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Semiclassics of the Quantum Current in Weak and Strong Magnetic Fields

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Preface

The present document is my thesis for the Ph.D. in mathematics from the University of Aarhus. It consists of the two research articles [Fou98] (reprinted below as Chapter 2) and a revised version of [Fou99] (given below as Chapter 3) and of an overview article, given below as Chapter 1.

My thesis marks the end of more than 8 happy years as a student of mathematics in the University of Århus. I would like to profit from this opportunity to thank as many as possible of those who made this time so great. I admit, that the list below can be tideous and long, but many people have gone out of their way in order to help me, and I feel that it is only fair to mention at least some of them.

I wish to thank Ebbe Thus Poulsen for giving me a very good first impression of university mathematics - he played a larger role, than he knows, in making me choose to study mathematics, Henrik Stetkær for answering many of my most stupid questions, Tage Bai Andersen for helping me go to study in France and Mexico, my inspiring instructors and the group around Eulers Venner.

As part of my Ph.D. I have spent some time abroad, special thanks goes to those persons who received me and from whose knowledge I have benefitted:

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- **ESI March-April 98 + September-October 99:** T. and M. Hoffmann-Ostenhof.
- **Paris October 98:** B. Helffer.

My supervisor since 1995 has been J.P.Solovej. I cannot imagine, that anyone could have been a better advisor for me, and I thank him for always finding good and exciting mathematical problems for me and for knowing that personal variables enter into the equation of a mathematical carreer. Since 1997 he has been in Copenhagen, though, and my formal supervisor has been E. Skibsted. I am very grateful that he has always had his door open for me and has taken time and interest in my problems. He has often been a great help to me.

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Contents

Chapter 1

Introduction

Contents

1.1 Semiclassical analysis of the current

Semiclassical analysis has grown into a large and highly developed area in mathematical physics. One can express the philosophy behind the works as trying to understand quantum systems in terms of their classical analogues, a typical example being that the energy of a non-interacting electron gas is given (to highest order in the semi-classical parameter h) by the phase space integral of the classical Hamiltonian. At this point it might be appropriate to invoke the famous 'correspondence principle' by N. Bohr, which, for our purposes, we will state as:

When Planck's constant tends to zero, the behaviour of a quantum mechanical system tends to that of its classical counterpart.

It can therefore be said that semiclassical analysis aims at finding to what extent the correspondence principle is true, i.e. how much influence classical mechanics has on the quantum system.

Semiclassical analysis in its rigorous mathematical form also has applications to other areas of mathematical physics, let me only mention here Helffer and Robert's famous counterexample to the Lieb-Thirring conjecture [HR90] and the semiclassical analysis needed in the study of large atoms (see [LSY94] for the study in the case of large magnetic fields).

The objects traditionally studied semi-classically are the energy (or more general Riesz means) and the density.

If a physical system in external magnetic vector potential $\vec{A}$ and electric scalar potential V has energy $E(h,\vec{A},V)$, then the density ρ is given as $\frac{\delta E}{\delta V}$ i.e. ρ is the distribution

$$\int \rho\psi = \frac{dE(h,\vec{A},V+t\psi)}{dt}|_{t=0}.$$

In the same way the current $\vec{j}$ is given as $\frac{\delta E}{\delta \vec{A}}$ i.e.

$$\int \vec{j}\cdot\vec{a} = \frac{dE(h,\vec{A}+t\vec{a},V)}{dt}|_{t=0}.$$

Notice here that gauge invariance implies that E does not depend on $\vec{A}$ but only on the magnetic field $\vec{B} = \nabla\times\vec{A}$ generated by $\vec{A}$. One can therefore define the magnetisation $\vec{M} = \frac{\delta E}{\delta \vec{B}}$. It is easy to see that $\vec{j} = \nabla\times\vec{M}$.

Let us introduce the specific models that have been analysed in this thesis:

We consider an electron gas of non-interacting electrons in external electromagnetic potentials $(\vec{A},V)$ and set the chemical potential to zero. At zero temperature the energy will be given as:

$$E(h,\vec{A},V) = \mathrm{tr}[H(h,\vec{A},V)1_{(-\infty,0]}(H(h,\vec{A},V))], \tag{1.1.1.1}$$

where $H(h,\vec{A},V)$ is the Hamiltonian operator of a single electron in the external fields $(\vec{A},V)$ and depending on the parameter h ('Planck's constant'). The choice of Hamiltonian will depend slightly on the kind of limit we consider:

- **Weak magnetic fields:**
  For the weak magnetic field i.e. normal semiclassics, we take the Schrödinger

Hamiltonian:

$$H = (-ih\nabla - \vec{A})^2 + V(x).$$

In this case we do the analysis in any dimension.

- **Strong magnetic fields:**
In a strong magnetic field, it is important to include the interaction of the electron spin with the external magnetic field. Therefore we use the Pauli Hamiltonian:

$$\mathbf{P} = \mathbf{P}(h, \vec{A}, V) = (-ih\nabla - \vec{A})^2 + V(x) - h\vec{\sigma}\cdot\vec{B},$$

acting in $L^2(\mathbb{R}^3;\mathbb{C}^2)$. Here $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli spin matrices:

$$\begin{aligned}\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\end{aligned}$$

and $\vec{B} = \nabla \times \vec{A}$. The analysis in this case is only in 3 dimensions.

1.1.1 The weak magnetic field

If we take a (formal) derivative of (1.1.1.1) with respect to $\vec{A}$, we get:

$$\int \vec{a}\cdot\vec{j} = -\mathrm{tr}(J 1_{(-\infty,0]}(H(h, \vec{A}, V)),$$

where

$$J = \vec{a}\cdot(-ih\nabla - \vec{A}) + (-ih\nabla - \vec{A})\cdot\vec{a}.$$

This is the expression that we take as the definition of the current. Notice, that if $0 \notin \mathrm{Spec}(H(h, \vec{A}, V))$, then this definition is equivalent to the original one.

We expect the semiclassical limit of the current to be related to the classical current. Therefore we need to make a (very) short digression to classical mechanics.
A classical electron gas in an electromagnetic field is described by a Hamiltonian function

$$h(x, p) = (p - \vec{A})^2 + V(x).$$

The ground state, using Fermi-Dirac statistics in the chemical potential λ, is given by the set

$$\{(x, p) | h(x, p) \leq \lambda\}$$

in phase space $\mathbf{R}^3_x \times \mathbf{R}^3_p$. The energy of the gas is;

$$E_{cl} = \int_{\{h(x,p)\leq\lambda\}} h(x, p)\,dx\,dp, \tag{1.1.1.2}$$

and the density and current are again defined by

$$\rho_{cl} = \frac{\delta E_{cl}}{\delta V},$$

and

$$\vec{j}_{cl} = \frac{\delta E_{cl}}{\delta \vec{A}}.$$

There is an important difference between the classical and the quantum gas:

A classical gas in equilibrium has no current.

This is a consequence of another fact:

There is no classical diamagnetism, i.e.

$$E_{cl}(h, \vec{A}, V) = E_{cl}(h, \vec{0}, V).$$

This is easily seen by a change of variables in (1.1.1.2).

Since the density obviously cannot vanish in the classical gas we are led to expect the following:

Observation 1.1.1. If the density is of order h^{α} as h tends to zero, then we have $\vec{j} = o(h^{\alpha})$ in the same limit.

This is indeed what we will see below. Let us notice that the current does <u>not</u> vanish in a quantum mechanical gas. A free quantum gas in a constant magnetic field has a nonvanishing current. This result goes at least back to Peierls [Pei56].

Thus we are naturally led to the second remark: We need to know how the density behaves in the semi-classical limit to know where to search for the current. This question has been answered by several people, here we should at least mention [HR83, PR85, Ivr98, Sob95]. The work that has been our source of inspiration is [HR83](see also [Rob87]).

We need the following assumption:

Assumption 1.1.2.

- $V \in C^{\infty}(\mathbf{R}^n)$ and $\min_{x \in \mathbf{R}^n} V(x) > 1$.
- $\forall \alpha \in \mathbf{N}^n, \exists c_{\alpha} > 0 : \forall x \in \mathbf{R}^n$
  $|\partial_x^{\alpha} V(x)| \leq c_{\alpha} V(x)$.
- $\exists C, M > 0$ such that: $\forall x, y \in \mathbf{R}^n$
  $|V(x)| \leq C V(y)(1 + |x - y|)^M$.
- $\vec{A} \in C^{\infty}(\mathbf{R}^n, \mathbf{R}^n)$.
- $\forall \alpha \in \mathbf{N}^n$ with $|\alpha| \geq 1 \, \exists c_{\alpha} \in \mathbf{R} : ||\partial^{\alpha} A|| \leq c_{\alpha} V^{1/2}$.

We will use the notation

$$a_0(x, p) = (p - \vec{A})^2 + V(x).$$

Then it can easily be seen that the assumption above was made to guarantee that

$$|\partial^{\alpha} a_0| \leq c_{\alpha} a_0.$$

Let now λ_0 be chosen so that $a_0^{-1}((-\infty,\lambda_0])$ is compact.

The result about the density can now be formulated as follows ([Rob87]).

Theorem 1.1.3. *Let $\phi \in C_0^\infty(\mathbf{R}^n)$ and let H satisfy Assumption 1.1.2. Suppose that $\lambda < \lambda_0$ is not a critical value for a_0. Then*

$$(2\pi h)^n \int \rho(x)\phi(x)dx = \iint \phi(x)1_{(-\infty,\lambda]}((a_0(x,p))dxdp + O(h).$$

The strong interplay with classical mechanics is illustrated by the following result:

Theorem 1.1.4. *We keep the assumptions from Theorem 1.1.3.*
Let

$$\Sigma_\lambda = \{(x,p)|a_0(x,p)=\lambda\}$$

and let $\Phi^t_{a_0}$ be the Hamiltonian flow generated by a_0. Suppose

$$S_\lambda(\{(x,p)\in\Sigma_\lambda|\exists t>0 \text{ such that } \Phi^t_{a_0}(x,p)=(x,p)\})=0,$$

where S_λ is the surface measure on Σ_λ. Then

$$(2\pi h)^n \int \rho(x)\phi(x)dx = \iint \phi(x)1_{(-\infty,\lambda]}(a_0(x,p))dxdp + o(h).$$

Remark 1.1.5. Assumption 1.1.2 is very restrictive since given a $\phi \in C_0^\infty$ the integral $\int \rho(x)\phi(x)dx$ should only depend on $\vec{A}$ and V in a neighborhood of $\operatorname{supp}\phi$. This is indeed the case as shown by Sobolev in [Sob95] (see also [Ivr98]).

It is in the context of the above results for the density that the results ([Fou98]) concerning the current should be seen. The main results from that paper follow here.

Theorem 1.1.6. *Let $H(h)$ satisfy Assumption 1.1.2. Let $\lambda < \lambda_0$ and assume λ not to be a critical value for a_0* [1]. *Then*

$$(2\pi h)^n \operatorname{tr}[J1_{]-\infty,\lambda]}(H(h))] = \iint_{\{a_0(x,p)\le\lambda\}} 2\vec{a}\cdot(p-\vec{A})\,dxdp + O(h). \qquad (1.1.1.3)$$

We thus have $\vec{j} = O(h^{1-n})$ because the integral on the right hand side in eq. (1.1.1.3) vanishes. By the change of variables $q = p - \vec{A}(x)$, we get indeed

$$\begin{aligned}\iint_{\{(p-\vec{A})^2+V(x)\le\lambda\}} \vec{a}\cdot(p-\vec{A})\,dxdp &= \iint_{\{q^2-V(x)\le\lambda\}} \vec{a}\cdot q\,dxdq \\ &= \int_{x\in\mathbf{R}^n} \vec{a}(x)\cdot\Big(\int_{\{q^2\le\lambda-V(x)\}} q\,dq\Big)\,dx \\ &= 0.\end{aligned}$$

The difficulty in proving Theorem 1.1.6 in comparison to Theorem 1.1.3, is that the function

$$\tau \mapsto \operatorname{tr}[J1_{]-\infty,\tau]}(H(h))],$$

[1] This is independent of $\vec{A}$ because the critical values of a_0 are the critical values of V. Notice also that the assumption on λ_0, $a_0^{-1}((-\infty,\lambda_0])$ is compact, is independent of $\vec{A}$.

is not monotone and therefore the standard Tauberian argument does not work. Our argument uses the relative boundedness of J with respect to $H(h)$ to reduce to the counting function $\mathrm{tr}[1_{]-\infty,\tau]}(H(h))]$ where standard arguments can be applied.
Just as for the density we get a better result in the case where there are "few" closed classical orbits on the energy surface.

Theorem 1.1.7. *We maintain the assumptions from Theorem 1.1.6. Suppose*

$$S_\lambda(\{(x,p)\in\Sigma_\lambda | \exists t\neq 0\, s.t.\, \Phi^t_{a_0}(x,p)=(x,p)\})=0, \tag{1.1.1.4}$$

where S_λ is the surface measure on Σ_λ and Φ_{a_0} is the Hamiltonian flow associated with a_0. Then

$$(2\pi h)^n \mathrm{tr}[(\vec{a}\cdot(-ih\nabla-\vec{A})1_{(-\infty,\lambda]}(H(h))] = h\gamma_1 + o(h) \quad \textit{for } h\searrow 0,$$

where γ_1 is:

$$\gamma_1(\lambda)=\iint_{\{a_0(x,p)\leq\lambda\}} \frac{i}{2}\mathrm{div}(\vec{a})\, dxdp + \frac{1}{2i}\int_{\Sigma_\lambda}\{a_0,b_0\}_P \frac{dS_\lambda}{|\nabla a_0|}.$$

Here $b_0=\vec{a}\cdot(p-\vec{A})$ and $\{a_0,b_0\}_P=\partial_p a_0(x,p)\partial_x b_0(x,p)-\partial_x a_0(x,p)\partial_p b_0(x,p)$, is the Poisson bracket.

Thus $\Re(\gamma_1)=0$ and therefore the current is of order $o(h^{1-n})$ in this case.

Of course, with the comparison with classical mechanics given above, it seems reasonable to get this kind of results. But from a purely mathematical point of view, the result could be thought of as simply not being clever enough! It seems we just used too weak techniques and therefore lost all the details we looked for. Illustratively speaking, it could have been, that we looked for currents with a too weak looking glass: maybe we could see a current of order $O(h^{1-n})$. Now it rapidly becomes clear, that it is very difficult to go to higher powers in h. Therefore, for some time it seemed easier to come up with a counterexample, i.e. to construct a specific model, where the current is of *order* h^{1-n}, but is not a pure power of h - i.e. an example, where the current is *oscillating* to power h^{1-n}.

The model, that first comes to mind, is the harmonic oscillator in a constant magnetic field. This is an explicitly solvable model, and it has the further advantage, that the density has an oscillating term of order h^{1-n}. Unfortunately, this does not gives the desired conclusion, since one can prove:

The current of a harmonic oscillator in a constant magnetic field is $o(h^{1-n})$

This was (almost) proved by an explicit calculus in my progres-report (presented at my qualifying exam). More precise techniques permit one to almost get the next term in the current (see [But99] and [Doz94]), and give a way of proving oscillations in non-explicitly solvable quantum models. One has to know a great deal about the corresponding classical system though, so the analysis has, as yet, been unconclusive.

1.1.2 Strong magnetic fields

When we use the Pauli operator

$$\mathbf{P}=\mathbf{P}(h,\vec{A},V)=(-ih\nabla-\vec{A})^2+V(x)-h\vec{\sigma}\cdot\vec{B},$$

with $\vec{A} = \mu(-x_2, 0, 0)$, we get an extra term in the 'current operator':
By formal differentiation of (1.1.1.1), we get:

$$\int \vec{a} \cdot \vec{j} = -\mathrm{tr}[\mathbf{J} 1_{(-\infty,0]}(\mathbf{P})],$$

where

$$\mathbf{J} = 2\vec{a} \cdot (-ih\nabla - \vec{A}) - ih\mathrm{div}\vec{a} + h\sigma_3(\partial_{x_1} a_2 - \partial_{x_2} a_1).$$

The last term, which is the new one, comes from the interaction of the electron spin with the magnetic field. This we will refer to as the *spin-current*, as opposed to the *persistent current*, which is the rest of the current.

When the magnetic field can be strong, in the sense that $\mu h \geq c > 0$ as h tends to zero, a new kind of semiclassical behaviour occurs. This was studied in [LSY94], where an expression for the energy was found, such that

$$\frac{E}{E_{scl}} \to 1$$

as $h \to 0$, and where the limit is uniform with respect to the magnetic field. Here the expression for E_{scl} is:

$$E_{scl} = -\frac{2}{3\pi h^2} \int \sum_{n=0}^{\infty} d_n |\vec{B}| [2nh|\vec{B}| + V(x)]_-^{3/2} dx,$$

with $d_0 = \frac{1}{2\pi}$ and $d_n = \frac{1}{\pi}$ for $n \geq 1$.

If we take the derivative of this expression, with respect to $\vec{A}$, we get something, which would be the natural guess of the semiclassical limit of the current in strong magnetic fields:

$$\begin{aligned} & \int \vec{j}_{scl} \cdot \vec{a}\, dx \\ \stackrel{def}{=} \; & \frac{d}{dt} E_{scl}(\vec{A} + t\vec{a})|_{t=0} \\ = \; & \frac{-2}{3\pi h^2} \sum_{n=0}^{\infty} d_n \int (\partial_{x_1} a_2 - \partial_{x_2} a_1) \\ & \times \left([2nh\mu + V(x)]_-^{3/2} - 3nh\mu [2nh\mu + V(x)]_-^{1/2} \right) dx. \end{aligned} \tag{1.1.1.5}$$

The result, which we prove, is that this guess is correct to highest order:

Theorem 1.1.8. *Suppose*

$$V(x) = \frac{q}{|x|} + o(|x|^{-1}) \tag{1.1.1.6}$$

as $x \to 0$, and suppose that $\forall m \in \mathbb{N}^3$ $\exists C_{m,V} \in \mathbb{R}_+$ such that:

$$|\partial^m V(x)| \leq C_{m,V} |x|^{-1-|m|}, \tag{1.1.1.7}$$

$\forall x \in B(8)$.
Suppose furthermore that $\exists C = C(h,\mu)$ *such that*

$$\mathbf{P}(h,\mu,V) \geq -C. \tag{1.1.1.8}$$

Suppose finally that

- $\exists c_{\mu,1} > 0$ *such that* $\mu h \geq c_{\mu,1}$,
- $\exists c_{\mu,2} > 0$ *such that* $\mu h^3 \leq c_{\mu,2}$.

Then

$$\vec{j} \overset{h\to 0}{\to} \vec{j}_{scl},$$

in the sense of distributions in the coordinates orthogonal to the magnetic field, i.e.:

$$\int \vec{j} \cdot \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} dx \overset{h\to 0}{\to} \int \vec{j}_{scl} \cdot \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} dx,$$

for all $a_1, a_2 \in C_0^\infty(B(1))$, *where* $B(1)$ *is the closed ball of unit radius in* $\mathbb{R}^3$.

The difficulty in obtaining such a result is best illustrated by the following calculation: The spin-current can be calculated from known results ([Sob94]):

$$\begin{aligned}
&\int \vec{j}_{spin} \cdot \vec{a}\, dx \\
= \;& h\left(\operatorname{tr}\left[(\partial_{x_1} a_2 - \partial_{x_2} a_1) 1_{(-\infty,0]}(H)\right] - \operatorname{tr}\left[(\partial_{x_1} a_2 - \partial_{x_2} a_1) 1_{(-\infty,0]}(H + 2\mu h)\right]\right),
\end{aligned}$$

where $H = (-ih\nabla - \vec{A})^2 + V(x) - \mu h$ acting in $L^2(\mathbb{R}^3)$. This is known to be:

$$\begin{aligned}
\approx \;& h\Bigg(\frac{\mu}{4\pi^2 h^2} \sum_{k=0}^{\infty} \int (\partial_{x_1} a_2 - \partial_{x_2} a_1)[2nh\mu + V(x)]_-^{1/2}\, dx \\
& - \frac{\mu}{4\pi^2 h^2} \sum_{k=0}^{\infty} \int (\partial_{x_1} a_2 - \partial_{x_2} a_1)[2nh\mu + V(x) + 2\mu h]_-^{1/2}\, dx\Bigg) \\
= \;& \frac{\mu}{4\pi^2 h} \int (\partial_{x_1} a_2 - \partial_{x_2} a_1)[V(x)]_-^{1/2}\, dx.
\end{aligned}$$

This term is of order $\frac{\mu}{h}$, which is much larger than the total current (at least in the case where $\mu h \to \infty$). Thus there is a huge cancellation effect in the current, which has to be taken into account if we want to obtain a good result.

Theorem 1.1.8 only gives the current in the direction orthogonal to the magnetic field - the method applied is simply unable to pick up the current in the other direction. In fact, we run into somewhat the same problem as in the weak magnetic field case, if we look at the parallel current: The highest order term of the parallel current vanishes, but the error term is of order $\frac{1}{h^2}$. Therefore, one needs additional work to get the parallel current.

The key to prove Theorem 1.1.9 below, is gauge invariance, i.e.

$$\int \vec{j} \cdot \vec{a}\, dx,$$

does only depend on the magnetic field $\vec{b} = \nabla \times \vec{a}$ generated by $\vec{a}$. Therefore, if

$$\vec{a} = \begin{pmatrix} 0 \\ 0 \\ a_3 \end{pmatrix},$$

we might find

$$\tilde{a} = \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} \in C_0^\infty,$$

such that

$$\nabla \times \vec{a} = \nabla \times \tilde{a}.$$

This is possible if and only if $\int a_3(x_1,x_2,x_3)\, dx_3 = 0$ for all (x_1,x_2). By using this, one can 'move a_3 along the x_3 axis' and prove the following:

Theorem 1.1.9. *Let the assumptions be as in Theorem 1.1.8. Assume furthermore that $V(x_1,x_2,x_3) \geq c_V > 0$, for $1 \leq |x_3| \leq 3$, and that V is infinitesimally bounded with respect to $-\Delta$, then*

$$\vec{j} \overset{h\to 0}{\to} \vec{j}_{scl},$$

in the sense of distributions, i.e.:

$$\int \vec{j} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} dx \overset{h\to 0}{\to} \int \vec{j}_{scl} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} dx,$$

for all $a_1,a_2,a_3 \in C_0^\infty(B(1))$.

Notes on future work

The results on current in strong magnetic fields are not completely satisfactory due to the following two drawbacks:

- We need extra conditions to find the parallel current to order $\frac{1}{h^2}$.
- The semiclassical result on the orthogonal current uses all the heavy machinery devellopped by Ivrii and Sobolev ([Ivr98] and [Sob94]). A shorter and more easily understandable proof is desireable.

The first drawback worries me less than the second. My feeling, as described above, is that mathematically the problem with the parallel current is comparable to the problem about the current in standard semiclassics.

The second drawback is more serious: My proof relies heavily on the work [Sob94]. This work is purely 3-dimensional (though there should be no additional difficulty in going through the same arguments in 2-dimensions) and therefore my results on the current are only stated in 3-dimensions, though it would be physically (at least) as interesting to prove the corresponding semiclassical formula in 2-dimensions. A better understanding of the current might also permit us to go to physical models that are not of mean field type - here I think of large atoms in strong magnetic fields, as studied in [LSY94].

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Chapter 2

Current in weak magnetic fields

Contents

2.1 Abstract

We give the semi-classical asymptotics of the quantum current in 2 cases: First for $T > 0$ (T is the absolute temperature). Here we get a complete asymptotics. Then for $T = 0$. Here it vanishes to the accessible orders.

2.2 Introduction

Since the classical work of Weyl estimating the counting function for the number of eigenvalues has been one of the central themes in semiclassical analysis. In the present paper we study Schrödinger operators of the form $H(h) = H(h,\vec{A},V) = (-ih\nabla - \vec{A})^2 + V$, where V is a potential and $\vec{A}$ is a vector potential generating a magnetic field $\vec{B} = \nabla \times \vec{A}$. In this case, the Weyl estimate corresponds to the fact that the *counting function* $\operatorname{tr}\left[1_{(-\infty,\lambda]}(H(h))\right]$ is given to leading order, in the limit $h \to 0$, by the classical phase space integral $\iint 1_{(-\infty,\lambda]}\left((hp - \vec{A}(x))^2 + V(x)\right) dp\, dx$.

More detailed information is given by the *density* or *local counting function* ρ, that is given, as a distribution, by

$$\int \rho\psi\, dx = \operatorname{tr}[\psi 1_{(-\infty,\lambda]}(H(h))],$$

for all $\psi \in C_0^\infty$. Clearly the counting function is the total integral $\int \rho\, dx$.

The physical interpretation of ρ is that it is the density of a non-interacting Fermi gas in the fields V and $\vec{B}$, and with chemical potential λ. An equally important physical quantity is the *quantum current* $\vec{j}$. The density is the response in the total Fermi energy to a variation in the potential, i.e.,

$$\int \rho\psi\, dx = \frac{\partial}{\partial\varepsilon}\operatorname{tr}[(H(h,\vec{A},V+\varepsilon\psi) - \lambda)1_{(-\infty,\lambda]}(H(h,\vec{A},V+\varepsilon\psi))]\bigg|_{\varepsilon=0}.$$

Likewise, the current is the response in the total Fermi energy to a variation in the vector potential, i.e.,

$$\int \vec{j}\cdot\vec{a}\, dx = \frac{\partial}{\partial\varepsilon}\operatorname{tr}\left[(H(h,\vec{A}+\varepsilon\vec{a},V) - \lambda)1_{(-\infty,\lambda]}\left(H(h,\vec{A}+\varepsilon\vec{a},V)\right)\right]\bigg|_{\varepsilon=0}.$$

The quantum current has largely been ignored in semiclassical analysis. The reason is quite simply that classically there is no current. The reason being that there is no classical diamagnetism, i.e., the classical energy

$$\iint (hp - \vec{A}(x))^2 + V(x)\, dp\, dx$$

is independent of $\vec{A}$, which is easily seen by a change of variables.

The aim of this paper is to prove that the semiclassical limit of the quantum current indeed vanishes. Moreover, we explore to which order in the semiclassical parameter h the current vanishes.

Formally we have (and it can be justified under the assumptions that we will impose below):

$$\begin{aligned}\int \vec{a}\cdot\vec{j}\,dx &= \frac{\partial}{\partial\varepsilon}\mathrm{tr}\left[(H(h,\vec{A}+\varepsilon\vec{a},V)-\lambda)1_{(-\infty,\lambda]}\left(H(h,\vec{A}+\varepsilon\vec{a},V)\right)\right]\Big|_{\varepsilon=0}\\ &= 2\mathrm{tr}\left[(\vec{a}\cdot(-ih\nabla-\vec{A})-\frac{ih}{2}\mathrm{div}(\vec{a}))1_{(-\infty,\lambda]}\left(H(h,\vec{A},V)\right)\right].\end{aligned}$$

Notice that since the operator multiplying the characteristic function is symmetric, we get,

$$\begin{aligned}\frac{1}{2}\int \vec{a}\cdot\vec{j}\,dx &= \Re\{\mathrm{tr}\left[\vec{a}\cdot(-ih\nabla-\vec{A})1_{(-\infty,\lambda]}\left(H(h,\vec{A},V)\right)\right]\}\\ &= \Re\{\mathrm{tr}\left[B(h)1_{(-\infty,\lambda]}\left(H(h,\vec{A},V)\right)\right]\},\end{aligned}$$

where $B(h)=\vec{a}\cdot(-ih\nabla-\vec{A})$, and where $\Re(z)$ denotes the real part of z. This is the quantity we will study semi-classically.
We will denote by a_0 the Weyl-symbol of $H(h)$. It is easy to see that

$$a_0(x,p)=(p-\vec{A}(x))^2+V(x). \tag{2.2.2.1}$$

To assure that a_0 satisfies:

$$|\partial^\alpha a_0|\le c_\alpha|a_0|,$$

for all $\alpha\in\mathbf{N}^n$, we will impose the following assumption on $\vec{A}$ and V.

Assumption 2.2.1.

- $V\in C^\infty(\mathbf{R}^n)$ and $\min_{x\in\mathbf{R}^n}V(x)>1$.
- $\forall\alpha\in\mathbf{N}^n,\ \exists c_\alpha>0:\ \forall x\in\mathbf{R}^n$
  $|\partial_x^\alpha V(x)|\le c_\alpha V(x)$.
- $\exists C,M>0$ such that: $\forall x,y\in\mathbf{R}^n$
  $|V(x)|\le CV(y)(1+|x-y|)^M$.
- $\vec{A}\in C^\infty(\mathbf{R}^n,\mathbf{R}^n)$.
- $\forall\alpha\in\mathbf{N}^n$ with $|\alpha|\ge 1\,\exists c_\alpha\in\mathbf{R}:\ \|\partial^\alpha A\|\le c_\alpha V^{1/2}$.

These will be standing assumptions throughout the paper. Notice that this includes the case of constant magnetic field. That $V\ge 1$ means in reality only that it is bounded below, for we can add a constant to the potential without changing the current - we only have to change λ correspondingly.

Under the above assumptions we know from [Rob87, Thm III-4] that $\exists h_0>0$ so that $H(h)$ is essentially self-adjoint on $\mathcal{S}(\mathbf{R}^n)$ (the Schwartz space of rapidly decreasing smooth functions) and uniformly bounded below for $h\in(0,h_0]$. These rather strong

asumptions on V, $\vec{A}$ (compare [AHS78]) are made to be in a position to apply the functional calculus of Helffer and Robert [HR83] (see also [Rob87]).
For notational convenience we will introduce the operator:

$$P_A = -ih\nabla - \vec{A}.$$

and write

$$B(h) = \vec{a} \cdot P_A.$$

Then $B(h)$ has Weyl symbol

$$\sigma[B(h)](x,p) = \vec{a} \cdot (p - \vec{A}) - \frac{h}{2i}\mathrm{div}(\vec{a}), \tag{2.2.2.2}$$

which we will write as $b_0 + hb_1$.

Let us take λ_0 such that $a_0^{-1}((-\infty,\lambda_0])$ is compact. This λ_0 will be fixed throughout the text. Our plan is to calculate the asymptotics of

$$\mathrm{tr}[B(h)f(H(h))] \quad \text{for } h \searrow 0, \tag{2.2.2.3}$$

first for $f \in C_0^\infty(-\infty,\lambda_0)$ and later for $f = 1_{(-\infty,\lambda]}$ with $\lambda < \lambda_0$.

The organization of the paper is the following: In sections 2.3 and 2.4 we prove that the operator $B(h)$ can be included in the functional calculus by Helffer and Robert. Thus we get for $f \in C_0^\infty(-\infty,\lambda_0)$ that $B(h)f(H(h))$ is h-admissible, and we get a complete expansion of $\mathrm{tr}[B(h)f(H(h))]$ in powers of h. These two sections follow very closely the presentation in Robert's book [Rob87].
As a reasonably easy application of the results in section 2.4, we calculate the semiclassical asymptotics for the quantum current $\vec{j}$, for a Fermi gas at $T > 0$ (T absolute temperature) in section 2.5. Here $\vec{j}$ is given by

$$\int_{\mathbf{R}^n} \vec{a} \cdot \vec{j}\, dx = \Re\{\mathrm{tr}[B(h)f(H(h))]\},$$

where $V(x)$ is supposed to go to infinity for $|x| \to \infty$, and f is the function:

$$f(t) = \frac{e^{-\beta(t-\alpha)}}{1 + e^{-\beta(t-\alpha)}},$$

with β playing the role of an inverse temperature.

Theorem 2.2.2. *Let $H(h)$ satisfy Assumption 2.2.1 and let furthermore $V(x)$ satisfy:*

$$\exists c, s > 0 \textit{ so that } V(x) \geq c|x|^s.$$

Let $c(h) = \sum h^j c_j$ be the symbol (calculated formally) of the operator $B(h)f(H(h))$. Then the c_j are in $L^1(\mathbf{R}_x^n \times \mathbf{R}_\xi^n)$ and we have:

$$(2\pi h)^n \mathrm{tr}[B(h)f(H(h))] = \sum_{j=0}^{N} h^j \iint c_j(x,\xi)\, dx d\xi + O(h^{N+1}) \quad \textit{for } h \searrow 0.$$

Remark 2.2.3. The first term in the expansion is zero, and the next is purely imaginary, so we have to go to order h^{2-n} to get a contribution to the current.

In section 2.6 we go on to calculate the semiclassical asymptotics for the quantum current at zero temperature. Here $\vec{j}$ is defined by the equation:

$$\int_{\mathbf{R}^n} \vec{a} \cdot \vec{j}\, dx = \Re\{\mathrm{tr}[B(h) 1_{(-\infty,\lambda]}(H(h))]\}. \tag{2.2.2.4}$$

Theorem 2.2.4. *Let $H(h)$ satisfy Assumption 2.2.1. Let $\lambda < \lambda_0$ and assume λ not to be a critical value for a_0* [1]*. Then*

$$(2\pi h)^n \mathrm{tr}[B(h) 1_{]-\infty,\lambda]}(H(h))] = \iint_{\{a_0(x,p)\leq\lambda\}} \vec{a}\cdot(p-\vec{A})\, dx\, dp + O(h). \tag{2.2.2.5}$$

We thus have $\vec{j} = O(h^{1-n})$ because the integral on the right hand side in eq. (2.2.2.5) vanishes. By the change of variables $q = p - \vec{A}(x)$, we get indeed

$$\begin{aligned}
\iint_{\{(p-\vec{A})^2+V(x)\leq\lambda\}} \vec{a}\cdot(p-\vec{A})\, dx dp &= \iint_{\{q^2-V(x)\leq\lambda\}} \vec{a}\cdot q\, dx dq \\
&= \int_{x\in\mathbf{R}^n} \vec{a}(x)\cdot\Big(\int_{\{q^2\leq\lambda-V(x)\}} q\, dq\Big)\, dx \\
&= 0.
\end{aligned}$$

The problem in proving Theorem 2.2.4 is that the function

$$\tau \mapsto \mathrm{tr}[B(h) 1_{]-\infty,\tau]}(H(h))],$$

is not monotone and therefore the standard Tauberian argument does not work. Our argument uses the relative boundedness of $B(h)$ with respect to $H(h)$ to reduce to the counting function $\mathrm{tr}[1_{]-\infty,\tau]}(H(h))]$ where standard arguments can be applied.
In section 2.7 we will improve Theorem 2.2.4 in the case where the set of classical closed orbits on the energy surface

$$\Sigma_\lambda = \{(x,p) | a_0(x,p) = \lambda\}$$

is of surface measure zero.

Theorem 2.2.5. *We maintain the assumptions from Theorem 2.2.4. Suppose:*

$$S_\lambda(\{(x,p)\in\Sigma_\lambda | \exists t \neq 0\, s.t.\, \Phi^t_{a_0}(x,p) = (x,p)\}) = 0, \tag{2.2.2.6}$$

where S_λ is the surface measure on Σ_λ and Φ_{a_0} is the Hamiltonian flow associated with a_0. Then

$$(2\pi h)^n \mathrm{tr}[B(h) 1_{(-\infty,\lambda]}(H(h))] = h\gamma_1 + o(h) \quad \textit{for } h \searrow 0,$$

where γ_1 is:

$$\gamma_1(\lambda) = \iint_{\{a_0(x,p)\leq\lambda\}} b_1\, dx dp + \frac{1}{2i}\int_{\Sigma_\lambda} \{a_0, b_0\}_P \frac{dS_\lambda}{|\nabla a_0|}.$$

Here $\{a_0,b_0\}_P = \partial_p a_0(x,p)\partial_x b_0(x,p) - \partial_x a_0(x,p)\partial_p b_0(x,p)$, is the Poisson bracket.

[1] This is independent of $\vec{A}$ because the critical values of a_0 are the critical values of V. Notice also that the assumption on λ_0, $a_0^{-1}((-\infty,\lambda_0])$ is compact, is independent of $\vec{A}$.

As γ_1 is purely imaginary the conclusion of this paper is that the quantum current at zero temperature vanishes to all accessible orders. We believe that the next term in the expansion of the current (the term of order $O(h^{1-n})$) may be oscillating. A simple calculation shows that for the counting function the term of order $O(h^{-1})$ is indeed oscillating for the harmonic oscillator in dimension 2 (i.e. $\vec{a}=0, V=x^2+1$). Unfortunately in the case of the harmonic oscillator with a constant magnetic field in dimension 2, i.e. $\vec{A}=(-x_2,0)$, $V(x)=x^2+1$, the contributions to the current within an energy level cancel to such an extent that the total current is $O(1)$ - not $O(h^{-1})$. One thus has to search for strong current in more complicated settings.

