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Modular Functors, TQFTs, and Moduli Spaces of Vector Bundles

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Preface

This thesis is a contribution to an ongoing effort to understand the Witten invariant of 3-manifolds and more generally topological quantum field theories. It attacks this problem from two angles: from the general perspective where it is proved that a modular functor defines a one-parameter family of topological quantum field theories, and on a more particular level where we work towards an explicit calculation of (what is conjecturally) the Witten invariant of finite order mapping tori. Hence, the thesis is divided into two parts devoted to these two subjects.

The manuscript is organized as follows: First, I give a general introduction to the topic of topological quantum field theory and the associated 3-manifold invariants. Then follows the two parts which are described in greater detail below. Finally is given a list of references and a list of notation. To make looking up notation as easy as possible, the list is compiled into two separate lists, one for each part. Both lists are placed in the back for easy access. The entries were printed in order of appearance. For the list of references, on the other hand, it was decided to make only one single list of references due to the significant overlap between references of the two parts and the general introduction.

In *Part I* I substitute the category of framed manifolds with the so called e-category due to K. Walker, [69], where the manifolds are equipped with a more convenient extra structure called the e-structure, which in the closed case is equivalent to the framing. In this category I prove that any 2 dimensional modular functor (with $S_{1,1} \neq 0$) induces a one-parameter family of $2+1$ dimensional topological quantum field theories. In fact I do this for two kinds of modular functors, namely modular functors on the category of extended surfaces and a modular functor on the category of extended surfaces with marked points and directions. Using slicings of 3-manifolds the approach is very similar to that of [69], but unlike Walker I work entirely with smooth manifolds and proofs left out in [69] are provided here.

In *Part II* I embark on a journey towards an explicit calculation of the Witten invariant of finite order mapping tori. The starting point is the rigorous definition of an invariant of 3 dimensional finite order mapping tori, considered by J.E. Andersen in [1], which is conjectured to be the Witten 3-manifold invariant. Knowing the

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fixed point set in the moduli space of flat connections under the action of the surface diffeomorphism that defines the mapping torus, it is essential to calculate the invariant. The main result of this part is a description of the fixed point set in terms of certain parabolic bundles on the quotient surface. Still, too little is known about these moduli spaces in general for me to arrive arrive at my destination: an explicit expression for the invariant, but in special cases more can be said. Thus, I look at the case of hyperelliptic involutions. This part is joint work with my advisor J.E. Andersen.

The manuscript is written using basic AMS-TeX and a set of macros devised by the author to take care of cross-referencing and automatic compilation of contents and list of notation etc. Most of the diagrams are made with XY-pic (though a few were typed using the intrinsic macros of AMS-TeX), and the drawings with TeXdraw.

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Not least I would like to acknowledge my indebtedness to my advisor Jørgen Ellegaard Andersen, who suggested I worked on these problems, for his immense patience, help, guidance and inspiration, and for his careful reading of parts of the manuscript and the numerous suggestions for improvements that followed (all of which I did not have the time to follow up). I very much appreciate his work on my behalf.

There remains to be made but one more acknowledgement: to my dear wife, Trine Grove for her loving support and encouragement, and for her careful help in proofreading and implementing corrections.

Aarhus, October, 1998. Jakob Grove

About the Revised Version. This is a slightly revised version of the submitted dissertation. Some typos has been corrected and most notably the final proof of appendix II.B has been implemented in such a way that it now covers the general case of ramified coverings as presented at the defense. This proves the result added to Theorem II.4.16 that $\overline{\mathcal{EM}}$ is a morphism of varieties. The method of the proof is somewhat different from the strategy outlined in the first version. The layout of the dissertation has also been slightly modified.

Stockholm, March, 1999. Jakob Grove

Contents

General Introduction

In 1988 E. Witten considered the Chern-Simons action on 3-manifolds and defined on the physical level of rigor some new invariants of 3-manifolds using generalized Feynman path integrals

$$Z_k(M) \stackrel{\text{def}}{=} \int_{\mathcal{A}} e^{\frac{ik}{4\pi} CS(A)} DA,$$

where $k \in \mathbb{Z}$ is the so-called quantum level, $\mathcal{A}$ the space of G connections on M, G being a reasonably nice Lie group, and $CS \colon \mathcal{A} \to \mathbb{R}$ the Chern-Simons functional

$$CS(A) \stackrel{\text{def}}{=} \int_M \langle A \wedge dA \rangle + \tfrac{2}{3} \langle A \wedge [A \wedge A] \rangle$$

with A interpreted as an element in the space of 1-forms with values in the Lie algebra $\mathfrak{g}$ of G, $\Omega^1(M; \mathfrak{g})$ (see [72]). His inspiration came from theoretical physics and in fact his intention was not only to get a 3-manifold invariant but to find a 3 dimensional Yang-Mills interpretation of the Jones polynomial, [38]. This he did. Not by using the Yang-Mills functional which depends on a choice of metric, but using the Chern-Simons functional above "taking the expectation value" of the "observable" W_L defined by any framed link $L = \coprod_{i=1}^{l} L_i$ in M colored by representations $R_1, \ldots R_l$ by

$$W_{(L,R)}(A) \stackrel{\text{def}}{=} \prod_{i=1}^{l} \operatorname{Tr}\big(R_i(\operatorname{Hol}_A(L_i))\big), \quad A \in \mathcal{A}.$$

In terms of path integrals this expectation value is written

$$Z_k(M, L, R) \stackrel{\text{def}}{=} \int_{\mathcal{A}} e^{\frac{ik}{4\pi} CS(A)} W_{(L,R)}(A) DA,$$

so the 3-manifold invariant is obtained by taking the empty link, and it is claimed that the invariants of the Jones theory is captured by taking $M = S^3$. Witten also introduced the notion of a topological quantum field theory (TQFT) of which the invariants are part. This associates to each surface a Hilbert space and to each manifold an element of the Hilbert space associated to the boundary. This was of course also defined using path integrals, but soon after the theory was axiomatized by M.F. Atiyah in [6] employing the axioms for a conformal field theory by G. Segal, [61]. These axioms have later been written up more precisely by other people, one of which is K. Walker (see [69]). It is his axioms that are used in the present presentation.

Since the birth of TQFTs in 1988, a lot of work has been put into rigorizing the theory. Most notorious is the construction of N. Reshetikhin and V.G. Turaev (cf. [58]. For every root of unity q they considered the braided tensor category defined by the representation of the quantum group $U_q\big(\mathfrak{sl}_2(\mathbb{C})\big)$. Linking this up with earlier results, [57], on invariants of ribbon graphs derived from quantum groups, they achieved a rigorously defined TQFT for every such q. In fact, for framed links in S^3 they got the (parameterized) Jones polynomial evaluated in q. Another invariant was defined by V.G. Turaev and O.Y. Viro in [67] by the use of $6j$-symbols; this was later discovered to be the norm squared of the Reshetikhin-Turaev invariant. Apart from the examples constructed by V.G. Turaev and H. Wenzl, [68], using other classical Lie algebras, these two theories are so far the only TQFTs known (to mathematics).

There are many other explorations of this field. To mention but a few: R. Kirby and P. Melvin, [40], W.B.R. Lickorish, [45 and 46], S.E. Cappell, R. Lee and E.Y. Miller, [22], L. Crane, [23], M. Kontsevich, [43], T. Kohno, [42], M. Polyak, [55], K. Walker, [69], C. Blanchet, N. Habagger, G. Masbaum and H. Vogel, [18], H. Wenzl, [71], and more recently R. Gelca, [28 and 29] — any omissions are unintentional, however, most likely unavoidable.

This work has been carried out in many ways and in varying generality. The most common approach to the rigorization problem has been to construct an invariant by some other input, and then prove that it satisfies the axioms. The input for these constructions has been gadgets like modular Hopf algebras as mentioned above, conformal field theories, or modular functors, and the method has been to represent the 3-manifolds combinatorially. Most commonly, the technique has been to represent the 3-manifolds by links in S^3 using surgery (cf. [44]) and to construct the theory so that it becomes invariant under Kirby moves (see [39]) as e.g. [58]. Another method is Heegaard splitting, where the theory is defined on handlebodies as e.g. [22]. Finally, an approach relying on slicing up the manifolds using special Morse functions was sketched by M. Kontsevich in [43] on manifolds without framing, and was later extended to manifolds with framing by K. Walker in [69] (actually, he considered manifolds with corners).

Unfortunately, the preprint [69] of K. Walker was never completed, and in its latest version it still contains a few mistakes and some proofs are left out. Part I of this thesis tries to remedy that. The invariant of Witten is in fact not an invariant of 3-manifolds but an invariant of *framed* 3-manifolds. A framing of a closed 3-manifold is a choice of homotopy class of trivialization of the tangent bundle. It is possible to define a framing also for manifolds with boundary but making sense of gluing of framings becomes rather cumbersome. Therefore, we substitute the framing for a more convenient extra structure, which we in accordance with Walker call the e-structure. The e-structure is equivalent to the framing in the case of closed manifolds. Our definition of the e-structure is the same as Walker's except that we are able to specify precisely which extra structures are relevant for our theory.

Actually, we follow the program of Walker quite closely. There is one fundamental difference in the setup: Walker has carried out everything in a special PL-category allowing corners, to be able to use Moore's and Seiberg's basic data to (re-)capture the modular functor (see [49]). We, on the other, hand work entirely in the smooth category. This difference raises some interesting questions about classification of TQFTs which we will return to by the end of section I.5.

Another important difference of course is that we try to give detailed proofs of the statements that now for long have been part of the folklore of the subject. Some of the proofs are probably more detailed than some readers would like them to be and they may seem to state the obvious, but we have tried to leave out as little as possible in order that no stone should be left unturned. Though, admittedly, there are exceptions.

A considerable part of Part I is devoted to defining the modular functors, stating the axioms of a TQFT, and extracting the necessary data from that. As mentioned earlier we use the method used by Kontsevich and Walker of cutting up the manifolds, using special Morse functions, into sufficiently simple pieces on which we are able to define the theory Z. In our case these *slicing functions* need to be especially nice as they are not allowed to have level surfaces crossing the boundary since that would create corners. These special Morse functions are closely related to the so-called framed functions of K. Igusa, [35 and 36].

The main results of this part are Theorem I.5.13, which is stated below, and the analogous Theorem I.6.10, where a family of TQFTs is constructed from a modular functor on the category of extended surfaces and the extended surfaces with marked points and direction respectively.

Theorem I.5.13. *A modular functor V on the category of e-surfaces satisfying that $S_{1,1} \neq 0$, induces a complex one-parameter family of $2+1$ dimensional TQFTs defined by solutions $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ to $\lambda_0\lambda_1 = 1$, $\lambda_1\lambda_2 S_{1,1} = 1$ and $\lambda_2\lambda_3 = 1$. Any two members Z and Z' of this family corresponding to solutions λ_i and λ_i' are related by*

$$Z'(\boldsymbol{M}) = \kappa^{\chi(M)} Z(\boldsymbol{M}),$$

for any e-3-manifold $\boldsymbol{M}$, where $\kappa = \frac{\lambda_0'}{\lambda_0}$. In particular, Z' and Z agree on closed manifolds.

Moreover, if V is unitary we get a family of unitary TQFTs parameterized by S^1, with the extra condition that $\lambda_0 = \bar{\lambda}_3$ and $\lambda_1 = \bar{\lambda}_2$. $\square$

The proof of this theorem uses surprisingly little advanced technology. It is, however, rather technical. The idea is simply to use the Morse flow to pass from one slicing to another, thereby getting the relations needed to ensure that the construction is independent of the slicing. This, however, does not suffice so one needs to deform and extend the flow, and that is where it gets technical.

One should compare this result with the upcoming paper [4] by J.E. Andersen and K. Ueno, where existence of a modular functor on e-surfaces with marked points and directions is proved. Hence, together they prove that to any suitably nice Lie group there is a family of TQFTs all giving the same 3-manifold invariant. The work of Andersen and Ueno builds on the fundamental paper by A. Tsuchiya, K. Ueno and Y. Yamada, [66].

The vector space V constructed in [4] is believed to be the same as the vector spaces devised by a number of other people: S. Axelrod, D. Della Pietra and E. Witten, [12], G. Segal, [61], M.F. Atiyah, [6], N.J. Hitchin, [34], A.A. Beilinson and D. Kazhdan, [14], A.A. Beilinson and V.V. Schechtman, [15], S.E. Cappel, R. Lee and E.Y. Miller, [22], and G. Faltings, [26]. We give a brief description of that construction in the introduction to Part II; for a more detailed exposition tailored for our purpose, see [1]. Here, we shall only say that it is the covariantly constant sections with respect to a projectively flat connection in a vector bundle Z over the Teichmüller space, T, whose fibers are $H^0\big(M(X_\sigma), \mathcal{L}^k\big)$, $\mathcal{L}$ being the determinant line bundle over the moduli space $M(X_\sigma)$ of semistable holomorphic vector bundles, $\sigma \in T$, and k is the so-called quantum level. This vector space is conjectured to be a modular functor. Hence, by the result of Part I it should give a TQFT, and, moreover, this TQFT is believed to be the Reshetikhin-Turaev theory. It follows by the axioms of a TQFT that the partition function $Z(X_\tau)$ of the mapping torus X_τ is $\operatorname{Tr} V(\tau)$.

In his paper, [1], J.E. Andersen considers the number $Z(X_\tau)$ for any orientation preserving a diffeomorphism, τ, of finite order on a closed, compact, and oriented surface, X. By the above, this number is conjecturally the Reshetikhin-Turaev invariant of 3-manifolds. The key idea was that by looking at only finite order diffeomorphisms he was guaranteed that there would be fixed points in the Teichmüller space. Thus, one need not consider covariant constant sections but merely the fiber $H^0\big(M(X_\sigma), \mathcal{L}^k\big)$ over the τ-invariant complex structure σ. In fact, things are a little bit more complicated because there is also a framing correction, so the general expression is

$$Z(\boldsymbol{X_\tau}) = \operatorname{Tr}\big(\tau\big|H^0(M(X_\sigma), \mathcal{L}^k)\big) \cdot \operatorname{Tr}\big(\boldsymbol{\tau}\big|\mathcal{L}_{D,\sigma}^{-\frac{1}{2}c_c}\big).$$

But the framing correction $\operatorname{Tr}\big(\boldsymbol{\tau}\big|\mathcal{L}_{D,\sigma}^{-\frac{1}{2}c_c}\big)$ was calculated in full detail in [1] so that is a matter that we shall disregard completely in this exposition. The bold character symbolizes objects and morphisms in the extended category which is to be defined in Part I.

Using the Lefschetz-Riemann-Roch theorem for finite automorphisms of projective varieties, J.E. Andersen also derives formulae for the first factor of the trace formula, [1, Theorems 9.2 and 9.3]. The invariant is expressed as a sum over the components of the fixed point set in the moduli space, and each term is given by a root of unity to the power of k times a polynomial in k, where each coefficient is expressed as pairings of certain cohomology classes on the fixed point set in $M(X_\sigma)$. On every smooth component Andersen is able to write these coefficients in terms of known generators of the cohomology ring of that component, thereby getting an explicit expression for the contribution to the sum of the smooth components. More precisely, he gets an expression of the form

$$Z_k(X_\tau) = \det(\tau)^{-\frac{1}{2}c_c} \sum_{c \in C} e^{2\pi i k CS(X_\tau, c)} k^{d_c} P_c(k),$$

where C parameterizes the connected components $M(X_\tau)_c$ of the moduli space $M(X_\tau)$ of flat connections over X_τ, d_c is the dimension

$$d_c = \operatorname*{generic\,max}_{A \in M(X_\tau)_c} \frac{1}{2}\big(\dim H^1(X_\tau, d_A) - \dim H^0(X_\tau, d_A)\big),$$

where "generic max" means the maximum obtained over Zariski open subsets, $\det(\tau)^{-\frac{1}{2}c_c}$ is the framing correction, and P_c is the perturbative contribution which is a polynomial in $\frac{1}{k}$ of degree d_c calculated through the Lefschetz fixed point formula. This compares well with the perturbative expansion of the Witten invariant mentioned below.

Part II deals with refining the calculations of J.E. Andersen in [1]. The mapping tori of finite order diffeomorphisms are Seifert fibered 3-manifolds (see [1] and [54]), and the ultimate aim of this project is to give explicit formulae for $Z_k(X_\tau)$ in terms of the Seifert invariants of X_τ. Clearly, this is of little topological interest as these manifolds are well-known and are classified by the Seifert invariants (at least when X_τ is a "large Seifert manifold"). But it may give insight into the invariant Z_k. For instance, one could speculate that it could give a hint on how to define finite type invariants for manifolds which are not homology spheres. More concretely, it will give us the opportunity to compare with calculations derived through other means and using other definitions which are believed to give the same invariant (see below).

To be able to apply algebraic geometry to the problems, we switch the picture to that of semistable holomorphic vector bundles over a Riemann surface and the action of an automorphism.

The major part of Part II deals with determining the fixed point set in the moduli space under the action of a finite order diffeomorphism. We first consider the case where $\pi\colon X \to Y = X/\langle\tau\rangle$ is unramified. In that setting we establish an identification of the fixed point set $|M(X)|$ with a certain quotient of a moduli space $M_{\langle L\rangle}(Y)$ over Y (Proposition II.3.6), and we establish in Proposition II.3.2 that $\pi^*\colon M_{\langle L\rangle}(Y) \to |M(X)|$ is a surjective morphism of varieties.

In order to generalize this to ramified coverings, we have to introduce parabolic structures and use those to transform the pullbacks under π by elementary modifications. This way we get a partial (however, well-understood) identification of this moduli space $Ma(Y)$ of parabolic bundles to a space $Lift(X,\tau)$ of lifts of τ defined in section II.3:

Theorem II.4.16. *Let $\tau\colon X \to X$ be an automorphism of the Riemann surface X. There is an injective map*

$$\mathcal{EM}\colon \mathrm{Ma}(Y) \longrightarrow \mathit{Lift}(X,\tau)$$

between the moduli space of admissible, semistable, parabolic bundles over $Y = X/\langle\tau\rangle$ and isomorphism classes of lifts $(W,\tilde{\tau})$ of τ to semistable holomorphic vector bundles W. The map is one-to-one from the subset of stable points in $\mathrm{Ma}(Y)$, not fixed by the group $\mathcal{L}_\pi$, onto the stable points in $Lift(X,\tau)$, and from the subset of semistable points onto the subset of semistable points represented by invariant line bundles L_1 and L_2. The inverse of $\mathcal{EM}$ on this subset is the restriction of $\mathcal{P}$.

Moreover, the surjective map

$$\overline{\mathcal{EM}}\colon \mathrm{Ma}(Y) \longrightarrow |M(X)|$$

is a morphism of varieties. □

To prove that $\overline{\mathcal{EM}}$ is morphism of varieties, it is necessary to set up the whole construction on universal bundles over the Grothendieck scheme and use the GIT construction of Mumford. That is done in appendix II.B.

There is a well-defined equivalence relation $\sim$ on $Ma(Y)$, and the main result of Part II is:

Corollary II.4.17. *Under the assumptions of the previous theorem, we get an identification*

$$\overline{\mathcal{EM}}\colon \mathrm{Ma}(Y)/\!\sim\; \longrightarrow |M(X)|,$$

under which the stable locus of $|M(X)|$ is identified with the stable points in $\mathrm{Ma}(Y)$ which are not fixed by $\mathcal{L}_\pi$.

Of course we will eventually have to prove that this identification is an isomorphism of varieties.

To do the Lefschetz fixed point formula calculation over all components, we lift the problem to *certain* moduli spaces of parabolic bundles, which are known to be smooth Kähler manifolds. Then it is possible to carry out calculations similar to those of [1], but this time leading to an explicit formula for the complete expression. Currently however, this is far from done. The present text deals only with setting up the machinery to do these calculations. We sketch how the Lefschetz fixed point calculation comes into play. But it seems that we will have to derive a lot of the necessary theory about the cohomology rings of the moduli spaces of parabolic bundles we encounter, before we can go on and do the actual calculations. We also briefly discuss the calculation of d_c and $CS(X_\tau, c)$.

A different approach to the rigorization problem is pertubation theory, where asymptotic behaviour as $k \to \infty$ is studied to find a way around the generalized Feynman path integrals. This technique, originally invented by physicists to be applied in quantum field theory, reduces the problem to studying finite dimensional integrals represented by the so-called Feynman diagrams. Dror Bar-Natan has an introduction to this method in [13]. It is expected that the coefficients to the asymptotic series can be obtained as integrals over the components of the moduli space of flat connections and that it will be of the form

$$Z_k(N) \sim \sum_{c \in C} \int_{A \in M(N)_c} e^{\frac{\pi i}{4}\eta(D_A)} e^{2\pi i k CS(A)} \tau_A^{\frac{1}{2}} k^{d_c} \left(\sum_{i=1}^{\infty} \alpha_c^i(A) \left(\frac{1}{k}\right)^i \right). \qquad (*)$$

(We shall abstain from introducing the terminology not already defined as it is not directly relevant for this discussion.) The so-called finite type invariants are expected to come into the picture as such coefficients and thereby connecting the theory to the study of knots and links.

What might be considered a hybrid calculation is the work of Lev Rozansky ([59] and [60]). He has for a large class of Seifert manifold transformed the Reshetikhin-Turaev theory into a continuous theory where he can apply some heavy machinery from analysis to study its asymptotic behaviour. This way he has apparently calculated the invariant in these cases. However, his calculations use many physics arguments but, recently, Rozansky's calculations have been rigourized by Søren K. Hansen, [32].

It is our hope to compare our results with Rozansky's and Søren K. Hansen's, as this would be a good indication that the Reshetikhin-Turaev theory and the one defined through geometric quantization are the same. And though our manifolds satisfy the relation that $h = 0$ and $b \neq 0$ for the two Seifert invariants h and b, and Rozansky's methods do not apply directly under those conditions, it is claimed by Rozansky that they can be stretched to apply. This is now confirmed through the work of Søren K. Hansen. Hence, the road should be cleared for the comparison of results derived through completely different paths. At the moment, we can say little more than that both methods predict that the asymptotic expressions are actually exact, i.e. we get polynomials and not power series.

Other people besides Lev Rozansky and ourselves have considered similar problems. Lisa Jeffrey has by using an approach similar to ours calculated the invariant in the genus 1 case (and for lens spaces as well), [37]. Because the mapping class group for the torus is so simple, this can be done very explicitly. However, in our calculations we have to assume that genus$(X) \geq 2$, so her setup is disjoint from ours. There is also work from the side of physics due to Matthias Blau, Ian Jermyn and George Thompson, [19], also regarding torus bundles. All of the aforementioned calculations confirm the form asymptotic expression $(*)$ predicted by pertubation theory.

Part I

Constructing TQFTs from Modular Functors

1. Introduction

In this part of the thesis, we consider the problem of constructing a TQFT from a modular functor. This is done for modular functors defined on two different categories of surfaces: the e-category of surfaces and the e-category of surfaces with marked points and directions.

Synopsis. This part of the thesis is organized as follows: *Section 1* is this introduction.

Section 2 motivates and defines the category of extended manifolds. Some properties of Lagrangian subspaces are needed which are defined in appendix A and readers less familiar with this subject may want to consult appendix A first.

Section 3 defines the modular functors on the category of extended surfaces and TQFTs, and derives fundamental observations that we will need in our later construction of the TQFT.

Section 4 deals with the technique of slicing 3-manifolds. The most technical proofs are put off to appendix B.

Section 5 contains the construction of the TQFT, the main result, Theorem 5.13, and its proof.

Section 6 introduces the category of extended surfaces with marked points and directions and modular functors there upon. We then go on and establish a correspondence between these modular functors and those of section 3 and thereby prove the main theorem in this context, Theorem 6.10.

Appendix A is devoted to Lagrangian subspaces and the Maslov index and provides the results required about those in this paper.

Appendix B gives the the proofs of some of the more technical propositions of sections 4. They are quite elementary but are nevertheless included for the paper to be self contained.

Notational remarks. Throughout the paper we use the convention that 3-manifolds have names like M and N, surfaces Y or Σ and 4-manifolds X and W. All homology groups have $\mathbb{R}$-coefficients unless stated otherwise. And all diffeomorphisms are orientation preserving if it is not explicitly written otherwise.

2. THE CATEGORY OF EXTENDED MANIFOLDS

The invariant described by E. Witten in [72] is an invariant of a 3-manifold M with framing, i.e. a choice of homotopy class of trivialization of the tangent bundle T. Later Atiyah suggested in [7], that considering a 2-framing would be more appropriate. A 2-framing is a homotopy class of $Spin(6)$-trivializations of $T \oplus T$.[1] This section will be devoted to the definition of a category which is more convenient to work with, and which can replace the category of (2-)framed manifolds.

We start by motivating the definition of extended manifolds: To any closed 3-manifold M there exists a 4-manifold W such that $M = \partial W$. It is proved in [69] and [7] that any manifold M with 2-framing a is determined via the relative Pontrjagin class by a pair (M, X), where X is a 4-manifold with boundary $\partial X = M$. Two 4-manifolds X and X' determine the same framing if and only if they have the same signature $\sigma(X) = \sigma(X')$. Thus, we may substitute the framing of a closed 3-manifold with an integer. Call this integer the framing number of M. How would one make sense of this if M had a boundary ∂M?

A crucial point is the ability to glue together extended manifolds. Suppose we want to glue together pairs (M_1, n_1) and (M_2, n_2) of manifolds with framing numbers via a diffeomorphism $f : \partial M_1 \to -\partial M_2$. To use the idea above we first choose manifolds M_i^+, $i = 1, 2$, such that $M_i \cup M_i^+$ are closed; and then we may choose 4-manifolds X_i such that $\partial X_i = M_i \cup M_i^+$ and $\sigma(X_i) = n_i$. The gluing $M_1 \cup_f M_2$ is diffeomorphic to $M_1 \cup I_f \cup M_2$, where I_f is the mapping cylinder[2] of f. Choose a 4-manifold W with boundary $\partial W = (-M_1^+) \cup I_f \cup (-M_2^+)$ and some signature $\sigma(W)$. Our task is to assign an integer to the glued manifold $M_1 \cup_f M_2$ in a way that is consistent with the above. To this end we need C.T.C. Wall's theorem about non-additivity of the signature (see appendix A or [70]):

$$\begin{aligned}\sigma(X_1 \cup W \cup X_1) &= \sigma(X_1 \sqcup X_2) + \sigma(W) - \sigma(K_1, K_0, K_2)\\ &= \sigma(X_1) + \sigma(X_2) + \sigma(W) - \sigma(K_1, K_0, K_2),\end{aligned}$$

[1] $2T$ has a natural $Spin(6)$-structure arising via the lift of $SO(3) \overset{\Delta}{\to} SO(3) \times SO(3) \to SO(6)$.
[2] $I_f = (\partial M_1 \times I) \sqcup \partial M_2 / \sim$, where $(x, 1) \sim f(x)$.

where $\sigma(K_1, K_0, K_2)$ is the Maslov index of

$$\begin{aligned} K_0 &= \ker\big(H_1(\partial M_1 \sqcup \partial M_2) \longrightarrow H_1(M_1^+ \sqcup M_2^+)\big), \\ K_1 &= \ker\big(H_1(\partial M_1 \sqcup \partial M_2) \longrightarrow H_1(M_1 \sqcup M_2)\big), \\ K_2 &= \ker\big(H_1(\partial M_1 \sqcup \partial M_2) \longrightarrow H_1(I_f)\big). \end{aligned}$$

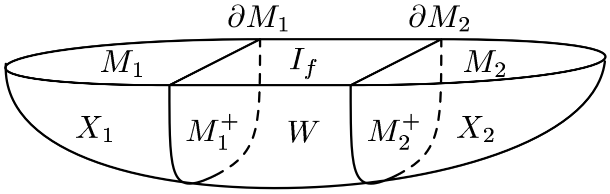

Figure 1. Constructing a 4-manifold with $M_1 \cup_f M_2$ as boundary.

It is well known that all subspaces K in $H_1(Y)$ that can be written as

$$K = \ker\big(H_1(\partial M) \longrightarrow H_1(M)\big)$$

for some 3-manifold M with $\partial M = Y$, are Lagrangian subspace with respect to the intersection pairing, and they are all spanned by their counterpart in $H_1(Y;\mathbb{Z})$. We notice in the above that the only information needed to calculate the framing number of the gluing was the framing numbers of the manifolds participating (including the mapping cylinder) and some Lagrangian subspaces in first homology of their boundaries. This motivates the following definition:

Definition 2.1. By an *extended 3-manifold* (an e-3-manifold) $\boldsymbol{M} = (M, L, n)$ we mean a compact, smooth and oriented 3-manifold M together with a choice of Lagrangian subspace $L \subseteq H_1(\partial M)$ compatible with the integer lattice $H_1(\partial M;\mathbb{Z})$ and a framing number $n \in \mathbb{Z}$.

This definition differs from Walker's as we only need to consider Lagrangian subspaces compatible with the integer lattice. One can think of L as being the kernel in first homology of the embedding of ∂M into some closing-off manifold M^+ (cf. Lemma 2.9) and n as the signature of some 4-manifold X with $\partial X = M \cup M^+$. Throughout the paper we use the convention that elements of the e-category will be written with bold characters and symbols or written as the whole tuple like $\boldsymbol{M} = (M, L, n)$. From now on all maximal isotropic and Lagrangian subspaces are compatible with the integer lattice unless we write otherwise.

To be able to define a modular functor on the e-category, we must allow for e-surfaces with boundary (in which case the intersection pairing may be degenerate). However, an extra choice is needed to make gluing well-defined namely parameterizations of the boundary — i.e. a choice of an orientation preserving map from S^1 to each boundary component. Thus, we define:

Definition 2.2. An *e-surface* $\boldsymbol{Y} = (Y, L)$ is a compact, smooth and oriented surface Y with parameterized boundary and a maximal isotropic subspace[3] $L \subseteq H_1(Y)$ compatible with the integer lattice $H_1(Y;\mathbb{Z})$. If $\boldsymbol{M} = (M, L, n)$ is an e-3-manifold we define the boundary to be $\partial\boldsymbol{M} \overset{\text{def}}{=} (\partial M, L)$. And we say that (Y, L) *contains* (Y_0, L_0), which we write $(Y_0, L_0) \subset (Y, L)$, if $i : Y_0 \hookrightarrow Y$, $i_* L_0 \subset L$, each component of $\partial Y_0 \cap \partial Y$ is a component of ∂Y_0 and ∂Y, and the parameterizations agree.

Notice that $i_* H_1(\partial Y_0) \subseteq L$ since $H_1(\partial Y_0)$ is orthogonal to $H_1(Y)$ with respect to the intersection pairing and therefore $H_1(\partial Y_0) \subseteq L_0$.

Definition 2.3. An *e-morphism* $\boldsymbol{f}\colon \boldsymbol{Y}_1 \to \boldsymbol{Y}_2$ of e-surfaces $\boldsymbol{Y}_i = (Y_i, L_i)$ is an isotopy class of orientation preserving diffeomorphisms $f : Y_1 \to Y_2$ that preserves boundary parameterizations together with an integer n. Hence, we write $\boldsymbol{f} = (f, n)$.

The integer in an e-morphism gives a way of defining an e-mapping cylinder:

Definition 2.4. Let $\boldsymbol{f} = (f, n) : \boldsymbol{Y}_1 \to \boldsymbol{Y}_2$ be a morphism of closed e-surfaces $\boldsymbol{Y}_1 = (Y_1, L_1)$ and $\boldsymbol{Y}_2 = (Y_2, L_2)$. Then the *e-mapping cylinder* $\boldsymbol{I_f}$ of $\boldsymbol{f}$ is the e-3-manifold $\boldsymbol{I_f} \overset{\text{def}}{=} (I_f, L_1 \oplus L_2, n)$.

Since we want $\boldsymbol{I}_{\boldsymbol{f}_2 \circ \boldsymbol{f}_1} = \boldsymbol{I}_{\boldsymbol{f}_2} \cup \boldsymbol{I}_{\boldsymbol{f}_1}$, Wall's theorem tells us how to define the composition of e-morphisms:

Definition 2.5. Let $\boldsymbol{f}_1 = (f_1, n_1) : \boldsymbol{Y}_1 \to \boldsymbol{Y}_2$ and $\boldsymbol{f}_2 = (f_2, n_2) : \boldsymbol{Y}_2 \to \boldsymbol{Y}_3$ be morphisms of e-surfaces $\boldsymbol{Y}_i = (Y_i, L_i)$. Then the *composition*[4] of $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ is

$$(f_2, n_2)(f_1, n_1) \overset{\text{def}}{=} \big(f_2 f_1, n_2 + n_1 - \sigma((f_2 f_1)_* L_1, f_{2*} L_2, L_3)\big).$$

Clearly, $(\mathrm{Id}, 0)$ is the identity e-morphism and by using basic properties of the signature correction term it is easily seen that $(f, n)^{-1} = (f^{-1}, -n)$. The mapping class groups $\Gamma(\boldsymbol{Y})$ of an e-surface $\boldsymbol{Y} = (Y, L)$ is the group of e-automorphisms of $\boldsymbol{Y}$. One can prove that $\Gamma(\boldsymbol{Y})$ is a central extension of the mapping class group $\Gamma(Y)$ of the base surface Y defined by the 2-cocycle $c : \Gamma(Y) \to \mathbb{Z}$, $c(f_1, f_2) = -\sigma\big((f_1 f_2)_* L, f_{1*} L, L\big)$ (see appendix A). Hence, we also call it the e-mapping class group and write it as $\boldsymbol{\Gamma}(Y)$. One can in fact prove that this cocycle is equivalent to the Shale-Weil cocycle induced from the framing (see [7] and [1]). The set of

[3] The distinction between maximal isotropic and Lagrangian subspaces is only due to convention (Lagrangian is used only for non-degenerate forms corresponding here to closed surfaces), since maximal isotropic implies $L^{\perp} = L$.

[4] Please notice that there is a sign error in [69].

e-morphisms of some collection of e-surfaces form the *mapping class groupoid* of that set.

Notice also that for any e-morphism $(f,n):(Y_1,L_1)\to(Y_2,L_2)$, one can factor

$$\begin{aligned}(f,n) &= \{(\mathrm{Id},k):(Y_2,L_2)\to(Y_2,L_2)\}\circ(f,n-k)\\ &= (f,n-k)\circ\{(\mathrm{Id},k):(Y_1,L_1)\to(Y_1,L_1)\}.\end{aligned}$$

In particular, $(\mathrm{Id},m):\boldsymbol{Y}\to\boldsymbol{Y}$ is $(\mathrm{Id},1)^m$.

Definition 2.6. The *disjoint union* of e-3-manifolds is defined as $(M_1,L_1,n_1)\sqcup(M_2,L_2,n_2)\stackrel{\mathrm{def}}{=}(M_1\sqcup M_2,L_1\oplus L_2,n_1+n_2)$. The disjoint union of e-surfaces is $(Y_1,L_1)\sqcup(Y_2,L_2)\stackrel{\mathrm{def}}{=}(Y_1\sqcup Y_2,L_1\oplus L_2)$. E-morphisms on disjoint unions are accordingly $(f_1,n_1)\sqcup(f_2,n_2)\stackrel{\mathrm{def}}{=}(f_1\sqcup f_2,n_1+n_2)$.

Definition 2.7. Let $\boldsymbol{Y}=(Y,L)$ be an e-surface and $g:C\to C'$ a parameterization reversing diffeomorphism of disjoint, closed, codimension 0 submanifolds C and C' of ∂Y. Then we define the *gluing* of $\boldsymbol{Y}$ by g as $\boldsymbol{Y}_g=(Y,L)_g\stackrel{\mathrm{def}}{=}(Y_g,L_g)$, where Y_g is Y glued with g and $L_g=q_*L$ is the image under the quotient map $q:Y\to Y_g$. If $\boldsymbol{f}=(f,n)$ is an e-automorphism of $\boldsymbol{Y}$ such that f commutes with g we define the induced e-morphism $\boldsymbol{f}_g:\boldsymbol{Y}_g\to\boldsymbol{Y}_g$ by $\boldsymbol{f}_g=(f,n)_g\stackrel{\mathrm{def}}{=}(qfq^{-1},n)$.

In the beginning of this section we saw how to glue together two particularly simple e-3-manifolds. Let us now consider the e-3-manifold $\boldsymbol{M}=(M,L,n)$. Assume that the boundary decomposes as $(\partial M,L)=(Y_1,L_1)\sqcup(-Y_2,L_2)\sqcup(Z,J)$. Assume furthermore that there exist 3-manifolds M_1^+, M_2^+ and M^+ such that $\partial M_1^+=-Y_1$, $\partial M_2^+=Y_2$ and $\partial M^+=-Z$ and such that

$$\begin{aligned}L_i &= \ker\big(H_1(Y_i)\to H_1(M_i^+)\big),\\ J &= \ker\big(H_1(Z)\to H_1(M^+)\big).\end{aligned}$$

Then there is a 4-manifold X with boundary $\partial X=M\cup M_1^+\cup M_2^+\cup M^+$ and $\sigma(X)=n$ (see fig. 2).

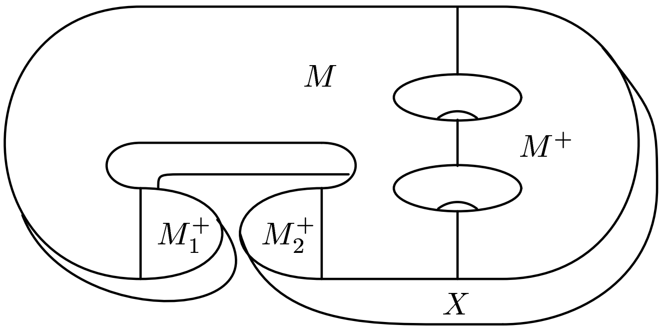

FIGURE 2. The manifold X with boundary $\partial X=M\cup M_1^+\cup M_2^+\cup M^+$.

Suppose that $\boldsymbol{f} = (f, m) : (Y_1, L_1) \to (Y_2, L_2)$ is an e-morphism. Then we can glue M by f to get $M_f \cong M \cup_{Y_1 \sqcup Y_2} I_f$. From the definition of the e-mapping cylinder, $I_{\boldsymbol{f}}$ has framing number m and there is a 4-manifold W such that $\partial W = (-M_1^+) \cup I_f \cup (-M_2^+)$ and $\sigma(W) = m$. This enables us to construct a 4-manifold X' with boundary $\partial X' = M_f \cup M^+$ and the framing number of M_f should be defined to be $\sigma(X')$. We construct X' by gluing in W in X as illustrated in fig. 3.

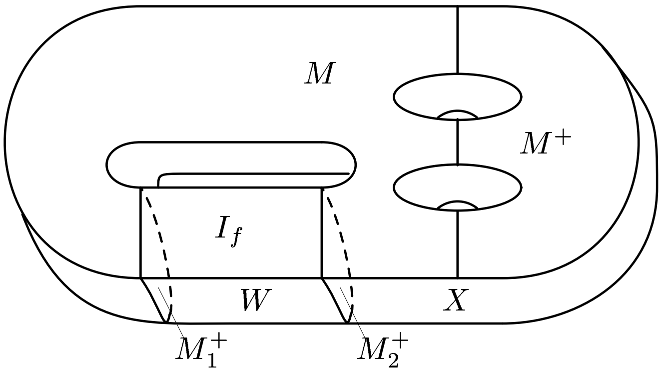

FIGURE 3. The manifold X' with boundary $\partial X = M_f$.

Thus, put

$$\begin{aligned} K_0 &= \ker\big(H_1(Y_1 \sqcup Y_2) \to H_1(M_1^+ \sqcup M_2^+)\big), \\ K &= \ker\big(H_1(Y_1 \sqcup Y_2) \to H_1(M \cup M^+)\big), \\ K_2 &= \ker\big(H_1(Y_1 \sqcup Y_2) \to H_1(I_f)\big). \end{aligned}$$

Then we have that $K_0 = L_1 \oplus L_2$, and $K_2 = \Delta_f^- \overset{\text{def}}{=} \{(x, -f_* x) \mid x \in H_1(Y_1)\}$ is the anti-diagonal under f. By Wall's theorem

$$\sigma(X') = \sigma(X) + \sigma(W) - \sigma(K, L_1 \oplus L_2, \Delta_f^-).$$

Since $H_1(Y_1)$ and $H_1(Y_2)$ project to 0 under the gluing, the subspace L_1 and L_2 vanishes and we have little choice but to define gluing of e-3-manifolds as follows:

Definition 2.8. Let $\boldsymbol{M} = (M, L, n)$ be an e-3-manifold with a decomposition of its boundary $(\partial M, L) = (Y_1, L_1) \sqcup (-Y_2, L_2) \sqcup (Z, J)$ and an e-morphism $\boldsymbol{f} = (f, m) : (Y_1, L_1) \to (Y_2, L_2)$. Then we define the *gluing* of $\boldsymbol{M}$ by $\boldsymbol{f}$ as

$$\boldsymbol{M_f} = (M, L, n)_{(f,m)} \overset{\text{def}}{=} \big(M_f, J, n + m - \sigma(K, L_1 \oplus L_2, \Delta_f^-)\big).$$

Of course one notices that a priori this definition depends on the choice of M^+; however:

Lemma 2.9. *Let Z be a closed surface. To any Lagrangian subspace J of $H_1(Z)$ there is a 3-manifold M^+ with boundary $\partial M^+ = Z$ such that*

$$J = \ker\big(H_1(Z) \to H_1(M^+)\big).$$

If Z furthermore is a closed codimension 0 submanifold of the boundary ∂M of some 3-manifold M, the Lagrangian subspace

$$K = \ker\big(H_1(\partial M - Z) \to H_1(M \cup M^+)\big)$$

is independent of the choice of M^+. Hence, K only depends on J.[5]

Proof. To construct M^+ let $\alpha_1, \dots, \alpha_g$ be a basis for $L_{\mathbb{Z}} = L \cap H_1(Z;\mathbb{Z})$. By Lemma A.4 every α_i is represented by a simple closed curve c_i in Z. That is because we can extend $\alpha_1, \dots, \alpha_g$ to a symplectic basis of $H_1(Z;\mathbb{Z})$ which can be hit by the standard basis under the action of $Sp(2g,\mathbb{Z})$. On the other hand, it is a classical result that the mapping class group $\Gamma(Z)$ maps surjectively onto $Sp(2g,\mathbb{Z})$ (see e.g. [21]). Hence, there is a diffeomorphism of Z that sends generating curves for the a-cycles $a_1, \dots, a_g$ of the standard basis to generating curves for $\alpha_1, \dots, \alpha_g$, which can then consequently be chosen simple and closed.

Cut up Z along $c_1, \dots, c_g$. Since non of the curves are separating, the resulting surface is a $2g$-punctured sphere. Now glue in disks D_i and D_i' in every hole and let the sphere be the boundary of a 3-ball. Gluing together D_i and D_i' for all i gives a topological 3-manifold M'. Up to diffeomorphism there is a unique smoothing M^+ of M'. This 3-manifold M^+ has Z as boundary, and under the embedding $j : Z \hookrightarrow M^+$ we have $j_*\alpha_i = 0$. Thus, by dimension count, $L = \ker\big(H_1(Z) \to H_1(M^+)\big)$.

Suppose M_1^+ and M_2^+ are two 3-manifolds with $\partial M_\nu^+ = Z$ and $L = \ker\big(H_1(Z) \to H_1(M^+{}_\nu)\big)$. Let M be a 3-manifold with boundary $\partial M = Y \sqcup Z$ and let $i : Y \hookrightarrow M$, $j : Z \hookrightarrow M$, $i_\nu : Y \hookrightarrow M \cup M_\nu^+$ and $j_\nu : Z \hookrightarrow M_\nu^+$ be inclusive maps. Consider the diagram formed over the Mayer-Vietoris sequences

$$\begin{tikzcd}
 & H_1(Y) \arrow[rr, equal] \arrow[d, "i_*\oplus 0"'] & & H_1(Y) \arrow[d, "i_{\nu *}"] & \\
\arrow[r] & H_1(Z) \arrow[r, "I_\nu"] & H_1(M) \oplus H_1(M_\nu^+) \arrow[r, "J_\nu"] & H_1(M \cup M_\nu^+) \arrow[r] & {}
\end{tikzcd}$$

where $I_\nu = j_* \oplus j_{\nu *}$ and $J_\nu = k_* - k_{\nu *}$, k and k_ν being the embeddings of M and M_ν^+ into $M \cup M_\nu^+$ respectively. If $x \in \ker i_{1*}$, $(i_*x, 0) \in H_1(M) \oplus H_1(M_1^+)$ is in the kernel $\ker J_1 = \operatorname{Im} I_1$. Let $y \in H_1(Z)$ be such that $I_1(y) = (i_*x, 0) \in H_1(M) \oplus H_1(M_1^+)$. This means that $y \in L$ and therefore, $I_2(y) = (i_*x, 0) \in H_1(M) \oplus H_1(M_2^+)$. Consequently, $x \in \ker i_{2*}$. Reversing the roles or counting dimensions gives the required result that $K = \ker\big(H_1(Y) \to H_1(M \cup M_\nu^+)\big)$ for $\nu = 1, 2$. ☐

[5] In section 5 another interpretation of K is introduced via the Morse flow. The Lagrangian subspace K is then the "flow" Φ of L as discussed in section 5 and appendix A for slices.

3. Modular Functors and TQFTs

In this section we give the axioms for a 2 dimensional modular functor and a $2+1$ dimensional topological quantum field theory and prove some very basic facts derived from the axioms.

Definition 3.1. A *label set* $\mathcal{L}$ is a finite set furnished with an involution $a \mapsto \hat{a}$ and some trivial element 1 such that $\hat{1} = 1$.

For example the set $\mathcal{L}$ could be a set of representations, the involution the operation of taking the dual representation, and 1 corresponding to the trivial representation.

Definition 3.2. Let $\mathcal{L}$ be a label set. A *labeling* of an e-surface $\boldsymbol{Y}$ is an assignment of elements x_i of $\mathcal{L}$ to each component ∂Y_i of ∂Y (this may also be called a labeling of ∂Y). We denote the le-surface by $(\boldsymbol{Y}, l)$, where $l = (x_1, \ldots, x_n)$ is the tuple of assigned labels from $\mathcal{L}$. The set of labelings of $\boldsymbol{Y}$ is denoted by $\mathcal{L}(\boldsymbol{Y})$ (or $\mathcal{L}(\partial Y)$). The category of *$\mathcal{L}$-labeled e-surfaces*, le-surfaces, consists of le-surfaces and e-morphisms preserving the labeling.

We notice that since a closed e-surface cannot carry any labeling it is simply an e-surface.

Definition 3.3. A *modular functor* based on the label set $\mathcal{L}$ is a functor V from the category of le-surfaces to the category of finite-dimensional complex vector spaces and linear isomorphisms satisfying the axioms MF1 to MF6 below.

MF1. *Disjoint union axiom*: For the disjoint union of any pair of le-surfaces there is the identification

$$V\big((\boldsymbol{Y}_1, l_1) \sqcup (\boldsymbol{Y}_2, l_2)\big) = V(\boldsymbol{Y}_1 \sqcup \boldsymbol{Y}_2, l_1, l_2) = V(\boldsymbol{Y}_1, l_1) \otimes V(\boldsymbol{Y}_2, l_2).$$

The identification is associative and compatible with the action of the mapping class groupoid in the sense that $V(\boldsymbol{f}_1 \sqcup \boldsymbol{f}_2) = V(\boldsymbol{f}_1) \otimes V(\boldsymbol{f}_2)$.

MF2. *Gluing axiom*: Let $\boldsymbol{Y} = (Y, L)$ be an e-surface and assume we are given the gluing data of Definition 2.7: Let $g : C \to C'$ be a parametrization reversing diffeomorphism of disjoint, closed, codimension 0 submanifolds C and C' of ∂Y, and let $\boldsymbol{Y}_g$ denote $\boldsymbol{Y}$ glued with g. If l is a labeling of $\boldsymbol{Y}_g$ and x a labeling of C then $(l, x, \hat{x})$ is correspondingly a labeling of $\boldsymbol{Y}$. Then there is an identification

$$V(\boldsymbol{Y}_g, l) = \bigoplus_{x \in \mathcal{L}(C)} V\big(\boldsymbol{Y}, (l, x, \hat{x})\big),$$

which is associative and compatible with the action of the mapping class groupoid.

MF3. *Duality axiom*: For any le-surface $(\boldsymbol{Y}, l)$ we identify the associated vector space

$$V(\boldsymbol{Y}, l) = V(-\boldsymbol{Y}, \hat{l})^*,$$

with the dual of the vector space associated to $(\boldsymbol{Y}, l)$ with opposite orientation. This identification is compatible with orientation reversal, the action of the mapping class groupoid, MF1, and MF2 in the following manner:

- The identifications

$$V(\boldsymbol{Y}) = V(-\boldsymbol{Y})^*,$$
$$V(-\boldsymbol{Y}) = V(\boldsymbol{Y})^*$$

  are mutually adjoint.
- For an le-morphism $\boldsymbol{f} = (f, n) : (\boldsymbol{Y}_1, l_1) \to (\boldsymbol{Y}_2, l_2)$ let $\boldsymbol{f}^- \stackrel{\text{def}}{=} (f, -n) : (-\boldsymbol{Y}_1, \hat{l}_1) \to (-\boldsymbol{Y}_2, \hat{l}_2)$ be the induced le-morphism between the le-surfaces with opposite orientation. Then

$$\langle \alpha, \beta \rangle = \big\langle V(\boldsymbol{f})\alpha, V(\boldsymbol{f}^-)\beta \big\rangle$$

  for all $\alpha \in V(\boldsymbol{Y}_1, l_1)$ and $\beta \in V(-\boldsymbol{Y}_1, \hat{l}_1)$, i.e. $V(\boldsymbol{f}^-)$ is the adjoint inverse of $V(\boldsymbol{f})$.
- For vectors

$$\alpha_1 \otimes \alpha_2 \in V(\boldsymbol{Y}_1 \sqcup \boldsymbol{Y}_2) = V(\boldsymbol{Y}_1) \otimes V(\boldsymbol{Y}_2),$$
$$\beta_1 \otimes \beta_2 \in V(-\boldsymbol{Y}_1 \sqcup -\boldsymbol{Y}_2) = V(-\boldsymbol{Y}_1) \otimes V(-\boldsymbol{Y}_2)$$

  associated to the disjoint union of e-surfaces we have

$$\langle \alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2 \rangle_{\boldsymbol{Y}_1 \sqcup \boldsymbol{Y}_2} = \langle \alpha_1, \beta_1 \rangle_{\boldsymbol{Y}_1} \langle \alpha_2, \beta_2 \rangle_{\boldsymbol{Y}_2},$$

  where the subscript on the brackets refers to the surface to whose vector space the pairing is associated.
- When gluing

$$\oplus_x \alpha_x \in V(\boldsymbol{Y}_g, l) = \bigoplus_x V\big(\boldsymbol{Y}, (l, x, \hat{x})\big),$$
$$\oplus_x \beta_x \in V(-\boldsymbol{Y}_g, \hat{l}) = \bigoplus_x V\big(-\boldsymbol{Y}, (\hat{l}, \hat{x}, x)\big)$$

  we get

$$\langle \oplus_x \alpha_x, \oplus_x \beta_x \rangle_{(\boldsymbol{Y}_g, l)} = \sum_x \hat{S}(x) \langle \alpha_x, \beta_x \rangle_{(\boldsymbol{Y}, (l, x, \hat{x}))},$$

  where $\hat{S}(x) = \hat{S}(x_1) \cdot \ldots \cdot \hat{S}(x_k)$, $x = (x_1, \ldots, x_k)$, and $\hat{S} : \mathcal{L} \to \mathbb{C} - \{0\}$ is a certain function which is part of the data for V.

MF4. *Empty surface axiom*: Let $\emptyset$ denote the empty le-surface. Then

$$V(\emptyset) \cong \mathbb{C}.$$

MF5. *Disc axiom*: Let D be an e-disk (there is only one possible choice of e-structure). Then

$$V(D, a) \cong \begin{cases} \mathbb{C}, & \text{for } a = 1, \\ 0, & \text{for } a \neq 1. \end{cases}$$

MF6. *Annulus axiom*: Let A be an e-annulus, $A \cong S^1 \times I$. (Also here there is only one e-structure.) Then

$$V\big(A, (a, b)\big) \cong \begin{cases} \mathbb{C}, & \text{for } a = \hat{b}, \\ 0, & \text{for } a \neq \hat{b}. \end{cases}$$

A few remarks: The disjoint union axiom MF1 implies that $V(\emptyset) = 0$ or $V(\emptyset) \cong \mathbb{C}$, so MF4 is clearly a non-triviality axiom. MF1 also gives a canonical identification of $V(\emptyset)$ with $\mathbb{C}$, namely the isomorphism that sends $1 \otimes 1 \in V(\emptyset \sqcup \emptyset)$ to $1 \in V(\emptyset)$. Hence, we may actually write $V(\emptyset) = \mathbb{C}$.

By axiom MF2

$$V\big(A, (a, \hat{a})\big) \cong V\big(A \cup A, (a, \hat{a})\big) = \bigoplus_x V\big(A, (a, x)\big) \otimes V\big(A, (\hat{x}, \hat{a})\big)$$

— see fig. 8. Hence, we must demand that $V\big(A, (a, x)\big) = 0$ if $x \neq \hat{a}$ and either $V\big(A, (a, \hat{a})\big) = 0$ or $V\big(A, (a, \hat{a})\big) \cong \mathbb{C}$. If $V\big(A, (a, \hat{a})\big) = 0$, any le-surface $(\boldsymbol{Y}, l)$ with a boundary component labeled by a would have $V(\boldsymbol{Y}, l) = 0$, so MF6 is also a non-triviality axiom.

In addition to the above axioms one may have an extra property namely unitarity:

MF-U. A *unitary modular functor* is a modular functor such that every associated vector space $V(\boldsymbol{Y})$ is furnished with a hermitian inner product

$$(-,-) : V(\boldsymbol{Y}) \otimes \overline{V(\boldsymbol{Y})} \longrightarrow \mathbb{C}$$

so that each morphism is unitary. The hermitian structure must satisfy compatibility properties like the ones in the duality axiom MF3 and commutativity of:

$$\begin{array}{ccc} V(\boldsymbol{Y}) & \xrightarrow[\cong]{} & V(-\boldsymbol{Y})^* \\ \downarrow{\cong} & & {\cong}\downarrow \\ \overline{V(\boldsymbol{Y})^*} & \xrightarrow{\cong} & \overline{V(-\boldsymbol{Y})}, \end{array} \tag{3.4}$$

where the vertical identifications come from the hermitian structure and the horizontal from the duality.

We notice here in particular that for a unitary modular functor $\hat{S}(a)$ is real and positive for all $a \in \mathcal{L}$, since $(\oplus_a \alpha_a, \oplus_a \beta_a)_{(\boldsymbol{Y}_g, l)} = \sum_a \hat{S}(a)(\alpha_a, \beta_a)_{(\boldsymbol{Y},(l,x,\hat{x}))}$.

Observation 3.5. Before we go on and define a topological quantum field theory, we make a trivial however important observation: The morphism $(\mathrm{Id}, 1) : \boldsymbol{Y} \to \boldsymbol{Y}$ of any e-surface is just multiplication by a fixed complex number $\lambda \neq 0$ given as part of the data of V. To see this notice first that this certainly is the case when $\boldsymbol{Y} = D$. Following the argument in the proof of Lemma A.6 we see that any e-surface $\boldsymbol{Y}$ allows a factorization $\boldsymbol{Y} = \boldsymbol{Y}' \cup D$ and using the definition of disjoint union of e-morphisms and the gluing axiom, we may write

$$V(\mathrm{Id}, 1) = V(\mathrm{Id}, 0) \otimes V(\mathrm{Id}, 1) : V(\boldsymbol{Y}', 1) \otimes V_1 \longrightarrow V(\boldsymbol{Y}', 1) \otimes V_1.$$

We call λ the *framing factor* for V.

Definition 3.6. A $2+1$ dimensional *topological quantum field theory* (TQFT) consists of a functor V from closed e-surfaces to complex vector spaces satisfying that $V(\boldsymbol{Y}_1 \sqcup \boldsymbol{Y}_2) = V(\boldsymbol{Y}_1) \otimes V(\boldsymbol{Y}_2)$ and $V(-\boldsymbol{Y}) = V(\boldsymbol{Y})^*$ and an association of a vector

$$\boldsymbol{M} \mapsto Z(\boldsymbol{M}) \in V(\partial \boldsymbol{M})$$

to every e-3-manifold $\boldsymbol{M}$, such that Z satisfies the axioms TQFT1 to TQFT4 below.

