SECURITIZATION OF INSURANCE RISK

CLAUS VORM CHRISTENSEN
DEPARTMENT OF THEORETICAL STATISTICS AND OPERATIONS RESEARCH
UNIVERSITY OF AARHUS
DK-8000 AARHUS C, DENMARK

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UNBELIEVABLE, WILLARD! LOOK-- IT'S A NEW KIND OF SECURITY INDEXED TO WEATHER AND NATURAL DISASTERS!
SO CALLED "CATASTROPHE BONDS" PAY OFF BIG AS LONG AS HURRICANES OR EARTHQUAKES DON'T HIT!
WOW. WHAT DO YOU THINK THIS'LL DO TO THE WEATHER CHANNEL?
Sebastian
AND FOR TOMORROW, CHANCE OF RAIN UP 2 1/4, HUMIDITY YIELD SLIPPING 3 BASIS POINTS...
WINNERS
LOSERS
32 1/2
5 7/8
+3/4

Preface

This thesis is the outcome of my PhD study and develops around securitization of insurance risks. In particular it considers products related to catastrophe insurance. The thesis is split into three major sections. The first section gives a short summary of each of the five manuscripts included in the thesis, there is one in English and one in Danish. The second section gives a longer review of the new main results contained in the thesis and discusses how these results relates to existing theories in the literature. The remainder of the thesis consists of the five manuscripts. Cynics would say that the second part exists so that you will not have to read the manuscripts.

Some of the work in the thesis was developed while I was visiting the financial and insurance mathematics group at the ETH, Zürich 1998-1999 and Laboratory of Actuarial Mathematics, University of Copenhagen 2000. I am grateful to Paul Embrechts and Ragnar Norberg respectively for making these stays possible.

The work in this thesis has benefited from stimulating discussions, suggestions and comments from a number of persons. Credits are due to Susan Black (PCS), Amy Casey (CBoT), Bent Jesper Christensen, Sam Cox, Freddy Delbaen, Paul Embrechts, Damir Filipovic, Asbjørn Trolle Hansen, Jørgen Hoffmann-Jørgensen, Dena Karras (CBoT), Randi Mosegaard, Thomas Møller, Jørgen Aase Nielsen, Ragnar Norberg, Jesper Lund Pedersen, Rolf Poulsen, Tina H. Rydberg, Uwe Schmock, Mogens Steffensen, Michael K. Sørensen, Jim Welsh (PCS), and of course my supervisor Hanspeter Schmidli.

Contents

Manuscripts

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Paper III: A new model for pricing catastrophe insurance derivatives, 14 pages.

Paper IV: Implied loss distributions for catastrophe insurance derivatives, 19 pages.

Paper V: How to hedge unknown risk, 23 pages.

Summaries of Manuscripts in the Thesis (English)

This thesis consists of the following five papers:

- Christensen, Claus Vorm and Hanspeter Schmidli, "Pricing catastrophe insurance products based on actually reported claims", *(to appear in Insurance: Mathematics and Economics).*
- Christensen, Claus Vorm, "The PCS-option, an improvement of the CAT-future", *(Manuscript, University of Aarhus).*
- Christensen, Claus Vorm, "A new model for pricing catastrophe insurance derivatives", *(Working Paper Series No. 28, Centre for Analytical Finance).*
- Christensen, Claus Vorm, "Implied loss distributions for catastrophe insurance derivatives", *(Manuscript, University of Aarhus.)*
- Christensen, Claus Vorm, "How to hedge unknown risk", *(Manuscript, University of Aarhus.)*

Below I give a short summary of the content of each paper.

<u>Refereed Publications</u>

Christensen, Claus Vorm and Hanspeter Schmidli (1998), "Pricing catastrophe insurance products based on actually reported claims", *(to appear in Insurance: Mathematics and Economics).*

Abstract: This article deals with the problem of pricing a financial product relying on an index of reported claims from catastrophe insurance. The problem of pricing such products is that, at a fixed time in the trading period, we do not know the total claim amount from the catastrophes occurred. Therefore we have to price these products solely from knowing the aggregate amount of the reported claims at the fixed time point. This article will propose a way to handle this problem by introducing a model taking reporting lags into account. The main idea of the article is to model the aggregate claim from a single catastrophe as a compound (mixed) Poisson model. We thereby obtain the possibility to separate the individual claims and to model the reporting times of the claims. This new model and an illustration of how price calculations can be done in the model is the main purpose of the article.

Working Papers

"The PCS-option, an improvement of the CAT-future", *(Manuscript, University of Aarhus).*
In 1992, Chicago Board of Trade (CBoT) introduced the CAT-future as an alternative to catastrophe reinsurance. But the product never became very popular, so in 1995 it was replaced by a new product the PCS-option. In relation to my PhD project, I therefore started collecting information about the PCS-option. The basic information was obtained by reading [14] and [55]. But some was also received by mailing with people from the PCS and the CBOT. After having received this information it seemed natural to gather it in a paper. This is done here, together with an explanation of why the CAT-future was replaced by the PCS-option, and why the PCS-option is an improvement. The paper also explains how to hedge catastrophe risk with PCS-options and it compares the PCS-option with traditional reinsurance.

"A new model for pricing catastrophe insurance derivatives", *(Working Paper Series No. 28, Centre for Analytical Finance.)*
Since the introduction of the insurance derivatives in 1992, it has been a problem how to price these products. The two main problems have been the following. First if we choose a realistic model for the underlying loss process the market will be incomplete and there will exist many equivalent martingale measures. Hence there will exist several arbitrage free prices of the product. Second we want a Pareto like tail for the underlying loss index, but heavy tails often give computational problems. It is therefore natural to look for a model that solves both these problems. In this paper we present a model which in some sense takes care of both problems. The model is inspired by results from Gerber and Shiu [43]. In [43] it is shown that the Esscher transform is a unique and transparent technique for valuing derivative securities if the logarithms of the underlying process are governed by a certain stochastic process with stationary and independent increments (a Levy process). In this paper we propose such a model, and by way of example we calculate prices for the PCS-option, but the approach can also be used for pricing other securities relying on a catastrophe loss index.

"Implied loss distributions for catastrophe insurance derivatives", *(Manuscript, University of Aarhus).*
In this paper we also price catastrophe insurance derivatives, but here we lead our analysis in another direction than in the previous papers. We follow a procedure familiar to the conventional option market which also is suggested by Lane and Movchan in [46], namely rather than

estimating volatilities and calculate consistent prices using, say the Black Scholes model, take the traded prices and extract the volatilities consistent with those prices, i.e. find the implied volatility. We cannot use the same procedure on the insurance derivative market directly since we are not able to characterize the price by a single parameter. But we can do something similar. We can choose a model for the implied loss distribution and then estimate the implied parameters from the observed prices.

This analysis can be used to evaluate cheapness and dearness among different prices and different insurance derivative products. We simply calculate implied prices from the implied loss distributions and compare them to the observed prices. There are two main problems in this analysis. First, what kind of distribution should be chosen for the implied losses and second, how should the involved parameters be estimated? In this paper we analyse those two problems and end up by recommending a new implied loss distribution and a new objective function for estimating the parameters.

"How to hedge unknown risk", *(Manuscript, University of Aarhus).* In this paper we are considering risk with more than one prior estimate of the frequency, e.g. environmental health risk of new and little known epidemics, or risk induced by scientific uncertainty in predicting the frequency and severity of catastrophic events. It is not possible to hedge this kind of risk using only traditional insurance practice. A new method is called for.

This problem was first mentioned by Chichilnisky and Heal in [15] (a non mathematical paper), where they argued that this unknown risk should be managed by using traditional insurance practice and by trading in the security market simultaneously. In this article we will continue and extend the ideas from [15]. The main purpose is to build a mathematical model that is able to handle this and related problems.

In the paper we extend the ideas from [15] by considering both complete and incomplete markets. Furthermore we consider the case where the premium charged by the insurance company is restricted. In this case the insurance company has to choose an allocation of the restricted premium corresponding to the states of the world. We propose four different methods for solving this problem. These four methods are then analysed and evaluated; advantages and disadvantages are illustrated by examples.

Summaries of Manuscripts in the Thesis (Danish)

Afhandlingen indeholder følgende fem artikler:

- Christensen, Claus Vorm and Hanspeter Schmidli, "Prisfastsættelse af katastrofeforsikrings produkter baseret på aktuelle, rapporterede skader", *(udkommer i Insurance, Mathematics and Economics).*
- Christensen, Claus Vorm, "PCS-optionen, en forbedring af CAT-futuren" *(Manuskript, Aarhus Universitet).*
- Christensen, Claus Vorm, "En ny model til prisfastsættelse af katastrofeforsikrings derivater", *(Working Paper Serie No. 28, Center for Analytisk Finansiering.)*
- Christensen, Claus Vorm, "Markedsudledte skadesfordelinger for katastrofeforsikrings derivater", *(Manuskript, Aarhus Universitet).*
- Christensen, Claus Vorm, "Hvordan man afdække ukendt risiko", *(Manuskript, Aarhus Universitet).*

Herunder følger et kort referat af indholdet i de fem artikler.

Accepterede artikler

Christensen, Claus Vorm and Hanspeter Schmidli (1998), "Prisfastsættelse af katastrofeforsikrings produkter baseret på aktuelle rapporterede skader", *(udkommer i Insurance, Mathematics and Economics).*

Resumé: Denne artikel omhandler problemer ved prisfastsættelse af et finansielt aktiv, som beror på et index af rapporterede skader fra katastrofeforsikringer. Problemet ved at prisfastsætte sådanne produkt er, at man på et givet tidspunkt i handelsperioden ikke kender det totale skadesbeløb fra allerede indtrufne katastrofer. Man skal derfor prisfastsætte produktet udelukkende ud fra kendskabet til beløbet af de rapporterede skader til et givet tidspunkt. Denne artikel anviser en måde hvorpå man kan løse dette problem. Hovedideen i artiklen er at modellere de samlede skader fra en katastrofe som en sammensat (blandet) Poisson model. Vi opnår derved muligheden for at separere de individuelle skader og modellere rapporteringstidspunkterne for skaderne. Denne nye model og en illustration af hvordan man kan beregne priser i modellen er hovedbudskabet i artiklen.

Working Papers

"PCS-optionen, en forbedring af CAT-futuren", *(Manuskript, Aarhus Universitet).*
I 1992 indførte CBoT CAT-futuren som et alternativ til katastrofe genforsikring. Men produktet blev aldrig rigtigt populært, så i 1995 blev det erstattet af PCS-optionen. I relation til mit PhD studie, startede jeg derfor med at indsamle information om PCS-optionen. Den mest basale information stammer fra artiklerne [14] og [55]. Derudover blev en del information indhentet via personlig kontakt ved ansatte ved PCS og CBoT. Efter at have indhentet denne information fandt jeg det naturligt, at samle den i en artikel. Det er gjort i denne artikel, som også forklarer hvorfor CAT-futuren blev erstattet af PCS-optionen og hvorfor PCS-optionen er en forbedring. Artiklen forklarer også hvordan man afdækker katastroferisiko og sammenligner PCS-optionen med almindelig genforsikring.

"En ny model til prisfastsættelse af katastrofeforsikrings derivater", *(Working Paper Serie No. 28, Center for Analytisk Finansiering).*
Siden finansielle katastrofe forsikringsprodukter blev indført i 1992, har det været et problem hvordan disse produkter skulle prisfastsættes. Der har været følgende to hovedproblemer. For det første, hvis vi vælger en realistisk model for den underliggende tabsproces, så bliver markedet ufuldstændigt og som følge heraf eksisterer der mange ækvivalente martingalmål. Dette betyder at der eksisterer mange arbitrage frie priser på produktet. Det andet problem er at vi gerne vil have at fordelingen for det underliggende tabs index har en tung hale, men tunge haler giver ofte beregningsmæssige problemer. Det har derfor været naturligt at lede efter en model der løser begge disse problemer. I denne artikel præsenterer vi en model, der i en vis henseende tager højde for begge disse probelmer. Modellen er inspireret af resultater fra Gerber og Shiu [43]. I [43] er det vist at Esscher transformen er en entydig og umiddelbar teknik for værdifastsættelse af aktiver, hvor logaritmen af den underliggende proces er styret af en bestemt stokastisk proces med stationære og uafhængige tilvækster (en Levy proces). I denne artikel foreslår vi en sådan model, og som eksempel beregner vi prisen for PCS-optionen. Metoden kan også benyttes til prisfastsættelse af andre aktiver, der beror på et katastrofetabs index.

"Markeds udledte skadesfordelinger for katastrofeforsikrings derivater", *(Manuskript, Aarhus Universitet).*
I denne artikel forsøger vi også at prisfastsætte katastrofe forsikrings derivater, men vi drejer nu vores analyse i en anden retning end i

de tidligere artikler. Vi benytter en metode, som også er kendt fra det almindelige optionsmarked, og som også er foreslået af Lane and Movchan in [46]. I stedet for at estimere volatiliteter og beregne konsistente priser ved hjælp af for eksempel Black Scholes, så tages her udgangspunkt i de handlede priser og derfra udledes den volatilitet, der er konsistent med disse priser, dvs. den markedsudledte volatilitet. Vi kan ikke bruge den samme procedure direkte på katastrofeforsikrings derivater, idet det typisk ikke er muligt at karakterisere prisen for disse ved hjælp af en enkelt parameter. Vi kan derimod gøre noget tilsvarende. Vi kan vælge en model for de markedsudledte tabsfordelinger, og så estimere de markedsudledte parametre hørende til disse fordelinger ud fra de observerede priser.

Denne analyse kan så bruges til at evaluere forskellige priser på forskellige katastrofeforsikrings derivater. Vi beregner simpelthen bare de implicitte priser ud fra de markedsudledte skadesfordelinger, og sammenligner dem med de observerede priser. Der er to hovedproblemer i denne analyse. For det første, hvilken fordeling skal vi vælge for den markedsudledte skadesfordeling og for det andet, hvordan skal de involverede parametre estimeres? I denne artikel analyserer vi disse to problemer og foreslr en ny skadesfordeling samt en ny objektfunktion til parameterestimation.

"Hvordan man afdækker ukendt risiko", *(Manuskript, Aarhus Universitet).*

I denne artikel betragter vi risiko med mere end et indledende estimat af risikoens frekvens, eksempelvis miljømæssig sundhedsrisiko ved nye og ukendte epidemier, eller risikoen ved den videnskabelige usikkerhed forbundet med at forudsige frekvensen og styrken af en given naturkatastrofe. Det er ikke muligt at afdække denne risiko udelukkende ved at benytte traditionel forsikringspraksis. Der er behov for en ny metode.

Dette problem blev første gang nævnt af Chichilnisky og Heal i [15] (en ikke matematisk artikel), hvor de argumenterede for, at det skulle være muligt at styre denne form for risiko ved simultant at benytte traditionel forsikringspraksis og handle på det finansielle marked. I denne artikel vil vi tage udgangspunkt i [15] og videreudvikle ideerne derfra. Hovedformålet med artiklen er at konstruere en matematisk model som er i stand til at løse dette og tilhørende problemer.

I artiklen udvider vi ideerne fra [15] ved at betragte både fuldstændige og de ufuldstændige finansielle markeder. Derudover betragter vi også den situation, hvor forsikringsselskabet kun kan forlange en begrænset præmie. I det tilfælde er forsikringsselskabet nødt til at

vælge en allokering af den begrænsede præmie hørende til de forskellige mulige tilstande. Vi foreslår fire forskellige måder at løse dette problem på. Disse fire metoder bliver så analyseret og evalueret og ved eksempler bliver fordele og ulemper skitseret.

1. Introduction

During the nineties a highly discussed theme among academics has been the interplay between insurance and finance. Some of the general issues have been: the increasing collaboration between insurance companies and banks; the discussions about risk management methodologies for financial institutions, and the emergence of finance related insurance products, e.g. catastrophe futures and options, PCS options, index-linked policies,........

This thesis develops around the interplay between insurance and finance and especially around the pricing of finance related insurance products (insurance derivatives). The insurance derivative was developed as a financial product which should work as an alternative or replacement of reinsurance. This meant that companies that would normally reduce their risk by reinsurance, could now consider these new financial products as alternatives. One of the main differences between the traditional concept of reinsurance and these new products, is the way they are priced. Reinsurance contracts are priced using traditional actuarial methods, whereas derivatives should be priced by financial methods of no arbitrage. To give an impression of the differences between these two methods of pricing, we start this second major section with a description of the two methods of pricing and their interaction.

The rest of this second section of the thesis, gives a chronological description of how the market for catastrophe insurance products has developed, with a special focus on the pricing approach. Here relevant literature is described and it is explained how my work relates to, contributes to, and extends various fields.

2. The difference between financial and actuarial pricing

As mentioned above I found it appropriate to stress how financial and actuarial pricing are related to one another. This is done by first describing the pricing procedures in insurance, then the pricing procedures in finance and finally by making some remarks on the interaction. The following subsections are closely related to the nice paper [30], but are also based on theory from [28], [32], [52] and [53].

2.1. **Insurance pricing.** Let the annual premium for a certain risk be π, denote by X_i the losses in year i and assume that the X_i's are independent and identically distributed. The company has a certain initial capital u. Then the capital of the company after year i is

$$C_i = u + \pi i - \sum_{j=1}^{i} X_j$$

It is well-known, see e.g. [53], that only if $\pi > E_P[X_i]$ there is a positive probability that $C_i \geq 0$ for all $i \in \mathbb{N}$, i.e. one should be prepared to pay more than $E[X_i]$ (a safety loading is added). That agents are in fact willing to pay more than $E_P[X_i]$ can be shown by looking at utility theory. Consider an agent who is going to buy an insurance to cover the losses X_i. Assume that the agent has initial capital k and utility function w, where $w' > 0$ (more is better) and $w'' < 0$ (decreasing marginal utility) and that $\mathrm{Var}_P[X_i] \neq 0$. Then the agent is willing to pay the premium $\tilde{\pi}$ defined by the equation

$$w(k - \tilde{\pi}) = E_P[w(k - X_i)].$$

The definition of w implies that $-w$ is convex and therefore Jensen's inequality immediately leads to

$$\tilde{\pi} > E_P[X_i].$$

An insurance contract between the agent and the insurer is now called feasible whenever

$$\tilde{\pi} \geq \pi > E_P[X_i].$$

One can now choose among various well-known premium principles for the valuation of the premium. We will now describe some of these premium calculation principles for the risk X.

- The expected value principle

$$\pi = E_P[X] + \delta E_P[X]$$

- The variance principle

$$\pi = E_P[X] + \delta \mathrm{Var}_P[X]$$

- The standard deviation principle

$$\pi = E_P[X] + \delta (\mathrm{Var}_P[X])^{1/2}$$

The loading factor δ is often determined by setting sufficiently protective solvency margins which may be derived from ruin estimates of the underlying risk process over a given (finite) period of time.

