogonal groups (by deformations).

Therefore, we need to consider the case $\ell \geq 1$ and $m = k+\ell \geq 2$. Condition (c) implies that for all $j = 1, 2, \ldots, \ell-1$, between β_j and β_{j+1} is at least one $1-\alpha_i$. From this follows $\ell - 1 \leq k$.

Suppose that the cardinality in (b) is strictly positive. Recall that it is even. Therefore $1 - \alpha_{k-1}, 1 - \alpha_k < \beta_1$. Remove from π the factors corresponding to α_k and α_{k-1}, and denote the obtained representation by π'. Recall that $l = 0$. Consider the family

$$\nu^x \times \nu^{\alpha_{k-1}} \rtimes \pi', \quad \alpha_{k-1} \leq x \leq \alpha_k.$$

This family contains π. Further, this is a continuous family of irreducible Hermitian representations. This follows from Proposition 5.1, 4.2 and 4.3. For example, by (4-2) reducibility could happen if $x+\beta_j = 1$, which imply $\alpha_{k-1} \leq 1-\beta_j \leq \alpha_k$, and in particular $\beta_j \leq 1 - \alpha_{k-1}$. This implies $\beta_1 < 1 - \alpha_{k-1}$ which is impossible (since we have obtained above that $1 - \alpha_{k-1} < \beta_1$). Other conditions in Proposition 5.1 are obvious (they follow from (a), $0 < \alpha_i \leq 1/2$, $1/2 < \beta_j < 1$ and 4.2, 4.3).

Now π' is unitarizable by inductive assumption. Therefore, $\nu^{\alpha_{k-1}} \times \nu^{\alpha_{k-1}} \rtimes \pi' \cong (\nu^{-\alpha_{k-1}} \times \nu^{\alpha_{k-1}}) \rtimes \pi'$ is unitarizable. So the whole family consists of unitarizable representations. Therefore, π is unitarizable.

It remains to consider the case when the set in (b) is empty, i.e. $\beta_1 < 1 - \alpha_k$ (this implies also $a_k < 1/2$ since $1/2 < \beta_1$). Similarly as above, remove from π the factors corresponding to α_k and β_1, and denote the obtained representation by π'. Consider the family of representations

$$\nu^{\alpha_k} \times \nu^x \rtimes \pi', \quad \alpha_k \leq x \leq \beta_1$$

containing π. From $\beta_1 < 1 - \alpha_k$ and Proposition 5.1, 4.2 and 4.3 one gets easily that this is irreducible family. For example if $x+\alpha_i = 1$, then $1-\alpha_i \leq \beta_1$, and so $1-\alpha_k \leq \beta_1$ which is impossible. If $x + \beta_j = 1$ for $j \geq 2$, then $\alpha_k \leq 1 - \beta_j$. Note that by (c) there exists some α_i satisfying $\beta_1 < 1 - \alpha_i < \beta_j$. All this implies $\alpha_k \leq 1 - \beta_j < \alpha_i$, which is again impossible since α_k is maximal among α_i's. Other conditions for applying Proposition 5.1 are obviously satisfied.

So we have continuous family of irreducible Hermitian representations. Now in the same way as in the previous case unitarizability follows.

Now we shall prove the exhaustion in (ii) of the theorem. It remains to see that each irreducible generic unitarizable representations π satisfies conditions in (i) of Theorem 5.1. By Vogan's result [V], each irreducible generic representations is irreducible standard module (in our case this is irreducible principal series), so we can write π as

$$\pi \cong \nu^{c_1}\chi_1 \times \nu^{c_2}\chi_2 \times \cdots \times \nu^{c_s}\chi_m \rtimes \tau.$$

where $\chi_i \in (\mathbb{C}^\times)\hat{}$, $c_i > 0$ and $\tau \in T$. Since π is Hermitian, formula (5-1) implies that

$$\pi \cong \nu^{a_1} \times \cdots \times \nu^{a_m} \times (\nu^{b_1}\varphi_1 \times \nu^{b_1}\bar{\varphi}_1) \times \cdots \times (\nu^{b_l}\varphi_l \times \nu^{b_l}\bar{\varphi}_l) \rtimes \tau$$

for some $\varphi_i \in (\mathbb{C}^\times)\hat{}\backslash\{1_{\mathbb{C}^\times}\}$, $a_i, b_j > 0$ and $\tau \in T$. Now (2) of above lemma implies that $\nu^{a_i} \rtimes 1$ is unitarizable. Now 4.3 implies $a_i < 1$. Further, (1) of the same lemma implies $b_j < 1/2$. Since $0 < a_i < 1$, we can write exponents $a_1, \ldots, a_m$ as a sequence

$$0 < \alpha_1 \leq \cdots \leq \alpha_k \leq 1/2 < \beta_1 \leq \cdots \leq \beta_\ell < 1$$

Proposition 5.1 and 4.2 (together with relations of the second and third sections) imply that (a) holds, since π is irreducible.

Further, using unitary parabolic reduction we get that $\nu^{a_1} \times \cdots \times \nu^{a_m} \rtimes 1$ is irreducible and unitarizable. Denote this representation again by π.

Consider the case $\ell \geq 2$. Suppose that the cardinality in (c) is ≥ 2 for some j. Take $\alpha_u \leq \alpha_v$ from that set $(u \neq v)$. Instead of ν^{α_v} in π put ν^x with $\alpha_u \leq x \leq \alpha_v$ and denote this representation by π_x. Since α_u and α_v belong to the same set in (c), representations π_x are irreducible. Therefore $\pi_x, \alpha_u \leq x \leq \alpha_v$ is a continuous family of irreducible Hermitian representations. Now (D) implies that π_{α_u} is unitarizable. Now $\pi_{\alpha_u} \cong (\nu^{\alpha_u} \times \nu^{-\alpha_u}) \rtimes \pi'$ (we get π' from π by removing ν^{α_u} and ν^{α_v}). Using parabolic reduction we get that π' is unitarizable.

In this way we have got a unitary representation with cardinality in (c) decreased for two. Continuing this procedure, we can suppose that this cardinality is 0 or 1. Suppose that it is 0. Then in the same way as above, where we have deformed α_v to α_u , we can deform irreducibly β_{j+1} to β_j in π (it is deformation by irreducible representations since between β_j and β_{j+1} there is no $(1-\alpha_i)$'s), and get a representation π'', which is unitarizable by (D). Applying unitary parabolic reduction to π'', we would get that $\nu^{\beta_j} \times \nu^{-\beta_j}$ is unitarizable. This is impossible since $\beta_j > 1/2$ (see 4.2). Therefore, cardinalities in (c) are odd. Note that with odd cardinalities in (c), we have also obtained strict inequalities between β_j's.

Suppose that the cardinality in (b) is ≥ 2. Let α_u, α_v $(u \neq v)$ be in that set. Now we can deform α_v to α_u in π (it is deformation by irreducible representations since α_u and α_v both belong to the set in (b)). Again using unitary parabolic reduction, we get a representation which has the cardinality in (b) decreased for two. Continuing this procedure, we can come to the case when this cardinality is 0 or 1. To finish the proof, we need to show that this number is not 1.

Let the cardinality in (b) be 1. Then $1-\alpha_k < \beta_1$. Further by our assumption $1-\alpha_i > \beta_1$ for all $i \leq k-1$. Let $\alpha_k \leq y \leq \beta_1$. Denote by π_y the representation that we get if we put ν^y instead of ν^{α_k} in π. Applying Proposition 5,1 we shall check that π_y is irreducible. Suppose $y + \alpha_i = 1$ for some $i \leq k-1$. Then $1 - \alpha_i \leq \beta_1$, which is impossible. Suppose $y + \beta_j = 1$ for some j. Then $\alpha_k \leq 1-\beta_j$, which implies $\beta_1 \leq \beta_j \leq 1-\alpha_k$. This contradicts to our assumption $1-\alpha_k < \beta_1$. From this we conclude irreducibility of π_y. Thus, we have continuous family of irreducible Hermitian representations, which contains π. So π_{β_1} is unitarizable. Now $\pi_{\beta_1} \cong \nu^{\beta_1} \times \nu^{-\beta_1} \rtimes \pi'$ where π' is Hermitian. Using unitary parabolic reduction we conclude that $\nu^{\beta_1} \times \nu^{-\beta_1}$ is unitarizable. This is impossible since $\beta_1 > 1/2$ (see 4.2).

This ends the proof of the exhaustion, and completes the proof of the theorem. $\square$

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SOME SMALL UNIPOTENT REPRESENTATIONS OF INDEFINITE ORTHOGONAL GROUPS AND THE THETA CORRESPONDENCE

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ABSTRACT. We locate a family of small unitary representations of the orthogonal groups in the theta correspondence for the dual pairs $(\mathrm{Sp}(2n,\mathbb{R}), \mathrm{O}(p,q))$, identifying them as double stable range lifts of the trivial representation of $\mathrm{O}(n)$. We simultaneously study the corresponding lifts of the determinant character, and show that the two lifts are irreducible constituents of unipotent derived functor modules on the "edge" of the weakly fair range.

1. INTRODUCTION

Consider a sequence of reductive dual pairs $(G_1, G_2), (G_2, G_3), (G_3, G_4), \ldots, (G_k, G_{k+1})$. An old idea is that under suitable hypotheses one should obtain interesting small unitary representations of G_k by beginning with interesting small unitary representations of G_1 and performing a sequence of iterated theta lifts. In particular, one could begin with the simplest unitary representations of G_1, the one-dimensional ones. This framework of iterated theta-lifting then provides a way to organize unitary representations of classical groups. It is natural to ask to what extent theta lifting preserves finer invariants of unitary representations. Questions of this sort have been studied by many authors. The best results of this kind are due to Howe and Li ([19]) and characterize low-rank unitary representations; see also Przebinda's paper [26] for other successes. Our interest here is somewhat more qualitative: suppose that π and π' are two closely related unitary representations of G_1 — for instance, suppose π and π' differ by tensoring with a character of G_1 — then how are their iterated lifts related? This is often very difficult to make explicit. The purpose of this paper is to establish some results in this direction. The virtue of their formulation is that they immediately suggest generalizations.

We work with the following sequence of pairs: $(\mathrm{O}(s,0), \mathrm{Sp}(2m,\mathbb{R}))\,, (\mathrm{Sp}(2m,\mathbb{R}), \mathrm{O}(2m,r))$ with $s \leq m \leq r/2$; these latter inequalities correspond to the stable range. Given a representation π of $\mathrm{O}(s,0)$, we let $\theta^2(\pi)$ denote the corresponding double lift to $\mathrm{O}(2m,r)$. (Because of the covers involved, this notation is imprecise but adequate for the introduction; more complete details are given in Section 2.2.) Let $\mathbf{1}_s$ and $\det_s$ denote the trivial and determinant representations of $\mathrm{O}(s,0)$. Our first result (Theorem 1.2) identifies the double lift $\theta^2(\mathbf{1}_s)$ as the special unipotent representation π'_s introduced by Knapp in [13] and [14] and studied further in [29]. We remark that a study of all double lifts of compact groups has recently been completed by Loke and Nishiyama-Zhu. We return to this below.

Before stating the theorem, we recall the definition of the representations of [14]. Let G be the identity component $\mathrm{SO}_e(2m,r)$ and assume $m \leq r/2$. Write $\mathfrak{g}$ for the complexified Lie algebra of G and τ for the complexified Cartan involution. Let $l = \left[\frac{2m+r}{2}\right]$, the rank of $\mathfrak{g}$. Fix an integer $s \geq 0$ whose parity matches that of r. (This latter condition may

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be dropped if we pass to the nonlinear cover of G, but since those groups do not arise in the theta correspondence we impose the parity condition.) Consider a τ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ in $\mathfrak{g}$ whose Levi factor corresponds to the subgroup

$$L = \mathrm{U}(m, 0) \times \mathrm{SO}_e(0, r) \subset G.$$

Let $\mathbb{C}_{\lambda_s}$ be the one-dimensional representation of L which is the $(-l + \frac{s}{2})$th power of the determinant representation on the $\mathrm{U}(m, 0)$ factor of L and trivial on the SO factor, and consider the derived functor module

$$\pi_s = A_{\mathfrak{q}}(\lambda_s). \tag{1.1}$$

Our notation follows that of [15]; in particular there is a ρ-shift involved in the infinitesimal character. The definition is arranged so that π_s is in the weakly fair range whenever $s \geq m + 1$, and it turns out (see [29]) that indeed π_s is irreducible whenever $s \geq m$. As soon as $s < m$, π_s becomes reducible, however, and we let π'_s denote the unique irreducible constituent of π_s whose lowest K-type matches the lowest K-type of π_s; see Sections 2.6 and 2.7 for more precise details. (This construction is an algebraic version of the kinds of analytic continuations first considered by Wallach in [36] and later taken up by other authors, e.g. [5], [8].) Knapp proves that whenever $0 \leq s_1 < s_2 \leq m$, π'_{s_1} is a unitary representation whose Gelfand-Kirillov dimension is strictly less than that of π'_{s_2}. Thus for $s > m$, the representation π'_s is a unitary cohomologically induced representation of the form $A_{\mathfrak{q}}(\lambda)$. But for $s < m$, π'_s is an interesting small unitary representation that is not obviously cohomologically induced. These latter representations arise very naturally in the theta correspondence. (A minor complication is that π'_s is defined as a representation of the identity component of $\mathrm{O}(2m, r)$ while it is representations of the full orthogonal group that arise in the correspondence.)

Theorem 1.2. *Fix integers $s \leq m \leq r/2$ so that the parity of r matches that of s. When restricted to the identity component $\mathrm{SO}_e(2m, r)$, the iterated lift of the trivial representation of $\mathrm{O}(s, 0)$ to $\mathrm{Sp}(2m, \mathbb{R})$ to $\mathrm{O}(2m, r)$ contains the representation π'_s as a summand.*

Because of some innocuous choices involved in defining the theta correspondence, Theorem 1.2 (and Proposition 1.3 below) are stated slightly imprecisely. See Section 2.2 and the statement of Theorem 2.1 (and Proposition 2.2).

As we mentioned above, Loke and Nishiyama-Zhu have also recently studied double lifts of representations of a compact group [20], [22], [23]. For instance, they give explicit formulas for the restriction of such double lifts to a maximal compact subgroup, say K. Since such formulas are also available for Knapp's representations [29], Theorem 1.2 may be proved by combining [20], [23], and [29]. But our interest here is somewhat different: we seek to identify certain double lifts as special unipotent representations and interpret them in terms of cohomological induction. The virtue of this formulation is that our results suggest generalization beyond double lifts from compact groups. Since it is difficult to recognize singular derived functor modules from their K-spectrums, we develop an alternative route to Theorem 1.2 based on the uniqueness statement given in Proposition 4.1. That approach makes the relationship between the double lifts $\theta^2(\mathbf{1}_s)$ and $\theta^2(\det_s)$ more transparent. In particular, it is closely connected to determining their Langlands parameters (which we do in Section 5).

We now discuss in more detail how the double lift $\theta^2(\det_s)$ is related to $\theta^2(\mathbf{1}_s)$. As we explain in Section 4, this matter is eventually reduced (in the notation of Theorem 1.2) to the case of $s = m$ and $r = 2m$ (or $r = 2m + 1$). In this case, $\pi'_m = \pi_m$ and $\pi'_{m+2} = \pi_{m+2}$;

that is, the full cohomologically induced representations π_m and π_{m+2} are both irreducible. In fact, π_m and π_{m+2} "straddle" the edge of the weakly fair range in the sense that π_k for $k \geq m+1$ is in the weakly fair range, but π_m is not. Moreover, π_m and π_{m+2} are even more closely related in that they have the same infinitesimal character, annihilator, and associated variety. (They are both special unipotent representations attached to the same dual nilpotent orbit.) We have:

Proposition 1.3. *Fix an integer $m > 0$ and let $r = 2m$ if m is even and $r = 2m+1$ if m is odd. When restricted to $\mathrm{SO}_e(2m, r)$, the iterated lift of the determinant representation of $\mathrm{O}(m, 0)$ to $\mathrm{Sp}(2m, \mathbb{R})$ to $\mathrm{O}(2m, r)$ contains the representation π'_{m+2}. (Thus the representations $\theta^2(\mathbf{1}_m)$ and $\theta^2(\det_m)$ straddle the edge of the weakly fair range in the sense described above.)*

As mentioned above, a general (but slightly more technical) statement analogous to Proposition 1.3 for $s \neq m$ is given in Proposition 4.6. In more detail, we understand the relationship between $\theta^2(\mathbf{1}_s)$ and $\theta^2(\det_s)$ by inducing them up to well-understood special unipotent representations of a larger group. (The technique of understanding unipotent representations of a smaller group in terms of those of a larger one is an old idea; for instance, it is one of the main tools in the inductive description of the results of [3].) In the end, loosely speaking, one may say that for all s, $\theta^2(\mathbf{1}_s)$ and $\theta^2(\det_s)$ again straddle the edge of the weakly fair range.

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2. Explicit details concerning π'_s

In this section, we introduce some auxiliary notation and recall some properties of the representation π'_s defined in the introduction.

