Essays on Contingent Claim Pricing

by

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Ph.D. Thesis

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Preface

This thesis consists of a summary, an introduction, and eight papers on different topics in contingent claim pricing. Whereas the summary only briefly describes the contributions of the papers, the introduction goes in more detail with the topics of the papers and link their results to the existing literature.

The thesis is written partly in LATEX, Publisher, and Word, reflecting the different preferences of coauthors and myself. This has unfortunately resulted in different typographical styles. We kindly ask the reader to bare over with this.

In my opinion, discussions and exchange of ideas is an essential input to good research. I would like to thank the following for excellent comments and discussions: Leif Andersen, Simon Babbs, Marc Chesney, Bent Jesper Christensen, Peter Ove Christensen, Michel Crouhy, Anders Damgaard, Darrell Duffie, Bernard Dumas, Pierre Collin Dufresne, Raul Espejel, Brian Fuglsbjerg, Bob Goldstein, Barbara Gruenewald, Asbjørn Hansen, Hua He, David Heath, Peter Honore, Farshid Jamshidian, Bjarke Jensen, Peter Løchte Jørgensen, Bill Keirstead, David Lando, Hayne Leland, 2×Jesper Lund, Terry Marsh, Stewart Mayhew, Kristian Miltersen, Rolf Poulsen, Mark Rubinstein, Wei Shi, Erik Schloegl, Eduardo Schwartz, Klaus Toft, Jørgen Warncke, and particularly my advisor Jørgen Aase Nielsen.

Summary

This thesis consists of an introduction and eight papers on different topics in contingent claim pricing. The papers are:

1. "The Pricing of Discretely Sampled Asian and Lookback Options: A Change of Numeraire Approach".
2. "The Passport Option", with Leif Andersen and Rupert Brotherton-Ratcliffe, General Re Financial Products, New York.
3. "American Option Pricing in the Jump-Diffusion Model", with Barbara Gruenewald, Johannes Gutenberg University of Mainz, Germany.
4. "Implied Modelling: Stable Implementation, Hedging, and Duality".
5. "New Skin for the Old Ceremony: Eight Different Derivations of the Black-Scholes Formula", with Bjarke Jensen and Rolf Poulsen, Department of Operations Research, University of Aarhus.
6. "An Arbitrage Term Structure Model of Interest Rates with Stochastic Volatility", with Pierre Collin Dufresne, HEC, Paris, and Wei Shi, Salomon Brothers, Tokyo..
7. "Pricing by Arbitrage in An International Economy".
8. "A Gaussian Exchange Rate and Term Structure Model".

Below we summarize the results and the content of these papers.

1. It is demonstrated that within the Black and Scholes (1973) model it is possible to reduce a number of discretely sampled Asian and lookback option pricing problems to problems of dimension one. The options considered are Asians and lookback options with fixed and floating strike prices. The main idea is to use the stock as numeraire for the martingale measure instead of the bank-account. Doing so we obtain that the stock price deflated option prices are functions of current time and position of a single one-dimensional Markov process. Due to the discrete sampling this process is discontinuous at the sampling dates, but continuous between the sampling points. This means that the option pricing problem can be solved by numerically solving a sequence, in the time domain, of standard partial differential equations (PDEs), where the first PDE generates the terminal boundary conditions of the second, which generates the boundary conditions of third, and so on. We use a Crank-Nicholson scheme to numerically solve this system of PDEs and by examples we demonstrate that this technique, in terms of speed and accuracy, is superior to Monte Carlo methods. We also illustrate that continuous versus discrete sampling can have substantial effects on the resulting option prices. Finally we show that the technique extends to the case when the underlying stock exhibits jumps and we illustrate the effect of jumps on Asian and lookback option prices.

2. The passport option is an OTC traded contract that grants its holder the right to continuously switch between short and long positions in the underlying stock. If positive, the accumulated gain is paid to the holder at the maturity of the option. In the framework of Black and Scholes (1973), it is shown that the option pricing problem can be solved as a stochastic control problem with one state variable. We derive the optimal

switching strategy and a partial differential equation for the stock price deflated option price. A closed-form solution is presented for the special case when the underlying is a martingale under the risk-neutral probability measure. For the general case we present an implicit finite difference scheme for the numerical solution. Various extensions of the basic passport option are considered, including American exercise features and discrete rather than continuous switching rights. We illustrate several features of the passport option by numerical examples.

3. The paper considers the pricing of American options when the underlying stock exhibits discontinuous returns. Our model of the stock is a "risk-neutralized" version of the jump-diffusion model of Merton (1976), where jumps are triggered by a Poisson process and the sizes of the jumps are displaced lognormal distributed. The paper presents three new results for the American option pricing problem in this modelling framework. First, it is shown that if the parameters of the model are constant over time then the American option prices satisfy a forward partial integro differential equation in maturity date and strike price. This implies that risk-adjusted parameters of the model can be inferred directly from observed option prices. Second, it is well-known that in the Black and Scholes (1973) model, the early exercise boundary of the American options might exhibit a discontinuity at expiration of the option. In the paper it is shown that this discontinuity is larger for the jump-model and the size of the discontinuity is quantified. By numerical examples it is illustrated that the difference to the Black-Scholes case can be quite large. Third, the paper gives closed form solutions for the perpetual option prices when the underlying exhibits constant jumps. A final section of the paper compares different numerical procedures for calculation of American option prices in a jump-diffusion economy.

4. Under the assumption that the underlying stock evolves according to a continuous Markov process, European call option prices satisfy a forward partial differential equation (PDE), where the variables are maturity date and strike price, whereas current time and spot are fixed. In the paper it is shown that the "Greeks" of European option, i.e. the option prices' partial derivatives with respect to the spot price, time, etc., also satisfy forward PDEs. This means that given a surface of local volatilities in time and spot, we are able to solve for the "Greeks" of European options of all strikes and maturities by simultaneously solving a small number of PDEs numerically. If the local volatility surface is inferred from observed option prices, the resulting "Greeks" will be "implied" by the option prices themselves. The forward equations also imply an interesting duality, namely that the European option pricing problem can be solved in a "dual" economy where time is reversed, the role of the spot and the strike are interchanged, etc.

We derive a closed form relation between implied Black-Scholes volatilities and local volatilities and use this relation to back out the local volatility surface from a set of observed S&P 500 index option bid/ask prices. We then numerically solve the forward equations to identify the Greeks of the input options. This is done using an implicit finite difference method that insures stability even though the model has to fit option "smiles" of several different maturities. The example illustrates that the implied "Greeks" might differ substantially from those of the Black-Scholes model.

5. The paper surveys eight different derivations of the Black and Scholes (1973) European option pricing formula. From pure arbitrage reasoning we have six different derivations:

i. The classical hedge argument leads to the fundamental partial differential equation for option prices, that is solved by pure non-probabilistic techniques.
ii. The martingale approach where we derive the Black-Scholes formula as a risk-adjusted expectation.
iii. The change of numeraire technique that enables us to solve for the option price without calculating a single integral.
iv. A *stop-loss start-gain* strategy replicates the option, but this strategy is not self-financing and the value of the required external financing is shown to equal the Black-Scholes price minus the initial price of the stop-loss start-gain strategy.
v. The European option price also solves a *forward* partial differential equation. This leads to a duality of the option pricing problem.
vi. Introducing the change of numeraire approach in the standard binomial model leads to a very short proof that the European option price of the binomial model converges to the Black-Scholes value.

The last two derivations put the Black-Scholes formula in an equilibrium context. The Black-Scholes formula is shown to be consistent with:

vii. The continuous-time capital asset pricing model.
viii. A single period representative investor economy, where the representative investor has constant relative risk-aversion and is endowed with lognormal distributed terminal wealth.

6. The paper presents an arbitrage based continuous-time model of the term structure of interest rates with "true" stochastic volatility in the sense that the volatilities of the bonds do not exhibit a one-to-one correspondence with the shape and the level of the yield curve. Most existing "stochastic volatility" term structure models have the drawback that a change in volatility is immediately reflected in the shape or the level of the yield curve. The model has two random factors, represented by two Brownian motions. The level of the yield curve is only directly affected by the first Brownian motion whereas the volatility is driven by both Brownian motions. Using the methodology of Heath, Jarrow, and Morton (1992), risk-adjusted dynamics of both factors are derived in a preference-free way. In order to do so we assume the existence of futures contracts on the volatility of the bonds in addition to the existence of a full continuum of initially observed zero-coupon bonds. A hedging argument shows that the volatility futures contracts can be replicated by a static trading strategy in *yield* futures contracts. Given the initial yield curve and the volatility futures prices, the model is fully specified by defining the volatility structure of the volatility futures contracts. The paper provides a simple specification of this volatility structure that only requires the estimation of

three parameters. For numerical implementation a discrete non-recombining trinomial approximation of the model is presented. We illustrate convergence, speed, and accuracy of this approximation by numerical examples and we analyze the effect of stochastic volatility on bond option prices.

7. In the paper we consider a general international multicurrency economy. Under the assumption of dynamic completeness and absence of arbitrage, we formalize the relationship between change of numeraire, exchange rates, and change of martingale measure. We show that knowledge of the nominal risk-premia of two currencies to a large extend enables us to fix the stochastic evolution of the exchange rate between the two economies. We demonstrate that use of the implicit change of numeraire induced by the exchange rates can reduce the complexity of the calculation of currency options and futures prices.

8. The paper presents a tractable model for the valuation of contingent claims on the term structures of two currencies and the associated exchange rate. The modelling framework is a deterministic version of the general Heath, Jarrow, and Morton (1992) framework that takes the initial yield curves as direct input. In this modelling framework we show that hedge portfolios of European style claims based on the underlying instruments can be identified by an integral representation. Further, we illustrate that closed-form solutions can be obtained for a general class of cross-currency derivatives and we identify a class of deterministic volatility structures that imply Markov representation by a three-dimensional Gaussian process. For this class of volatility structures all types of European style contingent claim prices and hedge ratios can be obtained by numerical integration in maximum three dimensions.

Introduction

Though the general theory of contingent claim pricing applies to contingent claims on all traded assets it is common to treat term structure derivatives separately because arbitrage consistent modelling of the random behavior of the term structure of interest rates is a great deal more complicated than the stochastic modelling of shares of equity, commodities, and foreign exchange. We follow this praxis and divide the introduction into two main sections, one for "stock" options and one for term structure contingent claims. Each of these sections contain subsections for the different topics considered in the papers. But first, we briefly describe the general assumptions, style, and notation of the thesis.

Assumptions, Style, and Notation

We consider stylized financial markets where there are

i. No frictions: No taxes, transactions costs, or short-selling constraints, perfectly divisible assets and continuous trading is possible.
ii. Price taking behavior: Trading decisions do not influence the market prices.
iii. Absence of arbitrage.

The assumptions might not seem reasonable from the perspective of each and every individual investor but they might be a good *approximation* for how financial markets work on an aggregate level. If we suppose that a market contains a large number of equally taxed investors with negligeable transactions cost and the ability to trade in such volume that the issue of divisibility can be ignored, then it is likely that these investors will drive the market prices towards a competitive equilibrium where the market prices will be *as if* we were in a situation of no market imperfections. Absence of arbitrage is a necessary condition for the existence of a "no-frictions" equilibrium and is therefore a natural "minimal" requirement for a model of the market.

In most research on contingent claim pricing and in most of our work the perfect market assumptions are combined with the assumption that asset prices evolve continuously. This lead to the celebrated result of Black and Scholes (1973), that we can construct a self-financing portfolio of marketed assets which replicates the pay-off of the considered contingent claim and hence if there are no arbitrage opportunities, the value of the claim must equal the initial cost of the replicating portfolio. This is both good and bad news for the contingent claims analysis. On one hand it means that contingent claim pricing is *relative* rather than *absolute*. That is, given a model of random evolution of the underlying asset evolution the contingent claim prices can be determined from the prices of the underlying assets observed in the market and we do not need to specify the expected return or the preferences of the investors in order to come up with the price of the considered claim. On the other hand, the result also means that contingent claims are redundant securities that are already spanned by the market of underlying assets. This gives rise to the natural question: Why do investors trade derivatives? The justification could be that many investors face significant transaction costs and other imperfections limiting their market access. Such agents have natural incentives to engage in contracts with investors with access to hedge the contracts in the market. This is typically what we observe in derivatives markets: Investors engage in OTC contracts with investment banks that use a combination of marketed options and dynamic trading to hedge their portfolio of derivatives.

Concerning style, we will in general work under the paradigm of "sufficient mathematical regularity". We are for example going to assume the existence of martingale measures instead of imposing mathematical conditions on the allowed trading strategies in order to guarantee that absence of arbitrage imply their existence. Likewise, if we differentiate something we are implicitly assuming that this "something" is appropriately differentiable and if we integrate something we are going to assume that this "something" is appropriately integrable.

Since pricing and hedging are the main objects of our papers we will generally specify the models directly under a martingale measure.

The notation varies a little from paper to paper. In general our time axis is the nonnegative part of the real line. Claims are issued at time 0 and mature at time $T > 0$ and current time is $t \in [0, T]$.

We generally use the term "a stock" as a common term for a share of equity, a foreign exchange rate, or a commodity. The time t price of a stock is $S(t)$. The stock is generally assumed to be paying a continuous "dividend yield" of $q(t)$, with the interpretation that the dividend yield of a currency is the foreign interest rate and for a commodity q represents minus the net convenience yield. We let $B(t)$ be the value of a bank-account that continuously accumulates the spot interest rate $r(t)$. We will let Q denote the martingale measure with the bank account as numeraire and Q' the martingale measure with the stock price as numeraire.

For term structure models we let $P(t, T)$ be the time t price of a zero-coupon bond maturing a time T and $f(t, T)$ be the time t instantaneous forward rate for deposit at time T. Q is again the martingale measure with the bank account as numeraire and Q^T is the martingale measure with the time T maturity zero-coupon bond as numeraire.

Stock Options

This section is divided into six subsections. The first three subsections are based on the papers "The Pricing of Discretely Sampled Asian and Lookback Options: A Change of Numeraire Approach" and "The Passport Option" and consider the pricing of Asian and lookback options and a new over-the-counter (OTC) traded contract termed "the passport option". The modelling framework is here the standard Black-Scholes model where the stock and the bank-account evolve according to the stochastic differential equations (SDEs)

$$\begin{aligned}\frac{dS(t)}{S(t)} &= (r-q)dt + \sigma dW(t) \\ &= \left(r - q + \sigma^2\right)dt + \sigma dW'(t) \\ \frac{dB(t)}{B(t)} &= r dt\end{aligned} \tag{1}$$

where r, q are constant, σ is the constant volatility, and W, W' are Brownian motions under Q, Q' respectively.

Next, a subsection discusses the pricing of exotic and American options when the underlying exhibits jumps. This subsection is partly based on the paper "The Pricing of Discretely Sampled Asian and Lookback Options: A Change of Numeraire Approach" and partly on "American Option Pricing in the Jump-Diffusion Model".

The following subsection treats another alternative to the Black-Scholes model: The "implied" modelling approach, where observed option prices are used to fix a diffusion model. This subsection is based on the paper "Implied Modelling: Stable Implementation, Hedging, and Duality".

The section on stock options is concluded by a subsection on the survey paper "New Skin for the Old Ceremony: Eight Different Derivations of the Black-Scholes Formula".

Asian Options

Asian options is a general term for options on the arithmetic average of the underlying stock over a subperiod of the life of the option. In the Black-Scholes framework there are no simple closed-form solutions for such contracts.[1] Various transforms and approximations of Asian option prices have been described in the literature, see for example Geman and Yor (1993) and Rogers and Shi (1995). However, these expressions are all derived under the assumption that the arithmetic average is sampled over continuous intervals, and this might be a poor approximation of the actually traded options. In the paper "The Pricing of Discretely Sampled Asian and Lookback Options: A Change of Numeraire Approach", we consider efficient numerical methods for the pricing of options depending on the discretely sampled arithmetic averages. The main trick applied in the paper is a change of numeraire technique that enables us to describe the problems as solutions to simple one-dimensional Markov problems. The change

[1] By the term "simple closed-form" we here mean expressions that are not written in terms of infinite sums and integrals over non-standard functions.

of numeraire technique is to our knowledge first applied to Asian options by Ingersoll (1987). But Ingersoll only considers Asian options with floating strike (often termed "the average strike option") and only in the case when the average is sampled continuously. We will now describe the technique under discrete sampling of the underlying.

The fixed strike Asian option on a discretely sampled average has the terminal pay-off

$$\left(\frac{1}{n}\sum_{i=1}^{n} S(t_i) - K\right)^+ \tag{2}$$

where $0 \leq t_1 < \ldots < t_n \leq T$ are the sampling points and K is a constant.

In a standard Black-Scholes framework the fair price is given by

$$F(t) = \mathrm{E}_t^Q \left[e^{-r(T-t)} \left(\frac{1}{n}\sum_{i=1}^{n} S(t_i) - K\right)^+ \right] \tag{3}$$

At a first glance this seems to be a rather complicated problem to solve numerically, because a Markov representation under Q would involve two state variables: the current stock price and the current average. But the pricing can also be performed using the martingale measure with the stock price as numeraire:

$$F(t) = S(t)\mathrm{E}_t^{Q'} \left[e^{-q(T-t)} \left(\frac{\frac{1}{n}\sum_{i=1}^{n} S(t_i) - K}{S(T)} \right)^+ \right] \tag{4}$$

Define

$$\begin{aligned} x(t) &= \frac{\frac{1}{n}\sum_{i:t_i \leq t} S(t_i) - K}{S(t)} \\ m(t) &= \max\{i : t_i \leq t\} \end{aligned} \tag{5}$$

Using Ito's lemma we find that[2]

$$dx(t) = -(r-q)x(t-)dt - \sigma x(t-)dW'(t) + \frac{1}{n}dm(t) \tag{7}$$

where W' is a standard Brownian motion under Q'. The SDE for x reveals that x is a Markov process under Q', and we may write $F(t)/S(t) = f(t, x(t))$. Using that $e^{-qt}f$ is a Q'-martingale, applying Ito expansion and the martingale representation theorem lead to a system of PDEs. Specifically, on $t \notin \{t_1, \ldots, t_n\}$, f solves the PDE

$$qf = f_t - (r-q)xf_x + \frac{1}{2}\sigma^2 x^2 f_{xx} \tag{8}$$

[2] By the notation $z(t-)$ and $z(t+)$ we mean

$$\begin{aligned} z(t-) &= \lim_{\epsilon \to 0} z(t - |\epsilon|) \\ z(t+) &= \lim_{\epsilon \to 0} z(t + |\epsilon|) \end{aligned} \tag{6}$$

subject to the boundary conditions

$$
\begin{aligned}
f(t_i-, x) &= f\left(t_i+, x+\frac{1}{n}\right) \\
f(T, x) &= x^+
\end{aligned}
\tag{9}
$$

This system can be solved numerically by backward recursion, where the first PDE generates the boundary conditions for the second, which generates the boundary conditions for the third and so on.

In the paper we speed up the procedure by noting that x can only cross the level $x = 0$ at the observation points, $\{t_i\}$ and that the expectation of x can be computed in closed form. Numerically we apply the implicit Crank-Nicholson scheme for the actual solution. We prefer the implicit schemes for stability and the Crank-Nicholson version because of its second order accuracy in the time domain.[3]

The floating strike Asian has the terminal pay-off

$$
\left(\frac{1}{n}\sum_{i=1}^{n} S(t_i) - \alpha S(T)\right)^+ \tag{10}
$$

where α is a constant. Again we apply the change of numeraire technique and consider

$$
x(t) = \frac{\sum\limits_{i:t_i \le t} S(t_i)}{S(t)} \tag{11}
$$

Using Ito's lemma yields

$$
\begin{aligned}
dx(t) &= -(r-q)x(t-)dt - \sigma x(t-)dW'(t) + dm(t) \quad , t \ge t_1 \\
x(t_1) &= 1
\end{aligned}
\tag{12}
$$

which again is a Markov process. We get that the stock price deflated option price satisfies the same set of PDE as the deflated fixed strike Asian option price, but this time subject to the boundary conditions

$$
\begin{aligned}
f(t_i-, x) &= f(t_i+, x+1) \\
f(T, x) &= \left(\frac{1}{n}x - \alpha\right)^+
\end{aligned}
\tag{13}
$$

In this case we can also treat American style features by introducing the free-boundary condition

$$
f(t, x) \ge \left(\frac{1}{m(t)}x - \alpha\right)^+ \tag{14}
$$

The numerical solution method is essentially the same as for the fixed strike Asian option.

3 For the properties of finite difference methods, see for example Mitchell and Griffiths (1980).

For both of the Asian options, hedge ratios can be identified on the (t, x) grid by use of the fact that the hedge ratios are given by

$$f(t, x(t)) - x(t) f_x(t, x(t)) \tag{15}$$

In the paper we compare this solution technique to Monte-Carlo solutions, and find that this method is about 100–1000 times faster than Monte-Carlo for the same accuracy, even though the Monte Carlo method is speeded up by use of a rather sophisticated control variate technique. We also illustrate that discrete rather than continuous sampling actually has effects on the resulting option prices. For realistic parameters, going from continuous to monthly sampling has an effect of roughly 5 per cent for options with one year to maturity.

Lookback Options

"Lookback options" is a common term for contingent claims depending on extrema of the underlying. Lookback options were first analyzed in Gatto, Goldman, and Sosin (1979), and Goldman, Sosin, and Shepp (1979). In a Black-Scholes setting these papers present closed form solutions for a couple of types of lookback options. Conze and Viswanathan (1991) derive closed form solutions for several more types of lookback options and Shepp and Shirayev (1993) derive a closed form solution for a special type of perpetual American style lookback option, termed the Russian option. All of this work is based on the assumption that the maximum is taken over a continuum of sampling points.

Babbs (1992) utilizes the change of numeraire approach to device a simple binomial scheme for the numerical evaluation of the standard (floating strike) lookback option. Babbs considers the discrete sampling case and illustrates that continuous versus discrete sampling may have a dramatic effect on lookback option prices. In the paper "The Pricing of Discretely Sampled Asian and Lookback Options: A Change of Numeraire Approach" we show how the change of numeraire technique can be applied to establish an efficient numerical valuation technique of fixed strike lookback options (also termed options on extrema) under discrete sampling. The technique of Babbs does not immediately apply to this case, instead we transform the option pricing problem into a barrier option problem and solve the pricing problem in two steps. Below we summarize how this is done.

The fixed strike lookback option promises the holder the terminal pay-off

$$\left(\overline{S}(T) - K\right)^+ \tag{16}$$

where

$$\overline{S}(t) = \max_{i:t_i \leq t} S(t_i) \tag{17}$$

and $0 \leq t_1 < \ldots < t_n \leq T$ are the sampling dates.

Suppose the current maximum is greater or equal to the strike of the option, i.e. $\overline{S}(t) \geq K$. We can now evaluate the option by[4]

$$\begin{aligned} F(t) &= \mathrm{E}_t^Q\Big[e^{-r(T-t)}\big(\overline{S}(T)-K\big)^+\Big] \\ &= \mathrm{E}_t^Q\Big[e^{-r(T-t)}\overline{S}(T)\Big] - e^{-r(T-t)}K \\ &= S(t)\mathrm{E}_t^{Q'}\left[e^{-q(T-t)}\frac{\overline{S}(T)}{S(T)}\right] - e^{-r(T-t)}K \\ &\equiv S(t)f(t) - e^{-r(T-t)}K \end{aligned} \tag{18}$$

Define

$$x(t) = \overline{S}(t)/S(t) \tag{19}$$

Applying Ito's lemma yields

$$\begin{aligned} &dx(t) = -(r-q)x(t-)dt - \sigma x(t-)dW'(t) + (1-x(t-))^+ dm(t) \quad , t \geq t_1 \\ &x(t_1) = 1 \end{aligned} \tag{20}$$

where $m(\cdot)$ is the sampling date counter.
We see that x is a Markov process and thereby that $f = f(t, x(t))$. Further we see that for $t \notin \{t_1, \ldots, t_n\}$, f solves

$$qf = f_t - (r-q)xf_x + \frac{1}{2}\sigma^2 x^2 f_{xx} \tag{21}$$

subject to the boundary conditions

$$\begin{aligned} &f(t_i-, x) = \begin{cases} f(t_i+, x) & , x \geq 1 \\ f(t_i+, 1) & , x < 1 \end{cases} \\ &f(T, x) = x \end{aligned} \tag{22}$$

This solves the problem for $\overline{S}(t) \geq K$. For the case $\overline{S}(t) < K$, we define the first passage time among the sampling points to the level K by

$$\tau = \inf\{t_i : S(t_i) \geq K\} \tag{23}$$

with the natural convention $\inf \emptyset = \infty$. Doing so we have that we can write[5]

$$\begin{aligned} F(t) &= \mathrm{E}_t^Q\Big[e^{-r(\tau-t)}\Big\{S(\tau)f(\tau, x(\tau)) - e^{-r(T-\tau)}K\Big\}\mathbf{1}_{\tau \leq T}\Big] \\ &= \mathrm{E}_t^Q\Big[e^{-r(\tau-t)}\Big\{S(\tau)f(\tau, 1) - e^{-r(T-\tau)}K\Big\}\mathbf{1}_{\tau \leq T}\Big] \end{aligned} \tag{24}$$

We see that this is a Markov problem, this time in S, and that we can identify F as the solution to the (standard) PDE

$$rF = F_t + (r-q)SF_S + \frac{1}{2}\sigma^2 S^2 F_{SS} \tag{25}$$

4 Note that time t information now includes the event $\{\overline{S}(t) \geq K\}$.

5 We let $\mathbf{1}_A$ denote the indicator function on the set A.

on $t \notin \{t_1, \ldots, t_n\}$, subject to the boundary conditions

$$F(t_i-, S) = \begin{cases} F(t_i+, S) & , S < K \\ Sf(t_i, 1) - e^{-r(T-t_i)}K & , S \geq K \end{cases} \tag{26}$$
$$F(t_n+, S) = 0$$

So numerically solving for the price of the lookback option with fixed strike is a two step procedure. First, we numerically solve for the function f on a (t, x)-grid. We then solve for the actual price on a grid in (t, S) using the numerical solution for $\{f(t_i, 1)\}_{i=1,\ldots,n}$. Hedge ratios are natural by-products of the second grid.

The numerical accuracy of this scheme turns out to be a little lower than for the Asian options. This is probably due to the fact that the numerical solution here involves a two-step procedure that accumulates numerical errors. But the solution technique is still much faster than Monte Carlo methods for the same precision. Our numerical results confirm the results by Babbs (1992) that discrete versus continuous sampling has dramatic effects on lookback option prices. Under realistic parameter assumptions, going from daily to continuous sampling increases option prices by roughly 8 per cent for option with one year to maturity.

Options with pay-offs of the type

$$\left(\overline{S}(T) - \alpha S(T)\right)^+ \tag{27}$$

can be treated more directly. In fact, all one has to do is to replace the terminal boundary condition in (22) by

$$f(T, x) = (x - \alpha)^+ \tag{28}$$

then the price of the option is given by $F(t) = S(t)f(t, x(t))$. If we want to consider this type of option with an American exercise feature we just have to add the free-boundary condition

$$f(t, x) \geq (x - \alpha)^+ \tag{29}$$

The Passport Option

A passport option is an OTC traded contract that grants its holder to repeatedly switch between short and long positions in an underlying asset. The gains on the stream of short/long positions are accumulated and, if positive, paid to the holder at maturity of the option. Holding a long (short) position over the whole life of the option makes the passport option equivalent to an at-the-money European call (put) option. The price of the passport option is thus bounded from below by the maximum of the call and the put option prices, and we might view the passport option as a contract that entitles the owner to continuously switch between a call and a put option.

The paper "The Passport Option" is to our knowledge the first theoretical treatment of the pricing of this contract.

The passport option pay-off can be represented by

$$\left(\int_0^T \pi(u)dS(u)\right)^+ \tag{30}$$

where $\{\pi(u)\}_{0\leq u\leq T}$ is an adapted strategy chosen by the holder with the property that $\pi(u) \in [-1, +1]$ for all u, assumed sufficiently regular to admit the above stochastic integral to have weak solutions.

At a first glance it seems that solving the pricing problem involves a stochastic control problem of two dimensions: the stochastic integral and the stock price. But, as we saw in the previous section, problems of this type are often easier to handle after a change of numeraire. Define

$$x(t) = \frac{\int_0^t \pi(u)dS(u)}{S(t)} \tag{31}$$

In the Black-Scholes economy we have that

$$dx(t) = (\pi(t) - x(t))\big((r-q)dt + \sigma dW'(t)\big) \tag{32}$$

We can now formulate the option pricing problem as a stochastic control problem:

$$\begin{aligned} &F(t) = S(t) \sup_{\{\pi(u)\}_{u\in[t,T]}} \mathrm{E}_t^{Q'}\left[e^{-q(T-t)}x(T)^+\right] \\ &\text{subject to} \\ &dx(u) = (\pi(u) - x(u))\left[(r-q)du + \sigma dW'(u)\right] \quad , u \geq t \end{aligned} \tag{33}$$

Following Øksendahl (1995), the structure of the SDE for x implies that we can restrict the search for optimal strategies to strategies with $\pi = \pi(t, x(t))$, hence we have that $f \equiv F/S = f(t, x(t))$. Further, the optimal strategy solves the Bellman equation

$$0 = \max_{\pi}\left\{-qf + f_t + (r-q)(\pi - x)f_x + \frac{1}{2}\sigma^2(\pi - x)^2 f_{xx}\right\} \tag{34}$$

This is a second order polynomial in π, and the convexity of f (which is shown in the paper) implies that the optimal strategy is given by

$$\pi^* = \mathrm{sign}\big((r-q)f_x - \sigma^2 x f_{xx}\big) \tag{35}$$

Hence, we get an optimal control of the "bang-bang" type.

Plugging this back into the Bellman equation yields the non-linear PDE

$$qf = f_t - (r-q)xf_x + \frac{1}{2}\big(1+x^2\big)f_{xx} + \big|(r-q)f_x - \sigma^2 x f_{xx}\big| \tag{36}$$

with boundary condition

$$f(T,x) = x^+ \tag{37}$$

Application of Ito's lemma implies that the replicating portfolio is given by

$$f + (\pi - x) f_x \tag{38}$$

number of stocks and the remaining amount

$$-S(\pi - x) f_x \tag{39}$$

on the bank-account. So even though f is a smooth function the replicating portfolio will have discontinuous weights in the stock and the bank-account and the option will have infinite "gamma" when the strategy changes.

In the special case $r = q$, it is possible to come up with a closed form solution. The optimal strategy reduces to[6]

$$\pi^* = -\text{sign}(x) \tag{40}$$

and the PDE becomes linear:

$$qf = f_t + \frac{1}{2}\sigma^2(1 + |x|)^2 f_{xx} \tag{41}$$

Note that the strategy $\pi = -\text{sign}(x)$ is the strategy that maximizes the local volatility of x.

The solution to the PDE is[7][8]

$$f(t,x) = e^{-q(T-t)}\left\{ x^+ + \Phi(z(t)) - (1+|x|)\Phi\left(z(t) - \sigma\sqrt{T-t}\right) + \frac{\sigma^2}{4}\int_t^T \Phi(z(s))ds \right\}$$

$$z(s) = \frac{-\ln(1+|x|)}{\sigma\sqrt{T-s}} + \frac{1}{2}\sigma\sqrt{T-s} \tag{42}$$

In the above formula the first three terms under the bracket represent the maximum of holding either $\pi = -1$ or $\pi = 1$ for the remaining of the life of the option. The last term, the time integral, represents the value of the right to switch strategy at future dates.

In the paper we describe two implicit finite difference schemes for the numerical solution of the passport option price in the general case $r \neq q$. The first scheme is constructed to price passport options where the strategy can be changed continuously, whereas the second scheme is designed to price passport options with only a discrete number of switching dates. Both schemes can also be used to price passport options with American or Bermudan exercise features.[9]

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6 We let $\text{sign}(z) = \mathbf{1}_{z\geq 0} - \mathbf{1}_{z<0}$.

7 $\Phi(\cdot)$ is the distribution function for a standard normal random variable.

8 Even though the formula involves non-differentiable functions, f is in fact at least twice continuous differentiable in x. This is shown in the paper.

9 As an alternative to the implicit schemes Jamshidian (1997) suggests numerical integration to price the passport option.

Our numerical examples show several interesting features of the passport option, among those:

i. Depending on the drift of the underlying asset, the optimal strategy will change at either zero, one or two levels of x. This behavior can be described by

$$\pi^*(t,x) = \begin{cases} -\text{sign}(x - x_*(t)) & , r \leq q \\ 1 - 2 \cdot \mathbf{1}_{x_*(t) \leq x \leq x^*(t)} & , r > q \\ 1 & , r \gg q \end{cases} \tag{43}$$

where $x_*(t) \leq x^*(t)$ are two curves, depending on the parameters, but decreasing (increasing) in t for $r > q$ $(r < q)$.

ii. The discrete passport option prices converge rapidly to the continuous passport option prices when the number of switching dates is increased.

iii. When an American exercise feature is added to the basic passport option and the drift of the underlying is positive, the early exercise boundary cuts through both the upper long region and the short region.

Discontinuous Returns of the Underlying Stock

The papers "The Pricing of Discretely Observed Asian and Lookback Options: A Change of Numeraire Approach" and "American Option Pricing in the Jump Diffusion Model" treat different aspects of contingent claim pricing when the underlying stock exhibits discontinuous dynamics.

When the underlying asset exhibits discontinuous returns the economy consisting of the risky stock and the money market account is no longer dynamically complete, which means that we can no longer replicate any pay-off measurable with respect to the filtration generated by the stock by a self-financing trading strategy. This has two effects:

i. The martingale measure with the bank-account as numeraire is no longer unique.

ii. The introduction of derivative securities might change the stock price and the interest rate.

(i.) implies that the price of a derivative security becomes non-unique and relative to what martingale measure is chosen. (ii.) is a far more serious problem to the option pricing issue; introducing new contingent claims can change the existing equilibrium and thereby the market prices of the underlying.

These two problems are circumvented in our papers by introducing two assumptions. First, we simply fix a martingale measure with the bank-account as numeraire and use this measure as our pricing functional. In fact, we only specify the asset price dynamics under the equivalent martingale measure and do not describe the asset dynamics under the original measure. Second, we assume that the existing market price of the stock and the interest rate are unaffected by the introduction of the considered derivative securities.

Specifically we assume that under the martingale measure with the bank-account as numeraire, $\mathcal{Q}$, the stock evolves according to the SDE

$$\frac{dS(t)}{S(t-)} = (r - q - k\lambda)dt + \sigma dW(t) + I(t)dN(t) \tag{44}$$

where

W is a standard Brownian motion.
N is a Poisson process with constant intensity λ.
$\{I(t)\}_{t\geq 0}$ is a sequence of independent and identically distributed random variables. Their distribution is given by

$$\ln(1 + I(t)) \underset{\mathcal{Q}}{\sim} \mathrm{N}\left(\gamma - \frac{1}{2}\delta^2, \delta^2\right) \tag{45}$$

for all t.
k is the mean jump in the instantaneous return

$$k = \mathrm{E}^{\mathcal{Q}}[I(t)] = e^{\gamma} - 1 \tag{46}$$

The processes W, I, N are assumed independent.

We thus have a model where the stock price exhibits random proportional jumps triggered by a constant intensity Poisson process. Between jumps the stock price evolves continuously according to a geometric Brownian motion.

The interest rate and the dividend yield, r, q, are assumed to be strictly positive.[10] The bank-account is as usual given by

$$B(t) = e^{rt} \tag{47}$$

This is in essence a "risk-neutral" version of the Merton (1976) model. Merton starts under the original measure and assumes that the jump-risk is non-systematic. Doing so, Merton is able to fix a martingale measure and obtains a closed form formula for European call options on the stock. Amin (1993) and Bates (1991) also start under the original measure and use an argumentation based on a representative investor equilibrium to fix a martingale measure. However, only the risk-adjusted dynamics affect derivative prices. This is the reason why we do not specify the relation between the original and the risk-adjusted probability measures.

Once the martingale measure with the bank account as numeraire, $\mathcal{Q}$, is fixed so is the martingale measure with the stock as numeraire, $\mathcal{Q}'$, and under $\mathcal{Q}'$ the stock evolves according to

$$\frac{dS(t)}{S(t-)} = \left(r - q - k\lambda + \sigma^2\right)dt + \sigma dW'(t) + I'(t)dN'(t) \tag{48}$$

where

$\{W'(t) = W(t) - \sigma t\}_{t\geq 0}$ is a standard Brownian.

[10] The strict positivity is essential because we are going to consider American call options, and these options will not be exercised prematurely in case of no dividends.

$\{N'(t)\}_{t\geq 0}$ is a Poisson process with intensity

$$\lambda' = \lambda(1+k) \tag{49}$$

$\{I'(t)\}_{t\geq 0}$ is a sequence of independent random variables with distribution

$$\ln\left(1+I'(t)\right) \underset{Q'}{\sim} \mathrm{N}\left(\gamma+\frac{1}{2}\delta^2, \delta^2\right) \tag{50}$$

for all t.

The processes W', N', I' are independent.

The paper "American Option Pricing in the Jump-Diffusion Model" presents three new results on the American option pricing problem in this type of setting:

i. Let $C(T,K)$ be the initial time 0 price of an American call option with strike K and expiration date T. Then the surface of these prices can be found as the solution to the partial integro differential equation (PIDE)[11]

$$\begin{aligned}(q+\lambda')C = &-C_T-(r-q-k\lambda)KC_K+\frac{1}{2}\sigma^2K^2C_{KK}\\ &+\lambda'\int_{-\infty}^{+\infty} C\left(T, Ke^{-\gamma-\delta^2/2+\delta z}\right)\phi(z)dz\end{aligned} \tag{51}$$

on

$$\{(T,K):\ C(T,K)>S(0)-K\} \tag{52}$$

subject to the free-boundary

$$C(T,K)\geq (S(0)-K)^+ \tag{53}$$

and the initial boundary condition

$$C(0,K)=(S(0)-K)^+ \tag{54}$$

ii. Let S_0^* be the solution to the implicit equation

$$S_0^* = K\frac{r+\lambda\left(1-\Phi\left(\frac{\ln(S_0^*/K)+\gamma}{\delta}-\frac{1}{2}\delta\right)\right)}{q+\lambda'\left(1-\Phi\left(\frac{\ln(S_0^*/K)+\gamma}{\delta}+\frac{1}{2}\delta\right)\right)} \tag{55}$$

when time tends to expiration of the option the limit of early exercise boundary for an American call option is given by

$$\max(K, S_0^*) \tag{56}$$

[11] $\phi(\cdot)$ is the standard normal density function.

iii. If the jumps are constant, i.e. $\delta = 0$, and the jumps are positive ($\gamma > 0$) then the early exercise boundary of a perpetual call option is given by

$$S^*_\infty = K\frac{1 - r\int\limits_0^\infty e^{-rv}\sum\limits_{n=0}^\infty \frac{e^{-\lambda v}(\lambda v)^n}{n!}\Phi\left(\frac{(r-q-k\lambda)v+n\gamma}{\sigma\sqrt{v}} - \frac{1}{2}\sigma\sqrt{v}\right)dv}{1 - q\int\limits_0^\infty e^{-qv}\sum\limits_{n=0}^\infty \frac{e^{-\lambda' v}(\lambda' v)^n}{n!}\Phi\left(\frac{(r-q-k\lambda)v+n\gamma}{\sigma\sqrt{v}} + \frac{1}{2}\sigma\sqrt{v}\right)dv} \tag{57}$$

and the price is given by

$$\begin{aligned} C(t) = {} & qS(t)\int\limits_0^\infty e^{-qv}\sum_{n=0}^\infty \frac{e^{-\lambda' v}(\lambda' v)^n}{n!}\Phi(d_n(v, S(t), S^*_\infty))dv \\ & - rK\int\limits_0^\infty e^{-rv}\sum_{n=0}^\infty \frac{e^{-\lambda v}(\lambda v)^n}{n!}\Phi\left(d_n(v, S(t), S^*_\infty) - \sigma\sqrt{v}\right)dv \\ d_n(u, x, y) = {} & \frac{\ln(x/y) + (r - q - k\lambda)u + n\gamma}{\sigma\sqrt{v}} \end{aligned} \tag{58}$$

If the jumps are constant and negative the early exercise boundary of the perpetual call option is given by[12]

$$S^*_\infty = K\frac{\beta}{\beta - 1} \tag{59}$$

where β is the positive solution to the equation

$$0 = \lambda(1+k)^\beta + \frac{1}{2}\sigma^2\beta^2 + \left(r - q - k\lambda - \frac{1}{2}\sigma^2\right)\beta - (r + \lambda) \tag{60}$$

and the price is given by

$$C(t) = \frac{S^*_\infty - K}{(S^*_\infty)^\beta}S(t)^\beta \tag{61}$$

The first result (i.) has some interesting applications. First, given parameters it is possible to generate a double continuum of American option prices by only solving one partial integro differential equation numerically. Second, if a surface of American option prices in the time-to-maturity and strike dimensions can be interpolated from a set of discrete market option prices then it is possible to directly infer the risk-adjusted parameters for this type of model. Unfortunately, this forward relationship is only valid if parameters are independent of time. So the forward equation can not be used to specify a model that exactly fits any surface of American option prices.

Using the put-call parity it is quite easy to verify that the limit, as time tends to expiration, of a early exercise boundary for an American call option has to be greater or equal to

$$K\max(1, r/q) \tag{62}$$

[12] In the paper we show that this equation has only one positive root and this root is strictly bigger than one.

in *any* model. To our knowledge van Moerbeke (1976) is the first to show that this is the exact limit in the Black-Scholes model. Our result (ii.) quantifies the limit of the early exercise boundary at expiration for the jump-model. In the paper we use numerical examples to demonstrate that this limit can be quite a lot bigger than it is the case for the Black-Scholes model. In fact, when $r \leq q$ and there is no discontinuity of the early exercise boundary at expiration for the Black-Scholes model, the early exercise boundary for the jump model might exhibit a quite large discontinuity at expiration of the option. The intuition behind this is that the jump component in the stock price implies that the uncertainty of the terminal pay-off does not vanish as time tends to expiration of the option. The result also illustrates that one should be careful about approximating the early exercise boundary of American options by the one of the Black-Scholes in the presence of jump risk.

The third result gives the prices of perpetual options under jump risk. These results can be generalized to the case of random sized, but uniformly signed, jumps. The second result (the negative jump case), is also found by Chesney (1995), that uses the result to device a numerical approximation for the American option. Another application of these results would be optimal capital structure problems in corporate finance where the firm value process is discontinuous due to technological innovations.

The last section of the paper considers numerical methods for the pricing of American options under jump risk. We derive a general finite difference grid that covers both the explicit scheme, used by Amin (1993) as well as implicit schemes. The implicit schemes are preferable because of their precision and stability properties. For the constant jump case we consider two alternative numerical procedures and their accuracy.

The paper "The Pricing of Discretely Observed Asian and Lookback Options: A Change of Numeraire Approach" shows that the change of numeraire approach can also be applied to reduce the complexity of Asian and lookback option pricing problems in the case of discontinuous dynamics of the underlying stock. As an example let us consider the discretely sampled Asian option considered in a previous section. As before define

$$x(t) = \frac{\frac{1}{n} \sum\limits_{i:t_i \leq t} S(t_i) - K}{S(t)} \tag{63}$$

With the discontinuous dynamics that we have assumed in this section we have that x evolves according to

$$\begin{aligned} dx(t) = &-(r - q - k\lambda)x(t-)dt - \sigma x(t-)dW'(t) \\ &- x(t-)\frac{I'(t)}{1+I'(t)}dN'(t) + \frac{1}{n}dm(t) \end{aligned} \tag{64}$$

where $m(\cdot)$ is the sampling point counter.

We see that x is again a Markov process under Q'. This implies that if we set $f = F/S$, then f is the solution to the PIDE

$$(q + \lambda')f = f_t - (r - q - k\lambda)xf_x + \frac{1}{2}\sigma^2 x^2 f_{xx} + \lambda' \int_{-\infty}^{+\infty} f\left(t, xe^{-\gamma-\delta^2/2+\delta z}\right)\phi(z)dz \tag{65}$$

on $\{(t,x) : t \notin \{t_1, \ldots, t_n\}\}$ subject to the boundary conditions

$$\begin{aligned} f(t_i-, x) &= f(t_i+, x + 1/n) \\ f(T, x) &= x^+ \end{aligned} \tag{66}$$

We solve the PIDEs numerically by use of implicit finite difference schemes similar to the one derived in "American Option Pricing in the Jump-Diffusion Model".

For parameters ensuring that the local second moment of the underlying is the same we compare the exotic option prices of the jump model with those of the standard Black-Scholes model. We generally find that the more peaked and fat-tailed distribution of the jump model implies that at-the-money options have lower prices and deep in-the-money and out-of-the-money options have higher prices than in the Black-Scholes framework.

"Implied" Modelling

Option prices on today's option markets often deviate considerably from the option prices implied by the Black-Scholes model. For standard European options it is not unusual to see implied Black-Scholes volatilities for the same maturity ranging from 10 to 30 per cent for different strikes.[13] The graph of implied volatilities as function of the strike is often termed the "smile", though the "smile" for equity options typically looks more like a "skew" with high implied volatility for low strikes and low implied volatility for high strikes. Pronounced smile effects have become increasingly common on equity option markets after the 1987 stock market crash and on the foreign exchange option markets after the European currency crisis in 1992. In other words, markets have put more (risk-adjusted) probability mass to extreme events after these shocks.

The smile effects have lead to the development of option pricing models that match the observed option smile. The main purpose of these models is to be able to price and hedge other options in accordance with the marketed option prices. Thereby the title of this chapter: models implied by market prices.

Breeden and Litzenberger (1978) are among the first to notice, or at least to quantify, the information content of observed option prices. Breeden and Litzenberger note that if $C(T, K)$ is the current price of a call option with maturity T and strike K then

$$\frac{\partial^2 C(T,K)}{\partial K^2} \tag{67}$$

[13] By the term "implied (Black-Scholes) volatility" we mean the volatility that would make the Black-Scholes formula produce the observed option price.

is the market state price density in the point $\{S(T) = K\}$. Put differently, a butterfly spread around a certain strike approximates the state price in that point.[14] The state prices density is sufficient to price all types of European style option with the same maturity and the curve of options of different strikes is sufficient to statically hedge it. However, the state price density, not even of all maturities, is not sufficient to price for example American style or barrier type of options. For this purpose we need a model for the dynamic evolution of the underlying stock.

The simplest continuous-time model, that (potentially) has sufficient degree of freedom to match a full surface of option prices in the time-to-maturity and strike dimensions, is

$$\frac{dS(t)}{S(t)} = (r(t) - q(t))dt + \sigma(t, S(t))dW(t) \tag{68}$$

where W is a Q-Brownian motion, $r(\cdot), q(\cdot)$ are respectively the deterministic time-dependent interest rate and dividend yield, and $\sigma(\cdot,\cdot)$ is the local volatility, a deterministic function of time and current spot.

Such a model can not be an appropriate model for the long term stock price behavior. Suppose for example that σ is monotonic in S. If we then consider the stock price evolution under two different equivalent probability measures, the exponential structure of the stock price SDE, implies that the stock price volatility will either vanish or explode as time tends to infinity under at least one of the probability measures. However, for short term horizons the model is able to capture important stylized facts of equity markets, such as for example that when the stock price drops the volatility rises.

Dupire (1993a) shows that the relation between the surface of European call options and the local volatility is given by the forward PDE[15]

$$0 = \left[-qC - C_T - (r-q)KC_K + \frac{1}{2}\sigma^2 K^2 C_{KK} \right](T, K) \tag{69}$$

subject to $C(0, K) = (S(0) - K)^+$.

Given a full surface of option prices, the current forward interest rates and forward dividend yields, it is possible to back out a unique local volatility surface supporting the option prices, by plugging the derivatives into the above formula. It is not immediately clear that *any* smooth surface of option prices consistent with absence of static arbitrage can be supported by a valid local volatility function, but Andersen (1996) shows that this is actually the case.

Rubinstein (1994) and Derman and Kani (1994) go along the same lines as Dupire, but in a discrete-time (binomial) setting.

It should be mentioned that the forward equation is also valid if the interest rate and the dividend yield are allowed to depend on the current spot and it would in principle be possible to come up with interest rates and dividend yields functions that made it possible to match at least some types of option smiles. But first, volatility rather than

[14] A butterfly spread around the strike K is a position consisting of long one option with strike $K + \Delta K$ and one option with strike $K - \Delta K$, and short two options with strike K.

[15] In order to avoid confusion it should be stressed that when we write $\sigma(T, K)$ we simply mean $\sigma(t, S(t))|_{t=T;S(t)=K}$.

drift seems to be the most important factor for short term options so such an approach does not seem to be a sound direction for equity, currency and commodity option pricing. Secondly, it would lead to grotesque models where for example the interest rate would be a direct function of a stock index (and the other way around).

In the paper "Implied Modelling: Stable Implementation, Hedging and Duality" it is noted that the forward equation can be differentiated with respect to the initial stock price in which case we get

$$\begin{aligned} q\Delta &= -\Delta_T - (r-q)K\Delta_K + \frac{1}{2}\sigma^2 K^2 \Delta_{KK} \\ \Delta(0,K) &= \mathbf{1}_{K \leq S(0)} \end{aligned} \tag{70}$$

where

$$\Delta(T,K) = \frac{\partial C(T,K)}{\partial S(0)} \tag{71}$$

is the initial hedge ratio of an option with maturity T and strike K. Further we have that[16]

$$\begin{aligned} q\Gamma &= -\Gamma_T - (r-q)K\Gamma_K + \frac{1}{2}\sigma^2 K^2 \Gamma_K \\ \Gamma(0,K) &= \delta(S(0)-K) \end{aligned} \tag{73}$$

where

$$\Gamma(T,K) = \frac{\partial^2 C(T,K)}{\partial S(0)^2} \tag{74}$$

is the initial "gamma" of a call option with maturity T and strike K. The gamma of an option represents the sensitivity of the replicating portfolio to a change in the underlying stock price. Staying within the lingo of option traders (or modern Greek) there is also a forward relation for the "Vegas" of the options

$$\begin{aligned} q\Psi - \sigma K^2 C_{KK} &= -\Psi_T - (r-q)K\Psi_K + \frac{1}{2}\sigma^2 K^2 \Psi_{KK} \\ \Psi(0,K) &= 0 \end{aligned} \tag{75}$$

where we define

$$\Psi(T,K) = \lim_{\epsilon \to 0} \frac{C(T,K)|_{\{\sigma(t,S)+\epsilon\}} - C(T,K)}{\epsilon} \tag{76}$$

This quantity can be interpreted as the marginal effect on option prices if we perform a small parallel shift of the full local volatility surface. The Vega is of course to some extend "inconsistent" with the model, where we have assumed that volatility only

16 $\delta(\cdot)$ is here the Dirac delta function, that has the formal properties

$$\begin{aligned} \delta(x) &= 0 \,, x \neq 0 \\ \int_{-\epsilon}^{\epsilon} \delta(x)dx &= 1 \end{aligned} \tag{72}$$

for any $\epsilon > 0$.

changes as time evolves or the stock price changes. However, the Vega is important because it reflects sensitivity to misspecification of the model.

Using the "Greek" forward equations one can generate the partial sensitivities for *all* marketed options by only solving *one* PDE numerically, and if the local volatility surface is calibrated to match observed option prices, the resulting "Greeks" are implied by the option prices themselves. This is clearly advantageous to the conventional alternative that would be to obtain the Greeks from the standard backward PDEs. Doing so would require the numerical solution of at least one PDE for each marketed option.

The forward PDEs also reveal an interesting duality. That is, if we revert time, and let $S(0)$ be a known quantity, and invent a backward running Brownian motion Z and let the "strike" follow the diffusion

$$\begin{aligned} \frac{dK(t)}{K(t)} &= (r-q)d(-t) + \sigma(t, K(t))dZ(-t) \\ K(T) &= K \end{aligned} \tag{77}$$

then the forward equations are the *backward* equations related to the initial value problems

$$\begin{aligned} C(T,K) &= \mathrm{E}\Big[e^{-qT}(K(0)-S(0))^{+}|K(T)=K\Big] \\ \Delta(T,K) &= \mathrm{E}\Big[e^{-qT}\mathbf{1}_{K(0)<S(0)}|K(T)=K\Big] \\ \Gamma(T,K) &= \mathrm{E}\Big[e^{-qT}\delta(K(0)-S(0))|K(T)=K\Big] \end{aligned} \tag{78}$$

We have here that the European call option pricing problem can be represented as a put option pricing problem where time is reverted, interest rates and dividend yields are swapped, the strike is the underlying, and the underlying is the strike. Secondly we obtain integral representations for two differentials. At a first sight this looks as a trivial consequence of a change of numeraire, however this is *not* the case due to the reversed time scale and the fact that the strike and the spot change places in the volatility term of the "underlying".

For the numerical implementation Dupire (1993a), Rubinstein (1994), and Derman and Kani (1994) apply binomial or trinomial trees, i.e. explicit schemes. The explicit schemes have the drawback that they may give rise to stability problems: the node probabilities can become negative when one tries to fit an arbitrary local volatility function induced by observed option prices. This is particularly a problem for the binomial tree that is very poorly suited for fitting smiles of different maturities. To overcome this problem we apply an implicit finite difference scheme for the numerical solution. The implicit finite difference scheme is unconditionally stable and convergent, see for example Mitchell and Griffiths (1980). We convert the observed bid and ask option prices into implied bid and ask volatilities and fit a smooth implied volatility surface in between these discrete points. We then use a closed form relation between local and implied volatility, derived in the paper, to convert the implied volatility surface into a local volatility surface. Using the implicit finite difference scheme we can now solve for example American and exotic option prices. We can also use the implicit scheme and the forward equation to back out the "Greeks" implied by the option prices themselves.

In the paper we apply this technique to a "snapshot" of actual prices on S&P 500 index option prices. All in all we have a table of input option prices of 21 strikes and 3 different maturities, all observed at the same time. We find that the implied risk-neutral density is dramatically left skewed and peaked.[17] The smile, or rather the "skew", of local volatility surface is much more pronounced than the one of the implied volatilities. This causes the implied Deltas and Gammas to deviate considerably from what is implied by combining the Black-Scholes model and the option's implied volatilities. For the Vega and for other sensitivities the effect is not that pronounced. The call options Deltas are much lower for the implied model than for the Black-Sholes benchmark. The intuitive explanation of this is the following:[18] The option Delta can be approximated by

$$\frac{1}{2\Delta S}[C(S+\Delta S) - C(S-\Delta S)] \tag{79}$$

When the stock rises, the option price increases but since volatility at same time falls the option prices does not increase as much as would be the case in the Black-Scholes model. On the other hand when the stock falls the option price drops less than under the Black-Scholes model because the volatility rises. This explains why the implied model imply smaller hedge ratios than the Black-Scholes model when there is a volatility skew.

"Eight Ways to Skin a Cat"

As the title indicates, the paper "New Skin for the Old Ceremony: Eight Different Derivations of the Black-Scholes Formula" surveys different techniques to obtain the Black-Scholes formula. The paper serves as an account for the different techniques, applied in continuous-time finance over the past twenty years and it can be seen as an introduction to the area. It can also be seen as a tribute to the path-breaking work of Fisher Black and Myron Scholes. In our humble opinion, their main result, that contingent claims can be prices relative to existing securities, and their option pricing formula stand among the most important economic contributions of this century.

The paper gives six pure arbitrage based derivations:

i. The classical dynamic hedging argument by Black and Scholes (1973) that leads to the fundamental partial differential equation of derivative security prices. We show how Fourier transforms can be applied to constructively solve the PDE and obtain the Black-Scholes formula.
ii. We derive the martingale approach of Harrison and Kreps (1979) and Harrison and Pliska (1981), discuss the connection to the fundamental PDE, and obtain the Black-Scholes formula as a risk-adjusted expectation of the discounted pay-off.

17 This observation has been confirmed by more thorough empirical studies, see for example Jackwert and Rubinstein (1996).
18 This formulation is due to Derman et al (1996).

iii. A change of numeraire approach is introduced and two martingale measures with two different numeraires, the bank-account and the stock, are used to derive the Black-Scholes formula without calculating a single integral.
iv. The concept of pricing by arbitrage hinges on the ability to construct a self-financing portfolio that replicates the option pay-off. Following Carr and Jarrow (1990) we show that a *stop-loss start-gain strategy* replicates the option contract, but the strategy is *not* self-financing, and the initial value of required external financing exactly equals the Black-Scholes price minus the initial cost of the stop-loss start-gain strategy. This constitutes a fifth proof of the Black-Scholes formula.
v. In the spirit of Dupire (1993a) we use the forward Fokker-Planck equation to derive a *forward* PDE for the call option prices, where the variables are time to maturity and strike rather than time and spot as in fundamental (backward) PDE. The forward PDE reveals an interesting duality:[19] the call option can be priced in a dual economy where time is reversed, the call option becomes a put with strike equal to the current spot, the strike is the underlying, etc. Using this duality we derive the Black-Scholes formula and the option delta as respectively a put option price and a digital option price in the dual economy.
vi. We consider the binomial model of Cox, Ross and Rubinstein (1979) and introduce the change of numeraire approach in discrete-time. Using this approach we are able to derive the convergence of the binomial option pricing formula to the Black-Scholes formula in a few lines.

We give two derivations of the Black-Scholes formula in the context of economic equilibria:

vii. We show that the Black-Scholes formula is consistent with the continuous-time CAPM of Merton (1971).
viii. Like Rubinstein (1976), we show that the formula can be obtained in a single period, representative investor economy, where the aggregate endowment is lognormal distributed and the representative investor has constant relative risk-aversion.

A final section of the paper briefly discusses the stability of the Black-Scholes formula if imperfections, such as transactions costs and non-continuous trading, are present in the economy.

[19] See also the discussion in the previous subsection.

The Term Structure of Interest Rates

This section is split into four subsections. The first subsection reviews the relations between different rates and zero-coupon bonds and describes the general Heath, Jarrow and Morton (1992) (HJM) approach used in our papers on term structure modelling. The second subsection describes the continuous-time stochastic volatility term structure model of the paper "An Arbitrage Term Structure Model of Interest Rates with Stochastic Volatility", and the third subsection describes a general discretezation scheme applied in that paper. The section is concluded by a subsection on the modelling of several term structures of different currencies. This subsection is based on the papers "Pricing by Arbitrage in An International Economy" and "A Gaussian Exchange Rate and Term Structure Model".

The HJM Approach

The relations between zero-coupon bond prices and interest rates, forward rates, the spot rate, and the bank account are

$$
\begin{aligned}
R(t,T) &= -\frac{\ln P(t,T)}{T-t} \Leftrightarrow P(t,T) = e^{-(T-t)R(t,T)} \\
f(t,T) &= -\frac{\partial \ln P(t,T)}{\partial T} \Leftrightarrow P(t,T) = e^{-\int_t^T f(t,y)dy} \\
r(t) &= f(t,t) = R(t,t) = -\frac{\partial \ln P(t,T)}{\partial T}\bigg|_{T=t} \\
B(t) &= e^{\int_0^t r(u)du}
\end{aligned}
\tag{80}
$$

where

$P(t,T)$ is the time t price of a zero-coupon bond maturing at time T. The boundary condition is of course $P(T,T) = 1$, for all T.
$R(t,T)$ is the time t interest rate or yield-to-maturity of a zero-coupon bond maturing at time T.
$f(t,T)$ is the time t instantaneous forward rate for deposit in the interval $[T, T + dT]$.
$r(t)$ is the time t spot interest rate, hence the return of holding a bond or the bank account over the interval $[t, t + dt]$.

Here and in the following we will assume that the curve of zero-coupon bonds $\{P(t,T)\}_{T\geq t}$ can be observed for all times t.
The object is now to specify models with arbitrage consistent stochastic dynamics for the evolution of the term structure in order to be able to price and hedge contingent claims on the yield curve, such as for example options on coupon bonds and interest rates. What makes this more difficult than setting up a model for a single stock is that the term structure of zero-coupon bond prices, $\{P(t,T)\}_{T\geq t}$, can be interpreted as a

full continuum of derivative assets each promising a payment of \$1 at maturity. This means that any model of the term structure has to specified so that the zero-coupon bonds mature at par. But there is more to arbitrage consistent term structure modelling. Using the general pricing equation we get that

$$P(t,T) = \mathrm{E}_t^Q \left[e^{-\int_t^T r(u)du} \right] \tag{81}$$

From this it is seen that the risk-adjusted dynamics of the spot rate define the risk-adjusted dynamics of the *whole* term structure, and that it is hard to independently specify the risk-adjusted dynamics of two bonds with different maturities and avoid arbitrage.[20]

The classical approach, followed by Merton (1970), Vasicek (1977), Cox, Ingersoll and Ross (1985), Beaglehole and Tenney (1991), and many others, is to specify the evolution of the term structure through a specification of the spot rate dynamics under Q, either directly or through a specification of the short rate process under the original measure and equilibrium considerations that establish the relation between the original measure and the martingale measure Q.

A more direct approach was pioneered by Ho and Lee (1986) in discrete-time and later developed in a general continuous-time setting by Heath, Jarrow and Morton (1992). The idea is to represent the stochastic evolution of the term structure by a continuum of stochastic differential equations

$$\left\{ \begin{array}{rl} \frac{dP(t,T)}{P(t,T)} = & r(t)dt + \sigma(t,T)dW(t) \\ r(t) = & -\left.\frac{\partial \ln P(t,T)}{\partial T}\right|_{T=t} \end{array} \right\}_{0 \leq t \leq T} \tag{82}$$

where W is a d-dimensional Brownian motion under Q, and $\{\sigma(t,T)\}_{0\leq t\leq T}$ is a family of $(1 \times d)$-dimensional adapted processes. Given the initial term structure $\{P(0,T)\}_{T\geq 0}$ and a specification of the bond volatilities we have specified the term structure evolution.[21] In order to ensure that the zero-coupon bonds mature at par HJM show that it is sufficient to require that $\sigma(t,\cdot)$ is absolutely continuous and $\sigma(t,t) = \mathbf{0}'$ for all t, i.e.

$$\sigma(t,T) = -\int_t^T \sigma_f(t,y)dy \tag{83}$$

for some family of well-behaved $(1 \times d)$-dimensional processes $\{\sigma_f(t,y)\}_{0\leq t\leq y}$. In fact, HJM specify their modelling framework in terms of forward rates and obtain the

[20] Famous examples of models that violate this fact and run into circularity problems are Brennan and Schwartz (1979) and Schaefer and Schwartz (1987).

[21] Of course, the bond volatilities have to be sufficiently regular for the family of processes (82) to be well-defined. As discussed in the original HJM paper, a direct insertion of the bond volatilities of the Cox, Ingersoll and Ross (1985) model into (82) will lead to undefined bond price processes for some initial term structures, and setting $\sigma(t,T) = \text{constant} \cdot \ln P(t,T)$ will lead to explosive forward rates.

following risk-adjusted dynamics of the forward rates

$$df(t,T) = \left[\sigma_f(t,T)\int_t^T \sigma_f(t,y)'dy\right]dt + \sigma_f(t,T)dW(t) \tag{84}$$

The term structure evolution is now described by the family of stochastic differential equations

$$\left\{df(t,T) = \left[\sigma_f(t,T)\int_t^T \sigma_f(t,y)'dy\right]dt + \sigma_f(t,T)dW(t)\right\}_{0\le t\le T} \tag{85}$$

and given the initial forward rates $\{f(0,T)\}_{T\ge 0}$ and a specification of $\{\sigma_f(t,T)\}_{0\le t\le T}$ we have specified the term structure evolution under $\mathcal{Q}$.
If we have a term structure contingent claim that promises a cash flow of

$$\alpha(u) = \alpha(u, P(u,T_1), \ldots, P(u,T_n)) \tag{86}$$

at time $u \in [0,T]$ and a terminal payment of

$$\beta = \beta(P(T,T_1), \ldots, P(T.T_n)) \tag{87}$$

the claim will have a fair value of

$$F(t) = \mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T \frac{\alpha(u)}{B(u)/B(t)}du + \frac{\beta}{B(T)/B(t)}\right] \tag{88}$$

The advantage of this approach is that the model automatically fits the initial term structure and that the volatilities of the zero-coupon bonds can be specified so that our model matches the empirical local covariance structure of the yield curve.

The downside is that we seldom get a finite dimensional Markov representation of the term structure and thereby a pricing PDE. This means that all pricing has to be done by either Monte-Carlo simulations or non-recombining tree approximations. Monte-Carlo techniques do not apply to American or Bermudan style claims, and non-recombining tree-structures are due to their explosive nature not applicable for long maturity contracts. Despite these numerical intractabilities, this approach remains the current state-of-the-art, and the term structure models that we have considered in the papers "An Arbitrage Term Structure Model with Stochastic Volatility", "Pricing by Arbitrage in an International Economy", and "A Gaussian Exchange Rate and Term Structure Model" are all formulated within the HJM framework.

Stochastic Volatility

Some term structure derivatives might be equally sensitive to changes in the volatility of the yield curve as to changes in the shape and the level of the yield curve itself. A hedging strategy that only accounts for changes in the level and the shape of the yield curve might therefore exhibit considerable tracking errors for very volatility dependent contingent claims. This suggest two things. First that we consider hedging strategies that involve other interest rate volatility dependent claims such as for example yield options. Secondly, that we develop models that incorporate random changes in the volatility that are not direct functions of the level of the yield curve, so that we can compute arbitrage consistent hedge ratios in derivative instruments.

Most existing models that involve stochastic interest rate volatility, such as for example Longstaff and Schwartz (1992), Fong and Vasicek (1991) and Chen (1995), have the drawback that changing volatility is directly linked to changing level and shape of the yield curve. These models are therefore qualitatively equivalent to other multifactor models where a number of yields serve as instruments for the driving stochastic processes. In fact, all of the mentioned models can be nested in the general exponential affine model described by Duffie and Kan (1993).

In the paper "An Arbitrage Term Structure Model of Interest Rates with Stochastic Volatility" we suggest a model where the uncertainty is driven by a standard two-dimensional Brownian motion. The first Brownian motion only affects the level of the yield curve, whereas the second Brownian motion only directly influences the volatility of the yield curve. Hence we have that

$$\frac{dP(t,T)}{P(t,T)} = \mu(t,T)dt + \sigma(t,T)dW_1^{\mathcal{P}}(t) \tag{89}$$

where $\left(W_1^{\mathcal{P}}, W_2^{\mathcal{P}}\right)$ is a standard two-dimensional Brownian motion under the original measure $\mathcal{P}$ and $\{\mu(t,T)\}_{0\leq t\leq T}$, $\{\sigma(t,T)\}_{0\leq t\leq T}$ are families of processes adapted to the filtration generated by the two-dimensional Brownian motion, with the volatilities satisfying the condition $\sigma(t,T) \underset{t\uparrow T}{\rightarrow} 0$ for all T.

Since the yield curve is only directly affected by the first Brownian motion, the bond market is incomplete in itself and the set of possible martingale measures with the bank-account as numeraire is not a singleton. To fix the martingale measure, we assume that there in addition to the full continuum of zero-coupon bonds exists a market for continuously resettled futures contracts on the instantaneous proportional variance of the zero-coupon bonds. We let $V\left(t,\bar{t},T\right)$ be the time t futures price for delivery of the instantaneous variance of the bond with maturity T at time $\bar{t}$, i.e. the futures contract is on the quantity $\sigma\left(\bar{t},T\right)^2$.

Volatility risk is now traded which implies that the martingale measure is unique, and we have that

$$V\left(t,\bar{t},T\right) = \mathrm{E}_t^{\mathcal{Q}}\left[\sigma\left(\bar{t},T\right)^2\right] \tag{90}$$

By the martingale representation theorem there exists some family of (1×2)-dimensional processes, $\left\{\sigma_V\left(t,\bar{t},T\right)\right\}_{0\leq t\leq \bar{t}<T}$, so that

$$\frac{dV\left(t,\bar{t},T\right)}{V\left(t,\bar{t},T\right)} = \sigma_V\left(t,\bar{t},T\right)dW(t) \tag{91}$$

for $0 \leq t \leq \bar{t} < T$, where $W = (W_1, W_2)'$ is a standard two-dimensional Brownian motion under Q. Given the initial term structure, the initial structure of futures prices, $\{V(0,t,T)\}_{0 \leq t < T}$, satisfying the restriction $V(0,t,T) \underset{t \uparrow T}{\rightarrow} 0$, and a specification of the family $\{\sigma_V(t,\bar{t},T)\}$, the evolution of the term structure is now specified by the system

$$\begin{aligned} \frac{dP(t,T)}{P(t,T)} &= r(t)dt + \sigma(t,T)dW_1(t) \\ \frac{dV(t,\bar{t},T)}{V(t,\bar{t},T)} &= \sigma_V(t,\bar{t},T)dW(t) \\ r(t) &= -\frac{\partial \ln P(t,T)}{\partial T}\Big|_{T=t} \\ \sigma(t,T)^2 &= \begin{cases} V(t,t,T) & ,T > t \\ 0 & ,T = t \end{cases} \end{aligned} \tag{92}$$

So far we have relied on the existence of futures contracts on the local variance of the zero-coupon bonds. We will now show that these contracts can be replicated by a static hedge strategy in yield futures contracts. Consider a futures contract for delivery of the yield $R(\bar{t},T)$ at time $\bar{t}$. The time t settlement price for such a contract is given by

$$Y(t,\bar{t},T) = \mathrm{E}_t^Q[R(\bar{t},T)] \tag{93}$$

Hence we have that

$$(T-t)Y(0,t,T) = -\ln P(0,T) - \mathrm{E}^Q\left[\int_0^t r(u)du\right] + \frac{1}{2}\mathrm{E}^Q\left[\int_0^t \sigma(u,T)^2 du\right] \tag{94}$$

Differentiating with respect to t and rearranging yield

$$V(0,t,T) = 2Y(0,t,t) + 2\frac{\partial}{\partial t}[(T-t)Y(0,t,T)] \tag{95}$$

So the futures contract for delivery of the instantaneous variance of a bond can be replicated by a limiting position in yield futures contracts. The hedging argument is not unlike the one applied to forward contracts in Dupire (1993b). However, Dupire's analysis of forward contracts does not apply in a stochastic interest rate environment as the one considered here.

The paper proposes the following simple specification of the model

$$\sigma(t,\bar{t},T) = e^{-\kappa(\bar{t}-t)}\begin{pmatrix}\nu_1 \\ \nu_2\end{pmatrix} \tag{96}$$

and we obtain the following dynamics of the bond volatilities

$$\sigma(t,T)^2 = V(0,t,T)e^{x(t)} \tag{97}$$

where x follows the Ornstein-Uhlenbeck process

$$\begin{aligned} dx(t) &= (\pi(t) - \kappa x(t))dt + \nu_1 dW_1(t) + \nu_2 dW_2(t) \\ x(0) &= 0 \\ \pi(t) &= -\frac{1}{4}(\nu_1^2 + \nu_2^2)(1 + e^{-2\kappa t}) \end{aligned} \tag{98}$$

We see that this specification implies that the volatility curve of the bonds shifts parallely. Litterman, Scheinkman and Weiss (1991) provide some empirical support for this feature.

In order to be able to analyze the effect of stochastic volatility on bond option prices we specify the initial bond variance futures prices to be of the same form as the local bond variance in the extended Vasicek (1977) model.[22] We set

$$V(0,t,T) = \beta^2 \frac{\left(1 - e^{-a(T-t)}\right)^2}{a^2} \tag{99}$$

where a, β are constants, and we obtain the following representation of the spot interest rate in our model

$$\begin{aligned} dr(t) &= (\phi_r(t) - ar(t))dt - \beta\sigma_r(t)dW_1(t) \\ d\ln\sigma_r(t) &= \left(\frac{\pi(t)}{2} - \kappa\ln\sigma_r(t)\right)dt + \nu_1 dW_1(t) + \nu_2 dW_2(t) \\ \phi_r(t) &= \frac{\partial f(0,T)}{\partial T}\bigg|_{T=t} + \beta^2 \int_0^t e^{-2a(t-u)}\sigma_r(u)^2 du + af(0,t) \end{aligned} \tag{100}$$

We see that the model as expected reduces to the extended Vasicek model when the volatility of the volatility, $\left(\nu_1^2 + \nu_2^2\right)^{1/2}$, goes to zero. We also note that we actually have a Markov representation of the term structure by the triple (r, σ_r, ϕ_r). This is not surprising since we have defined the volatility structure of the bonds to be of the separable form identified and termed "quasi-Gaussian" by Jamshidian (1991) and further analyzed by Cheyette (1995) and Ritchken and Sankarasubrahmanyam (1995).

Using a general discrete approximation of the model, described below, we now compare prices of call options on zero-coupon bonds in the stochastic volatility model to option prices generated by an extended Vasicek model with similar parameters. As Heston (1993), that considers stock options, we find that the effect of stochastic volatility on option prices depends on the correlation between the underlying (in this case the bonds) and the stochastic volatility. The general result is that stochastic volatility increases at-the-money option prices. The intuition must be that stochastic volatility makes the distribution of the underlying more peaked and fat-tailed. Negative correlation decreases deep in-the-money call option prices and increases out-of-the-money call option prices. Positive correlation has the opposite effect. The intuition behind this is that the presence of correlation between the level and the volatility "skews" the distribution of the underlying in the direction of the sign of the correlation.

Discrete Term-Structure Models

We make use of a discrete versions of the continuous-time HJM modelling framework in the paper "An Arbitrage Term Structure Model of Interest Rates with Stochastic Volatility". In the following we summarize the main ideas behind the scheme.

22 See Hull and White (1990).

Consider a discrete economy where time is indexed by $\{t_k\}_{k=0,\ldots,N}$. At each time step, t_k, the current state is given by the k sequence $\{s_0, \ldots . s_{k-1}\}$ with $s_j \in \{1, \ldots, d+1\}$ for all j. Following the time step t_k there are $(d+1)$ new states each with conditional $\mathcal{Q}$-probability $\theta^s(t_k)$, $s = 1, \ldots, d+1$. The bond prices in these new states are related to the current bond prices by the relation[23]

$$P^s(t_{k+1}, t_n) = \frac{P(t_k, t_n)}{P(t_k, t_{k+1})} h^s(t_k, t_n) \quad .0 \leq k < n \leq N \tag{101}$$

for $s = 1, \ldots, d+1$. We impose the following conditions on the perturbation function $h^{\cdot}(\cdot)$:

$$\begin{aligned} &\sum_{s=1}^{d+1} h^s(t_k, t_n)\theta^s(t_k) = 1 \\ &h^s(t_k, t_{k+1}) = 1 \\ &h^s(t_k, t_n) > 0 \end{aligned} \tag{102}$$

for all s, k, n.

The first restriction ensures that the bond prices deflated by the (discrete-time) bank-account are martingales under $\mathcal{Q}$. The second restriction ensures that the bond prices mature at par and the third condition is an arbitrage condition that implies that bond prices remain positive.

We note that this is a very general specification that applies to all kinds of traded assets where a martingale restriction is satisfied. In the stochastic volatility paper we apply a similar scheme to volatility futures contracts and it also possible to apply the scheme to stocks and foreign exchange rates. When we apply the scheme to assets without a terminal boundary condition, the second condition above drops out.

Also note that we have not restricted the perturbation function $h^{\cdot}(\cdot)$ to be deterministic. It might depend on current or past realized prices in the tree along the path $\{s_0, \ldots, s_{k-1}\}$. The same goes for the risk-neutral probabilities, but for convergence reasons we usually set $\theta^s(t_k) = 1/(d+1)$ for all s, k.

The perturbation functions here can be interpreted as discrete Radon-Nikodym derivatives in the sense that

$$h^s(t_k, t_n)\theta^s(t_k) \tag{103}$$

is the conditional probability that state s occurs over the next time step under the probability measure with the t_n-maturity bond as numeraire, also denoted the "t_n-forward risk adjusted" probability measure.

In order to mimic the stochastic behavior of a continuous-time model with a d-dimensional Brownian motion driving the uncertainty, we have to specify the perturbation function appropriately. To do so, specify the $(d+1)$-dimensional vectors

[23] The superscripts here denote the realization of s_k. The dependence of the current state is ignored to reduce the notational complexity.

$\epsilon_1, \ldots, \epsilon_d$ so that

$$\begin{aligned}
&\sum_{s=1}^{d+1} \theta^s(t_k)\epsilon_{i,s}(t_k) = 0 \\
&\sum_{s=1}^{d+1} \theta^s(t_k)\epsilon_{i,s}(t_k)\epsilon_{j,s} = \begin{cases} 0 & , i \neq j \\ 1 & , i = j \end{cases}
\end{aligned} \tag{104}$$

for $i, j = 1, \ldots, d$.
Loosely speaking the vector

$$\epsilon_{\cdot,s}\sqrt{t_{k+1} - t_k} \equiv \left(\epsilon_{1,s}, \ldots, \epsilon_{d,s}\right)\sqrt{t_{k+1} - t_k} \tag{105}$$

is supposed to "approximate" a realization of an increment of the d-dimensional Brownian motion W. We therefore impose the above conditions. The first condition ensures that each of the ϵ_i's have zero mean and the second condition ensures that the ϵ_i's are uncorrelated and have unit variance. For $d = 2, \theta = 1/3$ an example of this is

$$\begin{aligned}
\epsilon_{\cdot,1} &= (0.0 \quad \sqrt{2.0}\,) \\
\epsilon_{\cdot,2} &= (\sqrt{1.5} \quad -\sqrt{0.5}\,) \\
\epsilon_{\cdot,3} &= (-\sqrt{1.5} \quad -\sqrt{0.5}\,)
\end{aligned} \tag{106}$$

Note that the vectors define symmetric points on the unit circle scaled by $\sqrt{d}$.
We now define the perturbation functions to be

$$h^s(t_k, t_n) = \frac{e^{\sigma(t_k, t_{n-1})\epsilon_{\cdot,s}(t_k)\sqrt{t_{k+1}-t_k}}}{\sum_v \theta^v(t_k)e^{\sigma(t_k, t_{n-1})\epsilon_{\cdot,v}(t_k)\sqrt{t_{k+1}-t_k}}} \tag{107}$$

Note that the denominator of the perturbation function is the $\mathcal{Q}$-mean of the numerator if we consider $\epsilon_{\cdot,\cdot}$ as a random vector. It might seem curious why we let the perturbation function for the t_n maturity bond depend on the t_{n-1} maturity continuous volatility. The reason is that this specification ensures that the bonds mature at par, that the discrete model remains right-continuous, and that the volatility of a bond decreases in "uniform" steps as time tends to the maturity.
If we define $\Delta t = t_{k+1} - t_k$, $\Delta x(t_k) = x(t_{k+1}) - x(t_k)$, and the discrete spot interest rate $R(t_k) = -\ln P(t_k, t_{k+1})/\Delta t$, Taylor expansion of the above equations yields

$$\begin{aligned}
\Delta \ln P(t_k, t_n) = &\left(R(t_k) - \frac{1}{2}\|\sigma(t_k, t_{n-1})\|^2\right)\Delta t + \sigma(t_k, t_{n-1})\bar{\epsilon}(t_k)\sqrt{\Delta t} \\
&+ \mathcal{O}\Big((\Delta t)^{3/2}\Big)
\end{aligned} \tag{108}$$

We see that first and second moment of the local evolution of the bond prices match the ones of the continuous-time model. If we had a finite dimensional Markov representation of the continuous-time model this would essentially be sufficient to guarantee that the price processes of the discrete model converge to the ones of the discrete model under the probability measure Q.[24] For the stochastic volatility model the convergence issue is an unresolved mathematical question but numerical examples indicate that the discrete model converges relatively fast.

The scheme is generally not recombining, not even when there exists a finite dimensional Markov representation of the continuous-time model. The non-recombining property has the obvious disadvantage that the number of nodes grows exponentially. On the other hand this also means that we need fewer time steps to cover a large number of points and that the scheme converges after a low number of time steps, typically 10–15 time steps are sufficient which we illustrate in the stochastic volatility paper. But of course the exploding nature makes the non-recombining tree poorly suited for handling long maturity claims.

