Aspects of Lie theory

Pd.D. thesis by Steen Ryom-Hansen

1. Preface

This thesis deals with different aspects of Lie theory. The main themes are the modular representation theory of algebraic groups and the more recent theory of quantum groups. The thesis contains 5 sections which however are logically rather independent of each other.

The work presented here has mostly been carried out at the mathematical institute at Aarhus University. It is a pleasure to thank all members of the group of representation theorists there for providing a nice atmosphere for doing mathematics.

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2. Contents

3. Introduction

1.1. In chapter 4 we investigate themes dealing with the modular Lusztig conjecture. Using translation techniques we show a consequence of that conjecture.

1.2. In chapter 5 the setting is that of a highest weight category. We give a short proof of a theorem of Cline, Parshall and Scott.

1.3. In chapter 6 we investigate the piecewise linear function of Lusztig in terms of the combinatorics of crystals.

1.4. In chapter 7 we show that the refined Demazure formula Littelmann can be proved by purely combinatorial methods.

1.5. In chapter 8 we use the refined Demazure formula to show a quantum version of the Kempf vanishing theorem.

4. Some remarks on Ext groups.

§ 1. The setup.

1.1. Let us start out by defining the basic objects of modular representation theory.
So we are considering a simply connected simple algebraic group G over an algebraically closed field k of characteristic $p > 0$. We fix a maximal torus T and a Borel subgroup B of G containing T. The root system is denoted R and we let B correspond to the negative roots R^- of R. The character group of T is called $X(T)$ and the set of dominant characters $X(T)_+$:

$$X(T)_+ := \{ \lambda \in X(T) \mid <\lambda, \alpha^\vee> \geq 0 \text{ for all } \alpha \in R^+ \}$$

We have $R \subset X(T)$ and a partial order $\leq$ on $X(T)$ given by

$$\lambda \leq \mu \text{ iff } \mu - \lambda \in \mathbb{Z}_{\geq 0}[R^+]$$

1.2. For $\lambda \in X(T)$ we let $H^0(\lambda)$ denote the G module obtained from λ by induction from B to G. Then $H^0(\lambda) \neq 0$ iff $\lambda \in X(T)_+$ and in that case it has a simple socle $L(\lambda)$; all other compostion factors of $H^0(\lambda)$ are on the form $L(\mu)$ for $\mu \leq \lambda$. In fact the $L(\lambda)$ for $\lambda \in X(T)_+$ exhaust the set of finite dimensional simple G-modules.

For V a G-module V we write $\operatorname{ch} V$ for its formal character $\in \mathbb{Z}[X(T)]$. The subgroup in $\mathbb{Z}[X(T)]$ spanned by $\operatorname{ch} V$ for V a G-module is isomorphic to the Grothendieck group of the category of G-modules and thus has a natural basis consisting of $\operatorname{ch} L(\lambda)$ for $\lambda \in X(T)_+$. By the above it also has the set of $\operatorname{ch} H^0(\lambda)$s as a basis. The equality

$$\operatorname{ch} H^0(\lambda) = \sum_{\mu \leq \lambda} [H^0(\lambda): L(\mu)] \operatorname{ch} L(\mu) \tag{1.2.1}$$

where $[H^0(\lambda):L(\mu)]$ is the composition factor multiplicity of $L(\mu)$ in $H^0(\lambda)$ can therefore be inverted formally. We write the inverted expression as

(1.2.2) $$\operatorname{ch} L(\lambda) = \sum_{\mu \leq \lambda} (L(\lambda), H^0(\mu)) \operatorname{ch} H^0(\mu)$$

where $(L(\lambda), H^0(\mu)) \in \mathbb{Z}$. These numbers are given by the following formula

(1.2.3) $$(L(\lambda), H^0(\mu)) = \sum_i (-1)^i \dim \operatorname{Ext}^i_G(L(\lambda), H^0(\mu))$$

(From Frobenius reciprocity and Kempf vanishing we have that

$$\operatorname{Ext}^i_G(L(\lambda), H^0(\mu)) = \operatorname{Ext}^i_B(L(\lambda), \mu)$$

and by the standard resolution of μ this vanishes for i large and for $\mu \not< \lambda$. Hence the sum in (1.2.3) is finite. We also see that for $i > 0$

(1.2.4) $$\operatorname{Ext}^i_G(L(\lambda), H^0(\mu)) \neq 0 \Rightarrow \mu < \lambda \text{)}$$

Actually we have for any G-module M that

(1.2.5) $$(M, H^0(\mu)) = \sum_i (-1)^i \dim \operatorname{Ext}^i_G(M, H^0(\mu))$$

Let us outline a proof of this. Let $V(\lambda)$ denote the Weyl module with highest weigth λ; it is dual to $H^0(\lambda)$ under the duality that fixes $L(\lambda)$; thus it has the same character as $H^0(\lambda)$ and its unique simple image is $L(\lambda)$. From [CPSvdK] one has

(1.2.6) $$\dim \operatorname{Ext}^i_G(V(\lambda), H^0(\mu)) = \delta_{\lambda,\mu}\delta_{i,0}$$

So at least for $\mu = \lambda$ (1.2.5) is true. Now both sides of (1.2.5) are additive in M, so (1.2.5) would follow from (1.2.3). Let us thus show (1.2.3) by induction on μ. If λ is minimal in $X(T)_+$ then $V(\lambda) = L(\lambda)$ and we already know (1.2.3). Otherwise we consider the exact sequence

$$0 \longrightarrow Q \longrightarrow V(\lambda) \longrightarrow L(\lambda) \longrightarrow 0$$

All composition factors of Q have highest weight $< \lambda$. Hence (1.2.5) holds for the first two modules in the sequence and thus also for $L(\lambda)$.

1.3. For $\alpha \in R$ and $n \in \mathbb{Z}$ let $H_{\alpha,n}$ denote the hyperplane in $E = X(T)\otimes\mathbb{R}$:

$$H_{\alpha,n} := \{ \nu\in E \mid <\alpha^\vee, \nu+\rho> = np \}$$

The connected components in $E \setminus \bigcup_{\alpha,n} H_{\alpha,n}$ are called alcoves. The affine Weyl W_p group is the group generated by the reflections in all the $H_{\alpha,n}$. Let A^+ be the fundamental dominant alcove, i.e. the one given by

$$-\rho\in\overline{A^+} \quad \text{and} \quad A^+ + \rho \subset X(T)_+\otimes\mathbb{R}$$

And let $A^- := w_0 A^+$. We assume in the following that $p \geq h$ (the Coxeter number) and may thus find a $\lambda \in A^- \cap X(T)$. If A is an alcove we denote by λ_A the mirror image of λ in A. Then W_p acts simply transitively on the set of λ_A. For $w \in W_p$ the image of λ under this action is denoted $w.\lambda$.

The following is the main conjecture of the theory:

Conjecture 1.3. (Lusztig [L1]). Let $y,\ w \in W_p$ such that $y.\lambda,\ w.\lambda \in X(T)_+$. Suppose moreover that $w.\lambda$ belongs to the Jantzen region, i.e. that $<\check{\alpha_0}, w.\lambda> < p(p-h+2)$. Then we have

$$(L(w.\lambda), H^0(y.\lambda)) = (-1)^{l(w)-l(y)} P_{y,w}(1) \tag{1.3.1}$$

Here α_0 is the highest short root of R and $P_{y,w}(q)$ is the Kazhdan-Lusztig polynomial for the Coxeter group W_p.

By (1.2.4) the assertion that

$$P_{y,w}(q) = \sum_i q^{l(w)-l(y)-2i} \dim \mathrm{Ext}^i_G(L(w.\lambda), H^0(y.\lambda)) \tag{1.3.2}$$

with w and y as before would imply (1.3.1). And in section 2 of [A] it is shown that also the reverse implication holds; thus (1.3.2) could be used instead of (1.3.1) in the formulation of the conjecture. (Actually section 2 of [A] contains a whole series of equivalent formulations of the conjecture).

There has been much progress in recent years towards a solution to the conjecture; see [KL1, AJS]; however it is not yet completely settled.

Now from the inductive formula [KL2, 2.2.c] for the polynomials $P_{y,w}(q)$ one sees that their constant term is 1 for $y \leq w$. Hence believing in (1.3.2) we would have

(1.3.3) $$\dim \operatorname{Ext}_G^{l(w)-l(y)}(L(w.\lambda), H^0(y.\lambda)) = 1$$

for $y \leq w$ as in the conjecture. In this chapter we wish to demonstrate that (1.3.3) holds independently of the Lusztig conjecture and for all w, y such that $w.\lambda$, $y.\lambda \in X(T)_+$.

In [C] there is a proof of the corresponding statement for the category $\mathcal{O}$ of Bernstein, Gelfand and Gelfand. However the argument there relies in a desicive way on the Verma embedding theorem, i.e. the fact that homomorphisms between Verma modules are injective. This property in general fails for Weyl modules so the proof does not carry over.

§ 2. Translation arguments.

2.1. Let $\lambda \in A^- \cap X(T)$ be as in section 1. Take $\mu \in \in \overline{A^-} \cap X(T)$ and suppose that $\{ s \in W_p \mid s.\mu = \mu \} = \{1, s\}$. Let T_μ^λ and T_λ^μ be the Jantzen translation functors. They are exact and adjoint to each other. Assuming $y.\lambda \in X(T)_+$ we have

(2.1.1) $$\operatorname{ch} T_\lambda^\mu H^0(y.\lambda) = H^0(y.\mu)$$

(2.1.2) $$T_\lambda^\mu L(y.\lambda) = L(y.\mu) \quad \text{if } sy > y$$

(2.1.3) $$T_\lambda^\mu L(y.\lambda) = 0 \quad \text{if } sy < y$$

If $y < sy$ there is an exact sequence of G modules as follows

(2.1.4) $$0 \longrightarrow H^0(y.\lambda) \longrightarrow T_\mu^\lambda H^0(y.\mu) \longrightarrow H^0(sy.\lambda) \longrightarrow 0$$

If $w.\mu \in X(T)_+$ has $sw < w$ then by (2.1.2) and the adjointness of T_μ^λ and T_λ^μ we get

$$\mathrm{Hom}_G(L(sw.\lambda), T_\mu^\lambda L(w.\mu)) = \mathrm{Hom}_G(T_\lambda^\mu L(sw.\lambda), L(w.\mu)) = \mathrm{Hom}_G(L(w.\mu), L(y.\mu)) = k$$

so $L(sw.\lambda)$ is contained in the socle of $T_\mu^\lambda L(w.\mu)$ and a similar reasoning shows that it is also contained in the head of $T_\mu^\lambda L(w.\mu)$. (Actually there is equality in both cases). For $sw < w$ we can therefore define modules $Q(w.\lambda)$, $R(w.\lambda)$ and $U(w.\lambda)$ by the following exact sequences

(2.1.5) $$0 \longrightarrow L(sw.\lambda) \longrightarrow T_\mu^\lambda L(w.\mu) \longrightarrow Q(w.\lambda) \longrightarrow 0$$

(2.1.6) $$0 \longrightarrow R(w.\lambda) \longrightarrow T_\mu^\lambda L(w.\mu) \longrightarrow L(sw.\lambda) \longrightarrow 0$$

(2.1.7) $$0 \longrightarrow U(w.\lambda) \longrightarrow Q(w.\lambda) \longrightarrow L(sw.\lambda) \longrightarrow 0$$

(2.1.8) $$0 \longrightarrow L(sw.\lambda) \longrightarrow U(w.\lambda) \longrightarrow R(w.\lambda) \longrightarrow 0$$

The module $U(w.\lambda)$ plays an important role in [A]; it is shown that conjecture (1.3.1) holds if and only if $U(w.\lambda)$ is semisimple.

2.2. The following result is well known

Lemma 2.1. For $w,\ y \in W_p$ such that $w.\lambda,\ y.\lambda \in X(T)_+$ we have for $i > l(w) - l(y)$ that

$$\mathrm{Ext}^i_G(L(w.\lambda), H^0(y.\lambda)) = 0$$

Proof. We proceed by induction on w. If $w = w_0$ then the lemma holds by (1.2.4). We may then assume the vanishing of the lemma for $w' \in W_p$ such that $l(w') < l(w)$. The application of $\mathrm{Hom}_G(-, H^0(y.\lambda))$ to the short exact sequence of G modules

$$0 \longrightarrow K \longrightarrow V(w.\lambda) \longrightarrow L(w.\lambda) \longrightarrow 0$$

leads to a long exact sequence of Ext groups. From (1.2.6) this gives for $i > 0$ that

$$\text{(2.1.1)} \qquad \mathrm{Ext}^{i-1}_G(K,\ H^0(y.\lambda)) \simeq \mathrm{Ext}^i_G(L(w.\lambda), H^0(y.\lambda))$$

Now the composition factors of K are all on the form $L(w'.\lambda)$ with $l(w') < l(w)$ and will therefore by induction hypothesis satisfy

$$\mathrm{Ext}^{i-1}_G(L(w'.\lambda), H^0(y.\lambda)) = 0$$

But then also (2.1.1) vanishes and we are done. □

2.3. The next two lemmas relate the Ext groups that we are interested in to an Ext group involving $U(w.\lambda)$.

Lemma 2.3.1. Assume $sy < y$. Then the following holds

$$\mathrm{Ext}^{l(w)-l(y)}_G(U(w.\lambda), H^0(y.\lambda)) \simeq \mathrm{Ext}^{l(w)-l(y)}_G(L(sw.\lambda), H^0(sy.\lambda)) \text{ if } sy.\lambda \in X(T)_+$$

$$\mathrm{Ext}^{l(w)-l(y)}_G(U(w.\lambda), H^0(y.\lambda)) \simeq \mathrm{Ext}^{l(w)-l(y)-1}_G(L(sw.\lambda), H^0(y.\lambda)) \text{ if } sy.\lambda \notin X(T)_+$$

Proof. By lemma 2.6 of [A] we have that

$$\mathrm{Ext}^i_G(Q(w.\lambda), H^0(y.\lambda)) \simeq \mathrm{Ext}^i_G(L(sw.\lambda), H^0(sy.\lambda)) \quad \text{if } sy.\lambda \in X(T)_+$$

$$\mathrm{Ext}^i_G(Q(w.\lambda), H^0(y.\lambda)) \simeq \mathrm{Ext}^{i-1}_G(L(sw.\lambda), H^0(y.\lambda)) \quad \text{if } sy.\lambda \notin X(T)_+$$

We insert this information in the long exact sequence that arises from the application of $\mathrm{Hom}_G(-, H^0(y.\lambda))$ to (2.1.7). If $sy.\lambda \in X(T)_+$ part of the resulting sequence is

$$\longrightarrow \mathrm{Ext}^{l(w)-l(y)}_G(L(sw.\lambda), H^0(y.\lambda)) \longrightarrow \mathrm{Ext}^{l(w)-l(y)}_G(L(sw.\lambda), H^0(sy.\lambda)) \longrightarrow$$

$$\mathrm{Ext}^{l(w)-l(y)}_G(U(w.\lambda), H^0(y.\lambda)) \longrightarrow \mathrm{Ext}^{l(w)-l(y)+1}_G(L(sw.\lambda), H^0(y.\lambda)) \longrightarrow$$

while for $sy.\lambda \notin X(T)_+$ part of the resulting sequence is

$$\longrightarrow \mathrm{Ext}^{l(w)-l(y)}_G(L(sw.\lambda), H^0(y.\lambda)) \longrightarrow \mathrm{Ext}^{l(w)-l(y)-1}_G(L(sw.\lambda), H^0(y.\lambda)) \longrightarrow$$

$$\mathrm{Ext}^{l(w)-l(y)}_G(U(w.\lambda), H^0(y.\lambda)) \longrightarrow \mathrm{Ext}^{l(w)-l(y)+1}_G(L(sw.\lambda), H^0(y.\lambda)) \longrightarrow$$