Notations

Since the first sections are very close to the book [Rob87], we will use the notations therein. Thus $Op_h^w a$ denotes the h-pseudodifferential operator with Weyl symbol a. We will also use the symbol classes and theorems on continuity and composition from the same book. Finally we write

$$\int_{\mathbf{R}^n} f(p)d_h p = \frac{1}{(2\pi h)^n}\int_{\mathbf{R}^n} f(p)dp.$$

2.3 Functional Calculus

Let $f \in C_0^\infty(-\infty,\lambda_0)$ real valued. We want to prove that $B(h)f(H(h))$ is h-admissible. That will be the result of Prop. 2.3.5 below.

Lemma 2.3.1. *We have the following inclusion of domains (as closures of quadratic forms starting from $\mathcal{S}(\mathbf{R}^n)$):*

- $\mathcal{D}(V) \supseteq \mathcal{D}(H(h))$,
- $\mathcal{D}(P_A^2) \supseteq \mathcal{D}(H(h))$.

Lemma 2.3.2.

$$\|B(h)f(H(h))\|_{\mathcal{L}(L^2)} = O(1) \textit{ for } h \searrow 0,$$

where $\mathcal{L}(L^2)$ is the space of bounded operators on L^2.

Proof. We know [Rob87, Thm.III-4, Prop.III-13] that $f(H(h))$ is self-adjoint, bounded and of finite rank.

Let $v \in L^2$, $\|v\|=1$. Then we write

$$v = \sum_{j=1}^{N(h)} \alpha_j(h)\phi_j(h) + w(h),$$

where $\langle \phi_j,\phi_k\rangle = \delta_{j,k}$, $H(h)\phi_j(h) = \lambda_j(h)\phi_j(h)$ and $f(H(h))w(h)=0$.

Then

$$
\begin{aligned}
\sum_{l=1}^{n} \|(P_A)_l f(H(h))v\|^2 &\leq \langle \sum_{j=1}^{N} \alpha_j f(\lambda_j)\phi_j, (P_A)^2 \sum_{j=1}^{N} \alpha_j f(\lambda_j)\phi_j \rangle \\
&= \sum_{j,k} \bar{\alpha}_j \alpha_k f(\lambda_k)\overline{f(\lambda_j)} \langle \phi_j, (\lambda_k - V(x))\phi_k \rangle \\
&= \sum |\alpha_j|^2 f(\lambda_j)^2 \lambda_j - \langle f(H(h))v, V(x) f(H(h))v \rangle \\
&\leq \sum |\alpha_j|^2 f(\lambda_j)^2 \lambda_j - \inf V \|f(H(h))v\|^2 \\
&\leq \|f\|_\infty^2 \lambda_0,
\end{aligned}
$$

where we have used Lemma 2.3.1. □

Let $f \in C_0^\infty(-\infty, \lambda_0)$. Then $f(H(h))$ is h-admissible by the functional calculus (see [Rob87]). Therefore for all $N \in \mathbf{N}$ sufficiently big there exist $a_{f,1}, a_{f,2}, ..., a_{f,N}$ such that:

$$
f(H(h)) = \sum_{j=0}^{N} h^j Op_h^w a_{f,j} + h^{N+1} D_{f,N+1}(h),
$$

where $D_{f,N+1}$ is uniformly bounded in $\mathcal{L}(L^2)$. Moreover we have:

Lemma 2.3.3. *$B(h)D_{f,N+1}$ is uniformly bounded in $\mathcal{L}(L^2)$ and satisfies*

$$
\|B(h)D_{f,N+1}\|_{\mathcal{L}(L^2)} \leq \|\vec{a}\|_\infty p(f),
$$

where p is a seminorm in $\mathcal{S}$. Here p does not depend on h, $\vec{a}$ and f.

Proof. We recall that $1 \leq \inf\{\bigcup_{h\in(0,h_0]} \mathrm{Spec}(H(h))\}$. By the spectral theorem we may thus assume that $f \in C_0^\infty(0, \lambda_0)$. We define the continuous function $\theta(s)$ for $s \in \mathbf{C}$ as:

$$
\theta(s) = \begin{cases} \frac{1}{|\Im(s)|} & \text{if } |\Im(s)| \geq 2/\pi \\ \pi/2 & \text{otherwise} \end{cases},
$$

where $\Im(s)$ denotes the imaginary part of s. Let us define, for $s \in \mathbf{C}$, the path $Z_{\theta(s)}$ in the complex plane as the union of the following three paths:

- $\gamma_\pm = \{(t+1/2)e^{\pm i\theta(s)} | t \geq 0\}$
- $\gamma_0 = \{e^{iv}/2 | -\theta(s) \leq v \leq \theta(s)\}$.

When we integrate over $Z_{\theta(s)}$ we will look upon it as a continuous path starting with γ_-. We will denote the Mellin transform of f by $\mathcal{M}[f]$:

$$
\mathcal{M}[f](s) = \int_0^{+\infty} t^{s-1} f(t)\, dt,
$$

defined on all of $\mathbf{C}$. We remember the following properties of the Mellin transform:

- $\mathcal{M}[f](s)$ is holomorhic on all of $\mathbf{C}$.

- $\forall \rho \in \mathbf{R}$ we have: $\mathcal{M}[f](s)$ decreases rapidly on the line $\{s \in \mathbf{C} | \Re(s) = \rho\}$.
- $\forall \rho \in \mathbf{R}$ we have: $f(t) = \frac{1}{2\pi i}\int_{\rho - i\infty}^{\rho + i\infty} \mathcal{M}[f](s) t^{-s} ds.$

Now we are ready to start the proof. We have by construction ([Rob87, eq.(46),p.143 and Thm.III-10)])

$$D_{f,N+1} = \frac{1}{(2\pi)^2}\int_{\rho - i\infty}^{\rho + i\infty} \mathcal{M}[f](s) \int_{Z_{\theta(s)}} z^{-s}(H(h) - z)^{-1}\Delta_{z,N+1}(h)\, dz\, ds, \qquad (2.3.2.1)$$

where $\Delta_{z,N+1}(h)$, again by construction, satisfies

$$\|\Delta_{z,N+1}(h)\|_{\mathcal{L}(L^2)} = O\Big(\Big(\frac{|z|}{d(z)}\Big)^{q(N)}\Big)$$

([Rob87, Thm. III-9]), where $d(z) = \mathrm{dist}(z, [1, \infty[)$, $\rho \in \mathbf{R}$ and $q(N)$ is a sufficiently big integer independent of z and h. Because of Lemma 2.3.4 we can apply the differential operator $B(h)$ under the two integral signs, and we obtain the desired estimate, because for ρ sufficiently big the double integral becomes absolutely convergent:

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} |\mathcal{M}[f](\rho + is)| \\
&\qquad \times \int_{Z_{\theta(\rho+is)}} |z^{-(\rho+is)}| \|B(h)(H(h) - z)^{-1}\Delta_{z,N+1}(h)\|_{\mathcal{L}(L^2)}\, dz\, ds \\
&\le \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} |\mathcal{M}[f](\rho + is)| \Big\{ c\|\vec{a}\|_\infty e^{\theta|s|} \\
&\qquad + c\sin(\theta(\rho + is))^{-q(N)-1/2} e^{\theta|s|}\int_0^\infty (r + 1/2)^{-\rho}\, dr \Big\}\; ds \\
&\le c\|\vec{a}\|_\infty \int_{-\infty}^{\infty} |\mathcal{M}[f](\rho + is)| [1 + (\sin(\theta(s)))^{-q(N)-1/2}] ds \\
&\le c\|\vec{a}\|_\infty \int_{-\infty}^{\infty} |\mathcal{M}[f](\rho + is)(1 + s^2)^M| \frac{[\theta(s) + (\sin(\theta(s)))^{-q(N)-1/2}]}{(1 + s^2)^M} ds.
\end{aligned}$$

As $\frac{1}{\sin(\theta(\rho+is))} = O(s)$ for $s \to \infty$ and $\mathcal{M}[f](\rho + is)$ decreases rapidly on the line of integration, we get that the integral converges (to assure convergence of the r integral we can take $\rho = 2$). If we investigate the f dependence, which comes from the term

$$\sup_s |\mathcal{M}[f](\rho + is)(1 + s^2)^M|,$$

we get the seminorm of f that we wanted. $\square$

Lemma 2.3.4. *For $z \in \gamma_0$ we have:*

$$\|B(h)(H(h) - z)^{-1}\|^2_{\mathcal{L}(L^2)} \le 4n\|\vec{a}\|^2_\infty.$$

For $z \in \gamma_\pm$ we have:

$$\|B(h)(H(h) - z)^{-1}\|^2_{\mathcal{L}(L^2)} \le \frac{n\|\vec{a}\|^2_\infty}{\sin\theta}.$$

Proof. Because $H(h)$ is essentially self-adjoint on $\mathcal{S}$, $\{\phi \in L^2 | (H(h)-z)^{-1}\phi \in \mathcal{S}\}$ is dense in L^2 for $z \notin \mathbf{R}$. Let ϕ be in the above set, $z \in Z_{\theta(s)}$, and let $\psi = (H(h)-z)^{-1}\phi$. Then we have:

$$\begin{aligned} \|B(h)(H(h)-z)^{-1}\phi\|^2 &= n\|\vec{a}\|_\infty^2 \langle \psi, P_A^2 \psi \rangle \\ &\leq n\|\vec{a}\|_\infty^2 \langle \psi, H(h)\psi \rangle \\ &= n\|\vec{a}\|_\infty^2 \langle \phi, [H(h)-\bar{z}]^{-1} H(h) [H(h)-z]^{-1}\phi \rangle. \end{aligned}$$

So now we just need to prove that the function $g(x,z) = \frac{x}{(x-\bar{z})(x-z)} = \frac{x}{|x-z|^2}$ is bounded by 4 for $(x,z) \in [1,\infty) \times \gamma_0$ and by $\frac{1}{\sin\theta}$ for $(x,z) \in [1,\infty) \times \gamma_\pm$. The inequality for $z \in \gamma_0$ is obvious, and, for $z \in \gamma_\pm$, we get by simple geometric considerations that:

$$\frac{x}{|x-z|^2} \leq \frac{x}{x^2 \sin\theta} \leq \frac{1}{\sin\theta}.$$

$\square$

We conclude the following:

Proposition 2.3.5. *The operator $B(h)f(H(h))$ is h-admissible, and its symbol can be calculated as follows:*

$$\sigma[B(h)f(H(h))] \equiv \sigma[B(h)H_{f,N}] \quad (\text{mod } h^{N+1}),$$

where $H_{f,N} = \sum_{j=0}^N h^j Op_h^w(a_{f,j})$ is the sum of the first $N+1$ terms in the expansion of $f(H(h))$ as an h-admissible operator.

Proof. This is an easy consequence of the Lemma 2.3.3. Let us indeed write $f(H(h)) = H_{f,N} + h^{N+1} D_{N+1}(h)$. Then $B(h)H_{f,N}$ is h-admissible and strongly h-admissible because the symbol of $H_{f,N}$ has compact support. Thus, we only have to prove that $B(h)D_{N+1}(h)$ is bounded in $\mathcal{L}(L^2)$, and that is the statement of Lemma 2.3.3. $\square$

2.4 Traces with smooth functions

Our task in this section will be to find the asymptotics of

$$\operatorname{tr}(B(h)f(H(h))) \quad \text{for } h \searrow 0, \tag{2.4.2.1}$$

for $f \in C_0^\infty(-\infty, \lambda_0)$. This is given by Theorem 2.4.4 below.

Lemma 2.4.1.

$$\|B(h)f(H(h))\|_{tr} = O(h^{-n}).$$

Proof. Let $g \in C_0^\infty(-\infty, \lambda_0)$, $g \equiv 1$ on $\operatorname{supp} f$. We write:

$$\begin{aligned} g(H(h)) &= H_{g,N} + h^{N+1} D_{g,N+1}(h), \\ \text{where } H_{g,N} &= \sum_{j=0}^N h^j Op_h^w(a_{g,j}). \end{aligned}$$

We can then calculate:

$$\begin{aligned} B(h)f(H(h)) &= B(h)f(H(h))g(H(h)) \\ &= B(h)f(H(h))H_{g,N} + B(h)f(H(h))h^{N+1}D_{g,N+1}(h). \end{aligned}$$

Thus

$$B(h)f(H(h))[I-h^{N+1}D_{g,N+1}(h)] = B(h)f(H(h))H_{g,N}(h). \tag{2.4.2.2}$$

Here [...] is invertible for h small enough. So we get:

$$\begin{aligned} \|B(h)f(H(h))\|_{tr} \leq\ & \|B(h)f(H(h))\|_{\mathcal{L}(L^2)} \\ & \|I-h^{N+1}D_{g,N+1}(h)\|^{-1}_{\mathcal{L}(L^2)}\|H_{g,N}\|_{tr}. \end{aligned}$$

The first factor on the right is estimated by Lemma 2.3.2 and gives a constant. The second factor converges to 1, and the third factor is estimated by [Rob87, Thm. II-49] and gives an $O(h^{-n})$-term. □

Lemma 2.4.2. *We take the expansion of $f(H(h))$ as an h-admissible operator, as well, and we get:*

$$\|B(h)f(H(h)) - B(h)H_{f,N}H_{g,N}\|_{tr} = O(h^{N+1-n}).$$

Proof. Since $f(H(h))g(H(h)) = f(H(h))$, we get

$$\begin{aligned} \|\dots\|_{tr} \leq h^{N+1}\Big[& \|B(h)f(H(h))\|_{tr}\|D_{g,N+1}\|_{\mathcal{L}(L^2)} \\ & + \|B(h)D_{f,N+1}(h)\|_{\mathcal{L}(L^2)}\|H_{g,N}\|_{tr}\Big]. \end{aligned}$$

Here the trace norms on the right are known to be $O(h^{-n})$. The $\mathcal{L}(L^2)$-norms are known to be bounded in h, see Lemma 2.3.3. □

Lemma 2.4.3.

$$\|B(h)H_{f,N}H_{g,N} - B(h)H_{f,N}\|_{tr} = O(h^{N+1-n}). \tag{2.4.2.3}$$

Proof. We have:

$$f(H(h))g(H(h)) = f(H(h)) = H_{f,N} + h^{N+1}D_{f,N+1}.$$

But on the other hand we have as well:

$$\begin{aligned} f(H(h))g(H(h)) &= [H_{f,N} + h^{N+1}D_{f,N+1}][H_{g,N} + h^{N+1}D_{f,N+1}] \\ &= H_{f,N}H_{g,N} + h^{N+1}[\text{bounded terms}], \end{aligned}$$

so $H_{f,N} = H_{f,N}H_{g,N}$ modulo $O(h^{N+1})$.
NOTE: Here we have used that the symbols of $H_{f,N}$ and $H_{g,N}$ have compact supports and thus give rise to uniformly (in h) bounded operators on L^2 .
We can therefore write:

$$B(h)[H_{f,N}H_{g,N} - H_{f,N}] = h^{N+1}B(h)Op_h^w(\delta(h)).$$

Where $\delta(h)$ is an h-admissible function in the class denoted by $\Sigma_1^{(1+x^2+p^2)^{1/2}}$ in [Rob87] . This comes from using the composition theorem for strongly admissible operators on $H_{f,N}H_{g,N}$ and then noticing that the first N terms must be killed by the terms in $H_{f,N}$. The composition $B(h)Op_h^w(\delta(h))$ is thus a composition of strongly admissible operators with symbols in $\Sigma_1^{(1+x^2+p^2)^{1/2}}$. We find:

$$\|B(h)Op_h^w(\delta(h))\|_{tr} = O(h^{-n}), \tag{2.4.2.4}$$

by [Rob87, Thm.II-49] and the composition theorem. □

We now have:

Theorem 2.4.4. *Let the h-admissible operator $B(h)f(H(h))$ have the (formal) symbol $\sum_{j=0}^{\infty} h^j c_j$. Then we have the following asymptotics:*

$$(2\pi h)^n \mathrm{tr}[B(h)f(H(h))] = \sum_{j=0}^{\infty} h^j \iint_{\mathbf{R}^{2n}} c_j(x,p)\, dx dp.$$

Proof. The proof is easy given the observations, because we have:

$$\|B(h)f(H(h)) - B(h)H_{f,N}(h)\|_{tr} = O(h^{N+1-n}), \tag{2.4.2.5}$$

and the trace of $B(h)H_{f,N}(h)$ is easy to calculate because both operators in the composition are strongly h-admissible. □

Remark 2.4.5. We can easily calculate the first terms of the symbol $\sum h^j c_j$:

$$\begin{aligned} \sigma[B(h)] &= b_0 + hb_1, \\ \sigma[f(H(h))] &= f(a_0) - h^2 \frac{1}{24} f''(a_0) \sum_{j,k} \left(\frac{\partial^2 a_0}{\partial p_j \partial p_k} \frac{\partial^2 a_0}{\partial x_j \partial x_k} - \frac{\partial^2 a_0}{\partial x_j \partial p_k} \frac{\partial^2 a_0}{\partial p_j \partial x_k}\right) \\ &+ O(h^3). \end{aligned}$$

Thus

$$c_0 = b_0 f(a_0), \tag{2.4.2.6}$$

$$c_1 = b_1 f(a_0) + \frac{1}{2i}\{b_0, f(a_0)\}_P. \tag{2.4.2.7}$$

2.5 Current for positive temperature

We define

$$f(x) = \frac{e^{-\beta(x-\alpha)}}{1+e^{-\beta(x-\alpha)}},$$

where β is to be understood as an inverse temperature, and α is a constant[2].

[2] α is included because we have translated our operator to have $\inf V(x) > 1$.

To assure that the expression $\mathrm{tr}[B(h)f(H(h))]$ makes sense, we make the following assumption:

$$\exists c, s > 0 \text{ so that } V(x) \geq c|x|^s. \tag{2.5.2.1}$$

Under this assumption the spectrum of $H(h)$ is purely discrete. We write $\lambda_k(h)$ (resp. $\phi_k(h)$) for the sequence of eigenvalues (resp. eigenvectors).

Observation 2.5.1. Because we only need f on $\mathrm{Spec}(H(h))$ for the abstract definition of $f(H(h))$, we may assume that $\mathrm{supp} f \subseteq (0,\infty)$, $f \in \mathcal{S}$ and $\exists u \in \mathcal{S}$ so that $f = u^2$.

Then we want to prove Theorem 2.2.2.

Lemma 2.5.2. *There exists $N \in \mathbf{N}$ only depending on s in (2.5.2.1) so that*

$$\|B(h)f(H(h))\|_{tr} = c \sup_{\lambda}\{(|u(\lambda)| + |u'(\lambda)|)(1+\lambda^2)^N\}h^{-n}.$$

Proof. We write

$$B(h)f(H(h)) = [B(h)u(H(h))]u(H(h)),$$

and we will prove that the two operators are Hilbert-Schmidt.We have,

$$\begin{aligned}
\|B(h)u(H(h))\|_{HS}^2 &= \sum_{k=1}^{\infty} \|B(h)\phi_k(h)\|^2 f(\lambda_k(h)) \\
&\leq \sum_{k=1}^{\infty} c(\sum_l \|P_{A,l}\phi_k(h)\|)^2 f(\lambda_k(h)) \\
&\leq \sum_{k=1}^{\infty} c(1 + \langle P_A^2\phi_k(h), \phi_k(h)\rangle) f(\lambda_k(h)) \\
&\leq \sum_{k=1}^{\infty} c(1+\lambda_k(h)) f(\lambda_k(h)).
\end{aligned}$$

So we only have to prove, that if $\tilde{u}$ is Schwartz, then $\sum_{k=0}^{\infty} \tilde{u}(\lambda_k(h))$ converges and is $O(h^{-n})$. But this follows easily from the CLR-estimate([Sim79]):
$\exists c > 0$ so that:

$$N_h(\lambda) \leq ch^{-n} \iint_{\{a_0(x,p)\leq\lambda\}} dx dp,$$

where $N_h(\lambda)$ is the number of eigenvalues of $H(h)$ smaller than λ. From this (remember that $V(x) \geq c|x|^s$) we get

$$N_h(\lambda) \leq ch^{-n}\lambda^{n\frac{2+s}{2s}}.$$

This implies that $N_h(\cdot)$ defines a tempered distribution. We have

$$N_h'(\lambda) = \sum_k \delta(\lambda - \lambda_k(h)),$$

which implies:

$$\begin{aligned}
\sum_{k=0}^{\infty} \tilde{u}(\lambda_k(h)) &= \int N_h'(\lambda)\tilde{u}(\lambda)\, d\lambda \\
&= -\int \tilde{u}'(\lambda) N_h(\lambda)\, d\lambda.
\end{aligned}$$

This finishes the proof. $\square$

Let us take a partition of unity and multiply it by f to get a family of functions $\{f_j^2\}_{j=0}^{\infty}$ so that $f_j^2 \equiv f$ on $[j+1/4, j+3/4]$, $\operatorname{supp} f_j \subseteq [j-1/4, j+5/4]$. Then it is clear from the above lemma that $\sum B(h) f_j^2(H(h))$ converges to $B(h)f(H(h))$ in trace norm. The idea is now to try to take the limit $h \searrow 0$ inside the sum. To be able to do this we need the following lemma:

Lemma 2.5.3. *$B(h)f_j^2(H(h))$ is h-admissible, so we write*

$$B(h)f_j^2(H(h)) = \sum_{k=0}^{N} h^k Op_h^w(c_{j,k}) + h^{N+1} R_{j,N+1}(h),$$

where $\{R_{j,N+1}(h) | h \leq h_0\}$ is bounded in $\mathcal{L}(L^2)$, then

$$\|R_{j,N+1}(h)\|_{tr} \leq c h^{-n} p(f_j^2) P(j)$$

where $P(j)$ is a polynomial in j and where $p(f)$ is a seminorm of f in the Schwartz space.

Proof. We take $g \in C_0^\infty$, $g \equiv 1$ on $[-1/4, 5/4]$. Let $g_j(t) = g(t-j)$, then $g_j f_j^2 = f_j^2$. We write

$$\begin{aligned} h^{N+1}\|R_{j,N+1}(h)\|_{tr} \leq\ & \|B(h)f_j^2(H(h)) - B(h)H_{f_j,N}H_{g_j,N}\|_{tr} \\ & + \|B(h)H_{f_j,N}H_{g_j,N} - B(h)H_{f_j,N}\|_{tr} \\ & + \|B(h)H_{f_j,N} - C_{j,N}(h)\|_{tr}, \end{aligned} \tag{2.5.2.2}$$

where

$$\begin{aligned} f_j^2(H(h)) &= H_{f_j,N} + h^{N+1} D_{f_j,N+1} \\ g_j(H(h)) &= H_{g_j,N} + h^{N+1} D_{g_j,N+1} \\ C_{j,N} &= \sum_{k=0}^{N} h^k Op_h^w(c_{j,k}). \end{aligned}$$

We try to estimate each of the three terms on the right hand side in (2.5.2.2)
<u>The first:</u>

$$\begin{aligned} & \|B(h)f_j^2(H(h)) - B(h)H_{f_j,N}H_{g_j,N}\|_{tr} \\ \leq\ & h^{N+1}\|B(h)f_j^2(H(h)D_{g_j,N+1}(h)\|_{tr} \\ & + h^{N+1}\|B(h)D_{f_j,N+1}(h)H_{g_j,N}(h)\|_{tr} \\ \leq\ & h^{N+1}\|B(h)f_j^2(H(h))\|_{\mathcal{L}}\|D_{g_j,N+1}(h)\|_{\mathcal{L}}\|1_{\operatorname{supp} f_j}(H(h))\|_{tr} \\ & + h^{N+1}\|B(h)D_{f_j,N+1}(h)\|_{\mathcal{L}}\|H_{g_j,N}(h)\|_{tr}. \end{aligned}$$

We now remember the following inequalities:

$$
\begin{aligned}
\|B(h)f_j^2(H(h))\|_{\mathcal{L}} &\le c\|f_j^2\|_\infty(j+1) && \text{from the proof of Lemma 2.3.2}\\
\|D_{g_j,N+1}(h)\|_{\mathcal{L}} &\le c.p(g_j) && \text{from [Rob87, Eq.(51) p.144] .}\\
\|1_{\operatorname{supp} f_j}(H(h))\|_{tr} &\le N_h(j+1) &&\\
&\le ch^{-n}(j+1)^{n\frac{2+s}{2s}} &&\\
\|B(h)D_{f_j,N+1}(h)\|_{\mathcal{L}} &\le c.p(f_j^2) && \text{from Lemma 2.3.3}\\
\|H_{g_j,N}(h)\|_{tr} &\le c.p(g_j) && \text{from [Rob87, II.49]}
\end{aligned}
$$

Now we only have left to notice that $p(g_j) \le (1+j^2)^M p(g)$.
<u>The second:</u>
Here we just use the composition theorem for operators with symbols in the class $\Sigma_1^{(1+x^2+p^2)^{1/2}}$ and notice that

$$H_{f_j,N}H_{g_j,N} - H_{f_j,N} = h^{N+1}Op_h^w(\delta(h)).$$

Since

$$\|B(h)Op_h^w(\delta(h))\|_{tr} \le c.p(f_j^2)$$

by the composition theorem and [Rob87, II-49] we get the desired estimate for this term.
<u>The third:</u>

$$\|B(h)H_{f_j,N} - C_{j,N}(h)\|_{tr} = h^{N+1}\|Op_h^w(\delta(h)\|_{tr},$$

where $B(h)H_{f_j,N} = \sum h^k Op_h^w(c_{j,k}) + h^{N+1}Op_h^w(\delta(h))$. By the composition theorem for strongly admissible operators in $\Sigma_1^{(1+x^2+p^2)^{1/2}}$ and [Rob87, II-49] we get:

$$\|Op_h^w(\delta(h))\|_{tr} \le c.p(f_j^2).$$

□

Now we can prove the theorem of this section (Theorem 2.2.2):

Proof.

$$\lim_{h\searrow 0} h^{-(N+1)}[(2\pi h)^n \operatorname{tr}[B(h)f(H(h))] - \sum_{k=0}^{N} h^k \iint c_k(x,\xi)\,dxd\xi] =$$

$$\lim_{h\searrow 0} \sum_{j=0}^{\infty} (2\pi h)^n h^{-(N+1)} \operatorname{tr}[B(h)f_j(H(h)) - C_{j,N}]$$

and by the lemma

$$\sum_{j=0}^{\infty} (2\pi h)^n h^{-(N+1)} \|B(h)f_j(H(h)) - C_{j,N}\|_{tr}$$

converges, so we get:

$$\begin{aligned}
\lim_{h\searrow 0} h^{-(N+1)}[(2\pi h)^n \mathrm{tr}[B(h)f(H(h))] - \sum_{k=0}^{N} h^k \iint c_k(x,\xi)\,dxd\xi] \\
&= \sum_{j=0}^{\infty} \lim_{h\searrow 0} (2\pi h)^n h^{-(N+1)} \mathrm{tr}[B(h)f_j(H(h)) - C_{j,N}] \\
&= \sum_{j=0}^{\infty} \iint c_{j,N+1}(x,\xi)\,dxd\xi \\
&= \iint c_{N+1}(x,\xi)\,dxd\xi.
\end{aligned}$$

In the first and the last line we used that $\sum c_{j,N+1}$ converges to c_{N+1} in L^1. This is because we have (by the functional calculus and the composition theorem) that:

$$\begin{aligned}
c_{j,N} &= \sum_k Q_k f_j^{(k)}(a_0), \\
c_N &= \sum_k Q_k f^{(k)}(a_0),
\end{aligned}$$

where Q_k is a polynomial in b_0, b_1, a_0 and their derivatives (b_j are the principal and sub-principal symbols of $B(h)$). Now

$$(p_j - A_j) = \frac{1}{2}\partial_{p_j} a_0,$$

and we have the estimate:

$$|\partial^\alpha a_0| \leq c_\alpha |a_0|,$$

so we can dominate Q_k by a polynomial in a_0. Then the result follows by dominated convergence. □

2.6 Current for $T = 0$

We now fix a $\lambda < \lambda_0$ We suppose that λ is not a critical value for the principal symbol a_0 of $H(h)$. We will now prove Theorem 2.2.4.

Let $\phi_j(h)$ and $\lambda_j(h)$ be the eigenvectors and eigenvalues of $H(h)$ respectively with $\lambda_j \leq \lambda_0$. There will be a finite number of these for each h.

Lemma 2.6.1.

$$||B(h)1_{(-\infty,\lambda]}(H(h))|| \leq \lambda_0.$$

Proof. The proof is exactly as for Lemma 2.3.2. □

Lemma 2.6.2.

$$||B(h)1_{(-\infty,\lambda]}(H(h))||_{tr} = O(h^{-n}). \tag{2.6.2.1}$$

Proof.

$$\begin{aligned}\|B(h)1_{(-\infty,\lambda]}(H(h))\|_{tr} &\leq \lambda_0\|1_{(-\infty,\lambda]}(H(h))\|_{tr}\\ &= \lambda_0 N_h(\lambda),\end{aligned}$$

and the lemma follows from the CLR estimate. □

We will now take a partition of unity such that:

- $f_1^2+f_2^2=1$ on $[0,\lambda\,]$, $f_1,f_2\in C_0^\infty(\mathbf{R})$.
- supp $f_2\subset[\lambda-\varepsilon/2,\lambda+\varepsilon/2]$
  where $\lambda+\varepsilon<\lambda_0$ and $a_0^{-1}(\lambda-\varepsilon,\lambda+\varepsilon)$ is noncritical for a_0.
- supp $f_1\subset(-\infty,\lambda-\varepsilon/4]$.

We can then split our problem in the analysis of two terms, by:

$$\begin{aligned}\mathrm{tr}[B(h)1_{(-\infty,\lambda]}(H(h))] &= \mathrm{tr}[B(h)f_1^2(H(h))]\\ &+ \sum_{\lambda_j(h)\leq\lambda}\langle\phi_j(h),B(h)\phi_j(h)\rangle f_2^2(\lambda_j(h)).\end{aligned} \tag{2.6.2.2}$$

The first term on the right is nicely taken care of by Theorem 2.4.4. It thus remains to handle the second term which I will denote $M(\lambda,h)$. It is obvious that $M(\cdot,h)$ extends to a function on $\mathbf{R}$ constant "below" and "above" the support of f_2. We will now smoothen out the function $M(\lambda,h)$ by convolution with W_h (see below) and then use a Tauberian theorem to compare $M(\lambda,h)$ with $[M(\cdot,h)*W_h](\lambda)$ (see Theorem 2.6.5). This technique was also used in [HR83].

Definition 2.6.3. Let $\theta\in C_0^\infty(\mathbf{R})$, supp $\theta\subset(-T',T')$, where T' is sufficiently small[3], $\theta\geq 0$ and $\theta(0)=1$. We assume: θ even and $\hat\theta(0)>0$. We now define $W_h(\tau)=\frac{1}{2\pi h}\hat\theta(\frac{-\tau}{h})$.

We calculate:

$$\begin{aligned}\frac{d}{d\tau}[M(\cdot,h)*W_h](\tau) &= \sum\langle\phi_j(h),B(h)\phi_j(h)\rangle f_2^2(\lambda_j(h))[\delta_{\lambda_j(h)}*W_h](\tau)\\ &= \frac{1}{2\pi h}\mathrm{tr}\Big[B(h)f_2^2(H(h))\int e^{ih^{-1}t\tau}\theta(t)U_h(t)\,dt\Big],\end{aligned} \tag{2.6.2.3}$$

where:

$$U_h(t)=exp[-ih^{-1}tH(h)]. \tag{2.6.2.4}$$

We get:

[3] The hypothesis on T' are the following:

- $T'<T$ where T is the time interval in which the Hamilton- Jacobi equation associated a_0 to has a unique solution.
- Further, if $(x,p)\in a_0^{-1}(suppf_2)$, $|t|<T'$ and $\Phi^t_{a_0}(x,p)=(x,p)$ (where Φ_{a_0} is the Hamiltonian flow associated to a_0) then $t=0$.

It is standard to show that such a T' exists.

Proposition 2.6.4. *The leading term in the asymptotics of*

$$\operatorname{tr}[B(h)f_2^2(H(h))\int e^{ih^{-1}t\tau}\theta(t)U_h(t)\,dt] = J(\tau,h)+O(h^{2-n}), \tag{2.6.2.5}$$

for $\tau \in [\lambda-\varepsilon/4, \lambda+\varepsilon/4]$, *is:*

$$J(\tau,h) = (2\pi h)^{1-n}f_2^2(\tau)\int_{\{a_0=\tau\}}\vec{a}\cdot(\vec{q}-\vec{A})\frac{dS_\tau}{|\nabla a_0|}+O(h^{2-n}). \tag{2.6.2.6}$$

Proof. The proof is standard and will therefore be omitted (See for example [Rob87]). □

Upon integrating $\frac{d}{d\tau}[M(\cdot,h)*W_h](\tau)$ with respect to τ, we get:

$$\begin{aligned}&[M(\cdot,h)*W_h](\tau)\\ &\quad = (2\pi h)^{-n}\iint_{\{a_0(x,q)\leq\tau\}}\vec{a}\cdot(\vec{q}-\vec{A})f_2^2(a_0(x,q))\,dxdq+O(h^{1-n}).\end{aligned} \tag{2.6.2.7}$$

Now we know a lot about the asymptotics of $[M(\cdot,h)*W_h](\lambda)$. The next result relates this to the asymptotics of $M(\lambda,h)$.

Theorem 2.6.5.

$$|M(\lambda,h)-[M(\cdot,h)*W_h](\lambda)| = O(h^{1-n}).$$

Proof. Remember that we have chosen $\lambda<\lambda_0$. We have:

$$\begin{aligned}
& |M(\lambda,h)-[M(\cdot,h)*W_h](\lambda)|\\
\leq\ & \frac{1}{2\pi h}\int_{\{|\mu|<\lambda_0-\lambda\}}|M(\lambda,h)-M(\lambda-\mu,h))|\hat{\theta}(\frac{-\mu}{h})\,d\mu\\
& +\frac{1}{2\pi h}\int_{\{|\mu|\geq\lambda_0-\lambda\}}|M(\lambda,h)-M(\lambda-\mu,h))|\hat{\theta}(\frac{-\mu}{h})\,d\mu\\
\leq\ & \frac{1}{2\pi}\int_{\{|\mu|<\frac{\lambda_0-\lambda}{h}\}}\sum_{\lambda<\lambda_j(h)\leq\lambda+h\mu}|\langle\phi_j(h),B(h)\phi_j(h)\rangle|f_2^2(\lambda_j(h))|\hat{\theta}(\mu)|\,d\mu\\
& +ch^{-1-n}\int_{\{|\mu|\geq\lambda_0-\lambda\}}\hat{\theta}(\frac{-\mu}{h})\,d\mu\\
\leq\ & \|B(h)1_{(-\infty,\lambda_0]}(H(h))\|_{\mathcal{L}(L^2)}\int_{\mathbf{R}}|\sigma_h(\lambda+h\mu)-\sigma_h(\lambda)||\hat{\theta}(\mu)|\,d\mu\\
& +O(h^\infty),
\end{aligned}$$

where $\sigma_h(\tau)=\sum_{\lambda_j(h)\leq\tau}f_2^2(\lambda_j(h))$.
Now $\sigma_h(\tau)$ satisfies the hypothesis of [Rob87, Thm.V-13] (in fact it is the family of functions he wants to study there), so we conclude from his Lemma V-14:

$$\exists\Gamma>0: |\sigma_h(\lambda+h\mu)-\sigma_h(\lambda)|\leq\Gamma(1+|\mu|)h^{1-n}.$$

Inserting this inequality in the equation above and using the fact that $\hat{\theta}\in\mathcal{S}(\mathbf{R})$, we get the Theorem 2.6.5. □

Now we get Theorem 2.2.4 from Theorem 3.7.2 and equation (2.6.2.7) combined with Theorem 2.4.4.