- The exponential principle: Assume that the insurance company has initial capital h and utility function v, where $v' > 0$ and

$v'' < 0$. The premium π for the insurance company can be defined by the equation

$$v(h) = E_P[v(h + \pi - X)].$$

If the insurance company has an exponential utility function, i.e. $v(x) = 1 - e^{-\delta x}$ we, obtain by the above definition the exponential principle

$$\pi = \frac{1}{\delta} \log E_P[e^{\delta X}].$$

- The percentage principle: The company wants to keep the probability that the risk exceeds the premium income small. The company therefore chooses a parameter ϵ and defines the premium by

$$\pi = \inf\{y > 0 : P[X > y] \leq \epsilon\}.$$

- The Esscher principle:

$$\pi = \frac{E_P[Xe^{\delta X}]}{E_P[e^{\delta X}]}.$$

for an appropriate $\delta > 0$. An economic foundation for the Esscher principle, using risk exchange and equilibrium pricing has been given by Bühlmann in [9].

2.2. **Pricing in finance.** When we change from the pricing in insurance to the pricing in finance, the risk X typically becomes a contingent claim. Let us consider some examples. Let $(S_t)_{0\leq t\leq T}$ denote the underlying price process of some traded asset. The risk of a European call with strike K and maturity T is then

$$X = (S_T - K)^+$$

Note that this contract is similar to an excess-of-loss reinsurance treaty with priority K. Another example is the Asian option with strike price K, which specifies the payoff

$$X = (\frac{1}{T}\int_0^T S_u du - K)^+.$$

This contract is similar to the stop-loss treaty in reinsurance.

In the finance context, the argument against using $E_P[X]$ as the premium is based on the notion of no-arbitrage. The correct price at time t of a contingent claim X, in a no-arbitrage framework with risk-free interest rate r, is

$$v_t = E_Q[e^{-r(T-t)}X|\mathcal{F}_t]$$

and the premium to be charged at time $t = 0$

$$v_0 = E_Q[e^{-rT)}X],$$

see [28] for further details. We calculate the fair premium with respect to another probability measure Q. This risk neutral probability measure Q changes the original measure P in order to give more weight to unfavorable events in a risk averse environment. In financial economics this leads to the concept of "the market price of risk" and in insurance mathematics it should explain the safety loading.

In complete markets Q is the unique P-equivalent probability measure which turns $e^{-rt}S_t$ into a martingale. In incomplete models Q is not unique, and without further information on investor specific preferences only bounds on prices can be given.

Examples of complete models:

- Geometric Brownian motions (Black-Scholes),
- multi-dimensional geometric Brownian motions,
- $(N_t - \lambda t)_{t \geq 0}$ with N_t a homogeneous Poisson process with intensity λ, and
- square integrable point process martingales $(N_t - \int_0^t \lambda_s ds)_{t \geq 0}$, for deterministic λ.

Examples of incomplete models:

- Stochastic volatility models with unhedgable volatility risk, and
- processes with jumps of random size (e.g. compound Poisson processes and general jump diffusions).

If one chooses a martingale measure in the incomplete market one will at the same time choose the weights for the different risks and then implicitly the markets utility function, i.e. one could argue that it would be natural to work out the incomplete market price in a utility maximization framework. If one does so, a unique measure emerges in a very natural way, see [32] and references therein. Other important references for readers interested in incomplete markets are [37] and [38].

2.3. **The intersection between insurance and finance.** The classical risk process, being defined as a compound Poisson process, is traditionally used as a model for insurance business. And as we have seen in Subsection 2.1 the premium to be asked per unit of time is defined as the expectation plus some loading factor. It could then be interesting to investigate whether the financial approach from the previous subsection could be used to calculate premiums for risk processes, and how these premiums are related to the premiums obtained in Subsection 2.1. This is exactly the aim of the paper [27] by Delbaen and Haezendonck. Let us now outline the method introduced in this paper.

For a given finite time horizon $[0;T]$ we consider a company holding the risk process L given by a compound Poisson process

$$L_t = \sum_{i=1}^{N_t} Y_i$$

where the Y_i's are independent and identically distributed positive claims with common distribution function F. N_t is a homogeneous Poisson process with intensity $\lambda > 0$ and independent of $(Y_i)_{i \geq 1}$.

Let $(\tilde{S}_t)_{0 \leq t \leq T}$ be the discounted price process for the claim L_T. Delbaen and Haezendonck then at this point conclude, that the liquidity of the market makes it reasonable to assume that the market is arbitrage free, i.e. there exists a risk neutral probability measure under which the discounted price process $(\tilde{S}_t)_{0 \leq t \leq T}$ is an $\mathcal{F}_t$-martingale. Thus:

$$\tilde{S}_t = E_Q[L_T | \mathcal{F}_t], \quad 0 \leq t \leq T.$$

Suppose that at each time t, the company can sell the remaining risk of the period $]t;T]$ for a given (predictable) premium π_t. Since π_t is a premium that admits no arbitrage, it is determined as

$$\begin{aligned} \pi_t &= E_Q[L_T - L_t | \mathcal{F}_t], \quad 0 \leq t \leq T, \\ &= E_Q[L_T | \mathcal{F}_t] - L_t, \quad 0 \leq t \leq T. \end{aligned}$$

Hence, the underlying price process $(\tilde{S}_t)_{0 \leq t \leq T}$ can be decomposed into

$$\tilde{S}_t = \pi_t + L_t \quad 0 \leq t \leq T.$$

or in other words, the company's liabilities $\tilde{S}_t$ at time t consist of the claims up to time t and the premium for the remaining risk $L_T - L_t$.

If one further more imposes that

$$\pi_t = \eta(T - t) \quad 0 \leq t \leq T,$$

where η is a premium density, i.e. the premium is linear in time, then we obtain that under Q the risk process L_t remains a compound Poisson process. We therefore consider only those equivalent martingale measures Q that preserve the compound Poisson property of L_t within this non-arbitrage insurance context. A viable premium density then takes the form

$$\eta_Q = E_Q[L_1] = E_Q[N_1] E_Q[Y_1],$$

resulting in change of both claim-size and claim-intensity of the underlying process. It is then shown that the Q-measures giving rise to such viable premium principles have the following properties (formulated in terms of distribution functions):

$$F_Q^{(\beta)}(x) = \frac{1}{E_P[\exp(\beta(Y_1))]} \int_0^x e^{\beta(y)} dF(y), x \geq 0$$

where $\beta : \mathbb{R}^+ \to \mathbb{R}$ is increasing so that

$$E_P[\exp(\beta(Y_1))] < \infty \text{ and } E_P[Y_1 \exp(\beta(Y_1))] < \infty$$

The resulting premium density then satisfies for $\beta(y) \geq 0$,

$$\eta_P = E_P[N_1]E_P[Y_1] < \eta_{Q(\beta)} < \infty$$

hence taking safety loading into account. Special choices of β now lead to special premium principles, all consistent within the no-arbitrage set-up. Examples are:

- $\beta(x) = \alpha > 0$, then
  $\eta_{Q(\beta)} = e^\alpha E_P[N_1]E_P[Y_1] = e^\alpha \lambda E_P[Y_1]$ (the expected value principle);
- $\beta(x) = \log(a + bx)$, $0 < b < (E_P[Y_1])^{-1}$ and $a = 1 - bE_P[Y_1] > 0$, then
  $\eta_{Q(\beta)} = \lambda(E_P[Y_1] + b\mathrm{Var}_P[Y_1])$ (the variance principle);
- $\beta(x) = \alpha x - \log E_P[e^{\alpha Y_1}]$, $\alpha > 0$, then
  $\eta_{Q(\beta)} = \lambda \frac{E_P[Y_1 \exp(\alpha Y_1)]}{E_P[\exp(\alpha Y_1)]}$ (the Esscher principle).

So, in a sufficiently liquid insurance market, classical insurance premium principles can be reinterpreted in a standard no-arbitrage pricing set-up. The main results from [27] are generalized in [47] to be valid for mixed Poisson and doubly stochastic Poisson processes as well.

Consider now a contingent claim based on a loss process. One could then derive arbitrage-free prices of such contingent claims on the basis of the risk-neutral probability measure derived above. This is done in [4] where prices are calculated for different risk-neutral measures. The prices are calculated by solving integro-differential equations for the contingent claims numerically. But the risk neutral measure is not unique, so there still exist several possibilities to price these contingent claims excluding arbitrage opportunities. A natural way to choose a specific measure could then be, as argued above, to work out the incomplete market price in a utility maximization framework. And as stated above a unique measure emerges in a very natural way. We will discuss this further in the next section.

3. Securitization, the CAT-future

Let us start this section by listing the 10 most costly insurance losses in the period 1970-1999. The data can be found in sigma [58] (the insured losses are in USD millions at 1999 prices).

Date	Event	loss
24.08.1992	Hurricane “Andrew”, USA	19086
17.01.1994	Northridge earthquake, Calif.	14122
27.09.1991	Typhoon “Mireille”, Japan	6906
25.01.1990	Winter storm “Daria”, Europe	5882
15.09.1989	Hurricane “Hugo”, Puerto Rico	5664
25.12.1999	Winter storm “Lothar”, Europe	4500
15.10.1987	Autumn storm, Europe	4415
26.02.1990	Winter storm “Vivian”, Europe	4088
20.08.1998	Hurricane “George”, USA	3622
22.09.1999	Typhoon “Bart” hits south Japan	2980

Table 3.1. The 10 most costly insurance losses (1970-1999).

From table 3.1 it is seen that all 10 events have happened during the second half of the period. One therefore gets the impression that the frequency and severity of large losses has increased, which is also confirmed in [58]. This increase is due to higher population densities, more insured values in endangered areas and higher concentration of values in industrialized countries.

The insurance industry already got the impression of this increase in the early nineties after hurricane Andrew and the Northridge earthquake. And they soon realized that the reinsurance industry might lack the capital to cover the huge catastrophes of the future. To solve this problem *securitization of catastrophe risk* was invented. The idea of securitization was to transfer some of the risk from the insurance market to the financial market, where the risk-bearing capacity is much larger. This transfer should then be done by use of financial instruments such as options and futures on indices of catastrophe losses or catastrophe bonds. Catastrophe bonds are bonds where the payment of the coupon and/or the return of the principal of the bond is linked to the non-occurrence of a specified catastrophic event. In the rest of the thesis we will denote these financial instruments as *catastrophe insurance derivatives.*

We now give a short description of one of the first products of this kind, namely the CAT-future introduced by CBoT in 1992.

3.1. **Description of the CAT-future.** CAT-futures are traded at CBoT on a quarterly cycle, with contract months March, June, September and December. A contract for a calendar quarter (called the event quarter) is based on losses occurring in the listed quarter and being reported to the participating companies by the end of the following quarter. A contract also specifies an area and the type of claim to be taken into account. The additional three months following the reporting period is attributable to data processing lags. The six months period following the start of the event quarter is called reporting period. The three reporting months following the event quarter are to allow for settlement lags that are usual in insurance. The contracts expire on the fifth day of the fourth month following the end of the reporting period. Let $T_1 < T_2$ be the end of the event quarter and the end of the reporting period, respectively.

The settlement value of the contract is determined by a loss index; the ISO-index. Let us now consider the index. Each quarter approximately 100 American insurance companies report property loss data to the ISO (Insurance Service Office, a well-known statistical agent). ISO then selects a pool of at least ten of these companies on the basis of size, diversity of business, and quality of reported data. The ISO-index is calculated as the loss-ratio of this pool

$$\text{ISO-index} = \frac{\text{reported incurred losses}}{\text{earned premiums}}.$$

The list of companies included in the pool is announced by the CBoT prior to the beginning of the trading period for that contract. The CBoT also announces the premium volume for companies participating in the pool prior to the start of the trading period. Thus the premium in the pool is a known constant throughout the trading period, and price changes are attributable solely to changes in the market's expectation of loss liabilities.

The settlement value for the CAT-futures is

$$F_{T_2} = 25\,000 \times \min(I_{T_2}, 2)$$

where I_{T_2} is the ISO-index at the end of the reporting period, i.e. the ratio between the losses incurred during the event quarter and reported up till three months later and the premium volume for the companies participating in the pool.

Example 3.1. The June contract covers losses from events occurring in April, May and June and are reported to the participating companies by the end of September. The June contract expires on January 5th the following year. The contract is illustrated by Figure 3.1.

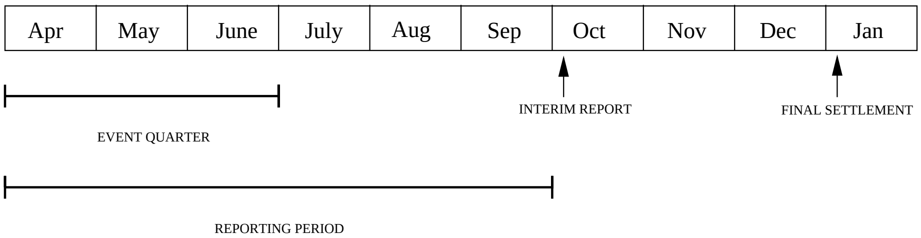

FIGURE 3.1. June CAT-Future contract

3.2. **The CAT-future pricing problem.** Since the introduction of the CAT-future in 1992, it has been a highly discussed theme among academics how these catastrophe insurance derivatives should be priced. It has not been possible to find a unique model like the Black-Scholes model since the underlying index cannot be described by a distribution as simple as the log-normal, and furthermore, the underlying is not a traded asset.

One of the first attempts to price these derivatives was done by Cummins and Geman in [24]. This paper develops an Asian option model for the pricing of the CAT-future. They argue that the Asian approach is appropriate since most insurance contracts, including the CAT-future, have pay-offs defined in terms of claims accumulations rather than the end-of-period values of the underlying state variables. For the underlying index [24] use the model

$$L_T = \int_0^T S(s)ds$$

where

$$dS(t) = \alpha S(t_-)dt + \sigma S(t_-)dW(t) + kdN(t).$$

In order to price the CAT-future they then assume that there exist two other securities on the market from which one can derive the behavior of the processes $W(t)$ and $N(t)$ under the Q-measure. Under these assumptions they are able to price the CAT-future by use of techniques arising from pricing Asian options.

The model, however, seems to be far from reality. At times where a catastrophe occurs or shortly thereafter, one would expect a strong increase of the loss index. It is therefore preferable to use a marked point process as it is popular in actuarial mathematics. But if we look after such a model we will also have to look for another pricing approach, since we are then no longer able to price by no arbitrage.

A model of this type was suggested by Aase in [1]. Here a compound Poisson model is used. This can be seen as catastrophes occurring at certain times and claims being reported immediately. In such a model there would be no need for the prolonged reporting period. In [1] a closed form pricing model is derived in the framework of general economic equilibrium theory under uncertainty.

An improvement of this model is the model suggested by Embrechts and Meister in [32]. In this case a doubly stochastic Poisson model is introduced. Here, a high intensity level will occur shortly after a catastrophe, where more claims are expected to be reported. For the pricing of the derivative, a utility and risk-minimization approach is used, and in the case of the exponential utility function, we obtain the following pricing formula. Let F_t be the price of the future, L_t the value of the losses occurred in the event quarter and reported till time t, $\mathcal{F}_t$ the information at time t, Π premiums earned and let $c = 25\,000/\Pi$. Then the price at time t, is (see [32, p.19])

$$F_t = c\frac{E_P[\exp(\alpha L_\infty)\,(L_{T_2} \wedge 2\Pi) \mid \mathcal{F}_t]}{E_P[\exp(\alpha L_\infty) \mid \mathcal{F}_t]}. \tag{3.1}$$

In particular, $E_P[\exp(\alpha L_\infty)]$ has to exist. The market will determine the risk aversion coefficient α. This pricing formula is also the one we use in Christensen and Schmidli [17]. In the next section we outline the main results from [17].

3.3. **Pricing based on actually reported claims.** One of the problems with pricing these financial products relying on indices of reported claims from catastrophe insurance is that, at a fixed time in the trading period, the total claim amount from the catastrophes occurred is not known. One therefore has to price these products solely from knowing the aggregate amount of the reported claims at the fixed time point. The idea of the manuscript [17] was to derive a pricing model that was only based on the actually reported claims and thereby extend the existing pricing models for products of this kind.

The main idea of [17] is to model the aggregate claim from a single catastrophe as a compound (mixed) Poisson model. We thereby obtain the possibility to separate the individual claims and to model the reporting times of the claims. In [17] it is shown how prices can be calculated within this new model. For pricing the catastrophe insurance futures and options we use the exponential utility approach of [32]. This approach will only work for aggregate claims with an exponentially decreasing tail. But data give evidence that the distribution tail of the aggregate claims is heavy tailed. In our model a heavy-tail can be obtained by a heavy-tailed distribution for the number of individual claims of a single catastrophe. For pricing we approximate the claim number distribution by a negative binomial distribution; more precisely, by a mixed Poisson distribution with a $\Gamma(\gamma, \eta)$-mixing distribution. Choosing γ and η small a heavy-tail behaviour can be approximated. The reader should note that the value of the security is based on a capped index and therefore has an upper bound. This justifies the light-tail approximation.

3.3.1. *The model.* The model looks as follows. Catastrophes occur at times $\tau_1 \leq \tau_2 \leq \tau_3 \ldots$ The i-th catastrophe produces M_i claims with sizes Y_{ij}. The j-th claim is then reported with lag D_{ij} ($D_{ij} \sim F_D$), i.e. at time $\tau_i + D_{ij}$. Furthermore let $M_i(t)$ be the number of claims from catastrophe i reported until time t. In our model the claims Y_{ij} from the i-th catastrophe are randomly ordered. This simplifies the modeling of the reporting lags D_{ij}. Let $(D_{i:j} : 1 \leq j \leq M_i)$ be the order statistics of the $(D_{ij})_{1 \leq j \leq M_i}$, and $Y_{i:j}$ be the claim corresponding to $D_{i:j}$. Then the claims occurred before T_1 and reported till $t \leq T_2$ amount to

$$L_t = \sum_{i=1}^{N_{t \wedge T_1}} \sum_{j=1}^{M_i(t)} Y_{i:j} .$$

In particular, the final aggregate amount, L_{T_2}, can be represented as

$$L_{T_2} = L_t + \sum_{i=1}^{N_{t \wedge T_1}} \sum_{j=M_i(t)+1}^{M_i(T_2)} Y_{i:j} + \sum_{i=N_{t \wedge T_1}+1}^{N_{T_1}} \sum_{j=1}^{M_i(T_2)} Y_{i:j} ,$$

i.e. the final aggregate amount L_{T_2} can at time t be represented as a sum of the claims that has already occurred and already been reported, plus a sum of the claims that already occurred and will be reported before the end of the reporting period, plus a sum of claims that will occur before the end of the loss period and be reported before the end of the reporting period. Based on the assumption that M_i is mixed Poisson distributed we can show that $M_i(t)$ also is mixed Poisson distributed but with a different parameter dependent on the old parameter and the distribution of the reporting lags, see [17] for further details.