2.1. **General notation.** Throughout G will denote a reductive Lie group with Lie algebra $\mathfrak{g}_\circ$ and complexification $\mathfrak{g}$. We let $G_{\mathbb{C}}$ denote the connected complex adjoint group associated to $\mathfrak{g}$. Recall that $G_{\mathbb{C}}$ acts on the nilpotent cone in $\mathfrak{g}$ with finitely many orbits; we let $\mathcal{N}$ denote the set of these orbits. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let W denote the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$. We let $\mathfrak{g}^\vee$ denote the complex dual Lie algebra; with $\mathfrak{h}$ fixed, $\mathfrak{g}^\vee$ comes equipped with a Cartan subalgebra $\mathfrak{h}^\vee$ which is canonically isomorphic to $\mathfrak{h}^*$, the linear dual of $\mathfrak{h}$. We let $\mathcal{N}^\vee$ denote orbits of $G^\vee_{\mathbb{C}}$ (the connected complex adjoint group defined by $\mathfrak{g}^\vee$) on the nilpotent cone in $\mathfrak{g}^\vee$.

Let K denote the maximal compact subgroup of G and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the complexified Cartan decomposition. Write $K_{\mathbb{C}}$ for the complexification of K. Then $K_{\mathbb{C}}$ acts with finitely many orbits on the set of nilpotent elements in $\mathfrak{p}$. We denote this set of orbits by $\mathcal{N}_{\mathfrak{p}}$.

2.2. **Notation for the theta correspondence.** Suppose (G, G') is a reductive dual pair in $\mathrm{Sp}(2n, \mathbb{R})$. Let $\mathrm{Mp}(2n, \mathbb{R})$ denote the connected double cover of $\mathrm{Sp}(2n, \mathbb{R})$. Let $\widetilde{G}$ and $\widetilde{G}'$ denote the preimages of G and G' in $\mathrm{Mp}(2n, \mathbb{R})$. Let $\mathrm{Irr}_{\mathrm{gen}}(\widetilde{G})$ denote the set of equivalence classes of irreducible Harish-Chandra modules for $\widetilde{G}$ that do not factor to G and adopt similar notation for $\mathrm{Irr}_{\mathrm{gen}}(\widetilde{G}')$. The theta correspondence is a map

$$\theta \;:\; \mathrm{Irr}_{\mathrm{gen}}(\widetilde{G}) \longrightarrow \mathrm{Irr}_{\mathrm{gen}}(\widetilde{G}') \cup \{0\};$$

here if $\pi \in \mathrm{Irr}_{\mathrm{gen}}(\widetilde{G})$ does not occur in the correspondence we write $\theta(\pi) = 0$. The map θ depends on a choice of oscillator for $\mathrm{Mp}(2n, \mathbb{R})$.

It is often desirable to work directly with Harish-Chandra modules for G rather than genuine representations of $\widetilde{G}$. This is possible only if the cover $\widetilde{G}$ splits. In that case, there exist genuine characters of $\widetilde{G}$. Fix one such η. Then tensoring with η provides a bijection between $\mathrm{Irr}_{\mathrm{gen}}(\widetilde{G})$ and $\mathrm{Irr}(G)$.

We introduce some further notation in one special case. For $s \leq m \leq r/2$, consider the dual pairs $(\mathrm{O}(s,0), \mathrm{Sp}(2m,\mathbb{R}))$ and $(\mathrm{Sp}(2m,\mathbb{R}), \mathrm{O}(2m,r))$. The covers $\widetilde{\mathrm{O}}(s,0)$ and $\widetilde{\mathrm{O}}(2m,r)$ both split, so fix genuine characters η_1 and η_2 of them. Write

$$\theta_1 \;:\; \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{O}}(s,0)) \longrightarrow \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{Sp}}(2m,\mathbb{R})),$$

and

$$\theta_2 \;:\; \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{Sp}}(2m,\mathbb{R})) \longrightarrow \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{O}}(2m,r)).$$

(The conditions that $s \leq m \leq r/2$, i.e. that each pair is in the stable range, dictates that all lifts are nonzero.) Consider the following composition

$$\mathrm{Irr}(\mathrm{O}(s,0)) \longrightarrow \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{O}}(s,0)) \longrightarrow \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{Sp}}(2m,\mathbb{R})) \longrightarrow \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{O}}(2m,r)) \longrightarrow \mathrm{Irr}(\mathrm{O}(2m,r))$$

defined by

$$X \longrightarrow X \otimes \eta_1 \longrightarrow \theta_1(X \otimes \eta_1) \longrightarrow \theta_2\left[\theta_1(X \otimes \eta_1)\right] \longrightarrow \theta_2\left[\theta_1(X \otimes \eta_1)\right] \otimes \eta_2.$$

We denote this composition by θ^2. It depends on choices of η_1, η_2, and of the oscillators defining θ_1 and θ_2. We make the standard choices, i. e., those used, e. g., in [21], [1], [24], and the references found in these papers. For η_1 this means that it is the unique character of $\widetilde{\mathrm{O}}(s,0)$ which occurs in the correspondence for every dual pair $\big(\mathrm{O}(s,0), \mathrm{Sp}(2k,\mathbb{R})\big)$ (for which $\mathrm{O}(s,0)$ has the same cover $\widetilde{\mathrm{O}}(s,0)$). The possible choices for η_2 differ by a character of $\mathrm{O}(2m,r)$; this choice does not affect the restriction of a representation to the identity component $\mathrm{SO}_e(2m,r)$.

With these choices in mind, we now restate Theorem 1.2 and Proposition 1.3 from the introduction.

Theorem 2.1. *Retain the notation introduced above. Fix integers $s \leq m \leq r/2$ so that the parity of r matches that of s. Then $\theta^2(\mathbf{1}_s)$ restricted to $\mathrm{SO}_e(2m,r)$ contains π'_s as a summand.*

Proposition 2.2. *Retain the notation introduced above. Fix an integer $m > 0$ and let $r = 2m$ if m is even and $r = 2m+1$ if m is odd. Then $\theta^2(\det_m)$ restricted to $\mathrm{SO}_e(2m,r)$ contains π'_{m+2} as a summand.*

2.3. **Primitive ideals: generalities.** A two-sided ideal in the enveloping algebra $\mathrm{U}(\mathfrak{g})$ is called primitive if it is the annihilator of a simple $\mathrm{U}(\mathfrak{g})$ module. Since each such simple module has an infinitesimal character, i.e. is annihilated by a unique codimension-one ideal in the center $\mathrm{Z}(\mathfrak{g})$, it is clear that each primitive ideal also contains a unique codimension-one ideal in $\mathrm{Z}(\mathfrak{g})$. Such ideals are parametrized (via the Harish-Chandra isomorphism) by W orbits on $\mathfrak{h}^*$. Let $\mathrm{Prim}(\mathfrak{g})$ denote the set of primitive ideals in $\mathrm{U}(\mathfrak{g})$ and $\mathrm{Prim}_\chi(\mathfrak{g})$ those with infinitesimal character $\chi \in \mathfrak{h}^*/W$. Duflo proved that $\mathrm{Prim}_\chi(\mathfrak{g})$ is finite and (in the inclusion partial order) contains a unique maximal element $\mathrm{J}_{\max}(\chi)$.

The associated variety $\mathrm{AV}(I)$ of $I \in \mathrm{Prim}(\mathfrak{g})$ is defined as follows. The ideal I inherits a grading from the obvious grading on $\mathrm{U}(\mathfrak{g})$. The associated graded $\mathrm{gr}(I)$ is a two-sided ideal in $\mathrm{gr}(\mathrm{U}(\mathfrak{g})) = S(\mathfrak{g})$. Thus $\mathrm{gr}(I)$ cuts out a subvariety, denoted $\mathrm{AV}(I)$, of $\mathfrak{g}^* \simeq \mathfrak{g}$. According to a well-known result of Borho-Brylinski, $\mathrm{AV}(I)$ is the closure of a unique element of $\mathcal{N}$.

2.4. **Special unipotent representations.** Given an orbit $\mathcal{O}^\vee$ in $\mathcal{N}^\vee$, we may construct a Jacobson-Morozov triple $\{e^\vee, h^\vee, f^\vee\}$ with $e^\vee \in \mathcal{O}^\vee$ and $h^\vee \in \mathfrak{h}^\vee$. We define $\chi(\mathcal{O}^\vee) = \frac{1}{2}h^\vee \in \mathfrak{h}^\vee \simeq \mathfrak{h}$. Different choices in this construction lead to at most a Weyl group translate of $\chi(\mathcal{O}^\vee)$. Hence $\chi(\mathcal{O}^\vee)$ is a well-defined element of $\mathfrak{h}^*/W$, and thus defines an infinitesimal character. We write $\mathrm{J}_{\max}(\mathcal{O}^\vee)$ for the maximal ideal $\mathrm{J}_{\max}(\chi(\mathcal{O}^\vee))$. Recall the Spaltenstein duality map

$$d\ : \mathcal{N} \longrightarrow \mathcal{N}^\vee,$$

as treated in the appendix to [2]. According to [2, Corollary A3],

$$\mathrm{AV}(\mathrm{J}_{\max}(\mathcal{O}^\vee) = \overline{d(\mathcal{O}^\vee)}.$$

A Harish-Chandra module X for G is called *integral special unipotent* if there exists an orbit $\mathcal{O}^\vee$ such that $\chi(\mathcal{O}^\vee)$ is integral and $\mathrm{Ann}(X) = \mathrm{J}_{\max}(\mathcal{O}^\vee)$ (with notation as in Section 2.3).

2.5. **A family of nilpotent orbits.** Suppose $\mathfrak{g} = \mathfrak{so}(2l, \mathbb{C})$. (In applications below we will take $2l = 2m+r$.) Then $\mathcal{N}$ is parametrized by partitions of $2l$ in which each even part occurs with even multiplicity. (In the case where all even parts have even multiplicity, there is an additional complication; it does not arise for us, however, and we ignore it.) Fix s even so that $2s \leq 2m \leq l$. We let $\mathcal{O}(s)$ denote the orbit parametrized as follows:

$$\mathcal{O}(s) = 3^s 2^{2m-2s} 1^{2l-4m+s} \qquad \text{if } (l,s) \neq (2m, 0) \tag{2.3}$$

$$\mathcal{O}(s) = 2^{2m-2s-2} 1^4 \qquad \text{if } (l,s) = (2m, 0); \tag{2.4}$$

Next suppose $\mathfrak{g} = \mathfrak{so}(2l+1, \mathbb{C})$. (In applications below we will take $2l+1 = 2m+r$.) Then $\mathcal{N}$ is again parametrized by partitions of $2l+1$ in which each even part occurs with even multiplicity. Fix s odd so that $2s \leq 2m \leq l$. We let $\mathcal{O}(s)$ denote the orbit parametrized as follows:

$$\mathcal{O}(s) = 3^s 2^{2m-2s} 1^{2l-4m+s+1}. \tag{2.5}$$

2.6. **Explicit details of the representation π_s': even orthogonal groups.** Fix integers s, m, and l such that s is even, $m \leq l/2$, and $0 \leq s \leq m$. Recall the definition of π_s' given in the introduction. In the notation of Sections 2.4 and 2.5, set $\mathcal{O}^\vee(s) = d(\mathcal{O}(s))$. The main result [29, Theorem 1.1] shows that π_s' is special unipotent attached to $\mathcal{O}^\vee(s)$. More precisely,

$$\mathrm{Ann}(\pi_s') = \mathrm{J}_{\max}(\mathcal{O}^\vee(s)) \text{ and } \mathrm{AV}(\pi_s') = \overline{\mathcal{O}(s)}; \tag{2.6}$$

here we are using the notation of Section 2.3.

We recall the infinitesimal character ν_s of π_s'. In the usual coordinates we have,

$$\nu_s = \chi(\mathcal{O}^\vee(s)) = (\overbrace{0, 1, \ldots, l-m-1}^{l-m}, \overbrace{\left|\frac{s}{2} - m\right|, \left|\frac{s}{2} - m + 1\right|, \ldots, \left|\frac{s}{2} - 1\right|}^{m}). \tag{2.7}$$

(The use of ν_s to represent both an infinitesimal character and a particular representative causes no confusion in practice.) Notice ν_s is integral since s is even.

Finally we recall the lowest K-type of π'_s. Retain the notation of the introduction and, in particular, let λ_s denote the differential of the character $\det^{-l+\frac{s}{2}} \otimes \mathbf{1}$ of $L_{\mathbb{R}}$. Our hypotheses guarantee that

$$\Lambda_s = \lambda_s + 2\rho(\mathfrak{u} \cap \mathfrak{p}) = (l - 2m + \tfrac{s}{2}, \ldots, l - 2m + \tfrac{s}{2}; 0, \ldots, 0) \qquad (2.8)$$

is dominant, and hence parametrizes (cf. Section 2.9) the lowest K-type in π_s. Therefore, by definition, Λ_s is the lowest K-type of π'_s

2.7. **Explicit details of the representation π'_s: odd orthogonal groups.** Fix integers s, m, and l such that s is odd, $m \leq l/2$, and $0 \leq s \leq m$. In the notation of Section 2.5 and 2.4, set $\mathcal{O}^\vee(s) = d(\mathcal{O}(s))$. Again [29, Theorem 1.1] shows that π'_s is special unipotent attached to $\mathcal{O}(s)$, and the conclusions of Equation (2.6) again hold. We recall the infinitesimal character ν_s of π'_s. In the usual coordinates we have,

$$\nu_s = \chi(\mathcal{O}^\vee(s)) = (|-1 + \tfrac{s}{2}|, |-2 + \tfrac{s}{2}|, \ldots, |-m + \tfrac{s}{2}|; l - m - \tfrac{1}{2}, l - m - \tfrac{3}{2}, \ldots, \tfrac{1}{2}). \quad (2.9)$$

Notice ν_s is integral since s is odd.

Finally we recall the lowest K-type of π_s and π'_s. It is parametrized (cf. Section 2.9) by

$$\Lambda_s = \lambda_s + 2\rho(\mathfrak{u} \cap \mathfrak{p}) = (l - 2m + 1 + \frac{1}{2}(s-1), \ldots, l - 2m + 1 + \frac{1}{2}(s-1); 0, \ldots, 0). \quad (2.10)$$

2.8. **Two auxiliary representations.** The discussion in Sections 2.6 and 2.7 made use of the fact that $m \leq l/2$. We now relax that condition and assume only that $m \leq l$. The definition π_s of Equation (1.1) still makes sense. The module π_s is in the weakly fair range if and only if $s \geq m + 1$. Now take $s = m$ or $m + 2$ and assume m has the same parity as l. So π_m and π_{m+2} "straddle" the weakly fair range as discussed in the introduction. It is easy to check that they have the same infinitesimal character and associated variety. In fact, the methods of [29] show that π_m and π_{m+2} are both irreducible and special unipotent attached to $\mathcal{O}^\vee(m)$ where $\mathcal{O}^\vee(m) = d(\mathcal{O}(m))$ and $\mathcal{O}(m)$ is defined by the explicit partitions given in Section 2.5 (which still make sense even though m is now only assumed to be weakly less than l). Notice also that the equations defining Λ_m and Λ_{m+2} in (2.8) and (2.10) still give a dominant K-type.

2.9. **Representations of $O(p,q)$ and $SO_e(p,q)$, and K-types.** If p and q are positive integers, let $p_0 = [\frac{p}{2}]$ and $q_0 = [\frac{q}{2}]$. We list the irreducible representations of $O(p)$ by parameters $\mu = (\mu_0; \epsilon)$, where $\mu_0 = (a_1, a_2, \ldots, a_{p_0})$ with the a_i non-increasing non-negative integers, and $\epsilon = \pm 1$, as described in §2.2 of [24]. The $O(n)$-type parametrized by $(\mu_0; -\epsilon)$ is obtained from that given by $(\mu_0; \epsilon)$ by tensoring with the determinant character, and the parameters $(\mu_0; \epsilon)$ and $(\mu_0; -\epsilon)$ correspond to the same representations of $O(p)$ if and only if p is even and $a_{p_0} > 0$. In this case, the restriction to $SO(p)$ is the sum of two representations with highest weights μ_0 and $(a_1, a_2, \ldots, a_{p_0-1}, -a_{p_0})$, respectively. In all other cases, the restriction to $SO(p)$ is irreducible and has highest weight μ_0. A K-type for $O(p,q)$ can be specified by a parameter of the form $(a_1, a_2, \ldots, a_{p_0}; \epsilon) \otimes (b_1, \ldots, b_{q_0}; \eta)$ with the a_i and b_i non-decreasing non-negative integers, and $\epsilon, \eta = \pm 1$. In many cases in this paper, the signs (ϵ, η) may be chosen to be $(+1, +1)$; in that case we will often omit them and simply write the K-type by giving its highest weight $(a_1, a_2, \ldots, a_{p_0}; b_1, \ldots, b_{q_0})$.

The irreducible admissible representations of $O(p,q)$ may be obtained by induction from irreducible admissible representations of the identity component $SO_e(p,q)$; since $SO_e(p,q)$ has index four in $O(p,q)$, the resulting induced representation can have one, two, or four irreducible summands, resulting in four, two, or one non-equivalent irreducible admissible representations of $O(p,q)$ containing a given representation of $SO_e(p,q)$ as a summand in its restriction. (Two such representations differ by tensoring with one of the one-dimensional representations of $O(p,q)$; see §3.2 of [24] for more details.) For our representations π'_s, we are typically in the intermediate situation: there are two non-equivalent representations of $O(2m,r)$ (distinguished by signs as indicated in §5), each having two summands when restricted to the identity component, one of which is π'_s. The second summand is a representation of $SO_e(2m,r)$ whose lowest K-type is obtained from Λ_s by changing the sign of the mth entry. Only one of the two representations of $O(2m,r)$ occurs as a stable range theta lift.