TQFT1. *Multiplicativity axiom*: For e-3-manifold $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ we have

$$Z(\boldsymbol{M}_1 \sqcup \boldsymbol{M}_2) = Z(\boldsymbol{M}_1) \otimes Z(\boldsymbol{M}_2) \in V(\partial \boldsymbol{M}_1) \otimes V(\partial \boldsymbol{M}_2).$$

TQFT2. *Naturality axiom*: Let $\boldsymbol{M}_1 = (M_1, L_1, n)$ and $\boldsymbol{M}_2 = (M_2, L_2, n)$ be e-3-manifolds with boundary such that there is an orientation preserving diffeomorphism $f : M_1 \to M_2$ with $(f|_{\partial M_1})_* L_1 = L_2$. Then

$$Z(\boldsymbol{M}_2) = V(f|_{\partial M_1}, 0) Z(\boldsymbol{M}_1).$$

For an e-3-manifold $\boldsymbol{M}$ with a decomposition of its boundary $\partial \boldsymbol{M} = \boldsymbol{Y}_1 \sqcup -\boldsymbol{Y}_2 \sqcup \boldsymbol{Z}$ as in Definition 2.8,

$$V(\partial \boldsymbol{M}) = V(\boldsymbol{Y}_1) \otimes V(\boldsymbol{Y}_2)^* \otimes V(\boldsymbol{Z}),$$

and we may write

$$Z(\boldsymbol{M}) = \textstyle\sum_j \alpha_j \otimes \beta_j \otimes \gamma_j,$$

for $\alpha_j \in V(\boldsymbol{Y}_1)$, $\beta_j \in V(\boldsymbol{Y}_2)^*$, and $\gamma_j \in V(\boldsymbol{Z})$. Assume $\boldsymbol{f} : \boldsymbol{Y}_1 \to \boldsymbol{Y}_2$ is an e-morphism and let $\boldsymbol{M}_{\boldsymbol{f}}$ be $\boldsymbol{M}$ glued with $\boldsymbol{f}$. Then $\partial \boldsymbol{M}_{\boldsymbol{f}} = \boldsymbol{Z}$ and we require:

TQFT3. *Gluing axiom*:

$$Z(\boldsymbol{M_f}) = \textstyle\sum_j \big\langle V(\boldsymbol{f})\alpha_j, \beta_j \big\rangle \gamma_j \in V(\partial \boldsymbol{M_f}) = V(\boldsymbol{Z}).$$

TQFT4. *Mapping cylinder axiom*: Let $I_{\mathbf{Id}}$ be the mapping cylinder of $\mathbf{Id} = (\mathrm{Id}, 0) : \boldsymbol{Y} \to \boldsymbol{Y}$. Then $V(\partial I_{\mathbf{Id}}) = V(\boldsymbol{Y}) \otimes V(\boldsymbol{Y})^*$ and we require that

$$Z(I_{\mathbf{Id}}) = V(\mathbf{Id}) = \mathrm{Id}_{V(\boldsymbol{Y})}\,.$$

When $\boldsymbol{M}$ is closed, the number $Z(\boldsymbol{M})$ is called the *partition function* of $\boldsymbol{M}$.

Again a few remarks are appropriate: Obviously,

$$Z(I_{\mathbf{Id}})^2 = Z(I_{\mathbf{Id}} \cup I_{\mathbf{Id}}) = Z(I_{\mathbf{Id}\circ\mathbf{Id}}) = Z(I_{\mathbf{Id}})$$

so if $\partial\boldsymbol{M} = \boldsymbol{Y}$ we have $Z(\boldsymbol{M}) \in \operatorname{Im} Z(I_{\mathbf{Id}})$. Thus, by restricting the theory to the subspace of $V(\boldsymbol{Y})$ spanned by $Z(\boldsymbol{M})$ for all $\boldsymbol{M}$ with $\partial\boldsymbol{M} = \boldsymbol{Y}$, we loose no information about e-3-manifolds. Hence, axiom TQFT4.

One notices that if $\boldsymbol{M}_1$ is a bordism from $\boldsymbol{Y}_1$ to $\boldsymbol{Y}_2$ and $\boldsymbol{M}_2$ is a bordism from $\boldsymbol{Y}_2$ to $\boldsymbol{Y}_3$, we get

$$\begin{aligned} Z(\boldsymbol{M}_1) &\in V(\boldsymbol{Y}_1)^* \otimes V(\boldsymbol{Y}_2) = \operatorname{Hom}\big(V(\boldsymbol{Y}_1), V(\boldsymbol{Y}_2)\big),\\ Z(\boldsymbol{M}_2) &\in V(\boldsymbol{Y}_2)^* \otimes V(\boldsymbol{Y}_3) = \operatorname{Hom}\big(V(\boldsymbol{Y}_2), V(\boldsymbol{Y}_3)\big), \end{aligned}$$

and it directly follows from the gluing axiom TQFT3 that

$$Z(\boldsymbol{M}_1 \cup \boldsymbol{M}_2) = Z(\boldsymbol{M}_2) \circ Z(\boldsymbol{M}_1).$$

Thus, if $\partial\boldsymbol{M}_1 = -\partial\boldsymbol{M}_2$ ($\boldsymbol{Y}_2 = \boldsymbol{Y}_3 = \emptyset$ in the above) $\boldsymbol{M}_1 \cup \boldsymbol{M}_2$ is closed and

$$Z(\boldsymbol{M}_1 \cup \boldsymbol{M}_2) = \big\langle Z(\boldsymbol{M}_1), Z(\boldsymbol{M}_2)\big\rangle.$$

As with the modular functor we may have a unitary structure in addition:

TQFT-U. A *unitary TQFT* is a TQFT whose associated vector spaces V have Hermitian structures compatible with the disjoint union relation and the orientation reversal relation. Moreover, V satisfies the commutative diagram (3.4), and

$$Z(-\boldsymbol{M}) = \overline{Z(\boldsymbol{M})}$$

under the identification $V(\partial(-\boldsymbol{M})) = \overline{V(\partial\boldsymbol{M})}$. Another way of putting it is that

$$\big\langle Z(\boldsymbol{M}_1), Z(-\boldsymbol{M}_2)\big\rangle = \big(Z(\boldsymbol{M}_1), Z(\boldsymbol{M}_2)\big),$$

whenever $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are e-3-manifolds with a common boundary.

Naturality, gluing and the mapping cylinder axiom combined strengthens the naturality and mapping cylinder axioms to the following two results, which we nevertheless choose to call axioms as they will take the place of TQFT2 and TQFT4:

TQFT4'. *Strong mapping cylinder axiom*: Let $\boldsymbol{f} : \boldsymbol{Y}_1 \to \boldsymbol{Y}_2$ be an e-morphism. Then

$$Z(I_{\boldsymbol{f}}) = V(\boldsymbol{f}).$$

Proof. By the mapping cylinder axiom and the remark above $Z(I_{\boldsymbol{f}}) = Z(I_{\mathbf{Id}_1} \cup I_{\boldsymbol{f}} \cup I_{\mathbf{Id}_2})$, where $\mathbf{Id}_i = (\mathrm{Id}, 0) : \boldsymbol{Y}_i \to \boldsymbol{Y}_i$, $i = 1, 2$. But $I_{\mathbf{Id}_1} \cup I_{\boldsymbol{f}} \cup I_{\mathbf{Id}_2}$ is just the gluing of $I_{\mathbf{Id}_1}$ and $I_{\mathbf{Id}_2}$ via $\boldsymbol{f}$, hence, if $\mathbf{Id}_1 = \sum_j \alpha_j \otimes \alpha_j^*$ and $\mathbf{Id}_2 = \sum_k \beta_k \otimes \beta_k^*$ we have from TQFT3 that

$$Z(I_{\boldsymbol{f}}) = \sum_{j,k} \left\langle V(\boldsymbol{f})\alpha_j, \beta_k^* \right\rangle \alpha_j^* \otimes \beta_k = V(\boldsymbol{f}).$$

□

TQFT2'. *Strong naturality axiom*: Let $\boldsymbol{M}_1 = (M_1, L_1, n_1)$ and $\boldsymbol{M}_2 = (M_2, L_2, n_2)$ be e-3-manifolds with boundary and assume $f : M_1 \to M_2$ is an orientation preserving diffeomorphism. Put $K = \ker(H_1(\partial M_2) \to H_1(M_2))$. Then

$$Z(\boldsymbol{M}_2) = V\big(f|_{\partial M_1}, n_2 - n_1 + \sigma(K, (f|_{\partial M_1})_* L_1, L_2)\big) Z(\boldsymbol{M}_1).$$

Proof. Naturality immediately gives that

$$Z\big(M_2, (f|_{\partial M_1})_* L_1, n_1\big) = V(f, 0) Z(\boldsymbol{M}_1).$$

Let $\boldsymbol{g} = (\mathrm{Id}, m)\colon \big(\partial M_2, (f|_{\partial M_1})_* L_1\big) \to (\partial M_2, L_2)$ and consider the e-3-manifold $\boldsymbol{M}$ constructed as $\big(M_2, (f|_{\partial M_1})_* L_1, n_2\big)$ glued with $I_{\boldsymbol{g}}$ along ∂M_2. The framing number of $\boldsymbol{M}$ is

$$m - \sigma\big(K, (f|_{\partial M_1})_* L_1, L_2\big),$$

so if we let $m = n_2 - n_1 + \sigma\big(K, (f|_{\partial M_1})_* L_1, L_2\big)$, $\boldsymbol{M}$ can be identified with $\boldsymbol{M}_2$. Thus,

$$\begin{aligned} Z(\boldsymbol{M}_2) &= Z\big((M_2, (f|_{\partial M_1})_* L_1, n_2) \cup I_{\boldsymbol{g}}\big) \\ &= V(\boldsymbol{g}) Z\big(M_2, (f|_{\partial M_1})_* L_1, n_1\big) = V(\boldsymbol{g}) V(f|_{\partial M_1}, 0) Z(\boldsymbol{M}_1), \end{aligned}$$

where $V(\boldsymbol{g}) V(f|_{\partial M_1}, 0) = V\big(f|_{\partial M_1}, n_2 - n_1 + \sigma(K, (f|_{\partial M_1})_* L_1, L_2)\big)$. □

We notice that if $\boldsymbol{M}_1$ is a bordism from $\boldsymbol{Y}_1$ to $\boldsymbol{Y}_2$ then

$$V(f|_{\partial M_1}, n_1 + n_2) Z(\boldsymbol{M}_1) = V(f|_{Y_2}, n_2) \circ Z(\boldsymbol{M}_1) \circ V(f|_{Y_1}, n_1)^{-1}.$$

Remark 3.7. If V is part of a modular functor, the strong naturality extends to closed e-3-manifolds in the sense that if $\boldsymbol{M}_1 = (M_1, n_1)$ and $\boldsymbol{M}_2 = (M_2, n_2)$ are two closed e-3-manifolds with an orientation preserving diffeomorphism $f\colon M_1 \to M_2$, then

$$Z(\boldsymbol{M}_2) = \lambda^{n_2-n_1} \cdot Z(\boldsymbol{M}_1),$$

where λ is the framing factor for V.

To see this, cut up M_1 into M_1^0 and M_1^1 and put $M_2^\mu = f(M_1^\mu)$, $\mu = 0, 1$. Then choose e-structures such that $\boldsymbol{M}_\nu = (M_\nu^0, L_\nu^0, n_\nu^0) \cup (M_\nu^1, L_\nu^1, n_\nu^1)$. Out of convenience we put $L_\nu^\mu = \ker\bigl(H_1(\partial M_\nu^\mu) \to H_1(M_\nu^\mu)\bigr)$, because then $L_2^\mu = (f|_{M_1^\mu})_* L_1^\mu$ and $n_\nu = n_\nu^0 + n_\nu^1$. Now, strong naturality implies that

$$Z(\boldsymbol{M}_2^\mu) = V(f|_{\partial M_1^\mu}, n_2^\mu - n_1^\mu)\bigl(Z(\boldsymbol{M}_1^\mu)\bigr) = \lambda^{n_2^\mu - n_1^\mu} \cdot V(f|_{\partial M_1^\mu}, 0)\bigl(Z(\boldsymbol{M}_1^\mu)\bigr)$$

by Observation 3.5, and we get

$$\begin{aligned} Z(\boldsymbol{M}_2) &= \bigl\langle Z(\boldsymbol{M}_2^0), Z(\boldsymbol{M}_2^1)\bigr\rangle \\ &= \Bigl\langle \lambda^{n_2^0-n_1^0} \cdot V(f|_{\partial M_1^0}, 0)\bigl(Z(\boldsymbol{M}_1^0)\bigr), \lambda^{n_2^1-n_1^1} \cdot V(f|_{\partial M_1^1}, 0)\bigl(Z(\boldsymbol{M}_1^1)\bigr)\Bigr\rangle \\ &= \lambda^{n_2-n_1} \cdot \bigl\langle Z(\boldsymbol{M}_1^0), Z(\boldsymbol{M}_1^1)\bigr\rangle = \lambda^{n_2-n_1} \cdot Z(\boldsymbol{M}_1). \end{aligned}$$

Fundamental observations. Since any le-surface can be build out of the fundamental bricks disks, annuli and pair of pants, it is no surprise that these play a special role. For the remaining part of the paper let D denote a fixed disk, A a fixed annulus and P our favorite pair of pants.

Fix a numbering of the boundary components of P. When listing a labeling of P we assume it to be listed in this order. To trace the morphisms of these surfaces we introduce what K. Walker calls seams: Let $0 < \varepsilon < \pi$ be fixed and choose three disjoint properly embedded curves in P joining the point $e^{i\varepsilon}$ on the j'th boundary component with $e^{-i\varepsilon}$ on the $j+1$'th — see fig. 4. Do likewise for D and A. Notice that the surfaces P, A, and D have unique e-structures.

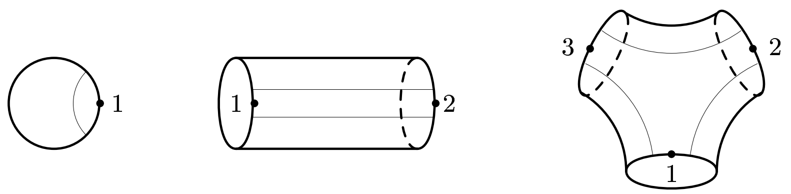

FIGURE 4. Standard surfaces with numberings and seams.

As an abbreviation, define

$$\begin{aligned} V_{abc} &\overset{\text{def}}{=} V\bigl(P, (a, b, c)\bigr), \\ V_{ab} &\overset{\text{def}}{=} V\bigl(A, (a, b)\bigr), \\ V_a &\overset{\text{def}}{=} V(D, a), \end{aligned}$$

for all $a, b, c \in \mathcal{L}$.

Define the standard orientation reversing maps ψ as illustrated in fig. 5. Notice that requiring that ψ fixes the seams, fixes the isotopy class of ψ (as it does with any diffeomorphism). In fact ψ should be considered as an orientation preserving map from the surface to the surface with the opposite orientation.

Notice that $(\psi, 0)$ is its own inverse and induces the identifications

$$V_{abc} = V^*_{\hat{a}\hat{c}\hat{b}},$$
$$V_{ab} = V^*_{\hat{b}\hat{a}},$$
$$V_a = V^*_{\hat{a}},$$

for all $a, b, c \in \mathcal{L}$. In this case we have that for any e-morphism $\boldsymbol{f} = (f, n)$ of D, A or P, $\boldsymbol{f}^- = (\psi f \psi, -n)$.

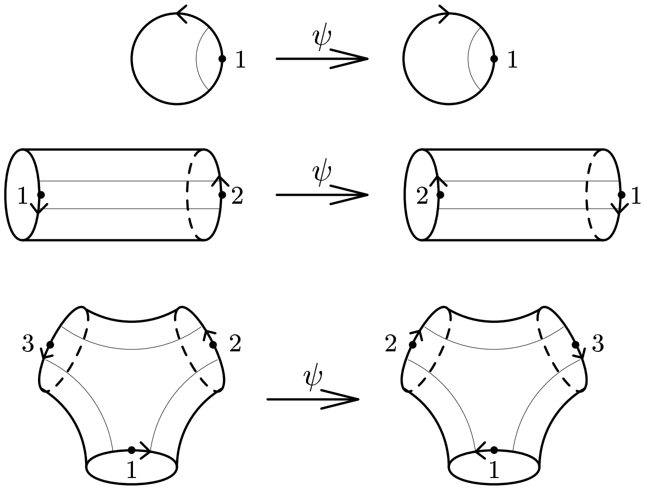

FIGURE 5. Standard orientation reversal.

Now consider the two self-gluings of P (fig. 6) and the map between them inducing the important isomorphism

$$S^a : \bigoplus_x V_{ax\hat{x}} \longrightarrow \bigoplus_y V_{ay\hat{y}}.$$

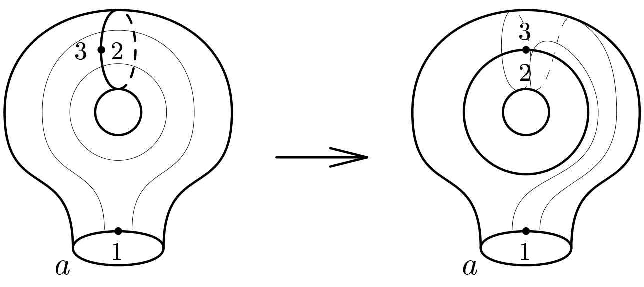

FIGURE 6. Changing the gluing of a pair of pants.

Similarly, we have the isomorphism induced from the two gluings of A (fig. 7)

$$S : \bigoplus_x V_{x\hat{x}} \longrightarrow \bigoplus_y V_{y\hat{y}}.$$

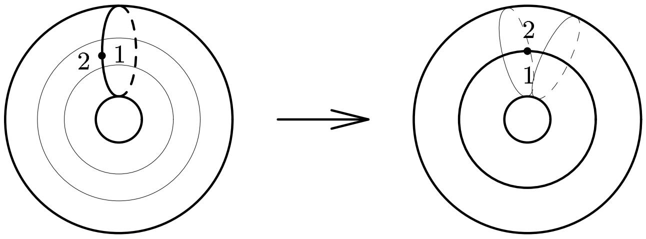

FIGURE 7. Changing the gluing of an annulus.

Notice that S is what we get from the map S^1 when we glue in a disk in fig. 6.

The gluing of A with itself as depicted in fig. 8 gives an isomorphism

$$V_{a\hat{a}} \longrightarrow V_{a\hat{a}} \otimes V_{a\hat{a}}.$$

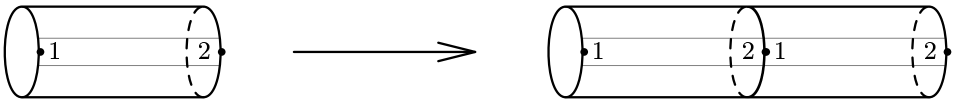

FIGURE 8. Diffeomorphic cylinders with numberings and seams.

This isomorphism is canonical after the seams have been chosen since they fix the isotopy class of the product structure of $A \cong S^1 \times I$. It is easily seen that there is a unique non-zero element $\beta_{a\hat{a}} \in V_{a\hat{a}}$, such that

$$\beta_{a\hat{a}} \mapsto \beta_{a\hat{a}} \otimes \beta_{a\hat{a}}$$

under this isomorphism.

In fact, the preferred elements $\beta_{a\hat{a}}$ satisfy an even stronger property: Assume that $(\boldsymbol{Y}, l)$ is an le-surface with a distinguished boundary component labeled by a and let $\boldsymbol{Y} \cup A$ denote the gluing of this boundary component onto boundary component number 2 of A. Then there is a diffeomorphism

$$f : Y \longrightarrow Y \cup A,$$

such that f is isotopic to $\mathrm{Id} : Y \to Y \subseteq Y \cup A$ via the preferred product structure on A. (By this we mean that there is a collar C of this particular boundary component of Y such that the isotopy f_t is the identity on $Y - C$ and $f_t|_{C\cup A}(\lambda, s) = \big(\lambda, \kappa_t(s)\big)$

for $\lambda \in S^1$ and $s \in]-\varepsilon, 1]$, where κ_t is some nice family of contractions of $]-\varepsilon, 1]$ to $]-\varepsilon, t]$.) This fixes the isotopy class of f. Put $\boldsymbol{f} = (f, 0)$ and there is an induced isomorphism

$$V(\boldsymbol{f}) : V(\boldsymbol{Y}, l) \longrightarrow V(\boldsymbol{Y}, l) \otimes V_{a\hat{a}}.$$

Clearly, there is an automorphism F of $V(\boldsymbol{Y}, l)$ such that $V(\boldsymbol{f})(x) = F(x) \otimes \beta_{a\hat{a}}$, since $\beta_{a\hat{a}}$ is a generator of $V_{a\hat{a}}$. By associativity of gluing we have that

$$F^2(x) \otimes \beta_{a\hat{a}} \otimes \beta_{a\hat{a}} = F(x) \otimes \beta_{a\hat{a}} \otimes \beta_{a\hat{a}},$$

since we can pull out one copy of A from the boundary of Y and then another between them, or we can pull out one copy and then use the identification of fig. 8. This means that F is an idempotent. On the other hand, F was an automorphism, hence, we conclude that

$$V(\boldsymbol{f})(x) = x \otimes \beta_{a\hat{a}}.$$

The isomorphism $S : \bigoplus_x V_{x\hat{x}} \to \bigoplus_y V_{y\hat{y}}$ has a matrix with respect to the two bases $\beta_{x\hat{x}}$ and $\beta_{y\hat{y}}$ in $V(T^2)$ coming from the two annuli decompositions of T^2:

$$S\beta_{x\hat{x}} = \sum_y S_{xy}\beta_{y\hat{y}}.$$

We also get preferred generators of V_1 by using the standard orientation reversing le-morphism on D to get a non-degenerate pairing

$$V_1 \otimes V_1 \longrightarrow \mathbb{C}.$$

Now we can choose $\beta_1 \in V_1$ such that

$$\beta_1 \otimes \beta_1 \mapsto 1.$$

There are exactly two possibilities of which we choose one, but there will be no ambiguities where we are going to use them, since they will occur in pairs at all crucial places.

Gluing in a disk in the pair of pants P results in the isomorphism

$$V_{1a\hat{a}} \otimes V_1 \longrightarrow V_{a\hat{a}}.$$

This introduces generators $\beta_{1a\hat{a}}$ as being the unique elements mapped to $\beta_{a\hat{a}}$. As noted before the isomorphism S^1 corresponds to S under the gluing of a disk. Hence, we have that $S^1 : \bigoplus_x V_{1x\hat{x}} \to \bigoplus_y V_{1y\hat{y}}$ has the matrix $S^1_{xy} = S_{xy}$ such that

$$S^1\beta_{1x\hat{x}} = \sum_y S_{xy}\beta_{1y\hat{y}}.$$

By allowing manifolds with corners (or PL structure), Walker easily proves in [69], that if V is part of a TQFT with corners, then in fact

$$\hat{S}(x) = S_{1x}.$$

Thus, by proving that a modular functor induces a TQFT it is apparent that this relation is an intrinsic part of the modular functor. Likewise for the isomorphism F associated to the changing of the gluing of the two pairs of pants in fig. 9, Walker gets the relation

$$F(\beta_{1\hat{a}a} \otimes \beta_{1\hat{b}b}) = \bigoplus_c \frac{\mathrm{Id}_{V_{cab}}}{\hat{S}(a)\hat{S}(b)},$$

needed to be satisfied in order to get the TQFT. One could speculate that maybe these relations are part of a difference between Walker's theory with corners and the smooth theory discussed here.

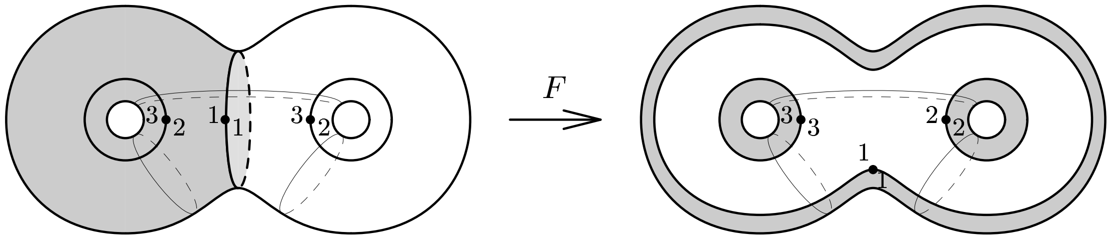

FIGURE 9. Changing the glueing of two pairs of pants.

Now, let $(S^2, x_1, \ldots, x_n) = S^2_n$ be the n-punctured sphere with numbered, parametrized boundaries, where the x_i's represents the center of the deleted disks. Let T_i be the Dehn twist along the i'th boundary for $i = 1, \ldots, n$ and consider the mapping classes ω_i, $i = 1, \ldots, n-1$, generated by diffeomorphisms h_i. We define h_i by choosing a twice-punctured disk D_i covering a neighbourhood of x_i and x_{i+1} in such a way that it does not meet x_j, $j \notin \{i, i+1\}$. We then let h_i be a diffeomorphism that is the identity on $S^2 - \{x_1, \ldots, x_n\} - D_i$ and which interchanges x_i and x_{i+1} preserving boundary parameterizations (cf. $\omega_2 : P \to P$ in fig. 10).

Proposition 3.8. *For $n \geq 2$ the mapping class group $\Gamma(S^2_n)$ of the n-punctured sphere $(S^2, x_1, \ldots, x_n)$ is generated by $T_1, \ldots, T_n$ and $\omega_1, \ldots, \omega_{n-1}$. They satisfy the properties:*

(1) $T_i T_j = T_j T_i$, *for all* i, j,
(2) $T_i \omega_j = \omega_j T_i$, *for* $i \neq j, j+1$,
(3) $T_i \omega_i = \omega_i T_{i+1}$ *and* $T_{i+1}\omega_i = \omega_i T_i$, *for all* i,
(4) $\omega_i \omega_j = \omega_j \omega_i$, *for* $|i-j| \geq 2$,
(5) $\omega_i \omega_{i+1} \omega_i = \omega_{i+1} \omega_i \omega_{i+1}$, *for all* i,
(6) $\omega_1 \ldots \omega_{n-2} \omega_{n-1}^2 \omega_{n-2} \ldots \omega_1 \in \langle T_1, \ldots, T_n \rangle$,
(7) $(\omega_1 \omega_2 \ldots \omega_{n-1})^n \in \langle T_1, \ldots, T_n \rangle$.

If $n = 0$ *or* $n = 1$, $\Gamma(S^2, x_1, \dots, x_n) = \{1\}$.

Proof. For $n = 0$ or 1 the result follows directly from [17, Theorem 4.5], since the Dehn twist is trivial. (Actually it also follows from the earlier proof of Smale in [64], since his explicit retraction of the group of diffeomorphisms of S^2 behaves well with respect to one fixed point.)

For $n \geq 2$ Birman calculates the mapping class group of S^2_n, consisting of diffeomorphisms not necessarily preserving boundary parameterizations. She arrives at the result that this is generated by $\omega_1, \dots, \omega_{n-1}$ satisfying the defining relations

(a) $\omega_i\omega_j = \omega_j\omega_i$, for $|i-j| \geq 2$,
(b) $\omega_i\omega_{i+1}\omega_i = \omega_{i+1}\omega_i\omega_{i+1}$, for all i,
(c) $\omega_1 \dots \omega_{n-2}\omega_{n-1}^2\omega_{n-2}\dots\omega_1 = 1$,
(d) $(\omega_1\omega_2\dots\omega_{n-1})^n = 1$.

Let $T = \langle T_1, \dots, T_n\rangle$ be the subgroup of $\Gamma(S^2_n)$ generated by the boundary Dehn twist. To see that $\Gamma(S^2_n)$ is generated by T and the ω_i's, we must prove that two diffeomorphisms that are isotopic in the sense of Birman, are isotopic in our sense modulo T:

First we write up an explicit isotopy taking any smooth family of diffeomorphisms $f_t \colon S^1 \to S^1$ to the identity family. Choose a parameterization of S^1 and let 0 be the initial point. If we visualize f as a diffeomorphism of $I \times S^1$, the curve $f_t(0)$ for $t \in I$ may spiral along the annulus, twisting several times. Let $\tau(t)$ be the net amount of twisting at t and let $\rho_s(f)_t \colon S^1 \to S^1$ be the diffeomorphism you get by first acting with f_t and then rotating backwards by $s\tau(t)$. Then $\rho_0(f)_t = f_t$ and $\rho_1(f)_t$ is a diffeomorphism that preserves the initial point for all t. Such diffeomorphisms can be identified with the diffeomorphisms h of the unit interval that satisfy: $h(0) = 0$, $h(1) = 1$ and $\frac{d^k}{dx^k}\big|_{x=0} h = \frac{d^k}{dx^k}\big|_{x=1} h$, for all $k \geq 1$. But they are linearly isotopic to the identity by $L_s(h)(x) = sx + (1-s)h(x)$. It must be checked that $L_s(h)$ is a diffeomorphism satisfying the above properties, but that is elementary and is left to the reader. Now, choose a smooth function $\psi \colon I \to I$ so that ψ is 0 in a neighbourhood of 0 and 1 in a neighbourhood of 1 and so that $\frac{d}{ds}\psi \geq 0$ for all s. Then we define the smooth isotopy $H \colon I \times I \times \mathrm{Diff}(S^1) \to I \times \mathrm{Diff}(S^1)$ as

$$H_s(f)_t = \begin{cases} \rho_{\psi(2s)}(f)_t, & \text{for } 0 \leq s \leq \frac{1}{2}, \\ L_{\psi(2s-1)}(\rho_1(f)_t), & \text{for } \frac{1}{2} \leq s \leq 1. \end{cases} \tag{3.9}$$

Now, suppose $f_t \colon S^2_n \to S^2_n$ is an isotopy not necessarily fixing the boundary parameterization. Assume however that f_0 and f_1 do. Choose a closed collar $C \cong \partial S^2_n \times I$ (such that ∂S^2_n corresponds to $\partial S^2_n \times \{1\} \subset C$) of the boundary small enough so that we can isotope f_t to act on $B = \partial S^2_n \times \{0\}$. Call it f_t again. Put

$$\tilde{f}_t(x) = \begin{cases} f_t(x), & \text{for } x \in S^2_n - C, \\ H_s(f|_B)_t(\lambda), & \text{for } (\lambda, s) \in C. \end{cases}$$

Then $\tilde{f}_t$ fixes ∂S^2_n pointwise and $\tilde{f}_0 = f_0$, but since there may have been twistings going on along f_t at the boundaries that were untwisted in $\tilde{f}_t$, we only have that $\tilde{f}_1 \equiv f_1 \mod T$.

This means that the kernel of the natural map $\Gamma(S^2_n) \to \Gamma_{\text{Birman}}(S^2_n)$ is T, and there are no other generators for $\Gamma(S^2_n)$ besides T and $\omega_1, \ldots, \omega_{n-1}$. We now start calculating the relations. Of course all of Birman's relations (a) to (d) apply modulo T, which accounts for item (6) and item (7). But we can do better than that. Clearly, the boundary Dehn twists commute, so item (1) is evident, and so is item (2) and item (4), since ω_j is the identity outside a neighbourhood of the j'th and the $(j+1)$'th boundary component. To verify item (3) and item (5) it is enough to check them on $S^2_3 = P$ as we can cut out a pair of pants so the mapping classes act trivially on the complement. This is left to the reader. □

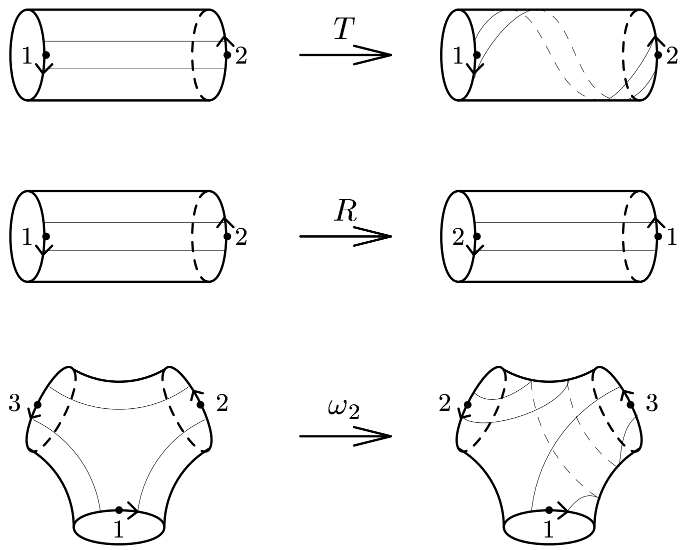

FIGURE 10. Mapping class group generators for A and P.

Of course it should be possible to specify which elements of T you get in item (6) and item (7), but we will refrain from doing so and only calculate it in the cases that we need, $n = 1, 2$ and 3. Thus, the e-mapping class group $\mathbf{\Gamma}(D) = \mathbb{Z}$ of the disk is generated by $(\text{Id}, 1)$. The e-mapping class group $\mathbf{\Gamma}(A)$ of the annulus is generated by $(\text{Id}, 1)$, the left-handed Dehn twist T, and the rotation R taking one end to the other (see fig. 10) clearly satisfying that $RTR = T$. Its action in the associated vector spaces is easily computed (in fact we shall only need the action on the vector spaces labeled by 1): The action of $(\text{Id}, 1)$ is determined in Observation 3.5. Since $V_{a\hat{a}}$ is one dimensional, T acts as multiplication by a non-zero complex number, call it t_a. Now that $RTR = T$ and $V(R) : V_{a\hat{a}} \to V_{\hat{a}a}$ we have $t_{\hat{a}} = V(R)t_aV(R) = t_aV(R)V(R) = t_a$. Most importantly we see that $t_1 = 1$ because the Dehn twist is trivial on the disk, hence, also on the gluing

$V(D \cup A, 1) = V_1 \otimes V_{11}$, $\mathrm{Id} \otimes T = T_D \otimes \mathrm{Id} = \mathrm{Id}$. Now, $V(R)\beta_{a\hat{a}} = r\beta_{\hat{a}a}$ for some $r \in \mathbb{C} - \{0\}$, but under the gluing giving the isomorphism $\beta_{a\hat{a}} \mapsto \beta_{a\hat{a}} \otimes \beta_{a\hat{a}}$ (fig. 8) we have

$$r\beta_{\hat{a}a} \otimes r\beta_{\hat{a}a} = r(\beta_{\hat{a}a} \otimes \beta_{\hat{a}a}).$$

Hence, $V(R)\beta_{a\hat{a}} = \beta_{\hat{a}a}$. In particular, $V(R)\beta_{11} = \beta_{11}$.

Notice that because we can factor out an annulus at every boundary component of a surface Y, the action of a Dehn twist along a boundary component coloured by a is given by t_a.

The e-mapping class group $\mathbf{\Gamma}(P)$ of the pair of pants is generated by $(\mathrm{Id}, 1)$, the Dehn twists T_i, $i = 1, 2, 3$, around each boundary component (as with A), and ω_j, $j = 1$ and 2, as in the proposition above and illustrated in fig. 10. Using Proposition 3.8 and one extra calculation or drawing we get that these generators satisfy the relations

$$\begin{gathered}
[(\mathrm{Id}, 1), T_i] = [(\mathrm{Id}, 1), \omega_k] = [T_i, T_j] = (\mathrm{Id}, 0), \quad i, j = 1, 2, 3, \quad k = 1, 2, \\
T_1\omega_1 = \omega_1 T_2, \quad T_2\omega_1 = \omega_1 T_1, \quad T_3\omega_1 = \omega_1 T_3, \\
T_1\omega_2 = \omega_2 T_1, \quad T_2\omega_2 = \omega_2 T_3, \quad T_3\omega_2 = \omega_2 T_2, \\
\omega_1^2 = T_1 T_2 T_3^{-1}, \quad \omega_2^2 = T_2 T_3 T_1^{-1}, \\
\omega_1\omega_2\omega_1 = \omega_2\omega_1\omega_2.
\end{gathered}$$

The twists act trivially on V_{111} and clearly ω_j must act as either Id or $-\mathrm{Id}$. However, consider ω_2 and the gluing of a disk to boundary component number 1 of P: Then $V(\omega_2) \otimes V(\mathrm{Id}) = V(T) = \mathrm{Id}_{V_{11}}$, hence, $V(\omega_2)$ acts as Id on V_{111}, and thus also $V(\omega_1)$ is Id on V_{111}.

4. Slicings of 3-manifolds

The purpose of this section is to provide the necessary technical background for our construction of the TQFT, namely the slicings of 3-manifolds possibly with boundary. For the proof of the main result of this section (Remark 4.13), we refer to K. Igusa, [35] and [36]. Although most of what goes on here could easily be generalized to other dimensions we specialize to dimension 3.

Definition 4.1. A *slicing function* f on a smooth compact 3-manifold M with boundary ∂M is a smooth function $f : M \to \mathbb{R}$ such that

(1) the restriction $f|_{\dot{M}}$ to the interior is a Morse function with distinct critical values,
(2) f has no critical points on the boundary,
(3) f is locally constant on the boundary.

We denote the set of slicing functions on M by $\mathcal{S}(M)$. Two slicing functions are said to be *unidirectional* if their tangent vector fields points in the same direction on every boundary component of M (either in- or outwards).

Often we will write a slicing function f together with a set of real numbers $t_0, \dots, t_k$ as $(f; t_0, \dots, t_k)$, indicating a choice of regular values of f such that

- there are no critical values below t_0,
- there are no critical values above t_k,
- there are at most one critical point between t_i and t_{i+1} for $0 \leq i \leq k-1$.

Note that this implies that $M \cong f^{-1}([t_0, t_k])$.

We furnish the space of smooth functions on M with the Whitney C^∞ topology (see e.g. [30]). As with ordinary Morse functions the slicing functions appear to occupy most of the space of smooth functions that are constant and non-singular on the boundary:

$$C^\infty_\partial(M) \stackrel{\text{def}}{=} \{\, f \in C^\infty(M) \mid d(f|_{\partial M}) = 0, d_x f \neq 0 \text{ for all } x \in \partial M \,\}.$$

Lemma 4.2. *Let M be a compact manifold with boundary. The set $\mathcal{S}(M)$ of slicing functions forms a dense open subset of $C^\infty_\partial(M)$.*

Proof. There is an open inclusion $i : C^\infty(M) \hookrightarrow C^\infty(\dot{M})$ into the smooth functions on the interior of M. For any slicing function $f \in \mathcal{S}(M)$, $i(f)$ is a Morse function with distinct critical values and therefore there is an open neighbourhood W of $i(f)$ in $C^\infty(\dot{M})$ containing only such functions. Now, $i^{-1}(W) \cap C^\infty_\partial(M)$ is an open neighbourhood of f in $C^\infty_\partial(M)$ consisting of functions whose restriction to $\dot{M}$ are Morse functions with distinct critical values. We need to make sure that they have no critical points on the boundary.

For all $x \in \partial M$, $d_x f \neq 0$, so by compactness we can find an open neighbourhood U around the section $j^1 f$ in the 1-jet bundle $J^1(M)$ so that $j^1 g(x) \neq (x, g(x), 0)$ for all $x \in \partial M$ and $j^1 g \in U$. Hence, for a member $M_1(U) = \{\, f \in C^\infty(M) \mid j^1 f(M) \subset U \,\}$ of the basis for the topology on $C^\infty(M)$, $i^{-1}(W) \cap C^\infty_\partial(M) \cap M_1(U)$ is an open neighbourhood of f in $C^\infty_\partial(M)$ consisting of slicing functions alone and consequently $\mathcal{S}(M)$ is open in $C^\infty_\partial(M)$.

Now, let $f \in C^\infty_\partial(M)$ and let Ω be an open neighbourhood of f in $C^\infty_\partial(M)$. There is an open neighbourhood $\tilde{\Omega}$ of f in $C^\infty(M)$ such that $\Omega = \tilde{\Omega} \cap C^\infty_\partial(M)$ and then $i(\tilde{\Omega})$ is an open neighbourhood of $i(f)$ in $C^\infty(\dot{M})$. This means that there is a $g \in C^\infty(M)$ such that $g \in \tilde{\Omega}$ and $g|_{\dot{M}}$ are Morse functions and have distinct critical values. Provided Ω is small enough $d_x g \neq 0$ for all $x \in \partial M$; in fact by choosing Ω sufficiently small we can make sure that dg is directed outwards or inwards in accordance with df on ∂M (though $d_x(g|_{\partial M})$ may be different from zero). What is left is to straighten g out to become constant on the boundary:

Assume that g is directed outwards on ∂M — if not, use the following on each component of ∂M and change signs when needed. Choose a collar C on M such that g has no critical points therein. Assume for simplicity that $C = \partial M \times]0,1]$ on M. Then choose a smooth function $\varphi\colon]0,1] \to [0,1]$ such that $\varphi\big(]0,\varepsilon[\big) = 0$, $\varphi\big(]1-\varepsilon,1]\big) = 1$ for some $0 < \varepsilon < \frac{1}{2}$ and $\varphi'(t) \geq 0$ for all t. Define $\bar{g}\colon \partial M \times]0,1] \to \mathbb{R}$ by $\bar{g}(x,t) = t + g_{\max}$, where $g_{\max} = \sup_{\partial M \times]0,1]}\{g(x,t)\}$ and then let

$$\tilde{g}(x,t) = \varphi(t)\bar{g}(x,t) + \big(1 - \varphi(t)\big)g(x,t)$$

on the collar and define it to be g elsewhere. Clearly, $\tilde{g}$ is smooth, and on the collar the derivative in the t-direction is

$$\varphi'(t)\big(t + g_{\max} - g(x,t)\big) + \varphi(t) + \big(1 - \varphi(t)\big)\frac{\partial g}{\partial t}(x,t) > 0,$$

since $\varphi(t) \geq 0$ and $\big(1-\varphi(t)\big) \geq 0$ are never zero at the same time and the remaining terms are always larger that zero. Thus, $\tilde{g}$ has the same critical point structure as g and the restriction to the boundary is constant. $\square$

Definition 4.3. A *slice* is a compact 3-manifold N with a decomposition $\partial N = \partial_0 N \sqcup \partial_l N \sqcup \partial_u N \sqcup \partial_1 N$ of the boundary into bottom-, lower-, upper- and top-components (possibly empty) such that there exits a slicing function $f : N \to [0,1]$ with at most one critical point, with $\partial_0 N = f^{-1}(0)$ and $\partial_1 N = f^{-1}(1)$, and with the gradient field of f directed inwards on $\partial_l N$ and outwards on $\partial_u N$.

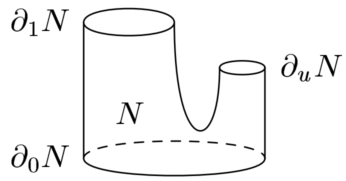

FIGURE 11. A slice in dimension 2.

Notice that the gradient vector field is directed inwards on $\partial_0 N$ and outwards on $\partial_1 N$. When we do not care about direction of the gradient field on the lower and upper boundary we may simply call them middle boundary and denote it by $\partial_m N$ (most of the time we will forget about $\partial_m N$ altogether). The slices come in 5 different types corresponding to the 4 possible indices of the critical point and the case where there is no critical point. The last type we call a trivial slice, since it is simply diffeomorphic to a cylinder by a well-known result, cf. [48]. Apart from that the slices can of course be cataloged by the different combinations of boundary surfaces.

Definition 4.4. A *slicing* s of a compact 3-manifold M is a decomposition

$$M = N_1 \cup \cdots \cup N_k$$

of M into slices N_i such that $N_i \cap N_{i+1} = \partial_1 N_i = \partial_0 N_{i+1}$ for $1 \leq i \leq k-1$ and $N_i \cap N_j = \emptyset$ otherwise. Define the bottom, lower, upper and top boundary of M with respect to the slicing s as $\partial_0(M,s) = \partial_0 N_1$, $\partial_l(M,s) = \cup_{i=1}^k \partial_l N_i$, $\partial_u(M,s) = \cup_{i=1}^k \partial_u N_i$ and $\partial_1(M,s) = \partial_1 N_k$. Two slicings $N_1 \cup \cdots \cup N_k$ and $N_1' \cup \cdots \cup N_l'$, $k \leq l$, are said to be *equivalent* if there exist $i_1 < \cdots < i_k \in \{1,\ldots,l\}$ such that N_j is diffeomorphic to N_{i_j}' (preserving the boundary type) for all $j \in \{1,\ldots,k\}$ and any other N_m' is trivial.

Often we shall view a slicing s as a sequence $\{N_i\}_{i=1}^k$ with the required gluing data. Before we devote our attention to Morse theory we state a couple of seemingly obvious facts about slicings.

Lemma 4.5. *Let M_1 and M_2 be compact 3-manifolds with slicing functions $(f_1; t_0, \ldots, t_k)$ and $(f_2; t_0', \ldots, t_l')$. Suppose $f_1^{-1}(t_k) = f_2^{-1}(t_0')$. Then we can glue M_1 and M_2 together along $f_1^{-1}(t_k) = f_2^{-1}(t_0')$ to form $M = M_1 \cup M_2$. Furthermore, f_1 and f_2 induces a slicing function f on M by smoothing the function*

$$\tilde{f}(x) = \begin{cases} f_1(x) & \text{for } x \in M_1, \\ t_k - t_0' + f_2(x) & \text{for } x \in M_2 \end{cases}$$

without introducing any new critical points.

Lemma 4.6. *Let Y be a surface and consider the trivial slice $Y \times I$. If $\partial_0(Y \times I) = Y$ then $\partial_1(Y \times I) = Y$, and there is a slicing function $(f; t_0, \ldots, t_k)$ (for some k) on $Y \times I$ with $f^{-1}(t_0) = Y \sqcup Y$ and $f^{-1}(t_k) = \emptyset$.*

The elementary however rather technical proofs of these lemmas can be found in appendix B on technical trickery.

Proposition 4.7. *Let M be a compact 3-manifold and let $(f; t_1, \ldots, t_k)$ be a slicing function of M. Then*

$$f^{-1}([t_0, t_1]) \cup \cdots \cup f^{-1}([t_{k-1}, t_k])$$

is a slicing of M. In fact, every slicing of M up to equivalence is got from a slicing function in this way. Furthermore, for any choice of decomposition $\partial M = Y_0 \sqcup Z_l \sqcup Z_u \sqcup Y_1$ of the boundary of M there is a slicing s such that $\partial_0(M,s) = Y_0$, $\partial_l(M,s) = Z_l$, $\partial_u(M,s) = Z_u$ and $\partial_1(M,s) = Y_1$.

Proof. That $f^{-1}([t_0, t_1]) \cup \cdots \cup f^{-1}([t_{k-1}, t_k])$ is a slicing of M is obvious. On the other hand, a slicing $M = N_1 \cup \cdots \cup N_k$ gives rise to a function

$$f(x) = i - 1 + f_i(x), \quad \text{for } x \in N_i.$$

By Lemma 4.5 this induces a slicing function on M with the desired properties.

We need to prove that slicing functions exist. By Lemma 4.2 it is enough to show that smooth functions on M that are constant on ∂M but non-singular on ∂M exist. This can be seen by choosing a collar $C_i \cong (\partial M)_i \times [0,1[$ on every boundary component $(\partial M)_i$, any smooth function f on M, and a smooth bump function φ supported inside $M - \cup_i (\partial M)_i \times [0, \frac{1}{2}[$ such that $\varphi\big(M - \cup_i(\partial M)_i \times]0,1[\big) = 1$. Let t denote the function taking the value of the second coordinate on the collar when defined. Then $\varphi(x)f(x) + \big(1-\varphi(x)\big)t(x)$ is a function with the right properties.

From *one* slicing function we can get a slicing function with any other direction of the gradient field on the boundary by attaching cylinders to the boundary components and bending those where the opposite direction is wanted. We can do this by Lemmas 4.5 and 4.6. Thus, by gluing on extra trivial slices if necessary we can match any decomposition of ∂M with a slicing. □

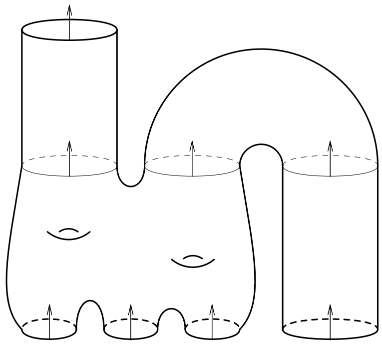

FIGURE 12. Changing direction of a slicing function.

The distinction between bottom and lower, and upper and top component is only relevant when gluing together slicings, as it by convention does not involve the middle boundary. In particular, we can pass from a slicing with non-empty middle boundary to one with an empty middle boundary without changing the critical point structure by "elongation". Thus, we will usually refrain from that cumbersome distinction.

To handle transition between different slicing function and slicings we introduce the notion of a *framed* function. For a more detailed exposition see the papers [35] and [36] of K. Igusa.

Definition 4.8. Let (M, g) be a Riemannian manifold. A *framing* of a singularity x_0 of a function $f \in C^\infty(M)$ is defined to be an orthonormal framing ξ of the negative eigenspace of the Hessian of f at x_0.

Definition 4.9. A *framed function* on M is a pair (f, ξ) consisting of a smooth function f on M having at most $A2$-singularities in the interior of M and no singular points on the boundary ∂M, and a framing ξ of each of the critical points of f. By a *family of framed functions* on M parameterized by a manifold $\mathcal{P}$ we mean a pair (f_t, ξ_t), where f_t is a framed function for each $t \in \mathcal{P}$ and ξ_t is a function giving a framing $\xi_t(x)$ of each critical point x of f_t. The family (f_t, ξ_t) is required to satisfy:

(1) f_t is generic in the sense that its 3-jet $j^3 f : M \times \mathcal{P} \to J^3(M, \mathbb{R})$ is transverse to the set of degenerate singular 3-jets. Similarly, the restriction $f_t|_{\partial M}$ to the boundary must be generic.

(2) Denoting the individual basis vectors of the framing $\xi_t(x)$ by $\xi_t^1(x), \ldots, \xi_t^i(x)$, we require that each function ξ^j is smooth on the subset of $M \times \mathcal{P}$ where it is defined.

(3) Suppose that at each $A2$-point (x, t) we extend the framing $\xi_t(x) = (\xi_t^1(x), \ldots, \xi_t^i(x))$ to an $(i+1)$-framing by defining $\xi_t^{i+1}(x)$ to be the unique unit vector lying in the one dimensional null space of $D^2 f_t(x)$ which points in the positive cubic direction (the x_1-direction in our local picture). If $\bar{\xi}_t(x)$ denotes this extended framing, then the extended functions $\bar{\xi}_t$ are smooth in their domains.

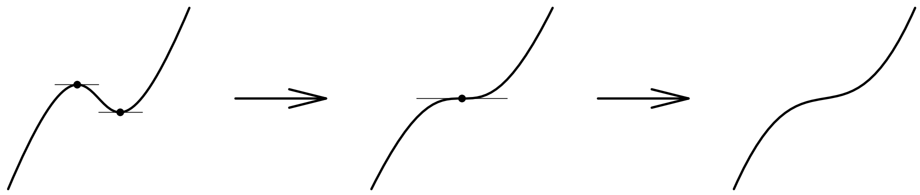

FIGURE 13. A one dimensional $A2$-singularity.

Notice in item (1) that the requirement that f_t must be generic means that each $A2$-point is universally unfolded, i.e. it has the local description as

$$x_1^3 + t x_1 \pm x_2^2 \pm \cdots \pm x_n^2, \tag{4.10}$$

as illustrated in fig. 13.

In particular, the $A2$-singularities are discrete in $M \times P$. Notice also that when an index i- and an index $i+1$-singularity cancel at an $A2$-point, the last framing vector of the index $i+1$-point must converge to the unit vector in the positive cubic direction of the $A2$-point (the x_1-direction in the local description above).

Theorem 4.11. (*Framed function theorem,* [35]) *Let M be a compact smooth n-manifold with a smooth family of metrics parameterized by the k-ball D^k and let (f_t, ξ_t) be a smooth family of framed functions on M parameterized by S^{k-1},* *where $k \leq n$. Suppose also that V is a neighbourhood of the boundary ∂M and that $f_t|_V = g$ for all $t \in S^{k-1}$ for some fixed function g on V. Then (f_t, ξ_t) extends to a family of framed functions on M parameterized by D^k so that $f_t|_V = g$ for all $t \in D^k$.*

Given two slicing functions f_0 and f_1 it is of course possible to find a neighbourhood V of the boundary where they have no critical points, and an isotopy from f_1 to and f_1' such that $f_1'|_V = f_0|_V$. Now, the framed function theorem almost immediately gives the desired result. One thing the theorem doesn't mention, however, is if the members of the family have distinct critical values. But this is also a generic property and by making a C^∞ small deformation of f_t, it can be arranged that critical values meet only in isolated points.

Remark 4.12. Thus, any two unidirectional slicing functions can be connected by a path with only finitely many non-slicing functions. At any of these non-slicing functions it can happen that

(1) the ordering (by values) of two critical points is exchanged or
(2) two critical points whose indices differ by one cancel in an $A2$-singularity.

Hence, an obvious combinatorial corollary for slices is:

Remark 4.13. Let M be a smooth compact 3-manifold possibly with boundary. Any slicing of M can be obtained from any other slicing corresponding to the same decomposition of the boundary by a series of finitely many of the following three moves or their inverses:

(1) Insertion of a trivial slice.
(2) Two adjacent slices whose indices differ by 1 corresponding to an $A2$-singularity are substituted with a trivial slice.
(3) Two adjacent slices are transformed in accordance with item number 1 above (see figure 14).

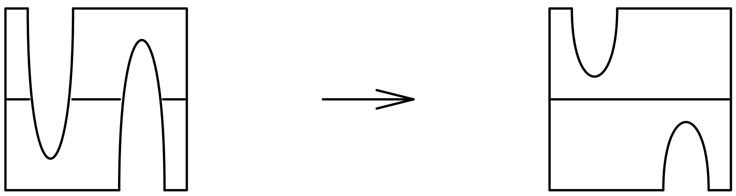

Figure 14. Conversion of a pair of slices to reorder the critical points.

e-Slicings. Now that we understand the slicings of 3-manifolds we can start defining the analogies in the extended category. However, there are no surprises: By an e-slice $\boldsymbol{N}$ we mean an e-3-manifold whose base manifold N is a slice. Thus, it can be written $\boldsymbol{N} = (N, L_0 \oplus L_l \oplus L_u \oplus L_1, n)$. An e-slicing $\boldsymbol{s}$ of the e-3-manifold $\boldsymbol{M}$ is a sequence $\{\boldsymbol{N}_i\}_{i=1}^k$ of e-slices with the required gluing data such that

$$\boldsymbol{M} = \boldsymbol{N}_1 \cup \cdots \cup \boldsymbol{N}_k,$$

where $\cup$ means gluing with some e-morphism. In particular, $s = \{N_i\}_{i=1}^k$ is a slicing of M.

5. Constructing the TQFT

We have now reached the point where we can begin defining the TQFT Z from a given modular functor V. The strategy applied here is to define Z on e-slices and subsequently on e-slicings. By doing this in a way such that it is independent of the choice of slicing of the e-manifold and so that it obeys the axioms TQFT1–4, we get a TQFT.

When considering Z on a slice N it is of no significance whether a boundary component belongs to the middle boundary $\partial_m N$. What is important is if the gradient field is directed inwards or outwards as on $\partial_0 N$ and $\partial_1 N$ respectively. Hence, until we start gluing slices together in general, we will refrain from this cumbersome notation. Also, we only need to consider connected slices.

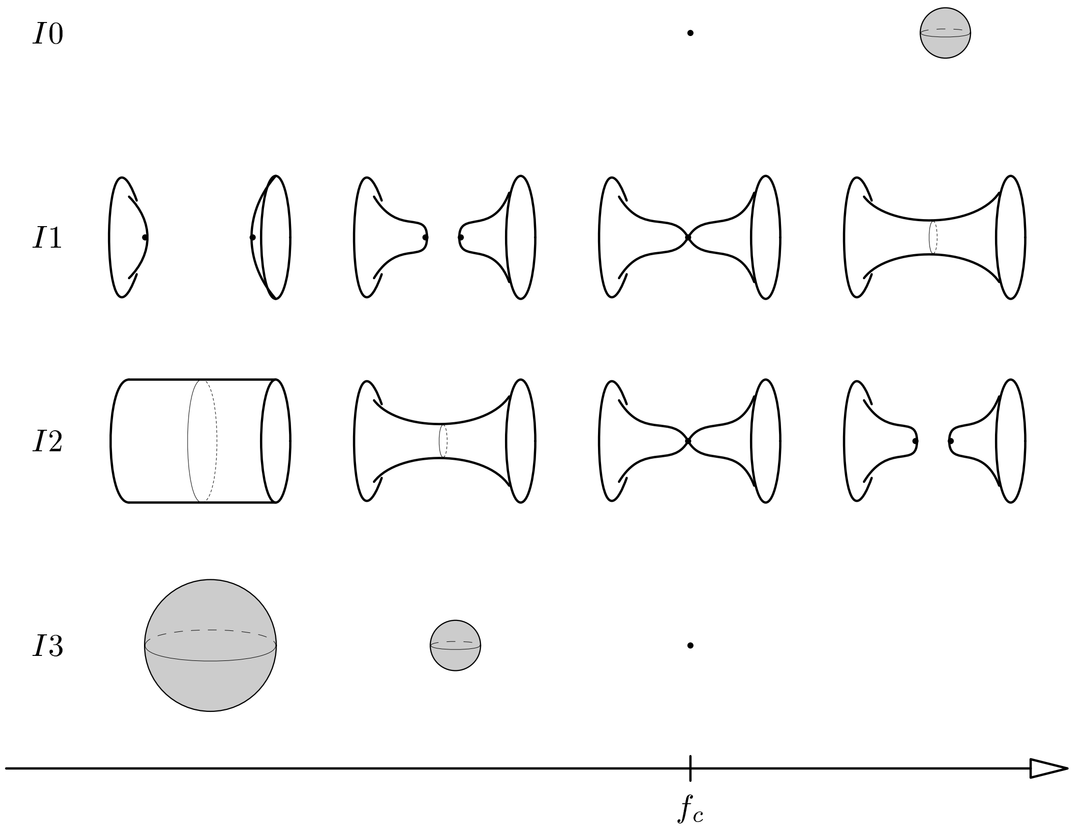

FIGURE 15. Local picture of level surfaces around the critical value f_c.