But by lemma 2.1 the first and the last terms in the two sequences are zero; the lemma is proved. □

Lemma 2.3.2. Assume $sy > y$. Then the following holds

$$\mathrm{Ext}^{l(w)-l(y)}_G(U(w.\lambda), H^0(y.\lambda)) \simeq \mathrm{Ext}^{l(w)-l(y)-1}_G(L(sw.\lambda), H^0(y.\lambda))$$

Proof. From the sequence (2.1.6) defining $R(w.\lambda)$ we get the long exact sequence

$$\longrightarrow \mathrm{Ext}^i_G(L(sw.\lambda), H^0(y.\lambda)) \longrightarrow \mathrm{Ext}^i_G(T^\lambda_\mu L(w.\mu), H^0(y.\lambda)) \longrightarrow \mathrm{Ext}^i_G(R(w.\lambda), H^0(y.\lambda)) \longrightarrow$$

And (2.1.4) gives the sequence

$$\longrightarrow \mathrm{Ext}^i_G(L(sw.\lambda), H^0(y.\lambda)) \longrightarrow \mathrm{Ext}^i_G(L(sw.\lambda), T^\lambda_\mu H^0(y.\lambda)) \longrightarrow \mathrm{Ext}^i_G(L(sw.\lambda), H^0(sy.\lambda)) \longrightarrow$$

We see that the first terms in these sequences are equal. And also the middle terms are isomorphic: by the adjointness of T^λ_μ and T^μ_λ together with (2.1.1), (2.1.2) both are isomorphic to

$$\mathrm{Ext}^i_G(L(w.\mu), H^0(y.\mu))$$

As the resulting diagrams commute we conclude that also the last terms are equal, i.e.

$$\mathrm{Ext}^i_G(R(w.\lambda), H^0(y.\lambda)) \simeq \mathrm{Ext}^i_G(L(sw.\lambda), H^0(sy.\lambda))$$

We insert this information in the long exact sequence that arises from the application of $\mathrm{Hom}_G(-, H^0(y.\lambda))$ to (2.1.8). A part of the resulting sequence is

$$\longrightarrow \mathrm{Ext}_G^{l(w)-l(y)-1}(L(sw.\lambda), H^0(sy.\lambda)) \longrightarrow \mathrm{Ext}_G^{l(w)-l(y)-1}(L(sw.\lambda), H^0(y.\lambda)) \longrightarrow$$

$$\mathrm{Ext}_G^{l(w)-l(y)}(U(w.\lambda), H^0(y.\lambda)) \longrightarrow \mathrm{Ext}_G^{l(w)-l(y)}(L(sw.\lambda), H^0(sy.\lambda)) \longrightarrow$$

By lemma 2.1 the first and the last terms are zero, whence the middle terms are isomorphic: the lemma is proved. □

2.4. After these preparatory lemmas we can now prove our main result.

Theorem 2.4. For all $y, w \in W_p$ such that $y \leq w$ and $y.\lambda, w.\lambda \in X(T)_+$ we have

$$\dim \mathrm{Ext}_G^{l(w)-l(y)}(L(w.\lambda), H^0(y.\lambda)) = 1$$

Proof. We proceed by induction on $l(w)$. If $w = w_0$ then also $y = w_0$. But

$$\dim \mathrm{Hom}_G(L(w_0.\lambda), H^0(w_0.\lambda)) = 1$$

We then assume the result for w' with $l(w') < l(w)$ and choose s with $sw < w$ and $sw.\lambda \in X(T)_+$. Then the theorem holds for sw and we get from lemma 2.3.1 and lemma 2.3.2 that

$$\dim \mathrm{Ext}_G^{l(w)-l(y)}(U(w.\lambda), H^0(y.\lambda)) = 1$$

So the theorem would be a consequence of the isomorphism

$$(2.4.1) \qquad \mathrm{Ext}_G^{l(w)-l(y)}(U(w.\lambda), H^0(y.\lambda)) \simeq \mathrm{Ext}_G^{l(w)-l(y)}(L(w.\lambda), H^0(y.\lambda))$$

We now claim that

$$(2.4.2) \qquad [U(w.\lambda), L(z.\lambda)] \neq 0,\ z \neq w \ \Rightarrow\ l(w) - l(z) \geq 2$$

Believing this we would from lemma 2.1 have that

$$(2.4.3) \qquad \mathrm{Ext}_G^{l(w)-l(y)-k}(L(z.\lambda), H^0(y.\lambda)) = 0 \ \text{ for } k = 0,\ 1$$

Now it is easy to see, (proof of prop. 2.8 (ii) of [A]) that

$$(2.4.4) \qquad [U(w.\lambda), L(w.\lambda)] = 1$$

And then (2.4.1) would follow by filtering $U(w.\lambda)$ and considering the terms of index $l(w) - l(y)$ and $l(w) - l(y) - 1$ in the long exact sequence given by the application of $\mathrm{Hom}_G(-,\ H^0(y.\lambda))$. So we aim at proving (2.4.2).

Assume on the contrary that $l(w) - l(z) = 1$, i.e. that $l(z) = l(sw)$. Choose $a_{sw,w'} \in \mathbb{Z}$ such that

$$(2.4.5) \qquad \mathrm{ch}\, L(sw.\lambda) = \sum_{w'} a_{sw,w'} \mathrm{ch}\, H^0(w'.\lambda)$$

Then $a_{sw,w'} = 0$ unless $w' \leq sw$ and $a_{sw,sw} = 1$. We apply $T_\mu^\lambda \circ T_\lambda^\mu$ to (2.4.5) and get by the exactness of translation and (2.1.1) – (2.1.4) the expression

$$\begin{aligned} \mathrm{ch}\, T_\mu^\lambda L(w.\mu) &= \sum_{w'} a_{sw,w'}\ \mathrm{ch}\, H^0(sw'.\lambda) + \sum_{w'} a_{sw,w'}\ \mathrm{ch}\ H^0(w'.\lambda) \\ &= \mathrm{ch}\, L(sw.\lambda) + \sum_{w'} a_{sw,w'} \mathrm{ch}\, H^0(sw'.\lambda) \end{aligned}$$

It is known, see e.g. the proof of theorem 2.16 in [A], that $sz < z$ for $[U(w.\lambda), L(z.\lambda)] \neq 0$; especially $z \neq sw$ for such z. By this and by the assumption on $l(z)$, we only need to consider the contributions from $w' = sw$ and $w' = sz$ in the sum to count the composition factor multiplicity of $L(z.\lambda)$ in $U(w.\lambda)$. We get

(2.4.6) $[U(w.\lambda), L(z.\lambda)] = a_{sw,sw}[H^0(w.\lambda), L(z.\lambda)] + a_{sw,sz}[H^0(z.\lambda), L(z.\lambda)]$

$$= a_{sw,sw}[H^0(w.\lambda), L(z.\lambda)] + a_{sw,sz} = [H^0(w.\lambda), L(z.\lambda)] + a_{sw,sz}$$

However we have from corollary 6.24 in Jantzens book [J] and the remark following it that

(2.4.7) $$[H^0(w.\lambda), L(z.\lambda)] = 1$$

Concerning $a_{sw,sz}$ we have by (1.2.3)

$$a_{sw,sz} = \sum_i (-1)^i \dim \mathrm{Ext}^i_G(L(sw.\lambda), H^0(sz.\lambda))$$

However $l(sw) = l(w) - 1$ and $l(sz) = l(z) - 1$ because $sz < z$ so we have $l(sw)$ - $l(sz) = l(w) - l(z) = 1$. And then we get by induction that

(2.4.8) $$a_{sw,sz} = -1$$

By (2.4.6), (2.4.7) and (2.4.8) we conclude that $[U(w.\lambda), L(z.\lambda)] = 0$ which is the desired contradiction. The theorem is proved. □

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5. Concrete Kazhdan-Lusztig theory.

§ 1. Highest weight categories.

1.1. In order to provide an axiomatic setting for many situations that arise naturally in representation theory, Cline Parshall and Scott have introduced the concept of a highest weight category. The basic ingredient of such a category is a weight poset Λ parametrizing three distinguished classes of modules, namely the simple modules $L(\lambda)$, the induced modules $A(\lambda)$ and the Weyl modules $V(\lambda)$. Let us mention some of the most important rules that these objects should satisfy: $A(\lambda)$ should have simple socle $L(\lambda)$ and all other composition factors on the form $L(\mu)$ with $\mu < \lambda$ while $V(\lambda)$ should have simple head $L(\lambda)$ and all other composition factors on the form $L(\mu)$ with $\mu < \lambda$. There should be enough projectives in $\mathfrak{C}$ so that we can speak about $\mathrm{Ext}^i_{\mathfrak{C}}(-,-)$ and with respect to this our objects should satisfy, (see (1.2.4) and (1.2.6) of chapter *4*)

$$\mathrm{Ext}^i_{\mathfrak{C}}(L(\lambda), A(\mu)) \neq 0 \Rightarrow \mu < \lambda \quad \text{if } i > 0$$

$$\mathrm{Ext}^i_{\mathfrak{C}}(V(\lambda), A(\mu)) = \delta_{i,0}\delta_{\lambda,\mu}k$$

where k is the ground field.

1.2. Let $\mathfrak{C}$ be a highest weight category. Then the projective cover $P(\lambda)$ of $L(\lambda)$ in $\mathfrak{C}$ has a filtration with quotients consisting of Weyl modules $V(\mu)$; the number of times $\mathrm{V}(\mu)$ appears in such a filtration is denoted $(P(\lambda),\mathrm{V}(\mu))$. This number is independent of the choice of filtration and satisfies the Brauer Humphreys reciprocity law

$$(1.2.1) \qquad (P(\lambda), V(\mu)) = [V(\mu), L(\lambda)]$$

Let now Γ be an ideal in Λ, that is a subset satisfying

$$\mu \leq \lambda \text{ and } \lambda \in \Gamma \Rightarrow \mu \in \Gamma$$

Then the subcategory $\mathfrak{C}[\Gamma]$ of $\mathfrak{C}$ whose objects are the objects in $\mathfrak{C}$ that have only composition factors on the form $L(\mu)$ for $\mu \in \Gamma$ is again a highest weight category. If we for a fixed $\lambda \in \Lambda$ let Γ be the set of $\mu \in \Lambda$ such that $\mu \leq \lambda$ then Γ is an ideal and the projective cover of $L(\lambda)$ in $\mathfrak{C}[\Gamma]$ will be $V(\lambda)$: this follows from (1.2.1). We say that $\mathfrak{C}[\Gamma]$
arises from $\mathfrak{C}$ by truncation.

1.3. Let us now more specifically consider a variation of the setup of Cline, Parshall and Scott in [CPS]. So Λ is assumed to be finite and equipped with an involutive duality functor $\mathcal{D}$ mapping $V(\lambda)$ to $A(\lambda)$ and fixing $L(\lambda)$. Furthermore there should be a set of operators $\{ s \mid s \in S \}$ acting on Λ and a length function $l{:}\Lambda \longrightarrow \mathbb{Z}_{\geq 0}$ satisfying $l(\lambda s) = l(\lambda) + 1$ for $\lambda s > \lambda$. We finitely assume that there exists a functor $\theta_s : \mathfrak{C} \longrightarrow \mathfrak{C}$ having the usual properties of *translation across the wall.* This means for instance that for $\tau s > \tau$ we have $\mathrm{soc}(\theta_s L(\tau)) = \mathrm{cap}(\theta_s L(\tau)) = L(\tau)$; we let $U_s(\tau)$ be the middle part of $\theta_s(\tau)$. In [CPS] the existence of θ_s is formulated as the existence of a so called *Hecke operator*.

1.4. We now define

$$P_{\mu,\nu}(q) := \sum_{i \geq 0} q^i \dim \mathrm{Ext}_{\mathfrak{C}}^{l(\mu)-l(\nu)-2i}(L(\nu), A(\mu))$$

The main theorem of [CPS] can then be formulated in the following way.

Theorem 1.4. In the above setup the following conditions are equivalent:

i) $P_{\mu,\nu}(q)$ can be calculated by the Kazhdan-Lusztig algoritm.

ii) $U_s(\tau)$ is semisimple for all τ, s such that $\tau s > \tau$.

iii) $L(\tau s)$ is a summand of $U_s(\tau)$ for all τ, s such that $\tau s > \tau$.

iv) $\mathrm{Hom}_{\mathfrak{C}}(U_s(\tau), L(\tau s)) \neq 0$ *for* all τ, s such that $\tau s > \tau$.

v) $\mathrm{Ext}_{\mathfrak{C}}^1(L(\tau), L(\tau s)) \neq 0$ for all τ and s.

vi) $\mathrm{Ext}_{\mathfrak{C}}^i(L(\nu), A(\mu)) = 0$ for all ν, μ such that $l(\nu) - l(\mu) \not\equiv i \pmod 2$.

vii) $\mathrm{Ext}_{\mathfrak{C}}^1(L(\nu), A(\mu)) = 0$ for all ν, μ such that $l(\nu) \equiv l(\mu) \pmod 2$.

viii) The analogous statements with $A(\mu)$ replaced by $V(\mu)$.

Survey of proof. The duality functor gives viii). The equivalence of i) and ii) were already proved in [A] (see also [V]) as well as iii) ⇒ vi). And the equivalence of iii), iv) and v) were also mentioned in that paper. Finally, in [CPS] it was settled that the even-odd vanishing of vii) implies the other conditions of the theorem, this was achieved through a derived category argument. □

1.5. The purpose of chapter is to prove that the conditon vii) implies the other conditions without using derived categories.

1.6. I wish to express my gratitude to Cline, Parshall and Scott for stimulating conversations and for pointing out the relevance of lemma 2.2 below. Thanks are also due to Henning Haahr Andersen for stimulating discussions.

§ 2. The reduction.

2.1. The next lemma could be compared with the calculation of $\mathrm{Ext}^i_{\mathfrak{C}}(L(\nu), L(\mu))$ in [CPS] and [BGS]. However, we are here working under a weaker assumption than in [*loc.cit.*] and also avoid the use of derived categories.