Under, some additional assumptions, see [17], we then show that $L_{T_2} - L_t$ is a compound Poisson sum, and we find the distribution of the parameter. This parameter depends on the (non-observable) mean claim number of the i-th catastrophe (λ_i). We therefore work with two models. First we work with a simple model assuming that λ_i is deterministic, i.e. all catastrophes have the same mean claim number. Next we work with a model where the λ_i is stochastic, i.e. we allow the parameter to depend on the information from the claims having occurred in the past.

We now know the distribution of the underlying loss process, and in order to price the CAT-future we follow the approach of Embrechts and Meister [32], or more precisely the pricing formula (3.1). The term $\exp(\alpha L_\infty)/E_P[\exp(\alpha L_\infty)]$ from (3.1) is strictly positive and integrates to one. Thus it is the Radon-Nikodym derivative dQ/dP of an equivalent measure. In the specific model we will consider, the process (L_t) follows the same model (but to be considered with different parameters) under the measure Q as under the measure P. For the exact behavior of L_t under Q, see [17]. We will use this fact to calculate the price.

3.3.2. *The pricing of the CAT-future.* Denoting by $\tilde{F}_L(\cdot;t)$ the distribution function of $L_{T_2} - L_t$ under Q conditioned on $\mathcal{F}_t$. We can then express the price at time t of the CAT-future as

$$
\begin{aligned}
& cE_Q[L_{T_2} \wedge 2\Pi | \mathcal{F}_t] \\
&\quad = c(L_t + E_Q[(L_{T_2} - L_t) - ((L_{T_2} - L_t) - (2\Pi - L_t))^+ \mid \mathcal{F}_t]) \\
&\quad = c\Big(L_t + E_Q[(L_{T_2} - L_t) \mid \mathcal{F}_t] - \int_{2\Pi - L_t}^{\infty} (1 - \tilde{F}_L(x;t))\, dx\Big). \qquad (3.2)
\end{aligned}
$$

The problem with the above expression, however, is that we have to find the n-fold convolutions of $F_D[T_2 - \tilde{\tau}]$ ($\tilde{\tau}$ is uniformly distributed on (t, T_1), in order to calculate the last term. To find an explicit expression seems to be hard.

Historical data show that so far, the cap 2 in the definition of the CAT-future has not been reached. The largest loss ratio was hurricane Andrew with $L_\infty = 1.79\Pi$. Under the measure P we have that $\{L_{T_2} > 2\Pi\}$ is a rare event. Since we are dealing with catastrophe insurance, the market risk aversion coefficient α cannot be large. Otherwise catastrophe insurance would not be possible. We therefore assume that $\{L_{T_2} > 2\Pi\}$ is also a rare event with respect to the measure Q, see also [32]. The light tail approximation to our model then assures that the tail of $\tilde{F}_L(\cdot;t)$ is exponentially decreasing. That is $\int_{2\Pi - L_t}^{\infty}(1 - \tilde{F}_L(x;t))\, dx$ will be small as long as $Q(L_{T_2} > 2\Pi)$ is small. The latter of course depending on the risk aversion coefficient α, being small in order to be able to neglect the last term. As in [32] we therefore propose the approximation $c(L_t + E_Q[(L_{T_2} - L_t) \mid \mathcal{F}_t])$ to the price of the CAT-future. For the exact calculation of this expression, see [17]. The final result is also stated here in Theorem 3.2.

In the extended model we assume that the λ_i's are stochastic. This can be seen as a measure of the severity of the catastrophe. For simplicity of the model, we assume that λ^i can be observed via reported claims only. Of course, in reality other information as TV-pictures or reports from the affected area will be available. Then for claims occurring before t we have some information on the intensity parameter λ^i. We therefore have to work with the posterior distribution of λ^i given $\mathcal{F}_t$. It would be desirable if the prior and the posterior distribution would belong to the same class, see the discussion in [21, Ch.10]. We therefore choose λ_i to be Γ distributed. Let $\lambda^i \sim \Gamma(\gamma, \eta)$. Under these assumptions we again calculate an approximation for the price of the CAT-future, see [17] for an exact calculation. The final result is stated here in Theorem 3.6.

The above results are both approximations, so it is relevant to ask how good these approximations are. This question is also considered in [17]. From equation 3.2 we know that the approximation error is given by the following expression:

$$c\Big(\int_{2\Pi - L_t}^{\infty} (1 - \tilde{F}_L(x;t))\, dx\Big) \tag{3.3}$$

where $\tilde{F}_L(\cdot;t)$ denotes the distribution function of $L_{T_2} - L_t$ under Q conditioned on $\mathcal{F}_t$. The reason for omitting this term was that it is hard to calculate $\tilde{F}_L(\cdot;t)$. In [17] we find an approximation to the expression above by using some of the approximations to $L_{T_2} - L_t$ known from actuarial mathematics, namely the translated gamma approximation and the Edgeworth approximation. We use these two approximations as examples which both are shown to be useful but other approximations may also possible.

The results in [17] are derived specially for the CAT-futures, even though an improved financial catastrophe insurance product, the PCS-option, was introduced in 1995. In the next section we will consider the PCS-option and discuss why the CAT-future was replaced. The results from [17] cannot directly be used for pricing PCS-options since they have a different structure. However, because of a strong correlation between claims reported and the PCS-index, some of the ideas may be used. We now turn to the PCS-option.

4. Securitization, the PCS-option

In 1995 CBoT introduced the PCS-option as a replacement of the CAT-future. Let us briefly describe the PCS-option before we explain why modifications of the CAT-future was needed.

4.1. **Description of the PCS-option.** In this section the definitions of the keywords that specify the PCS-options are given. For a more detailed description of the PCS-option see [14] or [16].

Property Claim Services (PCS), a division of American Insurance Services Group, is the recognized industry authority for catastrophe property damage estimates. PCS is a not-for-profit organization serving the insurance industry.

The PCS-options are traded by Chicago Board of Trade and are regional contracts whose value is tied to the so-called PCS Index. The PCS index tracks PCS estimates for insured industry losses resulting from catastrophic events (as identified by PCS) in the area and loss period covered. The options are traded as capped contracts, i.e. the cap limits the amount of losses that can be included under each contract. The value of a PCS call option at expiration day T, with exercise price A and cap value K is given by

$$C(T, L_T) = \min(\max(L_T - A, 0), K - A)$$

where L_T is the value of the PCS index at time T.

PCS-options can be traded as European calls, puts or spreads. Most of the trading activity occurs in call-spreads, since they essentially work like aggregate excess-of-loss reinsurance agreements, see [16] for further explanations.

The option includes both a loss period and a development period. The **loss period** is the time during which a catastrophic event must occur in order for resulting losses to be included in a particular index. During the loss period, PCS provides loss estimates as catastrophes occur. The **development period** is the time after the loss period during which PCS continues to estimate and reestimate losses from catastrophes occurred during the loss period. The reestimations may result (and have resulted historically) in adjustments upwards and downwards. PCS-option users can choose either a six-month or twelve-month development period. The settlement value for each index represents the sum of then-current PCS insured loss estimates provided and revised over the loss and development periods.

4.2. **Why the PCS-option was an improvement.** The main problem of the CAT-future was caused by the construction of the underlying index, the ISO index. Let I_t be the value of the ISO index at time t. One of the problems was that I_t was only published once before the settlement date. This took place just after the end of the reporting period (the Interim report, see Figure 3.1). This meant that the companies participating in the pool had a possibility of knowing at least part of the data used to form the index before the settlement date, while it was certainly more difficult for other insurers. This created an information asymmetry which was a potential factor preventing people from entering the market of CAT-futures. This problem was solved for the PCS-option, because PCS reports the PCS loss indices on each CBoT trading day (the index is only changed if there are new catastrophes or if the index is adjusted) and neither American Insurance Services Group nor any person employed by American Insurance Services Group will disclose any estimate of total insured losses following a catastrophe to any person prior to its official publication. This means that all investors receive the same information at the same time. Thereby the problem of asymmetric information is eliminated.

Another problem was the Moral Hazard problem. A company from the pool could manipulate data by delaying the report of a big loss so that it first would be included in the next reporting period and thereby never affect the index. The company's intension for doing so, could be that the company had agreed to a short position of a future contract[1]. That this possibility existed, could also have prevented people from entering the market of CAT-futures. This problem was also solved by PCS conducting surveys of the market, when they estimate the loss indices. These surveys are confidential and they are not used directly in the estimation of the indices. So it is extremely difficult for insurance companies to affect the indices, and thereby the Moral Hazard Problem was eliminated.

A more serious problem could occur due to the reporting period being too short. If a late quarter catastrophe occurs and claims are slow in developing, then the final claims ratio for the purpose of deciding the future payoff could be low relative to the actual final claims ratio. This problem occurred in the March 1994 contract period, the period of the Northridge earthquake. The settlement ratio was low and the contract pay-off did not truly reflect the actual claim loss. The construction of the PCS-option also solves this problem. The PCS index does not directly depend on a number of reported claims and the time from the end of the event period to the time the index is settled is also longer for the PCS-option than it was for the CAT-future.

These problems being solved, was probably the main reason for the PCS-options higher trading activity compared to the CAT-future. But the fact that the new product was more logically constructed than the old one, could also have had an effect. Hereby we mean that a

[1]When a company, at time t and at a price F_t, enters a short position of a future contract, it means that the company should pay $(F_T - F_t)$ to the other part of the contract at time T. The company will then like F_T to be as small as possible.

construction using options instead of futures and "options on futures", seems more logical when all the trading activities are in options.

In fact, the CBoT market has never progressed to a well launched market. However, the market is still developing, and competitors such as The Bermuda Commodities Exchange and The Catastrophe Risk Exchange are beginning to have some trading success, see [46]. J.A. Tilley [60] is mentioning the following four reasons why this market is emerging so slowly. First, since 1994 there has been a generally favorable catastrophe loss experience and as a result of this the reinsurance prices have decreased. This becomes a problem because many cedents of risk, both primary writers and reinsurers, have considered securitization as an alternative to reinsurance rather than complementary to reinsurance. Second, insurers are unwilling to be pioneers due to the high development cost. Third, the fact that the products are uncorrelated to other financial products is not a good enough selling story for investors. Investors want to understand the nature of the risk, and this takes time. And finally, there still remains unanswered questions about what form and structure of insurance linked securities and derivatives will be viewed most favorable by investors.

4.3. **New pricing models for the PCS-option.** Many of the models derived for the CAT-future can, with some adjustments, be used to price the PCS-option. In addition, models have been derived, e.g. the paper by Geman and Yor [39]. In [39] they assume that the dynamics of the claim index (L_t) under Q is driven by the following stochastic differential equation:

$$dL_t = W_t dt + \Delta dN_t$$

where W_t is a geometric Brownian motion, N_t is a Poisson process and Δ is a positive constant representing the magnitude of the jumps. In the development period the last term is excluded. Subsequently, they then obtain quasi-completeness of the insurance derivative market by applying the Delbaen and Hazendonk [27] methodology to the class of layers of reinsurance replicating the call-spreads. The pricing of the call-spreads is then done by using stochastic time change, see [39] for further details.

The model can, as the model by Cummins and Geman [24], be criticized for being unrealistic, because it is too light-tailed. As discussed earlier a compound Poisson model will be more realistic, but using the approach from the previous section we will still not be able to handle heavy-tailed distributions for the jumps. An interesting question is therefore the following. Does there exist a model for the loss index which has a heavy-tail and still allows for a unique pricing formula?

To present such a model was the aim of the manuscript “A new model for pricing catastrophe insurance derivatives” Christensen [18] and is to our knowledge the first paper with this agenda. We now give a short presentation of this manuscript.

4.3.1. *The model.* The model we present below is inspired by Gerber and Shiu [43]. In [43] they show how one can obtain a risk neutral Esscher measure in a unique and transparent way if the logarithms of the value of the underlying security is a Levy process. The idea is now to choose such a model.

Let L_t be the underlying loss index for a catastrophe insurance derivative, $[0, T_1]$ be the loss period and $[T_1, T_2]$ be the development period. We then assume that L_t for all $t \in [0, T_2]$ is described by

$$L_t = L^0 \exp(X_t)$$

where X_t is a Levy process, and $L^0 \in \mathbb{R}^+$. We model X_t differently in the loss period and the development period, for a similar model see [55]. The question is then how to model X_t for the loss period and for the development period.

For $t \in [0, T_1]$ we will model X_t by a compound Poisson process

$$X_t = \sum_{i=1}^{N_t} Y_i \qquad \forall t \in [0, T_1]$$

where N_t is a Poisson process with a fixed parameter λ_1, and Y_i is exponentially distributed with parameter α. We hereby obtain one of the desired properties namely as mentioned above the heavy-tail for L_t, e.g. when $X_t \sim$Exp(α) then $L_t - L^0$ is Pa(α, L^0) distributed. But as mentioned above this model is not chosen because it is the most obvious one but because it has a heavy-tail, allows for fluctuation and gives a the possibility to express the price in closed-form. Therefore, the model has some disadvantages compared to a more natural model, firstly the fact that late catastrophes become more severe than earlier ones and secondly $L_0 = L^0 > 0$. For this reason this model should only be used as a first “crude” approximation to the real world. We have tried to work out these problems, but this seems to be hard.

We have to choose a model for X_t for the development period. We know that the adjustments are done both upwards and downwards we will therefore again describe X_t as a compound Poisson process for $t \in [T_1, T_2]$

$$X_t = X_{T_1} + \sum_{i=1}^{\tilde{N}_{t-T_1}} \tilde{Y}_i$$

where $\tilde{N}_t$ is a Poisson process with a fixed parameter λ_2 and $\tilde{Y}_i$ is normally distributed ($\mathrm{N}(\mu, \sigma)$), where the most natural choice of μ is $\mu = 0$ (unbiased previous estimates). In order to use the results from Gerber and Shiu [43] we need to assume that the process X_t for $t \in [0, T_1]$ is independent of the process $X_t - X_{T_1}$ for $t \in [T_1, T_2]$. In the real world one will expect some dependence but the assumption is invariable in order to use the results form [43].

The value of the option at time t is then

$$C(t, L_t) = \exp(-r(T_2 - t))E^*[C(T_2, L_{T_2})|\mathcal{F}_t]$$

where r is the risk free interest rate and E^* is the mean value according to a risk neutral measure. Before we can proceed further in the calculation of the option price we will have to choose a risk neutral measure. This is done in the next sections.

4.3.2. *The pricing of the PCS-option.* This section describes how to compute a risk neutral measure using the Esscher Transform. The theory was introduced by Gerber and Shiu [43]. However we need to make some adjustments in order to use their results in our context.

Let L_t be the value of the PCS index at time t

$$L_t = L^0 \exp(X_t), \qquad \forall t \geq 0, \tag{4.1}$$

where X_t is a Levy process. Let $M(z, t)$ be the moment generating function defined by:

$$M(z, t) := E[\exp(zX_t)] = \int_{-\infty}^{\infty} \exp(zx)F(dx, t) \tag{4.2}$$

provided the integral is finite, where F denotes the distribution function for X_t. Because of the independent stationary increments we then have, (see [34], section IX.5) , that

$$M(z, t) = (M(z, 1))^t \tag{4.3}$$

For any $h \in \mathbb{R}$ the Esscher-Transformation $F(dx, t; h)$ is defined as:

$$F(dx, t; h) = \frac{\exp(hx)F(dx, t)}{M(h, t)}. \tag{4.4}$$

From this transformed density we define the Esscher-transformed moment generating function as:

$$M(z, t; h) = \int_{-\infty}^{\infty} \exp(zx)F(dx, t; h) = \frac{M(z + h, t)}{M(h, t)}. \tag{4.5}$$

Then it follows from (4.3) and (4.5) that

$$M(z, t; h) = (M(z, 1; h))^t \tag{4.6}$$

The idea of Gerber and Shiu [43] is to choose $h = h^*$ such that the discounted underlying process here $\{e^{-rt}L_t\}$ becomes a martingale under the Esscher transformed measure. But absence of arbitrage arguments do not apply because the underlying process L_t is a loss index and not a price process, i.e. it gives no meaning to derive the risk neutral measure under the conditions that $\{e^{-rt}L_t\}$ should be a martingale. So we have to consider another process. Let P_t be the deterministic premium paid till time t to receive the value L_t at time t and assume that the index $\{L_t/P_t\}$ is a traded asset. We then use the idea of Gerber and Shiu by choosing $h = h^*$ such that the process $\{e^{-rt}L_t/P_t\}$ is a martingale under the Esscher transformed measure.

The question now is how to model P_t. We have to consider the loss period and the development period separately. Therefore we first consider the loss period. We would like the premium to be arbitrage free, so we will calculate the premium according to the adjusted parameter principle suggested by Venter [61]. See the latter paper for a description of the premium principle and a discussion of why this premium principle is arbitrage free. Let now $\tilde{\lambda_1}$ and $\tilde{\alpha}$ be the adjusted parameters and let $\tilde{X_t}$ be the adjusted process, i.e. $\tilde{X_t}$ is a compound Poisson process with Poisson parameter $\tilde{\lambda_1}$ and with marks that are exponentially distributed with parameter $\tilde{\alpha}$. . The premium is then:

$$\begin{aligned} P_t = E_P[\tilde{L}_t] & \\ &= E_P[L^0 \exp(\tilde{X_t})] \\ &= L^0 \exp(\frac{\tilde{\lambda_1} t}{(\tilde{\alpha} - 1)}) \end{aligned}$$

Motivated by this, we will use the following model for P_t

$$P_t = L^0 \exp(\beta_1 t)$$

We are now ready to find the parameter h_l^* and thereby derive the risk neutral measure in the loss period. h_l^* is chosen such that the process $\{e^{-rt}L_t/P_t\}$ is a martingale under the Esscher transformed measure

$$\begin{aligned} & E^*[\exp(-rt)L_t/P_t] = 1 \\ \Rightarrow \quad & \exp((r + \beta_1)t) = E^*[\exp(X_t)] \\ \Rightarrow \quad & \exp((r + \beta_1)t) = \int_{-\infty}^{\infty} \exp(x) F(dx, t; h_l^*) \\ \Rightarrow \quad & \exp((r + \beta_1)t) = M(1, t; h_l^*) \end{aligned} \tag{4.7}$$

By (4.6) it follows that the condition for h_l^* in the loss period is:

$$\exp(r+\beta_1) = M_l(1,1,h_l^*) \tag{4.8}$$

where M_l denotes that it is the moment generating function with respect to the distribution in the loss period. For the development period the situation is similar, the model for X_t is just different, see [18] for further details. The condition for h_d^* in the development period is:

$$\exp(r+\beta_2) = M_d(1,1,h_d^*) \tag{4.9}$$

where M_d denotes that it is the moment generating function with respect to the distribution in the development period.