As noted before, there are 5 different types of slices: 4 types corresponding to the 4 different indices of the critical points, and one with no critical points (i.e. the mapping cylinders). In a neighbourhood of the critical point, the level surfaces develop as displayed in the movies in fig. 15; the shaded spheres, thin circles and "fat points" illustrates points flowing to or from the critical points.

Before we go on, let us introduce some notation. For a given choice of a slicing function f and a metric g on a slice N, let φ be the induced Morse flow. For a given flow parameter value t and point $x \in N$ we usually write it $\varphi(t,x) = \varphi(t)(x)$. Then we denote by φ_ν, $\nu = 0,1$, the flow from $\partial_\nu N$ to the critical level Σ^c (along $\operatorname{grad} f$ and $-\operatorname{grad} f$ respectively). The union of flow-lines to the critical point we call a critical system, and we often denote it $\mathfrak{C} = \mathfrak{C}_f$.

Now consider 1- and 2-slices. Outside $\mathfrak{C}$ the flow acts diffeomorphically and we may choose tubular neighbourhoods U_ν in $\partial_\nu N$ of $\mathfrak{C} \cap \partial_\nu N$ (being either a pair of disks on an annulus), such that the complements, Σ_ν, of these neighbourhoods satisfy that $\Sigma_1 = \varphi|_{\Sigma_0}(\Sigma_0)$, where $\varphi|_{\Sigma_0}$ means the diffeomorphic flow from the bottom to the top. Let $i_\nu \colon \Sigma_\nu \hookrightarrow \partial_\nu N$ be the natural embedding.

For the proofs and details of the following facts about Lagrangian subspaces, see appendix A. For any $x \in H_1(\partial_0 N)$, define the subset $\Phi(x) = \varphi_{1*}^{-1}\big(\varphi_{0*}(x)\big)$ of $H_1(\partial_1 N)$.

Definition 5.1. A pair $(L_0, L_1) \subset H_1(\partial_0 N) \oplus H_1(\partial_1 N)$ of Lagrangian spaces is called *natural* if

$$L_1 = \Phi(L_0).$$

Furthermore, e-structures consisting of a natural pair and framing number 0 will be named a *natural e-structure.*

On the other hand, we know that under the right circumstances $\Phi(L)$ is Lagrangian when L is, and $\Phi(L)$ is independent of the choice of both metric and slicing function.

Given a Lagrangian subspace $L_\nu \subset H_1(\partial_\nu N)$, we define the subspace $L'_\nu = i_{\nu*}^{-1}(L_\nu)$ which turns out to be maximal isotropic. When U_ν is glued back in, the canonical Lagrangian subspace of $H_1(U_\nu)$ together with L'_ν reproduces L_ν. In fact, gluing the canonical Lagrangian subspace of $H_1(U_1)$ together with $(\varphi|_{\Sigma_0})_*(L'_\nu)$ gives $\Phi(L_0)$ (cf. Proposition A.8).

Now we can define Z on all connected slices with natural e-structures. The field theory Z on trivial slices is simply defined to satisfy the mapping cylinder axiom.

A 0-slice. Let N be a 0-slice. Having no choice of e-structure on neither D nor S^2 clearly we must define Z on N as

$$\begin{aligned} Z(N,0,0) \colon \mathbb{C} &\to V(S^2) \\ x &\mapsto \lambda_0 x \cdot \beta_1 \otimes \beta_1, \end{aligned}$$

for some $\lambda_0 \in \mathbb{C}$.

A 1-slice. Choose a metric g on an index 1-slice N and a slicing function f with one critical point and with an associated Morse flow φ. Then Σ_0 is $\partial_0 N$ with the two small disks removed and Σ_1 is $\partial_1 N$ with an annulus cut out, the boundary parameterizations are chosen so the corresponds under φ, and there are identifications $V(\partial_0 N, L) = V\big((\Sigma_0, L_0'), (1,1)\big) \otimes V_1 \otimes V_1$, $V\big(\partial_1 N, \Phi(L)\big) = \bigoplus_a V\big((\Sigma_1, L_1'), (a, \hat{a})\big) \otimes V_{a\hat{a}}$. For $x \in V\big((\Sigma_0, L_0'), (1,1)\big)$ we put

$$\begin{aligned} Z\big(N, L \oplus \Phi(L), 0; f, g\big) &: V(\partial_0 N, L) \to V\big(\partial_1 N, \Phi(L)\big) \\ & x \otimes \beta_1 \otimes \beta_1 \mapsto \lambda_1 \cdot V(\varphi|_{\Sigma_0}, 0)(x) \otimes \beta_{11}, \end{aligned}$$

where $\lambda_1 \in \mathbb{C}$.

A 2-slice. Likewise choose a metric g and a single critical point slicing function f on an index 2-slice N. Now, Σ_0 is $\partial_0 N$ with the appropriate annulus removed and Σ_1 is $\partial_1 N$ with the two disks cut out as above. Then $V(\partial_0 N, L) = \bigoplus_a V\big((\Sigma_0, L_0'), (a, \hat{a})\big) \otimes V_{a\hat{a}}$ and $V\big(\partial_1 N, \Phi(L)\big) = V\big((\Sigma_1, L_1'), (1,1)\big) \otimes V_1 \otimes V_1$, so for $x_a \in V\big((\Sigma_0, L_0'), (a, \hat{a})\big)$ we define Z on N as

$$\begin{aligned} Z\big(N, L \oplus \Phi(L), 0; f, g\big) &: V(\partial_0 N, L) \to V\big(\partial_1 N, \Phi(L)\big) \\ & \textstyle\sum_a x_a \otimes \beta_{a\hat{a}} \mapsto \lambda_2 \cdot V(\varphi|_{\Sigma_0}, 0)(x_1) \otimes \beta_1 \otimes \beta_1 \end{aligned}$$

for some $\lambda_2 \in \mathbb{C}$.

A 3-slice. Finally, for the 3-slice N

$$\begin{aligned} Z(N, 0, 0) &: V(S^2) \to \mathbb{C} \\ & x \cdot \beta_1 \otimes \beta_1 \mapsto \lambda_3 x, \end{aligned}$$

for every $x \in \mathbb{C}$, where $\lambda_3 \in \mathbb{C}$ is fixed.

However, the λ_i's cannot be *any* complex numbers. As we shall see later (in Lemma 5.8), Remark 4.13 forces relations on the λ_i's.

Remark 5.2. Before we go on, we observe that Z is independent of the choice of diffeomorphism from the disks we cut out to our standard disks and likewise for the annuli. This follows from the fact that the action of the mapping class groups of disks and annuli are trivial on vector spaces associated to surfaces labeled by the trivial label. Consider for instance a 2-slice and suppose that we change the annuli diffeomorphism by the Dehn twist T. Then

$$\textstyle\sum_a x_a \otimes \beta_{a\hat{a}} \mapsto \sum_a t_a x_a \otimes \beta_{a\hat{a}} \mapsto \lambda_2 V(\varphi)(t_1 x_1) \otimes \beta_1 \otimes \beta_1 = \lambda_2 V(\varphi)(x_1) \otimes \beta_1 \otimes \beta_1,$$

since $t_1 = 1$.

It is also clear that the definition does not depend on the choice of boundary parameterizations as long as we use parameterizations that corresponds on the two sides of the cut under the flow.

Before we go on, we also point out that no metric or slicing functions were used in the definition of Z on 0- and 3-slices, so it is only necessary to prove independence of these for 1- and 2-slices.

Lemma 5.3. *Given a fixed slicing function f and a fixed metric g, the definition of Z does not depend on the choice of factorization compatible with f and g.*

Proof. Suppose the index is 1, and let $D_1^\nu \sqcup D_2^\nu$ be two choices of disk neighbourhoods around the two isolated points $\partial_0 N \cap \mathfrak{C}$ and let A^ν be the corresponding annuli neighbourhoods of the contracting circle $\partial_1 N \cap \mathfrak{C}$. As they are closed neighbourhoods we may choose an $\varepsilon > 0$ such that the closed disk neighbourhoods $D_{\mu,\varepsilon}$ of radius ε around the points are contained in the interior of $D_\mu^1 \cap D_\mu^2$, and such that the corresponding annulus A_ε is contained in the interior of $A^1 \cap A^2$. Now the closure of $D_\mu^\nu - D_{\mu,\varepsilon}$ is diffeomorphic to A and the closure of $A^\nu - A_\varepsilon$ to $A \sqcup A$.

Hence we can write $x_\varepsilon \in V\big((\partial_0 N - (D_{1,\varepsilon} \sqcup D_{2,\varepsilon}), L_0^\varepsilon), (1,1)\big)$ as $x_\varepsilon = x^\nu \otimes \beta_{11} \otimes \beta_{11} \in V\big((\partial_0 N - (D_1^\nu \sqcup D_2^\nu), L_0^\nu), (1,1)\big) \otimes V_{11} \otimes V_{11}$ (where L_0^ε a L_0^ν are the induced maximal isotropic subspaces). Then according to the definition of $Z(N, L \oplus \Phi(L), 0, ; f, g)$ using the excition of the ε-neighbourhoods

$$\begin{aligned}
x^\varepsilon \otimes \beta_1 \otimes \beta_1 \mapsto \lambda_1 \cdot V(\varphi|_{\partial_0 N - (D_{1,\varepsilon} \sqcup D_{2,\varepsilon})}, 0)(x^\varepsilon) \otimes \beta_{11} \\
= \lambda_1 \cdot V(\varphi|_{\partial_0 N - (D_1^\nu \sqcup D_2^\nu)}, 0)(x^\nu) \otimes \beta_{11} \otimes \beta_{11} \otimes \beta_{11} \\
= \lambda_1 \cdot V(\varphi|_{\partial_0 N - (D_1^\nu \sqcup D_2^\nu)}, 0)(x^\nu) \otimes \beta_{11},
\end{aligned}$$

where we used the fundamental property of $\beta_{a\hat{a}}$ and the fact that any diffeomorphism acts trivially on β_{11}. The last expression is the definition of $Z(N, L \oplus \Phi(L), 0, ; f, g)$ using the factorization by D_μ^ν and A^ν. The index 2 case can be proved in the same way. □

Lemma 5.4. *The Z defined above is independent of the choice of metric.*

Subsequently we will eliminate the metric in the notation.

Proof. For simplicity assume that N is a 1-slice; index 2 follows in the same way. Let g and $\tilde{g}$ be two metrics on the slice N and choose a slicing function f. A path between g and $\tilde{g}$ induces an isotopy φ^t between the Morse flows φ defined by g and $\tilde{\varphi}$ defined by $\tilde{g}$. Now, let Σ_ν^t be the complement in $\partial_\nu N$ of the points $\partial_\nu N \cap \mathfrak{C}^t$ flowing to the critical point via φ^t. Then define a family $\gamma_\nu^t\colon \Sigma_\nu^0 \to \Sigma_\nu^t$ of embeddings into $\partial_\nu N$ by $\gamma_\nu^t(x) = (\varphi_\nu^t)^{-1} \circ (\varphi_\nu^0)|_{\Sigma_\nu^0}(x)$.

We claim that by maybe perturbing γ_ν^t slightly in a neighbourhood of $\partial_\nu N \cap \mathfrak{C}^t$ we can extend this to a family of homeomorphisms $\hat{\gamma}_\nu^t\colon \partial_\nu N \to \partial_\nu N$ that is smooth on Σ_ν. To see this, first observe that the extension of γ_0^t to all of $\partial_0 N$ is obvious. Then choose a 2-sided collar around $\partial_1 N \cap \mathfrak{C}^0$ and consider the longitudinal curves $\alpha_x(s) = (x, s)$, $x \in \partial_1 N \cap \mathfrak{C}^0$, in the collar. For every α_x there is exactly one curve β_x^t (possibly not smooth in 0) such that $\varphi_1^0 \circ \alpha_x = \varphi_1^t \circ \beta_x^t$. Define $\hat{\gamma}_1^t(x) = \beta_x^t(0)$. Now, $\hat{\gamma}_1^t$ might be neither smooth nor injective on $\partial_1 N \cap \mathfrak{C}^0$. But the singularities in $\partial_1 N \cap \mathfrak{C}^t$ corresponding to these singularities look no worse than pictured in fig. 16 and are easily dissolved, e.g. by applying a bump function to a family like $\psi_t\colon [-2,2]^2 \to [-2,2]^2$, where $\psi_t(x,y) = \big(x, ((1-t)|x| + t)y\big)$. I.e. by deforming $\hat{\gamma}_\nu^1$ slightly in a neighbourhood of $\partial_\nu N \cap \mathfrak{C}^0$ through a path $\tilde{\gamma}_\nu^s$ we can arrive at a diffeomorphism $\tilde{\gamma}_\nu^1\colon \partial_\nu N \to \partial_\nu N$ that takes $\partial_\nu N \cap \mathfrak{C}^0$ to $\partial_\nu N \cap \mathfrak{C}^1$ in such a way that the restriction $\tilde{\gamma}_\nu^s|_{\Sigma_\nu^0}$ is an isotopy. On the other hand, results of among others Earle and McMullen, [25][6], state that since $\hat{\gamma}_\nu^0 = \mathrm{Id}_{\partial_\nu N}$ is homotopic (through $\hat{\gamma}_\nu^t$ and $\tilde{\gamma}_\nu^s$) to the diffeomorphism $\tilde{\gamma}_\nu^1$, then they are in fact isotopic.

To summarize: the embeddings $\gamma_\nu^0\colon \Sigma_\nu^0 \to \partial_\nu N$ and $\gamma_\nu^1\colon \Sigma_\nu^0 \to \partial_\nu N$ are isotopic through a path using the flows, γ_ν^1 is again isotopic to $\tilde{\gamma}_\nu^1|_{\Sigma_\nu^0}\colon \Sigma_\nu^0 \to \partial_\nu N$ through another specific path, and $\hat{\gamma}_\nu^0 = \mathrm{Id}_{\partial_\nu N}$ is abstractly isotopic to $\tilde{\gamma}_\nu^1\colon \partial_\nu N \to \partial_\nu N$.

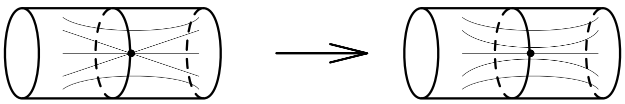

FIGURE 16. A slight deformation of $\hat{\gamma}_\nu^1$.

Choose a decomposition of the boundary components $\partial_\nu N = \Sigma_\nu \cup U_\nu$ with respect to φ; i.e. let U_ν be large enough so that they contain the collars and neighbourhoods above. Then $\tilde{\gamma}_\nu^1$ induces a suitable decomposition with respect to $\tilde{\varphi}$ and we are able to compare the results of $Z(\boldsymbol{N}; f, g)$ and $Z(\boldsymbol{N}; f, \tilde{g})$.

Recall that on Σ_0 the flow of f from $\partial_0 N$ to $\partial_1 N$ is $\varphi|_{\Sigma_0} = \varphi_1^{-1} \circ \varphi_0|_{\Sigma_0}$ so for $x \in V\big((\Sigma_0, L_0'), (1,1)\big)$ and using Remark 5.2, we get

$$\begin{aligned} V(\tilde{\gamma}_1^1, 0) \circ Z(\boldsymbol{N}; f, g)(x \otimes \beta_1 \otimes \beta_1) &= \lambda_1 \cdot V(\gamma_1^1, 0) V(\varphi_1^{-1}\varphi_0|_{\Sigma_0^0}, 0)(x) \otimes \beta_{11} \\ &= \lambda_1 \cdot V(\tilde{\varphi}_1^{-1}\varphi_1\varphi_1^{-1}\varphi_0|_{\Sigma_0^0}, 0)(x) \otimes \beta_{11} \\ &= \lambda_1 \cdot V(\tilde{\varphi}_1^{-1}\varphi_0|_{\Sigma_0^0}, 0)(x) \otimes \beta_{11}, \end{aligned}$$

[6]The author wishes to thank C. McMullen for bringing this reference to his attention.

$$
\begin{aligned}
Z(\boldsymbol{N}; f, \tilde{g}) \circ V(\tilde{\gamma}_0^1, 0)(x \otimes \beta_1 \otimes \beta_1) &= \lambda_1 \cdot V(\tilde{\varphi}_1^{-1}\tilde{\varphi}_0|_{\Sigma_0^1}, 0)V(\gamma_0^1, 0)(x) \otimes \beta_{11} \\
&= \lambda_1 \cdot V(\tilde{\varphi}_1^{-1}\tilde{\varphi}_0\tilde{\varphi}_0^{-1}\varphi_0|_{\Sigma_0^0}, 0)(x) \otimes \beta_{11} \\
&= \lambda_1 \cdot V(\tilde{\varphi}_1^{-1}\varphi_0|_{\Sigma_0^0}, 0)(x) \otimes \beta_{11},
\end{aligned}
$$

where we have used that $\tilde{\gamma}_\nu^1 \simeq \gamma_\nu^1 = \tilde{\varphi}_\nu^{-1} \circ \varphi_\nu|_{\Sigma_0^0}$. Thus, as $\tilde{\gamma}_\nu^1 \simeq \mathrm{Id}_{\partial_\nu N}$ we get that $Z(\boldsymbol{N}; f, \tilde{g}) = V(\tilde{\gamma}_1^1, 0) \circ Z(\boldsymbol{N}; f, g) \circ V(\tilde{\gamma}_0^1, 0)^{-1} = Z(\boldsymbol{N}; f, g)$. □

Partly out of convenience and partly out of necessity we postpone the remaining parts of the proof of well-definedness of Z on slices till later. In the meantime we extend Z to slices with any e-structure by forcing strong naturality. Next let $\boldsymbol{s} = \{\boldsymbol{N}_i\}$ be an e-slicing of an e-manifold $\boldsymbol{M}$ with corresponding slicing function f decomposing into f_i on each slice N_i. Define Z on the slicing by composing the $Z(\boldsymbol{N}_i; f_i)$'s using the gluing data for $\boldsymbol{M}$ in accordance with the gluing axiom of a TQFT. We write this

$$
Z(\boldsymbol{s}; f) \stackrel{\text{def}}{=} \prod_i Z(\boldsymbol{N}_i; f_i),
$$

where the product means composition according to the gluing data. That Z on slicings is well defined will be proved through the following series of lemmas.

Lemma 5.5. *If $F\colon N \to \tilde{N}$ is an orientation preserving diffeomorphism, then it takes natural pairs to natural pairs, and choosing a slicing function f on N and the induced one, $\tilde{f} = f \circ F^{-1}$, on $\tilde{N}$, we get*

$$
\begin{aligned}
&Z\big(\tilde{\boldsymbol{N}}, (F|_{\partial N})_*(L \oplus \Phi(L)), 0; \tilde{f}\big) \\
&\qquad = V(F|_{\partial_1 N}, 0) \circ Z\big(\boldsymbol{N}, L \oplus \Phi(L), 0; f\big) \circ V(F|_{\partial_0 N}, 0)^{-1} \\
&\qquad = V(F|_{\partial N}, 0)\big(Z(\boldsymbol{N}, L \oplus \Phi(L), 0; f)\big).
\end{aligned}
$$

Proof. Suppose $F : N \to \tilde{N}$ is an orientation preserving diffeomorphism. Given a slicing function f and a metric g on N we choose the slicing function $\tilde{f} = f \circ F^{-1}$ and the metric $\tilde{g} = (F^{-1})^* g$ on $\tilde{N}$. This implies that if φ is the Morse flow of f, the flow of $\tilde{f}$ is $F \circ \varphi \circ F^{-1}$. This means that F preserves natural pairs

$$
(F|_{\partial N})_*\big(L \oplus \Phi(L)\big) = (F|_{\partial_0 N})_* L \oplus (F|_{\partial_1 N})_* \Phi(L) = (F|_{\partial_0 N})_* L \oplus \tilde{\Phi}\big((F|_{\partial_0 N})_* L\big).
$$

If we have a decomposition of ∂N suitable for f, then its image under $F|_{\partial N}$ will be suitable for $\tilde{f}$. Now the identity for Z in the lemma follows immediately by the fact that $\tilde{\varphi} = F|_{\partial_1 N} \circ \varphi \circ F|_{\partial_0 N}^{-1}$. We see for instance for a 1-slice that

$$
\begin{aligned}
V(F|_{\partial_1 N}, 0) \circ Z\big(N, L \oplus \Phi(L), 0; f\big) &\circ V(F|_{\partial_0 N}, 0)^{-1}(\tilde{x} \otimes \beta_1 \otimes \beta_1) \\
&= \lambda_1 \cdot V\big(F|_{\Sigma_1} \circ \varphi \circ F|_{\Sigma_0}^{-1}, 0\big)(\tilde{x}) \otimes \beta_{11} = \lambda_1 \cdot V(\tilde{\varphi}, 0)(x) \otimes \beta_{11} \\
&= Z\big(\tilde{N}, (F|_{\partial N})_*(L \oplus \Phi(L)), 0; \tilde{f}\big)(\tilde{x} \otimes \beta_1 \otimes \beta_1),
\end{aligned}
$$

for $\tilde{x} \in V\big((\tilde{\Sigma}_0, (F|_{\Sigma_0})_* L'), (1,1)\big)$. □

As soon as the independence of slicing functions is established, this will prove naturality. To prove that Z satisfies strong naturality it is now enough to see that

$$Z\big(N, \tilde{L} \oplus \Phi(\tilde{L}), 0\big) = \lambda^{\sigma(K, L\oplus\Phi(L), \tilde{L}\oplus\Phi(\tilde{L}))} \cdot V(\widetilde{\mathbf{Id}}) Z\big(N, L \oplus \Phi(L), 0\big),$$

where $\widetilde{\mathbf{Id}} = (\mathrm{Id}, 0)\colon \big(\partial N, L \oplus \Phi(L)\big) \to \big(\partial N, \tilde{L} \oplus \Phi(\tilde{L})\big)$, $K = \ker\big(H_1(\partial N) \to H_1(N)\big)$, and λ is the non-zero complex number defined by $(\mathrm{Id}, 1)\colon \boldsymbol{Y} \to \boldsymbol{Y}$. Strong naturality follows because $\sigma\big(K, L \oplus \Phi(L), \tilde{L} \oplus \Phi(\tilde{L})\big) = 0$ by Lemma A.9.

Lemma 5.6. *If f_0 and f_1 are isotopic single-critical-point slicing functions on a slice N, then $Z(\boldsymbol{N}; f_0) = Z(\boldsymbol{N}; f_1)$ for any natural e-structure on N.*

Proof. Assume that N is an index 1-slice (index 2 is just the mirror image) and consider an isotopy $f_t \in \mathcal{S}(N)$ of slicing functions with one critical point on the slice N. Let $\mathfrak{C}^t$ denote the system of flow lines to the critical point x_t of f_t, $\mathfrak{C}^t_\nu = \mathfrak{C}^t \cap \partial_\nu N$, and let $\Sigma^t_\nu = \partial_\nu N - \mathfrak{C}^t_\nu$, $\nu = 0, 1$, as usual.

We can find an isotopy $F^t_0\colon \partial_0 N \to \partial_0 N$ of diffeomorphisms such that $F^0_0 = \mathrm{Id}_{\partial_0 N}$ and F^t_0 takes $\mathfrak{C}^0_0$ to $\mathfrak{C}^t_0$. Using the flows $\varphi^t\colon \Sigma^t_0 \to \Sigma^t_1$ from the bottom to the top, we define an isotopy of embeddings $F'^t_1\colon \Sigma^0_1 \to \Sigma^t_1 \subset \partial_1 N$ as $F'^t_1 = \varphi^t \circ F^t_0 \circ (\varphi^0|_{\Sigma^0_0})^{-1}$. As the surface to the two sides of $\mathfrak{C}^t_1$ may get twisted and distorted with respect to each other for $t > 0$, F'^t_1 does not necessarily extend to $\partial_1 N$. Therefore we close the boundaries of Σ^0_1 to form $\bar{\Sigma}^0_1$, so that the boundary circles are included in $\bar{\Sigma}^0_1$; let B_1 and B_2 denote the two boundary components, which can of course be naturally identified with $\mathfrak{C}^0_1$. Extend F'^t_1 to $\bar{F}'^t_1$ on $\bar{\Sigma}^0_1$. We can think of $\bar{F}'^t_1$ as an immersion with double points along $\mathfrak{C}^t_1$, so it is possible to compare $\bar{F}'^t_1|_{B_1}$ and $\bar{F}'^t_1|_{B_2}$; define $g_t\colon \mathfrak{C}^0_1 \to \mathfrak{C}^0_1$ as $g_t(\mu) = (\bar{F}'^t_1|_{B_2})^{-1} \circ \bar{F}'^t_1|_{B_1}(\mu)$, $\mu \in \mathfrak{C}^0_1$. Clearly, $g_0 = \mathrm{Id}_{\mathfrak{C}^0_1}$.

Choose small collars $C_1 \cong B_1 \times I$ and $C_2 \cong B_2 \times I$ of B_1 and B_2 in Σ^0_1 (B_i corresponding to $B_i \times \{1\}$). Recall the isotopy H from (3.9) and define $G^t\colon \bar{\Sigma}^0_1 \to \bar{\Sigma}^0_1$ to be

$$G^t(x) = \begin{cases} x, & \text{for } x \in \bar{\Sigma}^0_1 - C_2, \\ H_{1-s}(g)_t(\mu), & \text{for } x = (\mu, s) \in C_2. \end{cases}$$

Then define $\tilde{F}^t_1\colon \bar{\Sigma}^0_1 \to \partial_1 N$ as $\bar{F}'^t_1 \circ G^t$, and deform $\tilde{F}^t_1$ in a small neighbourhood $B_1 \times]\frac{1}{2}, 1]$ of B_1 in C_1 to a new diffeomorphism which we also denote $\tilde{F}^t_1$, so that $\tilde{F}^t_1(\mu, s) = \tilde{F}^t_1(\mu, 1)$ for all $\mu \in \mathfrak{C}^0_1$ and $s \in]\frac{1}{4}, 1]$. Then $\tilde{F}^t_1|_{B_1}$ and $\tilde{F}^t_1|_{B_2}$ glue together to form $F^t_1\colon \partial_1 N \to \partial_1 N$. Clearly, $F^0_1 = \mathrm{Id}_{\partial_1 N}$, F^t_1 takes $\mathfrak{C}^0_1$ to $\mathfrak{C}^t_1$, and $F^t_1|_{\partial_1 N - C} = \varphi^t \circ F^t_0 \circ (\varphi^0|_{\Sigma^0_0})^{-1}|_{\partial_1 N - C}$, where $C = C_1 \cup C_2$.

Now choose a decomposition of ∂N appropriate for f_0 with the annulus large enough to contain C. Then F^1_ν takes that decomposition to a decomposition appropriate for f_1 and a calculation like the one for independence of metrics ends the proof. □

Lemma 5.7. *Let f_0 be a trivial slicing function on M and f_1 a slicing function on M with two critical points of indices differing by 1. Suppose there is a path with precisely one A2-singularity connecting them. Then there is a diffeomorphism $F\colon M \to M$ and a disk $D_0 \subset \partial_0 M$ such that the flow $\tilde{\varphi}$ of $\tilde{f} = f_1 \circ F^{-1}$ preserves the tube $\{\, \varphi^0(t)(\partial D_0) \mid 0 \le t \,\}$ generated by the flow of f_0 and such that the two critical points of $\tilde{f}$ are inside that tube. Furthermore, F can be chosen so that $F|_{\partial M}$ is isotopic to the identity.*

Proof. First notice that it suffices to consider the case $M = \Sigma \times I$, since the trivial slicing function f_0 gives a preferred diffeomorphism $\Sigma \times I \to M$. By that simplification, $\varphi^0(t)(x, s) = (x, s + t)$.

The point of the local description (4.10) is that we can choose a disk D_0 in $\partial_0 M$ such that the singular points of f_t are inside the tube generated by the flow of f_t and ∂D_0 (for visualization try combining the local pictures in fig. 15). E.g. we see that for a 0-slice followed by a 1-slice, what happens is that a sphere is born and attached to the main surface to be "swallowed up". A 2-slice followed by at 3-slice is just that situation turned upside down. The 1-2 situation is displayed in fig. 17.

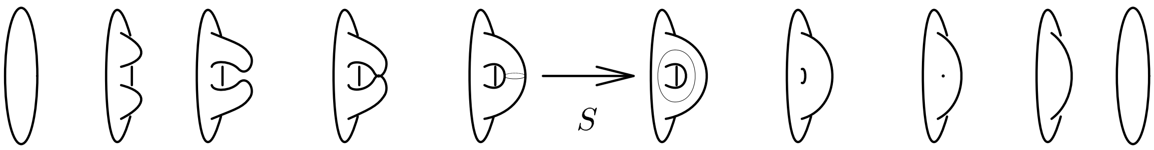

FIGURE 17. A movie describing 1-2-cancellation.

Let $F'\colon (\Sigma - \dot{D}_0) \times I \hookrightarrow \Sigma \times I$ be the embedding defined by the flow of f_1 and choose a small $\varepsilon > 0$, closed ε- and 2ε-tubular neighbourhoods D_ε and $D_{2\varepsilon}$ of D_0 in Σ, and an embedding $\tilde{F}\colon D_{2\varepsilon} \times I \hookrightarrow \Sigma \times I$ onto the complement of $F'((\Sigma - D_{2\varepsilon}) \times I)$ such that $F'(\partial D_0 \times I) = \tilde{F}(\partial D_0 \times I)$. (Such an embedding $\tilde{F}$ onto $F'((\Sigma - D_{2\varepsilon}) \times I)$ exists by Schönflies Theorem,[7] as we may embed $\Sigma \times I$ into S^3 as part of a handle body decomposition, and $\tilde{F}$ can be made to fit F' in the desired way by scaling.) Now define $G\colon S^1 \times [0, 2\varepsilon] \times I \to S^1 \times [0, 2\varepsilon] \times I$ as $G = \tilde{F}^{-1} \circ F'|_{S^1 \times [0,2\varepsilon] \times I}$. Any orientation preserving diffeomorphism of the annulus is isotopic to the identity. Using this isotopy for $G|_{S^1 \times \{0\} \times I}$ we can extend G to a homeomorphism $\bar{G}$ of $D_{2\varepsilon} \times I$. Smoothing $\bar{G}$ on $D_\varepsilon \times I$ gives a diffeomorphism $\tilde{G}$ of $D_{2\varepsilon} \times I$. Now, $\bar{F} = \tilde{F} \circ \tilde{G}$ is a new embedding of $D_{2\varepsilon} \times I$ which agrees with F' on $(D_{2\varepsilon} - D_\varepsilon) \times I$. Hence, it makes sense to glue together $\bar{F}$ and F' to form a diffeomorphism $F\colon \Sigma \times I \to \Sigma \times I$ with the required properties. To see that $F|_{\partial_1 M}$ is isotopic to $F|_{\partial_0 M}$, notice that $F|_{((\Sigma - \dot{D}_{2\varepsilon}) \times \{1\})}\colon (\Sigma - \dot{D}_{2\varepsilon}) \to \Sigma$ is isotopic as embedding to $F|_{((\Sigma - \dot{D}_{2\varepsilon}) \times \{0\})}\colon (\Sigma - \dot{D}_{2\varepsilon}) \to \Sigma$. Then an argument similar to the

[7] We thank H. Murakami for making us aware of this theorem.

one above extends this to an isotopy between $F|_{\partial_1 M}$ and $F|_{\partial_0 M}$. As any choice in the construction of F can be made so that $F|_{\partial_0 M} = \mathrm{Id}_{\partial_0 M}$, we are done. $\square$

Lemma 5.8. *Let f_0 be a trivial slicing function on M and f_1 a slicing function on M with two critical points of indices differing by 1. If there is a disk $D_0 \subset \partial_0 M$ so that the flow φ^1 of f_1 preserves the tube $\{\, \varphi^0(t)(\partial D_0) \mid 0 \leq t \,\}$ generated by the flow of f_0 and so that the critical points of f_1 are inside the tube, then $Z(\boldsymbol{M}; f_0) = Z(\boldsymbol{M}; f_1)$ for any natural e-structure on N provided that $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ satisfy the equations*

$$\left\{ \begin{matrix} \lambda_0\lambda_1 = 1, \\ \lambda_1\lambda_2 S_{1,1} = 1, \\ \lambda_2\lambda_3 = 1. \end{matrix} \right\} \tag{5.9}$$

Proof. Again we can assume $M = \Sigma \times I$. Under the given circumstances f_0 and f_1 allow the same partition of the boundary surfaces and by assumption $Z(M, L \oplus L, 0; f_0) = \mathrm{Id}_{V(\Sigma,L)}$. Let f_1^i denote the restriction of f_1 to the index i-slice. Remember that everything acts trivially on β_1. Thus, in the case 0-1 we get the equation

$$\begin{aligned} x \otimes \beta_1 &= Z\left(N^1, \Phi(L \oplus 0) \oplus L, 0; f_1^1\right) \\ &\qquad \circ \left(Z\left(\Sigma \times I, L \oplus L, 0; f_1^0\right) \otimes Z(N^0, 0, 0)\right)(x \otimes \beta_1) \\ &= \lambda_0\lambda_1 \cdot V(\varphi_1^1, 0) \circ V(\varphi_0^1, 0)(x) \otimes \beta_{11} \otimes \beta_1 \\ &= \lambda_0\lambda_1 \cdot V(\varphi^1, 0)(x) \otimes \beta_{11} \otimes \beta_1 = \lambda_0\lambda_1 \cdot V(\varphi^1, 0)(x) \otimes \beta_1 \end{aligned}$$

where φ_i^1 are the restrictions to the index i-slice N^i of the flow φ^1 of f_1 on $(\Sigma - D_0) \times I$. If the genus of Σ is 0, then φ^1 is clearly isotopic to the identity and we end up with the equation $\lambda_0\lambda_1 = 1$. Otherwise we have a homotopy relative to the boundary from φ^1 to $\mathrm{Id}_{\Sigma - D_0}$ via $h_s(x) = \mathrm{pr}_1 \circ \varphi^1\big(t(x,s), (x,s)\big)$, where $t(x,s)$ is the time from $(x,s) \in \Sigma \times I$ to the top via φ^1 and pr_1 is the projection on the first factor. Since the genus of Σ is 1 or greater, the result from [25] states that φ^1 is isotopic to $\mathrm{Id}_{\Sigma - D_0}$. Thus, $V(\varphi^1, 0) = \mathrm{Id}_{V(\Sigma - D_0, L)}$ and we end up with the equation $\lambda_0\lambda_1 = 1$ again.

A 1-2 cancellation occurs when we glue together the two slices via the diffeomorphism which is S from fig. 6 on the handle, H, generated in the 1-slice, and the identity on the complement, Σ' — see fig. 17 for a local picture and compare with fig. 6. We also notice that the flow acts trivially on β_{111} as does every diffeomorphism of P (cf. page 21). Thus, arguing along the same lines as above, we get

$$\begin{aligned}
&x \otimes \beta_{111} \otimes \beta_1 \otimes \beta_1 \\
&\qquad = Z\left(\left(N^2, \Phi(L) \oplus L, 0\right) \cup_S \left(N^1, L \oplus \Phi(L), 0\right); f_1\right)(x \otimes \beta_{111} \otimes \beta_1 \otimes \beta_1) \\
&\qquad = \lambda_1 \cdot Z\left(N^2, \Phi(L) \oplus L, 0; f_1^2\right) \circ \\
&\qquad\qquad S^1\big(V(\varphi_1^1|_{\Sigma'}, 0)(x) \otimes V(\varphi_1^1|_H, 0)(\beta_{111}) \otimes \beta_{11}\big) \\
&\qquad = \lambda_1 \cdot Z\big(N^2, \Phi(L) \oplus L, 0; f_1^2\big) \circ S^1\big(V(\varphi_1^1|_{\Sigma'}, 0)(x) \otimes \beta_{111}\big) \\
&\qquad = \lambda_1 \cdot Z\big(N^2, \Phi(L) \oplus L, 0; f_1^2\big)\big(V(\varphi_1^1|_{\Sigma'}, 0)(x) \otimes \textstyle\sum_b S^1_{1,b} \beta_{1b\hat{b}}\big) \\
&\qquad = \lambda_1 \cdot Z\big(N^2, \Phi(L) \oplus L, 0; f_1^2\big)\big(V(\varphi_1^1|_{\Sigma'}, 0)(x) \otimes \textstyle\sum_b S_{1,b} \beta_{1b\hat{b}} \otimes \beta_{b\hat{b}}\big) \\
&\qquad = \lambda_1 \lambda_2 S_{1,1} \cdot V(\varphi^1|_{\Sigma'}, 0)(x) \otimes \beta_{111} \otimes \beta_1 \otimes \beta_1,
\end{aligned}$$

implying that $\lambda_1\lambda_2 S_{1,1} = 1$ by the previous argument. The slicing function f_1^2 above is of course the restriction of f_1 to the 2-slice N^2.

The 2-3-cancellation is actually the reverse of 0-1-cancellation:

$$\begin{aligned}
x \otimes \beta_{11} \otimes \beta_1 &= \big(Z\left(\Sigma \times I, L \oplus L, 0\right) \otimes Z(N^3, 0, 0)\big) \\
&\qquad \circ Z\left(N^2, L \oplus \Phi(L), 0; f_1^2\right)(x \otimes \beta_{11} \otimes \beta_1) \\
&= \lambda_2\lambda_3 \cdot V(\varphi^1, 0)(x) \otimes \beta_1 = \lambda_2\lambda_3 \cdot x \otimes \beta_{11} \otimes \beta_1,
\end{aligned}$$

so that $\lambda_2\lambda_3 = 1$. □

The set of equations (5.9) has solutions when ever $S_{1,1} \neq 0$. Let in the following λ_i, $i = 0, 1, 2, 3$, be a fixed solution to (5.9). Later on we will study the relation between theories defined by different solutions.

Proposition 5.10. *The function Z on slices is independent of slicing functions and it satisfies strong naturality. In particular, the assignment to slicings is well-defined.*

As a consequence of this we no longer include the slicing function in the notation.

Proof. From Remark 4.12 we know that any two unidirectional slicing functions f_0 and f_1 can be joined by a path containing at most $A2$-singularities and changings of the order of the critical points. As we saw in Lemma 5.6, isotopy does not change Z. So let us verify that Z is also unchanged when passing through a singularity.

Suppose that f_0 is a trivial slicing function and f_1 a slicing function with two critical points of indices differing by 1, such that they are connected by a path with a single $A2$-singularity. Then by Lemma 5.7 there is a diffeomorphism $F\colon M \to M$, such that f_0 and $\tilde{f} = f_1 \circ F^{-1}$ fits into Lemma 5.8 so that $Z(\boldsymbol{M}; f_0) = Z(\boldsymbol{M}; \tilde{f})$. But from Lemma 5.5 we have

$$Z(\boldsymbol{M}; \tilde{f}) = V(F|_{\partial_1 M}, 0) \circ Z(\boldsymbol{M}; f_1) \circ V(F|_{\partial_0 M}, 0)^{-1} = Z(\boldsymbol{M}; f_1),$$

where we have used the property of F from Lemma 5.7. Thus, Z on slices is invariant under the passing through an $A2$-singularity as in move number (2) of Remark 4.12.

This leaves only move number (1) in Remark 4.12 to be checked. This is a question of Z being invariant when two adjacent slices are changed to reorder the critical values: Such reordering can occur when no flow lines to the first critical point intersects flow lines to the second. Obviously, the definition of Z on natural slices generalizes to the case with a function that is a slicing function except for having two critical points with the same value, as long as the respective flow lines are disjoint in the sense of the above. Then one can construct an isotopy of the two slices taking the first slicing function to the second passing through functions no more singular than the one described above (and having only that one singularity). □

Proposition 5.11. *Let $\boldsymbol{M}$ be an e-manifold and let $\boldsymbol{s}$ and $\boldsymbol{s}'$ be any two e-slicings of $\boldsymbol{M}$. Then $Z(\boldsymbol{s}) = Z(\boldsymbol{s}')$.*

Proof. It suffices to show that the statement is true

- when the base slicings $\{N_i\} = \{N_i'\}$ agree, but the e-structure is changed,
- when using natural e-structures (which also means gluing with the identity), Z is unchanged by the moves of Remark 4.13.

To see the latter we first notice that $Z(M, L, n) = \lambda^n \cdot Z(M, L, 0)$ no matter what slicing is used because of Remark 3.7. Now, let $\boldsymbol{s}$ and $\boldsymbol{s}'$ have natural e-structures. By the mapping cylinder axiom, move number (1) in Remark 4.13, i.e. insertion of a trivial slice, does obviously not change $Z(\boldsymbol{s})$. Step number (2) and (3) of Remark 4.13 follows from Proposition 5.10.

Now, let $\boldsymbol{s}$ be an e-slicing of the e-manifold $\boldsymbol{M}$. We must check that changing the e-structure of the slicing without changing the e-structure of $\boldsymbol{M}$ does not effect Z. Instead we check the equivalent statement that $Z(\boldsymbol{M}, \boldsymbol{s})$ changes according to strong naturality when the e-structure of the slicing $\boldsymbol{s}$ is changed. It suffices to check this when the e-structure of one slice is changed. Clearly, changing the e-structure of a slice is equivalent to gluing on trivial slices on each end, and hence, it is sufficient to check the effect of inserting a trivial slice (with some odd e-structure).

Let $\boldsymbol{M} = \cup_{j=1}^k \boldsymbol{N}_j$, where gluing may be via non-identity maps like $\boldsymbol{g} = (\mathrm{Id}, 0) : (\partial_1 N_i, L_1) \to (\partial_0 N_{i+1}, L_0)$. Suppose that we want to insert the trivial slice $I_{\boldsymbol{f}}$ between $\boldsymbol{N}_i$ and $\boldsymbol{N}_{i+1}$, where

$$\boldsymbol{f} = (\mathrm{Id}, m) : (\partial_1 N_i, L_1') \longrightarrow (\partial_0 N_{i+1}, L_0').$$

Let $\boldsymbol{N} = \cup_{j=1}^{i} \boldsymbol{N}_j$ and $\boldsymbol{N}' = \cup_{j=i+1}^{k} \boldsymbol{N}_j$. Then because gluing is associative $\boldsymbol{M} = \boldsymbol{N} \cup \boldsymbol{I_g} \cup \boldsymbol{N}'$, where $\cup$ here means gluing with the real identity. If $\boldsymbol{N}$ and $\boldsymbol{N}'$ has framing number n and n' respectively, then according to the gluing law the framing number of $\boldsymbol{M}$ is

$$n + n' - \sigma(K, L_1, L_0) - \sigma(K, L_0, K'),$$

where $K = \ker\big(H_1(\partial_1 N) \to H_1(\hat{N})\big)$ and $K' = \ker\big(H_1(\partial_0 N') \to H_1(\hat{N}')\big)$ are the Lagrangian subspaces got by flowing through N (resp. N') of the Lagrangian subspace associated to the complementary boundary of N (resp. N'), i.e. $\hat{N}$ and $\hat{N}'$ are the capped off manifolds provided by Lemma 2.9.

When inserting $\boldsymbol{I_f}$ we must glue via $\boldsymbol{g}' = (\mathrm{Id}, 0) : (\partial_1 N, L_1) \to (\partial_1 N, L_1')$ and $\boldsymbol{g}'' = (\mathrm{Id}, 0) : (\partial_0 N', L_0') \to (\partial_0 N', L_0)$, and then $\boldsymbol{M}' = \boldsymbol{N} \cup \boldsymbol{I_{g'}} \cup \boldsymbol{I_f} \cup \boldsymbol{I_{g''}} \cup \boldsymbol{N}'$. The framing number of $\boldsymbol{M}'$ is

$$n + n' + m - \sigma(K, L_1, L_1') - \sigma(K, L_1', L_0') - \sigma(K, L_0', L_0) - \sigma(K, L_0, K')$$

and therefore the changing Δ of the framing number is

$$\begin{aligned}
\Delta &= m - \sigma(K, L_1, L_1') - \sigma(K, L_1', L_0') - \sigma(K, L_0', L_0) + \sigma(K, L_1, L_0) \\
&= m - \sigma(L_1, L_1', L_0') - \sigma(K, L_1, L_0') - \sigma(L_0', L_0, L_1) - \sigma(K, L_0', L_1) \\
&= m - \sigma(L_1, L_1', L_0') - \sigma(L_0', L_0, L_1)
\end{aligned}$$

by Lemma A.5. So if strong naturality applies we should get that $Z(\boldsymbol{M}', \boldsymbol{s}') = \lambda^{\Delta} \cdot Z(\boldsymbol{M}, \boldsymbol{s})$.

But using the given slicing of $\boldsymbol{M}'$, $Z(\boldsymbol{M}', \boldsymbol{s}') = Z(\boldsymbol{N}) \circ Z(\boldsymbol{I_{g'}} \cup \boldsymbol{I_f} \cup \boldsymbol{I_{g''}}) \circ Z(\boldsymbol{N}')$. Now, $\boldsymbol{I_{g'}} \cup \boldsymbol{I_f} \cup \boldsymbol{I_{g''}} = \boldsymbol{I_{g'' \circ f \circ g'}}$ and we notice that $\boldsymbol{g}'' \circ \boldsymbol{f} \circ \boldsymbol{g}' = (\mathrm{Id}, m') \circ \boldsymbol{g}$ for $(\mathrm{Id}, m') : (\partial_1 N', L_0) \to (\partial_1 N', L_0)$, where m' can be calculated by the composition law to be $m' = m - \sigma(L_1, L_1', L_0') - \sigma(L_0', L_0, L_1) = \Delta$. This means that

$$\begin{aligned}
Z(\boldsymbol{M}', \boldsymbol{s}') &= Z(\boldsymbol{N}) \circ V(\boldsymbol{g}'' \circ \boldsymbol{f} \circ \boldsymbol{g}') \circ Z(\boldsymbol{N}') \\
&= Z(\boldsymbol{N}) \circ V(\mathrm{Id}, \Delta) \circ V(\boldsymbol{g}) \circ Z(\boldsymbol{N}') \\
&= \lambda^{\Delta} \cdot Z(\boldsymbol{M}, \boldsymbol{s}),
\end{aligned}$$

which was exactly what we wanted. □

Finally we are able to define the TQFT:

Definition 5.12. The TQFT Z induced by the modular functor V is defined for every e-3-manifold $\boldsymbol{M}$ as

$$Z(\boldsymbol{M}) \stackrel{\text{def}}{=} Z(\boldsymbol{M}, \boldsymbol{s}),$$

where $\boldsymbol{s}$ is any e-slicing of $\boldsymbol{M}$.

It is left to the reader to check that Z satisfies the axioms TQFT1–4.

Also observe that if V is a unitary modular functor, then Z becomes a unitary TQFT provided the λ_i's satisfy the extra condition that $\lambda_0 = \bar{\lambda}_3$ and $\lambda_1 = \bar{\lambda}_2$. This implies that solution's look like

$$\lambda_0 = S_{1,1}^{\frac{1}{2}} e^{\sqrt{-1}\mu}, \qquad \lambda_1 = S_{1,1}^{-\frac{1}{2}} e^{-\sqrt{-1}\mu},$$
$$\lambda_2 = S_{1,1}^{-\frac{1}{2}} e^{\sqrt{-1}\mu}, \qquad \lambda_3 = S_{1,1}^{\frac{1}{2}} e^{-\sqrt{-1}\mu},$$

for some $\mu \in \mathbb{R}$. Thus, there is now a *real* one parameter family of solutions whereas there were a *complex* one-parameter family when we did not require the theory to be unitary. This ends the definition and consistency check of the TQFT.

Finally let us investigate how TQFTs from different solutions to (5.9) relate. Let λ_i', $i = 0, 1, 2, 3$ be another solution to (5.9) and denote the associated field theory by Z'. Putting $\kappa \overset{\text{def}}{=} \frac{\lambda_0'}{\lambda_0}$ we have

$$\frac{\lambda_i'}{\lambda_i} = \kappa^{(-1)^i}, \qquad \text{for } i = 0, 1, 2, 3.$$

Choose a slicing $\{\boldsymbol{N}_j\}$ for the e-3-manifold $\boldsymbol{M}$. Then on each slice, Z and Z' give a homomorphism differing only by a scaling by $\kappa^{(-1)^i}$, i being the index of the slice. Hence,

$$Z'(\boldsymbol{M}) = \prod_{i=0}^{3} \kappa^{(-1)^i c_i} Z(\boldsymbol{M}) = \kappa^{\sum_{i=0}^{3}(-1)^i c_i} Z(\boldsymbol{M}) = \kappa^{\chi(M)} Z(\boldsymbol{M}),$$

where c_i is the number of critical points of index i and $\chi(M)$ is the Euler characteristic of M. In particular, if $\boldsymbol{M}$ is closed, the Euler characteristic $\chi(M)$ of M is zero, and the partition function invariant for 3-manifolds, that we get, will be independent of which solution to (5.9) we choose.

We may conclude:

Theorem 5.13. *A modular functor V on the category of e-surfaces satisfying that $S_{1,1} \neq 0$, induces a complex one-parameter family of $2+1$ dimensional TQFTs defined by solutions $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ to $\lambda_0\lambda_1 = 1$, $\lambda_1\lambda_2 S_{1,1} = 1$ and $\lambda_2\lambda_3 = 1$. Any two members Z and Z' of this family corresponding to solutions λ_i and λ_i' are related by*

$$Z'(\boldsymbol{M}) = \kappa^{\chi(M)} Z(\boldsymbol{M}),$$

for any e-3-manifold $\boldsymbol{M}$, where $\kappa = \frac{\lambda_0'}{\lambda_0}$. In particular, Z' and Z agree on closed manifolds.

Moreover, if V is unitary we get a family of unitary TQFTs parameterized by S^1, with the extra condition that $\lambda_0 = \bar{\lambda}_3$ and $\lambda_1 = \bar{\lambda}_2$. □

Example 5.14. As an example we do the easy calculations of the invariant of S^3 and $S^1 \times S^2$:

$$Z(S^3,0) = Z(N^3,0,0) \circ Z(N^0,0,0) = \lambda_3\lambda_0 = \lambda_3\lambda_2\lambda_2^{-1}\lambda_0 = \lambda_1 S_{1,1}\lambda_0 = S_{1,1},$$
$$\begin{aligned} Z(S^1 \times S^2, 0) &= Z(N^3,0,0) \circ Z(N^1,0,0) \circ Z(N^2,0,0) \circ Z(N^0,0,0) \\ &= \lambda_3\lambda_1\lambda_2\lambda_0 = 1, \end{aligned}$$

so we notice that the theory is normalized as Witten's in [72].

An interesting and to the author's knowledge unsolved problem is the classification of $2+1$ dimensional TQFTs. If one were to consider TQFTs with corners as Walker does in [69], one can cut up any slice into a trivial mapping torus part and a part containing the critical point, the latter part being the basic blocks with boundary for instance $A \sqcup (D \sqcup D)$ as illustrated in fig. 18. Hence, any TQFT would therefore be constructed by assigning a complex number to each of the four blocks corresponding to the four Morse indices in a way similar to the one described in this section. It is conjectured that using Moore's and Seiberg's basic data, one should be able to define a modular functor from every TQFT. Proving Theorem 5.13 in this context would then tell us that every TQFT is constructed as described in this section.

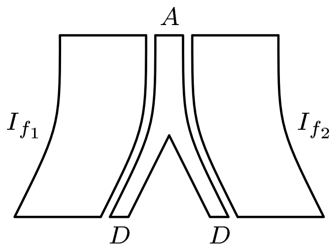

FIGURE 18. Decomposing a PL-slice of index 1.

It is, however, not possible to argue in the same way in our smooth case, since we are not allowed to decompose the slice in that way. So it is an open question whether there are any TQFTs that do not arise in the way described in this section. Given that we can get a modular functor for any TQFT (using Moore's and Seiberg's basic data), this is of course equivalent to the question if TQFTs on our smooth category are the same as TQFTs on Walker's manifolds with corners. The indications from physics are that the two theories should be the same.

In Walker's setup it is easy to prove that *if* V gives a TQFT then $S_{1,x} = \hat{S}(x)$, and he speculates that this may be true for any modular functor. One might add that any modular functor with $\hat{S}(1) = 0$ would be completely trivial since any factorization by a disk would yield a trivial pairing, $\langle -, - \rangle : V(\boldsymbol{Y}) \to V(\boldsymbol{Y})^*$, meaning that $V(\boldsymbol{Y})^* = 0 = V(\boldsymbol{Y})$. So far we have failed to reproduce the relation

$S_{1,1} = \hat{S}(1)$, and the author knows of no reason a priori why there could not be a modular functor with $S_{1,1} = 0$. This lack of an S-relation is not the only thing in our setup that is seemingly different to Walker's; as mentioned before in section 3, Walker also finds a relation for the map F in fig. 9.

6. Surfaces with Marked Points and Directions

Up till now we have considered modular functors from the category of e-surfaces possibly with boundary. In this section we will define another category, the category of (e-)surfaces with marked points and directions, and we will show that a modular functor on this category defines a TQFT as well.

Marked point and directions. For a vector space V we denote by PV the *space of directions* $(V - \{0\})/\mathbb{R}_{>0}$.

Definition 6.1. A *pointed e-surface* (or simply pe-surface) $\boldsymbol{Y} = \big(Y, L; (Q_1, v_1), \ldots, (Q_N, v_N)\big)$ is a closed, compact, smooth and oriented surface Y with N marked points Q_i and tangent directions $v_j \in PT_{Q_j}Y$ and a Lagrangian subspace $L \subseteq H_1(Y)$ compatible with the integer lattice $H_1(Y; \mathbb{Z})$.

Notice that the Lagrangian subspaces in $H_1(Y)$ are in one to one correspondence with the maximal isotropic subspaces of $H_1(Y - \{Q_1, \ldots, Q_N\})$. Hence, we may think of the marked points as corresponding to the boundary components in our previous definition. In particular, when considering boundary surfaces of e-3-manifolds there will be no marked points. To compactify the notation we write $\big((Q_1, v_1), \ldots, (Q_N, v_N)\big) = (\underline{Q}, \underline{v})$.

Definition 6.2. We say the e-surface $\boldsymbol{Y} = \big(Y, L, (\underline{Q}, \underline{v})\big)$ *contains* $\boldsymbol{Y}_0 = \big(Y_0, L_0, (\underline{Q}_0, \underline{v}_0)\big)$, denoted $\boldsymbol{Y}_0 \subset \boldsymbol{Y}$, if $i\colon Y_0 \hookrightarrow Y$, $i_* L_0 \subset L$, the marked points of Y_0 are marked points of Y, $\underline{Q}_0 \subseteq \underline{Q}$, and the corresponding directions agree.

Definition 6.3. A *pe-morphism* $\boldsymbol{f} : \boldsymbol{Y}_1 \to \boldsymbol{Y}_2$ is an isotopy class of orientation-preserving diffeomorphisms $f : Y_1 \to Y_2$ holding the set of marked points invariant and mapping directions to directions. We write $\boldsymbol{f} = (f, n)$.

Definition 6.4. Let $\boldsymbol{f}_1 = (f_1, n_1) : \boldsymbol{Y}_1 \to \boldsymbol{Y}_2$ and $\boldsymbol{f}_2 = (f_2, n_2) : \boldsymbol{Y}_2 \to \boldsymbol{Y}_3$ be morphisms of pe-surfaces $\boldsymbol{Y}_i = \big(Y_i, L_i, (\underline{Q}_i, \underline{v}_i)\big)$. Then the *composition* of $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ is defined by

$$(f_2, n_2)(f_1, n_1) \stackrel{\text{def}}{=} \big(f_2 f_1, n_2 + n_1 - \sigma((f_2 f_1)_* L_1, f_{2*} L_2, L_3)\big),$$

where σ is the Maslow index described in appendix A.