Lemma 2.1. Assume $\mathrm{Ext}^1_{\mathfrak{C}}(L(\nu), A(\mu)) = 0$ for $l(\nu) \equiv l(\mu)$ (mod 2). Then

$$\mathrm{Ext}^1_{\mathfrak{C}}(L(\lambda), A(\mu)) = \mathrm{Ext}^1_{\mathfrak{C}}(L(\lambda), L(\mu)) \quad \text{for } \mu < \lambda$$

Proof. We may assume that $l(\lambda) \not\equiv l(\mu)$ mod 2, because otherwise both sides of the lemma are 0; the left hand side by the assumption, and the right hand side by corollary 2.10 of [A]. Now define Q by

$$0 \longrightarrow Q \longrightarrow V(\lambda) \longrightarrow L(\lambda) \longrightarrow 0$$

Set $\Gamma := \{ \nu \in \Lambda \mid \nu \leq \lambda \}$. Then Γ is an ideal in Λ and we may replace $\mathfrak{C}$ by $\mathfrak{C}[\Gamma]$. By 1.2 we then get that $V(\lambda)$ is projective. Any element of $\mathrm{Ext}^1_{\mathfrak{C}}(L(\lambda), A(\mu))$ can then be described as a homomorphism

$$\varphi: \quad Q \longrightarrow A(\mu)$$

We need to show that $im\varphi = L(\mu)$. Assume to the contrary that $im\varphi / L(\mu) \neq 0$. Choose then τ maximal among the ν such that $[\mathrm{im}\varphi / L(\mu),\ L(\nu)] \neq 0$. Then there is a morphism $\psi \neq 0$

$$\psi: \quad im\varphi / L(\mu) \longrightarrow A(\tau)$$

because $A(\tau)$ is injective in the truncated subcategory consisting of weights $\leq \tau$: this is
dual to 1.2. Denote the composition $\psi \circ \varphi$ by $\overline{\psi}$. Then $\overline{\psi}$ is a morphism

$$\overline{\psi} : \; Q \longrightarrow A(\tau)$$

and thus represents a nonzero element of $\mathrm{Ext}^1_{\mathfrak{C}}(L(\lambda), A(\mu))$; hence $l(\tau) \not\equiv l(\lambda)$ mod 2. From the maximality of τ we also get a morphism $\phi \neq 0$

$$\phi\colon \; V(\tau) \longrightarrow \mathrm{im}\varphi / L(\mu) \subseteq A(\mu)/L(\mu)$$

If $l(\mu) \equiv l(\tau)$ mod 2 then $\mathrm{Ext}^1_{\mathfrak{C}}(V(\tau), L(\mu)) = \mathrm{Ext}^1_{\mathfrak{C}}(L(\mu), A(\tau))$ and ϕ can be extended to a morphism $\phi \neq 0$:

$$\phi : \; V(\tau) \longrightarrow A(\mu)$$

But then we must have $\tau = \mu$ which is a contradiction. Thus we have that $l(\tau) \not\equiv l(\mu)$ mod 2 and thereby $l(\mu) \equiv l(\lambda)$ mod 2 and another contradiction. The lemma is proved. □

2.2. The next lemma together with the previous one proves the promised implication vii) ⇒ v).

Lemma 2.2. Assume $\mathrm{Ext}^1_{\mathfrak{C}}(L(\lambda), A(\mu)) = \mathrm{Ext}^1_{\mathfrak{C}}(L(\lambda), L(\mu))$ for $\mu < \lambda$. Then

$$\mathrm{Ext}^1_{\mathfrak{C}}(L(\tau), L(\tau s)) \neq 0 \text{ for all } \tau \text{ and } s.$$

Proof. This follows from the translation principle [5;§II.7.18]. □

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6. Crystal/canonical bases in type A_2 and A_3.

§ 1. Introduction.

1.1. The aim of this chapter is to describe the combinatorial structure of the crystal basis in terms of the piecewise linear map R of Lusztig. One might think of this as a realization of the abtract concept of a crystal. We shall especially focus on those aspects of the combinatorial structure that are used in the proof of the refined Demazure character formula.

We only work with types A_2 and A_3; however we expect the description to carry over to more general types.

§ 2. The function R and the crystal operators $\tilde{e}_i$ and $\tilde{f}_i$.

2.1. Let us first consider type A_2. In that case there are two reduced expressions of w_0, namely $w_0 = s_2 s_1 s_2 = s_1 s_2 s_1$. The piecewise linear map $R\colon \mathbb{R}^3 \mapsto \mathbb{R}^3$ defined by Lusztig in [L1] is then

$$R\colon (a, b, c) \mapsto (b+c-\min(a, c),\ \min(a, c), a+b-\min(a, c))$$

R is a bijection and $R^{-1} = R$. It has two regions of linearity, namely the subsets of $\mathbb{R}^3$ given by $a \leq c$ and $a \geq c$.

2.2. The elements of the weight lattice will be written as 2-tuples with $\mathbb{Z}$ entries according to their coordinates in the basis of fundamental weights; the dominant weights thus have nonnegative coordinates. For $d = (d_1, d_2)$ a dominant weight we have from section 6 of [L1] a description of the canonical basis of the Weyl module $V(d) = V((d_1, d_2))$ as follows: let $Z \subset \mathbb{N}^6$ be the set

$$Z := \{\ (c', c'') \mid c' \in \mathbb{N}^3,\ c'' \in \mathbb{N}^3,\ R(c') = c''\ \}$$

Then by [L2] Z is in one to one correspondence with the crystal basis of $U_q^-(\mathfrak{sl}_3)$ by the map that takes a $(c', c'') \in Z$ to the element of $B(\infty)$ which is equivalent to the PBW basis element

$$F_{(1,2,1)}^{(c_1', c_2', c_3')}$$

Here the PBW basis is the one defined by Lusztig in [L1] and the equivalence relation is given by q times the $\mathbb{Z}[q]$-lattice generated by the PBW basis. (Notice that Lusztig works with $U_q^+(\mathfrak{sl}_3)$, when changing matters to $U_q^-(\mathfrak{sl}_3)$ the quantum parameter q becomes q^{-1}).

For (c', c'') in Z write $c' = (c_1', c_2', c_3')$ and $c'' = (c_1'', c_2'', c_3'')$ and define $Z_d \subset Z$ as

$$Z_d := \{ (c', c'') \in Z \mid \; c'_1 \le d_1 \text{ and } c''_3 \le d_2 \; \}$$

The set Z_d is then in one to one correspondence with the crystal basis of $V(d)$, this goes via the map

$$\pi_\lambda \colon U_q^-(\mathfrak{sl}_3) \to V(d); \qquad u \mapsto u v_\lambda$$

where v_λ is the highest weight vector in $V(d)$.

2.3. The crystal operators $\tilde{f}_i$ have by [L2] on Z the form

$$\tilde{f}_1 : (c', c'') \mapsto (\bar{c}', R(\bar{c}')) \text{ where } \bar{c}' = c' \text{ except for } \bar{c}'_1 = c'_1 + 1$$

$$\tilde{f}_2 : (c', c'') \mapsto (R(\bar{c}''), \bar{c}'') \text{ where } \bar{c}'' = c'' \text{ except for } \bar{c}''_1 = c''_1 + 1$$

And the operators $\tilde{e}_1$ and $\tilde{e}_2$ are given by

$$\tilde{e}_1 : (c', c'') \mapsto (\bar{c}', R(\bar{c}')) \text{ where } \bar{c}' = c' \text{ except for } \bar{c}'_1 = c'_1 - 1$$

$$\tilde{e}_2 : (c', c'') \mapsto (R(\bar{c}''), \bar{c}'') \text{ where } \bar{c}'' = c'' \text{ except for } \bar{c}''_1 = c''_1 - 1$$

where a tuple containing a negative number corresponds to the ideal basis element 0 that does not belong to the crystal. This gives the following ϵ-functions

$$\epsilon_1((c'_1, c'_2, c'_3), R(c'_1, c'_2, c'_3)) = c'_1$$

$$\epsilon_2(R(c''_1, c''_2, c''_3), (c''_1, c''_2, c''_3)) = c''_1$$

2.4. The reduced expression $w_0 = s_1 s_2 s_1$ induces the ordering of the positive roots as $\alpha_1 \prec \alpha_1 + \alpha_2 \prec \alpha_2$, and the expression $w_0 = s_2 s_1 s_1$ induces the opposite order. Then from the definition of PBW-elements we see that

$$wt_1((c'_1, c'_2, c'_3), R(c'_1, c'_2, c'_3)) = -2c'_1 - c'_2 + c'_3$$

$$wt_2(R(c_1'',c_2'',c_3''),(c_1'',c_2'',c_3'')) = -2c_1'' - c_2'' + c_3''$$

And the formula $\varphi_i(b) = wt_i(b) + \epsilon_i(b)$ implies that

$$\varphi_1((c_1',c_2',c_3'),R(c_1',c_2',c_3')) = -c_1' - c_2' + c_3'$$

$$\varphi_2(R(c_1'',c_2'',c_3''),(c_1'',c_2'',c_3'')) = -c_1'' - c_2'' + c_3''$$

2.5. Let us consider the antihomomorphism $*$ of $U_q^-(\mathfrak{sl}_3)$, in Lusztigs papers $*$ is denoted Ψ. By theorem 2.1.1 of [K1] $*$ leaves $B(\infty)$ invariant, so we would like to describe the induced map in the above language. We have that $w_0 s_1 w_0 = s_2$ and $w_0 s_2 w_0 = s_1$. By 2.11 of [L1] there is then the following relation between basis elements in $U_q^-(\mathfrak{sl}_3)$ of PBW-type:

$$\Psi(F_{(1,2,1)}^{(c_1,c_2,c_3)}) = F_{(2,1,2)}^{(c_3,c_2,c_1)}$$

We thus conclude the following formula for $*$ as a map on Z.

$$\begin{aligned} *: \quad &((c_1',c_2',c_3'),(c_1'',c_2'',c_3'')) \mapsto \\ &((c_3'',c_2'',c_1''),(c_3',c_2',c_1')) \end{aligned}$$

We used the fact $** = Id$ in order to determine the first three coordinates in the image of $*$. It is now possible to describe the operators $\tilde{f}_i^* = *\tilde{f}_i*$ introduced by Kashiwara [K1]:

$$\tilde{f}_1^* : (c',c'') \mapsto (R(\bar{c}''),\bar{c}'') \text{ where } \bar{c}'' = c'' \text{ except for } \bar{c}_3'' = c_3'' + 1$$

$$\tilde{f}_2^* : \ (c',c'') \mapsto (\bar{c}',R(\bar{c}')) \text{ where } \bar{c}' = c' \text{ except for } \bar{c}_3' = c_3' + 1$$

There are similar formulas for the operators $\tilde{e}_i^*$. From these formulas it is obvious that $\tilde{f}_1\tilde{f}_2^* = \tilde{f}_2^*\tilde{f}_1$ and $\tilde{f}_2\tilde{f}_1^* = \tilde{f}_1^*\tilde{f}_2$ etc. Kashiwara has shown this result in general, see corollary 2.2.2 of [K1].

§ 3. The strict morphisms Ψ_i in type A_2.

3.1. We now consider the crystals B_1 and B_2 defined by Kashiwara in [K1] example 1.2.6 of [K1]. We claim that the strict morphisms Ψ_1 and Ψ_2 from 2.2.1 in [loc. cit.] can be described in the following way where we have written $c', c'' = (c'_1, c'_2, c'_3)$, (c''_1, c''_2, c''_3):

$$\Psi_1 : B(\infty) \to B(\infty) \otimes B_1$$
$$((c', c'')) \mapsto (R(c''_1, c''_2, 0), (c''_1, c''_2, 0)) \otimes b_1(-c''_3)$$

$$\Psi_2 : B(\infty) \to B(\infty) \otimes B_2$$
$$((c', c'')) \mapsto ((c'_1, c'_2, 0), R(c'_1, c'_2, 0)) \otimes b_2(-c'_3)$$

Let us check this. It is clear that Ψ_i is injective and maps $u_\infty = (0, 0, 0)$ to $u_\infty \otimes b_i = (0, 0, 0) \otimes (0)$. Hence by the uniqueness statement in 2.2.1 of [loc. cit.] we only have to show that Ψ_i is a morphism, i.e. that it commutes with $\tilde{f}_i$ and $\tilde{e}_i$ on the elements that are not mapped to 0 and that it is strict. The action on $B(\infty) \otimes B_i$ is the usual action on a tensor product. Let us check commutativity of Ψ_1 and $\tilde{f}_1$. Set

$$b = ((c_1, c_2, c_3), R(c_1, c_2, c_3))$$

Then we have that

$$\Psi_1(b) := b_0 \otimes b_1(m) =$$

$$(\min(c_1, c_3), 0, c_2 + c_3), R(\ldots)) \otimes b_1(-c_1 - c_2 + \min(c_1, c_3))$$

From this it follows that if $c_1 < c_3$ then $\Psi_1(\tilde{f}_1 b) = \tilde{f}_1 b_0 \otimes b_1(m)$ and if $c_1 \geq c_3$ then $\Psi_1(\tilde{f}_1 b) = b_0 \otimes b_1(m-1)$. However $\varphi(b_0) = c_2 + c_3 - \min(c_1, c_3)$ while $\epsilon_1(b_1(m)) = -m = c_1 + c_3 - \min(c_1, c_3)$. Thus $\varphi_1(b_0) > \epsilon_1(b_1(m)) \Leftrightarrow c_3 > c_1$. Comparing this with the definition of the action of $\tilde{f}_1$ on a tensorproduct, e.g. 1.3 in

[K1], one sees that the assertion is true.

Finally strictness means that Ψ_i commutes with $\tilde{e}_j$ and $\tilde{f}_j$ on all crystal elements; it can be checked in the same manner as the above.

§ 4. The strict morphisms Ψ_i in type A_3.

4.1. We now turn to type A_3. The simple roots are denoted α_1, α_2 and α_3 where $<\alpha_1, \alpha_3> = 0$. We consider the two reduced expressions of w_0 given by

(4.1.1) $$w_0 = s_1 s_3 s_2 s_1 s_3 s_2$$

(4.1.2) $$w_0 = s_2 s_1 s_3 s_2 s_1 s_3$$

In order to determine the function R: $\mathbb{R}^6 \to \mathbb{R}^6$ we must find a sequence of braids and commutations that carries (4.1.1) to (4.1.2); for each braid in this sequence we should perform an A_2 transformation on the relevant coordinates and for each commuation we should commute the coordinates. A sequence of braids and commutations taking (4.1.1) to (4.1.2) is given on page 3 of [C3]. It looks as follows:

$$s_1 s_3 s_2 s_1 s_3 s_2 \rightsquigarrow s_3 s_1 s_2 s_1 s_3 s_2 \rightsquigarrow s_3 s_2 s_1 s_2 s_3 s_2 \rightsquigarrow$$

(4.1.3) $$s_3 s_2 s_1 s_3 s_2 s_3 \rightsquigarrow s_3 s_2 s_3 s_1 s_2 s_3 \rightsquigarrow s_2 s_3 s_2 s_1 s_2 s_3 \rightsquigarrow$$

$$s_2 s_3 s_1 s_2 s_1 s_3 \rightsquigarrow s_2 s_1 s_3 s_2 s_1 s_3$$

As one sees it involves 4 braids and 3 commutations; thus the R function is quite a lot more complicated in the A_3 case than in A_2 case; for example it does not satisfy $R = R^{-1}$ and has 10 regions of linearity.