2.7 The case of no periodic orbits

It is known from various sources that the classical hamiltonian flow has an influence on the spectrum of the quantized operator. On compact manifolds results in this direction have been proved by Duistermaat and Guillemin [DG75] and Colin de Verdiere [dV79]. We use the following assumption where we write $\Sigma_\tau = \{(x,p) \in \mathbf{R}^{2n} : a_0(x,p) = \tau\}$ for $\tau \in \mathbf{R}$.

Assumption 2.7.1. Let $\lambda < \lambda_0$ be a non-critical value for a_0. Suppose:

$$S_\lambda(\{(x,p) \in \Sigma_\lambda | \exists t \neq 0 \, s.t. \, \Phi^t_{a_0}(x,p) = (x,p)\}) = 0, \tag{2.7.2.1}$$

where S_λ is the surface measure on Σ_λ and Φ_{a_0} is the Hamiltonian flow associated with a_0.

Petkov and Robert [PR85] used this assumption to obtain good error estimates in the semi-classical limit on the number of eigenvalues in an interval. This assumption is also used in some of V. Ivrii's work [Ivr98]. The following is very much inspired by [PR85]. Under the above additional assumption we want to prove:

$$(2\pi h)^n \mathrm{tr}[B(h) 1_{(-\infty,\lambda]}(H(h))] = h\gamma_1 + o(h), \tag{2.7.2.2}$$

where γ_1 will be calculated explicitly as:

$$\gamma_1(\lambda) = \iint_{\{a_0(x,p)\le\lambda\}} b_1 \, dxdp - \frac{1}{2i}\int_{\Sigma_\lambda} \{a_0, b_0\}_P \frac{dS_\lambda}{|\nabla a_0|}.$$

Here $\{a_0, b_0\}_P = \partial_p a_0(x,p)\partial_x b_0(x,p) - \partial_x a_0(x,p)\partial_p b_0(x,p)$, is the Poisson bracket.This is the statement of Theorem 2.2.5.

We take the partition of unity f_1^2 and f_2^2 as in section 2.6, and write:

$$\mathrm{tr}[B(h) 1_{(-\infty,\lambda]}(H(h))] = \mathrm{tr}[B(h) f_1^2(H(h))] + M(\lambda, h), \tag{2.7.2.3}$$

where

$$M(\lambda, h) = \sum_{\lambda_j(h)\le\lambda} \langle \phi_j(h), B(h)\phi_j(h)\rangle f_2^2(\lambda_j(h)). \tag{2.7.2.4}$$

The term $\mathrm{tr}[B(h) f_1^2(H(h))$ has a complete asymptotics given by Theorem 2.4.4, so we are left with the analysis of $M(\lambda, h)$. This will be the subject of the next two subsections. In the first we prove that $M(\lambda, h)$ and it's smoothed out version $[M(\cdot,h) * W_h](\lambda)$ (where W_h was defined in Definiton 2.6.3) are equal up to an error of size $o(h^{1-n})$. Since $[M(\cdot,h) * W_h](\tau)$ is a continuous function in τ it is uniquely determined as a function once it is known as a distribution. This will be used in the last subsection to calculate γ_1.

2.7.1 The technical part

In this subsection we shall prove the following improvement of Theorem 2.6.5:

Theorem 2.7.2. *Under the assumptions 2.2.1 and 2.7.1 we have*

$$|M(\lambda,h)-[M(\cdot,h)*W_h](\lambda)|=o(h^{1-n}), \tag{2.7.2.5}$$

where W_h was defined in Definition 2.6.3.

We define for $L>1$:

$$K_L=\{(x,p)\in a_0^{-1}([\lambda-\varepsilon_0,\lambda+\varepsilon_0]):\exists t\neq 0,|t|\leq LT';\Phi^t_{a_0}(x,p)=(x,p)\}. \tag{2.7.2.6}$$

Here T' comes from the definition of W_h. We will now make a partition of unity in a neighborhood of the energy surface Σ_λ such that one part contains the closed orbits (i.e. K_L) but is of very small measure.

Observation 2.7.3. We may assume ε_0 so small that K_L is compact.

We take Ω_L to be an open neighborhood of K_L. One should think of Ω_L as being very "thin", i.e. of very small measure. Then we take functions $\omega_1,\omega_2\in C_0^\infty(\mathbf{R}^n)$ with

- $\omega_1^2+\omega_2^2=1$ on $a_0^{-1}([\lambda-\varepsilon/2,\lambda+\varepsilon/2])$,
- $\mathrm{supp}(\omega_1^2+\omega_2^2)\subset a_0^{-1}(\lambda-\varepsilon,\lambda+\varepsilon)$,
- $\omega_1\equiv 1$ on K_L and $\mathrm{supp}(\omega_1)\subset\Omega_L$.

Now, for k=1,2, we set

$$\begin{aligned}
M_k(\tau,h) &= \sum_{\lambda_j(h)\leq\tau}\langle\phi_j(h),Op_h^w(\omega_k)B(h)Op_h^w(\omega_k)f_2^2(H(h))\phi_j(h)\rangle\\
&= \sum_{\lambda_j(h)\leq\tau}\langle Op_h^w(\omega_k)\phi_j(h),B(h)Op_h^w(\omega_k)\phi_j(h)\rangle f_2^2(\lambda_j(h)).
\end{aligned}$$

Lemma 2.7.4.

$$|M(\lambda,h)-M_1(\lambda,h)-M_2(\lambda,h)|=O(h^{2-n}).$$

Proof. The proof is straight forward using the symbolic calculus. □

To know the first terms of $M(\lambda,h)$ it thus suffices to know the first terms of the $M_k(\lambda,h)$. This is what we will calculate next. We use the function W_h, as above, but with a parameter:

Definition 2.7.5. Let $\theta\in C_0^\infty(\mathbf{R})$, supp $\theta\subset(-T',T')$ be the function used in Definition 2.6.3. For $l\geq 1$ we define $W_{h,l}(\tau)=\frac{l}{2\pi h}\hat{\theta}(\frac{-l\tau}{h})$.

We will now analyze $[M_k(\cdot,h)*W_{h,l}](\tau)$. This analysis and a Tauberian result will lead us to the proof of Theorem 2.7.2.
As in section 2.6 we calculate:

$$\begin{aligned}&\frac{d}{d\tau}[M_k(\cdot,h)*W_{h,l}](\tau)\\ =\ &\int_{\mathbf{R}} \mathrm{tr}[Op_h^w(\omega_k)B(h)Op_h^w(\omega_k)f_2^2(H(h))e^{-i\frac{lt}{h}H(h)}]\frac{l}{2\pi h}e^{\frac{ilt\tau}{h}}\theta(t)\,dt. \quad (2.7.2.7)\end{aligned}$$

As usual, we take a $\chi\in C_0^\infty(\mathbf{R})$, $\chi\equiv 1$ on supp f_2 and we may then use

$$\tilde{U}_h(t)=\exp(-i\frac{t}{h}H(h)\chi(H(h))),$$

instead of $e^{-i\frac{t}{h}H(h)}$.
We have that

$$C_k(h)\equiv Op_h^w(\omega_k)B(h)Op_h^w(\omega_k),$$

is a strongly h-admissible operator with symbol $\sum h^j c_j^{(k)}$ where each term has support in the compact set: supp ω_k.

We end up having to calculate:

$$\frac{1}{2\pi h}\int \mathrm{tr}[C_k(h)f_2^2(H(h))\tilde{U}_h(t)]e^{-ih^{-1}t\tau}\theta(\frac{t}{l})\,dt. \quad (2.7.2.8)$$

Proposition 2.7.6. *We have the following asymptotics:*

$$\begin{aligned}(2\pi h)^n\frac{d}{d\tau}[M_1(\cdot,h)*W_{h,1}](\tau) &= f_2^2(\tau)\int_{\{a_0(x,p)=\tau\}} c_0^{(1)}(x,p)\frac{dS_\tau(x,p)}{|\nabla a_0(x,p)|}\\ &\quad +h\gamma_1^{(1)}(\tau)+O(h^2), \quad (2.7.2.9)\end{aligned}$$

Where the O is uniform in τ for $\tau\in\operatorname{supp} f_2^2$. And $\gamma_1^{(1)}\in C_0^\infty(\mathbf{R})$ with support contained in the support of f_2.

Proof. The proof is as for Prop. 2.6.4 (see [PR85]). □

Proposition 2.7.7. *We have, $\forall\, l$: $1\le l\le L$:*

$$\frac{d}{d\tau}[M_2(\cdot,h)*(W_{h,1}-W_{h,l})](\tau)=O(h^{2-n}) \quad (2.7.2.10)$$

Where the O is uniform in τ for $\tau\in\operatorname{supp} f_2^2$.

Proof. The proof is the same as the proof of [PR85, Prop.4.3]. □

We can write the statement of Proposition 2.7.7 like this:

$$(2\pi h)^n\frac{d}{d\tau}[M_2(\cdot,h)*W_{h,l}](\tau)=\gamma_0^{(2)}(\tau)+h\gamma_1^{(2)}(\tau)+O(h^{2-n}), \quad (2.7.2.11)$$

where the γ's are continuous and independent of l: $1\le l\le L$, and where the O is uniform for $\tau\in\operatorname{supp} f_2$. We now need a finer version of the Tauberian theorem:

Theorem 2.7.8. *There exists a constant $c \in \mathbf{R}$ so that:*

$$|M_2(\tau,h)-[M_2(\cdot,\tau)*W_{h,l}](\tau)| \leq \frac{c}{l}\gamma_0^{(2)}(\tau)h^{1-n}+c_2(l,\tau)h^{2-n},$$

where c_2 is continuous.

The proof is based on the following lemma:

Lemma 2.7.9. *$\exists k_0, k_1 \in \mathbf{R}$ such that for all $\phi_j(h)$ eigenvectors with $\lambda_j(h) \in [\lambda-\varepsilon, \lambda-\varepsilon]$ we have:*

$$|\langle \phi_j(h), Op_h^w(\omega_2)B(h)Op_h^w(\omega_2)\phi_j(h)\rangle| \leq k_0\|Op_h^w(\omega_2)\phi_j(h)\|^2+k_1 h.$$

Proof. Let $\chi \in C_0^\infty(\mathbf{R})$, $\chi \equiv 1$ on $[\lambda-\varepsilon,\lambda+\varepsilon]$, and $\operatorname{supp}\chi \subset [\lambda-2\varepsilon,\lambda+2\varepsilon]$. Let $g=\chi(a_0)$. Then

$$\chi(H(h)) = Op_h^w(g)+h^2R(h),$$

where $\{\|R(h)\| \,|h \in (0,h_0]\}$ is a bounded set. Here we used that $H(h)$ has no sub-principal symbol. Therefore we get:

$$\phi_j(h) = \chi(H(h))\phi_j(h) = Op_h^w(g)\phi_j(h)+h^2R(h)\phi_j(h).$$

So

$$\begin{aligned}
& \langle \phi_j(h), Op_h^w(\omega_2)B(h)Op_h^w(\omega_2)\phi_j(h)\rangle \\
= \; & \langle \phi_j(h), Op_h^w(\omega_2)B(h)Op_h^w(\omega_2)Op_h^w(g)\phi_j(h)\rangle + O(h^2) \\
= \; & \langle \phi_j(h), Op_h^w(\omega_2)(B(h)Op_h^w(g))Op_h^w(\omega_2)\phi_j(h)\rangle \\
& + \langle \phi_j(h), Op_h^w(\omega_2)B(h)[Op_h^w(\omega_2);Op_h^w(g)]\phi_j(h)\rangle + O(h^2) \\
= \; & \langle Op_h^w(\omega_2)\phi_j(h), (B(h)Op_h^w(g))Op_h^w(\omega_2)\phi_j(h)\rangle + O(h),
\end{aligned}$$

where we have used the composition rules for strongly h-admissible operators to conclude that the commutator $[Op_h^w(\omega_2);Op_h^w(g)]$ is a bounded operator of norm $O(h)$ and that $B(h)Op_h^w(g)$, $B(h)Op_h^w(\omega_j)$ are bounded operators, with operator norms uniformly bounded in h. $\square$

Now we can prove Theorem 2.7.8:

Proof. If we define:

$$\begin{aligned}
\sigma_h(\tau) &= \sum_{\lambda_j(h)\leq\tau} \langle \phi_j(h), Op_h^w(\omega_2)^2\phi_j(h)\rangle f_2^2(\lambda_j(h)) \\
&= \sum_{\lambda_j(h)\leq\tau} \|Op_h^w(\omega_2)\phi_j(h)\|^2 f_2^2(\lambda_j(h),
\end{aligned}$$

then $\sigma_h(\tau)$ satisfies the hypothesis of [PR85, Prop.3.2]:

1. $\sigma_h(\tau)$ is monotone, increasing in τ.
2. $\sigma_h(\tau) \equiv 0$ for $\tau \leq \lambda-\varepsilon$.
   $\sigma_h(\tau)$ is constant in τ for $\tau \geq \lambda+\varepsilon$.

3. $\sigma_h(\tau) \leq Kh^{-n}$ for all h sufficiently small and all τ.

4. $h^n \frac{d}{d\tau}[\sigma_h * W_{h,l}](\tau) = \gamma_0(\sigma,\tau) + \gamma_2(\sigma,\tau)h + O_{\tau,l}(h^2)$,
   where the O is (locally) uniform in τ and where
   $\tau \mapsto \gamma_0(\sigma,\tau)$ and $\tau \mapsto \gamma_1(\sigma,\tau)$ are of class C^1 on $\mathbf{R}$.

Only the last statement is not obvious. The proof of this statement follows the same lines as the proof of Proposition 2.7.7 and will thus be omitted. Therefore we can conclude from [PR85, Prop.3.2], that there exists $\tilde{\gamma} \geq 0$, and $\tilde{C}(L,\tau)$ a locally bounded, positive function of such that:

$$|\sigma_h(\tau + h\mu) - \sigma_h(\tau)| \leq \tilde{\gamma}(1/L + |\mu|)h^{1-n}\gamma_0(\sigma,\tau) + \tilde{C}(L,\tau)(1+|\mu|^2)h^{2-n}. \qquad (2.7.2.12)$$

Thus we get:

$$\begin{aligned}
& |M_2(\tau,h) - [M_2(\cdot,h) * W_{h,l}](\tau)| \\
\leq\ & \frac{l}{2\pi h}\int |M_2(\tau,h) - M_2(\tau-\mu,h)||\hat{\theta}(\frac{-l\mu}{h})|\,d\mu \\
=\ & \frac{l}{2\pi}\int |M_2(\tau,h) - M_2(\tau+h\mu,h)||\hat{\theta}(l\mu)|\,d\mu \\
=\ & \frac{l}{2\pi}\int |\sum_{\tau\leq\lambda_j(h)\leq\tau+h\mu} \langle\phi_j(h), Op_h^w(\omega_2)B(h)Op_h^w(\omega_2)\phi_j(h)\rangle f_2^2(\lambda_j(h))||\hat{\theta}(l\mu)|\,d\mu \\
\leq\ & \frac{k_0 l}{2\pi}\int |\sigma_h(\tau+h\mu) - \sigma_h(\tau)||\hat{\theta}(l\mu)|\,d\mu + O(h^{2-n}) \\
\leq\ & \frac{k_0 l}{2\pi}\int [\tilde{\gamma}(1/L+|\mu|)h^{1-n}\gamma_0(\sigma,\tau) + \tilde{C}(L,\tau)(1+|\mu|^2)h^{2-n}|\hat{\theta}(l\mu)|]\,d\mu \\
& +O(h^{2-n}).
\end{aligned}$$

So modulo an error of order h^{2-n} we get:

$$\begin{aligned}
& |M_2(\tau,h) - [M_2(\cdot,h) * W_{h,l}](\tau)| \\
\leq\ & c.l\int_{|\mu|\leq 1/l}(1/l + |\mu|)\gamma_0(\sigma,\tau)|\hat{\theta}(l\mu)|\,d\mu.h^{1-n} \\
+\ & c.l\int_{|\mu|\geq 1/l}(1/l + |\mu|)\gamma_0(\sigma,\tau)|\hat{\theta}(l\mu)|\,d\mu.h^{1-n} \\
\equiv\ & Term1 + Term2.
\end{aligned}$$

It is easy to handle $Term1$:

$$Term1 \leq \frac{c}{l}\gamma_0(\sigma,\tau)\sup(|\hat{\theta}|)h^{1-n}.$$

$Term2$ is calculated as (by the change of variables:$\mu' = l\mu$):

$$\begin{aligned}
Term2 &= c\gamma_0(\sigma,\tau)\int_{|\mu|\geq 1}\frac{1+|\mu|}{l}|\hat{\theta}(\mu)|\,d\mu h^{1-n} \\
&= c\frac{\gamma_0(\sigma,\tau)}{l}h^{1-n}\int_{|\mu|\geq 1}(1+|\mu|)|\hat{\theta}(\mu)|\,d\mu.
\end{aligned}$$

□

We can now prove Theorem 2.7.2:

Proof. We have:

$$\begin{aligned}|M(\lambda,h)-[M(\cdot,h)*W_h](\lambda)| &\leq |M(\lambda,h)-M_1(\lambda,h)-M_2(\lambda,h)|\\ &+ |[M(\cdot,h)-M_1(\cdot,h)-M_2(\cdot,h)]*W_h(\lambda)|\\ &+ |M_1(\lambda,h)-[M_1(\cdot,h)*W_h](\lambda)|\\ &+ |M_2(\lambda,h)-[M_2(\cdot,h)*W_h](\lambda)|.\end{aligned}$$

Here the first two terms on the right are taken care of by Lemma 2.7.4.
By def. of ω_1, Prop. 2.7.6, Prop. 2.7.7 and Thm 2.7.8:

$$\begin{aligned}& |M_1(\lambda,h)-[M_1(\cdot,h)*W_h](\lambda)|+|M_2(\lambda,h)-[M_2(\cdot,h)*W_h](\lambda)|\\ = & |M_1(\lambda,h)-[M_1(\cdot,h)*W_{h,1}](\lambda)|\\ & +|M_2(\lambda,h)-[M_2(\cdot,h)*W_{h,L}](\lambda)|+O(h^{2-n})\\ \leq & \gamma\gamma_0(1,\lambda)h^{1-n}+C_1(1,1,\lambda)h^{2-n}\\ & +\frac{\gamma}{L}\gamma_0(2,\lambda)h^{1-n}+C_1(1,2,\lambda)h^{2-n}+O(h^{2-n}).\end{aligned} \tag{2.7.2.13}$$

Let $\varepsilon>0$. Let us choose L so big that

$$\frac{\gamma}{L}\int_{\Sigma_\lambda}\frac{dS_\lambda}{|\nabla a_0|}\leq \varepsilon/2.$$

Now we construct an open set $U_\lambda \subset \mathbf{R}^{2n}$ satisfying:

- $K_L\cap\Sigma_\lambda\subset U_\lambda$.
- $\int_{\Sigma_\lambda\cap U_\lambda}\frac{dS_\lambda}{|\nabla a_0|}\leq\varepsilon/2$.
- $\exists\alpha>0$ such that $K_L\cap\Sigma_\tau\subset U_\lambda$ for all τ satisfying $|\tau-\lambda|\leq\alpha$.

(This construction is made in [PR85].) We can now define Ω_L by:

$$\Omega_L=U_\lambda\cup a_0^{-1}(\lambda-\varepsilon_0,\lambda-\alpha)\cup a_0^{-1}(\lambda+\alpha,\lambda+\varepsilon_0). \tag{2.7.2.14}$$

We thus see that the h^{1-n} term in (2.7.2.13) can be made as small as we want (Ω_L was defined just below Observation 2.7.3). □

2.7.2 The calculation of γ_1

Let us write:

$$(2\pi h)^n[M(\cdot,h)*W_h](\tau)=C_0(\tau)+hC_1(\tau)+O(h^2),$$

where the O is locally uniform i τ. We will now prove:

Theorem 2.7.10.

$$\begin{aligned} C_1(\tau) &= \iint_{\{a_0 \le \tau\}} f_2^2(a_0) b_1 \, dxdp + \frac{1}{2i} f_2^2(\tau) \int_{\Sigma_\tau} \{a_0, b_0\}_P \frac{dS_\tau}{|\nabla a_0|} \\ & -\frac{1}{2i} \int_{\{a_0 \le \tau\}} (f_2^2)'(a_0) \{a_0.b_0\}_P \, dxdp, \end{aligned} \tag{2.7.2.15}$$

for all $\tau \in (\lambda - \varepsilon/2, \lambda + \varepsilon/2)$.

Since $C_0(\tau)$ and $C_1(\tau)$ are continuous, they are determined uniquely by the distributions they define. That will make it easy for us to find C_1. Let $\phi \in C_0^\infty(\lambda - \varepsilon/2, \lambda + \varepsilon/2)$.

Lemma 2.7.11.

$$\int \phi(\tau)[M(\cdot,h) * W_h](\tau) \, d\tau = \mathrm{tr}[B(h) f_2^2(H(h)) \Phi(H(h))] + O(h^{2-n}), \tag{2.7.2.16}$$

where $\Phi(\mu) = \int_\mu^\infty \phi(t) \, dt$.

Proof. This is standard calculus (see [PR85]). □

We will now prove Theorem 2.7.10.

Proof. We can calculate $\mathrm{tr}[B(h) f_2^2(H(h)) \Phi(H(h))]$ using Theorem 2.4.4:

$$\begin{aligned} (2\pi h)^n \mathrm{tr}[B(h) f_2^2(H(h)) \Phi(H(h))] &= \iint b_0 (f_2^2 \Phi)(a_0) \, dxdp \\ +h \iint [b_1 (f_2^2 \Phi)(a_0) &+ \frac{1}{2i} \{b_0, (f_2^2 \Phi)(a_0)\}_P] \, dxdp + O(h^2) \end{aligned}$$

The rest of the proof is now an easy comparison of terms. □

Theorem 2.7.10, Lemma 2.7.11, Theorem 2.7.2 together with equation (2.7.2.3) now prove Theorem 2.2.5.

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Chapter 3

Current in strong magnetic fields

Contents

3.1 Abstract

We study the current of the Pauli operator in a strong constant magnetic field. We prove that in the semi-classical limit the persistent current and the current from the interaction of the spin with the magnetic field cancel, in the case where the magnetic field is very strong. Furthermore we calculate the next term in the asymptotics and estimate the error. Finally, we discuss the connection between this work and the semi-classical estimate of the energy in strong magnetic fields proved by Lieb, Solovej and Yngvason [LSY94].

3.2 Introduction

In recent years physicists have been very interested in understanding the current in quantum systems such as the quantum Hall systems and different types of nanostructures that experimental advances have made possible. In contrast, the current has been studied very little in the mathematics litterature. The current, however, is as natural a quantity as the density which has been studied to a great extent in the mathematics litterature, in particular, the integral of the density, i.e. the particle number (for fixed chemical potential), obeys the celebrated Weyl law in the semiclassical limit. In the semiclassical limit one cannot expect to see a static current since there is no classical, persistent or diamagnetic current. In quantum mechanics, however, there may be a static current. In [Fou98] the semiclassical limit of this current was studied and it was indeed found that the first term in the semiclassical expansion vanishes. This might be the reason why the quantum current has not attracted much attention in the mathematics community.

In this paper we study a different type of semiclassical limit in which the magnetic field strength may vary as the semiclassical parameter h tends to zero. If the field strength increases when h decreases in such a way that the magnetic length scale is comparable to the Planck scale, one should expect to see the effect of the current. In fact, in quantum Hall systems one has magnetic field strengths that make the magnetic field length of the order of the Planck scale. This type of semiclassical limit was studied by Lieb, Solovej and Yngvason in [LSY94] and [LSY95], where the limits of the energy and the density were studied. The purpose of this paper is to extend this analysis to include the persistent quantum current.

It should be noted that this paper deals solely with static situations. This is different from the situation in quantum Hall systems, where a constant voltage drop creates a stationary and not just static situation.

The object of study in this paper is the Pauli operator:

$$\mathbf{P} = \mathbf{P}(h,\vec{A},V) = (-ih\nabla - \vec{A})^2 + V(x) - h\vec{\sigma}\cdot\vec{B},$$

acting in $L^2(\mathbb{R}^3;\mathbb{C}^2)$. Here $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli spin matrices:

$$\begin{aligned}
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{aligned}$$

and $\vec{B} = \nabla \times \vec{A}$. This operator has, in general, infinitely many negative eigenvalues, even for V smooth and compactly supported (and negative), but it was proved in [LSY94] (see also [ES97] for the case of non-constant magnetic fields) that the sum of the negative eigenvalues $\mathrm{tr}[\mathbf{P}1_{(-\infty,0]}(\mathbf{P})]$ is finite. The sum of the negative eigenvalues represents the energy E of a noninteracting electron gas (of chemical potential 0) in the external electric potential V and magnetic potential $\vec{A}$. Furthermore, they proved a semi-classical formula for the energy, uniformly in the magnetic field strength, i.e. an expression $E_{scl} = E_{scl}(h, \vec{A}, V)$ (see (3.2.3.1) below), such that if $E = E(h, \vec{A}, V) = \mathrm{tr}[\mathbf{P}1_{(-\infty,0]}(\mathbf{P})]$ then

$$\frac{E(h, \vec{A}, V)}{E_{scl}(h, \vec{A}, V)} \to 1,$$

uniformly in $\vec{A}$, as $h \to 0$.

Given the energy, two quantities can be calculated: the *density* and the *current*. The density ρ is defined, as a distribution, as the variational derivative of E with respect to V, i.e.

$$\int \rho\phi\, dx = \frac{d}{dt} E(h, \vec{A}, V + t\phi)|_{t=0}.$$

In the context of strong magnetic fields, this has been studied in [Sob94], and a formula for the highest order term in the semi-classical limit was given, with good control of the error term.

The current $\vec{j}$ is the variational derivative of E with respect to the vector potential $\vec{A}$:

$$\int \vec{j} \cdot \vec{a}\, dx = \frac{d}{dt} E(h, \vec{A} + t\vec{a}, V)|_{t=0},$$

where the left hand side is to be understood in the sense of distributions. It will be the objective of this paper to obtain a semi-classical formula for this quantity when the magnetic field is strong, but constant. By a strong, constant magnetic field we mean that we take the limit $h \to 0$ but with a magnetic field $\vec{B} = (0, 0, \mu)$ so strong that $\mu h \geq c > 0$ as $h \to 0$.

The new mathematical challenge that presents itself in an analysis of the current in comparison with the density is that the highest order term vanishes. This phenomenon already appears in the standard semiclassical problem (weak magnetic field) analysed in [Fou98]. Due to this, we have to make a somewhat finer analysis, i.e. include lower order terms, than what is needed to find the density.

To get an idea of what to expect, let us first look at the semiclassical energy:

The semi-classical formula for the energy given in [LSY94] is:

$$E_{scl} = -h^{-3} \int P(h|\vec{B}(x)|, [V(x)]_{-}) dx, \tag{3.2.3.1}$$

where

$$P(B,W) = \frac{2}{3\pi} \sum_{n=0}^{\infty} d_n B [2nB - W]_{-}^{3/2},$$

and

$$[x]_{-} = \begin{cases} 0 & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

Here $d_0 = \frac{1}{2\pi}$ and $d_n = \frac{1}{\pi}$ for $n \geq 1$. If this semiclassical formula contains most of the *physics* of the problem then it should also give the current to highest order, so we try to calculate its functional derivative with respect to the vector potential. Let thus $\vec{a}$ be a test function. Then we have:

$$\begin{aligned} & \int \vec{j}_{scl} \cdot \vec{a}\, dx \\ \stackrel{def}{=} \; & \frac{d}{dt} E_{scl}(\vec{A} + t\vec{a})|_{t=0} \\ = \; & \frac{-2}{3\pi h^2} \sum_{n=0}^{\infty} d_n \int (\partial_{x_1} a_2 - \partial_{x_2} a_1) \\ & \times \left([2nh\mu + V(x)]_{-}^{3/2} - 3nh\mu [2nh\mu + V(x)]_{-}^{1/2} \right) dx, \end{aligned} \tag{3.2.3.2}$$

or

$$\begin{aligned} & \vec{j}_{scl} \\ = \; & \frac{1}{\pi h^2} \sum_{n=0}^{\infty} d_n \left([2nh\mu + V(x)]_{-}^{1/2} - nh\mu [2nh\mu + V(x)]_{-}^{-1/2} \right) \begin{pmatrix} \partial_{x_2} V \\ -\partial_{x_1} V \\ 0 \end{pmatrix} \end{aligned} \tag{3.2.3.3}$$

In the special case where $\mu h \to \infty$ we get:

$$\vec{j}_{scl} = \frac{1}{2\pi^2 h^2} [V(x)]_{-}^{1/2} \begin{pmatrix} \partial_{x_2} V \\ -\partial_{x_1} V \\ 0 \end{pmatrix}.$$

We will prove that the above formulas for the current are correct to highest order, and we will estimate the error.

Remark 3.2.1. The corresponding formulas in 2-dimensions are:

$$\begin{aligned} E_{scl}^{(2)} &= -h^{-1} \int \sum_{n=0}^{\infty} d_n |\vec{B}(x)| [2nh|\vec{B}(x)| + V(x)]_{-} \, dx, \\ \int \vec{a} \cdot \vec{j}_{scl}^{(2)} dx &= -h^{-1} \int \sum_{n=0}^{\infty} d_n (\partial_{x_1} a_2 - \partial_{x_2} a_1) \\ & \quad \times \left([2nh\mu + V(x)]_{-} - 2nh\mu [2nh\mu + V(x)]_{-}^{0} \right) dx. \end{aligned}$$

3.2.1 Statement of the results

We will fix $\vec{A} = \mu(-x_2, 0, 0)$ in the rest of this paper. We will thus write $\mathbf{P} = \mathbf{P}(h, \mu, V)$ instead of $\mathbf{P}(h, \vec{A}, V)$. A formal computation gives

$$\frac{d}{dt}E(h, \vec{A} + t\vec{a}, V)|_{t=0} = -\mathrm{tr}[\mathbf{B}1_{(-\infty,0]}(\mathbf{P})], \tag{3.2.3.4}$$

where

$$\mathbf{B} = 2\vec{a}\cdot(-ih\nabla - \vec{A}) - ih\mathrm{div}\vec{a} + h\sigma_3(\partial_{x_1}a_2 - \partial_{x_2}a_1).$$

We will take this as our starting point i.e. *define* the current as

$$\int \vec{j}\cdot\vec{a}\,dx = -\mathrm{tr}[\mathbf{B}1_{(-\infty,0]}(\mathbf{P})].$$

We shall allow $V \in C^\infty(B(8)\setminus\{0\})$ to have a Coulomb singularity, i.e. suppose

$$V(x) = \frac{q}{|x|} + o(|x|^{-1}) \tag{3.2.3.5}$$

as $x \to 0$, and suppose that $\forall m \in \mathbb{N}^3\ \exists C_{m,V} \in \mathbb{R}_+$ such that:

$$|\partial^m V(x)| \leq C_{m,V}|x|^{-1-|m|}, \tag{3.2.3.6}$$

$\forall x \in B(8)$.
Suppose furthermore that $\exists C = C(h,\mu)$ such that

$$\mathbf{P}(h,\mu,V) \geq -C. \tag{3.2.3.7}$$

Then we have the following:

Theorem 3.2.2. *Let the above conditions (3.2.3.5)-(3.2.3.7) on V be satisfied. Suppose*

- $\exists c_{\mu,1} > 0$ *such that* $\mu h \geq c_{\mu,1}$,
- $\exists c_{\mu,2} > 0$ *such that* $\mu h^3 \leq c_{\mu,2}$,

then

$$\vec{j} \overset{h\to 0}{\to} \vec{j}_{scl},$$

in the sense of distributions in the coordinates orthogonal to the magnetic field, i.e.:

$$\int \vec{j}\cdot\begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} dx \overset{h\to 0}{\to} \int \vec{j}_{scl}\cdot\begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} dx,$$

for all $a_1, a_2 \in C_0^\infty(B(1))$, where $B(1)$ is the closed ball of unit radius in $\mathbb{R}^3$.

Remark 3.2.3. $\mathbf{B}$ contains a term $h\sigma_3(\partial_{x_1}a_2 - \partial_{x_2}a_1)$ which comes from the term $h\vec{B}\cdot\vec{\sigma}$ in $\mathbf{P}$. If we define the *spin-current* $\vec{j}_{spin}$ to be

$$\int \vec{a}\cdot\vec{j}_{spin}\,dx = -\mathrm{tr}[h\sigma_3(\partial_{x_1}a_2 - \partial_{x_2}a_1)1_{(-\infty,0]}(\mathbf{P})],$$

then $\vec{j}_{spin}$ can easily be calculated from the known results about the density and it is seen to be of order $O(\mu/h)$. Thus a part of our result is that the *persistent current*

$$\int \vec{a}\cdot\vec{j}_{persistent}\,dx = -\mathrm{tr}[\Big(2\vec{a}\cdot(-ih\nabla - \vec{A}) - ih(\mathrm{div}\vec{a})\Big)\,1_{(-\infty,0]}(\mathbf{P})],$$

and the spin-current are equal with opposite sign to order $O(\mu/h)$.

Remark 3.2.4. The condition $\exists c_{\mu,2} > 0$ such that $\mu h^3 \leq c_{\mu,2}$ is only necessary if we have a singularity. In the case where V is smooth we can allow μ to be of any order in h, see Theorem 3.3.3 or its improvement Theorem 3.9.1.

If the potential is confining in the direction parallel to the magnetic field, we can also calculate the current in that direction:

Theorem 3.2.5. *Let the assumptions be as in Thm 3.2.2. Assume furthermore that $V(x_1,x_2,x_3) \geq c_V > 0$, for $1 \leq |x_3| \leq 3$, and that V is infinitesimally bounded with respect to $-\Delta$, then*

$$\vec{j} \overset{h\to 0}{\to} \vec{j}_{scl},$$

in the sense of distributions, i.e.:

$$\int \vec{j}\cdot\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} dx \overset{h\to 0}{\to} \int \vec{j}_{scl}\cdot\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} dx,$$

for all $a_1, a_2, a_3 \in C_0^\infty(B(1))$.

Apart from its obvious physical relevance, the Coulomb potential is mathematically interesting in this kind of problem, since a correct analysis demands asymptotic estimates in both weak and strong magnetic fields. To see this, one has to realise, that magnetic effects are important if $\frac{[V(x)]_-}{\mu h} \ll 1$ and neglectable if $\frac{[V(x)]_-}{\mu h} \gg 1$. This can, for example, be seen from the semiclassical formula for the energy. Thus we will need to split in two regions, one, close to the singularity, where $\frac{[V(x)]_-}{\mu h}$ is big, and one outside, where the ratio is small. In the first region, we have standard semi-classics, and the analysis from [Fou98] suffices. In the outer region no analysis of the current exists, therefore the main part of this paper, Sections 3.3 - 3.10 will deal with finding the correct estimates in this region. Finally, in Section 3.11, we will prove a more precise version of Theorem 3.2.2 above. The proof of Theorem 3.2.5 is identical to the proof of Theorem 3.3.7 below given in Section 3.10 and will therefore be omitted.

3.2.2 Notations

It will be convenient to use the functions:

$$\begin{aligned} g_0(\tau) &= 1_{(-\infty,0]}(\tau), \\ g_1(\tau) &= (-\tau)g_0(\tau), \end{aligned}$$

and to write $B(r)$ for the closed ball of radius r. For shortness we will sometimes write the current trace as

$$\mathrm{tr}[\mathbf{B}g_0(\mathbf{P})] = \mathcal{J}(h,\mu,\vec{a},V),$$

and the asymptotic term, as:

$$\begin{aligned} \mathcal{A}(h,\mu,\vec{a},V) &= \frac{2}{3\pi h^2}\sum_{n=0}^{\infty} d_n \int (\partial_{x_1}a_2(x) - \partial_{x_2}a_1(x)) \\ &\quad \times \left([2n\mu h + V(x)]_-^{3/2} - 3n\mu h[2nh\mu + V(x)]_-^{1/2} \right) dx. \end{aligned}$$

We will write $\hat{x} = (x_1, x_3)$ and $\hat{\xi} = (\xi_1, \xi_3)$.
Apart from the parameters h,μ we will need two other scales:

$$\alpha = h/\mu, \qquad\qquad \varepsilon = \frac{1}{\mu^2}.$$

We will denote by $\mathcal{B}^\infty(\Omega)$ the set of smooth functions f on the open set Ω satisfying

$$|\partial^m f| \leq C_m.$$

for all m.