Definition 6.5. The *disjoint union* of pe-surfaces is

$$\big(Y_1, L_1; (\underline{Q}^1, \underline{v}^1)\big) \sqcup \big(Y_2, L_2; (\underline{Q}^2, \underline{v}^2)\big) \stackrel{\text{def}}{=} \big(Y_1 \sqcup Y_2, L_1 \oplus L_2; (\underline{Q}^1, \underline{v}^1), (\underline{Q}^2, \underline{v}^2)\big).$$

Now, let $\boldsymbol{Y} = \big(Y, L; (\underline{Q}, \underline{v})\big)$ and $\boldsymbol{Y}_p = \big(Y_p, L_p; (P_1, w_1), (P_2, w_2), (\underline{Q}, \underline{v})\big)$ be pe-surfaces. Let Y' be the singular surface with double point (P, v) obtained from Y_p by identifying P_1 with P_2 and $T_{P_1}Y_p$ with $T_{P_2}Y_p$ while reversing the orientation such that the image of w_1 and w_2 is v (see fig. 19). Let $\nu : Y_p \to Y'$ be the normalization map. Suppose that there is a simple, closed curve c with base point P_0 in Y such that Y' can be obtained from Y by contracting c.

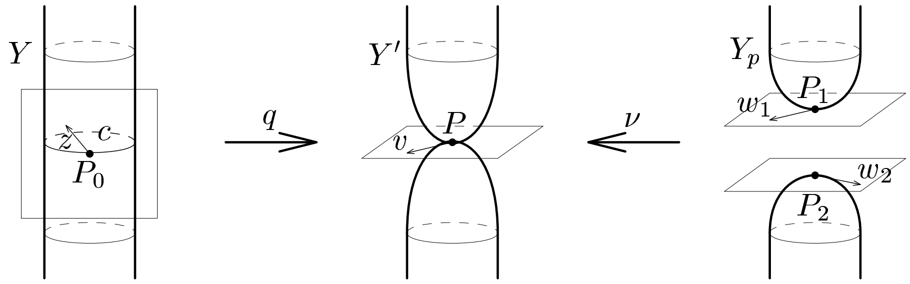

FIGURE 19. Pinching Y along (c, P_0).

Definition 6.6. Under the just mentioned conditions we say that $\boldsymbol{Y}_p$ is got from $\boldsymbol{Y}$ by *pinching along* (c, P_0) if there is a smooth map $q : Y \to Y'$ such that

(1) $q^{-1}(P) = c$,
(2) if $z \in PT_{P_0}Y$ and the orientation of c at P_0 induces the orientation of Y at P_0, then $q_*(z) = v$,
(3) $q_*(L) = \nu_*(L_p)$.

If $\boldsymbol{f}_1 = (f_1, n)$ is a pe-automorphism of $\boldsymbol{Y}$ such that f preserves (c, P_0), we define the induced pe-automorphism $\boldsymbol{f}_p = (f_p, n)$ on $\boldsymbol{Y}_p$ by $f_p \stackrel{\text{def}}{=} \nu^{-1} q f q^{-1} \nu$.

Notice that (3) implies that the homology class $[c]$ of c is in L (look for instance at at the dimension), so with respect to the Lagrangian subspaces it is just like surfaces with boundaries. The definitions of the category of e-3-manifolds are unchanged.

Modular functors and TQFTs. Let $\mathcal{L}$ be at label set as defined in section 2. The definition of a modular functor on the pe-surfaces is completely analogous to the definition given in section 3, but for clarity and for the benefit of the reader it is provided here also.

Definition 6.7. Let $\mathcal{L}$ be a label set. The category of *$\mathcal{L}$-labeled pe-surfaces* (lpe-surfaces) consists of pe-surfaces with an element of $\mathcal{L}$ assigned to each marked point and morphisms being pe-morphisms preserving the labeling. Such an assignment of elements of $\mathcal{L}$ to the marked points of $\boldsymbol{Y}$ is called a labeling of $\boldsymbol{Y}$ and we denote the lpe-surface by $(\boldsymbol{Y}, l)$, where l is the tuple of elements of $\mathcal{L}$. We denote the set of labelings of $\boldsymbol{Y}$ by $\mathcal{L}(\boldsymbol{Y})$ (or $\mathcal{L}(C)$ if C is the set of marked points).

Definition 6.8. A *modular functor* from the labeled pe-category of surfaces with marked points and directions based on the label set $\mathcal{L}$ is a functor $\dot{V}$ from the category of lpe-surfaces to the category of finite-dimensional complex vector spaces and linear isomorphisms satisfying the axioms MF1' to MF6' below.

MF1'. *Disjoint union axiom*: For the disjoint union of any pair of lpe-surfaces there is the identification

$$\dot{V}\big((\boldsymbol{Y}_1, l_1) \sqcup (\boldsymbol{Y}_2, l_2)\big) = \dot{V}(\boldsymbol{Y}_1 \sqcup \boldsymbol{Y}_2, l_1 \sqcup l_2) = \dot{V}(\boldsymbol{Y}_1, l_1) \otimes \dot{V}(\boldsymbol{Y}_2, l_2).$$

The identification is associative and compatible with the action of the mapping class groupoid in the sense that $\dot{V}(\boldsymbol{f}_1 \sqcup \boldsymbol{f}_2) = \dot{V}(\boldsymbol{f}_1) \otimes \dot{V}(\boldsymbol{f}_2)$.

MF2'. *Factorization axiom*: Let $\boldsymbol{Y}$ and $\boldsymbol{Y}_p$ be pe-surfaces. Assume that $\boldsymbol{Y}_p$ is obtained from $\boldsymbol{Y}$ by a series of pinchings along $(c_1, P_1), \ldots, (c_k, P_k)$, $c_i \cap c_j = \emptyset$, whenever $i \neq j$, and denote by C the set of basepoints $P_1, \ldots, P_k$. Then there is an identification

$$\dot{V}(\boldsymbol{Y}, l) = \bigoplus_{x \in \mathcal{L}(C)} \dot{V}\big(\boldsymbol{Y}_p, (l, x, \hat{x})\big),$$

which is associative and compatible with the action of the mapping class groupoid.

MF3'. *Duality axiom*: For any lpe-surface $(\boldsymbol{Y}, l)$ we identify the associated vector space

$$\dot{V}(\boldsymbol{Y}, l) = \dot{V}(-\boldsymbol{Y}, \hat{l})^*$$

with the dual of the vector space associated to $(\boldsymbol{Y}, l)$ with opposite orientation. This identification is compatible with orientation reversal, the action of the mapping class groupoid, MF1', and MF2' in the following manner:

- The identifications

$$\dot{V}(\boldsymbol{Y}) = \dot{V}(-\boldsymbol{Y})^*,$$
$$\dot{V}(-\boldsymbol{Y}) = \dot{V}(\boldsymbol{Y})^*$$

  are mutually adjoint.
- For an lpe-morphism $\boldsymbol{f} = (f, n) : (\boldsymbol{Y}_1, l_1) \to (\boldsymbol{Y}_2, l_2)$ let $\boldsymbol{f}^- \stackrel{\text{def}}{=} (f, -n) : (-\boldsymbol{Y}_1, \hat{l}_1) \to (-\boldsymbol{Y}_2, \hat{l}_2)$ be the induced one between the surfaces with opposite orientation. Then

$$\langle x, y \rangle = \big\langle \dot{V}(\boldsymbol{f})x, \dot{V}(\boldsymbol{f}^-)y \big\rangle$$

for all $x \in \dot{V}(\boldsymbol{Y}_1, l_1)$ and $y \in \dot{V}(-\boldsymbol{Y}_1, \hat{l}_1)$, i.e. $\dot{V}(\boldsymbol{f}^-)$ is the adjoint inverse of $\boldsymbol{f}$.

- For vectors

$$\alpha_1 \otimes \alpha_2 \in \dot{V}(\boldsymbol{Y}_1 \sqcup \boldsymbol{Y}_2) = \dot{V}(\boldsymbol{Y}_1) \otimes \dot{V}(\boldsymbol{Y}_2),$$
$$\beta_1 \otimes \beta_2 \in \dot{V}(-\boldsymbol{Y}_1 \sqcup -\boldsymbol{Y}_2) = \dot{V}(-\boldsymbol{Y}_1) \otimes \dot{V}(-\boldsymbol{Y}_2)$$

associated to the the disjoint union of pe-surfaces we have

$$\langle \alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2 \rangle_{\boldsymbol{Y}_1 \sqcup \boldsymbol{Y}_2} = \langle \alpha_1, \beta_1 \rangle_{\boldsymbol{Y}_1} \langle \alpha_2, \beta_2 \rangle_{\boldsymbol{Y}_2}.$$

- When factoring

$$\oplus_x \alpha_x \in \dot{V}(\boldsymbol{Y}, l) = \bigoplus_x \dot{V}\big(\boldsymbol{Y}_p, (l, x, \hat{x})\big),$$
$$\oplus_x \beta_x \in \dot{V}(-\boldsymbol{Y}, \hat{l}) = \bigoplus_x \dot{V}\big(- \boldsymbol{Y}_p, (\hat{l}, \hat{x}, x)\big)$$

we get

$$\langle \oplus_x \alpha_x, \oplus_x \beta_x \rangle_{(\boldsymbol{Y}, l)} = \sum_x \hat{S}(x) \langle \alpha_x, \beta_x \rangle_{(\boldsymbol{Y}_p, (l, x, \hat{x}))},$$

where $\hat{S}(x) = \hat{S}(x_1) \dots \hat{S}(x_k)$, $x = (x_1, \dots, x_k)$, is a function $\hat{S} : \mathcal{L} \to \mathbb{C} - \{0\}$ given as part of the data for $\dot{V}$.

MF4'. *Empty surface axiom*: Let $\emptyset$ denote the empty lpe-surface. Then

$$\dot{V}(\emptyset) \cong \mathbb{C}.$$

MF5'. *Once punctured sphere axiom*: Let $\boldsymbol{S}_1^2 = \big(S^2, 0; (Q, v)\big)$ be a pe-sphere with one marked point and the only possible choice of e-structure. Then

$$\dot{V}(\boldsymbol{S}_1^2, a) \cong \begin{cases} \mathbb{C}, & \text{for } a = 1, \\ 0, & \text{for } a \neq 1. \end{cases}$$

MF6'. *Twice punctured sphere axiom*: Let $\boldsymbol{S}_2^2 = \big(S^2, \mathbb{R}; (Q_1, v_1), (Q_2, v_2)\big)$ be a pe-sphere with two marked points and the one possible pe-structure. Then

$$\dot{V}\big(\boldsymbol{S}_2^2, (a, b)\big) \cong \begin{cases} \mathbb{C}, & \text{for } a = \hat{b}, \\ 0, & \text{for } a \neq \hat{b}. \end{cases}$$

As in section 3 one may have the additional property of unitarity:

MF-U'. A *unitary modular functor* is a modular functor such that every associated vector space $\dot{V}(\boldsymbol{Y})$ is furnished with a hermitian inner product

$$(-,-) : \dot{V}(\boldsymbol{Y}) \otimes \overline{\dot{V}(\boldsymbol{Y})} \longrightarrow \mathbb{C}$$

so that each morphism is unitary. The hermitian structure must satisfy compatibility properties like the ones in the duality axiom MF3' and commutativity of:

$$\begin{tikzcd}
\dot{V}(\boldsymbol{Y}) \arrow[r, "\cong"'] \arrow[d, "\cong"] & \dot{V}(-\boldsymbol{Y})^* \arrow[d, "\cong"'] \\
\overline{\dot{V}(\boldsymbol{Y})^*} \arrow[r, "\cong"] & \overline{\dot{V}(-\boldsymbol{Y})},
\end{tikzcd}$$

where the vertical identifications come from the hermitian structure and the horizontal from the duality.

It is a modular functor like this that is constructed in [4].

It is possible to do an analysis analogous to the one in the last part of section 3. In particular, we get the matrix S_{ab} corresponding to switching between two pinchings of the torus.

Correspondence with e-surfaces. It is a tempting idea that one should be able to construct a TQFT from this modular functor by using the apparent similarity between what happens when passing through a slice of index 2 and pinching. However, it turns out somewhat more complicated and not at all as natural a construction as one might think. On the other hand as we shall see in this section a modular functor $\dot{V}$ on lpe-surfaces induces a modular functor V on le-surfaces. Hence, by Theorem 5.13 we get our TQFT anyway (Theorem 6.10).

Proposition 6.9. *A modular functor $\dot{V}$ on the category of lpe-surfaces induces a modular functor V on the category of le-surfaces.*

Proof. In order to prove this we must show that to any e-surface $\boldsymbol{Y}$, we can associate a pe-surface $\dot{\boldsymbol{Y}}$ in such a way that by associating the vector space $\dot{V}(\dot{\boldsymbol{Y}})$ to $\boldsymbol{Y}$ a modular functor on the e-surfaces is formed.

Let $\dot{D}$ be the unit disk with the center marked with a point P_D and let $\delta\colon I \to \dot{D}$ be the radial curve $\delta(t) = (t,1)$, $1 \in S^1$. A direction in $PT_{P_D}\dot{D}$ is thereby defined. From any surface Y with parameterized boundary, we associate the pointed surface $\dot{Y}$ constructed by gluing on $\dot{D}$ on all the boundary components using the parameterization. This construction extends to the e-category simply by letting $\dot{\boldsymbol{Y}} = (\dot{Y}, L)$ for $\boldsymbol{Y} = (Y, L)$ and by extending e-morphisms to act as the identity on the glued-on disks. Clearly, $\dot{-}$ is a functor that is the identity functor on closed surfaces sending them to non-pointed surfaces and that preserves the properties of being a subsurface as well as being a disjoint union.

One should check that gluing of boundaries correspond to pinching: Under the conditions of Definition 2.7 let c_i be the parameterized boundary curves in C, and let $q\colon Y \to Y_g$ be the identification map. Then what should be checked is that $\dot{Y}$ is got from $\dot{Y}_g$ by pinching along $(q \circ c_i, q \circ c_i(0))$. But that is left to the reader.

Now we simply define V for all le-surfaces as $V(\boldsymbol{Y}, l) = \dot{V}(\dot{\boldsymbol{Y}}, l)$ and for all e-morphisms as $V(\boldsymbol{f}) = \dot{V}(\dot{\boldsymbol{f}})$. It is easy to see that the axioms of a modular functor on the category of le-surfaces are satisfied. □

Now, using Theorem 5.13 we get the main theorem of this section:

Theorem 6.10. *A modular functor $\dot{V}$ from the category of pe-surfaces with marked points and directions satisfying that $S_{1,1} \neq 0$, induces a complex 1-parameter family of $2+1$ dimensional TQFTs defined by solutions $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ to $\lambda_0\lambda_1 = 1$, $\lambda_1\lambda_2 S_{1,1} = 1$ and $\lambda_2\lambda_3 = 1$. Any two members Z and Z' of this family corresponding to solutions λ_i and λ'_i are related by*

$$Z'(\boldsymbol{M}) = \kappa^{\chi(M)} Z(\boldsymbol{M}),$$

for any e-3-manifold $\boldsymbol{M}$, where $\kappa = \frac{\lambda'_0}{\lambda_0}$. In particular, Z' and Z agrees on closed manifolds.

Moreover, if $\dot{V}$ is unitary we get a family of unitary TQFTs parameterized by S^1, with the extra condition that $\lambda_0 = \bar{\lambda}_3$ and $\lambda_1 = \bar{\lambda}_2$. □

Appendix A. About Lagrangian Subspaces

This appendix gives a brief overview of C.T.C. Wall's theorem about the non-additivity of the signature along with the definition and some properties of the correction term (the Maslow index) in the theorem. Moreover, we state and prove a series of auxiliary results for Lagrangian subspaces in the first homology of the boundary of a 3-manifold.

The situation is as follows: Suppose X is a compact 4-manifold decomposed along an embedded 3-manifold M_0 into X_1 and X_2 as depicted with artistic freedom in fig. 20 such that

$$\begin{aligned}\partial X_1 &= (-M_1) \cup M_0,\\ \partial X_2 &= (-M_0) \cup M_2,\\ \partial M_0 = \partial M_1 &= \partial M_2 = Y.\end{aligned}$$

The problem is to calculate the signature of X in terms of the signature of X_1 and X_2. It appears that:

Theorem A.1. [70] *If X is a 4-manifold as described above then*

$$\sigma(X) = \sigma(X_1) + \sigma(X_2) - \sigma(K_1, K_0, K_2),$$

where $K_i = \ker\big(H_1(Y) \to H_1(M_i)\big)$ for $i = 0, 1, 2$ and $\sigma(K_1, K_0, K_2)$ is defined below. □

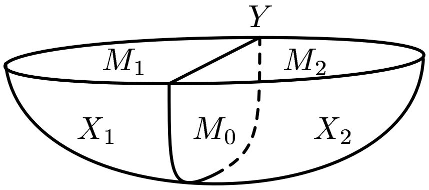

FIGURE 20. Decomposing the 4-manifold $X = X_1 \cup X_2$.

It calls for a little digression to explain the signature correction: Let A, B, C be subspaces of a finite-dimensional real vector space V and consider the additive relation between A and B

$$a \doteqdot b, \qquad \text{if there exists a } c \in C \text{ with } a + b + c = 0.$$

The domain is the set of $a \in A$ that can be written as $-b - c$, $b \in B$, $c \in C$, thus it is $A \cap (B + C)$. We also see that $a \doteqdot 0$ if and only if $a \in A \cap C$. This of course is symmetric in A and B and we get an isomorphism between

$$\frac{A \cap (B + C)}{A \cap C} \quad \text{and} \quad \frac{B \cap (C + A)}{B \cap C}.$$

This isomorphism preserves $A \cap B$, which we can then factor out. Now, there is symmetry also with respect to C so in fact we have identifications

$$W = \frac{A \cap (B + C)}{(A \cap B) + (A \cap C)} = \frac{B \cap (C + A)}{(B \cap C) + (B \cap A)} = \frac{C \cap (A + B)}{(C \cap A) + (C \cap B)}.$$

Let ω be a skew-symmetric bilinear form on V and denote by $A^\perp$ the orthogonal complement of A with respect to ω. One may verify that for subspaces A and B, $A^\perp \supseteq B^\perp$ if $A \subseteq B$, $(A^\perp)^\perp = A$, $(A + B)^\perp = A^\perp \cap B^\perp$, and $(A \cap B)^\perp = A^\perp + B^\perp$. We say that A is isotropic if $A \subseteq A^\perp$. A subspace A is maximal isotropic if the existence of a subspace B, such that $B \supseteq A$, implies that $B = A$. One can prove that if A is maximal isotropic then in fact $A^\perp = A$. If ω is non-degenerate, spaces for which $A^\perp = A$ are called Lagrangian.

Assume that A, B, C are isotropic with respect to ω. Suppose $a + b + c = a' + b' + c' = 0$ so that a and a' represent elements $[a]$ and $[a']$ in W. Define a symmetric bilinear form on W by

$$\Psi([a], [a']) = \omega(a, b').$$

This is well-defined by isotropicity of the subspaces. We check symmetry:

$$\begin{aligned}\Psi([a],[a']) - \Psi([a'],[a]) &= \omega(a,b') - \omega(a',b) = \omega(a,b') + \omega(b,a') \\ &= \omega(a+b,a'+b') - \omega(a,a') - \omega(b,b') = \omega(c,c') = 0.\end{aligned}$$

Now we define the signature correction term — also called the Maslov index — as the signature of this bilinear form on W

$$\sigma(A,B,C) \stackrel{\text{def}}{=} \operatorname{sign}(\Psi).$$

Lemma A.2. [70] *Permuting A, B and C changes $\sigma(A,B,C)$ by the sign of the permutation. In particular, $\sigma(A,B,C) = 0$ if two of the subspaces are identical. Moreover, if ω is replaced by $-\omega$ the sign of $\sigma(A,B,C)$ changes as well.*

Lemma A.3. *Let A_i, B_i and C_i be isotropic subspaces of (V_i, ω_i), $i = 1,2$. Then $A_1 \oplus A_2$, $B_1 \oplus B_2$ and $C_1 \oplus C_2$ are isotropic subspaces of $(V_1 \oplus V_2, \omega_1 \oplus \omega_2)$ and*

$$\sigma(A_1 \oplus A_2, B_1 \oplus B_2, C_1 \oplus C_2) = \sigma(A_1,B_1,C_1) + \sigma(A_2,B_2,C_2).$$

Proof. The proofs of these lemmas are easy and are left to the reader. □

Consider a surface Y and a maximal isotropic subspace $L \subseteq H_1(Y)$ with respect to the intersection pairing. Define

$$c(f_1,f_2) = -\sigma\big((f_1f_2)_*L, f_{1*}L, L\big),$$

where f_1 and f_2 are elements of the mapping class group $\Gamma(Y)$. If we let $\Gamma(Y)$ act trivially on $\mathbb{R}$, c becomes a 2-cocycle (see e.g. [20]):

$$(d_2c)(f_1,f_2,f_3) = c(f_2,f_3) - c(f_1f_2,f_3) + c(f_1,f_2f_3) - c(f_1,f_2) = 0.$$

This means that c determines a central extension of the mapping class group. This central extension is of course identical to the extended mapping class group $\Gamma(\boldsymbol{Y})$ defined in section 2, and it can be proved that it corresponds to Atiyah's extension of the mapping class group in [7] (see [1]).

Lemma A.4. *Let Y be a smooth, compact, closed, oriented surface and let $L_{\mathbb{Z}} \subset H_1(Y;\mathbb{Z})$ be a Lagrangian submodule. Assume that $\alpha_1, \dots, \alpha_g$ is a basis for $L_{\mathbb{Z}}$. Then there exist $\beta_1, \dots, \beta_g \in H_1(Y;\mathbb{Z})$ such that $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ is a symplectic basis for $H_1(Y;\mathbb{Z})$.*

Strictly speaking $\alpha_1, \dots, \beta_g$ cannot be a *basis* of $H_1(Y,\mathbb{Z})$ as $H_1(Y,\mathbb{Z})$ is only a $\mathbb{Z}$-module and not a vector space. However, since there is no torsion and $H_1(Y,\mathbb{Z})$ spans all of the vector space $H_1(Y,\mathbb{R})$ this abuse of terminology can be justified.

It appears to be common knowledge that the mapping class group $\Gamma(Y)$ maps surjectively onto the symplectic group $Sp(2g,\mathbb{Z})$ (cf. e.g. [21]). Hence, the above lemma has the immediate result that $\Gamma(Y)$ acts transitively on the set of Lagrangian subspaces of $H_1(Y)$ compatible with the integer lattice $H_1(Y;\mathbb{Z}) \subset H_1(Y)$, since $Sp(2g,\mathbb{Z})$ acts transitively on the space of symplectic bases. This together with the fact that c is a 2-cocycle proves the following:

Lemma A.5. *Let L_1, L_2, L_3 and L_4 be Lagrangian subspaces. Then*

$$\sigma(L_1, L_2, L_3) + \sigma(L_1, L_3, L_4) = \sigma(L_2, L_3, L_4) + \sigma(L_1, L_2, L_4).$$

□

Proof of Lemma A.4. First notice that because ω is unimodular (i.e. every functional $H_1(Y;\mathbb{Z}) \to \mathbb{Z}$ can be represented by ω and if $\omega(\alpha, -) = 0$ then $\alpha = 0$) we can always find β_i such that $\omega(\alpha_i, \beta_i) = 1$, since α_i is a generator. Thus, we proceed as follows:

Choose $\beta_1 \in (\mathrm{Span}_{\mathbb{Z}}\{\alpha_2, \dots, \alpha_g\})^{\perp}$ such that $\omega(\alpha_1, \beta_1) = 1$, and put $V_1 = \mathrm{Span}_{\mathbb{Z}}\{\beta_1\}$. Assume by induction that we have $\beta_1, \dots, \beta_i$ such that

$$\begin{gathered} \beta_j \in (\mathrm{Span}_{\mathbb{Z}}\{\alpha_1, \dots, \hat{\alpha}_j, \dots, \alpha_g\})^{\perp}, \\ \beta_j \in V_1^{\perp} \cap \dots \cap V_{j-1}^{\perp}, \\ \omega(\alpha_j, \beta_j) = 1, \quad \text{for } j = 1, \dots, i, \end{gathered}$$

where $V_j = \mathrm{Span}_{\mathbb{Z}}\{\beta_j\}$. While we always have that $\omega(\beta_j, \beta_j) = 0$ we see that $\omega(\beta_j, \beta_k) = 0$ for all $j, k \le i$, $j \ne k$. Notice also that $V_j \cap V_k = 0$ for all $j, k \le i$ and $(V_1 \oplus \dots \oplus V_i) \cap L_{\mathbb{Z}} = 0$.

We want to find

$$\beta_{i+1} \in (\mathrm{Span}_{\mathbb{Z}}\{\alpha_1, \dots, \hat{\alpha}_{i+1}, \dots, \alpha_g\})^{\perp} \cap V_1^{\perp} \cap \dots \cap V_i^{\perp}$$

so that $\omega(\alpha_{i+1}, \beta_{i+1}) = 1$. I.e. let $\varphi \colon H_1(Y,\mathbb{Z}) \to \mathbb{Z}$ be a homomorphism with $\varphi(\alpha_j) = 0$ for $j \ne i+1$, $\varphi(\beta_j) = 0$ for $j \le i$ and $\varphi(\alpha_{i+1}) = -1$ and choose β_{i+1} so that $\varphi(-) = \omega(\beta_{i+1}, -)$. The induction ends at β_g. Notice that $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ generate all of $H_1(Y,\mathbb{Z})$ and not only a sublattice, since $\omega(\alpha_i, \beta_i) = 1$ for all $i = 1, \dots, g$. □

As already defined in section 2, we say that an (isotropic or Lagrangian) subspace of $H_1(\Sigma)$ respects the integer lattice if it is generated by a subspace of $H_1(\Sigma;\mathbb{Z})$.

Lemma A.6. *Let Σ be a genus g surface, let Σ_0 be Σ with two small disks cut out, and let Σ_1 be Σ with a small annulus cut out. Assume L is a Lagrangian subspace of $H_1(\Sigma)$ respecting the integer lattice. If $i_\nu \colon \Sigma_\nu \hookrightarrow \Sigma$ is the natural inclusion, $L_\nu = i_*^{-1}(L)$ is a maximal isotropic subspace of $H_1(\Sigma_\nu)$ respecting the integer lattice.*

Proof. We can assume that the genus is greater than 0. Suppose first that Σ is connected. Following Lemma A.4 we can choose a symplectic basis $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ such that L is spanned by $\alpha_1, \ldots, \alpha_g$. Choose curves c_i, d_i in $\Sigma_0 \subset \Sigma$ so that $\alpha_i = [c_i]$ and $\beta_i = [d_i]$. Then $H_1(\Sigma_0)$ has a basis $[c_1], [d_1], \ldots, [c_g], [d_g], \gamma$, where γ is coming from one of the boundary components. Writing up L_0 and $L_0^\perp$ in this basis gives the result.

Similarly for Σ_1 if the annulus is non-separating, we chose the basis such that the surface is cut along a curve, say c_1, representing α_1. Then $\alpha_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g$ constitute a basis for $H_1(\Sigma_1)$, and thus we get the result for L_1.

If we are cutting the disks out of two different components or cutting out a separating annulus, $H_1(\Sigma_0) \cong H_1(\Sigma)$ and $H_1(\Sigma_1) \cong H_1(\Sigma)$ respectively and it is easily seen that $L_\nu = L_\nu^\perp$. □

We notice that any index 2-slice is an index 1-slice turned upside down (i.e. taking the slicing function f to $1-f$) and obviously pairs of Lagrangian subspaces that we get in the above procedure agree under that correspondence. Notice also that all subspaces considered in homology are compatible with the integer lattice. We have proved:

Proposition A.7. *For any index 1-slice N and any Lagrangian subspace $L \in H_1(\partial_0 N)$ there is a naturally induced maximal isotropic subspace $L' \subseteq H_1\big(\partial_0 N - (D \sqcup D)\big) \cong H_1(\partial_1 N - A)$ and a naturally induced Lagrangian subspace $L'' \subseteq H_1(\partial_1 N)$, such that L and L'' are the images of L' under the respective gluings. Symmetrically for an index 2-slice and a Lagrangian subspace containing the subspace generated by the collapsing curve. These constructions obviously agree when an index 1-slice is turned into an index 2-slice or visa versa by sending the slicing function f to $1-f$.*

Denote by Σ^c the level surface of the critical value for a slicing function on N. Let $\varphi_0 : \partial_0 N \to \Sigma^c$ be the gradient flow of the slicing function and let $\varphi_1 : \partial_1 N \to \Sigma^c$ be the flow backwards along the gradient field. This induces maps in homology and for $x \in H_1(\partial_0 N)$ defines the subset $\Phi(x) \stackrel{\text{def}}{=} \varphi_{1*}^{-1}\big(\varphi_{0*}(x)\big)$ of $H_1(\partial_1 N)$. Let therefore

$$\Phi(L) = \varphi_{1*}^{-1}\big(\varphi_{0*}(L)\big)$$

be the subspace obtained by flowing the Lagrangian L through N. Below we shall see that $\Phi(L)$ is in fact also Lagrangian. As it was introduced in Definition 5.1 such a pair of Lagrangian subspaces $\big(L, \Phi(L)\big)$ is called natural.

We will see in the next proposition, that (L, L'') is a natural pair of Lagrangian subspaces. It is of course also possible to talk of natural pairs of Lagrangians for general 3-manifolds with a slicing, by using the construction successively. From the proposition below it follows that the natural e-structure of a 3-manifold in general is independent of slicing.

Proposition A.8. *Let N be a slice and L a Lagrangian subspace of $H_1(\partial_0 N)$; if N is of index 2 assume also that L contains the subspace generated by the collapsing curve. Then $\Phi(L)$ is a Lagrangian subspace in $H_1(\partial_1 N)$ and $\Phi(L)$ agrees with L'' obtained by the cutting- and pasting-construction above.*

Moreover, if $L = \ker\big(H_1(\partial_0 N) \to H_1(N^+)\big)$ for some 3-manifold N^+ with boundary $\partial N^+ = \partial_0 N$, then $\Phi(L) = \ker\big(H_1(\partial_1 N) \to H_1(N \cup N^+)\big)$. In particular, $\Phi(L)$ is independent of both metric and slicing function.

Proof. We prove that $\Phi(L)$ agrees with spaces we get by cutting and pasting and thereby we get that it is in fact a Lagrangian subspace.

Notice now that for Σ' being $\partial_0 N$ with the appropriate bits cut out we have the commutative diagram[8]

$$\begin{array}{ccc} \Sigma' & \xrightarrow{i_0} & \partial_0 N \\ {\scriptstyle i_1}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_0} \\ \partial_1 N & \xrightarrow{\varphi_1} & \Sigma^c, \end{array}$$

since $\varphi_j i_j$ is the natural embedding in both cases. Therefore, in first homology we have that $\varphi_{0*} i_{0*} = \varphi_{1*} i_{1*}$ and

$$\varphi_{1*}^{-1}\varphi_{0*}(\operatorname{Im} i_{0*} \cap L) = \varphi_{1*}^{-1}\varphi_{0*}(i_{0*} i_{0*}^{-1} L) = i_{1*} i_{0*}^{-1} L + \ker \varphi_{1*}.$$

However, in all cases considered $L \subseteq \operatorname{Im} i_{0*}$, $i_{1*} i_{0*}^{-1} L \supseteq \ker \varphi_{1*}$ and $i_{1*} i_{0*}^{-1} L = L''$.

To prove that $\Phi(L) = \ker\big(H_1(\partial_1 N) \to H_1(N \cup N^+)\big)$ look at the retraction $r : N \to \Sigma^c$ induced by the Morse flow. This gives a homotopy equivalence $R = r \cup \mathrm{Id} : N \cup N^+ \to N_c^+$ to a singular manifold N_c^+. Let i denote the composition of $i^+ : \partial_0 N \hookrightarrow N^+$ and $I^+ : N^+ \hookrightarrow N \cup N^+$ and let i_1^+ and i^c be the embeddings

[8]In fact the diffeomorphism that identifies $\Sigma' \subset \partial_0 N$ and $\Sigma' \subset \partial_1 N$ is $\varphi_1^{-1}\varphi_0$, so $i_1 = \varphi_1^{-1}\varphi_0 i_0$.

$\partial_1 N \hookrightarrow N \cup N^+$ and $\Sigma^c \hookrightarrow N_c^+$. Then there is a commutative diagram

$$\begin{array}{ccc}
H_1(\partial_0 N) & \xrightarrow{i_*} & H_1(N \cup N^+) \\
\varphi_{0*}\big\downarrow & & \cong\big\downarrow R_* \\
H_1(\Sigma_c) & \xrightarrow{i^c_*} & H_1(N_c^+) \\
\varphi_{1*}\big\uparrow & & \cong\big\uparrow R_* \\
H_1(\partial_1 N) & \xrightarrow{i^+_{1*}} & H_1(N \cup N^+),
\end{array}$$

which implies that $\Phi(L) = \varphi_{1*}^{-1}\varphi_{0*}(L) = (i^+_{1*})^{-1}\big(I^+_* i^+_*(L)\big) = \ker i^+_{1*}$. All Lagrangian subspaces are kernels like this by Lemma 2.9, thus independence of the extra choices follows immediately. $\square$

Lemma A.9. *Let N be a slice, let L and $\tilde{L}$ be Lagrangian subspaces of $H_1(\partial_0 N)$, and let $K = \ker\big(H_1(\partial N) \to H_1(N)\big)$. Then*

$$\sigma\big(K, L \oplus \Phi(L), \tilde{L} \oplus \Phi(\tilde{L})\big) = 0.$$

Proof. It suffices to show this for a connected slice. Let $F \subset H_1(\partial N) = H_1(\partial_0 N) \oplus H_1(\partial_1 N)$ be defined as

$$\begin{aligned}
F &= \{(x, -y) \mid x \in H_1(\partial_0 N),\ y \in \Phi(x) \subseteq H_1(\partial_1 N)\} \\
&= \{(x, y) \mid x \in H_1(\partial_0 N),\ y \in H_1(\partial_1 N),\ \varphi_{0*}x = -\varphi_{1*}y\}.
\end{aligned}$$

Had N been a mapping cylinder of some mapping class f, $K = F$ would be the anti-diagonal under the map f. We will prove that in fact $F = K$ in general for slices.

Notice that Σ^c is a retract of N; actually this retraction $r : N \to \Sigma^c$ can be realized by the Morse flow from each side of Σ^c. Thus, if $i : \partial N \to N$, $ri|_{\partial_\nu N} = \varphi_\nu$ and $\varphi_{0*} + \varphi_{1*}$ is the composition

$$H_1(\partial_0 N) \oplus H_1(\partial_1 N) = H_1(\partial N) \xrightarrow{i_*} H_1(N) \xrightarrow[\cong]{r_*} H_1(\Sigma^c).$$

Therefore,

$$K = \ker i_* = \ker(\varphi_{0*} + \varphi_{1*}) = F.$$

Now, recall that σ on three maximal isotropic subspaces L_1, L_2, L_3 is defined as the signature of the bilinear form on

$$U = \frac{L_1 \cap (L_2 + L_3)}{(L_1 \cap L_2) + (L_1 \cap L_3)}$$

induced from the form on W. Hence, we have to look at

$$U = \frac{K \cap \big(L \oplus \Phi(L) + \tilde{L} \oplus \Phi(\tilde{L})\big)}{\big(K \cap L \oplus \Phi(L)\big) + \big(K \cap \tilde{L} \oplus \Phi(\tilde{L})\big)}.$$

But $L \oplus \Phi(L) + \tilde{L} \oplus \Phi(\tilde{L}) = (L + \tilde{L}) \oplus \Phi(L + \tilde{L})$ so from the above we have that $K_L \overset{\text{def}}{=} K \cap \big(L \oplus \Phi(L)\big) = \{(x, -y) \mid x \in L,\ y \in \Phi(x)\}$ and thus $U = K_{L+\tilde{L}}/(K_L + K_{\tilde{L}})$. Now, clearly $K_{L+\tilde{L}} = K_L + K_{\tilde{L}}$ and therefore $U = 0$ and subsequently $\sigma\big(K, L \oplus \Phi(L), \tilde{L} \oplus \Phi(\tilde{L})\big) = 0$. □

APPENDIX B. TECHNICAL TRICKERY

In this appendix we will prove the technical Lemmas 4.5 and 4.6.

Proof of Lemma 4.5. The problem is to smooth the function

$$\tilde{f}(x) = \begin{cases} f_1(x) & \text{for } x \in M_1, \\ t_k - t_0' + f_2(x) & \text{for } x \in M_2 \end{cases}$$

without introducing any further critical points or changing the original ones.

There is a tubular neighbourhood of ∂M_1 where there are no critical points of f_1. Through the (backwards) gradient flow of f_1 we construct a collar of M_1 at the boundary component $f_1^{-1}(t_k)$ on which f_1 has the form $f_1(x, t) = f_1(t)$ for $(x, t) \in f_1^{-1}(t_k) \times]\varepsilon, 0]$, $\varepsilon > 0$ and $f_1(\cdot)$ is some smooth function without critical points. Do similarly for M_2 and thus the problem is reduced to a one dimensional problem.

Assume therefore that we have two smooth functions f_1 and f_2 defined on $]-1, 0]$ and $[0, 1[$ resp. Assume furthermore that $0 < f_i'(x)$ everywhere for $i = 1, 2$ and that we have Taylor expansions around 0, $f_1(x) = a_0 + a_1 x + R_1(x)$ defined on all of $]-1, 0]$ and $f_2(x) = a_0 + a_1' x + R_2(x)$ on $[0, 1[$, where $R_i(x)$ at least is of order 2. We want to join them to a smooth function $f\colon]-1, 1[\to \mathbb{R}$ with $f'(x) > 0$ for all x.

First step is to reduce to a piecewise linear problem. We wish to choose $\varepsilon > 0$ and $\varphi\colon]-\varepsilon, 0] \to [0, 1]$ such that $\varphi\big(]-\frac{\varepsilon}{3}, 0]\big) = 0$, $\varphi\big(]-\varepsilon, -\frac{2\varepsilon}{3}[\big) = 1$ and $\varphi'(x) \leq 0$ everywhere, and such that

$$\bar{f}_1(x) = a_0 + a_1 x + \varphi(x) R_1(x),$$

is a perturbation of f_1 near 0 with $\bar{f}_1'(x) > 0$. Now, $\bar{f}_1'(x) = a_1 + \varphi'(x) R_1(x) + \varphi(x) R_1'(x)$. We can choose ε such that $|R_1'(x)| < \frac{a_1}{2}$ for $x \in]-\varepsilon, 0]$. Let $k \geq 2$ be some constant and decrease ε until $|R_1(x)| < \frac{a_1|x|}{6k}$ for $x \in]-\varepsilon, 0]$; this is possible since R_1 is of order 2. Then choose φ with the additional properties that $|\varphi'(x)| \leq \frac{3k}{\varepsilon}$. Then

$$\bar{f}_1'(x) > a_1 - \frac{3k}{\varepsilon}\frac{a_1\varepsilon}{6k} - \frac{a_1}{2} = 0$$

and on $]-\frac{\varepsilon}{3}, 0]$, $\bar{f}_1(x) = a_0 + a_1 x$. Similarly, we can construct $\bar{f}_2(x) = a_0 + a_1' x + \psi(x)R_2(x)$ such that $\bar{f}_2(x) = a_0 + a_1' x$ on $[0, \frac{\varepsilon'}{3}[$ for $\varepsilon' > 0$ and ψ chosen similar to φ.

To simplify the notation we now assume that $f_1(x) = a_0 + a_1 x$ on $]-K, 0]$ and $f_2(x) = a_0 + a_1' x$ on $[0, K[$ for $K > 1$. Without loss of generality we can also assume that $a_1 < a_1'$ and $a_1 < a_0$. Then choose a partition of unity φ_i such that $\varphi_1(x) = 1$ on $]-K, -1]$, $\varphi_2(x) = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $\varphi_3(x) = 1$ on $[1, K[$ and such that φ_1 is supported on $]-K, -\frac{1}{2}[$, φ_2 on $]-1, 1[$, φ_3 on $]\frac{1}{2}, K[$, and $\varphi_1'(x) + \varphi_2'(x) + \varphi_3'(x) = 0$. Put $\alpha_0 = a_0 + \frac{a_1' - a_1}{2}$, $\alpha_1 = \frac{a_1 + a_1'}{2}$ and $g(x) = \alpha_0 + \alpha_1 x$; then $g(-1) = f_1(-1)$ and $g(1) = f_2(1)$. Furthermore, let

$$\tilde{f}(x) = \varphi_1(x)(a_0 + a_1 x) + \varphi_2(x)g(x) + \varphi_3(x)(a_0 + a_1' x).$$

Clearly, $\tilde{f}$ is smooth and agrees with f_1 and f_2 outside $]-1, 1[$ and

$$\begin{aligned}\tilde{f}'(x) = a_1\varphi_1(x) + \varphi_1'(x)(a_1 x + a_0) + \alpha_1\varphi_2(x)\\ + \varphi_2'(x)(\alpha_0 + \alpha_1 x) + a_1'\varphi_3(x) + \varphi_3'(x)(a_0 + a_1' x).\end{aligned}$$

Observe that

$$\begin{aligned}
a_1\varphi_1(x) &\begin{cases} > 0 & \text{for } x < -\frac{1}{2}, \\ = 0 & \text{for } x \geq -\frac{1}{2}, \end{cases}\\
\alpha_1\varphi_2(x) &\begin{cases} > 0 & \text{for } -\frac{1}{2} < x < 1, \\ = 0 & \text{for } x \leq -1, x \geq 1, \end{cases}\\
a_1'\varphi_3(x) &\begin{cases} = 0 & \text{for } x \leq \frac{1}{2}, \\ > 0 & \text{for } x > \frac{1}{2}, \end{cases}\\
\varphi_1'(x)(a_1 x + a_0) &\begin{cases} < 0 & \text{for } -1 < x < -\frac{1}{2}, \\ = 0 & \text{for } x \leq -1, x \geq -\frac{1}{2}, \end{cases}\\
\varphi_2'(x)(\alpha_0 + \alpha_1 x) &\begin{cases} > 0 & \text{for } -1 < x < -\frac{1}{2}, \\ < 0 & \text{for } \frac{1}{2} < x < 1, \\ = 0 & \text{for } x \leq -1, -\frac{1}{2} < x < \frac{1}{2}, x \geq 1, \end{cases}\\
\varphi_3'(x)(a_0 + a_1' x) &\begin{cases} > 0 & \text{for } \frac{1}{2} < x < 1, \\ = 0 & \text{for } x \leq \frac{1}{2}, x \geq 1, \end{cases}
\end{aligned}$$

hence, $\tilde{f}'(x) > 0$ for $x \leq -1$, $-\frac{1}{2} \leq x \leq \frac{1}{2}$ and $x \geq 1$. A little more work is required for $-1 < x < -\frac{1}{2}$:

$$\begin{aligned}\tilde{f}'(x) &= a_1\varphi_1(x) + \varphi_1'(x)(a_1x + a_0) + \alpha_1\varphi_2(x)\varphi_2'(x)(\alpha_0 + \alpha_1 x)\\ &= \tfrac{1}{2}\big(a_1\big(1+\varphi_1(x)\big) + a_1\big(1-\varphi_1(x)\big) + (a_1 - a_1')(x+1)\varphi_1'(x)\big) > 0.\end{aligned}$$

The interval $\frac{1}{2} < x < 1$ follows in the same way. This ends the proof of lemma 4.5. $\square$

Proof of Lemma 4.6. Choose an embedding of Y into $\mathbb{R}^3$ such that the first coordinate is a Morse function. We have a map Φ of the ambient space $\mathbb{R}^3 \times \mathbb{R}$, defined as

$$\Phi(x_1, x_2, x_3, t) = (x_1 \cos \pi t, x_2, x_3, x_1 \sin \pi t),$$

that restricts to a diffeomorphism

$$\Phi : Y \times I \longrightarrow \Phi(Y \times I).$$

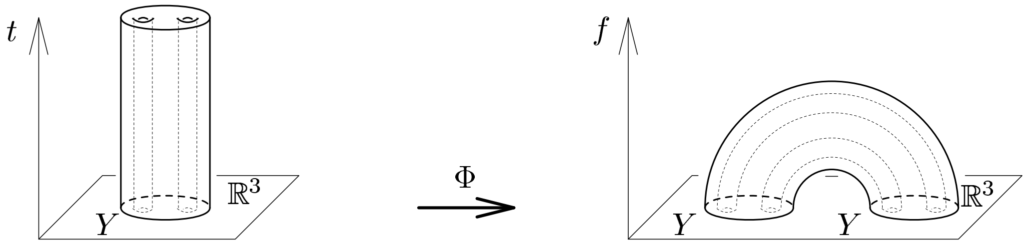

FIGURE 21. Bending diffeomorphism.

Let

$$f : Y \times I \longrightarrow \mathbb{R}$$

be the first coordinate of Φ, $f(x,t) = x_1 \sin \pi t$. By construction $f^{-1}(0) = Y \sqcup Y$ and $f^{-1}(t_k) = \emptyset$, for t_k large enough. Now, given the (inverse of) local coordinates $\varphi : U \to Y \subset \mathbb{R}^3$ we calculate the differential

$$d_{(u,v,t)}\big(f \circ (\varphi \times \mathrm{Id}_I)\big) = \left(\frac{\partial \varphi_1}{\partial u} \sin \pi t, \frac{\partial \varphi_1}{\partial v} \sin \pi t, \pi \cos \pi t\right).$$

Thus, (u,v,t) is a critical point provided $t = \frac{1}{2}$ and $\frac{\partial \varphi_1}{\partial u}(u,v) = \frac{\partial \varphi_1}{\partial v}(u,v) = 0$ meaning that the critical points are the critical points of the first coordinate function x_1 embedded in the top meridian as we would expect. The determinant of the Hessian is $-\pi^2 d^2_{(u,v)}\varphi_1 \sin^3 \pi t$ hence, $d^2_{(u,v,t)}\big(f \circ (\varphi \times \mathrm{Id}_I)\big)$ is invertible if and only if $d^2_{(u,v)}\varphi_1$ is invertible. But that is the case since the first coordinate of φ was a Morse function. Thus, f is a slicing function with the required properties. $\square$

Part II

Towards a Calculation of the Witten Invariant of Finite Order Mapping Tori

1. Introduction

Let $f\colon X \to X$ be an orientation preserving diffeomorphism of a closed, smooth, and oriented surface. Then we define the mapping torus X_f as the compact, smooth, oriented 3-manifold obtained by gluing the ends of the cylinder $X \times I$ by f; i.e. $X_f = X \times I/\sim$, where $(x,0) \sim (f(x),1)$. If $\boldsymbol{f} = (f,m)\colon \boldsymbol{X} \to \boldsymbol{X}$ is an e-morphism of an e-surface $\boldsymbol{X} = (X,L)$ (see section I.2 for definitions), we see by the gluing rule Definition I.2.8 that (in the notation of Part I) the e-mapping torus is $\boldsymbol{X_f} = (X_f, m - \sigma(\Delta_{\mathrm{Id}}^-, L \oplus L, \Delta_f^-))$. We assume that the genus g of X is greater than 1.

Suppose first that we have a TQFT, Z, as the one defined in section I.3. Then it follows from the gluing axiom that the invariant of the e-mapping torus $\boldsymbol{X_f}$ is

$$Z(\boldsymbol{X_f}) = \operatorname{Tr} V(\boldsymbol{f}).$$

There is on the other hand a geometric construction of a vector space $V(\boldsymbol{X})$ to each e-surface $\boldsymbol{X}$ which is conjectured to give a modular functor and therefore through Theorem I.5.13, a TQFT. We do however not need to know if this constitute a TQFT in order to associate a number to each mapping torus; we may merely define it as

$$Z(\boldsymbol{X_f}) \stackrel{\text{def}}{=} \operatorname{Tr} V(\boldsymbol{f}),$$

where $V(\boldsymbol{f})$ denotes the natural action of $\boldsymbol{f}$ on the vector space $V(\boldsymbol{X})$ (which exists by construction of $V(\boldsymbol{X})$). This definition was considered by J.E. Andersen in [1]. As there may be several e-surfaces $\boldsymbol{X}$ and e-morphisms $\boldsymbol{f}$ giving the same mapping torus $\boldsymbol{X_f}$ we have a priori no way of telling that this is an invariant of mapping tori.

But it is definitely an invariant of pairs $(\boldsymbol{X}, \boldsymbol{f})$ of e-surfaces and e-morphisms. That the construction of V leads to a modular functor is a widely accepted conjecture; in fact as mentioned in Part I, a proof may be close at hand (see [4]). In that case, V is part of a TQFT and $Z(\boldsymbol{X}_{\boldsymbol{f}})$ becomes an invariant of the e-mapping torus. (That is why we dare abuse the notation writing $Z(\boldsymbol{X}_{\boldsymbol{f}})$ for something which is strictly speaking not an invariant of $\boldsymbol{X}_{\boldsymbol{f}}$.)

We only give a brief description of $V(\boldsymbol{X})$ here; for a more detailed exposition tailored for our purpose, see [1]. Let X be a smooth, closed, oriented surface of genus $g \geq 2$ and let G be a simple, simply connected Lie group (in our case it will be $SU(2)$). The vector space $V(\boldsymbol{X})$ associated to $\boldsymbol{X}$ is defined as the covariant constant sections of a certain bundle Z over the Teichmüller space T_X with respect to a certain flat connection. As we are interested here only in finite order diffeomorphisms τ of X, we may assume that τ has a fixed point σ in T_X, and since τ (or rather its extended counterparts $\boldsymbol{\tau} = (\tau, m)$) preserves this vector bundle over T_X and its flat connection we need only to concern ourselves with the action of $\boldsymbol{\tau}$ on the fiber Z_σ. Thus, let X_σ denote the Riemann surface X with holomorphic structure σ and let $M(X_\sigma)$ be the moduli space of semistable $G^{\mathbb{C}}$ bundles over X_σ. Then there is a determinant line bundle $\mathcal{L}$ over $M(X_\sigma)$ whose first Chern class is represented by the Kähler form on $M(X_\sigma)$. In fact $\mathcal{L}$ is the ample generator of $\operatorname{Pic}\big(M(X_\sigma)\big) \cong \mathbb{Z}$, and because τ preserves the ample cone, $\tau^*\mathcal{L} \cong \mathcal{L}$. Hence, the action of τ lifts to $\mathcal{L}$ in the manner described in section 3. There is also a determinant line bundle $\mathcal{L}_D$ over T_X with a specified action of $\boldsymbol{\tau}$ on every real power $\mathcal{L}_D^\alpha$, $\alpha \in \mathbb{R}$. The space Z_σ at level $\kappa \geq 1$ is defined as

$$Z_\sigma = H^0\big(M(X_\sigma), \mathcal{L}^\kappa\big) \otimes \mathcal{L}_{D,\sigma}^{-\frac{1}{2}c_c},$$

where c_c is the central charge given by

$$c_c = \frac{|G|\kappa}{\kappa + h},$$

and h is the dual Coxeter number of G.

Hence, we may write

$$Z(\boldsymbol{X}_{\boldsymbol{\tau}}) = \operatorname{Tr}\big(\tau \big| H^0(M(X_\sigma), \mathcal{L}^\kappa)\big) \cdot \operatorname{Tr}\big(\boldsymbol{\tau} \big| \mathcal{L}_{D,\sigma}^{-\frac{1}{2}c_c}\big), \tag{1.1}$$

where we employ the notation $\operatorname{Tr}(\tau|H)$ of e.g. [9] for the trace of the (natural) action of τ on the space H.

According to [7] there is a natural choice of 2-framing for a closed 3-manifold; the corresponding e-structure, we will call the Atiyah e-structure, and in [1] it is proved that using this e-structure the second factor in the above expression for $Z(\boldsymbol{X}_{\boldsymbol{\tau}})$ simplifies to an expression containing only local data for τ's action on X, namely the Seifert invariants of X_f. We will employ the notation $Z(X_\tau)$ for $Z(\boldsymbol{X}_{\boldsymbol{\tau}})$, where the Atiyah e-structure is chosen. Of course if Z and V are part of a TQFT, then for any e-structure, the naturality axiom determines $Z(\boldsymbol{X}_{\boldsymbol{\tau}})$ in terms of $Z(X_\tau)$; so does (1.1).

Synopsis. This part of the thesis is organized as follows: This *section 1* is the introduction.

Section 2 contains the computation of $\ker\{\pi^*\colon \operatorname{Pic}(X/\langle\tau\rangle) \to \operatorname{Pic}(X)\}$ and related results on pushforward of $\mathcal{O}_X$ and character line bundles.

Section 3 introduces the fixed point problem by considering the unramified case. The concept of lifts is defined.

Section 4 considers the general case and establishes the main results Theorem 4.16 and Corollary 4.17.

Section 5 sketches in which directions we plan for further developments.

Section 6 deals with the example where $\tau = J$ is the hyperelliptic involution on a hyperelliptic surface X. We compute the space C of components of the fixed point set in the moduli space, the dimension d_c and $e^{CS(X_\tau,c)}$.

Appendix A gives proofs of results used in section 4.

Appendix B gives a brief introduction to geometric invariant theory and proves that $\overline{\mathcal{EM}}\colon Ma(Y) \to |M(X)|$ is a morphism of varieties.

Some notes on notation. We are going to encounter several kinds of moduli spaces so it might be worth taking a little time to introduce a strategy for giving names to these spaces:

$M(X)$ Denotes the moduli space of semistable holomorphic rank 2 vector bundles over X with trivial determinant *unless* a determinant condition is specified by a subscript (see below).

$M(X;\star)$ Is the moduli space of parabolic bundles, where the $\star$ indicates parabolic data; the $\star$ could mean one of the following:

- (χ, a, P) where P is the set of parabolic points, χ the multiplicities, and a the weights. We may omit P and write only (χ, a). Specifying only (a, P) or (a) means we are considering all multiplicities conforming to the dimension restriction dictated by a.
- i if $P = \{x_0, \ldots, x_N\}$ and the weights a are specified for all points in P. Then this indicates bundles with parabolic points $\{x_0, \ldots, x_{i-1}\}$ and parabolic structure $a|_{\{x_0,\ldots,x_{i-1}\}}$.

$Ma(\star)$ Denotes the *admissible* bundles for (X, τ) (see page 79); $\star$ can be either Y or (X, τ) depending on whether (X, τ) is implicit or not.

These moduli spaces may further be decorated with sub- and superscripts. Let $\mathcal{M}$ be any of the moduli spaces above.

$\mathcal{M}^\star$ can indicate stability conditions or the standard notation for fixed points, depending on whether

	$\star = s$	which denotes the stable locus,
	$\star = ss$	which is the complement of the stable locus,
	$\star = G$	when G is a group acting on $\mathcal{M}$, means the fixed points.
$\mathcal{M}_\star$	indicates	conditions on the determinant of the bundles:
	$\star = d$	where $d \in \mathbb{Z}$, means bundles with degree d,
	$\star = L$	where L is a line bundle, means bundles with determinant L,
	$\star = G$	where $G \subset \operatorname{Pic}$, denotes the bundles with determinant in G.

2. The Kernel of π^* : $\operatorname{Pic}(Y) \to \operatorname{Pic}(X)$

For a compact Riemann surface X, an automorphism $\tau\colon X \to X$ of order n gives a possibly ramified covering $\pi\colon X \to Y = X/\langle\tau\rangle$ of Riemann surfaces. In this section we calculate the kernel $\ker\{\,\pi^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)\,\}$; from this point on referred to as $\ker\pi^*$. Finally, we compute the push forward $\pi_*\mathcal{O}_X$ of the trivial bundle.

There is a map (see [5]) $\mathrm{Nm} = \mathrm{Nm}_\pi\colon \mathcal{M}(X) \to \mathcal{M}(Y)$ called the *norm map* and defined by

$$\mathrm{Nm}(f)(y) \stackrel{\text{def}}{=} \prod_{x\in\pi^{-1}(y)} f(x)^{\nu(x)},$$

where $\nu(x)$ is the multiplicity of x and $y \in Y$. There is a corresponding map $\mathrm{Nm}_\pi = \pi_*\colon \operatorname{Div}(X) \to \operatorname{Div}(Y)$ which is

$$\mathrm{Nm}\Big(\sum_i a_i\cdot x_i\Big) \stackrel{\text{def}}{=} \sum_i a_i\cdot\pi(x_i),$$

and which satisfies that $\mathrm{Nm}\big((f)\big) = \big(\mathrm{Nm}(f)\big)$. The norm map Nm_π is related to pullback π^* through $\mathrm{Nm}_\pi\circ\pi^*(D) = \deg\pi\cdot D$ for any $D \in \operatorname{Div}(Y)$.

Assume that $D \in \operatorname{Div}(Y)$ is a divisor for which the associated line bundle pulls back to the trivial bundle, $\mathcal{O}_X$, on X, i.e. $\pi^*D = (f)$ for some meromorphic function $f \in \mathcal{M}(X)$. Then using the relation between pullback and norm map,

$$n\cdot D = \mathrm{Nm}\circ\pi^*(D) = \mathrm{Nm}\big((f)\big) = \big(\mathrm{Nm}(f)\big),$$

we se that every element of the kernel has order at most n; hence, the kernel is a subgroup of the n-torsion points $\operatorname{Pic}_0^{(n)}(Y)$ in $\operatorname{Pic}(Y)$. The analysis of which subgroup, is divided into several cases.