The crystal basis of $U_q^-(\mathfrak{sl}_4)$ is in one to one correspondence with the set

$$Z = \{ (c', c'') \in \mathbb{N}^6 \times \mathbb{N}^6 \mid R(c') = c'' \}$$

Write $c' = (c'_1, c'_2, \ldots, c'_6)$ and $c'' = (c''_1, c''_2, \ldots, c''_6)$. Then the operators $\tilde{f}_i$ have the following form on Z

$$\tilde{f}_1 : (c', c'') \mapsto (\bar{c}', R(\bar{c}')) \text{ where } \bar{c}' = c' \text{ except for } \bar{c}'_1 = c'_1 + 1$$

$$\tilde{f}_3 : (c', c'') \mapsto (\bar{c}', R(\bar{c}')) \text{ where } \bar{c}' = c' \text{ except for } \bar{c}'_2 = c'_2 + 1$$

$$\tilde{f}_2 : (c', c'') \mapsto (R^{-1}(\bar{c}''), \bar{c}'') \text{ where } \bar{c}'' = c'' \text{ except for } \bar{c}''_1 = c''_1 + 1$$

and the $\tilde{e}_i$s act similarly. The ϵ functions can then be easily described. Now (4.1.1) corresponds to ordering the positive roots as $\alpha_1 \prec \alpha_3 \prec \alpha_1 + \alpha_2 + \alpha_3 \prec \alpha_2 + \alpha_3 \prec \alpha_1 + \alpha_2 \prec \alpha_2$ while (1.4.2) corresponds to the opposite order. Thus the definition of the PBW basis and the formula $\varphi_i(b) = wt_i(b) + \epsilon_i(b)$ give that

$$\varphi_1(c', c'') = -c'_1 - c'_3 + c'_4 - c'_5 + c'_6$$

$$\varphi_2(c', c'') = -c''_1 - c''_2 - c''_3 + c''_5 + c''_6$$

$$\varphi_3(c', c'') = -c'_2 - c'_3 - c'_4 + c'_5 + c'_6$$

4.2. We now claim that the Ψ_i are given by the formulas

$$\Psi_1 : B(\infty) \to B(\infty) \otimes B_1$$
$$\Psi_1 : (c', c'') \mapsto (R^{-1}(\bar{c}''), \bar{c}'') \otimes b_1(-c'_6) \quad \text{where } \bar{c}'' = c'' \text{ except for } \bar{c}'_6 = 0$$

$$\Psi_3 : B(\infty) \to B(\infty) \otimes B_3$$
$$\Psi_3 : (c', c'') \mapsto (R^{-1}(\bar{c}''), \bar{c}'') \otimes b_1(-c'_5) \quad \text{where } \bar{c}'' = c'' \text{ except for } \bar{c}'_5 = 0$$

$$\Psi_2 : B(\infty) \to B(\infty) \otimes B_2$$
$$\Psi_2 : (c', c'') \mapsto (\bar{c}', R(\bar{c}')) \otimes b_1(-c'_5) \quad \text{where } \bar{c}' = c' \text{ except for } \bar{c}'_1 = 0$$

Let us check that Ψ_1 is a morphism, i.e. that it commutes with $\tilde{e}_i$ and $\tilde{f}_i$ for all i. The case $i = 1$ is the most difficult so let us concentrate on that. Write

$$\bar{c}'' = (\bar{c}''_1, \ldots, \bar{c}''_5, 0)$$

Then running through the sequence (4.1.3) backwards and doing a few calculations one sees that $R^{-1}(\bar{c}'')$ has as its first coordinate

$$\bar{c}'_1 = \bar{c}''_2 + \bar{c}''_4 - \min(\bar{c}''_1, \bar{c}''_4) \tag{4.2.1}$$

This is hence ϵ_1 of $(R^{-1}(\bar{c}''), \bar{c}''))$. By the definition of the PBW basis we find that

$$wt_1(c', c'') = c''_1 - c''_2 + c''_3 - c''_4 \tag{4.2.2}$$

Summing (4.2.1) and (4.2.2) we then obtain

$$\varphi_1(R^{-1}(\bar{c}''), \bar{c}'')) = \bar{c}''_1 + \bar{c}''_3 - \min(\bar{c}''_1, \bar{c}''_4)$$

We have that $\epsilon_1(b_1(-c''_6)) = c''_6$. Thus the condition $\varphi_1(R^{-1}(\bar{c}''), \bar{c}'')) \leq \epsilon_1(b_1(-c''_6))$ is equivalent to

$$\bar{c}''_3 \leq \bar{c}''_6 \wedge \bar{c}''_1 + \bar{c}''_3 \leq \bar{c}''_4 + \bar{c}''_6 \tag{4.2.3}$$

4.3. Following [C3] we define for $c' = (c'_1, c'_2, \ldots, c'_6) \in \mathbb{R}^6$ the functions α_{A}, α_{B}, α_{C} and $\alpha_{\mathrm{D}} : \mathbb{R}^6 \to \mathbb{R}$ through

$$\alpha_{\mathrm{A}}(c') = c'_1 - c'_4, \ \ \alpha_{\mathrm{B}}(c') = \alpha_{\mathrm{D}}(c') = c'_3 - c'_6, \ \ \alpha_{\mathrm{C}}(c') = c'_2 - c'_5$$

In [*loc. cit.*] these functions are called the vertex vectors. Furthermore we define the function $g : \mathbb{R}^6 \to \mathbb{R}^6$ by

$$g(c'_1, c'_2, \ldots, c'_6) = (c'_3, c'_4, c'_5, c'_6, c'_4, c'_5)$$

Then in the papers of Carter there is constructed a function $\epsilon : \mathbb{R}^6 \to \mathbb{R}^6$ such that R

$= g + \epsilon$. The shape of this function depends of the region of linearity of R; this can in turn be expressed through inequalities involving the vertex vectors. The pages 6 and 7 of [C3] contain tables that describe all of this in the A_3 case (note that the function R is denoted f in Carter's work).

We now focus on the table on page 7 of [C3] describing ϵ. The first row corresponds to the region of linearity given by

$$\alpha_A \geq 0,\ \alpha_A + \alpha_B \geq 0,\ \alpha_C \geq 0,\ \alpha_B + \alpha_C \geq 0$$

According to the table R there takes the form

$$(4.3.1) \qquad R(c_1', c_2', \dots, c_6') = (c_3', c_4', c_5', c_6', c_4' + \alpha_B + \alpha_C, c_5' + \alpha_A + \alpha_B)$$

When c_1' is increased by one $\alpha_A(c')$ is also increased by one and hence the last coordinate of $R(c_1', c_2', \dots, c_6')$ is increased by one. Hence in that region the action of $\tilde{f}_1$ is on the second factor of the tensor product. But (4.3.1) satisfies (4.2.3), hence $\varphi_1(R^{-1}(\bar{c}''), \bar{c}'')) \leq \epsilon_1(b_1(-c_6''))$ and the action is as it should be.

In the same way one verifies for all other rows in the table that the action is OK.

4.4. Of course one might hope for more conceptual proofs. Anyway from the above results it seems reasonable to think of the R- function as encoding the data of a crystal.

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7. Littelmann's refined Demazure character formula revisited.

§1. The refined Demazure character formula.

1.1. Let the notation be that of Kashiwara [K1]. Especially for $w = s_{i_n} s_{i_{n-1}} \cdots s_{i_1}$ a reduced expression of $w \in W$ the subcrystals $B_w(\infty) \subset B(\infty)$ and $B_w(\lambda) \subset B(\lambda)$ are defined in the following recursive way

$$B_w(\infty) := \bigcup_k \tilde{f}_{i_n}^k B_{s_{i_n} w}(\infty), \qquad B_1(\infty) := u_\infty$$

$$B_w(\lambda) := \bigcup_k \tilde{f}_{i_n}^k B_{s_{i_n} w}(\lambda), \qquad B_1(\lambda) := u_\lambda$$

1.2. Let $\mathcal{D}_i$ be the additive operator on $\mathbb{Z}[B(\lambda)]$ given by

$$\mathcal{D}_i b = \sum_{0 \le k \le wt_i(b)} \tilde{f}_i^k b \quad \text{if } wt_i(b) \ge 0$$

$$\mathcal{D}_i b = -\sum_{1 \le k < -wt_i(b)} \tilde{e}_i^k b \quad \text{if } wt_i(b) < 0$$

Then the refined Demazure character formula of Littelmann, see [L] is the following equality in $\mathbb{Z}[B(\lambda)]$

$$\sum_{b \in B_w(\lambda)} b = \mathcal{D}_{i_n} \mathcal{D}_{i_{n-1}} \cdots \mathcal{D}_{i_1} u_\lambda \tag{1.2}$$

The purpose of this chapter is to review Kashiwara's proof in [K1] of the formula and to give a couple of short cuts.

1.3. The $\mathcal{D}_i$s commute with the usual Demazure operators on the group ring of the weight lattice $\mathbb{Z}[P]$ under the weight map w: $\mathbb{Z}[B(\lambda)] \to \mathbb{Z}[P]$. Hence, taking $w = w_0$ the longest element of the Weyl group, (1.2) generalizes the classical Demazure expression of the Weyl character, see e.g. [A].

1.4. Let us have a closer look at Kashiwaras proof of (1.2). The idea is to reduce to the verification of the following three properties of $B_w(\lambda)$

i) $B^*_w(\infty) = B_{w\text{-}1}(\infty)$

ii) $\tilde{e}_i B_w(\infty) \subset B_w(\infty) \sqcup \{0\} \ \forall\ i \in [1,n]$

iii) $\tilde{f}_j b \in B_w(\infty) \ \Rightarrow \tilde{f}^k_j b \in B_w(\infty) \ \forall$ k

1.5. Let us outline how (1.2) follows from these properties. An i-string S is defined to be a subset of $B(\infty)$ (or of $B(\lambda)$) on the form

(1.5.1) $$S = \{\ \tilde{f}^k_i b \mid k \geq 0, \text{ where } b \in B(\lambda) \text{ has } \tilde{e}_i b = 0\ \}$$

We call b the highest weight vector of S. Then for any i-string $S \subset B(\infty)$ the following statement about $B_w(\infty)$ is true

(1.5.2) $$B_w(\infty) \cap S \text{ is either } S \text{ or } \{b\} \text{ or the empty set}$$

This is seen by combining ii) and iii).

1.6. Now for $\lambda \in P$ let T_λ be the crystal on one element t_λ given by

$$wt_i(t_\lambda) = <\lambda, \alpha_i>, \quad \epsilon_i(t_\lambda) = -\infty, \quad \varphi_i(t_\lambda) = -\infty$$

$$\tilde{e}_i(t_\lambda) = 0, \quad \tilde{f}_i(t_\lambda) = 0$$

Let $\lambda \in P^+$. Then $u_\lambda \mapsto u_\infty \otimes t_\lambda$ defines an embedding of crystals $\imath : B(\lambda) \to B(\infty) \otimes T_\lambda$ that commutes with the $\tilde{e}_i$. Let namely $\bar{\pi}_\lambda$: $B(\infty) \to B(\lambda)$ be the map from theorem 5 iv) of [K2]. Then $\imath$ is given by

$$\imath : \bar{\pi}_\lambda(b) \mapsto b \otimes t_\lambda \text{ when } \bar{\pi}_\lambda(b) \neq 0$$

This is a well defined morphsm by ii) and iii) of the same theorem and commutes with the $\tilde{e}_i$s by iv) of the theorem. Finally it is clearly injective.

Now $B_w(\lambda)$ is the inverse image of $B_w(\infty)\otimes T_\lambda$ under $\imath$. Furthermore the inverse image under $\imath$ of an i-string for $B(\infty)$ must be an i-string for $B(\lambda)$. Thus (1.5.2) implies

$$(1.6.1) \qquad B_w(\lambda) \cap S \text{ is either } S \text{ or } \{b\} \text{ or the empty set}$$

for $S \subset B(\lambda)$ any i-string.

1.7. If $\tilde{e}_i b = 0$ for $b \in B(\lambda)$ then clearly $\mathcal{D}_i b$ is an i-string having b as its highest weight vector. And an easy calculation shows that $\mathcal{D}_i S = S$ for S any i-string. Now theorem 2 of [K2] says that

$$(1.7.1) \qquad B(\lambda) = \bigcup_{k_i \geq 0,\, j_i \in I,\, m \geq 0} \tilde{f}_{j_m}^{k_m} \tilde{f}_{j_{m-1}}^{k_{m-1}} \dots \tilde{f}_{j_1}^{k_1} u_\lambda$$

Hence $B(\lambda)$ is the disjoint union of i-strings for any $i \in I$: i-strings are either disjoint or coincide.

We now prove (1.2) by induction on $l(w)$. We thus assume the formula for $s_{i_n} w = s_{i_{n-1}} s_{i_{n-2}} \dots s_{i_1}$ and need to check the equality

$$(1.7.2) \qquad \sum_{b \in B_w(\lambda)} b = \mathcal{D}_{i_n}\Big(\sum_{b \in B_{s_{i_n} w}(\lambda)} b \Big)$$

As $\mathcal{D}_i$ leaves any i-string S invariant it is enough to verify the following equality

$$(1.7.3) \qquad \sum_{b \in B_w(\lambda) \cap S} b = \mathcal{D}_{i_n}\Big(\sum_{b \in B_{s_{i_n} w}(\lambda) \cap S} b \Big)$$

where S is an arbitrary i_n-string. Now (1.5.2) severely restricts the shape of these intersections; and even further restrictions are imposed by the condition

$$B_w(\lambda) \cap S = \bigcup_k \tilde{f}_{i_n}^k (B_{s_{i_n} w}(\lambda) \cap S)$$

which is a consequence of the definitions. All together we are left with only three possibilities, namely

(a) $$B_w(\lambda)\cap S = B_{s_{i_n}w}(\lambda)\cap S = \varnothing$$

(b) $$B_w(\lambda)\cap S = B_{s_{i_n}w}(\lambda)\cap S = S$$

(c) $$B_w(\lambda)\cap S = S \quad \text{and} \quad B_{s_{i_n}w}(\lambda)\cap S = \{b\} \text{ where } \tilde{e}_i b = 0$$

And in all three cases (1.5.5) can easily be checked.

1.8. We have thus reduced ourselves to the verification of i), ii) and iii) of 1.4. Kashiwara proves i) and ii) by realizing the $B_w(\lambda)$s as crystals of certain $U_q(\mathfrak{b})$-submodules of $V(\lambda)$ whereas the proof of the string property iii) relies on the combinatorial properties of the operators $\tilde{e}_i^*$ and $\tilde{f}_i^*$ together with i) and ii).

Here we shall demonstrate that i) and ii) can be obtained in the same combinatorial spirit that is employed for iii), that is without relying on an interpretation of the $B_w(\lambda)$s as crystals for any modules. This also gives a somewhat shorter proof of the refined Demazure character formula.

§ 2. Properties of $B_w(\infty)$.

2.1. Recall the injective morphism $\Psi_i : B(\infty) \to B(\infty)\otimes B_i$ defined in [K1]; B_i is the crystal defined in example 1.2.6 in [K1]. It satisfies among other the following conditions

(2.1.1) $$\Psi_i\colon u_\infty \mapsto u_\infty \otimes b_i$$

(2.1.2) $$\Psi_i(\tilde{f}_i^* b) = b_0 \otimes \tilde{f}_i b_1 \text{ where } \Psi_i(b) = b_0 \otimes b_1$$

(2.1.3) $$\tilde{f}_i \Psi_i(b) = \Psi_i(\tilde{f}_i b) \text{ and } \tilde{e}_i \Psi_i(b) = \Psi_i(\tilde{e}_i b)$$

where $B(\infty)\otimes B_i$ has the usual crystal structure of a tensor product. Using these properties one can obtain information about the commutation of $\tilde{f}_i$ and $\tilde{f}_i^*$; this is illustrated by the following lemma.