It is an elementary fact that:

$$L^2(\mathbb{R}^3) = L^2(\mathbb{R}^2_{(x_1,x_3)}) \otimes L^2(\mathbb{R}_{x_2}).$$

It is this splitting of $L^2(\mathbb{R}^3)$ that all tensor products will refer to.

We will freely use results on pseudodifferential operators as described, for example, in [Rob87]. The quantisation we use will be the Weyl-quantisation i.e. the symbol $s(x,\xi)$ is quantised as

$$Op_h^w(s)\phi(x) = \frac{1}{(2\pi h)^3}\int\int e^{ih^{-1}(x-y)\xi} s(\frac{x+y}{2},\xi)\phi(y)\,dyd\xi.$$

In asymptotic calculation we will sometimes use the shorthand

$$A(h) \simeq B(h),$$

if the two expressions $A(h)$ and $B(h)$ agree to highest order in h. Finally, it should be pointed out that the notation $\partial^\alpha f(z)$ is shorthand for $\partial^\alpha f|_z$ all through this paper.

3.3 The local asymptotics

Our strategy to prove the main theorems of this paper will be that of V. Ivrii: obtain good local results in regions where everything is smooth, and then use ”scaling” to put the pieces together. This last ”cutting-and-pasting” technique has been refined (by Ivrii and others, see [Ivr98], [IS93], [Sob95]) into what is usually called the ”multiscaling” technique and will be discussed in the last sections of the paper. Here we will just remind the reader that it is *absolutely crucial* for the technique to work, that the estimates obtained are indeed *local* i.e. depend only on local bounds on, for instance, the potential. The only *global* assumption, we need, (and are allowed to impose) is the semi-boundedness (and self-adjointness) of the operator in question, end even here it is important that the local estimates only depend on the existence of a lower bound, *not* on the size of it.

The local result is:
Let $E \in \mathbb{R}_+$, $\vec{a} \in C_0^\infty(B(E/4))$. Let furthermore $H_0 = (-ih\nabla - \vec{A})^2 - \mu h$. Assume V satisfies:

Assumption 3.3.1. (See [Sob94, Assumption 1.1])

- V is a real-valued function such that the self-adjoint operator $H = H_0 + V$ is well defined on the domain $\mathcal{D}(H) = \mathcal{D}(H_0)$ and is semibounded from below;
- $V \in C^\infty(B(4E))$.

Remark 3.3.2. The introduction of this kind of assumption in semi-classical problems is due to Ivrii [Ivr98].

Let finally

$$\mathbf{B} = 2\vec{a}\cdot(-ih\nabla - \vec{A}) - ih\mathrm{div}\vec{a} + h\sigma_3(\partial_{x_1}a_2 - \partial_{x_2}a_1).$$

Then we have:

Theorem 3.3.3. *Let $\vec{a} = (a_1, a_2, 0)$. Suppose that*

$$|\partial_{x_1}V(x)|^2 + |\partial_{x_3}V(x)|^2 + |V(x)| \geq c_{N.C.} > 0 \tag{3.3.3.1}$$

for all $x \in B(2E)$. Suppose further that $0 < h \leq h_0$, $\mu \leq C_\mu h^{-\zeta}$ for some $\zeta > 0$ and that there exists $\rho \in (0,1]$ such that $\mu \geq c_\rho h^{-\rho}$. Suppose finally that

$$|\partial^m \vec{a}(x)| + |\partial^m V(x)| \leq C_m$$

on $B(8E)$. Then

$$\begin{aligned}\mathrm{tr}[\mathbf{B}g_0(\mathbf{P})] &= \frac{2}{3\pi h^2}\sum_{n=0}^{\infty} d_n \int (\partial_{x_1}a_2(x) - \partial_{x_2}a_1(x)) \\ &\quad\times\left([2n\mu h + V(x)]_-^{3/2} - 3n\mu h[2nh\mu + V(x)]_-^{1/2}\right)dx \\ &\quad + O(h^{-1}\mu^{-1} + h^{-3}\mu^{-2} + h^{-1}),\end{aligned}$$

where O is uniform in the constants $\{C_m\}, c_{N.C.}, c_\rho, C_\mu, \rho, \zeta, E$.

Remark 3.3.4. Uniformity, here and in the rest of the paper, means that if the functions $\vec{a}, V$ are replaced by other functions satisfying the same bounds with the same constants, then the asymptotic estimate remains true with the same constant in the error bound.

Remark 3.3.5. The theorem is still true without the "non-critical" condition (3.3.3.1). This will be proved in Section 3.9.

First we want to prove this in the case where $\mu h \geq C$ where C is some sufficiently big constant (i.e. $\rho = 1$ and c_ρ sufficiently big). This is mainly for pedagogical reasons. When μh is big we only have to consider the lowest Landau level. This implies a greater simplicity in the exposition. Since furthermore, the persistent current and the spin-current cancel on the lowest Landau level, it becomes clear, why we have to make a somewhat finer analysis, than what is needed to find the density and the energy.
Thus, we will first prove Theorem 3.3.6 below, then, in Section 3.8, we will put in the few remaining arguments to prove Theorem 3.3.3.

Theorem 3.3.6. *Let $\vec{a} = (a_1, a_2, 0)$. Suppose that*

$$|\partial_{x_1} V(x)|^2 + |\partial_{x_3} V(x)|^2 + |V(x)| \geq c_{N.C.} > 0$$

for all $x \in B(2E)$. Suppose further that $0 < h \leq h_0$, $\mu \leq C_\mu h^{-\zeta}$ for some $\zeta > 0$ and that $\mu h \geq C$. Suppose finally that

$$|\partial^m \vec{a}(x)| + |\partial^m V(x)| \leq C_m$$

on $B(8E)$. Then there exists C_0 such that if $C > C_0$ we get

$$\operatorname{tr}[\mathbf{B} g_0(\mathbf{P})] = \frac{1}{3\pi^2} \frac{1}{h^2} \int (\partial_{x_1} a_2 - \partial_{x_2} a_1)[V(x)]_-^{3/2} dx + O(h^{-1}),$$

where O is uniform in the constants $\{C_m\}, c_{N.C.}, C_\mu, \zeta, E$.

Let us also state a version of 3.2.5 in the setup of the two theorems above:

Theorem 3.3.7. *Let $a_3 \in C_0^\infty(B(E))$ and define $\vec{a} = (0, 0, a_3)$. Suppose $V \in C^\infty(\mathbb{R}^3)$ and that there exists $\gamma > 0$ such that $\liminf_{|x| \to \infty} V(x) > \gamma$. Suppose further that $\mu h \geq c > 0$ as $h \to \infty$. Then*

$$\operatorname{tr}[\mathbf{B}(h, \mu, \vec{a}) g_0(\mathbf{P})] = O(h^{-1}).$$

Finally a few words about the following sections. Sections 3.4 and 3.6 below recall the results from [Sob94] that we will need in the rest of the paper. Sections 3.5 and 3.7 contain the proof of Theorem 3.3.6.

3.4 The Birkhoff normal form

In a strong magnetic field the Landau levels remain separated in energy as $h \to 0$. Therefore the variables defining the Landau levels do not approach their classical behaviour i.e. a standard semiclassical approximation is wrong. As has been realised in the studies of the energy and density, one can define (modified) Landau levels so that H, to a large

extent, preserves these levels. One can then treat the splitting into Landau levels quantum mechanically and make a standard semiclassical analysis on each level.
In this section we summarize the results on these modified Landau levels that we will need, and then in Section 3.6, we review the results about semiclassics on each level.

Let $W \in C_0^\infty(\mathbb{R}^3)$, $W(x) = V(x)$ on $B(3E)$. We will perform some reductions on

$$H_W = (-ih\partial_{x_1} + \mu x_2)^2 - h^2\partial_{x_2}^2 - h^2\partial_{x_3}^2 - \mu h + W(x),$$

for $\mu \geq \mu_0$ and $h \in (0, h_0]$. We will later in this section also have to use H from Assumption 3.3.1, which we will then write as H_V. Outside this section H will always refer to H_V.

3.4.1 1st reduction

Let

$$(\Phi_0 f)(x) = \frac{1}{(2\pi\alpha)^3}\int e^{i/\alpha[(x-y)\xi+\xi_1\xi_2]} f(y)dyd\xi.$$

Then it is easy to see that Φ_0 is unitary and that

$$\varepsilon\Phi_0^* H_W \Phi_0 = Op_\alpha^w h_\varepsilon - \alpha,$$

where

$$h_\varepsilon(x,\xi) = (\xi_2^2 + x_2^2) + \xi_3^2 + \varepsilon W(x_1 - \xi_2, x_2 - \xi_1, x_3).$$

In general:

$$\Phi_0^* Op_\alpha^w a \Phi_0 = Op_\alpha^w \tilde{a}, \tag{3.4.3.1}$$

where $\tilde{a}(x,\xi) = a(x_1 - \xi_2, x_2 - \xi_1, x_3, \xi)$.

3.4.2 2nd reduction

Without the potential W our operator would split into Landau levels i.e. after the 1st reduction the variables (x_2, ξ_2) would only appear in the symbol as the combination $(\xi_2^2 + x_2^2)$, which is the symbol of a harmonic oscillator. The idea now is to find a canonical transformation κ such that this is 'almost' true for $h_\varepsilon \circ \kappa(x,\xi)$, and then, by an Egorov type theorem, find an 'almost unitary' transformation T which realizes κ on the symbol level. This is done in the two theorems below. Before we state them we need a bit more notation.
Let K_α be the operator on $L^2(\mathbb{R})$:

$$(K_\alpha u)(t) = (-\alpha^2\partial_t^2 + t^2)u(t),$$

below K_α will be acting in the x_2 variable. We denote by u the three variables (x_1, x_3, ξ_1), and by v the remaining variables i.e. $v = (x_2, \xi_2, \xi_3)$. We denote by $\tau_N = \tau_N(x,\xi,\varepsilon)$ any function in $\mathcal{B}^\infty(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3 \times [-\varepsilon_0, \varepsilon_0])$, which satisfies

$$|\partial_u^{m_1}\partial_v^{m_2}\partial_\varepsilon^{m_3}\tau_N(x,\xi,\varepsilon)| \leq C_N(|\varepsilon| + v^2)^{(N-|m_2|/2-|m_3|)_+}.$$

Finally, we choose a $C_0^\infty(\mathbb{R})$ function $\sigma(t)$, satisfying $\sigma(t) = 1$ for $|t| \leq 1/4$ and $\sigma(t) = 0$ for $|t| \geq 1$, for $R > 0$ we write $\sigma_R(t) = \sigma(t/R)$.

Theorem 3.4.1. *[Sob94, Thm 7.4] Let* $\kappa_j : \mathbb{R}^3_x \times \mathbb{R}^3_\xi \to \mathbb{R}^3_x \times \mathbb{R}^3_\xi$, $j = 0, 1, \cdots$ *be a sequence of canonical transformations associated with the generating functions*

$$\phi_j(x,\xi) = x\xi + \varepsilon\sigma(\xi_3^2)\sigma(\xi_2^2 + x_2^2)A_j(x,\xi),$$

where

$$\begin{aligned} A_0(x,\xi) &= \frac{\xi_2}{2}\partial_z W(x_1,z,x_3)|_{z=-\xi_1} + \frac{x_2}{2}\partial_{x_1} W(x_1,-\xi_1,x_3), \\ A_j(x,\xi) &= \sum_{2j\le 2m+l+n+k\le 2j+1} a^{m,l}_{n,k}(u)\varepsilon^m \xi_2^n x_2^k \xi_3^l, \quad j \ge 1. \end{aligned}$$

Then there exist functions $a^{m,l}_{n,k} \in \mathcal{B}(\mathbb{R}^3)$ *such that for any integer* $M > 0$ *the composition* $\kappa = \kappa_0 \circ \kappa_1 \circ \cdots \circ \kappa_M$ *satisfies*

$$\begin{aligned} (h_\varepsilon \circ \kappa)(x,\xi) &= \xi_3^2 + \xi_2^2 + x_2^2 + \varepsilon\sigma(\xi_3^2)\sigma(\xi_2^2 + x_2^2) \\ &\quad \times \sum_{0\le m+n+l\le M} [Y^{(0)}_{m,n,l}(u)\xi_3^{2l} + Z^{(0)}_{m,n,l}(u)\xi_3^{2l+1}]\varepsilon^m(\xi_2^2 + x_2^2)^n \\ &\quad + \varepsilon\tau_{M+1}(x,\xi;\varepsilon). \end{aligned}$$

Here $Y^{(0)}_{m,n,l}, Z^{(0)}_{m,n,l} \in \mathcal{B}(\mathbb{R}^3)$. *In particular we get, where we write* $\nabla_2 = (\partial_{x_1}, \partial_{x_2})$, $\Delta_2 = \partial^2_{x_1} + \partial^2_{x_2}$*:*

$$\begin{aligned} Y^{(0)}_{0,0,0}(u) &= W(x_1,-\xi_1,x_3), \\ Y^{(0)}_{1,0,0}(u) &= -\frac{1}{4}(\partial_{x_2} W(x_1,-\xi_1,x_3))^2, \\ Y^{(0)}_{0,1,0}(u) &= \frac{1}{4}\partial^2_{x_2} W(x_1,-\xi_1,x_3), \\ Y^{(0)}_{0,0,1}(u) &= Z^{(0)}_{0,0,0}(u) = 0. \end{aligned}$$

The transformation T corresponding to κ is given in the following theorem:

Theorem 3.4.2. *[Sob94, Thm 7.6] For any positive integers* N, M, L *there exists an operator* $T = T_{N,M,L}(\alpha,\varepsilon)$ *satisfying the following properties:*
(1) It is almost unitary:

$$\begin{aligned} T^*(\alpha,\varepsilon)T(\alpha,\varepsilon) &= I + O(\alpha^L), \\ T(\alpha,\varepsilon)T^*(\alpha,\varepsilon) &= I + O(\alpha^L). \end{aligned}$$

(2) The representation

$$T^* Op^w_\alpha h_\varepsilon T = B = B_0 + B_1 \tag{3.4.3.2}$$

holds. Here

$$\begin{aligned} B_0 &= B_0(\alpha,\varepsilon) \\ &= -\alpha^2\partial^2_{x_3} + (I \otimes K_\alpha) + \varepsilon\sum_{n=0}^{N}\alpha^n \\ &\quad \times \sum_{0\le m+l+j\le M} \varepsilon^m\{Op^w_\alpha W^{(n)}_{mlj} \otimes K^l_\alpha \sigma(K_\alpha)\}, \\ W^{(n)}_{mlj} &= \sigma(\xi_3^2)[Y^{(n)}_{mlj}(u)\xi_3^{2j} + Z^{(n)}_{mlj}(u)\xi_3^{2j+1}], \end{aligned}$$

with some $Y^{(n)}_{mlj}, Z^{(n)}_{mlj} \in \mathcal{B}^\infty(\mathbb{R}^3)$. *In particular, the functions* $Y^{(0)}_{mlj}, Z^{(0)}_{mlj}$ *are defined in Theorem 3.4.1, and*

$$Y^{(1)}_{mlj} = Z^{(1)}_{mlj} = 0$$

for all m, l, j.
The operator $B_1 = B_1(\alpha, \varepsilon)$ *in (3.4.3.2) has the form* $B_1 = \varepsilon B_2 + \alpha^{N+1} B_3$. *Here* $B_2 = B_2(\alpha, \varepsilon) = Op^w_\alpha \tau_{M+1}$ *and the operator* $B_3 = B_3(\alpha, \varepsilon)$ *can be represented for any integer* $N_1 > 0$ *as*

$$\begin{aligned} B_3(\alpha, \varepsilon) &= \varepsilon \sum_{n=0}^{N_1} \alpha^n Op^w_\alpha b_{3n} + O(\alpha^{N_1+1}), \\ b_{3n} &\in \mathcal{B}^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_\xi \times [-\varepsilon_0, \varepsilon_0]). \end{aligned}$$

(3) Let κ *be the canonical transformation constructed in Theorem 3.4.1. Then for any symbol* $\psi \in \mathcal{B}^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$

$$T^* Op^w_\alpha \psi T = Op^w_\alpha(\psi \circ \kappa) + O(\varepsilon^2) + O(\alpha^2).$$

(4) Let $\psi_1, \psi_2 \in \mathcal{B}^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$ *be two symbols* $\psi_j = \psi_j(x, \xi, \varepsilon), j = 1, 2, \varepsilon \in [-\varepsilon_0, \varepsilon_0]$, *such that*

$$\mathrm{dist}\{\mathrm{supp}\,\psi_1, \mathrm{supp}\,\psi_2\} \geq c > 0,$$

when ε_0 *is small enough. Then for any* $N_1 > 0$

$$Op^w_\alpha \psi_1 T Op^w_\alpha \psi_2 = O(\alpha^{N_1}).$$

Remark 3.4.3. The idea of reducing our operator to this form is due to Ivrii (see [Ivr98]).

3.4.3 3rd reduction

We define

$$(\mathcal{U}_\mu f)(x) = \mu^{1/4} f(x_1, \frac{x_2}{\sqrt{\mu}}, x_3).$$

For any symbol a we then have

$$\mathcal{U}^*_\mu Op^w_\alpha a \mathcal{U}_\mu = Op^w_h \tilde{a}, \tag{3.4.3.3}$$

where

$$\tilde{a}(x, \xi) = a(x_1, \frac{x_2}{\sqrt{\mu}}, x_3, \frac{\xi_1}{\mu}, \frac{\xi_2}{\sqrt{\mu}}, \frac{\xi_3}{\mu}).$$

With the T from Theorem 3.4.2 above we define

$$\Phi = \Phi_0 T \mathcal{U}_\mu,$$

and we get ([Sob94, Theorem 7.7]):

Theorem 3.4.4. *Let $R > 0$ be an arbitrary number. Suppose $\mu \geq \mu_1 \equiv \max\{\mu_0, R\}$. Then*

$$\Phi^* H_W \Phi = P = P_0 + P_1,$$

and for any $g \in C_0^\infty(\mathbb{R})$,

$$\Phi^* g(H_W)\Phi = g(P) + O(\alpha^L).$$

Here

$$P_0 = P_0(h,\mu) = -h^2\partial_{x_3}^2 + \mu K_h - \mu h + \mathcal{W}_{M,N}(h,\mu),$$

where

$$\begin{aligned} & \mathcal{W}_{M,N}(h,\mu) \\ = & \sum_{n=0}^{N} (h/\mu)^n \sum_{0\leq m+l+j\leq M} \mu^{-2m-l-2j}\{Op_h^w(p_{mlj}^{(n)} + \mu^{-1} q_{mlj}^{(n)}) \otimes K_h^l \sigma(\mu^{-1}K_h)\}, \end{aligned}$$

with

$$\begin{aligned} p_{mlj}^{(n)}(\hat{x},\hat{\xi}) &= Y_{mlj}^{(n)}(\hat{x},\mu^{-1}\xi_1)\xi_3^{2j}\sigma_R(\xi_3^2), \\ q_{mlj}^{(n)}(\hat{x},\hat{\xi}) &= Z_{mlj}^{(n)}(\hat{x},\mu^{-1}\xi_1)\xi_3^{2j+1}\sigma_R(\xi_3^2), \end{aligned}$$

where $Y_{mlj}^{(n)}, Z_{mlj}^{(n)}$ are from Theorem 3.4.2. In particular we get, where we write $\nabla_2 = (\partial_{x_1}, \partial_{x_2})$, $\Delta_2 = \partial_{x_1}^2 + \partial_{x_2}^2$:

$$\begin{aligned} p_{000}^{(0)}(\hat{x},\hat{\xi}) &= W(x_1, -\mu^{-1}\xi_1, x_3)\sigma_R(\xi_3^2), \\ p_{100}^{(0)}(\hat{x},\hat{\xi}) &= -\frac{1}{4}(\nabla_2 W(x_1, -\mu^{-1}\xi_1, x_3))^2\sigma_R(\xi_3^2), \\ p_{010}^{(0)}(\hat{x},\hat{\xi}) &= \frac{1}{4}\Delta_2 W(x_1, -\mu^{-1}\xi_1, x_3)\sigma_R(\xi_3^2), \\ p_{001}^{(0)}(\hat{x},\hat{\xi}) &= q_{000}^{(0)}(\hat{x},\hat{\xi}) = 0, \end{aligned}$$

and

$$p_{mlj}^{(1)} = q_{mlj}^{(1)} = 0,$$

for all m, l, j. The operator $P_1 = P_1(h,\mu)$ above has the form $P_1(h,\mu) = P_2 + P_3 + O(\alpha^{N+1}) + O(\alpha^{L-1})$. Here $P_2 = P_2(h,\mu) = Op_h^w p_2$ is an operator whose symbol $p_2 \in \mathcal{B}^\infty(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$ satisfies the bound

$$|\partial_x\partial_\xi p_2(x,\xi,\alpha,\varepsilon)| \leq C_M \mu^{-2(M+1)},$$

for $x_3^2 + x_2^2 + \xi_2^2 \leq C$, and $P_3 = P_3(h,\mu)$ is an operator which can be presented in the form $P_3 = (Op_h^w\zeta)\tilde{P}_3$, where $\|\tilde{P}_3\| \leq C$ and $\zeta \in \mathcal{B}^\infty(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$ is a function such that $\zeta(x,\xi) = 0$ for $|\xi_3| \leq R/2$.

3.4.4 Consequences

Of course, this reduction is not worth anything if the operator P_1 is not "small" in some sense. This is indeed the case:

Theorem 3.4.5. *([Sob94, Cor.8.5])*
Let Φ and H_W be as above and let $g \in C_0^\infty(\mathbb{R})$. Then

$$\Phi^* g(H_W)\Phi = g(P_0) + \omega(h,\mu),$$

where we have introduced the notation

$$\omega(h,\mu) = O(\mu^{-2(M+1)} + (h/\mu)^{N+1} + (h/\mu)^{L-1} + h^{N_1})$$

for all $N_1 > 0$.

Before we state the next and most important theorem of this section, we need to introduce some more notation.

Lemma 3.4.6. *[Sob94, Lemma 8.1] Let $g \in C_0^\infty(\mathbb{R})$ and let $a \in \mathcal{B}^\infty(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$ be a symbol such that*

$$\operatorname{supp} a \subset \{(x,\xi) : x_2^2 + \xi_2^2 + \xi_3^2 \geq E^2\}.$$

Then there exists $\hat{E} = \hat{E}(g) > 0$ such that

$$\|g(P_0)Op_h^w a\| \leq C_{N_1} h^{N_1},\ C_{N_1} = C_{N_1}(E),\ \forall N_1 > 0,$$

whenever $E \geq \hat{E}$.

Let now $g_1 \in C_0^\infty(\mathbb{R})$ and define functions ρ, η satisfying

- $\rho \in C_0^\infty(\mathbb{R}^3)$, $\eta \in C_0^\infty(\mathbb{R})$,
- $\rho(x) = 1, |x| \leq 3E/2$,
  $\rho(x) = 0, |x| \geq 2E$,
- $\eta(t) = 1, |t| \leq 2\hat{E}$,
  $\eta(t) = 0, |t| \geq 3\hat{E}$,
  where $\hat{E} = \hat{E}(g_1)$ is the constant defined in Lemma 3.4.6.

Finally, we define $\theta(\hat{x},\hat{\xi}) = \rho(x_1,\xi_1,x_3)\eta(\xi_3)$ and write

$$\theta^{(\mu)}(\hat{x},\hat{\xi}) = \theta(\hat{x},\mu^{-1}\xi_1,\xi_3),$$

and

$$\Theta^{(\mu)} = Op_h^w \theta^{(\mu)} \otimes I.$$

Theorem 3.4.7. *([Sob94, Theorem 10.2])*
Let $\psi \in C_0^\infty(B(E/2))$ and $g_1 \in C_0^\infty(\mathbb{R})$. Suppose $\mu \geq \mu_1, h \in (0,h_0]$ and $\mu \leq ch^{-\zeta}$ for some $\zeta \geq 1$. Then there exists $T > 0$ such that for all $|t| \leq T$,

$$\|\psi g_1(H_V)e^{-itH_V/h} - \psi\Phi g_1(P_0)e^{-itP_0/h}\Theta^{(\mu)}\Phi^*\|_1 \leq Ch^{-\frac{3}{2}(1+\zeta)}\omega(h,\mu).$$

Finally we notice [Sob94, (8.5)]

$$g(P_0) = \sum_{0\leq k\leq C/(\mu h)} \oplus(g(P_0^{(k)}) \otimes \Pi_k),$$

where Π_k is the projection in $L^2(\mathbb{R}_{x_2})$ on the k-th eigenvalue of K_h. In particular, we get when $\mu h \to \infty$:

$$g(P_0) = g(P_0^{(0)}) \otimes \Pi_0.$$

3.4.5 Order of magnitude estimates

We will also need some order of magnitude estimates on some of the quantities appearing over and over again in the calculations. This is the content of the lemmas below:

Lemma 3.4.8. *[Sob94, Cor.2.14] Let $\chi \in C_0^\infty(B(3E))$, $g \in C_0^\infty(\mathbb{R})$. Let furthermore $\phi \in \mathcal{B}^\infty(\mathbb{R})$ satisfy*

$$\operatorname{dist}(\operatorname{supp}\chi, \operatorname{supp}\phi) \geq c > 0,$$

then

$$\begin{aligned} \|\chi g(H)\|_1 &\leq Ch^{-3}(1+\mu h)^{3/2} \\ \|\chi g(H)\phi\|_1 &\leq C_N(1+\mu h)^{3/2}h^N \text{ for all } N \in \mathbb{N}. \end{aligned}$$

Lemma 3.4.9. *[Sob94, Thm. 10.4] Let $\tilde{g} \in C_0^\infty(\mathbb{R})$, $g = \tilde{g}g_0$ and let $g_1 \in C_0^\infty(\mathbb{R})$, $g_1 \equiv 1$ on a neighborhood of* $\operatorname{supp} g$. *Suppose that*

$$|\partial_{x_1}V(x)|^2 + |\partial_{x_3}V(x)|^2 + |V(x)-\lambda| \geq c > 0$$

for all $x \in B(2E)$ and all $\lambda \in \operatorname{supp} g$, then

$$\|\psi g_1(H)\chi_h(H-\lambda)\|_1 \leq C(\mu h^{-2} + h^{-3}),$$

for all $\lambda \in \operatorname{supp} g$.

We will also need the following semiclassical localisation result:

Lemma 3.4.10. *Let $\vec{a} \in C_0^\infty(B(E/2))$, let $V, \tilde{V}$ both satisfying Assumption 3.3.1 be such that $V(x) = \tilde{V}(x)$ for all $x \in B(4E)$ and let $H, \tilde{H}$ be the corresponding magnetic Schrödinger operators. Let finally $g \in C_0^\infty(\mathbb{R})$. Then*

$$\|\vec{a}\cdot(-i\nabla - \mu\vec{A})(g(H-\mu h) - g(\tilde{H}-\mu h))\|_1 = O(h^\infty).$$

3.5 An equivalent operator on the lowest Landau level

In this section we assume that $\mu h > C$, where C is some sufficiently big constant. This assures that only the lowest Landau level plays a role. We find an equivalent operator on this level which has much nicer a priori properties than **B**. This is the statement of Lemma 3.5.2 below. The whole section is devoted to the proof of this lemma, which is the key to the calculation of the current.

Since

$$\mathbf{P} = \begin{pmatrix} H_0 + V & 0 \\ 0 & H_0 + V + 2\mu h \end{pmatrix},$$

we get:

$$g_0(\mathbf{P}) = \begin{pmatrix} g_0(H_0+V) & 0 \\ 0 & g_0(H_0+V+2\mu h) \end{pmatrix}. \tag{3.5.3.1}$$

Now, $H_0 + V$ is assumed to be bounded below, thus $H_0 + V + 2\mu h > 0$ when μh is sufficiently big. Therefore,

$$\operatorname{tr}[\mathbf{B}g_0(\mathbf{P})] = \operatorname{tr}[B(\mu,h)g_0(H)], \tag{3.5.3.2}$$

where $B(\mu,h) = Op_h^w(2\vec{a}(\xi - \vec{A}) + h(\partial_{x_1}a_2 - \partial_{x_2}a_1)$, and $H = H_0 + V$.

Let $\psi \in C_0^\infty(\mathbb{R}^3), \psi \equiv 1$ on a neighborhood of $\operatorname{supp}\vec{a}$. We may choose it such that $\operatorname{supp}\psi \subset B(E/2)$. Choose also $f \in C_0^\infty(\mathbb{R})$, $f \equiv 1$ on a neighborhood of 0.

Remember that W is C_0^∞ and $W(x) = V(x)$ for all $x \in B(4E)$.

Lemma 3.5.1. *Let*

$$h_\alpha(x,\xi) = (\xi_1 + x_2)^2 + \xi_2^2 + \xi_3^2 + \varepsilon W(x) - \alpha,$$

then

$$\operatorname{tr}[B(\mu,h)g_0(H)] = \operatorname{tr}[\psi B(\mu,h)f(Op_\alpha^w h_\alpha)\psi g_0(H)] + O(h^\infty).$$

Proof. Since $\psi \equiv 1$ on a neighborhood of $\operatorname{supp}\vec{a}$ we have $B(\mu,h) = \psi B(\mu,h)$. Furthermore $g_0(ct) = g_0(t)$ for all $c > 0$ and all t, therefore

$$\operatorname{tr}[B(\mu,h)g_0(H)] = \operatorname{tr}[\psi B(\mu,h)f(H/\mu^2)g_0(H)],$$

when μ is sufficiently big. From Theorem A.1 we now get:

$$\operatorname{tr}[\psi B(\mu,h)f(H/\mu^2)g_0(H)] = \operatorname{tr}[\psi B(\mu,h)f(Op_\alpha^w h_\alpha)g_0(H)] + O(h^\infty),$$

so we only have left to prove that

$$\operatorname{tr}[\psi B(\mu,h)f(Op_\alpha^w h_\alpha)(1-\psi)g_0(H)] = O(h^\infty).$$

But this is easy, since the composition $B(\mu,h)f(Op_\alpha^w h_\alpha)(1-\psi)$ is an α-admissible operator (in the sense of [Rob87]) and has vanishing symbol. ☐

Lemma 3.5.2. *Suppose $\vec{a} = (a_1,a_2,0)$. Let*

$$b(x,\xi) = [a_2(x)\partial_{x_1}V(x) - a_1(x)\partial_{x_2}V(x)]\,f\left((\xi_1 + x_2)^2 + \xi_2^2 + \xi_3^2 + \frac{V(x)}{\mu^2}\right).$$

Then

$$\operatorname{tr}[B(\mu,h)g_0(H)] = \frac{1}{\mu}\operatorname{tr}[\psi(Op_\alpha^w b)\psi g_0(H)] + O(1/h).$$

Remark 3.5.3. The assumption $a_3 \equiv 0$ is very important for the Lemma.

Remark 3.5.4. Notice, that we could replace V with W in the definition of b, since $b(x,\xi) = 0$ when $\vec{a}(x) = 0$.

Proof. We write again

$$h_\alpha(x,\xi) = (\xi_1 + x_2)^2 + \xi_2^2 + \xi_3^2 + \varepsilon W(x) - \alpha.$$

Since

$$B(\mu,h) = \mu\Big(Op^w_\alpha(2\vec{a}\cdot(\xi_1+x_2,\xi_2,\xi_3)) + \alpha(\partial_{x_1}a_2 - \partial_{x_2}a_1)\Big),$$

we have from the symbolic and functional calculus in the Weyl quantisation

$$B(\mu,h)f(Op^w_\alpha h_\alpha) = \mu Op^w_\alpha(\gamma_0 + \alpha\gamma_1 + \alpha\gamma_2 + \alpha^2\gamma_3) + O(\mu\alpha^3),$$

Here

$$\begin{aligned}
\gamma_0 &= 2\vec{a}\cdot(\xi_1+x_2,\xi_2,\xi_3)f(h_\alpha+\alpha),\\
\gamma_1 &= 2\Big(\vec{a}\cdot(\xi_1+x_2,\xi_2,\xi_3) + \frac{1}{2i}\big[\nabla_\xi(\vec{a}\cdot(\xi_1+x_2,\xi_2,\xi_3))\cdot\nabla_x(h_\alpha)\\
&\quad -\nabla_x(\vec{a}\cdot(\xi_1+x_2,\xi_2,\xi_3))\cdot\nabla_\xi(h_\alpha)\big]\Big)f'(h_\alpha+\alpha),\\
\gamma_2 &= (\partial_{x_1}a_2 - \partial_{x_2}a_1)f(h_\alpha+\alpha)\\
&\text{and}\\
\gamma_3 &= \gamma_{3,1}(x,\xi)f'(h_\alpha+\alpha) + \gamma_{3,2}(x,\xi)f''(h_\alpha+\alpha),
\end{aligned} \tag{3.5.3.3}$$

where $\gamma_{3,1},\gamma_{3,2}\in C_0^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)$. Since $\|\psi g_0(H)\|_1 = O((\mu/h)^{3/2})$ (see Lemma 3.4.8), we get by using Lemma 3.5.1:

$$\mathrm{tr}[B(\mu,h)g_0(H)] = \mathrm{tr}[\mu\psi Op^w_\alpha(\gamma_0+\alpha\gamma_1+\alpha\gamma_2+\alpha^2\gamma_3)\psi g_0(H)] + O(\mu\alpha^{3/2}). \tag{3.5.3.4}$$

Let now $g\in C_0^\infty(\mathbb{R})$ such that

$$g(H)g_0(H) = g_0(H),$$

i.e. $g\equiv 1$ on $[\inf\operatorname{Spec}H, 0]$. Then

$$\begin{aligned}
&\mathrm{tr}[\mu\psi Op^w_\alpha(\gamma_0+\alpha\gamma_1+\alpha\gamma_2+\alpha^2\gamma_3)\psi g_0(H)]\\
=\ &\mathrm{tr}[\mu\psi Op^w_\alpha(\gamma_0+\alpha\gamma_1+\alpha\gamma_2+\alpha^2\gamma_3)\psi g(H)g_0(H)g(H)]\\
=\ &\mu\mathrm{tr}[g(H)\psi Op^w_\alpha(\gamma_0+\alpha\gamma_1+\alpha\gamma_2+\alpha^2\gamma_3)\psi g(H)g_0(H)].
\end{aligned}$$

Notice, that this can be done uniformly in V, i.e. depending only on local data, by appealing to Lemma 3.4.10.
According to Theorem 3.4.7:

$$\|\psi g(H) - \psi\Phi g(P_0)\Theta^{(\mu)}\Phi^*\|_1 \le ch^{-\frac{3}{2}(1+\zeta)}\omega(h,\mu), \tag{3.5.3.5}$$

and, when μh is sufficiently big:

$$g(P_0) = g(P_0^{(0)})\otimes\Pi_0.$$

Thus

$$\begin{aligned}
&\mathrm{tr}[B(h,\mu)g_0(H)]\\
=\ &\mu\mathrm{tr}\Big[\Phi\left(Op^w_h(\theta^{(\mu)})g(P_0^{(0)})\otimes\Pi_0\right)\Phi^*\psi Op^w_\alpha(\gamma_0+\alpha\gamma_1+\alpha\gamma_2+\alpha^2\gamma_3)\psi\Phi\\
&\times\left(g(P_0^{(0)})Op^w_h(\theta^{(\mu)})\otimes\Pi_0\right)\Phi^* g_0(H)\Big] + O(\mu\alpha^{3/2}).
\end{aligned} \tag{3.5.3.6}$$

We therefore have to calculate the composition

$$(I\otimes\Pi_0)\Phi^*Op^w_\alpha(\gamma_0+\alpha\gamma_1+\alpha\gamma_2+\alpha^2\gamma_3)\Phi(Op^w_h\theta^{(\mu)}\otimes\Pi_0).$$

This is a long and detailed calculation, where we need detailed knowledge of Φ, in particular, of the canonical transformation κ from Theorem 3.4.1. Due to its length, this calculation has been split into a series of lemmas below, the result of which is Cor. 3.5.13, from which we conclude:

$$\begin{aligned}\mathrm{tr}[B(h,\mu)g_0(H)] &= \frac{1}{\mu}\mathrm{tr}[\Phi\left(Op^w_h(\theta^{(\mu)})g(P_0^{(0)})\otimes\Pi_0\right)(Op^w_h(r)\otimes\Pi_0)\\ &\quad\times\left(g(P_0^{(0)})Op^w_h(\theta^{(\mu)})\otimes\Pi_0\right)\Phi g_0(H)]\\ &\quad+O(1+\frac{1}{\mu h^2}),\end{aligned}\tag{3.5.3.7}$$

where

$$\begin{aligned}&r(\hat{x},\hat{\xi})=\\ &\left(a_2(x_1,\frac{-\xi_1}{\mu},x_3)\partial_{x_1}V(x_1,\frac{-\xi_1}{\mu},x_3)-a_1(x_1,\frac{-\xi_1}{\mu},x_3)\partial_{x_2}V(x_1,\frac{-\xi_1}{\mu},x_3)\right)\\ \times\quad &f\left(\frac{\xi_3^2+V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2}\right)\end{aligned}\tag{3.5.3.8}$$

Here the error term was estimated using the fact that

$$\|Op^w_h(\theta^{(\mu)})\otimes\Pi_0\|_1=O(\mu/h^2),$$

an estimate which comes from standard results on pseudodifferential operators. In the same way we can calculate:

$$\begin{aligned}\mathrm{tr}[\psi(Op^w_\alpha b)\psi g_0(H)] &= \mathrm{tr}[g(H)\psi(Op^w_\alpha b)\psi g(H)g_0(H)]\\ &\simeq \mathrm{tr}[\Phi\left(Op^w_h(\theta^{(\mu)})g(P_0^{(0)})\otimes\Pi_0\right)\Phi^*\psi Op^w_\alpha(b)\psi\Phi\\ &\quad\times\left(g(P_0^{(0)})Op^w_h(\theta^{(\mu)})\otimes\Pi_0\right)\Phi^*g_0(H)],\end{aligned}$$

using (3.5.3.5). We apply Lemma 3.5.6 and get:

$$\begin{aligned}\mathrm{tr}[\psi(Op^w_\alpha b)\psi g_0(H)] &= \mathrm{tr}[\Phi\left(Op^w_h(\theta^{(\mu)})g(P_0^{(0)})\otimes\Pi_0\right)(Op^w_h(r)\otimes\Pi_0)\\ &\quad\times\left(g(P_0^{(0)})Op^w_h(\theta^{(\mu)})\otimes\Pi_0\right)\Phi^*g_0(H)]+O(h^{-1}).\end{aligned}$$

Comparing with (3.5.3.7) we get the lemma. □

Lemma 3.5.5. *Let $\nu\in C_0^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)$, and let κ be the canonical transformation constructed in Theorem 3.4.1, then*

$$\nu(\kappa(x,\xi))=\nu(x,\xi)+\varepsilon\nu_1+O(\varepsilon^2),$$

where

$$\nu_1 = \begin{pmatrix} \partial_x \nu \\ \partial_\xi \nu \end{pmatrix} \cdot \begin{pmatrix} -\partial_\xi A \\ \partial_x A \end{pmatrix}.$$

Here A is given in (3.5.3.9) below[1].