Before we proceed we shall need to prove the following general statement:

Lemma 2.1. *Let $\tau\colon X \to X$ be an automorphism of order n of a Riemann surface X. Then for any n'th root of unity, λ, there exists a meromorphic function $h \in \mathcal{M}(X)$ such that*

$$h \circ \tau = \lambda \cdot h.$$

Proof. Let $D' \in \mathrm{Div}(Y)$ and put $L' = [D'] \in \mathrm{Pic}(Y)$, $D = \pi^* D'$, and $L = \pi^* L' = [D]$. We will produce a function with the required properties by studying the action of τ on the vector space

$$\mathcal{L}(D) \stackrel{\text{def}}{=} \{\, f \in \mathcal{M}(X) \mid (f) + D \geq 0 \,\}.$$

Recall that there is an element $s_0' \in \mathcal{M}(L')$ so that $D' = (s_0')$; let $s_0 = \pi^* s_0'$. Then $D = (s_0)$, and this induces an isomorphism

$$\mathcal{L}(D) \longrightarrow H^0(X, L)$$

defined by sending f to $f \cdot s_0$.

The automorphism τ acts on the finite dimensional vector space $H^0(X, L)$ by pullback. Since s_0 is τ-invariant, the action by pullback: $(f \circ \tau) \cdot s_0 = (f \circ \tau) \cdot (s_0 \circ \tau)$ on $\mathcal{L}(D)$ is natural with respect to this identification.

This means that if there is an $s = f \cdot s_0 \in H^0(X, L)$ such that $\tau^* s = e^{\frac{2\pi i}{n}} \cdot s$, then $f \circ \tau = e^{\frac{2\pi i}{n}} \cdot f$ which is what we are seeking.

Therefore consider the action on $H^0(X, L)$. Here $\lambda^n - 1$ is an annihilating polynomial for τ, so the minimal polynomial for τ is a divisor in $\lambda^n - 1$. Hence, the minimal polynomial looks like

$$\mu_\tau(\lambda) = (\lambda - \lambda_1) \cdot \ldots \cdot (\lambda - \lambda_k),$$

where $\lambda_1, \ldots, \lambda_k$ are k different roots of unity. Therefore τ can be diagonalized to the form

$$\tau = \mathrm{Diag}(\lambda_1, \ldots, \lambda_1, \ldots, \lambda_k, \ldots, \lambda_k).$$

Let $d_\lambda = \dim E_\lambda$ be the dimension of the eigenspace associated to the eigenvalue λ, and let $d_\lambda = 0$ if the root of unity, λ, is not an eigenvalue. Notice that $E_1 = \pi^* H^0(Y, L')$, so from Riemann-Roch we get that

$$d_1 = d' - \gamma + 1,$$

where $d' = \deg(L')$, and γ is the genus of Y.

Now, the trace of powers of τ is given by $\mathrm{Tr}(\tau^j) = \sum_{i=1}^k d_{\lambda_i} \lambda_i^j$. We claim that

$$d_\lambda = \frac{1}{n} \sum_{j=0}^{n-1} \mathrm{Tr}(\tau^j) \lambda^{-j}.$$

The verification of this is a straightforward calculation: let $\varepsilon = e^{\frac{2\pi i}{n}}$ and assume $\lambda = \varepsilon^m$, $0 \le m < n$. Then

$$\begin{aligned}
\frac{1}{n}\sum_{j=0}^{n-1} \mathrm{Tr}(\tau^j)\lambda^{-j} &= \frac{1}{n}\sum_{j=0}^{n-1}\Big(\sum_{i=1}^{k} d_{\lambda_i}\lambda_i^j\Big)\lambda^{-j} \\
&= \frac{1}{n}\sum_{j=0}^{n-1}\Big(\sum_{l=0}^{n-1} d_{\varepsilon^l}\varepsilon^{lj}\Big)\varepsilon^{-jm} \\
&= \frac{1}{n}\sum_{j=0}^{n-1}\Big(d_{\varepsilon^m} + \sum_{l\neq m} d_{\varepsilon^l}\varepsilon^{j(l-m)}\Big) \\
&= d_{\varepsilon^m} + \frac{1}{n}\sum_{l\neq m} d_{\varepsilon^l}\sum_{j=0}^{n-1}(\varepsilon^{l-m})^j = d_{\varepsilon^m} = d_\lambda,
\end{aligned}$$

since $\sum_{j=0}^{n-1}(\varepsilon^{l-m})^j = 0$ for $l \neq m$.

To calculate $\mathrm{Tr}(\tau^j)$ we use the Lefschetz fixed point formula, [8, Theorem 4.12], which states that if $\varphi\colon (\tau^j)^*L \to L$ is a bundle isomorphism then

$$\mathrm{Tr}\, H^0(\tau^j,\varphi) - \mathrm{Tr}\, H^1(\tau^j,\varphi) = \sum_{x\in X, \tau^j(x)=x} \frac{\mathrm{Tr}\,\varphi_x}{\det_{\mathbb{C}}(1-d_x\tau^j)},$$

for $j \neq 0 \mod n$. We are free to choose L' such that the degree of L is high enough to guarantee $H^1(X,L) = 0$, and since L is invariant, we may take $\varphi = \mathrm{Id}$. Let $x_{1,j},\dots,x_{l_j,j}$ be the fixed points for τ^j in X. Then around every $x_{i,j}$ there are local coordinates $z_{i,j}$ such that

$$\tau^j(z_{i,j}) = \eta_{i,j}\cdot z_{i,j},$$

where $\eta_{i,j} = e^{\frac{2\pi i}{n} j m_{i,j}}$ for some $0 < m_{i,j} < \frac{n}{j}$, $(m_{i,j}, \frac{n}{j}) = 1$ when $j|n$. If j does not divide n we notice that τ^j can only have fixed points which are fixed points for τ^k for some $k|n$ and orbit length. In that case $m_{i,j} = j'm_{i,k}$ where $j'k = j$. (Notice (though we will not use this) that there are relations among the $m_{i,j}$'s: if $j = lj'$ then we can choose $z_{i,j'} = z_{i,j}$ and $\tau^j(z_{i,j}) = (\tau^{j'})^l(z_{i,j}) = \eta^l_{i,j'}\cdot z_{i,j}$.) This reduces the Lefschetz fixed point formula to

$$\mathrm{Tr}(\tau^j) = \sum_{i=1}^{l_j}\frac{1}{1-\eta_{i,j}}, \tag{2.2}$$

for $j \neq 0 \mod n$. Now we get

$$\begin{aligned} d_\lambda &= \frac{1}{n}\left(\dim_{\mathbb{C}} H^0(X, L) + \sum_{j=1}^{n-1} \sum_{i=1}^{l_j} \frac{\lambda^{-j}}{1-\eta_{i,j}} \right) \\ &= d' - \gamma + 1 + \frac{1}{n} \sum_{j=1}^{n-1} \sum_{i=1}^{l_j} \frac{\lambda^{-j}}{1-\eta_{i,j}} \\ &= d' - \gamma + 1 + A(\tau, \lambda), \end{aligned}$$

where $A(\tau, \lambda)$ only depends on λ and local data for τ's action on X.[1] Therefore, we can choose L' with a degree high enough so that

$$d_\lambda > 0, \quad \text{for all } n\text{'th roots of unity } \lambda.$$

□

Now we calculate the kernel in the unramified case:

Lemma 2.3. *Let $\tau\colon X \to X$ be an automorphism of order n without any special orbits, so that the covering projection $\pi\colon X \to Y = X/\langle \tau \rangle$ is unramified. Then there is a line bundle $L_\pi \in \mathrm{Pic}(Y)$ of order n such that*

$$\ker \pi^* = \langle L_\pi \rangle.$$

Proof. Let $h \in \mathcal{M}(X)$ be the function from Lemma 2.1 such that $h \circ \tau = e^{\frac{2\pi i}{n}} \cdot h$. Then the norm map of h is given by

$$\begin{aligned} \mathrm{Nm}(h)(y) = \prod_{i=0}^{n-1} h\big(\tau^i(x)\big) &= e^{\frac{2\pi i}{n} \sum_{i=0}^{n-1} i} h(x)^n \\ &= e^{\frac{2\pi i}{n} \frac{n(n-1)}{2}} h(x)^n = (-1)^{n-1} h(x)^n, \end{aligned}$$

for any $x \in \pi^{-1}(y)$. We see that the divisor $(\mathrm{Nm}(h))$ is divisible by n. Notice also that the pullback is given by $\pi^* \mathrm{Nm}(h)(x) = \mathrm{Nm}(h)(\pi(x)) = (-1)^{n-1} h(x)^n$. Let $D \in \mathrm{Div}(Y)$ be such that $nD = (\mathrm{Nm}(h))$, then

$$n\pi^* D = \big(\pi^* \mathrm{Nm}(h)\big) = \big((-1)^{n-1} h^n\big) = (h^n) = n(h),$$

and thus $\pi^* D$ is a principal divisor with $\pi^* D = (h)$; i.e. $[D] \in \ker \pi^*$.

[1] Of course if τ has no special orbits, we have $A(\tau, \lambda) = 0$, as we would expect.

Suppose that $E \in \mathrm{Div}(Y)$ with $[E] \in \ker \pi^*$. Then $\pi^* E = (f)$ for some $f \in \mathcal{M}(X)$, and

$$\pi^*(D - E) = (h) - (f) = \left(\frac{h}{f}\right).$$

As $\pi^* E$ is τ-invariant, there is a root of unity $c = e^{\frac{2\pi i}{n} j}$, $0 \leq j \leq n-1$, so that $f \circ \tau = c \cdot f$. Consider $\pi^*(jD - E) = \left(\frac{h^j}{f}\right)$:

$$\frac{h^j}{f} \circ \tau = \frac{e^{\frac{2\pi i}{n} j} \cdot h^j}{e^{\frac{2\pi i}{n} j} \cdot f} = \frac{h^j}{f},$$

so there must be a $g \in \mathcal{M}(Y)$ with $\pi^* g = \frac{h^j}{f}$, and in that case

$$jD - E = (g).$$

This means that $\ker \pi^* = \langle [D] \rangle$.

We know that $[D]$ has order at most n since $nD = (\mathrm{Nm}(h))$, so suppose that $jD = (g)$ for some $g \in \mathcal{M}(Y)$. Then $(\pi^* g) = (h^j)$ in which case there must exist a $c \in \mathcal{O}^*(X) = \mathbb{C}^*$ such that $\pi^* g = c \cdot h^j$. But $\pi^* g$ is τ-invariant while $h^j \circ \tau = e^{\frac{2\pi i}{n} j} \cdot h^j$, so $\pi^* g = c \cdot h^j$ if and only if $j = 0 \mod n$. This means $L_\pi = [D]$ is indeed of order n. $\square$

Notice that the meromorphic function h in the proof is not unique, since we can modify it by the pullback of any meromorphic function on Y. Hence, neither D is unique, but there is a unique class of linearly equivalent divisors, and thus as we proved, L_π is in fact unique.

Notice also that if $\varphi \colon U \to X$ is a local section of π, then φ identifies $D|_U$ with $(h)|_{\varphi(U)}$.

As a final remark to this we observe that if $n = n_1 \cdot n_2$ then π factors through $\pi_1 \colon X \to X_1 = X/\langle \tau^{n_2} \rangle$ and the induced $\pi_2 \colon X_1 \to Y$, and there is a divisor D_1 in X_1 defined via $h \in \mathcal{M}(X)$ so that $\langle [D_1] \rangle = \ker \pi_1^*$. As $(\mathrm{Nm}_{\pi_2} \circ \mathrm{Nm}_{\pi_1}(h)) = (\mathrm{Nm}_\pi(h)) = n_1 n_2 D$ we get $n_2(\mathrm{Nm}_{\pi_1}(h)) = (\pi_2^* \circ \mathrm{Nm}_{\pi_2} \circ \mathrm{Nm}_{\pi_1}(h)) = n_1 n_2 \pi_2^* D$ and we get $\pi_2^* D = D_1$. In particular, D_1 is τ_1-invariant. Similarly, we see that there is a divisor D_2 of Y defined by h^{n_1} so that $\langle [D_2] \rangle = \ker \pi_2^*$ and $D_2 = n_1 D$.

Now we continue our quest for the kernel $\ker \pi^*$ by describing the ramified case.

Lemma 2.4. *Let $\tau \colon X \to X$ be an automorphism of prime order p such that the projection $\pi \colon X \to Y = X/\langle \tau \rangle$ is a ramified covering. Then the pullback*

$$\pi^* \colon \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X)$$

is injective.

Proof. The proof of this statement is essentially due to Mumford in [52]. Let $q\colon L \to Y$ be a line bundle in the kernel of π^*, and define the Riemann surface

$$X_L \overset{\text{def}}{=} \{\, s \in L \mid s^p = 1 \in \mathcal{O}_Y \cong L^p \,\}$$

and the unramified covering

$$\pi_L\colon X_L \longrightarrow Y.$$

Notice that a global non-zero section s of L, if it exits, can be scaled to satisfy $s^p = 1$, so that it gives a global section of X_L. On the other hand, a global section of X_L is a global section of L. Hence, L is a trivial bundle if and only if X_L is a trivial covering. Since

$$\begin{aligned}
\pi^* X_L &= \{\, (x, \hat{x}) \in X \times X_L \mid \pi(x) = \pi_L(\hat{x}) \,\} \\
&= \{\, (x, s) \in X \times L \mid \pi(x) = q(s),\ s^p = 1 \,\} \\
&= \{\, \hat{s} \in \pi^* L \mid \hat{s}^p = 1 \,\} = X_{\pi^* L}
\end{aligned}$$

it follows by the same argument that $\pi^* L$ is a trivial line bundle if and only if $\pi^* X_L$ is a trivial covering of X.

Suppose now that $L \in \ker \pi^*$ so that $\pi^* X_L$ is trivial and $\varphi\colon X \times \{\, 1, \ldots p \,\} \to \pi^* X_L$ is a trivialization. That gives rise to a commutative diagram

$$\begin{tikzcd}
X \times [p] \arrow[r, "\varphi", "\cong"'] \arrow[d, "\mathrm{pr}"'] & \pi^* X_L \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & X_L \arrow[d, "\pi_L"'] \\
X \arrow[r, equal] & X \arrow[r, "\pi"] & Y
\end{tikzcd}$$

where $[p] = \{\, 1, \ldots, p \,\}$. The projection pr has obvious sections $\sigma\colon X \to X \times [p]$, and the composition $\psi = \mathrm{pr}_2 \circ \varphi \circ \sigma$ is a morphism of coverings

$$\begin{tikzcd}
X \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & X_L \arrow[dl, "\pi_L"] \\
& Y &
\end{tikzcd}$$

In fact, $\pi^* X_L \cong X \times [p]$ if and only if there exists a morphism $\psi\colon X \to X_L$ making the above triangle commutative, because we may produce a section $\sigma\colon X \to \pi^* X_L$ by letting $\sigma(x) = (x, \psi(x))$.

Assuming that $\pi^* L \cong \mathcal{O}_X$, we get $\pi^* X_L \cong X \times [p]$, and from the existence of ψ we have that

$$p = \deg \pi = \deg(\pi_L|_{\mathrm{Im}\,\psi}) \cdot \deg \psi.$$

So if $L \not\cong \mathcal{O}_Y$, π_L must be of full degree, $\deg \pi_L = p$, and consequently $\deg \psi = 1$ in which case ψ is an isomorphism contradicting that π is ramified and π_L is not.$\square$

Lemma 2.5. *Let $\tau\colon X \to X$ be an automorphism of order n with a fixed point such that the projection $\pi\colon X \to Y = X/\langle\tau\rangle$ is a ramified covering with a one-point fiber. Then the pullback*

$$\pi^*\colon \operatorname{Pic}(Y) \longrightarrow \operatorname{Pic}(X)$$

is injective.

Proof. Let $n = p_1 \dots p_t$ be the prime factorization of n and let $x \in X$ be a point fixed by τ. Then $\tau^{p_2\cdots p_t}$ has order p_1 and keeps x fixed. Hence, for

$$\pi_1\colon X \longrightarrow X_1 \overset{\text{def}}{=} X/\langle\tau^{p_2\cdots p_t}\rangle$$

it follows from Lemma 2.4 that π_1^* is injective. Let τ_1 be the automorphism on X_1 induced by τ. Then $\operatorname{ord}(\tau_1) = p_2 \dots p_t$ and $\tau_1(x_1) = x_1$ for $x_1 = \pi_1(x)$.

Thus, the process continues by defining the p_k-fold covering with

$$\pi_k\colon X_{k-1} \longrightarrow X_k \overset{\text{def}}{=} X_{k-1}/\langle\tau_{k-1}^{p_{k+1}\cdots p_t}\rangle.$$

Obviously, π_k has the 1-point fiber $x_{k-1} = \pi_{k-2}(x_{k-2})$ and the induced automorphism τ_k has order $\operatorname{ord}(\tau_k) = p_{k+1} \dots p_t$ and keeps $x_k = \pi_k(x_{k-1})$ fixed. The process ends with the p_t-fold covering

$$\pi_t\colon X_{t-1} \longrightarrow X_t = Y.$$

Now, $\pi = \pi_t \circ \dots \circ \pi_1$ so the pullback is a composition $\pi^* = \pi_1^* \circ \dots \circ \pi_t^*$ of injective homomorphisms and is therefore itself injective. $\square$

Finally, we have arrived at the concluding statement on the kernel of the pullback of covering projections:

Proposition 2.6. *Let $\tau\colon X \to X$ be an automorphism of order n with possible special orbits of lengths $n_1, \dots, n_N$ and let $\pi\colon X \to Y = X/\langle\tau\rangle$ be the induced covering. Then there is a line bundle L_π over Y such that*

$$\ker \pi^* = \langle L_\pi\rangle,$$

and the order of L_π is the greatest common divisor, $\gcd(n_1, \dots, n_N, n)$, of the orbit lengths.

Proof. If τ has no special orbits, we are done by Lemma 2.3. Assume therefore that there are ramefication points. Let $k_1 = \min\{n_1, \dots, n_N\}$ so that there exits a point $x \in X$ with $\tau^{k_1}(x) = x$, and define the $\frac{n}{k_1}$-fold covering

$$\pi_1\colon X \longrightarrow X_1 \overset{\text{def}}{=} X/\langle\tau^{k_1}\rangle.$$

By Lemma 2.5 the pullback π_1^* is injective.

There is an induced automorphism $\tau_1 \colon X_1 \to X_1$ of order $\mathrm{ord}(\tau_1) = k_1$ and whose special orbits have lengths

$$n_{1,i} = \gcd(k_1, n_i), \quad i = 1, \ldots, N.$$

Let $k_2 = \min\{n_{1,1}, \ldots, n_{1,N}\}$. If $k_2 = k_1$, τ_1 has no special orbits and $\pi_2 \colon X_1 \to X_2/\langle \tau_1 \rangle = Y$ is an unramified covering. Otherwise continue the process with the $\frac{k_1}{k_2}$-fold covering

$$\pi_2 \colon X_1 \longrightarrow X_2 \stackrel{\text{def}}{=} X_1/\langle \tau_1^{k_2} \rangle.$$

(This case also covers the special case where $k_2 = 1$ so that π_2 has a 1-point fiber and $X_2 = Y$.) In general, define

$$\begin{aligned} n_{j,i} &\stackrel{\text{def}}{=} \gcd(k_j, n_{j-1,i}), \quad i = 1, \ldots, N, \\ k_{j+1} &\stackrel{\text{def}}{=} \min\{n_{j,1}, \ldots, n_{j,N}\} \end{aligned}$$

and as long as $k_j \neq k_{j-1}$ we get a $\frac{k_{j-1}}{k_j}$-fold covering

$$\pi_j \colon X_{j-1} \longrightarrow X_j \stackrel{\text{def}}{=} X_{j-1}/\langle \tau_{j-1}^{k_j} \rangle,$$

which has a 1-point orbit. The process ends when either $k_j = 1$ and $X_j = Y$, or when $k_j = k_{j-1}$ and we can define an unramified covering $\pi_j \colon X_{j-1} \to Y$. Let s denote the final step. In both cases $\pi = \pi_s \circ \cdots \circ \pi_1$, so in the former case all the pullbacks π_j^* are injective, as is π^*, and in the latter case all but π_s^* are injective and $\ker \pi_s^* = \langle L_{\pi_s} \rangle$, where L_{π_s} is the line bundle of order k_s defined in Lemma 2.3. Hence, put $L_\pi = L_{\pi_s}$. Then

$$\ker \pi^* = \langle L_\pi \rangle.$$

Denote by $m = \gcd(n_1, \ldots, n_N, n)$ the greatest common divisor of the orbit lengths. Then it is easily seen by induction that $m | n_{j,i}$ for all $i = 1, \ldots, N$ and all $j \leq s$. In particular, $m | k_j$. If $k_s = 1$, then $m = 1$, which corresponds to the fact that $\ker \pi^* = \{\mathcal{O}_Y\}$ is of order 1. If on the other hand $k_s = k_{s-1}$ then

$$k_{s-1} = k_s = \min_i \{n_{s-1,i}\} \leq n_{s-1,i} = \gcd(k_{s-1}, n_{s-2,i}) \leq k_{s-1},$$

so $k_s = k_{s-1} = n_{s-1,i}$ for all i. But since $n_{j,i} | n_i$ for all j and i, then $k_s | n_i$ for all i, and therefore $k_s | m$. Hence, we have proved that $k_s = m = \gcd(n_1, \ldots, n_N, n)$. $\square$

Remark 2.7. Notice that this shows that any such ramified cover coming from the action of a single automorphism τ of X factors into a ramified part $\pi_r \colon X \to X/\langle \tau^m \rangle$

for which the fiber lengths are co-prime, and an unramified part $\pi_u\colon X/\langle\tau^m\rangle \to Y$ whose degree is the greatest common divisor m of the orbits lengths of τ. Clearly, if $h \in \mathcal{M}(X/\langle\tau^m\rangle)$ is the meromorphic function from Lemma 2.3, such that $(\mathrm{Nm}_{\pi_u}(h)) = mD$ and $[D]$ generates $\ker \pi_u^*$, then $h_\pi = \pi_r^* h \in \mathcal{M}(X)$ satisfies that $(\mathrm{Nm}_\pi(h_\pi)) = nD$ and h_π is invariant under τ^m; in fact $h_\pi \circ \tau = \lambda \cdot h_\pi$, where $\lambda = e^{\frac{2\pi i}{m}}$.

Observation 2.8. We observe that if c is an m'th root of unity (m the common divisor of the orbit lengths), then there is a lift $\tilde{c}$ of $\tau_u\colon \bar{X} \to \bar{X}$, $\bar{X} = X/\langle\tau^m\rangle$, to $\mathcal{O}_{\bar{X}}$ defined as $\tilde{c}(g) = c^{-1}\cdot g\circ\tau_u$, and there is a line bundle L_c over Y defined as the line bundle associated to the locally free invertible sheaf $\mathcal{O}_{\bar{X}}^{\langle\tilde{c}\rangle}$ of the $\tilde{c}$-invariant sections. Now, $\mathcal{O}_{\bar{X}}^{\langle\tilde{c}\rangle}$ is locally trivial as we may choose U small enough that there exists a holomorphic function f with $f\circ\tau = c\cdot f$ (this can be done using local coordinates), such that f generates $\mathcal{O}_{\bar{X}}^{\langle\tilde{c}\rangle}(U) = \mathcal{O}_{\bar{X}}(\pi^{-1}(U))^{\langle\tilde{c}\rangle}$ as an $\mathcal{O}_Y(U)$-module. Take $0 \le j \le m-1$ such that $e^{\frac{2\pi i}{m}j} = c$, and $h \in \mathcal{M}(\bar{X})$ from Lemma 2.1 such that $h\circ\tau = e^{\frac{2\pi i}{m}}\cdot h$. We may identify the global meromorphic sections $\mathcal{M}(L_c)$ of L_c with $\mathcal{M}(\bar{X})^{\langle\tilde{c}\rangle}$. Let s_c correspond to h^j under this identification; that is if U_α is a suitable covering of Y with generators f_α for $\mathcal{O}_{\bar{X}}(\pi^{-1}(U))^{\langle\tilde{c}\rangle}$, then $h^j|_{U_\alpha} = \pi^* s_\alpha\cdot f_\alpha$, where $s_\alpha \in \mathcal{M}(U_\alpha)$, so s_c has the local presentation s_α (where $s_\alpha = g_{\alpha\beta}\cdot s_\beta$ on $U_\alpha\cap U_\beta$, $\pi^* g_{\alpha\beta} = (f_\beta|_{U_\alpha\cap U_\beta})/(f_\alpha|_{U_\alpha\cap U_\beta})$). Then $(s_c)|_{U_\alpha} = (s_\alpha)$, but clearly $(s_\alpha) = jD_{\pi_u}|_{U_\alpha}$, so in fact $(s_c) = jD_{\pi_u}$ and thus $L_c = [(s_c)] = L_\pi^j$. As in the last remark, this generalizes to the ramified case also, where the meromorphic function h_π will be the pullback under π_r of h.

Notice also that the characters χ of the induced automorphism, τ_u, on $X/\langle\tau^m\rangle$ are exactly lifts like the ones just described, so for any such character there is a j so that $L_\chi = L_\pi^j$, and the orders agree.

Suppose E is a holomorphic vector bundle with a lift $\tilde{\tau}$, and assume that $\tilde{\tau}_x^{k(x)} = \mathrm{Id}_{E_x}$ for every $x \in X$, where $k(x)$ denotes the length of the orbit through x. Then the quotient $\tilde{E} = E/\langle\tilde{\tau}\rangle$ is a well-defined holomorphic vector bundle. Also $\tilde{\tau}_c = c\cdot\tilde{\tau}$ acts on E, and it gives a vector bundle $\tilde{E}_c = E/\langle\tilde{\tau}_c\rangle$. If we identify E with $\mathcal{O}_X\otimes E$, $\tilde{\tau}_c$ can be realized in the following way: The local section e' is in the orbit of $\tilde{\tau}$ through e if there is a q so that $\tilde{\tau}^q(e) = e'\circ\tau^q$. Hence, if s is a local section of L_c, then s may be interpreted as an $s \in \mathcal{O}_X$ such that $s\circ\tau = c\cdot s$, so that $s\cdot e'$ is a local section of E for which $(s\cdot e')\circ\tau^q = c^q\cdot\tilde{\tau}^q(e) = \tilde{\tau}_c^q(e)$. This gives an isomorphism

$$L_c\otimes\tilde{E} \longrightarrow \tilde{E}_c$$

of holomorphic vector bundles over Y. Alternatively, this may be done in the two steps π_r and π_u, and in that case the induced bundles $\bar{E} = E/\langle\tilde{\tau}_r^m\rangle$ and $\bar{E}_c = E/\langle\tilde{\tau}_{r,c}^m\rangle$ over $\bar{X} = X/\langle\tau^m\rangle$ are the same, since $\tilde{\tau}_{r,c}^m = \tilde{\tau}_r^m$.

Proposition 2.9. *Let $\tau\colon X \to X$ be an automorphism of the compact Riemann surface X, and denote by $\pi\colon X \to Y = X/\langle\tau\rangle$ the canonical projection. Then the pushforward of the trivial bundle $\mathcal{O}_X$ over X is given by*

$$\pi_*\mathcal{O}_X = \bigoplus_{i=0}^{n-1} L_\pi^i = \frac{n}{m}\bigoplus_{i=0}^{m-1} L_\pi^i,$$

where m is the greatest common divisor of the lengths of orbits as in Proposition 2.6.

Proof. Consider first the ramified part $\pi_r\colon X \to \bar{X} = X/\langle\tau^m\rangle$. We may identify $\pi_{r*}\mathcal{O}_X \cong \big(\mathcal{O}_X \oplus\cdots\oplus (\tau^{(\frac{n}{m}-1)m})^*\mathcal{O}_X\big)/\langle\tau^m\rangle$. On the other hand, there is a splitting $\big(\mathcal{O}_X \oplus\cdots\oplus (\tau^{(\frac{n}{m}-1)m})^*\mathcal{O}_X\big)(\pi_r^{-1}(U)) = E_0(U)\oplus\cdots\oplus E_{\frac{n}{m}-1}(U)$ into eigenspaces for the action of τ^m each of which is a $\mathcal{O}_{\bar{X}}(U)$-module, and obviously τ is diagonal with respect to this splitting. Hence, there is a splitting $\pi_{r*}\mathcal{O}_X = L_1\oplus\cdots\oplus L_{\frac{n}{m}}$, and as $\pi_r^*\pi_{r*}\mathcal{O}_X = \frac{n}{m}\mathcal{O}_X$ we have by Krull-Schmidt-Atiyah that $L_i \in \ker \pi_r^*$; so from Proposition 2.6 it follows that $L_i \cong \mathcal{O}_{\bar{X}}$.

The pushforward $\pi_{u*}\mathcal{O}_{\bar{X}}$ is $\mathcal{O}_{\bar{X}}$ interpreted as a sheaf of $\mathcal{O}_Y$-modules. The action of τ_u on $\mathcal{O}_{\bar{X}}(\pi_u^{-1}(U))$ splits $\mathcal{O}_{\bar{X}}(\pi_u^{-1}(U))$ into a sum $E_0(U)\oplus\cdots\oplus E_{m-1}(U)$ of eigenspaces, where $E_j(U)$ is the eigenspace belonging to the eigenvalue $e^{\frac{2\pi i}{m}j}$. Clearly, the sheaf E_j is the sheaf $\mathcal{O}_X^{\langle\tilde{c}\rangle}$ from Observation 2.8, so $\pi_{u*}\mathcal{O}_{\bar{X}} = \mathcal{O}_Y \oplus L_\pi\oplus\cdots\oplus L_\pi^{m-1}$, which gives the desired result. $\square$

3. Fixed Points in the Moduli Space, the Unramified Case

Let X be a compact Riemann surface with an automorphism τ of order n. We will consider the action of τ in the moduli space $M(X)$ of semistable rank 2 holomorphic vector bundles with trivial determinant line bundle[2]. Denote the fixed point set for τ by $|M(X)|$. In this section we will expose the unramified case.

Assume that there are no special orbits for τ in X. In that case

$$\pi\colon X \longrightarrow Y = X/\langle\tau\rangle$$

is an unramified n-fold cover. Now, the kernel of $\pi^*\colon \operatorname{Pic}(Y)\to\operatorname{Pic}(X)$ is as we saw in Lemma 2.3 a cyclic group of order n generated by a bundle $L = L_\pi$. If $M_{\langle L\rangle}(Y)$ denotes the moduli space of semistable holomorphic rank 2 vector bundles over Y with determinant in the group $\langle L\rangle$, it turns out that there is a map

$$\pi^*\colon M_{\langle L\rangle}(Y) \longrightarrow |M(X)|.$$

We shall see in the following that $|M(X)|$ can be identified with a certain quotient of $M_{\langle L\rangle}(Y)$.

[2]Recall that this means strong equivalence classes of such bundles, where E and E' are strongly equivalent (written $E \equiv E'$) if the graded objects $\operatorname{Gr} E = \operatorname{Gr} E'$ are isomorphic.

Lemma 3.1. *There is a well-defined map $\pi^*\colon M_{\langle L\rangle}(Y) \to |M(X)|$. The pre-image under π^* is generated by the groups $\langle L\rangle$ and $\langle L\rangle \times \langle L\rangle$ in the following way:*

$$(\pi^*)^{-1}(\pi^*\tilde{E}) = \{\, \tilde{E}, \tilde{E}\otimes L, \dots \tilde{E}\otimes L^{n-1} \,\} = \tilde{E}\cdot\langle L\rangle,$$

for all stable bundles $\tilde{E} \to Y$ with $\det \tilde{E} \in \langle L\rangle$, *and*

$$\begin{aligned}(\pi^*)^{-1}\big(\pi^*(\tilde{L}_1 \oplus \tilde{L}_2)\big) &= \{\, \tilde{L}_1 \otimes L^{k_1} \oplus L_2 \otimes L^{k_2} \mid L^{k_\nu} \in \langle L\rangle \,\} \\ &= (\tilde{L}_1 \oplus \tilde{L}_2)\cdot(\langle L\rangle \times \langle L\rangle),\end{aligned}$$

for all line bundles $\tilde{L}_\nu \to Y$ satisfying the determinant relation that $\tilde{L}_1 \otimes \tilde{L}_2 \in \langle L\rangle$.

Proof. We start by looking at the pre-image $(\pi^*)^{-1}(\pi^*\tilde{E})$. Suppose $\tilde{E}$ and $\tilde{F}$ are bundles over Y so that $\pi^*\tilde{E} \cong \pi^*\tilde{F}$. Then of course $\pi_*(\pi^*\tilde{E}) \cong \pi_*(\pi^*\tilde{F})$, but $\pi_*(\pi^*\tilde{E}) \cong \pi_*(\pi^*\tilde{E}\otimes\mathcal{O}_X) \cong \tilde{E}\otimes\pi_*\mathcal{O}_X \cong \tilde{E}\otimes\bigoplus_{k=0}^{n-1} L^k$ by the projection formula, and likewise for $\tilde{F}$.

Now, if $\tilde{E}$ is stable, the Krull-Schmidt-Atiyah theorem gives us that so is $\tilde{F}$ and there is a $k \in \{\,0,\dots,n-1\,\}$ such that

$$\tilde{F} \cong \tilde{E}\otimes L^k.$$

If $\tilde{E}$ is not stable, the graded object is $\operatorname{Gr}(\tilde{E}) = \tilde{L}_1 \oplus \tilde{L}_2$ and again by Krull-Schmidt-Atiyah, $\operatorname{Gr}(\tilde{F}) = L_1' \oplus L_2'$, where

$$L_\nu' \cong \tilde{L}_\nu \otimes L^{k_\nu}$$

for some $k_1, k_2 \in \{\,0,\dots,n-1\,\}$.

Finally we prove that the map is well-defined; i.e. we check that the pull back $\pi^*\tilde{E}$ of a semistable bundle $\tilde{E}$ is actually again semistable. Let E_1 be the unique maximal semistable subbundle of $E = \pi^*\tilde{E}$ and assume $E_1 \neq E$. Now, $\tau^*E_1 \subset \tau^*E = E$ is also maximal semistable so by uniqueness $\tau^*E_1 = E_1$ and the canonical action of τ on E restricts to E_1. Hence, we get bundles $E_1/\langle\tau\rangle \subset E/\langle\tau\rangle$ over Y satisfying that $\pi^*(E_1/\langle\tau\rangle) = E_1$ and $\pi^*(E/\langle\tau\rangle) = E$. As $E/\langle\tau\rangle = \tilde{E}\otimes L^k$ for some k, $E/\langle\tau\rangle$ is semistable, so the slopes are related as $\mu(E_1/\langle\tau\rangle) \le \mu(E/\langle\tau\rangle)$ which gives us that $\mu(E_1) = n\cdot\mu(E_1/\langle\tau\rangle) \le n\cdot\mu(E/\langle\tau\rangle) = \mu(E)$ thereby proving that E is semistable. □

Proposition 3.2. *The map $\pi^*\colon M_{\langle L\rangle}(Y) \to |M(X)|$ is a surjective morphism of varieties.*

Proof. Consider first the case where $E \to X$ is a stable bundle. It is fixed by τ so there exists an isomorphism $\psi\colon \tau^*E \to E$ enabling us by the diagram below to construct a bundle map $\tilde{\tau}$ covering τ:

$$\begin{array}{ccccc} E & \xrightarrow{\psi} & \tau^* E & \xrightarrow{\mathrm{pr}_2} & E \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle \mathrm{pr}_1}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ X & = & X & \xrightarrow{\tau} & X. \end{array} \tag{3.3}$$

Since E is stable, it is simple and we can choose $\tilde{\tau}$ so that $\tilde{\tau}^n = \mathrm{Id}_E$. Hence, there is an induced vector bundle $\tilde{p}\colon \tilde{E} \to Y$, with $\tilde{E} = E/\langle\tilde{\tau}\rangle$ and the canonical projection $\tilde{\pi}\colon E \to \tilde{E}$. Clearly, the vector bundle homomorphism $\varphi\colon E \to \pi^*\tilde{E}$ defined by $\varphi(e) = \big(p(e), \tilde{\pi}(e)\big) \in \pi^*\tilde{E} = \{\, (x, \tilde{e}) \in X \times \tilde{E} \mid \pi(x) = \tilde{p}(\tilde{e}) \,\}$ is an isomorphism. Now, $\deg(\pi^* F) = n \cdot \deg(\tilde{F})$ for all bundles $\tilde{F}$ over Y, so the slopes are related by $\mu(\pi^* F) = n \cdot \mu(\tilde{F})$, hence, if $\tilde{F} \subset \tilde{E}$ is a proper subbundle, $\mu(\tilde{F}) = \frac{1}{n} \cdot \mu(\pi^* F) < \frac{1}{n} \cdot \mu(\pi^*\tilde{E}) = \mu(\tilde{E})$. I.e. $\tilde{E}$ is stable.

Now, suppose $E \to X$ is semistable but not stable. Then the graded object of E can be written $\mathrm{Gr}(E) = L_1 \oplus L_2$, where L_ν is a degree 0 line bundle over X for $\nu = 1, 2$. That E is a fixed point means that $\tau^* \mathrm{Gr}(E) = \mathrm{Gr}(\tau^* E) = \mathrm{Gr}(E)$, i.e. $\tau^* L_1 \oplus \tau^* L_2 \cong L_1 \oplus L_2$. Any non-zero homomorphism between stable bundles of the same degree is an isomorphism, so considering $i_{\nu,\nu'}\colon \tau^* L_\nu \to L_1 \oplus L_2 \to L_{\nu'}$, we see that either $\tau^* L_\nu \cong L_\nu$ or $\tau^* L_1 \cong L_2$ and vice versa, since $i_{\nu,\nu'}$ cannot be zero for both ν'.

That means that the isomorphism ψ, in diagram (3.3) with $L_1 \oplus L_2$ substituted for E, can be chosen to preserve the splitting. If $L_1 \not\cong L_2$, the bundle isomorphism $\tilde{\tau}^n$ has to be diagonal with respect to the splitting, since $H^0(\mathrm{End}(L_1 \oplus L_2)) = \mathbb{C} \oplus \mathbb{C}$. Thus in particular, we see that in cases of $\tau^* L_1 \cong L_2 \not\cong L_1$, n has to be even. And in both the case of $\tau^* L_\nu \cong L_\nu$ and $\tau^* L_1 \cong L_2$, it is easy to find a diagonal matrix Λ so that $(\Lambda\tilde{\tau})^n = \mathrm{Id}_{L_1 \oplus L_2}$. If $L_1 \cong L_2 = L_0$, apply diagram (3.3) to L_0 thereby getting a lift $\tilde{\tau}_0$ to L_0 which in turn gives a diagonal lift $\tilde{\tau}$ to $L_0 \oplus L_0$, and by the above argument there is a Λ so that $(\Lambda\tilde{\tau})^n = \mathrm{Id}_{L_0 \oplus L_0}$.

Hence, assume $\tilde{\tau}\colon L_1 \oplus L_2 \to L_1 \oplus L_2$ covers τ and $\tilde{\tau}^n = \mathrm{Id}_{L_1 \oplus L_2}$. Then there is a well-defined bundle $\tilde{E} = (L_1 \oplus L_2)/\langle\tilde{\tau}\rangle$ over Y. This bundle satisfies that $\pi^*\tilde{E} \cong L_1 \oplus L_2 = \mathrm{Gr}(E)$, and with the same argument as before we get that $\tilde{E}$ is at least semistable.

That π^* is a morphism is a a consequence of lemma B.14 in the case when there is no parabolic structure, but it is seen directly also on page 120. $\square$

Remark 3.4. Notice that an upshot of this proof is that any fixed point $[E]$ (not only for the unramified case) has a representative E', $\mathrm{Gr}\, E' = \mathrm{Gr}\, E$, for its strong equivalence class such that E' has a lift

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{\tau}} & E' \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ X & \xrightarrow{\tau} & X \end{array}$$

and that this can always be chosen so that $\tilde{\tau}^n = \mathrm{Id}_{E'}$. A lift makes E' into a $\langle\tau\rangle$-bundle. For future reference:

Definition 3.5. A *lift* to a fixed point of τ is choice of representative E', $\mathrm{Gr}\,E' = \mathrm{Gr}\,E$, and a lift $\tilde{\tau}$ to E' satisfying that $\tilde{\tau}^n = \mathrm{Id}_{E'}$. We denote by $\mathit{Lift}(X, \tau)$ be the space of isomorphism classes of lifts.

Recall that $M^s(X)$, $M^s_{\langle L\rangle}(Y)$ and $M^{ss}(X)$, $M^{ss}_{\langle L\rangle}(Y)$ denote the subsets in the moduli spaces consisting of stable points and the complement of the stable points respectively.

From Lemma 3.1 and Proposition 3.2 we conclude that there is a 1-1 correspondence between the quotients $M^s_{\langle L\rangle}(Y)/\langle L\rangle \sqcup M^{ss}_{\langle L\rangle}(Y)/(\langle L\rangle \times \langle L\rangle)$ and $|M(X)|$. Since $\det(\tilde{E} \otimes L) = \det \tilde{E} \otimes L^2 = L^{k+2}$ if $\det \tilde{E} = L^k$, we see that tensoring with L cycles through the components $M^s_{L^k}(Y)$ of $M^s_{\langle L\rangle}(Y)$ in steps of 2.[3] Thus, if n is odd, $M^s_{\langle L\rangle}(Y)/\langle L\rangle \cong M^s_{\mathcal{O}}(Y)$. If n is even, tensoring with L only identifies $M^s_{L^k}(Y)$ for odd powers and for even powers of k respectively, but $L^{\frac{n}{2}}$ acts internally in each $M^s_{L^k}(Y)$ so we have that $M^s_{\langle L\rangle}(Y)/\langle L\rangle \cong M^s_{\mathcal{O}}(Y)/\langle L^{\frac{n}{2}}\rangle \sqcup M^s_L(Y)/\langle L^{\frac{n}{2}}\rangle$. In the unstable case we notice that $\det(\tilde{L}_1 \otimes L^{k_1} \oplus L_2 \otimes L^{k_2}) = \tilde{L}_1 \otimes \tilde{L}_2 \otimes L^{k_1+k_2}$, so all the strata $M^{ss}_{L^k}(Y)$ of $M^{ss}_{\langle L\rangle}(Y)$ are identified under the action of $\langle L\rangle \times \langle L\rangle$. But there is also an internal action of $\langle L\rangle$ acting anti-diagonally as (L^k, L^{n-k}), hence, $M^{ss}_{\langle L\rangle}(Y)/(\langle L\rangle \times \langle L\rangle) \cong M^{ss}_{\mathcal{O}}(Y)/\langle L\rangle$.

There may however be stable bundles over Y that pulls back to form a semistable but *not* stable bundle: Let $\mathrm{Gr}(E) = L_1 \oplus L_2$. If $\tau^* L_\nu \cong L_\nu$, $\tilde{\tau}$ can be chosen so that the action is diagonal. That means that we can define bundles $\tilde{L}_\nu = L_\nu/\langle\tilde{\tau}\rangle \to Y$ with the property that $\pi^*\tilde{L}_\nu \cong L_\nu$, so we end up with a non-stable bundle over Y. This goes whether or not $L_1 \ncong L_2$. If on the other hand $\tau^* L_1 \cong L_2 \ncong L_1$, we don't have this splitting, at least not a priori. More precisely:

We know that in this case n is even, say $n = 2m$, and we see that $(\tau^*)^2 L_\nu \cong L_\nu$. Therefore we can chose the map $\tilde{\tau}$ covering τ such that $\tilde{\tau}^2$ acts diagonally. Then of course $\tilde{\tau}^2$ covers τ^2 and $(\tilde{\tau}^2)^m = 1$. Define a new Riemann surface $\bar{X} = X/\langle\tau^2\rangle$

[3] It is a direct consequence of the GIT construction that $M_L(Y)$ is connected for any line bundle L (except for genus 0 and 1).

and consider the line bundles $\bar{L}_\nu = L_\nu/\langle\tilde{\tau}^2\rangle$. The automorphism on $\bar{X}$ induced by τ, we denote by $\bar{\tau}$ with $\bar{\tau}^2 = 1$. Since $L_2 \cong \tau^* L_1$ we have that $\bar{L}_2 \cong (\tau^* L_1)/\langle\tilde{\tau}^2\rangle = \bar{\tau}^*(L_1/\langle\tilde{\tau}^2\rangle) = \bar{\tau}^*\bar{L}_1$ and, if $\bar{\pi}\colon X \to \bar{X}$ is the canonical projection, $\bar{\pi}^*\bar{L}_\nu \cong L_\nu$, hence in particular, $\bar{L}_1$ and $\bar{L}_2$ cannot be isomorphic.

Let $\tilde{\pi}\colon \bar{X} \to Y$ be the 2-fold cover. There is an isomorphism $\tilde{\pi}_*\bar{L}_1 \cong (\bar{L}_1 \oplus \bar{\tau}^*\bar{L}_1)/\langle\bar{\tau}\rangle$ and since $\bar{L}_1 \not\cong \bar{L}_2$, [53, Proposition 3.1] tells us that $\tilde{\pi}_*\bar{L}_1$ is stable. According to [53, Lemma 2.1] any such direct image satisfies that there is a canonical isomorphism $L_\chi \otimes \tilde{\pi}_*\bar{L}_1 \to \tilde{\pi}_*\bar{L}_1$ for all characters χ of the group $\langle\bar{\tau}\rangle$. Now, $\pi^*\tilde{\pi}_*\bar{L}_1 = \bar{\pi}^*\tilde{\pi}^*\big((\bar{L}_1 \oplus \bar{\tau}^*\bar{L}_1)/\langle\bar{\tau}\rangle\big) \cong \bar{\pi}^*(\bar{L}_1 \oplus \bar{\tau}^*\bar{L}_1) \cong L_1 \oplus \tau^*L_1 \cong \mathrm{Gr}(E)$. By Lemma 3.1 any bundle $\tilde{E} \to Y$ for which $\pi^*\tilde{E} \cong \pi^*\tilde{\pi}_*\bar{L}_1$ satisfies that there exist a $k = 0, \ldots, n-1$ such that $\tilde{E} \cong \tilde{\pi}_*\bar{L}_1 \otimes L^k$. In particular, any such $\tilde{E}$ is stable and has a canonical isomorphism $L_\chi \otimes \tilde{E} \to \tilde{E}$ for both characters χ of $\langle\bar{\tau}\rangle$. Clearly, these characters are in 1-1 correspondence with the characters χ of $\langle\tau\rangle$ satisfying that $\chi^2 = 1$.

If on the other hand $\tilde{E}$ satisfies that $L_\chi \otimes \tilde{E} \cong \tilde{E}$ then there is a line bundle $\bar{L}_1 \to \bar{X}$ such that $\tilde{E} = \pi_*\bar{L}_1 \cong (\bar{L}_1 \otimes \bar{\tau}^*\bar{L}_1)/\langle\bar{\tau}\rangle$ by [53, Proposition 2.6]. Hence, we conclude that a stable bundle $\tilde{E}$ over Y pulls back to a semistable bundle over X if and only if n is even and $L_\chi \otimes \tilde{E} \cong \tilde{E}$ for all order 2-characters of $\langle\tau\rangle$. Furthermore, a semistable bundle $L_1 \oplus L_2$ over X is the pullback of a stable bundle over Y if and only if n is even and $\tau^*L_1 \cong L_2 \not\cong L_1$.

This justifies the further division of the moduli spaces: Let $\mathcal{L}_\pi$ denote the order 2 subgroup of $\ker\pi$ generated by $L_\chi = L_\pi^{\frac{n}{2}}$ and denote by $M^{s,\mathcal{L}_\pi}_{\langle L\rangle}(Y)$ the subset of $M^s_{\langle L\rangle}(Y)$ consisting of bundles invariant under the multiplication by $L_\chi \in \mathcal{L}_\pi$ (i.e. the bundles that are non-stable under π^*) and let $M'_{\langle L\rangle}(Y) = M^s_{\langle L\rangle}(Y) - M^{s,\mathcal{L}_\pi}_{\langle L\rangle}(Y)$. If $n > 2$ and even, the action of L on $M'_{\langle L\rangle}(Y)$ is fixed point free, since no bundle is fixed by $L^{\frac{n}{2}}$; then certainly no bundle is fixed by L. We have proved:

Proposition 3.6. *Let X be a compact Riemann surface and $\tau \in \mathrm{Aut}(X)$ of order n and with no special orbits. Then $\pi\colon X \to Y$ is an unramified cyclic covering and there are 1-1 correspondences*

$$M'_{\langle L\rangle}(Y)/\langle L\rangle \xrightarrow{\pi^*} |M^s(X)|,$$
$$M^{s,\mathcal{L}_\pi(Y)}_{\langle L\rangle}/\langle L\rangle \sqcup M^{ss}_{\langle L\rangle}(Y)/(\langle L\rangle \times \langle L\rangle) \xrightarrow{\pi^*} |M^{ss}(X)|,$$

where

$$M'_{\langle L\rangle}(Y)/\langle L\rangle \cong \begin{cases} M^s_{\mathcal{O}}(Y), & \text{for } n \text{ odd,} \\ M'_{\mathcal{O}}(Y)/\mathcal{L}_\pi \sqcup M'_L(Y)/\mathcal{L}_\pi, & \text{for } n \text{ even,} \end{cases}$$

$$M^{s,\mathcal{L}_\pi}_{\langle L\rangle}(Y)/\langle L\rangle \cong \begin{cases} \emptyset, & \text{for } n \text{ odd,} \\ M^{s,\mathcal{L}_\pi}_{\mathcal{O}}(Y) \sqcup M^{s,\mathcal{L}_\pi}_{L}(Y), & \text{for } n \text{ even,} \end{cases}$$
$$M^{ss}_{\langle L\rangle}(Y)/(\langle L\rangle \times \langle L\rangle) \cong M^{ss}_{\mathcal{O}}(Y)/\langle L\rangle,$$

and the action of L in the last line on the right is anti-diagonal as (L^k, L^{n-k}) on (L_1, L_2).

As the moduli spaces $M_{L'}(Y)$ are irreducible for any line bundle L', the smooth part $M^s_{L'}(Y)$ is connected. Hence, the proposition gives us right away a description of the set of connected components of the smooth part of the fixed point set which we can compare to the description of J.E. Andersen in [1]. This namely has one component in the case of n odd and two when n is even. This agrees with J.E. Andersens calculation in [1] and the varieties are the same.

In this section we have only considered the case of fixed points with trivial determinant. The arguments also covers any other determinant. However, it is obvious that in order to get a non-empty fixed point set there are the constraints that the determinant should be invariant under τ and have a degree divisible by n.

4. Fixed Points in the Moduli Space, the General Case

This section is devoted to finding the correspondence between fixed points in the moduli space of semistable rank 2 holomorphic bundles over X and the moduli space of certain parabolic bundles over Y. In order not to interrupt the line of thought, some of the longer proofs of results related to elementary modifications are postponed until appendix A.

In the sequel we employ the convention that the set of special points for τ is denoted $\bar{P} \subset X$ and the branch points for π by $P = \pi(\bar{P})$. For every point $y \in P$ and $w \in \pi^{-1}(y) \subseteq \bar{P}$ we denote the length of the orbit through w by $k = k(w) = k(y)$, so the ramification number is $\frac{n}{k}$. There exists a neighbourhood $U = U_w$ of w in which there are local coordinates z centered around w such that τ^k can be expressed as $\tau^k(z) = e^{\frac{2\pi i}{n}kj} \cdot z$, for a unique $j = j(w) = j(y)$ with $\gcd(j, \frac{n}{k}) = 1$. (In particular, there are no other special points in U_w.)

If $\tilde{\tau}$ is a lift of τ to the holomorphic vector bundle W of rank 2 and s_1, s_2 is a holomorphic frame for $W|_U$, then $\tilde{\tau}^k\colon W|_U \to W|_U$ can be represented by a matrix A of holomorphic functions

$$A(z) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}(z),$$

such that we may write

$$\tilde{\tau}^k(s_1(z)) = a_{11}(z) \cdot s_1 \circ \tau^k(z) + a_{21}(z) \cdot s_2 \circ \tau^k(z),$$
$$\tilde{\tau}^k(s_2(z)) = a_{12}(z) \cdot s_1 \circ \tau^k(z) + a_{22}(z) \cdot s_2 \circ \tau^k(z).$$

The corresponding morphism of the sheaf of sections is $s \mapsto \tilde{\tau} \circ s \circ \tau^{-1}$ which translates to

$$\begin{aligned}
\tilde{\tau}^k(f_1 \cdot s_1)(z) &= f_1 \circ \tau^{-k}(z) \cdot \tilde{\tau}^k(s_1)(z) \\
&= f_1 \circ \tau^{-k}(z) \cdot \big(a_{11} \circ \tau^{-k}(z) \cdot s_1(z) + a_{21} \circ \tau^{-k}(z) \cdot s_2(z)\big), \\
\tilde{\tau}^k(f_2 \cdot s_2)(z) &= f_2 \circ \tau^{-k}(z) \cdot \tilde{\tau}^k(s_2)(z) \\
&= f_2 \circ \tau^{-k}(z) \cdot \big(a_{12} \circ \tau^{-k}(z) \cdot s_1(z) + a_{22} \circ \tau^{-k}(z) \cdot s_2(z)\big),
\end{aligned}$$

where we abuse the notation and write $\tilde{\tau}$ for both the bundle map and the map of sheaves. We can pass back to a bundle map from the sheaf map by expressing each $v \in W$ as $v = s(z)$ which is sent to $\tilde{\tau}(s) \circ \tau(z)$.

Since $\tilde{\tau}^n = \mathrm{Id}_W$ we have a relation

$$A\big((\tau^k)^{\frac{n}{k}-1}(z)\big) \circ \cdots \circ A\big(\tau^k(z)\big) \circ A(z) = \mathrm{Id}_{W_z}\,.$$

In particular, for $z = 0$ corresponding to w the relation reads $A(0)^{\frac{n}{k}} = \mathrm{Id}_{W_w}$. Hence, from general theory (see page 61) $A(0)$ is diagonalizable. Let us therefore choose the frame s_1, s_2 so that

$$A(0) = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix},$$

where $\theta_\nu^{\frac{n}{k}} = 1$, $\nu = 1, 2$. Pulling back a bundle $\tilde{E}$ from Y yields only lifts with $A(0) = \mathrm{Id}$, so stronger measures need to be applied. The answer is elementary modifications.

Write $\theta_\nu = e^{\frac{2\pi i}{n}kk_\nu}$ for $0 \le k_\nu < \frac{n}{k}$ and let $0 \le k'_\nu < \frac{n}{k}$ so that $k'_\nu = j^{-1} \cdot k_\nu \mod \frac{n}{k}$. Then order the frame s_1, s_2 such that $k'_1 \le k'_2$, and define $m_1 = m_1(w) = k'_2$. Hence, $e^{\frac{2\pi i}{n}k(k_2 - jm_1)} = 1$. Now, if $\sigma(z) = z$, σ^{m_1} generates the restriction $[-m_1 \cdot w]|_U$ of the sheaf $[-\sum_{i=0}^{\frac{n}{k}-1} m_1 \cdot \tau^i(w)]$. On any subset U' not intersecting the orbit through w we choose the constant frame for $[-m_1 \cdot w]|_{U'}$. Then the induced action $\tilde{\tau}$ on $W \otimes [-\sum_{i=0}^{\frac{n}{k}-1} m_1 \cdot \tau^i(w)]$ has the same presentation as $\tilde{\tau}$ on $W|_{U'}$. Around w (and any other member of the orbit through w) $\tilde{\tau}^k$ on the sheaf is calculated as

$$\begin{aligned}
\tilde{\tau}^k(\sigma^{m_1} \cdot s_\nu)(z) &= (\sigma \circ \tau^{-k})^{m_1}(z) \cdot \big(a_{1\nu} \circ \tau^{-k}(z) \cdot s_1(z) + a_{2\nu} \circ \tau^{-k}(z) \cdot s_2(z)\big) \\
&= e^{\frac{2\pi i}{n}kj(-m_1)} \cdot \big(a_{1\nu} \circ \tau^{-k}(z) \cdot (\sigma^{m_1} \cdot s_1)(z) + a_{2\nu} \circ \tau^{-k}(z) \cdot (\sigma^{m_1} \cdot s_2)(z)\big).
\end{aligned}$$

Thus, in every point of the orbit through w $\tilde{\tau}^k$ is represented by

$$A(0) = \begin{pmatrix} e^{\frac{2\pi i}{n}k(k_1 - k_2)} & 0 \\ 0 & 1 \end{pmatrix}.$$

We define a divisor D_1 on X as $D_1|_{U_w} = -m_1(w) \cdot w$ for every $w \in \bar{P}$ and $D_1|_U = 0$ for $U \not\ni w$. Notice that D_1 is τ-invariant. Now, put $V = W \otimes [D_1]$. The induced action $\tilde{\tau}$ on V has in every special point the simple diagonal form defined by $A(0)$ above.