Lemma 2.1. For $i,j \in [1,n]$ and $b \in B(\infty)$ we have

$$\bigcup_{k,n} \tilde{f}_i^n \tilde{f}_j^{*k} b = \bigcup_{k,n} \tilde{f}_j^{*k} \tilde{f}_i^n b$$

Proof. If $i \neq j$ then by corollary 2.2.2 of [K1] $\tilde{f}_i$ and $\tilde{f}_j^*$ commute and there is nothing to prove. So we assume $i = j$. Write

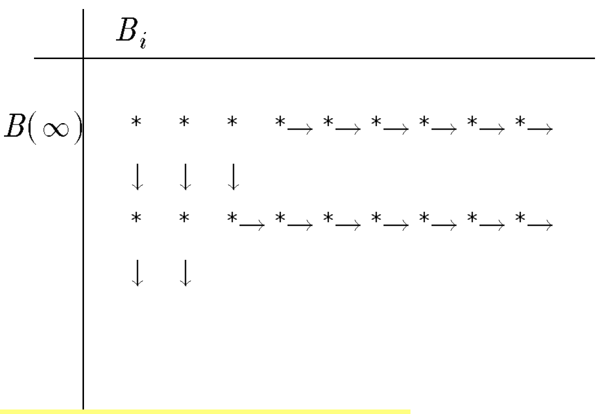
$$\Psi_i(b) = b_0 \otimes \tilde{f}_i^m b_i$$

and let $\varphi := \varphi(b_0)$ and $\epsilon := m$. Now Ψ_i is an embedding so to show the equality of the lemma it is enough to see that both sides have the same image under Ψ_i. So we replace b by $b_0 \otimes \tilde{f}_i^m\ b_i$ and keep in mind that the action of $\tilde{f}_j^*$ is on the right factor while $\tilde{f}_i$ acts as on a tensor product.

Let now $\Psi_i(b) = b_0 \otimes \tilde{f}_i^m b_i$ be represented as a point in the crystal graph associated to $B(\infty)\otimes B_i$. The crystal graph is a representation of the action of $\tilde{f}_i$ on $B(\infty)\otimes B_i$, so there is an arrow between two points in the graph if $\tilde{f}_i$ carries the corresponding crystal elements to each other.

If $\varphi \leq m$ the action of $\tilde{f}_i$ is on the second factor and there is a horizontal arrow leaving $b_0 \otimes \tilde{f}_i^m b_i$ and if $\varphi > m$ there is a vertical arrow leaving $b_0 \otimes \tilde{f}_i^m b_i$.

One typically gets a picture as the following



$$\begin{array}{l} * \quad *\rightarrow *\rightarrow *\rightarrow *\rightarrow *\rightarrow *\rightarrow *\rightarrow *\rightarrow \\ \downarrow \\ *\rightarrow *\rightarrow *\rightarrow *\rightarrow *\rightarrow *\rightarrow *\rightarrow *\rightarrow *\rightarrow \end{array}$$

The subset of $B(\lambda)$

$$\bigcup_k \tilde{f}_i^k(b_0 \otimes \tilde{f}_i^m b_i)$$

is represented by the points in the graph that can be hit by a sequence of arrows starting in $b_0 \otimes \tilde{f}_i^m b_i$.

On the other hand the action of $\tilde{f}_i^*$ is always on the second factor of the tensor product, so $\tilde{f}_i^*$ always takes a point in the graph to its right neighbour. Using this information one can now calculate the two sides of the lemma; in both cases on gets the infinite rectangle whose upper left corner is $\Psi_i(b) = b_0 \otimes \tilde{f}_i^m b_i$ and whose lower left corner is the point below $b_0 \otimes \tilde{f}_i^m b_i$ in which the arrows change direction. The lemma is proved. □

2.2. We can use the above to show the following result.

Theorem 2.2. $B_w(\infty) = \bigcup_{k_1 \dots k_n} \tilde{f}_{i_1}^{*k_1} \dots \tilde{f}_{i_n}^{*k_n} u_\infty$.

Proof. By definition $\tilde{f}_i^{*k} u_\infty = \tilde{f}_i^k u_\infty$ for all k and all i. So we get that

$$B_w(\infty) = \bigcup_{k_1 \dots k_n} \tilde{f}_{i_n}^{k_n} \dots \tilde{f}_{i_2}^{k_2} \tilde{f}_{i_1}^{*k_1} u_\infty$$

Using lemma 2.1 we can move $\tilde{f}_{i_1}^{*k_1}$ to the front position. And then we proceed with $\tilde{f}_{i_2}^{k_2}$ etc. The theorem is proved. □

2.3. And then we can deduce the property i) of $B_w(\infty)$:

Corollary 3. $B_w^*(\infty) = B_{w^{-1}}(\infty)$.

Proof. Let $b \in B_w(\infty)$, i.e. $b = \tilde{f}_{i_n}^{k_n} \dots \tilde{f}_{i_1}^{k_1} u_\infty$ for some $k_1, \dots k_n$. The definition of $\tilde{f}_i^*$ then gives that

$$b^* = \tilde{f}_{i_n}^{*k_n} \tilde{f}_{i_{n-1}}^{*k_{n-1}} \dots \tilde{f}_{i_1}^{*k_1} u_\infty$$

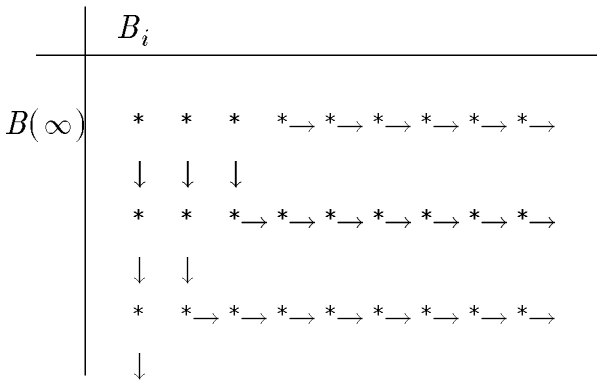
But from theorem 2.2 we see that $b^* \in B_{w^{-1}}(\infty)$ and the corollary is proved. □

2.4. We shall now consider the property ii). To that end we prove the following lemma.

Lemma 2.4. For all $i,\ j \in [1, n]$ and for all $b \in B(\infty)$ we have that

$$\tilde{e}_i \bigcup_k \tilde{f}_j^{*k} b \subset \bigcup_k \tilde{f}_j^{*k} \tilde{e}_i b \cup \bigcup_k \tilde{f}_j^{*k} b \cup \{0\}$$

Proof. Again only the case $i = j$ is nontrivial; otherwise $\tilde{e}_i$ and $\tilde{f}_j^*$ commute. We apply the morphism Ψ_i to both sides of the lemma and can then check the inclusion in the crystal graph:



The graph is infinte in downwards and to the right. We understand that $\tilde{e}_i(b) = 0$ if there is no arrow ending at the point corresponding to b. And again $\tilde{f}_i^*$ acts by shifting a b to the right while $\tilde{f}_i$ follows the arrows (and hence $\tilde{e}_i$ follows the arrows in negative direction).

Let us start out by verifying that there are no points missing in the picture

above.
So we must check that if the arrow leaving b is vertical and there is no arrow ending at b then neither should there be any arrow ending at b's right neighbour.

Let thus b be as indicated and write $\Psi_i(b) = b_0 \otimes \tilde{f}_i^m b_i$. Then $\varphi(b_0) > \epsilon(\tilde{f}_i^m b_i)$ $= m$ because the arrow leaving b is vertical. Now $\tilde{e}_i(b) = 0$ implies that $\tilde{e}_i(b_0) = 0$ because Ψ_i commutes with $\tilde{e}_i$ and no element of B_i is mapped to 0 under $\tilde{e}_i$.We get from these things that $\varphi(b_0) \geq \epsilon(\tilde{f}_i^{m+1} b_i)$ and as wanted

$$\tilde{e}_i(\Psi_i(\tilde{f}_i^* b)) = \tilde{e}_i(b_0 \otimes \tilde{f}_i^{m+1} b_i) = \tilde{e}_i b_0 \otimes \tilde{f}_i^{m+1} b_i) = 0$$

We now split the verification of the lemma in 3 cases. Firstly we consider the case of a b like above, i.e. with $\tilde{e}_i(b) = 0$ and with a vertical arrow leaving it. Then from the above considerations the left hand side of the lemma consists of those points in the row of b from which a horizontal arrow is leaving. But this is contained in the right hand side of the lemma (even with the first union omitted).

Then we consider the case where a horizontal arrow is leaving b. In that case the left hand side consists of the points that can be obtained from by following the arrows starting in b together with b itself and its immediate predeceesor (if any). And this is contained in the right hand side (only the $k = 0$ part of the first union is needed).

Finally we consider the case where there is both a vertical arrow entering and leaving b. Then the left hand side of the lemma consists of all points that are positioned to the right of b (including b itself) together with the points in the row above b that have an arrow leading into one of the first points. And the right hand side consists of the first points together with their upper neighbours. Thus the inclusion also holds in this case and the lemma is proved. □

2.5. We can now show the property ii) $B_w(\infty)$:

Theorem 2.5. For $i \in [1,n]$ we have that $\tilde{e}_i B_w(\infty) \subset B_w(\infty) \sqcup \{0\}$

Proof. We argue by induction on $l(w)$ and thus assume the theorem for $l(w)-1$. By theorem 2.2 $B_w(\infty)$ satisfies the equality

$$B_w(\infty) = \bigcup_{k_1} \tilde{f}_{i_1}^{*k_1} B_{ws_{i_1}}(\infty)$$

By induction hypothesis $\tilde{e}_i B_{ws_{i_1}}(\infty) \subset B_{ws_{i_1}}(\infty) \sqcup \{0\}$. Combining this with lemma 2.4 we obtain the induction step. The theorem is proved. □

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8. A q-analogue of Kempf's vanishing theorem.

§ 0. Introduction.

0.1. Let G be a reductive connected algebraic group and let B be a Borel subgroup. One of the central themes of the representation theory of G is the study of the induction functor H^0 from B-representations to G-representations. Many of the features of H^0 in the characteristic zero case also hold in the modular case, e.g. the properties that $\mathrm{H}^0(\lambda) \neq 0$ if and only if $\lambda \in P^+$, and that the weights of $\mathrm{H}^0(\lambda)$ are all less than or equal to λ. On the other hand the Borel-Weil-Bott theorem fails in general in the modular case, and hence the simplicity of $\mathrm{H}^0(\lambda)$ also breaks down in general. Still, we consider the $\mathrm{H}^0(\lambda)$'s to be fundamental objects of study, the reason being that their characters, like in the characteristic zero case, are given by the Weyl character formula. This fact in turn relies on the Kempf vanishing theorem, i.e that

$$\mathrm{H}^i(\lambda) = 0 \text{ for } i > 0 \text{ and } \lambda \in P^+$$

0.2. In 1979 Andersen and Haboush independently found a short proof of this vanishing, see [A] and [H]. Their idea was to show the following isomorphism

$$\mathrm{H}^i((p^r-1)\rho+p^r\lambda) \simeq St_r \otimes \mathrm{H}^i(\lambda)^{(r)}$$

where St_r is a Steinberg module and the superscript denotes the r-order Frobenius twist. Because of ampleness properties the left hand side is 0 for r sufficiently big; hence $\mathrm{H}^i(\lambda)^{(r)}$ must be zero, and thus also $\mathrm{H}^i(\lambda)$.

0.3. In [APW 1,2] and [AW] an induction functor H^0 for quantum groups is constructed and studied in great detail. Many of the results in these papers rely on specialization to the modular case. In the mixed case however, i.e. the case where the ground field is of positive characteristic prime to l, these methods fail to give a generalization of the Kempf vanishing theorem when $l<h$, the Coxeter number. And as higher ordered Frobenius twists do not exist for quantum groups, also the classical method sketched above fails.

0.4. Our approach to the quantum Kempf vanishing theorem is based on some properties of the crystal basis proved by Kashiwara in order to obtain the refined Demazure character formula in [K2]. In section 1 and 2 we discuss these results and in section 3 and 4 we show how they can be applied to deduce the Kempf vanishing for the quantum H^0 in all cases.

0.5. It is a pleasure to thank H. H. Andersen for providing crucial input to this work. Thanks are also due to G. Lusztig and to M. Kashiwara for hosting my stays at MIT in the fall 1992 and at RIMS in the spring 1993.

§ 1. Notation and some fundamental results.

1.0. Let $\mathfrak{g}$ be a complex finite dimensional Lie-algebra. In [K1] $\mathfrak{g}$ is allowed to be
a general Kac-Moody algebra, but otherwise we shall more or less follow the terminology of that paper. In particular, $\{\alpha_i\}_{i\in I}$ is the set of simple roots of $\mathfrak{g}$, $\{h_i\}_{i\in I}$ is the set of simple coroots, P is the weight lattice, $U_q(\mathfrak{g})$ is the quantized $\mathbb{Q}(q)$-algebra generated by e_i, f_i where $i\in I$ and q^h where $h \in P^*$, A is the subring of $\mathbb{Q}(q)$ consisting of rational functions regular at $q = 0$, $V(\lambda)$ is the Weyl module for $U_q(\mathfrak{g})$ with v_λ as a highest weight vector.

1.1. Assume that $\mathfrak{g}$ has rank one. Then the weight lattice P is equal to $\mathbb{Z}$. For $\lambda \geq$
-1 the dimension of $V(\lambda)$ is $\lambda+1$: a basis is $\{ f_i^{(k)} v_\lambda \mid 0 \leq k \leq \lambda \}$. The action of $U_q(\mathfrak{g})$ on this is given by the following formulas

(1.1.1) $$f f^{(k)} v_\lambda = [k+1] f^{(k+1)} v_\lambda$$

(1.1.2) $$e f^{(k)} v_\lambda = [\lambda-k+1] f^{(k-1)} v_\lambda$$

(1.1.3) $$q^h f^{(k)} v_\lambda = q^{\lambda-2k} f^{(k)} v_\lambda$$

where by convention $f^{(-1)}v_\lambda = f^{(\lambda+1)}v_\lambda = 0$.

1.2. By $U_q(\mathfrak{sl}_2)$-theory any $v \in V(\lambda)$ can be written uniquely in the form

$$v = \sum_n f_i^{(n)} u_n$$

where $u_n \in V(\lambda)$ is a weight vector and satisfies $e_i u_n = 0$. Then the operators $\tilde{e}_i$ and $\tilde{f}_i$ on $V(\lambda)$ are defined in the following way

$$\tilde{e}_i v := \sum_n f_i^{(n-1)} u_n, \qquad \tilde{f}_i v := \sum_n f_i^{(n+1)} u_n$$

The A-lattice $L(\lambda) \subset V(\lambda)$ is then defined as

$$L(\lambda) := A< \tilde{f}_{i_1}\tilde{f}_{i_2} \dots \tilde{f}_{i_k} v_\lambda \mid i_j \in I,\ k \geq 0 >$$

And $B(\lambda)$ is defined as follows

$$B(\lambda) := \pi(\{ \tilde{f}_{i_1}\tilde{f}_{i_2} \dots \tilde{f}_{i_k} v_\lambda \mid i_j \in I,\ k \geq 0 \}) \subset L(\lambda)\big/ qL(\lambda)$$

where π is the canonical map π: $L(\lambda) \to L(\lambda)\big/qL(\lambda)$. One of the main results of [K1] is then that $(L(\lambda), B(\lambda))$ forms a <u>lower</u> <u>crystal</u> <u>basis</u> of $V(\lambda)$; this means among other things that $\tilde{e}_i$ and $\tilde{f}_i$ induce operators on $B(\lambda) \cup \{0\}$, see theorem 2 of [K1].