Proof. We have from Theorem 3.4.1 that if $(y,\eta)=\kappa(x,\xi)$, then

$$\begin{aligned} y &= x - \varepsilon \sum_{j=0}^{M} \partial_\xi \left(\sigma(\xi_3^2)\sigma(x_2^2+\xi_2^2)A_j(x,\xi)\right) + O(\varepsilon^2), \\ \eta &= \xi + \varepsilon \sum_{j=0}^{M} \partial_x \left(\sigma(\xi_3^2)\sigma(x_2^2+\xi_2^2)A_j(x,\xi)\right) + O(\varepsilon^2), \end{aligned}$$

where the A_j are given in the definition of κ. So

$$\begin{aligned} y &= x - \varepsilon \partial_\xi A(x,\xi) + O(\varepsilon^2), \\ \eta &= \xi + \varepsilon \partial_x A(x,\xi) + O(\varepsilon^2), \end{aligned}$$

where

$$\begin{aligned} & A(x,\xi) \\ = \; & \sigma(\xi_3^2)\sigma(x_2^2+\xi_2^2) \times \\ & \left(\frac{\xi_2}{2}\partial_z W(x_1,z,x_3)|_{z=-\xi_1} + \frac{x_2}{2} W(x_1,-\xi_1,x_3) + \sum_{2\le l+n+k\le 2M+1} a_{n,k}^{0,l}(\hat{x},\xi_1)\xi_2^n x_2^k \xi_3^l \right). \end{aligned} \tag{3.5.3.9}$$

Thus the lemma follows by taking a Taylor expansion to second order. □

Lemma 3.5.6. *Let* $\nu \in C_0^\infty(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3)$, *then*

$$(I\otimes\Pi_0)\Phi^* Op_\alpha^w(\nu)\Phi(I\otimes\Pi_0) = Op_h^w(e)\otimes\Pi_0 + \varepsilon Op_h^w(e_1)\otimes\Pi_0 + O(\alpha^2),$$

where

$$\begin{aligned} e(\hat{x},\hat{\xi}) &= \nu(x_1, -\frac{\xi_1}{\mu}, x_3, \frac{\xi_1}{\mu}, 0, \frac{\xi_3}{\mu}) \\ & \quad + \frac{h}{4\mu}(\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{\xi_2}^2 - 2\partial_{x_1}\partial_{\xi_2})\nu(x_1, -\frac{\xi_1}{\mu}, x_3, \frac{\xi_1}{\mu}, 0, \frac{\xi_3}{\mu}), \end{aligned}$$

and where

$$\begin{aligned} & e_1(\hat{x},\hat{\xi}) \\ = \; & \Bigg[\frac{1}{2}\left(-\partial_{x_1}\nu(x_1, \frac{-\xi_1}{\mu}, x_3, \frac{\xi_1}{\mu}, 0, \frac{\xi_3}{\mu}) + \partial_{\xi_2}\nu(x_1, \frac{-\xi_1}{\mu}, x_3, \frac{\xi_1}{\mu}, 0, \frac{\xi_3}{\mu}) \right) \\ & \times \partial_{x_1} W(x_1, \frac{-\xi_1}{\mu}, x_3) \\ & - \frac{1}{2}\partial_{x_2}\nu(x_1, \frac{-\xi_1}{\mu}, x_3, \frac{\xi_1}{\mu}, 0, \frac{\xi_3}{\mu}) \partial_z W(x_1,z,x_3)|_{z=\frac{-\xi_1}{\mu}} \Bigg] \sigma(\frac{\xi_3^2}{\mu^2}) \\ & + \frac{\xi_3}{\mu}\zeta(\hat{x}, \frac{\hat{\xi}}{\mu}), \end{aligned}$$

[1] Remember that $W(x)=V(x)$ on $B(3E)$, and that $W\in C_0^\infty$.

where $\zeta \in C_0^\infty(\mathbb{R}^2_{\hat{x}} \times \mathbb{R}^2_{\hat{\xi}})$.

Remark 3.5.7. As is easily seen, this is the fundamental result that we needed in the proof of Lemma 3.5.2. The proof is rather long, though elementary: it consists essentially of an application of Lemma 3.5.5 and of the stationary phase formula.

Proof. From (3.4.3.1) it follows that

$$\Phi^* Op^w_\alpha \nu \Phi = U^*_\mu T^* Op^w_\alpha \tilde{\nu} T U_\mu$$

where $\tilde{\nu}(x,\xi) = \nu(x_1 - \xi_2, x_2 - \xi_1, x_3; \xi)$. Part (3) of Theorem 3.4.2 now tells us that:

$$T^*(Op^w_\alpha \tilde{\nu})T = Op^w_\alpha(\tilde{\nu} \circ \kappa) + O(\alpha^2),$$

and from Lemma 3.5.5 we get

$$\tilde{\nu}(\kappa(x,\xi)) = \tilde{\nu}(x,\xi) + \varepsilon \tilde{\nu}_1(x,\xi) + O(\varepsilon^2),$$

Because of (3.4.3.3) we therefore conclude that

$$\Phi^* Op^w_\alpha \nu \Phi = Op^w_h \bar{\nu} + \varepsilon Op^w_h \bar{\nu}_1 + O(\alpha^2),$$

where

$$\begin{aligned}
\bar{\nu}(x,\xi) &= \tilde{\nu}(x_1, \frac{x_2}{\sqrt{\mu}}, x_3; \mu^{-1}\xi_1, \frac{\xi_2}{\sqrt{\mu}}, \mu^{-1}\xi_3) \\
\bar{\nu}_1(x,\xi) &= \tilde{\nu}_1(x_1, \frac{x_2}{\sqrt{\mu}}, x_3; \mu^{-1}\xi_1, \frac{\xi_2}{\sqrt{\mu}}, \mu^{-1}\xi_3).
\end{aligned}$$

So

$$(I \otimes \Pi_0)\Phi^* Op^w_\alpha(\nu)\Phi(I \otimes \Pi_0) = (Op^w_h(e) + \varepsilon Op^w_h(e_1)) \otimes \Pi_0 + O(\alpha^2),$$

where e, e_1 have symbols

$$e(\hat{x},\hat{\xi}) = \frac{1}{2\pi h} \iiint \mathcal{H}_0(x_2) e^{ih^{-1}(x_2 - y_2, \xi_2)} \bar{\nu}(\hat{x}, \hat{\xi}, \frac{x_2 + y_2}{2}, \xi_2) \mathcal{H}_0(y_2)\ dx_2\, dy_2\, d\xi_2, \tag{3.5.3.10}$$

and

$$e_1(\hat{x},\hat{\xi}) = \frac{1}{2\pi h} \iiint \mathcal{H}_0(x_2) e^{ih^{-1}(x_2 - y_2, \xi_2)} \bar{\nu}_1(\hat{x}, \hat{\xi}, \frac{x_2 + y_2}{2}, \xi_2) \mathcal{H}_0(y_2)\ dx_2\, dy_2\, d\xi_2.$$

Let us first analyze e:

We can look upon the expression (3.5.3.10) as the expectation value of the operator $Op^w_h s(x_2,\xi_2)$ in the state $\mathcal{H}_0$. Here the symbol s depends on the parameters $(\hat{x},\hat{\xi})$ in the sense that

$$s(x_2,\xi_2) = \bar{\nu}(x,\xi).$$

Since $\bar{\nu}$ depends on (x_2, ξ_2) in the form $(\frac{x_2}{\sqrt{\mu}}, \frac{\xi_2}{\sqrt{\mu}})$, we get from the laws for changing symbol types (see [Rob87]):

$$\begin{aligned} e(\hat{x}, \hat{\xi}) &= \langle \mathcal{H}_0, Op_{h,0} s \mathcal{H}_0 \rangle + h \langle \mathcal{H}_0, Op_{h,0} s_1 \mathcal{H}_0 \rangle + O(\alpha^2) \\ &= I_1 + I_2 + O(\alpha^2), \end{aligned}$$

where

$$s_1(x, \xi) = \frac{1}{2i} \partial_{x_2} \partial_{\xi_2} \bar{\nu}.$$

Let us remember that

$$\begin{aligned} \mathcal{H}_0(x) &= \frac{1}{\sqrt[4]{\pi h}} e^{-x^2/(2h)}, \\ \mathcal{H}_0(\xi) &= \frac{1}{\sqrt{2\pi h}} \int e^{-ix\xi/h} \mathcal{H}_0(x) dx. \end{aligned}$$

So if we look at

$$I_1 = \langle \mathcal{H}_0, Op_{h,0} s \mathcal{H}_0 \rangle,$$

we get

$$\begin{aligned} I_1 &= \frac{1}{2\pi h} \iiint \mathcal{H}_0(x) e^{ih^{-1}(x-y,\xi)} s(x,\xi) \mathcal{H}_0(y) \;\; dx dy d\xi \\ &= \frac{1}{\sqrt{2\pi h}} \iint e^{-x^2/(2h)} e^{ih^{-1} x\xi} s(x,\xi) e^{-\xi^2/(2h)} dx d\xi \\ &= \frac{1}{\sqrt{2\pi h}} \iint e^{ih^{-1}(2x\xi + i(x^2+\xi^2)/2)} s(x,\xi) dx d\xi \\ &= \frac{1}{\sqrt{2\pi h}} \iint e^{ih^{-1}\langle (x,\xi), A(x,\xi)\rangle/2} s(x,\xi) dx d\xi, \end{aligned}$$

where A is the matrix:

$$A = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}.$$

From the theorem of stationary phase ([Hö90, Lemma 7.7.3]) we get:

$$I_1 \approx \frac{1}{\sqrt{2\pi h}} \left(\sqrt{\det(h^{-1}A/(2\pi i))} \right)^{-1} \sum_{j=0}^{\infty} \frac{h^j}{(2i)^j} \frac{(A^{-1}D, D)^j}{j!} s|_{(0,0)},$$

in the sense of an asymptotic series. We easily see that

$$A^{-1} = -\frac{1}{2} \begin{pmatrix} i & -1 \\ -1 & i \end{pmatrix},$$

and therefore

$$(A^{-1}D, D) = \frac{1}{2}[i\Delta - 2\partial_x \partial_\xi],$$

and

$$\left(\sqrt{\det(h^{-1}A/(2\pi i))} \right)^{-1} = \sqrt{2}\pi h.$$

So we get:

$$I_1 = s(0,0) + \frac{h}{4i}[i\Delta - 2\partial_x\partial_\xi]s|_{(0,0)} + O(\alpha^2).$$

By the same method

$$\begin{aligned} I_2 &= \langle \mathcal{H}_0, hOp_{h,0}s_1\mathcal{H}_0\rangle \\ &= hs_1(0,0) + O(\alpha^2), \end{aligned}$$

so

$$I_1 + I_2 = s(0,0) + \frac{h}{4}\Delta s(0,0) + O(\alpha^2).$$

Thus

$$\begin{aligned} e(\hat{x},\hat{\xi}) &= \tilde{\nu}(x_1,0,x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}) + \frac{h}{4\mu}\Delta_{(x_2,\xi_2)}\tilde{\nu}(x_1,0,x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}) + O(\alpha^2) \\ &= \nu(x_1,-\frac{\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}) \\ &\quad + \frac{h}{4\mu}(\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{\xi_2}^2 - 2\partial_{x_1}\partial_{\xi_2})\nu(x_1,-\frac{\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}) \\ &+ O(\alpha^2). \end{aligned} \tag{3.5.3.11}$$

In the same way we get:

$$e_1(\hat{x},\hat{\xi}) = \tilde{\nu}_1(x_1,0,x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}) + O(\alpha)$$

Here

$$\tilde{\nu}_1 = \begin{pmatrix} \partial_x\tilde{\nu} \\ \partial_\xi\tilde{\nu} \end{pmatrix} \cdot \begin{pmatrix} -\partial_\xi A \\ \partial_x A \end{pmatrix}.$$

We can thus calculate:

$$\begin{aligned} &\tilde{\nu}_1|_{x_2=\xi_2=0} \\ = &-\partial_{x_1}\tilde{\nu}\sum_{2\le l\le 2M+1}\xi_3^l\partial_{\xi_1}a_{0,0}^{0,l}(\hat{x},\xi_1)\sigma(\xi_3^2) \\ &-\partial_{x_2}\tilde{\nu}\left(\frac{1}{2}\partial_z W(x_1,z,x_3)|_{z=-\xi_1} + \sum_{2\le l+1\le 2M+1}\xi_3^l a_{1,0}^{0,l}(\hat{x},\xi_1)\right)\sigma(\xi_3^2) \\ &-\partial_{x_3}\tilde{\nu}\left(\sum_{2\le l\le 2M+1} l\xi_3^{l-1}a_{0,0}^{0,l}(\hat{x},\xi_1)\sigma(\xi_3^2) + \sigma'(\xi_3^2)2\xi_3\sum_{2\le l\le 2M+1}\xi_3^l a_{0,0}^{0,l}(\hat{x},\xi_1)\right) \\ &+\partial_{\xi_1}\tilde{\nu}\sum_{2\le l\le 2M+1}\xi_3^l\partial_{x_1}a_{0,0}^{0,l}(\hat{x},\xi_1)\sigma(\xi_3^2) \\ &+\partial_{\xi_2}\tilde{\nu}\left(\frac{1}{2}\partial_{x_1}W(x_1,-\xi_1,x_3) + \sum_{2\le l+1\le 2M+1}\xi_3^l a_{1,0}^{0,l}(\hat{x},\xi_1)\right)\sigma(\xi_3^2) \\ &+\partial_{\xi_3}\tilde{\nu}\sum_{2\le l\le 2M+1}\partial_{x_3}a_{0,0}^{0,l}(\hat{x},\xi_1)\xi_3^l\sigma(\xi_3^2) \\ = &\left(\frac{1}{2}(\partial_{\xi_2}\tilde{\nu})\partial_{x_1}W(x_1,z,x_3) - \frac{1}{2}(\partial_{x_2}\tilde{\nu})\partial_z W(x_1,z,x_3)|_{z=-\xi_1}\right)\sigma(\xi_3^2) \\ &+\xi_3\zeta(\hat{x},\hat{\xi}), \end{aligned}$$

where $\zeta \in C_0^\infty(\mathbb{R}_x^2 \times \mathbb{R}_\xi^2)$. Now $\tilde{\nu}(x,\xi) = \nu(x_1-\xi_2, x_2-\xi_1, x_3, \xi)$ so we get:

$$\begin{aligned}
&\tilde{\nu}_1|_{x_2=\xi_2=0} = \\
&\Big[\frac{1}{2}(-\partial_{x_1}\nu(x_1,-\xi_1,x_3,\hat{\xi}) + \partial_{\xi_2}\nu(x_1,-\xi_1,x_3,\hat{\xi}))\partial_{x_1}W(x_1,-\xi_1,x_3) \\
&-\frac{1}{2}(\partial_{x_2}\nu(x_1,-\xi_1,x_3,\hat{\xi}))\partial_z W(x_1,z,x_3)|_{z=-\xi_1}\Big]\sigma(\xi_3^2) + \xi_3\zeta(\hat{x},\hat{\xi}).
\end{aligned}$$

Finally, we get

$$\begin{aligned}
&\tilde{e_1}(\hat{x},\hat{\xi}) = \\
&\Big[\frac{1}{2}(-\partial_{x_1}\nu(x_1,\frac{-\xi_1}{\mu},x_3,\frac{\hat{\xi}}{\mu}) + \partial_{\xi_2}\nu(x_1,\frac{-\xi_1}{\mu},x_3,\frac{\hat{\xi}}{\mu}))\partial_{x_1}W(x_1,-\xi_1,x_3) \\
&-\frac{1}{2}(\partial_{x_2}\nu(x_1,\frac{-\xi_1}{\mu},x_3,\frac{\hat{\xi}}{\mu}))\partial_z W(x_1,z,x_3)|_{z=\frac{-\xi_1}{\mu}}\Big]\sigma(\frac{\xi_3^2}{\mu^2}) + \frac{\xi_3}{\mu}\zeta(\hat{x},\frac{\hat{\xi}}{\mu}),
\end{aligned}$$

where $\zeta \in C_0^\infty(\mathbb{R}_{\hat{x}}^2 \times \mathbb{R}_{\hat{\xi}}^2)$. ∎

The following corollaries now apply Lemma 3.5.6 on each of the symbols in (3.5.3.3).

Corollary 3.5.8. *Suppose $\vec{a} = (a_1, a_2, 0)$. Then*

$$(I\otimes\Pi_0)\Phi^* Op_\alpha^w(\gamma_0)\Phi(I\otimes\Pi_0) = \frac{h}{\mu}Op_h^w(\tilde{d})\otimes\Pi_0 + \varepsilon Op_h^w\tilde{d}_1\otimes\Pi_0 + O(\alpha^2),$$

where

$$\tilde{d} = \left(\partial_{x_2}a_1(x_1,-\frac{\xi_1}{\mu},x_3) - \partial_{x_1}a_2(x_1,-\frac{\xi_1}{\mu},x_3)\right)f(\frac{\xi_3^2 + V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2}),$$

and

$$\begin{aligned}
&\tilde{d}_1(\hat{x},\hat{\xi}) = \\
&\left(a_2(x_1,\frac{-\xi_1}{\mu},x_3)\partial_{x_1}W(x_1,\frac{-\xi_1}{\mu},x_3) - a_1(x_1,\frac{-\xi_1}{\mu},x_3)\partial_{x_2}W(x_1,\frac{-\xi_1}{\mu},x_3)\right) \\
&\times f(\frac{\xi_3^2 + V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2}) + \frac{\xi_3}{\mu}\zeta(\hat{x},\frac{\hat{\xi}}{\mu}).
\end{aligned}$$

Proof. From (3.5.3.3) we know that

$$\gamma_0 = 2\left(a_1(\xi_1+x_2) + a_2\xi_2\right)f(h_\alpha+\alpha),$$

so

$$\begin{aligned}
\gamma_0(x_1,-\frac{\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}) &= 0, \\
\partial_{x_1}^2\gamma_0(x_1,-\frac{\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}) &= 0, \\
\partial_{\xi_2}^2\gamma_0(x_1,-\frac{\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}) &= 0,
\end{aligned}$$

and

$$\begin{aligned}
&\partial^2_{x_2}\gamma_0(x_1,-\frac{\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu})\\
= \ & 4\partial_{x_2}a_1(x_1,-\frac{\xi_1}{\mu},x_3)f(\frac{\xi_3^2+V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2})\\
&+4\varepsilon a_1(x_1,-\frac{\xi_1}{\mu},x_3)f'(\frac{\xi_3^2+V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2})\partial_{x_2}V(x_1,-\frac{\xi_1}{\mu},x_3),\\
&-2\partial_{x_1}\partial_{\xi_2}\gamma_0(x_1,-\frac{\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu})\\
= \ & -4\partial_{x_1}a_2(x_1,-\frac{\xi_1}{\mu},x_3)f(\frac{\xi_3^2+V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2})\\
&-4\varepsilon a_2(x_1,-\frac{\xi_1}{\mu},x_3)f'(\frac{\xi_3^2+V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2})\partial_{x_1}V(x_1,-\frac{\xi_1}{\mu},x_3).
\end{aligned}$$

Thus

$$\tilde{d}=\left(\partial_{x_2}a_1(x_1,-\frac{\xi_1}{\mu},x_3)-\partial_{x_1}a_2(x_1,-\frac{\xi_1}{\mu},x_3)\right)f(\frac{\xi_3^2+V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2}).$$

We can also calculate:

$$\begin{aligned}
\partial_{x_2}\gamma_0(x_1,\frac{-\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}) &= 2a_1(x_1,\frac{-\xi_1}{\mu},x_3)f(\frac{\xi_3^2+V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2})\\
\partial_{\xi_2}\gamma_0(x_1,\frac{-\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}) &= 2a_2(x_1,\frac{-\xi_1}{\mu},x_3)f(\frac{\xi_3^2+V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2})\\
\partial_{x_1}\gamma_0(x_1,\frac{-\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}) &= 0,
\end{aligned}$$

so we get:

$$\begin{aligned}
&\tilde{d}_1(\hat{x},\hat{\xi})=\\
&\left(a_2(x_1,\frac{-\xi_1}{\mu},x_3)\partial_{x_1}W(x_1,\frac{-\xi_1}{\mu},x_3)-a_1(x_1,\frac{-\xi_1}{\mu},x_3)\partial_{x_2}W(x_1,\frac{-\xi_1}{\mu},x_3)\right)\\
&\times f(\frac{\xi_3^2+V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2})+\frac{\xi_3}{\mu}\zeta(\hat{x},\frac{\hat{\xi}}{\mu}).
\end{aligned}$$

□

Remark 3.5.9. Notice that if a_3 had not been zero then $\gamma_0(x_1,-\frac{\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu})$ would not have vanished.

Lemma 3.5.10. *Let $\tilde{\theta}\in C_0^\infty(\mathbb{R}^4_{\hat{x},\hat{\xi}})$, and let $\tilde{\theta}^{(\mu)}(\hat{x},\hat{\xi})=\tilde{\theta}(\hat{x},\frac{\xi_1}{\mu},\xi_3)$, then*

$$Op_h^w(\frac{\xi_3}{\mu}\zeta(\hat{x},\frac{\hat{\xi}}{\mu}))Op_h^w(\tilde{\theta}^{(\mu)})=O(1/\mu).$$

Proof. This is an easy consequence of the symbolic calculus and the compactness of the support of $\tilde{\theta}$. $\square$

Corollary 3.5.11. *Let $\tilde{\theta} \in C_0^\infty(\mathbb{R}^4_{\hat{x},\hat{\xi}})$, and let $\tilde{\theta}^{(\mu)}(\hat{x},\hat{\xi}) = \tilde{\theta}(\hat{x}, \frac{\xi_1}{\mu}, \xi_3)$, then*

$$(I \otimes \Pi_0)\Phi^* Op^w_\alpha(\gamma_1)\Phi(Op^w_h(\tilde{\theta}^{(\mu)}) \otimes \Pi_0) = O(\varepsilon)$$

and

$$(I \otimes \Pi_0)\Phi^* Op^w_\alpha(\gamma_3)\Phi(Op^w_h(\tilde{\theta}^{(\mu)}) \otimes \Pi_0) = O(\varepsilon).$$

Proof. This follows easily from Lemma 3.5.6 because $f' \equiv 0$ on a neighborhood of 0, and therefore

$$f'(\frac{\xi_3^2 + V(x_1, -\frac{\xi_1}{\mu}, x_3)}{\mu^2}) \equiv 0$$

on the support of $\tilde{\theta}^{(\mu)}$ for μ sufficiently big. Same argument works for f''. $\square$

Corollary 3.5.12.

$$(I \otimes \Pi_0)\Phi^* Op^w_\alpha(\gamma_2)\Phi(I \otimes \Pi_0) = Op^w_h(\tilde{d}_2) \otimes \Pi_0 + O(\varepsilon),$$

where

$$\tilde{d}_2(\hat{x},\hat{\xi}) = \left(\partial_{x_1} a_2(x_1, -\frac{\xi_1}{\mu}, x_3) - \partial_{x_2} a_1(x_1, -\frac{\xi_1}{\mu}, x_3)\right) f(\frac{\xi_3^2 + V(x_1, -\frac{\xi_1}{\mu}, x_3)}{\mu^2}).$$

To summarize the content of the above we have:

Corollary 3.5.13.

$$\begin{aligned} & (I \otimes \Pi_0)\Phi^* Op^w_\alpha(\gamma_0 + \alpha\gamma_1 + \alpha\gamma_2 + \alpha^2\gamma_3)\Phi(Op^w_h \theta^{(\mu)} \otimes \Pi_0) \\ = \quad & \varepsilon(Op^w_h(r) Op^w_h(\theta^{(\mu)})) \otimes \Pi_0 + O(\alpha^2 + \varepsilon/\mu), \end{aligned}$$

where

$$\begin{aligned} & r(\hat{x},\hat{\xi}) = \\ & \left(a_2(x_1, \frac{-\xi_1}{\mu}, x_3)\partial_{x_1} W(x_1, \frac{-\xi_1}{\mu}, x_3) - a_1(x_1, \frac{-\xi_1}{\mu}, x_3)\partial_{x_2} W(x_1, \frac{-\xi_1}{\mu}, x_3)\right) \\ & \times f(\frac{\xi_3^2 + V(x_1, -\frac{\xi_1}{\mu}, x_3)}{\mu^2}). \end{aligned}$$

Remark 3.5.14. Notice that the leading terms from γ_0 and γ_2 cancel. This is easily seen to be the cancellation of the spin-current and the persistent current.

3.6 Semiclassics on each Landau level

On each of the (modified) Landau levels we have a standard 2-dimensional semi-classical problem corresponding to the operators $P_0^{(k)}$. In this section we will state the semiclassical formulas that we need about those. Proofs of the statements can be found in [Sob94, Sect. 9].

Lemma 3.6.1. *Let* $\theta \in C_0^\infty(\mathbb{R}^2_{\hat{x}} \times \mathbb{R}^2_{\hat{\xi}})$ *satisfy*

$$\theta(\hat{x},\hat{\xi}) = 0,\ x_1^2+\xi_1^2+x_3^2 \geq (2E)^2,\ E>0.$$

Suppose that $R>0$ *is large enough and* $h \in (0,h_0]$, $\mu \geq \mu_1(R)$, *where* $\mu_1(R)$ *was defined in Thm 3.4.4.*
If θ *is even in* ξ_3 *then where we write* $\nabla_2 = (\partial_{x_1}, \partial_{x_2})$, $\Delta_2 = \partial_{x_1}^2 + \partial_{x_2}^2$*:*

$$\begin{aligned}
& \operatorname{tr}[Op_h^w \theta^{(\mu)} g(P_0^{(k)})] \\
= \;& \frac{\mu}{(2\pi h)^2}\int \theta(\hat{x},\hat{\xi}) g(\xi_3^2 + W(x_1,-\xi_1,x_3) + 2\mu hk)\, d\hat{x} d\hat{\xi} \\
& + \frac{1}{16\mu(\pi h)^2}\int \theta(\hat{x},\hat{\xi})[2\mu hk\Delta_2 W - (\nabla_2 W)^2](x_1,-\xi_1,x_3) \\
& \times g'(\xi_3^2 + W(x_1,-\xi_1,x_3) + 2\mu hk)\, d\hat{x} d\hat{\xi} \\
& + O(h^{-1} + \mu^{-3}h^{-2} + \mu).
\end{aligned}$$

If θ *is odd in* ξ_3 *then*

$$\operatorname{tr}[Op_h^w \theta^{(\mu)} g(P_0^{(k)})] = O(\mu^{-2}h^{-2} + \mu).$$

As is usual in this kind of semiclassical problem we need more than just smooth functions g, we also need to consider an h-dependent smoothing out (by means of a convolution) of our function g_0. The result in this context is given in the next lemma. Before we can state that, we will need some further notation and a non-criticality assumption of the usual type:

$$|\partial_{x_1} W(x)|^2 + |\partial_{x_3} W(x)|^2 + |W(x) + 2\mu hk - \lambda| \geq \delta > 0\ \ \forall x \in B(2E). \tag{3.6.3.1}$$

Let T be the number from Thm 3.4.7. Let $\hat{\chi} \in C_0^\infty(-T,T)$ satisfy

- $\hat{\chi}(t) = \hat{\chi}(-t)$,
- $\hat{\chi}(t) = 1/\sqrt{2\pi}$,
- $\hat{\chi} \geq 0$.

Then we define

$$\chi(\tau) = \frac{1}{\sqrt{2\pi}}\int \hat{\chi}(t) e^{i\tau t} dt.$$

We assume that $\chi \geq 0$, this is possible since we could have replaced $\hat{\chi}$ by $\hat{\chi} * \hat{\chi}$. Finally, we define

$$\chi_h(\tau) = \frac{1}{h}\chi(\frac{\tau}{h}),$$

and

$$g^{(h)}(\tau) = \int g(\sigma)\chi_h(\tau-\sigma)d\sigma,$$

for such functions g, where the integral converges.

Lemma 3.6.2. *Let θ be as in Lemma 3.6.1.*

1. *Let k be a number such that (3.6.3.1) is fulfilled for some $\lambda \in \mathbb{R}$. Then*

$$\|Op_h^w \theta^{(\mu)} \chi_h(P_0^{(k)} - \lambda)\|_1 \leq C\mu h^{-2}.$$

2. *Let $\tilde{g} \in C_0^\infty(\mathbb{R})$ and let $g = \tilde{g}g_0$. Let k be a number such that (3.6.3.1) is fulfilled for all $\lambda \in \operatorname{supp} g$. If θ is even in ξ_3, then*

$$\begin{aligned}
& \operatorname{tr}[Op_h^w \theta^{(\mu)} g^{(h)}(P_0^{(k)})] \\
= \ & \frac{\mu}{(2\pi h)^2} \int \theta(\hat{x}, \hat{\xi}) g(\xi_3^2 + W(x_1, -\xi_1, x_3) + 2\mu hk)\, d\hat{x} d\hat{\xi} \\
& + \frac{1}{16\mu(\pi h)^2} \int \theta(\hat{x}, \hat{\xi}) [2\mu hk \Delta_2 W - (\nabla_2 W)^2](x_1, -\xi_1, x_3) \\
& \times g'(\xi_3^2 + W(x_1, -\xi_1, x_3) + 2\mu hk)\, d\hat{x} d\hat{\xi} \\
& + O(h^{-1} + \mu^{-1-2s} h^{-2} + \mu).
\end{aligned}$$

If θ is odd in ξ_3, then

$$\operatorname{tr}[Op_h^w \theta^{(\mu)} g(P_0^{(k)})] = O(\mu^{-2} h^{-2} + \mu + \mu^{-1-2s} h^{-2}).$$

3.7 Calculation of the current

With the reduced operator it is rather easy to calculate the current by standard techniques: Choose $f_1, f_2 \in C_0^\infty(\mathbb{R})$ such that:

- $(f_1^2(H) + f_2^2(H))g_0(H) = g_0(H)$.
- $f_1^2(H)g_0(H) = f_1^2(H)$.
- $|\partial_{x_1} V(x)|^2 + |\partial_{x_3} V(x)|^2 + |V(x) - \lambda| \geq c > 0$ for all $(x, \lambda) \in B(2E) \times \operatorname{supp} f_2$.

Then

$$\begin{aligned}
\operatorname{tr}[\psi(Op_\alpha^w b)\psi g_0(H)] &= \operatorname{tr}[\psi(Op_\alpha^w b)\psi f_1^2(H) g_0(H)] \\
& \quad + \operatorname{tr}[\psi(Op_\alpha^w b)\psi f_2^2(H) g_0(H)].
\end{aligned}$$

The first part, $\operatorname{tr}[\psi(Op_\alpha^w b)\psi f_1^2(H)g_0(H)]$, will be calculated directly in Theorem 3.7.1 below. To handle the second term, $\operatorname{tr}[\psi(Op_\alpha^w b)\psi f_2^2(H)g_0(H)]$, we need a Tauberian argument. Theorems 3.7.2 and 3.7.4 will carry this through. From Theorems 3.7.1, 3.7.2 and 3.7.4 together we get Theorem 3.3.6 by a simple integration by parts.

Theorem 3.7.1.

$$
\begin{aligned}
&\frac{1}{\mu}\mathrm{tr}[\psi(Op^w_\alpha b)\psi f_1^2(H)g_0(H)] = \\
&\frac{1}{4\pi^2h^2}\times \\
&\iint (a_2(x_1,-\xi_1,x_3)\partial_{x_1}V(x_1,-\xi_1,x_3) - a_1(x_1,-\xi_1,x_3)\partial_{x_2}V(x_1,-\xi_1,x_3)) \\
&\times f_1^2(\xi_3^2+V(x_1,-\xi_1,x_3))d\hat{x}d\hat{\xi}+O(h^{-1}).
\end{aligned}
$$

Proof. Notice that $f_1^2(H)g_0(H) = f_1^2(H)$. Remember the definition of r in (3.5.3.8) and we get:

$$
\begin{aligned}
&\frac{1}{\mu}\mathrm{tr}[\psi(Op^w_\alpha b)\psi f_1^2(H)] \\
= \;&\frac{1}{\mu}\mathrm{tr}[f_1(H)\psi(Op^w_\alpha b)\psi f_1(H)] \\
= \;&\frac{1}{\mu}\mathrm{tr}[\Phi\left((Op^w_h\theta^{(\mu)})f_1(P_0^{(0)})Op^w_h(r)f_1(P_0^{(0)})(Op^w_h\theta^{(\mu)})\otimes\Pi_0\right)\Phi^*] \\
&+O(h^{-1}) \\
= \;&\frac{1}{\mu}\mathrm{tr}[(Op^w_h\theta^{(\mu)})^2 f_1(P_0^{(0)})Op^w_h(r)f_1(P_0^{(0)})]+O(h^{-1}) \\
= \;&\frac{1}{(2\pi h)^2}\iint r(\hat{x},\mu\hat{\xi})f_1^2(\xi_3+V(x_1,-\xi_1,x_3))d\hat{x}d\hat{\xi}+O(h^{-1}),
\end{aligned}
$$

where we used Lemma 3.6.1 to get the last equality. ☐

Now we can state:

Theorem 3.7.2. *Suppose that (3.3.3.1) is satisfied, then*

$$\mathrm{tr}[\psi(Op^w_\alpha b)\psi f_2^2(H)g_0(H)] = \mathrm{tr}[\psi(Op^w_\alpha b)\psi f_2^2(H)g_0^{(h)}(H)]+O(\mu/h).$$

We will need the following lemma:

Lemma 3.7.3. $\exists\varepsilon>0$ *such that*

$$|g_0(\tau)-g_0(\tau-\rho)|\le ch\chi_h(\tau),$$

for all $|\rho|\le h\varepsilon$. *Here* c *is a constant independent of* h *and* τ.