The Radon-Nikodym derivative for the risk neutral Esscher measure on the σ-algebra $\mathcal{F}_t$ can now be characterized

$$\frac{dQ}{dP}|\mathcal{F}_t = \begin{cases} \frac{e^{h_l^* X_t}}{M_l(h_l^*,t)} & t\in[0,T_1] \\ \frac{e^{h_l^* X_{T_1}}}{M_l(h_l^*,T_1)}\frac{e^{h_d^*(X_t-X_{T_1})}}{M_d(h_d^*,t-T_1)} & t\in[T_1,T_2] \end{cases}$$

where h_l^* and h_d^* are given by equation 4.8 and equation 4.9 respectively.

Although there is more than one equivalent measure, the risk neutral Esscher measure provides a unique and transparent answer. In [18] the measure is justified by looking at a representative investor with a power utility function, see [18] for further details.

The above results is then used to calculate the concrete risk neutral Esscher measures for both the loss period and the development period, see [18] for further details. Let us now consider the PCS call option with exercise price A, cap K and expiring date T_2. Let $v_1(t) := \ln(A/(L^0\exp X_t))$ and $v_2(t) = \ln(K/(L^0\exp X_t))$. The value of the option at time $t\in[T_1,T_2]$ can then be expressed as:

$$\begin{aligned} C(t,L_t) &= e^{-r(T_2-t)}\Big(L_t\exp(\beta_2(T_2-t))\Big(F(v_2(t),T_2-t,h_d^*+1) \\ &\quad -F(v_1(t),T_2-t,h_d^*+1)\Big) + K(1-F(v_2(t),T_2-t,h_d^*)) - \\ &\quad A(1-F(v_1(t),T_2-t,h_d^*))\Big) \end{aligned} \tag{4.10}$$

If $t\in[0,T_1]$ then the value of the call option at time t is given by

$$\begin{aligned} C(t,L_t) &= e^{-r(T_2-t)}E^*[C(T_2,L_{T_2})|\mathcal{F}_t] \\ &= e^{-r(T_2-t)}E^*[E^*[C(T_2,L_{T_2})|\mathcal{F}_{T_1}]|\mathcal{F}_t] \end{aligned}$$

the value of $E^*[C(T_2,L_{T_2})|\mathcal{F}_{T_1}]$ is given by (4.10). At time $t<T_1$ the values of $L_{T_1}, v_1(T_1)$ and $v_2(T_1)$ are stochastic. The conditional mean value at time t is therefore obtained by integrating the expression with respect to the distribution function for the risk neutral Esscher

transformed process in the loss period, see [18] for further details. Here we calculate the exact price of the PCS-option.

5. Implied loss distributions

As we have seen in the previous section, it is hard to find realistic models for the loss index, which also allows one to express consistent prices in a closed form.

In Christensen [19] we therefore lead our analysis in another direction. We follow a procedure familiar to the conventional option market which is also suggested by Lane and Movchan in [46]. Rather than estimating volatilities and calculate consistent prices using, say the Black Scholes model, they take the traded prices and extract the volatilities consistent with those prices, i.e. find the implied volatility. We cannot use the exact same procedure on the insurance derivative market, since we are not able to characterize the price by a single parameter. However we can do something similar. We can choose a model for the implied loss distribution and then estimate the implied parameters from the observed prices.

This analysis can be used to evaluate cheapness and dearness among different prices and insurance derivative products. We simply calculate implied prices from the implied loss distributions and compare them to the observed prices. There are two main problems related to this analysis. First what kind of distribution should be used for the implied loss distribution, and second, how should the involved parameters be estimated. These two questions are answered in the manuscript [19], where we base our analysis on data for the PCS-option. We now give a summary of this manuscript.

5.1. **The models.** In [19] we consider six different models for the implied losses which are all presented in this section. Five of these models are new. Before we present these models we first give a general description of the price for a PCS call-spread expressed by the implied loss distribution.

Consider now a PCS call-spread expiring at time T with upper and lower strike K_u and K_l, respectively. Let $\widetilde{F}_{L_T}$ and $\widetilde{f}_{L_T}$ be the implied distribution function and the implied density function for the aggregate PCS loss index (L_T) at time T. The value of the PCS call-spread at time 0 is then given by

$$
\begin{aligned}
P_{K_u,K_l}(L_0,0) &= \widetilde{E}[\min(\max(L_T - K_l, 0), K_u - K_l)] \\
&= \int_{K_l}^{K_u} (x - K_l)\widetilde{f}_{L_T}(x)dx + (K_u - K_l)(1 - \widetilde{F}_{L_T}(K_u))
\end{aligned}
$$

The question is now, how is this implied loss distribution of the PCS index related to the real statistical distribution, i.e. the distribution under the P-measure? The implied distribution for the losses and the real distribution for the losses will in general not be the same. But based on the discussion in [19], we only use models which could reasonably be used to describe the real losses, when we model the implied losses. We now present the six models for the implied losses.

Model 1. The first model considered is similar to the one suggested by Lane and Movchan [46], namely a compound Poisson model with gamma distributed claims, i.e.

$$\widetilde{L}_T = \sum_{i=1}^{N_T} Y_i$$

where $\widetilde{L}_T$ is the implied losses, $N_T \sim \text{Pois}(\lambda)$ and $Y_i \sim \Gamma(\gamma, \alpha)$. The nice thing about this model is that we know the nth convolution of the Y's $(Y_1 + \ldots + Y_n \sim \Gamma(n\gamma, \alpha))$. This fact makes computations very simple. A disadvantage of the model is that the claims are light-tailed, whereas data give evidence that the distribution of the aggregate claims is heavy-tailed. In this model we can only approximate a heavy-tail by choosing low values of γ and α.

It is important to have this model in the analysis in order to see how the results from this model differ from the following new and more complicated models.

Model 2. Looking at the listed call-spreads we see, that the one with the lowest strikes is the 40/60 call-spread (see Table 5.1 below). The bid and ask for this call-spread is 12 and 15 respectively, which are relatively large values for a product that has a maximum pay-out of 20. These facts could therefore indicate that the market expects the loss index to be above a given threshold K_0 for sure. If this is true and $K_0 > 40$, then there is no market for a 20/40 call-spread, because the market will expect the call-spread to be worth 20 for sure. Based on these indications we now make an extension of model 1,

$$\widetilde{L}_T = K_0 + \sum_{i=1}^{N_T} Y_i$$

where $\widetilde{L}_T$ is the implied losses, K_0 is a constant indicating the threshold the market expects the losses to be above for sure, $N_T \sim \text{Pois}(\lambda)$ and $Y_i \sim \Gamma(\gamma, \alpha)$. According to the bid of the 40/60 call-spread (12), we will not allow K_0 to be above 52 (40+12). A possible interpretation of this model is to think of K_0 as the mean value of the "normal" claims and of the compound Poisson process as a model of the excesses.

Model 3. The next model will also rely on a light-tail distribution but we will now put more fluctuation into the model.

The PCS index can be viewed as the sum of losses from the individual catastrophes, and the losses from the individual catastrophe can be viewed as the sum of the individual claims corresponding to this particular catastrophe. We could therefore model the PCS index L_T as

$$L_T = \sum_{i=1}^{N_T} \sum_{j=1}^{M_i} Y_{ij}$$

where N_T is the number of catastrophes, M_i the number of claims from the ith catastrophe and Y_{ij} is the claim size number j from catastrophe number i. Recall that this model is similar to the one suggested in [17], where it was used to incorporate the reporting times of the claims.

The number of claims from a catastrophe is very large, so by the strong law of the large numbers it follows that $\sum_{j=1}^{M_i} Y_{ij} \approx M_i E[Y_{ij}]$. If the approximation should be good we will also need the $\mathrm{Var}(Y_{ij})$ to be small. If we use this approximation, we could describe L_T as $L_T \approx \sum_{i=1}^{N_T} M_i Y$ where $Y = E[Y_{ij}]$. Motivated by this, we now model the implied PCS index as

$$\widetilde{L}_T = \sum_{i=1}^{N_T} M_i Y$$

where $N_T \sim \mathrm{Pois}(\lambda)$, $M_i \sim \mathrm{NB}(\gamma,\mathrm{p})$ or more precisely by a mixed Poisson distribution with a Γ mixing parameter $\lambda_i \sim \Gamma(\gamma, \alpha)$ and Y is a constant. We now have a model allowing for more fluctuation but we also have four parameters to estimate.

Model 4. We simply use the same model as model 1, but we now choose a Pareto distribution for the Y's, i.e. $L_T = \sum_{i=1}^{N_T} Y_i$ where $N_T \sim \mathrm{Pois}(\lambda)$ and $Y_i \sim \mathrm{Pa}(\alpha, \beta)$. However, we do not know the nth convolution of the Y's. We solve this problem by the following approximation. The value of the PCS call-spread with strikes K_l and K_u at time 0 is here given by

$$\begin{aligned}
&P_{K_u,K_l}(L_0, 0) \\
&= \sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \Big(\int_{K_l}^{K_u} (x - K_l) f^{*n}(x) dx + (K_u - K_l) \int_{K_u}^{\infty} f^{*n}(x) dx \Big) \\
&\approx \sum_{n=1}^{4} e^{-\lambda} \frac{\lambda^n}{n!} \Big(\int_{K_u}^{\infty} (x - K_l) f^{*n}(x) dx + (K_u - K_l) \int_{K_l}^{K_u} f^{*n}(x) dx \Big)
\end{aligned}$$

where $f^{*n}(x)$ denotes the density for the nth convolution of the Pareto distribution.

The first 4 convolutions are then found by the well-known general formula for the Lebesgue convolution ($f^{*2}(x) = \int_0^x f(x-y)f(y)dy$, $f^{*3}(x) = \int_0^x f(x-y)(\int_0^y f(y-z)f(z)dz)dy$, $\ldots$). By only taking the first 4 convolutions in the sum it should be possible for a computer to calculate the expression. And if λ (the average number of catastrophes) is small, the approximation is good because the term $\sum_5^\infty e^{-\lambda}\lambda^n/n!$ will be small.

Model 5 This model is inspired by the volatility surface models trying to explain the volatility smile, i.e. models where the implied volatility depends on the strike of the option. We construct a similar model where the most explanatory parameter in the implied loss distribution is dependent on the strike. We assume that the implied loss index is Pareto distributed, i.e.

$$\tilde{L}_T \sim \mathrm{Pa}(\alpha, \beta)$$

β being just a scale parameter. The most explanatory parameter in this loss distribution is the α parameter. We therefore choose α to be the strike dependent parameter, i.e. we assume that α is a function of the strike ($\alpha(K) = f(K)$). The estimation of the parameters is done in four steps, because we have to choose the function f first. The four steps are as follows:

1. We assume that $\alpha = \alpha_0$, i.e. independent of the strike. We then let $\widetilde{L}_T \sim Pa(\alpha_0, \beta_0)$ and estimate the parameters α_0 and β_0.
2. We now keep β fixed as β_0 and then for each PCS call spread with strikes K_l^i and K_u^i, we estimate an α_i from the traded price or the bid/ask spread dependent of what is available. These values are then plotted. A possible picture could be the one given by figure 5.1.
3. From this plot we choose a function to describe our α, i.e. if we choose a function f with three parameters a, b and c, $a, b, c \in \mathbb{R}$. We can now describe α by $\alpha(K) = f(a, b, c, K)$. The implied loss distribution for a PCS call option with strike K is therefore given by
$$\widetilde{L}_T \sim \mathrm{Pa}(\alpha(K), \beta)$$
where $\alpha(K) = f(a, b, c, K)$.
4. Calculate the theoretical prices and estimate the parameters.

Model 6. The models 3, 4 and 5 could also be extended by including a threshold as was done for model 1 in model 2. But we will desist from doing this, as model 3 and 5 will be over parameterized and model 4

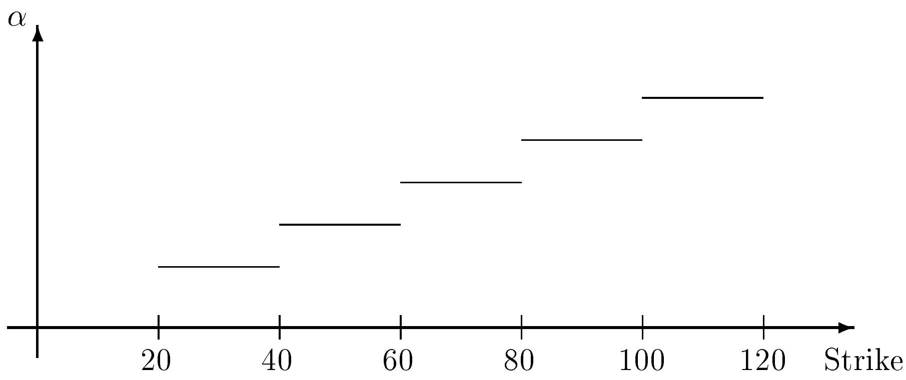

FIGURE 5.1. The α values.

will be too computationally heavy. We return to this discussion later. The last model we will consider is a very simple model, which we expect to be computationally very fast. It will be interesting to compare the results of this model with the results of the other more complicated models. We again include a threshold as we did for model 2 and then model L_T by

$$\widetilde{L}_T = K_0 + Y_T$$

where $\widetilde{L}_T$ is the implied losses, K_0 is a constant indicating the threshold the market expects the losses to be above almost surely, and $Y_T \sim \mathrm{Pa}(\alpha, \beta)$. Again we will not allow K_0 to be above 52.

The above six models are the models we are testing on the data available. The next step in the analysis is to describe the objective function from which the parameters should be found. The objective function that we purpose in the next section is new and different from the one used by Lane and Movchan in [46].

5.2. **The objective function.** Lane and Movchan [46] estimate the parameters for the implied loss distribution by the following procedure. *"The parameters are chosen such that they generate prices that are (i) lower than known offers; (ii) higher than known bids, and (iii) closest to actual traded prices. The optimization is two-tier. First, get inside the bid-offer spread. Second, get closest to actual traded prices. The two-tier effect is achieved by attaching (ideally non-Archimedean) weights to each of the two objective functions. "Closest" is defined as the absolute value of the difference between the actual traded price and the theoretical (or fitted) prices".*

We agree that it is desirable that the parameters are chosen such that the prices fulfill (i) and (ii), but we do not think that the requirements should be invariable, since it could be a problem to find a solution if the spreads are very small. And if the theoretical prices appear to be far away from the spread, it could be used to indicate that the chosen model may be wrong. We also agree on point (*iii*), i.e. if our data contains only traded prices, the parameters should be found by a least square fit. But the data primarily consists of spreads and single bids or asks, we therefore suggest the following objective function.

$$
\begin{aligned}
O &= \underbrace{\sum_{\text{bids}}\Big(\big(\frac{P_i^{\text{bid}} - P_i^{\text{th}}}{P_i^{\text{bid}}}\big)^+\Big)^2}_{\text{term 1}} + \underbrace{\sum_{\text{asks}}\Big(\big(\frac{P_i^{\text{th}} - P_i^{\text{ask}}}{P_i^{\text{ask}}}\big)^+\Big)^2}_{\text{term 2}} \\
&+ \underbrace{\delta_1\Big(\frac{1}{\#\text{spreads}}\sum_{\text{spreads}}\frac{(P_i^{\text{bid}} - P_i^{\text{ask}})}{(P_i^{\text{bid}} + P_i^{\text{ask}})/2}\Big)}_{\text{term 3}} \\
&\quad \underbrace{\sum_{\text{spreads}}\Big(\frac{P_i^{\text{th}} - (P_i^{\text{bid}} + P_i^{\text{ask}})/2}{P_i^{\text{bid}} - P_i^{\text{ask}}}\Big)^2 \wedge \frac{1}{4}}_{\text{term4}} \\
&+ \underbrace{\delta_2 \sum_{\text{single bids}}\Big(\big(\frac{P_i^{\text{th}} - 2P_i^{\text{bid}}}{P_i^{\text{bid}}}\big)^+\Big)^2}_{\text{term 5}} \\
&+ \underbrace{\delta_2 \sum_{\text{single asks}}\Big(\big(\frac{0.5P_i^{\text{ask}} - P_i^{\text{th}}}{P_i^{\text{ask}}}\big)^+\Big)^2\Big)}_{\text{term 6}}
\end{aligned}
$$

where the P_i^{th}'s are the theoretical prices, the P_i^{bid}'s are the observed bids and the P_i^{ask}'s are the observed asks. δ_1 and δ_2 are both constants. When P_i is a traded price, P_i is considered as both a bid and an ask where $P_i^{\text{bid}} = P_i^{\text{ask}}$.

Term 1 and 2 are included because, as mentioned above, we prefer the theoretical prices to be above the observed bids and below the observed asks. And as for the optimal case where we have only traded prices and no bid/ask spreads, these two terms alone will give us the commonly used least square fit.

As long as the average length of the spreads is small, we are close to the optimal case where we have only traded prices and term 1 and 2 will probably be sufficient to find a solution. But if the average length of the spread is large there is less information about the prices and we will probably be unable to find a unique solution. We therefore add term 4. In this term we value the information from the bid and the ask equally, i.e. we prefer the theoretical price to be in the middle of the bid/ask spread. We cap the single terms in the sum at 1/4, because if $P_i^{\text{th}} = P_i^{\text{bid}}$ or $P_i^{\text{th}} = P_i^{\text{ask}}$ the single term in the sum is equal to 1/4, and if $P_i^{\text{th}} > P_i^{\text{bid}}$ or $P_i^{\text{th}} < P_i^{\text{ask}}$ then it is punished in term 1 or 2. How much this fourth term should be valued compared to term 1 and 2 is adjusted by term 3. Term 3 is a constant δ_1 and a term denoting the average length of the spread. In agreement with the comments above, we thereby obtain that, if the average length of the spreads is small, we weight P_i^{th} being in the middle less than if the average length of the spreads is large.

The terms 5 and 6 are included in order to secure that the theoretical prices do not get too far away from the single bids or asks. By too far away we mean that a theoretical price is punished if it is lower than 50% of a single ask or higher than 200% of a single bid. By the term δ_2 we are able to adjust how much the fifth term should be valued compared to the other terms.

5.3. **The parameter estimation.** In this section we estimate the parameters and evaluate the six models described above. Before we start to estimate we first present the data.

The data material that we are going to use for this analysis are the prices for the National PCS call-spreads announced by the CBoT on January 7th 1999.

The National PCS call-spreads announced by the CBoT on January 7th 1999 is given by Table 5.1.