1.3. The functions ϵ_i, φ_i and w_i: $B(\lambda) \to \mathbb{Z}$ are defined in the following way:

$$\epsilon_i(b) := \max\{n \mid \tilde{e}_i b \neq 0\}, \quad \varphi_i(b) := \max\{n \mid \tilde{f}_i b \neq 0\}, \quad w_i(b) := <\mathrm{weight}(b), h_i>$$

We have the following relations between them

(1.3.1) $$w_i(b) = \varphi_i(b) - \epsilon_i(b)$$

$$(1.3.2) \qquad \epsilon_i(\tilde{f}_i b) = \epsilon_i(b) + 1, \qquad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$$

and $(L(\lambda), B(\lambda))$ is a normal crystal with respect to ϵ_i, φ_i and w_i.

1.4. The $\mathbb{Z}[q, q^{-1}]$ - subalgebra of $U_q(\mathfrak{g})$ generated by $f_i^{(n)}$, $e_i^{(n)}$ and q^h, $\{ {q^h \atop n} \}$ for $h \in P^*$ is denoted $U_q^{\mathbb{Z}}(\mathfrak{g})$ and the $U_q^{\mathbb{Z}}(\mathfrak{g})$-submodule $V_{\mathbb{Z}}(\lambda)$ of $V(\lambda)$ is by definition $U_q^{\mathbb{Z}}(\mathfrak{g})\, v_\lambda$. Furthermore $-$ is the $\mathbb{Q}$-automorphism of $U_q(\mathfrak{g})$ given by the formulas

$$\overline{e_i} = e_i, \qquad \overline{f_i} = f_i, \qquad \overline{q^h} = q^{-h}, \qquad \overline{q} = q^{-1}$$

Then Kashiwara has shown, $(G_l.2)$ in section 7.2. of [K1], that

$$(1.4.1) \qquad V_{\mathbb{Z}}(\lambda) \cap L(\lambda) \cap \overline{L(\lambda)} \overset{\pi}{\simeq} V_{\mathbb{Z}}(\lambda) \cap L(\lambda) \big/ V_{\mathbb{Z}}(\lambda) \cap qL(\lambda)$$

where π is the canonical map. The inverse of π is denoted G_λ.

1.5. It is known that the crystal $B(\lambda)$ is contained in and gives a $\mathbb{Z}$-basis of $V_{\mathbb{Z}}(\lambda) \cap L(\lambda) \big/ V_{\mathbb{Z}}(\lambda) \cap qL(\lambda)$. From (1.4.1) we now get the following results by applying lemma 7.1.2. of [K1]:

$$(1.5.1) \qquad V_{\mathbb{Z}}(\lambda) \cap L(\lambda) \cap \overline{L(\lambda)} \simeq \bigoplus_{b \in B(\lambda)} \mathbb{Z}\, G_\lambda(b)$$

$$(1.5.2) \qquad V_{\mathbb{Z}}(\lambda) \simeq \bigoplus_{b \in B(\lambda)} \mathbb{Z}[q, q^{-1}]\, G_\lambda(b)$$

$$(1.5.3) \qquad V(\lambda) \simeq \bigoplus_{b \in B(\lambda)} \mathbb{Q}(q)\, G_\lambda(b)$$

$$(1.5.4) \qquad L(\lambda) \simeq \bigoplus_{b \in B(\lambda)} A\, G(b), \qquad \overline{L(\lambda)} \simeq \bigoplus_{b \in B(\lambda)} \overline{A}\, G(b)$$

Due to these properties $\{ G_\lambda(b) \mid b \in B(\lambda) \}$ is said to be a global basis.

1.6. For Γ a $\mathbb{Z}[q, q^{-1}]$-algebra the quantum group $U_q^{\mathbb{Z}}(\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \Gamma$ is denoted $U_\Gamma(\mathfrak{g})$. We use the notation $U_\Gamma(\mathfrak{b})$ and $U_\Gamma(\mathfrak{b}^-)$ for the corresponding Borel subalgebras. Let $\mathcal{C}$ be the category of $U_q^{\mathbb{Z}}(\mathfrak{g})$-modules M satisfying

$$M = \underset{\lambda}{\oplus} M_\lambda$$

$$\forall m \in M \text{ and } i \in I: \quad e_i^{(n)} m = f_i^{(n)} m = 0 \quad \text{for } n \gg 0 \text{ etc}$$

The categories $\mathcal{C}^{\geq}$, $\mathcal{C}^{\leq}$, $\mathcal{C}_\Gamma$, $\mathcal{C}_\Gamma^{\geq}$, $\mathcal{C}_\Gamma^{\leq}$ are defined likewise. For M a $U_q^{\mathbb{Z}}(\mathfrak{g})$-module we define $\mathrm{F}(M)$ as the largest submodule of M which belongs to $\mathcal{C}$.

If $M \in \mathcal{C}$ we let $P(M)$ denote the set of weights of M.

1.7. Let $N \in \mathcal{C}^+$. Consider the $\mathbb{Z}[q, q^{-1}]$ - module $\mathrm{Hom}_{U_{\mathbb{Z}}(b)}(U_q^{\mathbb{Z}}(\mathfrak{g}), N) :=$

$$\{ f \in \mathrm{Hom}_{\mathbb{Z}[q,q^{-1}]}(U_q^{\mathbb{Z}}(\mathfrak{g}), N) \mid f(ub) = S^{-1}(b) f(u) \quad \forall u \in U_q^{\mathbb{Z}}(\mathfrak{g}), \forall b \in U_q^{\mathbb{Z}}(\mathfrak{b}) \}$$

where S is the antipode map of $U_q^{\mathbb{Z}}(\mathfrak{b})$. It has the structure of a $U_q^{\mathbb{Z}}(\mathfrak{g})$-module through $(uf)(m) := f(S(u)m)$. Then the APW induction $\mathrm{H}^0(N)$ is defined as

$$\mathrm{H}^0(N) := \mathrm{F}(\mathrm{Hom}_{U_q^{\mathbb{Z}}(b)}(U_q^{\mathbb{Z}}(\mathfrak{g}), N))$$

(Contrary to what APW do (and what is the tradition for algebraic groups) we are here inducing from positive Borel groups, this is the reason for the difference between our definition and the one in [APW]). The map $Ev\colon \mathrm{H}^0(N) \to N;\ f \mapsto f(1)$ is a $U_q^{\mathbb{Z}}(\mathfrak{b})$ - linear map; it induces the Frobenius reciprocity isomorphism:

$$\mathrm{Hom}_{U_q^{\mathbb{Z}}(b)}(E, N) = \mathrm{Hom}_{U_q^{\mathbb{Z}}(g)}(E, \mathrm{H}^0(N)) \qquad \forall N \in \mathcal{C}^{\geq}, E \in \mathcal{C}$$

This is the universal property of $\mathrm{H}^0(N)$. There is also a tensor product theorem for H^0.

For any $\mathbb{Z}[q, q^{-1}]$-algebra Γ we have likewise an induction functor $\mathrm{H}^0_\Gamma\colon \mathcal{C}_\Gamma^{\geq} \to \mathcal{C}_\Gamma$.

§ 2. W-filtrations and crystal bases.

2.1. In this section we shall improve on some of the results of [K2]. In that paper all theorems deal with the rings $\mathbb{Q}(q)$ and $\mathbb{Q}[q, q^{-1}]$; however we need the results to hold also for $\mathbb{Z}[q, q^{-1}]$.

2.2. Throughout the rest of this section we fix an $i \in I$ and consider the corresponding $\mathfrak{sl}_2$-component $\mathfrak{g}_i$ of $\mathfrak{g}$. Let $W^l(\lambda)$ be the sum of all $U_q(\mathfrak{g}_i)$-submodules of $V(\lambda)$ of dimension greater than or equal to l. Furthermore $W^l(B(\lambda))$ and $I^l(B(\lambda))$ are defined as the following subsets of $B(\lambda)$:

$$W^l(B(\lambda)) := \{ b \in B(\lambda) \mid \epsilon_i(b) + \varphi_i(b) \geq l \}$$

$$I^l(B(\lambda)) := \{ b \in B(\lambda) \mid \epsilon_i(b) + \varphi_i(b) = l \}$$

where $\epsilon_i(b)$ and $\varphi_i(b)$ are the functions mentioned in 1.3. Then (3.1.1) of [K2] says that

$$W^l(\lambda) = \bigoplus_{b \in W^l(B(\lambda))} \mathbb{Q}(q)\, G_\lambda(b) \qquad (2.2.1)$$

We can improve this to the following lemma:

Lemma 2.2 $W^l(\lambda) \cap V_{\mathbb{Z}}(\lambda) = \bigoplus_{b \in W^l(B(\lambda))} \mathbb{Z}[q, q^{-1}]\, G_\lambda(b)$

Proof. The inclusion $\supset$ follows from (2.2.1) together with (1.5.2). For the other inclusion assume $w \in W^l(\lambda) \cap V_{\mathbb{Z}}(\lambda)$. Then using (2.2.1) and (1.5.2) once more w can be written as

$$w = \sum_{b \in W^l(B(\lambda))} f_b\, G_\lambda(b) = \sum_{b \in B(\lambda)} g_b\, G_\lambda(b), \qquad f_b \in \mathbb{Q}(q),\ g_b \in \mathbb{Z}[q, q^{-1}]$$

But the $G_\lambda(b)$'s are independent so we can conclude that $f_b = g_b$. The lemma is proved. □

2.3. For $b \in I^l(B(\lambda))$ we have the following formulas, (3.1.2) of [K2]:

$$(2.3.1) \qquad f_i^{(k)} G_\lambda(b) \equiv \begin{bmatrix} \epsilon_i(b)+k \\ k \end{bmatrix} G_\lambda(\tilde{f}_i^k b) \quad \text{mod } W^{l+1}(\lambda)$$

$$(2.3.2) \qquad e_i^{(k)} G_\lambda(b) \equiv \begin{bmatrix} \epsilon_i(b)+k \\ k \end{bmatrix} G_\lambda(\tilde{e}_i^k b) \quad \text{mod } W^{l+1}(\lambda)$$

Lemma 2.3. The above formulas also hold mod $W^{l+1}(\lambda) \cap V_{\mathbb{Z}}(\lambda)$.

Proof. We have that $G_\lambda(b) \in V_{\mathbb{Z}}(\lambda)\ \forall b \in B(\lambda)$ and hence $f_i^{(k)} G_\lambda(b) - \begin{bmatrix} \epsilon_i(b)+k \\ k \end{bmatrix} G_\lambda(\tilde{f}_i^n b) \in V_{\mathbb{Z}}(\lambda)$. This proves the lemma. □

2.4. Let us now consider a $U_q^{\mathbb{Z}}(\mathfrak{b}_i)$-module $N \subset V(\lambda)$ and a $U_q^{\mathbb{Z}}(\mathfrak{b}_i)$-submodule $N_{\mathbb{Z}}$. Assume furthermore the existence of a $B_N \subset B(\lambda)$ such that

$$(2.4.1) \qquad N_{\mathbb{Z}} \simeq \bigoplus_{b \in B_N} \mathbb{Z}[q, q^{-1}]\, G_\lambda(b)$$

$$(2.4.2) \qquad N \simeq \bigoplus_{b \in B_N} \mathbb{Q}(q)\, G_\lambda(b)$$

According to the lemma 3.1.2. of [K2] B_N must then satisfy that

$$(2.4.3) \qquad \tilde{e}_i B_N \subset B_N \sqcup \{0\}$$

We now make the following definitions:

$$\tilde{N}_{\mathbb{Z}} = \sum_n f_i^{(n)} N_{\mathbb{Z}}, \qquad \tilde{N} = \sum_n f_i^{(n)} N, \qquad \tilde{B}_N = \bigcup_n f_i^{(n)} B_N$$

Then Kashiwara has shown, theorem 3.1.1 of [K2], that

$$(2.4.4) \qquad \tilde{N} = \bigoplus_{b \in \tilde{B}_N} \mathbb{Q}(q)\, G_\lambda(b)$$

We wish to improve this to a statement about $\tilde{N}_{\mathbb{Z}}$.

Lemma 2.4. $\tilde{N}_{\mathbb{Z}} = \bigoplus_{b \in \tilde{B}_N} \mathbb{Z}[q, q^{-1}]\, G_\lambda(b)$

Proof. The intersection of the right hand side of (2.4.4) with $V_{\mathbb{Z}}(\lambda) = \bigoplus_{b \in B(\lambda)} \mathbb{Z}[q, q^{-1}]\, G_\lambda(b)$ equals the right hand side of the lemma. Hence we must prove that

$$(2.4.5) \qquad \tilde{N} \cap V_{\mathbb{Z}}(\lambda) = \tilde{N}_{\mathbb{Z}}$$

The inclusion $\supset$ is clear. For the other inclusion choose an $n \in \tilde{N} \cap V_{\mathbb{Z}}(\lambda)$. Then we can write n in the following two ways

$$(*) \qquad n = \sum_{k \geq 0,\, b \in B_N} c_{k,b} f_i^{(k)} G_\lambda(b) = \sum_{\beta \in B(\lambda)} d_\beta\, G_\lambda(\beta)$$

where $c_{k,b} \in \mathbb{Q}(q)$ and $d_\beta \in \mathbb{Z}[q, q^{-1}]$.