3. The Correspondence and K-Types

We start by recalling the correspondence of K-types in the space of joint harmonics $\mathcal{H}$ for the dual pairs $\big(\mathrm{Sp}(2n,\mathbb{R}),\mathrm{O}(p,q)\big)$ [10]. We identify K-types for $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ (i. e., irreducible representations of $\widetilde{\mathrm{U}}(n)$) with their highest weights, and K-types for $O(p,q)$ as described in §2.9. Recall that each K-type μ which occurs in the Fock space $\mathcal{F}$ of the oscillator representation has associated to it a degree (the minimum degree of polynomials in the μ-isotypic subspace), and that if π and π' are representations of $\widetilde{G}$ and $\widetilde{G}'$, respectively, which correspond to each other, then each K-type for $\widetilde{G}$ which is of minimal degree in π will occur in $\mathcal{H}$ and correspond to a K-type for $\widetilde{G}'$ of minimal degree in π'. Since for a given choice of n, $\mathrm{Sp}(2n,\mathbb{R})$ is a member of many dual pairs, we refer to the degree of a K-type σ for $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ for the dual pair $\big(\mathrm{Sp}(2n,\mathbb{R}),\mathrm{O}(p,q)\big)$ as the (p,q)*-degree* of σ. As we will see below in Proposition 3.1, the degree of a K-type for $\mathrm{Sp}(2n,\mathbb{R})$ depends on the difference $p-q$ only, and the degree of a K-type for $\mathrm{O}(p,q)$ is independent of n. However, it depends not only on the highest weight but also on the signs. Consequently, two K-types with the same highest weight but different signs may have different degrees.

Proposition 3.1. *Let p, q, and n be non-negative integers, $p_0=[\frac{p}{2}]$, and $q_0=[\frac{q}{2}]$. The correspondence of K-types in the space of joint harmonics $\mathcal{H}$ for the dual pair $\big(\mathrm{Sp}(2n,\mathbb{R}),\mathrm{O}(p,q)\big)$ is given as follows.*

(1) *Let*
$$\mu=(a_1,a_2,\dots,a_x,0,\dots,0;\epsilon)\otimes(b_1,b_2,\dots,b_y,0,\dots,0;\eta) \tag{3.2}$$
be a K-type for $\mathrm{O}(p,q)$, with $a_x>0$ and $b_y>0$. Then μ occurs in $\mathcal{H}$ if and only if $n\ge x+\frac{1-\epsilon}{2}(p-2x)+\frac{1-\eta}{2}(q-2y)+y$. In that case, μ corresponds to
$$\big(\tfrac{p-q}{2},\tfrac{p-q}{2},\dots,\tfrac{p-q}{2}\big)+(a_1,\dots,a_x,\underbrace{1,\dots,1}_{\frac{1-\epsilon}{2}(p-2x)},0,\dots,0,\underbrace{-1,\dots,-1}_{\frac{1-\eta}{2}(q-2y)},-b_y,\dots,-b_1). \tag{3.3}$$

(2) *If a K-type μ for $\mathrm{O}(p,q)$ as in (3.2) occurs in the Fock space, then the degree of μ is*
$$\sum_{i=1}^{x}a_i+\sum_{i=1}^{y}b_i+\frac{1-\epsilon}{2}(p-2x)+\frac{1-\eta}{2}(q-2y). \tag{3.4}$$
For a K-type for $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ ξ which occurs in $\mathcal{F}$, write

$$\xi = \left(\tfrac{p-q}{2}, \tfrac{p-q}{2}, \dots, \tfrac{p-q}{2}\right) + (a_1, a_2, \dots, a_n). \tag{3.5}$$

Then the (p,q)-degree of ξ is $\sum_{i=1}^n |a_i|$.

Proof. This is well known and follows easily from the results of [16] and [10]. □

In many cases it turns out that the lowest K-type of a representation which occurs in the correspondence turns out to have minimal degree in the above sense. This is true in particular for a representation of the member of the dual pair which has greater rank.

Proposition 3.6. *Let p, q, and n be non-negative integers, and suppose π and π' are genuine irreducible admissible representations of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ and $\widetilde{\mathrm{O}}(p,q)$ which correspond to each other in the correspondence for the dual pair $\big(\mathrm{Sp}(2n,\mathbb{R}), \mathrm{O}(p,q)\big)$.*

(1) *If $p+q \le 2n+1$ then every lowest K-type of π is of minimal (p,q)-degree in π.*
(2) *Let $2n+1 \le p+q$, and let Λ_0 be the highest weight of a lowest K-type of π'. Then there exists a lowest K-type Λ of π' with highest weight Λ_0 such that Λ is of minimal degree in π'.*

Proof. For the case $p+q$ even this is Corollary 37 of [24]. If $p+q = 2n+1$ this follows from Corollary 5.2 of [1]. (In fact, here all lowest K-types of π' are of minimal degree in π'.) If $p+q$ is odd and $p+q < 2n+1$ let $k = \frac{1}{2}(2n+1-p-q)$ so that $(p+k)+(q+k) = 2n+1$. We know by the persistence principle (due to Kudla; this also follows from the induction principle, Theorem 8.4 of [1]) that π occurs in the correspondence for the dual pair $\big(Sp(2n,\mathbb{R}), O(p+k, q+k)\big)$, so that the lowest K-types of π are of minimal $(p+k, q+k)$-degree. Since the (p,q)-degree and the $(p+k, q+k)$-degree of a K-type for $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ coincide, the lowest K-types of π are of minimal degree for the original dual pair. The case $p+q > 2n+1$ is analogous. □

Now we compute lowest K-types for our double stable range lifts; they turn out to be unique up to signs. (The conclusion of Proposition 3.7 may be extracted from the techniques and results of [22] or [20]. We give an alternative approach.)

Proposition 3.7. *Let p, q, m, and r be non-negative integers such that $p+q \le m \le \frac{r}{2}$. Let χ be either the trivial or determinant representation of $\mathrm{O}(p,q)$. Recall from Section 2.2 the maps*

$$\theta \ : \ \mathrm{Irr}(\mathrm{O}(p,q)) \longrightarrow \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{Sp}}(2m,\mathbb{R}))$$

and

$$\theta' \ : \ \mathrm{Irr}_{\mathrm{gen}}(\widetilde{\mathrm{Sp}}(2m,\mathbb{R})) \longrightarrow \mathrm{Irr}(\mathrm{O}(2m,r)),$$

and set $\pi = \theta(\chi)$ and $\pi' = \theta'(\pi)$.

(1) *If $\chi = \mathbb{1}$ then the K-types of π are precisely those of the form*

$$\left(\frac{p-q}{2}, \dots, \frac{p-q}{2}\right) + (a_1, a_2, \dots, a_p, 0, \dots, 0, -b_q, \dots, -b_1), \tag{3.8}$$

where a_i and b_i are non-negative even integers for all i, and each K-type occurs with multiplicity one.

(2) *If* $\chi = \mathbb{1}$, *let* $a = \frac{p-q+r}{2} - m$. *Then* π' *has a lowest* K*-type* Λ, *unique up to signs, given by*

$$\Lambda = \begin{cases} (a, a, \ldots, a, \underbrace{0, \ldots, 0}_{q}; 0, \ldots, 0) & \text{if } a \geq 0 \text{ and even;} \\ (a, a, \ldots, a, \underbrace{1, \ldots, 1}_{q}; 0, \ldots, 0) & \text{if } a > 0 \text{ and odd;} \\ (0, \ldots, 0; \underbrace{-a, \ldots, -a}_{m-p}, 0, \ldots, 0) & \text{if } a \leq 0 \text{ and even;} \\ (0, \ldots, 0; \underbrace{-a, \ldots, -a}_{m-p}, \underbrace{1, \ldots, 1}_{p}, 0, \ldots, 0) & \text{if } a < 0 \text{ and odd.} \end{cases} \tag{3.9}$$

(3) *If* $\chi = \det$ *then the* K*-types of* π *are precisely those of the form (3.8) with* a_i *and* b_i *positive odd integers for all* i, *and each* K*-type occurs with multiplicity one.*

(4) *If* $\chi = \det$, *let* $a = \frac{p-q+r}{2} - m$. *Then* π' *has a lowest* K*-type* Λ, *unique up to signs, given by*

$$\Lambda = \begin{cases} (\underbrace{1, \ldots, 1}_{p}, 0, \ldots, 0; \underbrace{1, \ldots, 1}_{q}, 0, \ldots, 0) & \text{if } a = 0\ ; \\ (\underbrace{a+1, \ldots, a+1}_{p}, a, \ldots, a, \underbrace{0, \ldots, 0}_{q}; 0, \ldots, 0) & \text{if } a > 0 \text{ and odd;} \\ (\underbrace{a+1, \ldots, a+1}_{p}, a, \ldots, a, \underbrace{1, \ldots, 1}_{q}; 0, \ldots, 0) & \text{if } a > 0 \text{ and even;} \\ (0, \ldots, 0; \underbrace{-a+1, \ldots, -a+1}_{q}, \underbrace{-a, \ldots, -a}_{m-p-q}, 0, \ldots, 0) & \text{if } a < 0 \text{ and odd;} \\ (0, \ldots, 0; \underbrace{-a+1, \ldots, -a+1}_{q}, \underbrace{-a, \ldots, -a}_{m-p-q}, \underbrace{1, \ldots, 1}_{p}, 0, \ldots, 0) & \text{if } a < 0 \text{ and even.} \end{cases} \tag{3.10}$$

Proof. The first part of the proposition is Proposition 2.1, together with Corollary 2.7(c), of [17]. For (2), let

$$\sigma = \left(\tfrac{p-q}{2}, \ldots, \tfrac{p-q}{2}\right) + (a_1, a_2, \ldots, a_p, 0, \ldots, 0, -b_q, \ldots, -b_1), \tag{3.11}$$

be a K-type of π. Then

$$\sigma = \left(\tfrac{2m-r}{2}, \ldots, \tfrac{2m-r}{2}\right) + (a + a_1, \ldots, a + a_p, a, \ldots, a, a - b_q, \ldots, a - b_1), \tag{3.12}$$

and the $(2m, r)$-degree of σ is

$$d = \sum_{i=1}^{p} |a + a_i| + \sum_{i=1}^{q} |a - b_q| + (m - p - q)|a|. \tag{3.13}$$

Suppose a is even. Then the degree of σ is minimized if $a_i = 0$ and $b_i = a$ for all i if $a \geq 0$, and by choosing $a_i = -a$ and $b_i = 0$ for all i if a is negative. So π has a unique K-type σ_0 of minimal $(2m, r)$-degree. The K-type Λ corresponding to σ_0 in the space of joint harmonics for the dual pair $\big(Sp(2m,\mathbb{R}), O(2m,r)\big)$ (see Proposition 3.1) must then be a lowest K-type of π', unique up to signs, by Proposition 3.6.

If a is odd then π has more than one K-type of minimal $(2m,r)$-degree: for $a<0$ choose $b_i=0$ for all i, and $a_i=-a+1$ or $-a-1$ in such a way that the resulting weight is dominant. The corresponding K-types for $O(2m,r)$ are then those of the form

$$\Lambda_{k,t}=\big(\underbrace{1,\dots,1}_{k},0,\dots,0;\underbrace{-a,\dots,-a}_{m-p},\underbrace{1,\dots,1}_{t},0,\dots,0\big) \tag{3.14}$$

with $k+t=p$. By Proposition 3.6, the lowest K-types of π' are precisely those $\Lambda_{k,t}$ for which the Vogan-norm ([33] Definition 5.4.18) of $\Lambda_{k,t}$,

$$||\Lambda_{k,t}|| =<\Lambda_{k,t}+2\rho_c,\Lambda_{k,t}+2\rho_c> \tag{3.15}$$

is minimal. Since the quantity $<\Lambda_{k,t},\Lambda_{k,t}>$ only depends on $k+t=p$, we can minimize the quantity $n_{k,t}=\frac{1}{2}\big(||\Lambda_{k,t}||-<2\rho_c,2\rho_c>-<\Lambda_{k,t},\Lambda_{k,t}>\big)$ instead. If $n=[\frac{r}{2}]$ then $2\rho_c=(2m-2,2m-4,\dots,2,0;r-2,r-4,\dots,r-2n)$. So

$$n_{k,t}=\sum_{i=1}^{k}(2m-2i)+\sum_{i=1}^{m-p}(-a)(r-2i)+\sum_{i=1}^{t}(r-2m+2p-2i). \tag{3.16}$$

The second sum is independent of k and t, and it is clear that $n_{k,t}$ will be minimized by $k=p$, $t=0$ if $2m\le r-2m+2p$, and by $k=0$, $t=p$ if $2m\ge r-2m+2p$. It remains to show that we must have $4m>r+2p$. We know that $a=\frac{p-q+r}{2}-m<0$, so $p-q+r<2m$, Moreover, we have assumed that $p+q\le m$, so we get $2p+r<3m\le 4m$, and we are done with this case. The case $a>0$ is much easier and left to the reader.

Part (3) is Proposition 2.4 of [18], together with Proposition 2.1 of [11], and part (4) can be easily obtained using (3) and an argument similar to the one used for the case $\chi=\mathbb{1}$. We omit the details.

□

We conclude by introducing some additional notation. Fix integers $s\le m\le r/2$ so that the parities of s and r match. Recall the double lift θ^2 of Section 2.2 and assume that the choices made defining θ^2 match those in Proposition 3.7. Consider the set of highest weights of lowest K-types of the restriction of $\theta^2(\mathbf{1}_s)$ to $\mathrm{SO}_e(2m,r)$. In terms of the parametrization of Section 2.9, this set contains a unique highest weight (possibly occurring with multiplicity greater than one) whose coordinates are all nonnegative. Call the corresponding lowest K-type "positive". Then set

$\Lambda_s=$ the K-type that occurs as a "positive" lowest K-type in the restriction of $\theta^2(\mathbf{1}_s)$ to $\mathrm{SO}_e(2m,r)$ (3.17)

and

$\Lambda'_s=$ the K-type that occurs as a "positive" lowest K-type in the restriction of $\theta^2(\det_s)$ to $\mathrm{SO}_e(2m,r)$. (3.18)

Explicit formulas for them are given in Proposition 3.7. Notice that Λ_s has already been defined in Sections 2.6 and 2.7. Using Proposition 3.7, it is easy to check that the two definitions coincide.

4. Characterizing certain unipotent representations by their lowest K types: proof of Theorem 1.2 and Proposition 1.3

As alluded to in the introduction, the following is the key uniqueness result we need. It allows us to identify our special unipotent representations of interest by simply computing their associated varieties, infinitesimal characters, and lowest K-types.

Proposition 4.1. *Recall the notation of Sections 2.6 and 2.7, as well as that of Equations (3.17) and (3.18). There is a unique special unipotent representation of* $\mathrm{SO}_e(2m,r)$ *attached to* $\mathcal{O}^\vee(s)$ *with lowest* K*-type* Λ_s. *Similarly if* $s \neq 0$ *there is a unique special unipotent representation attached to* $\mathcal{O}^\vee(s)$ *with lowest* K*-type* Λ_s'.

Remark 4.2. The proof is rather technical, and we defer it to Section 5. Recall (from the discussion in Section 2.4 for instance) that a representation is a special unipotent representation if it has the right infinitesimal character and size. So the proposition roughly says that the infinitesimal character, size, and lowest K-types characterize certain representations uniquely. For many of the representations appearing in the proposition, the infinitesimal character and lowest K-types are enough to guarantee unicity; but there are some for which this is not enough and the further consideration of size must also be invoked.

As we shall see, the proof also extends to another case which will be important for us. Retain the relaxed setting of Section 2.8. Then π_m is the unique special unipotent representation attached to $\mathcal{O}^\vee(m)$ with lowest K-type Λ_m. Likewise π_{m+2} is the unique special unipotent representation attached to $\mathcal{O}^\vee(m)$ with lowest K type Λ_{m+2}. □

The next proposition says that the iterated lifts $\theta^2(\mathbf{1}_s)$ and $\theta^2(\det_s)$ from $\mathrm{O}(s,0)$ to $\mathrm{Sp}(2m,\mathbb{R})$ to $\mathrm{O}(2m,r)$ contain the unique representations of Proposition 4.1.

Proposition 4.3. *Fix integers* $s \leq m \leq r/2$ *so that the parity of* s *coincides with that of* r. *The double lift* $\theta^2(\mathbf{1}_s)$ *of the trivial representation lifted from* $\mathrm{O}(s)$ *to* $\mathrm{Sp}(2m,\mathbb{R})$ *and then to* $\mathrm{O}(2m,r)$ *restricted to* $\mathrm{SO}_e(2m,r)$ *contains the unique special unipotent representation of* $\mathrm{SO}_e(2m,r)$ *attached to* $\mathcal{O}^\vee(s)$ *with lowest* K*-type* Λ_s. *Similarly if* $s \neq 0$ *the restriction of the double lift* $\theta^2(\det_s)$ *of the determinant representation contains the unique special unipotent representation of* $\mathrm{SO}_e(2m,r)$ *attached to* $\mathcal{O}^\vee(s)$ *with lowest* K*-type* Λ_s'.

Proof. That Λ_s and Λ_s' are the indicated lowest K-types follows from the definitions of Equations (3.17) and (3.18). So all that remains to show is that the two double lifts are special unipotent attached to $\mathcal{O}^\vee(s)$. Using the correspondence of infinitesimal characters ([27]), it is easy to check that the two double lifts have the required infinitesimal character ν_s given in Sections 2.6 and 2.7. Hence it remains to verify only that the dense orbit (say $\mathcal{O}_s$) in the associated variety of the annihilator of the double lifts is indeed $\mathcal{O}(s)$. Since there are no representations with infinitesimal character ν_s with smaller associated variety (as follows from the general theory of special unipotent representations), it enough to show that that $\mathcal{O}_s \subset \mathcal{O}(s)$. Now the paper [26] gives an explicit upper bound on $\mathcal{O}_s$ in terms of certain moment map images. That upper bound can be computed explicitly and shown to coincide with $\mathcal{O}(s)$. (Computations of this sort are explained very carefully in [31].) The proof is complete. □

The main results of the introduction are now simple corollaries.