Call-Spreads	K_l/K_U	bid	ask
National	40/60	12.0	15.0
National	60/80	6.0	12.0
National	80/100	4.0	8.0
National	100/120	2.8	4.0
National	150/200	4.3	6.0
National	200/250	2.8	4.0
National	250/300		3.5
National	300/350		3.0

TABLE 5.1. The National PCS call-spread prices.

The first change in the underlying PCS index was made January 19th, where the index increased from 0 to 7.6. We have chosen the data from January 7th because the last changes in the bids and asks before January 19th were made here. If we take data from dates after January 19th we have to take the value of the index into account. If we consider data from a time point t where the PCS index is greater than 0, some adjustments have to be made. The implied losses at expiration time T can, at time t, be written as $\widetilde{L}_T = (\widetilde{L}_T - L_t) + L_t$, where L_t is a constant and $\widetilde{L}_T - L_t$ is the implied losses in the period from t to T. $\widetilde{L}_T - L_t$ can then be described by the same models as we used to describe $\widetilde{L}_T$, but the parameters will probably be changed. Even though we are looking at a model where $\widetilde{L}_T$ is a stationary process, we cannot expect the same parameters since the PCS index is influenced by some large seasonal effects.

The parameters are found by minimizing the objective function, with $\delta_1 = 0.001$ and $\delta_2 = 0.1$. We return to the discussion of these parameters later. The objective function is a function depending on a higher dimensional variable (the dimension is given by the number of parameters in the model). We therefore choose to minimize it by using a modification of the method of steepest descent described by Broyden see [8] and [35].

The parameters found by minimizing the objective function, the corresponding mean values and variances for the implied losses and the theoretical prices are listed in table 5.2, table 5.3 and table 5.4, respectively.

Model	par. 1	par. 2	par. 3	par. 4	value
1	$\lambda = 70$	$\alpha = 0.0123$	$\gamma = 0.0129$		0.058
2	$\lambda = 55$	$\alpha = 0.0050$	$\gamma = 0.0039$	$x = 47.2$	0.00015
3	$\lambda = 36$	$\alpha = 0.00019$	$\gamma = 0.0266$	$Y = 0.015$	0.086
4	$\lambda = 2.6$	$\alpha = 3.50$	$\beta = 90.7$		0.060
5	$\beta = 58$	$a = -0.117$	$b = -4.082$	$c = 0.596$	0.013
6	$\beta = 24$	$\alpha = 1.25$	$x = 40.0$		0.00010

TABLE 5.2. The estimated parameters.

	M1	M2	M3	M4	M5	M6
Mean value	74	90	77	96	(73;91)	139
Variance	6096	8453	6178	11652	∞	∞

TABLE 5.3. Mean value and variance of implied losses.

K_l/K_U	bid	M1	M2	M3	M4	M5	M6	ask
40/60	**12.0**	9.87	13.56	10.02	9.33	13.57	13.57	**15.0**
60/80	**6.0**	7.61	6.55	7.72	7.27	8.00	7.48	**12.0**
80/100	**4.0**	5.88	4.82	5.96	5.63	5.49	5.03	**8.0**
100/120	**2.8**	4.55	3.78	4.60	4.36	4.07	3.73	**4.0**
150/200	**4.3**	5.07	5.07	5.06	4.92	5.08	4.88	**6.0**
200/250	**2.8**	2.71	3.35	2.67	2.78	3.41	3.45	**4.0**
250/300		1.45	2.29	1.41	1.67	2.46	2.64	**3.5**
300/350		0.78	1.60	0.74	1.06	1.87	2.11	**3.0**

TABLE 5.4. The theoretical prices.

A detailed discussion of these results can be found in [19]. We now give the main conclusion. From Table 5.4 we see that model 1 is unable to generate prices that get into the bid/ask spread of the 40/60 and 200/250 call-spreads and we also see that it produces very low prices for the 250/300 and 300/350 call-spreads. This indicates that model 1 is a bad description of the implied losses. But recall that this is the model that was successfully suggested by Lane and Movchan in [46] so why now this difference? In [46] they consider market prices midyear 1998 where the PCS index was nearly 40 and this apparently makes a difference.

We also tried to model the midyear 1998 prices with model 1 and our objective function. The results are shown in Table 5.5 (The α parameter from [46] has been adjusted to correspond to the index value and not the Billion \$ value).

From Table 5.5 we see that model 1 in our objective function also generates reasonable results for the midyear 1998 prices. We therefore conclude that the reason for the bad fit of the 1999 prices is the model and not the objective function. Another important thing to note from Table 5.5 is that there are remarkable differences in the prices obtained by Lane and Movchan and the prices we obtain. We thereby see that the valuation of the bids and asks is highly dependent on the choice of the objective function.

Instead, we find that model 2 is a better model to use for the implied losses. However, it would be preferable to use model 6 also in order to support model 2. It is clear that none of the suggested models fit the implied losses perfectly, but we believe that model 2 supported by model 6 will be a good tool for investors analysing prices of catastrophe insurance derivatives. Models 3, 4 and 5 are all bad descriptions of the losses for various reasons, see [19] for further details.

K_l/K_U	bid	LM	CVC	ask
40/60	**11.0**	11.0	12.0	
60/80	**6.0**	7.5	8.2	**10.0**
80/100		5.7	6.1	**8.0**
100/120	**3.5**	4.4	4.6	**6.0**
100/150		9.4	9.5	**12.0**
120/140	**1.0**	3.5	3.5	**6.0**
250/300	**0.5**	1.9	1.4	**2.5**
100/200		14.7	14.4	**20.0**
150/200	**4.0**	5.4	4.9	**7.5**
180/200	**0.4**	1.8	1.6	**1.8**
λ		2.23	2.17	
γ		0.1887	0.2645	
α		0.0089	0.0124	

TABLE 5.5. The data from [46] contra our data.

In relation to how the parameters should be estimated we find that an improvement of the procedure from [46] was necessary for the following two reasons. Firstly we agree that it is desirable that the parameters are chosen such that the prices are lower than known offers and higher than known bids, but we do not think that the requirements should be invariable because, if the spreads are very small, it could be a problem to find a solution. And if the theoretical prices appear to be far away from the spread, it could be used to indicate that the chosen model may be wrong. Secondly we agree on point that the parameters should be chosen such that the prices gets closest to the actual traded prices, i.e. if our data contain only traded prices, the parameters should be found by a least square fit. But because the data primarily consists of spreads and single bids or asks, we find that this should be incorporated in the objective function. No matter what objective function one uses, it is clear from the discussion of model 1 that the choice of the objective function has a large effect on the derived prices, and it should therefore be chosen carefully.

5.4. **Other relevant references.** Let us end this section with a short description of some other interesting papers in relation to insurance derivatives. There is a huge amount of literature on the subject, a lot of it being non-mathematical, e.g. [11], [36], [57] and [60]. From the more mathematical papers let me shortly describe the following three.

Firstly, Brockett, Cox and Smith, [7] use a more actuarial pricing approach. They assume that the traders do not have complete information about the underlying loss process, but only information about a range of values for the loss process, i.e.

$$\mu_1 \leq E[L_T] \leq \mu_2 \quad \text{and} \quad \sigma_1^2 \leq Var[L_T] \leq \sigma_2^2$$

Based on this information they are then able to derive a range of prices for the insurance derivatives, see [7] for further details.

Second, Rasmussen, [50], develops along the line of Schweizer, [56], and uses the minimal martingale measure to price the PCS option. In [50] it is shown that the equivalent minimal martingale measure exists and it is shown how one can find the fair hedging price of a PCS option by choosing the equivalent minimal martingale measure as the pricing measure.

Finally, we mention the paper by Schmock, [54]. This paper considers catastrophe bonds issued by Winterthur. Several different models are presented in order to evaluate the value of the coupons, and it is shown how substantial the model risk, inherent in pricing such financial products, is.

6. The future of global reinsurance

In this section we will take a look into the future of global reinsurance. The results in this section are based on Christensen [20].

Risk related to natural phenomena such as various catastrophes has traditionally been distributed through the insurance and reinsurance system. Insurance companies accumulate the risk of individual entities and redistribute the risk to the global reinsurance industry. But, as discussed earlier, it will be insufficient to manage this risk in such a way in the future. A new way to managing such risk or unknown risk in general is called for.

When we talk about unknown risk, we refer to risk whose frequency we do not know, i.e. there is more than one estimate of the frequency of the risk. Examples of unknown risk are environmental health risk of new and little known epidemics, or risk induced by scientific uncertainty in predicting the frequency and severity of catastrophic events.

The problems related to unknown risk was first mentioned by Chichilnisky and Heal in [15] (a non mathematical paper), where they argued that unknown risk should be managed by using traditional insurance practice and by trading in the security market simultaneously. In the article [20] we continue and extend the ideas from [15]. The main purpose is to build a mathematical model that is able to handle these problems.

In the following we will describe the mathematical model from [20] and explain how we extend the ideas from [15] by considering both complete and incomplete markets and by considering the case where the premium charged by the insurance company is restricted.

In [20] we consider a general model for an insurance company, where the company faces n states of the world. For each of these states the insurance company is able to estimate the frequency of the risk, but the risk related to the states is unknown. We show how the company should handle this unknown risk. This is done by using the statistical approach to handle the known risk, i.e. the risk related to a given state, and by using the economic approach to handle the risk related to the different states.

6.1. **The model.** Let S denote the state of the world. We make the following assumptions:

- There are n states denoted by $\{s_1, \dots, s_n\}, \quad S \in \{s_1, \dots, s_n\}$.
- The probabilities corresponding to the n states are known
$$P(S = s_i) = p_i, \quad i = 1, \dots, n, \quad \sum_{i=1}^{n} p_i = 1$$
- F_i is known for all $i \in \{1, \dots, n\}$, where F_i denotes the distribution of the loss (L) of the insurance companies, given the state is i ($L|\{S = s_i\} \sim F_i$). Let $L_i = L|\{S = s_i\}$.
- If the insurance company knows the state S, then the statistical approach by adding a safety loading would work, i.e. if the insurance company knew that $S = s_i$, it would be reasonable to charge the premium P_i given by
$$P_i = E[L_i] + \delta_i$$
where δ_i is a safety loading calculated by a standard premium calculation principle.
- There exist n "state securities" traded on the n states. Security number j pays the amount c_{ij} if the state is i. Let c_i be the vector $c_i = (c_{i1}, \dots, c_{in})$ and let C be the matrix given by
$$C = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$
Let further $\tilde{c_j}$ be the jth column in C.
- The market is complete, i.e. the n columns in C are linearly independent.

- The market for these securities is arbitrage free and there exists a unique risk neutral measure. We denote the risk neutral probabilities by $q_1, \ldots, q_n$, and let q be the vector given by $q = (q_1, \ldots, q_n)$. From basic finance courses it is known that these risk neutral probabilities can be used to price the state securities, i.e. the price of state security number i is given by the discounted value of $q\tilde{c_j}$.
- There exists a risk free security and for simplicity we assume that the risk free interest rate is zero. This is no loss of generality, since we can discount all securities.

We now have a model where the insurance company exactly knows how they should handle the insurance risk if the state of the world is known. But because of the uncertainty about the state of the world, the general risk for the insurance company becomes unknown. In the next section we will show how the insurance company is able to handle this unknown risk.

6.2. **How to handle unknown risk in a complete market.** The expected loss for the insurance company is given by

$$E[L] = p_1 E[L_1] + \cdots + p_n E[L_n]$$

To cover these losses, the insurance company has to charge a premium P. But charging a premium is not enough, since we obtain a safety loading in state i given by $\tilde{\delta}_i = P - E[L_i]$ if the insurance company charges a premium P. The problem connected with this, is that we do not obtain the desired safety loading. For some i's we have that $\delta_i < \tilde{\delta}_i$ which means that the insurance company has been over charging. And for some i's we have that $\delta_i > \tilde{\delta}_i$, which means that the insurance company has been under charging, which could lead to a dangerous position. Before we solve this problem, we make the two following definitions.

Definition 6.1. A **trading strategy** for the insurance company is defined as a vector $m = (m_1, \ldots, m_n)^T$ where m_i denotes how many securities i the insurance company buys.

Definition 6.2. An **optimal trading strategy** for the insurance company is a costless trading strategy such that

$$P + c_i m - E[L_i] = \delta_i \qquad \forall i = 1, \cdots, n. \tag{6.1}$$

The questions are now whether it is possible to obtain this optimal strategy and if so, what premium should be charged in order to obtain it? These questions are answered in the following theorem.

Theorem 6.1. *An optimal trading strategy can be obtained if and only if*

$$P = q_1 P_1 + \cdots + q_n P_n.$$

In this case, the strategy m has to be chosen by

$$m = C^{-1} \begin{bmatrix} -P + E[L_1] + \delta_1 \\ \vdots \\ -P + E[L_n] + \delta_n \end{bmatrix}.$$

Remark 6.1. The problem can be simplified considered in the following way. The insurance company wants to obtain the premiums $(P_1, \ldots, P_n)$ corresponding to the n states. This could be obtained for all i if we for all i buy P_i of Arrow-Debreu (AD) security number i. AD security i is a security that pays 1 if the state is i and pays zero in all other states. These AD securities exist because the market is complete, and the price of AD security number i is given by q_i. The total price of this AD portfolio is therefore given by

$$\text{Total price} = \sum_{i=1}^{n} P_i q_i$$

So by charging a premium $P = \sum_{i=1}^{n} P_i q_i$, the insurance company can obtain the optimal strategy. This only works if the market is complete, we will return to the incomplete case later.

6.3. **The restricted premium case.** In the previous section we found the optimal premium to charge for the insurance company. But the insurance company may be unable to charge this premium for competition reasons. We therefore now assume that the premium which the insurance company can charge is fixed at P_0.

The insurance company should therefore choose a trading strategy which they find "optimal" under the restriction that the cost of the trading strategy equals P_0. What we mean by "optimal" is discussed later in this section. In this complete market case, choosing a trading strategy m is equivalent to choosing premiums $(P_1, \ldots, P_n)$. We have the following relation between $(P_1, \ldots, P_n)$ and m

$$(P_1, \ldots, P_n)^T = Cm.$$

The restriction can also be expressed in terms of the P_i's instead of m.

$$\begin{aligned} & vm = P_0 \\ \Rightarrow\ & qCC^{-1}(P_1, \ldots, P_n)^T = P_0 \\ \Rightarrow\ & q_1 P_1 + \cdots + q_n P_n = P_0. \end{aligned}$$

These observations allow us to reformulate the problem to a problem in terms of the premiums $(P_1, \dots, P_n)$ instead of a problem in terms of the trading strategy m.

The problem in this fixed premium case is therefore to find the "optimal" choice of the P_i's subject to the constraint $P_0 = P_1 q_1 + \cdots + P_n q_n$. In [20] we consider four different ways of solving this optimal premium choice (OPC), i.e. defining "optimal". The four OPC's are based on the following:

- OPC1: The goal here is to obtain the same risk quantity in all the states. To measure the risk quantity, we will use the mean divided by the standard deviation.
- OPC2: The goal here is to obtain the same ruin probabilities in all the states.
- OPC3: The goal here is to obtain the same expected utility in all the states.
- OPC4: The goal here is to obtain the maximal expected utility.

In [20] these four OPC's are solved, analysed and compared, see [20] for further details.

6.4. **The incomplete market case.** In this section we consider the incomplete market case, i.e. a market where the number of states n is larger than the number of securities. Now let k denote the number of securities. Let again v_i be the price of state security number i and let $v = (v_1, \dots, v_k)$. Because of the incompleteness in this market we are no longer able to construct the n AD securities. We therefore cannot construct the optimal trading strategy and set the premium by $P = q_1 P_1 + \cdots + q_n P_n$. So instead of constructing the optimal trading strategy an alternative could be to choose the cheapest strategy which assures that the premium in state i is greater than or equal to $P_i = E[L_i] + \delta_i$ for all states, i.e. choose a trading strategy that solve the following problem

$$\begin{aligned} &\min_m vm \\ &\text{st} \quad \sum_{j=1}^{k} m_j c_{ij} \geq E[L_i] + \delta_i \qquad i = 2, \dots, n. \end{aligned}$$

A problem of this strategy is that it could be very expensive. An alternative strategy is therefore to choose the premiums so that they get as close as possible to the optimal premiums $(P_1, \dots, P_n)$, i.e. choose the portfolio m that solves the following problem

$$\min_{m_1,\dots,m_k} \sum_{i=1}^{n} (\sum_{j=1}^{k} m_j c_{ij} - P_i)^2$$

or equivalently

$$\min_{m} \|Cm - \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix}\|^2$$

where C now is a $n \times k$ matrix. This is a well known problem and it is solved by the least square solution which is given by, (see [3] p. 318),

$$m = (C^T C)^{-1} C^T \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix}$$

After these considerations we now make the following definition

Definition 6.3. A **least square strategy** is a trading strategy such that the insurance company gets as close as possible to the desired n state premiums as possible in the least square sense, i.e. the **least square strategy** is obtained by the following portfolio of securities.

$$m = (C^T C)^{-1} C^T \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix}$$

The insurance company would of course prefer to follow the optimal trading strategy given by $P_i = E[L_i] + \delta_i$ but this is impossible in this market. But had it been possible the insurance company would have been willing to pay more for the optimal strategy than for the least square strategy. Therefore, if the insurance company follows the least square strategy they should charge a premium that is larger than the price of the least square strategy. They are thereby compensated for not having the optimal strategy but only the least square strategy.

Let us now as in the complete case consider the situation where the insurance company is unable to charge the desired premium for competition reasons. We again set the possible fixed premium that can be charged to P_0.

The problem now is that we want to set the P_i's according to OPC1, OPC2, OPC3 or OPC4 but the equation $P_0 = P_1 q_1 + \cdots + P_n q_n$ is no longer valid. We are no longer able to construct the n AD securities in this incomplete market. But instead of choosing the P_i's according to OPC1, OPC2, OPC3 or OPC4, we could choose the corresponding least square solution. We would then just have to replace the equation $P_0 = P_1 q_1 + \cdots + P_n q_n$ with an equation that makes sure that the price of the least square portfolio is equal to P_0. How this is done is described in [20], see [20] for further details.

7. Conclusion

What have we done in this thesis? Or perhaps more accurately: What are the contributions of the manuscripts included?

Christensen and Schmidli [17] present a model for insurance future pricing, which only relies on the information available. The products are priced solely from observing the reporting stream. Contrary to the existing literature we model the reporting times explicitly. We thereby obtain a more realistic model.

The results of this article rely on an approximation of the exact future price. One therefore has to be careful applying the results derived, because the results will be inaccurate if the cap-probability ($P(L_{T_2} > 2\Pi)$) or the risk aversion coefficient is "too large".

This paper suggests two ways of approximating the approximation error, the gamma approximation and the Edgeworth approximation. It is shown that they are both useful in the determination of the error level even though the gamma approximation seems to be the best.