One interesting application of the scheme is to combine it with Monte-Carlo simulations. If we consider term structure models without a finite dimensional Markov representation, we can not in general find the hedge ratios of a particular claim from pure Monte Carlo simulations because the diffusion of the claim is not given by a simple first order differential. The idea is now to take one step initially using the discrete scheme and then perform Monte-Carlo simulations from each of the resulting branches, possibly using the same realizations of the driving Brownian motion. In each of the branches at time t_1 we now have the price of the claim and approximations of the continuous-time hedge ratios can be obtained using simple matrix algebra.

The scheme can be seen as a generalization of the Ho and Lee (1986) binomial model, because Ho and Lee also use multiplicative perturbation functions. There are also certain similarities to the discrete schemes described in He (1990) and Heath, Jarrow and Morton (1990). Hua He also uses a $(d+1)$-multinomial process to mimic the evolution of a d-dimensional Brownian motion. However, He's approximation is not directly applicable to term structure models unless there is a finite-dimensional Markov representation and it is not necessarily arbitrage-free for all time-step length — the asset prices can go negative if the time-steps are long or the volatility is high. Our scheme, though, is applicable to any term structure model and the multiplicative structure ensures that asset prices always remain positive. The HJM discretization scheme is also by nature additive rather than multiplicative, and the direct discretization of the forward rate PDEs means that HJM need to correct the evolutions in order to ensure absence of arbitrage. Also, HJM use a splitting index of 2^d to approximate a d-dimensional Brownian motion which means that the discrete model becomes dynamically incomplete for $d > 1$.

Multicurrency Term Structure Modelling

The papers "Pricing by Arbitrage in An International Economy" and "A Gaussian Exchange Rate and Term Structure Model" examine the modelling of term structures in different currencies. The purpose is here to construct models that can be used for the pricing of exchange rate derivatives under interest rate risk and contingent claims written on bonds of different currencies, such as for example cross-currency swaptions.

[24] See for example Ethier and Kurtz (1986).

In the following the notation is the same as in the previous except that we let subscript i indicate that the considered quantity is of currency i where $i \in \{0, 1\}$. We will denote currency 0 "domestic currency" and currency 1 is "foreign currency". We suppose that the uncertainty in the economy consisting of the two countries is driven by a common d-dimensional Brownian motion $W^{\mathcal{P}}$ under the original probability measure $\mathcal{P}$. We let S be the exchange rate denoted in domestic currency units per foreign currency unit and we let σ_S be the proportional volatility term of the exchange rate, a $(1 \times d)$-dimensional process. We assume that the global economy is dynamically complete so that we can write

$$\mathrm{E}_t^{\mathcal{Q}_0}\left[\frac{x}{B_0(T)/B_0(t)}\right] = \frac{1}{S(t)}\mathrm{E}_t^{\mathcal{Q}_1}\left[\frac{x/S(T)}{B_1(T)/B_1(t)}\right] \tag{109}$$

for any time T measurable x for some unique martingale measures $\mathcal{Q}_0, \mathcal{Q}_1$. This relation implies that the relation between the martingale measures of the two different currencies must be given by the Radon-Nikodym derivative

$$\frac{d\mathcal{Q}_1}{d\mathcal{Q}_0} = \frac{B_1(t)}{B_0(t)}\frac{S(t)}{S(0)} \tag{110}$$

on the information at time t and it means that we can write the relation between the Brownian motions as

$$dW_1(t) = dW_0(t) - \sigma_S(t)'dt \tag{111}$$

Let η_i be the vector of nominal risk-premiums of currency i, in the sense that $dW_i = dW^{\mathcal{P}} + \eta_i' dt$. We can now write

$$\frac{dS(t)}{S(t)} = (r_0(t) - r_1(t))dt + (\eta_0(t) - \eta_1(t))dW_0(t) \tag{112}$$

If we suppose that we could measure the nominal risk-premium vector on each market, we would be able to fix the local covariance between the financial assets of each market and the exchange rate, and thereby to a large extend also be able to fix the stochastic evolution of the exchange rate.

After applying the HJM framework to each of the two bond markets and using the above we obtain the following modelling framework of the "global" bond market

$$\begin{aligned}
\frac{dP_0(t,T)}{P_0(t,T)} &= r_0(t)dt + \sigma_0(t,T)dW_0(t) \\
\frac{dP_1(t,T)}{P_1(t,T)} &= r_1(t)dt + \sigma_1(t,T)dW_1(t) \\
&= \left(r_1(t) - \sigma_1(t,T)\sigma_S(t)'\right)dt + \sigma_1(t,T)dW_0(t) \\
r_i(t) &= -\frac{\partial \ln P_i(t,T)}{\partial T}\Big|_{T=t}
\end{aligned} \tag{113}$$

with the restriction $\sigma_i(t,T) \underset{t\uparrow T}{\rightarrow} 0'$. After specifying the processes $\{\sigma_0(t,T), \sigma_1(t,T)\}_{0\le t\le T}$ we have specified the evolution of the two yield curves.

Except for the explicit use of different martingale measures this is the international HJM formulation derived in Amin & Jarrow (1991). The application of different martingale measures has some advantages for the calculation of derivative prices. We can, for example, represent the time 0 price of a European call option on the exchange rate as

$$P_1(0,T)Q_1^T(S(T) > K) - KP_0(0,T)Q_0^T(S(T) > K) \tag{114}$$

where Q_i^T is the currency i martingale measure with the maturity T discount bond as numeraire, under which $dW_i^T(t) = dW_i(t) - \sigma_i(t,T)'dt$ defines a vector Brownian motion. If we constrain the volatility terms of the bonds and the exchange rate to be deterministic we almost immediately obtain the exchange rate option pricing formula of Amin (1990).

Even though the volatility terms are deterministic, we are not sure that the resulting model of the exchange rate and the bond markets have a finite dimensional Markov representation. This means that if we for example want to calculate the price of a coupon bond option with M remaining coupons we have to evaluate an M-dimensional integral. The paper "A Gaussian Exchange Rate and Term Structure Model" identifies a class of deterministic bond volatility specifications that admit a Markov representation with the same dimension as the driving Brownian motion. This volatility structure is termed *Gaussian*. Under the Gaussian volatility specification the joint system of the bond prices and the exchange rate can for each martingale measure $Q \in \{Q_i, Q_i^T\}_{i=0,1;T\geq t}$ be represented by

$$\begin{aligned} P_i(t,T) &= \frac{P_i(0,T)}{P_i(0,t)} e^{A_i(t,T;Q)+B_i(t,T)x^Q(t)} \\ S(t) &= \frac{S(0)P_1(0,t)}{P_0(0,t)} e^{C(t;Q)+D(t)'x^Q(t)} \\ x^Q(t) &= \int_0^t G(u)dW^Q(u) \end{aligned} \tag{115}$$

where A, C are deterministic functions defined by the volatility structure and the chosen martingale measure (Q), B, D are deterministic three-dimensional vector functions defined by the volatility structure, and G is a deterministic 3×3 matrix function defined by the volatility structure, and finally W^Q is a three dimensional Brownian motion under the chosen martingale measure Q.[25] Moreover, we have that the volatility structure is Gaussian *if and only if* for each martingale measure $Q \in \{Q_i, Q_i^T\}_{i=0,1;T\geq t}$, the joint process $(r_0, r_1, \ln S)$ is a vector Gaussian process.[26] This class of volatility structures is a natural international extension of the volatility structures considered in Babbs (1990) and Jamshidian (1991).

We now have that any European style contingent claim with a time T domestic currency value given by the function

$$F(S(T), P_0(T,T_1), \ldots, P_0(T,T_M), P_1(T,T_1), \ldots, P_1(T,T_M)) \tag{116}$$

[25] The exact specifications of A, B, C, D, G are given in the paper.

[26] Arbitrage considerations restrict the possible specifications of the vector Gaussian process under each of the martingale measures. For brevity we omit the discussion here and refer to the paper.

has a time 0 currency 0 value of

$$\begin{aligned} &P(0,T)\mathrm{E}^{Q_0^T}[F(\cdot)] \\ &= P(0,T)\mathrm{E}^{Q_0^T}\left[F\left(x^{Q_0^T}(T)\right)\right] \\ &= P(0,T)\int_{\mathbb{R}^3} F(z)p(z)dz \end{aligned} \tag{117}$$

where $p(\cdot)$ is the time 0 density of $x^{Q_0^T}(T)$ under Q_0^T, a three-dimensional normal density function. In other words most European style contingent claims written on the term structures of the two currencies and/or the exchange rate can be priced by numerically evaluating a three-dimensional integral which is clearly more efficient than evaluating the M-dimensional integral considered previously.

Finally, we show that it is possible to give a similar three-dimensional integral representation of a hedge portfolio for the claim considered in (116). This hedge portfolio has the desirable feature that it is based on the underlying instruments:

$$S(\cdot), P_0(\cdot, T_1), \ldots, P_0(\cdot, T_M), P_1(\cdot, T_1), \ldots, P_1(\cdot, T_M) \tag{118}$$

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American Option Pricing in the Jump-Diffusion Model

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Abstract

In this paper we consider the American option pricing problem in a risk-neutralized version of the jump-diffusion model of Merton (1976). We derive a forward equation for the American option pricing problem where the strike and maturity date are the variables whereas the current spot and time are fixed. Among other applications the forward equation can also be used for static estimation of risk-adjusted parameters from observed American option prices. We show that the early exercise boundary in the jump model exhibits a larger discontinuity at maturity of the option than it is the case under the Black-Scholes model. We quantify the discontinuity and illustrate its magnitude by numerical examples. For the constant jump case we derive closed form expressions for the price and the optimal exercise boundary of the perpetual American option. A final section of the paper describes a general finite difference method for solving option valuation problems when the underlying exhibits discontinuous dynamics. For the constant jump case we present two alternative approximations of the American option price. We compare the approximations by numerical examples.

Introduction

Whereas the American option pricing problem is well examined within the context of the Black-Scholes model, little attention has been devoted to the problem under other assumptions on the stock price behavior. In this paper we consider the American call option pricing problem in a jump-diffusion setting and present some new results in this setting. Our model of the stock price is a risk-neutralized version of the Merton (1976) model, i.e. we fix a martingale measure and under this probability measure we assume stock price dynamics like in the Merton model.

We show that under time-homogeneity, the American option pricing problem can be solved by solving a *forward* partial integro differential equation. In this equation current time and stock price are kept constant whereas the expiration date and the strike price are variables. The forward equation might be used for simultaneously solving a double continuum of option prices by only solving one partial integro differential equation. The forward equation may also be applied for estimating the parameters of the model from observed American option prices.

We derive the limit of the early exercise boundary when time tends to expiration of the option. Due to the discontinuous dynamics of the stock price this limit shows to be higher than the analogue of the Black-Scholes model, and for realistic parameters this difference is quite substantial. This means that close to expiration, investors might deter the exercise even though the option is far in-the-money.

In the constant jump size case Chesney (1995) derived the price and the early exercise boundary of the perpetual American call option when the jumps in the return are negative. In this paper we give closed form expressions for the perpetual option price and boundary when the jumps in the return are positive. The closed form solutions for the perpetual option might show useful for analyzing optimal capital structure problems in corporate finance. We also show that in the constant positive jump and finite maturity case the American option price can be represented in an analytical form, that depends on the early exercise boundary.

In a final section we describe a general finite difference algorithm for option pricing problems in a jump-diffusion setting. For the constant jump case we derive two alternative numerical approximations of the American option price: one for positive jumps and one for negative jumps. In the positive jump case the approximation is based on the analytical representation of the American option price obtained in this paper. In the other case, when jumps are negative, the approximation is similar to the one of Baron-Adesi and Whaley (1987) and has earlier been described by Chesney (1995). We compare the different approximations by numerical examples.

The paper is organized as follows: the first section presents the model and restates the European option pricing formula by Merton (1976) in a slightly different form. The second section considers the general American option pricing problem in the jump-diffusion model and gives the forward equation for the option pricing problem and the limiting result for the early exercise boundary. In the third section we consider the special case of constant jump size. We derive the price of the perpetual option and its early exercise boundary in this case. The last section presents some possible numerical procedures and approximations for the American option price.

The Jump-Diffusion Model and European Options

We consider an economy that contains two assets: a dividend paying stock and a money market account.

We choose to start by assuming the existence of a martingale measure, $\mathcal{Q}$, under which all discounted security prices including accumulated dividends are martingales. The existence of a martingale measure implies absence of arbitrage. Under the martingale measure we assume that the underlying stock price evolves according to the stochastic differential equation:

$$\frac{dS(t)}{S(t-)} = (r - q - k\lambda)dt + \sigma dW(t) + I(t)dN(t) \tag{1}$$

where

r is the constant positive interest rate.[1]
q is the constant positive dividend yield of the underlying.
σ is the constant continuous volatility component.
W is a standard Brownian motion.
N is a Poisson process with constant intensity λ and arrival times $\{u_i\}_{i=1,2,\ldots}$ given by $u_i = \inf\{t|N(t) = i\}$.
$\{I(t)\}_{t\geq 0}$ is a sequence of independent identically distributed random variables. Their distribution is given by:

$$\ln(1 + I(t)) \underset{\mathcal{Q}}{\sim} \mathrm{N}\left(\gamma - \frac{1}{2}\delta^2, \delta^2\right) \tag{2}$$

for all t.
k is the mean jump in the instantaneous return:

$$k = \mathrm{E}^{\mathcal{Q}}[I(t)] = \mathrm{e}^{\gamma} - 1 \tag{3}$$

The processes W, N, I are assumed independent.
For currencies q denotes the foreign continuously compounded interest rate.
The second asset of the economy, the money market account, has deterministic dynamics:

$$\begin{aligned} \frac{dB(t)}{B(t)} &= r dt \\ B(0) &= 1 \end{aligned} \tag{4}$$

The discontinuous dynamics of the stock price implies that the market is incomplete. In other words, given the dynamics under the original measure, the martingale measure, $\mathcal{Q}$, is non-unique. However, once the martingale measure, $\mathcal{Q}$, is fixed the original

[1] By the term *positive* here and in the following we mean strict positivity, i.e. $r, q > 0$.

probability measure has no influence on derivative pricing. The relation between the original measure and the martingale measure does of course effect portfolio and hedging decisions, but the object of this paper is purely the pricing issue so for the rest of the paper we will ignore this discussion and simply assume the $\mathcal{Q}$-dynamics outlined above.[2] In the next section we will hint at how one identifies the risk-adjusted dynamics from observed American option prices.

The stock price behavior can be represented in the form:

$$\begin{aligned} S(t) &= S(0)\mathrm{e}^{(r-q)t} M^c(t) M^d(t) \\ &= S(0)\mathrm{e}^{(r-q)t} M(t) \end{aligned} \tag{5}$$

where M^c is the continuous martingale:

$$M^c(t) = \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W(t)\right) \tag{6}$$

and M^d is the pure discontinuous martingale:

$$M^d(t) = \mathrm{e}^{-k\lambda t} \prod_{i=1}^{N(t)} (1 + I(u_i)) \tag{7}$$

Lastly we define the martingale:

$$M = M^c \cdot M^d \tag{8}$$

It will show useful to introduce a second equivalent martingale measure. We define the probability measure $\mathcal{Q}'$ by the Radon-Nikodym derivative:

$$\frac{d\mathcal{Q}'}{d\mathcal{Q}}\bigg|_{[0,t]} = M(t) \tag{9}$$

$\mathcal{Q}'$ is the martingale measure with the stock price as numeraire. That is, under $\mathcal{Q}'$ any asset price discounted by the stock price including accumulated dividends is a martingale, as opposed to under $\mathcal{Q}$ where any asset price discounted by the money market account is a martingale.

Using the Girsanov theorem we get that the stock price evolution under $\mathcal{Q}'$ can be described by the stochastic differential equation:[3]

$$\frac{dS(t)}{S(t-)} = \left(r - q - k\lambda + \sigma^2\right)dt + \sigma dW'(t) + I'(t)dN'(t) \tag{10}$$

where W' is the $\mathcal{Q}'$-standard Brownian motion:

$$W'(t) = W(t) - \sigma t \tag{11}$$

[2] For a relation between the dynamics under the original measure and the dynamics under the martingale measure based on a representative investor equilibrium argumentation, see Bates (1993).

[3] One possible reference for a Girsanov Theorem general enough to handle our situation is Dellacherie and Meyer (1982).

and N' is a Q' Poisson process with intensity:

$$\lambda' = \lambda(1+k) \tag{12}$$

$\{I'(t)\}_{t\geq 0}$ is a sequence of independent identically distributed random variables with distribution:

$$\ln\left(1+I'(t)\right) \underset{Q'}{\sim} \mathrm{N}\left(\gamma + \frac{1}{2}\delta^2, \delta^2\right) \tag{13}$$

for all t.

Conditioning on the number of jumps over the interval $[0,t]$, $N(t) = n$ respectively $N'(t) = n$, we have that:

$$\begin{aligned}\frac{S(t)}{S(0)} &= \exp\left(\left(r - q - k\lambda - \frac{1}{2}\sigma^2\right)t + \sigma W(t) + \sum_{i=1}^{n} \ln\left(1 + I(u_i)\right)\right) \\ &= \exp\left(\left(r - q - k\lambda + \frac{1}{2}\sigma^2\right)t + \sigma W'(t) + \sum_{i=1}^{n} \ln\left(1 + I'(u_i)\right)\right)\end{aligned} \tag{14}$$

From this we see that the stock price is conditionally log-normal under both martingale measures:

$$\begin{aligned}\left(\ln\frac{S(t)}{S(0)} | N(t) = n\right) &\underset{Q}{\sim} \mathrm{N}\left((r - q - k\lambda)t + n\gamma - \frac{1}{2}(\sigma^2 t + n\delta^2), \sigma^2 t + n\delta^2\right) \\ \left(\ln\frac{S(t)}{S(0)} | N(t) = n\right) &\underset{Q'}{\sim} \mathrm{N}\left((r - q - k\lambda)t + n\gamma + \frac{1}{2}(\sigma^2 t + n\delta^2), \sigma^2 t + n\delta^2\right)\end{aligned} \tag{15}$$

Using that

$$\begin{aligned}Q(N(t) = n) &= \frac{\mathrm{e}^{-\lambda t}(\lambda t)^n}{n!} \\ Q'(N(t) = n) &= \frac{\mathrm{e}^{-\lambda' t}(\lambda' t)^n}{n!}\end{aligned} \tag{16}$$

the distribution functions of the stock price under the two martingale measures can be expressed explicitly in terms of sums of normal distribution functions. From this we almost immediately obtain the Merton (1976) formula for the European call option.

Lemma 1: The European Call Option Price.

Define:

$$\begin{aligned}\nu_n(s)^2 &= \sigma^2 s + n\delta^2 \\ d_n(s,x,y) &= \frac{\ln(x/y) + (r - q - k\lambda)s + n\gamma}{\nu_n(s)} + \frac{1}{2}\nu_n(s)\end{aligned} \tag{17}$$

Consider a European call option with expiration T and strike price K. Define the time to maturity to be $v = T - t$. The fair time t price of the option at the state space point

(t, S) *is:*

$$
\begin{aligned}
c(t,S) = {} & S\mathrm{e}^{-qv}\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda' v}(\lambda' v)^n}{n!}\Phi(d_n(v,S,K)) \\
& - K\mathrm{e}^{-rv}\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda v}(\lambda v)^n}{n!}\Phi(d_n(v,S,K)-\nu_n(v))
\end{aligned} \tag{18}
$$

where $\Phi(\cdot)$ denotes the cumulated standard normal distribution function.

Proof:

Note that:

$$
\begin{aligned}
c(t) &= \mathrm{E}_t^{Q}\left[\mathrm{e}^{-r(T-t)}(S(T)-K)^+\right] \\
&= \mathrm{E}_t^{Q}\left[\mathrm{e}^{-q(T-t)}S(t)\frac{M(T)}{M(t)}\mathbf{1}_{S(T)>K}\right] - \mathrm{E}_t^{Q}\left[\mathrm{e}^{-r(T-t)}K\mathbf{1}_{S(T)>K}\right] \\
&= \mathrm{e}^{-q(T-t)}S(t)Q'_t(S(T)>K) - \mathrm{e}^{-r(T-t)}KQ_t(S(T)>K)
\end{aligned} \tag{19}
$$

Using the distributional properties in (15) and (16) yields the result.

□

Instead of representing the option price as a sum of Black-Scholes prices as in Merton (1976) we choose to represent the option price by a decomposition based on the probabilities under the two different martingale measures in play here. This gives a straightforward interpretation of the Merton (1976) formula: the terminal pay-off is split into two components and these two components are valued under two different martingale measures. The first component is an uncertain component dependent on the terminal stock price. By choosing the martingale measure with the stock price as numeraire for evaluating this component one offsets "the stock price risk". The second component is a fixed amount equal to the strike price. Since the quantity is fixed the proper martingale measure to apply for valuation is the one with the bank-account as numeraire. Also note the power of the change of numeraire technique: we derived the Merton formula without calculating a single integral.

We now turn to the American option pricing problem.

The American Option Pricing Problem

Due to the positive dividend yield it might be rational to exercise the American call option prematurely. Each exercise strategy can be represented by a stopping time. It is therefore possible to characterize the American option pricing as a stopping time problem, where the object is to find the stopping time that maximizes the value of the option. To formalize this let T be the expiration date of an American call option with strike K, and let $\mathcal{T}_{t,T}$ be the set of all stopping times on the interval $[t, T]$ for the stock price process (1). Then the value of the American call can be written:

$$
C(t) = \sup_{\tau\in\mathcal{T}_{t,T}} \mathrm{E}_t^{Q}\left\{\mathrm{e}^{-r(\tau-t)}(S(\tau)-K)^+\right\} \tag{20}
$$

For fixed maturity and strike we see that due to the Markovian properties of the stock price process C must be a function of current time and stock price only, i.e. we can write:

$$C(t) = C(t, S(t)) \tag{21}$$

It is natural to divide the state space into two sets: *the continuation region,*

$$\{(t, S)|C(t, S) > S - K\} \tag{22}$$

and *the early exercise region,*

$$\{(t, S)|C(t, S) = S - K\} \tag{23}$$

We will take the following result as preliminary to our analysis.

Lemma 2: Properties of the American Option Pricing Problem.

i. *The American option price is the unique solution to the partial integro differential equation (PIDE):*

$$(r + \lambda)C = \frac{\partial C}{\partial t} + (r - q - k\lambda)S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \lambda \hat{\mathrm{E}}^{Q}[C(t, S(1 + I))] \tag{24}$$

on the continuation region, subject to the terminal boundary condition:

$$C(T, S) = (S - K)^+ \tag{25}$$

and the free boundary condition:

$$C(t, S) \geq (S - K)^+ \tag{26}$$

$\hat{\mathrm{E}}^{Q}[\cdot]$ *is the Q expectation operator over I.*

ii. *C as a function of (t, S) is continuous and the first derivative of the option price with respect to the stock price, $\partial C/\partial S$, exists and is continuous for $t < T$.*

iii. *The early exercise and the continuation regions are connected sets and the early exercise region is closed. The early exercise boundary defined by the function:*

$$S^*(t) = \inf_S \{C(t, S) = S - K\} \tag{27}$$

is decreasing for $t \leq T$.

The results (i.)-(ii.) can be found in Mastroeni and Mulinacci (1996) and Zhang (1993). The appendix contains a proof of (iii.).

The PIDE is the discontinuous counterpart to the fundamental partial differential equation of Black-Scholes (1973). The second preliminary result is the high-contact or smooth pasting condition saying that $\partial C/\partial S = 1$ at the boundary of the early exercise region. The third result states that for each point in time there is a unique stock price, S^*, so that for $S \geq S^*$ the option should be exercised.

Due to the time-homogeneity of the model the American option pricing problem can also be solved by solving a *forward* partial integro differential equation. This is stated in the following result.

Result 1: The Forward Equation of The American Option Pricing Problem.

Without loss of generality set current time to be zero. Let S be the current stock price and let $C(T,K)$ be the current price of an American call with expiration date T and strike price K.
The function $C(T,K)$ is the unique solution to the PIDE:

$$(q+\lambda')C = -\frac{\partial C}{\partial T} - (r-q-k\lambda)K\frac{\partial C}{\partial K} + \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 C}{\partial K^2} + \lambda' \hat{\mathrm{E}}^{Q'}\left[C\left(T, \frac{K}{1+I'}\right)\right] \quad (28)$$

on the set:

$$\{(T,K)|C(T,K) > S-K\} \quad (29)$$

subject to the initial boundary condition:

$$C(0,K) = (S-K)^+ \quad (30)$$

and the free boundary condition:

$$C(T,K) \geq (S-K)^+ \quad (31)$$

$\hat{\mathrm{E}}^{Q'}[\cdot]$ *is the expectation operator over I' under Q'.*
The forward equation turns the American option pricing problem into a problem where the spot and the current time are fixed and the strike and expiration date are the variables whereas the opposite is the case for the backward equation. The forward equation has two applications. First, it is possible to obtain a double continuum in strike and maturity of option prices by only solving one forward equation numerically. Second, we observe that given a double continuum of observed option prices it is possible to back out the parameters implied by the option prices. In other words equation (28) admits a static estimation of the risk-adjusted parameters of the model. It is important to note though that the derivation of the forward equation depends crucially on the assumption of time-homogeneity. In case the parameters depend on time the forward equation will not be valid.

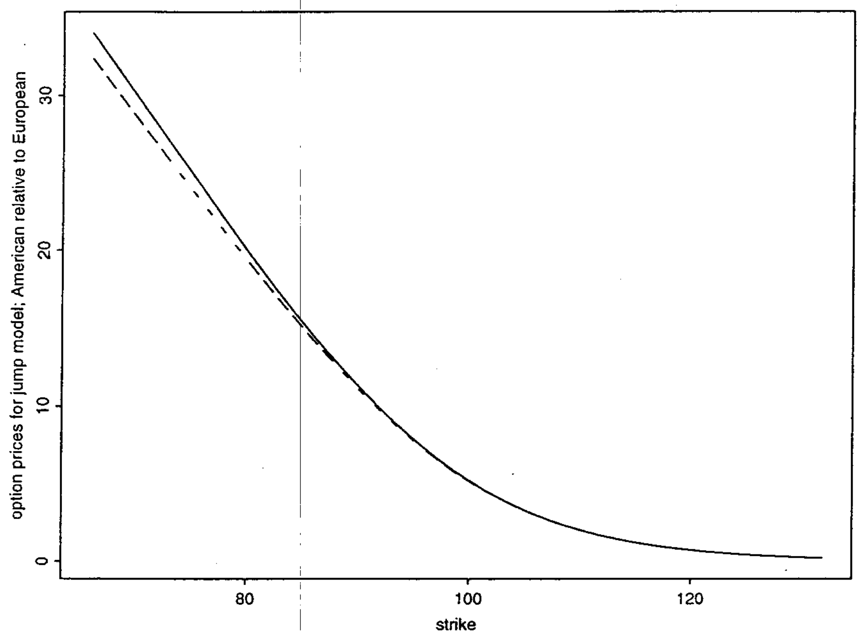
The forward equation without the free boundary conditions is also satisfied by the European option prices. This is easily seen from the proof in the appendix.

As an example of use of the forward equation, numerical solution of the forward PIDE yields the graphs of American and European option prices with time to maturity equal to one year shown in Figure 1. We refer the reader to the last section for the details on the numerical solution of PIDEs.

Note that the forward equation actually generates a full double continuum of American option prices for all maturity dates and strikes. So we could also show Figure 1 as a three dimensional graph.

It is natural to use the forward equation to generate the picture of critical *strike* prices, rather than critical stock prices, as function of the maturity date. That is, we fix current

Figure 1: Option Prices as Functions of Strike



Time 0 European (dashed line) and American (solid line) option prices as functions of strike generated by forward equation. Parameters: $r = 0.05, q = 0.05, \sigma = 0.1, \lambda = 1.0, \gamma = 0.0, \delta = 0.1, T = 1.0, S(0) = 100.0$.

time and stock price, (t, S), and for each maturity $u \geq t$ we find the maximal strike price $K^*(u)$ that solves

$$S - K^*(u) = C(t, S) \mid_{(T,K)=(u,K^*(u))} \tag{32}$$

Note that $K^*(\cdot)$ is a by-product when one numerically solves the free boundary problem of Result 1. Doing so we get the picture shown in Figure 2.

Figure 2 indicates a discontinuity of the critical strikes when time to maturity tends to zero. We will now show the existence of such a discontinuity and quantify its size.

It is well known that in the Black-Scholes model, the early exercise boundary in some cases exhibits a discontinuity at expiration of the option. Kim (1990) and van Moerbeke (1976) show that the limit of the early exercise boundary in the Black-Scholes is given by:

$$K \max\left(1, \frac{r}{q}\right) \tag{33}$$

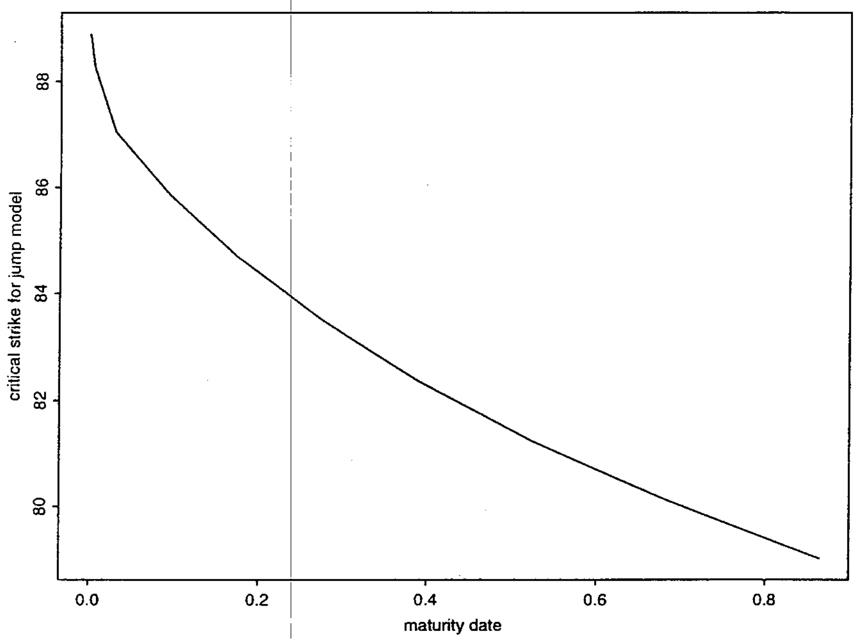
The following result gives the limit of the early exercise boundary for our model when time tends to expiration of the option.

Result 2: The Limiting Behavior of the Early Exercise Boundary.

Define S_0^* as the unique solution to the equation:

$$S_0^* = K \frac{r + \lambda\left(1 - \Phi\left(\frac{\ln(S_0^*/K)+\gamma}{\delta} - \frac{1}{2}\delta\right)\right)}{q + \lambda'\left(1 - \Phi\left(\frac{\ln(S_0^*/K)+\gamma}{\delta} + \frac{1}{2}\delta\right)\right)} \tag{34}$$

Figure 2: Critical Strike Price as Function of Time to Maturity



Critical strike price as function of maturity date seen from time 0. Parameters: $r = 0.05, q = 0.05, \sigma = 0.1, \lambda = 1.0, \gamma = 0.0, \delta = 0.1, S(0) = 100.0$.

The limit of the early exercise boundary when time tends to expiration is given by:

$$\lim_{t \uparrow T} S^*(t) = \max(K, S_0^*) \tag{35}$$

A proof is given in the Appendix.
From the put-call parity we have that:

$$\begin{aligned} &C(t,S) \geq c(t,S) \geq \mathrm{e}^{-q(T-t)}S - \mathrm{e}^{-r(T-t)}K \\ &\Downarrow \\ &S^* - K = C(t,S^*) \geq \mathrm{e}^{-q(T-t)}S^* - \mathrm{e}^{-r(T-t)}K \end{aligned} \tag{36}$$

Rearranging this and taking the limit yields:

$$S^* \geq K\frac{1-\mathrm{e}^{-r(T-t)}}{1-\mathrm{e}^{-q(T-t)}} \underset{t \uparrow T}{\longrightarrow} K\frac{r}{q} \tag{37}$$

But this is also satisfied for our result, since we have that:

$$S_0^* \geq K\frac{r}{q} \tag{38}$$

This can be seen by rearranging (33). We get:

$$\begin{aligned} &qS - rK \\ &= K\lambda\left(1 - \Phi\left(\frac{\ln(S/K)+\gamma}{\delta} - \frac{1}{2}\delta\right)\right) - S\lambda'\left(1 - \Phi\left(\frac{\ln(S/K)+\gamma}{\delta} + \frac{1}{2}\delta\right)\right) \\ &= \lambda\hat{\mathrm{E}}^Q\left[(K - S(1+I))^+\right] \end{aligned} \tag{39}$$

Note that the right hand side of the above is positive, decreasing, and concave in S and that the left hand side is positive for $S > Kr/q$. Hence S_0^* exists, is unique and (37) holds.

To give an intuitive argument why Result 1 holds, consider the pure jump case, i.e. $\sigma = 0$. Suppose that we are sitting a very small instant Δt before expiration of the option and that the current stock price is $S > K$. If we do not exercise the option we will with probability $\lambda \Delta t$ get the present value amount:

$$[S(1-(q+k\lambda)\Delta t + I) - (1-r\Delta t)K]^+ + \mathcal{O}(\Delta t^2) \tag{40}$$

and with probability $(1-\lambda\Delta t)$ we will get the present value amount:

$$S(1-(q+k\lambda)\Delta t) - K(1-r\Delta t) + \mathcal{O}(\Delta t^2) \tag{41}$$

This has current value:

$$\begin{aligned}
&\lambda \hat{\mathrm{E}}^{Q}\Big[(S(1+I)-K)^+\Big]\Delta t \\
&+ S\big(1-(q+\lambda')\Delta t\big) - K(1-(r+\lambda)\Delta t) + \mathcal{O}(\Delta t^2) \\
&= \lambda' S \Phi\left(\frac{\ln(S/K)+\gamma}{\delta} + \frac{1}{2}\delta\right) - \lambda K \Phi\left(\frac{\ln(S/K)+\gamma}{\delta} - \frac{1}{2}\delta\right) \\
&+ S\big(1-(q+\lambda')\Delta t\big) - K(1-(r+\lambda)\Delta t) + \mathcal{O}(\Delta t^2)
\end{aligned} \tag{42}$$

Setting this equal to the intrinsic value of the option, rearranging, and dropping terms of higher order than Δt yields equation (35) for the critical stock price. Of course the critical value has to be higher than the strike so we get (36) for the critical stock price. If we vary the mean jump size, i.e. γ, we get the limits of the early exercise boundaries and the option prices given in Table 1.

Table 1: Limit of the Early Exercise Boundary for Different Values of the Mean Jump Size

γ	S_0^*	C	c	σ_{BS}	C_{BS}	c_{BS}
-0.1	120.94	6.30	6.29	0.1761	6.74	6.68
0.0	112.48	5.24	5.20	0.1415	5.42	5.37
0.1	105.65	6.63	6.52	0.1704	6.52	6.46

Limits of early exercise boundaries, S_0^*, as function of mean jump size, γ; time 0 European, c, and American, C, option prices as functions of γ; corresponding Black-Scholes volatilites (see describtion below) and option prices. Parameters: $r = 0.05, q = 0.05, \sigma = 0.1, \lambda = 1.0, \delta = 0.1, T = 1.0, S(0) = 100.0,\ K = 100.0$.

In the jump-diffusion model the Q-variance of the log-stock price is given by:

$$\mathrm{var}_0^Q[\ln S(t)] = \left(\sigma^2 + \lambda\left(\left(\gamma - \frac{1}{2}\delta^2\right)^2 + \delta^2\right)\right)t \tag{43}$$

We will term the square root of this quantity $\sigma_{\text{BS}}\sqrt{t}$. Using this volatility we get the option prices under the Black-Scholes model given in Table 1.

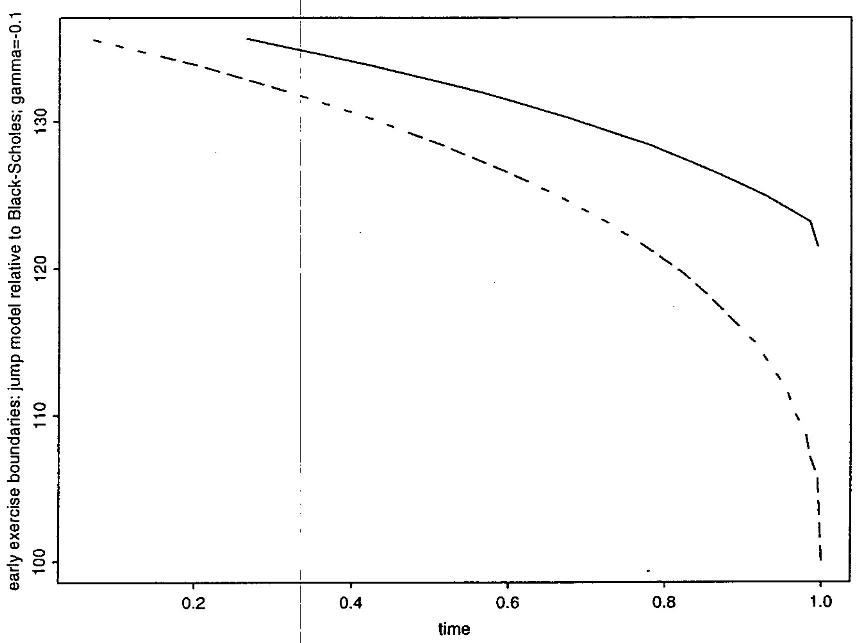
Noting that under Black-Scholes the limit of the early exercise boundary will in all of the three cases in Table 2 be equal to $K = 100.0$, we see that the differences of the limit for the jump model to the limit of the Black-Scholes are rather large, between 5 and 20 pct. We also note that this difference increases as the mean jump size decreases. Going back to the notion critical strike prices, we note the limit of the critical strike prices as time to maturity tends to zero is given by:

$$S(0)/S_0^* \tag{44}$$

For the case considered in Figure 2 we therefore get a limit of the critical strike price equal to 88.90, which is consistent with the picture shown in Figure 2.

Comparing the early exercise boundaries of the jump model to their Black-Scholes counterparts on the time scale $[0, 1]$ we get the pictures shown in Figure 3–5.

Figure 3: Early Exercise Boundary — Negative Mean Jump Size

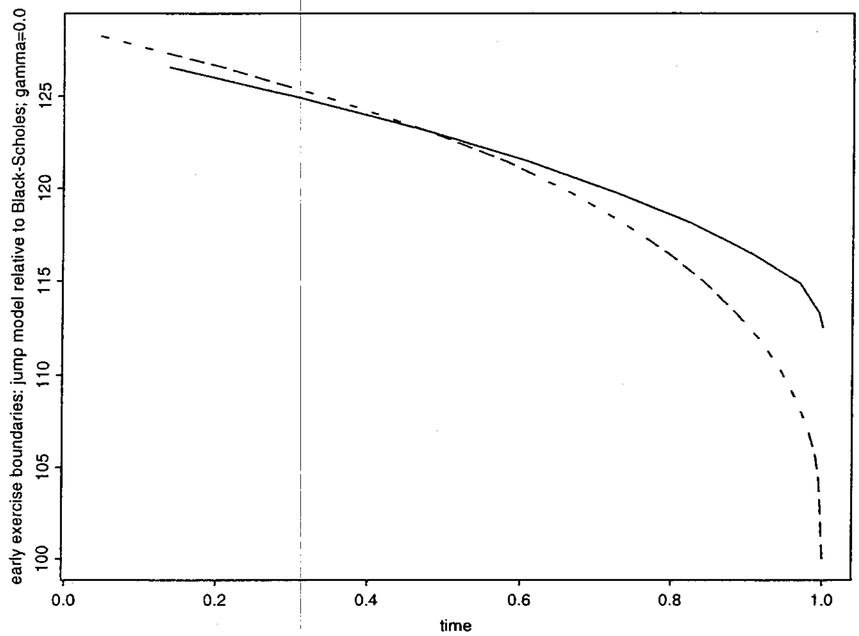


Early exercise boundary of jump model (solid line) compared to early exercise boundary of Black-Scholes model (dashed line). Parameters: $r = 0.05, q = 0.05, \sigma = 0.1. \lambda = 1.0, \gamma = -0.1, \delta = 0.1, T = 1.0, K = 100.0, \sigma_{\text{BS}} = 0.1761$

We note that the differences between the boundaries of the jump model and the ones of the Black-Scholes model are rather large. Bates (1993) argues that the Black-Scholes boundary might be used as an approximation of the jump model boundary. Figure 3–5 indicate that this might be a very poor approximation. Depending on the parameters of the model this might not have large consequences for the pricing of the options but it certainly does for the exercise decision.

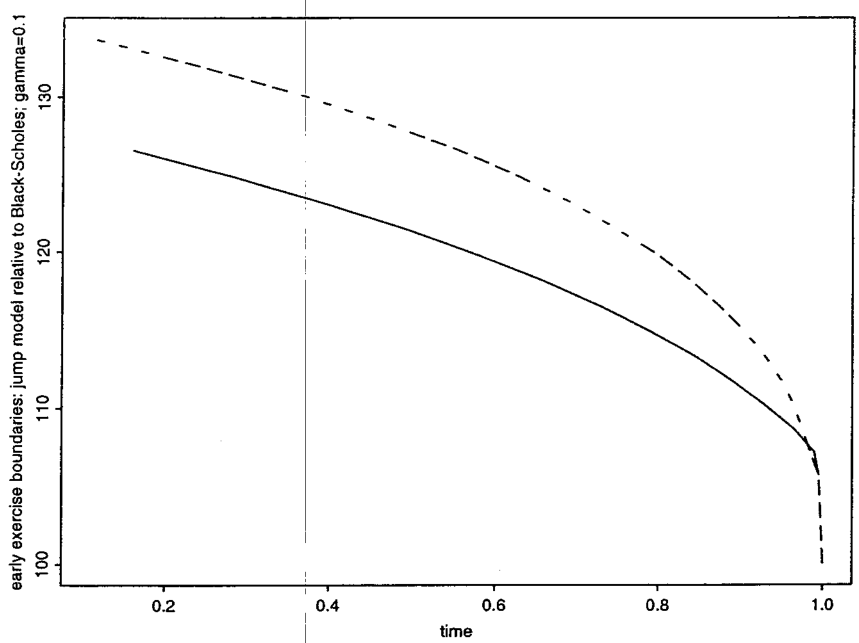
The next section considers a special case of the model namely when the jumps are constant. For this case we are able to obtain closed form solutions for the perpetual option and for the limit of the early exercise boundary.

Figure 4: Early Exercise Boundary — Zero Mean Jump Size



Early exercise boundary of jump model (solid line) compared to early exercise boundary of Black-Scholes model (dashed line). Parameters: $r = 0.05, q = 0.05, \sigma = 0.1, \lambda = 1.0, \gamma = 0.0. \delta = 0.1, T = 1.0, K = 100.0, \sigma_{\text{BS}} = 0.1415$.

Figure 5: Early Exercise Boundary — Positive Mean Jump Size



Early exercise boundary of jump model (solid line) compared to early exercise boundary of Black-Scholes model (dashed line). Parameters: $r = 0.05, q = 0.05, \sigma = 0.1, \lambda = 1.0, \gamma = 0.0, \delta = 0.1, T = 1.0, K = 100.0, \sigma_{\text{BS}} = 0.1704$

The Constant Jump Case

In this section we will assume that the jumps are constant i.e. $\delta = 0$, and thereby:

$$\frac{dS(t)}{S(t-)} = (r - q - k\lambda)dt + \sigma dW(t) + kdN(t) \tag{45}$$

The distributional properties of the model and the European option pricing formula are still valid with the minor modification that δ has to be set to zero in (2), (13), (15) and (17).

The partial integro differential equation (24) now reduces to:

$$(r + \lambda)C = \frac{\partial C}{\partial t} + (r - q - k\lambda)S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \lambda C(t, S(1 + k)) \tag{46}$$

The boundary conditions are of course still the same.

For the constant jump case we get a closed form expression for the limit of the early exercise boundary when time tends to expiration of the option.

Result 3: The Limiting Behavior of The Early Exercise Boundary in the Constant Jump Case.

In the constant jump case the limiting behavior of the early exercise boundary is given by:

$$\lim_{t \uparrow T} S^*(t) = K \max\left(1, \frac{r}{q}, \frac{r + \lambda}{q + \lambda'}\right) \tag{47}$$

A proof is given in the appendix.

Again, to gain insight in why this result holds consider the pure jump economy, i.e. $\sigma = 0$, at some time point just before expiration of the option. As in (42) we get the present value of keeping the option:

$$\begin{aligned} &\lambda(S(1 + k) - K)^+ \Delta t \\ &+ S\big(1 - (q + \lambda')\Delta t\big) - K(1 - (r + \lambda)\Delta t) + \mathcal{O}(\Delta t^2) \end{aligned} \tag{48}$$

Setting this equal to the intrinsic value and dropping terms of higher order than Δt yields the following equation for the critical stock price:

$$(q + \lambda')S - (r + \lambda)K = \big(\lambda' S - \lambda K\big)^+ \tag{49}$$

Since the dividend yield is positive the left hand side of this equation has a higher slope (in S) than the right hand side, and goes through 0 in the point $S = K(r + \lambda)/(q + \lambda')$. So this point is the unique solution if and only if

$$\frac{\lambda}{\lambda'}K \geq \frac{r + \lambda}{q + \lambda'}K \Leftrightarrow q \geq r(1 + k) \Leftrightarrow \frac{r + \lambda}{q + \lambda'} \geq \frac{r}{q} \tag{50}$$

If this is not the case the unique solution to (49) is bigger than $K\lambda/\lambda' = K/(1 + k)$ and equal to Kr/q. So we get what is stated in the result.

Setting $\delta = 0$ and varying γ we get the limits of the early exercise boundaries and the option prices shown in Table 2. We omit the trivial case $\gamma = 0$, where the model coincides with the Black-Scholes model.

We note that the limits are lower than for the model with random jump sizes.

In the case when the jumps are positive, the American call option has an analytical representation. Below we state this result.

Table 2: Limit of the Early Exercise Boundary for Different Values of the Constant Jump Size

γ	S_0^*	C	c	σ_{BS}	C_{BS}	c_{BS}
-0.1	109.97	5.29	5.27	0.1414	5.42	5.36
0.1	100.00	5.48	5.40	0.1414	5.42	5.36

Limits of early exercise boundaries, S_0^*, as function of mean jump size, γ; time 0 European, c, and American, C, option prices as functions of γ; corresponding Black-Scholes volatilites and option prices. Parameters: $r = 0.05, q = 0.05, \sigma = 0.1, \lambda = 1.0, \delta = 0.0, T = 1.0, S(0) = 100.0,\ K = 100.0$.

Result 4: Analytical Representation of the American Option.

Suppose the jumps are positive, i.e. $\gamma > 0$, then the American option price can be represented as:

$$\begin{aligned} C(t,S) = c(t,S) + qS \int_0^{T-t} \mathrm{e}^{-qv} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\lambda' v}(\lambda' v)^n}{n!} \Phi(d_n(v,S,S^*(t+v)))dv \\ - rK \int_0^{T-t} \mathrm{e}^{-rv} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\lambda v}(\lambda v)^n}{n!} \Phi\big(d_n(v,S,S^*(t+v)) - \sigma\sqrt{v}\big)dv \end{aligned} \tag{51}$$

A proof is given in the Appendix.

It is important to note that the decomposition of the American call price is only valid when the jumps are positive. The proof of the decomposition breaks down if the stock price can jump *out* of the early exercise region.

A similar decomposition can be derived for models where jumps are stochastic but only positive. As an example one could let $\ln I(t) \underset{Q}{\sim} \mathrm{N}(\cdot)$.

The decomposition can be used for numerical approximation of the price of the American call option with finite maturity when we have only positive jumps. The idea is to use (51) to obtain the early exercise boundary by a backward recursive procedure and then plug the early exercise boundary back into the pricing formula (51) to obtain the price of the American call. We will return to this in a subsequent section.

It is also worth noting that one could derive a decomposition like the one above with maturity and strike as the variables and the critical strike as function of maturity as the unknown. This would speed up the numerical calculations in the case where one wanted to calculate more than one American option price.

Now we turn to the perpetual case.

Due to the time-homogeneity of our model we have that the perpetual American option will only be a function of the current stock price and the optimal exercise boundary will be constant. In the following we give the prices and the early exercise boundaries for the American call option. The problem splits up into two cases: When jumps are positive and when they are negative.

Result 5: The Perpetual American Option.

i. *If the jumps are positive,* $\gamma > 0$, *we have that the early exercise boundary of the perpetual call is given by:*

$$S_{\infty}^{*} = K\frac{1 - r\int_{0}^{\infty} e^{-rv} \sum_{n=0}^{\infty} \frac{e^{-\lambda v}(\lambda v)^n}{n!}\Phi\left(\frac{(r-q-k\lambda)v+n\gamma}{\sigma\sqrt{v}} - \frac{1}{2}\sigma\sqrt{v}\right)dv}{1 - q\int_{0}^{\infty} e^{-qv} \sum_{n=0}^{\infty} \frac{e^{-\lambda' v}(\lambda' v)^n}{n!}\Phi\left(\frac{(r-q-k\lambda)v+n\gamma}{\sigma\sqrt{v}} + \frac{1}{2}\sigma\sqrt{v}\right)dv} \tag{52}$$

The perpetual American call price is given by:

$$\begin{aligned} C(S) = {} & qS\int_{0}^{\infty} e^{-qv}\sum_{n=0}^{\infty}\frac{e^{-\lambda' v}(\lambda' v)^n}{n!}\Phi(d_n(v,S,S_{\infty}^{*}))dv \\ & - rK\int_{0}^{\infty} e^{-rv}\sum_{n=0}^{\infty}\frac{e^{-\lambda v}(\lambda v)^n}{n!}\Phi\left(d_n(v,S,S_{\infty}^{*}) - \sigma\sqrt{v}\right)dv \end{aligned} \tag{53}$$

ii. *If the jumps are negative,* $\gamma < 0$, *then the early exercise boundary is given by:*

$$S_{\infty}^{*} = K\frac{\beta}{\beta - 1} \tag{54}$$

where β *is the positive solution to the equation:*

$$0 = \lambda(1+k)^{\beta} + \frac{1}{2}\sigma^2\beta^2 + \left(r - q - k\lambda - \frac{1}{2}\sigma^2\right)\beta - (r+\lambda) \tag{55}$$

The price of the perpetual call is given by:

$$C(S) = \frac{S_{\infty}^{*} - K}{(S_{\infty}^{*})^{\beta}}S^{\beta} \tag{56}$$

A proof is given in the appendix.

The result for negative constant jumps in the above result is also found by Chesney (1995).

It should again be noted that similar formulas could be derived for models with stochastic but exclusively positive or negative jumps.

It is rather surprising that the forms of the perpetual option price in these two cases are so different. It is worth noting though that in the case when the jump size and/or the jump intensity tend to zero the two solutions coincide. In fact the solution for the positive jumps tends to the Black-Scholes solution, and this solution can be integrated to give (56), as demonstrated by Kim (1990).

Let us now consider the difference between the perpetual option prices of the jump model with those of the Black-Scholes model. For the two cases, $\gamma = \pm 0.1$, we get the prices and the boundaries given in the table below.

Figure 6 shows the differences in the perpetual option prices for stock prices around at-the-money.

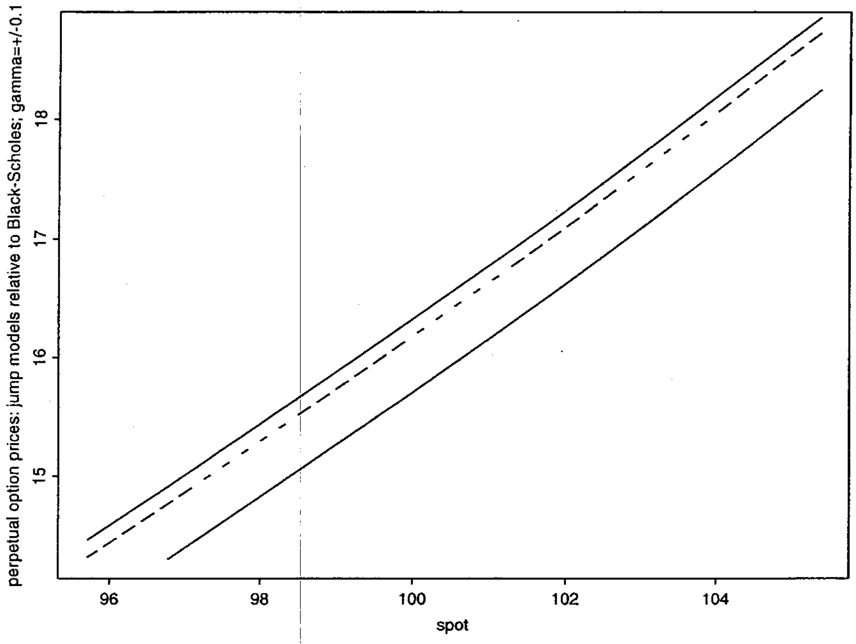
We note that even though the early exercise boundaries in the two cases $\gamma = \pm 0.1$ are very close, the prices of the perpetual options differ quite substantially in the at-the-money region.

Table 3: Perpetual Option Prices and The Early Exercise Boundaries

γ	C	S^*_∞	C_{BS}	$S^*_{\infty.\text{BS}}$
-0.1	15.71	153.78	16.19	155.83
0.1	16.33	153.66	16.19	155.83

Perpetual American option prices, C, and exercise boundaries, S^*_∞, as functions of γ; corresponding Black-Scholes perpetual American option prices. Parameters: $r = 0.05, q = 0.05, \sigma = 0.1, \lambda = 1.0, \delta = 0.0, T = 1.0, S(0) = 100.0,\ K = 100.0$.

Figure 6: Perpetual Option Prices Around At-The-Money



Perpetual option prices as function of current spot. Parameters: $\gamma = 0.1$ (top solid line), $\gamma = -0.1$ (bottom solid line), $\lambda = 0, \sigma_{\text{BS}} = 0.1414$ (dashed line), $r = 0.05, q = 0.05, \sigma = 0.1, \lambda = 1.0, \delta = 0.0, K = 100.0$.

Numerical Approximations

For the general case, when the jumps are random, the partial integro equation in (24) can be solved by finite difference techniques. In Amin (1993) this was done using an explicit finite difference approximation.[4] Below we describe a general finite difference method for solving PIDEs. As a special case it contains the explicit method by Amin (1993), but it also contains PIDE analogous to the Crank-Nicolson and the pure implicit approximation schemes. The main advantages of the implicit algorithms are that they are more likely to be stable and that their convergence in general are of higher orders. The basic idea is to approximate the derivatives and the integral in (24) by central differences and a Riemann sum respectively and thereby turn the partial integro differential equation

[4] Amin (1993) terms his approximation "a Markov chain approximation", because he approximates the continuous dynamics of the stock with a discrete process rather than discretizing the PIDE by explicit differences. However, the two approaches result in the same numerical valuation.

into a partial sum and difference equation. First we make the transformations:

$$\begin{aligned} x &= \ln S \\ y &= \ln(1+I) \end{aligned} \tag{57}$$

and redefine:

$$C(t,x) := C(t, \mathrm{e}^x) \tag{58}$$

The PIDE can now be expressed as:

$$\begin{aligned} &(r+\lambda)C = \\ &\frac{\partial C}{\partial t} + \left(r - q - k\lambda - \frac{1}{2}\sigma^2\right)\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial x^2} + \lambda\int C(t, x+y)\psi(y)dy \end{aligned} \tag{59}$$

where ψ is the normal density function:

$$\psi(y) = \frac{1}{\delta\sqrt{2\pi}}\exp\left(-\frac{\left(y - \gamma + \frac{1}{2}\delta^2\right)^2}{2\delta^2}\right) \tag{60}$$

We evaluate the PIDE at the point $(t+\theta\Delta t, x)$ for some fixed $\theta \in [0,1]$ and make the approximations:

$$\begin{aligned} C(t+\theta\Delta t, x) &\simeq (1-\theta)C(t,x) + \theta C(t+\Delta t, x) \\ \frac{\partial C(t+\theta\Delta t, x)}{\partial t} &\simeq \frac{1}{\Delta t}(C(t+\Delta t, x) - C(t,x)) \\ \frac{\partial C(t+\theta\Delta t, x)}{\partial x} &\simeq \frac{1-\theta}{2\Delta x}(C(t, x+\Delta x) - C(t, x-\Delta x)) \\ &+ \frac{\theta}{2\Delta x}(C(t+\Delta t, x+\Delta x) - C(t+\Delta t, x-\Delta x)) \\ \frac{\partial^2 C\left(t+\frac{\Delta t}{2}, x\right)}{\partial x^2} &\simeq \frac{1-\theta}{(\Delta x)^2}(C(t, x+\Delta x) - 2C(t,x) + C(t, x-\Delta x)) \\ &+ \frac{\theta}{(\Delta x)^2}(C(t+\Delta t, x+\Delta x) - 2C(t+\Delta t, x) + C(t+\Delta t, x-\Delta x)) \end{aligned} \tag{61}$$

The integral is approximated by:

$$\begin{aligned} &\int C\left(t + \frac{\Delta t}{2}, x+y\right)\psi(y)dy \simeq \\ &\sum_{h=-n}^{n}[(1-\theta)C(t, x+h\Delta x) + \theta C(t+\Delta t, x+h\Delta x)]\psi(h\Delta x)\Delta x \end{aligned} \tag{62}$$

for some number n. The choice of θ corresponds to different schemes: $\theta = 0$ corresponds to a pure implicit scheme, $\theta = 1/2$ corresponds to the Crank-Nicolson approximation, and $\theta = 1$ is equivalent to an explicit scheme.

Plugging these approximations into equation (59) and rearranging yields:

$$\sum_{h=-n}^{n} \alpha_h C(t, x + h\Delta x) = \sum_{h=-n}^{n} \beta_h C(t + \Delta t, x + h\Delta x) \tag{63}$$

with:

$$\begin{aligned}
\alpha_{-1} &= (1-\theta)\left(\frac{r - q - k\lambda - \frac{1}{2}\sigma^2}{2\Delta x} - \frac{\sigma^2}{2(\Delta x)^2} - \lambda \Delta x \psi(-\Delta x)\right) \\
\alpha_0 &= \frac{1}{\Delta t} + (1-\theta)\left(r + \lambda + \frac{\sigma^2}{\Delta x^2} - \lambda \Delta x \psi(0)\right) \\
\alpha_1 &= (1-\theta)\left(-\frac{r - q - k\lambda - \frac{1}{2}\sigma^2}{2\Delta x} - \frac{\sigma^2}{2(\Delta x)^2} - \lambda \Delta x \psi(\Delta x)\right) \\
\alpha_h &= -\lambda(1-\theta)\Delta x \psi(h\Delta x) \quad , |h| > 1 \\
\beta_0 &= \frac{1}{\Delta t} + \theta\left(-(r + \lambda) - \frac{\sigma^2}{(\Delta x)^2} + \lambda \Delta x \psi(0)\right) \\
\beta_h &= -\frac{\theta}{1-\theta}\alpha_h \quad , |h| > 0
\end{aligned} \tag{64}$$

We limit the state space to the grid:

$$\begin{aligned}
t_i &= i\Delta t \ , i = 0, \ldots, I; \ \Delta t = T/I \\
x_j &= \underline{x} + j\Delta x \ , j = 0, \ldots, J; \ \Delta x = (\overline{x} - \underline{x})/J
\end{aligned} \tag{65}$$

and supply the boundary conditions:

$$\begin{aligned}
C(t, x) &= e^x - K \ , x > \overline{x} \\
C(t, x) &= c(t, x) \ , x < \underline{x}
\end{aligned} \tag{66}$$

where the closed form solution of Lemma 2 is used for the European option price.

At each time point, t_i, we now have $(J+1)$ equations of the type (63) — one for each x_j. These equations can be arranged into a $(J+1) \times (J+1)$-dimensional matrix equation of the following type:

$$\mathbf{AC}(t_i) + \mathbf{a}(t_i) = \mathbf{BC}(t_{i+1}) + \mathbf{b}(t_{i+1}) \tag{67}$$

where $\mathbf{A}$ is a $(J+1) \times (J+1)$ matrix with rows consisting of the numbers:

$$(0, \ldots, 0, \alpha_{-n}, \ldots, \alpha_0, \ldots, \alpha_n, 0, \ldots, 0) \tag{68}$$

placed so that the element α_0 is on the diagonal. The matrix $\mathbf{B}$ is constructed in the same way but with elements taken from the β's. $\mathbf{C}(t_i)$ is the vector:

$$\mathbf{C}(t_i) = (C(t_i, x_0), \ldots, C(t_i, x_j), \ldots, C(t_i, x_J))' \tag{69}$$

and $\mathbf{a}(t_i), \mathbf{b}(t_{i+1})$ are the $(J+1)$-dimensional vectors with j'th elements:

$$\begin{aligned}
&\sum_{h \in \{|h| \le n | x_j + h\Delta x \notin [\underline{x}, \overline{x}]\}} \alpha_h C(t_i, x_j + h\Delta x) \\
&\sum_{h \in \{|h| \le n | x_j + h\Delta x \notin [\underline{x}, \overline{x}]\}} \beta_h C(t_{i+1}, x_j + h\Delta x)
\end{aligned} \tag{70}$$

By supplying the boundary conditions:

$$
\begin{aligned}
C(T, x_j) &= (\mathrm{e}^{x_j} - K)^+ \\
C(t_i, x_j) &\geq (\mathrm{e}^{x_j} - K)^+
\end{aligned}
\tag{71}
$$

the full grid option prices can be solved by backward recursion where one at each point in time solves the matrix system (67) and checks the free boundary condition. Zhang (1993) establishes the convergence of the option prices in a scheme like this.

Due to time-homogeneity the matrix $\mathbf{A}$ is the same at all time steps so it need only be inverted once whereafter the matrix equation (67) can be solved by multiplication. This makes the routine fairly fast. For the case $\theta = 1$ (the explicit method) the matrix $\mathbf{A}$ need not be inverted because it is proportional to the identity. This makes the explicit method marginally faster for a given number of state-space points. But the drawback of this method is that it is potentially unstable. Unless the choice of $\Delta t/(\Delta x)^2$ is bounded by a certain constant, the method will not converge or be stable. This is in fact true for all schemes with $\theta > 1/2$. Another way of speeding up the algorithm is to take two different weights for the differential approximations and for the integral approximation. One might for example set $\theta = 0$ on the differences and $\theta = 1$ on the sums. The matrix $\mathbf{A}$ would then be tridiagonal which would speed up the inversion and thereby the algorithm considerably. Combining this with an approximation of the integral that covers only a low number of points (say 10 points) would further increase the computational efficiency.

All prices and figures in the previous section were produced with $\theta = 1/2$ and

$$
\begin{aligned}
\underline{x} = \ln(S(0)/4),\ \ \overline{x} &= \ln(2S(0)) \\
I = J &= 200 \\
n &= 199
\end{aligned}
\tag{72}
$$

For constant jump sizes the grid is adjusted so that the jump falls on a grid point.

Instead of solving the backward PIDE (24) one might instead solve the forward PIDE (28). This is computationally advantageous if one has to calculate more than one option price. The same numerical technique applies to the forward PIDE.

Figure 7 illustrates the convergence of the finite difference scheme for the European option case for increasing $I = J = n + 1$. Parameters of the scheme are as above.

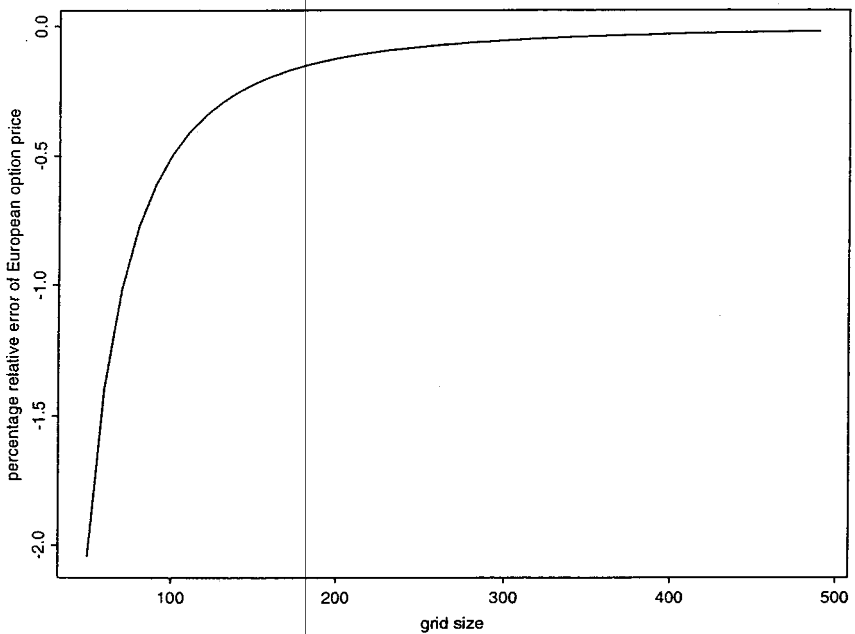
For the positive constant jump case there exists another approximation method. The idea is to use the decomposition in Result 4. First the time axis is divided by the time points:

$$
t_i = iT/I \ \ , i = 0, \ldots, I \tag{73}
$$

and the early exercise boundary is approximated by the sequence of discrete points:

$$
\{S_I^*(t_i)\} \tag{74}
$$

Figure 7: Relative Error of Finite Difference Solution for European Options



Relative percentage error of time 0 European option price as function of number of grid points in time and stock price dimension. Parameters: $r = 0.05, q = 0.05, \sigma = 0.1, \lambda = 1.0, \gamma = 0.0, \delta = 0.1, S = 100.0, K = 100, T = 1.0$.

The time integral in Result 4 is approximated by a right hand side Riemann sum and we get the following expression:

$$
\begin{aligned}
C(t,S) &\simeq C_I(t,S) \equiv c(t,S) \\
&+ qS \sum_{i:t<t_i} \mathrm{e}^{-(q+\lambda')(t_i-t)} \sum_{n=0}^{\infty} \frac{(\lambda'(t_i-t))^n}{n!} \Phi(d_n(t_i-t, S, S_I^*(t_i)))(t_i - t_{i-1}) \\
&- rK \sum_{i:t<t_i} \mathrm{e}^{-(r+\lambda)(t_i-t)} \sum_{n=0}^{\infty} \frac{(\lambda(t_i-t))^n}{n!} \Phi\big(d_n(t_i-t, S, S_I^*(t_i)) - \sigma\sqrt{t_i-t}\big)(t_i - t_{i-1})
\end{aligned}
\tag{75}
$$

By Result 3 the limit of the early exercise boundary at expiration is well-known so the approximate early exercise boundary can now be identified by backward recursion on the equation:

$$S_I^*(t_i) - K = C_I(t_i, S_I^*(t_i)) \tag{76}$$

starting at time t_{I-1}. Knowing the early exercise boundary the option price can be calculated using the approximation $C_I(t,S)$. One could also choose to approximate the time integral by a midpoint sum and linearly interpolate the early exercise boundary between the selected points. Another possible approximation is to use a trapezoid approximation. For simplicity we choose only to describe the right hand side approximation. It is worth noting that this approximation is approximating only the time integral. Therefore, one should expect that for a given set of time-points this approximation would be more precise than the finite difference approximation. Clearly this approximation will always converge as the number of time points I tends to infinity.

The following approximation is not a discretization but rather based on neglecting a term in the PIDE. It is therefore not clear how well this approximation works. Chesney (1995) suggests this approximation of the American option price for the negative constant jump case. The approximation has previously been applied in the Black-Scholes case by Baron-Adesi and Whaley (1987). The idea is the following. Since C and c satisfy the same linear PIDE on the continuation region so does $C - c$. Define $X(t) = 1 - \mathrm{e}^{-r(T-t)}$ and f by

$$Xf = C - c \tag{77}$$

then f solves:

$$(r + \lambda X)f = X\frac{\partial f}{\partial t} + (r - q - k\lambda)SX\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 X\frac{\partial^2 f}{\partial S^2} + \lambda X f(t, S(1+k)) \tag{78}$$

If we neglect the term $X\partial f/\partial t$ we obtain the following equation for f:

$$\left(\frac{r}{X} + \lambda\right)f = (r - q - k\lambda)S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} + \lambda f(t, S(1+k)) \tag{79}$$

A solution to this equation is

$$g(t)S(t)^{\beta(t)} \tag{80}$$

where $g(t)$ is a time-dependent function and $\beta(t)$ solves[5]

$$0 = \lambda(1+k)^{\beta(t)} + \frac{1}{2}\sigma^2\beta(t)^2 + \left(r - q - k\lambda - \frac{1}{2}\sigma^2\right)\beta(t) - \left(\frac{r}{X(t)} + \lambda\right) \tag{81}$$

defining:

$$C\left(t, \hat{S}(t)\right) = \hat{S}(t) - K \tag{82}$$

and imposing the high contact condition (see lemma 2(ii.)):

$$\frac{\partial C}{\partial S}\left(t, \hat{S}(t)\right) = 1 \tag{83}$$

we obtain the following approximation of the American call option price:

$$C_P(t, S) = c(t, S) + A(t)\left(S/\hat{S}(t)\right)^{\beta(t)} \tag{84}$$

where $\hat{S}(t)$ is the implicit solution to the equation:[6]

$$\hat{S}(t) - K = c\left(t, \hat{S}(t)\right) + \left(1 - \frac{\partial c\left(t, \hat{S}(t)\right)}{\partial S}\right)\frac{\hat{S}(t)}{\beta(t)} \tag{86}$$

---

5 Note that when $T \to \infty$, $\beta(t)$ tends to β of the perpetual option.

6 Note that

$$\frac{\partial c(t, S)}{\partial S} = \mathrm{e}^{-qv}\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda' v}(\lambda' v)^n}{n!}\Phi(d_n)$$

and $A(t)$ is given by:

$$A(t) = \left(1 - \frac{\partial c\left(t, \hat{S}(t)\right)}{\partial S}\right) \frac{\hat{S}(t)}{\beta(t)} \tag{87}$$

The curve $\left\{\hat{S}(t)\right\}$ approximates the early exercise boundary. Due to functional form of the approximating option price we will term this approximation the *power approximation.*

In Table 4 we compare the two alternative approximations to the finite difference solutions for at-the-money options for different dividend yields and jump intensities. The number of time points for the integral approximation is set to 100 and the finite difference method is applied with the parameters given in (72).

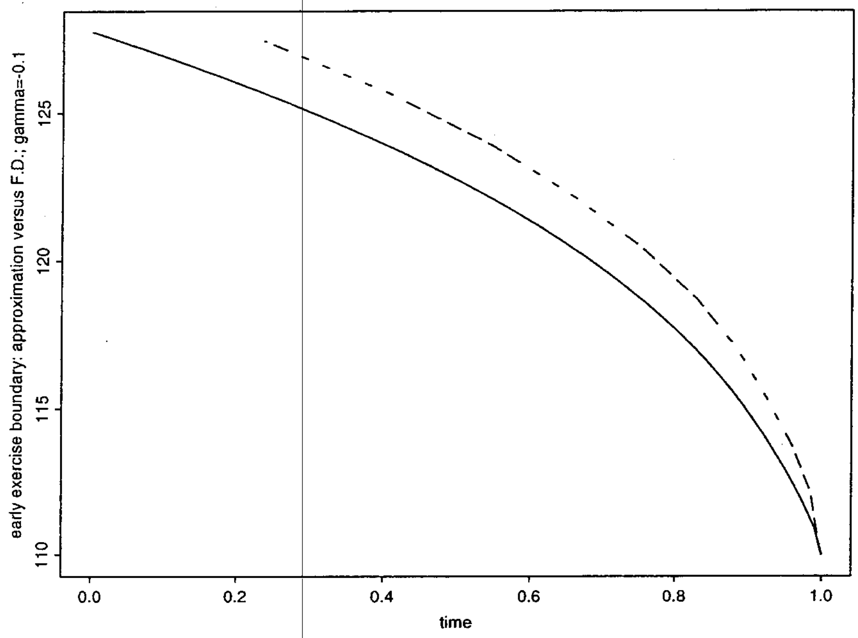
Table 4: The two Alternative Approximations Compared to Finite Difference

q	C_P $\gamma = -0.1$	C_{FD} $\gamma = -0.1$	C_I $\gamma = 0.1$	C_{FD} $\gamma = 0.1$
0.05	5.32	5.29	5.49	5.48
0.10	3.53	3.47	4.11	4.09
0.15	2.39	2.38	3.25	3.24
0.20	1.66	1.69	2.66	2.64
λ	C_P $\gamma = -0.1$	C_{FD} $\gamma = -0.1$	C_I $\gamma = 0.1$	C_{FD} $\gamma = 0.1$
0.5	4.63	4.60	4.72	4.71
2.0	6.50	6.47	6.77	6.76
5.0	9.17	9.13	9.62	9.61
10.0	12.39	12.34	13.02	13.00

Time 0 American option prices of power approximation, C_P, and integral approximation, C_I, compared to finite difference solution, C_{FD}, for different values of the dividend yield, q, and the jump-intensity, λ. Base case parameters: $r = 0.05, q = 0.05, \sigma = 0.1, \lambda = 1.0, \delta = 0.1, S = 100.0, K = 100, T = 1.0$.

We see that the deviations between the power approximation and the finite difference solution are rather large, with a maximum relative error of approximately 2 pct. Such magnitude of deviations might be too high for many applications. The time-integral approximation, on the other hand, is very close to the finite difference solution and the maximum relative deviation is less than 1 pct. The prices of the finite difference method seem to be a little bit lower than those of the time-integral approximation. Considering the higher theoretical precision of the time-integral approximation this might indicate that the finite difference approximation in our implementation slightly underprices the American options.

Figure 8: Early Exercise Boundaries — Power Approximation versus Finite Difference



Early exercise boundary of power approximation (solid line) compared to early exercise boundary of finite difference approximation (dashed line). Parameters: $r = 0.05, q = 0.05, \sigma = 0.1, \lambda = 1.0, \gamma = -0.1, \delta = 0.0, T = 1.0, K = 100.0$.

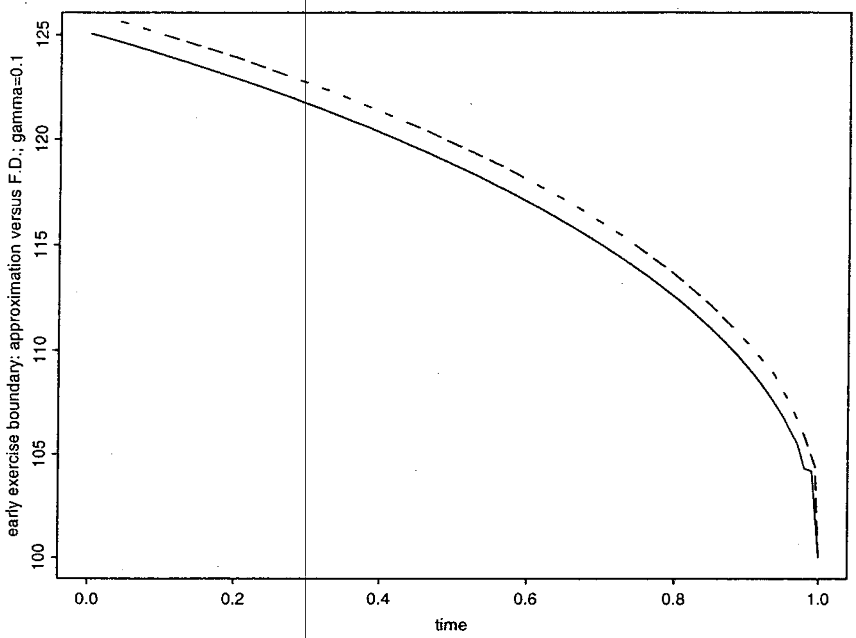
Figure 8 and 9 show the early exercise boundaries of the two alternative approximations compared to those of the finite difference solution.

The difference between the early exercise boundary of the power approximation and the finite difference solution is rather large and it seems to be increasing as time to maturity increases. The power approximation might therefore be too inaccurate for exercise decisions.

For the time-integral approximation we see that the two boundaries are fairly close and that the difference seems to be constant over time. The discrepancies might be attributed to the fact that the finite-difference approximation only considers discrete points in the stock price dimension and that on average the true boundary (in x terms) will lie $\Delta x/2$ below the boundary showed by the finite difference approximation. We note that there is a slight tendency of instability for the time-integral approximation for times close to maturity. In our implementation this was a persistent but not a serious problem for moderate jump intensities.

Regarding computer-time, the power approximation is much faster than the time-integral approximation which again in our implementation shows to be marginally faster than the finite difference approximation. But the finite difference approximation can also be used in the random jump-size case, and it applies to various other types of derivatives such as barrier options, lookbacks, and Asian options. We will return to the pricing of exotic options in a jump-diffusion context in another paper.

Figure 9: Early Exercise Boundaries — Time-Integral Approximation versus Finite Difference



Early exercise boundary of time integral approximation (solid line) compared to early exercise boundary of finite difference approximation (dashed line). Parameters: $r = 0.05, q = 0.05, \sigma = 0.1, \lambda = 1.0, \gamma = 0.1, \delta = 0.1, T = 1.0, K = 100.0$.

Conclusion

In this paper we have derived a forward equation for American option prices in a jump-diffusion setting. In the paper we have focused on its application to pricing a full continuum of American option prices. But as mentioned the forward equation has another interesting application. It admits a static estimation of the risk-adjusted parameters from observed American options prices. Moreover it is our belief that this type of forward equation also might be derived in other time-homogeneous models, such as recent models where the stock price follows hyperbolic distributions, see for example Chang and Madan (1995). This again allows static estimation of risk-adjusted parameters from American option prices in these models.

We have shown that the American option pricing problem in the jump-diffusion setting differs from the continuous path setting in the way that the limit of the early exercise boundary when time tends to expiration of the option might be considerably higher than in the continuous path case. Empirical observations of the exercise behavior of currency options on exchange rates in non-fully credible target zones might confirm this.

For the perpetual option we have obtained closed form solutions in the constant jump case. These solutions might be valuable for corporate finance models like Leland (1994) with the underlying value process following noncontinuous dynamics.

The last section has described a general finite difference method for the solution of PIDEs. This method will apply to the valuation of other types of options. Finally we have described two alternative approximations of the American option price when the jumps are constant.

Appendix

Proof of Lemma 2 (iii.):

Let τ^* be the optimal stopping time for the American option pricing problem. We have that:

$$C(t,S) = \mathrm{E}_t^Q\left[\mathrm{e}^{-r(\tau^*-t)}\left(S\mathrm{e}^{(r-q)(\tau^*-t)}\frac{M(\tau^*)}{M(t)} - K\right)^+\right] \tag{88}$$

From this it is easily verified that C is decreasing in t and increasing and convex in S. Since C is continuous, increasing, and convex in S, and the fact that:

$$S - K \leq C(t,S) \leq S \tag{89}$$

it follows that

$$C(t,S) = S - K \tag{90}$$

for all $S > S^*(t)$. Moreover since C is decreasing in t we have for all $S > S^*(t)$ that

$$S - K = C(t,S) \geq C(t+\epsilon, S) \tag{91}$$

for any ϵ satisfying $t < t+\epsilon \leq T$. This means that $S^*(\cdot)$ is decreasing. Combining this with (90) we have that the stopping region as well as the continuation regions are connected sets.