Proof of Theorem 1.2. As explained in Sections 2.6 and 2.7, [29] shows that π_s' is special unipotent attached to $\mathcal{O}(s)$ with lowest K-type Λ_s. So the theorem follows from Propositions 4.1 and 4.3.

Proof of Proposition 1.3. In the setting of Proposition 1.3, [29] shows that π_m and π_{m+2} are special unipotent representations attached to $\mathcal{O}^\vee(m)$ with respective lowest K-types Λ_m and Λ_m'. So the current proposition follows from Propositions 4.1 and 4.3. □

We now explain how to extend Proposition 1.3 to a more general setting. The idea, as mentioned in the introduction, is to induce $\theta^2(\mathbf{1}_s)$ and $\theta^2(\det_s)$ to special unipotent representations of a larger group G' that we can quickly recognize in terms of cohomological induction. The subtlety is to arrange the induction so that we indeed obtain (nonzero) special unipotent representations of G'. Here are the details.

Again fix integers $s \leq m \leq r/2$ so that the parity of s matches that of r. Consider the restrictions to $\mathrm{SO}_e(2m, r)$, say $\theta_e^2(\mathbf{1}_s)$ and $\theta_e^2(\det_s)$, of the double lifts $\theta^2(\mathbf{1}_s)$ and $\theta^2(\det_s)$ which contain the respective lowest K-types Λ_s and Λ_s'. Let $G' = \mathrm{SO}_e(4m - 2s, r)$ and maintain the notation of Section 2.1 with the addition of primes as appropriate. Let $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$ be a τ'-stable parabolic of $\mathfrak{g}'$ so that $\mathfrak{l}' \cap \bar{\mathfrak{l}}'$ corresponds to

$$L' = \mathrm{U}(m - s, 0) \times \mathrm{SO}_e(2m, r).$$

Suppose that $\mathfrak{u}'$ corresponds to a choice of positive roots so that the roots corresponding to the $\mathrm{U}(m - s, 0)$ factor appear before those in the SO factor. Consider the $(\mathfrak{l}', L' \cap K')$ module

$$\det{}^k \otimes \theta_e^2(\mathbf{1}_s),$$

where

$$k = -l + [s/2],$$

and $[s/2]$ is the greatest integer less than $s/2$. One may verify that $\det^k \otimes \theta_e^2(\mathbf{1}_s)$ is in the weakly fair range for $\mathfrak{q}$ ([15, Definition 0.35]). Let S be the middle degree $\dim(\mathfrak{u}' \cap \mathfrak{k}')$, and finally consider the derived functor module

$$\Gamma_{\mathbf{1}} = \mathcal{R}_{\mathfrak{q}'}^S(\det{}^k \otimes \theta_e^2(\mathbf{1}_s)). \tag{4.4}$$

Again we follow the normalization of [15] (and the notation of [33]) so that there is a ρ-shift in the infinitesimal character of Γ_1. Using the explicit formula for the lowest K-type Λ_s of $\theta_e^2(\mathbf{1}_s)$ given above, it is simple to check that the highest weight of the lowest K-type of $\det^k \otimes \theta_e^2(\mathbf{1}_s)$ shifted by $2\rho(\mathfrak{u}' \cap \mathfrak{p}')$ is still dominant and hence parametrizes the lowest K-type of Γ_1 (which, in particular, is thus nonzero). It seems likely that the methods of [35] could be applied to show that Γ_1 is irreducible. To be on the safe side, let Γ_1' denote the lowest K-type constituent of Γ_1. Similarly put

$$\Gamma_{\det} = \mathcal{R}_{\mathfrak{q}'}^S(\det{}^k \otimes \theta_e^2(\det{}_s)), \tag{4.5}$$

and let $\Gamma_{\det}'$ denote its lowest K-type constituent. (Again it is likely that $\Gamma_{\det}' = \Gamma_{\det}$.)

The following generalization of Proposition 1.3 states that the modules $\Gamma_{\mathbf{1}}'$ and $\Gamma_{\det}'$ induced from the double lifts of the trivial and determinant representation are special unipotent $A_{\mathfrak{q}}(\lambda)$ modules that straddle the weakly fair range.

Proposition 4.6. *Fix integers $s \leq m \leq r/2$ so that the parity of r matches that of s. Set $G' = \mathrm{SO}_e(4m - 2s, r)$. Consider the representations $\Gamma_{\mathbf{1}}'$ and $\Gamma_{\det}'$ of G' defined around Equations (4.4) and (4.5) above as cohomologically induced from the iterated lifts of the trivial and determinant representation of $\mathrm{O}(s, 0)$ to $\mathrm{Sp}(2m, \mathbb{R})$ to $\mathrm{O}(2m, r)$. Next recall* *the special unipotent representations π_{4m-2s} and $\pi_{4m-2s+2}$ of G' attached to $\mathcal{O}^\vee(4m-2s)$ discussed in Section 2.8. (These are cohomologically induced modules that straddle the edge of weakly fair range.) Then*

$$\Gamma'_{\mathbf{1}} = \pi_{4m-2s},$$

and

$$\Gamma'_{\det} = \pi_{4m-2s+2}.$$

In particular, when $s = m$ we recover Proposition 1.3.

Proof. We first show $\Gamma'_{\mathbf{1}} = \pi_{4m-2s}$. In the discussion around the definition of $\Gamma'_{\mathbf{1}}$, we mentioned that its lowest K-type is the lowest K-type of $\det^k \otimes \theta_e^2(\mathbf{1}_s)$ shifted by $2\rho(\mathfrak{u}' \cap \mathfrak{p}')$. It is easy to check that it matches the lowest K-type Λ_{4m-2s} of π_{4m-2s} described explicitly in Equation (2.8). As mentioned in Section 2.8, π_{4m-2s} is special unipotent attached to $\mathcal{O}^\vee(4m-2s)$. By the unicity discussed in Remark 4.2, it remains only to show that $\Gamma'_{\mathbf{1}}$ is special unipotent attached to $\mathcal{O}^\vee(4m-2s)$. It's easy to check that the infinitesimal character of $\Gamma'_{\mathbf{1}}$ matches that attached to $\mathcal{O}^\vee(4m-2s)$. So, just as in the proof of Proposition 1.3, it suffices to check that the dense orbit in the associated variety of the annihilator of $\Gamma'_{\mathbf{1}}$, say $\mathcal{O}$, matches $\mathcal{O}(4m-2s)$. Using the main results of [30], it is not difficult in fact to compute the associated variety of $\Gamma'_{\mathbf{1}}$ given the computation of the associated variety of $\theta_e^2(\mathbf{1}_s)$, which is known by combining Theorem 1.2 (identifying $\theta_e^2(\mathbf{1}_s)$ as a Knapp representation) and [29] (computing the associated variety of Knapp representations). One finds that indeed the associated variety of $\Gamma'_{\mathbf{1}}$ is a component of $\mathcal{O} \cap \mathfrak{p}'$. So indeed $\mathcal{O} = \mathcal{O}(4m-2s)$. Thus $\Gamma'_{\mathbf{1}} = \pi_{4m-2s}$.

The proof that $\Gamma'_{\det} = \pi_{4m-2s+2}$ follows in nearly the identical way. (A mild complication is that we have yet to compute the associated variety of $\theta_e^2(\det_s)$. It coincides with that of $\theta_e^2(\mathbf{1}_s)$, but we omit the details of that calculation here.) □

5. Langlands parameters and the proof of Proposition 4.1

In this section we prove Proposition 4.1. The Langlands parameters of the double lifts $\theta^2(\mathbf{1}_s)$ and $\theta^2(\det_s)$ are given in Theorem 5.33.

We start by defining representations $\pi_{\mathbb{1}}$ and $\pi_{\det}$ of $\mathrm{O}(2m, r)$ which will turn out to be the double theta lifts of the trivial and determinant representations of $\mathrm{O}(s, 0)$, respectively, by giving their Langlands parameters.

We use the notation established in [24]; following Vogan's version of the Langlands Classification [34], a set of Langlands parameters of an irreducible admissible representation π of $\mathrm{O}(p,q)$ consists of a Levi subgroup $MA \cong \mathrm{O}(p-2t-k, q-2t-k) \times \mathrm{GL}(2,\mathbb{R})^t \times \mathrm{GL}(1,\mathbb{R})^k$ of $\mathrm{O}(p,q)$ and data $(\lambda, \Psi, \mu, \nu, \epsilon, \kappa)$ with (λ, Ψ) the Harish-Chandra parameter and system of positive roots determining a limit of discrete series ρ of $\mathrm{O}(p-2t-k, q-2t-k)$, $\mu \in \mathbb{Z}^t$ and $\nu \in \mathbb{C}^t$ determining a relative limit of discrete series τ of $\mathrm{GL}(2,\mathbb{R})^t$, and the pair (ϵ, κ) with $\epsilon \in \{\pm 1\}^k$ and $\kappa \in \mathbb{C}^k$ determining a character χ of $\mathrm{GL}(1,\mathbb{R})^k$. (The group MA is of course implied by the other data.) The representation $\pi = \pi(\lambda, \Psi, \mu, \nu, \epsilon, \kappa)$ is then an irreducible quotient of an induced representation $\mathrm{Ind}_{MAN}^{\mathrm{O}(p,q)}(\rho \otimes \tau \otimes \chi \otimes \mathbb{1})$, with $P = MAN$ chosen such that certain positivity conditions are satisfied. Since $\mathrm{O}(p,q)$ is disconnected, there may be more than one such irreducible quotient which can be distinguished by signs (see §3.2 of [24]). Here, we will always mean the representation with all signs positive, so we omit them.

Let m, r, and s be integers as before, i. e., $m \geq 1$, $r \geq 2m$, $0 \leq s \leq m$, and $s \equiv r (mod\, 2)$. We define $\pi_{\mathbb{1}}$ to be the representation of $\mathrm{O}(2m, r)$ with the following Langlands parameters:

(1) If $r = 2m$ and $s = 0$ then $\pi_{\mathbb{1}}$ is the spherical representation with $MA \cong \mathrm{GL}(1,\mathbb{R})^{2m}$, given by $\pi_{\mathbb{1}} = \pi(0, \emptyset, 0, 0, \epsilon, \kappa)$, where $\epsilon = (1, \ldots, 1)$ and $\kappa = \nu_0$. (Here ν_0 is the infinitesimal character of π'_0 as in (2.7).)

(2) If $r \geq 2m + s$ let $\pi_{\mathbb{1}} = \pi(\lambda_d, \Psi, \mu, \nu, 0, 0)$ with $MA \cong \mathrm{O}(0, r-2m) \times \mathrm{GL}(2,\mathbb{R})^m$,

$$\lambda_d = \left(\frac{r}{2} - m - 1, \frac{r}{2} - m - 2, \ldots, 1, 0\right) \quad \text{or} \quad \lambda_d = \left(\frac{r}{2} - m - 1, \frac{r}{2} - m - 2, \ldots, \frac{3}{2}, \frac{1}{2}\right) \tag{5.1}$$

depending on whether r is even or odd,

$$\begin{aligned} \mu &= \left(\frac{r+s}{2} - m - 1, \frac{r+s}{2} - m - 1, \ldots, \frac{r+s}{2} - m - 1\right), \\ \nu &= \left(\frac{r-s}{2} + m - 1, \frac{r-s}{2} + m - 3, \ldots, \frac{r-s}{2} - m + 3, \frac{r-s}{2} - m + 1\right), \end{aligned} \tag{5.2}$$

and Ψ is the positive root system (uniquely) determined by λ_d.

(3) If $r \leq 2m + s$, and $r > 2m$ or $s > 0$ (so that we are not in the first case) we distinguish two cases, depending on the parity of $\frac{r-s}{2} + m$.

If $\frac{r-s}{2} + m$ is even then $\pi_{\mathbb{1}} = \pi(\lambda_d, \Psi, \mu, \nu, 0, 0)$ with $MA \cong \mathrm{O}(m - \frac{r-s}{2}, \frac{r+s}{2} - m) \times \mathrm{GL}(2,\mathbb{R})^{\frac{r-s}{4} + \frac{m}{2}}$,

$$\begin{aligned} \lambda_d &= \left(\frac{s}{2} - 1, \frac{s}{2} - 2, \ldots, \frac{r+s}{4} - \frac{m}{2}; \frac{r+s}{4} - \frac{m}{2} - 1, \frac{r+s}{4} - \frac{m}{2} - 2, \ldots, 1, 0\right) \quad \text{or} \\ \lambda_d &= \left(\frac{s}{2} - 1, \frac{s}{2} - 2, \ldots, \frac{r+s}{4} - \frac{m}{2}; \frac{r+s}{4} - \frac{m}{2} - 1, \frac{r+s}{4} - \frac{m}{2} - 2, \ldots, \frac{3}{2}, \frac{1}{2}\right) \end{aligned} \tag{5.3}$$

depending on whether r is even or odd,

$$\begin{aligned} \mu &= \left(\frac{r+s}{2} - m - 1, \frac{r+s}{2} - m - 1, \ldots, \frac{r+s}{2} - m - 1\right), \\ \nu &= \left(\frac{r-s}{2} + m - 1, \frac{r-s}{2} + m - 3, \ldots, 3, 1\right), \end{aligned} \tag{5.4}$$

and Ψ is the positive root system (uniquely) determined by λ_d.

If $\frac{r-s}{2} + m$ is odd then $\pi_{\mathbb{1}} = \pi(\lambda_d, \Psi, \mu, \nu, 0, 0)$ with $MA \cong \mathrm{O}(m - \frac{r-s}{2} + 1, \frac{r+s}{2} - m + 1) \times \mathrm{GL}(2,\mathbb{R})^{\frac{r-s}{4} + \frac{m}{2} - \frac{1}{2}}$,

$$\begin{aligned} \lambda_d &= \left(\frac{s}{2} - 1, \frac{s}{2} - 2, \ldots, \frac{r+s}{4} - \frac{m}{2} - \frac{1}{2}; \frac{r+s}{4} - \frac{m}{2} - \frac{1}{2}, \frac{r+s}{4} - \frac{m}{2} - \frac{3}{2}, \ldots, 1, 0\right) \quad \text{or} \\ \lambda_d &= \left(\frac{s}{2} - 1, \frac{s}{2} - 2, \ldots, \frac{r+s}{4} - \frac{m}{2} - \frac{1}{2}; \frac{r+s}{4} - \frac{m}{2} - \frac{1}{2}, \frac{r+s}{4} - \frac{m}{2} - \frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2}\right) \end{aligned} \tag{5.5}$$

depending on whether r is even or odd,

$$\begin{aligned}\mu &= \left(\frac{r+s}{2}-m-1, \frac{r+s}{2}-m-1, \ldots, \frac{r+s}{2}-m-1\right), \\ \nu &= \left(\frac{r-s}{2}+m-1, \frac{r-s}{2}+m-3, \ldots, 4, 2\right),\end{aligned} \tag{5.6}$$

and Ψ is chosen so that the corresponding limit of discrete series of $\mathrm{O}(m-\frac{r-s}{2}+1, \frac{r+s}{2}-m+1)$ is holomorphic.

Now assume $s \geq 1$. We define π_{det} to be the representation of $\mathrm{O}(2m, r)$ with the following Langlands parameters:

(1) If $r \geq 2m+s+2$ let $\pi_{\mathrm{det}} = \pi(\lambda_d, \Psi, \mu, \nu, 0, 0)$ with MA, λ_d, and Ψ as in the corresponding case for $\pi_{\mathbb{1}}$,

$$\begin{aligned}\mu &= \Big(\underbrace{\frac{r+s}{2}-m, \ldots, \frac{r+s}{2}-m}_{s \text{ entries}}, \underbrace{\frac{r+s}{2}-m-1, \ldots, \frac{r+s}{2}-m-1}_{m-s \text{ entries}}\Big), \\ &\text{and} \\ \nu &= \left(\frac{r+3s}{2}-m-2, \frac{r+3s}{2}-m-4, \ldots, \frac{r-s}{2}-m,\right. \\ &\qquad \left.\frac{r-s}{2}+m-1, \frac{r-s}{2}+m-3, \ldots, \frac{r-s}{2}-m+2s+1\right).\end{aligned} \tag{5.7}$$

(2) If $r \leq 2m+s+2$, we once again distinguish two cases, depending on the parity of $\frac{r-s}{2}+m$.