Christensen [16] is a gathering of information about the PCS-option. It explains why the PCS-option replaced the CAT-future and how the PCS-option is an improvement. The paper also explains how to hedge catastrophe risk with PCS-options and it compares the PCS-option with traditional reinsurance.

Christensen [18] derives a new model for pricing insurance derivatives which allows for heavy-tails, and also provides a unique pricing measure. The model is obtained by modeling the logarithms of the loss process as a compound Poisson process with exponential distributed marks in the loss period and with normal distributed marks in the development period. The price is found by evaluating the future pay-out of the insurance derivative under the risk neutral measure derived by the Esscher approach. In the article the exact price in the case of the PCS-option is calculated.

Christensen [19] analyses prices for catastrophe insurance derivatives by looking at the "implied loss distributions" embedded in the traded prices. And it gives answers to the two main problems in this analysis. First, what kind of distribution should be chosen for the implied losses and second, how should the involved parameters be estimated?

In relation to how the parameters should be estimated we have come up with a new objective function. It is not possible to prove that it is better than the one used by Lane and Movchan [46], but we find that the argumentation in the paper supports the choice of the proposed function.

In the paper it is also documented that the model suggested by Lane and Movchan [46] is unable to fit the PCS-option prices in general. Instead the manuscript suggests other models, some of which are shown to be more sufficient in the description of the implied losses.

Christensen [20] presents a model for managing unknown risk. The model is inspired by Chichilnisky and Heal [15] (a non mathematical paper), where they argued that unknown risk should be managed by using traditional insurance practice and by trading in the security market simultaneously. The model presented in [20] is new and it presents the ideas from [15] in a mathematical way, i.e. [20] show how unknown risk and related problems can be handled mathematically.

[20] also extends the ideas from [15] by considering both complete and incomplete markets. Furthermore it considers the case where the premium charged by the insurance company is restricted. In this case the insurance company has to choose an allocation of the restricted premium corresponding to the states of the world. We propose four different methods of solving this problem. These four methods are then analysed and evaluated, and by examples, advantages and disadvantages are illustrated.

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Manuscripts

Paper I: Pricing catastrophe insurance products based on actually reported claims, 17 pages.

Paper II: The PCS-option, an improvement of the CAT-future, 11 pages.

Paper III: A new model for pricing catastrophe insurance derivatives, 15 pages.

Paper IV: Implied loss distributions for catastrophe insurance derivatives, 19 pages.

Paper V: How to hedge unknown risk, 23 pages.

PRICING CATASTROPHE INSURANCE PRODUCTS BASED ON ACTUALLY REPORTED CLAIMS

CLAUS VORM CHRISTENSEN AND HANSPETER SCHMIDLI

ABSTRACT. This article deals with the problem of pricing a financial product relying on an index of reported claims from catastrophe insurance. The problem of pricing such products is that, at a fixed time in the trading period, the total claim amount from the catastrophes occurred is not known. Therefore one has to price these products solely from knowing the aggregate amount of the reported claims at the fixed time point. This article will propose a way to handle this problem, and will thereby extend the existing pricing models for products of this kind.

1. INTRODUCTION

Modelling claims from a catastrophe actuaries use heavy tailed distributions, such as the Pareto distribution. This means that the aggregate claim basically is determined by the largest claim, see [8] or [13]. This effect became clearly visible in the early 90's, when the insurance industry had to cover huge aggregate claims incurring from catastrophes. Because certain catastrophic events like earthquakes, hurricanes or flooding are typical for some areas, a properly calculated annual premium would be nearly as high as the loss insured. From an actuarial point of view, such events are not insurable. But people living in such areas need protection. One possibility would be the government (tax payer) to take over the risk, as it is the case for flooding in the Netherlands. Another possibility are futures or options based on a loss index. Here the risk is transfered to private investors. A description of these products can be found for example in [2] or [14].

In 1992 the Chicago Board of Trade (CBoT) introduced the CAT-futures. This future is based on the ISO-index, which measures the amount of claims occured in a certain period and reported to a participating insurance company until a certain time. The product never became popular among private investors. The reasons were that the index only was announced once before the settlement date, there was information asymmetry between insurers and investors, and that there

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was a lack of realistic models. In 1995 the CAT-future was replaced by the PCS-option. This option is based on a loss index — the PCS-index — estimated by an independent authority. The latter index is announced daily. In this paper we study a model for indices like the ISO-index or the PCS-index. In the case of the CAT-future where the information stream is generated by a delayed reporting of claims from the catastrophes, in the case of the PCS-option by more and more refined estimates. For simplicity we will formulate the model as a model for the ISO-index. More specifically, we study the case where the number of claims from a single catastrophe has a fixed distribution (Section 3.1) and thereafter the case where the number of claims depend on an unobserved "severity" of the catastrophe (Section 3.2).

The recently introduced PCS-options (see [5]) do not directly depend on reported claims. But there is a strong correlation between actually reported claims and the PCS-index. Because these options serve as a sort of reinsurance instrument, an insurance company exposed to catastrophic risk would have to estimate the PCS-index and its price in order to determine their hedging strategy. It therefore seems natural to use the information on the claims reported to this company. Therefore our (ISO-index) model may also be of interest for a company investing into the new catastrophe options.

The main purpose of this article is to introduce a model taking reporting lags into account. As illustration, how calculations can be done in this model, we will approximate the CAT-future price, even though this product is not traded anymore. For pricing the catastrophe insurance futures and options we use the exponential utility approach of [1], [4] or [9]. This approach will only work for aggregate claims with an exponentially decreasing tail. But data give evidence that the distribution tail of the aggregate claims is heavy tailed. In our model a heavy-tail can be obtained by a heavy-tailed distribution for the number of individual claims of a single catastrophe. For pricing we approximate the claim number distribution by a negative binomial distribution; more precisely, by a mixed Poisson distribution with a $\Gamma(\gamma, \eta)$-mixing distribution. Choosing γ and η small a heavy-tail behaviour can be approximated. The reader should note that the value of the security is based on a capped index and therefore has an upper bound. This justifies the light-tail approximation.

In the remaining parts of this introduction we describe the CAT-future. In Section 2 we introduce the model. In contrast to the existing literature, the reporting lags are explicitly taken into account. In Sections 3.1 and 3.2 we calculate approximations to the prices. Finally, in Section 4 we study the approximation error.

1.1. **Description of the CAT-futures.** CAT-futures are traded on a quarterly cycle, with contract months March, June, September, and

December. A contract for a calendar quarter (called the event quarter) is based on losses occurring in the listed quarter and being reported to the participating companies by the end of the following quarter. A contract also specifies an area and the type of claim to be taken into account. The additional three months following the reporting period is attributable to data processing lags. The six months period following the start of the event quarter is called reporting period. The three reporting months following the event quarter are to allow for settlement lags that are usual in insurance. The contracts expire on the fifth day of the fourth month following the end of the reporting period. We will use arbitrary times $T_1 < T_2$ for the end of the event quarter and the end of the reporting period, respectively. This will allow for redesigning the futures. As a matter of fact a longer reporting period would be much more suitable for the need of the insurance world.

The settlement value of the contract is determined by a loss index; the ISO-index. Let us now consider the index. Each quarter approximately 100 American insurance companies report property loss data to the ISO (Insurance Service Office, a well known statistical agent). ISO then selects a pool of at least ten of these companies on the basis of size, diversity of business, and quality of reported data. The ISO-index is calculated as the loss-ratio of this pool

$$\text{ISO-index} = \frac{\text{reported incurred losses}}{\text{earned premiums}}.$$

The list of companies which are included in the pool is announced by the CBoT prior to the beginning of the trading period for that contract. The CBoT also announces the premium volume for companies participating in the pool prior to the start of the trading period. Thus the premium in the pool is a known constant throughout the trading period, and price changes are attributable solely to changes in the market's expectation of loss liabilities.

The settlement value for the CAT-futures is

$$F_{T_2} = 25\,000 \times \min(I_{T_2}, 2)$$

where I_{T_2} is the ISO-index at the end of the reporting period, i.e. the ratio between the losses incurred during the event quarter and reported up till three months later and the premium volume for the companies participating in the pool.

Example 1.1. The June contract covers losses from events occurring in April, May and June and are reported to the participating companies by the end of September. The June contract expires on January 5th, the following year. The contract is illustrated by Figure 1.1.

1.2. **The CAT-future pricing problem.** Cummins and Geman [7] were the first to price the insurance futures. Their approach was quite

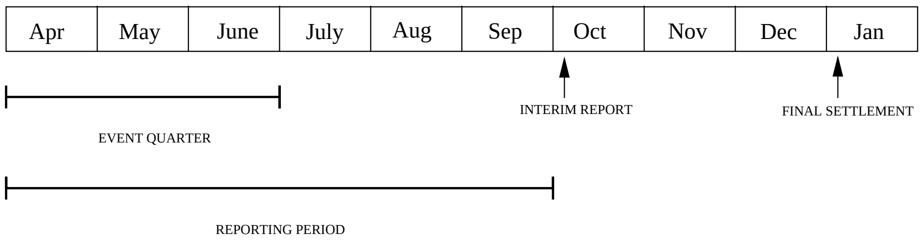

FIGURE 1.1. June CAT-Future contract

different from the approach used in this paper. As model they used integrated geometric Brownian motion. This allowed them to apply techniques arising from pricing Asian options. The model, however, seems to be far from reality. At times where a catastrophe occurs or shortly thereafter, one would expect a strong increase of the loss index. It therefore is preferable to use a marked point process as it is popular in actuarial mathematics.

The price to pay for the more realistic model is "non-uniqueness" of the market, see [1] and [9] for further details. In fact, the index (I_t) is not a traded asset. Thus markets cannot be complete. Moreover, as it is the case for term structure models, any equivalent measure may be used for no-arbitrage pricing. However, the preferences of the agents in the market will determine which martingale measure applies.

In this article we follow the approach of Embrechts and Meister [9]. There the general equilibrium approach is used, where all the agent's utility functions are of exponential type. More precisely, let F_t be the price of the future, L_t the value of the losses occured in the event quarter and reported till time t, $\mathcal{F}_t$ the information at time t, Π premiums earned and let $c = 25\,000/\Pi$. Then the price at time t, is (see [9, p.19])

$$F_t = c\frac{E_P[\exp(\alpha L_\infty)\,(L_{T_2} \wedge 2\Pi) \mid \mathcal{F}_t]}{E_P[\exp(\alpha L_\infty) \mid \mathcal{F}_t]}. \tag{1.2}$$

In particular, $E_P[\exp(\alpha L_\infty)]$ has to exist. The market will determine the risk aversion coefficient α.

The term $\exp(\alpha L_\infty)/E_P[\exp(\alpha L_\infty)]$ is strictly positive and integrates to one. Thus it is the Radon-Nikodym derivative dQ/dP of an equivalent measure. In the specific model we will consider, the process (L_t) follows under to the measure Q the same model (but with different parameters) as under P. We will use this fact to calculate the price of the CAT-future and the PCS-option. This change of measure is similar to the Esscher method described in [11] and [14]. If we assume that proportional reinsurance is possible, the premiums are fairly split between insurer and reinsurer, and that the proportion held in the portfolio can be changed at any time, then the index $(L_\infty(T) - \Pi(T))$ would become a traded asset, where $L_\infty(T)$ are all the claims occured till time T and

$\Pi(T)$ are the premiums earned to cover the claims occuring till time T. In our model $\Pi(T)$ would be a linear function. This would imply that the process $(L_\infty(T) - \Pi(T) : T \geq 0)$ is a martingale under the pricing measure. This condition will determine the risk aversion coefficient α, see for instance [15].

To proceed further in the calculation of the future price, one has to choose a model for (L_t). [1] used a compound Poisson model. This can be seen as catastrophes occurring at certain times and claims are reported immediately. In such a model there would not be a need for the prolonged reporting period. In [9] a doubly stochastic Poisson model is introduced. Here, a high intensity level will occur shortly after a catastrophe, where more claims are expected to be reported. In [12, Example 5.3] the asymptotic expected value and asymptotic variance for a general compound process are obtained.

The aim of this paper is to model the claims reported to the companies as individual claims with a reporting lag. This is done by modelling the aggregate claim from a single catastrophe as a compound (mixed) Poisson model. We thereby obtain the possibility to separate the individual claims and to model the reporting times of the claims. In Section 3.1 we calculate the future price using a compound Poisson model, whereas in Section 3.2 the results are extended by using a compound negative binomial model, represented as a mixed compound Poisson model. We thereby can estimate the mixing parameter from the reporting flow.

2. The model and assumptions

Let T_1 denote the end of the event period and $T_2 > T_1$ the end of the reporting period. We work on a complete probability space $(\Omega, \mathcal{F}, P)$ containing the following random variables and stochastic processes:

L_t	:	The aggregate amount of reported claims till time t,
N_t	:	The number of cat. occurred in the interval $[0, t]$,
M_i	:	The number of claims from the i-th cat.
$M_i(t)$	:	The number of claims from cat. i reported until t,
Y_{ij}	:	The claim size for the j-th claim from the i-th cat.
D_{ij}	:	The reporting lag for the j-th claim from the i-th cat.
τ_i	:	The occurrence time of the i-th cat.

We assume the following:

- $(\mathcal{F}_t)$ is the smallest right continuous complete filtration, such that the aggregate amount of reported losses L_t at time t is $(\mathcal{F}_t)$-adapted.
- (N_t) is a Poisson process with rate $\Lambda \in (0, \infty)$.
- $(M_i : i \in \mathbb{N})$, $(N_t : 0 \leq t \leq T_1)$, $(D_{ij} : i, j \in \mathbb{N})$, $(Y_{ij} : i, j \in \mathbb{N})$ are independent.

- M_i is mixed Poisson distributed with mixing distribution F_λ. That is, there are random variables (λ^i) with distribution F_λ such that, given λ^i, M_i is conditionally Poisson distributed with parameter λ^i. If the distribution F_λ is degenerated ($\lambda^i = \lambda$ for some constant λ) the (unconditional) distribution of M_i is Poisson with parameter λ. If F_λ is degenerate then M_i is Poisson distributed. We denote by λ^i the mixing parameter and by λ a generic random variable for λ^i.
- $(\lambda^i : i \in \mathbb{N})$ are iid and independent of (N_t), (D_{ij}), (Y_{ij}).
- $D_{ij} \sim F_D$, $Y_{ij} \sim F_Y$. We denote by Y (D, respectively) a generic variable for Y_{ij} (D_{ij}), and by $m_Y(r) = E[e^{rY}]$ the moment generating function of the claim sizes.
- The j-th claim Y_{ij} from the i-th catastrophe is reported at time $\tau_i + D_{ij}$.

We have $N_T - N_t \sim \text{Poi}(\Lambda(T-t))$ and $(\tau_{N_t+1}, \dots, \tau_{N_T} \mid N_T - N_t = n)$ has the same distribution as $(U_{(1)}, \dots, U_{(n)})$ where the (U_i) are iid uniformly distributed on the interval $[t, T]$ and $(U_{(i)})$ denotes the order statistics, see for instance [13, Thm 5.2.1]. Moreover, it can be shown, which may seem a little bit surprising, that the number of claims $M_i(T_2) - M_i(t)$ from catastrophe i reported in the period $[t, T_2]$ is, given λ^i, conditionally independent of the number of claims $M_i(t)$ reported in the period $[\tau_i, t]$. Moreover, for $1 \le i \le N_{T_1}$, given (τ_i), λ^i, we have

$$i \le N_t: \qquad M_i(t) \,\big|_{\lambda^i, \tau_i} \sim \text{Poi}(\lambda^i(F_D(t - \tau_i))) \tag{2.1}$$

and

$$\begin{aligned} & M_i(T_2) - M_i(t) \,\big|_{\lambda^i, \tau_i} \sim \text{Poi}(\lambda^i(F_D(T_2 - \tau_i) - F_D(t - \tau_i)))\,, \\ i > N_t: \ & M_i(T_2) - M_i(t) \,\big|_{\lambda^i, \tau_i} \sim \text{Poi}(\lambda^i(F_D(T_2 - \tau_i)))\,. \end{aligned}$$

In our model the claims Y_{ij} from the i-th catastrophe are randomly ordered. This simplifies the modelling of the reporting lags D_{ij}. Let $(D_{i:j} : 1 \le j \le M_i)$ be the order statistics of the $(D_{ij})_{1 \le j \le M_i}$, and $Y_{i:j}$ be the claim corresponding to $D_{i:j}$. Then the claims occured before T_1 and reported till $t \le T_2$ amount to

$$L_t = \sum_{i=1}^{N_{t \wedge T_1}} \sum_{j=1}^{M_i(t)} Y_{i:j}\,.$$

In particular, the final aggregate amount L_{T_2} can be represented as

$$L_{T_2} = L_t + \sum_{i=1}^{N_{t \wedge T_1}} \sum_{j=M_i(t)+1}^{M_i(T_2)} Y_{i:j} + \sum_{i=N_{t \wedge T_1}+1}^{N_{T_1}} \sum_{j=1}^{M_i(T_2)} Y_{i:j}\,.$$

For the rest of this section we work with the measure P conditioned on $\mathcal{F}_t$. Let

$$S_i = \sum_{j=M_i(t)+1}^{M_i(T_2)} Y_{i:j}\,.$$

For $i \leq N_t$, given λ^i, S_i is then compound Poisson distributed with intensity parameter $\lambda^i(F_D(T_2 - \tau_i) - F_D(t - \tau_i))$. At time t, N_t is known, so $S_1 + \cdots + S_{N_t}$ conditioned on $\lambda^1, \ldots, \lambda^{N_t}$ is again compound Poisson distributed with parameter $(\lambda^1(F_D(T_2 - \tau_1) - F_D(t - \tau_1)) + \cdots + \lambda^{N_t}(F_D(T_2 - \tau_{N_t}) - F_D(t - \tau_{N_t})))$. The latter is known from risk theory, see for instance [10, p. 13] or [13, Thm 4.2.2].