Since C is continuous the continuation region is open and the stopping region is closed.
□

Proof of Result 1:

Reconsider the stopping time problem at time 0:

$$C(0,S_0;T,K) = \sup_{\tau\in\mathcal{T}_{0,T}} \mathrm{E}^Q\left\{\mathrm{e}^{-r\tau}(S(\tau)-K)^+\right\} \tag{92}$$

Changing the martingale measure to Q' gives us the following representation:

$$\begin{aligned} C(0,S_0;T,K) &= \sup_{\tau\in\mathcal{T}_{0,T}} \mathrm{E}^{Q'}\left\{\mathrm{e}^{-q\tau}\left(S_0 - \frac{S_0K}{S(\tau)}\right)^+\right\} \\ &= \sup_{\tau\in\mathcal{T'}_{0,T}} \mathrm{E}^{Q'}\left\{\mathrm{e}^{-q\tau}(S_0 - Y(\tau))^+\right\} \end{aligned} \tag{93}$$

where $\mathcal{T}'_{0,T}$ is the set of all stopping times on $[0,T]$ for the process:

$$Y(t) = \frac{S_0K}{S(t)} \tag{94}$$

Using Ito's lemma yields:

$$\frac{dY(t)}{Y(t-)} = -(r - q - k\lambda)dt - \sigma dW'(t) - \frac{I'(t)}{1 + I'(t)} dN'(t) \qquad (95)$$
$$Y(0) = K$$

The problem is now turned into a put option pricing problem with a well-behaved underlying process and consequently we can write

$$C(0, S_0; T, K) = \psi(0, Y; T) \,|_{Y=K} \qquad (96)$$

where $\psi(\cdot)$ is the unique solution to the PIDE:

$$(q + \lambda')\psi =$$
$$\frac{\partial\psi}{\partial t} - (r - q - k\lambda)Y\frac{\partial\psi}{\partial Y} + \frac{1}{2}\sigma^2 Y^2 \frac{\partial^2\psi}{\partial Y^2} + \lambda' \mathrm{E}^{Q'}\left\{\psi\left(t, \frac{Y}{1+I'}; T\right)\right\} \qquad (97)$$

on

$$\left\{(t, Y) | \psi(t, Y; T) > (S_0 - Y)^+\right\} \qquad (98)$$

subject to the terminal boundary condition:

$$\psi(T, Y; T) = (S_0 - Y)^+ \qquad (99)$$

and the free boundary condition:

$$\psi(t, Y; T) \geq (S_0 - Y)^+ \qquad (100)$$

Due to the time-homogeneity of the backward equation we have that:

$$\psi(t, Y; T) = \psi(0, Y; T - t) \qquad (101)$$

Combining this with (97) we obtain the forward equation (28).
□

Proof of Result 2:

By Lemma 2 we that $S^*(\cdot)$ is decreasing and clearly $S^*(T) = K$, so if we let $\{t_n\}$ be a strictly increasing sequence with $t_n < T$ and $\lim_{n\to\infty} t_n = T$, the sequence $\{S^*(t_n)\}$ has a limit.
Define $S^{**}(t)$ as the implicit solution to the equation:

$$S^{**}(t) - K = c(t, S^{**}(t)) \qquad (102)$$

Since

$$c(t, S) - \left(\mathrm{e}^{-q(T-t)}S - \mathrm{e}^{-r(T-t)}K\right) \underset{S\to\infty}{\longrightarrow} 0 \qquad (103)$$

and

$$\mathrm{e}^{-q(T-t)}S - \mathrm{e}^{-r(T-t)}K < S - K \qquad (104)$$

for

$$S > K\frac{1 - \mathrm{e}^{-r(T-t)}}{1 - \mathrm{e}^{-q(T-t)}} \tag{105}$$

we have that a finite $S^{**}(t)$ exists for all t and since $\partial c/\partial S < 1$ for all $t < T$, S^{**} is unique.
Since $C \geq c$ we have that:

$$S^{**}(t_n) \leq S^*(t_n) \tag{106}$$

for all n. So:

$$\limsup_{n \to \infty} S^{**}(t_n) \leq \lim_{n \to \infty} S^*(t_n) \tag{107}$$

Now let τ^* be the optimal stopping time for the American option and fix S. We have that:

$$\begin{aligned}
0 &\leq C(t_n, S) - c(t_n, S) \\
&\leq \mathrm{E}_t^Q\left[\mathrm{e}^{-r(\tau^*-t_n)}(S(\tau^*) - K)^+\mathbf{1}_{\tau^*<T} + \mathrm{e}^{-r(T-t_n)}(S(T) - K)^+\mathbf{1}_{\tau^*=T}\right] \\
&- \mathrm{E}_t^Q\left[\mathrm{e}^{-r(T-t_n)}(S(T) - K)^+\mathbf{1}_{\tau^*<T} + \mathrm{e}^{-r(T-t_n)}(S(T) - K)^+\mathbf{1}_{\tau^*=T}\right] \\
&= S\mathrm{E}_t^{Q'}\left[\mathrm{e}^{-q(\tau^*-t_n)}\mathbf{1}_{\tau^*<T}\right] - K\mathrm{E}_t^Q\left[\mathrm{e}^{-r(\tau^*-t_n)}\mathbf{1}_{\tau^*<T}\right] \\
&- S\mathrm{E}_t^{Q'}\left[\mathrm{e}^{-q(T-t_n)}\mathbf{1}_{\tau^*<T}\right] + K\mathrm{E}_t^Q\left[\mathrm{e}^{-r(T-t_n)}\mathbf{1}_{\tau^*<T}\right] \\
&= S\mathrm{E}_t^{Q'}\left[\mathrm{e}^{-q(\tau^*-t_n)}\left(1 - \mathrm{e}^{-q(T-\tau^*)}\right)\mathbf{1}_{\tau^*<T}\right] \\
&- K\mathrm{E}_t^Q\left[\mathrm{e}^{-r(\tau^*-t_n)}\left(1 - \mathrm{e}^{-r(T-\tau^*)}\right)\mathbf{1}_{\tau^*<T}\right] \\
&\leq S\mathrm{E}_t^{Q'}\left[\mathrm{e}^{-q(\tau^*-t_n)}\left(1 - \mathrm{e}^{-q(T-t_n)}\right)\mathbf{1}_{\tau^*<T}\right] \\
&\leq S\left(1 - \mathrm{e}^{-q(T-t_n)}\right) \underset{n\to\infty}{\to} 0
\end{aligned} \tag{108}$$

Take $S > \liminf_{n \to \infty} S^{**}(t_n)$. Then $c(t_n, S) \leq S - K$ for infinitely many n. Suppose that:

$$S < \lim_{n \to \infty} S^*(t_n) \tag{109}$$

then there exists an n' so that $S < S^*(t_n)$ for all $n > n'$ and thus:

$$S - K < C(t_n, S) \tag{110}$$

for all $n > n'$. Thereby there exists an $\epsilon > 0$ so that

$$C(t_n, S) - c(t_n, S) > \epsilon \tag{111}$$

for infinitely many n. But this contradicts (108) so we conclude that

$$S \geq \lim_{n \to \infty} S^*(t_n) \tag{112}$$

and thereby that:

$$\liminf_{n\to\infty} S^{**}(t_n) \geq \lim_{n\to\infty} S^*(t_n) \tag{113}$$

We have now found that the sequence $\{S^{**}(t_n)\}$ has a limit and that this limit is equal to the limit of the sequence $\{S^*(t_n)\}$.
We now consider the limit of $S^{**}(t)$ as $t \to T$.
In the following we will in general use the notation $v = T - t$ for time to expiration and $\phi(\cdot)$ for the standard normal density function. We will also use the notation $f(x) = \mathcal{O}(g(x))$ near a point x_0 if $f(x)/g(x) \underset{x\to x_0}{\longrightarrow} h$ for some constant h and we will let $f(x) = o(g(x))$ denote the situation when $h = 0$.
Using Lemma 1 we get:

$$\begin{aligned}\frac{S^{**}(t)}{K} &= \frac{1 - \mathrm{e}^{-rv} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\lambda v}(\lambda v)^n}{n!} \Phi(d_n(v, S^{**}(t), K) - \nu_n(v))}{1 - \mathrm{e}^{-qv} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\lambda' v}(\lambda' v)^n}{n!} \Phi(d_n(v, S^{**}(t), K))} \\ &= \frac{1 - \mathrm{e}^{-(r+\lambda)v}\Phi(d_0(v, S^{**}(t), K) - \nu_0(v)) + \mathcal{O}(v)}{1 - \mathrm{e}^{-(q+\lambda')v}\Phi(d_0(v, S^{**}(t), K)) + \mathcal{O}(v)}\end{aligned} \tag{114}$$

In order to calculate the limit we distinguish the two cases:

$$\begin{aligned}\frac{\ln\left(S^{**}(t)/K\right)}{\sqrt{T-t}} &\underset{t\uparrow T}{\longrightarrow} k_0 \\ \frac{\ln\left(S^{**}(t)/K\right)}{\sqrt{T-t}} &\underset{t\uparrow T}{\longrightarrow} +\infty\end{aligned} \tag{115}$$

where k_0 is a nonnegative finite number. In the first case we have that

$$\lim_{t\uparrow T} d_0 = \lim_{t\uparrow T} d_0 - \nu_0 = k_0/\sigma \tag{116}$$

Hence

$$\lim_{t\uparrow T} \frac{S^{**}(t)}{K} = \frac{1 - \Phi(k_0/\sigma)}{1 - \Phi(k_0/\sigma)} = 1 \tag{117}$$

which again is consistent with (115) in the sense that $\ln\left(S^{**}(t)/K\right) \underset{t\uparrow T}{\longrightarrow} 0$.
Now consider the second case. $S^{**}(\cdot)$ is bounded from below by K and (114) implies that $S^{**}(\cdot)$ is analytic for $t < T$. For $t < T$ we can therefore write:

$$\ln\left(S^{**}(t)/K\right) = h_0 + h_1(t) \tag{118}$$

with h_0 being a nonnegative constant and $h_1(\cdot)$ being an analytic function on $[0, T]$ with $h_1(T) = 0$. If $h_0 = 0$ we must again have that:

$$\lim_{t\uparrow T} \frac{S^{**}(t)}{K} = 1 \tag{119}$$

If h_0 is positive need to apply l'Hospital's rule to gain the limit of (114). To do this we need to make some prior observations. We have that

$$\frac{\ln\left(S^{**}(t)/K\right)}{\sqrt{T-t}} = \frac{h_0 + h_1(t)}{\sqrt{T-t}} = \mathcal{O}\Big((T-t)^{-1/2}\Big) \tag{120}$$

near T. Moreover since h_1 is analytic and $h_1(T) = 0$ we have that:

$$0 = h_1(t) + h_1'(t)(T-t) + \mathcal{O}\Big((T-t)^2\Big) \tag{121}$$

for $t < T$. This means that:

$$h_1'(t) = o\Big((T-t)^{-1}\Big) \tag{122}$$

and thereby that:

$$\frac{\partial}{\partial t}\frac{\ln\left(S^{**}(t)/K\right)}{\sqrt{T-t}} = \frac{h_1'(t)}{\sqrt{T-t}} + \frac{1}{2}\frac{h_0 + h_1(t)}{(T-t)^{3/2}} = \mathcal{O}\Big((T-t)^{-3/2}\Big) \tag{123}$$

near T.

For $n \geq 1$ we have:

$$\begin{aligned} d_n - \nu_n &= \frac{\ln\left(S^{**}(t)/K\right) + (r-q-k\lambda)v + n\gamma}{\sqrt{\sigma^2 v + n\delta^2}} - \frac{1}{2}\sqrt{\sigma^2 v + n\delta^2} \\ &\underset{t\uparrow T}{\rightarrow} \frac{\ln\left(\lim S^{**}(t)/K\right) + n\gamma}{\sqrt{n}\delta} - \frac{1}{2}\sqrt{n}\delta \end{aligned} \tag{124}$$

and by (122):

$$\begin{aligned} &\frac{\partial(d_n - \nu_n)}{\partial t} \\ &= \frac{h_1'(t)}{\sqrt{\sigma^2 v + n\delta^2}} + \frac{1}{2}\frac{h_0 + h_1(t) + (r-q-k\lambda)v + n\gamma}{(\sigma^2 v + n\delta^2)^{3/2}}\sigma^2 - \frac{r-q-k\lambda-\frac{1}{4}\sigma^2}{\sqrt{\sigma^2 v + n\delta^2}} \\ &= o\big(v^{-1}\big) + \mathcal{O}(1) \end{aligned} \tag{125}$$

After these considerations we are ready to apply l'Hospital's rule to (114). For the numerator we get:

$$
\begin{aligned}
&(r+\lambda)\mathrm{e}^{-(r+\lambda)v}\sum_{n=0}^{\infty}\frac{(\lambda v)^n}{n!}\Phi(d_n-\nu_n)\\
&-\lambda\mathrm{e}^{-(r+\lambda)v}\sum_{n=1}^{\infty}\frac{(\lambda v)^{n-1}}{(n-1)!}\Phi(d_n-\nu_n)\\
&+\mathrm{e}^{-(r+\lambda)v}\sum_{n=0}^{\infty}\frac{(\lambda v)^n}{n!}\phi(d_n-\nu_n)\frac{\partial(d_n-\nu_n)}{\partial t}\\
&=(r+\lambda)\mathrm{e}^{-(r+\lambda)v}(\Phi(d_0-\nu_0)+\mathcal{O}(T-t))\\
&-\lambda\mathrm{e}^{-(r+\lambda)v}(\Phi(d_1-\nu_1)+\mathcal{O}(T-t))\\
&+\mathrm{e}^{-(r+\lambda)v}\phi(d_0-\nu_0)\frac{\partial(d_0-\nu_0)}{\partial t}\\
&+\mathrm{e}^{-(r+\lambda)v}\sum_{n=1}^{\infty}\frac{(\lambda v)^n}{n!}\phi(d_n-\nu_n)\frac{\partial(d_n-\nu_n)}{\partial t}\\
&=(r+\lambda)\Phi\left(+\mathcal{O}\left(v^{-1/2}\right)\right)\\
&-\lambda\Phi\left(\frac{\ln\left(\lim_{t\uparrow T}S^{**}(t)/K\right)+\gamma}{\delta}-\frac{1}{2}\delta+\mathcal{O}\left(v^{1/2}\right)\right)\\
&+\phi\left(+\mathcal{O}\left(v^{-1/2}\right)\right)\mathcal{O}\left(v^{-3/2}\right)\\
&+\sum_{n=1}^{\infty}\frac{(\lambda v)^n}{n!}\left(o(v^{-1})+\mathcal{O}(1)\right)\\
&+\mathcal{O}(v)\\
&\underset{t\uparrow T}{\rightarrow} r+\lambda\left(1-\Phi\left(\frac{\ln\left(\lim_{t\uparrow T}S^{**}(t)/K\right)+\gamma}{\delta}-\frac{1}{2}\delta\right)\right)
\end{aligned}
\tag{126}
$$

Applying the same analysis to the denominator yields:

$$
\lim_{t\uparrow T}\frac{S^{**}(t)}{K}=\frac{r+\lambda\left(1-\Phi\left(\frac{\ln\left(\lim_{t\uparrow T}S^{**}(t)/K\right)+\gamma}{\delta}-\frac{1}{2}\delta\right)\right)}{q+\lambda'\left(1-\Phi\left(\frac{\ln\left(\lim_{t\uparrow T}S^{**}(t)/K\right)+\gamma}{\delta}+\frac{1}{2}\delta\right)\right)}
\tag{127}
$$

So for the second case considered here we have that:

$$
\lim_{t\uparrow T}\frac{S^{**}(t)}{K}=\frac{S_0^*}{K}
\tag{128}
$$

We conclude that:

$$\lim_{t \uparrow T} S^{**}(t) = \max\left(K, S_0^*\right) \tag{129}$$

and the result follows.

□

Proof of Result 3:

As in the proof of Result 3 we have that the boundary for the American option and the one for the European, defined by (101) coincide in the limit. Using Lemma 1 as in the previous proof we have that:

$$\begin{aligned} \frac{S^{**}(t)}{K} &= \frac{1 - \mathrm{e}^{-rv} \sum\limits_{n=0}^{\infty} \frac{\mathrm{e}^{-\lambda v}(\lambda v)^n}{n!} \Phi(d_n(v, S^{**}(t), K) - \sigma\sqrt{v})}{1 - \mathrm{e}^{-qv} \sum\limits_{n=0}^{\infty} \frac{\mathrm{e}^{-\lambda' v}(\lambda' v)^n}{n!} \Phi(d_n(v, S^{**}(t), K))} \\ &= \frac{1 - \mathrm{e}^{-(r+\lambda)v} \Phi(d_0(v, S^{**}(t), K) - \sigma\sqrt{v}) + \mathcal{O}(v)}{1 - \mathrm{e}^{-(q+\lambda')v} \Phi(d_0(v, S^{**}(t), K)) + \mathcal{O}(v)} \end{aligned} \tag{130}$$

As in the proof of Result 3 we split the limit into two situations:

$$\begin{aligned} \frac{\ln\left(S^{**}(t)/K\right)}{\sqrt{T-t}} &\underset{t \uparrow T}{\longrightarrow} k_0 \\ \frac{\ln\left(S^{**}(t)/K\right)}{\sqrt{T-t}} &\underset{t \uparrow T}{\longrightarrow} +\infty \end{aligned} \tag{131}$$

where k_0 is a finite nonnegative number. The first situation corresponds to the case when:

$$\lim_{t \uparrow T} \frac{S^{**}(t)}{K} = 1 \tag{132}$$

We now consider the second case and use l'Hospital's rule on (130). For the numerator we get:

$$\begin{aligned} &(r+\lambda)\mathrm{e}^{-(r+\lambda)v} \sum_{n=0}^{\infty} \frac{(\lambda v)^n}{n!} \Phi\left(d_n - \sigma\sqrt{v}\right) \\ &- \lambda \mathrm{e}^{-(r+\lambda)v} \sum_{n=1}^{\infty} \frac{(\lambda v)^{n-1}}{(n-1)!} \Phi\left(d_n - \sigma\sqrt{v}\right) \\ &+ \mathrm{e}^{-(r+\lambda)v} \sum_{n=0}^{\infty} \frac{(\lambda v)^n}{n!} \phi\left(d_n - \sigma\sqrt{v}\right) \frac{\partial(d_n - \sigma\sqrt{v})}{\partial t} \\ &= (r+\lambda)\mathrm{e}^{-(r+\lambda)v} \left(\Phi\left(d_0 - \sigma\sqrt{v}\right) + \mathcal{O}(T-t)\right) \\ &- \lambda \mathrm{e}^{-(r+\lambda)v} \left(\Phi\left(d_1 - \sigma\sqrt{v}\right) + \mathcal{O}(T-t)\right) \\ &+ \mathrm{e}^{-(r+\lambda)v} \phi\left(d_0 - \sigma\sqrt{v}\right) \frac{\partial(d_0 - \sigma\sqrt{v})}{\partial t} \\ &+ \mathrm{e}^{-(r+\lambda)v} \sum_{n=1}^{\infty} \frac{(\lambda v)^n}{n!} \phi\left(d_n - \sigma\sqrt{v}\right) \frac{\partial(d_n - \sigma\sqrt{v})}{\partial t} \end{aligned} \tag{133}$$

By arguments similar to the ones in the proof of Result 3 we have for *all* $n \geq 0$:

$$\begin{aligned} d_n - \sigma\sqrt{T-t} &= \mathcal{O}\Big((T-t)^{-1/2}\Big) \\ \frac{\partial\big(d_n - \sigma\sqrt{T-t}\big)}{\partial t} &= \mathcal{O}\Big((T-t)^{-3/2}\Big) \end{aligned} \tag{134}$$

So the third term and the fourth term tend to zero as $t \uparrow T$.
The first term tends to:

$$r + \lambda \tag{135}$$

For the second term there are again two cases:

$$\lim_{t\uparrow T}\left(\ln\frac{S^{**}(t)}{K} + \gamma\right) < 0 \Rightarrow d_1 \underset{t\uparrow T}{\rightarrow} -\infty \tag{136}$$

$$\lim_{t\uparrow T}\left(\ln\frac{S^{**}(t)}{K} + \gamma\right) > 0 \Rightarrow d_1 \underset{t\uparrow T}{\rightarrow} +\infty \tag{137}$$

In the first case we can conclude that:

$$\frac{S^{**}(t)}{K} \underset{t\uparrow T}{\rightarrow} \frac{r+\lambda}{q+\lambda'} \tag{138}$$

This is consistent with (136) if and only if:

$$\frac{r+\lambda}{q+\lambda'}(1+k) < 1 \Leftrightarrow \frac{r}{q} < \frac{r+\lambda}{q+\lambda'} \tag{139}$$

In the second case we have that:

$$\frac{S^{**}(t)}{K} \underset{t\uparrow T}{\rightarrow} \frac{r}{q} \tag{140}$$

which again is consistent with (137) if and only if:

$$\frac{r}{q} > \frac{r+\lambda}{q+\lambda'} \tag{141}$$

Finally we see that the border case:

$$\lim_{t\uparrow T}\left(\ln\frac{S^{**}(t)}{K} + \gamma\right) = 0 \tag{142}$$

is exactly the situation when:

$$\frac{r}{q} = \frac{r+\lambda}{q+\lambda'} \tag{143}$$

Hence:

$$S^{**}(t) \underset{t\uparrow T}{\rightarrow} K\max\left(1, \frac{r}{q}, \frac{r+\lambda}{q+\lambda'}\right) \tag{144}$$

□

Proof of Result 4:

Since C and $\partial C/\partial S$ are continuous, decreasing in t and increasing in S, respectively, and

$$\begin{aligned} (S-K)^+ \leq C \leq S \\ 0 \leq \partial C/\partial S \leq 1 \end{aligned} \tag{145}$$

$C, \partial C/\partial S$ are absolutely continuous in the t and the S direction, respectively. Therefore the derivatives $\partial C/\partial t$ and $\partial^2 C/\partial S^2$ exist almost everywhere in the two dimensional Lebesque sense (the exception is the early exercise boundary) and we can therefore apply Ito's lemma to C/B and get:

$$\begin{aligned} &d\frac{C(t,S(t))}{B(t)} \\ &= \frac{\mathbf{1}_{S(t-)<S^*(t-)}}{B(t)}\left[\frac{\partial C}{\partial t} + (r-q-k\lambda)S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S\frac{\partial^2 C}{\partial S^2} + \lambda \mathrm{E}_t^{Q}[\Delta C] - rC\right]dt \\ &+ \frac{\mathbf{1}_{S(t-)>S^*(t-)}}{B(t)}\left[(r-q-k\lambda)S + \lambda \mathrm{E}_t^{Q}[\Delta C] - r(S-K)\right]dt + dH(t) \end{aligned} \tag{146}$$

where H is some Q-martingale and Δ is the difference operator

$$\Delta f(S) = f(S(1+I)) - f(S) \tag{147}$$

First note that by the PIDE (24), the first term in (146) is identical to zero. Next note that since jumps go only upwards $\Delta C = \Delta S$ on the stopping region.[7] We therefore get:

$$d\frac{C(t,S(t))}{B(t)} = -\frac{\mathbf{1}_{S(t-)>S^*(t-)}}{B(t)}[qS(t-) - rK]dt + dH(t) \tag{148}$$

Integrating from t to T and multiplying by $B(t)$ yields:

$$\begin{aligned} &\frac{B(t)}{B(T)}C(T,S(T)) - C(t,S(t)) \\ &= \mathrm{e}^{-r(T-t)}(S(T)-K)^+ - C(t,S(t)) \\ &= -\int_t^T \mathrm{e}^{-r(u-t)}(qS(u) - rK)\mathbf{1}_{S(u)>S^*(u)}du + H(T) - H(t) \end{aligned} \tag{149}$$

Taking conditional expectations, rearranging, using the Fubini Theorem, changing the measure and applying the distributional properties (15) and (16) yield the result.

□

---

7 This property is absolutely essential to the proof. If jumps could go downwards, equation (147) would have an extra term.

Proof of Result 5:

To prove (i.) we see from Lemma 1 that:

$$c(t) \underset{T \to \infty}{\longrightarrow} 0 \tag{150}$$

Using Result 4 we have at the early exercise boundary:

$$\begin{aligned} S^*_\infty - K = {} & qS^*_\infty \int_0^\infty \mathrm{e}^{-qv} \sum_{n=0}^\infty \frac{\mathrm{e}^{-\lambda' v}(\lambda' v)^n}{n!} \Phi(d_n(v, S^*_\infty, S^*_\infty))dv \\ & - rK \int_0^\infty \mathrm{e}^{-rv} \sum_{n=0}^\infty \frac{\mathrm{e}^{-\lambda v}(\lambda v)^n}{n!} \Phi\big(d_n(v, S^*_\infty, S^*_\infty) - \sigma\sqrt{v}\big)dv \end{aligned} \tag{151}$$

Rearranging yields the result for the early exercise boundary of the perpetual option. The price is obtained by inserting S^*_∞ in the pricing formula of Result 4.

To prove (ii.) note that the partial difference and differential equation for the perpetual option reduces to the time homogeneous equation:

$$(r+\lambda)C(S) = (r - q - k\lambda)SC'(S) + \frac{1}{2}\sigma^2 S^2 C''(S) + \lambda C(S(1+k)) \tag{152}$$

subject to the boundary conditions:

$$\begin{aligned} & S \geq C(S) \geq (S-K)^+ \\ & C(S^*_\infty) = S^*_\infty - K \\ & \frac{dC(S^*_\infty)}{dS} = 1 \\ & C(S) \underset{S \to 0}{\longrightarrow} 0 \\ & S^*_\infty = \arg\max_{S^*} C(S; S^*) \end{aligned} \tag{153}$$

The last boundary condition should be interpreted in the sense that the optimal early exercise boundary should maximize the call value for all stock prices.

The object is now to find the unique solution to (152) subject to the boundary conditions (153).

If β solves (55) then AS^β is a solution to (152) for any constant A. But the fourth boundary condition can only be satisfied when

$$\beta > 0 \tag{154}$$

Since the right hand side of (55) is a convex and increasing function in β with a negative values for $\beta \in \{0, 1\}$ and since it tends to infinity for $\beta \to \infty$, the equation (55) has exactly one solution that is strictly greater than 1.

The second boundary condition implies that:

$$A = \frac{S^*_\infty - K}{(S^*_\infty)^\beta} \tag{155}$$

Maximizing the call value:

$$\frac{S_\infty^* - K}{(S_\infty^*)^\beta} S^\beta \tag{156}$$

over all possible early exercise strategies yields:

$$S_\infty^* = \frac{\beta}{\beta - 1} K \tag{157}$$

for all S.

We note that the first boundary condition as well as the high contact condition are satisfied by the solution. This concludes the proof.

□

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The Passport Option

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The Passport Option

Abstract

A passport option[1] is a contract that grants its holder the right to repeatedly switch between short and long positions in an underlying asset. The gains on the stream of short/long positions are accumulated and paid at the option maturity, if positive. Working in the framework of geometric Brownian motion, we show that the value of the passport option is the solution to a Markov stochastic control problem. We derive the optimal switching strategy and a partial differential equation satisfied by the asset-deflated option price. A closed-form solution is derived for the special case where the underlying asset is a martingale under the risk-neutral measure. For the general case, the paper illustrates how a Crank-Nicholson finite difference scheme can be used for practical computation of the option value. The numerical scheme developed in the paper is applicable to a number of variations on the basic passport option contract.

1. The Passport Option

Consider a time horizon $[0,T]$ and $H+1$ discrete dates $t_0, t_1, t_2, \ldots, t_{H-1}, t_H$, where $t_0 = 0$ and $t_H = T$. At each t_i, $i = 0,1,2,\ldots, H-1$ an investor takes a position $u(t_i) \in [-1,1]$ in an asset S: the position is held over the interval $(t_i, t_{i+1}]$. The choice of $u(t_i)$ is determined by the investor at time t_i. The accumulated gain of the asset holdings over $[0,T]$ is

$$\sum_{i=0}^{H-1} u(t_i)[S(t_{i+1}) - S(t_i)]. \tag{1}$$

Under the terms of the *European passport option contract*, the option holder has the right, but not the obligation, to receive this gain at the option maturity T. The payout of the option at T is thus

$$\left(\sum_{i=0}^{H-1} u(t_i)[S(t_{i+1}) - S(t_i)]\right)^{+} \tag{2}$$

where we have used the notation $z^{+} = MAX(z,0)$.

The most salient feature of the passport option is obviously the H rights to alter the position in the underlying asset. While in practice the choice of the holding strategy u might be "irrational" and dictated by external hedging needs[2], the seller of the option clearly must assume that the buyer will attempt to maximize his financial gain on the contract. Unless the buyer of the option contractually agrees to a specific strategy, the passport option will consequently always command a price that corresponds to the strategy u for which the option value is maximized. Determination of this strategy is, as we shall see, generally a non-trivial problem.

The rest of this paper is organized as follows. In Section 2, we provide notation and state our process assumptions. Section 3 analyzes the pricing of European passport options for the case where changes in holding strategy takes place in continuous time. In Section 4, we address the numerical implementation of the results derived in Section 3. Section 5 contains various numerical results, and Section 6 discusses extensions of the numerical method to more complicated options. Finally, Section 7 contains our conclusions and some directions for further research.

2. A Continuous-Time Framework

Let us consider a standard Black-Scholes economy with two assets: a money-market account B and a dividend-paying stock[3] S. Assuming a constant risk-free interest rate r, B evolves deterministically

$$B(t) = \exp(rt)$$

We assume that S pays a constant dividend yield of γ and follows a one-factor geometric Brownian motion with constant volatility σ. According to Harrison and Pliska (1981), the absence of arbitrage dictates the existence of a probability measure Q (the "risk neutral" measure) under which the process for S can be written as the stochastic differential equation (SDE)

$$dS(t) / S(t) = (r - \gamma)dt + \sigma dW(t); \qquad (3)$$

here, W is a standard Brownian motion under Q.

To characterize the short/long strategy of the passport option holder, we now introduce an adapted process $\{u(t)\}_{0 \le t \le T}$, where $u(t) \in [-1,1]$. With this definition of u, we can introduce the gain process[4]

$$w(t) = \int_0^t u(s) dS(s) \qquad (4a)$$

or

$$dw(t) = u(t)dS(t), \ w(0) = 0. \qquad (4b)$$

Equation (4a-b) is a continuous-time version of the discrete gain process (1). For sufficiently small time increments $t_{i+1} - t_i$ in (2), (4a-b) is a reasonable approximation although the value of the passport option will be biased high[5] (see Section 6.2).

For a given holding strategy u, the payout at expiration T of the European passport option is defined as

$$V_u(T) = w(T)^+.$$

As V_u / B is a martingale under Q, the time t option value is

$$V_u(t) = e^{-r(T-t)} E_t^Q[w(T)^+] \tag{5}$$

where $E_t^Q[]$ denotes time t expectation under the probability measure Q.

We stress that the pricing equation (5) holds for any *fixed* strategy u. As discussed earlier, allowing the option buyer to chose u freely in [-1,1] implies that the fair value V of the European passport option is the maximum

$$V(t) = \sup_{u\in[-1,1]} V_u(t) = \sup_{u\in[-1,1]} e^{-r(T-t)} E_t^Q[w(T)^+] \tag{6}$$

where, from (4b),

$$dw(t) = u(t)S(t)(r-\gamma)dt + u(t)\sigma S(t)dW(t). \tag{7}$$

Equations (6) and (7) together form a stochastic control problem with $w(T)^+$ being the terminal *bequest function* and u the *controls*.

3. Solution

As the process (7) for w depends on S, we first attempt to introduce a shift of probability measure that will remove S from the SDE for w. Specifically, we use the Girsanov Theorem (Karatzas and Shreve (1994), Section 3.5) to introduce a new probability measure Q' under which

$$dW'(t) = dW(t) - \sigma dt \tag{8}$$

is a Brownian motion. We have

Proposition 1

Define $x(t) = w(t)/S(t)$, $v_u(t) = V_u(t)/S(t)$, and $v(t) = V(t)/S(t)$. (5) and (6) can then be written

$$v_u(t) = e^{-\gamma(T-t)} E_t^{Q'}[x(T)^+] \tag{9}$$

$$v(t) = \sup_{u\in[-1,1]} v_u(t) \qquad (10)$$

where

$$dx(t) = [u(t) - x(t)](r - \gamma)dt + [u(t) - x(t)]\sigma dW'(t). \qquad (11)$$

Proof:

First, notice that as $V_u(t)/B(t)$ is a martingale under Q, we can use the Martingale Representation Theorem (Duffie (1996), p. 287) to write

$$dV_u(t)/V_u(t) = rdt + \varphi(t)dW(t)$$

for some volatility function $\varphi(t)$. From Ito's lemma and (8)

$$\begin{aligned} dv_u(t)/v_u(t) &= (\gamma + \sigma^2 - \varphi(t)\sigma)dt + (\varphi(t) - \sigma)dW(t) \\ &= \gamma dt + (\varphi(t) - \sigma)dW'(t) \end{aligned}$$

whereby we conclude that $v_u(t)e^{-\gamma t}$ is a Q'-martingale. That is, we have

$$v_u(t) = e^{-\gamma(T-t)}E_t^{Q'}[v_u(T)] = e^{-\gamma(T-t)}E_t^{Q'}[x(T)^+]$$

which is (9). (10) follows directly from (6). (11) follows from (7) and Ito's lemma. ♠

With Proposition 1, we have reduced the stochastic control problem (5), (6), and (7) to a one-variable (x) setting. To proceed, we first need the following lemma:

Lemma 1

In the absence of arbitrage, the S-deflated option value v is an increasing function of x:

$$x^A \geq x^B \Rightarrow v(t, x^A) \geq v(t, x^B) \qquad (12)$$

Moreover, v is a convex function of x:

$$x^B = \lambda x^A + (1-\lambda)x^C \Rightarrow \lambda v(t,x^A) + (1-\lambda)v(t,x^C) \geq v(t,x^B), \tag{13}$$

for $\lambda \in [0,1]$.

Proof:

To prove (12), consider two investors A and B, each having purchased identical T-maturity passport options. The two investors have chosen different strategies u over $[0,t]$, resulting in different gains at time t. We assume that A did at least as well as B, i.e.

$$w^A(t) \geq w^B(t). \tag{14}$$

Now assume that B's optimal strategy going forward is $u_B^*(s),\ t \leq s \leq T$. From (5), we can write the fair value V^B of B's option as

$$V^B(t) = V_{u_B^*}^B(t) = e^{-r(T-t)} E_t^Q\left[\left(\Delta w^B(t,T) + w^B(t)\right)^+\right]$$

where $\Delta w^B(t,T) \equiv \int_t^T u_B^*(s)dS(s)$.

As B's optimal strategy may or may not be the same as A's optimal strategy, we have the following inequality for the fair value of A's option

$$V^A(t) = V_{u_A^*}^A(t) \geq V_{u_B^*}^A(t) = e^{-r(T-t)} E_t^Q\left[\left(\Delta w^B(t,T) + w^A(t)\right)^+\right] \geq V^B(t) \tag{15}$$

The last inequality follows from (14). Dividing (14) and (15) by $S(t)$ results in (12).

To show convexity, we introduce a third investor C and replace (14) with the condition

$$w^B(t) = \lambda w^A(t) + (1-\lambda)w^C(t) \tag{16}$$

With B's optimal strategy being $u_B^*(s),\ t \leq s \leq T$, obviously

$$V^A(t) \geq V_{u_B^*}^A(t),$$
$$V^C(t) \geq V_{u_B^*}^C(t).$$

We thus have

$$
\begin{aligned}
\lambda V^{A}(t)+(1-\lambda)V^{C}(t) &\geq \lambda V^{A}_{u^*_B}(t)+(1-\lambda)V^{C}_{u^*_B}(t) \\
&= e^{-r(T-t)}E_t^Q\left[\lambda\left(\Delta w^B(t,T)+w^A(t)\right)^+ +(1-\lambda)\left(\Delta w^B(t,T)+w^C(t)\right)^+\right] \\
&\geq e^{-r(T-t)}E_t^Q\left[\left(\Delta w^B(t,T)+\lambda w^A(t)+(1-\lambda)w^C(t)\right)^+\right] \\
&= V^B(t)
\end{aligned}
\tag{17}
$$

The last equality follows from (16). Dividing (16) and (17) by $S(t)$ gives the desired result.♠

Define now $sign(z)=\begin{cases}-1, & z<0\\ 1, & z\geq 0\end{cases}$. With Proposition 1 and Lemma 1, we can now state the main result of this section:

Proposition 2

The S-deflated option price $v(t)$ is a function of only t and $x(t)$, $v(t)=v(t,x(t))$. The optimal holding strategy $\{u^*(t)\}_{0\leq t\leq T}$, $u^*(t)\in[-1,1]$, is given by

$$u^*(t)=u^*(t,x)=sign\left((r-\gamma)\frac{\partial v}{\partial x}-x\sigma^2\frac{\partial^2 v}{\partial x^2}\right) \tag{18}$$

where $\frac{\partial v}{\partial x}\geq 0$ and $\frac{\partial^2 v}{\partial x^2}\geq 0$. v satisfies the partial differential equation (PDE)

$$\frac{\partial v}{\partial t}+(u^*-x)(r-\gamma)\frac{\partial v}{\partial x}+\frac{1}{2}(u^*-x)^2\sigma^2\frac{\partial^2 v}{\partial x^2}=\gamma v \tag{19a}$$

or, equivalently,

$$\frac{\partial v}{\partial t} - x(r-\gamma)\frac{\partial v}{\partial x} + \frac{1}{2}(1+x^2)\sigma^2\frac{\partial^2 v}{\partial x^2} + (r-\gamma)\frac{\partial v}{\partial x} - x\sigma^2\frac{\partial^2 v}{\partial x^2}\bigg| = \gamma v \qquad (19b)$$

with boundary condition

$$v(T) = x(T)^+ .$$

Proof:

By Theorem 11.3 in Øksendahl (1995), the form of the SDE (11) for x under Q' allows us to restrict the search for u^* to functions depending only on time and the current level of x, i.e.

$$u^*(t) = u^*(t, x(t)) .$$

Similarly, $v(t) = v(t, x(t))$. Applying the Hamilton-Jacobi-Bellman (HJB) equation (see e.g. Øksendahl (1995), Theorem 11.1) on (9) and (10), we get

$$\sup_{h\in[-1,1]}\left\{\frac{\partial v}{\partial t} + (h-x)(r-\gamma)\frac{\partial v}{\partial x} + \frac{1}{2}(h-x)^2\sigma^2\frac{\partial^2 v}{\partial x^2} - \gamma v\right\} = 0 . \qquad (20)$$

The supremum in (20) is obtained at $h = u^*(t, x(t))$.

Setting

$$C_1 \equiv (r-\gamma)\frac{\partial v}{\partial x},\ C_2 \equiv \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial x^2},\ C_3 \equiv \frac{\partial v}{\partial t} - x(r-\gamma)\frac{\partial v}{\partial x} + \frac{1}{2}x^2\sigma^2\frac{\partial^2 v}{\partial x^2} - \gamma v$$

the maximum in (20) can be written

$$\sup_{h\in[-1,1]}\{C_3 + h(C_1 - 2xC_2) + h^2 C_2\} . \qquad (21)$$

From the convexity properties outlined in Lemma 1, we know that $\partial^2 v / \partial x^2 \geq 0$. As consequently $C_2 \geq 0$, the maximum in (21) will always be obtained for either $h = 1$ or $h = -1$. The obvious result is

$$u^*(t,x) = sign(C_1 - 2xC_2) = sign\left((r-\gamma)\frac{\partial v}{\partial x} - x\sigma^2\frac{\partial^2 v}{\partial x^2}\right).$$

The result $\partial v / \partial x \geq 0$ follows directly from Lemma 1. (19b) follows from (19a) by insertion of (18) and using $(u^*)^2 = 1$. ♠

Not surprisingly, the optimal control is always either 1 or -1 (i.e. of the so-called "bang-bang" type); allowing the option holder to take "fractional" positions (i.e. $-1 < u < 1$) does not add any value to the option contract.

While we have used probabilistic techniques to derive the form of the PDE (19a-b), the equation can also be derived from a "classic" instantaneous hedge argument. Appendix A briefly discusses this approach and also shows that the instantaneous hedge ratio is given by

$$hedge\ ratio = \frac{\partial V}{\partial S} + u^*\frac{\partial V}{\partial w} = v + (u^* - x)\frac{\partial v}{\partial x}$$

We notice that the hedge ratio will jump whenever the sign of $u^*(t,x(t))$ is switched.

Corollary

In the special case when $r = \gamma$, the optimal holding strategy is

$$u^*(t,x(t)) = -sign(x(t)) = -sign(w(t)). \tag{22}$$

The valuation PDE becomes

$$\frac{\partial v}{\partial t} + \frac{1}{2}(1+|x|)^2\sigma^2\frac{\partial^2 v}{\partial x^2} = \gamma v, \quad v(T) = x(T)^+. \tag{23}$$

When $r=\gamma$, S is a martingale under Q (see (3)). This fact might appear to make the result (22) problematic: if S is a martingale (under Q) without any predictable trend, why is it possible to extract more value by following (22) than by using a naive strategy, say $u(t,x(t))=1$? To answer this question, set $r=\gamma$ and consider the case where the option holder must make his last change of strategy at some predetermined intermediate time $s\in(0,T)$. Let us denote the fair value of this option V^s. At s, the option holder has accumulated a gain of $w(s)$ and is faced with the decision of setting either $u(s,x(s))=1$ or $u(s,x(s))=-1$. Depending on whether a short or a long position is chosen, the terminal payout is

$$u(s,x(s))=1: \qquad V^s_{+1}(T)=\left(S(T)-[S(s)-w(s)]\right)^+ \tag{24a}$$

$$u(s,x(s))=-1: \qquad V^s_{-1}(T)=\left([S(s)+w(s)]-S(T)\right)^+ . \tag{24b}$$

(24a) is the payout of a regular call option with strike $S(s)-w(s)$, while (24b) is the payout of a put option with strike $S(s)+w(s)$. Applying the Black-Scholes formula (Black and Scholes (1973), Merton (1973)) with $r=\gamma$, we get

$$V^s_{+1}(s)e^{\gamma(T-s)}=\begin{cases} N(d_c)-[S(s)-w(s)]N\left(d_c-\sigma\sqrt{T-s}\right), & S(s)>w(s)\\ w(s), & S(s)\le w(s)\end{cases} \tag{25a}$$

$$V^s_{-1}(s)e^{\gamma(T-s)}=\begin{cases} [S(s)+w(s)]N(d_p)-S(s)N(d_p-\sigma\sqrt{T-s}), & S(s)>-w(s)\\ 0, & S(s)\le -w(s)\end{cases} \tag{25b}$$

where

$$d_c=\frac{\ln(S(s)/[S(s)-w(s)])+\frac{1}{2}\sigma^2(T-s)}{\sigma\sqrt{T-s}}$$

$$d_p=\frac{-\ln(S(s)/[S(s)+w(s)])+\frac{1}{2}\sigma^2(T-s)}{\sigma\sqrt{T-s}}$$

and $N()$ is the standard cumulative normal distribution.

Obviously, the option value at time s is

$$V^s(s)=MAX(V^s_{+1}(s),V^s_{-1}(s)). \tag{26}$$

It is easy to verify that the asymmetry of the Black-Scholes put and call formulas imply that

$$w(s) > 0 \Rightarrow V_{-1}(s) > V_{+1}(s) \tag{27a}$$
$$w(s) = 0 \Rightarrow V_{-1}(s) = V_{+1}(s) \tag{27b}$$
$$w(s) < 0 \Rightarrow V_{-1}(s) < V_{+1}(s). \tag{27c}$$

From (26) and (27a-c) we conclude that a) the right to shift at time s is, indeed, worth a positive amount; and b) the optimal strategy at time s is $u(s,x(s)) = -sign(x(s))$. Notice that (27a-c) is a consequence of the skew of the log-normal distribution of S. Had S instead been normally distributed, V_{-1}^{s} would equal V_{+1}^{s} for all $w(s)$, and the right to switch would have been worthless[6].

For later use, we now set $v^{s}(t,x(t)) \equiv V^{s}(t,x(t))/S(t)$, and notice, after straightforward manipulations, that (26) can be written as

$$v^{s}(s,x(s)) = e^{-\gamma(T-s)}\left\{x(s)^{+} + N(d) - (1+|x(s)|)N\left(d - \sigma\sqrt{T-s}\right)\right\}, \tag{28}$$
$$d = \frac{-\ln(1+|x(s)|) + \frac{1}{2}\sigma^{2}(T-s)}{\sigma\sqrt{T-s}}.$$

Returning to the case of infinitely many switching rights, but still $r = \gamma$, it turns out that the PDE (23) is sufficiently simple to allow for a closed form solution:

Proposition 3

In the special case of $r = \gamma$, the solution to (23) is given by

$$v(t,x(t)) = e^{-\gamma(T-t)}\left\{x(t)^{+} + N(d(t)) - (1+|x(t)|)N\left(d(t) - \sigma\sqrt{T-t}\right) + \tfrac{1}{4}\sigma^{2}\int_{t}^{T} N(d(s))ds\right\} \tag{29}$$

$$d(t) = \frac{-\ln(1+|x(t)|) + \frac{1}{2}\sigma^{2}(T-t)}{\sigma\sqrt{T-t}}.$$

Proof:
(29) can be verified directly by insertion into (23). See Appendix B for details about using the Laplace transform to derive (29).♠

By comparison with (28), we note that the value of the passport option can be decomposed into two terms: a term that captures the value of the option assuming time t is the last day on which a change in strategy is allowed, and a term that captures the value of having the right to modify the strategy in the future. The latter term, $\frac{1}{4}\sigma^2\int_t^T N(d(s))ds$, can easily be evaluated using numerical integration[7]; it is bounded by

$$0 < \frac{1}{4}\sigma^2\int_t^T N(d(s))ds < \frac{1}{4}\sigma^2(T-t)$$

Clearly the importance of this term decreases as t approaches T.

As a final comment, we notice that despite the presence of the non-differentiable functions $|x|$ and $(-x)^+$, (29) is in fact differentiable, even at $x = 0$: $v \in C^{1,2}([0,T) \times \Re)$. See Appendix B for further discussion of this issue and an analysis of the local behavior of v around $x = 0$. As discussed earlier, the hedge ratio will jump whenever x crosses 0.

4. Numerical Solution

While we have been able to solve the problem of valuing the European passport option in the special case of $r = \gamma$, the general PDE (18)-(19a) appears not to have a closed-form solution if $r \neq \gamma$. To handle this case, we turn to the development of a numerical scheme based on a finite difference discretization of the price PDE. First, define a uniform mesh (x_i, t_j) with

$$x_i = i\Delta_x, \quad t_j = j\Delta_t$$

for $0 \le i \le M$, $0 \le j \le N$ and $t_N = T$. x_0 and x_M represent $-\infty$ and ∞, respectively; reasonable choices for these numbers are, say, $\mp e^{4\sigma\sqrt{T-t}}$. Using the short notation $v_{i,j} = v(t_j, x_i)$ and $u^*(t_j, x_i) = u^*_{i,j}$, a mixed implicit-explicit finite difference discretization of (19a) is given by

$$
\begin{aligned}
&\theta\left[\sigma^2(u^*_{i,j}-x_i)^2-\Delta_x(r-\gamma)(u^*_{i,j}-x_i)\right]v_{i-1,j}-\\
&2\left[\theta\sigma^2(u^*_{i,j}-x_i)^2+\alpha(1+\theta\gamma\Delta_t)\right]v_{i,j}+\\
&\theta\left[\sigma^2(u^*_{i,j}-x_i)^2+\Delta_x(r-\gamma)(u^*_{i,j}-x_i)\right]v_{i+1,j}=\\
&-(1-\theta)\left[\sigma^2(u^*_{i,j}-x_i)^2-\Delta_x(r-\gamma)(u^*_{i,j}-x_i)\right]v_{i-1,j+1}+\\
&2\left[(1-\theta)\sigma^2(u^*_{i,j}-x_i)^2-\alpha(1-(1-\theta)\gamma\Delta_t)\right]v_{i,j+1}-\\
&(1-\theta)\left[\sigma^2(u^*_{i,j}-x_i)^2+\Delta_x(r-\gamma)(u^*_{i,j}-x_i)\right]v_{i+1,j+1}
\end{aligned}
\tag{30}
$$

where $0 \le \theta \le 1$ and $\alpha \equiv \Delta_x^2 / \Delta_t$. The boundary condition on this system of equations is

$$
v_{i,N} = x_i^- \tag{31}
$$

In (30), the parameter θ determines the time at which the partial derivatives w.r.t. x are evaluated. If $\theta = 1$, the x-derivatives are evaluated at time t_j and the differencing scheme gives rise to the *fully implicit finite difference method.* If $\theta = 0$, the x-derivatives are evaluated one time-step ahead, at t_{j+1}; and the differencing scheme is known as the *explicit finite difference method,* also sometimes referred to as a *trinomial tree*. Finally, when $\theta = \frac{1}{2}$, the x-derivatives are evaluated at $\frac{1}{2}(t_j + t_{j+1})$ and the scheme becomes the *Crank-Nicholson method.* Values of θ different from 0, 1/2, and 1 are possible, but little used in practice.

For the PDE (19a), the dependence of the coefficients on x makes the explicit scheme prone to instability[8]. We generally prefer the Crank-Nicholson scheme ($\theta = \frac{1}{2}$), which has the highest order of convergence in Δ_t (see Smith (1985) for further discussion).

The discretization scheme (30) must be complemented by an algorithm to find the optimal strategy $u^*_{i,j}$. Many such schemes are possible; here we present an algorithm based directly on the continuous-time result (18). In Section 6.2 we will discuss another approach particularly suitable for discrete passport options.

Step 1:
Make a guess for the values of $u^*_{i,j}$. For example, we can set $u^*_{i,j} = u^*_{i,j+1}$. The starting value of $u^*_{i,N-1}$ at the second-to-last time-step t_{N-1} can be computed analytically using the Black-Scholes formula (as in (26)).

Step 2:
Solve the tri-diagonal system (30) for $v_{i,j}$ in the usual way (see e.g. Press *et al* (1992), p. 50-51). In this step, we must also impose appropriate boundary conditions at x_0 and x_M. For example, we can set $v_{0,j} = 0$ and $v_{M,j} = v_{M-1,j} + (v_{M,j+1} - v_{M-1,j+1})/(1 + r\Delta_t)$, the latter being a discretization of the asymptotic behavior $\partial v / \partial x \to e^{-r(T-t)}$ for large x.

Step 3:
Update the values of $u^*_{i,j}$ using the following discretization of (18):

$$u^*_{i,j} = sign\left(\begin{array}{l} (r-\gamma)\dfrac{\theta(v_{i+1,j} - v_{i-1,j}) + (1-\theta)(v_{i+1,j+1} - v_{i-1,j+1})}{2\Delta_x} - \\ \sigma^2 x_i \dfrac{\theta(v_{i+1,j} - 2v_{i,j} + v_{i-1,j}) + (1-\theta)(v_{i+1,j+1} - 2v_{i,j+1} + v_{i-1,j+1})}{\Delta_x^2} \end{array} \right) \tag{32}$$

Return to Step 2 until the values of $u^*_{i,j}$ calculated in this step remain unchanged from the previous iteration.♠

As the switching boundary u^* normally changes very little from one time-step to the next, typically only one or two iterations are needed in Step 3. We note that the usage of (32) in Step 3 requires v to be convex. In cases where this condition cannot be guaranteed (e.g. for options with more complicated payouts than (31)) Step 3 must be modified. This issue will be discussed in Section 6.3.

5. Numerical Examples

As a specific example, we now introduce a stock S with a time t value of \$100 and a volatility of 30%. We first assume that interest rates and dividends are 0, i.e. $r = \gamma = 0$, and consider the pricing of a European passport option with one year left to maturity. Depending on the value of the current gain, $w(t)$, (29) gives the following theoretical option values $V(t)$[9]:

Theoretical Value of European Passport Option

$S(t) = \$100;\ r = \gamma = 0;\ \sigma = 0.3;\ T - t = 1$

$w(t)$	$V(t)$	$V_1'(t)$	$V_{-1}'(t)$	*Premium*
\$100	\$100.1566	\$100.0000	\$100.1493	\$0.0073
\$50	\$51.6456	\$50.0746	\$51.4859	\$0.1597
\$20	\$25.9063	\$23.5344	\$25.4406	\$0.4658
\$10	\$18.8846	\$17.0129	\$18.1410	\$0.7436
\$0	\$13.1381	\$11.9235	\$11.9235	\$1.2145
-\$10	\$8.8808	\$8.1410	\$7.0129	\$0.7398
-\$20	\$5.8876	\$5.4406	\$3.5344	\$0.4470
-\$50	\$1.5893	\$1.4859	\$0.0746	\$0.1034
-\$100	\$0.1566	\$0.1493	\$0.0000	\$0.0073

Table 1

In the table above, we have also listed the values of the at-the-money call ($V_1'(t)$) and the at-the-money put ($V_{-1}'(t)$) defined in (24a-b). The "premium" column in the table is defined as $V(t) - MAX\left(V_1'(t), V_{-1}'(t)\right)$ and, as mentioned earlier, can be interpreted as the value of the future rights to switch the position in S. As expected, this value is highest around $w(t) = 0$; i.e. when there is the most uncertainty about the future sign of $w(t)$ (and thus, from (23), the sign of the asset position $u(t)$).

In Figure 1 below we have graphed V, $V_1'(t)$, and $V_{-1}'(t)$ against w. The peak of the premium $V(t) - MAX\left(V_1'(t), V_{-1}'(t)\right)$ at $x = 0$ is evident.

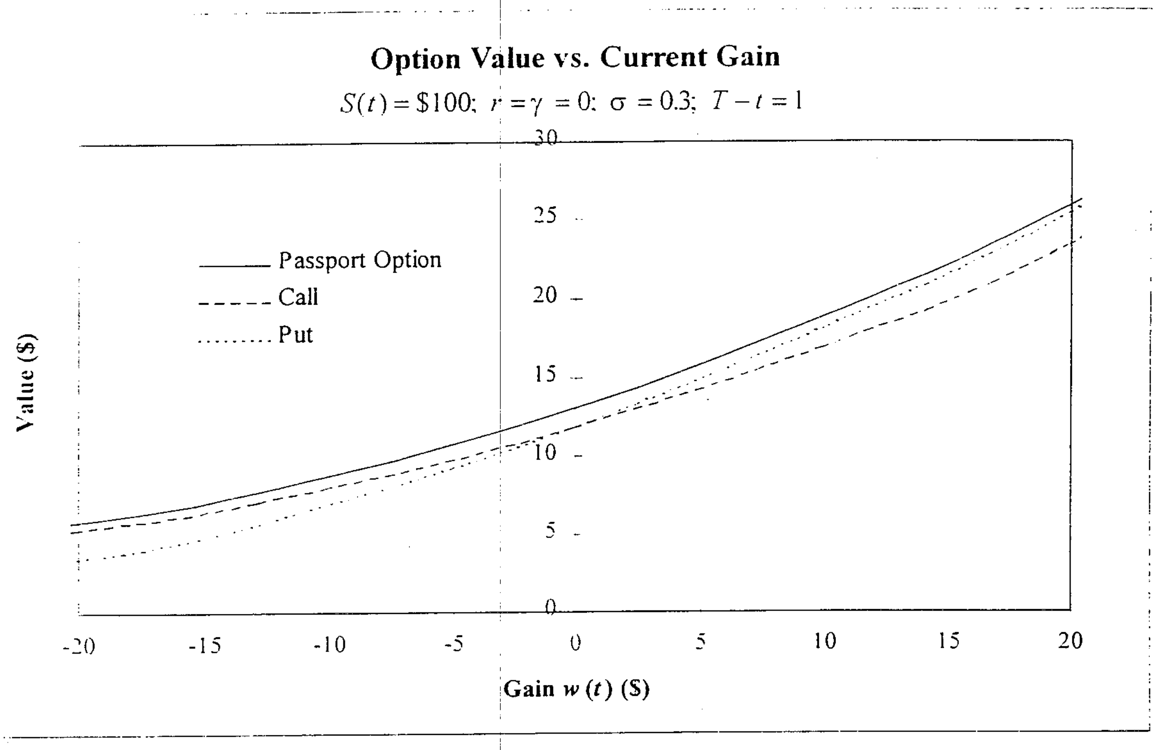

Figure 1

We now turn to the application of the finite difference scheme outlined in Section 4. Using a Crank-Nicholson scheme ($\theta = \frac{1}{2}$), the same process parameters as above, and $w(t) = 0$, we get the following prices as a function of the grid size:

Crank-Nicholson Numerical Value of European Passport Option

$S(t) = \$100$; $r = \gamma = 0$; $\sigma = 0.3$; $T - t = 1$; $w(t) = \$0$

Theoretical Value = $13.1381

x-steps (M)	*Time-steps (N)*		
	25	50	100
50	$13.0834	$13.0830	$13.0829
100	$13.1240	$13.1236	$13.1235
200	$13.1349	$13.1345	$13.1343
400	$13.1392	$13.1373	$13.1372

Table 2

In generating Table 2. we have used odd-even averaging (i.e. the prices reported in the table are the averages of the prices computed with M and M+1 x-steps). The CPU time when M=400 and N=100 was 0.85 seconds (166 MHz Pentium).

We see from the table above that i) the numerical scheme converges to the theoretical value, and ii) the number of x-steps is clearly more important than the number of time-steps. In Table 3 below, we have fixed the number of time-steps to N=50 and listed option prices for various values of w (with odd-even averaging). Again, it is clear that the numerical scheme converges to the correct values.

Crank-Nicholson Numerical Value of European Passport Option

$S(t) = \$100;\ r = \gamma = 0;\ \sigma = 0.3;\ T - t = 1$

$N = 50$

	Gain w(t)				
x-steps (M)	-\$20	-\$10	\$0	\$10	\$20
50	\$5.9074	\$8.9222	\$13.0830	\$18.9222	\$25.9074
100	\$5.8900	\$8.8745	\$13.1236	\$18.8745	\$25.8900
200	\$5.8884	\$8.8794	\$13.1345	\$18.8794	\$25.8884
400	\$5.8879	\$8.8807	\$13.1373	\$18.8807	\$25.8879
Theoretical	\$5.8876	\$8.8808	\$13.1381	\$18.8846	\$25.9063

Table 3

Now, let us turn to an example where $r \neq \gamma$. Specifically let us set[10] r = 5% and γ = 4.5%, and consider a 2-year passport option. With N = 100 and M = 800, the Crank-Nicholson scheme gives the following results (with odd-even averaging):

Crank-Nicholson Numerical Value of European Passport Option

$S(t) = \$100;\ r = 5\%;\ \gamma = 4.5\%;\ \sigma = 0.3;\ T - t = 2$

$N = 100;\ M = 800$

$w(t)$	$V(t)$	$V_1'(t)$	$V_{-1}'(t)$	*Premium*
$20	$28.2277	$25.2453	$26.7334	$1.4943
$10	$22.3741	$20.0208	$20.4249	$1.9492
$0	$17.4323	$15.7362	$14.8268	$1.6961
-$10	$13.5100	$12.2859	$10.0631	$1.2241
-$20	$10.4261	$9.5460	$6.2392	$0.8801

Table 4

In the figure below, we have graphed the optimal holding strategy u^* as observed in the finite difference lattice; notice the perhaps surprising presence of three distinctive regions.

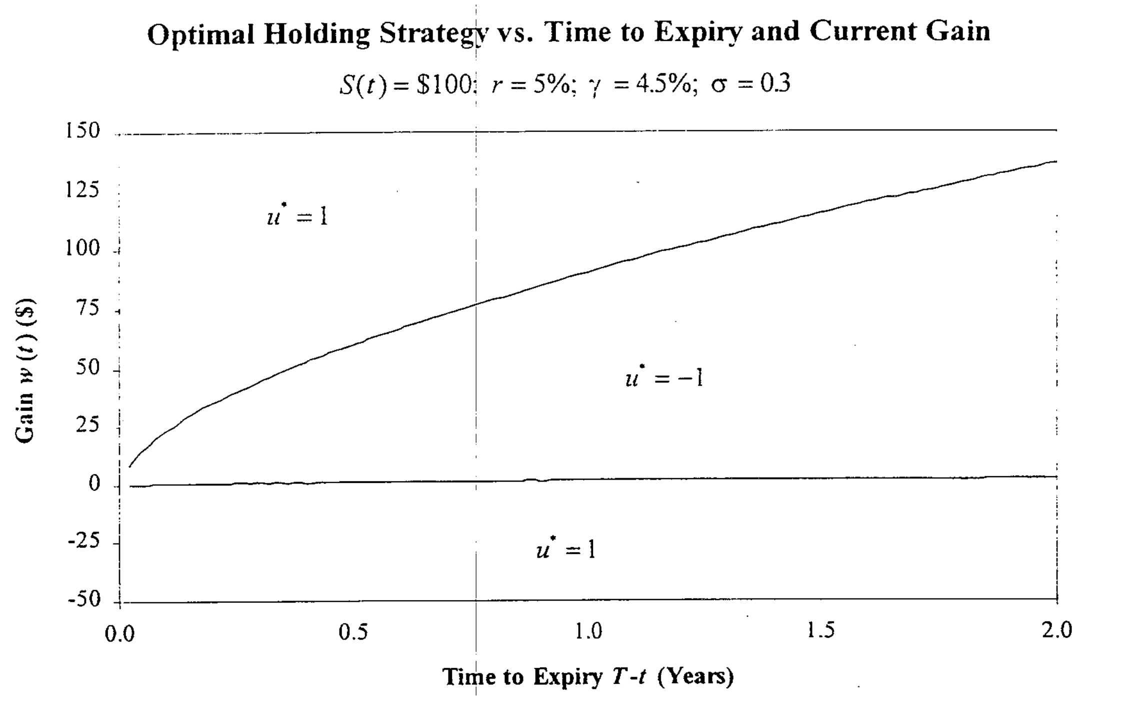

Figure 2

6. Extensions

6.1. American Exercise

So far we have limited our discussion of the passport option to the European variety where the payout can only take place at the final maturity date. In this section we will consider the pricing of the American passport option which can be exercised at any time before the final maturity. Using τ to denote the time of exercise, a rational exercise policy is characterized the following equation for the S-deflated option price

$$v(t) = \sup_{\tau,u} E_t^{Q'}\left[e^{-\gamma(\tau-t)}x(\tau)^+\right] \qquad (33)$$

where the supremum is taken over stopping times τ on $[t,T]$ and, as before, $u(s)$, $s \in [t,T]$, is an adapted process taking values in [-1,1]. (33) is a natural extension of equation (10), but is not suitable for actual computations. For this purpose, we state the following formulation of v as the solution to a free boundary problem:

Proposition 5

Let $A = \{(t,x) \in [0,T) \times \Re : v(t,x) > x^+\}$ denote the continuation region of the American passport option. The S-deflated value of the American passport option satisfies the PDE

$$\frac{\partial v}{\partial t} - (u^* - x)(r-\gamma)\frac{\partial v}{\partial x} + \frac{1}{2}(u^* - x)^2\sigma^2\frac{\partial^2 v}{\partial x^2} = \gamma v, \qquad (35a)$$

$$u^* = u^*(t,x) = sign\left((r-\gamma)\frac{\partial v}{\partial x} - \sigma^2 x\frac{\partial^2 v}{\partial x^2}\right) \qquad (35b)$$

on A, subject to the free boundary condition, for all (t,x),

$$v(t,x) \geq x^+ \qquad (35c)$$

and subject to the terminal boundary condition

$$v(T,x) = x^{+}. \tag{35d}$$

Proof:

As in the proof of Proposition 2, we first note that the structure of the SDE (11) for x allows us to only consider holding strategies of the form $u = u(s,x(s))$ and stopping times of the form

$$\tau = \inf\{s \geq t : (s,x(s)) \in B\}$$

where B is a stopping region, a subset of $[0,T] \times \Re$ (see e.g. Duffie (1996), p. 172-178). For a particular, not necessarily optimal, stopping time τ of the above form, we introduce

$$v^{\tau}(t,x(t)) = \sup_{u \in [-1,1]} E_t^{Q'}\left[e^{-\gamma(\tau-t)}x(\tau)^{+}\right] \tag{36}$$

such that

$$v(t,x(t)) = \sup_{\tau} v^{\tau}(t,x(t)) = v^{\tau^*}(t,x(t))$$

where τ^* is the optimal exercise strategy. If $(t,x(t)) \in B$ then, by definition, $v^{\tau}(t,x(t)) = x(t)^{+}$. Otherwise, when $(t,x(t)) \in ([0,T] \times \Re) \setminus B$, we use the fact that the HJB equation can be extended to cases where the bequest function (here: x^{+}) is realized at a stopping time of the form above (see Øksendahl (1993), p. 213-215). That is, the optimal holding strategy u_{τ}^{*}, given the exercise policy, is such that

$$\sup_{h \in [-1,1]}\left\{\frac{\partial v^{\tau}}{\partial t} - (h-x)(r-\gamma)\frac{\partial v^{\tau}}{\partial x} + \frac{1}{2}(h-x)^2\sigma^2\frac{\partial^2 v^{\tau}}{\partial x^2} - \gamma v^{\tau}\right\} \tag{37}$$

attains its maximum at $h = u_{\tau}^{*}(t,x(t))$. As the convexity property outlined in Lemma 1 can easily be verified to hold for v^{τ}, we get, as in Proposition 2, that u_{τ}^{*} is given by

$$u_{\tau}^{*} = sign\left((r-\gamma)\frac{\partial v^{\tau}}{\partial x} - \sigma^2 x\frac{\partial^2 v^{\tau}}{\partial x^2}\right) \tag{38}$$

for $(t, x(t)) \in ([0,T] \times \Re) \setminus B$. (37) and (38) hold for all exercise strategies, including the optimal strategy τ^*. For $\tau = \tau^*$, v^τ becomes v, u_τ^* becomes u^*, and the set $([0,T] \times \Re) \setminus B$ becomes the continuation region A. (35a-d) follow.♠

We notice that the PDE (35a) is similar to the one solved for the European option, equation (19a), although obviously the boundary conditions and the valid region for the PDE has changed. To incorporate early exercise into the numerical scheme in Section 4, we consequently just apply the usual technique of setting

$$v_{i,j} = MAX(\hat{v}_{i,j}, x_i^+) \qquad (39)$$

where $\hat{v}_{i,j}$ is the solution to the tri-diagonal system (30).

As an application of (39), consider the 2-year option from Section 5 where r = 5% and γ = 4.5%. In Table 5 below, we have listed the values of the American and European passport options, along with the values of corresponding American calls and puts. In the table, the "exercise premium" is the excess of the American passport option over the European passport option, and the "switching premium" is the excess of the American passport option over the largest of the American put and call. Not surprisingly the exercise premium is an increasing function of w.

Crank-Nicholson Numerical Value of European and American Passport Option

$S(t) = \$100$; $r = 5\%$; $\gamma = 4.5\%$; $\sigma = 0.3$; $T - t = 2$

$N = 100$; $M = 800$

$w(t)$	*European* $V(t)$	*American* $V(t)$	*American* $V_1'(t)$	*American* $V_{-1}'(t)$	*Exercise Premium*	*Switching Premium*
\$20	\$28.2277	\$29.1764	\$25.9756	\$27.8907	\$0.9487	\$1.2857
\$10	\$22.3741	\$23.0050	\$20.4814	\$21.1853	\$0.6309	\$1.8197
\$0	\$17.4323	\$17.8418	\$16.0218	\$15.2867	\$0.4095	\$1.8200
-\$10	\$13.5100	\$13.7776	\$12.4762	\$10.3297	\$0.2676	\$1.3014
-\$20	\$10.4261	\$10.6031	\$9.6682	\$9.6682	\$0.1170	\$0.9349

Table 5

Figure 3 illustrates how the early exercise boundary interacts with the regions for u^*. Comparison with Figure 2 shows that the right to early exercise in this case "cuts off" most of the upper holding strategy region.

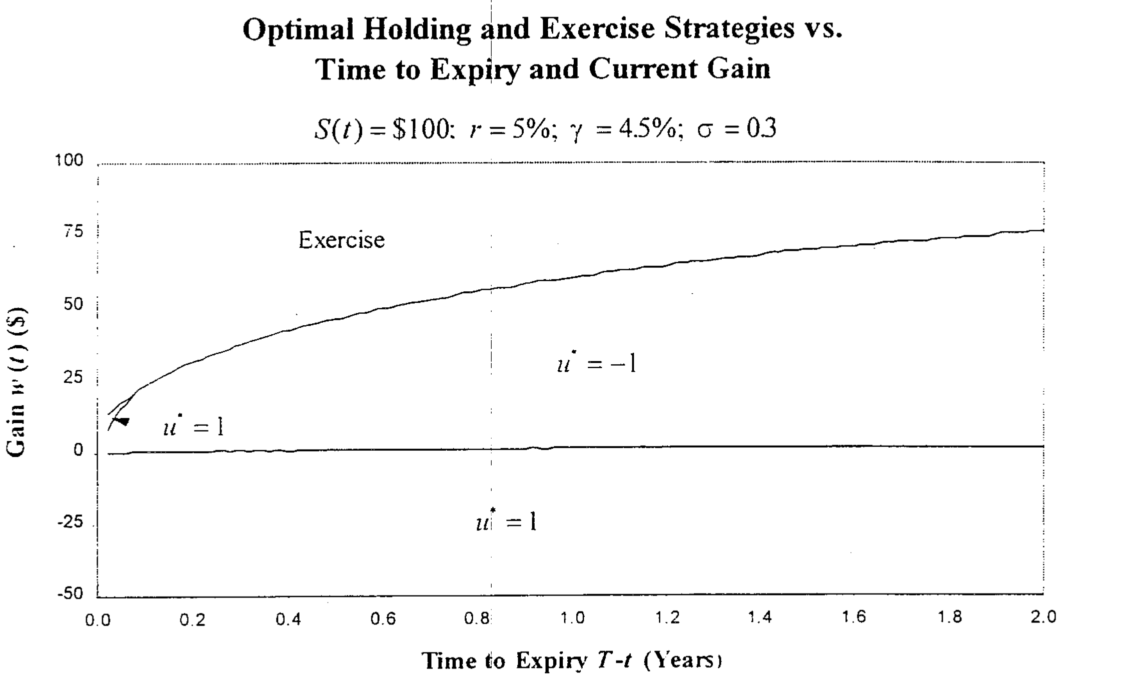

Figure 3

6.2. Discrete Passport Options

We now return to the case discussed in Section 1 where the holding strategy u can only be modified at H discrete times, $t_0, t_1, t_2, \ldots, t_{H-1}$, where $t_0 = 0$ and $t_H = T$. The S-deflated option price now solves the PDE

$$\frac{\partial v}{\partial t} - \left(u^*(t_i, x(t_i)) - x\right)(r - \gamma)\frac{\partial v}{\partial x} + \frac{1}{2}(u^*(t_i, x(t_i)) - x)^2 \sigma^2 \frac{\partial^2 v}{\partial x^2} = \gamma v \qquad (40a)$$

on $t \in (t_i, t_{i+1})$ subject to the boundary conditions

$$v(t_{i+1}-, x) = v(t_{i+1}+, x) \qquad (40b)$$

$$v(T,x) = x^- . \tag{40c}$$

Here we have used the notation $\lim_{\varepsilon\to 0}(z+|\varepsilon|) = z+$ and $\lim_{\varepsilon\to 0}(z-|\varepsilon|) = z-$. (40b) represents a set of so-called *jump conditions* which ensure that the option price is continuous for all t (see Wilmott *et al* (1993), p. 138-139 for a simple economical justification). Assume now that $v(t_{i+1}+,x)$ is known, and let $v_{-1}(t,x)$ and $v_{+1}(t,x)$ denote the solutions to (40a-d) on $t \in (t_i,t_{i+1})$ when $u(t_i,x(t_i)) = -1$ and $u(t_i,x(t_i)) = 1$, respectively. The value of the option at time t_i+ is then given by

$$v(t_i+,x) = MAX\left(v_{-1}(t_i+,x), v_{+1}(t_i+,x)\right) \tag{41}$$

with the optimal holding strategy

$$u^*\left(t_i,x(t_i)\right) = sign\left(v_{-1}(t_i+,x(t_i)) - v_{-1}(t_i+,x(t_i))\right) \tag{42}$$

To solve these equations numerically on $t \in (t_i,t_{i+1})$, we apply the finite difference scheme (30) on v_{-1} and v_{+1} with the known boundary condition (from (40b)):

$$v_{-1}(t_{i+1}-,x_j) = v_{+1}(t_{i+1}-,x_j) = v(t_{i+1}+,x_j)$$

After having solved for $v_{-1}(t_i+,x)$ and $v_{+1}(t_i+,x)$, (41) is applied and we can move on to the next interval $t \in (t_{i-1},t_i)$.

In the table and graph below, we display the results of applying the above numerical scheme to the case where $S(t) = \$100$, $r = \gamma = 0$, $\sigma = 0.3$, $T - t = 1$, and $w(t) = \$0$. We assume that the switching dates are equidistant, $t_i = iT/H$. For all values of H in the table, we have used a 500 by 500 finite difference lattice. As expected, the price of the discrete passport option is an increasing function of H.

Crank-Nicholson Numerical Value of Time-Discrete Passport Option

$S(t) = \$100;\ r = \gamma = 0;\ \sigma = 0.3;\ T - t = 1;\ w(t) = \$0$

$N = 500;\ M = 500$

H	$V(t)$
1	\$11.9225
2	\$12.3283
5	\$12.7408
10	\$12.9714
20	\$13.0186
50	\$13.0860
100	\$13.1102
250	\$13.1255
500	\$13.1308
$H = \infty$ (Analytic)	\$13.1381

Table 6

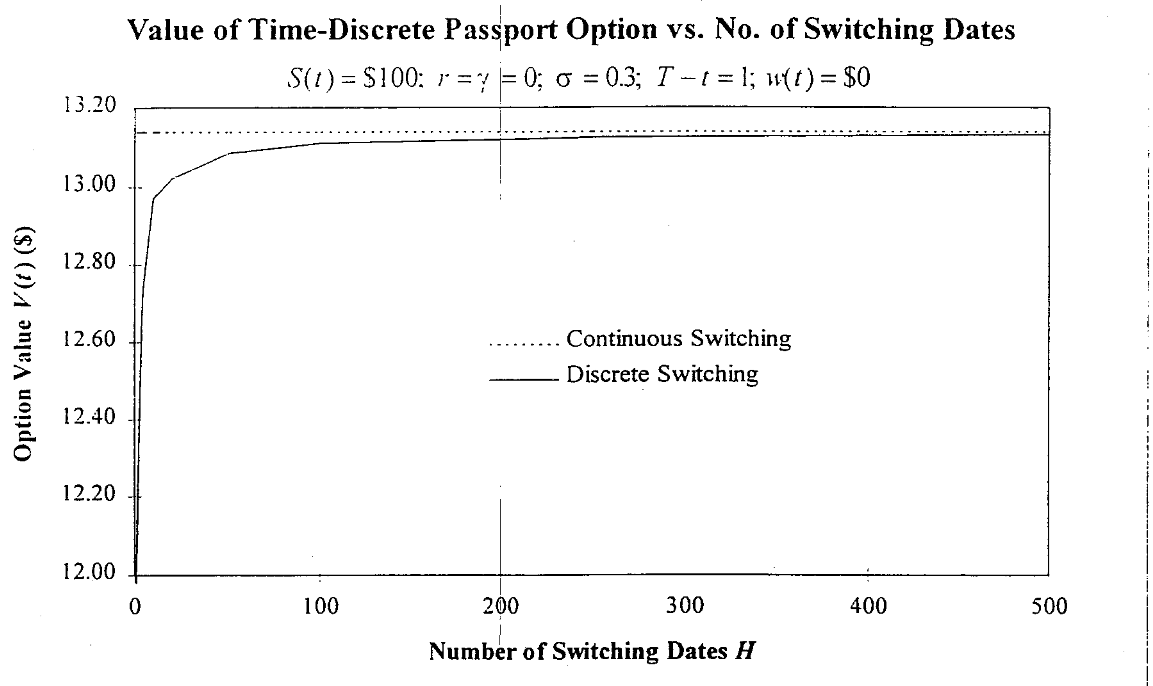

Figure 4

While the scheme above has been designed specifically with time-discrete switching in mind, we notice that application of extrapolation methods should allow us to price continuous passport options as well. We also note that American exercise can be incorporated in the scheme using the same techniques as in Section 6.1.

6.3. Non-convex Price Functions

So far we have exclusively considered the option payout $V(T) = w(T)^+$ and, as shown in Lemma 1, have always been able to rely on convexity of the S-deflated price function $v(t) = V(t)/S(t)$. To handle option payouts where convexity cannot be guaranteed, we first state the following generalization of Proposition 2:

Proposition 4:

Consider an option with a payout at time T,

$$V(T) = S(T)\phi\left(w(T)/S(T)\right)$$

for some sufficiently regular function $\phi(\cdot)$. $v = v(t,x)$ satisfies the PDE

$$\frac{\partial v}{\partial t} + (u^* - x)(r-\gamma)\frac{\partial v}{\partial x} + \frac{1}{2}(u^* - x)^2\sigma^2\frac{\partial^2 v}{\partial x^2} = \gamma v \qquad (42\backslash 3)$$

with boundary condition

$$v(T) = \phi\left(x(T)\right).$$

Defining $\psi(t,x) \equiv x - \frac{(r-\gamma)}{\sigma^2}\frac{\partial v}{\partial x} / \frac{\partial^2 v}{\partial x^2}$, the optimal strategy $u^*\left(t,x(t)\right) \in [-1,1]$ is given by

$$u^*(t,x) = \begin{cases} \psi(t,x), & \textit{if } \psi \in [-1,1] \textit{ and } \partial^2 v/\partial x^2 < 0 \\ sign\left(\psi(t,x)\right), & \textit{otherwise} \end{cases} \qquad (44)$$

Proof:

The proof of (43) is identical to the proof of (19a). (44) is obviously identical to (18) in case $\partial^2 v / \partial x^2 \geq 0$. For $\partial^2 v / \partial x^2 < 0$, consider equation (21) in the proof of (18),

$$\sup_{h \in [-1,1]} \{C_3 + h(C_1 - 2xC_2) + h^2 C_2\}$$

As C_2 now is negative, we differentiate w.r.t. h and obtain the first-order condition for the maximum

$$h = x - \frac{C_1}{2C_2}.$$

The definitions of C_1 and C_2 (in the proof of Proposition 2), and the restriction that $h \in [-1,1]$ lead to (44).♠

We see from (44) that when v is concave, the optimal control is no longer of the pure "bang-bang" type.

To adapt the numerical algorithm in Section 4 the options covered by Proposition 4, we note that the only changes necessary involve modifying the terminal payout condition (31) to

$$v_{i,N} = \phi(x_i) \tag{45}$$

and changing the computation of the optimal holding strategy (32) to

$$u^*_{i,j} = \begin{cases} \psi_{i,j}, & if \ \psi_{i,j} \in [-1,1] \ and \ \theta(v_{i+1,j} - 2v_{i,j} + v_{i-1,j}) + (1-\theta)(v_{i-1,j+1} - 2v_{i,j+1} + v_{i-1,j+1}) < 0 \\ sign(\psi_{i,j}), & otherwise \end{cases} \tag{46}$$

where

$$\psi_{i,j} = x_i - \frac{\Delta_x (r - \gamma)}{2\sigma^2} \frac{\theta(v_{i+1,j} - v_{i-1,j}) + (1-\theta)(v_{i+1,j+1} - v_{i-1,j+1})}{\theta(v_{i+1,j} - 2v_{i,j} + v_{i-1,j}) + (1-\theta)(v_{i+1,j+1} - 2v_{i,j+1} + v_{i-1,j+1})}.$$

Notice that the presence of the term $\frac{\partial v}{\partial x} / \frac{\partial^2 v}{\partial x^2}$ in the optimal holding strategy for concave v can potentially lead to numerical difficulties if both derivatives are close to zero. In general, the numerical properties of the above scheme must be examined on a case-by-case basis.

As an alternative to the scheme discussed above, we can also use the algorithm from Section 6.2, which also allows us to consider discretely sampled options. As the optimal holding strategy is no longer guaranteed to be either 1 or -1, we need to replace (42) with

$$v(t_i+,x) = \sup_{u\in[-1,1]}\{v_u(t_i+,x)\}$$

This is a simple univariate optimization problem which in most cases is easy to solve using standard methods.

Again, both schemes discussed above can be extended to American exercise using the techniques discussed in Section 6.1.

7. Summary

In this paper we have discussed numerical and analytical approaches to the pricing of passport options with continuous and discrete switching rights. While certain general analytical results are possible in the continuous case, only in special circumstances does it appear possible to derive closed-form pricing formulas; in most cases, one must instead rely on numerical schemes for price and hedge computations. As the usual binomial and trinomial lattices are ill-suited for the passport option price PDE, the numerical methods developed in this paper are instead based on a Crank-Nicholson finite difference scheme. The developed methods are stable and fast, and can readily handle both early (American) exercise, time-discrete holding strategies, and more complicated option payouts.

To conclude the paper, let us point out that while the paper was set in an idealized framework with constant interest rates, volatilities, and dividends, the numerical scheme as well as many of the analytical results apply equally well to the case where these parameters are functions of time. With time-varying parameters, the holding strategy boundaries are likely to be quite complicated. Another relevant extension that we leave to future research is the incorporation of possible penalty costs paid by the option holder whenever the holding strategy is reversed from short to long or long to short.