Proof. Choose 2ε such that $\chi_1(\tau)\ge\tilde{c}>0$ for $|\tau|\le 2\varepsilon$. Then, for $|\tau|\le 2\varepsilon h$ and $|\rho|\le h\varepsilon$ we have

$$|g_0(\tau)-g_0(\tau-\rho)|\le\frac{1}{\tilde{c}}\chi_1(\tau/h)=\frac{h}{\tilde{c}}\chi_h(\tau),$$

and for $|\tau|\ge 2\varepsilon h$ and $|\rho|\le h\varepsilon$ we have

$$|g_0(\tau)-g_0(\tau-\rho)|=0.$$

☐

Now we can prove Theorem 3.7.2.

Proof. In this proof (and only here) we will use the notation: $[x]$ = the integral part of x $= \max\{n \in \mathbb{Z} | n \leq x\}$. By cyclicity of trace it is enough to prove

$$\|Op^w_\alpha(b)\psi f_2(H)\left(g_0(H) - g_0^{(h)}(H)\right) f_2(H)\psi\|_1 = O(\mu/h).$$

Because $\|Op^w_\alpha(b)\| = O(1)$ it is thus enough to prove

$$\|\psi f_2(H)\left(g_0(H) - g_0^{(h)}(H)\right) f_2(H)\psi\|_1 = O(\mu/h).$$

We now estimate using the lemma above:

$$\begin{aligned}
& \|\psi f_2(H)\left(g_0(H) - g_0^{(h)}(H)\right) f_2(H)\psi\|_1 \\
= \; & \|\psi f_2(H)\int \chi_h(\rho)\,(g_0(H) - g_0(H-\rho))\,d\rho f_2(H)\psi\|_1 \\
= \; & \|\psi f_2(H)\int_{-\delta}^{\delta} \chi_h(\rho)\,(g_0(H) - g_0(H-\rho))\,d\rho f_2(H)\psi\|_1 + O(h^\infty) \\
\leq \; & \int_0^{\delta} \mathrm{tr}\,(\psi f_2(H)\chi_h(\rho)\,(g_0(H-\rho) - g_0(H))\,f_2(H)\psi)\,d\rho \\
& + \int_{-\delta}^{0} \mathrm{tr}\,(\psi f_2(H)\chi_h(\rho)\,(g_0(H) - g_0(H-\rho))\,f_2(H)\psi)\,d\rho + O(h^\infty) \\
\leq \; & \int_{-\delta}^{\delta} \chi_h(\rho)\mathrm{tr}\Bigg(\psi f_2(H) \\
& \left(\sum_{j=0}^{[\frac{|\rho|}{h\varepsilon}]} ch\chi_h(H - \mathrm{sign}(\rho)jh\varepsilon) + ch\chi_h(H + |\rho| - \mathrm{sign}(\rho)[\frac{|\rho|}{h\varepsilon}]h\varepsilon)\right) \\
& \times f_2(H)\psi\Bigg)\,d\rho + O(h^\infty) \\
\leq \; & c\frac{\mu}{h}\int \chi_h(\rho)(\frac{|\rho|}{h} + 1)d\rho + O(h^\infty) \\
= \; & O(\mu/h),
\end{aligned}$$

where we used

$$\|\psi f_2(H)\chi_h(H-\tau)\|_1 = O(\mu/h^2)$$

in the end. That inequality comes from Lemma 3.4.9. ☐

Theorem 3.7.4. *Suppose that (3.3.3.1) is satisfied. Then*

$$\begin{aligned}
& \frac{1}{\mu}\mathrm{tr}[\psi(Op^w_\alpha b)\psi f_2^2(H)g_0^{(h)}(H)] \\
= \; & \frac{1}{4\pi^2h^2}\times \\
& \int\Big(a_2(x_1,-\xi_1,x_3)\partial_{x_1}V(x_1,-\xi_1,x_3) - a_1(x_1,-\xi_1,x_3)\partial_{x_2}V(x_1,-\xi_1,x_3)\Big) \\
& \times(f_2^2g_0)(\xi_3^2 + V(x_1,-\xi_1,x_3))\,d\hat{x}d\hat{\xi} + O(h^{-1}).
\end{aligned}$$

Proof.

$$\frac{1}{\mu}\mathrm{tr}[\psi(Op^w_\alpha b)\psi f_2^2(H)g_0^{(h)}(H)] = \frac{1}{\mu}\int g_0(\tau)\mathrm{tr}[\psi(Op^w_\alpha b)\psi f_2^2(H)\chi_h(H-\tau)]d\tau$$
$$= \frac{1}{\sqrt{2\pi}\mu h}\iint g_0(\tau)\hat{\chi}(t)\mathrm{tr}[\psi(Op^w_\alpha b)\psi f_2^2(H)e^{-it(H-\tau)/h}]dt\,d\tau$$
$$= \frac{1}{\sqrt{2\pi}\mu h}\iint g_0(\tau)\hat{\chi}(t)e^{it\tau/h}\mathrm{tr}[f_2(H)\psi(Op^w_\alpha b)\psi f_2(H)e^{-itH/h}]dt\,d\tau.$$

Notice, that since χ_h is a Schwarz function, we can replace g_0 by $1_{[-E_0,0]}$. This will only introduce an error of order $O(h^\infty)$, and makes the integral absolutely convergent. Now we apply Theorem 3.4.7:

$$\|\psi f_2(H)e^{-itH/h} - \psi\Phi f_2(P_0)e^{-itP_0/h}\Theta^{(\mu)}\Phi^*\|_1 \leq ch^{\frac{3}{2}(1+\zeta)}\omega(h,\mu).$$

Since $\Phi^*\Phi \simeq I$ and μh is large, we thus get:

$$\begin{aligned}&\frac{1}{\mu}\mathrm{tr}[\psi(Op^w_\alpha b)\psi f_2^2(H)g_0^{(h)}(H)]\\ =\;&\frac{1}{\mu}\mathrm{tr}[(Op^w_h\theta^{(\mu)})^2 f_2(P_0^{(0)})(Op^w_h r)f_2(P_0^{(0)})g_0^{(h)}(P_0^{(0)})] + O(\frac{1}{\mu h}),\end{aligned}$$

and we conclude using Lemma 3.6.2. $\square$

Now we can prove Theorem 3.3.6:

Proof. From the Theorems 3.7.1, 3.7.2 and 3.7.4 together we get

$$\begin{aligned}&\frac{1}{\mu}\mathrm{tr}[\psi(Op^w_\alpha b)\psi g_0(H)] =\\ &\frac{1}{4\pi^2h^2}\iint\\ &\Big[a_2(x_1,-\xi_1,x_3)\partial_{x_1}V(x_1,-\xi_1,x_3) - a_1(x_1,-\xi_1,x_3)\partial_{x_2}V(x_1,-\xi_1,x_3))\Big]\\ &\times g_0(\xi_3^2+V(x_1,-\xi_1,x_3))d\hat{x}d\hat{\xi} + O(h^{-1}).\end{aligned}$$

Now we calculate:

$$\begin{aligned}&\frac{1}{4\pi^2h^2}\iint\\ &\Big[a_2(x_1,-\xi_1,x_3)\partial_{x_1}V(x_1,-\xi_1,x_3) - a_1(x_1,-\xi_1,x_3)\partial_{x_2}V(x_1,-\xi_1,x_3))\Big]\\ &\times g_0(\xi_3^2+V(x_1,-\xi_1,x_3))d\hat{x}d\hat{\xi}\\ =\;&\frac{1}{4\pi^2h^2}\int_{\{V(x)\leq 0\}}\Big[a_2(x)\partial_{x_1}V(x) - a_1(x)\partial_{x_2}V(x)\Big]2\sqrt{-V(x)}dx\\ =\;&\frac{-1}{2\pi^2h^2}\int_{\{V(x)\leq 0\}} a_2\frac{2}{3}\partial_{x_1}(\sqrt{-V(x)})^3 - a_1\frac{2}{3}\partial_{x_2}(\sqrt{-V(x)})^3dx\\ =\;&\frac{1}{3\pi^2h^2}\int_{\{V(x)\leq 0\}}(\partial_{x_1}a_2 - \partial_{x_2}a_1)(\sqrt{-V(x)})^3dx.\end{aligned}$$

This finishes the proof of Theorem 3.3.6. $\square$

3.8 The current for bounded μh.

In the case where $\mu h \leq C$, $\mu \geq ch^{-\rho}$ for a $\rho \in (0,1]$ we can use the same type of analysis as in the case of the very strong magnetic field.

3.8.1 Projection on the Landau levels

Lemma 3.8.1. *Let $\nu \in C_0^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$ and let $K \geq 0$. Then*

$$\begin{aligned}\sup_{k:k\mu h \leq K} \quad & \Big\| (I \otimes \Pi_k)\Phi^* Op^w_\alpha(\nu)\Phi(I \otimes \Pi_k) - Op^w_h(e^{(k)}) \otimes \Pi_k \\ & + \varepsilon Op^w_h(e_1^{(k)}) \otimes \Pi_k \Big\| \\ = \quad & O(\varepsilon^2 + \alpha^2),\end{aligned}$$

where

$$\begin{aligned} e^{(k)}(\hat{x},\hat{\xi}) &= \nu(x_1, -\frac{\xi_1}{\mu}, x_3, \frac{\xi_1}{\mu}, 0, \frac{\xi_3}{\mu}) \\ &+ \frac{(2k+1)h}{4\mu}(\partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{\xi_2} - 2\partial_{x_1}\partial_{\xi_2})\nu(x_1, -\frac{\xi_1}{\mu}, x_3, \frac{\xi_1}{\mu}, 0, \frac{\xi_3}{\mu}),\end{aligned}$$

and where

$$\begin{aligned} & e_1^{(k)}(\hat{x},\hat{\xi}) = \\ & \left[\frac{1}{2}\left(-\partial_{x_1}\nu(x_1, \frac{-\xi_1}{\mu}, x_3, \frac{\hat{\xi}}{\mu}) + \partial_{\xi_2}\nu(x_1, \frac{-\xi_1}{\mu}, x_3, \frac{\hat{\xi}}{\mu})\right)\partial_{x_1}W(x_1, \frac{-\xi_1}{\mu}, x_3)\right. \\ & \left. -\frac{1}{2}\partial_{x_2}\nu(x_1, \frac{-\xi_1}{\mu}, x_3, \frac{\hat{\xi}}{\mu})\partial_z W(x_1, z, x_3)\big|_{z=\frac{-\xi_1}{\mu}}\right]\sigma(\frac{\xi_3^2}{\mu^2}) + \frac{\xi_3}{\mu}\zeta(\hat{x}, \frac{\hat{\xi}}{\mu}),\end{aligned}$$

where $\zeta \in C_0^\infty(\mathbb{R}^2_{\hat{x}} \times \mathbb{R}^2_{\hat{\xi}})$.

Proof. As in the proof of Lemma 3.5.6 we get:

$$\Phi^* Op^w_\alpha \nu \Phi = Op^w_h \bar{\nu} + \varepsilon Op^w_h \bar{\nu}_1 + O(\alpha^2 + \varepsilon^2),$$

with notation as in that lemma. We now appeal to [Sob94][Lemma A.1] (stated below as Lemma 3.8.2) to conclude that

$$(I \otimes \Pi_k)\Phi^* Op^w_\alpha(\nu)\Phi(I \otimes \Pi_k) = \left(Op^w_h(\bar{\nu}_{0,k}) + \varepsilon Op^w_h(\bar{\nu}_{1,k})\right) \otimes \Pi_k + O(\varepsilon^2 + \alpha^2),$$

where

$$\bar{\nu}_{0,k}(\hat{x},\hat{\xi}) = \bar{\nu}_{sym}(x_1, \sqrt{\frac{(2k+1)h}{2\mu}}, x_3, \xi_1, \sqrt{\frac{(2k+1)h}{2\mu}}, \xi_3),$$

and

$$\bar{\nu}_{1,k}(\hat{x},\hat{\xi}) = \bar{\nu}_{1,sym}(x_1, \sqrt{\frac{(2k+1)h}{2\mu}}, x_3, \xi_1, \sqrt{\frac{(2k+1)h}{2\mu}}, \xi_3),$$

By a Taylor expansion of $\bar{\nu}_{0,k}$ we get

$$\begin{aligned}
&\bar{\nu}_{0,k}(\hat{x},\hat{\xi}) \\
=\ &\bar{\nu}(x_1,0,x_3,\xi_1,0,\xi_3) \\
&+\frac{(2k+1)h}{4\mu}(\partial^2_{x_2,x_2}+\partial^2_{\xi_2,\xi_2})\bar{\nu}(x_1,0,x_3,\xi_1,0,\xi_3)+O(h^4),
\end{aligned}$$

where the error was estimated using $O(k^2h^2/\mu^2)=O(h^4)$. If we compare this with eq.(3.5.3.11) we see that the expression for $e^{(k)}$ above is correct.
A Taylor expansion of $\bar{\nu}_{1,k}$ to first order and comparison with the proof of Lemma 3.5.6 finishes the proof. $\square$

We used the following lemma:

Lemma 3.8.2. *Let* $\nu\in\mathcal{B}(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)$, $\delta\in(0,1)$ *and define*

$$\nu^{(\delta)}(x,\xi)=\nu(x_1,\delta x_2,x_3,\xi_1,\delta\xi_2,\xi_3)$$

Then the following bound holds:

$$\sup_{k:hk\leq C\delta^2}\|(I\otimes\Pi_k)Op^w_h(\nu^{(\delta)})(I\otimes\Pi_k)-Op^w_h(\nu_k)(I\otimes\Pi_k)\|=O(\delta^8+h^4),$$

where

$$\nu_k(\hat{x},\hat{\xi})=\nu_{sym}((x_1,\sqrt{\frac{2k+1}{2}}\delta,x_3,\xi_1,\sqrt{\frac{2k+1}{2}}\delta,\xi_3).$$

Here we used the notation:

$$\begin{aligned}
a_{sym}(x,\xi)\ =\ &\frac{1}{4}\Big(a(x_1,x_2,x_3,\xi_1,\xi_2,\xi_3)+a(x_1,-x_2,x_3,\xi_1,\xi_2,\xi_3) \\
&+a(x_1,x_2,x_3,\xi_1,-\xi_2,\xi_3)+a(x_1,-x_2,x_3,\xi_1,-\xi_2,\xi_3)\Big).
\end{aligned}$$

We get the following corollary (compare with Cor. 3.5.8).

Corollary 3.8.3. *Suppose* $\vec{a}=(a_1,a_2,0)$. *Then*

$$(I\otimes\Pi_k)\Phi^*Op^w_\alpha(\gamma_0)\Phi(I\otimes\Pi_k)=\frac{h}{\mu}(2k+1)Op^w_h(\tilde{d})\otimes\Pi_k+\varepsilon Op^w_h\tilde{d}_1+O(\varepsilon^2+\alpha^2),$$

where

$$\tilde{d}=\left(\partial_{x_2}a_1(x_1,-\frac{\xi_1}{\mu},x_3)-\partial_{x_1}a_2(x_1,-\frac{\xi_1}{\mu},x_3)\right)f(\frac{\xi_3^2+V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2}),$$

and

$$\begin{aligned}
&\tilde{d}_1(\hat{x},\hat{\xi})= \\
&\left(a_2(x_1,\frac{-\xi_1}{\mu},x_3)\partial_{x_1}W(x_1,\frac{-\xi_1}{\mu},x_3)-a_1(x_1,\frac{-\xi_1}{\mu},x_3)\partial_{x_2}W(x_1,\frac{-\xi_1}{\mu},x_3)\right) \\
&\times f(\frac{\xi_3^2+V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2})+\frac{\xi_3}{\mu}\zeta(\hat{x},\frac{\hat{\xi}}{\mu}).
\end{aligned}$$

And we can conclude the projection by stating:

Corollary 3.8.4. *Suppose $\vec{a} = (a_1, a_2, 0)$, then we get the following estimate uniformly in k where $k\mu h \leq K$.*

$$\begin{aligned} & (I \otimes \Pi_k)\Phi^* Op^w_\alpha(\gamma_0 + \alpha\gamma_1 + \alpha\gamma_2 + \alpha^2\gamma_3)\Phi(Op^w_h\theta^{(\mu)} \otimes \Pi_k) \\ = & \left(\left(\frac{h}{\mu}2kOp^w_h(\tilde{d}) + \varepsilon Op^w_h(r)\right) Op^w_h(\theta^{(\mu)})\right) \otimes \Pi_k + O(\alpha^2 + \varepsilon/\mu), \end{aligned}$$

where $\tilde{d}$ was defined above, and where

$$\begin{aligned} & r(\hat{x}, \hat{\xi}) = \\ & \left(a_2(x_1, \frac{-\xi_1}{\mu}, x_3)\partial_{x_1} W(x_1, \frac{-\xi_1}{\mu}, x_3) - a_1(x_1, \frac{-\xi_1}{\mu}, x_3)\partial_{x_2} W(x_1, \frac{-\xi_1}{\mu}, x_3)\right) \\ & \times f(\frac{\xi_3^2 + V(x_1, -\frac{\xi_1}{\mu}, x_3)}{\mu^2}). \end{aligned}$$

3.8.2 Calculation of the current for the spin-down part

With the notation from Lemma 3.5.2 and section 3.7 we have:

$$\begin{aligned} & \mathrm{tr}[B(\mu, h)g_0(H)] \\ = & \mu\mathrm{tr}[\psi Op^w_\alpha(\gamma_0 + \alpha(\gamma_1 + \gamma_2) + \alpha^2\gamma_3)\psi f_1^2(H)] + \\ & \mu\mathrm{tr}[\psi Op^w_\alpha(\gamma_0 + \alpha(\gamma_1 + \gamma_2) + \alpha^2\gamma_3)\psi f_2^2(H)g_0(H)] + O(\mu\alpha^{3/2}). \end{aligned}$$

We now analyze each term separately.

Theorem 3.8.5. *Suppose $\vec{a} = (a_1, a_2, 0)$ then*

$$\begin{aligned} & \mu\mathrm{tr}[\psi Op^w_\alpha(\gamma_0 + \alpha(\gamma_1 + \gamma_2) + \alpha^2\gamma_3)\psi f_1^2(H)] \\ = & \sum_{0 \leq k \leq c/(\mu h)} \frac{1}{4\pi^2 h^2} \times \\ & \iint \Big(\mu h 2k[\partial_{x_2} a_1(x) - \partial_{x_1} a_2(x)] + [a_2(x)\partial_{x_1} W(x) - a_1(x)\partial_{x_2} W(x)]\Big) \\ & \times f_1^2(\xi_3^2 + 2k\mu h + W(x))dxd\xi_3 + O(h^{-1}\mu^{-1} + h^{-3}\mu^{-2}). \end{aligned}$$

Proof. We can calculate:

$$\begin{aligned}
& \mu\mathrm{tr}[\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi f_1^2(H)] \\
\simeq\ & \mu\mathrm{tr}[f_1(H)\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi f_1(H)] \\
\simeq\ & \mu\mathrm{tr}[\Phi\Theta^{(\mu)} f_1(H)\Phi^*\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi\Phi f_1(H)\Theta^{(\mu)}\Phi^*] \\
\simeq\ & \mu \sum_{0\le k\le c/(\mu h)} \mathrm{tr}[Op^w_h\theta^{(\mu)} f_1(P_0^{(k)})\Phi^* Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\Phi \\
& \times f_1(P_0^{(k)})Op^w_h\theta^{(\mu)}] \\
=\ & \mu \sum_{0\le k\le c/(\mu h)} \mathrm{tr}[Op^w_h\theta^{(\mu)} f_1(P_0^{(k)})\Phi^* \left(\frac{h}{\mu}2kOp^w_h(\tilde{d})+\varepsilon Op^w_h(r)\right)\Phi \\
& \times f_1(P_0^{(k)})Op^w_h\theta^{(\mu)}] \\
& +O(\mu\frac{c}{\mu h}(\alpha^2+\varepsilon/\mu)\mu/h^2)
\end{aligned}$$

Here we used $\tilde{d}, r$, which are defined in Cor. 3.8.3 and Cor. 3.8.4. We also used that $P_0^{(k)} \geq 2k\mu h - c$. The error can be written as $O(h^{-1}\mu^{-1}+h^{-3}\mu^{-2})$, so with the definitions of r and $\tilde{d}$ we get from Lemma 3.6.1

$$\begin{aligned}
& \mu\mathrm{tr}[\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi f_1^2(H)] \\
=\ & \mu \sum_{0\le k\le c/(\mu h)} \frac{\mu}{4\pi^2 h^2}\times \\
& \iint \left(\frac{h}{\mu}2k[\partial_{x_2}a_1(x)-\partial_{x_1}a_2(x)]+\varepsilon[a_2(x)\partial_{x_1}W(x)-a_1(x)\partial_{x_2}W(x)]\right) \\
& \times f_1^2(\xi_3^2+2k\mu h+W(x))dxd\xi_3+O(h^{-1}\mu^{-1}+h^{-3}\mu^{-2}).
\end{aligned}$$

□

Theorem 3.8.6.

$$\mu\mathrm{tr}[\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi f_2^2(H)(g_0(H)-g_0^{(h)}(H))] = O(h^{-1}+\mu).$$

Proof. We cannot use the argument from Theorem 3.7.2 right away, since $\|\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi\| = O(1)$, and using this estimate would lead to a too big error. Therefore we have to try to improve the estimate:
Let

- $\tilde{f} \in C_0^\infty(\mathbb{R})$, $\tilde{f} \equiv 1$ on $\operatorname{supp} f_2$.
- $\tilde{\psi} \in C_0^\infty(\mathbb{R}^3)$, $\tilde{\psi} \equiv 1$ on $\operatorname{supp}\psi$.

Then we have from Lemma 3.4.8 that $\|\psi\tilde{f}(H)(1-\tilde{\psi})\|_1 = O(h^\infty)$. Thus we get:

$$\begin{aligned}
& \mu\mathrm{tr}[\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi f_2^2(H)(g_0(H)-g_0^{(h)}(H))] \\
\simeq\ & \mu\mathrm{tr}[\tilde{f}(H)\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi\tilde{f}(H) \\
& \times\tilde{\psi} f_2(H)(g_0(H)-g_0^{(h)}(H))f_2(H)\tilde{\psi}] \\
\leq\ & \mu\|\tilde{f}(H)\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi\tilde{f}(H)\| \\
& \times\|\tilde{\psi} f_2(H)(g_0(H)-g_0^{(h)}(H))f_2(H)\tilde{\psi}\|_1.
\end{aligned}$$

The trace norm was estimated as $O(\mu/h)$ in the proof of Theorem 3.7.2, so let us look at the operator norm. Splitting into Landau levels as in the proof of the last theorem we get:

$$\begin{aligned}
& \|\tilde{f}(H)\psi Op_\alpha^w(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi\tilde{f}(H)\| \\
\simeq & \left\| \sum_{0\le k\le c/(\mu h)} \left(Op_h^w\theta^{(\mu)}\tilde{f}(P_0^{(k)})\Phi^*\left(\frac{h}{\mu}2kOp_h^w(\tilde{d})+\varepsilon Op_h^w(r)\right)\right.\right. \\
& \left.\left. \times\Phi\tilde{f}(P_0^{(k)})Op_h^w\theta^{(\mu)}\right)\otimes\Pi_k \right\| \\
= & O(h/\mu+\mu^{-2}).
\end{aligned}$$

This finishes the proof. □

Theorem 3.8.7. *Assume $\vec{a}=(a_1,a_2,0)$ and that (3.3.3.1) is satisfied, then*

$$\begin{aligned}
& \mu\mathrm{tr}[\psi Op_\alpha^w(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi f_2^2(H)g_0^{(h)}(H)] \\
= & \sum_{0\le k\le c/(\mu h)}\frac{1}{4\pi^2h^2}\times \\
& \iint\left(\mu h2k[\partial_{x_2}a_1(x)-\partial_{x_1}a_2(x)]+[a_2(x)\partial_{x_1}W(x)-a_1(x)\partial_{x_2}W(x)]\right) \\
& \times(f_2^2g_0)(\xi_3^2+2k\mu h+W(x))dxd\xi_3+O(h^{-1}\mu^{-1}+h^{-3}\mu^{-2}).
\end{aligned}$$

Proof. We calculate as usual:

$$\begin{aligned}
& \mu\mathrm{tr}[\psi Op_\alpha^w(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi f_2^2(H)g_0^{(h)}(H)] \\
= & \frac{\mu}{\sqrt{2\pi}h}\iint g_0(\tau)\hat{\chi}(\tau e^{it\tau/h}\mathrm{tr}[f_2(H)\psi Op_\alpha^w(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2c\gamma_3)\psi \\
& f_2(H)e^{-itH/h}]\ dt\,d\tau \\
\simeq & \mu\sum_{0\le k\le c/(\mu h)}\mathrm{tr}[(Op_h^w(\theta^{(\mu)})^2f_2(P_0^{(k)})\left(\frac{h}{\mu}2kOp_h^w\tilde{d}+\varepsilon Op_h^wr\right)f_2(P_0^{(k)})g_0^{(h)}(P_0^{(k)}) \\
\simeq & \mu\sum_{0\le k\le c/(\mu h)}\frac{\mu}{4\pi^2h^2}\times \\
& \iint\left(\frac{h}{\mu}2k[\partial_{x_2}a_1(x)-\partial_{x_1}a_2(x)]+\varepsilon[a_2(x)\partial_{x_1}W(x)-a_1(x)\partial_{x_2}W(x)]\right) \\
& \times(f_2^2g_0)(\xi_3^2+2k\mu h+W(x))dxd\xi_3+O(h^{-1}\mu^{-1}+h^{-3}\mu^{-2}).
\end{aligned}$$

□

We can thus conclude that for $\mu h\le C,\mu\ge ch^{-\rho}$ and the noncritical condition (3.3.3.1) satisfied, we get up to an error of order

$O(h^{-1}\mu^{-1}+h^{-3}\mu^{-2}+h^{-1}+\mu)$:

$$\begin{aligned}
& \operatorname{tr}[B(\mu,h)g_0(H)] \\
= & \sum_{0\leq k\leq c/(\mu h)} \frac{1}{4\pi^2h^2}\times \\
& \iint \Big(\mu h 2k[\partial_{x_2}a_1(x)-\partial_{x_1}a_2(x)]+[a_2(x)\partial_{x_1}W(x)-a_1(x)\partial_{x_2}W(x)]\Big) \\
& \times g_0(\xi_3^2+2k\mu h+W(x))dxd\xi_3 \\
= & \sum_{0\leq k\leq c/(\mu h)} \frac{1}{4\pi^2h^2}\Big(\int \mu h2k[\partial_{x_2}a_1(x)-\partial_{x_1}a_2(x)]2\sqrt{[V(x)+2\mu hk]_-}dx \\
& +\int [a_2(x)\partial_{x_1}W(x)-a_1(x)\partial_{x_2}W(x)]2\sqrt{[V(x)+2\mu hk]_-}dx\Big) \\
= & \sum_{0\leq k\leq c/(\mu h)} \frac{1}{4\pi^2h^2}\Big(\int \mu h4k[\partial_{x_2}a_1(x)-\partial_{x_1}a_2(x)]\sqrt{[V(x)+2\mu hk]_-}dx \\
& +\frac{4}{3}\int [\partial_{x_1}a_2(x)-\partial_{x_2}a_1(x)][V(x)+2\mu hk]_-^{3/2}dx\Big) \qquad (3.8.3.1)
\end{aligned}$$

3.8.3 The spin-up part

Remember from (3.2.3.4) and (3.5.3.1) that the current is given by:

$$\begin{aligned}
& \int \vec{j}\cdot\vec{a}dx \\
= & \operatorname{tr}[\Big(-2\vec{a}(-ih\nabla-\vec{A})+ih\operatorname{div}\vec{a}+h(\partial_{x_1}a_2-\partial_{x_2}a_1)\Big)g_0(H_0+V+2\mu h)] \\
+ & \operatorname{tr}[\Big(-2\vec{a}(-ih\nabla-\vec{A})+ih\operatorname{div}\vec{a}-h(\partial_{x_1}a_2-\partial_{x_2}a_1)\Big)g_0(H_0+V)].
\end{aligned}$$

When μh is finite we cannot disregard the first term. Having calculated the spin-down part of the current it is easy to treat the spin-up part though: Define $\tilde{V}=V+2\mu h$. Then we get

$$\begin{aligned}
& -\operatorname{tr}[\Big(-2\vec{a}(-ih\nabla-\vec{A})+ih\operatorname{div}\vec{a}+h(\partial_{x_1}a_2-\partial_{x_2}a_1)\Big)g_0(H_0+V+\mu h)] \\
= & \operatorname{tr}[\tilde{B}(\mu,h)g_0(H_0+\tilde{V}-\mu h)],
\end{aligned}$$

where

$$\tilde{B}(\mu,h)=\mu Op^w_{h/\mu}(2\vec{a}\cdot(\xi_1+x_2,\xi_2,\xi_3)-h/\mu(\partial_{x_1}a_2-\partial_{x_2}a_1)).$$

It is easy to see that this change of sign on $(\partial_{x_1}a_2-\partial_{x_2}a_1)$ (compare with (3.5.3.2)) only has as consequence that the factor k on the first term in (3.8.3.1) should be changed to

$(k+1)$. Therefore we get:

$$
\begin{aligned}
& \operatorname{tr}[\tilde{B}(\mu,h)g_0(H_0+\tilde{V}-\mu h)] \\
= \quad & \sum_{0\le k\le c/(\mu h)} \frac{1}{4\pi^2h^2}\times \\
& \left(\int \mu h 4(k+1)(\partial_{x_2}a_1(x)-\partial_{x_1}a_2(x))\sqrt{[\tilde{V}(x)+2\mu hk]_-}dx\right. \\
& \left.+\frac{4}{3}\int(\partial_{x_1}a_2(x)-\partial_{x_2}a_1(x))[\tilde{V}(x)+2\mu hk]_-^{3/2}dx\right) \\
& +O(h^{-1}\mu^{-1}+h^{-3}\mu^{-2}) \\
= \quad & \sum_{1\le k\le c/(\mu h)} \frac{1}{4\pi^2h^2}\times \\
& \left(\int \mu h 4k(\partial_{x_2}a_1(x)-\partial_{x_1}a_2(x))\sqrt{[V(x)+2\mu hk]_-}dx\right. \\
& \left.+\frac{4}{3}\int(\partial_{x_1}a_2(x)-\partial_{x_2}a_1(x))[V(x)+2\mu hk]_-^{3/2}dx\right) \\
& +O(h^{-1}\mu^{-1}+h^{-3}\mu^{-2}).
\end{aligned}
$$

Adding this to (3.8.3.1) we get the theorem 3.3.3.

3.9 Multiscaling: The non-critical condition

In this section we will prove that Theorem 3.3.3 holds without the non-critical condition (3.3.3.1):

Theorem 3.9.1. *Let* $\vec{a}=(a_1,a_2,0)$. *Suppose that* $0<h\le h_0$, $\mu\le C_\mu h^{-\zeta}$ *for some* $\zeta>0$ *and that there exists* $\beta\in(0,1]$ *such that* $\mu\ge c_\beta h^{-\beta}$. *Suppose finally that*

$$|\partial^m\vec{a}(x)|+|\partial^m V(x)|\le C_m$$

on $B(8E)$. *Then*

$$
\begin{aligned}
\operatorname{tr}[\mathbf{B}g_0(\mathbf{P})] \quad = \quad & \frac{2}{3\pi h^2}\sum_{n=0}^{\infty}d_n\int(\partial_{x_1}a_2(x)-\partial_{x_2}a_1(x)) \\
& \times\left([2n\mu h+V(x)]_-^{3/2}-3n\mu h[2nh\mu+V(x)]_-^{1/2}\right)dx \\
& +O(h^{-1}\mu^{-1}+h^{-3}\mu^{-2}+h^{-1}),
\end{aligned}
$$

where O *is uniform in the constants* $\{C_m\},c_\beta,C_\mu,\beta,\zeta,E$.

To prove this we will need the following version of Theorem 3.3.3, where the non-criticality assumption has been slightly modified:

Lemma 3.9.2 (Reference Problem 1). *Let $\vec{a} = (a_1, a_2, 0)$. Suppose that*

$$|\nabla V(x)|^2 + |V(x)| \geq c_{N.C.} > 0 \tag{3.9.3.1}$$

for all $x \in B(2E)$. Suppose further that $0 < h \leq h_0$, $\mu \leq C_\mu h^{-\zeta}$ for some $\zeta > 0$ and that there exists $\beta \in (0,1]$ such that $\mu \geq c_\beta h^{-\beta}$. Suppose finally that

$$|\partial^m \vec{a}(x)| + |\partial^m V(x)| \leq C_m$$

on $B(8E)$. Then

$$\begin{aligned} \operatorname{tr}[\mathbf{B}(h,\mu,\vec{a}) g_0(\mathbf{P})] &= \frac{2}{3\pi h^2} \sum_{n=0}^{\infty} d_n \int (\partial_{x_1} a_2(x) - \partial_{x_2} a_1(x)) \\ &\quad \times \left([2n\mu h + V(x)]_-^{3/2} - 3n\mu h [2nh\mu + V(x)]_-^{1/2} \right) dx \\ &\quad + O(h^{-1}\mu^{-1} + h^{-3}\mu^{-2} + h^{-1}), \end{aligned}$$

where O is uniform in the constants $\{C_m\}, c_{N.C.}, c_\beta, C_\mu, \beta, \zeta, E$.