Lemma 4.1. *Let X be a Riemann surface and $\tau\colon X \to X$ an order n automorphism. Suppose there is a lift $\tilde{\tau}$ of the action of τ to a holomorphic rank 2 vector bundle V. Let $y \in Y$ be a branch point for π and $w \in \pi^{-1}(y)$ a special point of ramification number k, and assume that there exists a local holomorphic frame s_1, s_2 in which $\tilde{\tau}_w^k\colon V_w \to V_w$ can be represented by the matrix*

$$\begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix}.$$

Then there is a unique locally free rank 2 sheaf E of $\mathcal{O}_X$-modules and a commutative diagram of short exact sequences and lifts

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "\iota"] \arrow[d, "\tilde{\tau}"'] & E \arrow[r, "\lambda"] \arrow[d, dotted, "\hat{\tau}"'] & \mathbb{C}_{\pi^{-1}(y)} \arrow[r] \arrow[d, "\hat{\theta}"'] & 0 \\
0 \arrow[r] & V \arrow[r, "\iota"] & E \arrow[r, "\lambda"] & \mathbb{C}_{\pi^{-1}(y)} \arrow[r] & 0
\end{tikzcd}$$

where $\mathbb{C}_{\pi^{-1}(y)}$ is the skyscraper sheaf with support $\pi^{-1}(y)$ and $\hat{\theta} = e^{\frac{2\pi i}{n}kj} \cdot \theta$, for j part of the local data for τ: $\tau^k(z) = e^{\frac{2\pi i}{n}kj} \cdot z$, $\gcd(j, \frac{n}{k}) = 1$. Locally around w, $E|_U$ is generated by a frame $(\hat{s}_1, \hat{s}_2)$ such that $\iota(s_1) = \sigma \cdot \hat{s}_1$ and $\iota(s_2) = \hat{s}_2$. The induced lift $\hat{\tau}$ is unique and $\hat{\tau}_w^k$ is represented by a holomorphic matrix function, $\hat{A}(z)$, with respect to $(\hat{s}_1, \hat{s}_2)$, such that

$$\hat{A}(0) = \begin{pmatrix} \hat{\theta} & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover, the determinant line bundles are related as $\det E = \det V \otimes [\pi^{-1}(y)]$. *In particular,* $\deg E = \deg V + k$.

Let us call this process (first order) *inverse elementary modification of V*.

Clearly, our $\tilde{\tau}$ and V fit into this lemma. For $w \in \pi^{-1}(y)$, define the non-negative integer $m_2 = m_2(w) = m_2(y) = k_2' - k_1'$, and denote by $\bar{P}_0$ the subset of $\bar{P}$ where $m_2(w) \neq 0$. Putting $(V, \tilde{\tau})$ through the process of this lemma $m_2(y)$ times for every branch point $y \in P$ results in a bundle $E \to X$ with a lift $\hat{\tau}\colon E \to E$ such that $\hat{\tau}_w^{k(w)} = \mathrm{Id}_{E_w}$ for every $w \in \bar{P}$, and we define a divisor D_2 in X with the restrictions $D_2|_{U_w} = m_2(w) \cdot w$, so that $\det E = \det V \otimes [D_2]$ and $\deg E = \deg V + \deg D_2$. Let us emphasize what we mean by putting $(V, \tilde{\tau})$ through this process m_2 times: Step one is a regular first order inverse elementary modification which gives a bundle E_1 equipped with a preferred frame (e_1^1, e_2^1) around the special point w such that the surjection λ_1 takes e_2^1 to zero. Step two is another inverse elementary modification along the first basis vector $e_1^1(w)$ which gives a bundle E_2 with a frame (e_1^2, e_2^2) satisfying that $\lambda_2(e_2^2) = 0$, and so on.

Now, as explained in Observation 2.8 there is a well defined holomorphic vector bundle

$$\tilde{p}\colon \tilde{E} = E/\langle\hat{\tau}\rangle \longrightarrow Y$$

with a natural identification of $\pi^*\tilde{E} \cong E$. Let $\tilde{\pi}\colon E \to \tilde{E}$ be the canonical projection. We want to give $\tilde{E}$ a parabolic structure thereby encoding the data necessary to recapture $\tilde{\tau}\colon W \to W$.

Recall that a non-trivial parabolic structure of a rank 2 holomorphic vector bundle $\tilde{E}$ in the point $y \in P$ is a filtration

$$\tilde{E}_y = \tilde{F}_{1,y} \supset \tilde{F}_{2,y} \supset \tilde{F}_{3,y} = \{0\}$$

and a pair of weights a_y:

$$0 \le a_y^1 < a_y^2 < 1.$$

A trivial parabolic point $y \in P$ is a point with the filtration

$$\tilde{E}_y = \tilde{F}_{1,y} \supset \tilde{F}_{2,y} = \{0\}$$

and a weight $0 \le a_y^1 < 1$. The number of weights is denoted n_y, and the multiplicities of the filtration is defined to be

$$k_y^i = \dim(\tilde{F}_{i,y}/\tilde{F}_{i+1,y})$$

for $1 \le i \le n_y$. For further details and a general definition see [47] or [62].

First define the filtrations by setting

$$\tilde{F}_{2,y} = \tilde{\pi}(\ker \lambda_w)$$

for some $w \in \pi^{-1}(y)$ and every $y \in P$. We notice that y is a non-trivial parabolic point if and only if $y \in \pi(\bar{P}_0)$. Therefore, let P_0 denote the non-trivial parabolic points, $P_0 = \pi(\bar{P}_0)$. Then define the weights

$$\begin{aligned} a_y^1 &= \tfrac{kk_1'}{n} = \tfrac{kk_2'}{n}, && \text{for } y \in P - P_0, \\ a_y^1 &= \tfrac{kk_1'}{n}, \quad a_y^2 = \tfrac{kk_2'}{n} && \text{for } y \in P_0. \end{aligned} \tag{4.2}$$

Definition 4.3. We say that the parabolic bundle $(\tilde{E}, \tilde{F}, a)$ is obtained from $(W, \tilde{\tau})$ through (higher order) *inverse elementary modifications* and we write

$$(\tilde{E}, \tilde{F}, a) = \mathcal{P}(W, \tilde{\tau})$$

($\mathcal{P}(W)$ if $\tilde{\tau}$ is implicit).

This is in fact abuse of language as elementary modifications strictly speaking is only the process of taking the bundle through the short exact sequences, whereas here we also throw in a pullback and a tensor product by a line bundle specified by the given data.

Let us see how we (re-)construct the bundle W with a lift $\tilde{\tau}$ covering τ. I.e. we are given the parabolic bundle $(\tilde{E}, \tilde{F}, a)$ as above; meaning that the weights are n'th roots of unity (other relations on both the parabolic structure and the determinant of $\tilde{E}$ will have to be imposed in order for this to give a fixed point upstairs). Define $E = \pi^*\tilde{E}$ and for every $w \in \pi^{-1}(y)$, $y \in P_0$, let $F_w = (\pi^*\tilde{F})_w$.

Lemma 4.4. *Let $y \in P_0$, and let E be a rank 2 holomorphic vector bundle over X equipped with a 1 dimensional subspace $F_w \subset E_w$ for every $w \in \pi^{-1}(y)$. Assume that there is a lift $\hat{\tau}$ covering τ and a local frame $\hat{s}_1, \hat{s}_2$ for $E|_{U_w}$ such that $\hat{s}_2(w) \in F_w$ and $\hat{\tau}^k$ has the matrix representation $\hat{A}(z)$ with*

$$\hat{A}(0) = \begin{pmatrix} \hat{\theta} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then there exist a surjective sheaf homomorphism $\lambda \colon E \to \mathbb{C}_{\pi^{-1}(y)}$ to the skyscraper sheaf supported over $\pi^{-1}(y)$, such that $\ker \lambda_w = F_w$ for $w \in \pi^{-1}(y)$. Furthermore, there is a unique locally free sheaf V of $\mathcal{O}_X$-modules that completes the rows in the diagram

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "\iota"] \arrow[d, dotted, "\tilde{\tau}"'] & E \arrow[r, "\lambda"] \arrow[d, "\hat{\tau}"'] & \mathbb{C}_{\pi^{-1}(y)} \arrow[r] \arrow[d, "\hat{\theta}"'] & 0 \\ 0 \arrow[r] & V \arrow[r, "\iota"] & E \arrow[r, "\lambda"] & \mathbb{C}_{\pi^{-1}(y)} \arrow[r] & 0 \end{tikzcd}$$

of short exact sequence and lifts, so that $V|_{U_w}$ is generated by local frames s_1, s_2 with $\iota(s_1) = \sigma \cdot \hat{s}_1$ and $\iota(s_2) = \hat{s}_2$. And there is a unique induced lift $\tilde{\tau} \colon V \to V$ whose k'th power $\tilde{\tau}^k$ has the representation $A(z)$ with respect to s_1, s_2 such that

$$A(0) = \begin{pmatrix} e^{\frac{2\pi i}{n}k(-j)} \cdot \hat{\theta} & 0 \\ 0 & 1 \end{pmatrix}.$$

Again, the determinant line bundles are related as $\det E = \det V \otimes [\pi^{-1}(y)]$ *and* $\deg E = \deg V + k$.

This is usually meant by saying that V is got through *elementary modifications of* E *by* λ.

Now return to our parabolic bundle $(\tilde{E}, \tilde{F}, a)$ and let $m_2(w) = m_2(y) = \frac{n}{k}(a^2_{\pi(w)} - a^1_{\pi(w)})$ for $w \in \pi^{-1}(y)$ and $y \in P_0$, and define the divisor $D_2 \in \mathrm{Div}(X)$ with $D_2|_{U_w} = m_2(w) \cdot w$ for $w \in \bar{P}_0$. Let D_1 be the divisor which locally is $D_1|_{U_w} = -m_1(w) \cdot w$ for $w \in \bar{P}$, where $m_1(w) = \frac{n}{k} a^2_{\pi(w)}$. Then the natural lift $\hat{\tau}\colon E \to E$ is simply defined by pullback and thus $\hat{\tau}^k_w\colon E_w \to E_w$ is the identity. Therefore, we can send $(E, F, \hat{\tau})$ through the elementary modifications of the above lemma $m_2(y)$ times for each $y \in P_0$: at each stage the induced lift splits V_w into two distinct eigenspaces (when $m_2(y) \neq 0$); choose the flag as the eigenspace which is not annihilated by ι and continue the process using this flag. This way we get a holomorphic vector bundle V and a lift $\tilde{\tau}\colon V \to V$ whose representatives over w are

$$\begin{pmatrix} e^{2\pi i j (a^1_{\pi(w)} - a^2_{\pi(w)})} & 0 \\ 0 & 1 \end{pmatrix}$$

Define the holomorphic vector bundle $W = V \otimes [-D_1]$. Then the induced lift $\tilde{\tau}$ of τ to W satisfies that $\tilde{\tau}^k_w\colon W_w \to W_w$ can be represented by

$$\begin{pmatrix} e^{2\pi i j a^1_{\pi(w)}} & 0 \\ 0 & e^{2\pi i j a^2_{\pi(w)}} \end{pmatrix},$$

and again $\det E = \det V \otimes [D_2]$ so that $\deg E = \deg V + \deg D_2$.

Definition 4.5. Correspondingly we say that $(W, \tilde{\tau})$ was obtained from $(\tilde{E}, \tilde{F}, a)$ through (higher order) *elementary modifications* which we write

$$(W, \tilde{\tau}) = \mathcal{EM}(\tilde{E}, \tilde{F}, a)$$

or simply $\mathcal{EM}(\tilde{E})$.

Furthermore, we say that a parabolic bundle, that allows such elementary modifications, and gives a well-defined bundle W with trivial determinant and a lift $\tilde{\tau}$, is *admissible*. We write $Ma(Y)$ for the moduli space of all semistable admissible parabolic bundles (if we want to specify (X, τ), we write $Ma(X, \tau)$ for the same space).

The admissible parabolic bundles $\tilde{E}$ for (W, τ) is precisely those satisfying that $\pi^* \det \tilde{E} = [2D_1 + D_2]$ and the weights are given by (4.2) for some lift.

Remark 4.6. Notice that the canonical action $\hat{\tau}$ on the pullback $\pi^*\tilde{E}$ is always diagonal around the special points (it is simply $\hat{\tau}(x,\tilde{e}) = (\tau(x),\tilde{e})$, which makes the isomorphism φ on page 71 equivariant). Thus, also $\tilde{\tau}\colon W \to W$ is diagonal in this sense. Hence, an upshot of this is that lifts $\tilde{\tau}$ of τ with $\tilde{\tau}^n = \mathrm{Id}_W$ are always diagonalizable locally around every point in X.

Remark 4.7. Remember from Remark 2.7 that we can factor $\pi\colon X \to Y$ into a ramified part

$$\pi_r\colon X \longrightarrow \bar{X} = X/\langle\tau^m\rangle,$$

which is a composition of coverings containing a one-point fiber, and an unramified part

$$\pi_u\colon \bar{X} \longrightarrow Y.$$

There is an induced automorphism $\tau_u\colon \bar{X} \to \bar{X}$. In the case $(\bar{E},\bar{F},\bar{a}) = \mathcal{P}(W,\tilde{\tau}^m)$, we claim that there is an induced lift $\tilde{\tau}_u\colon (\bar{E},\bar{F},\bar{a}) \to (\bar{E},\bar{F},\bar{a})$, and that $(\tilde{E},\tilde{F},a) \cong (\bar{E},\bar{F},\bar{a})/\langle\tilde{\tau}_u\rangle$, where $(\tilde{E},\tilde{F},a) = \mathcal{P}(W,\tilde{\tau})$. Conversely, we claim that the identity $\mathcal{EM}(\pi_u^*(\tilde{E},\tilde{F},a)) \cong (W,\tilde{\tau})$ holds.

To see this notice first that the special orbits for τ^m have lengths $\bar{k}(w) = \frac{k(w)}{m}$ and that the action in local coordinates still look like $(\tau^m)^{\frac{k}{m}}(z) = e^{\frac{2\pi i}{n}kj}\cdot z$. Now the rest of the data is easily calculated: If $\tilde{\tau}^k$ is represented by A then so is $(\tilde{\tau}^m)^{\frac{k}{m}}$ and consequently $\bar{k}_\nu = k_\nu$. Hence, for $0 \le \bar{k}'_\nu < \frac{n}{m}/\frac{k}{m} = \frac{n}{k}$ defined by $\bar{k}'_\nu = j^{-1}\cdot\bar{k}_\nu \mod \frac{n}{k}$ we have that $\bar{k}'_\nu = k'_\nu$, and thus $\bar{m}_2 = m_2$. Now, as the basic exact sequence (of Lemma 4.1) is the same as before, clearly the vector bundle constructed is E, and the induced lift $\bar{\hat{\tau}}\colon E \to E$ is $\hat{\tau}^m$. It follows that $(\tilde{E},\tilde{F},a) \cong (\bar{E},\bar{F},\bar{a})/\langle\tilde{\tau}_u\rangle$, where $\tilde{\tau}_u$ is the lift induced by $\hat{\tau}$. The converse is proved using similar arguments.

Stability. Let us contemplate the stability relations: Recall that stability respectively semistability is preserved under tensoring with line bundles, so W is stable (resp. semistable) if and only if V is stable (resp. semistable). Hence, we seek to relate the stability relations of the holomorphic bundle V and the parabolic bundle $\tilde{E}$

Lemma 4.8. *Suppose L is a $\tilde{\tau}$-invariant subbundle of V. Then there is a parabolic subbundle $\tilde{L}$ of $\tilde{E}$ induced through the inverse elementary modifications, and*

$$\operatorname{par}\mu(\tilde{E}) - \operatorname{par}\mu(\tilde{L}) = \frac{1}{n}\big(\mu(V) - \mu(L)\big).$$

Moreover, a parabolic subbundle $\tilde{L}$ of $\tilde{E}$ induces through elementary modifications an invariant subbundle L of V, and the same equality holds.

Corollary 4.9. *Under the conditions of the above lemma*

$$\mu(L) < \mu(V)$$

(resp. $\mu(L) \leq \mu(V)$) if and only if

$$\operatorname{par}\mu(\tilde{L}) < \operatorname{par}\mu(\tilde{E})$$

(resp. $\operatorname{par}\mu(\tilde{L}) \leq \operatorname{par}\mu(\tilde{E})$*).* □

Proof of Lemma 4.8. Let $L \subset V$ be a $\tilde{\tau}$-invariant subbundle. Using Lemma A.2 and Observation A.3 we get a $\hat{\tau}$-invariant line bundle $\hat{L} \subset E$. This gives a subbundle $\tilde{L} = \hat{L}/\langle \hat{\tau}|_{\hat{L}} \rangle \subset \tilde{E}$.

For any subbundle $\tilde{L}$ of $\tilde{E}$ the weights assigned, that makes it a parabolic subbundle, are as follows

$$a_y^1(\tilde{L}) = a_y^1 = a_y^2, \quad \text{for } y \in P - P_0,$$
$$a_y^1(\tilde{L}) = \begin{cases} a_y^1, & \text{for } \tilde{L}_y \neq \tilde{F}_y \text{ and } y \in P_0, \\ a_y^2, & \text{for } \tilde{L}_y = \tilde{F}_y \text{ and } y \in P_0. \end{cases}$$

For a parabolic subbundle $\tilde{L}$ of $\tilde{E}$, put

$$P(\tilde{L}) = \{\, y \in P_0 \mid \tilde{L}_y \neq F_y \,\}.$$

Then $\bar{P}(\tilde{L}) \stackrel{\text{def}}{=} \pi^{-1}P(\tilde{L}) = \bar{P}(\pi^*\tilde{L})$ of Lemma A.2, and there is a subbundle $L \subset V$ whose sheaf of sections fits into

$$0 \longrightarrow L \xrightarrow{\iota|_L} \pi^*\tilde{L} \xrightarrow{\lambda|_{\pi^*\tilde{L}}} F_{\pi^*\tilde{L}} \longrightarrow 0.$$

Now we calculate the parabolic slopes of $\tilde{E}$ and its subbundles: $\operatorname{par}\deg(\tilde{E}) = \deg\tilde{E} + \sum_{y\in P}\sum_{i=1}^{n_y} k_y^i a_y^i$, where $\deg\tilde{E} = \frac{1}{n}\deg E = \frac{1}{n}(2\deg D_1 + \deg D_2)$, so

$$\operatorname{par}\mu(\tilde{E}) = \frac{1}{2n}(2\deg D_1 + \deg D_2) + \sum_{y\in P-P_0} a_y^1 + \sum_{y\in P_0}\frac{a_y^1 + a_y^2}{2}.$$

Recall that $D_2|_{U_w} = m_2(w)\cdot w$, and $m_2(w) = m_2(y) = k_2' - k_1' = \frac{n}{k(y)}(a_y^2 - a_y^1)$, so we may write

$$\frac{1}{2n}\deg D_2 + \sum_{y\in P_0}\frac{a_y^1 + a_y^2}{2} = \sum_{y\in P_0}\left(\frac{1}{2n}k(y)m_2(y) + \frac{a_y^1 + a_y^2}{2}\right)$$
$$= \sum_{y\in P_0}\left(\frac{a_y^2 - a_y^1}{2} + \frac{a_y^1 + a_y^2}{2}\right) = \sum_{y\in P_0} a_y^2$$

and hence, the parabolic slope of $\tilde{E}$ is

$$\operatorname{par}\mu(\tilde{E}) = \frac{1}{n}\deg D_1 + \sum_{y\in P-P_0} a_y^1 + \sum_{y\in P_0} a_y^2. \tag{4.10}$$

From Lemma A.2 and Remark A.4 it follows that $\det \pi^*\tilde{L} = \det L \otimes [D_2|_{\bar{P}(\tilde{L})}]$ and subsequently $n\deg\tilde{L} = \deg L + \deg(D_2|_{\bar{P}(\tilde{L})})$. Hence,

$$\operatorname{par}\mu(\tilde{L}) = \frac{1}{n}\big(\deg L + \deg(D_2|_{\bar{P}(\tilde{L})})\big) + \sum_{y\in P-P_0} a_y^1 + \sum_{y\in P(\tilde{L})} a_y^1 + \sum_{y\in P_0 - P(\tilde{L})} a_y^2. \tag{4.11}$$

Using the relation between m_2 and $a_y^2 - a_y^1$ and arguments similar to the ones leading to formula (4.10), this implies

$$\operatorname{par}\mu(\tilde{E}) - \operatorname{par}\mu(\tilde{L}) = \frac{1}{n}(\deg D_1 - \deg L).$$

On the other hand, $V = W \otimes [D_1]$ so $\deg V = \deg W + 2\deg D_1 = 2\deg D_1$ and, thus, we get that

$$\operatorname{par}\mu(\tilde{E}) - \operatorname{par}\mu(\tilde{L}) = \frac{1}{n}\big(\mu(V) - \mu(L)\big).$$

$\square$

Suppose $(\tilde{E},\tilde{F},a)$ is parabolic semistable and let V_1 be the unique maximal semistable subbundle of V. As $\tilde{\tau}(V_1)$ is a bundle with the same properties, $\tilde{\tau}(V_1) = V_1$ by uniqueness. Assume that $V_1 \neq V$ so that V is not semistable. Then V_1 is an invariant line bundle in V and employing the previous lemma we get: $\mu(V_1) \leq \mu(V)$ by semistability of $\tilde{E}$, thus contradicting the assumption on V. Hence, we have proved that $W = \mathcal{EM}(\tilde{E})$ is semistable when $\tilde{E}$ is parabolic semistable.

If on the other hand W is stable (resp. semistable), then so is V, and using Corollary 4.9 it is evident that if $\tilde{E} = \mathcal{P}(W)$, $\operatorname{par}\mu(\tilde{L}) < \operatorname{par}\mu(\tilde{E})$ (resp. $\operatorname{par}\mu(\tilde{L}) \leq \operatorname{par}\mu(\tilde{E})$) for every parabolic subbundle $\tilde{L}\subset\tilde{E}$, i.e. $\tilde{E}$ is parabolically stable (resp. semistable).

It is obvious by construction that $\mathcal{P}$ is a well-defined map from $\mathit{Lift}(X,\tau)$ to $\mathit{Ma}(Y)$, and the only thing to check in order to make sure $\mathcal{EM}$ is a well-defined map the other way, is that if $\operatorname{Gr}(\tilde{E},F,a) = (\tilde{L}_1,a_1)\oplus(\tilde{L}_2,a_2)$ then $\operatorname{Gr}\mathcal{EM}(\tilde{E},F,a) = \mathcal{EM}\big((\tilde{L}_1,a_1)\oplus(\tilde{L}_2,a_2)\big)$. This amounts to the following: suppose that

$$0 \longrightarrow (\tilde{L}_1,a_1) \longrightarrow (\tilde{E},F,a) \longrightarrow (\tilde{L}_2,a_2) \longrightarrow 0\ .$$

Then the subbundle $\tilde{L}_1$ gives an invariant destabilizing subbundle L_1 of $W = \mathcal{EM}(\tilde{E}, F, a)$. Another line bundle L_2 is induced over X by the subbundle $(\tilde{L}_2, a_2) \subset (\tilde{L}_1, a_1) \oplus (\tilde{L}_2, a_2)$, and the claim is proved if $L_2 \cong W/L_1$. This can be seen by a chase round the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L_1 \arrow[r] \arrow[d] & \pi^* L_1 \arrow[r] \arrow[d] & \bar{F}|_{\bar{P}(L_1)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W \arrow[r] \arrow[dl] & \pi^* \tilde{E} \arrow[r] \arrow[d] & \bar{F} \arrow[r] \arrow[d] & 0 \\
W/L_1 \arrow[dr, dotted, "f"] & & \pi^* L_2 \arrow[r] \arrow[d] & \bar{F}|_{\bar{P}(L_2)} \arrow[r] \arrow[d] & 0 \\
 & L_2 \arrow[ur, hook] & 0 & 0 &
\end{tikzcd}$$

short exact sequences of sheaves, which gives a non-zero homomorphism f between line bundles of the same degree.

Let us record our progress so far in the following lemma:

Lemma 4.12. *Let $\tau \colon X \to X$ be an automorphism of a Riemann surface and denote by π the covering $X \to Y = X/\langle \tau \rangle$. There are well-defined maps*

$$\begin{aligned} \mathcal{EM} \colon \; & Ma(Y) \longrightarrow Lift(X, \tau), \\ \mathcal{P} \colon \; & Lift(X, \tau) \longrightarrow Ma(Y). \end{aligned}$$

Forgetting the lifts, $\mathcal{EM}$ gives a surjective map $\overline{\mathcal{EM}}$ onto the fixed points set $|M(X)|$.

Notice that parabolic bundles in $Ma(Y)$ coming from non-stable bundles on X have flags respecting the splitting $W = L_1 \oplus L_2$.

As in the unramified case there may be stable parabolic bundles that yield only *semi*-stable bundles (i.e. *not* stable).

Lemma 4.13. *Let $\tilde{E}$ be a stable parabolic bundle over Y. Then the underlying bundle W of $\mathcal{EM}(\tilde{E})$ is a semistable but not stable vector bundle over X if and only if the order of the group $\langle L_\pi \rangle$ is even, say $2l$, and $L_\pi^l \otimes \tilde{E} \cong \tilde{E}$.* $\square$

Again we denote by $\mathcal{L}_\pi$ the order 2 subgroup $\langle L_\pi^l \rangle$ and by $Ma^{\mathcal{L}_\pi}(Y)$ the moduli space of admissible parabolic bundles fixed by $\mathcal{L}_\pi$.

Proof. Suppose that V is semistable and not stable, and let $L_1 \subset V$ be a destabilizing line bundle, $\mu(L_1) = \mu(V)$. Following Lemma A.2 there is a line bundle $\hat{L}_1 \subset E$ such that $\iota(L_1) \subseteq \hat{L}_1$, and if L_1 is $\tilde{\tau}$-invariant, we may construct a line bundle $\tilde{L} \subset \tilde{E}$ with $\operatorname{par}\mu(\tilde{L}) = \operatorname{par}\mu(\tilde{E})$ contradicting the assumption on $\tilde{E}$. Hence, assume $\tau^* L_1 \ncong L_1$, and let L_2 be the quotient of V by L_1. Notice that $\mu(L_2) = \mu(L_1) = \mu(V)$.

If V is a semistable, fixed point and $\operatorname{Gr}(V) = L_1 \oplus L_2$, then $\tau^* L_1 \oplus \tau^* L_2 = L_1 \oplus L_2$. As L_ν and $\tau^* L_\nu$ all have the same degree, we conclude that either $\tau^* L_\nu \cong L_\nu$ (which we excluded above) or $\tau^* L_1 \cong L_2$ and vice versa.

Recall from Remark 2.7 that the covering π factors through a ramified covering π_r and an unramified covering π_u. The ramified covering π_r is obtained by dividing out by the action of τ^m, where m is the greatest common divisor of the orbit lengths for τ. Consider therefore first the case of a ramified covering for which the greatest common divisor of the lengths of the special orbits is 1. That means that there is at least one fiber of odd length.

Let $\pi_r^{-1}(\bar{x})$ be a fiber of odd length k and consider a neighbourhood U_w around $w \in \pi_r^{-1}(\bar{x})$. Since any lift, $\hat{\tau}$ of E, is locally diagonalizable (Remark 4.6) we may without loss of generality assume that the matrix representation of $\hat{\tau}^k$ is $A = \mathrm{Id}$ over U_w.

Let $\hat{L}_\nu \subset E$ be the line bundle constructed in Lemma A.2 from L_ν. Clearly, $\hat{\tau}\hat{L}_1 = \hat{L}_2$ and vice versa, and $\hat{L}_1$ and $\hat{L}_2$ spans E away from w (where they collapse to a line) by Observation A.3. The induced automorphism $\hat{\tau}^k$ interchanges these two bundles, $\hat{\tau}^k \hat{L}_1 = \hat{L}_2$, which contradicts the fact that $\hat{\tau}^k$ is represented by $A = \mathrm{Id}$ over U_w. Thus, no stable parabolic bundle $\tilde{E}$ over the target surface in a ramified covering with an orbit of odd length, can give rise to an only semistable bundle V upstairs.

If $\pi\colon X \to Y$ is an unramified covering, we know the result from section 3, so consider the unramified part $\pi_u\colon \bar{X} \to Y$ of a ramified covering $\pi\colon X \to Y$, and let us generalize the result of section 3 to parabolic bundles. Let $(\bar{E}, \bar{F}, \bar{a}) = \pi_u^*(\tilde{E}, \tilde{F}, a)$ be a semistable (not stable) parabolic pullback of a stable parabolic bundle. Now, $(\tilde{E}, \tilde{F}, a) = (\bar{E}, \bar{F}, \bar{a})/\langle \tilde{\tau}_u \rangle$ by Remark 4.7, so any invariant parabolic line bundle $\bar{L} \subset \bar{E}$ of parabolic slope $\operatorname{par}\mu(\bar{L}) = \operatorname{par}\mu(\bar{E})$ gives a parabolic subbundle $\tilde{L} = \bar{L}/\langle \tilde{\tau}_u | \bar{L} \rangle$ of $\tilde{E}$ with $\operatorname{par}\mu(\tilde{L}) = \operatorname{par}\mu(\tilde{E})$, thus contradicting stability of $\tilde{E}$.

Hence, assume $\operatorname{Gr}\bar{E} = \bar{L}_1 \oplus \bar{L}_2$ as a parabolic subbundle so that $\tilde{\tau}_u \bar{L}_1 \ncong \bar{L}_1$ and $\operatorname{par}\mu(\bar{L}_\nu) = \operatorname{par}\mu(\bar{E})$. Then the order of τ_u is even, $m = 2l$, and the covering factors through $\bar{\pi}\colon \bar{X} \to \check{X} = \bar{X}/\langle \tau_u^2 \rangle$ and the double cover $\check{\pi}\colon \check{X} \to Y$, the latter having the deck transformation $\check{\tau}\colon \check{X} \to \check{X}$ induced from τ_u.

Now, $\bar{L}_\nu$ is $\tilde{\tau}_u^2$-invariant and we get line bundles $\check{L}_\nu = \bar{L}_\nu/\langle \tilde{\tau}_u^2 \rangle$ over $\check{X}$. As in section 3, $\bar{\pi}^*\check{L}_\nu = \bar{L}_\nu$, $\check{L}_1 \not\cong \check{L}_2$, and $\check{\tau}^*\check{L}_1 \cong \check{L}_2$. Therefore, there is the identity

$$\tilde{E} \cong (\check{L}_1 \oplus \check{\tau}^*\check{L}_1)/\langle\check{\tau}\rangle \cong \check{\pi}_*\check{L}_1$$

of holomorphic bundles, and we recall that the vector bundles $\tilde{E}$ that come about as a push forward in this way are specified precisely as those invariant under the action of the group of characters of $\langle\check{\tau}\rangle$; i.e. there is a canonical isomorphism $L_\chi \otimes \tilde{E} \to \tilde{E}$ for every character χ of $\langle\check{\tau}\rangle$.

We check that $\tilde{E}$ is parabolically stable. Assume conversely that $\tilde{L} \subset \tilde{E}$ is a parabolic subbundle of parabolic slope $\operatorname{par}\mu(\tilde{L}) = \operatorname{par}\mu(\tilde{E})$. Then $\operatorname{par}\mu(\check{\pi}^*\tilde{L}) = 2\operatorname{par}\mu(\tilde{L}) = 2\operatorname{par}\mu(\tilde{E}) = \operatorname{par}\mu(\check{\pi}^*\tilde{E})$, and $\check{\pi}^*\tilde{E} = \check{L}_1 \oplus \check{L}_2$ so by Jordan-Hölder ([62, 3.IV.12]), $\check{\pi}^*\tilde{L} \cong \check{L}_1$ or $\check{\pi}^*\tilde{L} \cong \check{L}_2$. But $\check{\pi}^*\tilde{L}$ is invariant and neither of $\check{L}_\nu$ are, so $\tilde{E}$ has to be parabolically stable.

The group of character line bundles L_χ for $\langle\check{\tau}\rangle$ is exactly the order 2 subgroup $\langle L_\pi^l\rangle \subseteq \langle L_\pi\rangle$ (see Observation 2.8). The parabolic bundle $(\tilde{E}, \tilde{F}, a)$ is invariant under $\langle L_\pi^l\rangle$ if and only if the vector bundle $\tilde{E}$ is. □

Lemma 4.14. *The map $\mathcal{P}$ is injective on the subset of Lift(X,τ) consisting of the stable points and points whose underlying bundle can be represented by $L_1 \oplus L_2$, where L_1 and L_2 are invariant of equal degree and $L_1 \not\cong L_2$. In fact on this subset N*

$$\mathcal{EM} \circ \mathcal{P}|_N = \mathrm{Id}_N\,.$$

On the part of the complement of N where $\tau^*L_1 \cong L_2 \not\cong L_1$, the pre-image $\mathcal{P}^{-1}\big(\mathcal{P}(W,\tilde{\tau})\big)$ can be identified with $\mathbb{C}^*$

However, if $W \equiv L \oplus L$ and m is even the situation is even more degenerate and there are lots of lifts that do not come up using our construction. But we may exclude those extra lifts for out definition of *Lift*(X,τ) and in that case it turns out that as maps between semistable bundles with diagonal lifts and the semistable parabolic bundles of *Ma*(Y), $\mathcal{P}$ and $\mathcal{EM}$ are each others inverse.

Proof. Assume that $\mathcal{P}(W_1,\tilde{\tau}_1) = \mathcal{P}(W_2,\tilde{\tau}_2) = (\tilde{E}, F, a)$. Right away we see that the exact sequence defined by $\pi^*\tilde{E}$, π^*F, and a is unique, so we must have that $W_1 \cong W_2$ and in every special orbit $\tilde{\tau}_1^k = \tilde{\tau}_2^k$ is determined by a.

Suppose first that $W = W_1 = W_2$ is stable and $\tilde{\tau}$ is a lift. Then there are n other lifts $\tilde{\tau}_c$ parameterized by $c \in \mathbb{C}$, $c^n = 1$ such that $\tilde{\tau}_c = c \cdot \tilde{\tau}$. In every $w \in \bar{P}$, $\tilde{\tau}_c^{k(w)} = \tilde{\tau}^{k(w)}$, so c has to satisfy that $c^{k(w)} = 1$ for every $w \in \bar{P}$. This means exactly that $c^m = 1$ where m is the greatest common divisor $\gcd(n_1,\dots,n_N,n)$ of the lengths of the orbits.

Then induced lift $\hat{\tau}$ is scaled in the same way as $\tilde{\tau}$. If $\tilde{E} = E/\langle\hat{\tau}\rangle$, then using Observation 2.8 we identify

$$\tilde{E}_c = E/\langle\hat{\tau}_c\rangle = \tilde{E} \otimes L_c = \tilde{E} \otimes L_\pi^j$$

for a unique $0 \le j < m-1$ corresponding to c (actually $c = e^{\frac{2\pi i}{m}j}$). The canonical lift in E that makes the isomorphism $\varphi_c\colon E \to \pi^*\tilde{E}_c$ of page 71 equivariant is $\hat{\tau}_c$, so up to isomorphism we recapture $(W, \tilde{\tau})$ exactly by applying $\mathcal{EM}$.

Similarly, if W is semistable but not stable, it is strongly equivalent to a direct sum $L_1 \oplus L_2$ of two line bundles L_ν with $\mu(L_\nu) = \mu(W)$. First assume that $\tau^* L_\nu = L_\nu$ and $L_1 \not\cong L_2$. Then with respect to this splitting any lift is diagonal and can be written in terms of a fixed lift $\tilde{\tau}$ as

$$\tilde{\tau}_{(c_1,c_2)} = \begin{pmatrix} c_1 \cdot \alpha_1 & 0 \\ 0 & c_2 \cdot \alpha_2 \end{pmatrix}, \qquad \text{for } \tilde{\tau} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$$

where, by the same arguments as in the stable case, $c_\nu^m = 1$.

Given $(W, \tilde{\tau})$, then $(E, \hat{\tau})$, $\lambda\colon E \to F$, and a are uniquely determined. Scaling $\tilde{\tau}$ as described results in a similar scaling of $\hat{\tau}$, where we notice that the induced bundles $\hat{L}_\nu \subset E$ are invariant and $E = \hat{L}_1 \oplus \hat{L}_2$, as noted in Observation A.3.

In that case we get $\tilde{E}_{(c_1,c_2)} = E/\langle\hat{\tau}_{(c_1,c_2)}\rangle = (\tilde{L}_1 \otimes L_\pi^{j_1}) \oplus (\tilde{L}_2 \otimes L_\pi^{j_2})$ for $c_\nu = e^{\frac{2\pi i}{m}j_\nu}$ and $\tilde{E} = \tilde{L}_1 \oplus \tilde{L}_2$. Arguing as before we see that these m^2 parabolic bundles correspond uniquely (by pullback) to the m^2 lifts upstairs having the matrix presentations in the special points as determined by a.

Now, if $W \equiv L_1 \oplus L_2$ is semistable but $\tilde{\tau} L_1 = L_2 \not\cong L_1$, then $m = 2l$ is even and $\tilde{\tau}_{(c_1,c_2)}$ may be represented as

$$\tilde{\tau}_{(c_1,c_2)} = \begin{pmatrix} 0 & c_1 \cdot \alpha_2 \\ c_2 \cdot \alpha_2 & 0 \end{pmatrix}, \qquad \text{for } \tilde{\tau} = \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix}.$$

Therefore, as $\tilde{\tau}_{(c_1,c_2)}^{k(w)} = \tilde{\tau}^{k(w)}$ for every $w \in \bar{P}$, we have

$$\tilde{\tau}_{(c_1,c_2)}^m = \begin{pmatrix} (c_1c_2)^l(\alpha_1\alpha_2)^l & 0 \\ 0 & (c_1c_2)^l(\alpha_2\alpha_1)^l \end{pmatrix} = \begin{pmatrix} (\alpha_1\alpha_2)^l & 0 \\ 0 & (\alpha_2\alpha_1)^l \end{pmatrix},$$

so $(c_1c_2)^l = 1$. Hence, $c_2 = e^{\frac{2\pi i}{l}j} \cdot c_1^{-1}$ and there is a solution for every $0 \le j \le l-1$ and $c_1 \in \mathbb{C}^*$.

The line bundles $\bar{L}_\nu = \hat{L}_\nu / \langle \hat{\tau}_{(c_1,c_2)}^m|_{\hat{L}_\nu}\rangle \to \bar{X}$ are unaltered when changing c_i, and the lift $\tilde{\tau}_{u,(c_1,c_2)}$ of τ_u to $\bar{L}_1 \oplus \bar{L}_2$ induced by $\tilde{\tau}_{(c_1,c_2)}$ acts as

$$\begin{pmatrix} c_1c_2 \cdot \alpha_1\alpha_2 & 0 \\ 0 & c_1c_2 \cdot \alpha_2\alpha_1 \end{pmatrix}.$$

Therefore, we get line bundles $\check{L}_{\nu,(c_1,c_2)} = \check{L}_\nu \otimes \check{L}_{c_1c_2}$, where $\check{L}_{c_1c_2} = L^j_{\check{\pi}}$ is the line bundle over $\check{X}$ defined in Observation 2.8 and j is uniquely defined by $c_1c_2 = e^{\frac{2\pi i}{l}j}$. In the last step, $\breve{\pi}\colon \check{X} \to Y$, there is no ambiguity, as $(\breve{\pi}^*)^{-1}(\breve{\pi}^*\tilde{E}) = \{\tilde{E}, \tilde{E} \otimes L_{\breve{\pi}}\}$, but $\tilde{E} \cong \tilde{E} \otimes L_{\breve{\pi}}$. Hence, the l different parabolic bundles $\tilde{E} \otimes L^j_\pi$ correspond to l different families of lifts each of which is parameterized by $\mathbb{C}^*$. □

We now prove that the inverse elementary modifications constitute a left inverse for the elementary modifications.

Lemma 4.15. *The map $\mathcal{EM}$ is injective; in fact, for any admissible parabolic bundle $(\tilde{E}, \tilde{F}, a)$*

$$\mathcal{P} \circ \mathcal{EM}(\tilde{E}, \tilde{F}, a) = (\tilde{E}, \tilde{F}, a).$$

Proof. Suppose $\mathcal{EM}(\tilde{E}_1, \tilde{F}_1, a_1) = \mathcal{EM}(\tilde{E}_2, \tilde{F}_2, a_2) = (W, \tilde{\tau})$. As $\tilde{\tau}$ determines the exact sequence and the weights uniquely, $a_1 = a_2$ and the only possible ambiguity can be in the pullback. Hence, assume that $\pi^*\tilde{E}_1 \cong \pi^*\tilde{E}_2$. Then as $\pi_*(\pi^*\tilde{E}_\nu) \cong \pi_*(\pi^*\tilde{E}_\nu \otimes \mathcal{O}_X) \cong \tilde{E}_\nu \otimes \pi_*\mathcal{O}_X \cong \tilde{E}_\nu \otimes \bigoplus_{j=0}^{n-1} L^j_\pi$, we have that if the parabolic bundle $\tilde{E}_1$ is stable, then by Krull-Schmidt-Atiyah so is $\tilde{E}_2$ and there is a $0 \le q \le m-1$ so that

$$\tilde{E}_2 \cong \tilde{E}_1 \otimes L^q_\pi$$

as holomorphic bundles. But in order to give the same exact sequence the parabolic flag has to agree under this isomorphism as well.

When $\tilde{E}_1$ is not fixed by $\mathcal{L}_\pi$, the m different bundles $\tilde{E}_1 \otimes L^j_\pi$ correspond by the proof of the previous lemma precisely to the m different lifts with the given data in the special points, and $\mathcal{P}$ and $\mathcal{EM}$ are each others inverse as maps between the stable stratum in *Lift*(X, τ) and the stratum in *Ma*(Y) consisting of stable parabolic bundles which are not fixed by $\mathcal{L}_\pi$.

When $\tilde{E}_1$ is stable but fixed by $\mathcal{L}_\pi$, then the m bundles are pair wise isomorphic so there are l distinct bundles, and they correspond modulo $\mathbb{C}^*$ to the l different lifts upstairs.

If $\tilde{E}_1$ is semistable we may look at the graded object $\tilde{L}_{1,1} \oplus \tilde{L}_{1,2}$, and we get from Krull-Schmidt-Atiyah that also $\tilde{E}_2$ is semistable and the line bundles in the graded object have to satisfy that

$$\tilde{L}_{2,\nu} \cong \tilde{L}_{1,1} \otimes L^{q_\nu}_\pi \quad \text{or} \quad \tilde{L}_{1,2} \otimes L^{q_\nu}_\pi$$

for some $0 \le q_\nu \le q-1$ (i.e. $\tilde{L}_{2,2} \cong \tilde{L}_{1,2} \otimes L^{q_2}_\pi$ if $\tilde{L}_{2,1} \cong \tilde{L}_{1,1} \otimes L^{q_1}_\pi$ and vice versa). We conclude as above that the m^2 different parabolic bundles correspond to the m^2 different lifts of this type. So $\mathcal{P}$ is in fact left inverse to $\mathcal{EM}$. □

Since $\mathcal{P}$ is surjective, there is a well-defined action of $\mathbb{Z}/n$ respectively $\mathbb{Z}/n \times \mathbb{Z}/n$ on *Ma*(Y) defined by its action on *Lift*(X, τ). We shall study this action in the next subsection. Before that we collect the last series of results an more in:

Theorem 4.16. *Let $\tau\colon X \to X$ be an automorphism of the Riemann surface X. The map*

$$\mathcal{EM}\colon\ Ma(Y) \longrightarrow Lift(X,\tau)$$

between the moduli space of admissible, semistable, parabolic bundles over $Y = X/\langle\tau\rangle$ and isomorphism classes of lifts $(W,\tilde{\tau})$ of τ to semistable holomorphic vector bundles W is injective. The map is one-to-one from the subset of stable points in $Ma(Y)$, not fixed by $\mathcal{L}_\pi$, onto the stable points in $Lift(X,\tau)$, and from the subset of semistable points onto the subset of semistable points represented by invariant line bundles L_1 and L_2.[4] *The inverse of $\mathcal{EM}$ on this subset is the restriction of $\mathcal{P}$.*

Moreover, the surjective map

$$\overline{\mathcal{EM}}\colon\ Ma(Y) \longrightarrow |M(X)|$$

is a morphism of varieties.

Proof. The only thing there remains to be proved is that $\overline{\mathcal{EM}}$ is a morphism. That is the exact statement of lemma B.14. □

Recall from the way that $Lift(X,\tau)$ was defined that there is an action by $\mathbb{Z}/n \times \mathbb{Z}/n$ and $\mathbb{Z}/n$ on semistable respectively stable lifts which exactly determines the fixed points. Define an equivalence relation $\sim$ on $Ma(Y)$ by the action of $\mathbb{Z}/n\times\mathbb{Z}/n$ and $\mathbb{Z}/n$: $\tilde{E}_1 \sim \tilde{E}_2$ for $\tilde{E}_\nu \in Ma(Y)$ if

(1) $\tilde{E}_1$ and $\tilde{E}_2$ are both stable and $\tilde{E}_2 \in \mathbb{Z}/n \cdot \tilde{E}_1$, or
(2) $\tilde{E}_1$ and $\tilde{E}_2$ are both semistable and $\tilde{E}_2 \in (\mathbb{Z}/n \times \mathbb{Z}/n) \cdot \tilde{E}_1$.

Corollary 4.17. *Under the assumptions of the previous theorem, we get an identification*

$$\overline{\mathcal{EM}}\colon\ Ma(Y)/\!\sim\ \longrightarrow |M(X)|,$$

under which the stable locus of $|M(X)|$ is identified with the stable points in $Ma(Y)$ which are not fixed by $\mathcal{L}_\pi$.

Of course we still need to prove that this is in fact an isomorphism of varieties, which is something we have scheduled for the next stage of the project. But given that this gives a complete description of $|M(X)|$. Notice in particular that we have good control over connected components of $Ma(Y)$, since at least for higher genera fixing a determinant and a weight configuration gives a connected component of $Ma(Y)$. If the genus of Y is less than 2, other measures must be applied (see section 6).

[4] This is of course assuming that we limit lifts of $L \oplus L$ to diagonal lifts.

The action of $\mathbb{Z}/n\times\mathbb{Z}/n$ and $\mathbb{Z}/n$. Now, consider the action of $\mathbb{Z}/n\times\mathbb{Z}/n$ and $\mathbb{Z}/n$ induced on $Ma(Y)$ by $\mathcal{P}$ and the action on $Lift(X,\tau)$. In fact, the discussion above dealt with the action of the normal subgroup $\mathbb{Z}/m\times\mathbb{Z}/m$ respectively $\mathbb{Z}/m$ which is particularly important as it is this that determines which stable parabolic bundles are sent to stable bundles upstairs. We cannot give a nice unified presentation of this action. Instead we describe in detail what happens during the action of $\mathbb{Z}/n$ on a stable parabolic bundle giving a stable fixed point. The semistable case can be analyzed in the same way. In section 6 we explain also the action on semistable points in the case of a hyperelliptic involution.

The action on $Ma(Y)$ is defined through the action on lifts which is simply the multiplication by an n'th root of unity. Therefore, let W be a fixed point, $\tilde{\tau}=\tilde{\tau}_0$ a lift, and $k_{\nu,0}=k_\nu$, $k'_{\nu,0}=k'_\nu$, $m_{\nu,0}=m_\nu$ and j the data from before. And let $(\tilde{E}_0,F_0,a_0)=(\tilde{E},F,a)$ be the parabolic vector bundle corresponding to $(W,\tilde{\tau})$. Notice that tensoring by a locally free, invertible sheaf is an exact functor so if we define E' to be the vector bundle that extends $\mathbb{C}_{\pi^{-1}(y)}$ by W, i.e. E' is obtained by elementary modifications on W,

$$0\longrightarrow W\longrightarrow E'\longrightarrow \mathbb{C}_{\pi^{-1}(y)}\longrightarrow 0\ .$$

Therefore, for any line bundle L, we get a short exact sequence

$$0\longrightarrow W\otimes L\longrightarrow E'\otimes L\longrightarrow \mathbb{C}_{\pi^{-1}(y)}\longrightarrow 0\ .$$

Hence, by uniqueness of the extension, the bundle obtained by elementary modification on $W\otimes L$ is $E'\otimes L$.

For every $w\in\pi^{-1}(y)$ define $j'=j'(y)=j'(w)=j(y)^{-1}\mod\frac{n}{k}$ so that $0<j'<\frac{n}{k}$. Define e^1_w and e^2_w to be the two eigendirections in W_w corresponding to $k_{1,0}$ and $k_{2,0}$ respectively. The action on $\tilde{\tau}$ is simply $(l,\tilde{\tau})\mapsto e^{\frac{2\pi i}{n}l}\tilde{\tau}$, so put $k_\nu(l)=k'_\nu+j'\cdot l\mod\frac{n}{k}$, $0\le k_\nu(l)<\frac{n}{k}$ and define

$$\begin{aligned}k'_{1,l}&=\min\{k_1(l),k_2(l)\},\\ k'_{2,l}&=\max\{k_1(l),k_2(l)\}.\end{aligned}$$

Set $m_{1,l}=k'_{2,l}$ and let $D_{1,l}$ be the divisor defined by $m_{1,l}$ in the same way that m_1 defined D_1. Likewise, let $m_{2,l}=k'_{2,l}-k'_{1,l}$ and $D_{2,l}$ be the resulting divisor.

Notice that $m_{2,l}=k'_{2,l}-k'_{1,l}$ can at most take on two different values as a function of l. To see this, assume that $k'_{\nu,l}$ has been "flipped" with respect to $k'_{\nu,0}$, such that $k'_{1,l}=k'_{2,0}+j'\cdot l\mod\frac{n}{k}$ and vice versa, so $k'_{2,0}-k'_{1,0}+k'_{2,l}-k'_{1,l}=0\mod\frac{n}{k}$. Thus, $m_{2,l}$ assumes the values $m_{2,0}$ and $\frac{n}{k}-m_{2,0}$. When these "flips" occur and how many, is determined by $m_{2,0}$ and j. Such a flip also changes the direction in which we do the elementary modifications from first e^2_w to e^1_w and later back again.

On the other hand, $m_{1,l}$ will change for every step, and locally around w it will have a cycle going through $1,\dots,\frac{n}{k}$ as well. For the divisors $D_{\nu,l}$ we may observe also, that there is a global cycle coming from the fact that the greatest common divisor of the orbit lengths m divides k. That means that $\frac{n}{m}=0 \mod \frac{n}{k}$, so that $D_{\nu,q\frac{n}{m}+l}=D_{\nu,l}$.

Similarly, we see that if $c=e^{\frac{2\pi i}{m}}$ then $\tilde{\tau}_{\frac{n}{m}}=\tilde{\tau}_c$, where $\tilde{\tau}_c$ is the lift from the discussion of the action of $\mathbb{Z}/m$ above. Hence, for every multiple of $\frac{n}{m}$, an instance of L_π is multiplied to the parabolic bundle $\tilde{E}_0$.

Let us study in detail what happens in each step. These considerations are local in the sense that we consider one special orbit at a time (the global picture is somewhat more complicated). Denote by $\mathrm{em}_\nu^r(E)$ the r times iterated elementary modification of E in direction e_ν:

$$0 \longrightarrow E \xrightarrow{\ \iota_\nu\ } \mathrm{em}_\nu^1(E) \xrightarrow{\ \lambda_\nu\ } \mathbb{C}_{\pi^{-1}(y)} \longrightarrow 0$$

meaning that for the preferred frame (e_1,e_2), $\iota_\nu(e_\nu)(w)=0$. There is a well-defined inverse em_ν^{-1} which fits the short exact sequence

$$0 \longrightarrow \mathrm{em}_\nu^{-1}(E) \xrightarrow{\ \iota_\nu\ } E \xrightarrow{\ \lambda_\nu\ } \mathbb{C}_{\pi^{-1}(y)} \longrightarrow 0\ .$$

When we need to keep track of the induced lifts, we shall write $\mathrm{em}_\nu^r(W,\tilde{\tau})$.

Consider first the case where $\frac{n}{k}-k'_{\nu,l-1}<j'$ or $\frac{n}{k}-k'_{\nu,l-1}>j'$ so that no flip occurs in the next step. Then $m_{1,l}=m_{1,l-1}+j'$ and $m_{2,l}=m_{2,l-1}$, so

$$\begin{aligned}(E_{l-1},\hat{\tau}_{l-1}) &= \mathrm{em}_1^{m_{2,l-1}}(W,c^{l-1}\cdot\tilde{\tau})\otimes([-m_{1,l-1}\cdot w],\tilde{\tau}_{[-m_{1,l-1}\cdot w]}),\\ (E_l,\hat{\tau}_l) &= \mathrm{em}_1^{m_{2,l}}(W,c^{l}\cdot\tilde{\tau})\otimes([-m_{1,l}\cdot w],\tilde{\tau}_{[-m_{1,l-1}\cdot w]})\end{aligned}$$

around w, where $c=e^{\frac{2\pi i}{n}}$, and $\tilde{\tau}_{[-s\cdot w]}$ means (as allways) the canonical lift to $[-s\cdot w]$ defined on sheaves as the one that locally around $w\in\pi^{-1}(y)$ is $\tilde{\tau}^k_{[-s\cdot w]}(\sigma^s)=\sigma^s\circ\tau^{-k}$, where $\sigma(z)=z$ is the usual local generator of $[-w]$. We emphasize that the multiplication by c does not influence the elementary modifications, since we no longer use the eigenvalues of $\tilde{\tau}$ to specify the directions. Hence, $(W,c^l\cdot\tilde{\tau})=\mathrm{em}_1^{-m_{2,l-1}}(E_{l-1},\hat{\tau}_{l-1})\otimes([-D_{1,l-1}],c\cdot\tilde{\tau}_{[-D_{1,l-1}]})$ and we see that

$$(E_l,\hat{\tau}_l)=(E_{l-1},\hat{\tau}_{l-1})\otimes([-j'\cdot\pi^{-1}(y)],c\cdot\tilde{\tau}_{[-j'\cdot\pi^{-1}(y)]})$$

around $\pi^{-1}(y)$. Now, notice that $(c\cdot\tilde{\tau}_{[-j'\cdot\pi^{-1}(y)]})^k(\sigma^{j'})=e^{\frac{2\pi i}{n}k}\cdot(\sigma\circ\tau^{-k})^{j'}=e^{\frac{2\pi i}{n}k}\cdot(e^{-\frac{2\pi i}{n}kj}\cdot\sigma)^{j'}=\sigma^{j'}$, so we get a line bundle $\tilde{L}_j=[-\sum_{y\in P}j'\cdot\pi^{-1}(y)]/\langle c\cdot\tilde{\tau}_{[-\sum_{y\in P}j'\cdot\pi^{-1}(y)]}\rangle$ over Y, and therefore the above identity descends to an identity of parabolic bundles over Y,

$$\tilde{E}_l=\tilde{E}_{l-1}\otimes\tilde{L}_j.$$

To handle the flip, we need the following two lemmas:

Lemma 4.18. *In the above notation*

$$\mathrm{em}_\perp^{-1} \circ \mathrm{em}^{-1}(E, \tilde{\tau}) = (E, \tilde{\tau}) \otimes ([-\pi^{-1}(y)], \tilde{\tau}_{[-\pi^{-1}(y)]}),$$

where em *is any inverse elementary modification and* $\mathrm{em}_\perp$ *is inverse elementary modification in any other direction than the one that continues* em *(see below).*

The condition for this lemma to work is that if $\iota\colon \mathrm{em}^{-1}(E) \to E$ is the sheaf injection from the extension defining $\mathrm{em}^{-1}(E)$ and $F_w = \ker \iota_w$, for every $w \in \pi^{-1}(y)$, then $\iota_\perp\colon \mathrm{em}_\perp^{-1} \circ \mathrm{em}^{-1}(E) \to \mathrm{em}^{-1}(E)$ from the extension defining $\mathrm{em}_\perp^{-1}$ must have F in its image.