Appendix A

The Instantaneous Hedge

We consider a portfolio consisting of one passport option and -k units of stock. The time t value of this portfolio is simply

$$\Pi = V - kS\,. \tag{A.1}$$

Now, over the short interval (t, $t+dt$), the value of the portfolio changes by an amount

$$d\Pi = dV - k(dS + \gamma S dt)\,. \tag{A.2}$$

Under the optimal strategy u^*, $V = V(t, S, w)$. Assuming sufficient differentiability of V, we can write

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial w}dw + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{\partial^2 V}{\partial S \partial w}(dSdw) + \frac{1}{2}\frac{\partial^2 V}{\partial w^2}(dw)^2\,.$$

Using (3) and (4b) to write $dw = u^* dS$ and $(dS)^2 = \sigma^2 S^2 dt$, we get

$$dV = \frac{\partial V}{\partial t}dt + \left(\frac{\partial V}{\partial S} + u^*\frac{\partial V}{\partial w}\right)dS + \frac{1}{2}\left(\frac{\partial^2 V}{\partial S^2} + 2u^*\frac{\partial^2 V}{\partial S \partial w} + \left(u^*\right)^2\frac{\partial^2 V}{\partial w^2}\right)\sigma^2 S^2 dt\,. \tag{A.3}$$

Combining (A.2) and (A.3), we notice that we can make the portfolio Π instantaneously risk-free by selecting

$$k = \frac{\partial V}{\partial S} + u^*\frac{\partial V}{\partial w} \tag{A.4}$$

With the hedge ratio choice (A.4), the portfolio must earn an instantaneous return of r, i.e. $d\Pi = r\Pi dt$. Combination of (A.2), (A.3), and (A.4) then yields

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\left(\frac{\partial^2 V}{\partial S^2} + 2u^*\frac{\partial^2 V}{\partial S \partial w} + \left(u^*\right)^2\frac{\partial^2 V}{\partial w^2}\right) + (r-\gamma)S\left(\frac{\partial V}{\partial S} + u^*\frac{\partial V}{\partial w}\right) = rV\,. \tag{A.5a}$$

The payoff condition at $t = T$ is given by

$$V(T,S,w) = w^{+} . \tag{A.5b}$$

Equations (A.5a-b) allow us to write the solution as

$$V(t,S,w) = Sv(t,x), \quad x \equiv w/S . \tag{A.6}$$

Substitution of (A.6) into (A.5a-b) results in the following PDE for $v(t,x)$:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^{2}(u^{*} - x)^{2}\frac{\partial^{2} v}{\partial x^{2}} + (u^{*} - x)(r - \gamma)\frac{\partial v}{\partial x} = \gamma v, \quad v(T,x) = x^{+} .$$

Note that, in terms of the function $v(t,x)$, the required hedge ratio (A.4) is given by

$$k = v + (u^{*} - x)\frac{\partial v}{\partial x} . \tag{A.7}$$

Appendix B

Solution of Equation (23)

We first consider the homogenous case $r = \gamma = 0$. Let

$$\tau = \sigma^2 (T - t), \; v(t,x) = f(\tau, x) \tag{B.1}$$

so that (24)-(25) becomes

$$\frac{\partial f}{\partial \tau} = \frac{1}{2}(1+|x|)^2 \frac{\partial^2 f}{\partial x^2}, \quad f(0,x) = x^+. \tag{B.2}$$

The Laplace transform $\hat{f}(s,x)$ of $f(\tau,x)$, given by

$$\hat{f}(s,x) = \int_0^\infty e^{-s\tau} f(\tau,x) d\tau$$

satisfies the ordinary differential equation

$$\frac{1}{2}(1+|x|)^2 \frac{\partial^2 \hat{f}}{\partial x^2} - s\hat{f} = -x^{-} \tag{B.3}$$

with $\hat{f} \sim x/s$ as $x \to \infty$, and $\hat{f} \to 0$ as $x \to -\infty$. The solution to (B.3) is readily found to be

$$\hat{f}(s,x) = \frac{x^+}{s} + \frac{(1+|x|)^{(1-\sqrt{1+8s})/2}}{s\left(\sqrt{1+8s} - 1\right)}. \tag{B.4}$$

Both $\hat{f}$ and $\partial \hat{f} / \partial x$ are continuous at $x = 0$.

Using standard inversion formulae (see, for example, Abramowitz & Stegun (1972), p. 1020-1030), we obtain the solution

$$f(\tau,x) = x^{+} + N\big(z(\tau)\big) - (1+|x|)N\big(z(\tau) - \sqrt{\tau}\big) + \tfrac{1}{4}\int_0^{\tau} N\big(z(u)\big)du, \tag{B.5}$$

$$z(\tau) = \frac{-\ln(1+|x|)}{\sqrt{\tau}} + \frac{\sqrt{\tau}}{2}.$$

(B.5) holds for $r = \gamma = 0$. For $r = \gamma \neq 0$, we note that if v solves (23) when $\gamma = 0$, $ve^{-\gamma(T-t)}$ solves (23) for $\gamma \neq 0$. Using this result and reversing the variable shift (B.1) leads to equation (29).

To investigate the behavior of (B.5) around $x = 0$, we first notice that

$$f(\tau,0) = (1 + \tfrac{1}{4}\tau)N\big(\tfrac{1}{2}\sqrt{\tau}\big) + \tfrac{1}{2}\sqrt{\tau}N'\big(\tfrac{1}{2}\sqrt{\tau}\big) - \tfrac{1}{2}. \tag{B.6}$$

As $z(\tau) = \frac{1}{2}\sqrt{\tau} - \frac{|x|}{\sqrt{\tau}} + O(x^2)$ and $N\big(z(\tau)\big) = N\big(\tfrac{1}{2}\sqrt{\tau}\big) - \frac{|x|}{\sqrt{\tau}}N'\big(\tfrac{1}{2}\sqrt{\tau}\big) + O(x^2)$, we get from (B.5) and (B.6):

$$\begin{aligned}
f(\tau,x) &= f(\tau,0) + x^{+} + |x|\Big(N\big(\tfrac{1}{2}\sqrt{\tau}\big) - 1\Big) - \tfrac{1}{4}|x|\int_0^{\tau} u^{-1/2}N'\big(\tfrac{1}{2}\sqrt{u}\big)du + O(x^2) \\
&= f(\tau,0) + x^{+} - \tfrac{1}{2}|x| + O(x^2) \\
&= f(\tau,0) + \tfrac{1}{2}x + O(x^2)
\end{aligned}$$

where the first and second equalities require a few straightforward manipulations. We conclude that f (or v) is at least twice differentiable at $x = 0$.

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Endnotes

[1]The name of this option is, to the best of our knowledge, due to Peter Carr.

[2]Indeed, it appears that the option originally was created as a hedging tool for corporations uncertain about the day-to-day changes in the magnitude and sign of their exposure to FX rates. The passport option is also a useful tool for speculation in short-term movements of stock or FX levels.

[3]While we use the word "stock", S could equally well be an FX rate or a commodity. The "dividend" γ should then be interpreted as the foreign interest rate or the net of convenience yield and storage costs, respectively.

[4]This definition of the gain process implicitly assumes that the passport option holder is not entitled to any of the dividends paid by the asset. Reinvestment of dividends can, however, easily be accounted for by simply setting $\gamma = 0$.

[5]This is due to the fact that the representation (4a-b) implies an infinite number of rights to switch from long to short. In (2), the option holder has only a finite number (N) of such rights. Notice also that choosing a piecewise constant strategy u in (4a-b) allows us to replicate any discrete strategy.

[6]This result also holds in the case of infinitely many switch dates. When $dS(t) = \sigma dW(t)$ under Q, (20) becomes $\sup_{h\in[-1,1]}\left\{\frac{\partial v}{\partial t} + \frac{1}{2}h^2\sigma^2\frac{\partial^2 v}{\partial x^2}\right\} = 0$ which shows that $u^*(t,x(t)) = \pm 1$.

[7]In the perpetual case, $T = \infty$, the integral can be evaluated as a weighted sum of two Bessel functions.

[8]In the trinomial tree analogy, instability of the explicit scheme is basically equivalent to negative branching probabilities in the tree. An analysis of (30) reveals that negative probabilities are virtually impossible to avoid.

[9]A 200-step Simpson scheme was used to evaluate the integral in (29).

[10]We let r and γ be quite close to ensure that the optimal strategy boundary is reasonably interesting. If $r >> \gamma$, the "call" strategy, $u = 1$, tends to dominate the "put" strategy $u = -1$.

The Pricing of Discretely Sampled Asian and Lookback Options: A Change of Numeraire Approach[1]

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[1]I would like to thank Peter Honore for valuable suggestions on the Monte Carlo simulations and William Keirstead for discussions on this topic.

Abstract

This paper considers the pricing of four types of exotic options with terminal pay-offs that depend on discrete samples of the path of the underlying asset. These options are: the Asian option with fixed strike, the average strike option (an Asian option with floating strike), the lookback option with floating strike, and the lookback option with fixed strike. In a Black & Scholes (1973) framework we are able to describe these option prices as functions of current time and a non-continuous *one*-dimensional state variable. The main trick used to obtain this is an appropriate change of numeraire. The discontinuity of the state variable is limited to the dates over which the arithmetic average or the extrema is calculated. This means that the option prices can be calculated by numerically solving a sequence (in the time dimension) of partial differential equations in one variable in addition to time. Computationally this is not harder than numerically solving one standard partial differential equation in the Black-Scholes model. The valuation of the fixed strike Asian and lookback options are shown to have a similarity: It is possible to reformulate the pricing problems of these options as barrier option pricing problems. For the floating strike options it is also possible to treat the American exercise using our technique. We illustrate the speed and the accuracy of our technique by numerical examples where we compare our prices to prices generated by Monte Carlo simulations. By numerical examples we also illustrate that discrete rather than continuous sampling have quite a large effect on the prices of the options considered in this paper. This is particularly the case for the lookback options. Finally we show that our technique also can be applied to the case when the underlying has discontinuous dynamics. In this case we show that the option contracts can be found by numerical solving partial *integro* differential equations.

1 Introduction

Exotic options that have pay-offs that depend on the arithmetic average, the maximum, or the minimum of the underlying stock over a certain time period have become increasing popular hedging and speculation instruments over recent years. Parallel to that, a growing body of litterature has considered the pricing and hedging of such derivatives. Within the Black & Scholes (1973) model closed form solutions have been obtained for lookback option prices by Goldman, Sosin & Gatto (1979*a*), Goldman, Sosin & Shepp (1979*b*), and Conze & Viswanathan (1991). No closed form solution has yet been derived for Asian option prices, but various transforms and approximations have been obtained, see for example Geman & Yor (1993) and Rogers & Shi (1995).

The closed form solutions for the lookback options are based on the assumption that the maximum is taken over the whole continuous path of the underlying. But, for most traded lookback options the maximum is not based on daily highs of the underlying over the whole life of the option. The maximum is rather based on daily closing prices over either the whole life of the option or only over a discrete number of trading days. For such contract specifications the assumption of continuous observations seems as a poor approximation. The same goes for the average options. The approximations obtained for the options depending on the arithmetic average are also based on the assumption that the average is sampled over continuous intervals, typically the whole life of the option. However, all traded Asian options depend on averages sampled over a discrete and often a low number of trading days. The consequence is that in practice one has to resolve to Monte Carlo simulations in order to price these types of contracts.

In this paper we suggest a simple and computationally efficient alternative to Monte Carlo simulations for four types of path-dependent options: The Asian option, the average strike option, the lookback option with fixed strike, and the lookback option with floating strike. The idea is to make use of change of numeraire techniques to obtain that the option prices are functions of time and a *one*-dimensional Markov state variable. The technique has previously been applied to the pricing of lookback options with floating strike by Babbs (1992), but as indicated, this paper extends the methodology to the pricing of three other types of path-dependent options.

Due to the discrete observations, the state variables involved here exhibit jumps at the observation points with probability one. However, in between two observation points the state variable evolves continuously, so it is possible to describe the option price as the solution to a standard partial differential equation (PDE) in such a region. Letting the first PDE generate the terminal boundary condition of the second and so forth we obtain a sequence of PDEs that can be solved numerically by finite difference techniques. In the paper we employ Crank-Nicolson schemes for the numerical solution of the pricing problems. We could in fact also set up binomial or trinomial trees for the numerical solution but we choose not to for two reasons. First, the non-standard dynamics of the state variable yields problems with the stability of such trees. That is, one has to take unreasonable small time steps in order to insure the stability of the numerical solution. Second, the nature of the pricing problems are similar to barrier option pricing problems. Trees give rise to odd-even problems for such pricing problems, see for example Boyle & Lau (1994).

As mentioned the fixed strike option pricing problems for both the lookback and the Asian option can be converted into barrier option pricing problems. This is rather surprising given the nature of the original pricing problems. However, the determining state variables that we identify here have a "barrier" type of behavior, in the sense that if they go through a certain level, typically at-the-money, their dynamics become more tractable and it is possible to derive the risk-adjusted expectation of the terminal pay-off in closed form.

For the floating strike options it is also possible to apply the PDE technique to the pricing of options with an American feature. We illustrate that the American feature can have a quite dramatic effect on the prices of average strike options.

We provide numerical examples that illustrate the speed and the accuracy of our procedures. Our benchmark is Monte Carlo simulations with a large number of samples combined with a control variate technique. In most cases the finite difference solutions get within penny accuracy compared to the Monte Carlo solutions in less than one second of CPU time.

Further numerical examples show that the discrete rather than continuous sampling of either the average or the maximum has quite an effect on the option prices. This is especially the case when we consider the lookback options, but the effect is also significant for options on the arithmetic average.

Finally we show how the technique can be applied to the case when the underlying exhibits discontinuous dynamics. Our model is in this case a "risk-neutralised" version of the Merton (1976) model where the jumps are triggered by a Poisson process and the jumps in return are displaced lognormal distributed. In this case the sequence of PDEs is replaced by a sequence of partial integro differential equations (PIDEs), that also can be solved by finite difference techniques. We apply such an algorithm and illustrate the effect of jumps on the prices of the options considered here, by numerial examples.

The paper is organized as follows. The first section shortly describes the modelling framework and the main trick applied in this paper: the change of numeraire for the martingale measure. We then have a section for each of the considered options, in respective order these are: the Asian (fixed) strike option, the average strike option, the fixed strike lookback and the floating strike lookback option. Each section contains numerical examples of the accuracy and the speed of our solution procedure. Then a section follows that considers the effect of discrete versus continuous sampling. The final section of the paper shows how our technique also can be applied to non-continuous dynamics of the underlying stock.

2 The Model and Change of Numeraire

We start by considering the standard Black-Scholes economy with two assets: a dividend paying stock and a money market account. We will later extend the model to cover the case when the underlying exhibits discontinuous dynamics.

We assume the existence of an equivalent martingale measure, $\mathcal{Q}$, under which all discounted security prices (including accumulated dividends) are martingales. This assumption implies absence of arbitrage.

Under $\mathcal{Q}$ the stock is assumed to evolve according to the stochastic differential equation

$$\frac{dS(t)}{S(t)} = (r-q)dt + \sigma dW(t), \tag{1}$$

where r is the constant continuously compounded interest rate, q is the constant continuous dividend yield, σ is the instantaneous volatility of the stock return, and W is a standard $\mathcal{Q}$-Brownian motion.

If one considers the pricing of currency or commodity options, q denotes the foreign interest rate or minus the proportional cost-of-carry, respectively.

The money-market account evolves according to

$$\frac{dB(t)}{B(t)} = rdt, B(0) = 1$$

Suppose that a security promises a payment of H \$ at time T, where H is a random variable that can be represented by some well-behaved functional taken on the stock price up to time T, $\{S(u)\}_{0 \leq u \leq T}$. Then the fair price at time t of this claim can be represented as

$$F(t) = \mathrm{E}_t \left[e^{-r(T-t)} H \right], \tag{2}$$

where $\mathrm{E}_t[\cdot]$ denotes expectation taken under the measure Q conditional on the information at time t.

Moreover, the Martingale Reprentation Theorem guarantees that there exists a process γ adapted to the filtration generated by the stock price so that

$$dF(t) = rF(t)dt + \gamma(t)dW(t)$$

This implies that there exists a continuously rebalanced self-financing trading strategy in the stock and the money market account replicating the security. This trading strategy consists of $\Delta = \gamma/(\sigma S)$ number of stocks and the rest $F - \Delta S$ on the money market account. If we are able to write $F(t) = F(t, S(t), z(t))$ where z is some locally deterministic process (possibly multidimensional) then

$$\Delta(t) = \frac{\partial F}{\partial S}(t)$$

One might also solve the security valuation problem by applying a change of numeraire resulting in the alternative valuation equation

$$F(t) = S(t)\mathrm{E}'_t\left[e^{-q(T-t)}\frac{H}{S(T)}\right] \tag{3}$$

where $\mathrm{E}'_t[\cdot]$ denotes conditional expectation under Q', which is defined by

$$dQ' = \frac{S(T)}{S(t)e^{(r-q)(T-t)}}dQ \tag{4}$$

on $[t, T]$. By the Girsanov Theorem we have that under Q'

$$W'(t) = W(t) - \sigma t \tag{5}$$

and thereby

$$\frac{dS(t)}{S(t)} = (r - q + \sigma^2)dt + \sigma dW'(t)$$

When H depends on the whole path of the underlying up to the terminal date, T, we should in principle keep track of the whole path of the underlying up to current time, t, in order to calculate the expectation in the valuation equation (2) or the expectation in the alternative valuation equation (3). However, if we are able to come up with a *Markov* process, x, with evolution

$$dx(t) = \mu(t, x(t))dt + v(t, x(t))dW'(t)$$

so that

$$\frac{H}{S(T)} = \zeta(x(T))$$

for some function $\zeta(\cdot)$, then it is not neccesary to keep track of the whole path of the underlying. Due to the Markov property of x, the expectation in (3) can be evaluated by only keeping track of the current value of x. Hence, the deflated option price $f \equiv F/S$ will be a function of $(t, x(t))$ only, and f can be represented as the solution to the one-dimensional PDE

$$qf = \frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}v^2\frac{\partial^2 f}{\partial x^2}$$

subject to the terminal boundary condition $f(T, x) = \zeta(x)$. The PDE can be solved numerically by finite difference techniques, which, as will be demonstrated, is much faster than solving the expectation by Monte-Carlo methods.

We will now show that a Markov representation is indeed possible for the Asian options with fixed and floating strikes and for the lookback options with fixed and floating strike.

3 The Asian Option with Fixed Strike

Let

$$0 = t_0 \leq t_1 < ... < t_n \leq t_{n+1} = T$$

and define

$$\begin{aligned} A(t) &= \sum_{1 \leq i \leq n : t_i \leq t} S(t_i) \\ m(t) &= \max_{1 \leq i \leq n} \{i : t_i \leq t\} \end{aligned}$$

The Asian option with fixed strike promises the holder the time T payment[1]

$$(\frac{1}{n}A(T) - K)^+$$

where is K is some fixed amount - the strike price.

The object is now to evaluate the time t fair price of the option

$$F(t) = S(t)\mathrm{E}'_t\left[e^{-q(T-t)}(\frac{\frac{1}{n}A(T) - K}{S(T)})^+\right] \tag{6}$$

[1]We define $z^+ = \max(0, z)$.

Let

$$x(t) = \frac{\frac{1}{n}A(t) - K}{S(t)} \tag{7}$$

When we hit an observation point, t_i, the process x will jumps by $1/n$. To see this, note that for $1 \leq i \leq n$ we have[2]

$$\begin{aligned}
x(t_i+) &= \frac{\frac{1}{n}\sum_{j=1}^{i} S(t_j) - K}{S(t_i)} \\
&= \frac{\frac{1}{n}(\sum_{j=1}^{i-1} S(t_j) + S(t_i)) - K}{S(t_i)} \\
&= \frac{\frac{1}{n}\sum_{j=1}^{i-1} S(t_j) - K}{S(t_i)} + \frac{1}{n} \\
&= x(t_i-) + \frac{1}{n}
\end{aligned}$$

At times in between observations the process x evolves continuously, because only the denominator in (7) changes as time evolves. Hence, using Ito's lemma we have[3]

$$\begin{aligned}
dx(t) &= -(r-q)x(t-)dt - \sigma x(t-)dW'(t) + \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{t_i \leq t} \\
&= -(r-q)x(t-)dt - \sigma x(t-)dW'(t) + \frac{1}{n}dm(t)
\end{aligned}$$

We see that under Q', the evolution of x only depends on x itself, so x is a Markov process under Q'. This implies that evaluation of the expectation in (6) only requires the knowledge of the current position of x and we can write $F(t)/S(t) = f(t, x(t))$.

Further, we observe that if $x(t) \geq 0$, then $x(u) \geq 0$ for all $u \geq t$ with probability one. This implies that for all $x(t) \geq 0$

$$\begin{aligned}
f(t, x(t)) &= \mathrm{E}'\left[e^{-q(T-t)}x(T)^{+}|x(t)\right] \\
&= \mathrm{E}'\left[e^{-q(T-t)}x(T)|x(t)\right] \\
&= e^{-r(T-t)}x(t) + \frac{1}{n}\sum_{i:t<t_i\leq t_n} e^{-r(T-t_i)-q(t_i-t)} \qquad (8) \\
&\equiv g(t, x(t))
\end{aligned}$$

The third equality is shown in the Appendix.

We note that if $x(t) < 0$, the process x can only pass through the level $x = 0$ at the future

[2] We define $z(t-) = \lim_{\epsilon \to 0} z(t - |\epsilon|)$ and $z(t+) = \lim_{\epsilon \to 0} z(t + |\epsilon|)$.

[3] We define $\mathbf{1}_A$ to be the indicator function on the set A.

sampling points $\{t_i\}_{m(t)<i\leq n}$. Suppose x passes the level $x=0$ at some point t_i. ($i \leq n$). We then have

$$f(t_i, x(t_i)) = g(t_i, x(t_i)) = g(t_i, x(t_i-) + 1/n)$$

In case the level $x = 0$ is not passed for any $\{t_i\}_{i=1,\dots,n}$, the holder of option will receive nothing. To formalize this let us define τ to be the first passage time of the level $x=0$ for the process x, among the oberservation points $\{t_i\}_{i=1,\dots,n}$ of the level, i.e.

$$\begin{aligned} \tau &= \inf_{1\leq i\leq n}\{t_i : x(t_i) \geq 0\} \\ &= \inf_{1\leq i\leq n}\{t_i : x(t_i-) \geq -1/n\} \end{aligned}$$

We can then write

$$\begin{aligned} f(t, x(t)) &= \mathrm{E}'\left[e^{-q(\tau-t)}g(\tau, x(\tau))|x(t)\right] && (9) \\ &= \mathrm{E}'\left[e^{-q(\tau-t)}g(\tau, x(\tau-) + 1/n)|x(t)\right] && (10) \end{aligned}$$

Solving for f is then a first passage time problem for a Markov process. This demonstrates the parallel to an "up-and-in" barrier option: the stock price deflated value of the option, f, is the risk-adjusted expectation of the discounted value of a pay-off at the first passage-time to a certain level. This problem can be formulated as a PDE problem, that easily can be solved numerically as we will illustrate in the subsection below. Before we consider that, we note that by definition

$$F(t) = S(t)f(t, x(t))$$

and by Ito expansion

$$\Delta(t) = f(t, x(t)) - x(t)\frac{\partial f}{\partial x}(t, x(t))$$

We also have that

$$F(t)|_{K=K'} = S(t)f(t, \frac{\frac{1}{n}A(t) - K'}{S(t)})$$

Which gives us the possibility of solving for more than one option price once the function f is identified.

3.1 Numerical Solution and Numerical Results

One can now set up a system of trinomial or binomial trees that discretize the random evolution of x. Except from the observation points, $\{t_i\}_{i=1,\dots,n}$, x evolves as a geometric Brownian motion, so a Cox, Ross & Rubinstein (1979) binomial tree applied to the x process could be used on the regions in between observation points. However, the jumps at the observation points are additive and this works poorly with a standard binomial tree that normally is specified to be log linear. So we choose not to use this approach.

As yet an alternative one might use that $x(t_i)$ conditional on $x(t_{i-1})$ is lognormal under $\mathcal{Q}'$. Discretising the state space in the x dimension will therefore make it possible to solve for the option prices using numerical integration at each point t_i and backwards recursion to current time. Computationally this procedure is qudratic in the chosen mesh size. So is the implicit finite difference method. The implicit schemes, though, only involve simple operations and will thus in general be faster. We therefore choose to concentrate on the implicit finite difference techniques.

We now need to identify the PDE system for the numerical solution of the Asian option price or rather the function f. This can be done directly in (t, x), but we prefer to eliminate the discontinuous dynamics by introducing

$$y(t) = x(t) - \frac{m(t)}{n}$$

We now have

$$dy(t) = -(r-q)(y(t) + \frac{m(t)}{n}) - \sigma(y(t) + \frac{m(t)}{n})dW'(t) \tag{11}$$

Since $e^{-qt}f(t)$ is a $\mathcal{Q}'$-martingale, Ito's lemma and the Martingale Representation Theorem together imply that f is the solution to the PDE

$$qf = \frac{\partial f}{\partial t} - (r-q)(y + \frac{m(t)}{n})\frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2(y + \frac{m(t)}{n})^2\frac{\partial^2 f}{\partial y^2} \tag{12}$$

on $\{(t, y) : t_{i-1} < t < t_i, y < -(i-1)/n, i = 1, \dots, n\}$, subject to the boundary conditions

$$\begin{aligned} f(t_i-, y) &= f(t_i+, y), y < -i/n \\ f(t_i-, y) &= g(t_i, y + i/n), -(i-1)/n > y \geq -i/n \end{aligned} \tag{13}$$

and

$$f(t, y) = 0, y < -1 \tag{14}$$

for $t \geq t_n$.
When solving this numerically, we start at time t_n-. By the boundary conditions we get the value of $f(t_n-, y)$. We then numerically solve back to time $t_{n-1}+$. where the solution acts as terminal boundary condition for the numerical solution on $(t_{n-2}. t_{n-1})$. Recursing back to current time we get the current value of f and thereby the option price.
Note that the state space of the process y changes as time progresses. At each time we have that $y < -(i-1)/n$ when $t_{i-1} < t < t_i$. But the state space is constant for all t between two observation points, and running backwards in time, the new added regions have boundary conditions specified by the known function $g(\cdot, \cdot)$.
We solve this system on a linear grid where we at all times in the interval (t_{i-1}, t_i) set the upper point equal to $y_{max} = -(i-1)/n - \Delta y/2$ and the lower point to some $y_{min} < -1$. Typically we set $y_{min} = -2$ for maturities less than a year. The time points are chosen so that we hit all the points $\{t_i\}$. In order to solve the PDE numerically we need to specify some "artifical" boundary conditions at y_{min} and y_{max}. We choose to set $\partial^2 f/\partial y^2 = 0$ at the boundaries of the grid. The solution technique applied is a Crank-Nicolcon scheme. We refer the reader to Mitchell & Griffiths (1980) for a detailed description of the properties of the Crank-Nicolson scheme, but among the nice properties are that the scheme is uniformly stable and that its local precision is of order $(\Delta t)^2 + (\Delta y)^2$ which is maximal for standard finite difference schemes for partial differential equations of the parabolic type.
Table 1 compares option prices for various strikes generated by the finite difference algorithm with different grid sizes to option prices obtained by Monte Carlo simulations. For the reference we also report the CPU times accured for generating the option prices using the two different types of techniques.

	Asian Option Prices			
K	M.C.	F.D.. I=500	F.D., I=100	F.D..I=50
90.0	12.98	12.99	12.99	12.98
92.5	11.05	11.05	11.05	11.05
95.0	9.27	9.27	9.27	9.27
97.5	7.67	7.66	7.66	7.66
100.0	6.24	6.23	6.23	6.23
102.5	5.01	5.00	5.00	5.00
105.0	3.96	3.95	3.95	3.95
107.5	3.08	3.07	3.07	3.07
110.0	2.36	2.35	2.35	2.36
CPU	46.0s	0.65s	0.06s	0.04s

Table 1 : The parameters are: $r = 0.05, q = 0.0, \sigma = 0.2, T = 1.0, t = 0.0, n = 10, S(0) = 100.0, t_i = 0.1i$. "M.C." refers to Monte Carlo Solution, and "F.D." refers to finite difference solution. The different I's refer to the number of time steps. We used $I/10$ number of steps per jump size $1/n$ in the y direction. The Monte Carlo prices are based on 10^5 simulations with a control variate technique. The standard deviations of the Monte Carlo estimated prices is estimated to $3 \cdot 10^{-3}$. Reported CPU times are for all 9 strikes.

We see that the finite difference algorithm for this option is surprisingly accurate, and that the prices change very little as the grid size is changed. The maximum relative error compared to the Monte Carlo procedure is approximately 0.4 percent. Here it is important to note that the Monte Carlo price is not an absolute figure. It might vary slightly from one simulation to another; as mentioned in the legend of Table 1, the standard deviation of the option prices are approximately $3 \cdot 10^{-3}$.

For the reported CPU times, here and in the following, it should be noted that all programming was done in C and the hardware used was a Hewlett-Packard 9000 Unix system.

Let us briefly describe the Monte Carlo technique. We apply a control variate technique to our Monte Carlo simulations in order to decrease the number of necessary simulations. That is, we simulate a collection of paths, $\{(S(t_1), \ldots, S(t_n))(\omega)\}_\omega$, under Q and consider the regression equation

$$\begin{aligned} e^{-r(T-t)}(\frac{1}{n}A(T) - K)^+(\omega) &= \mathrm{E}_t\left[e^{-r(T-t)}(\frac{1}{n}A(T) - K)^+\right] \\ &+ \left[\sum_{i=1}^{n} a_i(S(t_i) - S(0)e^{(r-q)(t_i-t)})\right](\omega) \end{aligned}$$

where $a_1, \ldots, a_n$ are constants. Note that the regressors under the sum have zero Q-mean. We run an ordinary least squares regression on this and simultaneously estimate the coefficients $\{a_i\}$ and the Q-mean of the pay-off, i.e. the fair price of the option. This also gives us an estimate of the standard deviation of the estimate of the parameters. i.e. an estimate of the standard deviation of the Monte Carlo option prices. The properties of the procedure are described in detail in Davidson & Mackinnon (1993). One can also include different powers of the stock price minus its moments as control variates. We choose not to, because the stock prices alone give sufficient precision for our purpose and because the presence of additional parameters to be estimated makes the Monte Carlo procedure more computationally demanding.

4 The Average Strike Option

With the definitions in the previous section the pay-off terminal time T pay-off of the average strike (put) option can be written as

$$(\frac{1}{n}A(T) - \alpha S(T))^{+}$$

where α is a constant.

Using the valuation equation (3) we have that the time t price of the option is given by:

$$F(t) = S(t)\mathrm{E}'_t\left[e^{-q(T-t)}(\frac{1}{n}\frac{A(T)}{S(T)} - \alpha)^{+}\right]$$

For $t \geq t_1$ define $x(t)$ by

$$x(t) = \frac{A(t)}{S(t)} \tag{15}$$

Applying the same argument as in the previous section we get that for $t \geq t_1$:

$$\begin{aligned} dx(t) &= -(r - q)x(t-)dt - \sigma x(t-)dW'(t) + dm(t) \\ x(t_1) &= 1 \end{aligned}$$

This is a Markov process with domain on the positive part of the real line. The object is now to solve the initial value problem

$$F(t)/S(t) \equiv f(t, x(t)) = \mathrm{E}'\left[e^{-q(T-t)}(\frac{1}{n}x(T) - \alpha)^{+}|x(t)\right] \tag{16}$$

Due to the Markovian property of x this can be done by solving a sequence of PDEs, as we formally describe in the following section.

Suppose that we want to evaluate an average strike option with an American feature, i.e. the option might me exercised at some time, t, in the interval $[t_1, T]$ with resulting payout

$$(\frac{1}{m(t)}A(t) - \alpha S(t))^+$$

Finding the fair price of such a contract is a stopping time problem in the sense that we are supposed to find the exercise time that maximizes the value of the option. To formalize this let $\mathcal{T}$ be the set of stopping times on the interval $[t_1, T]$ with respect to the filtration generated by the stock price. Then the average strike option with the American feature has the fair value

$$\begin{aligned} F(t) &= \sup_{\tau \in \mathcal{T}} \mathrm{E}_t \left[e^{-r(\tau - t)} (\frac{1}{m(\tau)} A(\tau) - \alpha S(\tau))^+ \right] \\ &= S(t) \sup_{\tau \in \mathcal{T}} \mathrm{E}'_t \left[e^{-q(\tau - t)} (\frac{1}{m(\tau)} \frac{A(\tau)}{S(\tau} - \alpha)^+ \right] \\ &= S(t) \sup_{\tau \in \mathcal{T}} \mathrm{E}' \left[e^{-q(\tau - t)} (\frac{1}{m(\tau)} x(\tau) - \alpha)^+ | x(t) \right] \end{aligned} \quad (17)$$

This defines a Markovian stopping time problem for $f = F/S$ that can be treated in a free boundary formulation as we will illustrate in the following subsection.
Both in the American and the European style case we have that for $t \geq t_1$:

$$F(t) = S(t) f(t, x(t))$$

Applying the alternative valuation equation (3) to this quantity we get that for $t < t_1$:

$$F(t) = S(t) e^{-q(t_1 - t)} f(t_1, 1)$$

The hedge ratios are given by

$$\begin{aligned} \Delta(t) &= e^{-q(t_1 - t)} f(t_1, 1), t < t_1 \\ \Delta(t) &= f(t_1, x(t)) - x(t) \frac{\partial f}{\partial x}(t, x(t)), t \geq t_1 \end{aligned}$$

4.1 Numerical Solution and Results

As for the fixed strike case we introduce

$$y(t) = x(t) - m(t)$$

and we have

$$dy(t) = -(r-q)(y(t)+m(t))dt - \sigma(y(t)+m(t))dW'(t)$$

On $t_{i-1} < t < t_i, i > 1$, $e^{-qt}f$ is a Q'-matingale and therefore the solution to

$$qf = \frac{\partial f}{\partial t} - (r-q)(y+m(t))\frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2(y+m(t))^2\frac{\partial^2 f}{\partial y^2} \tag{18}$$

on $y > -i$, subject to the boundary conditions

$$\begin{aligned} f(t_i-, y) &= f(t_i+, y) \\ f(t_{n+1}+, y) &= f(T, y) = (\frac{1}{n}y + 1 - \alpha)^+ \end{aligned} \tag{19}$$

The American style average strike option can be handled by adding the free boundary condition

$$f(t, y) \geq (\frac{1}{m(t)}y + 1 - \alpha)^+ \tag{20}$$

We apply a linear grid to this problem, supply the same "artificial" boundary conditions as for the fixed strike, and again we use the Crank-Nicolson scheme.
Table 2 reports prices generated by the finite difference algorithm and compare these quantities to numbers generated by Monte Carlo simulations.

	Average Strike Option Prices			
α	M.C.	F.D., I=500	F.D., I=100	F.D.,I=50
0.900	8.98	8.98	8.98	8.99
0.925	7.18	7.18	7.17	7.18
0.950	5.61	5.60	5.60	5.58
0.975	4.28	4.27	4.26	4.26
1.000	3.18	3.18	3.18	3.18
1.025	2.31	2.31	2.30	2.30
1.050	1.65	1.64	1.64	1.63
1.075	1.14	1.14	1.14	1.14
1.100	0.78	0.77	0.77	0.77
CPU	48.0s	1.94s	0.12s	0.05s

Table 2 : The parameters are: $r = 0.05, q = 0.0, \sigma = 0.2, T = 1.0, t = 0.0, n = 10, S(0) = 100.0, t_i = 0.1i$. "M.C." refers to Monte Carlo Solution, and "F.D." refers to finite difference solution. The different I's refer to the number of time steps. We used $I/10$ number of steps per jump size $1/n$ in the y direction. The Monte Carlo prices are based on 10^5 simulations with a control variate technique. The standard error on the estimated Monte Carlo option prices is approximately $3.0 \cdot 10^{-3}$. Reported CPU times are for all 9 strikes.

As for the fixed strike Asian the precision of the finite difference solution is remarkable. Even though the grid size changes by a factor 10, the relative price changes are less than 0.6 percent for all strikes. The maximum relative deviation to the Monte Carlo solution is about 1.2 percent. But, it shall again be emphasized that the Monte Carlo solution need not be more accurate than the finite difference solutions and serves only as a benchmark. . The additional computer time accured here compared to the Asian option is due to the fact that here a finite difference algorithm has to be run for each α wheras for the Asian option one need only solve one finite difference grid to obtain the prices for all strikes.
Figure 1 illustrates the effect of adding the American feature to the average strike option when the parameters are given as in Table 2. As we see, the effect is quite large.

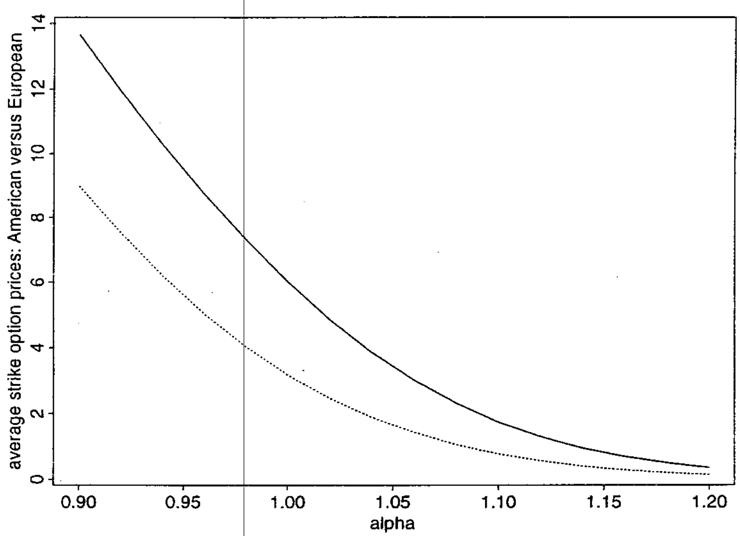

Figure 1: American (solid line) versus European style (dotted line) average strike option prices. Parameters are: $r = 0.05, q = 0.0, \sigma = 0.2, T = 1.0, t = 0.0, S(0) = 100.0$. Prices were generated using a 500×500 finite difference grid.

5 The Lookback Option with Fixed Strike

For $t \geq 0$ define

$$\overline{S}(t) = \sup_{1 \leq i \leq m(t)} S(t_i)$$

with the convention $\overline{S}(t) = 0$ for $t \leq t_1$.
The fixed strike lookback option promises the time T payment

$$(\overline{S}(t) - K)^+$$

The solution of this pricing problem is a two-step procedure. First, we solve the option price at time t when $\overline{S}(t) \geq K$. We then solve for the case $\overline{S}(t) < K$ by observing that in this case the option might be viewed as a first passage time problem of S to the level K where the reward is equal to the value of the option at $\overline{S}(t) \geq K$.
Suppose $\overline{S}(t) \geq K$. We then have

$$\begin{aligned} F(t) &= \mathrm{E}_t\left[e^{-r(T-t)}(\overline{S}(T) - K)^+\right] \\ &= \mathrm{E}_t\left[e^{-r(T-t)}(\overline{S}(T) - K)\right] \\ &= S(t)\mathrm{E}'_t\left[e^{-q(T-t)}\frac{\overline{S}(T)}{S(T)}\right] - e^{-r(T-t)}K \end{aligned} \tag{21}$$

Define

$$x(t) = \frac{\overline{S}(t)}{S(t)} \tag{22}$$

for $t \geq t_1$.
For $1 \leq i \leq n$ we have

$$\begin{aligned} x(t_i+) &= 1, x(t_i-) \leq 1 \\ x(t_i+) &= x(t_i-), x(t_i-) > 1 \end{aligned}$$

Elsewhere the evolution of x is continuous and for $t \geq t_1$ we have

$$\begin{aligned} dx(t) &= -(r-q)x(t-)dt - \sigma x(t-)dW'(t) + (1 - x(t-))^+ dm(t) \\ x(t_1) &= 1 \end{aligned}$$

So x is a Markov process with domain on $x > 0$.
Define

$$\begin{aligned} f(t) &= \mathrm{E}'_t\left[e^{-q(T-t)}\frac{\overline{S}(T)}{S(T)}\right] \\ &= \mathrm{E}'_t\left[e^{-q(T-t)}x(T)\right] \\ &= \mathrm{E}'\left[e^{-q(T-t)}x(T)|x(t)\right] \end{aligned} \tag{23}$$

where the last equality follows due to the Markov property of x.
We have that f can be written as $f(t) = f(t, x(t))$, and that this quantity can be found by numerically solving the PDE related to the initial value problem (23). We will show how this is done in the subsection below.
This establishes the option price at $t \geq t_1$, $\overline{S}(t) \geq K$, explicitly

$$F(t) = S(t)f(t, x(t)) - e^{-r(T-t)}K \tag{24}$$

Suppose we are sitting at time $t \geq t_1$ with $\overline{S}(t) < K$. The first time $t_i > t, (i \leq n)$, $S(t_i) \geq K$ we get a reward of

$$\begin{aligned} F(t_i) &= S(t_i)f(t_i, x(t_i)) - e^{-r(T-t_i)}K \\ &= S(t_i)f(t_i, 1) - e^{-r(T-t_i)}K \end{aligned} \tag{25}$$

The second equality is valid because in the above, t_i is the first time $\overline{S}(t)$ goes above K. If a level of K or above is not reached at any of the sampling times $t_i, i = 1, \ldots, n$, the holder of the option receives nothing.
(25) implies that we for $t \geq 0, \overline{S}(t) < K$ may write the option price as

$$\begin{aligned} F(t) &= \mathrm{E}_t\left[e^{-r(\tau-t)}(S(\tau)f(\tau, 1) - e^{-r(T-\tau)}K)\mathbf{1}_{\tau \leq t_n}\right] \\ &= \mathrm{E}\left[e^{-r(\tau-t)}(S(\tau)f(\tau, 1) - e^{-r(T-\tau)}K)\mathbf{1}_{\tau \leq t_n}|S(t)\right] \end{aligned} \tag{26}$$

where

$$\tau = \inf_{1 \leq i \leq n}\{t_i : S(t_i) \geq K\}$$

with the convention $\inf \emptyset = \infty$.
This shows the parallel to an up-and-in barrier option. When f is known, F can be found by numerically solving the first passage time problem (26). We illustrate how this is done in the subsection below. Finding the option price is therefore a two-step procedure. First we solve for $\{f(u, x)\}$ for all (u, x) with $u \geq \max(t, t_1)$. This is done by numerically solving a initial value problem from T down to t. If $\overline{S}(t) \geq K$ then the option price is given by (24). Else we keep $\{f(t_i, 1)\}_{1 \leq i \leq n : t_i > t}$ and solve the first passage time problem (26).
The hedge ratio is given by

$$\begin{aligned} \Delta(t) &= \frac{\partial F}{\partial S}(t, S(t)), \overline{S}(t) < K \\ \Delta(t) &= f(t, x(t)) - x(t)\frac{\partial f}{\partial x}(t, x(t)), \overline{S}(t) \geq K \end{aligned}$$

5.1 Numerical Solution and Results

We start by performing a log transformation. We define $y = \ln x$ and we get[4]

$$dy(t) = -(r - q + \frac{1}{2}\sigma^2)dt - \sigma dW'(t) + y(t-)^{-}dm(t)$$

Since $e^{-qt}f(t)$ is a Q'-martingale and y is Markovian, the solution to the initial value problem (23) can be found as the solution to the following system of PDEs. On $t_{i-1} < t < t_i, i > 1$, f solves

$$qf = \frac{\partial f}{\partial t} - (r - q + \frac{1}{2}\sigma^2)\frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial y} \tag{27}$$

subject to the boundary conditions

$$\begin{aligned} f(t_i-, y) &= f(t_i+, 0), y < 0 \\ f(t_i-, y) &= f(t_i+, y), y \geq 0 \\ f(t_{n+1}+, y) &= f(T, y) = e^y \end{aligned} \tag{28}$$

Now redefine y and let $y(t) = \ln(S(t)/K)$. The first passage time problem (26) can be handled by noting that for $\overline{S}(t) < K$, $g \equiv F/K$ is the solution to

$$rg = \frac{\partial g}{\partial t} + (r - q - \frac{1}{2}\sigma^2)\frac{\partial g}{\partial y} + \frac{1}{2}\sigma^2\frac{\partial^2 g}{\partial y} \tag{29}$$

on $\{(t, y) : t_{i-1} < t < t_i, i = 1, \ldots, n; y < 0\}$, subject to the boundary conditions:

$$\begin{aligned} g(t_i-, y) &= g(t_i+, y), y < 0 \\ g(t_i-, y) &= e^y f(t_i, 0) - e^{-r(T-t_i)}, y \geq 0 \\ g(t_n+, y) &= 0, y < 0 \end{aligned} \tag{30}$$

The $f(t, \cdot)$ in (30) should be interpretated as function of y as in (27). This means that we can treat f and g in the same grid and simultaneously solve for f and g, at each time step in that respective order. At current time t options of different strikes are generated by $F(t) = Kg(t, S(t)/K)$.

[4]The notation $(\cdot)^{-}$ is defined by

$$z^{-} = -\min(0, z)$$

We apply the Crank-Nicolson scheme to the numerical solution of this problem, where we supply the "artificial" boundary conditions[5]

$$\frac{\partial^2 f}{\partial y^2} - \frac{\partial f}{\partial y} = \frac{\partial^2 g}{\partial y^2} - \frac{\partial g}{\partial y} = 0$$

at the upper and the lower bound of the grid and we arrange the grid so that the level $y = 0$ is on the grid and the points $\{t_i\}$ are among the time points of the grid.
Below Table 3 shows option prices generated by finite difference and compares to option prices of Monte Carlo solution.

	Fixed Strike Lookback Option Prices			
K	M.C.	F.D., I=500	F.D., I=100	F.D.,I=50
90.0	24.41	24.39	24.27	23.81
92.5	22.07	22.06	21.93	21.47
95.0	19.78	19.77	19.64	19.18
97.5	17.57	17.56	17.43	16.96
100.0	15.48	15.47	15.34	14.87
102.5	13.53	13.52	13.39	12.95
105.0	11.75	11.74	11.62	11.22
107.5	10.14	10.14	10.03	9.67
110.0	8.70	8.71	8.62	8.30
CPU	46.0s	0.68s	0.06s	0.03s

Table 3 : The parameters are: $r = 0.05, q = 0.0, \sigma = 0.2, T = 1.0, t = 0.0, n = 10, S(0) = 100.0, t_i = 0.1i$. "M.C." refers to Monte Carlo Solution, and "F.D." refers to finite difference solution. The different I's refer to the number of time steps *and* also to the number of steps in the y direction. The Monte Carlo prices are based on 10^5 simulations with a control variate technique. The standard deviations of the Monte Carlo prices are approximately $3.0 \cdot 10^{-3}$. Reported CPU times are for all 9 strikes.

Comparing the finite difference solution on the 500×500 grid to the Monte Carlo solution shows a maximal relative error of approximately 0.1 percent which is clearly within any reasonable demands for precision. But the finite difference solutions for the two smaller grids do not show sufficient precision. This must be attributed the two step procedure involved here; numerical errors might be accumulated in the two steps. The conclusion is that this type of option requires a finer mesh than the options considered in the previous sections.

[5]These conditions are equivalent to the condition $\partial^2 f/\partial x^2 = \partial^2 F/\partial S^2 = 0$.

6 The Lookback Option with Floating Strike

With the definitions of the previous sections the time T pay-off of a floating strike lookback option can be expressed as

$$(\overline{S}(T) - \alpha S(T))^+$$

Of the options considered in this paper this is the easiest option to evaluate numerically. For $t \geq t_1$ the fair price is given by

$$\begin{aligned} F(t) &= \mathrm{E}'_t\left[e^{-q(T-t)}(\frac{\overline{S}(T)}{S(T)} - \alpha)^+\right] \\ &= \mathrm{E}'_t\left[e^{-q(T-t)}(x(T) - \alpha)^+ | x(t)\right] \end{aligned}$$

where x is defined as in (22). [6]

Letting $f = F/S$, f solves a Markovian initial boundary problem equivalent to (23). In the subsection below we supply the PDE with boundary conditions associated to this problem. If we want to consider a floating strike lookback option with an American feature, note that the fair price of such a contract can be represented as

$$\begin{aligned} F(t) &= \sup_{\tau \in \mathcal{T}} \mathrm{E}_t\left[e^{-r(\tau-t)}(\overline{S}(\tau) - \alpha S(\tau))^+\right] \\ &= S(t) \sup_{\tau \in \mathcal{T}} \mathrm{E}'_t\left[e^{-q(\tau-t)}(\frac{\overline{S}(\tau)}{S(\tau)} - \alpha)^+\right] \\ &= S(t) \sup_{\tau \in \mathcal{T}} \mathrm{E}'\left[e^{-q(\tau-t)}(x(\tau) - \alpha)^+ | x(t)\right] \end{aligned} \quad (31)$$

where $\mathcal{T}$ is the set of stopping times on $[t, T]$ adapted to the filtration generated by S. As in (17) this is a Markovian stopping time problem that can be reformulated as a free boundary problem for $f = F/S$. We formulate this as a PDE problem in the subsection below.

In both the European and the American style floating strike lookback option we have that

$$\begin{aligned} F(t) &= S(t)e^{-q(t_1-t)}f(t_1, 1), t < t_1 \\ F(t) &= S(t)f(t, x(t)), t \geq t_1 \end{aligned}$$

The hedge ratio is given by

$$\begin{aligned} \Delta(t) &= e^{-q(t_1-t)}f(t_1, 1), t < t_1 \\ \Delta(t) &= f(t, x(t)) - x(t)\frac{\partial f}{\partial x}(t, x(t)), t \geq t_1 \end{aligned}$$

[6] This is also observed by Babbs (1992), who treats the American style case in a way similar to what is outlined below. However, Babbs applies a binomial scheme for the numerical solution.

6.1 Numerical Solution and Results

As for the fixed strike lookback we choose to log-transform the state variable, and define $y = \ln x$. We now get that f solves the PDE

$$qf = \frac{\partial f}{\partial t} + (r - q + \frac{1}{2}\sigma^2)\frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial y^2}$$

when $t_{i-1} < t < t_i, i > 1$, subject to the boundary conditions

$$\begin{aligned} f(t_i-, y) &= f(t_i+, 0), y < 0 \\ f(t_i-, y) &= f(t_i+, y), y \geq 0 \\ f(t_{n+1}-, y) &= f(T, y) = (e^y - \alpha)^+ \end{aligned} \tag{32}$$

If we consider an American style option we have to add the free boundary condition

$$f(t, y) \geq (e^y - \alpha)^+$$

We apply the same "artificial" boundary conditions as in the previous section and again we use the Crank-Nicolson scheme for the numerical solution.

Table 4 reports option prices generated using the finite difference solution and compares to option prices found by Monte-Carlo simulations.

	Floating Strike Lookback Option Prices			
α	M.C.	F.D., I=500	F.D., I=100	F.D.,I=50
1.000	10.01	10.00	9.96	9.86
1.025	8.27	8.26	8.23	8.14
1.050	6.77	6.76	6.74	6.68
1.075	5.51	5.50	5.47	5.43
1.100	4.46	4.45	4.42	4.39
1.125	3.59	3.58	3.56	3.53
1.150	2.88	2.87	2.85	2.81
1.175	2.30	2.29	2.28	2.25
1.200	1.83	1.82	1.81	1.79
CPU	46.0s	2.43s	0.14s	0.06s

Table 4 : The parameters are: $r = 0.05, q = 0.0, \sigma = 0.2, T = 1.0, t = 0.0, n = 10, S(0) = 100.0, t_i = 0.1i$. "M.C." refers to Monte Carlo Solution, and "F.D." refers to finite difference solution. The different I's refer to the number of time steps *and* also to the number of steps in the y direction. The Monte Carlo prices are based on 10^5 simulations with a control variate technique. The standard deviation of the Monte Carlo option prices is approximately $3 \cdot 10^{-3}$. Reported CPU times are for all 9 strikes.

We choose only to show prices for values of α greater than one. This is because all options with $\alpha \leq 1$ are all "in-the-money" with probability one, due to the sampling of the maximum that we use here (we have $t_n = T$). That means that for all $\alpha < 1$ the option contract has a value that equals the value of the contract with $\alpha = 1$ plus $S(0)(1-\alpha)\epsilon^{-qT}$.
Comparing the finite difference solutions to the Monte Carlo solutions we have that the maximal relative error is about 0.5 percent for the 500×500 grid. 1 percent for the 100×100 grid, and approximately 2 percent for the 50×50 grid. This is acceptable but the example shows that one has to use a higher degree of precision for the lookback than for the Asian options.
Figure 2 illustrates the effect of adding an American exercise feature to the option when the parameters are given as for the prices of Table 4. The effect is significant, though not as large as for the average strike option.

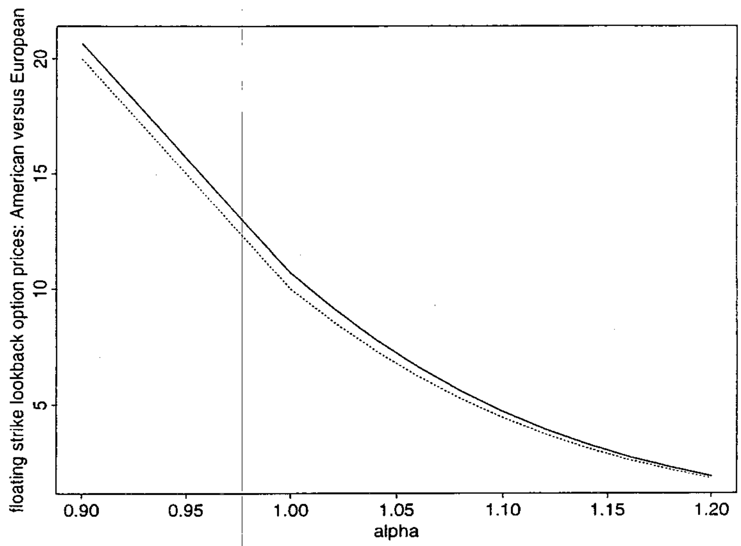

Figure 2: American style (solid line) versus European style (dotted line) floating strike lookback option prices. Parameters are: $r = 0.05, q = 0.0, \sigma = 0.2, T = 1.0, t = 0.0, S(0) = 100.0$. Prices are generated using a 500×500 finite difference grid.

7 Discrete versus Continuous Sampling

In this section we consider the difference between discrete and continuous sampling of the underlying. For the options depending on averages we extend our methodology to the continuous observation case and apply finite difference solutions to the resulting PDEs, whereas we apply the closed form solutions obtained by Conze and Viswanathan (1991) for the lookbacks.

7.1 Average Options

Suppose now that the average for the fixed strike Asian option is calculated over the full interval $[0, T]$. Redefining A and x as

$$\begin{aligned} A(t) &= \int_0^t S(u)du \\ x(t) &= \frac{\frac{1}{T}A(t) - K}{S(t)} \end{aligned}$$

yields

$$dx(t) = (\frac{1}{T} - (r-q)x(t))dt - \sigma x(t)dW'(t)$$

so x is a Markov process under Q'.
This implies that the deflated option price $f = F/S$ solves the PDE [7]

$$qf = \frac{\partial f}{\partial t} + (\frac{1}{T} - (r-q)x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}$$

subject to the terminal boundary condition

$$f(T, x) = x^+$$

Using this we can numerically solve for Asian option prices by a finite difference algorithm. Table 5 compares options with discrete observation average to those with continuous average.

[7] The methodology of identifying the Asian option pricing problem as a barrier option pricing problem can also be applied here since the process x as defined here also exhibits the property that $x(t) \geq 0$ implies that $x(u) \geq 0$ for all $u \geq t$ with probability one.

K	Asian Option Prices				
	$n=\infty$	$n=250$	$n=52$	$n=12$	$n=4$
90.0	12.59	12.61	12.67	12.92	13.58
92.5	10.62	10.64	10.71	10.98	11.69
95.0	8.81	8.84	8.91	9.19	9.95
97.5	7.19	7.21	7.28	7.58	8.36
100.0	5.76	5.78	5.85	6.16	6.94
102.5	4.53	4.55	4.62	4.92	5.69
105.0	3.51	3.52	3.59	3.87	4.61
107.5	2.67	2.68	2.74	3.00	3.69
110.0	1.99	2.00	2.06	2.29	2.92

Table 5 : The parameters are: $r = 0.05, q = 0.0. \sigma = 0.2, T = 1.0, t = 0.0. S(0) = 100.0, t_i = i/n$. n refers to the number of discrete observations, and $n = \infty$ refers to the continuous observation case. Prices are generated on a grid of dimension 500×500 points. For the $n = 250$ case we took 1250 time steps. Discrete option prices were cross checked with Monte Carlo simulations.

We see that the option prices increase as the number of observations decrease. This can be attributed the fact that the standard deviation of the average decreases as the number of observations increases.

Even when sampling is performed weekly the effect of discrete observations versus continuous observations is significant. This illustrates that the approximation that we sample continuously might be crude even when sampling is performed quite frequent.

For the average strike option we have that if

$$x(t) = A(t)/S(t)$$

then

$$\begin{aligned} dx(t) &= (1-(r-q)x(t))dt - \sigma x(t)dW'(t) \\ x(0) &= 0 \end{aligned}$$

and thereby that the deflated option price $f = F/S$ solves the PDE [8]

$$qf = \frac{\partial f}{\partial t} + (1-(r-q)x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} \tag{33}$$

subject to the terminal boundary condition

$$f(T,x) = (\frac{1}{T}x - \alpha)^+$$

[8]Ingersoll (1987) is to the author's knowledge the first to observe this.

and in case the option is American the PDE is also subject to the free boundary condition

$$f(t,x) \geq (\frac{1}{t}x - \alpha)^+$$

for all t.

We can now generate continuous average strike option prices by numerical solution of the PDE (33). Doing so we can compare to average strike option prices with discretely sampled average. This is done in Table 6.

α	Average Strike Option Prices				
	$n=\infty$	$n=250$	$n=52$	$n=12$	$n=4$
0.900	9.04	9.04	9.03	8.99	8.93
0.925	7.30	7.30	7.28	7.20	7.01
0.950	5.78	5.77	5.74	5.62	5.34
0.975	4.48	4.47	4.43	4.30	3.95
1.000	3.40	3.40	3.36	3.21	2.83
1.025	2.54	2.53	2.49	2.35	1.96
1.050	1.85	1.84	1.81	1.68	1.32
1.075	1.33	1.32	1.29	1.17	0.87
1.100	0.93	0.93	0.90	0.80	0.55

Table 6: The parameters are: $r = 0.05, q = 0.0, \sigma = 0.2, T = 1.0, t = 0.0, S(0) = 100.0, t_i = i/n$. n refers to the number of discrete observations, and $n = \infty$ corresponds to the continuous observation case. Prices were generated on a grid of dimension 500×500 points. For the $n = 250$ case we took 1250 time steps. Discrete option prices were cross checked with Monte Carlo simulations.

We see that the average strike option prices increase as the number of observations is increased. This can be explained by the observation that second moment of the average minus the terminal stock price increases as the number of observations increase.

The difference between continuous and discrete sampling is not as big as for the Asian option. It is still present, though, when sampling is performed on weekly basis, especially for the out-of-the-money options.

7.2 Lookback Options

Defining the maximum as

$$\overline{S}(t) = \sup_{0 \leq u \leq t} S(u)$$

Conze & Viswanathan (1991) derive closed form solutions for the floating and the fixed strike continuous lookback options. We use these formulas to compare discrete and continuous lookback options.[9] Table 7 and 8 illustrate the differences.

	Fixed Strike Lookback Option Prices				
K	$n=\infty$	$n=250$	$n=52$	$n=12$	$n=4$
90.0	28.68	27.80	26.80	24.77	22.01
92.5	26.30	25.42	24.42	22.42	19.80
95.0	23.92	23.04	22.05	20.11	17.67
97.5	21.55	20.67	19.69	17.87	15.65
100.0	19.17	18.31	17.41	15.75	13.76
102.5	16.89	16.10	15.27	13.77	12.01
105.0	14.80	14.07	13.31	11.97	10.41
107.5	12.91	12.24	11.55	10.34	8.98
110.0	11.20	10.60	9.98	8.89	7.70

Table 7 : The parameters are: $r = 0.05, q = 0.0, \sigma = 0.2, T = 1.0, t = 0.0, S(0) = 100.0, t_i = i/n$. n refers to the number of discrete observations, and $n = \infty$ refers to the continuous observation case. Prices were generated on a grid of dimension 500×500 points. For the $n = 250$ case we took 1250 time steps. Discrete option prices were cross checked with Monte Carlo simulations.

	Floating Strike Lookback Option Prices				
α	$n=\infty$	$n=250$	$n=52$	$n=12$	$n=4$
1.000	14.29	13.41	12.41	10.37	7.41
1.025	11.97	11.20	10.34	8.58	6.03
1.050	9.98	9.31	8.56	7.04	4.86
1.075	8.29	7.71	7.06	5.73	3.87
1.100	6.86	6.36	5.79	4.65	3.06
1.125	5.65	5.22	4.74	3.75	2.40
1.150	4.64	4.27	3.86	3.01	1.87
1.175	3.79	3.48	3.13	2.41	1.45
1.200	3.09	2.82	2.53	1.92	1.12

Table 8 : The parameters are: $r = 0.05, q = 0.0, \sigma = 0.2, T = 1.0, t = 0.0, S(0) = 100.0, t_i = i/n$. n refers to the number of discrete observations, and $n = \infty$ refers to the continuous observation case. Prices were generated on a grid of dimension 500×500 points. For the $n = 250$ case we took 1250 time steps. Discrete option prices were cross checked with Monte Carlo simulations.

[9] The solution method that we apply to the discrete lookbacks can also be applied to the continuous lookbacks. To see this, observe that for the continuous maximum we have that under Q', $\overline{S}/S$ is a geometric Brownian motion under with *reflecting boundary* (from above) in 1. Babbs (1992) observes this and applies it to the floating strike lookback in a binomial setting.

The differences between continuous and discrete lookback option prices are quite large which illustrate that for lookbacks there is a great distance between 250 and infinity. For fixed strike the relative differences between daily and continuous observations are between 3 and 10 percent. For floating strike options the same quantity is in between 7 and 10 percent. So taking the continuous maximum as a proxy for a discrete maximum is a rather poor approximation.

8 Discontinuous Returns of the Underlying

In this section we extend the model of the stock price to allow for discontinuous dynamics and show that the technique used in the previous sections also can be applied to this type of stock price behavior.

Under Q the stock is assumed to evolve according to the stochastic differential equation

$$\frac{dS(t)}{S(t-)} = (r - q - k\lambda)dt + \sigma dW(t) + I(t)dN(t) \tag{34}$$

where r, q, σ, W are defined as in Section (2). N is a Poisson process with intensity λ and $\{I(t)\}_{t\geq 0}$ is a sequence of independent and identically distributed random variables with the distributional property

$$\ln(1 + I(t)) \sim_Q \mathrm{N}(\gamma - \frac{1}{2}\delta^2, \delta^2)$$

and Q-mean

$$k = \mathrm{E}\,[I(t)] = e^{\gamma} - 1$$

W, I, N are assumed to be independent processes.

The economy is now incomplete, i.e. there exists no hedging strategy in the stock and the bond that perfectly replicates the pay-off of derivatives, and fixing the original measure means that the martingale measure Q is non-unique. However, this does not influence derivative pricing once a martingale measure, Q, is fixed like we do above by simply assuming the Q-dynamics for the stock given by (34). Of course the relation between the objective probability measure and the martingale measure matters if we are considering portfolio and hedging decisions, but that is beyond the scope of this paper, so we will ignore this dicussion for the remaining of the paper.

Defining Q' as in (4) the Girsanov Theorem implies that [10]

$$\frac{dS(t)}{S(t-)} = (r - q - k\lambda + \sigma^2)dt + \sigma dW'(t) + I'(t)dN'(t)$$

where W' is a Q'-Brownian Motion given as in (5), $\{I'(t)\}_{t\geq 0}$ is a sequence of independent identically distributed random variables with distribution given by

$$\ln(1 + I'(t)) \sim_{Q'} \mathrm{N}(\gamma + \frac{1}{2}\delta^2, \delta^2)$$

and N' is a Q' Poisson process with intensity

$$\lambda' = \lambda(1 + k) = \lambda e^{\gamma}$$

W', I', N' are also independent under Q'.
In this type of economy the valuation equations (2) and (3) are still valid.

8.1 Path Dependent Options under Jumps

The techniques applied to the pricing problems in the previous sections naturally extend to the case when the underlying exhibits jumps. To see this let us consider the options one by one.
The Asian option with fixed strike has the value given by (6) and if we define x as in (6) we now have that

$$\begin{aligned} dx(t) &= -(r - q - k\lambda)x(t-)dt - \sigma x(t-)dW'(t) \\ &- x(t-)\frac{I'(t)}{1 + I'(t)}dN'(t) + \frac{1}{n}dm(t) \end{aligned} \tag{35}$$

This is clearly a Markov process with the property that if $x(t) \geq 0$ then $x(u) \geq 0$ for all $u \geq t$ with probability one. This implies that if $x(t) \geq 0$ then the deflated option price is given by

$$F(t)/S(t) \equiv f(t) = \mathrm{E}'_t[x(T)] = g(t, x(t)) \tag{36}$$

where $g(\cdot, \cdot)$ is defined as in (8). The last equality is shown in the Appendix.
Now if $x(t) < 0$ the process x can still only pass the level $x = 0$ at the points $\{t_i\}_{i=1,\ldots,n}$, which again implies that for $x(t) < 0$ we may write the deflated price of the option, f, as

[10] Note that the Q' measure is uniquely related to Q. So once Q is fixed, so is Q'.

the solution to a first passage time problem as we did in (9). We will return to how this is solved numerically in the subsection below. Once f is obtained, the option price is given by

$$F(t) = S(t)f(t, x(t))$$

For the average strike option considered in Section 4 we get that if we define x as in (15), x now evolves according to

$$\begin{aligned} dx(t) &= -(r - q - k\lambda)x(t-)dt - \sigma x(t-)dW'(t) - x(t-)\frac{I'(t)}{1+I'(t)}dN'(t) + dm(t) \\ x(t_1) &= 1 \end{aligned}$$

for $t \geq t_1$. This is a Markov process with domain on $x > 0$. The solution to the deflated option price is now given as the solution to the Markovian initial value problem

$$f(t, x(t)) = \mathrm{E}'\left[e^{-q(T-t)}(\frac{1}{n}x(T) - \alpha)^+|x(t)\right]$$

We show how to handle this numerically in the following subsection. For the average strike option with an American exercise feature we obtain the same type of Markovian stopping time problem as in (17). This can be given a free boundary formulation that we will consider in the next subsection. Given f we have that

$$\begin{aligned} F(t) &= S(t)f(t, x(t)), t \geq t_1 \\ F(t) &= S(t)e^{-q(t_1-t)}f(t, 1), t < t_1 \end{aligned}$$

The lookback option with fixed strike can also be handled by the technique applied in Section 5. The key observations are the same. We first note that for $\overline{S}(t) \geq K$ the option price might be written as in (24). Defining $x = \overline{S}/S$ Ito's lemma implies that for $t \geq t_1$

$$\begin{aligned} dx(t) &= -(r - q)x(t-)dt - \sigma x(t-)dW'(t) \\ &- x(t-)\frac{I'(t)}{1+I'(t)}dN'(t) + (1 - x(t-))^+dm(t) \\ x(t_1) &= 1 \end{aligned} \tag{37}$$

This is clearly a Markov process with domain on $x > 0$. So

$$f(t) \equiv \mathrm{E}'_t\left[e^{-q(T-t)}x(T)\right]$$

is the solution to a Markov initial value problem. We will return to the numerical solution of this problem in the following. On the other hand, if $\overline{S}(t) < K$ we can write the option price as the solution to a Markovian first passage time problem as in (26), because S is still a Markovian process. So once f is obtained for the points $\{t_i\}_{i=1,...,n}$ the problem can be handled by numerically solving the first passage time problem. We will return to how this is done. To summarize, we have a two step procedure: If $\overline{S}(t) \geq K$. then the option price is given by

$$F(t) = S(t)f(t, x(t)) - Ke^{-r(T-t)} \tag{38}$$

else the option price is given by as the solution to a Markov first passage time problem like (26).

Consider now the floating strike lookback option. We have seen that if $x = \overline{S}/S$, x has the Markov stochastic evolution (37). So the European style option price is given as the solution to Markov initial value problem

$$F(t)/S(t) \equiv f(t, x(t)) = \mathrm{E}'\left[e^{-q(T-t)}(\frac{1}{n}x(T) - \alpha)^{+}|x(t)\right]$$

for $t \geq t_1$. and for $t < t_1$:

$$F(t) = S(t)e^{-q(t_1-t)}f(t_1, 1)$$

We will return to how this can be handled numerically in the subsection below. The American style option is handled as in (31). That is, we have to solve a Markov optimal stopping time problem. In the following subsection we do this numerically by reformulating the problem as a free boundary problem.

8.2 Numerical Solution and Results under Jumps

The Markovian nature of the refomulated pricing problems that we have seen in the previous subsection means that the pricing can be done by solving *partial integro differential equations* (PIDEs). The term *integro* is added because the PIDEs not only involve partial derivatives but also integrals since the processes considered here have discontinuities of random sizes at random times. The numerical solution of such equations can still be done on finite grids by applying finite difference techniques, but we need to supply additional "artificial" boundary conditions in order to make this machinery work. This is because the integrals in the PIDEs typically include terms outside the boundaries of a reasonably sized grid. We will in the following derive the PIDEs that need to be solved numerically and supply our choices of "artificial" boundary conditions.

In the following we will let y_{min} and y_{max} denote the lower and upper boundary, respectively. These quantities are in some of the cases dependent on the interval (t_{i-1}, t_i) that we are considering, but for brevity we will ignore this.

With $y(t) = x(t) - m(t)/n$ the PIDE analog to the PDE (12) for the fixed strike Asian option can be written as

$$\begin{aligned}(q+\lambda')f &= \frac{\partial f}{\partial t} - (r-q-k\lambda)(y+\frac{m(t)}{n})\frac{\partial^2 f}{\partial y} + \frac{1}{2}\sigma^2(y+\frac{m(t)}{n})^2\frac{\partial^2 f}{\partial y^2} \\ &+ \lambda' \mathrm{E}'_{I'}\left[f(t, \frac{y+m(t)/n}{1+I'} - \frac{m(t)}{n})\mathbf{1}_{y_{min}\leq \frac{y+m(t)/n}{1+I'} - \frac{m(t)}{n} \leq y_{max}}\right] + h(t,y) \quad (39)\end{aligned}$$

The operator $\mathrm{E}'_I[\cdot]$ is defined for any function $\nu(\cdot)$ by

$$\mathrm{E}'_{I'}[\nu(I')] = \int_{-1}^{\infty} \nu(\xi)\psi(\xi)d\xi \quad (40)$$

where $\psi(\cdot)$ is the density for I' under Q':

$$\psi(\xi) = \frac{1}{\sqrt{2\pi}\delta(1+\xi)}\exp(-\frac{1}{2}(\frac{\ln(1+\xi)-\gamma}{\delta} - \frac{1}{2}\delta)^2)$$

The function $h(\cdot,\cdot)$ is in turn defined as

$$h(t,y) = \lambda' \mathrm{E}'_{I'}\left[f(t, \frac{y+m(t)/n}{1+I'} - \frac{m(t)}{n})(\mathbf{1}_{\frac{y+m(t)/n}{1+I'} - \frac{m(t)}{n} < y_{min}} + \mathbf{1}_{\frac{y+m(t)/n}{1+I'} - \frac{m(t)}{n} > y_{max}})\right]$$

The PIDE (39) is to be solved subject to the boundary conditions (13) and (14), on the set

$$\{(t,y) : t_{i-1} < t < t_i, y < (i-1)/n, i = 1, \ldots, n\}$$

Before we can solve this numerically we need to make a reasonable approximation for $h(\cdot,\cdot)$. We set

$$\begin{aligned}f(t,y) &= g(t, y+m(t)/n), y > y_{max} \\ f(t,y) &= 0, y < y_{min}\end{aligned}$$

Straightforward calculations show that

$$\begin{aligned}h(t,y) &= \lambda e^{-r(T-t)}(y+\frac{m(t)}{n})\Phi(\frac{\ln\frac{y_{max}+m(t)/n}{y+m(t)/n} + \gamma}{\delta} - \frac{1}{2}\delta) \\ &+ \frac{\lambda'}{n}(\sum_{i\leq n: t_i > t} e^{-r(T-t_i)-q(t_i-t)})\Phi(\frac{\ln\frac{y_{max}+m(t)/n}{y+m(t)/n} + \gamma}{\delta} + \frac{1}{2}\delta)\end{aligned}$$

Substituting this into (39) and using the additional "artificial" boundary condition that $\partial^2 f/\partial y^2 = 0$ at the lower and the upper bounds we can now numerically solve for the Asian option price using the finite difference scheme described in Andreasen & Gruenewald (1996). Whitout affecting stabililty, the speed of the procedure might be increased by taking an explicit approximation for the integral and an implicit approximation for the partial derivatives.

For the floating strike option we introduce $y(t) = x(t) - m(t)$ as in Section 4, and we obtain the follwoing PIDE analog to the PDE (18)

$$\begin{aligned}(q+\lambda')f &= \frac{\partial f}{\partial t} - (r - q - k\lambda)(y+m(t))\frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2(y+m(t))^2\frac{\partial^2 f}{\partial y^2} \\ &+ \lambda' \mathrm{E}'_{I'}\left[f(t, \frac{y+m(t)}{1+I'} - m(t))\mathbf{1}_{y_{min} < \frac{y+m(t)}{1+I'} - m(t) < y_{max}}\right] + h(t,y) \qquad (41)\end{aligned}$$

that is valid on $\{(t,y) : t_{i-1} < t < t_i, y > -(i-1), i = 2,\ldots,n+1\}$ and has to be solved subject to the boundary conditions (19) and in case the option is American style additionally to the free boundary condition (20).

For $y > y_{max}$ we set

$$\begin{aligned}f(t,y) &= \mathrm{E}'_t\left[e^{-q(T-t)}(\frac{1}{n}y(T) + 1 - \alpha)\right] \\ &= \frac{y+m(t)}{n}e^{-q(T-t)} + \frac{1}{n}\sum_{i\le n: t_i > t} e^{-r(T-t_i)-q(t_i-t)} - \alpha e^{-q(T-t)}\end{aligned}$$

in the European case and for the American style option we let

$$f(t,y) = \frac{1}{m(t)}y + 1 - \alpha$$

For $y < y_{min}$ we set

$$f(t,y) = 0$$

in both cases. This result in the follwoing approximations for $h(\cdot,\cdot)$. For the European case we get

$$\begin{aligned}h(t,y) &= \lambda(y + \frac{m(t)}{n})\frac{e^{-r(T-t)}}{n}\Phi(\frac{\ln\frac{y+m(t)}{y_{max}+m(t)} - \gamma}{\delta} + \frac{1}{2}\delta) \\ &+ \lambda'(\frac{1}{n}\sum_{i\le n: t_i > t} e^{-r(T-t_i)-q(t_i-t)} - \alpha e^{-q(T-t)})\Phi(\frac{\ln\frac{y+m(t)/n}{y_{max}+m(t)} - \gamma}{\delta} - \frac{1}{2}\delta)\end{aligned}$$

In case the option is American we have the approximation

$$
\begin{aligned}
h(t,y) &= \lambda(\frac{y}{m(t)}+1)\Phi(\frac{\ln\frac{y+m(t)/n}{y_{max}+m(t)/n}-\gamma}{\delta}+\frac{1}{2}\delta) \\
&- \lambda'\alpha\Phi(\frac{\ln\frac{y+m(t)/n}{y_{max}+m(t)/n}-\gamma}{\delta}-\frac{1}{2}\delta)
\end{aligned}
$$

Substituting these equations into the PIDE (41) we can now numerically solve the average strike option prices using the algortihm described in Andreasen & Gruenewald (1996). Consider the Lookback option with fixed strike. With $y = \ln(\overline{S}/S)$ and we have that f defined as in (23) solves the PIDE

$$
\begin{aligned}
(q+\lambda')f &= \frac{\partial f}{\partial t}-(r-q-k\lambda+\frac{1}{2}\sigma^2)\frac{\partial f}{\partial y}+\frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial y^2} \\
&+ \mathrm{E}'_{I'}\left[f(t,y-\ln(1+I'))\mathbf{1}_{y_min<y-\ln(1+I')<y_{max}}\right]+h(t,y)
\end{aligned}
\tag{42}
$$

on the set $\{(t,y): t_{i-1}<t<t_i, i=2,\ldots,n+1\}$ subject to the boundary conditions (28). Let

$$
\begin{aligned}
f(t,y) &= f(t,y_{min}), y<y_{min} \\
f(t,y) &= e^{-r(T-t)+y}, y>y_{max}
\end{aligned}
$$

The last condition is obtained by taking the discounted conditional Q' expectation of $x(T)$ as if there were no jumps in x at the observation points $\{t_i\}_{i=1,\ldots,n}$. $h(\cdot,\cdot)$ is then approximated by

$$
\begin{aligned}
h(t,y) &= \lambda e^{-r(T-t)+y}\Phi(\frac{y-y_{max}+\gamma}{\delta}+\frac{1}{2}\delta) \\
&+ \lambda' f(t,y_{min})\Phi(\frac{y_{min}-y-\gamma}{\delta}+\frac{1}{2}\delta)
\end{aligned}
$$

The PIDE (42) can now be solved numerically on a grid. This gives us the solution for the option price when $\overline{S}(t)\geq K$. If this is not the case we proceed by noting that for $\overline{S}(t)<K$, with the definitions $y=\ln(S/K)$ and $g=F/K$, the PIDE equivalent to the PDE (29) is

$$
\begin{aligned}
(r+\lambda)g &= \frac{\partial g}{\partial t}+(r-q-k\lambda-\frac{1}{2}\sigma^2)\frac{\partial g}{\partial y}+\frac{1}{2}\sigma^2\frac{\partial^2 g}{\partial y^2} \\
&+ \lambda\mathrm{E}_I\left[g(t,y+\ln(1+I))\mathbf{1}_{y_{min}<y+\ln(1+I)<y_{max}}\right]+h(t,y)
\end{aligned}
\tag{43}
$$

on $\{(t,y): t_{i-1}<t<t_i, i=1,\ldots,n\}$ subject to the boundary conditions (30). The operator $\mathrm{E}_I[\cdot]$ is defined as in (40) with the modification that the Q' density is now

replaced by the $\mathcal{Q}$-density. Analogous to the previous, the function $h(\cdot.\cdot)$ is defined as

$$h(t,y) = \mathrm{E}_I\left[g(t,y+\ln(1+I))(\mathbf{1}_{y+\ln(1+I)<y_{min}} + \mathbf{1}_{y+\ln(1+I)>y_{max}})\right]$$

For $y < y_{min}$ we set

$$g(t,y) = 0$$

and for $y > y_{max}$ we let

$$\begin{aligned} g(t,y(t)) &= \mathrm{E}_t\left[e^{-r(t_{m(t)+1}-t)}g(t_{m(t)+1}, y(t_{m(t)+1}))\right] \\ &= \mathrm{E}_t\left[e^{-r(t_{m(t)+1}-t)}e^{y(t_{m}(t)+1)}f(t_{m(t)+1},1) - e^{-r(T-t)}\right] \\ &= e^{-q(t_{m(t)+1}-t)+y(t)}f(t_{m(t)+1},1) - e^{-r(T-t)} \end{aligned}$$

From this we obtain

$$\begin{aligned} h(t,y) &= \lambda' e^{-q(t_{m(t)+1}-t)+y}\Phi(\frac{y-y_{max}+\gamma}{\delta}+\frac{1}{2}\delta) \\ &- \lambda e^{-r(T-t)}\Phi(\frac{y-y_{max}+\gamma}{\delta}-\frac{1}{2}\delta) \end{aligned}$$

Using this we can numerically solve the PIDE (43) using the finite difference machinery. Finally, let us consider the lookback option with floating strike. Definiting $y = \ln(\overline{S}/S)$ and $f = F/S$ we get that f is the solution to (42) on $\{(t,y) : t_{i-1} < t < t_i, i = 2,\ldots,n+1\}$, subject to the boundary conditions (32) and if the option is American style also subject to the free boundary condition (33). What is left is to supply an approximation of $h(\cdot,\cdot)$ for this option. For $y < y_{min}$ we set

$$f(t,y) = f(t,y_{min})$$

for both the American and European style cases.
For $y > y_{max}$ we set

$$f(t,y) = e^{-r(T-t)+y} - e^{-q(T-t)}\alpha$$

for the European case. This corresponds to the dicounted $\mathcal{Q}'$-expected terminal payoff if we ignore that the option could go out-of-the-money and the (possible) jumps at the observation points $\{t_i\}_{i=1,\ldots,n}$. For the American style option we set

$$f(t,y) = e^y - \alpha$$

for $y > y_{max}$.