If $\frac{r-s}{2}+m$ is odd then $\pi_{\mathrm{det}} = \pi(\lambda_d, \Psi, \mu, \nu, 0, 0)$ with $MA \cong \mathrm{O}(m-\frac{r-s}{2}+1, \frac{r+s}{2}-m+1) \times \mathrm{GL}(2, \mathbb{R})^{\frac{r-s}{4}+\frac{m}{2}-\frac{1}{2}}$,

$$\begin{aligned}\lambda_d &= \left(\frac{s}{2}, \frac{s}{2}-1, \ldots, \frac{r+s}{4}-\frac{m}{2}+\frac{1}{2}; \frac{r+s}{4}-\frac{m}{2}-\frac{1}{2}, \frac{r+s}{4}-\frac{m}{2}-\frac{3}{2}, \ldots, 1, 0\right) \quad \text{or} \\ \lambda_d &= \left(\frac{s}{2}, \frac{s}{2}-1, \ldots, \frac{r+s}{4}-\frac{m}{2}+\frac{1}{2}; \frac{r+s}{4}-\frac{m}{2}-\frac{1}{2}, \frac{r+s}{4}-\frac{m}{2}-\frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2}\right)\end{aligned} \tag{5.8}$$

depending on whether r is even or odd,

$$\begin{aligned}\mu &= \Big(\underbrace{\frac{r+s}{2}-m, \ldots, \frac{r+s}{2}-m}_{\frac{r+3s}{4}-\frac{m}{2}-\frac{1}{2} \text{ entries}}, \underbrace{\frac{r+s}{2}-m-1, \ldots, \frac{r+s}{2}-m-1}_{m-s \text{ entries}}\Big), \\ \nu &= \left(\frac{r+3s}{2}-m-2, \frac{r+3s}{2}-m-4, \ldots, 3, 1,\right. \\ &\qquad \left.\frac{r-s}{2}+m-1, \frac{r-s}{2}+m-3, \ldots, \frac{r-s}{2}-m+2s+1\right),\end{aligned} \tag{5.9}$$

and Ψ is the positive root system (uniquely) determined by λ_d.

If $\frac{r-s}{2}+m$ is even then $\pi_{\mathrm{det}} = \pi(\lambda_d, \Psi, \mu, \nu, 0, 0)$ with $MA \cong \mathrm{O}(m-\frac{r-s}{2}+2, \frac{r+s}{2}-m+2) \times \mathrm{GL}(2, \mathbb{R})^{\frac{r-s}{4}+\frac{m}{2}-1}$,

$$\lambda_d = \left(\frac{s}{2}, \frac{s}{2}-1, \ldots, \frac{r+s}{4}-\frac{m}{2}; \frac{r+s}{4}-\frac{m}{2}, \frac{r+s}{4}-\frac{m}{2}-1, \ldots, 1, 0\right) \quad \text{or}$$
$$\lambda_d = \left(\frac{s}{2}, \frac{s}{2}-1, \ldots, \frac{r+s}{4}-\frac{m}{2}; \frac{r+s}{4}-\frac{m}{2}, \frac{r+s}{4}-\frac{m}{2}-1, \ldots, \frac{3}{2}, \frac{1}{2}\right) \tag{5.10}$$

depending on whether r is even or odd,

$$\mu = \Big(\underbrace{\frac{r+s}{2}-m, \ldots, \frac{r+s}{2}-m}_{\frac{r+3s}{4}-\frac{m}{2}-1\,\text{entries}}, \underbrace{\frac{r+s}{2}-m-1, \ldots, \frac{r+s}{2}-m-1}_{m-s\,\text{entries}}\Big),$$
$$\nu = \left(\frac{r+3s}{2}-m-2, \frac{r+3s}{2}-m-4, \ldots, 4, 2, \right. \tag{5.11}$$
$$\left. \frac{r-s}{2}+m-1, \frac{r-s}{2}+m-3, \ldots, \frac{r-s}{2}-m+2s+1\right),$$

and Ψ is chosen so that the corresponding limit of discrete series of $\mathrm{O}(m-\frac{r-s}{2}+1, \frac{r+s}{2}-m+1)$ is holomorphic.

Recall the $\mathrm{SO}(2m)\times\mathrm{SO}(r)$-types Λ_s and Λ'_s defined in (3.17) and (3.18). We use the same notation (Λ_s and Λ'_s) for the unique $\mathrm{O}(2m)\times\mathrm{O}(r)$-types with positive signs containing Λ_s and Λ'_s respectively in their restrictions to $\mathrm{SO}(2m)\times\mathrm{SO}(r)$.

Proposition 5.12. *Let m, r, s, $\pi_{\mathbb{1}}$ and $\pi_{\det}$ be as above. Then $\pi_{\mathbb{1}}$ and $\pi_{\det}$ both have infinitesimal character ν_s (see (2.7)), $\pi_{\mathbb{1}}$ has lowest K-type Λ_s, and $\pi_{\det}$ has lowest K-type Λ'_s.*

Proof. This amounts to a case-by-case calculation using the theory of [33] and [15] as described in detail and explicitly in §3.2 of [24] for r even (the odd case is very similar). If $\pi = \pi(\lambda_d, \Psi, \mu, \nu, \epsilon, \kappa)$ with $MA \cong \mathrm{O}(2m-2t-k, r-2t-k)\times\mathrm{GL}(2,\mathbb{R})^t\times\mathrm{GL}(1,\mathbb{R})^k$, write $p = [m-t-\frac{k}{2}]$, $q = [\frac{r-2t-k}{2}]$, $\lambda_d = (a_1, \ldots, a_p; b_1, \ldots, b_q)$, $\mu = (\mu_1, \ldots, \mu_t)$, $\nu = (n_1, \ldots, n_t)$, and $\kappa = (\kappa_1, \ldots, \kappa_k)$. Then the infinitesimal character of π is given by

$$\gamma = \left(a_1, \ldots, a_p, b_1, \ldots, b_q, \frac{\mu_1+n_1}{2}, \ldots, \frac{\mu_t+n_t}{2}, \frac{\mu_1-n_1}{2}, \ldots, \frac{\mu_t-n_t}{2}, \kappa_1, \ldots, \kappa_k\right). \tag{5.13}$$

To compute the lowest K-types of π, one assigns to π the Vogan parameter λ_a, an element of $\mathfrak{t}^*$ which is essentially the discrete part of the infinitesimal character, as follows: the parameter λ_a is the element of $\mathfrak{t}^*$ which is dominant with respect to a fixed positive system of compact roots and conjugate by the compact Weyl group to

$$\Big(a_1, \ldots, a_p, \frac{\mu_1}{2}, \ldots, \frac{\mu_t}{2}, \underbrace{0, \ldots, 0}_{m-t-p}; b_1, \ldots, b_q, \frac{\mu_1}{2}, \ldots, \frac{\mu_t}{2}, \underbrace{0, \ldots, 0}_{[\frac{r}{2}]-t-q}\Big); \tag{5.14}$$

the lowest K-types are then of the form

$$\lambda_a + \rho(\mathfrak{u}\cap\mathfrak{p}) - \rho(\mathfrak{u}\cap\mathfrak{k}) + \delta_L, \tag{5.15}$$

where $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is the theta stable parabolic subalgebra of $\mathfrak{g}$ determined by λ_a, $\rho(\mathfrak{u} \cap \mathfrak{p})$ and $\rho(\mathfrak{u} \cap \mathfrak{k})$ are one half the sum of the positive noncompact and compact roots in $\mathfrak{u}$, respectively, and δ_L is the highest weight of a fine K-type for L, a subgroup of $\mathrm{O}(2m, r)$ corresponding to $\mathfrak{l}$. The allowed choices for δ_L are determined by Ψ and ϵ (see Proposition 10 of [24]).

We omit the detailed calculation, but illustrate using an example below.

□

Example 5.16. Let $m = s = 2$ and $r = 6$. Then $\pi_{\mathbb{1}}$ has Langlands parameters $MA \cong \mathrm{O}(0,2) \times \mathrm{GL}(2,\mathbb{R})^2$, $\lambda_d = (0)$, $\mu = (1,1)$, and $\nu = (3,1)$ (using either case (2) or (3)). So $\lambda_a = (\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}, 0)$. We have the infinitesimal character of $\pi_{\mathbb{1}}$ given by

$$\left(\frac{1}{2} + \frac{3}{2}, \frac{1}{2} + \frac{1}{2}, \frac{1}{2} - \frac{3}{2}, \frac{1}{2} - \frac{1}{2}, 0\right) = (2, 1, -1, 0, 0), \tag{5.17}$$

which is Weyl group conjugate to ν_2. The lowest K-type is of the form

$$\begin{aligned} &\lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k}) + \delta_L \\ =&\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}, 0\right) + (2, 2; 1, 1, 0) - \left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}, 0\right) + (0, 0; 0, 0, 0) \\ =&(2, 2; 0, 0, 0) = \Lambda_2. \end{aligned} \tag{5.18}$$

For $\pi_{\det}$ we have $MA \cong \mathrm{O}(2,4) \times \mathrm{GL}(2,\mathbb{R})$, $\lambda_d = (1; 1, 0)$, $\mu = (2)$ and $\nu = (2)$ (using case (2)). So $\lambda_a = (1, 1; 1, 1, 0)$, and the infinitesimal character is

$$(1 + 1, 1, 1 - 1, 1, 0) = (2, 1, 0, 1, 0) \underset{W}{\sim} \nu_2. \tag{5.19}$$

The lowest K-type is

$$\begin{aligned} &\lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k}) + \delta_L \\ =&(1, 1; 1, 1, 0) + (2, 2; 1, 1, 0) - \left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}, 0\right) + \left(\frac{1}{2}, \frac{1}{2}; -\frac{1}{2}, -\frac{1}{2}, 0\right) \\ =&(3, 3; 0, 0, 0) = \Lambda_2'. \end{aligned} \tag{5.20}$$

Here the fine K-type δ_L is uniquely determined by Ψ which is such that the resulting limit of discrete series on $\mathrm{O}(2,4)$ is holomorphic.

We show below that in many cases, including most cases for $\pi_{\mathbb{1}}$ and the case $s = m$ for $\pi_{\det}$, this representation is uniquely determined by its lowest K-type and infinitesimal character. This then implies that the representation is indeed the double lift of the appropriate character of $\mathrm{O}(0, s)$, and in the case of $\pi_{\mathbb{1}}$, that it is the unique representation (with positive signs) of $\mathrm{O}(2m, r)$ having the Knapp representation π_s' as a summand in its restriction to the identity component $\mathrm{SO}_e(2m, r)$. Moreover, π_s' is then uniquely determined be its lowest K-type and infinitesimal character. In [13], Knapp conjectured that this should typically be the case for the representations π_s', and he gives a conjectural method for determining the Langlands parameters, which Friedman [6] proves to yield the correct parameters.

Proposition 5.21. *Let m, r, s, $\pi_{\mathbb{1}}$ and $\pi_{\det}$ be as above.*

(1) *If $\frac{r+s}{2} - m \neq 1$ then $\pi_{\mathbb{1}}$ is the unique irreducible admissible representation of* $\mathrm{O}(2m, r)$ *with infinitesimal character ν_s and lowest K-type Λ_s.*

(2) *If $s = m$ then $\pi_{\det}$ is the unique irreducible admissible representation of* $\mathrm{O}(2m, r)$ *with infinitesimal character ν_s and lowest K-type Λ'_s.*

Proof. The spherical case $\frac{r+s}{2} - m = 0$ for $\pi_{\mathbb{1}}$ follows from the well-known fact that any spherical representation of $\mathrm{O}(p, q)$ is determined by its infinitesimal character. For $\pi_{\mathbb{1}}$ and the case $\frac{r+s}{2} - m \geq 2$ or $\pi_{\det}$ with $m = s$, one computes the Vogan parameter λ_a associated to Λ_s (or Λ'_s) (see §5.3 of [33]), which essentially determines the Levi factor MA and the discrete part λ_d, μ of the parameters. Then one checks that there is only one parameter ν giving the given infinitesimal character, and the root system Ψ is then determined by the given unique lowest K-type. We give some details of the argument for $\pi_{\mathbb{1}}$ and the case $r \leq 2m + s$ with $\frac{r-s}{2} + m$ and $r = 2n$ even, and leave the remaining cases to the reader.

We have

$$\Lambda_s = \left(\frac{r+s}{2} - m, \frac{r+s}{2} - m, \dots, \frac{r+s}{2} - m; 0, \dots, 0 \right), \tag{5.22}$$

so if ρ_c is one half the sum of the positive compact roots, then

$$\begin{aligned}\Lambda_s + 2\rho_c = \Bigg(&\underbrace{\frac{r+s}{2} + m - 2, \frac{r+s}{2} + m - 4, \dots, r+2, r}_{\frac{m}{2} - \frac{r-s}{4}}, \underbrace{r-2, r-4, \dots, \frac{r+s}{2} - m}_{\frac{m}{2} + \frac{r-s}{4}}; \\ &\underbrace{r-2, r-4, \dots, \frac{r+s}{2} - m}_{\frac{m}{2} + \frac{r-s}{4}}, \frac{r+s}{2} - m - 2, \dots, 2, 0 \Bigg).\end{aligned} \tag{5.23}$$

We may choose ρ (so that $\Lambda_s + 2\rho_c$ is dominant) to be

$$\begin{aligned}\rho = \Bigg(&\underbrace{\frac{r}{2} + m - 1, \frac{r}{2} + m - 2, \dots, \frac{3r-s}{4} + \frac{m}{2}}_{\frac{m}{2} - \frac{r-s}{4}}, \\ &\underbrace{\frac{3r-2}{4} + \frac{m}{2} - 1, \frac{3r-2}{4} + \frac{m}{2} - 3, \dots, \frac{r+2}{4} - \frac{m}{2} + 1}_{\frac{m}{2} + \frac{r-s}{4}}; \\ &\underbrace{\frac{3r-2}{4} + \frac{m}{2} - 2, \frac{3r-2}{4} + \frac{m}{2} - 4, \dots, \frac{r+2}{4} - \frac{m}{2}}_{\frac{m}{2} + \frac{r-s}{4}}, \\ &\frac{r+2}{4} - \frac{m}{2} - 1, \frac{r+2}{4} - \frac{m}{2} - 2, \dots, 1, 0 \Bigg),\end{aligned} \tag{5.24}$$

so that

$$\Lambda_s + 2\rho_c - \rho = \Bigg(\frac{s}{2}-1, \frac{s}{2}-2, \dots, \frac{r+s}{4}-\frac{m}{2}, \underbrace{\frac{r+s}{4}-\frac{m}{2}-1, \dots, \frac{r+s}{4}-\frac{m}{2}-1}_{\frac{m}{2}+\frac{r-s}{4}};$$

$$\underbrace{\frac{r+s}{4}-\frac{m}{2}, \dots, \frac{r+s}{4}-\frac{m}{2}}_{\frac{m}{2}+\frac{r-s}{4}}, \frac{r+s}{4}-\frac{m}{2}-1, \frac{r+s}{4}-\frac{m}{2}-2, \dots, 1, 0 \Bigg). \tag{5.25}$$

Projection onto the dominant (w. r. t. ρ) Weyl chamber yields

$$\lambda_a = \Bigg(\frac{s}{2}-1, \frac{s}{2}-2, \dots, \frac{r+s}{4}-\frac{m}{2}, \underbrace{\frac{r+s}{4}-\frac{m}{2}-\frac{1}{2}, \dots, \frac{r+s}{4}-\frac{m}{2}-\frac{1}{2}}_{\frac{m}{2}+\frac{r-s}{4}};$$

$$\underbrace{\frac{r+s}{4}-\frac{m}{2}-\frac{1}{2}, \dots, \frac{r+s}{4}-\frac{m}{2}-\frac{1}{2}}_{\frac{m}{2}+\frac{r-s}{4}}, \frac{r+s}{4}-\frac{m}{2}-1, \frac{r+s}{4}-\frac{m}{2}-2, \dots, 1, 0 \Bigg). \tag{5.26}$$

All entries of λ_a that occur only once will be entries of the Harish-Chandra parameter λ_d, and since $\frac{r+s}{4} - \frac{m}{2} - \frac{1}{2} > 0$, the Levi factor MA will be of the form

$$MA \cong \mathrm{O}(2m-2t, r-2t) \times \mathrm{GL}(2, \mathbb{R})^t \tag{5.27}$$

for some $0 \le t \le \frac{m}{2} + \frac{r-s}{4}$.

Recall that the infinitesimal character (up to Weyl group action) is

$$\nu_s = \Bigg(\frac{s}{2}-1, \frac{s}{2}-2, \dots, \frac{r+s}{4}-\frac{m}{2}, \frac{r+s}{4}-\frac{m}{2}-1, \dots, \frac{s}{2}-m;$$

$$\frac{r}{2}-1, \frac{r}{2}-2, \dots, \frac{r+s}{4}-\frac{m}{2}, \frac{r+s}{4}-\frac{m}{2}-1, \dots, 1, 0 \Bigg). \tag{5.28}$$

Since ν_s does not contain $\frac{r+s}{4} - \frac{m}{2} - \frac{1}{2}$ as an entry, it can not be an entry in λ_d, so we must have $t = \frac{m}{2} + \frac{r-s}{4}$ and

$$\lambda_d = \left(\frac{s}{2}-1, \frac{s}{2}-2, \dots, \frac{r+s}{4}-\frac{m}{2}; \frac{r+s}{4}-\frac{m}{2}-1, \frac{r+s}{4}-\frac{m}{2}-2, \dots, 1, 0 \right). \tag{5.29}$$

A relative limit of discrete series of $\mathrm{GL}(2, \mathbb{R})^t$ parametrized by a pair (μ, ν) with $\mu = (\mu_1, \dots, \mu_t) \in \left(\mathbb{Z}_+\right)^t$ and $\nu = (n_1, \dots, n_t) \in \mathbb{C}^t$ has infinitesimal character $\frac{1}{2}(\mu_1 + n_1, \dots, \mu_t + n_t, -\mu_1 + n_1, \dots, -\mu_1 + n_t)$. In order to account for the remaining entries $(\frac{r}{2} - 1, \frac{r}{2} - 2, \dots, 1, 0, -1, \dots, \frac{s}{2} - m)$ of ν_s, we must have $\mu = (\frac{r+s}{2} - m - 1, \frac{r+s}{2} - m - 1, \dots, \frac{r+s}{2} - m - 1)$ and $\nu = (\frac{r+s}{2} + m - 1, \frac{r+s}{2} + m - 3, \dots, 3, 1)$. (Note that changing the sign on an entry of ν does not change the equivalence class of the corresponding representation of $\mathrm{O}(2m, r)$.) □

To illustrate how Proposition 5.21 fails in the other cases, we look at three examples; the first example deals with the ambiguities in Langlands parameters for representations with lowest K-type Λ_s when $\frac{r+s}{2} - m = 1$, the other two look at representations which have the same infinitesimal character and lowest K-type as π_{det}.