The proof follows essentially by a change of gauge:

Proof. Write $\mathbf{P} = \mathbf{P}(\vec{A}, V)$. Let U_1 be the unitary gauge transformation given by:

$$(U_1 f)(x) = e^{-i\mu x_1 x_2 / h} f(x),$$

and let U_2 be the unitary change of variables:

$$(U_2 f)(x) = f(x_2, -x_1, x_3).$$

Notice the following relations:

$$\begin{aligned} (U_2^* f)(x) &= f(-x_2, x_1, x_3) \\ U_2^* \nabla U_2 &= \begin{pmatrix} -\partial_{x_2} \\ \partial_{x_1} \\ \partial_{x_3} \end{pmatrix} \\ U_2^* V U_2 &= \tilde{V}, \end{aligned}$$

where $\tilde{V}(x) = V(-x_2, x_1, x_3)$. Then:

$$\begin{aligned} & U_2^* U_1^* \mathbf{P}(\vec{A}, V) U_1 U_2 \\ = \;& U_2^* \mathbf{P}(\vec{A} + \mu \begin{pmatrix} 0 \\ x_1 \\ 0 \end{pmatrix}, V) U_2 \\ = \;& U_2^* \left(-h^2 \partial_{x_1}^2 + (-ih\partial_{x_2} - \mu x_1)^2 - h^2 \partial_{x_3}^2 - \mu h \sigma_3 + V(x) \right) U_2 \\ = \;& -h^2(-\partial_{x_2})^2 + (-ih\partial_{x_1} + \mu x_2)^2 - h^2 \partial_{x_3}^2 - \mu h \sigma_3 + \tilde{V}(x) \\ = \;& \mathbf{P}(\vec{A}, \tilde{V}). \end{aligned}$$

Similarly

$$\begin{aligned}
U_2^*U_1^*\vec{a}\cdot(-ih\nabla-\vec{A})U_1U_2 &= U_2^*\vec{a}\cdot(-ih\nabla-\mu\begin{pmatrix}0\\x_1\\0\end{pmatrix})U_2 \\
&= U_2^*\vec{a}U_2\cdot\left(-ih\begin{pmatrix}-\partial_{x_2}\\ \partial_{x_1}+\mu x_2\\ \partial_{x_3}\end{pmatrix}\right) \\
&= \tilde{a}\cdot(-ih\nabla-\vec{A}),
\end{aligned}$$

where $\tilde{a}(x)=\begin{pmatrix}a_2(-x_2,x_1,x_3)\\ -a_1(-x_2,x_1,x_3)\\ a_3(-x_2,x_1,x_3)\end{pmatrix}$.
Let us finally notice that:

$$U_2^*(\partial_{x_1}a_2-\partial_{x_2}a_1)U_2=\partial_{x_1}\tilde{a}_2-\partial_{x_2}\tilde{a}_1.$$

Now we are ready to prove Lemma 3.9.2:
Choose a partition of unity $\{\phi_j\}$ on $B(E)$ such that $\operatorname{supp}\phi_j\subset B(x_j,E_j/2)$ and that on $B(x_j,8E_j)$ we have either

$$|\partial_{x_1}V(x)|^2+|\partial_{x_3}V(x)|^2+|V(x)|\geq c_{N.C.}/4, \tag{3.9.3.2}$$

or

$$|\partial_{x_2}V(x)|^2+|\partial_{x_3}V(x)|^2+|V(x)|\geq c_{N.C.}/4. \tag{3.9.3.3}$$

This can obviously be done uniformly in $c_{N.C.}$ and the C_m's. Now we write: $\mathcal{J}(h,\mu,\vec{a},V)=\operatorname{tr}[\mathbf{B}(h,\mu,\vec{a},V)g_0(\mathbf{P})]$, and notice that

$$\mathcal{J}(h,\mu,\vec{a},V)=\sum_j\mathcal{J}(h,\mu,\phi_j\vec{a},V).$$

Likewise, we write:

$$\begin{aligned}
\mathcal{A}(h,\mu,\vec{a},V) &= \frac{2}{3\pi h^2}\sum_{n=0}^{\infty}d_n\int(\partial_{x_1}a_2(x)-\partial_{x_2}a_1(x)) \\
&\quad\times\left([2n\mu h+V(x)]_-^{3/2}-3n\mu h[2nh\mu+V(x)]_-^{1/2}\right)dx,
\end{aligned}$$

and notice the same linearity:

$$\mathcal{A}(h,\mu,\vec{a},V)=\sum_j\mathcal{A}(h,\mu,\phi_j\vec{a},V).$$

Now, if (3.9.3.2) is satisfied on $\operatorname{supp}\phi_j$ we can use Theorem 3.3.3 to estimate:

$$\left|\mathcal{J}(h,\mu,\phi_j\vec{a},V)-\mathcal{A}(h,\mu,\phi_j\vec{a},V)\right|\leq O(h^{-1}\mu^{-1}+h^{-3}\mu^{-2}+h^{-1}).$$

On the other hand, if (3.9.3.3) is satisfied on $\operatorname{supp}\phi_j$ we conjugate by (U_1U_2), and find ourselves, once again, in a situation where Theorem 3.3.3 is applicable: The above calculation shows that

$$\mathcal{J}(h,\mu,\phi_j\vec{a},V) = \mathcal{J}(h,\mu,\widetilde{\phi_j\vec{a}},\tilde{V}),$$

and we see that

$$|\partial_{x_1}\tilde{V}|^2 + |\partial_{x_3}\tilde{V}|^2 + |V(x)| \geq c_{N.C.}/4$$

on $B(x_j, 8E_j)$. Thus we can apply Theorem 3.3.3. If we finally notice that

$$\mathcal{A}(h,\mu,\vec{a},V) = \mathcal{A}(h,\mu,\tilde{a},\tilde{V}),$$

we can put the pieces together and obtain Lemma 3.9.2. $\square$

Remark 3.9.3. Note that the lemma remains true if (3.9.3.1) is replaced by:

$$|\partial_x V|^2 + |V(x)| + h \geq c > 0. \tag{3.9.3.4}$$

This is the condition that we will use in the following.

Having cast the reference problem in this form we are facing very much the same problem as treated in [Sob95, Sections 5,6]. Our treatment will also be very similar.

Proof. We choose

$$f(x) = l(x) = A^{-1}\left[V(x)^2 + (\partial_x V)^4 + h^2\right]^{1/4},$$

where A is a sufficiently big constant to be determined below. Then

$$\begin{array}{rcl} f(x), l(x) & > & 0, \\ |\partial_x l(x)| \leq \rho & < & 1 \\ c \leq \dfrac{f(x)}{f(y)} & \leq & C \quad \forall x \in B(8) \cap B(y, l(y)), \end{array} \tag{3.9.3.5}$$

if A is sufficiently big. Furthermore, there exist constants c_α, independent of h, such that

$$\begin{array}{rcl} |\partial^\alpha V(x)| & \leq & c_\alpha f(x)^2 l(x)^{-|\alpha|} \\ |\partial^\alpha \vec{a}(x)| & \leq & c_\alpha l(x)^{-|\alpha|} \end{array}$$

on $B(8)$. Now, if we choose a sequence of points $\{x_k\}$ such that

- $B(1) \subset \cup_k B(x_k, l(x_k)) \equiv \cup_k B_k$,
- $\cup_k B(x_k, 8l(x_k)) \subset B(8)$,
- the intersection of more than $N = N(\rho)$ balls is empty (this is possible due to $\rho < 1$ in (3.9.3.5), see [H9̈0]),

and a corresponding partition of unity:

- $\psi_k \in C_0^\infty(B_k)$,

- $|\partial_\alpha \psi_k(x)| \leq c_\alpha l_k^{-|\alpha|}$, where $l_k = l(x_k)$,
- $\sum \psi_k \equiv 1$ on $B(1)$,

then

$$\mathcal{J}(h,\mu,\vec{a},V) = \mathcal{J}(h,\mu,\sum_k \psi_k\vec{a},V) = \sum_k \mathcal{J}(h,\mu,\psi_k\vec{a},V).$$

Since also the asymptotic term satisfies

$$\mathcal{A}(h,\mu,\vec{a},V) = \mathcal{A}(h,\mu,\sum_k \psi_k\vec{a},V) = \sum_k \mathcal{A}(h,\mu,\psi_k\vec{a},V),$$

we can write

$$\mathcal{J}(h,\mu,\vec{a},V) - \mathcal{A}(h,\mu,\vec{a},V) = \sum_k \left(\mathcal{J}(h,\mu,\psi_k\vec{a},V) - \mathcal{A}(h,\mu,\psi_k\vec{a},V)\right).$$

Now by scaling, dilatation and a gauge-transformation in the $\mathcal{J}$-term:

$$\begin{aligned}
&\mathcal{J}(h,\mu,\psi_k\vec{a},V) - \mathcal{A}(h,\mu,\psi_k\vec{a},V) \\
= \; & f_k\left[\mathcal{J}(\frac{h}{f_k l_k},\frac{\mu l_k}{f_k},\hat{a}_k,\hat{V}) - \mathcal{A}(\frac{h}{f_k l_k},\frac{\mu l_k}{f_k},\hat{a}_k,\hat{V})\right],
\end{aligned}$$

where $\hat{a}_k(x) = (\psi_k\vec{a})(l_k x + x_k)$ and $\hat{V}(x) = f_k^{-2}V(l_k x + x_k)$. We want to apply the reference problem to $\mathcal{J}(\frac{h}{f_k l_k},\frac{\mu l_k}{f_k},\hat{a}_k,\hat{V})$, so we have to check that this is allowed. Let us notice that by continuity of V; f,l are bounded on $B(8)$. Therefore it is easy to see that

$$\begin{aligned}
|\partial^\alpha \hat{a}(x)| &\leq C_\alpha \\
|\partial^\alpha \hat{V}(x)| &\leq C_\alpha,
\end{aligned}$$

where the C_α's are independent of k. Let us check the non-critical condition (3.9.3.4):

$$\begin{aligned}
|\partial_x \hat{V}|^2 + |\hat{V}(x)| + \frac{h}{l_k f_k} &= \frac{|(\partial_x V)(l_k x + x_k)|^2 + V(l_k x + x_k) + h}{f_k^2} \\
&\geq \frac{cA^2 f(l_k x + x_k)}{f_k^2} \\
&\geq c,
\end{aligned}$$

for $x \in B(1)$. We also have to check that $\frac{h}{f_k l_k}$ is bounded above, and that $\mu = \frac{\mu l_k}{f_k} \geq c_\mu(\frac{h}{f_k l_k})^{-\beta}$. This is easily seen to be the case.
Now, since we can use the reference problem, we get:

$$\begin{aligned}
&|\mathcal{J}(h,\mu,\psi_k\vec{a},V) - \mathcal{A}(h,\mu,\psi_k\vec{a},V)| \\
\leq \; & C f_k \left(\frac{f_k l_k}{h}\frac{f_k}{\mu l_k} + \frac{f_k^3 l_k^3}{h^3}\frac{f_k^2}{\mu^2 l_k^2} + \frac{f_k l_k}{h}\right) \\
= \; & C\int_{B_k} f_k \left(\frac{f_k l_k}{h}\frac{f_k}{\mu l_k} + \frac{f_k^3 l_k^3}{h^3}\frac{f_k^2}{\mu^2 l_k^2} + \frac{f_k l_k}{h}\right) l_k^{-3}\,dx \\
\leq \; & C\int_{B_k} \left(\frac{1}{h\mu} + \frac{l(x)^4}{h^3\mu^2} + \frac{1}{h}\right) dx,
\end{aligned}$$

where we used that $f(x) = l(x)$ in the last inequality.

Thus,

$$\begin{aligned} |\mathcal{J}(h,\mu,\vec{a},V) - \mathcal{A}(h,\mu,\vec{a},V)| &\leq CN_\rho \int_{B(8)} \left(\frac{1}{h\mu} + \frac{l(x)^4}{h^3\mu^2} + \frac{1}{h} \right) dx \\ &\leq C \left(\frac{1}{h\mu} + \frac{1}{h^3}\mu^2 + \frac{1}{h} \right). \end{aligned}$$

□

3.10 The current parallel to the magnetic field

In this section we prove Theorem 3.3.7. We will first prove that the current parallel to the magnetic field is constant in the x_3-variable (to highest order). This allows us to move the test-function a_3 out where the potential is positive, and here the current vanishes to all orders in h.

Lemma 3.10.1. *Suppose*

$$\int_{-\infty}^{\infty} a_3(x_1,x_2,x_3)\ dx_3 = 0,$$

for all (x_1,x_2). *Then*

$$\mathrm{tr}[\mathbf{B}(h,\mu,(0,0,a_3))g_0(\mathbf{P})] = O(h^{-1}).$$

Remark 3.10.2. The condition on a_3 is equivalent to the existence of $\tilde{a}_3 \in C_0^\infty(\mathbb{R}^3)$ such that $a_3 = \partial_{x_3}\tilde{a}_3$. Lemma 3.10.1 can thus be interpreted as stating, that the distribution $\partial_{x_3} j_3$ vanishes to all orders higher than h^{-1}.

Proof. Define $\vec{a} = (a_1,a_2,0) \in C_0^\infty(\mathbb{R}^3)$ as

$$\begin{aligned} a_1(x) &= -\int_{-\infty}^{x_3} \partial_{x_1} a_3(x_1,x_2,y)\ dy \\ a_2(x) &= -\int_{-\infty}^{x_3} \partial_{x_2} a_3(x_1,x_2,y)\ dy \end{aligned}$$

Then $\nabla \times \vec{a} = \nabla \times (0,0,a_3)$ and therefore we get by the result from Appendix D that:

$$\mathrm{tr}[\mathbf{B}(h,\mu,(0,0,a_3))g_0(\mathbf{P})] = \mathrm{tr}[\mathbf{B}(h,\mu,\vec{a})g_0(\mathbf{P})].$$

Theorem 3.3.6 now gives the conclusion of the lemma. □

Let now $a_{3,T}(x) \equiv a_3(x_1,x_2,x_3 - T)$. Lemma 3.10.1 above then says that

$$\mathrm{tr}[\mathbf{B}(h,\mu,(0,0,a_3))g_0(\mathbf{P})] = \mathrm{tr}[\mathbf{B}(h,\mu,(0,0,a_{3,T}))g_0(\mathbf{P})] + O(h^{-1}),$$

locally uniformly in T.

Let $T \in \mathbb{R}$ be so big that $V > \gamma/2$ on $B(4E) + T\vec{e}_3$. The next lemma proves that then $\mathrm{tr}[\mathbf{B}(h,\mu,(0,0,a_{3,T}))g_0(\mathbf{P})] = O(h^\infty)$, which finishes the proof of the theorem.

Lemma 3.10.3. *Suppose $\vec{a} \in C_0^\infty(B(E))$, that $V \geq \gamma > 0$ on $B(4E)$ and that the hypothesis of Theorem 3.3.6 are fullfilled, then*

$$\mathrm{tr}[\mathbf{B}(h,\mu,\vec{a})g_0(\mathbf{P})] = O(h^\infty).$$

Proof. Choose $\tilde{V}$ satifying

- $\tilde{V} \equiv V$ on $B(4E)$.
- $\tilde{V}(x) \geq \gamma$ for all x.
- $\tilde{V} - \gamma \in C_0^\infty(\mathbb{R}^3)$.

Choose furthermore $f \in C_0^\infty(\mathbb{R})$, with $\sup(\operatorname{supp} f) \leq \gamma/2$ such that

$$f(\mathbf{P})g_0(\mathbf{P}) = g_0(\mathbf{P}).$$

Let $\tilde{\mathbf{P}}$ denote the Pauli-operator with V exchanged with $\tilde{V}$. Then we get:

$$\begin{aligned} |\mathrm{tr}[\mathbf{B}(h,\mu,\vec{a})g_0(\mathbf{P})]| &= |\mathrm{tr}[\mathbf{B}(h,\mu,\vec{a})f(\mathbf{P})g_0(\mathbf{P})]| \\ &\leq \|\mathbf{B}(h,\mu,\vec{a})f(\mathbf{P})\|_1 \\ &= \|\mathbf{B}(h,\mu,\vec{a})f(\tilde{\mathbf{P}})\|_1 + O(h^\infty) \\ &= O(h^\infty). \end{aligned}$$

The last equality is due to the fact that $\tilde{\mathbf{P}} \geq \gamma$ and therefore $f(\tilde{\mathbf{P}}) = 0$. The next to last equality is a consequence of the localisation result in Lemma 3.4.10. ☐

3.11 Multiscaling

In this section we will finally prove the following more precise version of Theorem 3.2.2:

Theorem 3.11.1. *Suppose $V \in C^\infty(B(8)\setminus\{0\})$ satisfies:*

$$V(x) = \frac{q}{|x|} + o(|x|^{-1}) \tag{3.11.3.1}$$

as $x \to 0$, and

$$|\partial^m V(x)| \leq C_{m,V}|x|^{-1-|m|}, \tag{3.11.3.2}$$

$\forall x \in B(8)$.
Suppose furthermore that $\exists C = C(h,\mu)$ such that

$$\mathbf{P}(h,\mu,V) \geq -C.$$

Suppose

- $\exists c_{\mu,1} > 0$ *such that* $\mu h \geq c_{\mu,1}$,
- $\exists c_{\mu,2} > 0$ *such that* $\mu h^3 \leq c_{\mu,2}$

Let finally $\vec{a} = (a_1, a_2, 0) \in C_0^\infty(B(1))$ *satisfy*

$$|\partial^m \vec{a}| \leq C_{m,\vec{a}},$$

then for all $\nu > 0$

$$\begin{aligned}
\operatorname{tr}[\mathbf{B} g_0(\mathbf{P})] &= \frac{2}{3\pi h^2} \sum_{n=0}^{\infty} d_n \int (\partial_{x_1} a_2(x) - \partial_{x_2} a_1(x)) \\
&\quad \times \left([2n\mu h + V(x)]_-^{3/2} - 3n\mu h [2nh\mu + V(x)]_-^{1/2} \right) dx \\
&\quad + O(h^{-1} + \frac{1}{\mu^{1/3} h^{2+\nu}}),
\end{aligned}$$

where O is uniform in the constants $\{C_{m,V}\}, \{C_{m,\vec{a}}\}, c_{\mu,1}, c_{\mu,2}$.

Remark 3.11.2. The constants $\{C_{m,\vec{a}}\}, c_{\mu,1}, c_{\mu,2}$ do not depend on $\vec{a}, \mu$. The index is only there to distinguish them from each other and the other constants in the theorem.

Remark 3.11.3. The asymptotics does not depend on the lower bound $-C$ of $\mathbf{P}$.

For the parallel current the corresponding result is

Theorem 3.11.4. *Let the assumptions be as in Theorem 3.11.1, but with* $\vec{a} = (a_1, a_2, a_3)$. *Assume that* $V(x) \geq c_V > 0$, *for* $1 \leq |x| \leq 3$, *and that the spectrum of* $\mathbf{P}$ *below* 0 *is discrete, then for all* $\nu > 0$

$$|\mathcal{J}(h,\mu,\vec{a},V) - \mathcal{A}(h,\mu,\vec{a},V)| = O(h^{-1} + \frac{1}{\mu^{1/3} h^{2+\nu}}),$$

where O is uniform in the constants $\{C_{m,V}\}, \{C_{m,\vec{a}}\}, c_{\mu,1}, c_{\mu,2}, c_V$.

We are going to perform a so-called *multiscale analysis* invented by Ivrii et al.([Ivr98], [IS93], see also [Sob94]) Since our problem is very similar to the problem analyzed in [Sob96b] our choices of scaling functions will be the same.

We will divide space into several regions and obtain asymptotic extimates in each of them. This is due to the fact that, as far as magnetic effects are concerned, there is an enormous difference between the vicinity of the singularity and the rest of the space. Close to the singularity V is much bigger than μh and therefore magnetic effects are neglectable. In this region the analysis performed in [Fou98] is applicable. Further out, μh and V become comparable and we see a current.

Let us write $\vec{a} = \chi_1 \vec{a} + \chi_2 \vec{a} = \vec{a}_1 + \vec{a}_2$, where $\chi_1(x) = \chi(x/r^2)$ and $\chi_2 = 1 - \chi_1$ (here and in what follows χ will denote a standard smooth cut-off function around 0). The exact choice of r will be made in the end of this section, here we will just remark that we impose:

$$r^2 \leq \frac{1}{\mu h}, \tag{3.11.3.3}$$

which, in a sense, is the condition that, on the support of χ_1, the electric potential dominates.

3.11.1 The inner region $\{|x| \leq r^2\}$

In the innermost region, we do not see a current. This will be the result of Cor. 3.11.7 below.
We have to evaluate the trace $\mathrm{tr}[\mathbf{B}(h,\mu,\vec{a}_1)g_0(\mathbf{P})]$, with $\vec{a}_1$ supported on a region of radius r^2. This we can write as

$$\mathbf{B}(h,\mu,\vec{a}_1) = Op_h^w\left(2\tilde{a}(\frac{x}{r^2})\cdot(\xi-\mu\vec{A})\right) + h/r^2 b(\frac{x}{r^2})\sigma_3,$$

where $\tilde{a}$ and $b = \nabla\times\tilde{a}$ are now supported on a region of radius 1.

Lemma 3.11.5. *We have*

$$\mathrm{tr}[\mathbf{B}(h,\mu,\vec{a}_1)g_0(\mathbf{P})] = O(h^{-1} + \frac{\mu r^3}{h} + \frac{r}{h^2} + \frac{\mu r^6}{h^2} + \frac{r^3}{h^2}).$$

Lemma 3.11.5 follows upon collecting the results of the Lemmas 3.11.10, 3.11.12 and 3.11.13 below.

Let us look at the asymptotic term:

Lemma 3.11.6.

$$\mathcal{A}(h,\mu,\vec{a}_1,V) = O(\frac{r\mu}{h}).$$

Proof. We write $\vec{a}_1(x) = \tilde{a}(\frac{x}{r^2})$, and $V(x) = \frac{\Phi(x)}{|x|}$. Then we can calculate:

$$\begin{aligned}
&\mathcal{A}(h,\mu,\tilde{a}(\frac{x}{r^2}),V)\\
=\;& \frac{2}{3\pi h^2}\sum_{n=0}^{\infty} d_n \int \frac{1}{r^2}(\partial_{x_1}\tilde{a}_2(\frac{x}{r^2}) - \partial_{x_2}\tilde{a}_1(\frac{x}{r^2}))\\
&\times\left([2n\mu h + V(x)]_-^{3/2} - 3n\mu h[2nh\mu + V(x)]_-^{1/2}\right)dx\\
=\;& \frac{2}{3\pi h^2}\sum_{n=0}^{\infty} d_n r^4 \int (\partial_{x_1}\tilde{a}_2(y) - \partial_{x_2}\tilde{a}_1(y))\\
&\times\left([2n\mu h + \frac{\Phi(r^2y)}{r^2|y|}]_-^{3/2} - 3n\mu h[2nh\mu + [\frac{\Phi(r^2y)}{r^2|y|}]_-^{1/2}\right)dy\\
=\;& \frac{2r}{3\pi h^2}\sum_{n=0}^{\infty} d_n \int (\partial_{x_1}\tilde{a}_2(y) - \partial_{x_2}\tilde{a}_1(y))\\
&\times\left([2n\mu h r^2 + \frac{\Phi(r^2y)}{|y|}]_-^{3/2} - 3n\mu h r^2[2nh\mu r^2 + \frac{\Phi(r^2y)}{|y|}]_-^{1/2}\right)dy.
\end{aligned}$$

Now we use Prop. C.1 to conclude:

$$\begin{aligned}
&\mathcal{A}(h,\mu,\tilde{a}(\frac{x}{r^2}),V)\\
=\;& O\left(\frac{r}{h^2}\int|\partial_{x_1}\tilde{a}_2 - \partial_{x_2}\tilde{a}_1|(y)\left((h\mu r^2)^{3/2} + \sqrt{h\mu r^2}\frac{1}{|y|} + h\mu r^2\frac{1}{|y|}\right)dy\right)\\
=\;& O(\frac{r}{h^2}\sqrt{h\mu r^2}),
\end{aligned}$$

since $h\mu r^2 \leq 1$. $\square$

From Lemma 3.11.5 and Lemma 3.11.6 we get, upon noticing that $\mu r^2 \leq h^{-1}$ and $r^3 \leq r$:

Corollary 3.11.7.

$$|\mathcal{J}(h,\mu,\vec{a}_1,V) - \mathcal{A}(h,\mu,\vec{a}_1,V)| = O(h^{-1} + \frac{r}{h^2} + \frac{\mu r^6}{h^2}).$$

Remark 3.11.8. Notice that we prove that $|\mathcal{J} - \mathcal{A}|$ is small by proving that both $|\mathcal{J}|$ and $|\mathcal{A}|$ are small.

To prove Lemma 3.11.5 let us first look at the part of the trace involving b, i.e. the spin current. This has the form of 'a density trace' i.e. it is the trace of a smooth function composed with a projection. This has been studied in [Sob96a] and the result is given in the following lemma:

Lemma 3.11.9. *Let A be a self adjoint-operator in $L^2(\mathbb{R}^3)$, semibounded from below with form domain $\mathcal{D}[A]$ such that for any $\zeta \in C_0^\infty(B(4E))$ the following conditions are satisfied:*

- *For any $u \in \mathcal{D}[A]$ one has $\zeta u \in \mathcal{D}[A]$; there exists a function $\zeta_1 \in C_0^\infty(B(4E))$ such that*
$$A[u,\zeta v] = A[\zeta_1 u, \zeta v],$$
for all $u, v \in \mathcal{D}[A]$;
- *There exists a potential V, infinitesimally bounded with repect to $-\Delta$ such that if $H = (-ih\nabla - \mu\vec{A})^2 + V$, then $\zeta\mathcal{D}[A] \subset \mathcal{D}[H]$, $\zeta\mathcal{D}[H] \subset \mathcal{D}[A]$, and*
$$A[\zeta u, \zeta v] = H[\zeta u, \zeta v],$$
- *$V = \frac{\Phi(\frac{x}{|x|})}{|x|} + U$, where $\Phi, U \in C^\infty$ and U is bounded.*

Let finally $h \in (0, h_0]$, $\mu > 0$, $\mu h \leq C$ and $\psi \in C_0^\infty(B(E/2))$. Then the following asymptotics holds uniformly in V:

$$\mathrm{tr}[\psi g_0(A)] = \frac{1}{(2\pi h)^3}\int \psi(x) g_0(\xi^2 + V(x))\,dxd\xi + O((1+\mu)h^{-2}).$$

This lemma we can use to calculate the part of the trace involving b.

Lemma 3.11.10.

$$\mathrm{tr}[h/r^2 b(x/r^2)\sigma_3 g_0(\mathbf{P})] = O(h^{-1} + \frac{\mu r^3}{h} + \frac{r}{h^2}).$$

Proof. If we write $V(x) = \frac{\Phi(x)}{|x|}$ and make the change of variables $y = x/r^2$ we get, on the spin down subspace,

$$h/r^2 \mathrm{tr}\left[b(y) g_0\left((-ih/r\nabla - \mu r^3\vec{A}(y))^2 - (\mu r^3)h/r - \frac{\Phi(r^2 y)}{|y|}\right)\right],$$

and correspondingly on the spin up subspace. Let us first concentrate on the spin-down case. Since $(\mu r^3)h/r = \mu h r^2 \leq 1$ by (3.11.3.3), this trace is given in Lemma 3.11.9. Therefore we get:

$$
\begin{aligned}
& h/r^2 \mathrm{tr}\left[b(y) g_0\left((-ih/r\nabla - \mu r^3 \vec{A}(y))^2 - (\mu r^3)h/r - \frac{\Phi(r^2 y)}{|y|}\right)\right] \\
= \quad & c(h/r)^{-3}\int b(y)\left[-\mu h r^2 - \frac{\Phi(r^2 y)}{|y|}\right]_-^{3/2} dy + O((h/r)^{-2}(1+\mu r^3)),
\end{aligned}
$$

where the constant c is explicit. If we analyze the spin-up part in the same way, we get:

$$
\begin{aligned}
& \mathrm{tr}[h/r^2 b(x/r^2)\sigma_3 g_0(\mathbf{P})] \\
& = cr/h^2 \int b(y)\left(\left[-\mu h r^2 - \frac{\Phi(r^2 y)}{|y|}\right]_-^{3/2} - \left[\mu h r^2 - \frac{\Phi(r^2 y)}{|y|}\right]_-^{3/2}\right) dy \\
& + O\left(\frac{1+\mu r^3}{h}\right).
\end{aligned}
$$

□

Remark 3.11.11. Notice that the result depends only on how the potential V behaves on a region of size r^2.

Now we look at the remaining term in the trace. Here we have to split into two regions. This is not due to any fundamental difference between this part and the part considered above. In fact this splitting is essentially the same as should be used to prove Lemma 3.11.9, but in the case considered above we could just use the final result.
The two regions are:

$$\Omega_1 = \{|x| \leq h^2/\theta\}$$

and

$$\Omega_2 = \{h^2/\theta \leq |x| \leq r^2\},$$

where θ is a sufficiently small constant (independent of h, μ) which will be chosen below. Write $\vec{a}_1 = \phi_1 \vec{a}_1 + \phi_2 \vec{a}_1$, where ϕ_1, ϕ_2 are smooth cut-offs to the regions Ω_1, Ω_2, respectively.

On Ω_1 we have to analyze

$$\mathrm{tr}[Op_h^w(\bar{a}(\frac{x}{h^2/\theta}) \cdot (\xi - \mu\vec{A}))g_0(\mathbf{P})],$$

where $\bar{a}$ is supported on a ball of radius 1.

Lemma 3.11.12.

$$\mathrm{tr}[Op_h^w(\bar{a}(\frac{x}{h^2/\theta}) \cdot (\xi - \mu\vec{A}))g_0(\mathbf{P})] = O(h^{-1}).$$

Proof. We will only look at the spin down part, the other case follows easily. After the change of variable $y = \theta x/h^2$ the expression becomes:

$$h^{-1}\mathrm{tr}\left[Op_\theta^w\left(\bar{a}(\xi - \frac{\mu h^3}{\theta}\vec{A})\right) g_0\left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - \frac{\Phi(h^2y/\theta)}{|y|}\right)\right].$$

Choose a function $\psi \in C_0^\infty$, $0 \leq \psi$, $\psi\bar{a} = \bar{a}$. Then we get (using the spectral theorem and the cyclicity of trace):

$$\begin{aligned}
& \frac{1}{h}\mathrm{tr}\left[Op_\theta^w\left(\bar{a}\cdot(\xi - \frac{\mu h^3}{\theta}\vec{A})\right)\right. \\
& \left.\times g_0\left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - \frac{\Phi(h^2y/\theta)}{|y|}\right)\right] \\
= \ & \frac{1}{h}\mathrm{tr}\left[Op_\theta^w\left(\bar{a}\cdot(\xi - \frac{\mu h^3}{\theta}\vec{A})\right)\right. \\
& \times g_0\left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - \frac{\Phi(h^2y/\theta)}{|y|}\right) \\
& \left.\times g_0\left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - \frac{\Phi(h^2y/\theta)}{|y|}\right)\psi\right] \\
\leq \ & \frac{1}{h}\left\|Op_\theta^w(\xi - \frac{\mu h^3}{\theta}\vec{A})g_0\left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - \frac{\Phi(h^2y/\theta)}{|y|}\right)\right\| \\
& \times\left\|g_0\left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - \frac{\Phi(h^2y/\theta)}{|y|}\right)\psi\right\|_1 \\
\leq \ & \frac{1}{h}O(\theta^{-3}),
\end{aligned}$$

where we used the estimates:

$$\left\|Op_\theta^w(\xi - \frac{\mu h^3}{\theta}\vec{A})g_0\left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - \frac{\Phi(h^2y/\theta)}{|y|}\right)\right\| \leq C, \qquad (3.11.3.4)$$

and

$$\left\|\psi g_0\left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - \frac{\Phi(h^2y/\theta)}{|y|}\right)\right\|_1 \leq C\theta^{-3}. \qquad (3.11.3.5)$$

The second estimate (3.11.3.5) is well known and is known to depend only on properties of V on a region of size h^2/θ.

To prove (3.11.3.4) we take $W(y) = \zeta(y)\frac{\Phi(h^2y/\theta)}{|y|}$, where ζ is some C_0^∞ function, which is 1 on $B(1)$. Using results from Appendix B, we only have to prove the estimate with $\frac{\Phi(h^2y/\theta)}{|y|}$ replaced by W.

Now take $\phi \in \mathrm{Ran}(g_0\left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - W(y)\right)$ with $||\phi|| = 1$, and write

$$\begin{aligned}
&\left\|\left(-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y)\right)\phi\right\|^2 \\
\leq\ & \langle\phi, \left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - W(y)\right)\phi\rangle + \mu h^3 + \langle\phi, W(y)\phi\rangle \\
\leq\ & \mu h^3 + \langle\phi, W(y)\phi\rangle
\end{aligned}$$

and we finish using the infinitesimal boundedness of the potential. □

Lemma 3.11.13.

$$\mathrm{tr}[Op_h^w(\phi_2\vec{a}_1\cdot(\xi-\mu\vec{A}))g_0(\mathbf{P})] = O(\frac{\mu r^6}{h^2} + \frac{r^3}{h^2}).$$

In the proof below, we will write $\vec{a}$ instead of $\phi_2\vec{a}_1$. On Ω_2 we need to multiscale: We have the following reference problem:

Theorem 3.11.14 (Reference Problem). *If $\vec{a} \in C_0^\infty(B(0,1))$ and $\vec{a}$, V, satisfy the following bounds:*
$|\partial^\alpha\vec{a}| \leq C_\alpha$, $|\partial^\alpha V| \leq C_\alpha$ on $B(8)$, and $\mu h \leq 1$, $h \leq h_0$. Then we have

$$\mathrm{tr}[Op_h^w(\vec{a}\cdot(\xi-\mu\vec{A}))g_0(H)] = O((\mu+1)h^{-2}),$$

where the O is uniform in the constants bounding the derivatives of $\vec{a}$, V.

Using the localisation arguments in Appendix A this theorem is a consequence of the results of [Fou98]. The proof is identical to sections 4,5 in [Sob95] and will therefore be omitted.

We now define functions $f = \frac{1}{\sqrt{|x|}}$, $l(x) = \rho|x|$ where $\rho < 1/16$. Notice that $|\partial^\alpha\vec{a}| \leq c_\alpha l(x)^{-|\alpha|}$ and $|\partial^\alpha V| \leq c_\alpha f(x)^2 l(x)^{-|\alpha|}$, on Ω_2. Since $|\partial_x l(x)| \leq \rho < 1$ we can find a sequence of points (See [H\"90] or [Sob95]) $x_k \subset \Omega_2$ such that

$$\cup_{x\in\mathrm{supp}\,\vec{a}} B(x,l(x)) \subset \Omega_2 \subset \cup_k B(x_k, 8l(x_k))$$

and a number $N = N(\rho)$ (independent of h) such that the intersection of more than $N(\rho)$ balls is empty, and furthermore a corresponding partition of unity $\{\psi_k\}$ satisfying:

- $\psi_k \in C_0^\infty(B(x_k, 8l(x_k)))$,
- $|\partial^m\psi_k| \leq C(\rho)l(x_k)^{-|m|}$,
- $\sum\psi_k = 1$ on Ω_2.

Using this partition of unity we write

$$\begin{aligned}
\mathrm{tr}[Op_h^w(\vec{a}\cdot(\xi-\mu\vec{A}))g_0(H)] &= \sum \mathrm{tr}[Op_h^w(\psi_k\vec{a}\cdot(\xi-\mu\vec{A}))g_0(H)] \\
&\equiv \sum_k T_k.
\end{aligned}$$

Now we have

Lemma 3.11.15.
$$|T_k| \le C\int_{B_k}\left(\frac{\mu}{h^2}f(x)^2+\frac{f(x)^3}{l(x)h^2}\right)dx$$

This will be proved below. We first prove Lemma 3.11.13 using Lemma 3.11.15:

Proof. Because only a finite (fixed) number of balls can intersect we thus get that:

$$\begin{aligned}|\mathrm{tr}[Op_h^w(\vec{a}\cdot(\xi-\mu\vec{a}))g_0(H)]| &\le C\int_{\Omega_2}\left(\frac{\mu}{h^2}f(x)^2+\frac{f(x)^3}{l(x)h^2}\right)dx\\ &= Ch^{-2}\int_{h^2/\theta}^{r^2}(\mu|x|+\frac{1}{\sqrt{|x|}})d|x|\\ &= O(\frac{\mu r^4}{h^2}+\frac{r}{h^2}).\end{aligned}$$

In the final estimate we used that θ is a *constant*. This proves Lemma 3.11.13. □

Now we prove Lemma 3.11.15:

Proof. First we notice the following scaling relations: Let l, f be positive scalars, $z \in \mathbb{R}^3$ and define $\mathcal{U}_l u(x) = l^{3/2}u(lx)$, $T_z u(x) = u(x+z)$, then:

$$f^{-2}\mathcal{U}_l T_z H(A,V,h,\mu)T_z^*\mathcal{U}_l^* = H(\hat{A},\hat{V},\alpha,\nu),$$

where

- $\hat{A}(x) = l^{-1}A(lx+z) = (-x_2 - z_2/l, 0, 0)$
- $\hat{V}(x) = f^{-2}V(lx+z)$
- $\alpha = h/(fl)$, $\nu = \mu l/f$.

Let now Φ be the gaugetransformation

$$\Phi u(x) = e^{i\frac{h}{fl}x_1 z_2/l}u(x),$$

and let $\mathcal{U}(l,f,z)$ be the unitary transformation

$$\mathcal{U}(l,f,z) = \Phi\,\mathcal{U}_l T_z,$$

then
$$f^{-2}\mathcal{U}(l,f,z)H(A,V,h,\mu)\,\mathcal{U}(l,f,z)^* = H(A,\hat{V},h/(fl),\mu l/f).$$

Let
$$J(A,V,h,\mu,\vec{a}) = \mathrm{tr}[Op_h^{\alpha}(\vec{a}\cdot(\xi-\mu A)g_0(H(A,V,h,\mu))],$$

then the above proves that $J(A,V,h,\mu,\vec{a}) = fJ(A,\hat{V},\frac{h}{fl},\frac{\mu l}{f},\hat{a})$, where $\hat{a} = \vec{a}(lx+z)$. Now $T_k = J(A,V,h,\mu,\psi_k\vec{a})$, which thus means that:

$$T_k = f_k J\left(A,\frac{V(l\cdot+x_k)}{f_k^2},\frac{h}{f_k l_k},\frac{\mu l_k}{f_k},(\psi_k\vec{a})(l_k\cdot+x_k)\right).$$

The following conditions are satisfied:

- $\frac{h}{f_k l_k}\frac{\mu l_k}{f_k} = \frac{h\mu}{f_k^2} = h\mu|x_k| \leq h\mu r^2 \leq 1$.
- $\frac{h}{f_k l_k} = \frac{h}{\rho\sqrt{|x_k|}} \leq \frac{\sqrt{\theta}}{\rho} \leq h_0$ if θ, ρ are chosen properly.
- $|\partial^\alpha f_k^{-2} V(l_k \cdot + x_k)| \leq c_\alpha$, where c_α is some constant independent of f, l, k.
- $|\partial^\alpha (\psi_k \vec{a})(l_k \cdot + x_k)| \leq C_\alpha$ where the same remark applies to C_α.