We wish to modify the first sum so that the occurring b's all satisfy $\tilde{e}_i b = 0$. Assume that b occurs in the sum and that $\tilde{e}_i b \neq 0$. Choose l minimal such that $b \in W^l(B(\lambda))$. By (2.2.1) we then have $G_\lambda(b) \in W^l(\lambda)$ and thus $G_\lambda(b) \in W^l(\lambda) \cap N \subset W^l(\lambda) \cap \tilde{N} = W^l(\tilde{N})$; so it follows from $\mathfrak{sl}_2$-theory that

$$(2.4.6) \qquad G_\lambda(b) \equiv c\, f_i\, e_i\, G_\lambda(b) \mod W^{l+1}(\tilde{N})$$

where $c \in \mathbb{Q}(q)$. On the other hand (2.2.1) and (2.3.2) give that

$$(2.4.7) \qquad e_i G_\lambda(b) = c_1 G_\lambda(\tilde{e}_i b) + \sum_{b \in W^{l+1}(B(\lambda))} c_b\, G_\lambda(b)$$

with $c_1,\ c_b \in \mathbb{Q}(q)$. As $e_i G_\lambda(b) \in N$, it can also be written as a $\mathbb{Q}(q)$-combination of the $G_\lambda(b)$'s with $b \in B_N$; hence we get from the independence of the $G_\lambda(b)$'s that $\tilde{e}_i b$ along with the b's in the sum belong to B_N. (This is actually the proof of (2.4.3)). Thus, writing (2.4.7) in the form

$$e_i G_\lambda(b) \equiv c_1 G(\tilde{e}_i b) \mod W^{l+1}(\tilde{N})$$

we deduce that

$$c\, f_i\, e_i\, G_\lambda(b) \equiv c\, c_1 f_i\, G_\lambda(\tilde{e}_i b) \quad \bmod\ W^{l+1}(\tilde{N})$$

Combining this with (2.4.6) we obtain

$$\mathrm{G}_\lambda(b) \equiv c\, c_1 f_i\, G_\lambda(\tilde{e}_i b) \quad \bmod\ W^{l+1}(\tilde{N}) \tag{2.4.8}$$

Using this formula and descending induction on l we can write n as promised

$$n = \sum_{b \in B_N,\, \tilde{e}_i b = 0} c_{k,b} f_i^{(k)} G_\lambda(b)$$

Using $\mathfrak{sl}_2$-theory once more and an induction like the previous one we can furthermore ensure that the occuring $f_i^{(k)} G_\lambda(b)$ satisfy $k \leq w_i(b)$. Now from (2.3.1) and (2.4.4) we have that

$$f_i^{(k)} G_\lambda(b) \equiv G(\tilde{f}_i^k b) \quad \bmod\ W^{l+1}(\tilde{N})$$

Thus the set of vectors in $\tilde{N}$

$$\{\ f_i^{(k)} G_\lambda(b)\ |\ k \leq w_i(b),\ b \in B_N,\ \tilde{e}_i b = 0\ \}$$

is a $\mathbb{Q}(q)$-basis of $\tilde{N}$ and the base change matrix from $\{G_\lambda(b)|\ b \in \tilde{B}_N\}$ is triangular with ones on the diagonal with respect to a proper indexing of the basis. But then in (*) we must have that $c_{k,b} \in \mathbb{Z}[q, q^{-1}]$ and thus $n \in \tilde{N}_{\mathbb{Z}}$. This proves the lemma. □

2.5. For $w \in W$ the $U_q^{\mathbb{Z}}(\mathfrak{b})$-submodule $V_w^{\mathbb{Z}}(\lambda)$ of $V_{\mathbb{Z}}(\lambda)$ is defined in the following recursive way

$$\mathrm{V}_1^{\mathbb{Z}}(\lambda) := v_\lambda, \qquad V_w^{\mathbb{Z}}(\lambda) := U_q^{\mathbb{Z}}(\mathfrak{g}_s)\, V_{sw}^{\mathbb{Z}}(\lambda) \quad sw < w$$

It is shown in lemma 3.3.1 of [K2] that $\mathrm{V}_w^{\mathbb{Z}}(\lambda)$ also has the following description

$$V_w^{\mathbb{Z}}(\lambda) := U_q^{\mathbb{Z}}(\mathfrak{b})\, v_{w\lambda}$$

where $v_{w\lambda} \in V_{\mathbb{Z}}(\lambda)$ is defined in the following recursive way:

$$v_{1\lambda} := \mathrm{v}_{\lambda}, \qquad v_{w\lambda} := f_s^{(m)} v_{sw\lambda} \quad sw < w$$

where $m := <\alpha_i, sw\lambda>$. By the quantum Verma relations this is independent of the choice of reduced expression of w and hence also $V_w^{\mathbb{Z}}(\lambda)$ is independent of the reduced expression of w.

Applying now the above lemma to $N_{\mathbb{Z}} = V_{sw}^{\mathbb{Z}}(\lambda)$ we obtain the existence of a $B_w(\lambda) \subset B(\lambda)$ such that

(2.5.1) $$\mathrm{V}_w^{\mathbb{Z}}(\lambda) = \bigoplus_{b \in B_w(\lambda)} \mathbb{Z}[q, q^{-1}] \, G_{\lambda}(b)$$

Then $B_w(\lambda)$ has the following properties

(2.5.2) $$\tilde{e}_i B_w(\lambda) \subset B_w(\lambda) \sqcup \{0\}$$

(2.5.3) $$B_w(\lambda) = \bigcup_k \tilde{f}_s^k B_{sw}(\lambda)$$

The first property is a consequence of (2.4.3) and the second one follows from lemma 2.4.
Let S be an i-string, i.e a subset of $B(\lambda)$ on the form

$$S = \{ \, \tilde{f}_i^k b \mid k \geq 0, \; b \in B(\lambda), \; \tilde{e}_i b = 0 \, \}$$

where b is called the highest weight vector. Then $B_w(\lambda)$ has the following property

(2.5.4) $B_w(\lambda) \cap S$ is either S or $\{b\}$ or the empty set.

This is rather deep; it is the content of theorem 3.3.2 of [K2].

2.6. We shall investigate the consequences of these properties for the $V_w^{\mathbb{Z}}(\lambda)$:

Lemma 2.6. There is a $U_q^{\mathbb{Z}}(\mathfrak{b}_i)$-filtration of $V_w^{\mathbb{Z}}(\lambda)$

$$0 = W^l(\lambda) \cap V_w^{\mathbb{Z}}(\lambda) \subset W^{l-1}(\lambda) \cap V_w^{\mathbb{Z}}(\lambda) \subset \ldots \subset W^0(\lambda) \cap V_w^{\mathbb{Z}}(\lambda) = V_w^{\mathbb{Z}}(\lambda)$$

such that the quotients are direct sums of Weyl $U_q^{\mathbb{Z}}(\mathfrak{g}_i)$-modules restricted to $U_q^{\mathbb{Z}}(\mathfrak{b}_i)$ and of rank one $U_q^{\mathbb{Z}}(\mathfrak{b}_i)$-modules having dominant weights

Proof. We can construct a $U_q^{\mathbb{Z}}(\mathfrak{b}_i)$-filtration of $V_w^{\mathbb{Z}}(\lambda)$ in the following way

$$0 = W^l(\lambda) \cap V_w^{\mathbb{Z}}(\lambda) \subset W^{l-1}(\lambda) \cap V_w^{\mathbb{Z}}(\lambda) \subset \ldots \subset W^0(\lambda) \cap V_w^{\mathbb{Z}}(\lambda) = V_w^{\mathbb{Z}}(\lambda)$$

where l is chosen big enough for the first equality to hold. By lemma 2.2, (2.5.1) and the definition of I^l, the quotients are

$$W^k(\lambda) \cap V_w^{\mathbb{Z}}(\lambda) \Big/ W^{k+1}(\lambda) \cap V_w^{\mathbb{Z}}(\lambda) \simeq \bigoplus_{b \in I^k(B_w(\lambda))} \mathbb{Z}[q, q^{-1}]\, G_\lambda(b)$$

If S is an i-string then $\epsilon_i(b) + \varphi_i(b)$ is constant on S; this follows from (1.3.2). But then $I^k(S)$ is either S or the empty set and we conclude that $I^k(B_w(\lambda))$ inherits the string property (2.5.4). If the intersection is S, the formula (2.3.2) together with lemma 2.3 and the description of the Weyl $U_q^{\mathbb{Z}}(\mathfrak{g}_i)$-modules for $U_q^{\mathbb{Z}}(\mathfrak{g}_i)$ in 1.1 show that $\{G_\lambda(b),\ b \in S\}$ gives rise to an $U_q^{\mathbb{Z}}(\mathfrak{g}_i)$ Weyl module restricted to $U_q^{\mathbb{Z}}(\mathfrak{b}_i)$. If the intersection is a highest weight vector $\{b\}$ then it has the weight $l > 0$: $\epsilon_i(b) = 0$ whence $\varphi_i(b) = \varphi_i(b) + \epsilon_i(b) = l$ and then (1.3.1) gives $w_i(b) = \varphi_i(b) - \epsilon_i(b) = l$. The lemma is proved. □

Remark 2.6. For N a $U_q^{\mathbb{Z}}(\mathfrak{b}_i)$-module the filtration of it by the $N \cap W^l(\lambda)$ is denoted the W-filtration of N.

2.7. Let k be field of characteristic $p > 0$ which is made into a $\mathbb{Z}[q, q^{-1}]$-algebra by sending q to an l'th root of unity. (Later on we shall appeal to results from [AW], hence we should really impose the restrictions on l that occur in that paper. However, in [AP] it is shown that the Frobenius map of [L] can be employed to get rid of these restrictions). Then, as the filtration quotients in lemma 2.6 are free $\mathbb{Z}[q, q^{-1}]$-modules we get by tensoring a filtration of $V_w^k(\lambda) := V_w^k(\lambda) \otimes_{\mathbb{Z}[q,q^{-1}]} k$ having the same properties as the one of $V_w^{\mathbb{Z}}(\lambda)$.

§3. Josephs induction functor.

3.1. In this section we shall compare the induction functor $\mathcal{D}$ of Joseph, see [J], with the APW induction H^0. Let k be as in 2.7. Then $\mathcal{D}$ is defined in the following way

Definition 3.1. Let N be a finite dimensional $U_k(\mathfrak{b})$-module and let $U_k \supset U_k(\mathfrak{b})$ be a parabolic (in the sense of APW) quantum group. Then $\mathcal{D}N$ is

$$\mathcal{D}N := \mathrm{D}(U_k \otimes_{U_k(b)} N)$$

where the tensor product has structure as a U_k-module through left multiplication and D is the functor from U_k-modules to finite dimensional U_k-modules that takes an M to the largest finite dimensional quotient of M by a U_k-submodule.

Remark 3.2. Recall that if M is an integrable U_k-module then the (relevant) Weyl group acts on the weights of M, see [AW] proposition 1.7. Hence, arguing as in [APW], 1.14, we find that there is a unique submodule of $U_q \otimes_{U_k(b)} N$ such that the quotient is of maximal dimension, i.e. D is well defined.

The universel property of $\mathcal{D}$ is given by the following Frobenius reciprocity

$$\mathrm{Hom}_{U_k(b)}(N, E) = \mathrm{Hom}_{U_k}(\mathcal{D}N, E)$$

where E, N are finite dimensional U_k, $U_k(\mathfrak{b})$-modules. The isomorphism is induced by the natural $U_k(\mathfrak{b})$-map σ: $N \to \mathcal{D}N$. Furthermore $\mathcal{D}$ satisfies a tensor product theorem.

3.2. For M a U_k-module we set $M^{dual} := \mathrm{Hom}_k(M, k)$. Then M^{dual} has two structures of a U_k-module, namely M^* and M^t defined by

$$M^* : \qquad (uf)(m) := f(S(u)m)$$

$$M^t : \qquad (uf)(m) := f(S^{-1}(u)m)$$

where S is the antipodal map of the Hopf algebra U_k. When M is of finite dimension we have the isomorphisms $M^{*t} \simeq M^{t*} \simeq M$.

3.2. Using this we can deduce the following lemma

Lemma 3.2. Let N be a finite dimensional $U_k(\mathfrak{b})$-module. Then

$$(\mathcal{D}N)^* \simeq \mathrm{H}^0_k(N^*)$$

Proof. Let $\Phi \in \mathrm{Hom}_{U_k}((\mathcal{D}N)^*, \mathrm{H}^0_k(N^*))$ be the map corresponding to $\sigma \in \mathrm{Hom}_{U_k(g)}(N, \mathcal{D}N)$ under the isomorphisms

$$\mathrm{Hom}_{U_k(b)}(N, \mathcal{D}N) \simeq \mathrm{Hom}_{U_k(b)}((\mathcal{D}N)^*, N^*) \simeq \mathrm{Hom}_{U_k}((\mathcal{D}N)^*, \mathrm{H}^0_k(N^*))$$

The second isomorphism was Frobenius reciprocity for H^0_k. Let $\Psi \in \mathrm{Hom}_{U_k}(\mathrm{H}^0_k(N^*), (\mathcal{D}N)^*)$ be the map corresponding to $Ev \in \mathrm{Hom}_{U_k(b)}(H^0_k(N^*), N^*)$ under the isomorphisms

$$\mathrm{Hom}_{U_k}(\mathrm{H}^0_k(N^*), (\mathcal{D}N)^*) \simeq \mathrm{Hom}_{U_k}(\mathcal{D}N, \mathrm{H}^0_k(N^*)^t) \simeq$$

$$\mathrm{Hom}_{U_k(b)}(N, \mathrm{H}^0_k(N^*)^t) \simeq \mathrm{Hom}_{U_k(b)}(H^0_k(N^*), N^*)$$

Here the second isomorphism was Frobenius reciprocity for $\mathcal{D}$; it can be applied since $\mathrm{H}^0_k(N^*)$ according to [AW] is finite dimensional. We can describe Φ and Ψ in the following way

$$\Phi\colon f \mapsto [\, u \mapsto (\, n \mapsto f(u(\sigma n))\,)\,]$$

$$\Psi\colon f \mapsto [\, u\sigma(n) \mapsto f(u)(n)\,]$$

Using these descriptions one checks that $\Phi \circ \Psi = Id$ and $\Psi \circ \Phi = Id$. □

3.3. The functor $\mathcal{D}$ is coinvariant and right exact so we would like to introduce its derived functors. However, as the category of finite dimensional U_k-modules does not have enough projectives one cannot proceed as normal when defining these. To overcome this obstacle, Joseph proposes to use objects of the form $E\otimes\lambda$, where E is a finite dimensional and λ is dominant, as substitutes for projectives [J]. We shall show that this definition also makes sense in our context.

Lemma 3.3. Let M be a finite dimensional $U_k(\mathfrak{b})$-module. Then there exists a $\lambda \in P^+$ and a finite dimensional U_k-module E such that M is a quotient of $E\otimes\lambda$.

Proof. In [AW] the following ampleness property of H^0 is shown

(3.3.1) $$\mathrm{H}^i_k(\lambda) = 0 \quad \text{for } i > 0 \text{ and } \lambda \ll 0$$

(We are here inducing from positive Borel groups, hence dominant is replaced by anti-dominant). As M is finite dimensional, we can use the above to find a λ such that $P(M\otimes -\lambda) \subset P^-$ and such that

(3.3.2) $$\mathrm{H}^1_k(\mu) = 0 \qquad \forall\ \mu \in P(M\otimes -\lambda)$$

We now proceed by induction on the cardinality of $P(M\otimes -\lambda)$. Choose ν maximal in $P(M\otimes -\lambda)$; then $\nu \subset M\otimes -\lambda$ as $U_k(\mathfrak{b})$-modules. From this we obtain a commutative diagram as follows

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{H}^0_k(\nu) & \longrightarrow & \mathrm{H}^0_k(M\otimes -\lambda) & \longrightarrow & \mathrm{H}^0_k((M\otimes -\lambda)/\nu) & \longrightarrow & 0 \\ & \downarrow Ev & & \downarrow Ev & & \downarrow Ev & & \\ 0 \longrightarrow & \nu & \longrightarrow & M\otimes -\lambda & \longrightarrow & (M\otimes -\lambda)/\nu & \longrightarrow & 0 \end{array}$$

The rows are both exact, the first one by (3.3.2), the second by construction. The first vertical map is surjective by definition and the third is surjective by induction hypothesis. But then also the second vertical map must be surjective and we are done. □

3.4. We can now in the usual way construct resolutions of all finite dimensional U_k-modules N; these will be on the form $(E\otimes\lambda)^{\cdot} \twoheadrightarrow N$ and in general infinite. We can furthermore assume that in $(E\otimes\lambda)^{\cdot} \twoheadrightarrow N$ all $-\lambda$ satisfy $\mathrm{H}^i_k(\lambda) = 0$ for $i > 0$, this is possible by (3.3.1).