Proof. It is enough to consider the exact sequences around a point $w \in \pi^{-1}(y)$. In that case we observe that there is a commutative diagram of three exact sequences

$$\begin{tikzcd}
 & 0 & & & \\
 & \mathbb{C} \arrow[u] & & & \\
0 \arrow[r] & \mathcal{O} \oplus \mathcal{O} \arrow[u, "\lambda_2"] \arrow[r, "\left(\begin{smallmatrix} \sigma & 0 \\ 0 & 1 \end{smallmatrix}\right)"] & \mathcal{O} \oplus \mathcal{O} \arrow[r, "\lambda_1"] & \mathbb{C} \arrow[r] & 0 \\
 & \mathcal{O} \oplus \mathcal{O} \arrow[u, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & \sigma \end{smallmatrix}\right)"] \arrow[ur, "\left(\begin{smallmatrix} \sigma & 0 \\ 0 & \sigma \end{smallmatrix}\right)"'] & & & \\
 & 0 \arrow[u] & & &
\end{tikzcd}$$

where λ_ν is evaluation in $w = 0$ of the ν'th coordinate. As the sequence along the hypotenuse defines $E \otimes [-w]$, and it is clear that the lifts transform as described by the arguments of Lemma A.1, we are done. $\square$

Lemma 4.19. *When considering elementary modifications in the fiber* $\pi^{-1}(y)$ *of length* k*, we have*

$$\pi^* \mathrm{em}_\nu^1(\tilde{E}) = \mathrm{em}_\nu^{\frac{n}{k}}(\pi^* \tilde{E}, \hat{\tau}),$$

where $\hat{\tau}$ *is the natural lift induced by the pullback (the one making the isomorphism of page 71 equivariant).*

Proof. Notice first that it makes sense to talk of em_ν upstairs and downstairs along corresponding directions, since a frame (e_1, e_2) around y with preferred directions $e_1(w)$ and $e_2(w)$ lifts to a frame around $w \in \pi^{-1}(y)$ giving preferred directions there.

As with the previous lemma, it is enough to do a local examination. Consider a disk neighbourhood V of $w \in \pi^{-1}(y)$ such that the image under the projection is a biholomorphic disk U and the projection takes the local coordinates z in V to $w = z^{\frac{n}{k}}$. Then we get a commutative diagram with short exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \oplus \mathcal{O} \arrow[r, "\iota_X"] & \mathcal{O} \oplus \mathcal{O} \arrow[r, "\lambda_X"] & \mathbb{C}_w^{\frac{n}{k}} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O} \oplus \mathcal{O} \arrow[r, "\iota_Y"] \arrow[u, "\pi^*"] & \mathcal{O} \oplus \mathcal{O} \arrow[r, "\lambda_Y"] \arrow[u, "\pi^*"] & \mathbb{C}_w \arrow[r] \arrow[u, "\pi^*"] & 0
\end{tikzcd}$$

where $\pi^*\colon \mathbb{C} \to \mathbb{C}^{\frac{n}{k}}$ takes v to $(v, 0, \ldots, 0)$, $\iota_Y(f,g) = (\sigma \cdot f, g)$, $\lambda_Y(f,g) = f(0)$, $\iota_X(f,g) = (\sigma^{\frac{n}{k}} \cdot f, g)$, and $\lambda_X(f,g) = (f(0), f'(0), \ldots, f^{(\frac{n}{k}-1)}(0))$. This proves the statement. $\square$

Consider now the case where $\frac{n}{k} - k'_{1,l-1} > j'$ and $\frac{n}{k} - k'_{2,l-1} < j'$ so that a flip occurs in the next step. Then $k'_{1,l} = k'_{2,l-1} + j' - \frac{n}{k}$ and $k'_{2,l} = k'_{1,l-1} + j'$, and we get that $m_{2,l} = \frac{n}{k} - m_{2,l-1}$ and $m_{1,l} = m_{1,l-1} - m_{2,l-1} + j'$. Now, by the first lemma

$$\begin{aligned}
(E_l, \hat{\tau}_l) &= \mathrm{em}_2^{\frac{n}{k} - m_{2,l-1}}(W, c^l \cdot \tilde{\tau}) \otimes ([-m_{1,l} \cdot \pi^{-1}(y)], \tilde{\tau}_{[-m_{1,l}\cdot\pi^{-1}(y)]}) \\
&= \mathrm{em}_2^{\frac{n}{k} - m_{2,l-1}}\, \mathrm{em}_1^{-m_{2,l-1}}(E_{l-1}, \hat{\tau}_{l-1}) \\
&\qquad \otimes ([m_{2,l-1} \cdot \pi^{-1}(y)], \tilde{\tau}_{[m_{2,l-1}\cdot\pi^{-1}(y)]}) \otimes ([-j' \cdot \pi^{-1}(y)], c \cdot \tilde{\tau}_{[-j'\cdot\pi^{-1}(y)]}) \\
&= \mathrm{em}_2^{\frac{n}{k}}(E_{l-1}, \hat{\tau}_{l-1}) \otimes ([-j' \cdot \pi^{-1}(y)], c \cdot \tilde{\tau}_{[-j'\cdot\pi^{-1}(y)]})
\end{aligned}$$

From the second lemma we have that $\mathrm{em}_2^{\frac{n}{k}}(E_{l-1})/\langle \mathrm{em}_2^{\frac{n}{k}}(\hat{\tau}_{l-1})\rangle = \mathrm{em}_2(\tilde{E}_{l-1})$ so also this descends to Y as

$$\tilde{E}_l = \mathrm{em}_2(\tilde{E}_{l-1}) \otimes \tilde{L}_j,$$

where we notice that the flag in $\tilde{E}_{l,w}$ is the first direction whereas the elementary modifications are done in the second. So in a sense the flag is flipped with respect to $F_{l-1} \subset \tilde{E}_{l-1,w}$.

We observe that this makes sense also globally:

Proposition 4.20. *Let $\tilde{E}_l = \mathcal{P}(W, e^{\frac{2\pi i}{n} l} \cdot \tilde{\tau})$. There is a line bundle $\tilde{L}_\tau$ over Y and a well-defined series of elementary modifications $\mathrm{em}_{(l)}$ defined above for every special orbit, so that*

$$\tilde{E}_l = \mathrm{em}_{(l)}(\tilde{E}_{l-1}) \otimes \tilde{L}_\tau.$$

$\square$

The action of $\mathbb{Z}/n \times \mathbb{Z}/n$ in the semistable case may be analyzed in a similar fashion. In the case of semistable parabolic bundles, the analysis is in some ways simpler as the elementary modifications are done in the directions of the line bundles $L_1 \oplus L_2$, in which case the elementary modifications at a point w are done by tensoring $[w]$ onto the line bundles. We shall not bore the reader with yet another long description like the one above (nor consume the precious time of the author). Instead we refer to the calculation of the action in the hyperelliptic case of section 6.

Remark 4.21. If we added parabolic points to W outside $\bar{P}_0$, then they would be transferred to $V = W \otimes [D_1]$ and the output E of elementary modifications on V. If these parabolic structures were $\tilde{\tau}$ invariant, then they would give extra parabolic structures in $\tilde{E}$ on Y, and it is an easy exercise to see that every calculation and statement in this section carries over to this setting.

Remark 4.22. As in the unramified case this construction generalizes right away to bundles with other determinants than the trivial one. The relations determining the admissible bundles of course change, and one may actually observe that there will be cases (as for unramified coverings) where these equations have no solutions.

5. Towards a Calculation of the Witten Invariant

We have finally arrived at the point where we can begin explaining how to calculate $\operatorname{Tr}\{\tau\colon H^0(M(X),\mathcal{L}^\kappa) \to H^0(M(X),\mathcal{L}^\kappa)\}$ and its companions d_c and $CS(X_\tau, c)$. This section is meant to give a flavour of the work lying ahead.

Let $\mathcal{L} \in \operatorname{Pic}(M(X))$ be the ample generator of the cyclic group $\operatorname{Pic}(M(X))$ (cf. [56] or [24]). We will adapt the notation from [9] and write $\operatorname{Tr}(\tau | H^q(M(X),\mathcal{L}^\kappa))$ for $\operatorname{Tr}\{\tau\colon H^q(M(X),\mathcal{L}^\kappa) \to H^q(M(X),\mathcal{L}^\kappa)\}$. As the moduli space $M(X)$ is not in general smooth, we introduce a space over it which is smooth and in which we may do our calculations. Let $x \in X$ be outside the set $\bar{P}_0$ for every lift[5] and consider the moduli spaces $M(X;i)$, $0 \le i \le k(x)$, of parabolic bundles with parabolic points $x, \dots, \tau^{i-1}(x)$ and parabolic weights $a^{\nu}_{\tau^j(x)} = t_\nu$, for $\nu \in \{1, n_j\}$, $0 \le t_1 < t_{n_j} < 1$, $n_j \in \{1,2\}$. Notice that had we introduced different weights t^j on $x, \dots \tau^{k-1}(x)$, τ could not act in $M(X;k,(t^1,\dots,t^{k-1}))$.

If E is a bundle over X and L is a subbundle then

$$\operatorname{par}\mu(E) - \operatorname{par}\mu(L) = \big(\mu(E) - \mu(L)\big) + \left(\bar{P}(L)\frac{t_2 - t_1}{2} - \big(i - \bar{P}(L)\big)\frac{t_2 - t_1}{2}\right).$$

The summand in the first parenthesis is half-integer so if we make sure that the second parenthesis belongs to $]-\frac{1}{2}, 0[\cup]0, \frac{1}{2}[$ semistability of E will imply stability. That condition can be guaranteed by demanding $t_2 - t_1 < \frac{1}{i}$ and i odd. In that case $M(X;i)$ is smooth.

[5] In general this will imply that x is a generic point.

Similarly we see that if the $t_2 - t_1 < \frac{1}{i}$, there are well-defined maps

$$\varphi_i\colon M(X;i) \longrightarrow M(X;i-1)$$

got by forgetting the parabolic structure over $\tau^{i-1}(x)$ (of course stable points may degenerate to non-stable but semistable points). This makes $M(X;i)$ into a generically projective bundle over $M(X;i-1)$. Let $\Phi_i = \varphi_1 \circ \cdots \circ \varphi_i\colon M(X;i) \to M(X;0) = M(X)$. Then clearly there is an injective homomorphism

$$\Phi_i^*\colon H^0\big(M(X),\mathcal{L}^\kappa\big) \longrightarrow H^0\big(M(X;i),\Phi_i^*\mathcal{L}^\kappa\big).$$

On the other hand, since $M(X;i)$ is normal for all i (see [62, 3.IV.31]), then by [65, 5.12] $\varphi_{i*}\mathcal{O}_{M(X;i)} \cong \mathcal{O}_{M(X;i-1)}$. Inductively this leads to $\Phi_{i*}\mathcal{O}_{M(X;i)} \cong \mathcal{O}_{M(X)}$ so by the projection formula (see eg. [33])

$$\Phi_{i*}\Phi_i^*\mathcal{L}^\kappa \cong \Phi_{i*}(\mathcal{O}_{M(X;i)} \otimes \Phi_i^*\mathcal{L}^\kappa) \cong \Phi_{i*}\mathcal{O}_{M(X;i)} \otimes \mathcal{L}^\kappa \cong \mathcal{L}^\kappa,$$

and consequently we get an injective homomorphism

$$\Phi_{i*}\colon H^0\big(M(X;i),\Phi_i^*\mathcal{L}^\kappa\big) \longrightarrow H^0\big(M(X),\mathcal{L}^\kappa\big).$$

Hence, $\Phi_i^*\colon H^0\big(M(X),\mathcal{L}^\kappa\big) \to H^0\big(M(X;i),\Phi_i^*\mathcal{L}^\kappa\big)$ is an isomorphism.

Now we make a small digression into the realm of derived functors. Recall that if F is a left exact functor from an abelian category $\mathcal{A}$ with enough injectives, we may define the q'th *right derived functor* R^qF of F by taking for each object A of $\mathcal{A}$ an injective resolution

$$0 \to A \xrightarrow{\varepsilon} I^0 \longrightarrow I^1 \longrightarrow \dots$$

and defining $R^qF(A) = H^q\big(F(I^*)\big)$. There is an easy lemma:

Lemma 5.1. *If $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ are left exact functors from abelian categories and $R^qF(A) = 0$ for $q > 0$, then*

$$R^q(GF)(A) = R^qG\big(F(A)\big).$$

Proof. Under the assumption in the lemma, $F(I^*)$ is an injective resolution for $F(A)$, so $R^qG\big(F(A)\big) = H^q\big(GF(I^*)\big) = R^q(GF)(A)$. □

It follows from [33, Proposition 8.1] that for any sheaf E over a topological space M, $H^q(M,E) = R^q p_{M*}(E)$, where p_{M*} is the pushdown functor coming from the constant map $p_M\colon M \to *$ to a point $*$. Hence, if $f\colon M \to N$ satisfies that $R^q f_*(E) = 0$ for $q > 0$, it follows from the obvious commutative triangle

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "p_M"'] & & N \arrow[dl, "p_N"] \\ & * & \end{tikzcd}$$

and the above lemma that there is an identity

$$H^q(M, E) = R^q p_{M*}(E) = R^q(p_{N*}f_*)(E) = R^q p_{N*}(f_*E) = H^q(N, f_*E).$$

In particular, it is proved in [41][6] that if M is smooth, then $R^q f_*(\mathcal{O}_M) = 0$ for $q > 0$. Thus, we have that for i *odd*, $R^q\Phi_{i*}\mathcal{O}_{M(X;i)} = 0$ for $q > 0$. Hence, by the generalized projection formula, $R^q\Phi_{i*}(\Phi_i^*\mathcal{L}^\kappa) = 0$ for $q > 0$ and from the statement above, we have

$$H^q\big(M(X;i), \Phi_i^*\mathcal{L}^\kappa\big) \cong H^q\big(M(X), \mathcal{L}^\kappa\big),$$

when i is odd. According to [56], the canonical bundle $K_{M(X)}$ of $M(X)$ is $K_{M(X)} = \mathcal{L}^{-2}$, so $\mathcal{L}^\kappa \otimes K^*_{M(X)} = \mathcal{L}^{\kappa+2}$ is ample. Thus,

$$H^q\big(M(X), \mathcal{L}^\kappa\big) = H^q\big(M(X), K_{M(X)} \otimes \mathcal{L}^\kappa \otimes K^*_{M(X)}\big) = 0$$

for all $q > 0$ and $\kappa > 0$ by [63, Theorem 7.80].

If $k = k(x)$ is odd we have the equivariant map $\varphi = \Phi_k\colon M(X;k) \to M(X)$ giving the equivariant isomorphism $\varphi^*\colon H^0\big(M(X;k), \varphi^*\mathcal{L}^\kappa\big) \to H^0\big(M(X), \mathcal{L}^\kappa\big)$. Hence, $\operatorname{Tr}\big(\tau\big|H^0\big(M(X), \mathcal{L}^\kappa\big)\big) = \operatorname{Tr}\big(\tau\big|H^0\big(M(X;k), \varphi^*\mathcal{L}^\kappa\big)\big)$, which, since

$$\operatorname{Tr}\big(\tau\big|H^0\big(M(X;k), \varphi^*\mathcal{L}^\kappa\big)\big) = \sum_q (-1)^q \operatorname{Tr}\big(\tau\big|H^q\big(M(X;k), \varphi^*\mathcal{L}^\kappa\big)\big),$$

can be calculated using the Atiyah-Segal Lefschetz fixed point theorem, [9, Theorem 3.3],

$$\sum_q (-1)^q \operatorname{Tr}\big(\tau\big|H^q\big(M(X;k), \varphi^*\mathcal{L}^\kappa\big)\big) = \chi\left(|M(X;k)|, \frac{[(\varphi^*\mathcal{L}^\kappa)|_{|M(X;k)|}](\tau)}{\lambda_{-1}(N^*)(\tau)}\right),$$

where N is the complex normal bundle to $|M(X;k)|$ in $M(X;k)$ and where "(τ)" means evaluation in τ of the elements in the localized K-rings. Alternatively, one can use the Atiyah-Singer version, [10, (4.4)],

$$\begin{aligned}&\sum_q (-1)^q \operatorname{Tr}\big(\tau\big|H^q\big(M(X;k), \varphi^*\mathcal{L}^\kappa\big)\big)\\ &\qquad = \left\{\frac{\operatorname{ch}\big((\varphi^*\mathcal{L}^\kappa)|_{|M(X;k)|}\big)(\tau)\operatorname{Td}\big(|M(X;k)|\big)}{\lambda_{-1}(N^*)(\tau)}\right\}\big[|M(X;k)|\big].\end{aligned}$$

[6] The author is thankful to T.R. Ramadas and V.B. Mehta for pointing out this reference.

Another method which may be used instead of the extra parabolic points in the situation where τ has a 1-point orbit, x, is the Hecke correspondence. This is closely related to the $k=1$ case in the above. No weights are used but the flag is used to construct a morphism $\rho\colon M_{[x]}(X;1) \to M(X)$ via elementary modifications. This gives the Hecke diagram

$$\begin{tikzcd} & M_{[x]}(X;1) \arrow[dl, "\varphi"'] \arrow[dr, "\rho"] & \\ M_{[x]}(X) & & M(X) \end{tikzcd}$$

which by our construction is equivariant. It turns out as above that $H^q(M(X), \mathcal{L}^\kappa) \cong H^q(M_{[x]}(X), \varphi_*\rho^*\mathcal{L}^\kappa)$ and all cohomology vanishes for $q \geq 1$. This means we can use this for a Lefschetz fixed point calculation as above. For further details see [16].

Recall that the fundamental group of a mapping torus X_τ is the semi-direct product

$$\pi_1(X_\tau) = \langle (\alpha, n) \in \pi_1 X \times \mathbb{Z} \mid (*, 1)(\alpha, n)(*, -1) = (\tau_*\alpha, n) \rangle.$$

If we identify the moduli space $M(Z)$ with the representations of $\pi_1(Z)$ modulo conjugation, $\mathrm{Hom}\big(\pi_1(Z), SU(2)\big)/SU(2)$, this gives a natural surjective map

$$r\colon M(X_\tau) \longrightarrow |M(X)|$$

given by restriction. It turns out that the fiber $r^{-1}(\rho)$ can be identified with $Z_\rho \cdot \rho / Z_\rho$ where Z_ρ is the stabilizer of ρ which acts on $Z_\rho \cdot \rho$ by conjugation. In particular, r is a 2-sheeted cover over the irreducible representations. The map r also gives a map from the connected components C_τ of $M(X_\tau)$ to the connected components C of $|M(X)|$, and one can show that for $SU(2)$, the fiber has at most 2 points (see [1]).

Denote by $M(X_\tau)_c$ the space $r^{-1}(|M(X)|_c)$. J.E. Andersen derives through a Mayer-Vietoris argument that the dimension

$$d_c = \operatorname*{generic\,max}_{A \in M(X_\tau)_c} \frac{1}{2}\big(\dim H^1(X_\tau, d_A) - \dim H^0(X_\tau, d_A)\big)$$

is given by the dimension of the component in the fixed point set:[7]

$$d_c = \dim_{\mathbb{C}} |M(X)|_c.$$

[7]This argument uses the interpretation of $M(X_\tau)$ as the moduli space of flat connections.

As noted before, generic max, means the maximum obtained on Zariski open subsets.

He also shows that the Chern-Simons invariant (mod $\mathbb{Z}$) on $M(X_\tau)$ is constant on $r^{-1}(|M(X)|_c)$ and that it is given by

$$e^{2\pi i\kappa CS(X_\tau,c)} = \operatorname{Tr}\left(\tau\colon \mathcal{L}^\kappa_{[E]} \longrightarrow \mathcal{L}^\kappa_{[E]}\right)$$

for any class $[E] \in |M(X)|$. Now, the determinant line bundle can be written in E as

$$\mathcal{L}_{[E]} = \det H^1(X,E) \otimes \det H^0(X,E)^*,$$

so given our presentation of the fixed point set we should be equipped with the tools to calculate the exponent of the Chern-Simons:

First observe that $H^0(X,E) = 0$ when E is stable and of degree 0. To see this, assume conversely that $s \in H^0(X,E) - \{0\}$ and consider the divisor $Z(s)$ of zeros for s. Now, $\mathcal{O}(Z(s)) = \{\, f \in \mathcal{M}(X) \mid Z(s) + (f) \geq 0 \,\}$ and one sees that there is an injection of bundles $[Z(s)] \hookrightarrow E$ induced by the homomorphism $\mathcal{O}(Z(s)) \to E$ which sends f to $f \cdot s$. But the number of zeros for s is $|Z(s)| = \deg[Z(s)] < \mu(E) = 0$, which is a contradiction.

Therefore, the Lefschetz fixed point formula, [8, Theorem 4.12], gives

$$\operatorname{Tr} H^1(\tau^l, \tilde{\tau}^l) = -\sum_{x\in X,\tau^l(x)=x} \frac{\operatorname{Tr}\tilde{\tau}^l_x}{\det_{\mathbb{C}}(1-d_x\tau^l)},$$

which can be used to calculate $\det H^1(\tau,\tilde{\tau})$. Now, $\det H^1(\tau, c\cdot\tilde{\tau}) = c^{2(g-1)} \cdot \det H^1(\tau,\tilde{\tau})$, where $c^n = 1$ is a change of lift and $2(g-1) = \dim H^1(X,E)$ by Riemann-Roch. Hence, we cannot expect this to be independent of the lift! — We have ideas how one may solve this by making a globally consistent choice of lift, but time does not permit us to explore that presently. See section 6 for a simple example.

These numbers are calculated for all Seifert 3-manifolds by D.R. Auckly, [11], so they are known. But Auckly's formulae relies on a list of representations of the fundamental group which satisfies that there will be at least one member of the list in every component of the moduli space. From our perspective however, it is not clear which; so it will not only be the most beautiful construction but it will also be the most convenient if we can find an intrinsic way of calculating the Chern-Simons invariant.

6. The Hyperelliptic Involution

Let X be a compact, hyperelliptic Riemann surface of genus $g \geq 2$ with a hyperelliptic involution J. Then $\pi\colon X \to X/\langle J\rangle = \mathbb{P}^1$ and the set $\bar{P}$ of fixed points of J in X are the Weierstrass points $w_1, \dots w_{2g+2}$. We wish to compute the fixed point set in the moduli space $M(X)$ of semistable holomorphic bundles of trivial determinant under the action of J.

Our first step will be to calculate the admissible parabolic bundles for (X, J); Theorem 4.16 and Corollary 4.9 will do the rest. Every holomorphic bundle $\tilde{E}$ over $\mathbb{P}^1$ splits into a sum of line bundles $\tilde{L}_\nu$ each of which are isomorphic to some $\mathcal{O}(r_\nu)$ for some $r_\nu \in \mathbb{Z}$. Let H be the hyperelliptic bundle over X defined as $H = [x + J(x)]$ for any $x \in X$; in particular $H = [2w_j]$ for every $w_j \in \bar{P}$, and $\pi^*\mathcal{O}(1) = \pi^*[\pi(x)] = [x + J(x)] = H$ for some $x \in X$. Also, every fixed point w_j has a neighbourhood U_{w_j} with local coordinates z in which $J(z) = -z$. In particular, we notice that for every special orbit, $k = j = 1$.

Let $(\tilde{E}, \tilde{F}, a)$ be a parabolic bundle over $\mathbb{P}^1$, and define the subsets

$$\bar{P}_1 = \{\, w \in \bar{P} \mid a^1_{\pi(w)} \neq 0 \,\},$$
$$\bar{P}_0 = \{\, w \in \bar{P} \mid a^1_{\pi(w)} \neq a^2_{\pi(w)} \,\}.$$

Characterizing the admissible parabolic bundles $\tilde{E}$ means determining which subsets $\bar{P}_1$ and $\bar{P}_0$ of $\bar{P}$ yield a holomorphic bundle W with lift $\tilde{J}$, whose matrix in $\bar{P}$ has the values

$$A(w) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{for } w \in \bar{P} - (\bar{P}_0 \cup \bar{P}_1),$$
$$A(w) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{for } w \in \bar{P}_1,$$
$$A(w) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{for } w \in \bar{P}_0.$$

Put $k_\nu(w) = 2a^\nu_{\pi(w)}$, $m_1(w) = k_2(w)$ and $m_2(w) = k_2(w) - k_1(w)$. Then if such a lift $(W, \tilde{J})$ exists, the bundle is $W = V \otimes [\bar{P}_1]$ where V is a bundle that fits into an exact sequence

$$0 \longrightarrow V \xrightarrow{\ \iota\ } E \xrightarrow{\ \lambda\ } F \longrightarrow 0$$

from section 4, where $E = \pi^*\tilde{E}$. Now, $\pi^*\tilde{E} = \pi^*\big(\mathcal{O}(r_1) \oplus \mathcal{O}(r_2)\big) = H^{r_1} \oplus H^{r_2}$ and $\det \pi^*\tilde{E} = \det E = \det V \otimes [\bar{P}_0]$ so

$$\det W = \det V \otimes [2\bar{P}_1] = H^{r_1 + r_2 + |\bar{P}_1|} \otimes [-\bar{P}_0].$$

On the other hand, $\det W = \mathcal{O}_X$, hence, we get a numerical criterion that $\deg W = 2(r_1 + r_2 + |\bar{P}_1|) - |\bar{P}_0| = 0$. In particular, the numerical criterion forces that $|\bar{P}_0|$

is even. We must now determine the $\bar{P}_0$ for which $H^{\frac{|\bar{P}_0|}{2}} \otimes [-\bar{P}_0] = \mathcal{O}_X$. Since $H = [w_0]^2$ for any $w_0 \in \bar{P}$, we may write

$$H^{\frac{|\bar{P}_0|}{2}} \otimes [-\bar{P}_0] = [w_0]^{2\frac{|\bar{P}_0|}{2}} \otimes \bigotimes_{w \in \bar{P}_0} [-w] = \bigotimes_{w \in \bar{P}_0} [w_0 - w].$$

Notice that this line bundle $L = \bigotimes_{w \in \bar{P}_0} [w_0 - w]$ is of order 2 so that L is in the 2-torsion points $J_2(X)$ of the Jacobian $J(X)$.

According to [5], the line bundles $[w_i - w_j]$ correspond, under the isomorphism $J_2(X) \cong H_1(X, \mathbb{Z}/2)$, to the homology class η_{ij} constructed by connecting $\pi(w_i)$ and $\pi(w_j)$ by a curve λ_{ij} which does not intersect any other ramification points, and lifting it to form a closed curve γ_{ij} through w_i and w_j (see fig. 1 below).

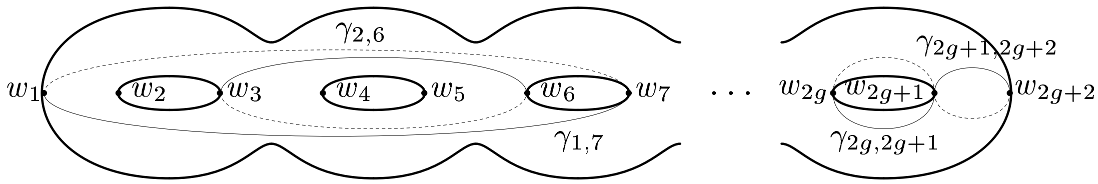

FIGURE 1. Weierstrass points and lifted curves γ_{ij} on X.

Let $a_1, b_1, \ldots, a_g, b_g$ be the symplectic standard basis for $H_1(X, \mathbb{Z}/2)$ displayed in fig. 2 below. To compactify the notation in the subsequent calculations define $a_0 = b_0 = a_{g+1} = b_{g+1} = 0$.

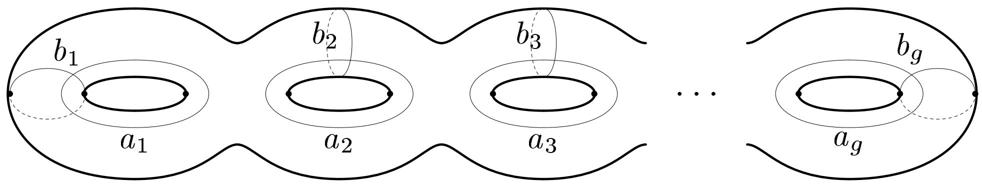

FIGURE 2. The standard basis for $H_1(X, \mathbb{Z}/2)$.

We observe the following formulae for intersection pairing

$$\eta_{2i,2j} \cdot a_k = \begin{cases} 0 \mod 2, & \text{for } k \neq i \text{ and } k \neq j, \\ 1 \mod 2 & \text{for } k = i \text{ or } k = j, \end{cases}$$

where $\eta_{2i,2j} \cdot a_k = \eta_{2i+1,2j} \cdot a_k = \eta_{2i,2j+1} \cdot a_k = \eta_{2i+1,2j+1} \cdot a_k$, and

$$\eta_{2i,2j} \cdot b_k = \begin{cases} 0 \mod 2, & \text{for } k < i \text{ and } k \geq j, \\ 1 \mod 2 & \text{for } i \leq k < j, \end{cases}$$
$$\eta_{2i+1,2j} \cdot b_k = \begin{cases} 0 \mod 2, & \text{for } k \leq i \text{ or } k \geq j, \\ 1 \mod 2 & \text{for } i < k < j. \end{cases}$$
$$\eta_{2i,2j+1} \cdot b_k = \begin{cases} 0 \mod 2, & \text{for } k < i \text{ or } k > j, \\ 1 \mod 2 & \text{for } i \leq k \leq j. \end{cases}$$
$$\eta_{2i+1,2j+1} \cdot b_k = \begin{cases} 0 \mod 2, & \text{for } k \leq i \text{ or } k > j, \\ 1 \mod 2 & \text{for } i < k \leq j. \end{cases}$$

Now, for $1 \leq j \leq 2g+2$ define the integer

$$[j] \stackrel{\text{def}}{=} \begin{cases} \frac{j}{2} & \text{for } j \text{ even}, \\ \frac{j-1}{2} & \text{for } j \text{ odd}. \end{cases}$$

Then if δ is the Krohnecker delta, we see from the intersection pairing formulea that

$$\eta_{j,k} = b_{[j]} + b_{[k]} + \delta_{2[j],j} \cdot a_{[j]} + \sum_{i=[j]+1}^{[k]-1} a_i + \delta_{2[k]+1,k} \cdot a_{[k]}.$$

We are looking for a characterization of the sets $\bar{P}_0$ of even length such that for any given k, $\bigotimes_{w_j \in \bar{P}_0} [w_k - w_j] = \mathcal{O}_X$ or equivalently, that $\sum_{w_j \in \bar{P}_0} \eta_{k,j} = 0$. Using the above expression for $\eta_{k,j}$, we see that the b-part of $\sum_{w_j \in \bar{P}_0} \eta_{k,j}$ is

$$\mathrm{pr}_b \left(\sum_{w_j \in \bar{P}_0} \eta_{k,j} \right) = 2N \cdot b_{[k]} + \sum_{w_j \in \bar{P}_0} b_{[j]} = \sum_{w_j \in \bar{P}_0} b_{[j]},$$

where $2N = \bar{P}_0$. Now, since this has to be zero, each $b_{[j]}$ in the sum has to occur an even number of times, i.e. twice, which means that the j's have to occur in pairs $2i$, $2i+1$, or in the case of $j = 1$ (or $j = 2g+2$ respectively) $a_{[j]} = b_{[j]} = 0$ so they too are allowed as a pair. Thus, with respect to the b-part admissible sets are of the forms

$$\bar{P}_0 = \{\, w_{2i_1}, w_{2i_1+1}, w_{2i_3}, \ldots, w_{2i_N}, w_{2i_N+1} \,\},$$
$$\bar{P}_0 = \{\, w_1, w_{2i_1}, w_{2i_1+1}, w_{2i_3}, \ldots, w_{2i_{N-1}}, w_{2i_{N-1}+1}, w_{2g+2} \,\}.$$

Notice that for such pairs we have that

$$\eta_{k,2i} + \eta_{k,2i+1} = 2\left(b_{[k]} + b_i + \delta_{2[k],k} \cdot a_{[k]} + \sum_{j=[k]+1}^{i-1} a_j\right) + \delta_{2i+1,2i+1} \cdot a_i = a_i,$$

$$\eta_{1,k} + \eta_{k,2g+2} = 2b_{[k]} + \sum_{i=1}^{[k]-1} a_i + \delta_{2[k]+1,k} \cdot a_{[k]} + \delta_{2[k],k} \cdot a_{[k]} + \sum_{i=[k]+1}^{g} a_i = \sum_{i=1}^{g} a_i,$$

so the a-part of the sum $\sum_{w_j \in \bar{P}_0} \eta_{k,j}$ is

$$\mathrm{pr}_a\left(\sum_{w_j \in \bar{P}_0} \eta_{k,j}\right) = \sum_{\substack{w_j \in \bar{P}_0 \\ j \text{ even}}} a_{[j]} + \begin{cases} \sum_{i=1}^{g} a_i, & \text{if } w_1, w_{2g+2} \in \bar{P}_0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, as the a_i's constitute half of a symplectic basis, the only possibilities for $\bar{P}_0$ is to be all of $\bar{P}$ or the empty set.

Now, if $\bar{P}_0 = \emptyset$, $W \otimes [-\bar{P}_1] = V = L_1 \oplus L_2$ where L_ν are invariant bundles (as there are one-point orbits and therefore no ramified part) with $\deg L_\nu = \mu(V) = -|\bar{P}_1|$. If $\mathcal{O}(r_\nu)$ are the line bundles over $\mathbb{P}^1$ such that $\pi^*\mathcal{O}(r_\nu) = L_\nu$ then we see that $|\bar{P}_1| = -2 \cdot r_\nu$ is even. Let us record our findings in the following proposition.

Proposition 6.1. *Let X be a hyperelliptic Riemann surface of genus $g \geq 2$ and let J be a hyperelliptic involution. Then the admissible parabolic bundles for (X, J) over $\mathbb{P}^1$ consist of*

(1) *holomorphic vector bundles with trivial parabolic structures in every image $y \in P$ of a Weierstrass point with an even number of each of the weights 0 and $\frac{1}{2}$ and degree $2r = -|\{a_y \mid a_y = \frac{1}{2}\}|$ (i.e. $\operatorname{par}\deg = 0$), and*
(2) *the parabolic bundles with the pair of weights $(0, \frac{1}{2})$ in every $y \in P$ and degree $d = g + 1$ (and $\operatorname{par}\deg = 2d$), where non-stable bundles have flags respecting the splitting into line bundles of the base vector bundle.* □

We denote the latter weight configuration $a = \{(0, \frac{1}{2}), \ldots, (0, \frac{1}{2})\}$ by $\underline{(0, \frac{1}{2})}$.

Without too much effort, we can extract information about these moduli spaces of parabolic bundles: First notice that if a is a trivial, parabolic structure then the stability condition is preserved when forgetting the parabolic structure; thus $M_d(Y, a) = M_d(Y)$. Hence, for each of the 2^{2g+1} trivial parabolic structures a of the proposition, we have

$$M_d(\mathbb{P}^1; a) = M_d(\mathbb{P}^1) = \{[\mathcal{O}(r) \oplus \mathcal{O}(r)]\},$$

as $r = -\frac{1}{2}|\{a_y \mid a_y = \frac{1}{2}\}|$ is even.

Then consider the moduli space corresponding to the weight configuration $\underline{a = (0, \frac{1}{2})}$. This is a bit harder, and we will settle for the components, their dimension, and the associated Chern-Simons invariant.

Let W be a fixed point for J and $\tilde{J}$ a lift of J. Let E be a bundle over X got through the elementary modifications dictated by $\tilde{J}$. Then $\deg E = 2g + 2$ and the orbit bundle $\tilde{E}$ has degree $d = \deg \tilde{E} = g + 1$ and $\operatorname{par} \mu(\tilde{E}) = g + 1$. Now, because we are over $\mathbb{P}^1$ there is an integer r such that $\tilde{E} \cong \mathcal{O}(r) \oplus \mathcal{O}(d - r)$. As we are only considering bundles up to isomorphism (and strong equivalence), we may assume that $r \geq \frac{1}{2}d \geq d - r$. If $\tilde{L}$ is a parabolic subbundle of $\tilde{E}$ then $\operatorname{par}\deg(\tilde{L}) = \deg(\tilde{L}) + d - \frac{1}{2}|P(\tilde{L})|$, so the semi-stability relation becomes

$$\mu(\tilde{L}) \leq \frac{|P(\tilde{L})|}{2}$$

with "$<$" substituted for "$\leq$" if we want stability. In particular, we notice that $\mu(\tilde{L}) \leq \frac{1}{2}|P(\tilde{L})| \leq g + 1 = d$, so as $\mathcal{O}(r)$ is a subbundle we must have $\frac{1}{2}d \leq r \leq d$.

Say we want to prove that for each $\frac{1}{2}d \leq r \leq d$, there exist stable parabolic vector bundles whose underlying vector bundle is $\mathcal{O}(r) \oplus \mathcal{O}(d - r)$. Then we need to prove that it is possible to choose a flag F in $(\mathbb{P}^1)^{2d}$ corresponding to the $2d$ parabolic points, such that $s < \frac{1}{2}|P(\varphi(\mathcal{O}(s)))|$ for every $s \leq r$ and every embedding φ of $\mathcal{O}(s)$.

The bundle $\tilde{L}$ may be written as $\mathcal{O}(s)$ for some integer s. As there are no homomorphism from a bundle into a bundle with lower degree, we see that $s \leq r$, and we notice that for $s < 0$ the relation is trivially fulfilled. We also notice that if $d - r < s \leq r$, s has to be r. Otherwise, the morphism $\mathcal{O}(s) \to \mathcal{O}(r) \oplus \{0\}$ will have zero-points and will not be an embedding. Hence, we need only consider $0 \leq s \leq d - r$. Under these circumstances we have

$$\operatorname{Hom}\big(\mathcal{O}(s), \mathcal{O}(t)\big) \cong H^0\big(\mathcal{O}(t - s)\big) \cong \mathbb{C}^{t-s+1},$$

for $t = d - r$ or $t = r$. For any $t \in \mathbb{Z}$, we can take $\mathcal{O}(t)$ to be the meromorphic functions f on $\mathbb{P}^1$ which have a pole of order at most t in ∞ and which are holomorphic elsewhere. In this picture $H^0\big(\mathcal{O}(t)\big)$ consists of polynomials $\rho(z) = a(z - z_1)\dots(z - z_{t'})$, $t' \leq t$, and any homomorphism $\varphi \in \operatorname{Hom}\big(\mathcal{O}(s), \mathcal{O}(t)\big)$ can be written as $\varphi(f) = \rho \cdot f$ for some polynomial ρ of the mentioned type. This means there are $d + 1 - 2s$ degrees of freedom to choose a degree s line bundle inside $\mathcal{O}(r) \oplus \mathcal{O}(d - r)$.

Given a flag F in $\mathcal{O}(r) \oplus \mathcal{O}(d - r)$ over the $2d$ points P, we ask ourselves if it is unstable, i.e. if there is a degree s line bundle (for some s) that hits this flag in at least $2(d - s) + 1$ points. This is a question of solving finitely many (actually

$\binom{2d}{2(d-s)+1}$ many) systems of $2(d-s)+1$ equations with $d+1-2s$ unknowns. This is over-determined by d equations, so as $d \geq 3$ there will generically be no solutions to this problem. Since we are considering only finitely many s's this implies that a generic flag F will be a stable flag, and the set of stable flags is connected.

Hence, for each $\frac{1}{2}d \leq r \leq d$ there is a non-empty moduli space of parabolic bundles whose underlying holomorphic bundle is $\mathcal{O}(r) \oplus \mathcal{O}(d-r)$, and these moduli spaces contain stable points. Notice that, as the codimension of the critical variety is at least 3, these moduli spaces are connected (for each fixed r). The dimension of these components is $k^2(\gamma-1)+1+\dim \mathcal{F}$, where k is the rank, γ the genus of the base surface, and $\dim \mathcal{F}$ is the dimension of the underlying flag-manifold ([47, Theorem 5.3]). Thus, in our case the dimension is $2g-1$. Denote the component of $M\big(X; \underline{(0,\frac{1}{2})}\big)$ with underlying bundle isomorphic to $\mathcal{O}(r) \oplus \mathcal{O}(d-r)$ by $M\big(X; \underline{(0,\frac{1}{2})}\big)_r$. There are $\binom{2d}{2r}$ non-stable points in $M\big(X; \underline{(0,\frac{1}{2})}\big)_r$ corresponding to the flags with $2r$ flags in $\mathcal{O}(d-r)$ and $2(d-r)$ in $\mathcal{O}(r)$ (though only $\frac{1}{2}\binom{2d}{d}$ non-stable flags when $r = d-r$).

Now, let us consider the action of $\mathbb{Z}/2 \times \mathbb{Z}/2$ and of $\mathbb{Z}/2$ which determine the fixed point set. Here $j = j' = 1$ for all points P.

Consider first the action of $(-1,-1)$ respectively -1. Clearly the isolated semi-stable points $\mathcal{O}(-r) \oplus \mathcal{O}(-r)$, $r = \frac{1}{2}|\bar{P}_1|$, are sent to $\mathcal{O}(r-d) \oplus (r-d)$.

For the components containing stable points, all the weights are $(0,\frac{1}{2})$, so we have that

$$\begin{aligned} k_1 &= k_1' = 0, & k_2 &= k_2' = 1, \\ m_1 &= 0, & m_2 &= 1 \end{aligned}$$

always and in every special point $w \in \bar{P}$, but we have a flip in every step. From Proposition 4.20 it is known that the two elements $\tilde{E}_1$ and $\tilde{E}_2$ of a $\mathbb{Z}/2$-orbit are related by

$$\tilde{E}_2 = \mathrm{em}_{2,P}(\tilde{E}_1) \otimes \tilde{L}_\tau,$$

where the elementary modifications are done in every point of P and in the opposite direction than those that are specified by the flag of $\tilde{E}_1$, and $\tilde{L}_\tau = \mathcal{O}(-d)$, $d = g+1$ is the line bundle of degree $-d$.

The parabolic moduli space with the given weight configuration and underlying bundle $\mathcal{O}(r) \oplus \mathcal{O}(d-r)$, $\frac{1}{2}d \leq r \leq d$, is connected, so by tracing the action on semistable points we gain information on how $\mathbb{Z}/2$ treats the component. Let $\tilde{E}_1$ be such a bundle, i.e. the flag sits with $2(d-r)$ lines in $\mathcal{O}(r)$ and $2r$ lines in $\mathcal{O}(d-r)$. The elementary modifications em_1 have the flag sitting as the image of the injection of the short exact sequence, so under em_2 it must be killed. Thus,

$$\tilde{E}_2 = \big(\mathcal{O}(r+2(d-r)) \oplus \mathcal{O}(d-r+2r)\big) \otimes \mathcal{O}(-d) = \mathcal{O}(d-r) \oplus \mathcal{O}(r)$$

with the flag flipped. Hence under the canonical isomorphism $\tilde{E}_1 \cong \tilde{E}_2$, the flags agree, so the semistable points of these components are fixed by $(-1,-1) \in \mathbb{Z}/2 \times \mathbb{Z}/2$.[8] Therefore, $\mathbb{Z}/2$ acts within the same component of the moduli space (when the component contains stable points), and the action is free on stable points.

We now investigate the action of $\zeta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ on non-stable points. Let us start with a bundle $\tilde{E}_1$ with trivial parabolic structure corresponding to the lift $(W, \tilde{J})$ on X with the following data: over the points $\bar{P} - \bar{P}_1$, $k'_{1,0} = k'_{2,0} = 0$ and over $\bar{P}_1$, $k'_{1,0} = k'_{2,0} = 1$, so $D_{1,0} = -\bar{P}_1$ and $D_{2,0} = 0$. Under the action by ζ, the data over points in $\bar{P}_1$ are sent to $k'_{1,1} = 0$ and $k'_{2,1} = 1$ with respect to first and second direction, and to $k'_{1,1} = 0$ and $k'_{2,1} = 1$ over $\bar{P} - \bar{P}_1$, but with respect to second and first direction. Thus, $D_{1,1} =$ and $D_{2,1} = \bar{P}$. This gives

$$
\begin{aligned}
(E_1, \hat{\tau}_1) &= (W, \tilde{\tau}) \otimes ([-\bar{P}_1], \tilde{\tau}_{[-\bar{P}_1]}), \\
(E_2, \hat{\tau}_2) &= \mathrm{em}_{1,\bar{P}-\bar{P}_1} \circ \mathrm{em}_{2,\bar{P}_1}(W, \zeta \circ \tilde{\tau}) \\
&= \mathrm{em}_{1,\bar{P}-\bar{P}_1} \circ \mathrm{em}_{2,\bar{P}_1}(E_1, \zeta \circ \hat{\tau}_1) \otimes ([\bar{P}_1], \tilde{\tau}_{[\bar{P}_1]}) \\
&= \mathrm{em}^2_{2,\bar{P}_1} \circ \mathrm{em}_{1,\bar{P}}(E_1, \zeta \circ \hat{\tau}_1).
\end{aligned}
$$

By the arguments of section 4 this descends to the holomorphic vector bundle $\tilde{E}_2 = \mathrm{em}_{1,\bar{P}_1}\big(\mathrm{em}_{2,\bar{P}}(E_1)/\,\mathrm{em}_{2,\bar{P}}(\zeta \circ \tilde{\tau}_1)\big)$ over $\mathbb{P}^1$. However, the underlying bundle for $\tilde{E}_1$ is $\mathcal{O}(-r) \oplus \mathcal{O}(-r)$ so $E_1 = H^{-r} \oplus H^{-r}$. The directions stated above tells us how to do the elementary modifications on E_1 so $\mathrm{em}_{2,\bar{P}}(E_1) = (H^{-r} \otimes [\bar{P}]) \oplus H^{-r}$ and

$$\mathrm{em}_{2,\bar{P}}(E_1)/\,\mathrm{em}_{2,\bar{P}}(\zeta \circ \tilde{\tau}_1) = \mathcal{O}(d-r) \oplus \mathcal{O}(-r).$$

Hence,

$$\zeta \cdot \big(\mathcal{O}(-r) \oplus \mathcal{O}(-r)\big) = \mathcal{O}(d-r) \oplus \mathcal{O}(r),$$

where $d = g + 1$ and the last bundle has a flag F with $F_w \subset \mathcal{O}(r)$ when $w \in \bar{P} - \bar{P}_1$, and $F_w \subset \mathcal{O}(d-r)$ when $w \in \bar{P}_1$. Having described the action by the two generators $(-1,-1)$ and ζ for $\mathbb{Z}/2 \times \mathbb{Z}/2$, we have a description for the complete action. We observe that all the one-point components of $Ma(\mathbb{P}^1)$ get identified with the semistable points in the larger components under the equivalence relation $\sim$. Hence, the connected components of $Ma(\mathbb{P}^1)/\sim$ are precisely parameterized by $\frac{1}{2}d \leq r \leq d$.

Following the recipe on page 14 in [1], we see that X_J is a Seifert 3-manifold with Seifert invariants:

$$\big(b, g', (\alpha_1, \beta_1), \ldots, (\alpha_s, \beta_s)\big) = \big(-(g+1), 0, (2,1), \ldots, (2,1)\big)$$

[8] This corresponds to the two lifts $(W, \tilde{\tau}) = (L, \tilde{\tau}_L) \oplus (L, -\tilde{\tau}_L)$ and $(W, -\tilde{\tau}) = (L, -\tilde{\tau}_L) \oplus (L, \tilde{\tau}_L)$ being equivalent.

and $s = 2g+2$.

Then, in order to calculate the Chern-Simons invariant, we only need to look at those of Auckly's representations $\omega(\varepsilon, n_1, \dots n_{2g+2})[g_1, \dots, g_{2g+2}]\colon \pi_1(X_J) \to Sp(1)$, where $n_j = 0$ or 1, and $\varepsilon = 0$ or $\frac{1}{2}$ (see [11]). Because of a relation in $\pi_1(X_J)$,

$$(e^{2\pi i \varepsilon})^{-(g+1)} e^{\pi i \sum_{j=1}^{2g+2}(n_j+\varepsilon)} = 1$$

so $e^{\pi i \sum_{j=1}^{2g+2} n_j} = (e^{2\pi i \varepsilon})^{g+1}$, and by the theorem page 232 in [11], we have

$$e^{2\pi i CS(\omega)} = e^{-2\pi i \sum_{j=1}^{2g+2}(n_j^2 + 2n_j\varepsilon)} = 1$$

for all $[\omega] \in M(X_J)$.

If on the other hand we use the formula $e^{2\pi i CS(X_J, c)} = \operatorname{Tr}(J|\mathcal{L}_{[E]})$, where $\mathcal{L}_{[E]} = \det H^1(X,E) \otimes \det H^0(X,E)^*$, we get a different result: Let $E \in |M^s(X)|$ and choose a lift $\tilde{J}$ of J to E. Then by the Lefschetz fixed point formula, [8, Theorem 4.12]

$$\operatorname{Tr} H^0(J,\tilde{J}) - \operatorname{Tr} H^1(J,\tilde{J}) = \sum_{w\in\bar{P}} \frac{\operatorname{Tr}\tilde{J}_w}{\det_{\mathbb{C}}(1-d_wJ)} = 0.$$

And Riemann-Roch tells us that

$$\dim H^0(X,E) - \dim H^1(X,E) = 2(1-g) + \deg E = 2(1-g).$$

Now, since E is stable, $H^0(X,E) = 0$, so we have that $\dim H^1(X,E) = 2(g-1)$. Clearly, $H^1(J,\tilde{J})$ has order 2 on $H^1(X,E)$ so we can assume it is diagonal with ± 1 on the diagonal. As $\operatorname{Tr} H^1(J,\tilde{J}) = 0$, there must be equally many $+1$'s as -1's. Hence

$$\operatorname{Tr}(J|\mathcal{L}_{[E]}) = \det H^1(J,\tilde{J}) = (-1)^{g-1}.$$

Notice since the order of the automorphism divides $\dim H^1(X,E)$, then $\operatorname{Tr}(J|\mathcal{L}_{[E]})$ is independent of the choice of lift $\tilde{J}$. That makes it even more puzzling that what seems like a natural lift to $\mathcal{L}$ does not give the right answer.

We conclude:

Proposition 6.2. *Let X be a hyperelliptic Riemann surface of genus $g \geq 1$ and let J be a hyperelliptic involution. The fixed point set $|M(X)|$ is decomposed into connected components*

$$|M(X)| = Ma(X,J)/\pm 1 = \coprod_{\frac{d}{2}\leq r\leq d} \left(M\big(X; \underline{(0,\tfrac{1}{2})}\big)_r / \pm 1 \right).$$

Each component $M\big(X;\underline{(0,\tfrac{1}{2})}\big)_r$ *has dimension*

$$d_r = 2g - 1.$$

There are $\binom{2d}{2r}$ *non-stable points corresponding to the flags with* $2r$ *flags in* $\mathcal{O}(d-r)$ *and* $2(d-r)$ *in* $\mathcal{O}(r)$ *when* $r > d-r$, *and* $\frac{1}{2}\binom{2d}{d}$ *non-stable flags when* $r = d-r$. *Moreover, on every component the exponent of the Chern-Simons functional is*

$$e^{2\pi i\kappa CS(X_J,r)} = 1.$$

APPENDIX A. PROOFS OF ELEMENTARY MODIFICATION RESULTS

In the present section we give the proofs of results regarding elementary modifications and inverse elementary modification. We prove the fundamental Lemma 4.1 and Lemma 4.4. First we consider the matter locally:

Lemma A.1. *Let* $U \subseteq \mathbb{C}$ *be a disk around* $0 \in \mathbb{C}$ *furnished with an automorphism* $\tau(z) = e^{\frac{2\pi i}{n}j}\cdot z$, *for some* j *prime to* n. *Then there exists a short exact sequence as rows in the diagram below, and if* $\tilde{\tau}\colon \mathcal{O}^2(U) \to \mathcal{O}^2(U)$ *is a lift represented by* A *with*

$$A(z) = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} \quad \text{and} \quad A(0) = \begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix},$$

in some frame e_1, e_2, *then it induces another lift* $\hat{\tau}$ *represented by*

$$\hat{A}(z) = \begin{pmatrix} \hat{a}_{11}(z) & \hat{a}_{12}(z) \\ \hat{a}_{21}(z) & \hat{a}_{22}(z) \end{pmatrix} \quad \text{with} \quad \hat{A}(0) = \begin{pmatrix} \hat{\theta} & 0 \\ 0 & 1 \end{pmatrix},$$

such that there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}^2(U) \arrow[r, "i"] \arrow[d, "\tilde{\tau}"'] & \mathcal{O}^2(U) \arrow[r, "p"] \arrow[d, "\hat{\tau}"'] & \mathbb{C} \arrow[r] \arrow[d, "\hat{\theta}"'] & 0 \\ 0 \arrow[r] & \mathcal{O}^2(U) \arrow[r, "i"] & \mathcal{O}^2(U) \arrow[r, "p"] & \mathbb{C} \arrow[r] & 0, \end{tikzcd}$$

where $\hat{\theta} = e^{\frac{2\pi i}{n}j}\cdot\theta$.

If on the other hand we have the lift $\hat{\tau}$, *then it gives a lift* $\tilde{\tau}$ *making the diagram commutative.*

Proof. Let e_1, e_2 be a holomorphic frame of $\mathcal{O}^2(U)$, and let $\sigma(z) = z$.

Look at the $\mathcal{O}$-module $\mathcal{O}(-1)(U) = \{ f \in \mathcal{O}(U) \mid f(0) = 0 \}$ and notice that there is an $\mathcal{O}$-isomorphism

$$\begin{aligned} \mathcal{O}(-1)(U) \oplus \mathbb{C} &\longrightarrow \mathcal{O}(U) \\ (f, v) &\longmapsto f + v \end{aligned}$$

with obvious inverse $h \mapsto (h \cdot h(0), h(0))$. (The module structure on $\mathcal{O}(-1) \oplus \mathbb{C}$ that makes this work is $h \cdot (f, v) = (h \cdot f + (h - h(0) \cdot v), h(0) \cdot v)$.) Using this we easily construct the short exact sequence in the lemma by first choosing the frame e_1, e_2 in which

$$A(z) = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} \quad \text{with} \quad A(0) = \begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix},$$

and then letting

$$\begin{gathered} i(f, g) = ((\sigma \cdot f, 0), g) \in \big(\mathcal{O}(-1)(U) \oplus \mathbb{C}\big) \oplus \mathcal{O}(U) = \mathcal{O}^2(U), \\ p((f, v), g) = v \in \mathbb{C}. \end{gathered}$$

Now, $\tilde{\tau}$ acts as shown on page 75 and commutativity, $\hat{\tau} \circ i = i \circ \tilde{\tau}$, forces

$$\begin{aligned} \hat{\tau} \circ i(e_1)(z) &= i\big(a_{11} \circ \tau^{-1}(z) \cdot e_1(z) + a_{21} \circ \tau^{-1}(z) \cdot e_2(z)\big) \\ &= \sigma(z) \cdot a_{11} \circ \tau^{-1}(z) \cdot e_1(z) + a_{21} \circ \tau^{-1}(z) \cdot e_2(z) \end{aligned}$$

so we define $\hat{\tau}$ by

$$\begin{aligned} \hat{\tau}(e_1)(z) &= e^{\frac{2\pi i}{n} j} \cdot a_{11} \circ \tau^{-1}(z) \cdot e_1(z) + \big(\sigma \circ \tau^{-1}(z)\big)^{-1} \cdot a_{21} \circ \tau^{-1}(z) \cdot e_2(z), \\ \hat{\tau}(e_2)(z) &= \hat{\tau} \circ i(e_2)(z) = i \circ \tilde{\tau}(e_2)(z) \\ &= \sigma(z) \cdot a_{12} \circ \tau^{-1}(z) \cdot e_1(z) + a_{22} \circ \tau^{-1}(z) \cdot e_2(z). \end{aligned}$$

Thus, the induced $\hat{\tau}$ has the representation

$$\hat{A}(z) = \begin{pmatrix} \hat{a}_{11}(z) & \hat{a}_{12}(z) \\ \hat{a}_{21}(z) & \hat{a}_{22}(z) \end{pmatrix} = \begin{pmatrix} e^{\frac{2\pi i}{n} j} \cdot a_{11}(z) & \sigma \circ \tau(z) \cdot a_{12}(z) \\ \sigma(z)^{-1} \cdot a_{21}(z) & a_{22}(z) \end{pmatrix}.$$

As $a_{21}(0) = 0$, $\hat{A}$ is well-defined also in $z = 0$,[9]

$$\hat{A}(0) = \begin{pmatrix} e^{\frac{2\pi i}{n} j} \cdot \theta & 0 \\ \hat{a}_{21}(0) & 1 \end{pmatrix},$$

[9]This is precisely the reason why we make these elementary modifications one at a time and not to higher order in one step.

therefore we only need to check $\hat{a}_{21}(0) = 0$ in order to get the diagonal form we wish for. But since $\hat{\tau}$ is an $\mathcal{O}$-morphism (though τ-twisted) and $\hat{\tau}^n|_{\operatorname{Im} i} = \operatorname{Id}_{\operatorname{Im} i}$, then $\hat{\tau}^n = \operatorname{Id}_{\mathcal{O}^2(U)}$. Hence, also for $\hat{A}$ we have the relation

$$\hat{A}\big(\tau^{n-1}(z)\big) \circ \cdots \circ \hat{A}\big(\tau(z)\big) \circ \hat{A}(z) = \operatorname{Id}.$$

Thus, as $\hat{A}(0)^n = \operatorname{Id}$ linear algebra (see page 61) tells us that $\hat{A}(0)$ is diagonalizable. However, as it is already triangular, it must be diagonal; i.e. $\hat{a}_{21}(0) = 0$.