For $N_t < i \leq N_{T_1}$, given λ^i and τ_i, S_i is then compound Poisson distributed with intensity parameter $\lambda^i(F_D(T_2 - \tau_i))$. We again have that $S_{N_t+1} + \cdots + S_{N_{T_1}}$ conditioned on N_{T_1}, $\lambda^{N_t+1}, \ldots, \lambda^{N_{T_1}}$ and $\tau^{N_t+1}, \ldots, \tau^{N_{T_1}}$, is compound Poisson distributed with intensity parameter $\sum_{i=N_t+1}^{N_{T_1}} \lambda^i(F_D(T_2 - \tau_i))$. So all in all we get that $L_{T_2} - L_t = S_1 + \cdots + S_{N_t} + S_{N_t+1} + \cdots + S_{N_{T_1}}$ given $N_{T_1}, \lambda^1, \ldots, \lambda^{N_{T_1}}, \tau^1, \ldots, \tau^{N_{T_1}}$ is compound Poisson distributed with intensity parameter

$$\begin{aligned} &\sum_{i=1}^{N_t} \lambda^i(F_D(T_2 - \tau_i) - F_D(t - \tau_i)) + \sum_{i=N_t+1}^{N_{T_1}} \lambda^i(F_D(T_2 - \tau_i)) \\ \overset{\mathrm{d}}{=} &\sum_{i=1}^{N_t} \lambda^i(F_D(T_2 - \tau_i) - F_D(t - \tau_i)) + \sum_{i=N_t+1}^{N_{T_1}} \lambda^i(F_D(T_2 - \tilde{\tau}_i)) \end{aligned} \tag{2.2}$$

where $\tilde{\tau}_i$ are iid uniformly distributed on (t, T_1) and independent of $\mathcal{F}_t$. Here $\overset{\mathrm{d}}{=}$ means equality in distribution. Thus for t fixed $L_{T_2} - L_t$ becomes a mixed compound Poisson model.

3. Calculation of the CAT-future price

3.1. **Deterministic λ^i.** In this section we will derive the future price (1.2) when $\lambda^i = \lambda$ is deterministic. We therefore first need the value $E_P[\exp\{\alpha L_\infty\}]$. We remark that $\sum_{j=1}^{M_i} Y_{ij}$ has a compound Poisson distribution with moment generating function $\exp\{\lambda(m_Y(r) - 1)\}$. This yields

$$E_P[\exp\{\alpha L_\infty\}] = \exp\{\Lambda(e^{\lambda(m_Y(\alpha)-1)} - 1)\}\,.$$

Let us consider now the process (L_t) under the measure Q. For an introduction to change of measure methods we refer to [13]. A simple calculation yields that under Q the process (L_t) is of the same type, only with different parameters. (N_t) is a Poisson process with rate

$$\tilde{\Lambda} = \Lambda E_P\Big[\exp\Big\{\sum_{j=1}^{M_1} Y_{1j}\Big\}\Big] = \Lambda \exp\{\lambda(m_Y(\alpha) - 1)\}\,.$$

The number of claims of the i-th catastrophe is Poisson distributed with parameter $\tilde{\lambda} = \lambda(m_Y(\alpha))$ and the individual claims have the distribution function $\tilde{F}_Y(x) = \int_0^x e^{\alpha y}\, dF_Y(y)/m_Y(\alpha)$. The lags D_{ij} have the same distribution as under P.

The price of a CAT-future is therefore $cE_Q[L_{T_2} \wedge 2\Pi]$. Denoting the distribution function of $L_{T_2} - L_t$ under Q conditioned on $\mathcal{F}_t$ by $\tilde{F}_L(\cdot;t)$ we can express the price as

$$\begin{aligned} &c(L_t + E_Q[(L_{T_2} - L_t) - ((L_{T_2} - L_t) - (2\Pi - L_t))^+ \mid \mathcal{F}_t]) \\ &\quad = \; c\Big(L_t + E_Q[(L_{T_2} - L_t) \mid \mathcal{F}_t] - \int_{2\Pi - L_t}^{\infty} (1 - \tilde{F}_L(x;t))\, dx\Big)\,. \end{aligned}$$

But the problem with the above expression is that we have to find the n-fold convolutions of $F_D[T_2 - \tilde{\tau}]$, in order to calculate the last term. To find an explicit expression seems to be hard.

Historical data show that, so far, the cap 2 in the definition of the CAT-future has not been reached. The largest loss ratio was hurricane Andrew with $L_\infty = 1.79\Pi$. Under the measure P we have that $\{L_{T_2} > 2\Pi\}$ is a rare event. Because we are dealing with catastrophe insurance, the market risk aversion coefficient α cannot be large. Otherwise, catastrophe insurance would not be possible. We therefore assume that $\{L_{T_2} > 2\Pi\}$ is also a rare event with respect to the measure Q, see also [9]. The light tail approximation to our model then assures that the tail of $\tilde{F}_L(\cdot;t)$ is exponentially decreasing. That is $\int_{2\Pi - L_t}^{\infty} (1 - \tilde{F}_L(x;t))\, dx$ will be small as long as $Q(L_{T_2} > 2\Pi)$ is small, see also the discussion in Section 4. The latter depends of course on the risk aversion coefficient α, which has to be small in order to be able to neglect the last term. As in [9] we therefore propose the approximation $c(L_t + E_Q[(L_{T_2} - L_t) \mid \mathcal{F}_t])$ to the price of the CAT-future, and we then make the following definition.

Definition 3.1. Let p_t be the price of the CAT-future at time t. The upper bound p_t^{approx} of p_t defined as

$$\begin{aligned} p_t^{\text{approx}} &= p_t + \int_{2\Pi - L_t}^{\infty} (1 - \tilde{F}_L(x;t))\, dx \\ &= c(L_t + E_Q[(L_{T_2} - L_t) \mid \mathcal{F}_t]) \end{aligned}$$

is used as an approximation to the future price p_t.

Theorem 3.2. *Let the assumptions be as in Section 2 with a fixed risk aversion coefficient α. Assume further that $\lambda^i = \lambda$ is deterministic.*

Then p_t^{approx} *for a given risk aversion coefficient* α *is given by*

$$\frac{25\,000}{\Pi}\Big(L_t+\Big(\sum_{i=1}^{N_t}(F_D(T_2-\tau_i)-F_D(t-\tau_i))\\ +\tilde{\Lambda}(T_1-t)E_Q[F_D(T_2-\tilde{\tau})]\Big)\tilde{\lambda}E_Q[Y]\Big).$$

for $t\in[0,T_1]$ *and*

$$\frac{25\,000}{\Pi}\Big(L_t+\sum_{i=1}^{N_{T_1}}(F_D(T_2-\tau_i)-F_D(t-\tau_i))\tilde{\lambda}E_Q[Y]\Big)$$

for $t\in[T_1,T_2]$.

Proof: We only consider $E_Q[L_{T_2}-L_t \mid \mathcal{F}_t]$. From the considerations in Section 2 we know that for $t<T_1$

$$\begin{aligned} &E_Q[(L_{T_2}-L_t)\mid\mathcal{F}_t]\\ &= \Big(E_Q\Big[\sum_{i=1}^{N_t}\tilde{\lambda}(F_D(T_2-\tau_i)-F_D(t-\tau_i))\;\Big|\;N_t,\tau_1,\dots,\tau_{N_t}\Big]\\ &\quad+E_Q\Big[\sum_{i=N_t+1}^{N_{T_1}}\tilde{\lambda}F_D(T_2-\tilde{\tau}_i)\Big]\Big)E_Q[Y_{ij}]\\ &= \Big(\sum_{i=1}^{N_t}(F_D(T_2-\tau_i)-F_D(t-\tau_i))\\ &\quad+\tilde{\Lambda}(T_1-t)E_Q[F_D(T_2-\tilde{\tau})]\Big)\tilde{\lambda}E_Q[Y]\,. \end{aligned} \tag{3.3}$$

Note that

$$E_Q[F_D(T_2-\tilde{\tau})]=E_P[F_D(T_2-\tilde{\tau})]=\frac{1}{T_1-t}\int_{T_2-T_1}^{T_2-t}F_D(s)\,ds\,, \tag{3.4}$$

provided $t<T_1$, and $E_Q[Y]=m_Y'(\alpha)/m_Y(\alpha)$. If $T_1\le t\le T_2$ we find

$$E_Q[(L_{T_2}-L_t)\mid\mathcal{F}_t]=\sum_{i=1}^{N_{T_1}}(F_D(T_2-\tau_i)-F_D(t-\tau_i))\tilde{\lambda}E_Q[Y]\,.$$

The approximation error will be discussed in Section 4.

3.2. **Stochastic λ^i.** We now assume that the λ^i's are stochastic and independent. This can be seen as a measure of the severity of the catastrophe. For simplicity of the model, we assume that λ^i can be observed via reported claims only. Of course, in reality other information as TV-pictures or reports from the affected area will be available. Then for claims occurring before t we have some information on the intensity parameter λ^i. We therefore have to work with the posterior distribution of λ^i given $\mathcal{F}_t$. It would be desirable if the prior and the posterior distribution would belong to the same class, see the discussion in [6, Ch.10]. We therefore choose λ^i to be Γ distributed. Let $\lambda^i \sim \Gamma(\gamma, \eta)$.

We find

$$E_P[\exp\{\alpha L_\infty\}] = \exp\left\{\Lambda\left(\frac{\eta}{\eta - m_Y(\alpha) + 1}\right)^\gamma - 1\right\}.$$

It again turns out that under the measure Q the process (L_t) is of the same type with different parameters. (N_t) is a Poisson process with rate $\tilde{\Lambda} = \Lambda\eta^\gamma(\eta - m_Y(\alpha) + 1)^{-\gamma}$, λ^i is $\Gamma(\gamma, \eta - m_Y(\alpha) + 1)$ distributed, M_i given λ^i is conditionally Poisson distributed with parameter $\tilde{\lambda}^i = \lambda^i m_Y(\alpha)$ and Y has distribution $\tilde{F}_Y(x) = \int_0^x e^{\alpha y}\, dF_Y(y)/m_Y(\alpha)$. As before the lags (D_{ij}) have the same distribution under Q as under P. Thus M_i has a mixed Poisson distribution where the mixing variable $\tilde{\lambda}^i$ is $\Gamma(\gamma, (\eta - m_Y(\alpha) + 1)/m_Y(\alpha))$ distributed. Let $\tilde{\gamma} = \gamma$ and $\tilde{\eta} = (\eta - m_Y(\alpha) + 1)/m_Y(\alpha)$.

We now fix the time t at which we want to find the CAT-future price. For $i \le N_t$ the posterior distribution of $\tilde{\lambda}^i$ at time t is then

$$\tilde{\lambda}^i \,|_{\mathcal{F}_t} \sim \Gamma(\gamma + M_i(t), F_D(t - \tau_i) + \tilde{\eta})\,. \tag{3.5}$$

Theorem 3.6. *Let the assumptions be as in Section 2 with a fixed risk aversion coefficient α. Assume further that $\lambda^i \sim \Gamma(\gamma, \eta)$. Then p_t^{approx} for a given risk aversion coefficient α, is given by*

$$\frac{25\,000}{\Pi}\Big(L_t + \Big(\sum_{i=1}^{N_t} E_Q[\tilde{\lambda}^i \mid \mathcal{F}_t](F_D(T_2 - \tau_i) - F_D(t - \tau_i)) + \frac{\gamma m_Y(\alpha)\tilde{\Lambda}(T_1 - t)}{\eta - m_Y(\alpha) + 1} E_Q[F_D(T_2 - \tilde{\tau})]\Big) E_Q[Y]\Big)$$

for $t \in [0, T_1]$ and

$$\frac{25\,000}{\Pi}\Big(L_t + \sum_{i=1}^{N_{T_1}} E_Q[\tilde{\lambda}^i \mid \mathcal{F}_t](F_D(T_2 - \tau_i) - F_D(t - \tau_i))E_Q[Y]\Big)$$

for $t \in [T_1, T_2]$.

Proof: For the calculation of $E_Q[L_{T_2} - L_t \mid \mathcal{F}_t]$ we again split $L_{T_2} - L_t$ into the terms occuring from catastrophes occurred before and catastrophes that will occur in the future. Consider first the case $t < T_1$. The first terms have expectation

$$\sum_{i=1}^{N_t} E_Q[\tilde{\lambda}^i \mid \mathcal{F}_t](F_D(T_2 - \tau_i) - F_D(t - \tau_i))E_Q[Y]\,.$$

Note that $E_Q[\tilde{\lambda}^i \mid \mathcal{F}_t] = (\gamma + M_i(t))/(F_D(t - \tau_i) + \tilde{\eta})$.

For the expectation of the second terms we obtain

$$\begin{aligned}&\tilde{\Lambda}(T_1 - t)E_Q[\tilde{\lambda}^i F_D(T_2 - \tilde{\tau})]E_Q[Y_{ij}] \\ = \;&\frac{\gamma m_Y(\alpha)\tilde{\Lambda}(T_1 - t)}{\eta - m_Y(\alpha) + 1} E_Q[F_D(T_2 - \tilde{\tau})]E_Q[Y]\,.\end{aligned}$$

The expectation was already calculated in (3.4). This yields the desired expressions. If $T_1 \leq t \leq T_2$ we find

$$E_Q[(L_{T_2} - L_t) \mid \mathcal{F}_t] = \sum_{i=1}^{N_{T_1}} E_Q[\tilde{\lambda}^i \mid \mathcal{F}_t](F_D(T_2 - \tau_i) - F_D(t - \tau_i))E_Q[Y]\,.$$

Note that with the exception of (3.4) the upper bound can be found explicitly.

4. The approximation error

The results in Theorems 3.2 and 3.6 are both approximations, so it is relevant to ask how good these approximations are. In this section we will investigate this question. We only consider the case where $\lambda^i = \lambda$ is constant. For the mixed Poisson case the results are similar.

4.1. **The approximation of the approximation error.** From Section 3.1 we know that the approximation error (AE) is given by the following expression:

$$c\Big(\int_{2\Pi - L_t}^{\infty} (1 - \tilde{F}_L(x;t))\,dx\Big) \tag{4.1}$$

where $\tilde{F}_L(\cdot;t)$ denotes the distribution function of $L_{T_2} - L_t$ under Q conditioned on $\mathcal{F}_t$. The reason for omitting this term was that it is hard to calculate $\tilde{F}_L(\cdot;t)$. In order to find an approximation to the expression above we will now try to use some of the approximations to $L_{T_2} - L_t$ known from actuarial mathematics. Namely the translated gamma approximation and the Edgeworth approximation.

The idea behind the translated gamma approximation is to approximate the distribution function by $k + Z$ where k is a constant and Z is $\Gamma(g,h)$ distributed, such that the first three moments of $L_{T_2} - L_t$ and $k + Z$ coincide. We already have calculated the mean value μ_L of $L_{T_2} - L_t$ in (3.3). Standard calculations yield also the (conditional) variance σ_L^2 and the (conditional) coefficient of skewness $s_L = E_Q[(L_{T_2} - L_t - E_Q[L_{T_2} - L_t])^3 \mid \mathcal{F}_t]\sigma_L^{-3}$. From this the parameters of the translated gamma distribution are found to be

$$g = \frac{4}{s_L^2}, \qquad h = \frac{2}{s_L \sigma_L}, \qquad k = \mu_L - \frac{2\sigma_L}{s_L}.$$

The approximation error therefore is approximated by

$$\begin{aligned}
\mathrm{AP(G)} &= c\Big(\int_{2\Pi-L_t}^{\infty}\int_{x-k}^{\infty}\frac{h^g}{\Gamma(g)}y^{g-1}e^{-hy}\,dy\,dx\Big)\\
&= c\Big(\frac{h^g}{\Gamma(g)}\Big(\int_{(2\Pi-L_t)-k}^{\infty} y^g e^{-hy}\,dy\\
&\quad -(2\Pi-L_t-k)\int_{(2\Pi-L_t)-k}^{\infty} y^{g-1}e^{-hy}\,dy\Big)\Big)\,.
\end{aligned}$$

The idea behind the Edgeworth approximation is to consider the corresponding standardized random variable Z and then to approximate its distribution. So consider the random variable

$$Z=\frac{L_{T_2}-L_t-E_Q[L_{T_2}-L_t]}{\sqrt{\mathrm{Var}_Q[L_{T_2}-L_t]}}\ .$$

The Taylor expansion of $\log M_Z(r)$ around $r=0$ has the form

$$\log M_Z(r)=a_0+a_1r+a_2\frac{r^2}{2}+a_3\frac{r^3}{6}+a_4\frac{r^4}{24}+\cdots$$

where

$$a_k=\frac{d^k\log M_Z(r)}{d_r^k}\bigg|_{r=0}\ .$$

Simple calculations show that $a_0=0$, $a_1=E[Z]=0$, $a_2=Var[Z]=1$, $a_3=s_L$ and $a_4=\frac{E[(L_{T_2}-L_t-E[L_{T_2}-L_t])^4]}{\mathrm{Var}[L_{T_2}-L_t]^2}-3$. In our case a_3 and a_4 are calculated under Q and conditioned on $\mathcal{F}_t$, and both the values can be found by standard calculations. We truncate the Taylor series after the term involving r^4. The moment generating function of Z can be written as

$$M_Z(r)\approx e^{r^2/2}e^{a_3r^3/6+a_4r^4/24}\approx e^{r^2/2}\left(1+a_3\frac{r^3}{6}+a_4\frac{r^4}{24}+a_3^2\frac{r^6}{72}\right)\ .$$

The inverse of $\exp\{r^2/2\}$ is easily found to be the normal distribution function $\Phi(x)$. For the other terms we derive

$$r^ne^{r^2/2}=\int_{-\infty}^{\infty}(e^{rx})^{(n)}\,\Phi'(x)\,dx=(-1)^n\int_{-\infty}^{\infty}e^{rx}\Phi^{(n+1)}(x)\,dx\ .$$

Thus the inverse of $r^ne^{r^2/2}$ is $(-1)^n$ times the n-th derivative of Φ. The approximation yields

$$\begin{aligned}
P[L_{T_2}-L_t\le x] &= P[Z\le z]\\
&\approx \Phi(z)-\frac{a_3}{6}\Phi^{(3)}(z)+\frac{a_4}{24}\Phi^{(4)}(z)+\frac{a_3^2}{72}\Phi^{(6)}(z)
\end{aligned}$$

where $z = (x - E[L_{T_2} - L_t])/\sqrt{\mathrm{Var}[L_{T_2} - L_t]}$. The approximation error therefore is approximated by

$$\mathrm{AP(E)} = c\Big(\sqrt{\mathrm{Var}[L_{T_2} - L_t]}$$

$$\int_{z_0}^{\infty} \Phi(z) - \frac{a_3}{6}\Phi^{(3)}(z) + \frac{a_4}{24}\Phi^{(4)}(z) + \frac{a_3^2}{72}\Phi^{(6)}(z)\, dz\Big)\ .$$

where $z_0 = ((2\Pi - L_t) - E[L_{T_2} - L_t])/\sqrt{\mathrm{Var}[L_{T_2} - L_t]}$.

We now have constructed two ways of approximating the AE. The question is then whether we obtain a better price if we correct the uncapped future price with these approximations, or we are better off just using the uncapped future price directly? We will now look at an example in order to answer this question.

4.2. **Example.** The capped future price (F_t^C) is calculated according to equation (1.2)

$$F_t^C = c\frac{E_P[\exp(\alpha L_\infty)\,(L_{T_2} \wedge 2\Pi) \mid \mathcal{F}_t]}{E_P[\exp(\alpha L_\infty) \mid \mathcal{F}_t]}$$

where a reliable value of the expression is obtained by Monte-Carlo simulations. In order to use MC we make the following assumptions:

- The claim sizes are exponentially distributed with parameter μ,
- The reporting lags are exponentially distributed with parameter β.