Lemma 3.4. Let $(E\otimes\lambda)^{\cdot} \twoheadrightarrow N$ be a resolution like the above. Then the cohomology $\mathcal{D}^{\cdot}$ of the complex $\mathcal{D}(E\otimes\lambda)^{\cdot} \twoheadrightarrow 0$ is independent of the choice of resolution. Furthermore there is an isomorphism of U_k-modules

$$\mathcal{D}^i(N)^* \simeq \mathrm{H}^i_k(N^*)$$

Proof. Dualizing the resolution $(E\otimes\lambda)^{\cdot} \twoheadrightarrow N$ we get the resolution $N^* \rightarrowtail (E^*\otimes -\lambda)^{\cdot}$. It is acyclic for H^0_k because the tensor identity gives that

$$\mathrm{H}^i_k(E^*\otimes\lambda) \simeq E^*\otimes\mathrm{H}^i_k(\lambda) \simeq 0$$

But then $\mathrm{H}^i_k(N^*)$ is the ith cohomology of $0 \rightarrowtail \mathrm{H}^0_k(E^*\otimes\lambda)^{\cdot}$, which by lemma 3.2 is the ith cohomology of $0 \rightarrowtail (\mathcal{D}(E\otimes\lambda))^{*\cdot}$. The lemma is proved. □

§4. The vanishing theorem.

4.1. Assume that we are in the rank one case, i.e. $\mathfrak{g} = \mathfrak{sl}_2$. We then have the following well known results

Lemma 4.1. i) Let $\lambda \geq -1$. Then $\mathcal{D}^j\lambda = 0$ for $j>0$, while $\mathcal{D}\lambda$ has dimension $\lambda+1$, a basis being $\{f^{(k)}\otimes v_\lambda \mid 0\leq k\leq\lambda\}$. The action of $U_q(\mathfrak{g})$ is as in 1.1.
ii) Let λ be as before and let Q be the $U_k(\mathfrak{b})$-module $\mathcal{D}\lambda/\lambda$. Then $\mathcal{D}Q = 0$ and $\mathcal{D}^j Q = 0$ for $j>0$.

Proof. i) follows from the corresponding proposition 4.2 in [APW] and the preceding lemma. As for ii) consider the long exact sequence of $U_k(\mathfrak{g})$-modules

$$\longrightarrow \mathcal{D}^1\mathcal{D}\lambda \longrightarrow \mathcal{D}^1Q \longrightarrow \mathcal{D}\lambda \xrightarrow{g} \mathcal{D}\lambda \longrightarrow \mathcal{D}Q \longrightarrow 0$$

which arises from the application of $\mathcal{D}$ to the sequence defining Q. The definition of derived functors in the last section gives that $\mathcal{D}^j\mathcal{D}\lambda = 0$ for $j > 0$; thus the $j>1$ case of ii) follows by combining with i). Now, g must be a nonzero scalar times the identity map on $\mathcal{D}\lambda$ because it is the map corresponding to $\sigma \neq 0$ under Frobenius reciprocity and

$$\mathrm{Hom}_{U_k(g)}(\mathcal{D}\lambda, \mathcal{D}\lambda) \simeq \mathrm{Hom}_{U_k(b)}(\lambda, \mathcal{D}\lambda) \simeq k$$

Hence $\mathcal{D}^1Q$ as well as $\mathcal{D}Q$ must be zero. The lemma is proved. □

4.2. We still consider the rank one case. For a copy of $\mathfrak{sl}_2$ in U_k corresponding to the simple reflection s we denote the corresponding Joseph induction $\mathcal{D}_s$.

Theorem 4.2. i). Let $V_w^k(\lambda)$ be as in 2.6. Then $\mathcal{D}_s^j V_w^k(\lambda) = 0$ for $j > 0$.
ii). Let $sw < w$. Then $\mathcal{D}_s V_{sw}^k(\lambda) = V_w^k(\lambda)$.

Proof. ad i). Let $i \in I$ be the index corresponding to s and consider the W-filtration of $V_w^k(\lambda)$ from lemma 2.5 with respect to this i. The quotients Q of this all satisfy $\mathcal{D}_s^j Q = 0$ for $j>0$: if Q is a dominant line this is because of lemma 4.1 and if Q is the restriction of an $\mathfrak{sl}_2$-module it follows from the definition of $\mathcal{D}^j$ in §3. Using induction on the filtration length we conclude that $\mathcal{D}_s^j V_w^k(\lambda) = 0$ for $j > 0$.
ad ii). Let the $U_k(\mathfrak{b}_i)$-module R be defined such that the following is exact

$$(4.2.1) \qquad 0 \longrightarrow V_{sw}^k(\lambda) \longrightarrow V_w^k(\lambda) \longrightarrow R \longrightarrow 0$$

We know from 2.5 that $V_w^k(\lambda)$ has a k-basis on the form $\{G(b) \mid b \in B_w(\lambda)\}$; hence R

has a basis on the form $\{ G(b) \mid b \in B_w(\lambda) \setminus B_{sw}(\lambda) \}$. In the W-filtration of R, the quotient between the lth and the $(l+1)$th term hence has basis $\{ G(b) \mid b \in I^l(B_w(\lambda)) \setminus I^l(B_{sw}(\lambda)) \}$.

We saw in the proof of lemma 2.6 that the $I^l(B_y(\lambda))$ satisfy string properties like (2.5.4). If S is an i-string and $S \cap I^l(B_{sw}(\lambda)) \neq \emptyset$, then by (2.5.4) we have $S \subset I^l(B_w(\lambda))$. Now $V_w^k(\lambda)$ is a $U_k(\mathfrak{g}_i)$-module, so $I^l(B_w(\lambda))$ is the disjoint union of i-strings; and if $S \subset I^l(B_w(\lambda))$ is such a string then from (2.5.4) we get $S \cap I^l(B_{sw}(\lambda)) \neq \emptyset$. Putting these things together we see that $I^l(B_w(\lambda)) \setminus I^l(B_{sw}(\lambda))$ is the union of i-strings with the highest weight vector omitted; this must furthermore be of $\mathfrak{sl}_2$-weight l. From the formula (2.3.2) we then see that the span of the global basis elements of $I^l(B_w(\lambda)) \setminus I^l(B_{sw}(\lambda))$ form a $U_k(\mathfrak{b}_i)$-module of type $\mathcal{D}\lambda / \lambda$. And then an induction on the filtration length, the induction start being provided by ii) of lemma 4.1, proves that $\mathcal{D}_s^1 R = \mathcal{D}_s R = 0$.

Now, $V_w^k(\lambda)$ is a $U_k(\mathfrak{g}_i)$-module so we have $\mathcal{D}_s V_w^k(\lambda) = V_w^k(\lambda)$; combining this with $\mathcal{D}^1 R = \mathcal{D}_s R = 0$ we may finish the proof of ii) by applying $\mathcal{D}$ to (4.2.1). □

4.3. In theorem 4.2 the induction $\mathcal{D}_s$ was induction from a rank one Borel subgroup. However, $V_w^k(\lambda)$ has a module structure for the full Borel group $U_k(\mathfrak{b})$. Denote by $U_k'(\mathfrak{g}_i)$ the minimal parabolic quantum group generated by this and the $f_i^{(n)}$'s and by $\mathcal{D}_s'$ the induction from $U_k(\mathfrak{b})$ to $U_k'(\mathfrak{g}_i)$. As k-vectorspaces and $U_k(\mathfrak{g}_i)$-modules $\mathcal{D}_s' V_w^k(\lambda)$ and $\mathcal{D}_s V_w^k(\lambda)$ are isomorphic, namely both equal to the largest f_i-finite quotient of

$$U_k(\mathfrak{g}_i) \otimes_{U_k(b_i)} V_{sw}^k(\lambda)$$

We have the following lemma.

Lemma 4.3. There is an isomorphism of $U_k'(\mathfrak{g}_i)$-modules

$$\mathcal{D}_s' V_{sw}^k(\lambda) \simeq V_w^k(\lambda)$$

Proof. There is a commutative diagram

$$\begin{array}{ccc} V^k_{sw}(\lambda) & \xrightarrow{Id} & V^k_{sw}(\lambda) \\ \sigma\downarrow & & \downarrow i \\ \mathcal{D}'_s V^k_{sw}(\lambda) & \xrightarrow{\varphi} & V^k_w(\lambda) \end{array}$$

where i is the inclusion map, σ is the canonical map - these are $U'_k(\mathfrak{g}_i)$-linear - and φ is the $U_k(\mathfrak{g}_i)$-linear map obtained from theorem 4.2 ii) together with the remarks in 4.3. Any element of $\mathcal{D}'_s V^k_{sw}(\lambda)$ can be written as a linear combination of elements on the form $u\sigma(v)$ where $u \in U^-_k(\mathfrak{g}_i)$, $v \in V^k_{sw}(\lambda)$. We must show that φ commutes with e_j for $j \neq i$; it suffices to do that for $u\sigma(v)$. On the one hand we have

$$\varphi(e_j(u\sigma(v))) = \varphi(ue_j\sigma(v)) = \varphi(u\sigma(e_jv)) = u\varphi(\sigma(e_jv)) = ui(e_jv) = e_jui(v)$$

where we used the commutativity of the diagram and of e_j and f_i. On the other hand

$$e_j\varphi(u\sigma(v)) = e_ju\varphi(\sigma(v)) = e_jui(v)$$

We see that two sides are equal. □

4.4. We omit from now on the primes on the $\mathcal{D}$, inductions will be from the full Borel subalgebra. For $w_0 = s_{i_1}s_{i_2}\dots s_{i_n}$ a reduced expression of the longest element of W we define $\mathcal{D}_{w_0}$ as $\mathcal{D}_{i_1}\mathcal{D}_{i_2}\dots\mathcal{D}_{i_n}$, (a priori this may depend on the chosen expression). The Weyl module $V_k(\lambda)$ is by definition $V_{\mathbb{Z}}(\lambda)\otimes k$. We now obtain the following theorem

Theorem 4.4. For $\lambda \in P^+$ there are isomorphisms of $U_k(\mathfrak{b})$-modules

$$\mathcal{D}_{w_0}\lambda \simeq \mathcal{D}\lambda \simeq V_k(\lambda)$$

Proof. As $V_k(\lambda)$ has finite dimension there is by definition of $\mathcal{D}$ a surjection φ: $\mathcal{D}\lambda \twoheadrightarrow$

$V_k(\lambda)$. Now, successive applications of theorem 4.2 and lemma 4.3 give the isomorphism $\mathcal{D}_{w_0}\lambda \simeq \mathcal{D}\lambda \simeq V_k(\lambda)$. And applying Frobenius reciprocity succesively, the canonical map σ: $\lambda \to \mathcal{D}\lambda$ induces a $U_k(\mathfrak{b})$-linear map ψ: $\mathcal{D}_{w_0}\lambda \simeq V_k(\lambda) \to \mathcal{D}\lambda$. But any $U_k(\mathfrak{b})$-linear map $V_k(\lambda) \to \mathcal{D}\lambda$ must also be $U_k(\mathfrak{g})$-linear as one sees from Frobenius reciprocity: $V_k(\lambda)$ and $\mathcal{D}\lambda$ are both $U_k(\mathfrak{g})$-modules. Thus the composition $\varphi\circ\psi$ is *a* $U_k(\mathfrak{g})$-linear endomorphism of $V_k(\lambda)$ and one checks that it is nonzero on the λ'th weight space. But $V_k(\lambda)$ is a highest weight module; hence $\varphi\circ\psi$ must be a nonzero scalar times the identity. This shows that $\mathcal{D}\lambda \simeq V_k(\lambda)\oplus M$ for M some $U_k(\mathfrak{g})$-module. Now, $\mathcal{D}\lambda$ is indecomposable being a highest weight module too, and we get a contradiction unless $M = 0$. The theorem is proved. □

4.5. We can now prove our main theorem.

Theorem 4.5. (Kempf vanishing). Let $\lambda \in P^{\cdot}$. Then $\mathrm{H}^i_k(\lambda) = 0$ for $i > 0$.

Proof. By lemma 3.4 the theorem is equivalent to $\mathcal{D}^i(-\lambda) = 0$. Let $(E\otimes\nu)^{\cdot} \longrightarrow -\lambda \longrightarrow 0$
be a resolution of $-\lambda$ as in 3.4. Then the complex $(\mathcal{D}_{s_n}(E\otimes\nu))^{\cdot} \longrightarrow \mathcal{D}_{s_n}(-\lambda) \longrightarrow 0$ is exact by lemma 4.1 i). It is also acyclic for $\mathcal{D}_{s_{n-1}}$ because by the the tensor identity and theorem 4.2 we have

$$\mathcal{D}^i_{s_{n-1}}(\mathcal{D}_{s_n}(E\otimes\nu)) \simeq E\otimes\mathcal{D}^i_{s_{n-1}}(\mathcal{D}_{s_n}(\nu)) \simeq E\otimes\mathcal{D}^i_{s_{n-1}}(\mathrm{V}^k_{s_n}(\nu)) \simeq 0$$

Thus the application of $\mathcal{D}_{s_{n-1}}$ to $(\mathcal{D}_{s_n}(E\otimes\nu))^{\cdot} \longrightarrow \mathcal{D}_{s_n}(-\lambda) \longrightarrow 0$ gives a complex that evaluates $\mathcal{D}^i_{s_{n-1}}(\mathcal{D}_{s_n}(-\lambda))$. But we know from theorem 4.2 that these cohomology groups are zero so the complex $\mathcal{D}_{s_{n-1}}(\mathcal{D}_{s_n}(E\otimes\nu))^{\cdot} \longrightarrow \mathcal{D}_{s_{n-1}}\mathcal{D}_{s_n}(-\lambda) \longrightarrow 0$ is exact. Furthermore the argument from before shows that it is acyclic for $\mathcal{D}_{s_{n-2}}$. Continuing we eventually reach the sequence $(\mathcal{D}_{w_0}(E\otimes\nu))^{\cdot} \longrightarrow \mathcal{D}_{w_0}(-\lambda) \longrightarrow 0$ which thus is exact. By theorem 4.4 it is isomorphic to $(\mathcal{D}(E\otimes\nu))^{\cdot} \longrightarrow \mathcal{D}(-\lambda) \longrightarrow 0$ and we are done. □

4.6. We have a couple of corollaries to theorem 4.5.

Corollary 4.6.1. (Demazure vanishing). For $\lambda \in P^{-}$ we have $\mathcal{D}^{i}_{s_n}(\mathcal{D}_{s_{n-1}}\mathcal{D}_{s_{n-2}}\ldots\mathcal{D}_{s_1}(\lambda)) \simeq 0.$

Proof. This is contained in the proof of theorem 4.5.

Corollary 4.6.2. The modular Kempf and Demazure vanishing theorems.

Proof. The results follow from theorem 4.5 and corollary 4.5.1 by specializing q to 1: there are base change theorems controling this.

Corollary 4.6.3. Demazures character formula in terms of the H^{i}.

Proof. The classical proof carries over, see [A1].

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Steen Ryom-Hansen (email ryom@mi.aau.dk)
Matematisk Institut
Aarhus Universitet
Ny Munkegade
8000 Aarhus C, Denmark.