Example 5.30. Let $m = 3$, $s = 2$, and $r = 6$, and consider $\pi_{\mathbb{1}}$, a representation of $\mathrm{O}(6,6)$ with lowest K-type $\Lambda_2 = (1,1,1;0,0,0)$ (a fine K-type) and infinitesimal character $\nu_2 = (2,2,1,1,0,0)$. The Langlands parameters are given by $MA \cong \mathrm{O}(2,2) \times \mathrm{GL}(2,\mathbb{R})^2$, $\lambda_d = (0;0)$, $\Psi = \{e_1 \pm f_1\}$, $\mu = (0,0)$, and $\nu = (4,2)$. Notice that the associated Vogan parameter $\lambda_a = (0,0,0;0,0,0)$, so $L = \mathrm{O}(6,6)$, $\rho(\mathfrak{u} \cap \mathfrak{p})$ and $\rho(\mathfrak{u} \cap \mathfrak{k})$ are zero as well, and any representation containing Λ_2 as a lowest K-type must be a constituent of a principal series representation. The full principal series will have a second lowest K-type $(0,0,0;1,1,1)$. The particular principal series which has $\pi_{\mathbb{1}}$ as a constituent is obtained by inducing the character given by $\epsilon = (1,1,1,-1,-1,-1)$ and $\kappa = (0,1,2,0,1,2)$ on $MAN = \mathrm{GL}(1,\mathbb{R})^6 \cdot N$ to $\mathrm{O}(6,6)$ (this is the character mapping $(r_1,r_2,r_3,r_4,r_5,r_6) \in (\mathbb{R}^\times)^6$ to $|r_2||r_3|^2 sign(r_4) sign(r_5)|r_5| sign(r_6)|r_6|^2$). The constituent containing the other lowest K-type has Langlands parameters which differ from those of $\pi_{\mathbb{1}}$ only by the choice of positive roots Ψ (the corresponding limit of discrete series is antiholomorphic). We can get other representations with these lowest K-types and the same infinitesimal character by matching the entries of κ with the entries of ϵ in different ways. In this way (and eliminating duplications by ensuring that the nonparity condition F-2 of [34] is satisfied) we get precisely three pairwise inequivalent irreducible admissible representations of $\mathrm{O}(6,6)$ with unique lowest K-type Λ_2 and infinitesimal character ν_2, namely $\pi_{\mathbb{1}}$, $\pi(\lambda_d, \Psi, 0, 0, \epsilon', \kappa')$ with $MA \cong \mathrm{O}(2,2) \times \mathrm{GL}(1,\mathbb{R})^4$, $\epsilon' = (1,1,-1,-1)$ and $\kappa' = (2,2,1,1)$, and $\pi(\lambda_d, \Psi, 0, 0, \epsilon', \kappa'')$ with MA as for the previous representation, and $\kappa'' = (1,1,2,2)$.

Once we assume that our Levi factor is $MA \cong \mathrm{O}(2,2) \times \mathrm{GL}(2,\mathbb{R})^2$, the remaining parameters are uniquely determined, and this is the only type of ambiguity which occurs for the Langlands parameters of representations which have the same infinitesimal character and lowest K-type as $\pi_{\mathbb{1}}$.

Example 5.31. An ambiguity similar to the one dealt with in Example 5.30 occurs with π_{det} when Λ'_s is small, i. e., of the form $(2,\ldots,2,1,\ldots,1;0,\ldots,0)$. Let $s = 2$, $m = 3$, and $r = 6$ as before. Then $\Lambda'_2 = (2,2,1;0,0,0)$, $\nu_2 = (2,2,1,1,0,0)$, and the Langlands parameters of π_{det} are given by $MA \cong \mathrm{O}(2,2) \times \mathrm{GL}(2,\mathbb{R})^2$, $\lambda_d = (1;0)$, $\mu = (1,0)$, and $\nu = (1,4)$. The Vogan parameter is given by $\lambda_a = (1,\frac{1}{2},0;\frac{1}{2},0,0)$, so there are two more irreducible admissible representations with this lowest K-type and infinitesimal character, namely with $MA \cong \mathrm{O}(2,2) \times \mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(1,\mathbb{R})^2$, λ_d as above, $\mu = (1)$, $\nu = (3)$, $\epsilon = (1,-1)$, and $\kappa = (2,0)$ or $(0,2)$.

Example 5.32. If Λ'_s is not small as in Example 5.31 then the Levi factor part of the Langlands parameters is uniquely determined by the lowest K-type and its uniqueness. We get a different kind of ambiguity from the fact that if $s \neq m$ then MA contains $\mathrm{GL}(2,\mathbb{R})$ factors with two different discrete parameters attached, and there may be more than one way to match continuous parameters and get the same infinitesimal character. For instance, let $s = 4$, $m = 6$, and $r = 14$. Then $\Lambda'_4 = (4,4,4,4,3,3;0,0,0,0,0,0,0)$, $\nu_2 = (6,5,4,4,3,3,2,2,1,1,1,0,0)$, and π_{det} has Langlands parameters $MA = \mathrm{O}(2,4) \times \mathrm{GL}(2,\mathbb{R})^5$, $\lambda_d = (2;1,0)$, $\mu = (3,3,3,2,2)$ and $\nu = (5,3,1,10,8)$. Continuous parameters

$(9, 5, 5, 8, 2)$ or $(7, 5, 3, 10, 4)$ give two more inequivalent representations with the same lowest K-type and infinitesimal character.

The next result shows how to introduce additional hypotheses to circumvent the ambiguities of the previous examples.

Theorem 5.33. *Let m, r, s, $\pi_{\mathbb{1}}$ and π_{det} be as above. Recall the maximal primitive ideal $\mathrm{J}_{\mathrm{max}}(\nu_s)$ with infinitesimal character ν_s (Section 2.3).*

(1) *$\pi_{\mathbb{1}}$ is the unique irreducible admissible representation of $\mathrm{O}(2m, r)$ with annihilator $\mathrm{J}_{\mathrm{max}}(\nu_s)$ and lowest K-type Λ_s.*
(2) *π_{det} is the unique irreducible admissible representation of $\mathrm{O}(2m, r)$ with annihilator $\mathrm{J}_{\mathrm{max}}(\nu_s)$ and lowest K-type Λ'_s.*

Sketch. We begin by recalling the τ-invariant of an irreducible $\mathrm{U}(\mathfrak{g})$ module π with regular integral infinitesimal character. For the purposes of this paragraph, $\mathfrak{g}$ may be taken to be an arbitrary complex semisimple Lie algebra. Fix a positive system of roots Δ^+ for a Cartan $\mathfrak{h}$ in $\mathfrak{g}$. Let λ denote a dominant (with respect to Δ^+) representative of the infinitesimal character of π. Fix a simple root $\alpha \in \Delta^+$ and let $\lambda_\alpha = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$. Then λ_α is integral and dominant (with respect to Δ^+); moreover, λ and λ_α differ by a weight of a finite dimensional representation of $\mathfrak{g}$. We may thus consider the translation functor ψ_α from λ to λ_α. The τ-invariant of π, denoted $\tau(\pi)$, is the set of simple α for which $\psi_\alpha(\pi) = 0$.

Return to our setting and let π be an irreducible admissible representation for $\mathrm{O}(2m, r)$. We now extract a consequence of the classification of primitive ideals in $\mathfrak{g} = \mathfrak{so}(2m+r, \mathbb{C})$. Assume π has infinitesimal character ν_s and make the standard choices (of Cartan, positive roots Δ^+, etc.) so that the formulas (2.7) and (2.9) give the dominant representative of the infinitesimal character of π. Since we are assuming the parity of r and s match, ν_s is integral. The translation principle dictates the following: there is a regular infinitesimal character with dominant representative ν_s^{reg} so that ν_s^{reg} and ν differ by the weight of a finite-dimensional representation of $\mathfrak{g}$; and there is a representation π^{reg} with infinitesimal character ν_s^{reg} so that if T denotes the translation functor from ν_s^{reg} to ν, then

$$T(\pi^{\mathrm{reg}}) = \pi.$$

Now let S_s denote the set of simple roots $\alpha \in \Delta^+$ for which ν_s is *not* singular, i.e. those simple roots so that $\langle \nu_s, \alpha \rangle \neq 0$. We claim that

$$\mathrm{J}_{\mathrm{max}}(\nu_s) \text{ annihilates } \pi \text{ iff } S_s = \tau(\pi^{\mathrm{reg}}).$$

This follows from the classification of primitive ideals in $\mathrm{U}(\mathfrak{g})$. In more detail, the paper [29] gives the explicit tableau parameters of the primitive ideal $\mathrm{J}_{\mathrm{max}}(\nu_s^{\mathrm{reg}})$, and from there it is simple to extract the τ-invariant statement.

Now we sketch how to use the τ-invariant criterion to rule out the ambiguities highlighted in Examples 5.30–5.32. Our task is to take a representation π (different from $\pi_{\mathbf{1}}$ and π_{det}) given by one of the Langlands parameters in those examples, compute the τ-invariant of π^{reg}, and show that it differs from S_s. Here is a sketch of how to do that. The papers [32] and [34] explain how to produce the Langlands parameters of π^{reg} and then to compute the τ-invariant of π^{reg}. (Theorem 4.12 in [32] is especially relevant.) Then one can see directly that the τ-invariant of π^{reg} is strictly *smaller* than S_s. For instance, suppose we encounter the ambiguity of the sort treated in Examples 5.30 and 5.31. In these cases the Levi factor contains more $\mathrm{GL}(1, \mathbb{R})$ factors than the Levi factor for π_1.

The real roots that arise from these additional GL(1) factors cannot satisfy the parity condition. This corresponds to saying that the real roots are *not* in the τ-invariant of π^{reg}, and thus the τ-invariant is smaller than possible and, in particular, smaller than S_s. The final kind of ambiguity treated in Example 5.32 is slightly more subtle. One must verify that the alternative matchings of continuous parameters always lead to a smaller τ-invariant for π^{reg}. Given [32, Theorem 4.12], this is a rather complicated combinatorial check. We omit the details. $\square$

As a corollary, we immediately obtain the Langlands parameters of the double lifts $\theta(\mathbf{1}_s)$ and $\theta(\det_s)$.

Corollary 5.34. *Let m, r, s, $\pi_{\mathbb{1}}$ and $\pi_{\det}$ be as above. Then*

$$\pi_{\mathbf{1}} = \theta^2(\mathbf{1}_s) \text{ and } \pi_{\det} = \theta^2(\det_s).$$

Proof. Since the double lifts $\theta(\mathbf{1}_s)$ and $\theta(\det_s)$ satisfy the hypothesis of the theorem (with the annihilator hypothesis explained in the proof of Proposition 4.3), the corollary follows. $\square$

The proof of Proposition 4.1 is also now a simple corollary: if there were more than one representation of the kind described in the proposition, there would be more than one of the kind described in Theorem 5.33. This contradiction completes the proof. $\square$

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Recent results on fractal analysis of trajectories of some dynamical systems

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1 Motivation

In this article we will give a survey of recent results by the authors concerning the computation of box dimension and Minkowski content of some spiral trajectories in $\mathbb{R}^2$ and $\mathbb{R}^3$. We also describe results dealing with trajectories of some discrete one-dimensional dynamical systems.

An important tool in the study of dynamical systems are fractal dimensions. Here we deal with box dimensions only, since the Hausdorff dimension of trajectories that we consider are always trivial. Box dimensions have a long history, going back to the very beginning of the 20th century (Minkowski 1903, Bouligand 1928). There are many other names for box dimension appearing in the literature, usually meaning the upper box dimension. One can encounter other equivalent names such as box counting dimension, Minkowski-Bouligand dimension, Kolmogorov dimension, Borel logarithmic rarefaction, Besicovitch-Taylor index, entropy dimension, fractal dimension, and limit capacity.

The notion of Minkowski content is not so widely known as the box dimension. We mention here the work by Lapidus and He [8] dealing with this notion, in the context of the generalized Weyl-Berry conjecture. Also B. Mandelbrot uses the notion of lacunarity, which is defined as the reciprocal value of the Minkowski content.

There are many important contributions by outstanding specialists dealing with the study of Hausdorff dimension of chaotic phenomena. Since 1970s thermodynamic formalism, developed by Sinai, Ruelle, and Bowen, resulted in Hausdorff dimension of the Smale horseshoe and in a lots of results about Hausdorff dimension of Julia and Mandelbrot sets. Since 1980 physicists started to estimate and compute fractal dimensions of strange attractors (Lorenz, Henon, Chua, etc.), in order to measure their complexity. Fractal dimensions are estimated also for attractors of infinite-dimensional dynamical systems. For applications of fractal dimensions to dynamics see a survey article Županović and Žubrinić [18] and the references therein.

We consider a different problem, namely, we compute box dimension of a trajectory itself and study its dependence on the bifurcation parameter. There are some well known bifurcations which can be described in the new way using fractal analysis. We have done it for the Hopf-Takens bifurcation and also for the saddle-node and period doubling bifurcations of some discrete one-dimensional systems.

An important motivation for our work was the monograph by Claude Tricot [12]. From this monograph we learned that due to the fact that a realistic smooth spiral has

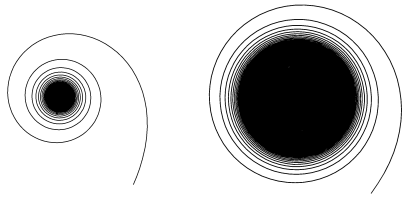

Figure 1: Spirals $r = \varphi^{-1/2}$ and $r = \varphi^{-1/6}$ have box dimensions 4/3 and 12/7 respectively.

a non-vanishing width, it looks as an object of almost a planar nature. According to the mentioned monograph, Hausdorff and Bouligand initiated the idea introducing noninteger dimensions at the beginning of the 20th century. Mendès France in his Foreword to Tricot's book cited the following opinion of a famous painter Kandinsky: "A straight line is the total negation of the plane whereas a curved line is potentially the plane in that it contains the essence of the plane within itself." See W. Kandinsky: *Point and Line to Plane*, Dover Books on Art History, 1926. As Mendès France nicely remarked, "With his artist's eye he has understood that a line can mimic a plane." Indeed, mathematicians can explain it in the more precise way.

As a simple example, we may consider two planar vector fields with spiral trajectories Γ as in Figure 1, having different "concentrations" at the origin. This difference of concentrations can be explained by the notion of box dimension $\dim_B \Gamma$ (to be defined below), which according to Tricot's formula, see [12, p. 121], is equal to $2/(1+\alpha)$ for spirals $r = m\varphi^{-\alpha}$, $\varphi \geq \varphi_1 > 0$, where $\alpha \in (0,1]$ and $m > 0$ are given constants. Here we use polar coordinates. So, the corresponding dimensions of spirals on Figure 1 are 4/3 and 12/7. As we see, the second spiral on Figure 1 is almost two dimensional in the sense of box dimension.

On Figure 2 we have two spirals with box dimensions both equal to 4/3, but still we clearly see that their "concentrations" at the origin are different. Therefore we use a subtler tool, the Minkowski content $\mathcal{M}^d(\Gamma)$ of spirals Γ, to be introduced in Section 2. For the spiral Γ defined by $r = m\varphi^{-\alpha}$, $\alpha \in (0,1)$, its value is

$$\mathcal{M}^d(\Gamma) = m^d \pi (\pi\alpha)^{-2\alpha/(1+\alpha)} \frac{1+\alpha}{1-\alpha},$$

see [14, Corollary 2]. The respective values of Minkowski content for spirals on Figure 2 are approximately 6.97 and 59.63, that is, the right-hand spiral has more than eight times larger Minkowski content than the left-hand spiral. As we have mentioned, Mandelbrot introduced the notion of lacunarity, which is just the reciprocal value of the Minkowski content. So, the left-hand spiral on Figure 2 has larger lacunarity than the spiral on the right.

Furthermore, we would like to illustrate the connection between box dimension of spiral trajectories of limit cycle type and the algebraic multiplicity of the corresponding limit cycles.

As an example we consider a planar vector field described with the following system

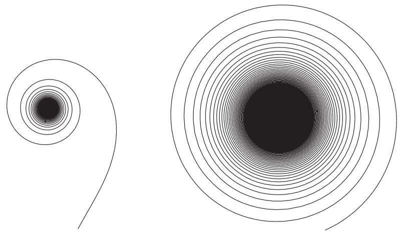

Figure 2: Spirals $r = \varphi^{-1/2}$ and $r = 5\varphi^{-1/2}$ both have box dimensions 4/3, but different Minkowski contents, see [18, p. 395]; reprinted with permission of Elsevier.

in polar coordinates:

$$\begin{cases} \dot{r} &= r(r^4 - 2r^2 + a_0), \\ \dot{\varphi} &= 1. \end{cases} \tag{1}$$

By varying parameter a_0 we can obtain the following phase portraits exhibited on Figures 3 and 4. Here we consider spirals Γ near the corresponding limit cycles, and we call them spirals of the limit cycle type. We notice that the spiral of the limit cycle type on Figure 4 tends slower to its limit cycle than the corresponding spirals of the limit cycle type on Figure 3. We point out that for $a_0 = 1$, Figure 4, we have the limit cycle of algebraic multiplicity two, while for $a_0 < 1$, Figure 3, the corresponding limit cycle has multiplicity one. Algebraic multiplicity of the limit cycle is defined as the multiplicity of the root $r = a$ of the right-hand side of (1). In the case of $a_0 = 1$ the box dimension of spirals of limit cycle type is 3/2, while in all the other cases it is equal to 1, see Theorem 2.