Doing so we get the following approximation for h when the option is European

$$\begin{aligned} h(t,y) &= \lambda e^{-r(T-t)+y}\Phi(\frac{y-y_{max}+\gamma}{\delta}+\frac{1}{2}\delta) \\ &- \lambda' e^{-q(T-t)}\alpha\Phi(\frac{y-y_{max}+\gamma}{\delta}-\frac{1}{2}\delta) \\ &+ \lambda' f(t,y_{min})\Phi(\frac{y_{min}-y-\gamma}{\delta}+\frac{1}{2}\delta) \end{aligned}$$

For the American style option we get the approximation

$$\begin{aligned} h(t,y) &= \lambda e^{y}\Phi(\frac{y-y_{max}+\gamma}{\delta}+\frac{1}{2}\delta) \\ &- \lambda'\alpha\Phi(\frac{y-y_{max}+\gamma}{\delta}-\frac{1}{2}\delta) \\ &+ \lambda' f(t,y_{min})\Phi(\frac{y_{min}-y-\gamma}{\delta}+\frac{1}{2}\delta) \end{aligned}$$

With this we can numerically solve for the price of the lookback option with floating strike. The Figures 3 through 6 illustrate the effect of adding a jump component to the stochastic evolution of the underlying. We set the parameters of the jump model so that the Q-local variance of the stock

$$\frac{\partial}{\partial u}\mathrm{Var}_t\left[\frac{S(u)}{S(t)}\right]|_{u=t} = 0.2^2$$

for both models.

The effect of adding the jump component is low for the Asian options. Probably due to the more fat tailed and peaked distribution of the jump model, there is a tendency of lower prices for the jump model around at-the-money and higher prices deeb in-the-money and out-of-the-money. The introduction of jumps reduces the prices of the average strike options at least around at-the-money. Deep-out-of-the money options seem to have higher prices than in the Black-Scholes case. The fixed strike lookbacks prices are also reduced when there are jumps in the dynamics of the underlying. This must again be attributed the more peaked distribution of the jump-model. For floating strike lookback option prices, we see a similar picture: Reduced prices except for the deep out-of-money case.

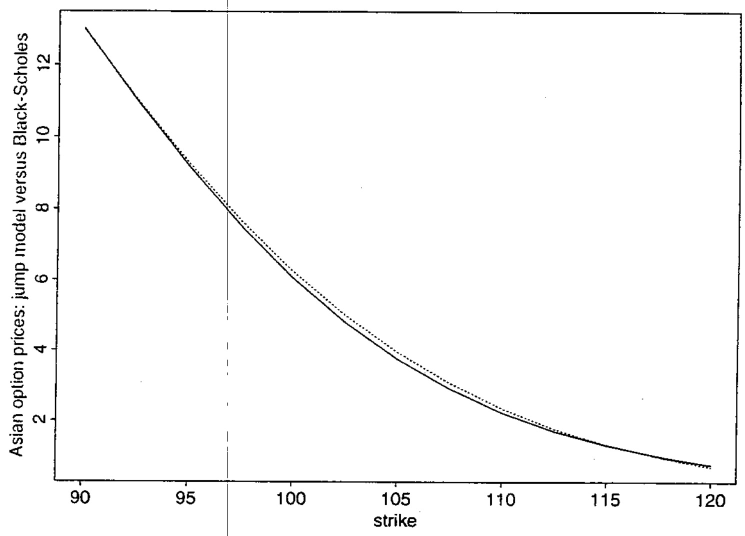

Figure 3: Asian option prices: Jump model (solid line) versus Black-Scholes model (dotted line). Parameters are: $r = 0.05, q = 0.0, \sigma = 0.2, T = 1.0, t = 0.0, S(0) = 100.0$ for both models, except for the jump model where $\sigma = 0.099$ and $\lambda = 3.0, \delta = 0.1, \gamma = 0.0$. Prices were generated using a 500×500 finite difference grid.

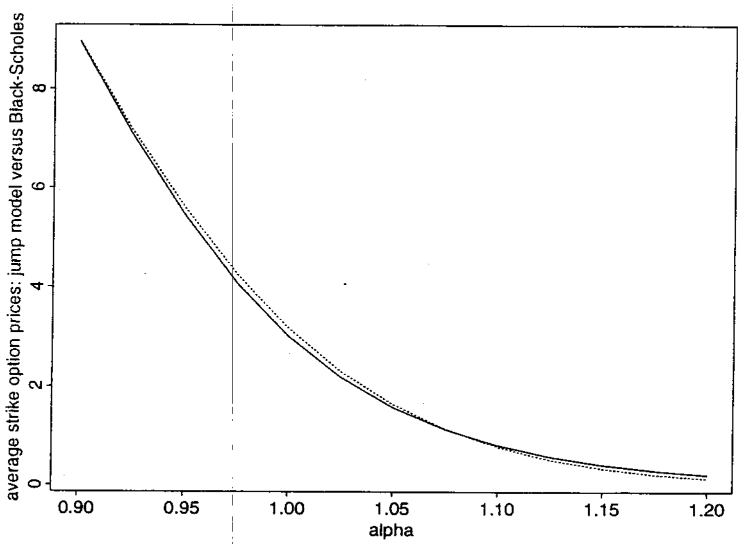

Figure 4: Average strike option prices: Jump model (solid line) versus Black-Scholes model (dotted line). Parameters are: $r = 0.05, q = 0.0, \sigma = 0.2, T = 1.0, t = 0.0, S(0) = 100.0$ for both models, except for the jump model where $\sigma = 0.099$ and $\lambda = 3.0, \delta = 0.1, \gamma = 0.0$. Prices were generated using a 500×500 finite difference grid.

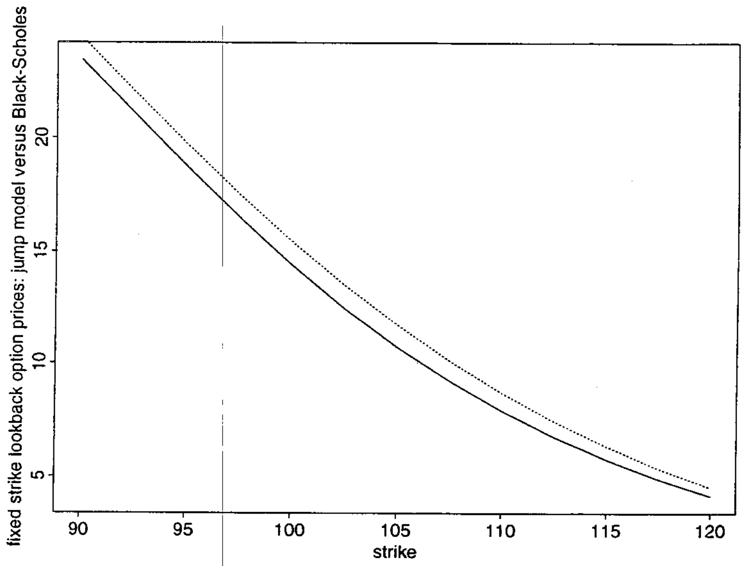

Figure 5: Fixed strike lookback option prices: Jump model (solid line) versus Black-Scholes model (dotted line). Parameters are: $r = 0.05, q = 0.0, \sigma = 0.2, T = 1.0, t = 0.0, S(0) =$ 100.0 for both models, except for the jump model where $\sigma = 0.099$ and $\lambda = 3.0, \delta = 0.1, \gamma =$ 0.0. Prices were generated using a 500×500 finite difference grid.

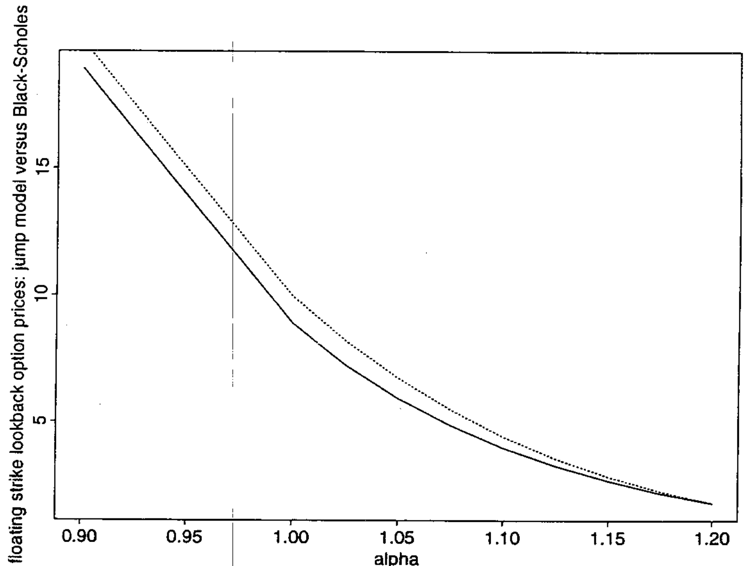

Figure 6: Fixed strike lookback option prices: Jump model (solid line) versus Black-Scholes model (dotted line). Parameters are: $r = 0.05, q = 0.0, \sigma = 0.2, T = 1.0, t = 0.0, S(0) =$ 100.0 for both models, except for the jump model where $\sigma = 0.099$ and $\lambda = 3.0, \delta = 0.1, \gamma =$ 0.0. Prices were generated using a 500×500 finite difference grid.

9 Conclusion

This paper has described an approach to the numerical pricing of discretely observed path-dependent options that is highly competitive in terms of accurracy and speed compared to Monte Carlo simulations. We have illustrated this by numerical examples for four types of path-dependent options.

A second advantage of this pricing technique compared to Monte Carlo techniques is the ability to price the floating strike American style options. This can not be done by standard Monte Carlo methods.

In the Black-Scholes and the jump framework the technique applies to most types of European options on the average and the maximum (or minimum). Among the types of options that have not been considered in this paper but can be priced using our approach are combinations of maximum, minimum, and average and digital options on the average and/or the maximum. Another application for our technique is the pricing of equity linked life insurance contracts.

10 Appendix

10.1 Derivation of the Equations (8) and (36)

Let x be defined as in (7) and let $\nu(\cdot)$ be a deterministic function. Using Ito's lemma and (35) we get

$$\begin{aligned} d\left[\nu(t)x(t)\right] = &= (\nu'(t)x(t-) - (r-q-k\lambda)\nu(t)x(t-))dt - \sigma x(t-)\nu(t)dW'(t) \\ &- \nu(t)x(t-)\frac{I'(t)}{1+I'(t)}dN'(t) + \nu(t)\frac{1}{n}dm(t) \end{aligned}$$

Inserting

$$\nu(t) = e^{(r-q)t}$$

we get that

$$d\left[\nu(t)x(t)\right] = -\sigma x(t-)\nu(t)dW'(t) - \nu(t)x(t-)(\frac{I'(t)}{1+I'(t)}dN'(t) - k\lambda dt) + \nu(t)\frac{1}{n}dm(t) \quad (44)$$

We have that

$$\int_t^{\cdot} x(u-)\nu(u)(\sigma dW'(u) + \frac{I'(u)}{1+I'(u)}dN'(u) - k\lambda du)$$

is a $\mathcal{Q}'$-martingale, so integrating (44) and taking $\mathcal{Q}'$ expectation yield

$$\begin{aligned} e^{(r-q)T}\mathrm{E}'_t\left[x(T)\right] &= e^{(r-q)t}x(t) + \frac{1}{n}\int_t^T e^{(r-q)u}dm(u) \\ &= e^{(r-q)}x(t) + \frac{1}{n}\sum_{i:t<t_i\geq t_n} e^{(r-q)t_i} \end{aligned}$$

So we get that

$$e^{-q(T-t)}\mathrm{E}'_t\left[x(T)\right] = e^{-r(T-t)}x(t) + \frac{1}{n}\sum_{i:t<t_i\leq t_n} e^{-r(T-t_i)-q(t_i-t)}$$

For $\lambda = 0$ we have equation (8) and for $\lambda > 0$ we have equation (36).

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Implied Modelling: Stable Implementation, Hedging, and Duality

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Abstract

Dupire (1993) derives a forward partial differential equation for European option prices. In this paper we go further and derive forward equations for all the "Greeks" of European option prices. The forward equations turn the option pricing problem into a problem in strikes and maturities with fixed spot and time rather than a problem in spot and time with fixed maturity and strike. Examination of the forward equations reveals that the European option pricing problem might be solved in a dual economy where every parameter of the option pricing problem is reversed: time is reversed, the underlying is now the strike, the strike price is the underlying, the call is a put, etc.

We set up a model that takes a set of marketed European option prices of different strikes and maturities as primitives, i.e. the model is constructed so that it automatically fits the current "volatility smiles" of different maturities. The numerical implementation of the model will always be stable given that there is absence of static arbitrage in the input option prices. The derived forward equations enable us to find the "Greeks" of the marketed options implied by the option prices themselves in a computational efficient way. In fact we are able to find the "Greeks" of all options on a 200×200 grid (in time to maturity and strike) in less than a second of computer time. A numerical example based on S&P 500 data shows that the "implied Greeks" can differ considerably from those of the Black-Scholes model. This suggests that in a world with "volatility smiles", hedging strategies based on the Black-Scholes model might exhibit large tracking errors.

Introduction

In recent years considerable attention among financial researchers as well as practitioners has been directed towards construction and implementation of option pricing models that automatically fit option prices observed in financial markets. This development has been pushed forward by very pronounced deviations of option prices observed in today's financial markets from prices calculated using the Black-Scholes (1973) model. It is not unusual to see implied volatilities, deduced by inverting the Black-Scholes formula, ranging from 10% for out-of-the-money stock call options to 30% for in-the-money calls. Dramatic "smiles" has also been observed in currency options markets.

The object of this line of research is mainly to derive models that can be used for the pricing and hedging of exotic options and OTC derivatives but also the hedging of marketed options, in a way that is consistent with market prices on standard options. In other words, the idea is to take a set of exchange traded options as primitives and then price and hedge other derivatives relative to the marketed options.

The theoretical basis for this research goes back to Breeden and Litzenberger (1978) who show that buying a butterfly spread in call options around a fixed strike price is approximately equivalent to buying a state contingent claim on the spot being in that point. So given a continuum over strikes of option prices the state price density or equivalently the risk-neutral density of the spot can be recovered by taking the second derivative of the option prices with respect to the strike.

In a discrete time and state setting Derman and Kani (1994) and Rubinstein (1994) go further and construct binomial models that fit a set of input option prices.

Dupire (1993) investigates the continuous-time case where the stock price is driven by a Markovian diffusion process. He shows that if European option prices are given for all strikes and all maturities then the local volatility as a function of time and spot can be determined uniquely from these prices. To show this Dupire uses the forward Fokker-Planck equation of the risk-neutral density of the stock-price process to derive a forward equation for European option prices in maturity date and strike price. From this he is able to deduce the volatility function of the spot from the derivatives of the option prices in the strike and time-to-maturity dimension. Dupire suggests a trinomial approximation scheme for the actual implementation.

In this paper we show that the forward equation for the option prices has other applications. First, we show that the Deltas and the Gammas of the marketed options also satisfy a forward running equation. Secondly, we show that the Vegas (the volatility sensitivities) and the option prices' sensitivities to other parameters can be deduced from similar equations. This enables us to compute the "Greeks" in a very efficient way: we get out *all* "the Greeks" of *all* the marketed options by running only *one* finite difference grid forward, simultaneously solving a small set of PDEs. And these "Greeks" are the ones implied by the prices of the marketed options.

Lastly we show that the forward equations for the option prices imply a duality: the problem of pricing and hedging of European options can be solved in a dual economy where the spot is the strike, the strike is the spot, the call is the put, the interest rate is the dividend yield, and the dividend yield is the interest rate. The hedge ratio will be the price of a digital option and the Gamma will be a state price.

For the actual numerical implementation, the binomial and the trinomial models of Derman and Kani (1994), Rubinstein (1994), and Dupire (1993) unfortunately all suffer from one or both of the following drawbacks:

i. The model only fits option prices at one maturity date.

ii. The model is not always stable.

(i.) implies that the model only matches the option prices across strikes at one maturity date only, i.e. information from option prices at other maturities is not incorporated in the model. (ii.) means that even though option prices given as input are consistent with absence of arbitrage the one period risk-neutral probabilities might go negative. This of course induce arbitrage and stability problems in the model. The stability problem is due to the fact that these implementations all try to push the evolution of the underlying, that in this case can have as arbitrary volatility function, into a binomial structure or an explicit finite difference scheme. This approach is bound to give trouble with the stability, since explicit finite difference and binomial lattice schemes can not approximate any Markovian diffusion process.

In this paper we suggest an implementation that incorporates information from all strikes and all maturities. Moreover our implementation is stable given that input option prices are consistent with absence of static arbitrage. This is obtained through first deducing the local volatility function from option prices observed in the market and then using an implicit finite difference scheme for calculating other option prices, hedge ratios, and parameter sensitivities. The implicit finite difference method has the advantage over that any continuous Markov process can be treated without problems with the stability. Furthermore it is computational efficient. As an input our implementation can take either observed European option prices or (Black-Scholes) implied volatilities.

Andersen (1996) also uses the implicit finite difference method in the context of implied modelling. In this paper the risk-neutral distribution is first estimated from a set of input option prices. Then by solving a constrained quadratic program the local volatilities are backed out from the risk-neutral densities of the different maturities. Contrary to this we directly infer the local volatilities from the input implied volatilities. This is done through an explicit formula that relates the surface of implied volatilities to the surface of local volatilities. And instead of solving a large set of backward equations to obtain the "Greeks" of the input options we calculate these quantities by simultaneously solving a small set of forward equations.

We illustrate our implementation by a numerical example, where the input data are extracted from a single day of S&P 500 index option quotes. Our numerical example shows that the implied local volatility smile is much more pronounced than the implied Black-Scholes volatility smile. Implied risk-neutral distributions seem to be peaked and highly skewed. We show that this implies that the call option hedge ratios of the implied model are substantially lower than those implied by the Black-Scholes model.

This suggests that in a world with "volatility smiles", hedging using the Black-Scholes Delta might induce significant "over hedging" for the calls and vice-versa for the puts. Gammas and other sensitivities also seem to differ significantly from those of the Black-Scholes model.

The paper is organized as follows. The first section presents the modelling framework and reviews the standard arbitrage pricing results for one-factor diffusion models. In the second section we present the forward equation of Dupire (1993) and give new forward equations for the Deltas and other option price sensitivities. The duality of the option pricing problem in our modelling framework is discussed in the third section. The fourth and last section considers the numerical implementation.

The Modelling Framework and Arbitrage Pricing

We start by presenting the type of model that we will consider in this paper and review some standard arbitrage pricing results.

We consider an economy where there is a single dividend paying stock and a money market account.[1] Suppose that the continuously compounded interest rate, r, and the stock's continuous dividend yield, q, are deterministic functions of time. Assume that the stock price evolves according to the stochastic differential equation

$$\frac{dS(t)}{S(t)} = (\mu(t, S(t)) - q(t))dt + \sigma(t, S(t))dw^{\mathcal{P}}(t) \tag{1}$$

where $w^{\mathcal{P}}$ is a standard Brownian motion under the objective probability measure $\mathcal{P}$, μ is the instantaneous mean return of holding the stock, and σ is the instantaneous standard deviation of the return, the so-called local volatility. We assume that μ, σ are well-behaved functions of time and stock price. Further we define

$$\begin{aligned} B(t;T) &= \exp\left(-\int_t^T r(y)dy\right) \\ D(t;T) &= \exp\left(-\int_t^T q(y)dy\right) \end{aligned} \tag{2}$$

$B(t;T)$ will be the time t price of a zero-coupon bond expiring at time T and $r(T)$ will be the time t continuously compounded maturity T forward rate. $D(\cdot)$ can be interpreted as a compounded dividend factor in the sense that $S(T)/(S(t)D(t;T))$ is the return over $[t, T]$ of the strategy of buying one stock at time t and then continuously reinvesting the dividends in new stocks.

Under absence of arbitrage and assumption of sufficient regularity of the stock price process there exists an equivalent probability measure, $\mathcal{Q}$, under which all discounted

[1] Currency options as well as options on commodities can also be considered in this framework. All one has to do is to replace the dividend yield by the foreign interest rate or minus the cost-of-carry of the commodity.

asset prices including accumulated dividends are martingales.[2] This is the so-called risk-neutral measure. Moreover the model is dynamically complete, in the sense that any pay-off measurable with respect to the filtration generated by the stock price can be replicated by a self-financing dynamic trading strategy in the stock and the bond. The completeness of the model means that the risk-neutral measure is unique. Under the risk-neutral measure the stock evolves according to the stochastic differential equation

$$\frac{dS(t)}{S(t)} = (r(t) - q(t))dt + \sigma(t, S(t))dw^Q(t) \tag{3}$$

where w^Q is a standard Brownian morion under Q. A European call option with maturity date T and strike K written on the stock will have the time t price[3]

$$B(t;T)\mathrm{E}_t^Q\left\{(S(T) - K)^+\right\} \tag{4}$$

Due to the Markovian properties of the stock price process the call option price can be written as the function

$$C(t, S; T, K) \tag{5}$$

Using Ito's lemma this implies that the valuation of the option can be performed by solving the backward partial differential equation

$$r(t)C = \frac{\partial C}{\partial t} + (r(t) - q(t))S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma(t, S)^2 S^2 \frac{\partial^2 C}{\partial S^2} \tag{6}$$

with the boundary condition

$$C(T, S; T, K) = (S - K)^+ \tag{7}$$

Moreover the self-financing dynamic trading strategy replicating the option consists of

$$\Delta(t, S(t); T, K) = \frac{\partial C(t, S(t); T, K)}{\partial S} \tag{8}$$

stocks and the remaining amount

$$C(t, S(t); T, K) - \Delta(t, S(t); T, K)S(t) \tag{9}$$

on the bank-account.

Forward Equations for European Option Prices

The previous section reviewed the well-known theory of pricing by arbitrage, initiated by Black and Scholes (1973) and further developed and refined by Harrison and Kreps (1979), and others. In this section we consider a less well-known result by Dupire (1993), and derive some implications of this result.

Dupire shows that under our assumptions, the option prices also satisfy a *forward* equation in maturity date and strike price. The result is restated below.

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[2] For sufficient regularity conditions see for example Rydberg (1996).

[3] In this paper we will only consider European calls. All the presented results can also be deduced for European puts by use of the put-call parity.

Result 1: The Forward Equation for European Option Prices.
The option price function $C(t,S;T,K)$ *is the solution to the forward partial differential equation*

$$q(T)C = -\frac{\partial C}{\partial T} - (r(T) - q(T))K\frac{\partial C}{\partial K} + \frac{1}{2}\sigma(T,K)^2K^2\frac{\partial^2 C}{\partial K^2} \tag{10}$$

subject to the boundary condition

$$C(t,S;t,K) = (S-K)^+ \tag{11}$$

Dupire derives Result 1 in a zero-interest rate and no-dividend economy. For completeness we show Result 1 in this slightly more general setting in the Appendix.[4]

Result 1 converts the option pricing problem from one where strike and maturity are fixed and spot and time are variables into a problem where spot and time are fixed whereas expiration and strike are variables.

This result has several implications. If we observe a double continuum of option prices in strike and maturity and this surface is sufficiently smooth to admit double differentiation in the strike direction and single differentiation in the maturity dimension, then the local volatility function is uniquely determined from (10). The trick is simply to isolate the local volatility function and then to plug in the derivatives of the observed option prices. Under the assumption that the surface of option prices is sufficiently smooth Andersen (1996) shows that if there are no static arbitrage opportunities in the option market then this procedure will always produce a valid local volatility function, i.e. $\sigma(t,S)^2 > 0$ for all (t,S). In practical implementations the described procedure requires extensive use of extrapolation and interpolation. This a nontrivial subject that we will return to in a subsequent section.

Result 1 also tells us that if we know the volatility function then a full grid, in maturity date and strike, of option prices can be computed by solving only one partial differential equation numerically. This is clearly advantageous to solving a full set of backward equations or alternatively solving the Fokker-Planck equation for the risk-neutral density and then obtaining the option prices from numerical integration. Forward equations also hold for the standard sensitivities of the option prices to changes in the spot and time. The result below demonstrates this.

First define:

$$\begin{aligned}\Gamma(t,S;T,K) &= \frac{\partial^2 C(t,S;T,K)}{\partial S^2}\\ \Theta(t,S;T,K) &= \frac{\partial C(t,S;T,K)}{\partial t}\end{aligned} \tag{12}$$

The Γ reflects the sensitivity of the hedge ratio to changes in the underlying or the "convexity" of the option, i.e the second order non-linear element of the option price's response to changes in the spot. The Θ reflects the change in the option price due to time elapsing.

4 The forward equation (10) is in fact also valid when r, q additionally are allowed to be functions of the underlying.

Result 2: Forward Equations for Δ, Γ, Θ.

i. *The hedge ratio for the option prices, $\Delta(t, S; T, K)$, is the solution to the forward partial differential equation*

$$q(T)\Delta = -\frac{\partial \Delta}{\partial T} - (r(T) - q(T))K\frac{\partial \Delta}{\partial K} + \frac{1}{2}\sigma(T, K)^2 K^2 \frac{\partial^2 \Delta}{\partial K^2} \tag{13}$$

subject to the boundary condition

$$\Delta(t, S; t, K) = \mathbf{1}_{K \leq S} \tag{14}$$

ii. *The convexity of the option prices, $\Gamma(t, S; T, K)$, is the solution to the forward partial differential equation*

$$q(T)\Gamma = -\frac{\partial \Gamma}{\partial T} - (r(T) - q(T))K\frac{\partial \Gamma}{\partial K} + \frac{1}{2}\sigma(T, K)^2 K^2 \frac{\partial^2 \Gamma}{\partial K^2} \tag{15}$$

subject to the boundary condition

$$\Gamma(t, S; t, K) = \delta(K - S) \tag{16}$$

where $\delta(\cdot)$ is the Dirac Delta function.[5]

iii. *The time sensitivity of the option prices, $\Theta(t, S; T, K)$, is given by*

$$\begin{aligned}&\Theta(t, S; T, K) = \\ &r(t)C(t, S; T, K) - (r(t) - q(t))S\Delta(t, S; T, K) - \frac{1}{2}\sigma(t, S)^2 S^2 \Gamma(t, S; T, K)\end{aligned} \tag{18}$$

The proof of this result is rather simple: To see (i.) differentiate (10) with respect to the stock price and change the orders of differentiation. Differentiating once again gives us (ii.). Finally (iii.) obtains from the backward PDE (6).

Say we have identified the local volatility function supporting the observed option prices. Then the hedge ratio and the other sensitivities implied by the option prices can be obtained by numerically solving two forward PDEs simultaneously in the same grid. We illustrate how to do this in practice by a numerical example in the last section.[6]

Option traders often like to consider the sensitivity of the option prices to a change in the volatility. This is clearly inconsistent with the standard Black-Scholes model as well as the kind of model considered here. In our framework volatility only changes

[5] The Dirac Delta function is the density with respect to the Lebesque measure of 0 (!), in the sense that

$$\begin{aligned}&\delta(x) = 0, \; x \neq 0 \\ &\int_{-\epsilon}^{\epsilon} \delta(x)dx = 1\end{aligned} \tag{17}$$

for any $\epsilon > 0$.

[6] One could also obtain these quantities by numerically solving 2 forward equations of the type given in Result 1, for two stock prices locally around the current spot, and then approximate the derivatives by first and second order central differences. This is computational almost as efficient as the described procedure.

as spot and time change. Nonetheless, for practical reasons it is often useful to deduce these sensitivities.[7] Consider a change in the local volatility surface of the additive type

$$\sigma(t,S) \curvearrowright \sigma(t,S)+\epsilon \tag{19}$$

for all (t,S) where ϵ is a small number. We will term the sensitivity of the option prices to such a change *Vega* and represent it by

$$\Psi(t,S;T,K) \tag{20}$$

Likewise one might want to consider changes of the type

$$\begin{aligned} r(t) &\curvearrowright r(t)+\epsilon \\ q(t) &\curvearrowright q(t)+\epsilon \end{aligned} \tag{21}$$

for all t. Let the associated option price sensitivities be denoted

$$\begin{aligned} &\Phi(t,S;T,K) \\ &\Lambda(t,S;T,K) \end{aligned} \tag{22}$$

respectively.

It is important to note that apart from giving us an ad-hoc based estimate of for example the effect of a random change in the interest rate, the above defined sensitivities also measure the sensitivity of our option pricing and hedging to the estimation error that we always will encounter in practical implementations due to bid-ask spreads and other market imperfections and rigidities.

The following result states forward equations for identification of these sensitivities.

Result 3: Forward Equations for Sensitivities to Parameters.

i. *The option price volatility sensitivity, $\Psi(t,S;T,K)$, is the solution to the forward partial differential equation*

$$\begin{aligned} &q(T)\Psi-\sigma(T,K)K^2\frac{\partial^2 C(t,S;T,K)}{\partial K^2} \\ &=-\frac{\partial\Psi}{\partial T}-(r(T)-q(T))K\frac{\partial\Psi}{\partial K}+\frac{1}{2}\sigma(T,K)^2K^2\frac{\partial^2\Psi}{\partial K^2} \end{aligned} \tag{23}$$

subject to the boundary condition

$$\Psi(t,S;t,K)=0 \tag{24}$$

ii. *The option price interest rate sensitivity, $\Phi(t,S;T,K)$, is the solution to the forward partial differential equation:*

$$\begin{aligned} &q(T)\Phi+K\frac{\partial C(t,S;T,K)}{\partial K} \\ &=-\frac{\partial\Phi}{\partial T}-(r(T)-q(T))K\frac{\partial\Phi}{\partial K}+\frac{1}{2}\sigma(T,K)^2K^2\frac{\partial^2\Phi}{\partial K^2} \end{aligned} \tag{25}$$

[7] Bond portfolio managers also compute and use duration and convexity of bonds as measures of bond price sensitivities to interest changes even though these measures are inconsistent with almost any term structure model, see Cox, Ingersoll and Ross (1979).

subject to the boundary condition

$$\Phi(t, S; t, K) = 0 \tag{26}$$

iii. *The dividend sensitivity of the option prices, $\Lambda(t, S; T, K)$, is the solution to the forward partial differential equation*

$$\begin{aligned} q(T)\Lambda + C(t, S; T, K) - K\frac{\partial C(t, S; T, K)}{\partial K} \\ = -\frac{\partial \Lambda}{\partial T} - (r(T) - q(T))K\frac{\partial \Lambda}{\partial K} + \frac{1}{2}\sigma(T, K)^2 K^2 \frac{\partial^2 \Lambda}{\partial K^2} \end{aligned} \tag{27}$$

subject to the boundary condition

$$\Lambda(t, S; t, K) = 0 \tag{28}$$

The proofs of these results are pretty straightforward: differentiate equation (10), change the order of differentiation, rearrange terms, and the results obtain.[8]

The forward equations of Result 3 require the knowledge of the derivatives of the option prices with respect to the strike, but these quantities are natural by-products when one solves (10). The idea is therefore to numerically solve the forward equations of Result 1 through 3 simultaneously in *one* finite difference grid and thereby obtain prices, hedge ratios and sensitivities for a double continuum of options in strike and expiration date simultaneously.

Duality of the Option Pricing Problem

The forward equations derived in the previous section reveal a duality, which we will now discuss.

Without loss of generality fix current time to be 0 and define

$$u = -T \tag{29}$$

Suppose K is no longer a fixed quantity but that it instead follows the process

$$\frac{dK(t)}{K(t)} = (q(-t) - r(-t))dt + \sigma(-t, K(t))dw^{Q'}(t) \tag{30}$$

on $t \leq 0$ where $w^{Q'}$ is a standard Brownian motion under some probability measure Q'. Further, suppose that S is a fixed and known quantity, and that we are sitting at time $u = -T$ evaluating the expectation

$$C(0, S; T, K) = D(0; -u)\mathrm{E}^{Q'}\left[(S - K(0))^+ | K(u) = K\right] \tag{31}$$

[8] The same technique could also be used to derive forward equations for sensitivities to non-parallel shifts in the parameters. For simplicity we choose not to.

Then (10) is the *backward* equation resulting from that problem. Moreover, utilizing Result 2 we see that

$$\begin{aligned} \Delta(0, S; T, K) &= D(0; -u)Q'[K(0) \leq S | K(u) = K] \\ \Gamma(0, S; T, K) &= D(0; -u)\mathrm{E}^{Q'}[\delta(K(0) - S) | K(u) = K] \end{aligned} \tag{32}$$

We conclude that the option pricing problem can equally well be solved in an economy dual to the one considered. Going into the dual economy time is reversed, the strike price becomes the underlying, the spot becomes the strike, the interest rate becomes the dividend yield, and the dividend yield becomes the interest rate. Furthermore the replicating portfolio position in the stock in the primal economy is given by the price of a digital option in the dual economy and the Gamma of the primal economy can be found as a state price in the dual economy. Interestingly, the Deltas and the Gammas now have integral rather than differential representations.

Going even further we see from Result 3 that the Vega of the primal economy can be found as the price of an asset that pays a continuous dividend stream of

$$\sigma(-t, K(t))K(t)^2 \frac{\partial^2 C(0, S; -t, K(t))}{\partial K^2} \tag{33}$$

on $t \in [u, 0]$ in the dual economy. But differentiating the forward equation of the primal economy with respect to $\sigma(T, K)$ is the same as differentiating the backward equation of the dual economy with respect to $\sigma(-t, K)$. So the Vega of the primal economy is also the Vega of the dual economy.

Similarly, the interest rate sensitivity of the primal economy is the dividend sensitivity of the dual economy and the dividend sensitivity of the primal economy is the interest rate sensitivity of the dual economy.

Result 4 summarizes the duality.

Result 4: Duality of the European Option Pricing Problem.

The European option pricing problem can be solved by considering a dual economy. The relations between the primal and the dual economy are described in the following table:

Economy	Primal	Dual
Underlying	S	K
Option	Call	Put
Strike	K	S
Interest rate	r	q
Dividend yield	q	r
Δ	Hedge ratio	Digital option
Γ	Gamma	State price

Ψ	Vega	Vega
Φ	Interest rate sensitivity	Dividend rate sensitivity
Λ	Dividend rate sensitivity	Interest rate sensitivity

After these considerations we now turn to the second object of the paper: the implementation of the described theory.

Numerical Implementation

We now consider the construction of a model that fits a discrete set of observed European option prices in the sense that the option prices of the model lie in between the bid/ask spread of the marketed options.

According to Result 1 we can derive a unique local volatility surface that supports a given smooth double continuum, in strike and maturity, of option prices. First we need this smooth double continuum of option prices. This requires interpolation between the observed strikes and maturities but in most cases it also requires extrapolation into areas where we do not have observations. We will for example never have observations of option prices with maturities all the way down to current time. This is not unlike estimating the first and the last part of the term structure of interest rates.

We prefer to do the interpolation and extrapolation in the space of implied (Black-Scholes) volatilities rather than directly in option prices for two reasons: First, it is easier to relate to implied volatilities than to the prices themselves, which is probably also why traders often prefer to quote option prices in terms of implied volatilities. Secondly, it is easier to smooth, interpolate, and extrapolate in a space where the function that has to be interpolated is rather (though not at all perfectly) flat. Presumably this will be the case for the surface of implied volatilities.

When the identification of the implied volatility surface is done we need to convert these into local volatilities. Defining $\hat{\sigma}(T, K)$ to be the time 0 (Black-Scholes) implied volatility function of strike, K, and maturity date, T, Result 5 gives the relation between implied Black-Scholes volatility and local volatility.[9]

Result 5: The Relation between Implied and Local Volatility.

The local volatility function is related to the Black-Scholes implied volatility by the equation

$$\frac{1}{2}\sigma(T,K)^2K^2 = \left(\frac{\frac{\hat{\sigma}}{2T} + \hat{\sigma}_T + (r-q)K\hat{\sigma}_K}{\frac{1}{\hat{\sigma}TK^2} + 2\frac{d}{\hat{\sigma}\sqrt{T}K}\hat{\sigma}_K + \frac{d(d-\hat{\sigma}\sqrt{T})}{\hat{\sigma}}(\hat{\sigma}_K)^2 + \hat{\sigma}_{KK}} \right)(T,K) \tag{34}$$

$$d(T,K) = \frac{1}{\hat{\sigma}(T,K)\sqrt{T}} \ln \frac{SD(0;T)}{KB(0;T)} + \frac{1}{2}\hat{\sigma}(T,K)\sqrt{T}$$

[9] Result 5 was simultaneously derived by Andersen (1996).

where subscripts denote partial derivatives.

The proof of Result 5 is in the appendix.

The local volatilities are given as differentials of the implied volatility surface, whereas if we want to convert local volatilities into Black-Scholes implied volatilities we have to perform (not straightforward) integration, or solve the partial differential equation (10). For the term structure of interest rates we have a similar relation: Forward rates can be obtained from yield-to-maturity rates by differentiation, but we have to perform integration to get the yield-to-maturity rates from forward rates.

Our implementation is rather simple and does not require any iteration or search for an optimal or feasible solution once we have identified or rather estimated the surface of implied volatilities of the market. It can be summarized in the following steps:

i. Convert bid and ask option prices into implied volatilities.
ii. Smooth a surface of implied volatilities in the (T, K) space between bid and ask volatilities.
iii. Use Result 5 to convert implied volatilities to local volatilities.
iv. Using Result 1 through 3 we simultaneously solve for all the "Greeks" of the marketed options by a forward running implicit finite difference scheme.
v. The scheme constructed in (iv.) can now be solved backwards to obtain prices and hedge ratios of non-marketed claims.

In the above we do not necessarily need to compute the "Greeks". Step (iv.) might be skipped and we might go directly from step (iii.) to step (v.).

We now illustrate our procedure by a numerical example.[10]

The input data are bid and ask option prices, interest rates, dividend yields and the spot. In the example below the option prices used as input are based on median bid/ask quotes of S&P 500 call option prices over the day 90.03.19. The interest rates and the dividend yields are backed out from the put-call parity.

Suppose we have a table of bid/ask option price implied volatilities, interest rates and dividend yields as the one below. An empty cell means that there is no observation for this particular time to maturity and strike.

The interest rates and dividend yields in the above table are related to the discount factors and the factors of accumulated dividends by

$$\begin{aligned} B(0;T) &= \exp\left(-TR(0;T)\right) \\ D(0;T) &= \exp\left(-TQ(0;T)\right) \end{aligned} \tag{35}$$

We perform cubic splines in the rates R, Q, and then find r, q by the relations

$$\begin{aligned} r(T) &= R(0;T) + T\frac{\partial R(0;T)}{\partial T} \\ q(T) &= Q(0;T) + T\frac{\partial Q(0;T)}{\partial T} \end{aligned} \tag{36}$$

10 The data set was kindly provided by Jens Jackwerth and Mark Rubinstein.

Table 1: Input Data.

$T_i \rightarrow$	0.2411	0.5096	0.7589
$K_j \downarrow$	$\hat{\sigma}_{bid}/\hat{\sigma}_{ask}$	$\hat{\sigma}_{bid}/\hat{\sigma}_{ask}$	$\hat{\sigma}_{bid}/\hat{\sigma}_{ask}$
250	0.2729/0.3907	0.2564/0.3054	0.2569/0.2860
275	0.2761/0.3261	0.2467/0.2738	0.2460/0.2642
300	0.2337/0.2596	0.2215/0.2385	0.2264/0.2362
305	0.2295/0.2531	0.2150/0.2301	-
310	0.2183/0.2408	0.2093/0.2229	-
315	0.2112/0.2265	0.2036/0.2165	-
320	0.2007/0.2138	0.1969/0.2089	-
325	0.1890/0.2021	0.1872/0.1995	0.1969/0.2054
330	0.1807/0.1908	0.1832/0.1920	0.1888/0.1975
335	0.1728/0.1816	0.1763/0.1846	0.1840/0.1920
340	0.1623/0.1690	0.1688/0.1769	0.1775/0.1856
345	0.1517/0.1589	0.1625/0.1701	0.1727/0.1794
350	0.1414/0.1482	0.1580/0.1655	0.1707/0.1745
355	0.1303/0.1393	0.1529/0.1595	0.1624/0.1688
360	0.1216/0.1321	0.1465/0.1523	0.1568/0.1633
365	0.1119/0.1271	-	-
370	0.0998/0.1303	-	-
375	0.0970/0.1297	0.1245/0.1351	0.1402/0.1460
380	0.1064/0.1220	-	-
385	0.1093/0.1273	-	-
400	-	-	0.0819/0.1124
Interest Rate $R(0;T)$	0.0803	0.0807	0.0802
Dividend Yield $Q(0;T)$	0.0378	0.0358	0.0353
Spot	341.18		

Letting $\underline{S}, \overline{S}$ be the lower and the upper bound of the range of stock prices that we want to consider and letting τ be our time horizon, the object is now to estimate a "smooth" surface of implied volatilities, $\{\hat{\sigma}(T,K)\}_{T\in[0,\tau],K\in[\underline{S},\overline{S}]}$, that lies in between the bid and ask input volatilities. As objective function for the "smoothness" we take the sums of squared second derivatives of the implied volatility surface in the (joint) time to maturity and strike direction, i.e. we minimize[11]

$$\int_0^{\tau}\int_{\underline{S}}^{\overline{S}} \left[\left(\frac{\partial^2\hat{\sigma}}{\partial K^2}\right)^2 + 2\left(\frac{\partial^2\hat{\sigma}}{\partial T\partial K}\right)^2 + \left(\frac{\partial^2\hat{\sigma}}{\partial T^2}\right)^2\right](T,K)dKdT \tag{37}$$

over the function $\{\hat{\sigma}(T,K)\}$ subject to the constraint

$$\hat{\sigma}_{bid}(T_i,K_j) \leq \hat{\sigma}(T_i,K_j) \leq \hat{\sigma}_{ask}(T_i.K_j) \tag{38}$$

for all strikes K_j and maturities T_i given in table 1.
We let

$$\underline{S} = S(0)/2, \overline{S} = 2S(0), \tau = 0.8 \tag{39}$$

and limit the state space to the discrete points given by the strikes in Table 1 plus 10 strikes between $K_0 \equiv \underline{S}$ and the lowest strike of Table 1 and 10 strikes between the highest strike and $K_{40} \equiv \overline{S}$. In the time to maturity dimension we only consider the maturities given in Table 1 plus the time points $T_0 = 0.0$ and $T_4 = \tau = 0.8$. All in all we now have a grid of 5×41 points or 204 variables, $\{\hat{\sigma}(T_i,K_j)\}$. For the implied volatility function to be well-behaved outside the interval of strikes $[250,400]$, we add the constraints

$$\hat{\sigma}_{bid}(T_i,250) \leq \hat{\sigma}(T_i,K_j) \leq \hat{\sigma}_{ask}(T_i.250) \tag{40}$$

for all $K_j < 250, i = 1,2,3$ and

$$\begin{aligned} &\hat{\sigma}_{bid}(0.2411,385) \leq \hat{\sigma}(0.2411,K_j) \leq \hat{\sigma}_{ask}(0.2411.385), \quad K_j > 385 \\ &\hat{\sigma}_{bid}(0.7589,400) \leq \hat{\sigma}(0.7589,K_j) \leq \hat{\sigma}_{ask}(0.7589.400), \quad K_j > 400 \end{aligned} \tag{41}$$

One might argue that these constraints are arbitrary, but some constraints on the implied volatilities outside the range of observed option prices are needed. Otherwise implied volatility might go negative which of course implies a static arbitrage. One has to be a bit careful about the constraints on the implied volatilities outside the frame of input strikes. If the implied volatility surface is bent too hard at the boundaries of the range of input strikes the outcome can be arbitrage opportunities and/or spikes in the local volatility surface.
The optimization program (37) has the following discrete counterpart

$$\begin{aligned} &\min_{\{\hat{\sigma}(T_i,K_j)\}_{ij}} \sum_{i=1}^{3}\sum_{j=1}^{39} \left([(\Delta_{KK}\hat{\sigma})^2 + 2(\Delta_{TK}\hat{\sigma})^2 + (\Delta_{TT}\hat{\sigma})^2](T_i,K_j)\right) \times M_{ij} \\ &\text{s.t.} \quad \hat{\sigma}_{bid}(T_i,K_j) \leq \hat{\sigma}(T_i,K_j) \leq \hat{\sigma}_{ask}(T_i,K_j), \quad i = 1.2,3; j = 0,\ldots,40 \end{aligned} \tag{42}$$

[11] Any norm of the 2×2 matrix $\left(\partial^2\hat{\sigma}/\partial(T,K)^2\right)$ can be used. We choose the simplest possible.

where Δ_{TT}, Δ_{TK} and Δ_{KK} are the discrete second derivative operators

$$\Delta_{TT}\hat{\sigma}(T_i,\cdot) = \frac{2}{T_{i+1}-T_{i-1}}\left(\frac{\hat{\sigma}(T_{i+1},\cdot)-\hat{\sigma}(T_i,\cdot)}{T_{i+1}-T_i} - \frac{\hat{\sigma}(T_i,\cdot)-\hat{\sigma}(T_{i-1},\cdot)}{T_i-T_{i-1}}\right)$$
$$\Delta_{TK}\hat{\sigma}(T_i,K_j) = \frac{1}{T_{i+1}-T_{i-1}}\left(\frac{\hat{\sigma}(T_{i+1},K_{j+1})-\hat{\sigma}(T_{i+1},K_{j-1})}{K_{j+1}-K_{j-1}} - \frac{\hat{\sigma}(T_{i-1},K_{j+1})-\hat{\sigma}(T_{i-1},K_{j-1})}{K_{j+1}-K_{j-1}}\right)$$
$$\Delta_{KK}\hat{\sigma}(\cdot,K_j) = \frac{2}{K_{j+1}-K_{j-1}}\left(\frac{\hat{\sigma}(\cdot,K_{j+1})-\hat{\sigma}(\cdot,K_j)}{K_{j+1}-K_j} - \frac{\hat{\sigma}(\cdot,K_j)-\hat{\sigma}(\cdot,K_{j-1})}{K_j-K_{j-1}}\right) \tag{43}$$

and M_{ij} is the areal

$$M_{ij} = (T_{i+1}-T_{i-1})(K_{j+1}-K_{j-1})/4 \tag{44}$$

This is a quadratic program with linear constraints. For the numerical solution of this program we apply Lempke's algorithm that exploits the linear quadratic structure of the problem, which makes it fast compared to a general non-linear optimization scheme.[12]
Instead of going through the procedure of estimating the "true market prices" of the options one might be tempted to simply smooth a surface through the midpoint volatilities and take this surface as direct input to the model. However, the local volatility surface implied by the input implied volatilities is extremely sensitive to small changes in the input implied volatilities and the outcome might be that "tick size noise" triggers a local volatility surface with a lot of spikes and dips.

We obtain the estimated implied volatilities and call option prices listed in Table 2.

By solving the optimization problem we obtain the estimated implied volatilities and call option prices listed in Table 2. If a cell is marked by (*) it means that there were no observed option price for that particular combination of strike and maturity. The numbers given in such cells are interpolated values obtained by the optimization program. For brevity we choose not to list the option prices or the implied volatilities outside the range of strikes of the input options, but Figure 1 shows the full surface of implied volatilities, which is generated by two dimensional cubic spline interpolation between the discrete points of the optimization program.

As noted, we obtain the full continuous surface of implied volatilities, $\{\hat{\sigma}(T,K)\}_{T\in[0,\tau];K\in[\underline{S},\overline{S}]}$, by interpolating between the discrete points of the optimization problem using a bicubic spline procedure. A straightforward bicubic spline does not guarantee sufficient smoothness of the implied volatility surface to apply Result 5 to back out the local volatilities. The problem is that one usually performs a bicubic spline by first splining in the first direction and then in the second. Twice continuous differentiability is thereby obtained in the last direction, whereas not even continuous differentiability is guaranteed in the first direction. To overcome this problem we take the approach described below.

[12] Lempke's algorithm is basically a modified simplex algorithm. For a more detailed description, see for example Bazaraa and Shetty (1979).

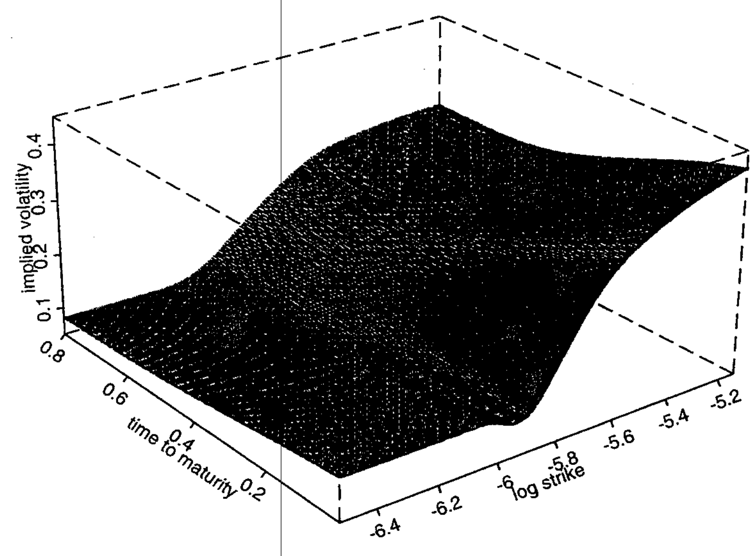
Table 2: Estimated Implied Volatilities and Call Option Prices

$T_i \rightarrow$	0.2411	0.5096	0.7589
$K_j \downarrow$	$\hat{\sigma}(\cdot,\cdot)$ /$C(\cdot,\cdot)$	$\hat{\sigma}(\cdot,\cdot)$ /$C(\cdot,\cdot)$	$\hat{\sigma}(\cdot,\cdot)$ /$C(\cdot,\cdot)$
250	0.3066/93.13	0.2634/95.89	0.2636/98.78
275	0.2799/69.23	0.2473/73.22	0.2504/77.39
300	0.2393/45.99	0.2216/51.52	0.2277/56.89
305	0.2301/41.48	0.2157/47.34	0.2224/52.93*
310	0.2207/37.04	0.2097/43.24	0.2169/49.03*
315	0.2112/32.69	0.2036/39.22	0.2114/45.19*
320	0.2016/28.44	0.1974/35.31	0.2059/41.44*
325	0.1919/24.33	0.1912/31.50	0.2002/37.77
330	0.1823/20.39	0.1849/27.82	0.1945/34.20
335	0.1728/16.67	0.1785/24.29	0.1887/30.73
340	0.1636/13.21	0.1721/20.92	0.1828/27.38
345	0.1547/10.10	0.1659/17.74	0.1769/24.16
350	0.1464/7.38	0.1593/14.77	0.1710/21.08
355	0.1388/5.13	0.1529/12.03	0.1650/18.16
360	0.1321/2.81	0.1465/9.56	0.1590/15.42
365	0.1264/2.08	0.1402/7.37*	0.1530/12.87*
370	0.1217/1.21	0.1339/5.48*	0.1470/10.53*
375	0.1178/0.66	0.1278/3.91	0.1410/8.41
380	0.1145/0.34	0.1218/2.65*	0.1351/6.54*
385	0.1118/0.16	0.1160/1.69*	0.1291/4.93*
400	0.1093/0.02*	0.1008/0.29*	0.1124/1.66

We want to apply a finite difference grid to our pricing problem. We therefore need the local volatility in say $M \times N$ grid points, $\{(t_i, S_j)\}_{i=1,\dots,N;j=1,\dots,M}$, on the box $[0,\tau] \times [\underline{S}, \overline{S}]$. To obtain this from Result 5 we need $\hat{\sigma}, \hat{\sigma}_T, \hat{\sigma}_K, \hat{\sigma}_{KK}$ in each of these points. By first performing a standard bicubic spline we obtain $\hat{\sigma}$ in all of the points $\{(t_i, S_j)\}_{i,j}$.

We then perform one-dimensional splines in the t-direction on the values $\{\hat{\sigma}(t_i, S_j)\}_i$ for each level S_j and obtain the values of $\hat{\sigma}$ for all the points $\left\{\left(\frac{1}{2}t_i + \frac{1}{2}t_{i+1}, S_j\right)\right\}_{i,j}$. By taking central differences we now have an approximation of $\hat{\sigma}_T$ for all points $\{(t_i, S_j)\}_{i,j}$.

Figure 1: The Implied Black-Scholes Volatilities.



Equivalently, we perform one-dimensional cubic splines in the S-dimension on $\{\hat{\sigma}(t_i, S_j)\}_j$ for each time t_i to obtain the values $\{\hat{\sigma}(t_i, \frac{1}{2}S_j + \frac{1}{2}S_{j+1})\}_{i,j}$. Combining this with central differences we obtain approximating values for $\hat{\sigma}_K, \hat{\sigma}_{KK}$ for all points on the considered grid $\{(t_i, S_j)\}_{i,j}$.

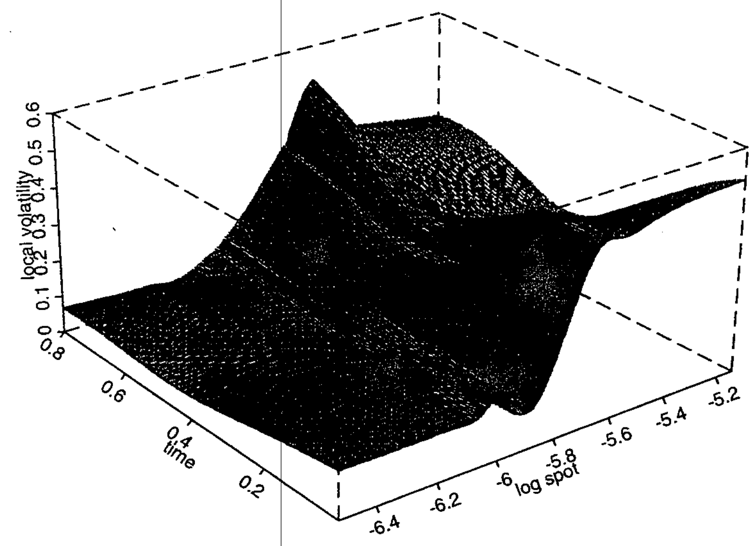
The smoothness of the one-dimensional splines guarantees that the central differences are well-behaved. Note however that it does not guarantee the existence of cross derivatives or continuous differentiability along *any* curve on the plane $\{(t, S)\}$, but we only need the existence of the derivatives along the lines in the S and t directions. The procedure is, of course, theoretically inferior to performing a real bicubic spline that guarantees twice continuous differentiability of the function $\hat{\sigma}(\cdot, \cdot)$ in any direction, but the procedure has the advantage that it only relies on one-dimensional spline procedures which can be found in many standard computer libraries or scientific computing books, for example in Flannery, Press, Teukolsky, and Vetterling (1992). Also important, is that it is computationally fast.

By performing this procedure and plugging the approximations for $(\hat{\sigma}, \hat{\sigma}_T, \hat{\sigma}_K, \hat{\sigma}_{KK})$ into Result 5 we get the surface of local volatilities shown in Figure 2.

The smile is much more pronounced in the local volatility domain than in the implied volatility domain. When a surface of implied volatilities is converted into local volatilities then every tendency of deviation from the Black-Scholes model or constant volatility case is inflated.

The surface of local volatilities is rather smooth for our input data. Whereas implied volatility is in the region of 8 to 40 percent, local volatility is in the region of 5 to 50 percent. A couple of observations deserve further comments though: there is a clear tendency of that the local volatility drops pretty dramatically on the high in the region where the stock price is in the interval $[250, 275]$. This can partly be attributed our smoothing procedure that starts levering out the implied volatilities in that region. But the fact that this gets more pronounced as we move out in time must be attributed input

Figure 2: Implied Local Volatilities.



data. Stale option prices in the long maturity/high strike corner might be the problem. We could partly avoid this by not bounding the implied volatilities, but the consequence would be that the local volatility would explode above the level of the highest strike. Another observation is that there is a slight tendency of a fold in the high strike and short time to maturity region. This must be caused by the fact that we used call options as input data, and that such options are extremely sensitive to "tick size noise" because their prices are very low.

We obtain the option prices of the model and the "Greeks" of the marketed options by numerically solving the partial differential equations of Result 1, 2, and 3. The machinery used here is the implicit finite difference method.[13] The method is uniformly stable as long as the supplied volatility function, interest rates, and dividends are continuous and bounded, see Mitchell and Griffiths (1980). The basic idea in this scheme is to approximate the derivatives in a partial differential equations by differences, turning the partial differential equation into a partial difference equation. To illustrate this let us consider the partial differential equation (10). After performing the transformation $x = \ln K$ we get the PDE

$$q(T)C = -\frac{\partial C}{\partial T} - \left(r(T) - q(T) + \frac{1}{2}\sigma(T,x)^2\right)\frac{\partial C}{\partial x} + \frac{1}{2}\sigma(T,x)^2\frac{\partial^2 C}{\partial x^2} \tag{45}$$

We have implicitly redefined the notation so that

$$\begin{aligned} \sigma(T,x) &:= \sigma(T,\mathrm{e}^x) \\ C(T,x) &:= C(0,S_0;T,\mathrm{e}^x) \end{aligned} \tag{46}$$

The logarithmic transformation is performed because of the exponential (or compounded) nature of the dynamics of the underlying stock. In most cases this increases the precision of the finite difference scheme.

13 One might expect that the Cranck-Nicholson algorithm would give higher precision here. However our analysis suggests that the straightforward implicit finite difference is sufficiently accurate for our purposes.

We make the following approximations

$$\begin{aligned}
\frac{\partial C(T,x)}{\partial T} &\simeq \frac{1}{\Delta T}(C(T,x) - C(T-\Delta T, x)) \\
\frac{\partial C(T,x)}{\partial x} &\simeq \frac{1}{2\Delta x}(C(T,x+\Delta x) - C(T,x-\Delta x)) \\
\frac{\partial^2 C(T,x)}{\partial x^2} &\simeq \frac{1}{(\Delta x)^2}(C(T,x+\Delta x) - 2C(T,x) + C(T,x-\Delta x))
\end{aligned} \tag{47}$$

Plugging these difference approximations into equation (45) and rearranging we get the partial difference equation

$$C(T-\Delta T, x) = \alpha(T,x)C(T,x-\Delta x) + \beta(T,x)C(T,x) + \gamma(T,x)C(T,x) \tag{48}$$

with

$$\begin{aligned}
\alpha(T,x) &= \Delta T\left(-\frac{r-q+\frac{1}{2}\sigma^2}{2\Delta x} - \frac{\sigma^2}{2(\Delta x)^2}\right)(T,x) \\
\beta(T,x) &= \left(1 + \Delta T\left(q + \frac{\sigma^2}{(\Delta x)^2}\right)\right)(T,x) \\
\gamma(T,x) &= \Delta T\left(\frac{r-q+\frac{1}{2}\sigma^2}{2\Delta x} - \frac{\sigma^2}{2(\Delta x)^2}\right)(T,x)
\end{aligned} \tag{49}$$

We now limit our state space to the grid

$$\begin{aligned}
T_i &= i\Delta T, \quad i = 0, \dots, N \\
x_j &= \underline{x} + j\Delta x, \quad j = 0, \dots, M \\
\Delta T &= 0.8/N \\
\Delta x &= (\overline{x} - \underline{x})/M \\
\underline{x} &= \ln(S_0/2) \\
\overline{x} &= \ln(2S_0)
\end{aligned} \tag{50}$$

and supply the boundary condition that

$$\left(K^2\frac{\partial^2 C}{\partial K^2}\right)(T_i, x_j) = \frac{\partial^2 C(T_i,x_j)}{\partial x^2} - \frac{\partial C(T_i,x_j)}{\partial x} \equiv 0 \tag{51}$$

for $j = 0, M$ for all i. On the boundaries we again approximate the derivatives by appropriate differences to get a linear equation as (48), this time with

$$\begin{aligned}
\alpha(T_i, x_0) &= \gamma(T_i, x_M) = 0 \\
\beta(T_i, x_0) &= \left(1 + \Delta T\left(q - \frac{r-q}{\Delta x}\right)\right)(T_i) \\
\gamma(T_i, x_0) &= \left(\frac{\Delta T(r-q)}{\Delta x}\right)(T_i) \\
\alpha(T_i, x_M) &= -\left(\frac{\Delta T(r-q)}{\Delta x}\right)(T_i) \\
\beta(T_i, x_M) &= \left(1 + \Delta T\left(q + \frac{r-q}{\Delta x}\right)\right)(T_i)
\end{aligned} \tag{52}$$

We can arrange all this into a tridiagonal set of linear equations of dimension $(M+1) \times (M+1)$ and write it on the form

$$\mathbf{C}(T_{i-1}) = \mathbf{A}(T_i)\mathbf{C}(T_i) \tag{53}$$

where $\mathbf{A}(\cdot)$ is a known tridiagonal matrix and $\mathbf{C}(\cdot)$ is the vector of option prices at the different strikes, $K_j = \mathrm{e}^{x_j}$, $j = 0, \ldots, M$. The idea is now to invert (47) to obtain $\mathbf{C}(T_i)$ given $\mathbf{C}(T_{i-1})$. Since each element in $\mathbf{C}(0)$ is known by the boundary condition

$$(S - \mathrm{e}^{x_j})^+ \tag{54}$$

we can solve the system recursively to obtain the initial option prices for all strikes and maturities in the grid.

The Delta and Gamma functions satisfy the same (log-transformed) PDE so for these quantities we have a similar tridiagonal matrix equation at each time step — the matrix $\mathbf{A}(\cdot)$ is the same as for the option prices. We only have to supply the initial boundary conditions

$$\mathbf{1}_{x_j \leq \ln S}$$

$$\frac{\mathbf{1}_{x_j \in [\ln S - \Delta x/2, \ln S + \Delta x/2]}}{S\Delta x} \tag{55}$$

for each element in the initial vectors of Deltas and Gammas.

For the more exotic "Greeks" we have boundary conditions saying that they should be equal to 0 at all initial nodes. Their associated PDEs are different from the PDEs considered above in that they have a term added at the left hand side of the equations. But these terms are known when the system of partial differential equations is solved simultaneously, so the linear system that has to be solved at each time step is for the Vega function

$$\mathbf{\Psi}(T_{i-1}) = \mathbf{A}(T_i)(\mathbf{\Psi}(T_i) - \mathbf{B}(T_i)) \tag{56}$$

where $\mathbf{\Psi}(\cdot)$ is at each time step the vector of Vegas for different strikes, $K_j = \mathrm{e}^{x_j}$, and $\mathbf{B}(T_i)$ is the vector with elements

$$\Delta T \sigma(T_i, x_j)\left(\frac{\partial^2 C(T_i, x_j)}{\partial x^2} - \frac{\partial C(T_i, x_j)}{\partial x}\right) \tag{57}$$

The above partial derivatives are approximated by central differences.

Dividend and interest rate sensitivities are handled in a similar way. For the interest sensitivity function, Φ, we substitute the vector $\mathbf{B}(T_i)$ in equation (50) by a vector with elements

$$-\Delta T \frac{\partial C(T_i, x_j)}{\partial x} \tag{58}$$

For Λ we substitute $\mathbf{B}(T_i)$ in equation (50) by a vector with elements:

$$\Delta T\left(-C(T_i, x_j) + \frac{\partial C(T_i, x_j)}{\partial x}\right) \tag{59}$$

Table 3: Option Prices and Local Volatilities of the Model.

$T_i \rightarrow$	0.2411	0.5096	0.7589
$K_j \downarrow$	$C(\cdot,\cdot)$, $(\sigma(\cdot,\cdot))$	$C(\cdot,\cdot)$, $(\sigma(\cdot,\cdot))$	$C(\cdot,\cdot)$, $(\sigma(\cdot,\cdot))$
250	93.14 (0.3407)	95.88 (0.2720)	98.78 (0.3542)
275	69.23 (0.4136)	73.22 (0.3414)	77.39 (0.4863)
300	45.99 (0.4260)	51.51 (0.2967)	56.89 (0.3810)
305	41.47 (0.3099)	47.33 (0.2754)	52.92 (0.3401)*
310	37.02 (0.2941)	43.23 (0.2691)	49.02 (0.3340)*
315	32.67 (0.2765)	39.21 (0.2594)	45.19 (0.3143)*
320	28.42 (0.2511)	35.29 (0.2475)	45.19 (0.2985)*
325	24.31 (0.2287)	31.49 (0.2355)	37.77 (0.2823)
330	20.36 (0.2070)	27.81 (0.2236)	34.19 (0.2650)
335	16.64 (0.1881)	24.27 (0.2110)	30.73 (0.2496)
340	13.19 (0.1683)	20.91 (0.1987)	27.38 (0.2333)
345	10.07 (0.1517)	17.72 (0.1862)	24.15 (0.2182)
350	7.37 (0.1372)	14.76 (0.1736)	21.09 (0.2038)
355	5.12 (0.1253)	12.03 (0.1608)	18.16 (0.1897)
360	3.37 (0.1150)	9.56 (0.1483)	15.42 (0.1769)
365	2.09 (0.1095)	7.37 (0.1352)*	12.87 (0.1646)*
370	1.23 (0.1041)	5.49 (0.1225)*	10.54 (0.1533)*
375	0.68 (0.0993)	3.92 (0.1106)	8.43 (0.1417)
380	0.35 (0.0962)	2.66 (0.0990)*	6.56 (0.1303)*
385	0.18 (0.0892)	1.71 (0.0888)*	4.95 (0.1226)*
400	0.02 (0.0940)*	0.31 (0.0638)*	1.68 (0.0933)

A first check of the accuracy of the scheme is to see if we hit the observed option prices in Table 2. We choose a grid of 200×200 points. The option prices of the model are reported in the table below. The bracketed value in each cell is the local volatility at time equal to the maturity date and spot equal to the strike.

The cells where numbers are marked by (*) are the ones where there were no initial observation. By running our scheme forward we get out these option prices as well. In fact we get out option prices and sensitivities in all the 200×200 points of our grid.

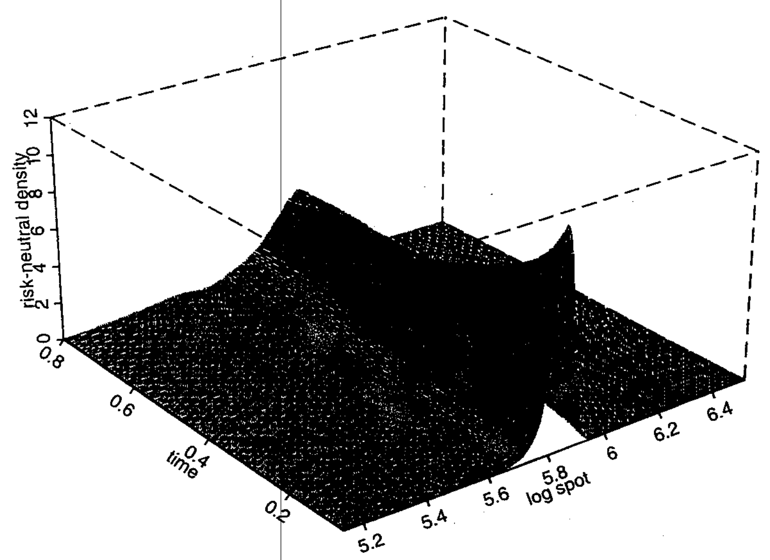
Comparing Table 3 with Table 2 we see that the pricing errors are relatively small; the highest absolute error is about 2 cents. It should be noted that the pricing error is due discretization error only. In a finer grid the pricing errors would in general be smaller. However, we fell that the precision of the scheme is sufficient for most practical purposes.

From the option prices we can back out the risk-neutral distribution of the model using that the time 0 risk-neutral density in (T, K) is given by:

$$\frac{1}{B(0;T)} \frac{\partial^2 C}{\partial K^2}(T, K) \qquad (60)$$

The resulting risk-neutral distribution on our grid of the logarithm of the stock price is shown in Figure 4.

Figure 4: The Implied Risk-Neutral Density.



The risk-neutral distribution is rather smooth, but it gets more skewed as we move out in time. Figure 5 shows the risk-neutral distribution of the underlying at the maturities of the input option prices.

We see that the risk-neutral distributions are very peaked, skewed and lower fat-tailed. There is a slight tendency of bimodularity at all maturities. The market's risk-neutral probability of a "crash" is obviously rather high, whereas the market's risk-neutral probability of a very bull market is low.[14] It should be stressed that this probability distribution reflects the investors' aggregate expectations towards the future stock price evolution *adjusted for risk*, and this is not necessarily the same as the actual stock price distribution. In fact the actual distribution as well as the investors' aggregate perceptions about it might differ substantially from what is shown in Figure 4–5 due to the risk-adjustment.

In Figure 6–8 we show the Deltas of the model compared to the Deltas of the Black-Scholes model. Dotted lines are Deltas of the model whereas dotted lines are Deltas of the Black-Scholes model.

14 See Jackwerth and Rubinstein (1996) for a detailed empirical investigation of this phenomenon

Figure 5: Implied Risk-Neutral Densities at the Maturities of the Input Options.

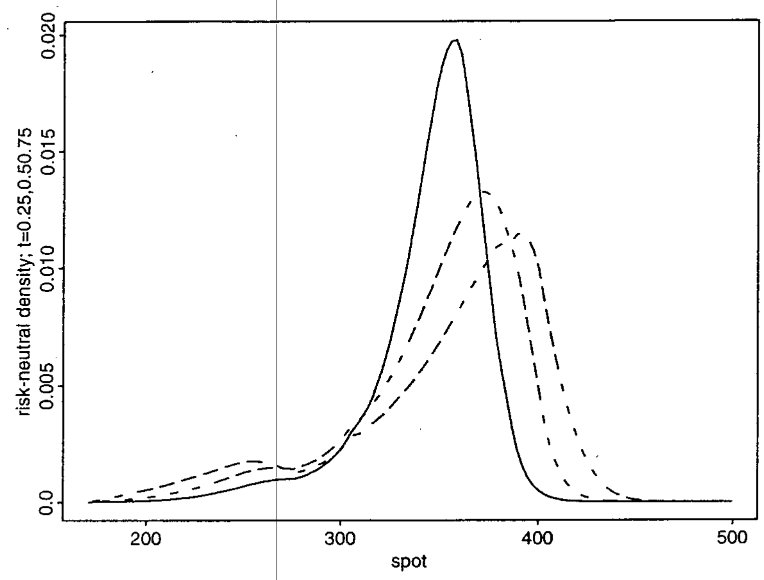
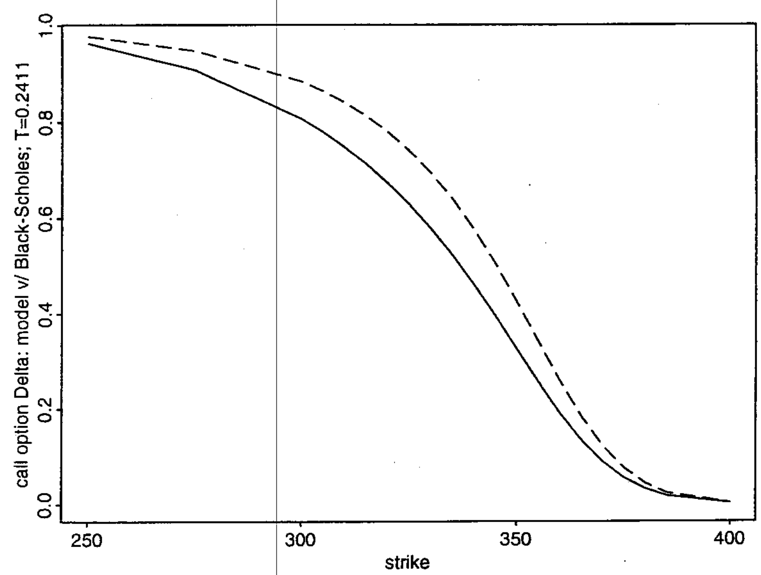


Figure 6: The Deltas of the Input Options — Short Maturity.



The Deltas shown in Figure 6–8 are depicted as functions of the strike and against the Black-Scholes Deltas calculated with the implied volatilities of the different strikes (see Table 1). Since in our case the implied volatilities vary over the strike prices the Black-Scholes Deltas as a function of strike do not look as in a standard textbook. The skewed risk-neutral distribution implies that the call Deltas of the model are significantly lower than those of the Black-Scholes model. The opposite is the case for the puts. The maximum absolute difference is at-the-money and of a magnitude of approximately 20 percent This means that if we used Black-Scholes (with the implied volatilities) for hedging the marketed call options, we would effectively "over-hedge" quite substantially relative to our model.

Figure 7: The Deltas of the Input Options — Mid Maturity.

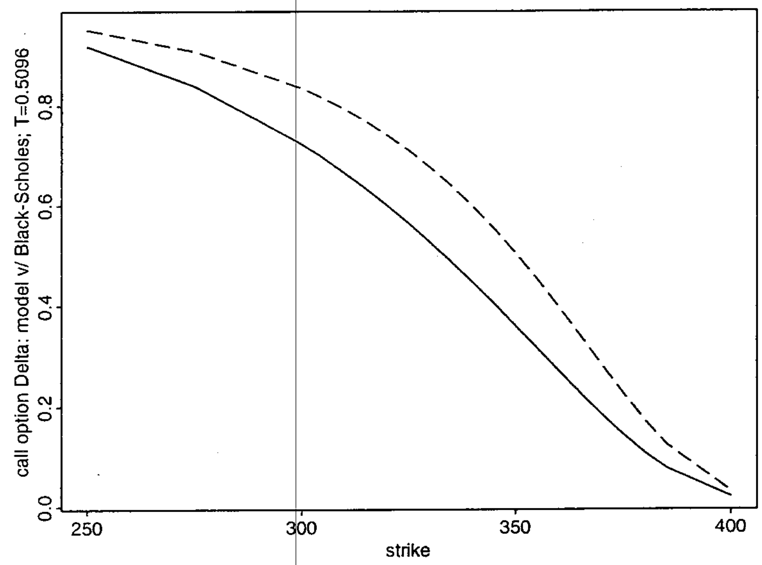
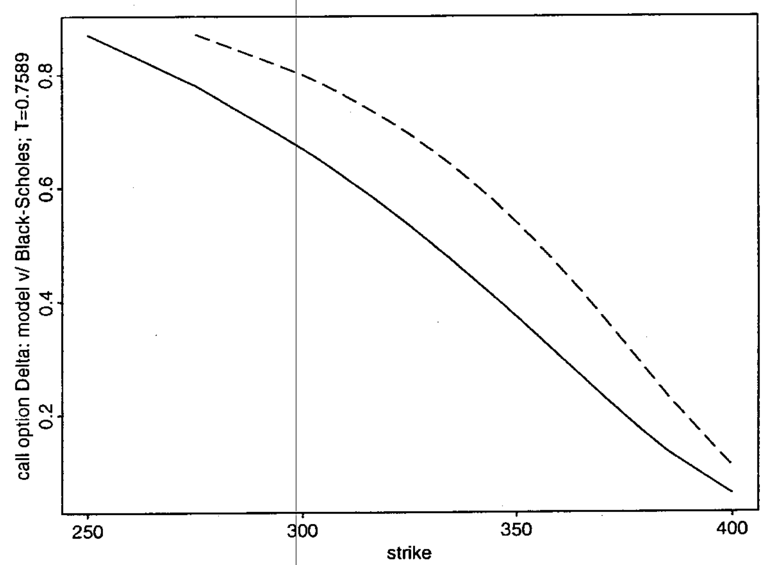


Figure 8: The Deltas of the Input Options — Long Maturity.



Since the European option prices are taken as primitives and inputs in this modelling framework one might ask what the Deltas of the marketed options are needed for. The Deltas serve at least three purposes: First, if one wants to replicate exotic claims by dynamic trading in the marketed options, one would need the Deltas to construct such a strategy. Secondly, the Deltas reflect the primal risk-exposure of options. This information is of high value to managers of books of options. Thirdly, one might be trying to arbitrage marketed options by dynamic trading strategies in other marketed options, the stock and/or futures contracts, in which case one again would need the Deltas.

Figure 9–10 show the Gammas of the marketed options implied by our model (solid lines) relative to those of the Black-Scholes model with volatility equal to the implied

Figure 9: The Gammas of the Input Options — Short Maturity.

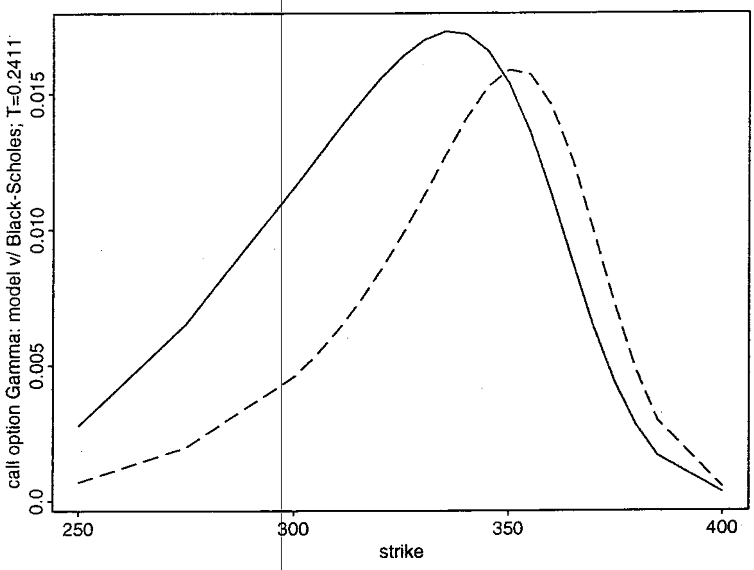
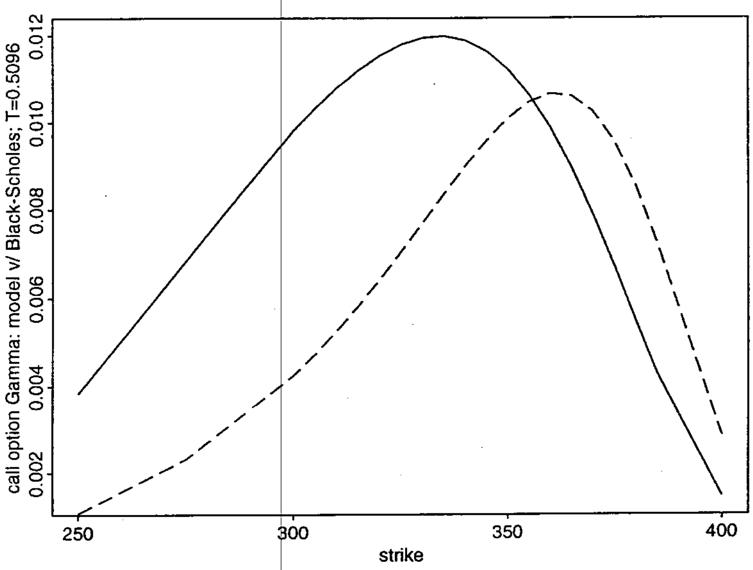


Figure 10: The Gammas of the Input Options — Mid Maturity.



volatility (dotted lines). For brevity we choose to show the Gammas for the two first maturities only. The differences between the Gammas of our model and the ones of the Black-Scholes model are rather large. We see that the Gammas are much bigger for in-the-money calls for our model and lower for the out-of-the-money options. This means that in our model the composition of the replicating portfolio change more rapidly than under Black-Scholes. This again implies potentially higher transaction costs. Considering the trade-off between tracking error and accumulated transaction costs this might induce that one would choose to rebalance hedging portfolios more infrequent under our model than under the Black-Scholes model.