Therefore we can apply the reference problem (3.11.14) to conclude that

$$\begin{aligned}
|T_k| &\leq f_k C\left((\frac{\mu l_k}{f_k}+1)\frac{f_k^2 l_k^2}{h^2}\right) \\
&= C' \int_{B_k} f_k \left((\frac{\mu l_k}{f_k}+1)\frac{f_k^2 l_k^2}{h^2}\right) l_k^{-3} dx \\
&\leq C'' \int_{B_k} f(x) \left((\frac{\mu l(x)}{f(x)}+1)\frac{f(x)^2 l(x)^2}{h^2}\right) l(x)^{-3} dx \\
&= C'' \int_{B_k} \left(\frac{\mu}{h^2} f(x)^2 + \frac{f(x)^3}{l(x)h^2}\right) dx.
\end{aligned}$$

□

3.11.2 The outer region

In the outer region the result is the following[2]

Lemma 3.11.16. *Let the assumption be as in Theorem 3.11.1. Then*

$$|\mathrm{tr}[\mathbf{B}(h,\mu,\vec{a}_2) g_0(\mathbf{P}(h,\mu,V))] - \mathcal{A}(h,\mu,\vec{a}_2,V(x))| = O(\frac{1+r^{-7}\mu^{-4}}{\mu h})$$

In the outer region, $\mathcal{D} = \{|x| \geq r^2\}$, magnetic effects become important and we see a current.

In $\mathcal{D}$ we perform a multiscaling with the same scaling functions $f(x) = |x|^{-1/2}$ and $l(x) = \rho|x|$, $\rho < \frac{1}{16}$ as in Ω_2, but now we use the asymptotics for the current in a strong magnetic field as reference problem.

We will write $\vec{a}$ instead of $\vec{a}_2$.

Theorem 3.11.17 (Ref. Problem in $\mathcal{D}$). *Let $\hat{a} \in C_0^\infty(B(0,1);\mathbb{R}^3)$, $A(x) = (-x_2, 0, 0)$, and V be a function such that*

$$\mathbf{P} = \mathbf{P}(h,\mu,A,V) = [\vec{\sigma}\cdot(-ih\nabla - \mu A)]^2 + V$$

is self adjoint and bounded below. Suppose that $\exists c_\alpha, m, M, \zeta, \beta, h_0 > 0$ such that

- $|\partial^\alpha \hat{a}| \leq c_\alpha$, $|\partial^\alpha V| \leq c_\alpha$ *on $B(0,8)$,*

[2] Remember that $\vec{a}_2$ is the testfunction $\vec{a}$ cut smoothly down to the region $\{|x| > r^2\}$

- $0 < h \leq h_0$,
- $h^\zeta \mu \leq m$,
- $h^\beta \mu \geq M$,

then

$$\operatorname{tr}[\mathbf{B}(h,\mu,\hat{a})g_0(\mathbf{P})] = \mathcal{A} + O(h^{-1}\mu^{-1} + h^{-3}\mu^{-2} + h^{-1}).$$

where

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}(h,\mu,\hat{a},V)\\
&= \frac{2}{3\pi h^2}\sum_{n=0}^{\infty} d_n \int (\partial_{x_1}\hat{a}_2(x) - \partial_{x_2}\hat{a}_1(x))\\
&\quad \times \left([2n\mu h + V(x)]_-^{3/2} - 3n\mu h[2nh\mu + V(x)]_-^{1/2} \right) dx.
\end{aligned}$$

This is the statement of Thm 3.9.1. We will use this with $\zeta = 3$, and β such that

$$\mu h^\beta r^{3-\beta} \geq 1. \tag{3.11.3.6}$$

That it is possible to find such a β for our choice of r will be proved at the end of this section.

On $\mathcal{D}$ we have

- $|\partial^\alpha \vec{a}| \leq c_\alpha l(x)^{-|\alpha|}$,
- $|\partial^\alpha V| \leq c_\alpha f(x)^2 l(x)^{-|\alpha|}$.

Again we can find a partition of unity $\{\psi_k\}$ as in the previous multiscaling. We write:

$$\begin{aligned}
\mathcal{I}(h,\mu,\vec{a},V) &= \mathcal{I}(h,\mu,\sum_k \psi_k \vec{a},V)\\
&= \sum_k \mathcal{I}(h,\mu,\psi_k\vec{a},V),
\end{aligned}$$

and also

$$\begin{aligned}
\mathcal{A}(h,\mu,\vec{a},V) &= \mathcal{A}(h,\mu,\sum_k \psi_k \vec{a},V)\\
&= \sum_k \mathcal{A}(h,\mu,\psi_k\vec{a},V).
\end{aligned}$$

We want to prove that

$$\begin{aligned}
&|\mathcal{I}(h,\mu,\psi_k\vec{a},V) - \mathcal{A}(h,\mu,\psi_k\vec{a},V)|\\
\leq\ & C\int_{B_k} f(x)\left(\frac{f(x)l(x)}{h}\frac{f(x)}{\mu l(x)} + \frac{f(x)^3 l(x)^3}{h^3}\frac{f(x)^2}{\mu^2 l(x)^2} + \frac{f(x)l(x)}{h}\right) l(x)^{-3}dx.
\end{aligned}$$

The proof is similar to the proof of Lemma 3.11.15 and will therefore be omitted. First we have to check:

$$\frac{\mu l}{f}\cdot\frac{h^3}{l^3 f^3} \approx \mu h^3 \frac{|x|^{1-3}}{|x|^{-(1+3)/2}} = \mu h^3 \leq m,$$

and

$$\frac{\mu l}{f}\cdot\frac{h^\beta}{l^\beta f^\beta}=\mu h^\beta\frac{l^{1-\beta}}{f^{1+\beta}}\approx\mu h^\beta\frac{|x|^{1-\beta}}{|x|^{-(1+\beta)/2}}=\mu h^\beta|x|^{3-\beta/2}\geq\mu h^\beta r^{3-\beta}$$

Thus we get:

$$\begin{aligned}|\mathcal{J}(h,\mu,\vec{a},V)-\mathcal{A}(h,\mu,\vec{a},V)| &= O\left(\int_{\mathcal{D}_1} f\left(\frac{fl}{h}\frac{f}{\mu l}+\frac{f^3l^3}{h^3}\frac{f^2}{\mu^2l^2}\right)l^{-3}dx\right)\\ &= O\left(\int_{r^2}^1\frac{1}{\mu h}|x|^{-5/2}+\frac{1}{h^3\mu^2}\frac{1}{|x|^3}d|x|\right)\\ &= O\left(\frac{1}{\mu h}+\frac{1}{h^3\mu^2}\frac{1}{\mu h}r^{-3}\frac{h^3}{\mu^2}r^{-4}\right)\\ &= \frac{1}{\mu h}O(1+r^{-7}\mu^{-4}).\end{aligned}$$

Finally, we can finish the proof of Theorem 3.11.1.

Proof. We have the following conditions on r i.e. equations (3.11.3.3) and (3.11.3.6):

$$\mu hr^2 \leq 1 \tag{3.11.3.7}$$

$$\exists\beta\in(0,3]\text{ such that }\mu h^\beta r^{3-\beta} \geq 1, \tag{3.11.3.8}$$

and since we want the error terms to be small we need

$$\begin{aligned} r &\ll 1\\ \mu r^6 &\ll 1\\ hr^{-7}\mu^{-5} &\ll 1 \end{aligned} \tag{3.11.3.9}$$

To make the optimal choice of r let $\delta>0$ and write

$$\begin{aligned}\mu &= h^{-\gamma}\\ r &= h^{\gamma/3-\delta(3-\gamma)}.\end{aligned}$$

This defines γ and r. Choose

$$\beta=\frac{9\delta}{1+3\delta}.$$

Then (3.11.3.8) is satisfied, since:

$$\begin{aligned}\mu h^\beta r^{3-\beta} &= h^{-\gamma+\beta+(3-\beta)(\gamma/3-\delta(3-\gamma))}\\ &= h^{-\gamma+\beta+\gamma-3\delta(3-\gamma)-\beta\gamma/3+\beta\delta(3-\gamma)}\\ &= h^{\beta(1-\gamma/3)+\beta3\delta(1-\gamma/3)-9\delta(1-\gamma/3)}\\ &= h^{(1-\gamma/3)(\beta(1+3\delta)-9\delta)}\\ &= 1.\end{aligned}$$

The other equation, (3.11.3.7), holds if just $\delta<1/6$ since:

$$\begin{aligned}\mu hr^2 &= h^{-\gamma+1+2\gamma/3-2\delta(3-\gamma)}\\ &= h^{1-\gamma/3-6\delta(1-\gamma/3)}\\ &= h^{(1-\gamma/3)(1-6\delta)}.\end{aligned}$$

The conditions (3.11.3.9) become

$$\begin{aligned} h^{\gamma/3-\delta(3-\gamma)} &\ll 1 \\ h^{\gamma-6\delta(3-\gamma)} &\ll 1 \\ h^{1+8\gamma/3+7\delta(3-\gamma)} &\ll 1. \end{aligned}$$

The first two of these get better for small δ, and the first is the largest term of the three. This finishes the proof of theorem 3.11.1. $\square$

A Some localisation arguments

In this section we will prove the following localisation result.

Theorem A.1. *Let $E > 0$ and let $\mathcal{H}$ be an operator satisfying:*

- *$\mathcal{H}$ is a self adjoint operator which is bounded below on $L^2(\mathbb{R}^d)$.*
- *$\exists\{a_l\}_{l=1}^d, V$ all in $C_0^\infty(\mathbb{R}^d)$, such that for all $u \in C_0^\infty(B(4E))$:*

$$\mathcal{H}u = Hu,$$

where we have used the notation

$$H = \sum(-ih\partial_l - a_l)^2 + V.$$

Let C_α be the constants such that

$$\begin{aligned} |\partial^\alpha V| &\le C_\alpha \\ |\partial^\alpha a_l| &\le C_\alpha, \end{aligned} \tag{A.3.1}$$

on $B(8E)$. Let finally $\chi \in C_0^\infty(B(E))$ and $g \in C_0^\infty(\mathbb{R})$. Then

$$\|\chi(-ih\partial_l - a_l)[g(\mathcal{H}) - g(H)]\|_1 = O(h^\infty),$$

where the O is uniform in E, g, χ and the constants C_α in (A.3.1).

Remark A.2. Let C_r be constants so that $|g^{(r)}| \le C_r$. By *uniform* we mean that if $\tilde{\mathcal{H}}$, $\tilde{H} = \sum(-ih\partial_l - \tilde{a}_l)^2 + \tilde{V}$ satisfy the above assumptions with the same constants C_α and the same E, χ, and if $\tilde{g} \in C_0^\infty(\mathbb{R})$ with $|\tilde{g}^{(r)}| \le C_r$ (the same constants as in the bounds on $|g^{(r)}|$) and $\sup\operatorname{supp}\tilde{g} \le \sup\operatorname{supp} g$, then

$$\|\chi(-ih\partial_l - \tilde{a}_l)[g(\tilde{\mathcal{H}}) - g(\tilde{H})]\|_1 \le C_N h^N,$$

$\forall N \in \mathbb{N}$, where the constants C_N are the same as in Theorem A.1. Observe, that we do not assume $\operatorname{supp}\tilde{g} \subset \operatorname{supp} g$.

Notation:
Let us introduce the following notation:
Let $\lambda_0 \ge 1 + 2\sup|V(x)|$ then we define $d(z) = \operatorname{dist}(z, [-\lambda_0, \infty))$.
Let furthermore $\langle z\rangle = (1+|z|^2)^{1/2}$. Finally we will write $Q_l = (-ih\partial_l - a_l)$.

We will use the following lemma:

Lemma A.3. *[Sob95, Lemma 3.6] Let $\chi \in C_0^\infty(\mathbb{R}^d)$ and $\phi \in \mathcal{B}(\mathbb{R}^d)$ be such that*

$$\operatorname{dist}(\operatorname{supp}\chi, \operatorname{supp}\phi) \geq c > 0,$$

and let $r, m = 0, 1$. Then for any $N > d/2$

$$\|\chi Q_l^r (H-z)^{-1} (Q_q^*)^m \phi\|_1 \leq C_N \frac{\langle z\rangle^{\frac{m+r}{2}}}{d(z)} \left(\frac{\langle z\rangle^{\frac{1}{2}}}{h}\right)^d \left(\frac{\langle z\rangle h^2}{d(z)^2}\right)^N.$$

We start the proof of Theorem A.1 with the following lemma:

Lemma A.4. *Suppose $\chi \in C_0^\infty(B(3E))$. Then for any $N > d/2$:*

$$\begin{aligned} &\left\|\chi Q_l \left\{(\mathcal{H}-z)^{-1} - (H-z)^{-1}\right\}\right\|_1 \\ \leq\ & C_N \left[\frac{\langle z\rangle^{1/2}}{h}\right]^d \left[\frac{h^2\langle z\rangle}{d(z)^2}\right]^{N+1/2} \left\{\frac{\langle z\rangle^{1/2}}{|\Im z|} + h^{-1}\right\}, \end{aligned}$$

where $\Im(z)$ is the imaginary part of z.

Proof. Define $\chi_1 \in C_0^\infty(B(20E/6))$ satisfying: $\chi_1(x) = 1$ on $|x| \leq 19E/6$. Thus $\chi_1\chi = \chi$. Furthermore we will write $\phi = 1 - \chi_1$. Writing $(H-z)^{-1} = (H-z)^{-1}\chi_1 + (H-z)^{-1}\phi$, we get

$$\begin{aligned} & \chi Q_l[(\mathcal{H}-z)^{-1} - (H-z)^{-1}] \\ =\ & \chi Q_l[\chi_1(\mathcal{H}-z)^{-1} - (H-z)^{-1}\chi_1] - \chi Q_l (H-z)^{-1}\phi \\ =\ & T_1 + T_2. \end{aligned}$$

The last term is easily estimated using Lemma A.3 as

$$\|T_2\|_1 = \|\chi Q_l (H-z)^{-1}\phi\|_1 \leq C_N \frac{\langle z\rangle^{1/2}}{d(z)} \left[\frac{\langle z\rangle^{1/2}}{h}\right]^d \left[\frac{h^2\langle z\rangle}{d(z)^2}\right]^N,$$

which is seen to fit the estimate we want to prove.
Using the identity:

$$\chi(\mathcal{H}-z)^{-1} = (H-z)^{-1}\chi - (H-z)^{-1} Z (\mathcal{H}-z)^{-1},$$

where

$$Z = -[H,\chi] = \sum_{j=1}^n ih(Q_j^*(\partial_j\chi) + (\partial_j\chi)Q_j),$$

we get that the first term is

$$T_1 = \sum_{j=1}^n \left(-\chi Q_l (H-z)^{-1} ih (Q_j^*(\partial_j\chi_1) + (\partial_j\chi_1)Q_j)(\mathcal{H}-z)^{-1}\right).$$

This we can estimate as:

$$
\begin{aligned}
\|T_1\|_1 &\leq \sum_{j=1}^{n} h\Big\{\big\|\chi Q_l(H-z)^{-1}Q_j^*(\partial_j\chi_1)(\mathcal{H}-z)^{-1}\big\|_1 \\
&\quad + \big\|\chi Q_l(H-z)^{-1}(\partial_j\chi_1)Q_j(\mathcal{H}-z)^{-1}\big\|_1\Big\} \\
&\leq \sum_{j=1}^{n} 2h\|\chi Q_l(H-z)^{-1}Q_j^*(\partial_j\chi_1)\|_1\frac{1}{|\Im z|} \\
&+ \sum_{j=1}^{n} h^2\|\chi Q_l(H-z)^{-1}(\partial_j^2\chi_1)\|_1\frac{1}{|\Im z|} \\
&\leq C_N h\frac{\langle z\rangle}{d(z)}\left[\frac{\langle z\rangle^{1/2}}{h}\right]^d\left[\frac{\langle z\rangle h^2}{d(z)^2}\right]^N\frac{1}{|\Im z|},
\end{aligned}
$$

where we used Lemma A.3 to get the last estimate. □

Now we can prove Theorem A.1:

Proof. We use the representation:

$$
\begin{aligned}
g(A) &= \sum_{j=0}^{m}\int(\partial^j g)(\lambda)\Im[i^j(A-\lambda-i)^{-1}]d\lambda \\
&+ \frac{1}{\pi(m-1)!}\int_0^1\tau^{m-1}\int_{\mathbb{R}}(\partial^m g)(\lambda)\Im[i^m(A-\lambda-i\tau)^{-1}]d\lambda\, d\tau,
\end{aligned}
$$

which holds for all self adjoint operators A, $g\in C_0^\infty$, $m\geq 2$ (See [AdMBG91]).
Writing

$$
\delta(\lambda,\tau)=(\mathcal{H}-\lambda-i\tau)^{-1}-(H-\lambda-i\tau)^{-1},
$$

we thus get:

$$
\begin{aligned}
&\chi Q_l\{g(\mathcal{H})-g(H)\} \\
=&\sum_{j=0}^{m}\frac{1}{\pi(m-1)!}\int_{\mathbb{R}}(\partial^j g)(\lambda)\chi Q_l\Im[i^j\delta(\lambda,1)]d\lambda \\
&+\frac{1}{\pi(m-1)!}\int_0^1\tau^{m-1}\int_{\mathbb{R}}(\partial^m g)(\lambda)\chi Q_l\Im[i^m\delta(\lambda,\tau)]d\lambda d\tau. \qquad \text{(A.3.2)}
\end{aligned}
$$

Choose $m=2N+3$. Using Lemma A.4 the first term is easily estimated by $O(h^{2N-d})$:

$$
\|\chi Q_l\Im(i^j\delta(\lambda,1)\|_1\leq ch^{-d-1}(\sqrt{2+\lambda^2})^{d/2+N+1/2}h^{2N+1}\frac{1}{(1+|\lambda|)^{2N+1}}.
$$

For N sufficiently big, this is integrable in λ, and we get

$$
\begin{aligned}
&\|\sum_{j=0}^{m}\frac{1}{\pi(m-1)!}\int_{\mathbb{R}}(\partial^j g)(\lambda)\chi Q_l\Im[i^j\delta(\lambda,1)]d\lambda\|_1 \\
\leq\ & c\sup_{j=0..m}\{|g^{(j)}|\}h^{2N-d}. \qquad \text{(A.3.3)}
\end{aligned}
$$

The second integral in (A.3.2) we split in two:

$$I_1 = \frac{1}{\pi(m-1)!}\int_0^1 \tau^{m-1}\int_{-2\lambda_0}^{\infty}(\partial^m g)(\lambda)\chi Q_l \Im[i^m\delta(\lambda,\tau)]d\lambda d\tau,$$

and

$$I_2 = \frac{1}{\pi(m-1)!}\int_0^1 \tau^{m-1}\int_{-\infty}^{-2\lambda_0}(\partial^m g)(\lambda)\chi Q_l \Im[i^m\delta(\lambda,\tau)]d\lambda d\tau.$$

Inside the integral in I_1 we estimate:

$$\|\chi Q_l \Im(i^j\delta(\lambda,\tau)\|_1 \leq ch^{-d-1}(\sqrt{2+\lambda^2})^{d/2+N+1/2}h^{2N+1}\tau^{-2N-2}.$$

Using our choice of m, I_1 is easily estimated. I_2 is estimated just like (A.3.3). □

As a corollary to Theorem A.1 we get the following generalisation of the result in [Fou98]:

Lemma A.5. *Let the notation be as above. Then the currents of $\mathcal{H}$ and of H on the set $B(E)$ are the same up to an error of order $O(h^{1-n})$, i.e. for all $\chi \in C_0^\infty(B(E))$ and for all l we have:*

$$\mathrm{tr}[\chi Q_l(g_0(\mathcal{H}) - g_0(H))] = O(h^{1-n}).$$

Again this is uniform in E,χ and the C_α's.

Proof. Choose $g \in C_0^\infty(\mathbb{R})$ such that $gg_0 = g_0$ on Spec $\mathcal{H}$. Notice, that the bounds on $|g^{(r)}|$ do not depend on $\inf \mathrm{Spec}\,\mathcal{H}$. Write, using the spectral theorem:

$$\begin{aligned}\mathrm{tr}[\chi Q_l g_0(\mathcal{H})] &= \mathrm{tr}[\chi Q_l g(\mathcal{H})g_0(\mathcal{H})] \\ &= \mathrm{tr}[\chi Q_l g(H)g_0(\mathcal{H})] + O(h^\infty).\end{aligned}$$

Now we get from [Fou98] that $\chi Q_l g(H)$ is h-admissible. By an expansion of this operator in powers of h we get:

$$\mathrm{tr}[\chi Q_l g_0(\mathcal{H})] = \mathrm{tr}[Op_h^w \theta g_0(\mathcal{H})] + O(h^{1-n}),$$

where $\theta(x,\xi) = \chi(\xi_l - a_l)g((\xi - a_l)^2 + V(x)) \in C_0^\infty(\mathbb{R}^n)$. That this is $O(h^{1-n})$ follows from [Sob95] and the Tauberian argument given in [Fou98]. □

B Localisation in a neighborhood of a singularity

In this appendix we will prove that to study the current close to, for example, a Coulomb singularity, only the local behaviour of the singularity matters. The result below can be rephrased as follows:
Let $\chi \in C_0^\infty(B(1))$ and let V be a potential, such that, if $\zeta \in C_0^\infty(B(2))$, then ζV is bounded relatively to the kinetic energy $(-i\nabla - \vec{A})^2$. Then $\exists C > 0$ such that:

$$\|\chi(-i\nabla - \vec{A})[g_0((-i\nabla - \vec{A})^2 + V) - g_0((-i\nabla - \vec{A})^2 + \zeta V)]\| \leq C,$$

where C only depends on local information, i.e. on ζV.

Let us now be more precise: Let V (playing the role of ζV in the discussion above) be a multiplication operator such that $\exists\, 0<\varepsilon<1$ and $M>0$:

$$\langle u, |V|u\rangle \leq \varepsilon\langle u, -\Delta u\rangle + M\|u\|^2, \tag{B.3.1}$$

for all $u \in C_0^\infty$. Observe that this implies, by the diamagnetic inequality, that

$$\langle u, |V|u\rangle \leq \varepsilon\langle u, (-i\nabla - \vec{A})^2 u\rangle + M\|u\|^2,$$

with the same constants ε, M. Denote by H the selfadjoint operator $(-i\nabla - \vec{A})^2 + V$.

Assumption B.1. Let $\mathcal{H}$ be a selfadjoint operator in $L^2(\mathbb{R}^3)$, $\mathcal{H} \geq -\lambda_0$ for some $\lambda_0 > 1$ and satisfying for all $\phi \in C_0^\infty(B(2))$:

- $\forall u \in \mathcal{D}[\mathcal{H}]$ (the form domain of $\mathcal{H}$) we have $\phi u \in \mathcal{D}[\mathcal{H}]$ and $\exists \phi_1 \in C_0^\infty(B(2))$ such that
  $\langle u, \mathcal{H}(\phi v)\rangle = \langle(\phi_1 u), H(\phi v)\rangle$ for all $u, v \in \mathcal{D}[\mathcal{H}]$.

Remark B.2. The application in this article is to decompose Coulomb singularities, but the assumption is by far more general.

The result is the following:

Lemma B.3. *Let $\chi \in C_0^\infty(B(1))$, then*

$$\|\chi(-i\partial_{x_j} - A_j)[g_0(\mathcal{H}) - g_0(H)]\| \leq C,$$

where C depends only on χ and on ε, M in (B.3.1).

Remark B.4. C does *not* depend on the lower bound λ_0.

The main ingredient to prove Lemma B.3 is the following:

Lemma B.5. *Let $\chi \in C_0^\infty(B(1))$, and let $z \in \mathbb{C}$ with $0 < |\Im(z)| \leq 1$, then for all $N > 0$ there exists $C_N > 0$ such that*

$$\|\chi(-i\partial_{x_j} - A_j)[(\mathcal{H}-z)^{-1} - (H-z)^{-1}]\| \leq C_N \frac{M+1+|z|}{d_M(z)} \left[\frac{M+|z|}{d_M(z)}\right]^N \frac{1}{|\Im(z)|},$$

where $d_M(z) = \operatorname{dist}(z, [-M, \infty))$.

Proof. Choose $\chi_1 \in C_0^\infty(B(2))$, $\chi_1 \equiv 1$ on $B(3/2)$, and write

$$\begin{aligned}
&\chi(-i\partial_{x_j} - A_j)[(\mathcal{H}-z)^{-1} - (H-z)^{-1}] \\
=\ &\chi(-i\partial_{x_j} - A_j)[\chi_1(\mathcal{H}-z)^{-1} - (H-z)^{-1}\chi_1] \\
&+\chi(-i\partial_{x_j} - A_j)(H-z)^{-1}\phi,
\end{aligned}$$

where $\phi = 1 - \chi_1$. Now the lemma follows from the identity

$$(\mathcal{H}-z)^{-1} - (H-z)^{-1}\chi_1 = -2i\sum_{l=1}^{3}(-i\partial_{x_l} - A_l)(\partial_{x_l}\chi_1) + \Delta\chi_1,$$

and the following result from [Sob96a, Lemma 3.3]:

$$\begin{aligned}
&\|\chi(-i\partial_{x_j}-A_j)^{m_1}(H-z)^{-1}(-i\partial_{x_l}-A_l)^{m_2}\phi\| \\
\leq\ & C\frac{(M+|z|)^{\frac{m_1+m_2}{2}}}{d_M(z)}\left[\frac{M+|z|}{d_M(z)^2}\right]^N,
\end{aligned}$$

where $m_1, m_2 \in \{0,1\}$. $\square$

The lemma B.3 now follows, using almost analytic extensions, just like in the previous appendix.

C A calculation with poisson summation

Let us write $t = \frac{[V(x)]_-}{2\mu h}$, and

$$\begin{aligned}
& S = S([V(x)]_-, h\mu) \\
=\ & \sum_{n=0}^{\infty} d_n\left([2h\mu n - [V(x)]_-]_-^{3/2} - 3/2(2h\mu)n[2h\mu n - [V(x)]_-]_-^{1/2}\right) \\
=\ & (2h\mu)^{3/2}\sum_{n=0}^{\infty} d_n\left([n-t]_-^{3/2} - \frac{3}{2}n[n-t]_-^{1/2}\right).
\end{aligned}$$

In this appendix we want to prove the following computational result:

Proposition C.1.

$$S([V(x)]_-, h\mu) = O((h\mu)^{3/2} + \sqrt{h\mu}[V(x)]_- + h\mu[V(x)]_-),$$

uniformly in x.

Proof. Let us write $F_t(\alpha) = \left([\alpha-t]_-^{3/2} - \frac{3}{2}\alpha[\alpha-t]_-^{1/2}\right)$, then

$$S = \frac{(2h\mu)^{3/2}}{\pi}\left(\frac{F_t(0)}{2} + \sum_{k=1}^{\infty} F_t(k)\right).$$

We use Poisson Summation and get:

$$S = \frac{(2h\mu)^{3/2}}{\pi}\left\{\int_0^\infty F_t(\alpha)d\alpha + 2\Re\left(\sum_{k=1}^{\infty}\int_0^\infty F_t(\alpha)e^{i2\pi k\alpha}d\alpha\right)\right\}.$$

Let us look at the first term:

$$\begin{aligned}
& \int_0^\infty F_t(\alpha)d\alpha \\
=\ & \int_0^t (t-\alpha)^{3/2}d\alpha - \frac{3}{2}\left\{[\alpha\frac{2}{3}(t-\alpha)^{3/2}]_{\alpha=0}^t - \frac{2}{3}\int_0^t (t-\alpha)^{3/2}d\alpha\right\} \\
=\ & 0.
\end{aligned}$$

One part of $\Re\left(\int_0^\infty F_t(\alpha)e^{i2\pi k\alpha}d\alpha\right)$ was calculated in [Sob96b, p.399]:

Lemma C.2.

$$\begin{aligned}
&\Re\left(\int_0^\infty (t-\alpha)^{3/2} e^{i2\pi k\alpha} d\alpha\right)\\
= \;& \frac{3}{8\pi^2 k^2} t^{1/2} - \frac{3}{16\pi^2 k^{5/2}}\left(\cos(2\pi kt)\mathcal{C}(2\sqrt{kt}) + \sin(2\pi kt)\mathcal{S}(2\sqrt{kt})\right),
\end{aligned}$$

where

$$\mathcal{C}(x) = \int_0^x \cos(\pi u^2/2) du,$$

and

$$\mathcal{S}(x) = \int_0^x \sin(\pi u^2/2) du.$$

What we have left to calculate is thus $-\frac{3}{2}\int_0^t \alpha\sqrt{t-\alpha} e^{i2\pi k\alpha} d\alpha$. This we do explicitly:

$$\begin{aligned}
&-\frac{3}{2}\int_0^t \alpha\sqrt{t-\alpha} e^{i2\pi k\alpha} d\alpha\\
= \;& -\frac{3}{4\pi i}\frac{d}{dk}\int_0^t \sqrt{t-\alpha} e^{i2\pi k\alpha} d\alpha\\
= \;& -\frac{1}{2\pi i}\frac{d}{dk}\int_0^t \frac{d}{dt}(t-\alpha)^{3/2} e^{i2\pi k\alpha} d\alpha\\
= \;& -\frac{1}{2\pi i}\frac{d}{dk}\left\{\frac{d}{dt}\left(\int_0^t (t-\alpha)^{3/2} e^{i2\pi k\alpha} d\alpha\right) - (t-\alpha)^{3/2} e^{i2\pi k\alpha}\Big|_{\alpha=t}\right\}\\
= \;& -\frac{1}{2\pi i}\frac{d}{dk}\frac{d}{dt}\int_0^t (t-\alpha)^{3/2} e^{i2\pi k\alpha} d\alpha\\
= \;& -\frac{1}{2\pi i}\frac{d}{dk}\frac{d}{dt}\left\{\frac{t^{3/2}}{2\pi i k} + \frac{3t^{1/2}}{8\pi^2 k^2} - \frac{3e^{i2\pi kt}}{16\pi^2 k^{5/2}}\int_0^{2\sqrt{kt}} e^{-i\pi u^2/2} du\right\}\\
= \;& \frac{1}{2\pi i} 3/2 \frac{t^{1/2}}{2\pi i k^2} + \frac{1}{2\pi i}\frac{3}{8\pi^2 k^3\sqrt{t}} +\\
& \frac{1}{2\pi i}\frac{d}{dk}\left(\frac{3e^{i2\pi kt}}{16\pi^2 k^{5/2}}\left\{2\pi i k\int_0^{2\sqrt{kt}} e^{-i\pi u^2/2} du + \sqrt{k} t^{-1/2} e^{-i\pi 2kt}\right\}\right).
\end{aligned}$$

Here we used Lemma C.2 to get the next to last equality. We calculate the real part and get:

$$\begin{aligned}
&-\frac{3t^{1/2}}{8\pi^2 k^2} + \frac{3}{16\pi^2}\Re\left\{\frac{d}{dk}\left(k^{-3/2} e^{i2\pi kt}\int_0^{2\sqrt{kt}} e^{-i\pi u^2/2} du\right)\right\}\\
= \;& -\frac{3t^{1/2}}{8\pi^2 k^2} + \frac{3}{16\pi^2}\frac{d}{dk}\left\{k^{-3/2}\left[\cos(2\pi kt)\mathcal{C}(2\sqrt{kt}) + \sin(2\pi kt)\mathcal{S}(2\sqrt{kt})\right]\right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
S \;=\; & \sum_{k=1}^{\infty}\frac{(2h\mu)^{3/2}}{\pi} 2\Big\{\frac{-15}{32\pi^2 k^{5/2}}\left[\cos(2\pi kt)\mathcal{C}(2\sqrt{kt}) + \sin(2\pi kt)\mathcal{S}(2\sqrt{kt})\right]\\
& + \frac{3}{16\pi^2 k^{3/2}}\frac{d}{dk}\left[\cos(2\pi kt)\mathcal{C}(2\sqrt{kt}) + \sin(2\pi kt)\mathcal{S}(2\sqrt{kt})\right]\Big\}.
\end{aligned}$$

Using, that $\mathcal{C}$ and $\mathcal{S}$ are bounded with bounded first derivatives, we thus see that

$$
\begin{aligned}
S &= O\left((h\mu)^{3/2}\sum_{k=1}^{\infty}(k^{-5/2}+\frac{t}{k^{3/2}}+\frac{\sqrt{t}}{k^2})\right)\\
&= O((h\mu)^{3/2}(1+t+\sqrt{t}))\\
&= O((h\mu)^{3/2}+\sqrt{h\mu}[V]_-+h\mu[V]_-).
\end{aligned}
$$

$\square$

D Gauge invariance of the current

In this appendix we will prove that the current $\mathcal{J}(h,\mu,\vec{a},V)$ as a function of $\vec{a}$ only depends on the magnetic field $\vec{b}=\nabla\times\vec{a}$ generated by $\vec{a}$, i.e. that if $\vec{a}=\tilde{a}+\nabla\phi$ then $\mathcal{J}(h,\mu,\vec{a},V)=\mathcal{J}(h,\mu,\tilde{a},V)$:

Lemma D.1. *Suppose V is relatively bounded with respect to $-h^2\Delta$ and that* $\mathrm{Spec}(\mathbf{P}(h,\mu,V))$ *below zero is discrete. Then $\forall\phi\in C_0^\infty(\mathbf{R}^3)$ we have $\mathcal{J}(h,\mu,\nabla\phi,V)=0$.*

Proof. Let ψ be an eigenfunction of $\mathbf{P}(h,\mu,V)$ with eigenvalue $\lambda<0$. We may, with a slight abuse of notation assume that

$$(H+W)\psi=\lambda\psi,$$

where $W=V\pm\mu h$ and $H=(-ih\nabla-\vec{A})^2$. We have to prove that

$$\langle\psi,(\nabla\phi)\cdot(-ih\nabla-\vec{A})\psi\rangle+\langle\psi,(-ih\nabla-\vec{A})\cdot(\nabla\phi)\psi\rangle=0,$$

or equivalently

$$\langle\psi,(-ih\nabla\phi)\cdot(-ih\nabla-\vec{A})\psi\rangle+\langle\psi,(-ih\nabla-\vec{A})\cdot(-ih\nabla\phi)\psi\rangle=0.$$

Notice that $(-ih\partial_{x_j}-A_j)\phi=\phi(-ih\partial_{x_j}-A_j)+(-ih\partial_{x_j}\phi)$, thus we get, using the self-adjointness of $(-ih\partial_{x_j}-A_j)$ and the relative boundedness of W:

$$
\begin{aligned}
&\langle\psi,(-ih\nabla\phi)\cdot(-ih\nabla-\vec{A})\psi\rangle+\langle\psi,(-ih\nabla-\vec{A})\cdot(-ih\nabla\phi)\psi\rangle\\
=\;&\langle\psi,\Big[(-ih\nabla-\vec{A})\phi-\phi(-ih\nabla-\vec{A})\Big]\cdot(-ih\nabla-\vec{A})\psi\rangle\\
&+\langle\psi,(-ih\nabla-\vec{A})\cdot\Big[(-ih\nabla-\vec{A})\phi-\phi(-ih\nabla-\vec{A})\Big]\psi\rangle\\
=\;&\langle(-ih\nabla-\vec{A})\psi,\phi(-ih\nabla-\vec{A})\psi\rangle-\langle\psi,\phi\lambda\psi\rangle+\langle\psi,\phi W\psi\rangle\\
&+\langle\lambda\psi,\phi\psi\rangle-\langle W\psi,\phi\psi\rangle-\langle(-ih\nabla-\vec{A})\psi,\phi(-ih\nabla-\vec{A})\psi\rangle.
\end{aligned}
$$

$\square$

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