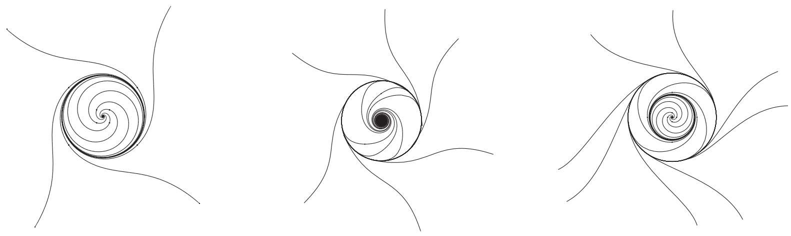

Figure 3: Portraits of the solutions of (1) for parameters $a_0 < 0$, $a_0 = 0$, $a_0 \in (0, 1)$.

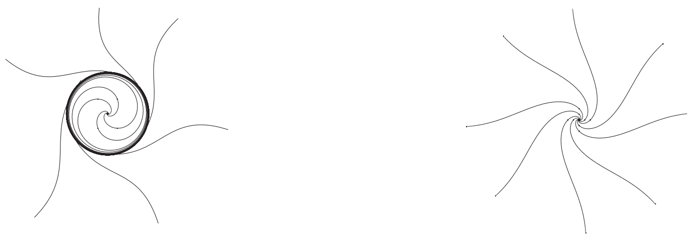

Figure 4: Portraits of the solutions of (1) for parameters $a_0 = 1$, $a_0 > 1$

In order to see more clearly the concentration of a spiral near its limit cycle, let us

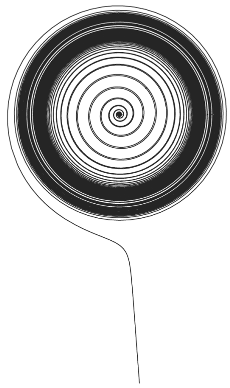

Figure 5: Two spiral trajectories of system $\dot{r} = r\left(r - \frac{3}{4}\right)^4\left(r + \frac{3}{4}\right)^4$, $\dot{\varphi} = 1$, tending to the limit cycle $r = 3/4$ (the cycle is within the narrow white strip).

consider the following system, see Figure 5:

$$\begin{aligned}\dot{r} &= r\left(r - \frac{3}{4}\right)^4\left(r + \frac{3}{4}\right)^4,\\ \dot{\varphi} &= 1.\end{aligned}$$

The box dimension in this case is 7/4, that is, the spirals converging to the limit cycle $r = 3/4$ are almost two-dimensional.

2 Box dimension and Minkowski content

Here we introduce some basic definitions that we shall need in the sequel. Assume that $A \subset \mathbb{R}^N$ is a bounded set. By the *Minkowski sausage* of radius ε around A we mean the ε-neighbourhood of A, and denote it by $A_\varepsilon := \{y \in \mathbb{R}^N : d(y, A) < \varepsilon\}$, where $d(y, A)$ is Euclidean distance from y to A. It represents a generalization of the notion of ball, in which case the center A is just a single point.

The *lower s-dimensional Minkowski content* of A, $s \geq 0$, is defined by:

$$\mathcal{M}_*^s(A) := \liminf_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^{N-s}},$$

and analogously the *upper s-dimensional Minkowski content* $\mathcal{M}^{*s}(A)$. Note that in general these values are in $[0, \infty]$. The corresponding lower box dimension of A is defined by

$$\underline{\dim}_B A := \inf\{s \geq 0 : \mathcal{M}_*^s(A) = 0\},$$

and analogously the upper box dimension,

$$\overline{\dim}_B A := \inf\{s \geq 0 : \mathcal{M}^{*s}(A) = 0\}.$$

If both of these values coincide, the common value is denoted by $\dim_B A$, and is called box dimension of A. For various properties of box dimensions see for example Falconer

[6]. Clearly, $\underline{\dim}_B A \le \overline{\dim}_B A$, and the inequality may be strict, see [6]. Moreover, it is possible to construct a class of fractal sets $A \subset \mathbb{R}^N$ for which $\underline{\dim}_B A = 0$ and $\overline{\dim}_B A = N$, see [13].

If $\mathcal{M}^s_*(A) = \mathcal{M}^{*s}(A)$ for some $s \ge 0$, this common value is called *s-dimensional Minkowski content* of A, and is denoted by $\mathcal{M}^s(A)$. If for some $d \ge 0$ we have that $\mathcal{M}^d(A) \in (0,\infty)$, then we say that the set A is *Minkowski measurable.* In this case clearly $d = \dim_B A$.

In the case when lower or upper d-dimensional Minkowski contents of A are *degenerate* that is, equal to 0 or ∞, where $d = \dim_B A$, we deal with generalized Minkowski contents, introduced by C. He and M. Lapidus [8]. The aim is to find explicit positive *gauge functions* $h_i : [0, \varepsilon_0) \to \mathbb{R}$, $i = 1, 2$, nondecreasing and $h_i(0) = 0$, such that the corresponding *generalized Minkowski contents*

$$\begin{aligned}\mathcal{M}_*(h_1, A) &:= \liminf_{\varepsilon\to 0} \frac{|A_\varepsilon|}{\varepsilon^N} \cdot h_1(\varepsilon),\\ \mathcal{M}^*(h_2, A) &:= \limsup_{\varepsilon\to 0} \frac{|A_\varepsilon|}{\varepsilon^N} \cdot h_2(\varepsilon),\end{aligned}$$

are nondegenerate, that is, different from 0 and ∞.

As a simple example, let A be a segment of length l. Then $|A_\varepsilon| = 2l\varepsilon + \varepsilon^2\pi$, and hence

$$\mathcal{M}^s(A) = \lim_{\varepsilon\to 0} \frac{|A_\varepsilon|}{\varepsilon^{2-s}} = \lim_{\varepsilon\to 0}(2l\varepsilon^{s-1} + \pi\varepsilon^s).$$

We have $\mathcal{M}^1(A) = 2l$ for $s = 1$, $\mathcal{M}^s(A) = \infty$ for $s < 1$, and $\mathcal{M}^s(A) = 0$ for $s > 1$. For rectifiable curves Γ we have that in general $\mathcal{M}^1(\Gamma) = 2l$, where l is the length of the curve. For more details see [8] and [13].

3 Classification of spirals in $\mathbb{R}^2$

By a planar spiral of focus type we mean the graph of a function $r = f(\varphi)$, $\varphi \ge \varphi_1$, such that $f(\varphi) \to 0$ as $\varphi \to \infty$ and $k \mapsto f(\varphi + 2k\pi)$ is decreasing for each φ.

Following [14] we classify spirals in three different ways.

(a) Spirals of *focus type* are defined as above. Spirals of *limit cycle type* are defined by $r = 1 - f(\varphi)$, or $r = 1 + f(\varphi)$, with f as above. Note that $r = 1 - f(\varphi)$ tends to the limit cycle $r = 1$ from inside, and $r = 1 + f(\varphi)$ from outside.

(b) We introduce *power, exponential, and logarithmic* spirals of focus type by $r = \varphi^{-\alpha}$, $r = e^{-\varphi}$, and $r = (\log\varphi)^{-1}$ respectively. Anologous spirals of limit cycle type are $r = 1 - \varphi^{-\alpha}$, $r = 1 - e^{-\varphi}$, $r = 1 - (\log\varphi)^{-1}$. Similarly for the spirals tending to the limit cycle from outside. It is clear that the definition can be given in the more general form.

(c) We say that a spiral is *nondegenerate* if its Minkowski content is nondegenerate. Analogously for degenerate spirals.

For example, $r = \varphi^{-1}$, $\varphi \ge \varphi_1 > 0$, defines the degenerate power spiral of focus type. It is nonrecifiable, and of box dimension 1. Nonrectifiability in this case is reflected in the fact that 1-dimensional Minkowski content of the spiral is infinite. By $r = 1 - e^{-\varphi}$ we obtain degenerate exponential spiral of limit cycle type.

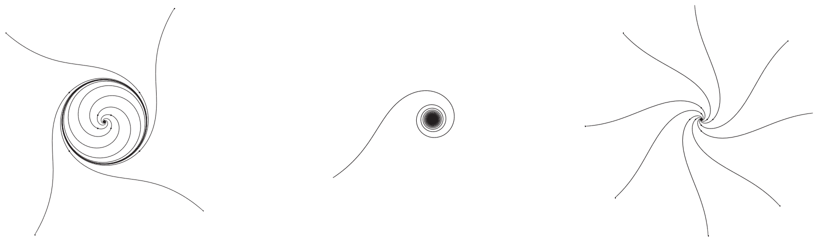

Figure 6: The cases of $a_0 < 0$, $a_0 = 0$, and $a_0 > 0$ for system (3).

By defining $r = \varphi^{-\alpha}$ we obtain nondegenerate power spirals of focus type for $\alpha \in (0,1)$, and $\dim_B \Gamma = 2/(1+\alpha)$. If $r = 1 - \varphi^{-\alpha}$, we obtain nondegenerate power spirals of limit cycle type, for any $\alpha > 0$, and $\dim_B \Gamma = \frac{2+\alpha}{1+\alpha}$. It can be shown that these spirals are not only nondegenerate, but Minkowski measurable, see [14].

4 Fractal analysis of Hopf-Takens bifurcation

We consider planar polynomial vector fields of the form

$$\begin{aligned} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y). \end{aligned}$$

Let us introduce the notion of *strong focus* as a singular point (equilibrium point) where the matrix of the linear part of the vector field has both real and imaginary parts of eigenvalues nonzero. *Weak focus* is a singular point where both eigenvalues are pure imaginary and nonzero.

A standard model where the Hopf bifurcation occurs, written in the form of vector field, is

$$\begin{aligned} X &= (-y + a_0 x + x(x^2+y^2))\frac{\partial}{\partial x} + \\ &\quad (x + a_0 y + y(x^2+y^2))\frac{\partial}{\partial y}, \end{aligned}$$

or equivalently,

$$\left\{ \begin{aligned} \dot{x} &= -y + a_0 x + x(x^2+y^2), \\ \dot{y} &= x + a_0 y + y(x^2+y^2). \end{aligned} \right. \tag{2}$$

In polar coordinates it has the form

$$\left\{ \begin{aligned} \dot{r} &= r(r^2 + a_0), \\ \dot{\varphi} &= 1. \end{aligned} \right.$$

Changing the parameter a_0 from positive values across zero to negative ones, the phase portrait changes from strong focus through weak focus to strong focus surrounded with a limit cycle. See Figure 6.

In the classical work by Takens [11] a more general situation is considered permitting multiple parameters and multiple limit cycles. Therefore it is called the *Hopf-Takens bifurcation.* Takens considers a $p-$parameter family of planar vector fields X such that $X(0) = 0$. It is required that when all parameters are equal to zero then the field X has eigenvalues on the imaginary axes and nonzero. He proved that if X is of codimension l (see Guckenheimer and Holmes [7]), then all possible nearby phase portraits and related bifurcations of X can be separated into two models $X_{\pm}^{(l)}$, where

$$\begin{aligned} X_{\pm}^{(l)} &= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) \pm \big((x^2+y^2)^l + \\ &\qquad a_{l-1}(x^2+y^2)^{l-1} + \ldots + a_0\big)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \end{aligned}$$

$(a_0, \ldots, a_{l-1}) \in \mathbb{R}^l$, is a normal form of X. This system is called the *standard model for Hopf Takens bifurcation.* In polar coordinates it reads as:

$$\begin{cases} \dot{r} &= r(r^{2l} + \sum_{i=0}^{l-1} a_i r^{2i}), \\ \dot{\varphi} &= 1. \end{cases} \tag{3}$$

The case of $l = 1$ corresponds to the classical situation of Hopf bifurcation.

Here we consider a standard model of Hopf-Takens bifurcation from the point of view of fractal geometry. We are interested in fractal properties of trajectories near singular points and limit cycles.

If a spiral trajectory is tending to a strong focus, it can be shown that it is of exponential type. We describe our results for standard Hopf-Takens model:

(1) any spiral trajectory tending to a weak focus is of power type,

(2) any spiral trajectory tending to a limit cycle of multiplicity one is of exponential type, while a spiral trajectory tending to a limit cycle of multiplicity $m > 1$ is of power type.

In order to state our results it is necessary to introduce a few additional notions. We say that a spiral $r = f(\varphi)$ of focus type is *comparable with the spiral* $r = \varphi^{-\alpha}$ *of power type* if

$$\underline{C}\varphi^{-\alpha} \le f(\varphi) \le \overline{C}\varphi^{-\alpha}$$

for some $\underline{C}, \overline{C} > 0$, and for all $\varphi \in [\varphi_1, \infty)$. Analogously for spirals with negative orientation, that is, $\underline{C}|\varphi|^{-\alpha} \le f(\varphi) \le \overline{C}|\varphi|^{-\alpha}$ for $\varphi \in (-\infty, \varphi_1]$.

A spiral $r = f(\varphi)$ of focus type is *comparable with the exponential spiral* $r = e^{-\beta\varphi}$ if

$$\underline{C}e^{-\beta\varphi} \le f(\varphi) \le \overline{C}e^{-\beta\varphi}$$

for some $\underline{C}, \overline{C} > 0$ and $\beta > 0$, and for all $\varphi \in [\varphi_1, \infty)$. Analogously for spirals with negative orientation, that is, for $\varphi \in (-\infty, \varphi_1]$ and $\beta < 0$.

Remark 1. Theorems 1 and 2 below can be extended to more general vector fields, that is, fields with normal form (3). For this it suffices to use Takens [11, Theorem 1.5 and Remark 1.6] with $p = l$. Indeed, in this case the whole configuration of closed integral curves near the singularity of p-parametric family X, $p = l$, is differentiably equivalent to the corresponding configuration in $X_{\pm}^{(l)}$, so that box dimensions of the corresponding trajectories are preserved.

Theorem 1 (The case of focus, [14]) *Let* Γ *be a part of a trajectory of (3) near the origin.*

(a) If $a_0 \neq 0$*, then the spiral* Γ *is of exponential type, that is, comparable with* $r = e^{a_0 \varphi}$*, and hence*

$$\dim_B \Gamma = 1.$$

(b) Let k *be a fixed integer,* $1 \leq k \leq l$*,* $a_l = 1$ *and* $a_0 = \ldots = a_{k-1} = 0$*,* $a_k \neq 0$*. Then* Γ *is comparable with the spiral* $r = \varphi^{-1/2k}$*, and*

$$d := \dim_B \Gamma = \frac{4k}{2k+1}.$$

Γ *is Minkowski measurable, and d-dimensional Minkowski content is equal to an explict constant.*

Theorem 1 shows the connection between the multiplicity k of focus of (3) and the box dimension of a spiral trajectory tending to the focus.

Remark 2. Notice that from Theorem 1 and the fact that the initial vector field X and its normal form $X_{\pm}^{(l)}$ are locally diffeomorphic, it follows that the Hopf-Takens bifurcation of codimension l, assuming additionally that $k = l$ (ie. when we have birth of l limit cycles), occurs with box dimension equal to $4l/(2l+1)$. In particular, for $l = 1$ we have classical Hopf bifurcation of two dimensional flows, for which at the moment of bifurcation the spiral trajectory has dimension equal to $4/3$. Remark also that the larger box dimension of a spiral trajectory at the moment of bifurcation, the more limit cycles are born.

Theorem 2 (The case of limit cycle, [14]) *Let the system (3) have a limit cycle* $r = a$ *of multiplicity* m*,* $1 \leq m \leq l$*. By* Γ_1 *and* Γ_2 *we denote the parts of two trajectories of (3) near the limit cycle from outside and inside respectively.*

(a) Then Γ_1 *and* Γ_2 *are comparable with exponential spirals* $r = a \pm e^{-\beta\varphi}$ *when* $m = 1$*,* $\beta \neq 0$ *(depending only on the coefficients* a_i*,* $0 \leq i \leq l-1$*);*

(b) Γ_1 *and* Γ_2 *are comparable with power spirals* $r = a \pm \varphi^{-1/(m-1)}$ *when* $m > 1$*.*

In both cases we have

$$d := \dim_B \Gamma_i = 2 - \frac{1}{m}, \quad i = 1, 2.$$

For $m = 1$ *we have degenerate spirals and* $\mathcal{M}^d(h, \Gamma_i) = 2/\beta$*,* $i = 1, 2$*, with the gauge function* $h(\varepsilon) := \varepsilon(\log(1/\varepsilon))^{-1}$*.*

For $m > 1$ *the spirals are Minkowski measurable.*

Theorem 2 shows the connection between multiplicity of limit cycles of (3) and the box dimension of the corresponding spiral trajectories.

Example 1. Let us consider system (3) for $l = 1$, see Figure 6.

(1) For $a_0 > 0$ we have exponential spirals of focus type with box dimension 1. In this case they are rectifiable.

(2) For $a_0 = 0$ we have power spirals of focus type with box dimension $4/3$.

(3) For $a_0 < 0$ a limit cycle appears, and box dimension of all trajectories are equal to 1.