In Figure 11–12 we show the Thetas of our model (solid lines) compared to those of the Black Scholes model (dotted lines) for the two first maturities. The Thetas turn out to be more similar to those of the Black-Scholes model than the Deltas and the Gammas, though the differences seem to increase as time to maturity increases.

Figure 11: The Thetas of the Input Options — Short Maturity.

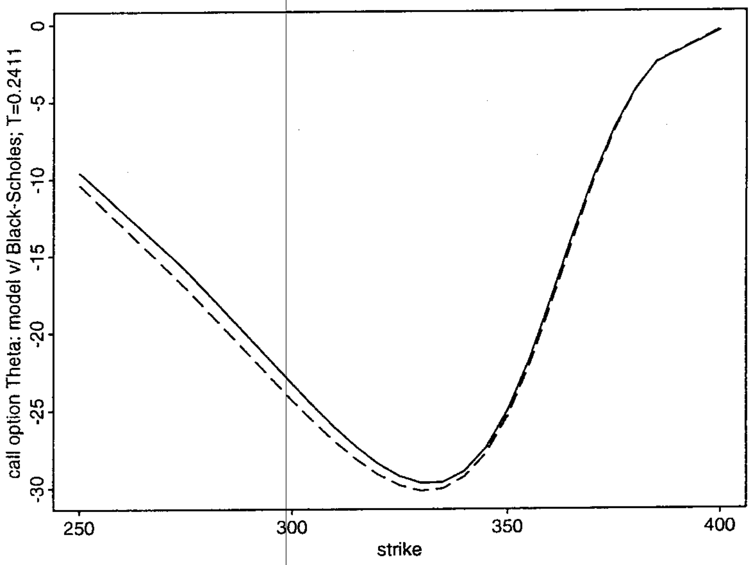
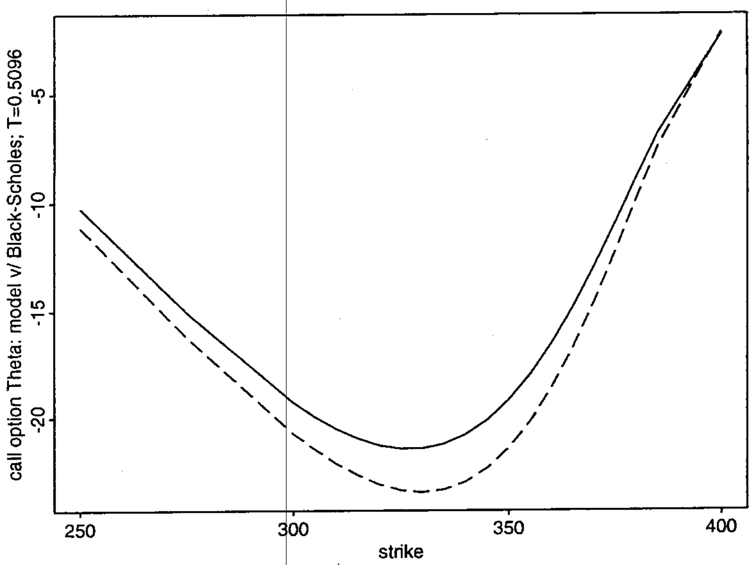


Figure 12: The Thetas of the Input Options — Mid Maturity.



Of the more exotic "Greeks" we will only consider the Vegas. These are depicted in Figure 13–14 below for the two first maturities.

Figure 15: The Vegas of the Input Options — Short Maturity.

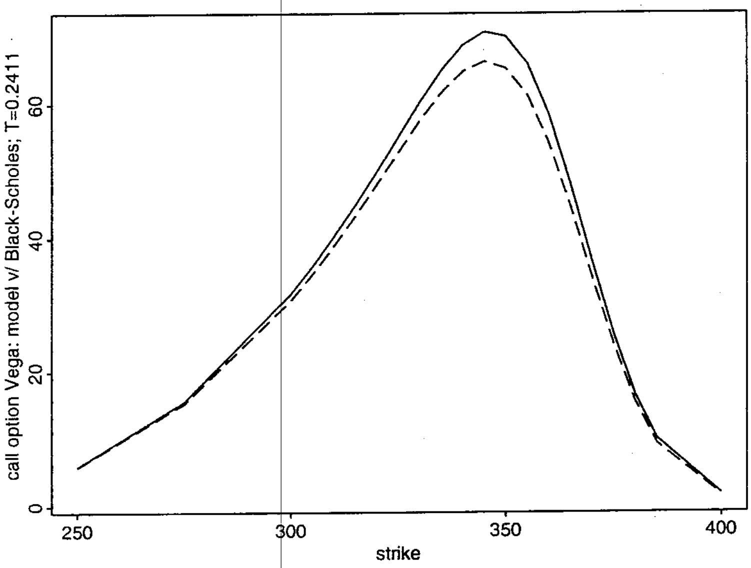
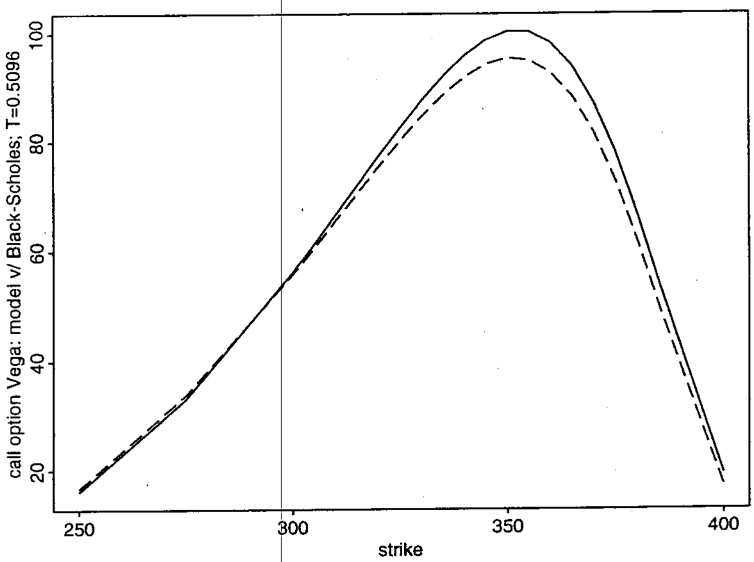


Figure 16: The Vegas of the Input Options — Mid Maturity.



The Vegas of our model (solid lines) and the Vegas of the Black-Scholes (dotted lines) shown in Figure 15–16 seem very similar, though they show different things. In fact the Black-Scholes Vegas show what would happen if the surface of implied volatilities was shifted parallely, in which case the option prices of our model would follow. The Vegas of our model show what would happen if the surface of local volatilities was shifted parallely. It is therefore rather surprising that the two different Vegas are so similar. Generally though the Vegas of the model are a little bit higher than those implied by the Black-Scholes model.

Regarding the precision of the calculated sensitivities and computer time: we first checked our scheme for the standard Black-Scholes case. The observed discrepancies were so small that they would not have any significant effect in practical applications.

The program that estimates the implied volatility surface runs in approximately 5 seconds of CPU time on a Hewlett-Packard 9000 unix system. The program that generates the local volatility surface, and calculates the risk-neutral distribution and the "Greeks" of all the options in the 200×200 grid of time to maturity and strikes runs in approximately 10 seconds of CPU time on the same system.[15] By far most of the computer time is used at generating the grid of local volatilities and exporting the output data. Once the grid of local volatilities is generated the actual calculation of sensitivities is done in less than 0.5 seconds of CPU time. This means that any exotic claim satisfying the backward PDE (6) could be priced within a split of a second. They way this is done is through backward finite difference in the constructed grid of local volatilities, interest rates and dividend yields. This for example applies to American options and barrier options. For more exotic path dependent claims such as Asian options and look-back options, the surface of local volatilities lends itself to Monte Carlo methods.

Conclusion

In this paper we have derived new forward equations for the sensitivities of European option prices. These forward equations admit fast numerical computation of the "Greeks" of European Options. We have shown that the European option pricing problem might be solved in a dual setting, where the economy is virtually turned upside-down. We have presented a numerical implementation of a model that incorporates observed option prices at different maturities and strike prices. Moreover our implementation shows to be computational fast and fairly precise. Our numerical example shows that "implied Greeks" might differ substantially from those implied by the Black-Scholes model.

Future research should be directed towards utilizing American style option prices as primitives, and towards testing whether the "implied" models actually outperform existing option pricing models when it comes to hedging standard options as well as exotics. Another interesting topic for future research is to see if stochastic volatility models could be implemented in a way so that they fit market option prices.

Acknowledgments

I would like to thank Leif Andersen for insightful comments on an earlier version of this paper. The paper has also benefitted from comments from the participants at the finance seminar at Johannes Gutenberg University of Mainz. Of course, I bare the full responsibility for any remaining errors in the paper.

15 The computer programs were written in C and FORTRAN.

Appendix

Proof of Result 1

We will show the result under the assumption of sufficient regularity. To be more precise we will assume that the time 0 risk-neutral density of $S(T)$ in the point x, $\phi(T,x)$, satisfies the conditions:

$$\begin{aligned} \phi(T,x) &\to 0 \\ x\phi(T,x) &\to 0 \\ \frac{\partial}{\partial x}\left[\sigma(T,x)^2 x^2 \phi(T,x)\right] &\to 0 \end{aligned} \tag{61}$$

when $x \to 0,\ x \to \infty$.

Without loss of generality let current time be 0. Under the assumption of sufficient regularity the risk-neutral density satisfies the forward Fokker-Planck equation:

$$0 = -\frac{\partial \phi}{\partial T} - \frac{\partial}{\partial x}[(r(T) - q(T))x\phi] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma(T,x)^2 x^2 \phi\right] \tag{62}$$

subject to the boundary condition

$$\phi(0,x) = \delta(x - S) \tag{63}$$

where S is the current spot.

Under our assumptions integration of the forward equation on the interval $[y, \infty[$ yields:

$$0 = -\frac{\partial}{\partial T}\int_y^\infty \phi(T,x)dx + (r(T) - q(T))y\phi(T,y) - \frac{1}{2}\frac{\partial}{\partial y}\left[\sigma(T,y)^2 y^2 \phi(T,y)\right] \tag{64}$$

We proceed by integrating once more, this time over $[K, \infty[$. Under our assumptions we have that:

$$\begin{aligned} 0 = &-\frac{\partial}{\partial T}\int_K^\infty\int_y^\infty \phi(T,x)dxdy \\ &+ (r(T) - q(T))\int_K^\infty y\phi(T,y)dy + \frac{1}{2}\sigma(T,K)^2 K^2 \phi(T,K) \end{aligned} \tag{65}$$

Integrating by parts we have for the initial call price:

$$\begin{aligned} C(T,K) := C(0,S;T,K) &= B(0;T)\int_K^\infty (y - K)\phi(T,y)dy \\ &= B(0;T)\int_K^\infty\int_y^\infty \phi(T,x)dydy \end{aligned} \tag{66}$$

By taking derivatives of the above formulae we get:

$$
\begin{aligned}
\frac{\partial C(T,K)}{\partial T} &= -r(T)C(T,K) + B(0;T)\frac{\partial}{\partial T}\int\limits_K^\infty\int\limits_y^\infty \varphi(T.x)dx \\
\frac{\partial C(T,K)}{\partial K} &= -B(0;T)\int\limits_K^\infty \phi(T,y)dy \\
\frac{\partial^2 C(T,K)}{\partial K^2} &= B(0;T)\phi(T,K)
\end{aligned}
\tag{67}
$$

By rearranging this and inserting in the forward equation the forward PDE obtains.

□

Proof of Result 5

Like in the previous proofs we fix current time to be 0 without loss of generality.

The implied volatilities satisfy:

$$
C(0,S;T,K) = W(T,K,\hat{\sigma}(T,K)) \tag{68}
$$

where $W(\cdot)$ is the Black-Scholes formula:

$$
\begin{aligned}
W(T,K;\nu) &= D(0;T)S\Phi(d) - B(0;T)K\Phi\left(d - \nu\sqrt{T}\right) \\
d &= \frac{1}{\nu}\ln\frac{SD(0;T)}{KB(0;T)} + \frac{1}{2}\nu\sqrt{T}
\end{aligned}
\tag{69}
$$

This means that we can write:

$$
\begin{aligned}
\frac{\partial C}{\partial T} &= \frac{\partial W}{\partial T} + \frac{\partial W}{\partial \nu}\hat{\sigma}_T \\
\frac{\partial C}{\partial K} &= \frac{\partial W}{\partial K} + \frac{\partial W}{\partial \nu}\hat{\sigma}_K \\
\frac{\partial^2 C}{\partial K^2} &= \frac{\partial^2 W}{\partial K^2} + 2\frac{\partial^2 W}{\partial \nu \partial K}\hat{\sigma}_K + \frac{\partial^2 W}{\partial \nu^2}(\hat{\sigma}_K)^2 + \frac{\partial W}{\partial \nu}\hat{\sigma}_{KK}
\end{aligned}
\tag{70}
$$

where subscripts denote partial derivatives.

By Result 1 we have that:

$$
q(T)W = -\frac{\partial W}{\partial T} - (r(T) - q(T))K\frac{\partial W}{\partial K} + \frac{1}{2}\nu^2 K^2 \frac{\partial^2 W}{\partial K^2} \tag{71}
$$

So by plugging the derivatives of $C(\cdot)$ into Result 1 we get:

$$
\begin{aligned}
&\frac{1}{2}\hat{\sigma}^2 K^2 \frac{\partial^2 W}{\partial K^2} + \frac{\partial W}{\partial \nu}\hat{\sigma}_T + (r-q)K\frac{\partial W}{\partial \nu}\hat{\sigma}_K \\
&= \frac{1}{2}\sigma(T,K)^2 K^2 \left[\frac{\partial^2 W}{\partial K^2} + 2\frac{\partial^2 W}{\partial \nu \partial K}\hat{\sigma}_K + \frac{\partial^2 W}{\partial \nu^2}(\hat{\sigma}_K)^2 + \frac{\partial W}{\partial \nu}\hat{\sigma}_{KK}\right] \\
&\Downarrow \\
&\frac{1}{2}\sigma(T,K)^2 K^2 = \left(\frac{\frac{1}{2}\hat{\sigma}^2 K^2 \frac{\partial^2 W}{\partial K^2} + \frac{\partial W}{\partial \nu}\hat{\sigma}_T + (r-q)K\frac{\partial W}{\partial \nu}\hat{\sigma}_K}{\frac{\partial^2 W}{\partial K^2} + 2\frac{\partial^2 W}{\partial \nu \partial K}\hat{\sigma}_K + \frac{\partial^2 W}{\partial \nu^2}(\hat{\sigma}_K)^2 + \frac{\partial W}{\partial \nu}\hat{\sigma}_{KK}}\right)(T,K) \\
&= \left(\frac{\frac{\hat{\sigma}}{2T} + \frac{\frac{\partial W}{\partial \nu}}{\hat{\sigma}TK^2\frac{\partial^2 W}{\partial K^2}}\hat{\sigma}_T + (r-q)K\frac{\frac{\partial W}{\partial \nu}}{\hat{\sigma}TK^2\frac{\partial^2 W}{\partial K^2}}\hat{\sigma}_K}{\frac{1}{\hat{\sigma}TK^2} + 2\frac{\frac{\partial^2 W}{\partial \nu \partial K}}{\hat{\sigma}TK^2\frac{\partial^2 W}{\partial K^2}}\hat{\sigma}_K + \frac{\frac{\partial^2 W}{\partial \nu^2}}{\hat{\sigma}TK^2\frac{\partial^2 W}{\partial K^2}}(\hat{\sigma}_K)^2 + \frac{\frac{\partial W}{\partial \nu}}{\hat{\sigma}TK^2\frac{\partial^2 W}{\partial K^2}}\hat{\sigma}_{KK}}\right)(T,K) \\
&= \left(\frac{\frac{\hat{\sigma}}{2T} + \hat{\sigma}_T + (r-q)K\hat{\sigma}_K}{\frac{1}{\hat{\sigma}TK^2} + 2\frac{d}{\hat{\sigma}\sqrt{T}K}\hat{\sigma}_K + \frac{d(d-\hat{\sigma}\sqrt{T})}{\hat{\sigma}}(\hat{\sigma}_K)^2 + \hat{\sigma}_{KK}}\right)(T,K)
\end{aligned}
\tag{72}
$$

The last equality follows since differentiation of the Black-Scholes formula shows that:

$$
\begin{aligned}
\frac{\frac{\partial W}{\partial \nu}}{\hat{\sigma}TK^2\frac{\partial^2 W}{\partial K^2}} &= 1 \\
\frac{\frac{\partial^2 W}{\partial \nu \partial K}}{\hat{\sigma}TK^2\frac{\partial^2 W}{\partial K^2}} &= \frac{d}{\hat{\sigma}\sqrt{T}K} \\
\frac{\frac{\partial^2 W}{\partial \nu^2}}{\hat{\sigma}TK^2\frac{\partial^2 W}{\partial K^2}} &= \frac{d\left(d-\hat{\sigma}\sqrt{T}\right)}{\hat{\sigma}}
\end{aligned}
\tag{73}
$$

This concludes the proof.

□

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New Skin For the Old Ceremony: Eight Different Derivations of the Black-Scholes Formula[1]

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[1]We would like to thank Asbjørn Hansen for sketching a derivation we did not know about.

Abstract

In this paper we present eight different derivations of the Black-Scholes formula. To the best of our knowledge this is the largest collection of derivations of the Black-Scholes formula. In these derivations we pass through almost all techniques applied in continuous-time arbitrage pricing. So the paper can also be seen as an introduction to continuous-time arbitrage pricing. Our first derivation is the classical hedging argument presented in the original article, Black & Scholes (1973). We derive the fundamental partial differential equation (PDE) and rather than just verifying that the formula solves the PDE, we give a constructive technique to derive the formula from the PDE. We then derive the martingale approach and discuss the connection to the fundamental PDE. We solve the risk-adjusted expectation and obtain the formula. In the following section we apply the change of numeraire technique, which gives intuition to the functional form of the formula, and derive the Black-Scholes formula without calculating a single integral. Since the self-financing argument is central in understanding the arbitrage pricing principle, we use another section to demonstrate that a "stop-loss and start-gain" strategy is not self-financing and that the value of the required external financing equals the Black-Scholes formula minus the intrinsic value of the option. A fifth derivation of the formula is based on the *forward* equation of European option prices, derived by Dupire (1993). We show that the forward equation gives rise to the definition of a dual economy where time is reversed, the option is a put rather than a call, etc. We solve for the option price in this economy which gives us the Black-Scholes formula, and show that the hedge ratio can be obtained as the price of a digital option in the dual economy. We then consider the discrete-time binomial model by Cox, Ross & Rubinstein (1979). We introduce the concept of change of numeraire in this setting, and obtain the convergence of the binomial option pricing formula to the Black-Scholes formula in a few lines. We feel that this type of derivation provides more intuition than many textbook derivations of the same result. The last two sections derive the Black-Scholes formula in the contexts of different equilibria. First we apply the continuous-time CAPM to the option pricing problem and obtain the fundamental PDE. We then show that the Black-Scholes formula might be obtained in a representative investor equilibrium without continuous trading.

1 Introduction

Along with the Modigliani-Miller theorem and the CAPM-pricing relation, the Black-Scholes formula for the price of a call-option is among the most famous results in financial economics. In fact its appearance in 1973 triggered the almost explosive development of the field commonly referred to as 'mathematical finance'. Almost every book in that field contains a proof and an alternative proof of the the result. Even the original paper by Black and Scholes contained an 'alternative derivation'. This has led to a range of proofs. These illustrate the techniques that are being used in the area today. This paper serves as an account of the quite sophisticated techniques that have been developed in mathematical finance during the last 20 years.

Our intent is to illustrate how these can be used to derive a result that most people with interest in finance are familiar with. The real power of the techniques naturally lies in their ability to cope with different generalizations of the simple set-up of Black and Scholes, but we know from our own experience that a 'general to specific'-approach can provide better understanding. By nature this paper is technical but we have a fairly high degree of detail and many references meaning that it should be possible for readers not previously familiar with the techniques to understand and gain insight from the paper. Furthermore a certain level of detail is necessary to illustrate how approaches differ, while at the same time indicating the parallels. To the best of our knowledge this is the largest collection of Black-Scholes proofs in the literature to this date.

The outline of the paper is as follows: In Section 2 we describe the model, some central concepts, present the result that the rest of the paper is going to evolve around: THE BLACK-SCHOLES FORMULA, and point out why it is, exactly, that this result is so brilliant. By the hedge argument (that was the ingenious insight of Black and Scholes) the fundamental partial differential equation (PDE) for the arbitrage-free price of a call-option is derived in Section 3. Here we also illustrate that their argument stands up to the test of today's more rigorous framework and indicate how to solve the PDE constructively by non-probabilistic methods. Section 4 shows how martingale techniques can be used to solve the pricing problem and stresses the relationship between means of solutions of stochastic differential equations (SDEs) and PDEs. Furthermore this section aims to take the mystery out of the concept of an 'equivalent martingale measure', something that is almost a mantra to mathematical finance. Section 5 shows how the seemingly neutral concept of using different numeraires can turn out to be a very powerful tool, something the focus in finance circles has recently been drawn to. In fact we derive the Black-Scholes formula without calculating a single integral. In Section 6 we initially try to 'mess with your head' by making a strategy that seemingly contradicts the previous results. But we show that a careful inspection and some advanced stochastic calculus not only resolves the paradox, but also provides an extra proof. It is shown in Section 7 that the price of the call-option also satisfies a PDE that runs in strike price and maturity date, *a forward equation.* We see that this not only gives another proof of the result but also has practical implications for numerical purposes. Section 8 derives the formula as a limiting case of discrete binomial model. The actual convergence proof is not only shorter than most others in the literature but also highlights an interesting similarity between numeraire/measure changes in discrete and continuous cases. Section 9 shows that we can also derive the formula from the continuous time CAPM model, which links together two of the most celebrated results in financial economics. Utility maximization of a representative agent with a utility function exhibiting constant relative risk aversion is shown also to do the trick in Section 10. Sections 9 and 10 thus demonstrate that we can restrict either the distributions of returns or the preferences of investors to derive interesting results, something that holds in other cases too. Finally, Section 11 sums up the contributions of the paper and discusses the results.

2 The Model and the Result

We consider a non-dividend paying stock the price of which is assumed to be the solution to the stochastic differential equation (SDE)

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^{\mathcal{P}}, \tag{1}$$

where μ and σ are constants and $(W_t^{\mathcal{P}})$ is a Brownian Motion on some filtered probability space $(\Omega, (\mathcal{F}_t), \mathcal{P})$.

Furthermore consider a bond with price dynamics given by

$$\frac{dB_t}{B_t} = rdt, \quad B_T = 1$$

where r is the (continuously compounded) interest rate which is assumed to be constant.[1]

Our aim is to price a European call-option on the stock with maturity date T and strike price K. This is a security that gives the bearer the right, but not the obligation, to buy

[1]Notice that this is an ordinary differential equation with solution given by: $B_t = e^{-r(T-t)}$.

one share of stock at time T (and only at that time) for a price of K \$. Hence the contract has a terminal pay-off of

$$\max(S_T - K, 0) \equiv (S_T - K)^+$$

and no intermediate payments.

Closely related to the call-option is the put-option which gives the bearer the right, but not the obligation, to sell the stock at a certain price at a certain date, i.e. the put-option has a pay-off of $(K - S_T)^+$. Let C_t and P_t denote, respectively, the price of a (European) Call and Put with the same contractual specifications. A simple static hedge argument then shows that to prevent arbitrage opportunities we must have the following relationship, known as the Put-Call-Parity:

$$P_t = C_t + B_t K - S_t.$$

As this indicates we will assume that there are no transactions costs, no shortselling constraints and that all assets are perfectly divisible. Furthermore we will allow investors to continuously readjust their portfolios.

A *trading strategy* (a_t, b_t) is a *predictable* (2-dimensional) stochastic process satisfying certain technical conditions.[2] To us, a_t will represent the number of stocks held at time t, while b_t is the bond holdings. $V_t = a_t S_t + b_t B_t$ is the *value process.*

A trading strategy is called *self-financing* if

$$dV_t = a_t dS_t + b_t dB_t$$

which means that we only have to make an investment today. The gains are reinvested, and we do not use extra funds to cover our losses. An *arbitrage opportunity* is a self-financing trading strategy such that either

$$V_0 \leq 0, \ \mathcal{P}(V_T \geq 0) = 1, \ \mathcal{P}(V_T > 0) > 0$$

or[3]

$$V_0 < 0, \ \mathcal{P}(V_T \geq 0) = 1$$

So, an arbitrage opportunity is 'a free lunch'. Reasonably, though one should not be tempted to intuition about stochastics, we can not have such strategies in the economic equilibrium.

[2]The trading strategy will be stochastic because it depends on the stock and bond whose price evolution is stochastic. However, at any given time t we will know how much to hold. Among other reasons the technical conditions the strategy has to fulfil is to exclude *doubling strategies.* See Duffie (1992) for details.

[3]In incomplete markets the two conditions are not equivalent.

There would be an infinite demand for the 'arbitrage strategy'. while no agent (without a serious financial deathwish) would be willing to supply it.
¿From pure static arbitrage considerations the only bounds that can be put on the call option price are:[4]

$$S_t \geq C_t \geq (S_t - B_t K)^+ .$$

The main contribution of Black & Scholes (1973) is that they close the gap and give an exact pricing formula by dynamic arbitrage arguments.

Result 1 (The Black-Scholes Formula) *If the setting is as described above then to prevent arbitrage opportunities we must have $C_t = C(S_t, t)$ where*

$$\begin{aligned} C(x,t) &= x\Phi(z) - e^{-r(T-t)} K \Phi(z - \sigma\sqrt{T-t}) \qquad (2) \\ z &= \frac{\ln(\frac{x}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

and Φ denotes the cumulative density of the standard normal distribution.
In order to replicate the pay-off of the call-option we should hold

$$a_t = \frac{\partial C}{\partial S}(S_t, t) = \Phi(z)$$

shares of stock and

$$C(S_t, t) - a_t S_t$$

in bonds.

This is a remarkable result that has been twisted and turned in the literature for more than twenty years. The most noteworthy thing is that the instanteneous expected return of the stock , μ, does not enter the expression. In other words: Two investors need not agree on the expected return of the stock in order to agree about the option price.[5]
Notice also that, even though short selling was allowed, we never sell short the stock. Moreover the position in the stock is bounded above by 1.
This method of pricing is *relative.* We price the option in terms of the stock and bond,

[4]This is given that the call option contract is the only existing derivative security written on the stock. If there exists several option contracts in the economy, say with different strikes, there would be static arbitrage bounds between these contracts.

[5]It is worth noting that increasingly frequent discrete sampling of the underlying stock gives an improved estimate of the volatility but high frequency sampling does not neccesarily improve the estimate of the drift. So if the stock price is only observed at frequent but discrete times it is likely that investors will agree on the volatility but not neccesarily on the drift. For a derivation of this see Ingersoll (1987).

whose prices are taken as given. We do not need any other general equilibrium constraints on the economy other than there being 'no free lunches'. We shall later see that we can arrive at the result from a general equilibrium model, but this is in a sense 'overkill': The above conditions are exactly what we need.

3 The Hedge Argument, The Fundamental PDE, and 'How to Solve'

The technique presented in this section is the one originally used by Black and Scholes to derive the formula that now bears their names. The result was simultaneously and independently derived by Merton. Let Y_t denote the price of a call option with strike K and maturity T. Now *assume* that Y_t can be written as a twice continuously differentiable function of S_t (hence. no dependence on past S_u's) and t. That is

$$Y_t = C(S_t, t)$$

Ito's Lemma applied to Y_t yields

$$dY_t = \left(\mu S_t \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2}\right) dt + \frac{\partial C}{\partial S}\sigma S_t dW_t^{\mathcal{P}} \quad (3)$$

where some of the dependences have been notationally suppressed.
Now *assume* that a self-financing trading strategy (a_t, b_t) exists such that

$$a_t S_t + b_t B_t = Y_t \quad \forall t \in [0, T] \quad (4)$$

so a_t is the number of shares of stock held and b_t is the number of bonds held at time t. Hence we have

$$\begin{aligned} dY_t &= d(a_t S_t + b_t B_t) \\ &= a_t dS_t + b_t dB_t \\ &= (a_t \mu S_t + b_t r B_t)dt + a_t \sigma S_t dW_t^{\mathcal{P}} \end{aligned}$$

where the second equality follows from the self-financing condition. So now we have two Ito-expressions for dY_t. This means (by the Unique Decomposition Theorem, see e.g. Duffie (1992)) that the drift- and diffusion terms in these must be equal.
Matching 'diffusion'-terms yields (since $S_t > 0$ $\mathcal{P}$ -a.s.)

$$a_t = \frac{\partial C}{\partial S}(S_t, t),$$

which gives us the shares of stock to hold.

On the other hand from (4)

$$S_t \frac{\partial C}{\partial S}(S_t, t) + b_t B_t = C(S_t, t)$$

so

$$b_t = \frac{1}{B_t}(C(S_t, t) - S_t \frac{\partial C}{\partial S}(S_t, t))$$

¿From the 'drift'-terms we get

$$rS_t \frac{\partial C}{\partial S}(S_t, t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2}(S_t, t) + \frac{\partial C}{\partial t}(S_t, t) = rC(S_t, t)$$

which holds if the function C satisfies the partial differential equation (PDE)

$$rx \frac{\partial C}{\partial x}(x, t) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2}(x, t) + \frac{\partial C}{\partial t}(x, t) = rC(x, t), \tag{5}$$

A similar argument holds for all types of assets whose prices only depend on S_t and t so (5) is often referred to as the fundamental PDE for arbitrage free asset pricing.

As we are considering a European call-option, C should further satisfy the boundary condition[6]

$$C(x, T) = (x - K)^+ \tag{6}$$

So to find the arbitrage free call price we have to solve (5)-(6). Is this price unique, one might ask. Yes it is. Given that (5)-(6) has been solved we have found a (hopefully) self-financing trading strategy that replicates the option. If the option had any other price than the initial investment in this replicating portfolio, we would just sell the option and buy the replicating portfolio (or the other way round, depending on which is the cheaper alternative). This would leave us with a risk-free profit. An arbitrage opportunity !

Now. the reader might argue: 'How do we know that the proposed trading strategy is indeed self-financing ?'

This is a fair point to make, we have just assumed so far, that this was the case, and made explicit use of it.

So let us see that the Black-Scholes trading strategy as introduced in the above is self-financing.

Notice that as

$$\frac{\partial C}{\partial x} = \Phi(z(x))$$

[6]In fact we should also require $x \geq C(x, t) \geq (x - e^{-r(T-t)}K)^+$, but this turns out to be automatically satisfied in this case.

and because Φ and z are twice differentiable in x and t (for $t < T$) we have no trouble using our standard Ito Formula. We have to show that

$$dY_t = a_t dS_t + b_t dB_t$$

¿From the definitions of Y_t. a_t, and b_t , Ito yields

$$\begin{aligned} dY_t &= d(a_t S_t) + d(b_t B_t) \\ &= (a_t dS_t + da_t S_t + da_t dS_t) + d(C - a_t S_t) \\ &= a_t dS_t + (da_t S_t + da_t dS_t + dC - da_t dS_t - da_t S_t - a_t dS_t) \\ &= a_t dS_t + (dC - a_t dS_t) \end{aligned}$$

Now, dC we know from (3) and as $a_t = \frac{\partial C}{\partial S}$ we have (by (1))

$$dY_t = a_t dS_t + (\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial S^2} S_t^2)dt \tag{7}$$

but as C solves (5) we have $\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial S^2} S_t = rC - rS_t \frac{\partial C}{\partial S}$. Plugging this into (7) gives

$$\begin{aligned} dY_t &= a_t dS_t + \underbrace{\frac{1}{B_t}(C - S_t \frac{\partial C}{\partial S})}_{b_t} \underbrace{(rB_t dt)}_{dB_t} \\ &= a_t dS_t + b_t dB_t \end{aligned}$$

As we see it was not difficult to show (the hardest part is convincing yourself that there is something to show), but it hinges on the properties of C. Later (in Section 6) we will meet a trading strategy that is obviously self-financing. Except: It isn't !
In the words of Lou Reed:

> *"Don't believe half of what you see and none of what you hear."*

The intention has been to highlight the central hedge-argument: The option can be replicated by continuously trading in the stock and the bond. It could seem from the original article that Black and Scholes do not fully realize what a powerful technique they have come up with. Nonetheless they are remarkably clear in their argumentation.
The next section may be more interesting in a historical context as we show a constructive technique for solving (5)-(6).

3.1 Solving the Black-Scholes PDE

Solving PDEs is usually a pretty tricky matter. Therefore most textbooks just encourage the reader to verify that the expression in the Black-Scholes Formula actually satisfies the PDE (5) and boundary condition. This is a straightforward calculation. the volume of which depends strongly on one's ability to differentiate.

A Feynman-Kac approach, which highly resembles the technique used in Section 4, can be used for a more constructive proof. In fact if we insist on taking a probabilistic view, Ito's Lemma can also be used for the construction of a solution.

An analytic way of solving the PDE is given by the following sketch:

1. By making the transformations (with $\tau = T - t$)

$$z = \ln(x) + (r - \frac{\sigma^2}{2})\tau \tag{8}$$

$$\theta = \sigma^2 \tau \tag{9}$$

$$u(z, \theta) = e^{r\tau} C(x, t) \tag{10}$$

   solving Equation (5) can be seen to be eqivalent to solving

$$\frac{1}{2} u_{zz} - u_\theta = 0$$

   with the boundary condition

$$u(z, 0) = (e^z - K)^+$$

   This is known to physisists as the heat equation. In the martingale approach this is equivalent to transforming the problem into one that only involves standard normal variables.

2. The Fourier Transform (FT), $\mathcal{F}$, is a mapping between function spaces defined by (the notation varies)

$$\hat{u}(\xi, \theta) \equiv \mathcal{F}(u, \xi) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, \theta) e^{\frac{i\xi x}{\sqrt{2\pi}}} dx \tag{11}$$

   where $u : \mathbf{R} \times \mathbf{R}_+ \mapsto \mathbf{R}$. Provided that the considered function is continuous (or normalized in points of discontinuity) an *inverse FT* exists and is explicitly calculated simply by replacing 'i' by '$-i$' in (11) (hence it acts very similar to a FT). By using the differentiation rule for Fourier Transforms (see e.g. Borchsenius (1985))

$$\mathcal{F}\left(\left(\frac{\partial}{\partial z}\right)^n u(z, \theta), \xi\right) = (-i\xi)^n \mathcal{F}(u, \xi)$$

it is seen that the FT of u, $\hat{u}$, satisfies the ordinary differential equation[7]

$$\hat{u}_\theta(\xi,\theta) + \frac{1}{2}\xi^2\hat{u}(\xi,\theta) = 0$$

with boundary condition $\hat{u}(\xi,0) = \mathcal{F}((e^z - K)^+,\xi)$. This is easily solved by:

$$\hat{u}(\xi,\theta) = \hat{u}(\xi,0)e^{-\frac{\xi^2}{2}\theta}$$

3. The problem is then transformed back. Here we need the convolution theorem for FT's[8]

$$\mathcal{F}(fg,\xi) = \mathcal{F}(f,\xi) * \mathcal{F}(g,\xi),$$

which obviously also appears to inverse FT's. We then have

$$u(z,\theta) = u(z,0) * K_\theta(z)$$

where K_θ is such that $\hat{K}_\theta(\xi) = e^{-\frac{\xi^2}{2}\theta}$. Fortunately any table of FTs will tell you that[9]

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}\theta}e^{-\frac{izx}{\sqrt{2\pi}}}dz = \frac{1}{\sqrt{\theta}}e^{-\frac{x^2}{2\theta}} = K_\theta(x).$$

Note that something that bears more than a vague resemblance to a normal density function has now turned up. The convolution integral that one has to evaluate thus looks like:

$$\begin{aligned}
u(z,\theta) &= \frac{1}{\sqrt{2\pi\theta}}\int_{-\infty}^{\infty}(e^{z-y} - K)^+e^{-\frac{y^2}{2\theta}}dy \\
&= \frac{1}{\sqrt{2\pi\theta}}\int_{-\infty}^{\ln(\frac{x}{K})+(r-\sigma^2/2)\tau}(e^{z-y} - K)e^{-\frac{y^2}{2\theta}}dy \\
&= \frac{e^z}{\sqrt{2\pi\theta}}\int_{-\infty}^{\ln(\frac{x}{K})+(r-\sigma^2/2)\tau}e^{-y}e^{-\frac{y^2}{2\theta}}dy - K\frac{1}{\sqrt{2\pi\theta}}\int_{-\infty}^{\ln(\frac{x}{K})+(r-\sigma^2/2)\tau}e^{-\frac{y^2}{2\theta}}dy \\
&= \frac{e^{z+\theta/2}}{\sqrt{2\pi\theta}}\int_{-\infty}^{\ln(\frac{x}{K})+(r-\sigma^2/2)\tau}e^{-\frac{(y+\theta)^2}{2\theta}}dy - K\Phi\left(\frac{\ln(\frac{x}{K})+(r-\sigma^2/2)\tau)}{\sqrt{\theta}}\right) \\
&= e^{z+\theta/2}\Phi\left(\frac{\ln(\frac{x}{K})+(r+\sigma^2/2)\tau)}{\sqrt{\theta}}\right) - K\Phi\left(\frac{\ln(\frac{x}{K})+(r-\sigma^2/2)\tau)}{\sqrt{\theta}}\right)
\end{aligned}$$

[7]Notice that in our case the order of differentiation w.r.t θ and integration can be interchanged.
[8]$*$ denotes convolution: $(f*g)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi - x)g(x)dx$. Notice the scaling.
[9]It might be listed under Fourier COSINE Transforms.

which by transforming (completely) back to the original variables yields[10]

$$\begin{aligned} C_{t*} &= \epsilon^{-r\tau} u(z,\tau) \\ &= \epsilon^{-r\tau} \epsilon^{\ln x + (r-\sigma^2/2)\tau} e^{(\sigma^2\tau)/2} \Phi(\cdot\cdot) - \epsilon^{-r\tau} K\Phi(\cdot) \\ &= x\Phi\left(\frac{\ln(\frac{x}{K}) + (r+\sigma^2/2)(T-t))}{\sigma\sqrt{(T-t)}}\right) - \epsilon^{-r(T-t)} K\Phi\left(\frac{\ln(\frac{x}{K}) + (r-\sigma^2/2)(T-t))}{\sigma\sqrt{(T-t)}}\right). \end{aligned}$$

Voila! The Black-Scholes formula!

This technique is the one originally used by Black & Scholes (1973)/Merton (1973*b*) although this might be somewhat hidden in the original papers.

4 The Martingale Approach

In the last section the partial differential equation and thereby the arbitrage pricing methodology were derived under the assumption that the stock price followed a Markov process. In this section we will derive the martingale pricing approach that does not depend on the Markov property of the stock and draw the analogy to the partial differential equation obtained in the previous section. Using the martingale valuation technique we calculate the Black-Scholes formula.

The martingale pricing technique was pioneered by Cox & Ross (1976) and later on further developed and refined by Harrison & Kreps (1979), Harrison & Pliska (1981), and others.

We start by defining the quantity

$$\eta = \frac{\mu - r}{\sigma}$$

and the Girsanov factor

$$\xi_t = \exp(-\frac{1}{2}\eta^2 t - \eta W_t^{\mathcal{P}})$$

The Girsanov factor is a positive $\mathcal{P}$-martingale with mean 1, so we can define a new probability measure $\mathcal{Q}$, equivalent to $\mathcal{P}$, by setting

$$d\mathcal{Q} = \xi_t d\mathcal{P} \tag{12}$$

on $\mathcal{F}_t$. According to the Girsanov Theorem we have that under $\mathcal{Q}$

$$W_t^{\mathcal{Q}} = W_t^{\mathcal{P}} + \eta t$$

[10] see (8)-(10).

is a Brownian motion.[11] Plugging this into (1) yields that under $\mathcal{Q}$, the stock price evolves according to the stochastic differential equation

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^{\mathcal{Q}} \tag{13}$$

Let us now define

$$\begin{aligned} G_t &= B_t \mathrm{E}_t^{\mathcal{Q}}\left[(S_T - K)^+\right] \\ &= B_t \mathrm{E}_t^{\mathcal{P}}\left[\frac{\xi_T}{\xi_t}(S_T - K)^+\right] \end{aligned}$$

We observe that from the definition $e^{-rt}G_t$ must be a $\mathcal{Q}$-martingale with respect to the filtration $(\mathcal{F}_t)$. By the Martingale Representation Theorem we therefore have that

$$d\left[e^{-rt}G_t\right] = \gamma_t dW_t^{\mathcal{Q}}$$

for some process γ.[12] Introducing $\Gamma = e^{rt}\gamma/G$, using Ito's lemma and reintroducing the $\mathcal{P}$ Brownian motion we get

$$\begin{aligned} dG_t &= G_t(rdt + \Gamma_t dW_t^{\mathcal{Q}}) \\ &= G_t((r + \Gamma_t \eta)dt + \Gamma_t dW_t^{\mathcal{P}}) \end{aligned}$$

Now consider a self-financing strategy with value V and no consumption flow before T consisting of a stocks and the remaining amount is put in $b = (V - aS)/B$ bonds. Such a strategy evolves according to

$$\begin{aligned} dV_t &= a_t dS_t + \frac{V_t - a_t S_t}{B_t} dB_t \\ &= rV_t dt + a_t \sigma S_t dW_t^{\mathcal{Q}} \\ &= (a_t S_t(\mu - r) + rV_t)dt + a_t \sigma S_t dW_t^{\mathcal{P}} \\ &= (a_t S_t \sigma \eta + rV_t)dt + a_t \sigma S_t dW_t^{\mathcal{P}} \end{aligned} \tag{14}$$

Choosing $V_0 = G_0$ and

$$a = \frac{\Gamma V}{\sigma S}$$

we see that G, V have the same evolution and the same starting value, hence $V_t = G_t$ for all t. Since $C_T = G_T$ we conlude that $C_t = G_t$ for all t. Otherwise there exists an arbitrage

[11] For the Girsanov Theorem in the context of financial economics see for example Duffie (1992).
[12] A possible reference for the Martingale Representation Theorem is Duffie (1992).

because G_t can be replicated through the dynamic trading strategy outlined above. We have now obtained the martingale pricing equation

$$C_t = e^{-r(T-t)} \mathrm{E}_t^{\mathcal{Q}} \left[(S_T - K)^+ \right] \tag{15}$$

Note that this relation does not depend on μ and/or σ being constants. In fact the measure $\mathcal{Q}$ is unique and the replication argument goes through as long as only the diffusion coefficient, but not necessarily the drift, of the stock is adapted to the filtration generated by the stock.[13] Investors need not have the same beliefs about the drift of the stock for the arbitrage pricing to be valid. All they have to agree about is the diffusion coefficient. In a continuous-time economy this means that all they have to agree about are the zero-sets for the stock price evolution.

Before we use the martingale valuation equation to calculate the Black-Scholes formula we note that the Markovian property of the stock price process under $\mathcal{Q}$ together with the above valuation equation implies that we can write the option price as a function of current time and stock price only, i.e. $C_t = C(S_t, t)$. Using Ito's lemma it follows that[14]

$$dC_t = \left(\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right)(S_t, t)dt + \frac{\partial C(S_t, t)}{\partial S} \sigma S_t dW_t^{\mathcal{Q}} \tag{16}$$

Comparing this to equation (14) and matching the diffusion terms yields

$$a_t \sigma S_t = \frac{\partial C(S_t, t)}{\partial S} \sigma S_t$$

and thereby that the replicating strategy is given by

$$a_t = \frac{\partial C(S_t, t)}{\partial S}, \quad b_t = \frac{C_t - a_t S_t}{B_t}$$

Matching the drift terms yields the partial differential equation

$$\begin{aligned} rC &= \frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \\ s.t \; C_T &= (S_T - K)^+ \end{aligned}$$

On the other hand we see that if S follows the process (13) under the probability measure $\mathcal{Q}$ the partial differential equation implies that the discounted option price, $e^{-rt}C_t$, must be

[13] For obscure stock price processes ξ might not be a $\mathcal{P}$-martingale, which implies that $\mathcal{Q}$ as defined by (12) is not an equivalent probability measure. But if $\mathcal{Q}$ is well-defined it is unique.

[14] Strictly speaking C need to be differentiable in t and twice differentiable in S for (16) to be valid. But both (2) and the derivations below show that it is.

a $\mathcal{Q}$-martingale. From this the pricing equation (15) follows. We see that there is a strict analogy between the two pricing methodologies.
To compute the Black-Scholes formula we use that sitting at time 0, $\ln S_T$ is normal under $\mathcal{Q}$ with mean and variance

$$m = \mathrm{E}^{\mathcal{Q}}[\ln S_T] = \ln S_0 + (r - \frac{1}{2}\sigma^2)T \tag{17}$$

$$v^2 = \mathrm{Var}^{\mathcal{Q}}[\ln S_T] = \sigma^2 T \tag{18}$$

so

$$\begin{aligned} C_0 &= e^{-rT}\int_{\frac{\ln K - m}{v}}^{\infty} (e^{m+vx} - K)\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx \\ &= e^{-rT}\int_{\frac{\ln K - m}{v}}^{\infty} \frac{e^{m+vx-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx - e^{-rT}K\int_{\frac{\ln K - m}{v}}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx \\ &= e^{-rT+m+\frac{1}{2}v^2}\int_{\frac{\ln K - m}{v}}^{\infty} \frac{e^{-\frac{1}{2}(x-v)^2}}{\sqrt{2\pi}}dx - e^{-rT}K\int_{\frac{\ln K - m}{v}}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx \\ &= S_0\Phi\left(\frac{\ln(S_0/K) + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right) - e^{-rT}K\Phi\left(\frac{\ln(S_0/K) + rT}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}\right) \end{aligned}$$

5 Change of Numeraire

This section reviews a popular technique for solving the valuation equation

$$C_0 = e^{-rT}\mathrm{E}^{\mathcal{Q}}\left[(S_T - K)^+\right]$$

The technique is often referred to as the change of numeraire technique because it involves a change of discounting factor from the bank-account to the underlying asset. The technique will show to be extremely powerful: in this section we will derive the Black-Scholes formula without evaluating a single integral. But more than being a technique the change of numeraire approach also exposes an intriquing interpretation of the probabilities in the Black-Scholes formula as will be demonstrated in this section.
The change of numeraire technique showed up in several papers in the late eighties but it was probably known in the financial research community long before.
The idea is the following. We note that the martingale approach in the last section does not depend on the bank-account being used as numeraire for the pay-offs. In fact we could choose S as the numeraire of another martingale measure, $\mathcal{Q}'$, and under this measure

$$\frac{C_t}{S_t}$$

would be a martingale. To see this observe that

$$e^{-rt}\frac{S_t}{S_0} = \exp(-\frac{1}{2}\sigma^2 t + \sigma W_t^{\mathcal{Q}})$$

is a postive $\mathcal{Q}$-martingale with mean 1. Hence, we can define a new equivalent probability measure related to $\mathcal{Q}$ and $\mathcal{P}$ by:

$$\begin{aligned} d\mathcal{Q}' &= e^{-rt}\frac{S_t}{S_0}d\mathcal{Q} \\ &= e^{-rt}\frac{S_t}{S_0}\xi_t d\mathcal{P} \end{aligned}$$

on $\mathcal{F}_t$. The Brownian motion under $\mathcal{Q}'$ is then given by

$$\begin{aligned} W_t^{\mathcal{Q}'} &= W_t^{\mathcal{Q}} - \sigma t \\ &= W_t^{\mathcal{P}} + \eta t - \sigma t \end{aligned}$$

Straightforward application of this yields the valuation equation

$$C_0 = S_0 \mathrm{E}^{\mathcal{Q}'}\left[\frac{(S_T - K)^+}{S_T}\right] \tag{19}$$

where

$$\frac{dS_t}{S_t} = (r + \sigma^2)dt + \sigma dW_t^{\mathcal{Q}'}$$

But this does not reduce the complexity of derivation of the Black-Scholes formula. We will still have to evaluate an integral like the one in the previous section. So instead we reconsider our initial valuation equation. We split up the pay-off to get:

$$\begin{aligned} C_0 &= \mathrm{E}^{\mathcal{Q}}\left[e^{-rT}S_T\mathbf{1}_{\{S_T>K\}}\right] - e^{-rT}K\mathrm{E}^{\mathcal{Q}}\left[\mathbf{1}_{\{S_T>K\}}\right] \\ &= S_0\mathcal{Q}'(S_T > K) - e^{-rT}K\mathcal{Q}(S_T > K) \end{aligned} \tag{20}$$

We feel that this equation has a very nice interpretation: Given that the European option finishes in-the-money, the option pay-off can be decomposed into two components, the first component is the uncertain amount S_T, and the second component is the fixed amount $-K$. The present value of receiving S_T at time T is of course S_0. But this has to be weighted with some risk-adjusted probability of finishing in-the-money. $\mathcal{Q}'$ is the right measure to use because it exactly off-sets the "S"-risk. In other words under $\mathcal{Q}'$ pay-offs are valued as if one were "risk-neutral" with respect to the risk of the underlying stock. The second component $-K$ is a fixed dollar amount. The proper probability measure to apply is therefore the

measure $\mathcal{Q}$ under which pay-offs are measured relative to the risk-less bond.
The formula is general, in the sense that it does not depend on the underlying stock following a geometric Brownian motion. In fact, if an equivalent martingale measure with the bank account as numeraire exists (and this measure need not be unique) then one can derive the above formula. In a subsequent section we will show that the European option price of the Cox et al. (1979) model has a similar interpretation.
To obtain the Black-Scholes formula we simply have to evaluate the two probabilities in the above equation. We observe that under $\mathcal{Q}'$, $\ln S_T$ is normal with mean and variance given by

$$\begin{aligned} \mathrm{E}^{\mathcal{Q}'}[\ln S_T] &= \ln S_0 + (r + \frac{1}{2}\sigma^2)T \\ \mathrm{Var}^{\mathcal{Q}'}[\ln S_T] &= \sigma^2 T \end{aligned}$$

Using this and the distribution of $S(T)$ under $\mathcal{Q}$ given in the previous section we immediately obtain

$$\begin{aligned} \mathcal{Q}'(S_T > K) &= \Phi\left(\frac{\ln(S_0/K) + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right) \\ \mathcal{Q}(S_T > K) &= \Phi\left(\frac{\ln(S_0/K) + rT}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}\right) \end{aligned}$$

and thereby the Black-Scholes formula.
The change of numeraire technique is a very powerful tool that can be applied to other types of option contracts and to more general models. In the fixed income literature this technique has elegantly been applied to option pricing problems under the name of "forward-risk-adjustment", see for example Jamshidian (1989) and El Karoui & Rochet (1989). In the context of exotic options the technique has shown useful in the evaluation of Asian options, lookback options, barrier options, and various other exotica. See for example Ingersoll (1987), Babbs (1992), Dufresne, Kierstad & Ross (1996), and Graversen & Peškir (1995).

6 Shaking your Foundation

In this section we will investigate a strategy that duplicates the payoff of the European call and costs nothing initially. Moreover it seems to be self-financing. However, as we shall see, this strategy is not self-financing and therefore does not qualify to be a "threat" against the Black-Scholes formula. However we will see that by taking care of the extra external financing we will actually reach the B.-S. formula. The idea of getting the Black-Scholes formula in this way is due to Carr & Jarrow (1990).The strategy has also been carefully analyzed in the literature by Seidenverg (1988) and Dybvig (1988) and is known to practitioners as the *stop-loss start-gain strategy.*

Consider the following trading strategy: If the present value of the strike price K is below the stock price hold one share of the stock. Finance this by using borrowed funds. If the stock price falls below the present value of the strike price liquidate the position. As we shall see below this strategy will at terminal date T be worth exactly the same as the call-option. Now if the stock price initially is worth less than the present value of the strike price the strategy costs nothing initially. Therefore if the strategy is self-financing this would create arbitrage-opportunities in the Black Scholes-economy. To analyze this strategy we proceed more formally. Let:

$$\begin{aligned} a_t &= \mathbf{1}_{\{S_t > KB_t\}}, \\ b_t &= -\mathbf{1}_{\{S_t > KB_t\}} K, \quad \forall t \in [0,T]. \end{aligned}$$

Then the value of the portfolio at time t, Y_t, is equal to:

$$\begin{aligned} Y_t &= a_t S_t + b_t B_t \\ &= \mathbf{1}_{\{S_t > KB_t\}} S_t - \mathbf{1}_{\{S_t > KB_t\}} K B_t \\ &= (S_t - KB_t)^+ \end{aligned} \tag{21}$$

Now we see that the value of the portfolio is always the lower bound for a call option and furthermore since $B_T = 1$ we see that we have duplicated the call's payoff.

Therefore if $\frac{S_0}{B_0} < K$ then the portfolio initially costs nothing. To examine if this portfolio is self-financing we notice that the self-financing condition:

$$Y_t = Y_0 + \int_0^t a_u dS_u + \int_0^t b_u dB_u, \quad \forall t \in [0,T].$$

can be reduced (by using Ito's lemma) to:

$$\frac{Y_t}{B_t} = \frac{Y_0}{B_0} + \int_0^t a_u d\left(\frac{S_u}{B_u}\right), \quad \forall t \in [0,T]. \tag{22}$$

If we ease notation by letting

$$F_t = \frac{S_t}{B_t}, \quad \forall t \in [0,T],$$

where F_t is the stock price with the bond as numeraire (the forward price) we see that (22) gives us that the stop-loss and start-gain strategy is self-financing if and only if:

$$(F_t - K)^+ = (F_0 - K)^+ + \int_0^t \mathbf{1}_{\{F_u > K\}} dF_u, \quad \forall t \in [0,T]. \tag{23}$$

Fortunately enough this is not the case - otherwise there could be arbitrage in the economy as described above. To see that the strategy is not self-financing we use the following lemma:[15]

Lemma 1 *Let X be a continuous semimartingale of the form: $X_t = X_0 + M_t + V_t$, where M is a local martingale.*

1. *Let $g: \mathbf{R} \to \mathbf{R}$ be a convex function (which is not necessarily twice differentiable). Then:*
$$g(X_t) = g(X_0) + \int_0^t D^- g(X_s) dX_s + \int_{-\infty}^{\infty} \Lambda_t(x) \mu(dx), \tag{24}$$
where $D^- g$ is the left-hand derivative. I.e.
$$D^- g(x) \equiv \lim_{h \to 0_-} \frac{g(x+h) - g(x)}{h},$$
and μ is the second derivative measure:
$$\mu(a,b) \equiv D^- g(b) - D^- g(a), \quad a < b.$$

2. *Let $k: \mathbf{R} \to [0, \infty)$ be a Borel-measurable function. Then:*
$$\int_0^t k(X_s(\omega)) d < M >_s (\omega) = 2 \int_{-\infty}^{\infty} k(x) \Lambda_t(x, \omega) dx, \quad 0 \le t < \infty \tag{25}$$
holds for P-a.e. $\omega \in \Omega$.

3. *$\Lambda_t(x, \omega) \ge 0$ P- a.s.*

Using (24) with the convex function $g(x) = (x - K)^+$, noticing that $D^- g(x) = \mathbf{1}_{\{x > K\}}$ and $\mu(dx) = \mathbf{1}_{\{x = K\}}$ give us:[16]

$$(F_t - K)^+ = (F_0 - K)^+ + \int_0^t \mathbf{1}_{\{F_u > K\}} dF_u + \Lambda_t(K), \quad \forall t \in [0, T]. \tag{26}$$

We see that the difference between (23) and (26) is the term $\Lambda_t(K)$ which is called the *local time* at K by time t in the stochastic calculus literature. Now we will show that $\Lambda_t(K)$ is positive with positive probability for any t, which shows us that the stop-loss and start-gain strategy is not self-financing.
If we take the "risk-neutral" expectation of (26) we get:

$$\mathrm{E}_0^Q \left[(F_t - K)^+ \right] = (F_0 - K)^+ + \mathrm{E}_0^Q \left[\Lambda_t(K) \right], \quad \forall t \in [0, T], \tag{27}$$

[15]see Karatzas & Shreve (1988) p. 218.
[16]This is also known as the Tanaka-Meyer-formula.

where

$$\mathrm{E}_0^{\mathcal{Q}}\left[\int_0^t \mathbf{1}_{\{F_u>K\}}dF_u\right] = 0,$$

because F is a $\mathcal{Q}$-martingale and thereby we have that the integral-term is a $\mathcal{Q}$-martingale. Now it is obvious from the results in the previous sections that:

$$\mathcal{Q}(F_t > K) > 0, \ \mathcal{Q}(F_t < K) > 0. \ \ \forall t \in (0,T].$$

Furthermore: Because $g(x) = (x-K)^+$ is strictly convex over an interval containing K we get that Jensen's inequality holds strictly for $g(x)$. I.e.:

$$\mathrm{E}_0^{\mathcal{Q}}\left[(F_t - K)^+\right] > \left(\mathrm{E}_0^{\mathcal{Q}}\left[F_t\right] - K\right)^+ = (F_0 - K)^+ \tag{28}$$

Now combining (27) with (28) we get $\mathrm{E}_0^{\mathcal{Q}}[\Lambda_t(K)] > 0$. Since $\Lambda_t(K) \geq 0$ it follows that $\mathcal{Q}(\Lambda_t(K) > 0) > 0$. $\forall t \in (0,T]$. Therefore $\mathcal{P}(\Lambda_t(K) > 0) > 0$, $\forall t \in (0,T]$ which shows us that the strategy is not self-financing.
In our setting $\Lambda_t(K)$ has a very nice interpretation. Suppose that we change our strategy in the following way:
Buy one share of stock each time F rises from K to $K+\epsilon$, $\epsilon > 0$. In this case we should also go short in K bonds. We see that every time the transaction takes place it requires an additional ϵ bonds. Furthermore we liquidate the portfolio every time F goes back to K.
Now let $U_t(\epsilon)$ denote the number of times F has risen from K to $K+\epsilon$ until time t. Then we see that with the above mentioned strategy we would have to invest in $\epsilon U_t(\epsilon)$ bonds at time t to handle the external financing. Now it can be shown that:

$$\lim_{\epsilon \downarrow 0} \epsilon U_t(\epsilon) = \Lambda_t(K).$$

That is: The additional local time term from equation (26) can be interpreted as the external financing required to trade by the stop-loss start-gain strategy.
Now we will show that evaluating (27) for $t = T$ and noticing (from the previous section) that:

$$\frac{C_0}{B_0} = E_0^Q\left[(F_T - K)^+\right] = E_0^Q\left[(S_T - K)^+\right]$$

yields the B.S.-formula. I.e.:

$$C_0 = (S_0 - e^{-rT}K)^+ + e^{-rT}E_0^Q\left[\Lambda_T(K)\right]. \tag{29}$$

equals the Black-Scholes price of the option.
(29) has a nice interpretation: The first term on the right-hand side is the option's intrinsic

value and is according to (21) equal to the initial investment required in the stop-loss start-gain strategy. The residual ($e^{-rT}E_0^Q[\Lambda_T(K)]$) is referred to as the option's time value which in this case is the present value of the expected external financial costs.
¿From the previous section we know that F is a Q-martingale. By Girsanov's theorem we therefore have that:

$$dF_t = \sigma F_t dW_t^Q.$$

That is:

$$F_t = F_0 \exp\{\sigma W_t^Q - \frac{1}{2}\sigma^2 t\}.$$

Therefore we get the following transition density for F:

$$\begin{aligned}\psi(F_t, t; F_0, 0) &= \frac{1}{F_t\sigma\sqrt{t}\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln F_t - (\ln F_0 - \frac{1}{2}\sigma^2 t)}{\sigma\sqrt{t}}\right)^2\right\} \\ &= \frac{1}{F_t\sigma\sqrt{t}}\phi\left(\frac{\ln\left(\frac{F_0}{F_t}\right) - \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}\right),\end{aligned} \tag{30}$$

where $\phi(z) \equiv \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right)$ is the standard normal density function.
Now taking the "risk-neutral" expectation of (25) with $X = F$ we get:

$$\mathrm{E}_0^Q\left[\int_0^T k(F_s)d<F>_s\right] = \mathrm{E}_0^Q\left[2\int_{-\infty}^{\infty} k(x)\Lambda_T(x)dx\right]. \tag{31}$$

Using that $d<F>_t = \sigma^2 F^2(t)dt$ on the left side and employing Fubini's theorem on the right side of (31) gives us:

$$\int_{-\infty}^{\infty} k(x)\int_0^T \sigma^2 x^2\psi(x,t;F_0,0)dt\,dx = \int_{-\infty}^{\infty} k(x)2\mathrm{E}_0^Q[\Lambda_T(x)]\,dx. \tag{32}$$

Now choose $k(x) = \mathbf{1}_{\{x\in A\}}$, where $A \in \mathcal{F}$. (32) then becomes:

$$\int_A\int_0^T \sigma^2 x^2\psi(x,t;F_0,0)dt\,dx = \int_A 2\mathrm{E}_0^Q[\Lambda_T(x)]\,dx.$$

Realizing that the integrands are nonnegative and that both integrals are equal for any $A \in \mathcal{F}$ we get:

$$\int_0^T \sigma^2 x^2\psi(x,t;F_0,0)dt = 2\mathrm{E}_0^Q[\Lambda_T(x)]. \tag{33}$$

Combining (30) with (33) yields:

$$\mathrm{E}_0^Q[\Lambda_T(K)] = \frac{\sigma K}{2}\int_0^T \frac{1}{\sqrt{t}}\phi\left(\frac{\ln\left(\frac{F_0}{K}\right) - \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}\right)dt. \tag{34}$$

If we substitute (34) back into (29) we get:

$$C_0 = \left(S_0 - e^{-rT}K\right)^+ + e^{-rT}\frac{\sigma K}{2}\int_0^T \frac{1}{\sqrt{t}}\phi\left(\frac{\ln\left(\frac{S_0}{Ke^{-rT}}\right) - \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}\right) dt.$$

Now changing variable by $\nu \equiv \frac{\sqrt{t}\sigma}{\sqrt{T}}$ gives us:

$$C_0 = \left(S_0 - e^{-rT}K\right)^+ + e^{-rT}K\sqrt{T}\int_0^\sigma \phi\left(\frac{\ln\left(\frac{S_0}{Ke^{-rT}}\right) - \frac{1}{2}\nu^2 T}{\nu\sqrt{T}}\right) d\nu.$$

Finally we have reached the Black-Scholes formula. This is seen by noticing that (2) differentiated with respect to σ is:

$$e^{-rT}K\sqrt{T}\phi\left(\frac{\ln\left(\frac{S_0}{Ke^{-rT}}\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right), \tag{35}$$

with the boundary condition that $C_0 = (S_0 - e^{-rT}K)^+$ when $\sigma = 0$.

7 The Forward Equation of European Option Prices

Compared to the technique applied in section 5 the following derivation of the Black-Scholes formula might seem cumbersome. On the other hand the result that we will derive will shed further light on the European option pricing problem and will expose an interesting duality of the pricing problem that we consider. In the spirit of Dupire (1993) we derive a forward partial differential equation for the European option prices. In this equation the variables are the strike and the maturity whereas the current spot and time are fixed. This is opposed to the standard backward partial differential equation derived in section 3, where the spot and time are the variables and strike and maturity are fixed. Examining this forward equation reveals that the option pricing problem can be solved in a dual economy where every parameter is turned upside down: the strike price is the underlying, the option is a put with strike equal to the current spot, time is reversed, etc.

Again we start from the valuation equation

$$\begin{aligned} C_0 &= e^{-rT}\mathrm{E}^{\mathbb{Q}}\left[(S_T - K)^+\right] \\ &= e^{-rT}\int_K^\infty (x - K)\psi(x, T)dx \end{aligned} \tag{36}$$

where $\psi(x,T)$ is the $\mathcal{Q}$-density of S_T in the point x given S_0 at time 0.
Due to the Markov property of the spot price we have that ψ solves the forward Fokker-Planck equation[17]

$$0 = -\frac{\partial\psi}{\partial T} - \frac{\partial}{\partial x}\left[rx\psi\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2 x^2 \psi\right]$$

subject to the initial boundary condition $\psi(x,0) = \delta(x - S_0)$, where $\delta(\cdot)$ is the Dirac Delta function.[18] We will use this to derive a forward equation for the option prices.
Assuming that

$$\begin{aligned} rx\psi(x,T) &\rightarrow 0 \\ \sigma^2 x^2 \psi(x,T) &\rightarrow 0 \\ \frac{\partial}{\partial x}\left[\sigma^2 x^2 \psi(x,T)\right] &\rightarrow 0 \end{aligned}$$

for $x \rightarrow \infty$, which is clearly satisfied in the Black-Scholes model, integration of the forward equation over the interval (y,∞) yields:

$$0 = -\frac{\partial}{\partial T}\int_y^\infty \psi(x,T)dx + ry\psi(y,T) - \frac{1}{2}\frac{\partial}{\partial y}\left[\sigma^2 y^2 \psi(y,T)\right]$$

Integrating once more, this time over (K,∞), yields

$$0 = -\frac{\partial}{\partial T}\int_K^\infty \int_y^\infty \psi(x,T)dx\ dy + r\int_K^\infty y\psi(y,T)dy + \frac{1}{2}\sigma^2 K^2 \psi(K,T)$$

Now we go back to the pricing equation. By partwise integration we have that

$$C_0 = e^{-rT}\int_K^\infty \int_y^\infty \psi(x,T)dx\ dy$$

so that

$$\frac{\partial}{\partial T}\int_K^\infty \int_y^\infty \psi(x,T)dx\ dy = re^{rT}C_0 + e^{rT}\frac{\partial C_0}{\partial T}$$

Further we have that

$$\begin{aligned} \int_K^\infty y\psi(y,T)dy &= e^{rT}C_0 - Ke^{rT}\frac{\partial C_0}{\partial K} \\ \psi(K,T) &= e^{rT}\frac{\partial^2 C_0}{\partial K^2} \end{aligned} \quad (37)$$

[17]See Revuz & Yor (1991) p 269 for the Fokker-Planck equation.
[18]The Dirac Delta function is defined by

$$\delta(x) = 0$$

for all $x \neq 0$ and

$$\int_{-\epsilon}^{\epsilon} \delta(x)dx = 1$$

for all $\epsilon > 0$.

If we let $C(K,T)$ denote the initial price of a European call with strike K expiring at time T, we obtain the following forward partial differential equation for the European call option prices

$$0 = -\frac{\partial C}{\partial T} - rK\frac{\partial C}{\partial K} + \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 C}{\partial K^2} \tag{38}$$

subject to the initial boundary condition $C(K,0) = (S_0 - K)^+$.

The forward equation can now be solved to yield the Black-Scholes formula.

The advanced reader might observe that the forward equation could be derived from the valuation equation under the $\mathcal{Q}'$ measure, (19), combined with the time homogeneity of the stock price process in the Black-Scholes model. Under the assumption of a positive dividend yield of the underlying stock, Andreasen & Gruenewald (1996) apply this technique to obtain a forward equation for American calls in the Black-Scholes model as well as in the jump-diffusion model of Merton (1976).[19] But the forward equation (38) is more general; under assumption of sufficient regularity it holds for all Ito processes of the type

$$\frac{dS_t}{S_t} = \mu(t;\omega)dt + \sigma(S_t,t)dW_t^{\mathcal{P}}$$

if additionally the interest rate is only a function of time and the stock price. For stock option pricing and short maturities it is in most cases reasonable to assume (at most) time-dependent interest rates. Given todays yield curve it is then possible to uniquely determine the function $\sigma(S,t)$ from a full double continuum of option prices, $C(K,T)$, by the forward equation (38). The trick is simply to estimate the derivatives in (38) and isolate the function $\sigma(\cdot,\cdot)$. In other words from a full set of marketed options we can "infer the option pricing model of the market". This was first observed by Dupire (1993), but already Breeden & Litzenberger (1978) noted the relation (37). This relation tells us that we can infer the stock's risk-adjusted distribution at a given maturity date from a continuum of option prices of different strikes.[20]

Another interesting implication of the forward equation is that when the volatility coefficient (now possibly a function of time and spot) is given we can price all options on the market by only solving one partial differential equation numerically. Andreasen (1996) observes that this also goes for the hedge ratios of the European options. To see this define

$$\Delta(K,T) = \frac{\partial C(K,T)}{\partial S}|_{S=S_0} \tag{39}$$

[19]The positive dividend yield implies that the American call might be exercised prematurely.

[20]For further exploration of this see for example Shimko (1991), Derman & Kani (1994), Rubinstein (1994), and Jackwerth & Rubinstein (1996).

Differentiation of the forward equation (38) now yields

$$0 = -\frac{\partial \Delta}{\partial T} - rK\frac{\partial \Delta}{\partial K} + \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 \Delta}{\partial K^2}$$

subject to the initial boundary condition $\Delta(K,0) = \mathbf{1}_{\{S_0 \geq K\}}$.
Similar forward equations might be derived for other "Greeks".
The last point to be stated is the duality of the option pricing problem implied by the forward equation. Suppose that the time axis is reversed, S_0 is a fixed quantity, and that we are sitting at time T evaluating

$$\mathrm{E}^{\mathcal{R}}\left[(S_0 - K_0)^+ | K_T = K\right] \tag{40}$$

for the process

$$\frac{dK_t}{K_t} = (-r)d(-t) + \sigma dW_t^{\mathcal{R}}$$

where $W^{\mathcal{R}}$ is some backward running Brownian motion under some probability measure $\mathcal{R}$. Then the forward equation (38) is the backward equation resulting for this problem. So we conclude that the option pricing problem might be solved in a dual economy where time is reversed, the strike is the underlying that pays a proportional dividend of r, the option is a put on the initial stock price, and finally the interest rate is equal to zero. Further we see that in this "space" the hedge ratio of the original economy will be a digital option. The option pricing can therefore be performed in a reversed binomial tree.[21]
We conlude this section by using (40) to calculate the Black-Scholes formula and (39) to calculate the hedge ratio. Note that under $\mathcal{R}$ we have that

$$K_0 = K\exp\left((-r - \frac{1}{2}\sigma^2)T + \sigma(W_0^{\mathcal{R}} - W_T^{\mathcal{R}})\right)$$

so

$$\begin{aligned}
&\mathrm{E}^{\mathcal{R}}\left[(S_0 - K_0)^+ | K_T = K\right] \\
= &\int_{-\infty}^{\frac{\ln(S_0/K)+rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}} \left(S_0 - Ke^{(-r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x}\right) \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx \\
= &S_0 \int_{-\infty}^{\frac{\ln S_0 + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx - e^{-rT}K \int_{-\infty}^{\frac{\ln(S_0/K)+rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}} \frac{e^{-\frac{1}{2}(x-\sigma\sqrt{T})^2}}{\sqrt{2\pi}} dx \\
= &S_0 \Phi\left(\frac{\ln(S_0/K) + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right) - e^{-rT}K\Phi\left(\frac{\ln(S_0/K) + rT}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}\right)
\end{aligned}$$

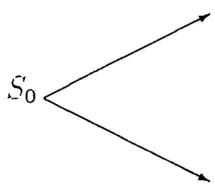
[21]For more on the duality see Dupire (1994) and Andreasen (1996).

For the hedge ratio we get:

$$\begin{aligned} \mathrm{E}^{\mathcal{R}}\left[\mathbf{1}_{\{K_0 \le S_0\}} | K_T = K\right] &= \mathcal{R}(K_0 \le S_0 | K_T = K) \\ &= \Phi\left(\frac{\ln(S_0/K) + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right). \end{aligned}$$

8 A Convergence Proof

Cox, Ross, and Rubinstein were the first to publish a paper with a formal convergence proof along the lines of this section. A much less known paper with the same result (and from the same year) is by Rendleman & Bartter (1979). But the use of binomial models for economic reasoning is much older, dating (at least) back to Arrow and Debreu in the 50'ies.
Let us consider the following situation: A stock today has a price of S_0 and can in the next period either go up to uS_0 or down to dS_0. This happens with probabilities p and $1-p$, respectively. In the economy there further exists a risk-free zero coupon bond maturing in the next period with (discretely compounded) interest rate r_d ($u > 1 + r_d > d > 0$, to avoid dominance), and a call-option on the stock with exercise price K. The situation is illustrated in Figure 1.



Stock	Bond	Call
uS_0	1	$C_u = (uS_0 - K)^+$
dS_0	1	$C_d = (dS_0 - K)^+$

Figure 1: The One-Period Binomial Model

We are interested in hedging the option by trading a shares of stock and b bonds. A perfect hedge, i.e. an exact replication of the option's pay-off in every possible future state, is achieved by letting

$$a = \frac{C_u - C_d}{(u-d)S_0}$$
$$b = \frac{uC_d - dC_u}{(u-d)}$$

(notice that $a \approx \frac{\partial C}{\partial S}$, so the analogy to the continuous case is striking).
To prevent arbitrage opportunities the price of the hedge portfolio must be equal to the present price of the call-option. Writing this out and reshuffling leads to

$$C_0 = R^{-1}(qC_u + (1-q)C_d) \tag{41}$$

where

$$q = \frac{R-d}{u-d}, \qquad R = 1 + r_d$$

So we see that the price of the call-option is the discounted expected future value where the expectation is under a measure that gives probability q of an 'up-jump'. Notice that the original probabilities do not enter the expression. This hedge argument is the key in Cox et al. (1979). Notice that we can write (41) as

$$\frac{C_0}{B_0} = \mathrm{E}_0^{\mathcal{Q}}\left(\frac{C_T}{B_T}\right), \quad B_0 = R^{-T}$$

with obvious subscript notation and $\mathcal{Q}$ denoting the measure naturally induced by q. So: Using the bond as numeraire, the call price is a $\mathcal{Q}$-martingale. In other words the notation is consistent with that of Section 5. At this point let us make an observation. If we consider

$$q' = \frac{u}{1+r_d}q$$

then direct inspection reveals that

$$\frac{C_0}{S_0} = \mathrm{E}_0^{\mathcal{Q}'}\left(\frac{C_T}{S_T}\right)$$

with $\mathcal{Q}'$ being the measure induced by q'. Again the notation is consistent: Using the stock as numeraire, the call price is a $\mathcal{Q}'$-martingale.
The argument is easily extended to a setting with n independent multiplicative binomial movements per unit of time ensuring us that the martingale pricing techniques of Section 4 carry over in a discrete setting. Using the arguments from Section 5 we can thus still write out the call price as

$$C_0 = S_0\mathcal{Q}'(S_T^{(n)} > K) - KB_0\mathcal{Q}(S_T^{(n)} > K) \tag{42}$$

where

$$\begin{aligned} S_T^{(n)} &= S_0 u^j d^{Tn-j} \\ j &\overset{Q}{\sim} \mathrm{bi}(Tn, q) \\ j &\overset{Q'}{\sim} \mathrm{bi}(Tn, q') \\ B_0 &= R^{-Tn} \end{aligned}$$

with 'bi' denoting the binomial distribution. Again we claim that the call price at any time can only be a function of current stock price and time. This claim is then justified by our ability to exactly replicate the final pay-off which only depends on S_T by a dynamic trading strategy in the stock and the bond.
For computational purposes (42) is often rewritten as

$$C_0 = S_0 \check{\Phi}(m; Tn, q') - K B_0 \check{\Phi}(m; Tn, q) \tag{43}$$

with m being the smallest non-negative integer greater than $\ln(K/(S_0 d^{Tn}))/\ln(u/d)$ and $\check{\Phi}(m; Tn, q')$ denoting the complementary binomial distribution function.
In the Black-Scholes setting we have that

$$\begin{aligned} \ln S_T &\overset{Q}{\sim} N(\ln S_0 + (r - \frac{\sigma^2}{2})T, \sigma^2 T) \\ \ln S_T &\overset{Q'}{\sim} N(\ln S_0 + (r + \frac{\sigma^2}{2})T, \sigma^2 T). \end{aligned}$$

Now let anything with an 'n' on it refer to a binomial model with n moves per time unit. Our aim is to show that as n approaches infinity the call-price in the n-model converges to that of the Black-Scholes model. Because of the decompositions and (20) and (42) our main task is to choose the parameters of the binomial model such that

$$\ln S_T^{(n)} \overset{Q}{\to} N(\ln S_0 + (r - \frac{\sigma^2}{2})T, \sigma^2 T) \tag{44}$$

$$\ln S_T^{(n)} \overset{Q'}{\to} N(\ln S_0 + (r + \frac{\sigma^2}{2})T, \sigma^2 T) \tag{45}$$

Regarding interest rates we don't have much choice but to let $R_n = e^{r/n}$. This means that the key parameters we have to choose are the sizes of the up and down moves, u_n and d_n. A good choice is

$$\ln u_n = \frac{\sigma}{\sqrt{n}} \tag{46}$$

$$\ln d_n = -\frac{\sigma}{\sqrt{n}} \tag{47}$$

With M_n and V_n denoting mean and variance of $\ln S_T^{(n)}$ we then have

$$\begin{aligned} M_n^Q &= \ln S_0 + Tn(q_n \ln u_n + (1-q_n)\ln d_n) \\ V_n^Q &= Tnq_n(1-q_n)(\ln u_n - \ln d_n)^2 \end{aligned}$$

and likewise for Q'. Remembering that

$$q_n = \frac{e^{r/n} - d_n}{u_n - d_n}$$

allows us to rewrite M_n and V_n by Taylor expanding the exponential function to the second order. This, when keeping +'s and -'s straight, reveals that

$$\begin{aligned} M_n^Q &\to \ln S_0 + (r - \frac{\sigma^2}{2})T \\ V_n^Q &\to \sigma^2 T \end{aligned}$$

and by similar calculations we get:

$$\begin{aligned} M_n^{Q'} &\to \ln S_0 + (r + \frac{\sigma^2}{2})T \\ V_n^{Q'} &\to \sigma^2 T \end{aligned}$$

So the first and second moments converge, under the respective measures, and the jumps vanish in the limit. This allows us to invoke (basically) a Lindeberg-Feller version of the Central Limit Theorem (see e.g. Duffie (1992)) to confirm the validity of (44) and (45). Finally dominated convergence ensures that the elements of the binomial decomposition (42) converge to their continuous counterparts, which establishes the desired result.
We have not used the original probabilities for anything (except that we have implicitly assumed them to be non-zero). Furthermore it can be shown that we can add any term of order n or higher in (46) and (47) and still have the same Q (and Q') convergence results. This could, if we were such inclined, help us establish convergence of the underlying process.
In this Section we have thus illustrated two things:

1. The Black-Scholes formula can be seen as the limiting case of a binomial model.

2. The change of numeraire technique also works nicely in a discrete setting.

9 The Continuous-Time CAPM

This section shows that one might also obtain the Black-Scholes formula in the continuous-time capital asset pricing model by Merton (1971). The derivation is basically taken from Ingersoll (1987) but a similar derivation appears in Cox and Rubinstein (1985). Suppose that the market in total contains N (non-dividend paying) risky assets that evolve according to the N-dimensional stochastic differential equation:

$$dS_t = I_{S_t}(\mu dt + \Sigma dW_t^{\mathcal{P}})$$

where I_S is the diagonal matrix with diagonal elements $(S_1, \dots, S_N)$, μ is an N-dimensional constant vector, Σ is a constant $N \times N$ matrix, and $W^{\mathcal{P}}$ is an N-dimensional Brownian motion under $\mathcal{P}$. For simplicity we will suppose that Σ has full rank. Suppose that there additionally exists a risk free asset paying a constant continuously compounded interest rate r. Consider an investor that maximizes expected additive utility on some time horizon $[0, \tau]$.

$$\mathrm{E}^{\mathcal{P}}\left[\int_0^\tau u(x_t, t)dt\right]$$

over consumption flow, x, and risky portfolio holdings vector, a, subject to the self-financing constraint or dynamic budget constraint

$$\begin{aligned} dV_t &= a_t' dS_t + (V_t - a_t' S_t) r dt - x_t dt \\ &= (V_t \theta_t'(\mu - r\mathbf{1}) + rV_t - x_t)dt + V_t \theta_t' \Sigma dW_t^{\mathcal{P}} \end{aligned}$$

where V_t is the current wealth and θ is the N dimensional vector with elements $\theta_i = a_i S_i / V$. θ_i is the fraction of the investor's wealth invested in risky asset i.
Defining the indirect utility as

$$J(V_t, t) = \max_{(\theta_s, x_s)_{s \geq t}} \mathrm{E}_t^{\mathcal{P}}\left[\int_t^\tau u(x_s, s)ds\right]$$

we get the Bellman-Hamilton equation[22]

$$0 = \max_{\theta, x} u + \frac{\partial J}{\partial t} + (V\theta'(\mu - r\mathbf{1}) + rV - x)\frac{\partial J}{\partial V} + \frac{1}{2}V^2 \theta' \Sigma \Sigma' \theta \frac{\partial^2 J}{\partial V^2}$$

The first order conditions imply that in optimum

$$\theta = -\frac{\partial J / \partial V}{V \partial^2 J / \partial V^2}(\Sigma\Sigma')^{-1}(\mu - r\mathbf{1})$$

[22]For an intuitive proof of the Bellman Hamilton equation see for example Ingersoll (1987).