To verify that the right square commutes, notice first that $p(e_1) = 1$ and $p(e_2) = 0$. Then

$$p \circ \hat{\tau}(e_\nu) = \hat{a}_{\nu 1}(0) = \begin{cases} \hat{\theta}, & \nu = 1, \\ 0, & \nu = 2. \end{cases}$$

The other way round is proved in the same way using the same calculations. $\square$

Proof of Lemma 4.1. Choose a covering $\{U_\alpha\}$ of X so that no U_α contains more than one special point, and so that those that do contain a special point are disks with local coordinates z with $\tau^k(z) = e^{\frac{2\pi i}{n} jk} \cdot z$ as before, and so that there are trivializations $\varphi_\alpha \colon V|_{U_\alpha} \to \mathcal{O}^2(U_\alpha)$. Let s_1^α, s_2^α be the special frame if U_α contains a special point and any frame if not. Then $e_\nu^\alpha = \varphi_\alpha(s_\nu^\alpha)$ is a frame for $\mathcal{O}^2(U_\alpha)$.

The cocycles $g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ define V:

$$V = \coprod_\alpha \mathcal{O}^2(U_\alpha)/\sim,$$

where $s_\alpha \sim s_\beta$ for $s_\alpha \in \mathcal{O}^2(U_\alpha)$ and $s_\beta \in \mathcal{O}^2(U_\beta)$ if $U_\alpha \cap U_\beta \neq \emptyset$ and $s_\alpha|_{U_\alpha \cap U_\beta} = g_{\alpha\beta}(s_\beta|_{U_\alpha \cap U_\beta})$. From now on let the restriction to $U_\alpha \cap U_\beta$ be implicit when we compare sections like s_α and s_β.

Inspired by Lemma A.1, define for every β with $U_\beta \cap \pi^{-1}(y) = w$ the homomorphisms i_β as i in Lemma A.1, and for α with $U_\alpha \cap \pi^{-1}(y) = \emptyset$, let $i_\alpha = \operatorname{Id}_{\mathcal{O}^2(U_\alpha)}$. Our goal is to define cocycles $\hat{g}_{\alpha\beta} \colon \mathcal{O}^2(U_\alpha \cap U_\beta) \to \mathcal{O}^2(U_\alpha \cap U_\beta)$ and isomorphisms $\hat{\tau}_\alpha \colon \mathcal{O}^2(U_\alpha) \to \mathcal{O}^2(U_{\alpha'})$, $U_{\alpha'} = \tau(U_\alpha)$, such that we get commutative cubes

$$\begin{tikzcd}
& \mathcal{O}^2(U_{\beta'}) \arrow[rr, "i_{\beta'}"] \arrow[dd, "g_{\alpha'\beta'}"] & & \mathcal{O}^2(U_{\beta'}) \arrow[dd, "\hat{g}_{\alpha'\beta'}"] \\
\mathcal{O}^2(U_\beta) \arrow[ur, "\tilde{\tau}_\beta"] \arrow[rr, "i_\beta"] \arrow[dd, "g_{\alpha\beta}"] & & \mathcal{O}^2(U_\beta) \arrow[ur, "\hat{\tau}_\beta"] \arrow[dd, "\hat{g}_{\alpha\beta}"] & \\
& \mathcal{O}^2(U_{\alpha'}) \arrow[rr, "i_{\alpha'}"] & & \mathcal{O}^2(U_{\alpha'}) \\
\mathcal{O}^2(U_\alpha) \arrow[ur, "\tilde{\tau}_\alpha"] \arrow[rr, "i_\alpha"] & & \mathcal{O}^2(U_\alpha) \arrow[ur, "\hat{\tau}_\alpha"] &
\end{tikzcd}$$

for all intersections $U_\alpha \cap U_\beta$. Then define the bundle E as

$$E = \coprod_\alpha \mathcal{O}^2(U_\alpha)/\sim,$$

where $s_\alpha \sim s_\beta$ if $s_\alpha = \hat{g}_{\alpha\beta}(s_\beta)$ on $U_\alpha \cap U_\beta$, and put $\iota_\alpha = i_\alpha \circ \varphi_\alpha$. They glue together to form the injective sheaf homomorphism

$$\iota\colon V \longrightarrow E,$$

and together with the induced lift $\hat{\tau}$, this would give existence and commutativity of the left square of the diagram in the lemma.

So for $U_\alpha \cap \pi^{-1}(y) = \emptyset$ and $U_\beta \cap \pi^{-1}(y) = w$, we want $\hat{g}_{\alpha\beta} \circ i_\beta = i_\alpha \circ g_{\alpha\beta} = g_{\alpha\beta}$. Hence,

$$\hat{g}_{\alpha\beta}(e_1^\beta) = \sigma^{-1} \cdot \hat{g}_{\alpha\beta}(\sigma e_1^\beta) = \sigma^{-1} \cdot \hat{g}_{\alpha\beta} \circ i_\beta(e_1^\beta) = \sigma^{-1} \cdot g_{\alpha\beta}(e_1^\beta),$$
$$\hat{g}_{\alpha\beta}(e_2^\beta) = g_{\alpha\beta}(e_2^\beta)$$

which forces

$$\hat{g}_{\alpha\beta} = \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & 1 \end{pmatrix} g_{\alpha\beta}$$

with respect to the chosen frames (remember that w corresponds to $z = 0$ which is not in $U_\alpha \cap U_\beta$, so σ^{-1} is holomorphic and $\hat{g}_{\alpha\beta}$ is well-defined). The $\hat{g}_{\alpha\beta}$'s constitute a a cocycle provided $g_{\alpha\beta}$ is diagonal when ever $U_\beta \ni w \in \pi^{-1}(y)$ and $U_\alpha \cap U_\beta \neq \emptyset$ (and $w \notin U_\alpha$ as the covering is chosen). This can be arranged by choosing the covering fine enough around w, say by subdividing U_β.

Then define $\tilde{\tau}_\alpha = \varphi_{\alpha'} \circ (\tilde{\tau}|_{U_\alpha}) \circ \varphi_\alpha^{-1}$ for any α, and let $\hat{\tau}_\alpha = \tilde{\tau}_\alpha$ for every α with $U_\alpha \cap \pi^{-1}(y) = \emptyset$. To define $\hat{\tau}_\beta$ when $U_\beta \cap \pi^{-1}(y) = w$, we introduce the notation of cyclically indexed coordinate patches, U_{β_j}, for $j \in \mathbb{Z}/k$, such that $U_{\beta_{j+1}} = \tau(U_{\beta_j})$ and $\beta_0 = \beta$. Let $e_1^{\beta_0}, e_2^{\beta_0}$ be the frame for $V|_{U_{\beta_0}}$ from Lemma A.1, and define frames $e_\nu^{\beta_j} = \tilde{\tau}_{\beta_0}^j(e_\nu^{\beta_0})$ for $V|_{U_{\beta_j}}$ when $0 \leq j < k$. Let $\hat{e}_\nu^{\beta_j}$ be the induced frames from Lemma A.1, and define $\hat{\tau}_{\beta_j}$ by $\hat{\tau}_{\beta_j}(\hat{e}_\nu^{\beta_j}) = \hat{e}_\nu^{\beta_{j+1}}$, for $0 \leq j \leq k-2$. Then the relations $\hat{\tau}_{\beta_j} \circ i_{\beta_j} = i_{\beta_{j+1}} \circ \tilde{\tau}_{\beta_j}$ and $\hat{g}_{\alpha_{j+1}\beta_{j+1}} \circ \hat{\tau}_{\beta_j} = \hat{\tau}_{\alpha_j} \circ \hat{g}_{\alpha_j\beta_j}$, for $0 \leq j \leq k-2$, are clearly satisfied; we only need to check that it behaves correctly with respect to the k'th power which is constructed using Lemma A.1.

First notice that by Lemma A.1, $\hat{\tau}_\beta^k \circ i_\beta = i_\beta \circ \tilde{\tau}_\beta^k$. As it makes no difference in the following calculation we may as well assume that $g_{\alpha\beta}(e_\nu^\beta) = e_\nu^\alpha$ as it simplifies the notation. The left face of the cube commutes, since $\tilde{\tau}$ is globally defined, therefore we conclude from

$$g_{\alpha\beta} \circ \tilde{\tau}_\beta^k(e_\nu^\beta)(z) = a_{\nu 1}^\beta \circ \tau^{-k}(z) \cdot e_1^\alpha(z) + a_{\nu 2}^\beta \circ \tau^{-k}(z) \cdot e_2^\alpha(z),$$
$$\tilde{\tau}_\alpha^k \circ g_{\alpha\beta}(e_\nu^\beta)(z) = a_{\nu 1}^\alpha \circ \tau^{-k}(z) \cdot e_1^\alpha(z) + a_{\nu 2}^\alpha \circ \tau^{-k}(z) \cdot e_2^\alpha(z),$$

that $A_\alpha = A_\beta$ on $U_\alpha \cap U_\beta$.

Hence, as $\hat{A}_\alpha = A_\alpha$, $\hat{g}_{\alpha\beta} \circ \hat{\tau}_\beta^k$ is represented by

$$\begin{aligned}
\begin{pmatrix} (\sigma\circ\tau^k)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \hat{a}_{11}^\beta & \hat{a}_{12}^\beta \\ \hat{a}_{21}^\beta & \hat{a}_{22}^\beta \end{pmatrix} &= \begin{pmatrix} (\sigma\circ\tau^k)^{-1}\cdot \hat{a}_{11}^\beta & (\sigma\circ\tau^k)^{-1}\cdot \hat{a}_{12}^\beta \\ \hat{a}_{21}^\beta & \hat{a}_{22}^\beta \end{pmatrix} \\
&= \begin{pmatrix} e^{\frac{2\pi i}{n}kj}\cdot(\sigma\circ\tau^k)^{-1}\cdot a_{11}^\beta & a_{12}^\beta \\ \sigma^{-1}\cdot a_{21}^\beta & a_{22}^\beta \end{pmatrix} = \begin{pmatrix} \sigma^{-1}\cdot a_{11}^\alpha & a_{12}^\alpha \\ \sigma^{-1}\cdot a_{21}^\alpha & a_{22}^\alpha \end{pmatrix} \\
&= \begin{pmatrix} \hat{a}_{11}^\alpha & \hat{a}_{12}^\alpha \\ \hat{a}_{21}^\alpha & \hat{a}_{22}^\alpha \end{pmatrix} \cdot \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned}$$

which again represents $\hat{\tau}_\alpha^k \circ \hat{g}_{\alpha\beta}$. Thus, $\hat{g}_{\alpha\beta}\circ\hat{\tau}_\beta = \hat{\tau}_\alpha\circ\hat{g}_{\alpha\beta}$, and existence of $\hat{\tau}\colon E \to E$ and commutativity of the first square in our diagram is in place.

Commutativity of the second square of the diagram in the lemma follows directly from the construction of $\hat{\tau}$ and Lemma A.1.

We must check that E and $\hat{\tau}$ are independent of the choice of local trivializations φ_α. But that is a straightforward calculation yielding that if (after a possible subdivision of the open cover) $\varphi'_\alpha = h_\alpha \circ \varphi_\alpha$, then the new cocycles for V are $g'_{\alpha\beta} = h_\alpha \circ g_{\alpha\beta} \circ h_\beta^{-1}$ and one gets that the new cocycles for E are $\hat{g}'_{\alpha\beta} = h_\alpha \circ \hat{g}_{\alpha\beta} \circ h_\beta^{-1}$, so that they define the same vector bundle. A similar calculation gives independence for the induced maps. Independence of the chosen frames follows from independence of local trivializations.

To prove that E is unique given a flag in V taking on the role as kernel for the bundle homomorphism $V \to E$, assume that we have two extensions

$$0 \longrightarrow V \xrightarrow{\iota_\nu} E_\nu \xrightarrow{\lambda_\nu} \mathbb{C}_{\pi^{-1}(y)} \longrightarrow 0\ ,$$

$\nu = 1, 2$. We are going to construct a homomorphism $\Psi\colon E_1 \to E_2$ and the 5-lemma will give the desired result.

Assume that E_1 is given as above with ι_1 having the local form $i_{1,\alpha} = \mathrm{Id}_{\mathcal{O}^2(U_\alpha)}$ and $i_{1,\beta}$ taking $e_1^\beta \mapsto \sigma\cdot e_1^\beta$, $e_2^\beta \mapsto e_2^\beta$ (where we use the convention as above that $U_\beta \cap \pi^{-1}(y) = \{w\}$ and $w \notin U_\alpha$ for all $\alpha \neq \beta$). Locally ι_2 is represented by

$$i_{2,\alpha} = \begin{pmatrix} c_{11}^\alpha & c_{12}^\alpha \\ c_{21}^\alpha & c_{22}^\alpha \end{pmatrix}$$

for all α. Notice that the condition that $\ker \iota_{1,w} = \ker \iota_{2,w}$ implies that we may write $c_{11}^\beta = \sigma\cdot\hat{c}_{11}^\beta$ $c_{21}^\beta = \sigma\cdot\hat{c}_{21}^\beta$ for $\hat{c}_{11}^\beta, \hat{c}_{21}^\beta \in \mathcal{O}(U_\beta)$.

Clearly there is a well defined homomorphism $\Psi\colon \operatorname{Im}\iota_1 \to \operatorname{Im}\iota_2$ by sending $e \in \operatorname{Im}\iota_1$ to $\iota_2 \circ \iota_1^{-1}(e)$. Notice that for any $e \in E_1(U_\beta)$, $\sigma\cdot e \in \ker\lambda_{1,\beta} = \operatorname{Im}\iota_{1,\beta}$.

Suppose $e = f \cdot e_1^\beta + g \cdot e_2^\beta$, then $\iota_{1,\beta}^{-1}(\sigma \cdot e) = f \cdot e_1^\beta + \sigma g \cdot e_2^\beta$ and $\iota_{2,\beta} \circ \iota_{1,\beta}^{-1}(\sigma \cdot e) = \sigma\big((f\hat{c}_{11}^\beta + g a_{12}^\beta) \cdot e_1^\beta + (f\hat{c}_{21}^\beta + g a_{22}^\beta) \cdot e_2^\beta\big)$. Therefore define

$$\Psi_\beta = \begin{pmatrix} \hat{c}_{11}^\beta & c_{12}^\beta \\ \hat{c}_{21}^\beta & c_{22}^\beta \end{pmatrix} \quad \text{and} \quad \Psi_\alpha = \begin{pmatrix} c_{11}^\alpha & c_{12}^\alpha \\ c_{21}^\alpha & c_{22}^\alpha \end{pmatrix} = \iota_{2,\alpha} \quad \text{for all other } \alpha$$

One checks that with transition functions $g_{\alpha\beta}^\nu$ for E_ν chosen as above with $g_{\alpha\beta}^1$ diagonal for $U_\alpha \cap U_\beta \neq \emptyset$, we have $\Psi_\alpha \circ g_{\alpha\beta}^1 = g_{\alpha\beta}^2 \circ \Psi_\beta$ so the Ψ_α's patch together nicely to form a homomorphism $\Psi\colon E_1 \to E_2$.

Finally we compute the determinants. The determinant of $\mathbb{C}_{\pi^{-1}(y)}$ is defined using the exact sequence

$$0 \longrightarrow \mathcal{O}(-\pi^{-1}(y)) \xrightarrow{\quad i \quad} \mathcal{O}_X \xrightarrow{\quad p \quad} \mathbb{C}_{\pi^{-1}(y)} \longrightarrow 0.$$

I.e. $\det \mathbb{C}_{\pi^{-1}(y)} = \det \mathcal{O}(\pi^{-1}(y)) = [\pi^{-1}(y)]$, and it follows that $\det E = \det V \otimes [\pi^{-1}(y)]$ and that $\deg E = \deg \det E = \deg \det V + \deg[\pi^{-1}(y)] = \deg V + k$. $\square$

Proof of Lemma 4.4. With respect to the lift and the determinant line bundles, this proof is very similar to the previous one. To kick things off however choose the canonical projection $\lambda_w\colon E_w \to E_w/F_w \cong \mathbb{C}$ for a fixed $w \in \pi^{-1}(y)$ so that we have $\ker \lambda_w = F_w$. Extend this invariantly to all of $\pi^{-1}(y)$ and then to a sheaf homomorphism $\lambda\colon E \to \mathbb{C}_{\pi^{-1}(y)}$ as $\lambda(U_w)(s) = \lambda_w(s(w))$ for all $s \in E(U_w)$ and $w \in \pi^{-1}(y)$. As such a λ is uniquely determined by its action on straws, it only depends on the initial choice of λ_w which is unique up to scaling which in turn does not change the kernel. Defining the action on $\mathbb{C}_{\pi^{-1}(y)}$ as multiplication by $\hat{\theta}$, yields commutativity of the second square in the diagram.

Define the locally free sheaf

$$V = \ker \lambda$$

of $\mathcal{O}$-modules; this only depends on F_w for $w \in \pi^{-1}(y)$, and not on the actual choice of λ.

Again we need to consider it locally to verify that the lift $\tilde{\tau}$ is represented as desired. Let $\{U_\alpha\}$ be a covering as before, and let $\hat{s}_1^\alpha, \hat{s}_2^\alpha$ be local frames for $E|_{U_\alpha}$ such that $\hat{\tau}^k$ has the stated matrix representation if $U_\alpha \cap \pi^{-1}(y) = w$. Then choose frames s_1^α, s_2^α for $V|_{U_\alpha}$ so that the inclusion is $\iota(s_1^\alpha) = \sigma \cdot \hat{s}_1^\alpha$ if $w = U_\alpha \cap \omega^{-1}(y)$ and $\iota(s_1^\alpha) = \hat{s}_1^\alpha$ otherwise, and $\iota(s_2^\alpha) = \hat{s}_2^\alpha$ always. A calculation similarly to the one in the proof of Lemma 4.1 gives the required result about $\tilde{\tau}$.

The relations for the determinant and degrees are a trivial consequence of the exact sequences as before. $\square$

Lemma A.2. *Consider the short exact sequence*

$$0 \longrightarrow V \xrightarrow{\ \iota\ } E \xrightarrow{\ \lambda\ } \mathbb{C}_{\pi^{-1}(y)} \longrightarrow 0$$

from Lemma 4.1 and Lemma 4.4 with lifts $\tilde{\tau}$ and $\hat{\tau}$ of τ to V and E respectively.

Let $L \subset V$ be a line bundle and let $\bar{P}(L)$ be the subset of $\bar{P}$ where $\iota(L)$ fails to be a vector bundle, $\bar{P}(L) = \{ w \in \bar{P} \mid (\iota|_L)_w = 0 \}$.[10] *Then L gives a line bundle $\hat{L} \subset E$ fitting into a short exact sequence*

$$0 \longrightarrow L \xrightarrow{\ \iota|_L\ } \hat{L} \xrightarrow{\ \lambda|_{\hat{L}}\ } F_L \longrightarrow 0,$$

where F_L is the skyscraper sheaf which is $F_L(U) = \mathbb{C}_{\pi^{-1}(y)}(U) = \mathbb{C}$ for $U \cap \bar{P}(L) \neq \emptyset$ and $F_L(U) = 0$ otherwise.

If on the other hand $\hat{L} \subset E$ is a line bundle and $\bar{P}(\hat{L}) = \{ w \in \bar{P} \mid \hat{L}_w \neq F_w = \ker \lambda_w \}$, then there is a line bundle $L \subset V$ fitting into the above exact sequence with an $F_{\hat{L}}$ having support $\bar{P}(\hat{L})$ in the place of F_L.

If $\hat{L}$ was constructed from L in this way or vice versa, $\bar{P}(L) = \bar{P}(\hat{L})$ and in fact the two constructions are each others inverse.

Proof. Let L be a subbundle of V and assume that it is trivializable over $\{U_\alpha\}$. Then for every α we may write a frame for $L|_{U_\alpha}$ as $e^\alpha = f_1^\alpha \cdot e_1^\alpha + \sigma^{m^\alpha} f_2^\alpha \cdot e_2^\alpha$ where $f_1^\alpha \in \mathcal{O}(U_\alpha)$, $f_2^\alpha \in \mathcal{O}^*(U_\alpha)$ (when $\{U_\alpha\}$ is fine enough), $m^\alpha \in \mathbb{Z}_{\geq 0}$, and e_ν^α is the usual frame for $\mathcal{O}^2(U_\alpha)$. When $U_\beta \cap \pi^{-1}(y) = w$, the image $\iota_\beta(e^\beta) = \sigma \cdot (f_1^\beta \cdot e_1^\beta + \sigma^{m^\beta - 1} f_2^\beta \cdot e_2^\beta)$ if $m^\beta > 0$, and $\iota_\beta(e^\beta) = \sigma f_1^\beta \cdot e_1^\beta + f_2^\beta \cdot e_2^\beta$ if $m^\beta = 0$. Hence, let the induced local frame be

$$\hat{e}^\beta = \begin{cases} f_1^\beta \cdot \hat{e}_1^\beta + \sigma^{m^\beta - 1} f_2^\beta \cdot \hat{e}_2^\beta, & \text{for } m^\beta > 0, \\ \sigma f_1^\beta \cdot \hat{e}_1^\beta + f_2^\beta \cdot \hat{e}_2^\beta, & \text{for } m^\beta = 0, \end{cases}$$

when $U_\beta \cap \pi^{-1}(y) = w$, and $\hat{e}^\alpha = e^\alpha$ when $U_\alpha \cap \pi^{-1}(y) = \emptyset$. The case $m^\beta > 0$ corresponds to when $w \in \bar{P}(L)$. Locally around a point $w \in \bar{P}(L)$, the sequence is described as

$$0 \longrightarrow \mathcal{O}(-1)(U) \xrightarrow{\ i\ } \mathcal{O}(U) \xrightarrow{\ p\ } \mathbb{C} \longrightarrow 0$$

and the cocycles $h_{\alpha\beta}$ for L is changed to $\hat{h}_{\alpha\beta} = \sigma^{-1} \cdot h_{\alpha\beta}$ when ever $w \in U_\beta$ and $w \notin U_\alpha$. The inclusions $j_\alpha \colon \mathcal{O}(U_\alpha) \to \mathcal{O}^2(U_\alpha)$, defined as $j_\alpha(g) = g \cdot e^\alpha$, make L into a subbundle of V (i.e. $g_{\alpha\beta} \circ j_\beta = j_\alpha \circ h_{\alpha\beta}$). Similarly, with $\hat{j}_\alpha(g) = g \cdot \hat{e}^\alpha$

[10] This may seem like a bombastic notation for something which is one point or the empty set, but it is in keeping with the notation in the general case of higher order elementary modifications.

we get that $\hat{g}_{\alpha\beta} \circ \hat{\jmath}_\beta = \hat{\jmath}_\alpha \circ \hat{h}_{\alpha\beta}$ so the engineered line bundle $\hat{L} = \coprod_\alpha \mathcal{O}(U_\alpha)/\sim$, where $s_\alpha \sim s_\beta$ if $s_\alpha|_{U_\alpha \cap U_\beta} = \hat{h}_{\alpha\beta}(s_\beta|_{U_\alpha \cap U_\beta})$ becomes a subbundle of E and is an extension by L of the skyscraper sheaf F_L supported over $\bar{P}(L)$ ($F_L(U) = F(U)$ if $U \cap \bar{P}(L) \neq \emptyset$ and $F_L(U) = 0$ otherwise):

$$0 \longrightarrow L \xrightarrow{\iota|_L} \hat{L} \xrightarrow{\lambda|_{\hat{L}}} F_L \longrightarrow 0.$$

If on the other hand $\hat{L}$ is a subbundle of E, define $F_{\hat{L}}$ to be the skyscraper sheaf supported over $\bar{P}(\hat{L})$ as before and look at the restriction of λ to $\hat{L}$: it is a surjection

$$\lambda|_{\hat{L}} \colon \hat{L} \longrightarrow F_{\hat{L}} \to 0.$$

Define the sheaf $L = \ker \lambda|_{\hat{L}}$. Since $\lambda_w(\hat{L}_w) = 0$ for $w \notin \bar{P}(\hat{L})$, $L \subset \ker \lambda = V$, so L is in fact a subbundle of V.

Clearly, if $\hat{L}$ comes from L using the above construction, $\bar{P}(L) = \bar{P}(\hat{L})$, and the bundle induced by $\hat{L}$ is L since $L = \ker \lambda|_{\hat{L}}$. If on the other hand L is induced by $\hat{L}$ and $\hat{L}$ is generated locally by sections $\hat{s}_\alpha$, then L is generated by either $\hat{s}_\alpha$ or $\sigma \cdot \hat{s}_\alpha$ depending on whether $\bar{P}(L) \cap U_\alpha = \emptyset$ or not. $\square$

Observation A.3. We observe that if one of the two line bundles L or $\hat{L}$ is invariant (with respect to $\tilde{\tau}$ and $\hat{\tau}$ respectively), then so is the other. In general we have in the language of the lemma above, that $\hat{\tau}\hat{L} = \widehat{\tilde{\tau}L}$; i.e. the image under $\hat{\tau}$ of a bundle $\hat{L}$ induced from L is the bundle induced from the image $\tilde{\tau}L$ of L under $\tilde{\tau}$.

Notice also from the proof that if L_1 and L_2 are invariant line bundles with $\mu(L_\nu) = \mu(V)$ so that $V = L_1 \oplus L_2$, then $E = \hat{L}_1 \oplus \hat{L}_2$. To see this we may choose the local frames (e_1, e_2) so that L_ν is spanned by either e_1 or e_2 depending on whether $w \in \bar{P}(L_\nu)$ or not. So if L_ν is spanned locally around $w \in \bar{P}_0$ by e_μ, then $\hat{L}_\nu$ is spanned by $\hat{e}_\mu$, so clearly $\hat{L}_1 \oplus \hat{L}_2 = E$.

If L_ν were not invariant the induced line bundles $\hat{L}_\nu$ in E would collapse in w to a single line (and in any other special point). But clearly $\hat{L}_1$ and $\hat{L}_2$ span E away from the special points.

Remark A.4. One may of course iterate the elementary modification and inverse elementary modification processes both supported over the same $y \in P_0$ and over other points in P_0. In that case we call it *higher order elementary modifications* respectively *higher order inverse elementary modifications*. The conclusions of the previous lemma and observation is also true for these iterations. However, the subset $\bar{P}(L)$ may vary through the iterations: say $L_1 = \hat{L}$, $L_2 = \hat{L}_1$ etc., then $\bar{P}(L_i)$ need not be the same as $\bar{P}(L_{i+1})$. This is of course true when changing the fiber $\pi^{-1}(y)$ over which we do the elementary modifications, but it may also be the case when we iterate over a fixed $\pi^{-1}(y)$. Now if L is invariant, then the $\bar{P}(L_i)$'s are the same. To see this it is enough to consider $m_2(y)$ consecutive modifications over $\pi^{-1}(y)$. Recall from Remark 4.6 that an upshot of Lemma 4.1 and Lemma 4.4 is that we can chose local eigenframes (e_1, e_2) for $\tilde{\tau}$. As L is invariant, this frame can be chosen so that L is spanned around w by either e_1 or e_2 depending on whether $w \in \bar{P}(L)$ or not. Then during successive elementary modifications,

$$0 \longrightarrow E_{i-1} \xrightarrow{\ \iota_i\ } E_i \xrightarrow{\ \lambda_i\ } \mathbb{C}_{\pi^{-1}(y)} \longrightarrow 0\ ,$$

where $E_0 = V$, it is clear that $w \in \bar{P}(L_i)$ if and only if $w \in \bar{P}(L_{i-1})$, where $L_i = \hat{L}_{i-1}$.

This means that if E is derived from V through elementary modifications determined by the effective divisor D_2, and $L \subset V$ and $\hat{L} \subset E$ are corresponding invariant line bundles, then the divisor $0 \leq D_L \leq D_2$ for which $\hat{L} = L \otimes [D_L]$ is exactly $D_2|_{\bar{P}(L)}$. Because clearly $\operatorname{Supp} D_L = \bar{P}(L)$, but on the other hand if $w \in \bar{P}(L)$, then also $w \in \bar{P}(L_i)$ through the $m_2(w)$ iterations and vice versa.

APPENDIX B. GEOMETRIC INVARIANT THEORY

In this section we will give a cursory exposition of the tools necessary to prove that $\overline{\mathcal{EM}}\colon Ma(Y) \to |M(X)|$ is a morphism of varieties; namely geometric invariant theory (GIT). For further details the reader is referred to [47], [62], [51], and the more courageous reader to [31]. Many results are stated here without proofs; the proofs can be found in the listed references. Some passages of this appendix on more general theory is borrowed from [2] thereby saving the author the labor of translating the text from French. It is regrettably necessary to make the transition from the language of Riemann surfaces to that of schemes.

The *Hilbert polynomial* $P_E(T)$ of a coherent sheaf E over X is defined by

$$P_E(n) = \chi(E(n)) = \dim H^0\big(X, E(n)\big) - \dim H^1\big(X, E(n)\big).$$

When E is locally free, Riemann-Roch implies that P_E is a polynomial of degree less than or equal to 1. In fact if E is a rank r bundle of degree $d > r(2g-1)$, then

$$P_E(T) = p + rT,$$

where $p = \dim H^0(X, E) = d - r(g-1)$. This is due to the fact:

Lemma B.1. [62, Lemma 1.III.20] *Let (r, d) be a pair of integers satisfying that $r \geq 2$ and $d > r(2g-1)$ and let E be a semistable holomorphic vector bundle of rank r and degree d. Then*

(1) *the bundle E is generated by its global sections, and*
(2) $H^1(X, E) = 0$.

In the sequel E will be a coherent sheaf with Hilbert polynomial $P_E = \rho_0$ of degree at most 1.

Definition B.2. A *family of qoutients* (S, G) of the coherent sheaf E with Hilbert polynomial ρ_0 is a $\mathbb{C}$-scheme S together with a flat coherent sheaf G on $S \times_{\mathbb{C}} X$ and a surjection

$$p_X^* E \longrightarrow G,$$

such that the Hilbert polynomial of the restriction G_s of G to $X_s = \{s\} \times_{\mathbb{C}} X$ is equal to ρ_0 for all closed points $s \in S$, where $p_X^* \colon S \times_{\mathbb{C}} X \to X$ is the projection.

Two families (S, G) and (S, G') of quotients of E are equivalent if there is an isomorphism $\varphi \colon G \to G'$ of sheaves over $S \times_{\mathbb{C}} X$ such that

$$\begin{tikzcd} & G \arrow[dd, "\varphi"] \\ p_X^* E \arrow[ur] \arrow[dr] & \\ & G' \end{tikzcd}$$

commutes.

A morphism of families (S, G) and (S', G') of quotients of E is a pair (f, φ) where $f \colon S \to S'$ is a morphism of schemes and $\varphi \colon G \to \bar{f}^* G'$ is an equivalence of families (S, G) and $(S, \bar{f}^* G')$. Here $\bar{f} \overset{\text{def}}{=} f \times \mathrm{Id}_X \colon S \times_{\mathbb{C}} X \to S' \times_{\mathbb{C}} X$ is the induced morphism of schemes.

Theorem B.3. (*Grothendieck* [31], [62]) *There is a functor from $\mathbb{C}$-schemes to sets which associates to any scheme S the set of isomorphism classes of families of quotients of E with Hilbert polynomial ρ_0. This functor is represented by a projective $\mathbb{C}$-scheme $Q_X = \mathrm{Quot}^{\rho_0}_{E/X/\mathbb{C}}$ (Grothendieck's quot scheme). I.e. there is a coherent sheaf $\mathcal{U}_X$ on $Q_X \times_{\mathbb{C}} X$ such that $(Q_X, \mathcal{U}_X)$ is a family of quotients of E with Hilbert polynomial ρ_0 satisfying the universal property, that for any other such family (S, G), there exists a unique morphism $\Phi \colon S \to Q_X$ so that $\bar{\Phi}^* \mathcal{U}_X$ is equivalent to G.*

Suppose that $\mathbb{E}$ is the trivial bundle of rank $p = d - r(g-1)$ and $\rho_0(T) = p + rT$, and assume that $d > r(2g-1)$. Then there is an open sub-scheme R_X of $Q_X = \mathrm{Quot}^{\rho_0}_{\mathbb{E}/X/\mathbb{C}}$ characterized by the property that R_X is exactly the points $q \in Q_X$ for which $\mathcal{U}_{X,q}$ is locally free over $X_q = \{q\} \times_{\mathbb{C}} X$ and the homomorphism $H^0(X_q, \mathbb{E}) \to H^0(X_q, \mathcal{U}_{X,q})$ given by the quotient morphism is an isomorphism. Then $\mathcal{U}_{R_X} \overset{\text{def}}{=} \mathcal{U}_X|_{R_X \times_{\mathbb{C}} X}$ is locally free. The sub-scheme R_X satisfies *local universality*:

Proposition B.4. [62, Proposition 1.III.21] *Given a $\mathbb{C}$-scheme S and a locally free sheaf F of rank r on $S \times_{\mathbb{C}} X$ such that*

(1) *F_s has degree d as a vector bundle on X_s,*
(2) *F_s is generated by its global sections, and*
(3) *$H^1(X_s, F_s) = 0$.*

Then for all $s_0 \in S$ there is a neighbourhood $S_0 \subseteq S$ of s_0 and a morphism $f\colon S_0 \to R_X$ so that $F|_{S_0 \times_{\mathbb{C}} X} \cong \bar{f}^* \mathcal{U}_{R_X}$.

Hence, if F is a rank r bundle of degree $d > r(2g-1)$ and $S = \{s_0\}$, then (1), (2), and (3) in the proposition are satisfied, and local universality gives an $f\colon \{s_0\} \to R_X$ so that $F \cong \bar{f}^* \mathcal{U}_{R_X}$. Therefore, we may think of R_X as having a point q over which $\mathcal{U}_{R_X, q} \cong F$.

Denote by $\mathrm{GL}(p)$ the group $\mathrm{Aut}(\mathbb{E})$. We define an action of $\mathrm{GL}(p)$ on Q_X in the following manner: Let $A \in \mathrm{GL}(p)$ and let

$$\alpha\colon p_X^* \mathbb{E} \longrightarrow \mathcal{U}_X$$

be the surjection from the definition of a family of quotients. Then there is an automorphism $\bar{A}^{-1} = p_X^*(A^{-1})\colon p_X^*\mathbb{E} \to p_X^*\mathbb{E}$, and this gives a surjection

$$\alpha_A \stackrel{\text{def}}{=} \alpha \circ \bar{A}^{-1}\colon p_X^* \mathbb{E} \longrightarrow \mathcal{U}_X,$$

making $(Q_X, \mathbb{E})$ into a new family of quotients. By the universality of Theorem B.3 there is a unique morphism $\sigma_A\colon Q_X \to Q_X$ and an isomorphism of sheaves $\varphi_A\colon \sigma_A^* \mathcal{U}_X \to \mathcal{U}_X$.

Clearly, if $A = \mathrm{Id}$, then $\sigma_A = \mathrm{Id}$, and a short calculation shows that $\sigma_{AB} = \sigma_A \circ \sigma_B$ for $A, B \in \mathrm{GL}(p)$, so this is really an action. Notice also that if $A \in \mathbb{C}^* \cdot \mathrm{Id}$ then $\sigma_A = \mathrm{Id}$ by uniqueness and φ_A is multiplication by a non-zero scalar, so in fact we have an action of $\mathrm{PGL}(p) = \mathrm{GL}(p)/\mathbb{C}^*$ on Q_X.

Proposition B.5. [62, Proposition 1.III.22]

(1) *The open sub-scheme R_X of Q_X is $\mathrm{PGL}(p)$-invariant.*
(2) *For every pair (q_1, q_2) of points in R_X, the vector bundles $\mathcal{U}_{R_X, q_1}$ and $\mathcal{U}_{R_X, q_2}$ are isomorphic if and only if q_1 and q_2 are in the same orbit of $\mathrm{PGL}(p)$.*
(3) *For any $q \in R_X$, the stabilizer of the action of $\mathrm{PGL}(p)$ is $\mathrm{Aut}(\mathcal{U}_{R_X, q})/\mathbb{C}^*$.*

The next essential definition requires the notion of some more algebro geometric properties:

Definition B.6. A morphism $f\colon Z \to S$ of schemes is *affine* if there exits an affine covering $\{V_i\}$ of S so that $f^{-1}(V_i)$ is affine.

The affine morphism f is of *finite type* if there exists a finite affine covering $\{V_i\}$ of S, where $V_i = \operatorname{Spec} A_i$ for some ring A_i, such that $f^{-1}(V_i) = \operatorname{Spec} B_i$, where B_i is a finitely generated algebra over A_i.

A scheme S over a ring A is of *finite type* if there exists a finite, affine covering $\{V_i\}$ of S with $V_i = \operatorname{Spec} A_i$, such that A_i is a finitely generated algebra over A.

Definition B.7. An algebraic group G is *geometrically reducible* if for all representations $\rho\colon G \to \mathrm{GL}(n)$, the following holds: for all $v \in \mathbb{C}^n - \{0\}$, there exits a homogeneous non-constant G-invariant polynomial f so that $f(v) \neq 0$.

From now on the groups acting will be geometrically reducible.

Definition B.8. A *good quotient* of S with geometrically reducible group G is a pair (M, f), where $f\colon S \to M$ is a morphism of $\mathbb{C}$-schemes such that

(1) f is affine, G-invariant and surjective,
(2) there is an induced isomorphism $f^*\colon H^0(U, \mathcal{O}) \to H^0\big(f^{-1}(U), \mathcal{O}\big)^G$ for all open subsets U of M,
(3) if S' is a G-invariant, closed subvariety of S, then $f(S')$ is a closed subvariety of M,
(4) and if S_1 and S_2 are closed subvarieties so that $S_1 \cap S_2 = \emptyset$, then $f(S_1) \cap f(S_2) = \emptyset$.

Good quotients possess a universal property too:

Theorem B.9. *Let (M, f) be a good quotient of S with G. If M' is a $\mathbb{C}$-scheme of finite type over $\mathbb{C}$, and $f'\colon S \to M'$ is a G-invariant morphism of finite type, then there exists a unique morphism $g\colon M \to M'$ making the diagram*

$$\begin{tikzcd} S \arrow[d, "f"'] \arrow[dr, "f'"] & \\ M \arrow[r, dotted, "g"'] & M' \end{tikzcd}$$

commutative. Consequently good quotients are unique when they exist.

We write $S//G$ for the good quotient of S with G.

Definition B.10. A linerization of the action of G on S is a line bundle L on Y equipped with an action of G that covers the action of G on Y.

Definition B.11. A point $s \in S$ is said to be semistable if there exists a positive integer m and a section g in L^m such that $g(s) \neq 0$ and such that the complement S_g of the zero-set of g, is an affine open subset of S. A point s is stable if it is semistable, $\dim(G \cdot s) = \dim G$, and the orbits of the closed points in S are closed subvarieties.

Proposition B.12. [62, p.34] *The moduli space $M_d(X)$ of semistable rank r and degree d vector bundles is a good quotient of the semistable part, R_X^{ss}, of R_X.*

Notice that $\tau^*\mathcal{U}_{X,q} \overset{\text{def}}{=} \big((\mathrm{Id}\times_{\mathbb{C}}\tau)^*\mathcal{U}_X\big)_q \cong \mathcal{U}_{X,\tau(q)}$ so $\mathcal{U}_{X,q}$ is a fixed point if and only if $\tau(q) \in \mathrm{PGL}(p)\cdot q$. Denote by $|R_X^{ss}|$ the closed sub-scheme of R_X^{ss} consisting of such points. This is clearly $\mathrm{PGL}(p)$ invariant. The restriction of the good quotient $(M_d(X), f)$ of R_X^{ss} with $\mathrm{PGL}(p)$ to $|R_X^{ss}|$ is a good quotient. To see this, simply use the definition; the only thing we need to check is that

$$f^*_{|R_X^{ss}|}\colon H^0\big(U \cap f(|R_X^{ss}|), \mathcal{O}\big) \longrightarrow H^0\big(f^{-1}_{|R_X^{ss}|}(U \cap f(|R_X^{ss}|)), \mathcal{O}\big)^{\mathrm{PGL}(p)}$$

is an isomorphism for every open subset U of $M_d(X)$. But this is true because the local ring of a subvariety Z of another variety S is the equivalence classes $\langle U, f\rangle$, where $U \subseteq S$ is open, $U \cap Z \neq \emptyset$ and $f \in \mathcal{O}(U)$. We say $\langle U, f\rangle$ is equivalent to $\langle V, g\rangle$ if $f = g$ on $U \cap V$ ([33, I.3.13]). Thereby we conclude that the good quotient $|R_X^{ss}|/\!/\,\mathrm{PGL}(p) = |M_d(X)|$ is the fixed point set in the moduli space of semistable holomorphic vector bundles.

In the parabolic case one can choose a $\tilde{d}_0$ so that any parabolic bundle E over Y of degree $\tilde{d} \geq \tilde{d}_0$ and having fixed parabolic weights a and multiplicities $\chi = \{k_y^i \mid y \in P, 1 \leq i \leq n_y\}$ over P, satisfies that

(1) the underlying vector bundle E is generated by its global sections, and
(2) $H^1(Y, E) = 0$.

Such bundles all have the same $\dim H^0(Y, E)$ which we denote by $\tilde{p}$, and their Hilbert polynomial will be equal.

Now, let $\tilde{\mathbb{E}}$ be the trivial holomorphic vector bundle of rank $\tilde{p}$ and let $\tilde{\rho}_0(T) = \tilde{p}+rT$ be a fixed Hilbert polynomial. Then let $Q_Y = \mathrm{Quot}^{\tilde{\rho}_0}_{\tilde{\mathbb{E}}/Y/\mathbb{C}}$ be the Grothendieck quot scheme representing families of quotients of $\tilde{\mathbb{E}}$ over Y with Hilbert polynomial $\tilde{\rho}_0$, $\mathcal{U}_Y$ its universal sheaf, and R_Y the open sub-scheme defined in the same way as R_X.

Let $\mathcal{F}(\chi)$ denote the flag-variety over $R_Y \times_{\mathbb{C}} P$ defined as all possible flags with multiplicities χ. Let $\tilde{R}_Y$ be the closed subvariety of $\mathcal{F}(\chi)$ whose points $\{F_x\}$ satisfy that for any pair of points (y, y') in Y, the two projections to R_Y from $\mathcal{F}(\chi)|_{R_Y\times_{\mathbb{C}}\{y\}}$ respectively $\mathcal{F}(\chi)|_{R_Y\times_{\mathbb{C}}\{y'\}}$ agree. This is precisely what ensures that we get a (canonical) projection

$$\Pi\colon \tilde{R}_Y \longrightarrow R_Y.$$

Notice that a $q \in \tilde{R}_Y$ defines a parabolic structure on vector bundle $\mathcal{U}_{R_Y,\Pi(q)}$ over Y. We shall write $\tilde{\mathcal{U}}_{\tilde{R}_Y,q}$ for this parabolic bundle and $\tilde{\mathcal{U}}_{\tilde{R}_Y}$ for the canonical bundle over $\tilde{R}_Y \times_{\mathbb{C}} Y$.

There are completely analogous statements for $\tilde{R}_Y$ as there were for R_X, in particular: it has the local universal property, it is invariant under the action of $\mathrm{PGL}(\tilde{p})$, ([62, p.83]) and the moduli space of parabolic bundles of degree $\tilde{d}$, weights a and multiplicities χ is a good quotient of $\tilde{R}_Y^{ss}(\chi, a)$ the subset consisting of points that are semistable with respect to (χ, a) ([62, p.84]).

As a corollary of the last statement, we have that fixing the weight configuration a, the multiplicities and the determinant for the underlying bundle, the moduli space $M_{\tilde{L}}(Y;\chi,a)$ of parabolic bundles of multiplicities χ, weights a and determinant $\tilde{L}$ is irreducible when the genus of Y is bigger than or equal to 2. This is because it is the good quotient of $\tilde{R}_{Y,\tilde{L}}^{ss}(\chi, a)$ which is an open subset of an irreducible variety ([62, p.84]). For low genus other considerations must be made.

Proving that $\overline{\mathcal{EM}}$ is a morphism. We start out with a rather general statement. Let Z be a compact Riemann surface and let $\tilde{R}_{Z,L}^{ss}(\chi, a)$ be the subvariety of semistable points of $\tilde{R}_Z$ with respect to the weight a and multiplicity configuration χ and with determinant L. Let $p' = \dim H^0(Z_q, \tilde{U}_{\tilde{R}_Z,q})$. Needles to say, we are assuming that $d' = \deg \tilde{U}_{\tilde{R}_Z,q}$ is large enough that we are in the domain of GIT (we will be more precise about this later).

Lemma B.13. *Given a map*

$$f \colon \tilde{R}_{Z,L}^{ss}(\chi, a) \longrightarrow M(X; \bar{\chi}, \bar{a}),$$

where $(\bar{\chi}, \bar{a})$ is the parabolic structure dictated by f. Suppose that f is presented by a locally free sheaf $\mathcal{V}$ over $\tilde{R}_{Z,L}^{ss}(\chi, a) \times_{\mathbb{C}} X$ (possibly with parabolic structure); i.e. that $f(q) = [\mathcal{V}_q]$ for all $q \in \tilde{R}_{Z,L}^{ss}(\chi, a)$, where $[\cdot]$ means the strong equivalence class. If $\mathcal{V}$ satisfies items (1) *to* (3) *in local universality then f is a morphism.*

Moreover, if f is $\mathrm{GL}(p')$-invariant it induces a morphism

$$\hat{f} \colon M_L(Z; \chi, a) \longrightarrow M(X; \bar{\chi}, \bar{a}).$$

Proof. By assumption, $\mathcal{V}$ conforms to the requirements of local universality so for every $q \in \tilde{R}_{Z,L}^{ss}(\chi, a)$ there is a neighbourhood V_q of q in $\tilde{R}_{Z,L}^{ss}(\chi, a)$ and a morphism $f_q \colon V_q \to \tilde{R}_X$ so that $\mathcal{V}|_{V_q \times_{\mathbb{C}} X} \cong \bar{f}^* \tilde{\mathcal{U}}_{\tilde{R}_X}$. Clearly $\operatorname{Im} f_q \subseteq \tilde{R}_X^{ss}(\bar{\chi}, \bar{a})$.

Let $g_X \colon \tilde{R}_X^{ss}(\bar{\chi}, \bar{a}) \to M(X; \bar{\chi}, \bar{a})$ be the good quotient. Then $f|_{V_q} = g_X \circ f_q \colon V_q \to M(X; \bar{\chi}, \bar{a})$ is a composition of morphisms. That means that f is a morphism locally around every point $q \in \tilde{R}_Z^{ss}(\chi, a)$, hence, it is a morphism.

Now if the morphism $f\colon \tilde{R}^{ss}_{Z,L}(\chi,a) \to M(X;\bar{\chi},\bar{a})$ is $\mathrm{GL}(p')$-invariant, it gives by universality of good quotients a morphism $\hat{f}$ so that

$$\begin{tikzcd}
\tilde{R}^{ss}_{Z,L}(\chi,a) \arrow[d, "g_X"'] \arrow[dr, "f"] & \\
M_L(Z;\chi,a) \arrow[r, dotted, "\hat{f}"'] & M(X;\bar{\chi},\bar{a})
\end{tikzcd}$$

commutes. □

By choosing $Z = X$ and $\mathcal{V} = \tilde{\mathcal{U}}_X|_{\tilde{R}^{ss}_{X,L}} \otimes p_X^* L'$ we verify that tensoring with a line bundle L' of high enough degree is a morphism[11] $M_L(X;\chi,a) \to M_{L\otimes L'^2}(X,\chi,a)$ as this is clearly a $\mathrm{GL}(p)$-invariant operation, since $\tilde{\mathcal{U}}_{A\cdot q} \cong \tilde{\mathcal{U}}_q$ for $A \in \mathrm{GL}(p)$. Similarly we see that pullback along $\pi\colon X \to Y$ is a morphism of varieties.

Now we specialize to rank $r = 2$ to study our elementary modification construction. Here the data χ becomes obsolete as the weights determine the multiplicities.

It is enough to consider the construction of $\overline{\mathcal{EM}}$ when we fix a determinant $\tilde{L}' \in \mathrm{Im}\{\det\colon Ma(Y) \to \mathrm{Pic}(Y)\}$ and a weight configuration a in $Ma(Y)$ and consider the restriction of $\overline{\mathcal{EM}}$ to $M_{\tilde{L}'}(Y;a) \subseteq Ma(Y)$ (i.e. when the genus of Y is greater than 1, we restrict to a single component of $Ma(Y)$, otherwise to several components).

In order to use GIT, we first need to bring ourselves into a situation where it is valid. Therefore, choose a generic fiber $\pi^{-1}(y)$ in X and an integer N so that $2Nn > 2(2g-1)$ (recall that $\deg(W\otimes[\pi^{-1}(y)]^N) = \deg W + 2N\deg[\pi^{-1}(y)] = 2Nn$). Let $d = 2Nn$ and $L_N = [\pi^{-1}(y)]^N$.

Given a lift $\tilde{\tau}$ which in turn gives the divisors D_1 and D_2, the degree of the corresponding parabolic vector bundle is $\tilde{d} = \tilde{d}(\tilde{\tau}) = \frac{1}{n}(d + 2\deg D_1 + \deg D_2) = 2N + \frac{1}{n}(2\deg D_1 + \deg D_2)$. The absolute value $|2\deg D_1 + \deg D_2|$ for any lift is bounded by n times the number of special points. By increasing N we can make sure that for any lift $\tilde{\tau}$, $\tilde{d}(\tilde{\tau})$ is bigger than the threshold $\tilde{d}_0$ that ensures that

(1) the underlying vector bundle $\tilde{E}$ is generated by its global sections, and
(2) $H^1(Y,\tilde{E}) = 0$.

This puts us in the realm of GIT.

Put $L = L_N^2$ and $\tilde{L} = \tilde{L}' \otimes L_N^2$. Since L_N is τ-invariant and $\pi^{-1}(y)$ does not intersect the special orbits, the constructions $\mathcal{P}$ and $\mathcal{EM}$ still apply, in fact they commute with tensoring by L_N: $\mathcal{P}(W\otimes L_N,\tilde{\tau}) = \mathcal{P}(W,\tilde{\tau})\otimes\tilde{L}_N$ and $\mathcal{EM}\big((\tilde{E},F,a)\otimes$

[11] This verifies that there is a well defined variety structure on the moduli space.

$\tilde{L}_N\big) = \mathcal{EM}(\tilde{E}, F, a) \otimes L_N$, where $\tilde{L}_N = [y]^N$. In fact the vertical maps in the commutative diagram

$$\begin{array}{ccc} M_{\tilde{L}'}(Y;a) & \xrightarrow{\overline{\mathcal{EM}}} & |M(X)| \\ {\scriptstyle \otimes \tilde{L}_N}\Big\downarrow & & {\scriptstyle \otimes L_N}\Big\downarrow \\ M_{\tilde{L}}(Y;a) & \xrightarrow{\overline{\mathcal{EM}}} & |M_L(X)| \end{array}$$

are variety isomorphisms by definition (that is how the variety structure is defined at low degree). Hence we must show that $\overline{\mathcal{EM}}\colon M_{\tilde{L}}(Y;a) \to |M_L(X)|$ is a morphism.

The map $\overline{\mathcal{EM}}\colon M_{\tilde{L}}(Y;a) \to |M_L(X)|$ is actually defined on the level of bundles so there is a natural lift to $\widetilde{\mathcal{EM}}\colon \tilde{R}^{ss}_{Y,\tilde{L}}(a) \to |M_L(X)|$. To use the lemma we need to construct a locally free sheaf $\mathcal{V}$ representing this $\widetilde{\mathcal{EM}}$. Choose skyscraper sheaves $\mathbb{C}_0, \ldots, \mathbb{C}_M$ of "rank" 1 such that $\dim\big((\bigoplus_{i=0}^M \mathbb{C}_i)_w\big) = m_2(w)$ everywhere. Let $\mathcal{F}$ be the sheaf over $\tilde{R}^{ss}_{Y,\tilde{L}}(a) \times_{\mathbb{C}} Y$ defined inside $\tilde{\mathcal{U}}_{\tilde{R}_{Y,\tilde{L}}(a)}$ by the flag variety, let $\mathcal{F}_0 = \bar{\pi}^*\mathcal{F}$, $\mathcal{V}_0 = \bar{\pi}^*\tilde{\mathcal{U}}_{\tilde{R}_{Y,\tilde{L}}(a)}$ and define the sheaf $\mathcal{G}_0 = (\mathcal{V}_0/\mathcal{F}_0) \otimes p_X^*\mathbb{C}_0$. This is a universal version of the skyscraper sheaf in the exact sequence giving the elementary modifications. There is a natural surjection

$$\lambda_0\colon \mathcal{V}_0 \longrightarrow \mathcal{G}_0 \to 0.$$

Let $\mathcal{V}_1 = \ker\lambda_0$, and $\iota_0\colon \mathcal{V}_1 \to \mathcal{V}_0$ the canonical sheaf inclusion. As in section 4 the natural lift $\tilde{\tau}$ to $\mathcal{V}_0$ induces a lift $\tilde{\tau}$ to $\mathcal{V}_1$ and as before, this gives a sheaf $\mathcal{F}_1 \subset \mathcal{V}_1$ defined as the eigenspaces for $\tilde{\tau}_w$, $w \in \bar{P}$, that are not annihilated by $\iota_{0,w}$. Then define $\mathcal{G}_1 = (\mathcal{V}_1/\mathcal{F}_1) \otimes p_X^*\mathbb{C}_1$, $\lambda_1\colon \mathcal{V}_1 \twoheadrightarrow \mathcal{G}_1$, and $\mathcal{V}_2 = \ker\lambda_1$. Continue the process through to $\mathcal{G}_M = (\mathcal{V}_M/\mathcal{F}_M) \otimes p_X^*\mathbb{C}_M$ and define

$$\mathcal{V} = \mathcal{V}_{M+1} = \ker\lambda_M.$$

We need to prove that $\mathcal{V}$ is locally free, then the constraints on the degree of $\mathcal{V}_q$ for every $q \in \tilde{R}^{ss}_{Y,\tilde{L}}(a)$ ensures that (1) through (3) of local universality is satisfied.

It is enough to prove that each step of the above construction yields a locally free sheaf. So assume that $\mathcal{V}_i$ is free. Let V be an open subset of $\tilde{R}^{ss}_{Y,\tilde{L}}(a)$ and U an open subset of X so that the restriction $\mathcal{V}_i|_{V\times_{\mathbb{C}}U}$ is trivial and $\mathcal{F}_i|_{V\times_{\mathbb{C}}U}$ is either $\mathcal{O}(V)$ or 0 depending on whether U intersects $\bar{P}$ or not. If the parabolic points $\bar{P} \subset X$ do not intersect U, then $\mathcal{G}_i|_{V\times_{\mathbb{C}}U} = 0$ so $\mathcal{V}_{i+1}|_{V\times_{\mathbb{C}}U} \cong \mathcal{V}_i|_{V\times_{\mathbb{C}}U} \cong \mathcal{O}^2(V \times_{\mathbb{C}} U)$. So assume $\bar{P} \cap U = w$, then $\mathcal{G}_i|_{V\times_{\mathbb{C}}U} \cong \mathbb{C}$. As in appendix A we may choose local frames e_1 and e_2 for $\mathcal{O}^2(V \times_{\mathbb{C}} U)$ that $e_2(v,w) \in \mathcal{F}_i|_{V\times_{\mathbb{C}}U}$ for all $v \in V$ and $\lambda_{i,(v,w)}(e_2(v,w)) = 1$. Then $\ker\lambda_i|_{V\times_{\mathbb{C}}U}$ is spanned as an $\mathcal{O}(V \times_{\mathbb{C}} U)$-module by $\bar{\sigma} \cdot e_1$ and e_2, where $\bar{\sigma}(v,z) = z$. We see that $\mathcal{V}_{i+1}$ is locally free.

Like we did previously, we see that the morphism represented by $\mathcal{V}$ is $\mathrm{GL}(\tilde{p})$-invariant because $\tilde{\mathcal{U}}_{\tilde{R}_{Y,\tilde{L}}(a),q} \cong \tilde{\mathcal{U}}_{\tilde{R}_{Y,\tilde{L}}(a),A\cdot q}$ for all $A \in \mathrm{GL}(\tilde{p})$. Therefore there is an induced morphism from the moduli space $M_{\tilde{L}}(Y;a)$ to $|M_L(X)|$ represented by $\mathcal{V}$. But clearly this morphism is $\overline{\mathcal{EM}}$. Hence we have proved $\overline{\mathcal{EM}}\colon M_{\tilde{L}}(Y;a) \to |M_L(X)|$ is a morphism of varieties. Thus from the above diagram we see that $\overline{\mathcal{EM}}\colon M_{\tilde{L}'}(Y;a) \to |M(X)|$ is a morphism, and ultimately we conclude:

Lemma B.14. *For any automorphism τ of a compact Riemann surface X, the map*

$$\overline{\mathcal{EM}}\colon \mathrm{Ma}(X,\tau) \longrightarrow |M(X)|$$

is a morphism of varieties. $\square$

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List of Notation

Part I

Part II