It is to be noted that the box dimension of any trajectory near the origin is nontrivial (that is, larger than one) only for $a_0 = 0$, since then a periodic orbit is born.

Example 2. Let us consider system (3) for $l = 2$ and $a_1 = -2$. Figures 3 and 4 show the following:

(1) If $a_0 < 0$ all box dimensions are equal to 1, and the spirals are of exponential type.

(2) If $a_0 = 0$ then $\dim_B \Gamma_1 = 4/3$, and the spirals are of power type; here Γ_1 is a part of any trajectory near the origin. The part near the limit cycle $r = \sqrt{2}$ has box dimension equal to 1, and it is of exponential type.

(3) If $a_0 \in (0, 1)$ we have two limit cycles of multiplicity one, and all box dimensions are equal to 1. Spirals are of exponential type.

(4) If $a_0 = 1$ then we have limit cycle $r = 1$ of multiplicity two, and all trajectories near the limit cycle (either inside or outside) have box dimensions equal to 3/2. These are spirals of power type. Trajectories inside the limit cycle, but near the origin, have box dimension equal to 1 (exponential case).

(5) If $a_0 > 1$ then box dimensions of all trajectories are equal to 1 (exponential case).

As above, here we also have box dimension of trajectory near the origin of nontrivial value only for $a_0 = 0$, since then a periodic orbit is born. We shall encounter the same phenomenon in a class of one-dimensional discrete systems.

5 Fractal analysis of trajectories of some vector fields in $\mathbb{R}^3$

Here we consider spiral trajectories in $\mathbb{R}^3$, which we assume to be contained in a two-dimensional surface. Their box dimension depends on properties of the surface as well.

We have noticed some new phenomena regarding box dimensions of spacial spiral trajectories. For example, the fact that for some systems the box dimension essentially depends on the coefficients of the systems, and not only on the exponents.

According to properties of the surface we distinguish the following two types of spirals. If the surface is defined by $z = r^\beta$, then for $\beta \geq 1$ we obtain Lipschitzian surface, while for $\beta \in (0, 1)$ we obtain Hölderian surface. The corresponding spiral in $\mathbb{R}^3$, defined in cylindrical coordinates (r, φ, z) by

$$r = \varphi^{-\alpha}, \quad z = r^\beta, \quad \varphi \geq \varphi_1 > 0,$$

is said to be *Lipschitz-focus* spiral if $\beta \geq 1$, and *Hölder-focus* if $\beta \in (0, 1)$, see Figure 7.

For a spiral defined by

$$r = 1 - \varphi^{-\alpha}, \quad z = |1 - r|^\beta$$

we say to be *Lipschitz-cycle spiral* if $\beta \geq 1$, and *Hölder-cycle spiral* if $\beta \in (0, 1)$, see Figure 8. For more general spirals in $\mathbb{R}^3$ see [16].

We deal with a class of systems such that the linear part in Cartesian coordinates has a pure imaginary pair and a simple zero eigenvalues. Its normal form in cylindrical

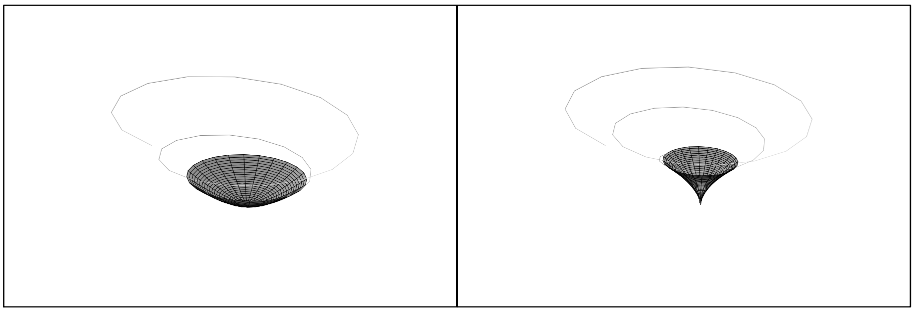

Figure 7: Lipschitz and Hölder spirals of focus type

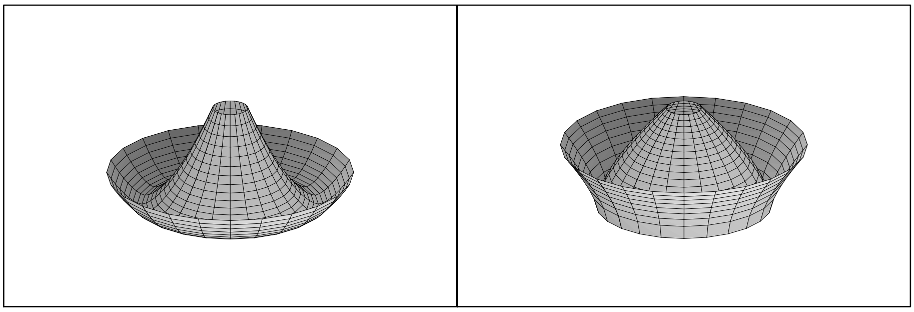

Figure 8: Surfaces containing cycle spirals of Lipschitz and Hölder types

coordinates is as follows (for normal forms see Guckenheimer and Holmes [7]):

$$\begin{aligned} \dot{r} &= c_1 r^3 + \ldots + c_m r^{2m+1} \\ \dot{\varphi} &= 1 \\ \dot{z} &= d_2 z^2 + \ldots + d_n z^n. \end{aligned} \tag{4}$$

Let us formulate two results from [16] in a simplified form, that appeared in [15].

Theorem 3 (The case of focus in $\mathbb{R}^3$, [15]) *Let Γ be a part of trajectory of (4) near the origin in $\mathbb{R}^3$. Assume that k and p are minimal positive integers such that $c_k \neq 0$ and $d_p \neq 0$. Assume also $c_k d_p > 0$.*

If $2k+1 \geq p$ then Γ is a Lipschitz-focus and nondegenerate spiral with

$$\dim_B \Gamma = \frac{4k}{2k+1},$$

If $2k+1 < p$ then Γ is a Hölder-focus and nondegenerate spiral with

$$\dim_B \Gamma = 2 - \frac{2k+p-1}{2kp}.$$

Theorem 4 (The case of limit cycle in $\mathbb{R}^3$, [15]) *Let the system (4) have limit cycle $r = a$ of multiplicity j, $1 \leq j \leq m$. By Γ_1 and Γ_2 we denote the parts of two trajectories of (4) near the limit cycle from outside and inside respectively.*

For $j = 1$ we have

$$\dim_B \Gamma_i = 1, \quad i = 1, 2.$$

For $j > 1$ we have the following alternative:
(a) if $j \geq p$ then Γ_i are Lipschitz-cycle nondegenerate spirals and

$$\dim_B \Gamma_i = 2 - \frac{1}{j}, \quad i = 1, 2;$$

(b) if $j < p$ then Γ_i are Hölder-cycle nondegenerate spirals and

$$\dim_B \Gamma_i = 2 - \frac{1}{p}, \quad i = 1, 2.$$

In the proof we use among others the fact that the box dimension and the nondegeneracy of a set are not affected by bi-Lipschitz mappings.

Example 3. We found a class of systems such that the box dimension of trajectories depends in nontrivial way on the coefficients of the system. Let us provide the following example:

$$\dot{r} = a_1 r z, \quad \dot{\varphi} = 1, \quad \dot{z} = b_2 z^2$$

The value

$$\dim_B \Gamma = \frac{2}{1 + a_1/b_2}, \quad \frac{a_1}{b_2} \in (0, 1)$$

is obtained by explicit computation of the solution, using Tricot's formula for planar spirals of the form $r = C_1 \cdot (-b_2\varphi + C_2)^{-a_1/b_2}$ near the origin, obtained by projecting the solution spiral into horizontal plane. We also use the fact that the solution spiral is contained on the Lipschitz surface $z = C_3 r^{b_2/a_1}$.

Remark 3. Let us consider the spiral Γ defined by $r = \varphi^{-\alpha}$, $z = r^{\beta}$. Only one of two projections of the spiral Γ onto horizontal and vertical planes, has box dimension equal to $\dim_B \Gamma$. More precisely, in the Lipschitz case, that is when $\beta \geq 1$, the horizontal projection spiral has the same box dimension as the initial spiral. In the Hölder case, that is when $\beta \in (0, 1)$, only the vertical projection has the same box dimension, see [15, 16].

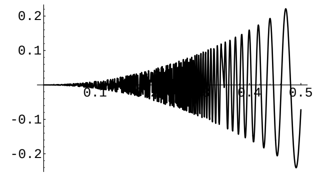

Figure 9: $(2, 4)$-chirp, $\alpha = \beta = 1/2$

The projection of the spiral

$$r = \varphi^{-\alpha}, \quad z = r^{\beta}, \quad \varphi \in [\varphi_1, \infty),$$

for fixed $\alpha \in (0,1)$, $\beta \in (0,1)$, onto the (y,z)-plane, is easily seen to be the curve

$$y = z^{1/\beta} \sin(z^{-1/\alpha\beta})$$

which is of chirp-type, see Figure 9. Box dimension of the graph of this function is equal to $2 - \frac{\alpha(1+\beta)}{1+\alpha\beta}$, see Tricot [12, p. 122]. For applications of chirp-type curves in the study of solutions of one-dimensional p-Laplace equations see Pašić [9], and Pašić, Županović [10].

6 Fractal analysis of trajectories of some discrete 1-dimensional dynamical systems

Our motivation for studying discrete systems is to be able to compute box dimension of spiral trajectories of more general continuous dynamical systems via their Poincaré mapping. Note that in previous sections all continuous systems had explicit solutions.

We would like to develop necessary tools for computation of box dimension of spiral trajectories even without solving a given system explicitely. For this we expect that the Poincaré map $P(x)$ and the displacement function $V(x) := x - P(x)$ will be crucial. Therefore we started to study box dimension of trajectories of discrete systems on the real line, and obtained results presented in [5].

We shall need the following notation. Any two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ of positive real numbers are said to be *comparable*, and we write $a_n \simeq b_n$ as $n \to \infty$, if

$$A \leq a_n/b_n \leq B,$$

for some positive constants A and B and n sufficiently large.

Analogously, for two positive functions $f, g : (0,r) \to \mathbb{R}$ we say to be *comparable* and we write $f(x) \simeq g(x)$ as $x \to 0$, if $A \leq f(x)/g(x) \leq B$ for x sufficiently small.

The following theorem deals with solution sequences $(x_n)_{n\geq 1}$ converging monotonically to zero for which it is possible to compute box dimension directly from the equation.

Theorem 5 ([5]) *Let $\alpha > 1$ and let $f : (0,r) \to (0,\infty)$ be a monotonically nondecreasing function such that $f(x) \simeq x^\alpha$ as $x \to 0$, and $f(x) < x$ for all $x \in (0,r)$. The sequence $S(x_1) := (x_n)_{n\geq 1}$ is defined by*

$$x_{n+1} = x_n - f(x_n), \quad x_1 \in (0,r).$$

Then

$$x_n \simeq n^{-1/(\alpha-1)} \quad \text{as } n \to \infty.$$

Furthermore,

$$\dim_B S(x_1) = 1 - \frac{1}{\alpha},$$

and the set $S(x_1)$ is Minkowski nondegenerate.

Now we consider what happens with box dimension when the saddle-node or the period doubling bifurcations occur. The following two bifurcation results are well known, see Devaney [3]. The novelty is additional information about the box dimension of trajectories.

Theorem 6 (Saddle-node bifurcation, [5]) *Suppose that a function $F : J \times (x_0 - r, x_0 + r) \to \mathbb{R}$, where J is an open interval in $\mathbb{R}$, is such that $F(\lambda_0, \cdot)$ is of class C^3 for some $\lambda_0 \in \mathbb{R}$, and $F(\cdot, x)$ of class C^1 for all x. Assume that*

$$\begin{aligned} F(\lambda_0, x_0) &= x_0, \\ \frac{\partial F}{\partial x}(\lambda_0, x_0) &= 1, \\ \frac{\partial^2 F}{\partial x^2}(\lambda_0, x_0) &< 0, \\ \frac{\partial F}{\partial \lambda}(\lambda_0, x_0) &\neq 0. \end{aligned} \tag{5}$$

Then λ_0 is the point where saddle-node bifurcation occurs. There exists $r_1 \in (0, r)$ such that for any sequence $S(\lambda_0, x_1) = (x_n)_{n \geq 1}$ defined by $x_{n+1} = F(\lambda_0, x_n)$, $\quad x_1 \in (x_0, x_0 + r_1)$, we have $|x_n - x_0| \simeq n^{-1}$ as $n \to \infty$,

$$\dim_B S(\lambda_0, x_1) = \frac{1}{2},$$

and $S(\lambda_0, x_1)$ is Minkowski nondegenerate. Analogous result holds if $x_1 \in (x_0 - r_1, x_0)$, where in (5) we have the opposite sign.

It is worth noticing that in the following theorem we have the same phenomenon as in the planar continuous case: the box dimension jumps just before the periodic orbit is born, see [5].

Theorem 7 (Period-doubling bifurcation, [5]) *Let $F : J \times (x_0 - r, x_0 + r) \to \mathbb{R}$ be a function of class C^2, where J is an open interval in $\mathbb{R}$, and $F(\lambda_0, \cdot)$ is of class C^4 for some $\lambda_0 \in J$. Assume that*

$$\begin{aligned} F(\lambda_0, x_0) &= x_0, \\ \frac{\partial F}{\partial x}(\lambda_0, x_0) &= -1, \\ \frac{\partial^2 F}{\partial x^2}(\lambda_0, x_0) &\neq 0, \end{aligned}$$
$$\frac{\partial^2 (F^2)}{\partial \lambda\, \partial x}(\lambda_0, x_0) \neq 0, \qquad \frac{\partial^3 (F^2)}{\partial x^3}(\lambda_0, x_0) \neq 0.$$

Then λ_0 is the point where period-doubling bifurcation occurs. There exists $r_1 \in (0, r)$ such that for any sequence $S(\lambda_0, x_1) = (x_n)_{n \geq 1}$ defined by

$$x_{n+1} = F(\lambda_0, x_n), \quad x_1 \in (x_0 - r_1, x_0 + r_1), \quad x_1 \neq x_0,$$

we have $|x_n - x_0| \simeq n^{-1/2}$ *as* $n \to \infty$,

$$\dim_B S(\lambda_0, x_1) = \frac{2}{3},$$

and $S(\lambda_0, x_1)$ *is Minkowski nondegenerate.*

The proofs of Theorems 6 and 7 are based on Theorem 5. These two theorems can be applied to the study of logistic map.

Corollary 1 (Logistic map) *Let* $F(\lambda, x) = \lambda x(1-x)$, $x \in (0,1)$, *and let* $S(\lambda, x_1) = (x_n)_{n\geq 1}$ *be a sequence defined by initial value* x_1 *and* $x_{n+1} = F(\lambda, x_n)$.

(a) For $\lambda_0 = 1$, *taking* $x_1 > 0$ *sufficiently close to* $x_0 = 0$, *we have that* $x_n \simeq n^{-1}$ *as* $n \to \infty$, *and*

$$\dim_B S(1, x_1) = \frac{1}{2}.$$

(b) (Onset of period-2 cycle) For $\lambda_0 = 3$ *the corresponding fixed point is* $x_0 = 2/3$. *For any* x_1 *sufficiently close to* x_0 *we have that* $|x_n - x_0| \simeq n^{-1/2}$, *and*

$$\dim_B S(3, x_1) = \frac{2}{3}.$$

(c) For any $\lambda \notin \{1, 3\}$ *and* x_1 *such that the sequence* $S(\lambda, x_1)$ *is convergent, we have that* $\dim_B S(\lambda, x_1) = 0$.

(d) (Onset of period-4 cycle) If $\lambda_0 = 1 + \sqrt{6}$ *then for any* x_1 *sufficiently close to period-2 trajectory* $A = \{a_1, a_2\}$ *we have that* $d(x_n, A) \simeq n^{-1/2}$ *as* $n \to \infty$, *and*

$$\dim_B S(1 + \sqrt{6}, x_1) = \frac{2}{3}.$$

(e) (Period-3 cycle) Let $\lambda_0 = 1 + \sqrt{8}$ *and let* a_1, a_2, a_3 *be fixed points of* F^3 *such that* $0 < a_1 < a_2 < a_3 < 1$, $F(a_1) = a_2$, $F(a_2) = a_3$, *and* $F(a_3) = a_1$. *Then there exists* $\delta > 0$ *such that for any initial value*

$$x_1 \in (a_1 - \delta, a_1) \cup (a_2 - \delta, a_2) \cup (a_3, a_3 + \delta)$$

we have $d(x_n, \{a_1, a_2, a_3\}) \simeq n^{-1}$ *as* $n \to \infty$, *and*

$$\dim_B S(1 + \sqrt{8}, x_1) = \frac{1}{2}.$$

All trajectories appearing in this corollary are Minkowski nondegenerate.

We conjecture that the values of box dimensions of trajectories corresponding to all period-doubling bifurcation parameters λ_k where 2^k-periodic points occur, are equal to 2/3 (see [5]).

Remark 4. Using the asymptotic behaviour of iterates of the Poincaré map and Theorem 5 we can obtain dimensional results for Hopf-Takens bifurcation described in Section 4 in a different way, which will be exposed in [17]. In this way we can compute box dimension of spiral trajectories with iterates of the Poincaré map of power type. We believe that it will be possible to apply this approach also to systems with iterates of the Poincaré map depending not only on powers but also on logarithmic terms.

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