Observe that for all i, j the ratio θ_i/θ_j is independent of wealth and utility. So if all investors have additive separable utility they will all hold the same portfolio of risky assets. This means that the market portfolio of risky assets will be given by

$$\theta_M = k(\Sigma\Sigma')^{-1}(\mu - r\mathbf{1})$$

for some one-dimensional process k. The expected instantaneous excess return of the (risky) market portfolio is therefore

$$\mu_M - r = k(\mu - r\mathbf{1})'(\Sigma\Sigma')^{-1}(\mu - r\mathbf{1})$$

and the local variance of the market return is

$$v_M^2 = k^2(\mu - r\mathbf{1})'(\Sigma\Sigma')^{-1}(\mu - r\mathbf{1})$$

The vector of local covariances between the market portfolio and the assets' instantaneous return is given by

$$c = k(\mu - r\mathbf{1})$$

Combining these equations we get

$$\mu_i = r + \frac{c_i}{v_M^2}(\mu_M - r)$$

Now suppose an option contract on S_i is introduced on the market in zero net supply. Since the market is dynamically complete the market equilibrium is not changed and the above expected return relation is still valid. If the option price is only a function of the underlying stock and time, Ito's lemma implies that the local covariance of the return of the option with the market return can be written as

$$\frac{1}{C}\frac{\partial C}{\partial S}c_i$$

Therefore the expected instantaneous return of the option contract is given by

$$r + \frac{1}{C}\frac{\partial C}{\partial S}\frac{c_i}{v_M^2}(\mu_M - r).$$

Using Ito's lemma on the option price, $C(S_i(t), t)$, yields that the instantaneous return of the option contract is given by

$$\frac{1}{C}\left[\frac{\partial C}{\partial t} + \mu_i S_i \frac{\partial C}{\partial S_i} + \frac{1}{2}\|\Sigma_i\|^2 S_i^2 \frac{\partial^2 C}{\partial S_i^2}\right]$$

Equating this to the return of the option and inserting the expected return of the underlying stock yields the partial differential equation

$$rC = \frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}$$

where we have omitted the subscript on the stock and introduced the notation $\sigma^2 = \|\Sigma_i\|^2$. We have thereby derived the Black-Scholes partial differential equation in the context of the continuous-time CAPM. The formula for the European call can be calculated as in Section 3.

Note that this approach relies on the assumption that the option price is a function of time and currrent stock only. As shown in Section 3 this assumption can be justified by the Black-Scholes hedging argument that uniquely fixes the option price given no arbitrage posibilities. But here our argumentation is not based on a hedging argument but rather a risk-return relation. It is therefore important to note that the above derivation only shows the consistency of the Black-Scholes formula with the CAPM pricing relation, the derivation does not *prove* the Black-Scholes formula.

¿From the last equations it is tempting to conclude that the Black-Scholes formula or a preference-free pricing formula can be derived in the context of any linear factor model of expected asset return like the continuous-time CAPM. This is only true though if the market additionally is dynamically complete or effectively complete. In general, if a new asset is introduced in an incomplete economy, a new equilibrium will be the outcome and the prices of the existing assets will change. As mentioned, an exception to this is when an incomplete market is effectively complete. An example of this situation is the consumption based capital asset pricing model described in Merton (1973*a*) where the state variables determining the investment opportunity set are spanned by the marketed assets.[23]

10 A Representative Investor Approach

In this section we show that the Black-Scholes formula can be derived in a model where the continuous-trade assumption is replaced by the assumption of a representative investor with power utility. The approach was introduced by Rubinstein (1976).

First, let us consider a one-period model where trading can be performed at the times $0, T$. Suppose an agent maximizes expected utility of terminal consumption

$$\mathrm{E}^{\mathcal{P}}\left[u(x_T)\right]$$

[23]See also Christensen, Graversen & Miltersen (1996).

subject to the budget constraint

$$x_T = a'S_T$$
$$V_0 = a'S_0$$

where a is the vector of portfolio holdings, S is the vector of prices of the marketed assets, and V_0 is the initial wealth. Forming the Lagrangian yields the first order condition

$$S_0 = \lambda^{-1}\mathrm{E}^{\mathcal{P}}\left[u'(x_T)S_T\right]$$

where the prime denotes the first derivative and λ is the Lagrange multiplier of the budget constraint. Specifically we get for the risk-free asset

$$e^{-rT} = \lambda^{-1}\mathrm{E}^{\mathcal{P}}\left[u'(x_T)\right]$$

Combining these equations we get the valuation equation

$$S_0 = e^{-rT}\mathrm{E}^{\mathcal{P}}\left[\frac{u'(x_T)}{\mathrm{E}^{\mathcal{P}}\left[u'(x_T)\right]}S_T\right]$$

With these preliminaries let us now assume that the market has a representative investor with power utility function

$$u(x) = \frac{x^{1+\gamma}}{1+\gamma}$$

with $\gamma < 0$, i.e. $-\gamma$ is the constant relative risk aversion of the representative investor.
Now we redefine the notation; let S be the price of one particular stock, and V_0 be initial aggregate wealth.
Let us assume that aggregate consumption at time T and the time T stock price are jointly log-normally distributed, so that we can write

$$S_T = S_0 e^{(\mu-\frac{1}{2}\sigma^2)T+\sigma W_T^{\mathcal{P}}}$$
$$x_T = V_0 e^{(\mu_x-\frac{1}{2}\sigma_x^2)T+\sigma_x W_{x,T}^{\mathcal{P}}}$$

where $W^{\mathcal{P}}, W_x^{\mathcal{P}}$ are $\mathcal{P}$-Brownian motions with constant correlation ρ.
Note that

$$\frac{u'(x_T)}{\mathrm{E}^{\mathcal{P}}\left[u'(x_T)\right]} = e^{-\frac{1}{2}\gamma^2\sigma_x^2 T+\gamma\sigma_x W_{x,T}^{\mathcal{P}}}$$

For the market to be in equilibrium we must have that the valuation equation holds for the stock. Inserting the above in the valuation equation yields

$$\begin{aligned} S_0 &= S_0 e^{-rT}\mathrm{E}^{\mathcal{P}}\left[e^{(\mu-\frac{1}{2}(\sigma^2+\gamma^2\sigma_x^2))T+\sigma W_T^{\mathcal{P}}+\gamma\sigma W_{x,T}^{\mathcal{P}}}\right] \\ &= S_0 e^{(\mu+\sigma\gamma\sigma_x\rho-r)T} \end{aligned}$$

so

$$\mu = r - \gamma\rho\sigma\sigma_x \tag{48}$$

Now we want to evaluate a call option on S_T with strike K. Using the valuation equation and the above derivations we get:

$$\begin{aligned} C_0 &= e^{-rT}\mathrm{E}^{\mathcal{P}}\left[e^{-\frac{1}{2}\gamma^2\sigma_x^2 T+\gamma\sigma_x W_{x,T}^{\mathcal{P}}}(S_T - K)^+\right] \\ &= S_0\mathrm{E}^{\mathcal{P}}\left[e^{-\frac{1}{2}(\sigma^2+2\gamma\rho\sigma\sigma_x+\gamma^2\sigma_x^2)T+\sigma W_T^{\mathcal{P}}+\gamma\sigma_x W_{x,T}^{\mathcal{P}}}\mathbf{1}_{\{S_T\geq K\}}\right] \\ &\quad -Ke^{-rT}\mathrm{E}^{\mathcal{P}}\left[e^{-\frac{1}{2}\gamma^2\sigma_x^2 T+\gamma\sigma_x W_{x,T}^{\mathcal{P}}}\mathbf{1}_{\{S_T\geq K\}}\right] \end{aligned}$$

By introducing the joint density of $(W^{\mathcal{P}}, W_x^{\mathcal{P}})_T$ we could calculate the expectations to give us the Black-Scholes formula. But it is much easier to make use of the change of measure induced by the Girsanov factors under the expectations.

Define two new equivalent probability measure $\mathcal{Q}'$ and $\mathcal{Q}$ by the Radon-Nikodyn derivatives

$$\begin{aligned} \frac{d\mathcal{Q}'}{d\mathcal{P}} &= e^{-\frac{1}{2}(\sigma^2+2\gamma\rho\sigma\sigma_x+\gamma^2\sigma_x^2)T+\sigma W_T^{\mathcal{P}}+\gamma\sigma_x W_{x,T}^{\mathcal{P}}} \\ \frac{d\mathcal{Q}}{d\mathcal{P}} &= e^{-\frac{1}{2}\gamma^2\sigma_x^2 T+\gamma\sigma_x W_{x,T}^{\mathcal{P}}} \end{aligned}$$

Using these probability measures we can write

$$C_0 = S_0\mathcal{Q}'(S_T > K) - e^{-rT}K\mathcal{Q}(S_T > K)$$

The Girsanov Theorem together with relation (48) imply that

$$\begin{aligned} S_T &= S_0 e^{rT+\frac{1}{2}\sigma^2 T+\sigma W_T^{\mathcal{Q}'}} \\ &= S_0 e^{rT-\frac{1}{2}\sigma^2 T+\sigma W_T^{\mathcal{Q}}} \end{aligned}$$

where $W^{\mathcal{Q}'}, W^{\mathcal{Q}}$ are some standard normal Brownian motions under the two respective probability measures.[24] Using this we immediately obtain the Black-Scholes formula.

In this section the assumption of continuous trade was replaced by the assumption of existence of a representative investor. Unless investors have identical or very similar preferences a representative investor is in general not guaranteed to exist in an incomplete market like the one analyzed. Even if a representative investor exists, the market equilibrium, prices of existing assets, and the representative preferences might change when a new asset (in this case the option) is introduced on an incomplete market. Despite these drawbacks this approach is widely used in models of incomplete markets.

[24]Notice: The correlation between the two coordinates is ρ no matter which of the two measures ($\mathcal{Q}$ or $\mathcal{Q}'$) we use.

11 Discussion

This paper has presented eight different ways to derive the Black-Scholes formula. We have thereby presented most of the techniques applied in today's continuous-time arbitrage pricing. Except for the last section the derivations have all relied on two seemingly crucial assumptions, namely the ability to trade continously and the absence of market frictions. The question is now how robust is the Black-Scholes formula when we start to relax these assumptions. Ingersoll (1987) considers a one period model like the one considered in the last section. Ingersoll does not assume log-normality of the aggregate consumption or the existence of a representative agent, but solely that the underlying asset has a log-normal distribution. Ingersoll shows that if investors reach an equilibrium where the pricing kernel is decreasing in the underlying stock then rather narrow bounds can be put on the call option price, and these bounds will (of course) lie around the Black-Scholes price. A pricing kernel that is decreasing in the stock is consistent with an equilibrium where the risk of the stock earns a positive risk premium.

Constantinides & Zariphopoulou (1995) introduce proportional transaction costs in the Black-Scholes model. They give an upper bound for the reservation bid price and a lower bound for the reservation ask price for any investor with concave utility of terminal wealth in this economy. For reasonable parameters these bounds are placed rather tight around the Black-Scholes price.

These findings indicate that the Black-Scholes formula is more robust to the introduction of market imperfections than one should think. In other words even though investors might not be able to perfectly replicate the option pay-off through dynamic trading they may still reach an equilibrium where the option is priced very close to the Black-Scholes price.

In todays option markets the implied volatilities differ quite substantially across strikes and maturities.[25] This of course means that the Black-Scholes model is not consistent with the observed option prices. But it doesn't necessarily invalidate the Black-Scholes *approach.* The Black-Scholes arbitrage pricing methodology holds as long as the underlying asset follows a continuous evolution. So if observed option prices do not confirm the Black-Scholes model, it just means that a more advanced model is needed. One could also note that it would be surprising if the most simple continuous-time model matched reality.

These final considerations only support that the Black-Scholes arbitrage pricing methodol-

[25] The implied volatility of an option is the volatility that makes the Black-Scholes price equal one of the options.

ogy stands as one of the finest economic contributions of this century and the Black-Scholes model will remain a benchmark for all further refinements of the derivative pricing theory.

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An Arbitrage Term Structure Model of Interest Rates with Stochastic Volatility[1]

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Abstract

We present here a continuous-time arbitrage model of the term structure of interest rates. There are two factors in our model. The level of the zero-coupon bond prices is driven by the first factor only, whereas the volatility of the zero-coupon bond prices is affected by both factors. The specification of our model is motivated by prior empirical studies which point out desirable features that are not captured by existing theoretical models. As in Heath, Jarrow and Morton (1990), we derive the risk neutral dynamics of both factors by arbitrage arguments. To derive the risk neutral dynamics of the bond volatilities we assume the existence of futures contracts written on the instantaneous variances of the zero-coupon bond prices. These contracts are obviously not traded in any market today, but we show that they can be replicated by a static trading strategy in interest rate futures. Since our model takes term structures of both interest rates and volatility as given, fitting problems are avoided. We provide a simple specification of the model, that exhibits mean reversion in the level of volatility and requires only three parameters to be estimated. We suggest a discrete approximation of the continuous-time model in the spirit of Ho and Lee (1986), which can be used to price European as well as American term structure derivatives. This approximation is based on a trinomial scheme and takes zero-coupon bond prices and volatility futures prices as direct input. We illustrate the convergence of our approximation scheme by numerical examples.

Finally, we show that for a given parametrization of the initial volatility curve and volatility futures price curve the short rate exhibits mean reversion as well. This allows us to draw a comparison with a simple extended Vasicek model in order to investigate the effects of stochastic volatility in the short rate on interest rate options. The impact of stochastic volatility appears to be similar as the one observed in Heston's (1993) paper for stock options. This is interesting given the different nature of the processes followed both by the short rate and the volatility. It also gives some support to the widespread assertion that eventhough stochastic volatility might not be essential for the modelization of the term structure, it might have an important impact on interest rate derivatives.

1 Introduction

Although there has been a recent upsurge in the development of pricing models incorporating stochastic volatility in the equity derivatives literature, there have been relatively few attempts to incorporate stochastic volatility in term structure models. We see three main possible reasons for that: (i) a "natural" lag of the term structure literature, (ii) theoretical impediments, and (iii) a lack of empirical support.

First, the term structure literature tends to "naturally" (probably because of its complexity) lag behind the equity derivatives literature.

Also, one may say that the fixed income literature has not as well an established paradigm as the equity derivatives literature with the Black and Scholes framework.

Hence, there appear to be theoretical impediments to the implementation of stochastic volatility models for pricing term structure derivatives. Given those theoretical limitations the empirical support for modelling volatility as a second stochastic factor might have seemed too small to search any further.

In the present paper we present a model that, as we argue, lifts some of the theoretical impediments and, at least, provide some theoretical support for why stochastic volatility might be important for term structure derivatives pricing. We discuss recent empirical evidence in favor of our modelization at the end of the introduction.

When it comes to interest rate derivatives pricing, two approaches are still opposing each other. For one there is the "equilibrium based" approach, according to which one is to specify one or more factors that are jointly Markov and drive the term structure. Given the process for these factors under the real measure, $\mathcal{P}$, and some specification for the "market price of risk" of each of these factors, one can define the so-called risk neutral measure, $\mathcal{Q}$, under which all dicounted asset prices are martingales (Harrisson and Kreps (1979)). Notice that the "market price of risk" specification may either be arbitrarily imposed (Vasicek (1977)) or derived under some restrictive preference and economic environment assumption (Cox, Ingersoll and Ross (1985)).[1]

More recently a second strand of literature has developed that avoids the crux of the explicit modelization fo the "market price of risk."

The "arbitrage"-approach initiated by Ho and Lee (1986) and generalized by

[1] Traditional analysis (see Marsh (1994) for a survey) tends to distinguish equilibrium models (who endogenously derive market price of risks) from arbitrage models (who exogenously specify the market price of risk). The point is that in practice there is hardly any way to determine the actual functional form of the market price of risk and it is usually hard to build the relevant "general equilibrium" model.

Heath, Jarrow and Morton (1990) (HJM), takes the initial term structure as given and, using the no arbitrage condition, derives some restrictions on the drift and diffusion terms of the process of the forward rates under the risk neutral probability measure $\mathcal{Q}$.

The result is that given an initial term structure and a specification of the volatility of the forward rates or equivalently of the volatility of the bond prices, all derivatives prices can be computed without any specification of the "market price of risk" in a way consistent with the no arbitrage condition. Moreover, the bond prices computed using the model match the ones observed in the market. Of course, the major problem of this "arbitrage" approach is to find a reasonable volatility specification (in terms of number of factors driving the term structure as well as their functional form) and very often also to propose a computationally efficient implementation.

In this paper, we propose to extend the HJM methodology to account for the presence of a term structure of instantaneous volatilities of bond prices. The idea is to make use of the information present in traded contracts other than the bond prices themselves, to back out the implicit term structure of instantaneous volatilities of zero coupon bond prices and derive the dynamics of the instantaneous volatility of bond prices under the risk neutral measure in a way consistent with the no arbitage condition. We achieve this result in two steps. First, we show that if a futures contract (with futures price $V(t, \bar{t}, T)$) for delivery of the instantaneous volatility of bond prices $(\sigma_P(\bar{t}, T)^2)$ at time $\bar{t}$ were traded on the market at time t, then the dynamics under the $\mathcal{Q}$ measure of the term structure of both the instantaneous volatilities of bond prices and the bond prices themselves would be completely determined by the instantaneous volatility specification of this futures price $V(t, \bar{t}, T)$. This is similar to the HJM result with one more "layer," namely we assume the existence of traded bond prices and the existence of traded futures contracts on the volatility of bond prices with futures price $V(t, \bar{t}, T)$. Second, we show that even though such a futures contract does not exist, it can be perfectly replicated by trading in an interest rate futures contract that does exist (a future on yields in our example). Our first assumption, thus, does not appear so far fetched.

Dupire (1993) makes a similar hedging argument in order to extend the Black-Scholes stock option pricing model to incorporate stochastic volatility in a preference free fashion. There are major differences, though, between his approach and ours. We address those in the third section.

Our model is thus a two factor model of the term structure. The first factor affects directly the level of interest rates and the second factor affects the bond price volatility. It is related to the stochastic volatility option pricing literature (Hull and White (1987), Scott (1987), Scott (1993), Chesney and Scott (1989), Heston (1993)) and the equivalent literature in the fixed income field namely Longstaff and Schwartz (1992) and Fong and Vasicek (1991). All those models have in common that they model the volatility as a second factor. They differ in the way they determine the "market price of risk" of volatility. Both Longstaff and Schwartz, and Scott (1993) use an equilibrium approach as in Cox, Ingersoll and Ross (1985), whereas the others simply assume that the "market price of risk" of volatility is a constant (equal to zero i.e. volatility risk is not priced for Hull and White, Chesney and Scott, Heston or positive for Fong and Vasicek).

It is not very realistic to assume that volatility risk is not priced, especially for interest rate sensitive assets.[2] The equilibrium approach appears to be theoretically the soundest, but probably, for practical purposes, not very relevant, since it is hardly possible to put the real world into a general equilibrium framework. The arbitrage approach we propose appears to be a good alternative in that it finds a consistent way to price assets without having to make assumptions about the market price of risk of volatility. The latter is "incorporated" in the term structure of instantaneous volatility futures prices. Of course, the market price of risk of the factor affecting the level of interest rates is handled just as in the traditional HJM model. Hence, the specificity of our model is that the market prices of risk of both the factor affecting the level of the yield curve and the volatility need not be specified. They are "incorporated" in the two observed term structures of yields and futures prices.

As for all pricing models the tradeoff is always between realism and computational complexity. The preferable HJM approach to pricing fixed income derivatives has often been dismissed on behalf of its complexity and difficulty to implement as it often gives rise to non Markovian models for the short rate (i.e. non-recombining lattices) and is strongly dependent on the specification of the volatility of forward rates. Adding a "layer" to the HJM model could appear to be even more formidable. We thus propose a simple discretization scheme and illustrate the convergence as well as speed of the procedure on some numerical examples.

Whether the added complexity is of importance or not will be a matter of empirical tests. There appears though to be a growing body of empirical literature supporting the effort to account for stochastic volatility in interest rate movements when trying to price interest rate sensitive derivatives.

Dybvig (1990) for example conducts historical data analysis to look into the specification of the model factors. He studies the forward rates of maturities up to

[2]Indeed it seems difficult to argue that interest rate risk is diversifiable or that it is not correlated with per capita consumption rates; and these are the two main arguments usually advanced to justify the zero market price of risk assumption.

five years, and shows that conditional on the information at a given point in time, the next period's innovations in log discounts are almost perfectly correlated across maturities. This suggests that any second factor in a term structure model should not be additive in the level of interest rates, but should be related to the variance of interest rates, which may well have a small effect on bond pricing but probably has a significant one on bond option pricing.

Litterman, Scheinkman and Weiss (1991) and Litterman and Scheinkman study the relationship between the level of interet rates and the volatility of interest rates and show that, for hedging of fixed income securities with option like features, stochastic volatility is an essential feature.

Finally, Brenner, Harjes and Kroner (1993) investigate a functional form for the dynamics of the short-term interest rate volatility, which nests the term structure models studied by Chan, Karolyi, Longstaff and Sanders (1992), as well as some popular GARCH models. Their empirical results suggest that it is not enough to model the variation of interest rate volatility through the level of the rate, as most models do. But, on the other hand, they show that GARCH type models rely too heavily on serial correlation in variances and thus fail to adequately model the relationship between interest rate levels and volatility. Their conclusion is that while it is important to model the dependence of interest rate volatility on interest rate level, a second factor driving the volatility of interests rates is probably more appropriate.

Although we have not empirically tested our model for goodness of fit using real world data, it nevertheless provides some theoretical support as for why stochastic volatility might be important for bond option pricing. We show that for a particular parametrization of the initial volatility structure the simple specification of our model boils down to an extended Vasicek Model with stochastic volatility. It thus lends itself nicely to some interesting comparisons with the homoskedastic case.

The results of our comparisons are similar in nature to the ones obtained by Heston (1993). Yet they are surprising because of the different driving processes considered (in Heston the underlying, which is the stock price and not the short rate - interest rates are constant in his model - follows a geometric Brownian motion, whereas here the short rate is mean reverting). The results suggest that stochastic volatility is important when one considers pricing contingent claims on the term structure. Indeed, significant pricing differences arise although both models fit the same initial term structure and long term volatility of the short rate.

The rest of the paper is organized as follows. Section 2 derives the general model assuming the existence of a futures contract on the instantaneous volatility of bond prices. Section 3 presents the hedging argument that motivates the modelization of the term structure of bond price volatilities. We provide a simple specification of the model in section 4. In section 5, we propose a trinomial tree discretization scheme that is computationally efficient to implement our model. We discuss the impact of introducing stochastic volatility on bond option prices in section 6 and, finally, conclude in section 7.

2 The Model

We consider a continuous-time economy where a continuum of zero coupon bonds $\{P(t,T)\}_{0\leq t\leq T}$ is traded for all maturities. The term structure of interest rates usually defined at time t as the mapping $(T-t) \rightarrow R(t,T)$ can be derived from these bond prices as follows:

$$R(t,T) = -\frac{1}{T-t}\ln P(t,T) \tag{1}$$

We also define the instantaneous forward rate as:

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}\,, \tag{2}$$

and the spot interest rate $r(t) \equiv f(t,t)$ as:

$$r(t) = -\left.\frac{\partial \ln P(t,T)}{\partial T}\right|_{T=t}. \tag{3}$$

The process under the real probability measure $\mathcal{P}$ is

$$\frac{dP(t,T)}{P(t,T)} = \mu_P(t,T)dt + \sigma_P(t,T)dw_1(t) \tag{4}$$

where w_1 is a one dimensional Brownian motion under $\mathcal{P}$. Since, by definition, of the zero coupon bonds we have the following boundary condition:

$$P(T,T) = 1\,, \tag{5}$$

we impose:[3]

$$\sigma_P(T,T) = 0. \tag{6}$$

Moreover, we assume that there exists a continuum of futures contracts on the instantaneous variance of the zero coupon bonds. We define $V(t,\bar{t},T)$, $t \leq \bar{t} \leq T$, to be the time t continuously marked to market futures price for delivery of $\sigma_P(\bar{t},T)^2$ at time $\bar{t}$. From (6) we have to impose the restriction that $V(t,T,T) = 0$.

[3] Of course σ_P also needs to be smooth in the last variable. Absolute continuity is a sufficient condition here.

This assumption might seem a little restrictive since these contracts are not traded on any market, but, as we show in the next section, they can be perfectly replicated by trading in futures on yields. If either of the above exist we can describe the dynamics of the futures price under $\mathcal{P}$ as :

$$\frac{dV(t,\bar{t},T)}{V(t,\bar{t},T)} = \mu_V(t,\bar{t},T)dt + \sigma_V(t,\bar{t},T)'dw(t)\,, \quad 0 \le t \le \bar{t} < T \tag{7}$$

where $w(t)' = (w_1(t)\ w_2(t))$ and $w_1(t)$ and $w_2(t)$ are two independent $\mathcal{P}$ Brownian motions (consequently σ_V is a $(2,1)$ column vector, the prime denotes transpose). Absence of arbitrage implies the existence of an equivalent probability measure $\mathcal{Q}$ under which discounted prices are martingales.

Thus we have under $\mathcal{Q}$:

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt + \sigma_P(t,T)dw_1^{\mathcal{Q}}(t) \tag{8}$$

By definition of futures contracts, the futures price is a martingale under the equivalent $\mathcal{Q}$ measure (see for example Duffie (1988)). Since by construction the final payoff of the futures contract is:

$$V(\bar{t},\bar{t},T) = \sigma_P(\bar{t},T)^2 \tag{9}$$

we must have:

$$V(t,\bar{t},T) = \mathrm{E}_t^{\mathcal{Q}}\left[\sigma_P(\bar{t},T)^2\right]. \tag{10}$$

From (7), the dynamics of the futures price under $\mathcal{Q}$ can then be expressed as follows:

$$\frac{dV(t,\bar{t},T)}{V(t,\bar{t},T)} = \sigma_V(t,\bar{t},T)'dw^{\mathcal{Q}}(t) \tag{11}$$

In (8) and (10) above the superscript $\mathcal{Q}$ denotes that we have changed measure and that $w^{\mathcal{Q}}$ is a $\mathcal{Q}$ Brownian motion.

It is now well known that it is perfectly equivalent to apply the martingale restriction on forward prices, as in HJM, or directly on bond prices, as we do it here. Notice that the martingale restriction, when applied to bond prices, simply states that under the $\mathcal{Q}$ measure, their instantaneous return should be the riskless rate. This is much less picturesque than the restriction imposed on the drift of forward prices (see HJM), yet equivalent.[4] The martingale restriction applied to the futures price states that its drift under $\mathcal{Q}$ should be zero.

Notice that equations (8) and (10) extend the HJM result in the following way: Given an initial term structure of instantaneous volatilities $\{V(0,t,T)\}_{0\le t\le T\le \mathcal{T}}$, an

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[4]This equivalence is easily seen by applying Itô's lemma to $P(t,T) = e^{\int_t^T -f(t,s)ds}$ under $\mathcal{Q}$.

initial term structure of interest rates characterized by $\{P(0,T)\}_{0\leq T\leq \tau}$ and a specification of the vector process of the instantaneous volatility of the futures prices $\{\sigma_V(t,\bar{t},T)\}_{0\leq t\leq \bar{t}\leq T}$, the risk neutral dynamics of the yield curve is fully characterized and we can price any interest rate contingent claim in a manner consistent with the observed term structure as well as with the implied volatility structure of interest rates.

As a matter of fact, a straightforward application of Itô's lemma shows that the prices of zero coupon bonds satisfy the following:

$$P(t,T) = \frac{P(0,T)}{P(0,t)} e^{-\frac{1}{2}\int_0^t \left(\sigma_P(s,T)^2 - \sigma_P(s,t)^2\right) ds + \int_0^t \left(\sigma_P(s,T) - \sigma_P(s,t)\right) dw_1^Q(s)} \tag{12}$$

$$V(t,\bar{t},T) = V(0,\bar{t},T) e^{-\frac{1}{2}\int_0^t \|\sigma_V(s,\bar{t},T)\|^2 ds + \int_0^t \sigma_V(s,\bar{t},T)' dw^Q(s)} \tag{13}$$

$$\sigma_P(t,T)^2 = V(t,t,T) \tag{14}$$

Since bond prices can actually be observed in the market, they are an easily found input to the model. On the other hand, the identification of the initial term structure of instantaneous futures prices on volatility ($\{V(0,t,T)\}_{0\leq t\leq T\leq \tau}$) and the specification of the instantaneous volatility of the futures price $V(t,\bar{t},T)$ are crucial to the model and less straightforward. In the next section we show how the initial term structure of instantaneous volatilities can be recovered from the futures prices on yields.

3 The Hedging Argument

Our main result in the previous section relies on the assumption of the existence of traded futures contracts on the instantaneous volatility of bond prices. Obviously such contracts are not traded on todays financial markets. For our purpose, it is enough, though, that this contract be perfectly replicable through a trading strategy in existing contracts. Its fair value, under the condition of absence of arbitrage, would thus be perfectly established by the initial value of this trading strategy. And the model's prediction would hold, just as if the contracts were actually traded. And such a strategy is possible as we now proceed to show.

Let us, for example, consider futures contracts on the continuously compounded yield $R(t,T)$ as defined in (1). Eurodollar yield futures are traded at the CME.

Using the equivalent martingale measure, the futures price for such a contract is simply given by:

$$Y(t,\bar{t},T) = \mathrm{E}_t^Q\left[R(\bar{t},T)\right] \tag{15}$$

Integrating (8) we get:

$$\ln(P(t,T)) = \ln(P(0,T)) + \int_0^t r(u)du - \frac{1}{2}\int_0^t \sigma_P(u,T)^2 du + \int_0^t \sigma_P(u,T) dw_1^{\mathcal{Q}}(u) \tag{16}$$

Hence, taking the expectation of this equation under the $\mathcal{Q}$ measure we get the following relation:

$$Y(0,t,T)(T-t) = -\ln(P(0,T)) - \mathrm{E}_0^{\mathcal{Q}}\left[\int_0^t r(u)du\right] + \frac{1}{2}\mathrm{E}_0^{\mathcal{Q}}\left[\int_0^t \sigma_P(u,T)^2 du\right] \tag{17}$$

Finally differentiating this equation with respect to t yields:

$$\frac{\partial}{\partial t}\{(T-t)Y(0,t,T)\} = -\mathrm{E}_0^{\mathcal{Q}}[r(t)] + \frac{1}{2}\mathrm{E}_0^{\mathcal{Q}}\left[\sigma_P(t,T)^2\right] \tag{18}$$

and we see that the knowledge of the futures contract on continuously compounded yields is completely equivalent to the knowledge of the futures contract we used in the previous section $V(t,\bar{t},T)$. Indeed, the term structure of instantaneous volatilities of bond prices defined in terms of the futures prices $V(t,\bar{t},T)$ can be backed out from the observed futures prices on yields as follows:

$$V(0,t,T) = 2Y(0,t,t) - 2Y(0,t,T) + 2(T-t)\frac{\partial}{\partial t}\{Y(0,t,T)\} \tag{19}$$

Notice that

$$\lim_{T\to t} V(0,t,T) = 0$$

This is consistent with (6) since this limit converges to the bond price instantaneous volatility at maturity ($\lim_{T\to t} V(0,t,T) = \sigma_P(t,t)^2$). In order to replicate the $V(0,t,T)$ contract we would thus have to take a static hedging position consisting of : (i) a long position in 2 futures contracts maturing at time t on the instantaneous yield, (ii) a short position in $2(1+(T-t)/\Delta t)$ futures contracts maturing at time t on the yield between time t and T, and (iii) a long position in $2(T-t)/\Delta t$ futures contract maturing at time $t+\Delta t$ on the yield between time $t+\Delta t$ and T.

Recently, Dupire (1993) has proposed a similar, arbitrage free approach to pricing options on equity derivatives when the underlying stock price volatility is stochastic.

His method requires a forward contract for delivery of the instantaneous volatility of the stock price to be traded. Since such a contract is not traded in the market, Dupire suggests that a static trading strategy in another contract which he calls the "log-contract" (because it pays the logarithm of the stock price at maturity) allows to perfectly replicate the needed forward contract.[5] But since those "log-contracts" are not traded either, Dupire has to assume the existence of an infinite number of

[5]The "log-contract" has also been advocated by Neuberger (1994) as an efficient way to hedge volatility on currency derivatives.

traded options for all maturities and strike prices in order to be able to derive the state prices (equal to Breeden and Litzenberger's (1978) butterflies) which finally give him the price of the log contract. Although this argument is theoretically very appealing it might be hard to implement in practice since the estimation of the risk neutral density from observed option prices using non-parametric methods is not yet fully mastered (see Rubinstein (1994), Mayhew (1995) for a survey).

More specifically, Dupire's hedging argument can hardly be applied directly to term structure derivatives. As a matter of fact, with stochastic interest rates the forward prices Dupire uses to back out the term structure of volatilities become formidable to compute, since all calculations have to be done under the so-called "forward-neutral" measure (see El Karaoui and Rochet (1989) and Jamshidian (1991)).[6]

The choice we make to deal with futures prices on the instantaneous variance is essential to provide a simple static hedge of volatility. Computations are much simpler because, by definition of futures contracts, the futures price is a martingale under the equivalent risk neutral measure.

Of course as a downside, this assumption might appear to be more stringent. Indeed, futures prices are continuously marked to market contracts and hence imply the existence of an organized marked. But, as it turns out, the only contract required to replicate our $V(t, \bar{t}, T)$ contract is a futures on a yield, a market which already exists.

Hence, once properly adapted, the idea of Dupire might work even better for pricing fixed income derivatives than for equity derivatives.[7]

4 A simple specification of the model

Given the level of generality of the model presented in (12)-(14), it is hard to discuss it any further without some assumptions on the diffusion of the futures contract on bond price volatilities. In this section we specify the functional form of the instantaneous volatility of the futures contract $V(t, \bar{t}, T)$ in order to provide a practical implementation of the model.

[6] In the case of stochastic interest rates the hedging argument proposed by Dupire can be shown to involve the computation of the price of $E_t^Q\left[e^{-\int_t^T r(s)ds}\int_t^T r(s)ds\right]$ which can be cumbersome to compute especially if the interest rate model chosen has to fit the current term structure, i. e. has a time varying drift and/or diffusion.

[7] Obviously the relevance of our argument will also depend on issues such as: how liquid is the market for futures on yield, how many maturities are traded on that market... But notice, that we have provided only one example of a contract that could be used to hedge bond price volatility (although this contract was probably not created for that purpose). There may be others. And perhaps this will also help develop the market for such contracts?

By analogy with previous work by Jamshidian (1991) and Babbs (1990) we choose a state independent, separable specification for the diffusion part of $V(t,\bar{t},T)$:

$$\sigma_V(t,\bar{t},T) = \kappa(\bar{t})\upsilon(t) \tag{20}$$

where κ and υ are deterministic functions defined on $\Re$ and $\Re^2$ respectively. This form of diffusion usually leads to a so called "generalized mean reverting" process for the underlying variable.[8]

Indeed, for $0 \leq t \leq T$ we have:

$$\ln V(t,t,T) = \ln V(0,t,T) - \frac{1}{2}\kappa(t)^2 \int_0^t ||\upsilon(u)||^2 du + \kappa(t) \int_0^t \upsilon(u)' dw^{\mathcal{Q}}(u) \tag{21}$$

For a fixed $\hat{T} > 0$ we also have:

$$\ln V(t,t,t+\hat{T}) = \ln V(0,t,t+\hat{T}) - \frac{1}{2}\kappa(t)^2 \int_0^t ||\upsilon(u)||^2 du + \kappa(t) \int_0^t \upsilon(u)' dw^{\mathcal{Q}}(u) \tag{22}$$

Combining the two equations above we have:

$$\frac{V(t,t,T)}{V(0,t,T)} = \frac{V(t,t,t+\hat{T})}{V(0,t,t+\hat{T})} \equiv e^{X(t)} \tag{23}$$

Notice that under this specification the return on futures prices is independent of the maturity of the underlying volatility. That is, if we consider the futures price $V(t,\bar{t},T)$ as the expectation of the volatility at time $\bar{t}$ for a bond maturing at time T (which it is, but with the expectation taken under the $\mathcal{Q}$ measure), then we see that in this model the expected shift in volatility between two dates (in the above between time 0 and t) is the same accross all maturities of the underlying bonds.

Alternatively, we can use the definition of $\sigma_P(t,T)^2 = V(t,t,T)$ and write the following:

$$\sigma_P(t,T)^2 = V(0,t,T)e^{X(t)} \tag{24}$$

where

$$\begin{aligned} dX(t) &= \left\{ -\frac{1}{2}\kappa(t)^2 ||\upsilon(t)||^2 - \frac{1}{2}\kappa(t)\dot{\kappa}(t) \int_0^t ||\upsilon(u)||^2 du + \frac{\dot{\kappa}(t)}{\kappa(t)} X(t) \right\} dt \\ &\quad + \kappa(t)\upsilon(t)' dw^{\mathcal{Q}}(t) \end{aligned} \tag{25}$$

Setting

$$\kappa(\bar{t}) = e^{-\kappa \bar{t}}, \quad \upsilon(t) = e^{\kappa t} \begin{pmatrix} \upsilon_1 \\ \upsilon_2 \end{pmatrix} \tag{26}$$

[8]The analogy is not perfect, since, here, it is the natural logarithm of the futures price that can be shown to follow such a process. This is because we are specifying the diffusion of the *return* on the futures price and not of the futures price itself.

we get:

$$dX(t) = (\pi(t) - \kappa X(t))\,dt + v_1 dw_1^Q(t) + v_2 dw_2^Q(t) \quad (27)$$

$$\pi(t) = -\frac{1}{4}(v_1^2 + v_2^2)(1 + e^{-2\kappa t}) \quad (28)$$

We see that with this simple specification the volatility curve shifts parallely. Litterman, Scheinkman and Weiss (1991) provide some empirical support in their study for parallely shifting bond price volatility curves. Also, in this simple version of the model, the volatility level exhibits mean-reversion. This is indeed a desirable property since absence of mean-reversion of the volatility curves might have induced instability. One might be interested to investigate the form of the process of the instantaneous interest rate level in our model. We will consider that issue in section 6 where we draw a comparison with a generalized Vasicek model. But notice that in general it is very hard to get a simple, explicit representation of the short rate process. In particular, it is highly unlikely that for general term structures of volatility futures prices the short rate would exhibit constant mean reversion as well. We will analyze this point further in section 6.

In the following section we show how for the given specification of the instantaneous volatility $\sigma_V(t, \bar{t}, T)$ of the futures price for delivery of the instantaneous volatility of bond prices at $\bar{t}$, a simple numerical algorithm can be implemented.

5 Numerical Implementation: A Discrete Approximation Scheme

The implementation of the "arbitrage" based models that fit the current term structure usually involves Monte Carlo simulation techniques. These techniques are very appropriate for European type contingent claims, but they usually fail when the option has early exercise features or is path dependent. Moreover they are very unefficient when it comes to finding the appropriate hedge ratios.

Following the lead of Ho and Lee (1986) and the approach of He (1990) we propose and alternative method that can be used to price American type of derivatives as well as hedge ratios.

Nevertheless, our approach differs from both cited papers. Ho and Lee (1986) propose a binomial recombining tree for a very special Markovian type of representation of the short rate that allows to fit the current term structure. Clearly this is not possible in our model since we have to fit two term structures at every node in the tree. Our model will in general not allow a Markovian representation of the short term rate and thus of bond prices. The tree will thus be non-recombining. Moreover, since we are handling two sources of noise, our tree has to be at least trinomial (see Ingersoll (1987)). We thus choose to discretize the model in a trinomial non-recombining discretization scheme similar to He (1990). Contrary to He, though, we do not discretize the stochastic differential equation under the $\mathcal{P}$ measure but directly under the risk neutral measure. Moreover, our dicretization scheme can be applied to the case of a continuum of asset prices that do not necessarily have a common finite dimensional Markovian representation (i.e. we can fit any initial term structure of yields and volatilities within our framework). Of course, the price of the latter extension is that, unlike He, we do not have a rigorous proof of the convergence of our discretization scheme. Nevertheless, we give a sketch of proof that suggests that the convergence is likely to occur, and rely on the numerical implementation for further evidence.[9]

The setup is the following:

At each time step t_k, $\forall k \in \{1, \cdots, N\}$ the current state is a k–sequence $\{s_{t_0}, s_{t_1}, \cdots, s_{t_{k-1}}\}$ with $s_{t_j} \in \{1, 2, 3\}\, \forall j \in \{1, 2, \cdots, k-1\}$. At each point $(t_k, \{s_{t_j}\}_{j \in \{0, \cdots k\}})$ the risk-neutral probability of state s occuring over the next time interval is given by $\theta^s(t_k)$ with the natural restriction: $\sum_{s=1}^{3} \theta^s(t_k) = 1$. We will drop the t_k argument since in our implementation we choose $\theta^s = 1/3\ \forall s$.

We assume that over each time step bond and futures prices are perturbed according to the following scheme for $s \in \{1, 2, 3\}$:

$$
\begin{aligned}
P(t_{k+1}, t_n) &= \frac{P(t_k, t_n)}{P(t_k, t_{k+1})} h^s(t_k, t_n), \quad 0 \le k < n \le N && (29)\\
V(t_{k+1}, t_l, t_n) &= V(t_k, t_l, t_n) g^s(t_k, t_l, t_n), \quad 0 \le k \le l < n \le N && (30)
\end{aligned}
$$

The functions $h(\cdot)$ and $g(\cdot)$ are possibly state dependent and have to fulfill the following arbitrage restrictions:

$$
\begin{aligned}
\sum_s h^s(t_k, t_n)\theta^s &= 1 && (31)\\
h^s(t_k, t_{k+1}) &= 1 && (32)\\
h^s(t_k, t_n) &> 0 && (33)\\
\sum_s g^s(t_k, t_l, t_n)\theta^s &= 1 && (34)\\
g^s(t_k, t_l, t_n) &> 0 && (35)
\end{aligned}
$$

Notice that these conditions insure that $P(t_k, t_n) = P(t_k, t_{k+1})\mathrm{E}^{\mathcal{Q}}[P(t_{k+1}, t_n)]$, which is the martingale restriction for discounted prices under the risk neutral measure. Notice also that the function h can be interpreted as a Radon Nykodim derivative. As a matter of fact it changes the measure from $\mathcal{Q}$ to the so called forward neutral

[9] We are not aware of existing mathematical results similar to the Ethier and Kurtz (1986) theorem that would allow to extend He's proof to the non-Markovian framework.

measure (see Jamshidian (1991) and El Karaoui and Rochet (1989)). A similar interpretation holds for g.

We then define two three dimensional vectors (ε_1 and ε_2) of constants $\{\varepsilon_{1,s}\}_{s\in\{1,2,3\}}$ and $\{\varepsilon_{2,s}\}_{s\in\{1,2,3\}}$ which represent the possible realisations of the two source of risk at each node in the tree. Hence they have to satisfy the zero mean and unit variance restrictions of Brownian motion which translate into the following restrictions on their "discretized" version:

$$\sum_s \theta^s \varepsilon_{i,s} = 0 \tag{36}$$

$$\sum_s \theta^s \varepsilon_{i,s}^2 = 1 \tag{37}$$

$$\sum_s \theta^s \varepsilon_{1,s}\varepsilon_{2,s} = 0 \tag{38}$$

for $i = 1, 2$.[10] Let us then define:

$$h^s(t_k, t_n) = \frac{e^{\sigma_P(t_k,t_{n-1})\varepsilon_{1,s}\sqrt{t_{k+1}-t_k}}}{\sum_{s'=1}^{3} \theta^{s'} e^{\sigma_P(t_k,t_{n-1})\varepsilon_{1,s'}\sqrt{t_{k+1}-t_k}}} \tag{39}$$

$$g^s(t_k, t_l, t_n) = \frac{e^{\sigma_V(t_k,t_{l-1},t_{n-1})'\varepsilon_{\cdot,s}\sqrt{t_{k+1}-t_k}}}{\sum_{s'=1}^{3} \theta^{s'} e^{\sigma_V(t_k,t_{l-1},t_{n-1})'\varepsilon_{\cdot,s'}\sqrt{t_{k+1}-t_k}}} \tag{40}$$

In the above $\varepsilon_{\cdot,s}$ denotes the s^{th} line of the horizontal concatenation of the two vectors ε_1 and ε_2.

Notice that the h and g functions are chosen so that (31)-(35) hold, i.e. so that the Martingale properties of the prices are satisfied and so that the bonds mature at par. One notational particularity might deserve some attention. Because we choose to translate (6) into its discretized version:

$$\sigma_P(t_{n-1}, t_{n-1}) = 0, \tag{41}$$

it becomes necessary to use $\sigma_P(t_k, t_{n-1})$ in the perturbation function for the bond price maturing at t_n. This is purely to insure that bond prices mature at par.[11]

Intuition for the functional form of h given above can easily be obtained by noting that using a discretized version of equations (12) one can (formally) write (for small $t_{k+1} - t_k$):

$$P(t_{k+1}, t_n) = \frac{P(t_k, t_n)}{P(t_k, t_{k+1})} e^{-\frac{1}{2}\left(\sigma_P(t_k,t_{n-1})^2 - \sigma_P(t_k,t_k)^2\right)(t_{k+1}-t_k) + (\sigma_P(t_k,t_{n-1}) - \sigma_P(t_k,t_k))\,(w_1^Q(t_{k+1}) - w_1^Q(t_k))} \tag{42}$$

[10]For our implementation we have chosen: $\varepsilon'_{1,\cdot} = \{0.0, \sqrt{1.5}, -\sqrt{1.5}\}$ and $\varepsilon'_{2,\cdot} = \{\sqrt{2.0}, -\sqrt{0.5}, -\sqrt{0.5}\}$. For different possible choices of the perturbation matrix see He (1990).

[11]Alternatively this notation can be seen as a translation of the fact that the discretized process is right-continuous.

But, using the condition (41) above and noting that:

$$e^{\frac{1}{2}\sigma_P(t_k,t_{n-1})^2(t_{k+1}-t_k)} = \mathrm{E}^{\mathcal{Q}}[e^{\sigma_P(t_k,t_{n-1})(w_1^{\mathcal{Q}}(t_{k+1})-w_1^{\mathcal{Q}}(t_k))}] = \sum_{s'=1}^{3} \theta^{s'}(t_k) e^{\sigma_P(t_k,t_{n-1})\varepsilon_{1,s'}\sqrt{t_{k+1}-t_k}},$$

we find the functional form for h in (29). A similar *heuristic* derivation can be given for g.

With $\sigma_P(t_k, t_{n-1})^2 = V(t_k, t_k, t_{n-1})$, the price structure of our non-recombining trinomial tree can be generated.

We now turn to the issue of convergence of this discretization scheme. As we already mentioned, we are not aware of mathematical results that would help prove the convergence. But looking at the moments of the bonds and futures prices shows that the model has desirable features. It also gives more insight as to why we chose this discretization scheme.

Taking the log of equation (29) and using the above we get:

$$\ln P(t_{k+1}, t_n) - \ln P(t_k, t_n) = -\ln P(t_k, t_{k+1}) + \ln h^s(t_k, t_n) =$$
$$-\ln P(t_k, t_{k+1}) - \ln\left(\sum_{s'=1}^{3} \theta^{s'} e^{\sigma_P(t_k,t_{n-1})\varepsilon_{1,s'}\sqrt{t_{k+1}-t_k}}\right) + \sigma_P(t_k, t_{n-1})\varepsilon_{1,s}\sqrt{t_{k+1}-t_k}$$

Recalling $R(t,T)$ from equation (1) and Taylor expanding the exponential and the logarithm around respectively 1 and 0 gives the following stochastic dynamics for our discretized model:[12]

$$\ln P(t_{k+1}, t_n) - \ln P(t_k, t_n) =$$
$$\left[R(t_k, t_{k+1}) - \frac{1}{2}\sigma_P(t_k, t_{n-1})^2\right](t_{k+1} - t_k) + \sigma_P(t_k, t_{n-1})\varepsilon_{1,s}\sqrt{t_{k+1}-t_k} + o\left((t_{k+1}-t_k)^{\frac{3}{2}}\right)$$

Applying the same techniques to the futures prices we get their dynamics:

$$\ln V(t_{k+1}, t_l, t_n) - \ln V(t_k, t_l, t_n) =$$
$$-\frac{1}{2}||\sigma_V(t_k, t_{l-1}, t_{n-1})||^2(t_{k+1} - t_k) + \sigma_V(t_k, t_{l-1}, t_{n-1})'\varepsilon_{\cdot,s}\sqrt{t_{k+1}-t_k} + o\left((t_{k+1}-t_k)^{\frac{3}{2}}\right)$$

Neglecting the terms of order $o\left((t_{k+1}-t_k)^{\frac{3}{2}}\right)$ and above we get the following moments:

$$\mathrm{E}_{t_k}^{\mathcal{Q}}[\Delta \ln P(t_k, t_n)] = \left(R(t_k, t_{k+1}) - \frac{1}{2}\sigma_P(t_k, t_{n-1})^2\right)(t_{k+1} - t_k) \quad (43)$$

$$\mathrm{E}_{t_k}^{\mathcal{Q}}[\Delta \ln V(t_k, t_l, t_n)] = -\frac{1}{2}||\sigma_V(t_k, t_{l-1}, t_{n-1})||^2(t_{k+1} - t_k) \quad (44)$$

$$\mathrm{Var}_{t_k}^{\mathcal{Q}}[\Delta \ln P(t_k, t_n)] = \sigma_P(t_k, t_{n-1})^2(t_{k+1} - t_k) \quad (45)$$

$$\mathrm{Var}_{t_k}^{\mathcal{Q}}[\Delta \ln V(t_k, t_l, t_n)] = \sigma_V(t_k, t_{l-1}, t_{n-1})^2(t_{k+1} - t_k) \quad (46)$$

$$\mathrm{Cov}_{t_k}^{\mathcal{Q}}[\Delta \ln P(t_k, t_n), \Delta \ln V(t_k, t_l, t_n)] = \sigma_P(t_k, t_{n-1})[\sigma_V(t_k, t_{l-1}, t_{n-1}]_{1,1}(t_{k+1} - t_k) \quad (47)$$

[12]Note that the discretization error appears only in the drift term and not in the volatility.

Noting from (1) and (3) that $r(t) = \lim_{T \to t} R(t,T)$, we see that the above is analogous to the continuous time model. This provides a starting point for showing the convergence in $\mathcal{Q}$ distribution of our discretized model to the continuous time limit when the time interval $t_{k+1} - t_k$ tends to zero. Below we give numerical examples that illustrate the convergence of the discrete approximation of the continuous-time model.

Note that for hedging purposes we need 3 (1 plus the number of sources of uncertainty) assets that are not perfectly correlated with each other. In our case all bond prices are perfectly instantaneously correlated. Thus we cannot hedge volatility risk using only bonds. The futures contract on interest rate yields - introduced in section 2, for the hedging argument - would be the ideal tool to hedge bond price volatility risk. If, for practical reasons such a contract were not available, we would suggest to define a "reference" option, the price of which could easily be obtained using our discretization scheme. Hedge ratios could then be computed using two bonds of different maturities and this "reference" option.

Table 1 gives some numerical results that illustrate the convergence of our discretization scheme. We consider American and European put options to sell 100 of the underlying 2 year zero-coupon bond. Maturity of the options is 1 year. The strikes were set equal to the initial forward prices. We used the simple specification of the model with the parameters set to $\kappa = 2.0$, $v_1 = -0.5$, and $v_2 = 0.5$. The yield curve was upward sloping with a short rate of 5% and a long rate of 7%. The parametrization of the initial yield curve was:

$$R(0,T) = 0.07 - 0.02e^{-2.0T} \tag{48}$$

We chose the initial volatility curve so that the forward rate volatility was 7% in the short end and 2% in the long end. Explicitly:

$$\frac{\partial \sigma_P(0,T)}{\partial T} = 0.02 + 0.05e^{-T} \tag{49}$$

The volatility futures price term structures were chosen to be equal to the initial volatility curve (i.e. $V(0,t,T) = \left(0.02 * (T-t) - 0.05(1 - e^{-(T-t)})\right)^2$). Of course, this was done for illustrative purposes only. Any other (parametric as well as non parametric) form of initial yield, volatility, and volatility futures price curves can be handled just as easily within our framework.

N	Discrete Approximation European	American	CPU time
2	1.88	2.05	<0.1s
3	1.57	2.01	<0.1s
4	1.57	2.16	<0.1s
5	1.56	2.10	<0.1s
6	1.49	2.09	0.2s
7	1.49	2.13	0.3s
8	1.49	2.13	0.6s
9	1.46	2.12	1.5s
10	1.46	2.14	4.5s
11	1.46	2.13	14.0s
12	1.44	2.12	42s
13	1.44	2.14	2m01s
14	1.44	2.14	6m01s
15	1.43	2.13	18m
16	1.43	2.14	54m
17	1.43	2.14	2h42m
M.C.	1.43	-	-

Table 1 : **European and American put prices and CPU-time as a function of time steps in the discrete approximation. The programming was done in C and the hardware used was a HP-9000 Unix system.**

We note that after 10 steps or 4.5 seconds of CPU time prices are within 1.5% relative error compared to the values of the longest tree. The European prices can also be compared to the value obtained from Monte-Carlo simulations denoted "M.C.." It is our feeling that the speed of the program could be increased by optimizing the program, for example by minimizing the call of logarithm and exponential functions. Another way of obtaining higher accuracy is by use of averaging and Richardson extrapolation.

A non-recombining tree structure will converge much faster than an otherwise identical recombining structure because the number of nodes in the tree increases exponentially (3^N in our case) versus polynomially for a recombining structure. This gives a finer mesh of points and the distribution of the underlying prices will, therefore, be approximated to a higher degree of accuracy. The drawback of the tree structure is of course the same as its advantage: the exponential increase of computer time required for evaluating a tree as the number of time steps increases. This limits the number of time steps in the tree that can be handled within reasonable time. One should therefore expect the tree to undervalue American options whereas European option prices should not be affected. The use of Richardson extrapolation could, to some extent, offset this effect on American option prices.

6 Effects of stochastic volatility on bond option prices: Comparison with the generalized Vasicek model.

It has been argued that stochastic volatility might be an important determinant of bond option prices. In this section we try to investigate the effects of stochastic volatility on bond option prices implied by our model. Recently Heston (1993) has shown that for options on stocks, stochastic volatility had very subtle effects. He compares a stochastic volatility option price with a regular Black Scholes model and shows that: (i) if the correlation between the stock price and the volatility is null an increase in the volatility of the volatility increases the kurtosis of the underlying stock price distribution and, hence, increases in and out of the money option prices, whereas it decreases at of the money option prices (relatively to the Black Scholes model).[13] (ii) If the correlation between stock price and volatility is positive (negative), this increases (decreases) the kurtosis - that is it increases the right (left) tail of the stock price distribution - and, consequently, it increases (decreases) out of the money options and decreases (increases) in of the money options.

We follow Heston and try to compare our model to a well known benchmark of one factor models used for pricing interest rate derivatives: the generalized Vasicek model (see Hull and White (1990)).

This model assumes a generalized mean reverting process for the short rate given by the following equation under Q:

$$dr(t) = (\phi(t) - a_{\text{HW}} r(t))\,dt + \sigma_{\text{HW}} dz(t) \tag{50}$$

where z_t is a one dimensional brownian motion.[14]

As in Heston, though, the two models will be comparable only if they have approximately the same structure, i.e. differ only in the stochastic volatility feature.

We thus investigate under what conditions the short rate process in our model (as specified in section 3), will exhibit similar mean reversion as the extended Vasicek process.

Integrating (2) we get:

$$f(t,T) = f(0,T) + \int_0^T \sigma_P(s,T)\frac{\partial \sigma_P(s,T)}{\partial T} ds - \int_0^T \frac{\partial \sigma_P(s,T)}{\partial T} dw_1^Q(s) \tag{51}$$

[13]Heston uses the long term mean of the volatility in the Black Scholes formula to keep the comparison relevant.

[14]This is not the most general presentation, since a_{HW} and σ_{HW} could be made time dependent as well (see Hull and White (1990)). But, for simplicity of exposition we shall specialize to the case where they are true constants.

Then, using (3), we get:

$$r_t = f(0,t) + \int_0^t \sigma_P(s,t) \left.\frac{\partial \sigma_P(s,T)}{\partial T}\right|_{T=t} ds - \int_0^t \left.\frac{\partial \sigma_P(s,T)}{\partial T}\right|_{T=t} dw_1^Q(s) \tag{52}$$

Taking the differential of the latter we find the process of the short rate:

$$dr_t = \left.\frac{\partial f(t,T)}{\partial T}\right|_{T=t} dt - \left.\frac{\partial \sigma_P(t,T)}{\partial T}\right|_{T=t} dw_1^Q(t) \tag{53}$$

with:

$$\begin{aligned}\left.\frac{\partial f(t,T)}{\partial T}\right|_{T=t} &= \left.\frac{\partial f(0,T)}{\partial T}\right|_{T=t} + \int_0^t \left.\left(\frac{\partial \sigma_P(s,T)}{\partial T}\right)^2\right|_{T=t} ds \\ &+ \int_0^t \sigma_P(s,t) \left.\frac{\partial^2 \sigma_P(s,T)}{\partial T^2}\right|_{T=t} ds - \int_0^t \left.\frac{\partial^2 \sigma_P(t,T)}{\partial T^2}\right|_{T=t} dw_1^Q(t)\end{aligned} \tag{54}$$

It is clear, that, even with our current, simple specification, the short rate process will not, in general, follow such a simple mean reverting process. But, comparing (54) to (52), it is easy to see that a sufficient condition for the short rate process to exhibit mean reversion is:

$$\frac{\partial^2 \sigma_P(t,T)}{\partial T^2} = -a * \frac{\partial \sigma_P(t,T)}{\partial T}, \tag{55}$$

with a being a constant. Hence, substituting (24) into (55), we find a sufficient condition for the short term rate to exhibit mean reversion, namely: when the initial bond price and futures prices volatility structure verify the following differential equation:

$$\frac{\partial V(0,t,T)/\partial T}{2V(0,t,T)} = \frac{\partial^2 V(0,t,T)/\partial T^2}{\partial V(0,t,T)/\partial T} - a \tag{56}$$

In the special case where the initial volatility structure verifies the latter sufficient condition, our model can be conveniently summarized by the following set of equations:

$$\begin{aligned} dr(t) &= (\phi_r(t) - ar(t))\, dt - \sigma_r^f(0,t)\sigma_r(t) dw_1^Q(t) && (57) \\ \phi_r(t) &= \left.\frac{\partial f(0,T)}{\partial T}\right|_{T=t} + \int_0^t \sigma_r^f(0,s,t)^2 \sigma_r(s)^2 ds + af(0,t) && (58) \\ d\ln\sigma_r(t) &= \left(\frac{\pi(t)}{2} - \kappa \ln \sigma_r(t)\right) dt + \sigma_\sigma dw_3^Q(t) && (59) \end{aligned}$$

where we have changed slightly some notations to make the relationship between the two models more apparent. The changes are given by the following:

$$\sigma_r^f(0,t,T) = \frac{\partial V(0,t,T)^{1/2}}{\partial T} \tag{60}$$

$$\sigma_r^f(0,t) \equiv \sigma_r^f(0,t,t) = \left.\frac{\partial V(0,t,T)^{1/2}}{\partial T}\right|_{T=t} \tag{61}$$

$$\sigma_r(t) = e^{X(t)/2} \tag{62}$$

$$\sigma_\sigma = \frac{\sqrt{v_1^2+v_2^2}}{2} \tag{63}$$

$$w_3^Q(t) = \frac{v_1 w_1^Q(t) + v_2 w_2^Q(t))}{\sqrt{v_1^2+v_2^2}} \tag{64}$$

All of these variables have straightforward interpretations. But notice that $w_3^Q(t)$ is a standard Brownian motion that is correlated with w_1^Q. The instantaneous correlation coefficient is:

$$\rho dt \equiv dw_1^Q(t) dw_3^Q(t) = \frac{v_1}{\sqrt{v_1^2+v_2^2}} \tag{65}$$

In order to be able to perform the comparison it remains to find a suitable parametrization of the inital volatility curve. One that fits the purpose nicely is the following:

$$V(0,t,T) = \beta^2 \frac{(1-e^{-a(T-t)})^2}{a^2} \tag{66}$$

In that case $\sigma_r^f(0,t,T) = \beta e^{-a(T-t)}$ and $\sigma_r^f(0,t) = \beta$.

With this specification of the initial volatility structure, the two models are now readily comparable.

We choose the same initial yield curve for both models, namely: $R(0,T) = 0.07 - 0.02e^{-2.0T}$. The instantaneous volatility for the HW model is chosen so as to match the long term mean of the volatility of the short rate, i.e. $\sigma_{\rm HW} = \beta e^{\pi(\infty)/(2*\kappa)}$ (with $\pi(\infty) = -(v_1^2+v_2^2)/4$).

The correlation between the short rate and the volatility can be changed by varying v_1 between $-\sqrt{v_1^2+v_2^2}$ and $+\sqrt{v_1^2+v_2^2}$. The following graphs 1, 2, 3 show the difference between the ACS model and the HW call option prices with a one year maturity on one year zero coupon bonds for various strike prices. The inital spot price is taken to be 86.99 \$ (this corresponds to 100 zero coupon bonds with a two year maturity). The inital forward price ($P(0,2)/P(0,1)$) is 93.06\$. The mean reversion parameter (κ) of the volatility of the short rate is set to 0.5 throughout the comparisons. The mean reversion parameter of the short rate (a) was set to 0.2. Table 2 below summarizes the various parameter choices made for the comparison.

Parameter choices	
$R(0,T) = 0.07 - 0.02e^{-2.0T}$	
$P^f(0,1,2) = P(0,2)/P(0,1) = 93.06$	
$V(0,t,T) = \beta^2 \frac{(1-e^{-a(T-t)})^2}{a^2}$	
$a = 0.2$	
$\beta = 0.05$	
$\kappa = 0.5$	
$\sigma_\sigma = 0.3$	$\sigma_{HW} = 0.0489$
$\sigma_\sigma = 0.6$	$\sigma_{HW} = 0.0457$

Table 2

Graph 1 shows the difference between the ACS model and the HW model prices for zero correlation and various σ_σ, (namely: $\sigma_\sigma = 0.3$, and $\sigma_\sigma = 0.6$).[15]

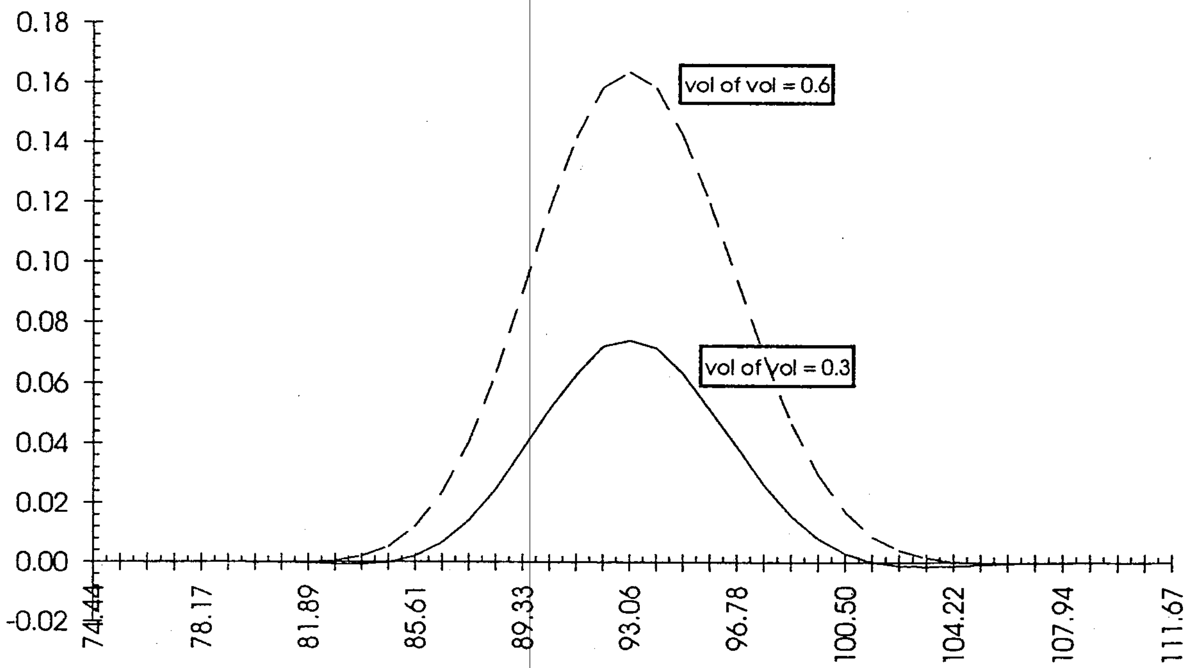

Figure 1: Difference ACS minus HW call option price for different σ_σ and various strike prices. The correlation between volatility and short rate (ρ) is set to zero.

As a result we see that an increase in the volatility of the volatility leads to an increase in prices of at the money call options. There is a slight decrease in far out and in the money option prices, but it is hardly perceptible on the graph. It is much less pronounced then in Heston's work, but, nevertheless, present.

[15]The transformation from σ_σ to values for v_1 and v_2 is straightforward given equations (63)and (65).

Notice that for $\sigma_\sigma = 0$, both models should give the same prices. The difference observed gives thus another idea of the convergence of our discretization scheme. Simulations were run with twelve time steps and give rise to a difference with respect to the HW model of *at most* 0.04 for option prices that are worth around 1.5$ (i.e. a relative error of around 2% in the worst case - the average relative error was 0.8% with a standard deviation of 2.2%). Of course these result can be improved by increasing the number of time steps at the expense of computation time as shown in our table 1 (previous section).

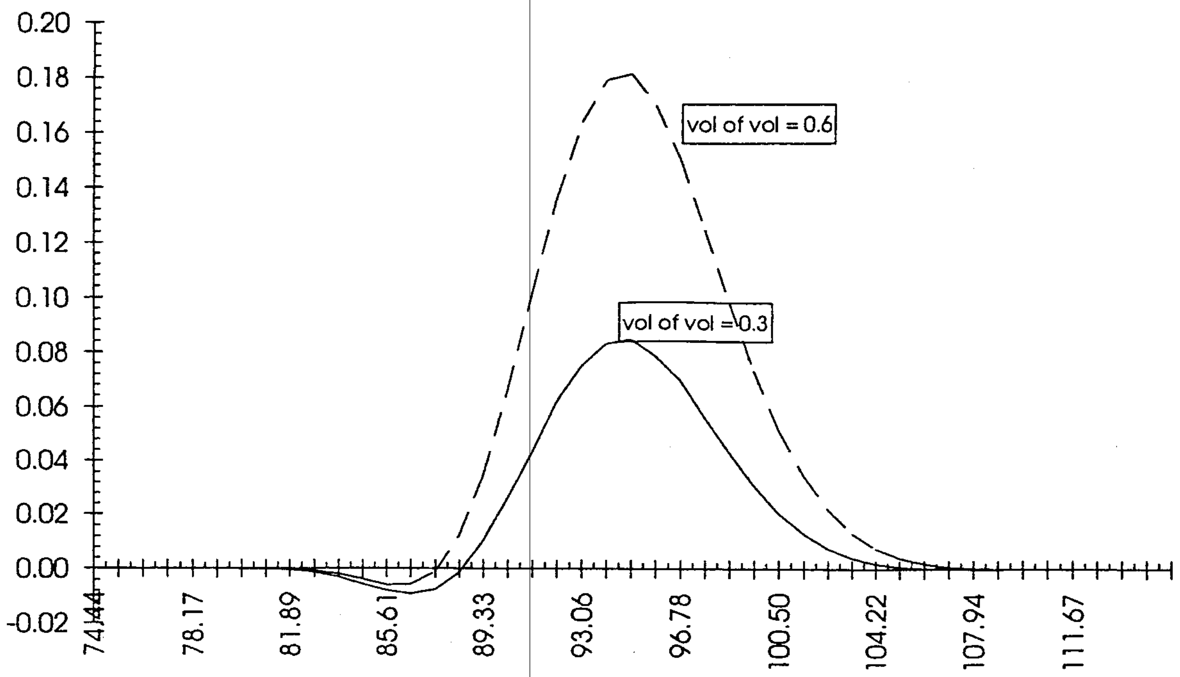

Figure 2: Difference ACS minus HW call option price for different σ_σ and various strike prices. The correlation between volatility and short rate (ρ) is set to +0.5.

Graphs 2 and 3 show the difference between the ACS and HW model call option prices when the correlation is respectively negative ($\rho = -0.5$) and positive ($\rho = 0.5$). The results show that for negative correlation the stochastic volatility model decreases out of the money option prices, and increases in the money option prices relative to the homskedastic model. For positive correlation the results are reversed. Moreover, for both cases, the difference increases as the volatility of the volatility increases. These results are similar in nature to Heston's (1993), which is rather interesting given the different nature of the two processes considered. The interest rate in our comparison is mean reverting as opposed to the process for the stock price

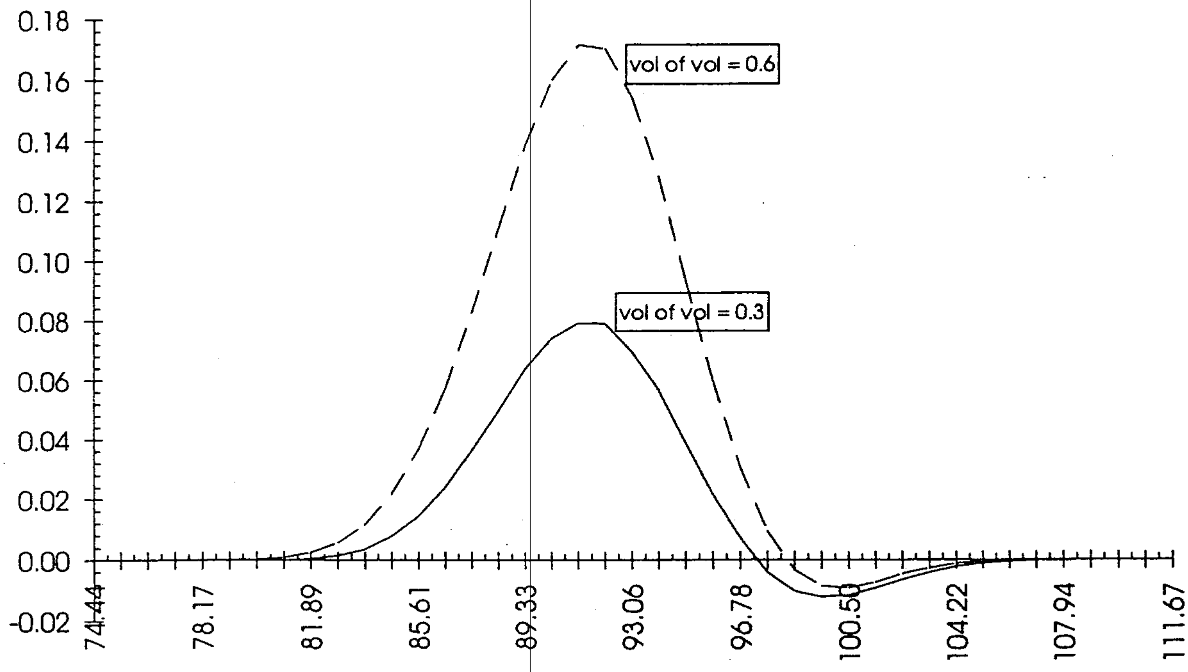

Figure 3: Difference ACS minus HW call option price for different σ_σ and various strike prices. The correlation between volatility and short rate (ρ) is set to -0.5.

which follows a geometric brownian motion in Heston's work (Heston sets interest rates to zero in his comparison).

Notice also that the intuition proposed by Heston applies to our framework when the proper correlation is considered, namely the correlation between bond prices and volatility.

Positive correlation ($\rho = dw_1^Q dw_3^Q > 0$) implies a negative correlation between the short rate and its instaneous volatility (as can be seen from equations (57) and (59)). Hence it implies an increase in the left tail of the distribution of the interest rate. One thus expects to see "more" low interest rates which in turn imply "more" high bond prices and hence an increase in out of the money option prices relative to in the money option prices. This is exactly what we observe in Figure 2.

Of course, for negative correlation between bond prices and instaneous volatility of the short rate ($\rho < 0$, see Figure 3), the results are reversed: the increase in the right tail of the distribution of interest rates (due to positive correlation of the short rate with the volatility) increases in of the money option prices with respect to out of the money option prices. It is interesting to observe that even with a mean reverting short rate as well as a mean reverting volatility these features remain present.

These results also show that the choice of the volatility process can have a big impact on bond option prices, although, by construction, the inital term structure of yields as well as long term volatilities are the same in both models. Moreover, they help to illustrate the convergence of the discretization scheme we proposed in the previous section.

7 Conclusion

This paper has presented a new approach to the pricing of term structure contingent claims in the presence of stochastic volatility. Unlike the previous models we do not need to explicitly model the market price of volatility risk. Our paper extends the Heath Jarrow and Morton methodology in that it takes as input the current term structure of both bond prices and instantaneous volatility of bond prices as well as the specification of the diffusion part of the volatility[16] and derives the risk neutral dynamics of interest rates and instantaneous volatility of interest rates. We can thus price any interest rate contingent claim in an arbitrage free way.

We propose a trinomial tree as a discrete approximation scheme to our model that should be usefull to price and hedge any European as well as American type of

[16]Notice that it is actually the term structure of futures prices on the underlying volatility $V(t, \bar{t}, T)$ in the paper that we take as input, but it is really a proxy for the volatility itself, that cannot be observed explicitly.

contingent claims on the term structure. We also show that for a simple parametrization of the initial volatility structure, and futures prices of volatility, the short rate, in our model, exhibits mean reversion just as in the extended Vasicek model. Thus, comparisons with the homoskedastic case are possible. These lead to similar results as in Heston (1993) for stock options. It demonstrates that the often claimed importance of the volatility specification for bond option prices is a valid concern and, also, is a further indication of the convergence of our discretization scheme.

Future work should be directed towards implementing this model and empirically testing its validity. Several methods have been proposed to estimate stochastic volatility models that could be applied to our model (see Taylor (1991) and Anderson (1991) for a survey and comparison).

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Pricing by Arbitrage in an International Economy[1]

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Abstract

This paper investigates the concept of arbitrage pricing in a complete multicurrency economy. It turns out that the exchange rate can be described by the pricing functionals of the different currencies, and thereby that the exchange rate dynamics to a large extend can be inferred from asset price dynamics. With this insight the pricing of currency derivatives can be simplified by use of the implicit change of martingale measure induced by the exchange rates. In the last section of the paper an exchange rate model, that incorporates stochastic term structure dynamics, is presented.

Introduction

Dynamics of exchange rates and asset prices of different currencies are highly interrelated. Investors buy currency to hold foreign assets, not to hold the currency itself. Due to this, restrictions are put on the stochastic evolution of the exchange rates. In this paper we find these restrictions and show that the stochastic evolution of the exchange rates to a large extend can be deduced from the dynamics of the asset prices. The key insight here is to consider the currencies for what they are, namely different numeraires for different investors. An exchange rate is thereby not only a risky asset paying a dividend stream equal to the foreign interest rate but also a tool for change of numeraire and pricing functional. We show that the exchange rates are explicitly given from the pricing functionals of the different currencies, and by this that the exchange rates, the term structures and the change of martingale measures are highly interrelated.

Our framework is an extension of the one developed in Amin & Jarrow (1991), in the sense that we require completeness of the global market and absence of arbitrage from every investor's perspective. This is used to deduce the explicit evolution of the exchange rates.

The first section presents the framework considered and the assumptions on which the results presented later are based. The second section presents the exact stochastic evolution of the exchange rate. The third section describes results for futures and option pricing that are immediate consequences of the exchange rate result. We also obtain a result for the pricing of futures on the exchange rate, that is valid for a broad class of term structure models. The last section presents a model that incorporates domestic and foreign term structure dynamics. An argument based on equilibrium considerations justifies that the stochastic evolution of the exchange rate can be entirely described by the term structure dynamics. This and our futures price result make the model presented easy to adjust to initial data.

The Framework

Consider an international economy with $I+1$ currencies each indexed by $i \in \{0, \dots, I\}$. Currency number 0 will be the basis or domestic currency in which all exchange rates are quoted. In each currency a set of assets is marketed. These sets are denoted $(H_i)_{i=0,\dots,I}$, and the set of all assets in the global economy is denoted $H = \cup_i H_i$. The uncertainty of the prices of the different assets and the exchange rates will be represented by the following standard continuous time and continuous state set-up.

Let $(\Omega, \mathcal{F}, Q)$ be a probability space with a complete right-continuous filtration $(\mathcal{F}_t)_{t\in[0,\tau]}$ generated by an M-dimensional standard Brownian motion $(w(t))_{t\in[0,\tau]}$ on $(\Omega, \mathcal{F}, Q)$. We will denote the Lebesque measure λ. When nothing else is stated, t will indicate a time point in the interval $[0, \tau]$, and i will refer to any of the currencies.
Suppose that there exists a locally risk-free interest rate, r_i, in each currency. Suppose that r_i is progressively measurable and $Q - a.s.$ integrable in the time domain, so we can define the bank account in the i'th currency as:

$$B_i(t) = \exp\left(\int_0^t r_i(u)du\right) \tag{1}$$

Now consider asset $k \in H_i$. Let $y_k(t)$ be equal to the time t currency i price of asset k + all dividends from asset k occurred in the interval $[0, t]$ accumulated in the i'th bank account. Suppose that y_k is an Ito process which evolves according to:

$$dy_k(t) = y_k(t)\mu_k(t,\omega)dt + y_k(t)\sigma_k(t,\omega)'dw(t) \tag{2}$$

where the volatility term is an M-dimensional process. We will denote y_k the value or the price of asset k.
The exchange rate S_i quotes the price of the i'th currency in terms of currency number 0. Of course $S_0 \equiv 1$, but for $i = 1, \ldots, I$ we will assume that S_i is an Ito process evolving according to:

$$dS_i(t) = S_i(t)\mu_{S_i}(t,\omega)dt + S_i(t)\sigma_{S_i}(t,\omega)'dw(t) \tag{3}$$

where the volatility term again is an M-dimensional process.
For $k \in H_j$ the value of asset k in currency i is given by

$$v_k^i = \frac{S_j y_k}{S_i} \tag{4}$$

Since j is arbitrary v_k^i is defined for all $k \in H$. Do also define the discounted value of asset $k \in H$ in currency i by

$$z_k^i = \frac{v_k^i}{B_i} \tag{5}$$

By earlier assumptions and Ito's lemma, this is an Ito process and its evolution can be described on the form:

$$dz_k^i(t) = z_k^i(t)\gamma_k^i(t,\omega)dt + z_k^i(t)\delta_k^i(t,\omega)'dw(t) \tag{6}$$

where again the volatility term is an M-dimensional process.

Complete Markets

We will assume complete markets in the sense that for each $i \in \{0,..,I\}$ there exist M assets $\{k_1(i),..,k_M(i)\} \subseteq H$ so that the $M \times M$ dimensional matrix process

$$\Delta_i = \begin{pmatrix} \left(\delta^i_{k_1(i)}\right)' \\ \vdots \\ \left(\delta^i_{k_M(i)}\right)' \end{pmatrix} \tag{7}$$

is non-singular $Q \times \lambda - a.s.$ on the interval $[0,\tau]$. Now let Γ_i be the corresponding excess return vector process

$$\Gamma_i = \begin{pmatrix} \gamma^i_{k_1(i)} \\ \vdots \\ \gamma^i_{k_M(i)} \end{pmatrix} \tag{8}$$

For each $i \in \{0,...,I\}$ define the vector process η_i as the $Q \times \lambda-$almost unique solution to

$$\Gamma_i + \Delta_i \eta_i = 0 \tag{9}$$

Arbitrage-free Markets

Assume that for each $i \in \{0,...,I\}$ and for each $k \in H, \eta_i$ is also the solution to the equation

$$\gamma^i_k + \left(\delta^i_k\right)' \eta_i = 0 \tag{10}$$

The above assumption can be based on an arbitrage argument similar to the one given in Vasicek (1977). From now on η_i will be referred to as the risk premium in the i'th currency. The m'th element of $(-\eta_i)$ can be seen as the premium given by the market for exposure to the m'th risk factor represented as the m'th element of the Brownian motion.

Measures and Martingale Pricing Processes.

Suppose that for all $i \in \{0,...,I\}, \eta_i$ is $Q - a.s.$ square integrable in the time domain and that

$$\xi_i(t) = \exp\left(\int_0^t \eta_i(u)' dw(u) - \frac{1}{2}\int_0^t \|\eta_i(u)\|^2 du\right) \tag{11}$$

and

$$\exp\left(\int_0^t \left(\eta_i(u) + \delta^i_k(u)\right)' dw(u) - \frac{1}{2}\int_0^t \|\eta_i(u) + \delta^i_k(u)\|^2 du\right) \tag{12}$$

are Q-martingales for all $k \in H$. For convenience we will assume that among the assets in H there exist zero-coupon bonds of all maturities in the interval $[0,\tau]$ in each currency. Let the volatility term of a zero-coupon bond in the i'th currency with maturity

T, be given by $a_i(t.T)$, $t \in [0, T]$. Denote the corresponding price by $P_i(t, T)$, and for $t \in [0, T]$ define:

$$\zeta_i^T(t) = \exp\left(\int_0^t (\eta_i(u) + a_i(u, T))' dw(u) - \frac{1}{2}\int_0^t \|\eta_i(u) + a_i(u, T)\|^2 du\right) \quad (13)$$

which is a Q-martingale by the above assumption. Define the risk-neutral measure in the i'th currency by the Radon-Nikodyn derivative

$$\frac{dQ_i^*}{dQ} = \xi_i(\tau) \quad (14)$$

The time T forward risk-adjusted measure in the i'th currency has domain on $[0, T]$ and is defined by

$$\frac{dQ_i^T}{dQ} = \zeta_i^T(T) \quad (15)$$

Finally define the martingale pricing process in the i'th currency by

$$\Lambda_i(t) = \frac{\xi_i(t)}{B_i(t)} \quad (16)$$

By use of the Girsanov Theorem we have the following results:

i. $Q, (Q_i^*)_{i=1,\dots,I}, (Q_i^T)_{i=0,\dots,I}^{T\in[0,\tau]}$ are equivalent probability measures on common domain.

ii. $w_i^*(t) = w(t) - \int_0^t \eta_i(u)du$ is a M-dimensional Brownian motion under Q_i^*.

iii. $z_k^i(t)$ is a Q_i^* martingale for all $k \in H$.

iv. $w_i^T(t) = w_i^*(t) - \int_0^t a_i(u, T)du$ is a M-dimensional Brownian motion under Q_i^T.

v. $\frac{v_k^i(t)}{P_i(t,T)}$ is a Q_i^T-martingale for all $k \in H$.

vi. $v_k^i(t)\Lambda_i(t)$ is a Q-martingale for all $k \in H$.

vii. If x_i is a non-negative random variable so that $x_i \in \mathcal{F}_t$, $x_i\Lambda_i(t) \in L^1(Q)$, then a derivative asset with a time t value of x_i in currency i and no intermediate pay-outs has a time 0 value of

$$V_i(0) = E\{x_i\Lambda_i(t)|\mathcal{F}_0\} = E_i^*\left\{\frac{x_i}{B_i(t)}|\mathcal{F}_0\right\} = P_i(0, t)E_i^t\{x_i|\mathcal{F}_0\} \quad (17)$$

where E, E_i^*, E_i^t denote the expectation operators under Q, Q_i^*, Q_i^t respectively[1].

[1] This result is due to Harrison & Pliska (1981).

It is important to note that each of the above pricing functionals, works only in the currency in which they are defined.

The Stochastic Evolution of the Exchange Rates

With the framework established we can state the first result.

Theorem.

In the environment described we have for all $i \in \{0, \dots, I\}$:
The drift term of the exchange rate is given by

$$\mu_{S_i} = r_0 - r_i - \sigma'_{S_i}\eta_0 \ , \quad \lambda \times Q - a.s. \tag{18}$$

The volatility term of the exchange rate can be written as:

$$\sigma_{S_i} = \eta_i - \eta_0 \ , \quad \lambda \times Q - a.s. \tag{19}$$

Up to indistinguishability the exchange rate can be written as[2]:

$$\begin{aligned} S_i(t) = S_i(0)\frac{\Lambda_i(t)}{\Lambda_0(t)} &= S_i(0)\frac{B_0(t)}{B_i(t)}\frac{\xi_i(t)}{\xi_0(t)} \\ &= S_i(0)\frac{P_i(0,t)}{P_0(0,t)}\frac{\zeta_i^t(t)}{\zeta_0^t(t)} \end{aligned} \tag{20}$$

The Radon-Nikodyn derivatives between the risk neutral measures and between the forward risk adjusted measures are given by:

$$\begin{aligned} \frac{dQ_i^*}{dQ_0^*} &= \frac{B_i(\tau)}{B_0(\tau)}\frac{S_i(\tau)}{S_i(0)} \ , \quad Q - a.s. \\ \frac{dQ_i^t}{dQ_0^t} &= \frac{P_0(0,t)}{P_i(0,t)}\frac{S_i(t)}{S_i(0)} \ , \quad Q - a.s. \end{aligned} \tag{21}$$

Proof

The first statement is an immediate consequence of Ito's lemma and the fact that

$$\frac{S_i(t)B_i(t)}{B_0(t)} \tag{22}$$

is a Q_0^*-martingale. The second statement is established by the following. Let $k \in H_j$ and Ito derive

$$z_k^i(t) = \frac{S_j(t)y_k(t)}{S_i(t)B_i(t)} \tag{23}$$

under Q_i^*. Using that a martingale must have zero drift $\lambda \times Q - a.s.$ and inserting $\mu_k = r_j - \sigma'_k\eta_j$ and the first result of the theorem we get the relation:

$$\begin{aligned} -(\sigma_{S_j} - \sigma_{S_i})'\eta_0 - \sigma'_k\eta_j + \sigma'_{S_j}\sigma_k - \sigma'_{S_i}\sigma_{S_j} + \sigma'_k\sigma_{S_i} \\ + \|\sigma_{S_i}\|^2 + \sigma'_{S_j}\eta_j + \sigma'_k\eta_i - \sigma'_{S_i}\eta_i = 0 \end{aligned} \tag{24}$$

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2 This relation appears implicitly in Dumas et al. (1993)

Separating terms yields:

$$\begin{aligned} \sigma_k'(\sigma_{S_i} - \sigma_{S_j}) &= \sigma_k'(\eta_i - \eta_j) \\ \sigma_{S_j}'(\sigma_{S_i} - \sigma_{S_j}) &= (\eta_j - \eta_0)'(\sigma_{S_i} - \sigma_{S_j}) \end{aligned} \tag{25}$$

which is true for all $i, j = 0, \ldots, I$ and $k \in H_j$ (remember that $\sigma_{S_0} = 0$). By symmetry and simple manipulations we get that for all $i \in \{0, \ldots, I\}$, $k \in H$:

$$\begin{aligned} \sigma_k'\sigma_{S_i} &= \sigma_k'(\eta_i - \eta_0) \\ \|\sigma_{S_i}\|^2 &= \sigma_{S_i}'(\eta_i - \eta_0) \end{aligned} \tag{26}$$

Completeness of the global economy now yields the second statement of the theorem. (20) and (21) follow directly from the two first results.□

Remark

The first result in the theorem states that the open interest parity will hold under the domestic risk neutralized probabilities. This is obviously not true under the original probability measure unless the two economies considered have identical risk premiums. Actually assuming that the risk premiums depend on the interest rates, the drift term of the exchange rate can depend positively or negatively of the difference in the interest rates.

The second statement of the theorem has the unpleasant consequence for the pricing of currency derivatives, that the use of an equivalent martingale measure does not mean that we can ignore the specification of risk premiums. Also we must stress that given the stochastic evolution of the prices, one can not arbitrarily specify the functional form of σ_{S_i} without possibly violating the principle of absence of arbitrage. This is illustrated in Ingersoll (1987) p. 400–401.
In principle all sources of uncertainty in domestic and foreign asset markets are reflected in the exchange rate. More than that, the volatility term of the exchange rate can reflect sources of uncertainty that are not present in any marketed asset. This is seen by the possibility that one of the columns in Δ_i could have only zero-elements except for the element representing $(S_jB_j)/(S_iB_i)$.
Until now we have worked under the implicit assumption that there exists some original ("true") measure (Q) that each investor agrees to. At a first glance at the theorem, one could fear that heterogeneous beliefs would induce different beliefs about the volatility coefficients of the exchange rates. This is not necessarily the case. Assume that Q represents the beliefs of one investor and $\overline{Q}$ represents the beliefs of another, and that Q and $\overline{Q}$ are equivalent probability measures so that the Radon-Nikodyn derivative can be written as a Q—martingale:

$$\frac{d\overline{Q}}{dQ} = \exp\left(\int_0^\tau \theta(u)'dw(u) - \frac{1}{2}\int_0^\tau \|\theta(u)\|^2 du\right) \tag{27}$$

Then

$$\overline{\Gamma}_i = \Gamma_i + \Delta_i\theta \ , \ \overline{\Gamma}_i + \Delta_i\overline{\eta}_i = 0 \tag{28}$$

define the expected excess return vectors and the risk premiums consistent with the beliefs of the second investor. Now it is easily seen that

$$\overline{\eta}_i = \eta_i - \theta \tag{29}$$

and thereby that the volatility terms of the exchange rates are unaffected by heterogeneous but "equivalent" beliefs.

The third statement of the theorem gives an intuitively appealing relation between the martingale pricing processes and the exchange rates. The exchange rate is simply a tool for change of numeraire and thereby also for change of pricing functional. This and the last statement point out that the term structure of interest rates, the change of measures, and the exchange rate are highly interrelated. As we will see in the next section this has consequences for the pricing of currency derivatives.

Observe that if the pricing functionals are uniquely determined from the asset prices alone, then the dynamics of the exchange rates are uniquely determined from the stochastic evolution of the asset prices. This is the case if each market is complete in itself.

Our exchange rate result is consistent in the sense that prices turn out to be the same no matter which pricing functional is used. To see this consider the asset in (vii). Due to the theorem we have that the domestic value of this asset is given by

$$S_i(0)V_i(0) = S_i(0)E(x_i\Lambda_i(t)|\mathcal{F}_0) = E(x_iS_i(t)\Lambda_0(t)|\mathcal{F}_0) \tag{30}$$

So investors based in different countries come to the same prices.

A last observation can be made from the theorem, namely that in general the exchange rates will be path-dependent, due to the fact that the bank accounts in the different currencies and Radon-Nikodyn derivatives in general will depend on the whole path of the Brownian motion. So the exchange rates can not be inverted from the asset prices even though it may be possible to represent the 'local' stochastic evolution of the exchange rates by the asset prices only.

The Pricing of Currency Derivatives

Currency derivatives can like other assets be valuated directly by the pricing functionals described in (17). But for a derivative asset that can be priced by a terminal value that is piecewise linear in $(S_0, \ldots, S_I)$, it is in some cases more convenient to make use of the change of measure and pricing functional induced by the exchange rates. We will demonstrate this by considering exchange rate options and futures marketed in the domestic currency.

European Currency Options

Consider a fixed basket of currencies with weight q_i in the i'th currency. A European call option with strike price K and maturity t has the time 0 value:

$$V^{call}(0) = \sum_{i=0}^{I} q_iS_i(0)P_i(0,t)Q_i^t(A) - KP_0(0,t)Q_0^t(A) \tag{31}$$

where

$$A = \left\{ \omega \in \Omega \;\middle|\; \sum_{i=0}^{I} q_i S_i(t) > K \right\} \tag{32}$$

The corresponding put value can be obtained from the put-call parity:

$$V^{put}(0) = V^{call}(0) - \sum_{i=0}^{I} q_i S_i(0) P_i(0,t) + K P_0(0,t) \tag{33}$$

Proof
Clearly

$$V^{call}(t) = \left(\sum_{i=0}^{I} q_i S_i(t) - K \right)^+ \tag{34}$$

is $\mathcal{F}_t$-measurable, nonnegative and $\Lambda_0(t) V^{call}(t) \in L^1(Q)$ so by (17):

$$\begin{aligned} V^{call}(0) &= P_0(0,t) E_0^t \left\{ \left(\sum_{i=0}^{I} q_i S_i(t) - K \right)^+ \;\middle|\; \mathcal{F}_0 \right\} \\ &= P_0(0,t) \sum_{i=0}^{I} q_i E_0^t \{ S_i(t) \mathbf{1}_A | \mathcal{F}_0 \} - K P_0(0,t) E_0^t \{ \mathbf{1}_A | \mathcal{F}_0 \} \\ &= \sum_{i=0}^{I} q_i S_i(0) P_i(0,t) Q_i^t(A) - K P_0(0,t) Q_0^t(A) \end{aligned} \tag{35}$$

The last equality follows by use of the theorem. The put-call parity can be obtained by evaluating a put-option in a similar fashion. □

Example
Fix $i \in \{1, \ldots, I\}$ and suppose that for all $u \in [0,t]$:

$$a_i(u,t) \,,\; a_0(u,t) \,,\; \sigma_i(u) \tag{36}$$

only depend on time and maturity. Let us consider the case where the basket only contains currency i and $q_i = 1$. Then the option described above has the value:

$$V^{call}(t) = S_i(0) P_i(0,t) Q_i^t(A) - K P_0(0,t) Q_0^t(A) \tag{37}$$

Under the two measures the exchange rate can be written as:

$$\begin{aligned} S_i(t) &= S_i(0) \frac{P_i(0,t)}{P_0(0,t)} \exp\left(-\frac{1}{2}\nu^2 + \nu x_0^t \right) \\ S_i(t) &= S_i(0) \frac{P_i(0,t)}{P_0(0,t)} \exp\left(\frac{1}{2}\nu^2 + \nu x_i^t \right) \end{aligned} \tag{38}$$

where

$$x_0^t \underset{Q_0^t}{\sim} N(0,1) \,,\; x_i^t \underset{Q_i^t}{\sim} N(0,1) \tag{39}$$

and

$$\nu = \left(\int_0^t \|a_i(u,t) + \sigma_{S_i}(u) - a_0(u,t)\|^2 du \right)^{1/2} \tag{40}$$

Using this we get[3]:

$$V^{call}(0) = S_i(0)P_i(0,t)N(d_+) - KP_0(0,t)N(d_-) \tag{41}$$

where

$$d_\pm = \frac{1}{\nu} \ln \left(\frac{S_i(0)P_i(0,t)}{KP_0(0,t)} \right) \pm \frac{1}{2}\nu \tag{42}$$

Finding a closed form solution for the price of an option on a full basket of currencies is difficult unless we make the unrealistic assumption that the log values of the exchange rates are perfectly correlated under the various forward risk adjusted measures.

Remark

The general option pricing equation (31) tells us that the value of a European currency option can be decomposed in two parts. The first term in (31) reflects the expected present value of the basket of currencies at the exercise. The second term states the expected present value (negative) of buying the basket of currencies at the strike price K. If one views the exchange rate as a risky asset paying a continuous dividend stream equal to the foreign interest rate it is clear that an American call option on the exchange rate might be exercised prematurely. So for American options the above formulas (31–33) can only be used for finding the "European component" of the American option price.

Futures Prices

Let us now consider the futures price for delivery of currency i at time $T \geq t$. Throughout this paper we will assume that the futures price is given by:

$$F^i(t,T) = E_0^*(S_i(T)|\mathcal{F}_t) \tag{43}$$

In the appendix we give a proof of this for the case where the bank account and the futures price are sufficiently regular Ito processes. A proof for the Gaussian case when η_0 is bounded can be found in Babbs (1990).

Assume that $B_0(T)\Lambda_i(T) \in L^1(Q)$ then by the above assumption and the theorem the futures price is given by:

$$\begin{aligned} F^i(t,T) &= S_i(t)E_i^*\left\{ \frac{B_0(T)/B_0(t)}{B_i(T)/B_i(t)} \mid \mathcal{F}_t \right\} \\ &= S_i(t)P_i(t,T)E_i^T\left\{ \frac{B_0(T)}{B_0(t)} \mid \mathcal{F}_t \right\} \end{aligned} \tag{44}$$

By linearity of futures prices the price for a futures on a basket of currencies is given by the weighted sum of the individual futures prices.

[3] This result was originally obtained in Amin & Jarrow (1991) by use of a slightly different technique.

Example
Under Q_i^T we can write the domestic bank account as:

$$\begin{aligned}\frac{B_0(T)}{B_0(t)} = \frac{1}{P_0(t,T)} \exp\left(\int_t^T a_0(u,T)'(a_0(u,T) - \sigma_{S_i}(u) - a_i(u,T))du\right) \\ \times \exp\left(-\frac{1}{2}\int_t^T \|a_0(u,T)\|^2 du - \int_t^T a_0(u,T)' dw_i^T(u)\right)\end{aligned} \tag{45}$$

If we assume non-stochastic volatility terms, we have that the first term is deterministic and that the second is a Q_i^T-martingale, and thereby that[4]:

$$F^i(t,T) = S_i(t)\frac{P_i(t,T)}{P_0(t,T)} \exp\left(\int_t^T a_0(u,T)'(a_0(u,T) - \sigma_{S_i}(u) - a_i(u,T))du\right) \tag{46}$$

Observe that the first term in the above equation is the forward price.

Remark
From the general futures pricing result (44) we see that the exchange rate futures price can be found from the current exchange rate and the foreign and domestic term structures. The dynamics of the exchange rate do not explicitly enter the futures price. The stochastic evolution does though implicitly enter the futures price equation because the change of measure from the domestic to the foreign risk neutral probabilities involves the stochastic evolution of the exchange rate. This is illustrated in (45).
We also observe from (44) that when the domestic term structure is non-stochastic, the futures price will equal the forward price.[5] The intuition behind this is that one may regard the exchange rate as a risky asset paying a continuous dividend stream equal to the foreign interest rate. Then we can apply the result of Black (1976) and Cox, Ingersoll & Ross (1981) stating that when the (domestic) interest rate is non-stochastic, forward and futures prices are equal.
A second observation can be made from the general futures pricing equation, namely that for a large class of term structure models the futures price will have a functional form similar to the one of the bond prices. To be more precise let $N \leq M$ and suppose that:

$$r_0(t) - r_i(t) = g(t)'x(t) + h(t) \tag{47}$$

where x is the unique solution to:

$$dx(t) = (a(t) + b(t)x(t))dt + C(t)D(t,x)d\hat{w}_i^*(t) \tag{48}$$

and g, h, a, b, C are time-dependent continuous functions on $\Re^N, \Re, \Re^N, \Re^{N\times N}, \Re^{N\times N}$ respectively. $D(t,x) = \text{Diag}\left(\sqrt{d_1(t,x)}, \ldots, \sqrt{d_N(t,x)}\right)$ with $(d_n(t,x))_{n=1,\ldots,N}$ being functions and continuous in time and affine in x. $\hat{w}_i^*$ is defined as $\hat{w}_i^* =$

[4] This was also originally found by Amin & Jarrow (1991).
[5] This observation was also made in Amin & Jarrow (1991) in the Gaussian frame work.

$\left(w_{i.1}^*, \ldots, w_{i.N}^*\right)'$. Assume that for all t : $x(t) \in U(t)$ $Q - a.s.$ for some open set $U(t) \subseteq \Re^N$[6].

Then the futures price is given by:

$$F^i(t,T) = S_i(t) \exp\left(G^i(t,T) + H^i(t,T)'x(t)\right) \tag{49}$$

where $G^i, H^i$are time and maturity dependent functions on $\Re, \Re^N$ respectively. G^i, H^i are continuous differentiable in both arguments.

The proof of the conjecture stated above, is given in the appendix and is equivalent to the one stated in Duffie & Kan (1993) for the exponential affine term structure models. Given that x is a general continuous Markovian on an open set and a certain non-degeneracy condition is fulfilled, one can show that if the futures price has the above functional form then the difference in the interest rates will be affine in x and that x will evolve according to a stochastic differential equation similar to (48) under Q_i^*. The proof of that is similar to the one given in Duffie & Kan (1993) for the exponential affine term structure models.

Observe that (47–48) include the term structure models in Merton (1973), Vasicek (1977), Cox, Ingersoll & Ross (1985b) and various time dependent and multidimensional extensions of these.

A Model

In the following we propose a model for the exchange rates that incorporates stochastic term structure dynamics in the different currencies and is consistent with the results of the theorem. So this model is not only a model for the exchange rates but also a model for the term structures in the different currencies. The model can embody the intially observed term structures and can be brought to yield only positive interest rates. This makes it a suitable tool for valuation of long term contingent claims on exchange rates and bonds in different currencies, such as swaps and swaptions.

Suppose that x is an M-dimensional Markovian state variable that 'drives' all prices and exchange rates in the global economy, in the sense that for all $k \in H$:

$$\mu_k(t,\omega) = \mu_k(t,x(t)) \ , \ \sigma_k(t,\omega) = \sigma_k(t,x(t)) \tag{50}$$

an for all $i \in \{0,\ldots,I\}$:

$$\mu_{S_i}(t,\omega) = \mu_{S_i}(t,x(t)) \ , \ \sigma_{S_i}(t,\omega) = \sigma_{S_i}(t,x(t)) \tag{51}$$

Assume that x under Q_i^* is the unique solution to the stochastic differential equation:

$$dx(t) = \left(a(t) + b^i(t)x(t)\right)dt + C(t)D(x(t))dw_i^*(t) \tag{52}$$

where $a : [0,\tau] \to \Re^M$, $b^i = \text{Diag}\left(b_1^i, \ldots, b_M^i\right)$, and $C = \text{Diag}(c_1, \ldots, c_M)$ are continuous functions of time. For all m : c_m does not change sign on $[0,\tau]$. D

[6] Duffie & Kan (1993) supply regularity conditions sufficient for existence of a unique solution to the stochastic differential equation on $\{(t,x(t)) | t \in [0,\tau], x(t) \in U(t)\}$ where $U(t)$ is the open set $U(t) = \bigcap_n \{q \in \Re^N | d_n(t,q) > 0\}$. Under these conditions for all t : $x(t) \in U(t)$ $Q - a.s.$

is defined as the function $D(x) = \text{Diag}(\sqrt{x_1}, \ldots, \sqrt{x_M})$. In order to ensure that the stochastic differential equation has a unique solution we impose the restriction $2a_m(t) > c_m(t)$, $m = 1, \ldots, M$, $t \in [0, \tau]$.
Assume that:

$$r_i(t) = \beta^i(t)' x(t) \ , \ i = 1, \ldots, I \ , \ t \in [0, \tau] \tag{53}$$

for some continuous vector function $\beta^i : [0, \tau] \to \Re^M$. To ensure non-negative interest rates we would require that $\beta^i_m \geq 0$ on the interval $[0, \tau]$ for all $m = 1, \ldots, M$.
The bond prices are given by

$$P_i(t, T) = \exp\left(A^i(t, T) + B^i(t, T)' x(t)\right) \tag{54}$$

where A^i, B^i are functions on $\Re, \Re^M$ respectively, continuous differentiable in the first argument and continuous in the last. A^i, B^i are given as the solutions to:

$$\begin{aligned} \beta^i_m(t) &= \frac{1}{2} B^i_m(t, T)^2 c_m(t)^2 + B^i_m(t, T) b^i_m(t) + \frac{\partial}{\partial t} B^i_m(t, T) \ , \\ B^i_m(T, T) &= 0 \ , \ m = 1, \ldots, M \\ A^i(t, T) &= \int_t^T a(u)' B^i(u, T) du \end{aligned} \tag{55}$$

for all $t \in [0, T]$.
This is in essence a multicurrency, multidimensional and time dependent extension of the model in Cox, Ingersoll & Ross (1985b). This specific type of model is in Jamshidian (1993b) termed "a separable multifactor CIR-square-root model", because the bond prices can be separated into a product of one-factor model prices.
Using the theorem and the stochastic differential equation for x we get that the volatility term of the exchange rates can be written as[7]:

$$\sigma_{S_i}(t) = [C(t) D(x(t))]^{-1} \left(b^i(t) - b^0(t)\right) x(t) \equiv D(x(t)) C(t) \rho^i(t) \tag{56}$$

For currency derivative pricing we can now write the i'th exchange rate at time t under Q^t_j as:

$$\begin{aligned} S_i(t) = S_i(0) \frac{P_i(0, t)}{P_0(0, t)} \exp\left(\sum_{m=1}^{M} \int_0^t \nu^i_m(u, t) \nu^j_m(u, t) c_m(u)^2 x_m(u) du \right) \\ \times \exp\left(-\frac{1}{2} \sum_{m=1}^{M} \int_0^t \nu^i_m(u, t)^2 c_m(u)^2 x_m(u) du + \int_0^t \nu^i(u, t)' C(u) D(x(u)) dw^t_j(u) \right) \end{aligned} \tag{57}$$

where

$$\nu^i(u, t) = B^i(u, t) + \rho^i(u) - B^0(u, t) \ , \ u \in [0, t] \tag{58}$$

[7] Except on a Q-zero-subset of Ω where $D(x(t))$ is singular.

and

$$
\begin{aligned}
dx(u) &= \big(a(u) + (b^j(u) + \hat{c}(u)B^j(u,t)')x(u)\big)du + C(u)D(x(u))dw_j^t(u) \\
\hat{c}(u) &= \left(c_1(u)^2, \ldots, c_M(u)^2\right)', \; u \in [0,t]
\end{aligned} \tag{59}
$$

Even though the bond prices in the different currencies at any time t are given by $x(t)$, this will not be the case for the exchange rates. The exchange rates at time t will depend on the whole path $(x(u))_{u\in[0,t]}$. In order to design a Markovian process that at all times describes the exchange rates and the bond prices we could introduce a suitable transformation of the exchange rates as additional state variables. The number of state variables then becomes $M+I$. If we let $s_i \equiv \ln S_i$, $i = 1,\ldots,I$ be the new state variables, s_i will under Q_0^* evolve according to:

$$
\begin{aligned}
ds_i(t) = &\sum_{m=1}^{M}\left(\beta_m^i(t) - \beta_m^0(t) - \frac{1}{2}\rho_m^i(t)^2 c_m(t)^2\right)x_m(t)dt \\
&+\rho^i(t)'C(t)D(x(t))dw_0^*(t), \; i = 1,\ldots,I
\end{aligned} \tag{60}
$$

We can now write the new $(M+I)$-dimensional state variable for the system of bonds and exchange rates as the solution to:

$$
dq(t) \equiv d\begin{pmatrix} x(t) \\ s(t) \end{pmatrix} \equiv \big(g(t) + h^0(t)x(t)\big)dt + K(t)D(x(t))dw_0^*(t) \tag{61}
$$

where g, h^0, K are time dependent functions on $\Re^{M+I}, \Re^{(M+I)\times M}, \Re^{(M+I)\times M}$ respectively.

Under assumption of sufficient regularity[8] we can do the following. Let $V(t,q(t))$ denote the time t price of a currency contingent claim marketed in domestic currency with no intermediate payments in the interval $[0,T[$ and a terminal value at time T given by $\hat{V}(q(T))$, then by (vii) and Ito's lemma $V(t,q(t))$ must fulfill the partial differential equation:

$$
\begin{aligned}
\beta^0(t)'x(t) = &\frac{1}{2}\mathrm{tr}\left(K(t)D(x(t))D(x(t))K(t)'\frac{\partial^2}{\partial q \partial q'}V(t,q(t))\right) \\
&+\Big(g(t) + h^0(t)x(t)\Big)'\frac{\partial}{\partial q}V(t,q(t)) + \frac{\partial}{\partial t}V(t,q(t))
\end{aligned} \tag{62}
$$

for all $t \in [0,T]$ subject to the boundary condition $V(T,q(T)) = \hat{V}(q(T))$. In principle we could now set up a finite difference scheme for solving the partial differential equation. This might be possible for the case where $I = 1, M = 2$, but for a higher dimensional system it probably will be almost impossible. In that case we will have to do Monte Carlo simulations for the pricing of currency derivatives. These simulations could be based on either (57)–(59) or (61).

8 For regularity conditions see Friedman (1975) ch. 6.

Equilibrium Considerations
Note that the functional form of the volatility term for the exchange rates is consistent with risk premiums of the form:

$$\eta_i(t) = D(x(t))\lambda_i(t) \tag{63}$$

for some time dependent vector function λ_i. Referring to Ingersoll (1987) p.400–401 this is consistent with absence of arbitrage in this type of model.

In the next section we show that if we use all M state variables for modelling the stochastic evolution of the term structures, then the model is uniquely parameterized from the observed term structures and exchange rate futures prices. It may not seem appropriate for practical use of the model to assume that all state variables that affect the exchange rates also affect at least one of the term structures, but from a theoretical point of view there is no reason to believe that this should not be the case. To see this let us consider a two country version of the equilibrium model described in Cox, Ingersoll & Ross (1985a). In the following let all parameters be constants and let us ignore time indices. Suppose that in each country there is a single production process, whose value in the numeraire of the i'th country evolves according to:

$$dK_i = K_i g_i' x dt + K_i h_i' D(x) dw \ , \ i = 0,1 \tag{64}$$

for some constant vectors g_i, h_i. Under the 'original' measure Q, x is assumed to be evolving according to a stochastic differential equation similar to (52) but with constant parameters. Now make the assumption that the two markets as well as the global market are complete and that there is no trade between the two countries[9]. Further, make the assumption that all investors only consume the products of their native country and that they have rational expectations and maximize the expected value of identical additive separable log utility functions over infinite investment horizons. Using the work of Cox, Ingersoll & Ross (1985a) we can write the equilibrium interest rates as:

$$r_i = \sum_{m=1}^{M} \left(g_{i,m} - (h_{i,m})^2 \right) x_m \ , \ i = 0,1 \tag{65}$$

The risk premiums become:

$$\eta_i = D(x) h_i \ , \ i = 0,1 \tag{66}$$

Because of the no-trade assumption the exchange rate between the two countries is only a 'shadow' exchange rate. This will have the stochastic evolution:

$$dS = S \left(\sum_{m=1}^{M} (g_{0,m} - g_{1,m} + h_{1,m}(h_{1,m} - h_{0,m})) x_m \right) dt + S(h_1 - h_0)' D(x) dw \tag{67}$$

We observe that only in the special case where there exists m so that $g_{i,m} = (h_{i,m})^2$, $i = 0,1$, there will be state variables that affect the volatility term of the

[9] In absence of the no-trade restriction the analysis would be much more complex. It is though the opinion of the author that introducing trade between the two countries would not alter the conclusion of the analysis.

exchange rate but not one of the term structures.

Calibration of the Model

Calibrating the described model can be done in several ways, we suggest the following. Assume that all state variables affecting the exchange rates also are present in the term structures, and let $M \geq 2$. For simplicity let $I = 1$, let all parameters of the model except a be constants, and let C be equal to the identity matrix. Assume that the risk premiums are of the form given in (63) and estimate β^0, β^1 from time series data. We need now to find the function $a(u)$ and the constant vectors b^0, b^1.
With constant parameters there exist closed form solutions for B^0, B^i infinitely differentiable in time and maturity[10]. Suppose that there exists a continuous differentiable initial forward curve in each currency. Let the forward curves be given by $(f_0(0,T))_{T\in[0,\tau]}$, $(f_i(0,T))_{T\in[0,\tau]}$. Using that $f_i(0,T) = -\frac{\partial}{\partial T}\ln P_i(0,T)$ we get that a must fulfill the integral equations:

$$
\begin{aligned}
f_0(0,T) &= -\int_0^T a(u)'\frac{\partial}{\partial T}B^0(u,T)du - \frac{\partial}{\partial T}B^0(0,T)'x(0) \\
f_1(0,T) &= -\int_0^T a(u)'\frac{\partial}{\partial T}B^1(u,T)du - \frac{\partial}{\partial T}B^1(0,T)'x(0)
\end{aligned}
\tag{68}
$$

$x(0)$ can be obtained by inversion of M different bond prices.
The above defines a Volterra integral equation in two dimensions. The integral equation has continuous solutions for $(a(t))_{t\in[0,\tau]}$ but only if $M = 2$ the solution will be unique. If no solution satisfies the restriction $a_m(t) > \frac{1}{2}$ for all t, m we can not fit the initial term structures. In order to fix a solution for a we could set $a_m = a_n$ for $m, n \leq M/2$ and $a_k = a_l$ for $k.l \geq (M+1)/2$, or one could let $a_3, \ldots, a_M$ be constants.
Let F_0, F_1 denote futures prices for futures on the exchange rate marketed in the domestic and the foreign market respectively. Given our futures price result and the symmetry we know that currency futures prices are given by:

$$
\begin{aligned}
F_0(0,T) &= S_1(0)\exp\left(G^0(0,T) + H^0(0,T)'x(0)\right) \\
F_1(0,T) &= \frac{1}{S_1(0)}\exp\left(G^1(0,T) + H^1(0,T)'x(0)\right)
\end{aligned}
\tag{69}
$$

where

$$
\begin{aligned}
\beta^0_m - \beta^1_m &= \frac{1}{2}H^0_m(t,T)^2 + H^0_m(t,T)b^1_m + \frac{\partial}{\partial t}H^0_m(t,T) , \\
H^0_m(T,T) &= 0 , \ m = 1, \ldots, M \\
G^0(t,T) &= \int_t^T a(u)'H^0(u,T)du
\end{aligned}
\tag{70}
$$

---

[10] See Cox, Ingersoll & Ross (1985b).

and

$$\beta_m^1 - \beta_m^0 = \frac{1}{2}H_m^1(t,T)^2 + H_m^1(t,T)b_m^0 + \frac{\partial}{\partial t}H_m^1(t.T) \ ,$$
$$H_m^1(T,T) = 0 \ , \ m = 1, \dots, M \tag{71}$$
$$G^1(t,T) = \int_t^T a(u)' H^1(u,T) du$$

for all $0 \leq t \leq T \leq \tau$. In this constant parameter case an explicit expression for H^i exists and it is of course on the same form as B^i.[11]

If we take a sufficient number of currency futures in each currency and the initial forward curves the parameters a, b^0, b^1 can now be found from the equations above using numerical techniques. Now also ρ^1 is determined.

Observe that we could also choose to parameterize b^0, b^i from bond volatility curves. The bond volatility curves could be obtained from finite difference valuation of term structure contingent claims[12], or by use of the closed form solutions for different term structure contingent claim prices described in Jamshidian (1993b). But if there exists m so that $\beta_m^0 = 0$ then b_m^0 and thereby ρ_m^1 can not be inferred from the bond volatility curves.

As another alternative for using the futures prices for calibration, we could use other currency derivatives such as options on the exchange rate. That would require Monte-Carlo simulations and thereby a lot more computer time. It is though advisable to check whether the model calibrated on the futures prices actually fits the observed option prices or not.

If we had assumed that some of the state variables were present in the volatility term of the exchange rate, but not affected the term structures, we could calibrate the common parameters for the term structures and exchange rate by the futures method. But we would have to do Monte Carlo simulations on exchange rate derivatives (other than futures) to calibrate the remaining parameters of the model.

Conclusion

In a multi currency and complete global market setting the stochastic evolution for the exchange rates has been found. We have developed formulas for European options and futures on the exchange rates, and found a closed form solution for the currency futures price that is consistent with a large class of term structure models. We have presented a model for the evolution of the exchange rate that precludes negative interest rates. This makes it a suitable tool for evaluating long term contingent claims on the exchange rate and on bonds in different currencies. Due to our futures price result it turns out that this model is easily calibrated to the observed initial prices.

What remains to be done is to see if the results presented for the stochastic evolution of the exchange rate extend to the general semi-martingale case[13]. Another interesting

[11] Again see Cox, Ingersoll & Ross (1985b).

[12] See Duffie & Kan (1993).

[13] It is the belief of the author that this is easily shown to be the case.

question is to see what can be said about the situation where the global market is incomplete.
Turning to the presented model, empirical testing is needed to see how well it fits actual term structure and exchange rate dynamics, and how well it predicts contingent claim prices.

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Appendix

Lemma
Let $F(t.T)$ be the time t futures price for delivery of an asset at time $T \geq t$. The asset has a time T domestic value of $V(T)$. Assume that $V(T) \in L^1(Q_0^*)$ and that the futures price under Q_0^* is an Ito process that evolves according to

$$dF(t) = \mu_F(t)du + \sigma_F(t)' dw_0^*(t) \ , t \in [0, T] \tag{72}$$

where $\sigma_F \in \mathcal{H}^1(Q_0^*)$.[14] Make the further assumption that $(B_0)^{-1}\sigma_F \in \mathcal{H}^1(Q_0^*)$.[15] Then

$$F(t, T) = E_0^*(V(T)|\mathcal{F}_t) \tag{74}$$

Proof
The following argument is similar to the one in Duffie (1992).
Consider the discounted cumulated dividend process for the futures:

$$\begin{aligned} & B_0(t) \int_t^s B_0(u)^{-1} dF(u) \\ = \ & B_0(t) \left(\int_t^s B_0(u)^{-1} \mu_F(u) du + \int_t^s B_0(u)^{-1} \sigma_F(u)' dw_0^*(u) \right) \ , s \in [t, T] \end{aligned} \tag{75}$$

The initial value of a futures is 0, so under absence of arbitrage we will require that the above is a Q_0^*-martingale. By assumptions $(B_0)^{-1}\sigma_F \in \mathcal{H}^1(Q_0^*)$, and thereby is the above a Q_0^*-martingale if and only if $\mu_F = 0 \ , \lambda \times Q_0^* - a.s.$ So under the assumptions the requirement is possible and under the requirement F must be a Q_0^*-martingale, which yields that:

$$F(t, T) = E_0^*(V(T)|\mathcal{F}_t) \tag{76}$$

Note that the result does not violate the assumption that the futures price is an Ito process. □

Proof of the Futures Price Conjecture
Define

$$\begin{aligned} \nu(t, x) &= a(t) + b(t)x \\ \sigma(t, x) &= C(t)D(t, x) \end{aligned} \tag{77}$$

14 $\mathcal{H}^p(Q_0^*)$ is defined as the set of progressively measurable processes X which fulfill the requirement:

$$E_0^*\left(\left(\int_0^T X(u)^2 du\right)^{p/2}\right) < \infty$$

15 This will be the case if for example r_0 is $Q - a.s.$ non-negative at all time points, because in that case would $B_0(u)^{-1} \in]0, 1] \ Q - a.s.$

The candidate functional form for the futures price is given by (48). By our futures price result we see that:

$$E_i^*\left\{ \frac{B_0(T)/B_0(t)}{B_i(T)/B_0(t)} \;\middle|\; \mathcal{F}_t \right\} = \exp\left(G^i(t,T) + H^i(t,T)'x(t)\right) \tag{78}$$

The left hand side of the above equation is continuous differentiable in T, so if solutions for G^i, H^i exist then G^i, H^i are continuous differentiable in T.
By Ito's lemma G^i, H^i must satisfy the partial differential equation:

$$\begin{aligned} 0 = -\sum_{n=1}^{N} g_n(t)x_n - h(t) + \frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N} H_m^i(t,T)H_n^i(t,T)\sigma_m(t,x)\sigma_n(t,x) \\ + \sum_{n=1}^{N} H_n^i(t,T)\nu_n(t,T) + \frac{\partial}{\partial t}G^i(t,T) + \sum_{n=1}^{N}\frac{\partial}{\partial t}H_n^i(t,T)x_n \end{aligned} \tag{79}$$

for all $t \leq T$ subject to the boundary condition $G^i(T,T) = H_n^i(T,T) = 0$. σ_n denotes the n'th row in σ. Note that the above PDE is affine in x. Since $U(t)$ is an open set we can separate terms to get the $N+1$ equations:

$$\begin{aligned} 0 = \frac{\partial}{\partial t}H_n^i(t,T) + \mathcal{K}_n^i\left(t, H^i(t,T)\right), \; H_n^i(T,T) = 0, \; n = 1,\ldots,N \\ 0 = \frac{\partial}{\partial t}G^i(t,T) + \mathcal{L}^i\left(t, H^i(t,T)\right), \; G^i(T,T) = 0 \end{aligned} \tag{80}$$

where $\mathcal{L}^i, \mathcal{K}_n^i : [0,\tau] \times \Re^N \to \Re$ are linear-quadratic in the last argument. The first N equations form an N-dimensional ordinary differential equation, in H^i. This type of differential equation is called a Ricatti equation and is known to have a unique solution.[16] The last differential equation then defines a unique solution for G^i. The continuity of the parameters of the stochastic differential equation ensures that G^i, H^i are continuous differentiable in t. □

16 By a solution f to a differential equation like

$$0 = \frac{\partial}{\partial t}f(t,T) + \mathcal{G}(t, f(t,T)), \; f(T,T) = 0 \tag{81}$$

we mean that for all $0 \leq t_1 \leq t_2 \leq T$:

$$f(t_2,T) - f(t_1,T) = -\int_{t_1}^{t_2} \mathcal{G}(u, f(u,T))du \;, \; f(T,T) = 0 \tag{82}$$

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A Gaussian Exchange Rate and Term Structure Model

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Abstract

This paper presents a tractable model for the valuation of contingent claims on the term structures of interest rates of two currencies and the exchange rate between them, such as for example cross currency swaptions. The modelling framework is a deterministic volatility version of the general Heath, Jarrow, and Morton (1992) model that takes the initial yields and volatility curves as direct input. We illustrate that closed-form formulas can be obtained for a general class of cross-currency derivatives and we identify a class of deterministic volatility structures that imply Markov representation by a three dimensional Gaussian process. For this class of volatility structures all types of European style contingent claim prices and hedge ratios can be obtained by numerical integration in maximum three dimensions.

Introduction

This paper presents an arbitrage based model for the term structures in two different currencies and the associated exchange rate. The model is constructed to take the yield curves of the different term structures as input, and thus no fitting is needed in order to match the observed term structures. Likewise, the joint local covariance structure of the bonds and the exchange rate are direct input to the model.

Under the assumption that the volatilities of the bonds are deterministic functions of time and time-to-maturity we illustrate by examples how closed form solutions can be obtained for a class of claims in this multicurrency economy. Our examples include options on yield spreads and options on exchange rate futures.

We also show that if volatilities are deterministic, it is possible to give an integral representation rather than a differential representation for a hedging portfoilo based on the underlying instruments.

Within the class of deterministic bond volatilities we identify a class of volatility structures that imply finite dimensional Markovian representation of the yield curves by a vector Gaussian process for the joint evolution of the spot interest rates of the two currencies and the logarithm of the exchange rate. This is a natural multicurrency extension of the work by Babbs (1990) and Jamshidian (1991). As Jamshidian (1991) we therefore term this class of volatilities "Gaussian".

Under the Gaussian volatility structures all European style claim prices and hedge ratios on the two term structures and the exchange rate can be computed by numerical integration in maximum three dimensions.

The paper is organized as follows. The first section describes the modelling framework that we use. The second section considers the pricing of contingent claims under deterministic volatilities and presents the Gaussian volatility structure. Proofs are given in the appendix.

The General Continuous-Time Framework

We consider an economy of two currencies and the exchange rate between them. Currency 0 is domestic currency and currency 1 is foreign currency, and we let S denote the exchange rate in terms of domestic currency units per foreign currency unit. We assume that there in each currency exists a full curve of zero-coupon bonds of all maturities. The currency i, time t price of a maturity T zero-coupon bond will be denoted $P_i(t,T)$ and the spot rate of currency i, r_i, is given by

$$r_i(t) = -\frac{\partial \ln P_i(t,T)}{\partial T}\Big|_{T=t} \tag{1}$$

The accumulating money market account in currency i, therefore has the value

$$B_i(t) = e^{\int_0^t r_i(u)du} \tag{2}$$

We assume that the uncertainty of the exchange rate and the yield curves are driven by a common d-dimensional Brownian motion, and that there for each currency exist equivalent martingale measures with different numeraires. To be precise we assume that there exist martingale measures $\mathcal{Q}_i, \mathcal{Q}_i^T$ so that a claim paying x domestic currency units at time $t \leq T$, with x being measurable with respect to the information at time t, has the initial domestic currency value

$$\begin{aligned} \mathrm{E}^{\mathcal{Q}_0}\left[\frac{x}{B_0(t)}\right] &= P_0(0.T)\mathrm{E}^{\mathcal{Q}_0^T}\left[\frac{x}{P_0(t,T)}\right] \\ &= S(0)\mathrm{E}^{\mathcal{Q}_1}\left[\frac{x/S(t)}{B_1(t)}\right] = S(0)P_1(0,T)\mathrm{E}^{\mathcal{Q}_0^T}\left[\frac{x/S(t)}{P_1(t,T)}\right] \end{aligned} \tag{3}$$

$\mathcal{Q}_i$ is the currency i martingale measure with the bank-account as numeraire and $\mathcal{Q}_i^T$ is the currency i martingale measure with the time T maturity zero-coupon bond as numeraire.

According to Heath, Jarrow and Morton (1992) we can represent the random evolution of the bonds and the exchange rate in such a framework by the system of stochastic differential equations

$$\begin{aligned} \frac{dP_i(t,T)}{P_i(t,T)} &= r_i(t)dt + a_i(t,T)dW_i(t) \\ \frac{dS(t)}{S(t)} &= (r_0(t) - r_1(t))dt + \sigma_S(t)dW_0(t) \end{aligned} \tag{4}$$

where $\{\sigma_S(t)\}_{t\geq 0}$ is a well-behaved $(1 \times d)$-dimensional processes, $\{a_0(t,T), a_1(t,T)\}_{0\leq t\leq T}$ are families of well-behaved $(1 \times d)$-dimensional processes, with the property that $a_i(t,T) \underset{t\uparrow T}{\rightarrow} 0$, and W_i is a d-dimensional standard Brownian motion under $\mathcal{Q}_i$.

As derived in Andreasen (1995) the relation between the Brownian motions under the two martingale measures $\mathcal{Q}_0$ and $\mathcal{Q}_1$ is given by

$$dW_1(t) = dW_0(t) - \sigma_S(t)'dt \tag{5}$$

and Brownian motions under $\mathcal{Q}_i^T$ are related to the Brownian motions under $\mathcal{Q}_i$ by

$$dW_i^T(t) = dW_i(t) - a_i(t,T)dt \tag{6}$$

for $t \leq T$.

We now turn to the issue of contingent claim pricing under the assumption that the volatility terms are deterministic.

Contingent Claim Pricing under Deterministic Volatilities

The following lemma states the pricing equation for domestic currency denominated contingent claims under the martingale measure $\mathcal{Q}_0^t$ and gives integral representations for the prices of the underlying assets under this probability measure.

Lemma 0: European Style Contingent Claim Pricing

Suppose a European style contingent claim promises a time t domestic currecny payment of

$$F(t; S(t), P_0(t,T_1), \ldots, P_0(t,T_M), P_1(t,T_{M+1}), \ldots, P_1(t,T_N)) \tag{7}$$

with $T_1 \ldots, T_N > t$. *Then the initial price of the claim is*

$$F(0) = P_0(0,t)\mathrm{E}^{\mathcal{Q}_0^t}[F(t;\cdot)] \tag{8}$$

where

$$S(t) = \frac{S(0)P_1(0,t)}{P_0(0,t)} e^{-\frac{1}{2}\int_0^t \|a_1(u,t)+\sigma_S(u)-a_0(u,t)\|^2 du + \int_0^t (a_1(u,t)+\sigma_S(u)-a_0(u,t))dW_0^t(u)}$$

$$P_0(t.T) = \frac{P_0(0,T)}{P_0(0,t)} e^{-\frac{1}{2}\int_0^t \|a_0(u,T)-a_0(u,t)\|^2 du + \int_0^t (a_1(u,T)-a_0(u,t))dW_0^t(u)}$$

$$P_1(t,T) = \frac{P_1(0,T)}{P_1(0,t)} e^{-\frac{1}{2}\int_0^t (\|a_1(u,T)+\sigma_S(u)-a_0(u,t)\|^2 - \|a_1(u,t)+\sigma_S(u)-a_0(u,t)\|^2)du}$$
$$\times e^{\int_0^t (a_1(u,T)-a_1(u,t))dW_0^t(u)} \tag{9}$$

By translating the pay-offs, using the exchange rate, we can also use Lemma 0 to price foreign currency denominated claims.

We now restrict ourselves to deterministic volatility structures. Specifically we make the following assumptions on our input data, the volatilities of the bonds and the exchange rate, and the initial term structures:

A1: The volatilities $a_0(\cdot,\cdot), a_1(\cdot,\cdot), \sigma_S(\cdot)$ are deterministic, and $a_i(t,\cdot)$ is a continuous differentiable function for all t and $i = 0, 1$.

A2: The initial term structures of bond prices $\{P_i(0,T)\}_{T\geq 0}$ are continuous differentiable in the maturity date.

The deterministic volatility assumption implies an interesting result for hedging which is stated in the following lemma.

Lemma 1: Hedging under Deterministic Volatility

Redefine the pay-off function considered in Lemma 0 so that

$$F(t;\cdot) = g(S(t), P_0(t,T_1), \ldots, P_0(t,t_M), S(t)P_1(t,T_{M+1}), S(t)P_1(t,T_N)) \tag{10}$$

and let $g_j(\cdot)$ *be the partial derivative of* $g(\cdot)$ *with respect to the* $(j+1)$*'th argument. If* $g(x_{-j},\cdot) \equiv g(x_1,\ldots,x_{j-1},\cdot,x_{j+1},\ldots,x_N)$ *is not differentiable in the point* x_j *define*[1]

$$g_j(x_{-j},\xi) = \frac{\partial g_j(x_{-j},\xi)}{\partial x_j}\mathbf{1}_{\xi\neq x_j} + \delta(\xi - x_j)(g(x_{-j},x_j+) - g(x_{-j},x_j-)) \tag{11}$$

[1] Here $\mathbf{1}_A$ denotes the indicator function on the set A, $\delta(\cdot)$ is the Dirac-Delta function, $f(x+) = \lim_{\epsilon\to 0} f(x+|\epsilon|)$ and $f(x-) = \lim_{\epsilon\to 0} f(x-|\epsilon|)$.

in the neighboorhood of x_j.[2]

Then a self-financing hedge portfolio is given by

$E^{Q_1^t}[g_0(\cdot)]$ *foreign zero-coupon bonds maturing at time t.*

For $j = 1, \ldots, M$: $E^{Q_0^{T_j}}[g_j(\cdot)]$ domestic zero-coupon bonds with maturity T_j.

For $j = M+1, \ldots, N$: $E^{Q_1^{T_j}}[g_j(\cdot)]$ foreign zero-coupon bonds with maturity T_j.

The remaining amount in domestic zero-coupon bonds maturing at time t.

The result is a generalization of the result derived by Babbs (1990) for a single factor Markovian term structure model and continuous pay-off functions. Here we do not necessarily need the continuity or a finite dimensional Markov representation. This is obtained because we use step functions and Dirac Delta functions to extend the notion of differentiability.

Lemma 1 states a "natural" way of composing the hedge portfolio in a term structure model, namely to use the underlying instruments directly. Also, the composition of the hedge portfolio is intuitive in the sense that we use the expected partial derivatives of the pay-off function as weights.

The distribution of the exchange rate and any basket of different currency zero-coupon bonds is now jointly lognormal under the martingale measures with the bank-account and the zero-coupon bonds of different currencies as numeraires and by applying the Lemma, it is in general possible to come up with closed form formulas for all European style claims with domestic or foreign currency pay-offs of the form

$$(k_1 x - k_2 y)^+ \tag{12}$$

where k_1, k_2 are constants and x, y are products of powers of zero-coupon bond prices and/or exchange rates. The trick is basically to apply Lemma 0, decompose the pay-off into two components, and make use of a change of measure to eliminate the random factors x and y from the two components of the pay-off.

To illustrate this let us consider a couple of examples. Let us first consider the price of a European call option on an exchange rate futures price. Suppose that the futures contract's maturity is date T, that the option expires at time $t \leq T$, and the strike is K, then the price of the option at time 0 is given by

$$P_0(0,t)\left[F(0)e^{\mu}\Phi\left(\frac{\ln(F(0)/K)+\mu}{\nu}+\frac{1}{2}\nu\right) - K\Phi\left(\frac{\ln(F(0)/K)+\mu}{\nu}-\frac{1}{2}\nu\right)\right]$$

$$\nu^2 = \int_0^t \|a_1(u,T) + \sigma_S(u) - a_0(u,T)\|^2 du$$

$$\mu = \int_0^t (a_1(u,T) + \sigma_S(u) - a_0(u,T))a_0(u,t)' du \tag{13}$$

[2] This definition does not apply to *any* pay-off function but for most realistic pay-off functions, the functions $\{g_j(\cdot)\}_{j=0,\ldots,N}$ are well-defined.

where $F(s)$ is the time s futures price

$$F(s) = \frac{S(s)P_1(s,T)}{P_0(s,T)} e^{-\int_s^T (a_1(u,T)+\sigma_S(u)-a_0(u,T))a_0(u,T)' du} \tag{14}$$

As another example let us consider an option on the spread of time t simple rates of the two currencies, i.e. a contract that at time t pays a domestic currency amount of

$$(R_1(t,T) - R_0(t,T))^+ \tag{15}$$

where

$$R_i(t,T) = \frac{1}{T-t}\left(P_i(t,T)^{-1} - 1\right) \tag{16}$$

The time 0 value of this contract is given by

$$\frac{P_0(0,t)}{T-t}\left\{\frac{P_1(0,t)}{P_1(0,T)} e^{\frac{1}{2}(\mu_T - \mu_t + \theta_1)}\Phi(z_1) - \frac{P_0(0,t)}{P_0(0,T)} e^{\theta_0}\Phi(z_0)\right\} \tag{17}$$

where

$$\begin{aligned}
\theta_i &= \int_0^t \|a_i(u,T) - a_i(u,t)\|^2 du \\
\mu_v &= \int_0^t \|a_1(u,v) + \sigma_S(u) - a_0(u,t)\|^2 du \\
\xi &= \int_0^t (a_1(u,T) - a_1(u,t))(a_0(u,T) - a_0(u,t))' du \\
\nu &= \int_0^t \|a_1(u,T) - a_1(u,t) - a_0(u,T) + a_0(u,t)\|^2 du \\
z_0 &= \frac{1}{\nu}\left[\ln \frac{P_1(0,t)/P_1(0,T)}{P_0(0,t)/P_0(0,T)} + \frac{1}{2}(\mu_T - \mu_t) + \xi - \frac{3}{2}\theta_0\right] \\
z_1 &= \frac{1}{\nu}\left[\ln \frac{P_1(0,t)/P_1(0,T)}{P_0(0,t)/P_0(0,T)} + \frac{1}{2}(\mu_T - \mu_t) + \theta_1 - \xi - \frac{1}{2}\theta_0\right]
\end{aligned} \tag{18}$$

The derivations of the pricing formulas can be found in the Appendix.

Hedge ratios for these types of claims can be found by differentiation of the formulas or by use of Lemma 1.

For more complicated claims it is generally not possible to come up with closed form solutions and numerical computation is hard. Consider for example a European option on a domestic currency coupon bond, i.e. a pay-off of the form

$$\left(\sum_{i=1}^{M} \alpha_i P_0(t,T_i) - K\right)^+ \tag{19}$$

where $\alpha_1, \ldots, \alpha_M, K$ are constants. Since there is not necessarily a finite dimensional Markov representation of the yield curves, calculating the price of the option involves numerical evaluation of an M-dimensional integral over the pay-off multiplied by the joint density of the zero-coupon bonds under the martingale measure with the domestic $\bar{t}$ maturity bond as numeraire.

We will now present a volatility structure that admits a d-dimensional representation of the bond prices of the two-currencies and the exchange rate. Under this structure all European style contingent claims can be evaluated by numerically evaluating a d-dimensional integral. For simplicity we make the assumption

A3: The Brownian motion driving the uncertainty of the economy has dimension three, i.e. $d = 3$.

Let ι_i be the i'th row of the 3×3 identity matrix.

Definition: The Gaussian Volatility Structure

Let

$$\Sigma, \phi : [0, \infty[\to \mathbb{R}^{3\times 3} \tag{20}$$

be continuous matrix functions, with

$$\iota_3' \phi(t) = (1 \quad -1 \quad 0) \tag{21}$$

for all t.
Define the matrix function

$$\Phi : [0, \infty[\to \mathbb{R}^{3\times 3} \tag{22}$$

as the unique solution to the differential equation

$$\frac{d}{dt}\Phi(t) = \phi(t)\Phi(t) \ , \ \Phi(0) = I \tag{23}$$

and assume that Φ *is always non-singular.*
If

$$\begin{aligned} a_i(t,T) &= -\iota_{i+1}' \left[\int_t^T \Phi(y)dy \right] \Phi(t)^{-1}\Sigma(t) \quad , i = 0, 1 \\ \sigma_S(t) &= \iota_3' \Sigma(t) \end{aligned} \tag{24}$$

for all $0 \le t \le T < \infty$*, we will say that the volatility structure is* Gaussian, *and we will term* ϕ *the* mean reversion parameter.

It will now become clear why we term ϕ the mean-reversion parameter and why this class of deterministic volatilities are denoted Gaussian.

Result 1: Term Structure and Exchange Rate Representation under the Gaussian Volatility Structure

i. *Under the Gaussian volatility structure, the bond prices and the exchange rate have the following representation*

$$
\begin{aligned}
P_0(t,T) &= \frac{P_0(0,T)}{P_0(0,t)} e^{-\frac{1}{2}\iota_1' B(t,T)A(t)B(t,T)'\iota_1 + \iota_1' B(t,T)x(t)} \\
P_1(t,T) &= \frac{P_1(0,T)}{P_1(0,t)} e^{-\frac{1}{2}(\iota_2' B(t,T)+2\iota_3'\Phi(t))A(t)B(t,T)'\iota_2 + \iota_2' B(t,T)x(t)} \\
S(t) &= \frac{S(0)P_1(0,t)}{P_0(0,t)} e^{-\frac{1}{2}\iota_3'\Phi(t)A(t)\Phi(t)'\iota_3 + \iota_3'\Phi(t)x(t)}
\end{aligned}
\tag{25}
$$

where

$$
\begin{aligned}
B(t,T) &= -\int_t^T \Phi(y)dy \\
A(t) &= \int_0^t \Phi(u)^{-1}\Sigma(u)\Sigma(u)'\left(\Phi(u)^{-1}\right)' du \\
x(t) &= \int_0^t \Phi(u)^{-1}\Sigma(u)dW_0^t(u)
\end{aligned}
\tag{26}
$$

ii. *The interest rates are given by*

$$
r_i(t) = r_i(t,x(t)) \equiv -\frac{\partial \ln P_i(0,T)}{\partial T}\bigg|_{T=t} + \iota_{i+1}'\Phi(t)x(t) \tag{27}
$$

iii. *If the volatility structure is Gaussian, the system* $(r_0, r_1, \ln S)$ *is a Gaussian process, and the SDE can be written on the form*

$$
d\begin{pmatrix} r_0(t) \\ r_1(t) \\ \ln S(t) \end{pmatrix} = \left(\theta(t) + \phi(t)\begin{pmatrix} r_0(t) \\ r_1(t) \\ \ln S(t) \end{pmatrix}\right)dt + \Sigma(t)dW_0(t) \tag{28}
$$

where $\theta(\cdot)$ *is a three dimensional deterministic function.*

iv. *Suppose the system* $(r_0, r_1, \ln S)$ *is a Gaussian process under* Q_0*, as in (28), satisfying the arbitrage restriction* $E_t^{Q_0}[dS(t)/S(t)] = (r_1(t) - r_0(t))dt$*. Then the volatility structure of the bonds and the exchange rate is Gaussian.*

Result 1 is derived in the Appendix.

We see how the Gaussian volatility structure relate to the property that the spot rates follow three-dimensional Gaussian process. The matrix ϕ relates to the mean reversion of the spot rates and Σ define their local covariance structure. We also see that condition (21) has to be satisfied in order avoid arbitrage. The relation between the volatility structure and the mean-reversion parameter is given by the differential equation (23). For constant ϕ we get that[3]

$$\Phi(t) = e^{\phi t} \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!}\phi^n \tag{29}$$

This matrix function can be determined in closed form after we have identified the eigenvalues and eigenvectors of ϕ. As an example consider the mean-reversion matrix

$$\phi = \begin{pmatrix} -k_1 & 0 & 0 \\ 0 & -k_2 & 0 \\ 1 & -1 & 0 \end{pmatrix} \tag{30}$$

where k_1, k_2 are positive constants. This mean-reversion matrix corresponds to a volatility structure defined by

$$\Phi(t) = e^{\phi t} = \begin{pmatrix} e^{-k_1 t} & 0 & 0 \\ 0 & e^{-k_2 t} & 0 \\ \frac{1-e^{-k_1 t}}{k_1} & -\frac{1-e^{-k_2 t}}{k_2} & 1 \end{pmatrix} \tag{31}$$

and the local covariance matrix $\Sigma(t)\Sigma(t)'$.

The exponential affine representation in (i.) and (ii.) is valid under any of the martingale measures Q_i, Q_i^T. This is due to the fact that the Brownian motions of the martingale measures are related by integrals of deterministic functions as stated in (5) and (6). This also goes for (iii.) of Result 1; the joint process of the spot rates becomes Gaussian under any of the measures Q_i, Q_i^T.

Result 1 is in essence an extension of the results by Babbs (1990) and Jamshidian (1991) to the multicurrency case. The result states that the Gaussian volatility structure obtains if and only if the spot rates are a Gaussian vector process, in which case the model becomes a multidimensional extended version of the Vasicek (1977) model or rather the Hull and White (1990) model, because this model automatically fits the initial term structures.

The affine representation of bond prices and the exchange rate obtained in Result 1 makes it easier to compute contingent claim prices. For European style claims we have the following result.

Result 2: European Style Contingent Claim Prices under the Gaussian Volatility Structure

Suppose the volatility structure is Gaussian. Then a European style contingent claim as the one considered in Lemma 0, has a pay-off that can be written as

$$F(t; \cdot) = F(t, x(t)) \tag{32}$$

[3] If A is a square matrix we define $A^n = \underbrace{A \cdot \ldots \cdot A}_{(n)}$ for $n \geq 1$ and let A^0 be equal to the identity matrix.

and its time 0 price is given by[4]

$$F(0) = P_0(0,t) \int_{\mathbb{R}^3} F\left(t; A(t)^{1/2} z\right) \psi(z) dz \tag{33}$$

where $\psi(z) = \psi(z_1, z_2, z_3)$ *is the normal density function:*

$$\psi(z) = \frac{e^{-\frac{1}{2}\|z\|^2}}{(2\pi)^{3/2}} \tag{34}$$

If we define g *as in Lemma 1 and write* $g(\cdot) = g(x(t))$, $g_j(\cdot) = g_j(x(t))$ *we obtain that the weights of the replicating portfolio of Lemma 1 are given by*

$$\begin{aligned}
\mathrm{E}^{Q_1^t}[g_0(x(T))] &= \int_{\mathbb{R}^3} g_0\left(A(t)\Phi(t)'\iota_3 + A(t)^{1/2} z\right) \psi(z) dz \\
\mathrm{E}^{Q_0^{T_j}}[g_j(x(T))] &= \int_{\mathbb{R}^3} g_j\left(A(t)B(t,T_j)'\iota_1 + A(t)^{1/2} z\right) \psi(z) dz \\
\mathrm{E}^{Q_1^{T_j}}[g_j(x(T))] &= \int_{\mathbb{R}^3} g_j\left(A(t)\left(B(t,T_j)'\iota_2 + \Phi(t)'\iota_3\right) + A(t)^{1/2} z\right) \psi(z) dz
\end{aligned} \tag{35}$$

This makes us able to calculate any European style contingent claim prices and hedge ratios by numerically evaluating (maximum) three dimensional integrals. Using an efficient routine for numerical integration one can obtain these prices and hedge ratios rather fast. Practical experiments suggest that penny accuracy can be reached within seconds.

For general contingent claims we have the partial differential equation stated below.

Result 3: The Fundamental Partial Differential Equation

Assume the volatility structure is Gaussian and consider a contingent claim paying a continuous dividend stream of

$$\alpha(t; S(t), P_0(t,T_1), \ldots, P_0(t,T_M), P_1(t,T_{M+1}), \ldots, P_1(t.T_N)) \tag{36}$$

on $[0,T]$ *and terminal value*

$$F(T; S(t), P_0(t,T_1), \ldots, P_0(t,T_M), P_1(t,T_{M+1}), \ldots, P_1(t,T_N)) \tag{37}$$

We can write

$$\begin{aligned}
\alpha(t;\cdot) &= \alpha(t; x(t)) \\
F(T;\cdot) &= F(T; x(T))
\end{aligned} \tag{38}$$

[4] If A is a covariance matrix then there exists an orthogonal matrix $\mathcal{O}$ and a diagonal matrix $\Lambda = \mathrm{Diag}(\{\lambda_n\})$ with non-negative elements, so that $A = \mathcal{O}\Lambda\mathcal{O}'$. In this case define $A^{1/2} = \mathcal{O}\mathrm{Diag}(\{\sqrt{\lambda_n}\})$.

and under assumption of sufficient regularity we have that the price at time $t \in [0, T]$ satisfies the partial differential equation[5]

$$r_0(t, x)G - \alpha(t, x) = G_t + \iota_1' \Phi(t) A(t) G_x + \frac{1}{2}\text{tr}\left[\dot{A}(t) G_{xx}\right] \tag{39}$$

subject to the boundary condition

$$G(T, x) = F(T, x) \tag{40}$$

Finite difference algorithms can be applied to numerical solution of the partial differential equation. One possibility is to use the implicit ADI method described in Mitchell and Griffiths (1980).

Conclusion

We have described a class of deterministic bond and exchange rate volatilities that lead to a finite dimensional Markov representation of the yield curves of the two currencies by a Gaussian vector process for the spot rates of the economy. The paper thereby extends the results of Babbs (1990) and Jamshidian (1991) to a multicurrency framework.

The resulting model is tractable and at the same time sufficiently general to handle the pricing of derivatives with very complicated pay-off structures.

[5] Note that $\dot{A} \equiv dA/dt = \Phi^{-1}\Sigma\Sigma'(\Phi^{-1})'$

Appendix

Proof of Lemma 0

The pricing equation (8) follows directly from the definition of the martingale measure $\mathcal{Q}_0^t$.
The pricing equation implies that under $\mathcal{Q}_0^t$ the forward prices

$$\frac{P_0(u,T)}{P_0(u,t)}, \qquad \frac{S(u)P_1(u,T)}{P_0(u,t)} \tag{41}$$

are martingales. This means that

$$\begin{aligned} d\frac{P_0(u,T)}{P_0(u,t)} &= \frac{P_0(u,T)}{P_0(u,t)}(a_0(u,T) - a_0(u,t))dW_0^t(u) \\ d\frac{S(u)P_1(u,T)}{P_0(u,t)} &= \frac{S(u)P_1(u,T)}{P_0(u,t)}(a_1(u,T) + \sigma_S(u) - a_0(u,t))dW_0^t(u) \end{aligned} \tag{42}$$

Integration and division now yield the results.

Proof of Lemma 1

Let $u < t$ and consider self-financing portfolio with domestic currency value $V(u)$ and weights $\{\pi_j(u)\}_{i=0,\dots,N}$ ordered as in Lemma 1. Using the proof of Lemma 0 this portfolio value evovles according to

$$\begin{aligned} d\frac{V(u)}{P_0(u,t)} &= \pi_0(u)d\frac{S(u)P_1(u,t)}{P_0(u,t)} + \sum_{j=1}^{M}\pi_j(u)d\frac{P_0(u,T_j)}{P_0(u,t)} \\ &+ \sum_{j=M+1}^{N}\pi_j(u)d\frac{S(u)P_1(u,T_j)}{P_0(u,t)} \\ &= \pi_0(u)\frac{S(u)P_1(u,t)}{P_0(u,t)}(a_1(u,t) + \sigma_S(u) - a_0(u,t))dW_0^t(u) \\ &+ \sum_{j=1}^{M}\pi_j(u)\frac{P_0(u,T_j)}{P_0(u,t)}(a_0(u,T_j) - a_0(u,t))dW_0^t(u) \\ &+ \sum_{j=1}^{M}\pi_j(u)\frac{S(u)P_1(u,T_j)}{P_0(u,t)}(a_1(u,T_j) + \sigma_S(u) - a_0(u,t))dW_0^t(u) \end{aligned} \tag{43}$$

We can write

$$\frac{F(u)}{P(u,t)} = \mathrm{E}_u^{\mathcal{Q}_0^t}[g(f_0(u)\xi_0, \dots, f_N(u)\xi_N)] \tag{44}$$

where

$$f_j(u) = \begin{cases} \frac{S(u)P_1(u,t)}{P_0(u,t)} & ,j=0 \\ \frac{P_0(u,T_j)}{P_0(u,t)} & ,j=1,\dots,M \\ \frac{S(u)P_1(u,T_j)}{P_0(u,t)} & ,j=M+1,\dots,N \end{cases}$$

$$\xi_j = \begin{cases} \Xi(a_1(\cdot,t)+\sigma_S(\cdot)-a_0(\cdot,t)) & ,j=0 \\ \Xi(a_0(\cdot,T_j)-a_0(\cdot,t)) & ,j=1,\dots,M \\ \Xi(a_1(\cdot,t)+\sigma_S(\cdot)-a_0(\cdot,t)) & ,j=M+1,\dots,N \end{cases} \tag{45}$$

$$\Xi(\nu(\cdot)) = e^{-\frac{1}{2}\int_u^t \|\nu(s)\|ds+\int_u^t \nu(s)dW_0^t(u)}$$

If $g(\cdot)$ is a function for which the partial derivatives $\{g_j(\cdot)\}$ interpretated as in (11) are well-defined, then $F(u)/P_0(u,t)$ is at least twice continuous differentiable in $f_j(u)$ for all $u < t$. Using that $F(u)/P_0(u,t)$ is a Q_0^t-martingale, that the joint distribution of $\{\xi_j\}$ is independent of the level of the yield curves or the exchange rate combined, and Ito expansion yield

$$\begin{aligned} d\frac{F(u)}{P_0(u,t)} &= \sum_{j=0}^{N}\left(\mathrm{E}_u^{Q_0^t}[g_j(f_0(u)\xi_0,\dots f_N(u)\xi_N)\xi_j]\right)df_j(u) \\ &= \left(\mathrm{E}_u^{Q_1^t}[g_0(\cdot)]\right)f_0(u)(a_1(u,t)+\sigma_S(u)-a_0(u,t))dW_0^t(u) \\ &+\sum_{j=1}^{M}\left(\mathrm{E}_u^{Q_0^{T_j}}[g_j(\cdot)]\right)f_j(u)(a_0(u,T_j)-a_0(u,t))dW_0^t(u) \\ &+\sum_{j=M+1}^{N}\left(\mathrm{E}_u^{Q_1^{T_j}}[g_j(\cdot)]\right)f_j(u)(a_1(u,T_j)+\sigma_S(u)-a_0(u,t))dW_0^t(u) \end{aligned} \tag{46}$$

The second equality obtains because of the relations (5) and (6) and the Girsanov Theorem. Comparing to (43) yields the result if $V(0) = F(0)$.

Derivation of Equations (13) and (14)

Using Cox, Ingersoll and Ross (1981), Lemma 0, and the definition of W_0^t yields that the futures price is given by

$$\begin{aligned} F(u) &= \mathrm{E}_u^{Q_0}[S(T)] \\ &= \frac{S(u)P_1(u,T)}{P_0(u,T)}e^{-\int_u^T a_0(s,T)(a_1(s,T)+\sigma_S(s)-a_0(s,T))ds} \end{aligned} \tag{47}$$

Hence

$$\begin{aligned} \frac{dF(u)}{F(u)} &= (a_1(u,T)+\sigma_S(u)-a_0(u,T))dW_0(u) \\ &= (a_1(u,T)+\sigma_S(u)-a_0(u,T))\big(dW_0^t(u)+a_0(u,t)'du\big) \end{aligned} \tag{48}$$

and thereby that the option price is given by

$$P_0(0,t)F(0)e^{\mu}\mathrm{E}^{Q_0^t}\left[e^{-\mu}\frac{F(t)}{F(0)}\mathbf{1}_{F(t)>K}\right] - P_0(0,t)\mathrm{E}^{Q_0^t}\left[\mathbf{1}_{F(t)>K}\right] \tag{49}$$

The result now obtains almost immediately after observing that $e^{-\mu}F(t)/F(0)$ is a Girsanov factor that defines a new Brownian motion given by

$$dW_0^t(u) - (a_1(u,T) + \sigma_S(u) - a_0(u,T))'du \tag{50}$$

Derivation of Equation (17)

We note that the price of the claim can be written as

$$\begin{aligned}
&P_0(0,t)\mathrm{E}^{Q_0^t}\left[(R_1(t,T) - R_0(t,T))^+\right] \\
&= \frac{P_0(0,t)}{T-t}\mathrm{E}^{Q_0^t}\left[\left(P_1(t,T)^{-1} - P_0(t.T)^{-1}\right)^+\right] \\
&= \frac{P_0(0,t)}{T-t}\left\{\mathrm{E}^{Q_0^t}\left[P_1(t,T)^{-1}\mathbf{1}_A\right] - \mathrm{E}^{Q_0^t}\left[P_0(t,T)^{-1}\mathbf{1}_A\right]\right\}
\end{aligned} \tag{51}$$

where $A = \left\{P_1(t,T)^{-1} > P_0(t,T)^{-1}\right\}$. From Lemma 0 we have that

$$\begin{aligned}
P_0(t,T)^{-1} &= \frac{P_0(0,t)}{P_0(0,T)}e^{\theta_0}e^{-\frac{1}{2}\theta_0 - \int_0^t (a_0(u,T)-a_0(u,t))dW_0^t(u)} \\
P_1(t,T)^{-1} &= \frac{P_1(0,t)}{P_1(0,T)}e^{\frac{1}{2}(\mu_T-\mu_t+\theta_1)}e^{-\frac{1}{2}\theta_1 - \int_0^t (a_1(u,T)-a_1(u,t))dW_0^t(u)}
\end{aligned} \tag{52}$$

Making use of the probability measures with Brownian motions given by

$$\begin{aligned}
&dW_0^t(u) + (a_0(u,T) - a_0(u,t))dt \\
&dW_0^t(u) + (a_1(u,T) - a_1(u,t))dt
\end{aligned} \tag{53}$$

yields the result.

Proof of Result 1

i. The representation of the domestic bond prices follows directly from Lemma 0. To obtain the relation for the foreign currency bonds and the exchange rate we observe that

$$\iota_3'\frac{d}{dt}\Phi(t) = \iota_3'\phi(t)\Phi(t) = (\iota_1 - \iota_2)'\Phi(t) \tag{54}$$

By integration we get:

$$\iota_3'\Phi(t) = \iota_3'\Phi(u) + (\iota_1 - \iota_2)'\int_u^t \Phi(y)dy \tag{55}$$

Hence, we have that

$$
\begin{aligned}
&a_1(u,T)' + \sigma_S(u)' - a_0(u,t)' \\
&= \Big(\iota_3'\Phi(u) + (\iota_1 - \iota_2)' \int_u^t \Phi(y)dy - \iota_2' \int_t^T \Phi(y)dy\Big)\Phi(u)^{-1}\Sigma(u) \\
&= \Big(\iota_3'\Phi(t) - \iota_2' \int_t^T \Phi(y)dy\Big)\Phi(u)^{-1}\Sigma(u)
\end{aligned} \tag{56}
$$

Inserting this in Lemma 0, yields the result.

ii. Using (i.) and the definition of the spot interest rates yields the result.

iii. From (ii.) we obtain that

$$
dr_i(t) = -\frac{\partial^2 \ln P_i(0,t)}{\partial t^2}dt + \iota_{i+1}'\dot{\Phi}(t)x(t)dt + \iota_{i+1}'\Phi(t)dx(t) \tag{57}
$$

Using the definition of x and the Brownian motion under $\mathcal{Q}_0^t$ we obtain

$$
dx(t) = \Phi(t)^{-1}\Sigma(t)dW_0(t) + A(t)\Phi(t)'\iota_1 \tag{58}
$$

Inserting this in the above stochastic differential equation and using (i.) and (ii.) to represent $x(t)$ as an affine function of $(r_0, r_1, \ln S)(t)$ yields the desired SDE for (r_0, r_1). Applying Ito's lemma to the logarithm of exchange rate yields

$$
d\ln S(t) = \Big(-\frac{1}{2}\left\|\iota_3'\Sigma(t)\right\|^2 + \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} r_0(t) \\ r_1(t) \\ \ln S(t) \end{pmatrix}\Big)dt + \iota_3'\Sigma(t)dW_0(t) \tag{59}
$$

This concludes the proof of (iii.).

iv. Assume that $(r_0(t), r_1(t), \ln S(t))$ follows the stochastic differential equation of (iv.). Observe that

$$
\begin{aligned}
&\begin{pmatrix} r_0(s) \\ r_1(s) \\ \ln S(s) \end{pmatrix} \\
&= \Phi(s)\Bigg(\Phi(t)^{-1}\begin{pmatrix} r_0(s) \\ r_1(s) \\ \ln S(s) \end{pmatrix} + \int_t^s \Phi(u)^{-1}\theta(u)du + \int_t^s \Phi(u)^{-1}\Sigma(u)dW_0(u)\Bigg) \\
&= \Phi(s)\Bigg(\Phi(t)^{-1}\begin{pmatrix} r_0(s) \\ r_1(s) \\ \ln S(s) \end{pmatrix} \\
&+ \int_t^s \Phi(u)^{-1}\big(\theta(u) + \Sigma(u)\Sigma(u)'\iota_3\big)du + \int_t^s \Phi(u)^{-1}\Sigma(u)dW_1(u)\Bigg)
\end{aligned} \tag{60}
$$

Using

$$P_i(t,T) = \mathrm{E}_t^{Q_i}\left[e^{-\int_t^T r_i(s)ds} \right] \tag{61}$$

we obtain

$$P_i(t,T) = g_i(t,T)e^{-\iota_{i+1}\left(\int_t^T \Phi(s)ds\right)\Phi(t)^{-1}z(t)} \tag{62}$$

where $g_i(\cdot,\cdot)$ is some deterministic function and $z = (r_0, r_1, \ln S)$. Ito expansion now yields the result.

Proof of Result 2

Result 1 implies that

$$F(t;\cdot) = F(t;x(t)) \tag{63}$$

Using that

$$x(t) \underset{Q_0^t}{\sim} N(\mathbf{0}, A(t)) \tag{64}$$

yields the first result. The result for the hedging portfolio holdings obtains by use of (5), (6), (56), and the definitions of Result 1.

Proof of Result 3

From the definition of x and the probability measure Q_0^t we have that

$$dx(t) = A(t)\Phi(t)'\iota_1 dt + \Phi(t)^{-1}\Sigma(t)dW_0(t) \tag{65}$$

The Markov property of x under Q_0 implies that the time t price is given by

$$\begin{aligned} G(t) &= \mathrm{E}_t^{Q_0}\left[\int_t^T \frac{\alpha(u;x(u))}{B_0(u)/B_0(t)}du + \frac{F(T;x(T))}{B_0(u)/B_0(t)}\right] \\ &= \mathrm{E}^{Q_0}\left[\int_t^T \frac{\alpha(u;x(u))}{B_0(u)/B_0(t)}du + \frac{F(T;x(T))}{B_0(u)/B_0(t)} \;\middle|\; x(t)\right] \\ &= G(t,x(t)) \end{aligned} \tag{66}$$

Observing that

$$\frac{G(t)}{B_0(t)} + \int_0^t \frac{\alpha(u;x(u))}{B_0(u)}du \tag{67}$$

is a Q_0-martingale, applying Ito's lemma and using the Martingale Representation Theorem yields the partial differential equation.

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