The uncapped future price (F_t^U) is calculated according to Theorem 3.2.

We will keep all the parameters fixed in the example, except from the premium Π and the risk aversion coefficient α in order to see how the approximations depend on these two parameters. We use the following parameters:

$$\begin{array}{lll}
T_1 = 1 & T_2 = 2 & t = 0.5 \\
\Lambda = 6 & N_t = 3 & \mu = 0.0005 \\
\tau_1 = 0.1 & \tau_2 = 0.25 & \tau_3 = 0.4 \\
M_1(t) = 698 & M_2(t) = 528 & M_3(t) = 259 \\
\Pi = (1+\theta)12 \times 10^6 & \lambda = 1000 & \\
L_t = E[L_t] = 2.97 \times 10^6 & \beta = 3 &
\end{array}$$

Π is calculated by the expected value principle with safety loading θ under the assumption that all the claims will be reported.

The parameters are chosen such that $P(L_{T_2} > 2\Pi)$ is consistent with the few data that we had. None out of the approximately 80 available settlement values exceeded the level 2Π. For dates before 1992 the ISO index had to be estimated from the final aggregate loss value (L_∞). The largest values of the ratio L_{T_2}/Π there have been seen so far is 1.7893 (the Eastern Loss Ratio from Hurricane Andrew, Sept. 1992) and 1.0508 (the Western Loss Ratio from Northridge Earthquake, 2nd March 1994). In our example with $\theta = 0.05$ we have that $P(L_{T_2}/\Pi > 1.79) \approx 0.01$.

For different values of α and θ, Table 4.1 shows the values of F_t^C, F_t^U, the approximation error AE $= F_t^U - F_t^C$ by using the uncapped future price, the approximation error $\text{AE(G)} = (F_t^U - \text{AP(G)}) - F_t^C$ if we correct F_t^U by the gamma approximation to (4.1), and finally the approximation error $\text{AE(E)} = (F_t^U - \text{AP(E)}) - F_t^C$ if we correct F_t^U by the Edgeworth approximation (4.1).

α	θ	F_t^C	F_t^U	AE	AE(G)	AE(E)
1×10^{-8}	0.05	23666.8	23668.3	1.5	-1.9	0.5
1×10^{-8}	0.10	22590.8	22592.5	1.7	0.2	1.4
1×10^{-8}	0.15	21608.8	21610.2	1.4	0.7	1.3
1×10^{-7}	0.05	25999.3	26009.7	10.4	-2.1	6.2
1×10^{-7}	0.10	24822.5	24827.5	5.0	-1.0	3.2
1×10^{-7}	0.15	23743.7	23748.0	4.3	1.5	3.6
2×10^{-7}	0.05	29106.7	29158.8	52.1	0.7	32.0
2×10^{-7}	0.10	27808.1	27833.4	25.3	-1.7	16.7
2×10^{-7}	0.15	26605.0	26623.2	18.2	4.3	14.3
3×10^{-7}	0.05	32817.4	33008.2	190.8	-6.0	62.4
3×10^{-7}	0.10	31402.1	31507.9	105.8	-7.4	47.2
3×10^{-7}	0.15	30052.8	30138.0	85.2	21.4	60.0

TABLE 4.1. The approximation errors.

From Table 4.1 we see, that for all the chosen parameters the uncapped future price seems to approximate the capped future price fairly well, and best when the risk aversion coefficient is small or the safety loading is large. But are the chosen parameters reasonable?

Let us first discuss the θ parameter. In insurance the safety loading is always positive, and looking at real data the safety loading seems to be "large" when we are considering catastrophe insurance. By "large" we mean that the event $\{L_{T_2} > 2\Pi\}$ never has occured.

The α parameter is the risk aversion coefficient for the single insurance company when pricing in a utility maximization framework (see [9] for further details), or the markets risk aversion when pricing in a general equilibrium model (see [9] for further details). The first thing to note on the α parameter is that the parameter is price defined, i.e. it depends on the way we price the losses. Here the values of the losses are "large" and therefore the α parameter becomes "small". The parameters are then chosen in such a way that different prices are represented, for $\theta = 0.15$ and $\alpha = 1 \times 10^{-8}$ the capped future price is

21608.8 and for $\theta = 0.05$ and $\alpha = 3 \times 10^{-7}$ the capped future price is 32812.7. An indication that the single insurance company or the market should have a low risk aversion coefficient, is the market conditions: The CAT-future pays a high profit with a small probability and a low profit with a high probability. After these remarks on the parameters we now turn to the figures.

From Table 4.1 we see that there is some variance on the figures from the MC simulations. This is observed in the column named AE, where the AE should be decreasing when the θ's are increasing. But apart from this it is clear that both approximations give reasonable values for the approximation error, i.e. if the uncapped future price is corrected with one of the approximations we in general obtain a more accurate price. From the values it seems like the AE(E) underestimates the AE, but even though that it is the case in this example this does not hold in general. Based on this example the gamma approximation gives the best approximations for nearly all the values. The only exception is for $\alpha = 1 \times 10^{-8}$, and $\theta = 0.05$, but this is probably caused by the variance in the MC. So based on this example the gamma approximation is the best one to use.

Finally we conclude that in our model under the above assumption the uncapped future price is a good approximation. But as mentioned above we obtain a more accurate price if we correct with one of the approximations, and in this example the gamma approximation is the best one to use.

5. Conclusion

This paper develops a model for insurance future pricing, which only relies on the information available. The products are priced solely from observing the reporting stream. Contrary to the existing literature we model the reporting times explicitly. We thereby obtain a more realistic model.

The results of this article rely on an approximation to the exact future price. One therefore has to be careful applying the results derived, because the results will be inaccurate if the cap-probability ($P(L_{T_2} > 2\Pi)$) or the risk aversion coefficient is "too large".

This paper suggest two ways to approximate the approximation error, the gamma approximation and the Edgeworth approximation. It is shown that they both are useful in the determination of the error level even though that the gamma approximation seems to be the best.

The results are derived specially for the CAT-futures, even though an improved financial catastrophe insurance product, the PCS-option, was introduced in 1995. For a description of the PCS-option and an explanation of why the CAT-future was improved see [5]. The results from this article cannot directly be used for pricing PCS-options because they have another structure. But, because of a strong correlation between claims reported and the PCS-index some of the ideas may be used. This is a topic for further research.

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(C. Vorm Christensen) Department of Theoretical Statistics and Operations Research, University of Aarhus, Ny Munkegade 116, 8000 Aarhus C, Denmark

E-mail address: vorm@imf.au.dk

(H. Schmidli) Department of Theoretical Statistics and Operations Research, University of Aarhus, Ny Munkegade 116, 8000 Aarhus C, Denmark

E-mail address: schmidli@imf.au.dk

THE PCS-OPTION, AN IMPROVEMENT OF THE CAT-FUTURE

CLAUS VORM CHRISTENSEN

Abstract. In 1992, CBoT introduced the CAT-future as an alternative to catastrophe reinsurance. But the product never became very popular. In 1995 it was replaced by a new product, the PCS-option. This article describes the PCS-option and attempts to explain why the new product is an improvement. The article also explains how to hedge a catastrophe risk with PCS-options and finally it compares the PCS-options with traditional reinsurance.

1. Introduction

The insurance industry has been hit very hardly in the 1990s by their catastrophe insurances. This has been caused by a record number of natural catastrophe losses, of which the insurance premiums only covered a small part. At the same time many of the catastrophe premiums are very large (approximately the value of the maximal losses), so there is hardly no room for increasing the capacity in the insurance market. Examples of risks which demand such large premiums are flooding in the Netherlands occuring every spring and earthquakes i L.A. also occuring regularly. Thus the search for new capacity has led to the prospect of trading insurance risk not only within the traditional insurance system but also transferring them to the more liquid financial markets.

On the December 11, 1992 CBoT made the first attempt to do so. They launched futures on catastrophe loss indices and related options (CAT-future and options). The CAT-option, also referred to as the future option, has as underlying instrument one catastrophe insurance future contract. Because of this relation the article will only consider the CAT-future in the section where these old product is under consideration). Following initial difficulties, which will be explained later, these standardized contracts have been improved, and on September 29, 1995 CBoT introduced the PCS-options. The underlying assets of the PCS-options are the PCS indices. These loss indices are provided daily to the CBoT by the Property Claim Services (PCS), which is the recognized industry authority for catastrophe property damage estimates.

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This article will first give a description of the PCS-option, then it will describe the ISO index which was the underlying index of the CAT-futures and also the main reason for the product's problems. Then the index for the PCS-options is described and it is explained how the PCS-options improved the CAT-future. It will then be shown how to hedge with PCS-options, and finally the article gives a description of PCS-options versus reinsurance.

2. Specification of the PCS-option

In this section the definitions of the keywords that specify the PCS-options are given. The information about the PCS-option is primarily taken from [2] and [5], but some was also obtained by mailing with people from the PCS a the CBOT. As mentioned above, the underlying asset of the PCS-options are the PCS indices. There are nine different type of indices, which are provided daily by the PCS. The nine indices are divided into one national index, five regional indices and three state indices. The five regional indices are: Eastern, Northeastern, Southeastern, Midwestern and Western. The three state indices are: Florida, Texas and California. Each loss index tracks PCS estimates for insured industry losses resulting from catastrophic events (as identified by PCS) in the area and loss period covered.

PCS-options can be traded as calls, puts, or spreads. Most of the trading activity occurs in call spreads, since they essentially work like aggregate excess-of-loss reinsurance agreements, see Section 4. PCS-options are traded both as "small cap" and as "large cap" contracts. These caps limit the loss that can be included under each contract. Small cap options track aggregate insured industry losses from \$ 0 to \$ 20 billion. Large cap options track aggregate insured industry losses from \$ 20 billion to \$ 50 billion.

The **loss period** is the time during which a catastrophic event must occur in order that resulting losses are included in a particular index. During the loss period, PCS provides loss estimates as catastrophes occur. Most PCS options have quarterly loss periods, with contracts listed for March, June, September and December. Western and California PCS-options have annual loss periods and are available only as annual contracts. The last day of the loss period is the calendar day of the quarter or year. Losses from catastrophes starting in one quarter or year and ending in the next will be included in the quarter or year in which the catastrophe started.

The **development period** is the time after the loss period during which PCS continues to estimate and reestimate losses from catastrophes occured during the loss period. PCS-option users can choose either a six-month or twelve-month development period. The development period begins immediately after the loss period ends. The PCS index value at the end of the chosen development period will be used for settlement purposes, even though PCS loss estimates may continue to change.

PCS-options settle in cash on the last business day of the development period. The settlement value (L_T) for each index represents the sum of then-current PCS insured loss estimates provided and revised over the loss and development periods. PCS-options are options of European type, that means that they can be exercised on the expiration day at the end of the development period only.

The value of the PCS-call option at expiration day T, exercise price X and cap value K can be expressed as

$$C(T, L(T)) = \min(\max(L(T) - X, 0), K - X)$$

Due to difficulties in trading options in industry loss dollar amounts, the CBoT has developed a pricing index to reflect dollar loss amounts raging from \$ 0 to \$ 50 billion. Each PCS loss index represents the sum of then-current PCS estimates for insured catastrophic losses in the area and loss period covered, divided by \$ 100 million and rounded to the nearest first decimal point. PCS-options prices or premiums, are quoted in points and tenths of a point. Each point equals \$ 200; each tenth of a point equals \$ 20. We will end this subsection with an example.

Example 2.1. Let us consider a reinsurer who buys a June Eastern small cap call PCS-option with strike value of 20 and a development period of six-months. This contract tracks losses from catastrophic events occuring in the Eastern region between April 1 and June 30 1997. The six-month development period runs from July 1 to December 31. The option will thus settle on December 31, 1997 according to the settlement value of the index.

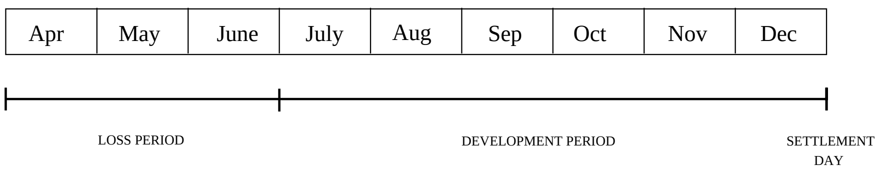

FIGURE 1. June PCS-option with development period of six-months

Let us assume that the losses have been estimated to \$ 3.565.270.000. The index value would then be 35.65 rounded to 35.7. The value of the call option is then

$$C(T, L(T)) = \min(\max(35.7 - 20, 0), 200 - 20) * \$200 = \$3140$$

So in this example the reinsurer receives \$ 3140. If the loss index has been estimated above \$ 20 billion let us say \$ 23 billion then the value of the call option would be

$$C(T, L(T)) = \min(\max(230 - 20, 0), 200 - 20) * \$200 = \$36000$$

and the reinsurer would then (only) have received \$ 36000 because it was a small cap option.

3. Loss estimation

As mentioned in the introduction, the PCS-options is an improvement of the CAT-futures. The main problems with CAT-futures were caused by the underlying asset, the so called ISO index. The improvements have therefore mainly been achieved by changing this underlying asset. This subsection will highlight some of the problems of the CAT-futures, and explain how the introduction of the PCS-option solved some of them. The information about the CAT-future is obtained from [1] and [3]. Let us first consider the ISO index.

3.1. **The ISO index.** Each quarter approximately 100 American insurance companies reported property loss data to the ISO (Insurance Service Office, a well known statistical agent). ISO then selected a pool of at least ten of these companies on basis of size, diversity of business, and quality of reported data. The ISO index was then calculated as the loss ratio of this pool.
The ISO index:

$$\text{ISO index} = \frac{\text{reported incurred losses}}{\text{earned premiums}}.$$

The list of companies included in the pool was announced by the CBoT prior to the beginning of the trading period for that contract. The CBoT also announced the premium volume of the companies participating in the pool prior to the start of the trading period. Thus the premium in the pool was a known constant throughout the trading period, and price changes were attributed solely to changes in the markets expectation of loss liabilities.

CAT-futures were traded on a quarterly cycle, with contract months March, June, September, and December. A contract for any given calendar quarter (the event quarter) was based on losses occuring in the listed quarter, and beeing reported to the participating companies by the end of the following quarter. The six month period following the start of the event quarter is known as the reporting period. The three additional reporting months following the close of the event quarter are to allow for loss settlement lags that are common in insurance. The contracts expire on the fifth day of the fourth month following the end of the reporting period. The additional three months following the reporting period is attributable to data processing lags. Trading was conducted from the date the contract was listed until the settlement date.

Example 3.1. The June contract covers losses from events occuring in April, May and June as reported to the participating companies by the end of September. The June contract expires on January 5th the following year. The contract is illustrated by the figure below.

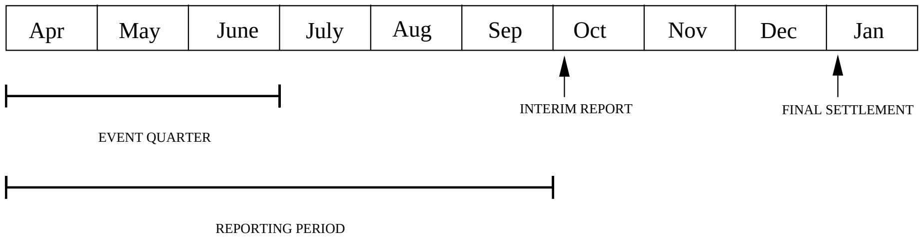

FIGURE 2. June CAT-future contract

Finally the settlement value for the CAT-futures was given by:

$$F_T = \$25000 \times \min(I_T, 2)$$

where I_T is the ISO index at time T, i.e. the ratio between the losses incurred during the event quarter, though reported up un till three months later, and the premium volume for the companies participating in the pool.

Let us now focus on the problems with the ISO index. Let I_t be the value of the ISO index at time t. One of the problems was that I_t was only published once before the settlement date. This took place just after the end of the reporting period (the Interim report see figure 2). This meant that the companies, participating in the pool, had a possibility of knowing at least part of the data used to form the index before the settlement date, while it was certainly more difficult for other insurers. This created a information asymmetry which was a potential factor preventing people from entering the market of CAT-futures.

Another problem was the Moral Hazard problem. A company from the pool could manipulate data by delaying the report of a big loss so it first would be included in the next reporting period and thereby never affect the index. The companys intension for doing so, could be that the company had agreed to a short position of a future contract[1]. That this possibility existed, could also have prevented people from entering the market of CAT-futures.

[1]When a company, at time t and at a price F_t, enters a short position of a future contract, it means that the company should pay $(F_T - F_t)$ to the other part of the contract at time T. The company will then like F_T to be as small as possible

As mentioned in [4] a more serious problem could occur because the reporting period was too short. If a late quarter catastrophe occurs and claims are slow in developing, then the final claims ratio for the purpose of deciding the future payoff could be low relative to the actual final claims ratio. This problem occurred in the March 1994 contract period, the period of the Northridge earthquake. The settlement ratio was low and the contract payoff did not truly reflect the actual claim loss. After this description of the ISO index and its problems, we now turn to the PCS index.

3.2. **The PCS index.** Property Claim Services (PCS), a division of American Insurance Services Group, is the recognized industry authority for catastrophe property damage estimates. PCS is a not-for-profit organization serving the insurance industry.

When PCS, in its sole judgement, estimates that a natural or man-made event within the United States is likely to cause more than $25 million in total insured property losses, and determines that such effect is likely to affect a significant number of policy holders and property/casualty insurance companies, PCS identifies the event as a catastrophe and assigns it a catastrophe serial number (a "PCS Identified Catastrophe"). The types of insured "perils" that have caused insured losses deemed catastrophic by PCS include, without limitation, tornadoes, hurricanes, storms, floods, ice and snow, freezing, wind, water damage, hail, earthquakes, fires, explosions, volcanic eruptions and civil disorders. PCS compiles three different types of estimates: The Flash Loss Estimates, the Preliminary Loss Estimates and the Resurvey Loss Estimates. Let us now focus on these.

Simultaneously with announcing that a catastrophe has been identified (generally within 48-72 hours after the occurrence of a PCS Identified Catastrophe), PCS generally provides a "flash" estimate anticipated industry insured property losses from such event. The Flash Estimates generally are based on PCS's initial meteorological or seismological information and/or initial telephonic information from industry personnel and public officials in the affected areas. Such estimates are expressed in terms of a range of estimated total insured property losses. These Flash Loss Estimates give the insurers and reinsurers an initial perspective on the catastrophe's severity, but are not included in the indices calculated for the CBoT.

The indices compiled for the CBoT, comprise Preliminary Loss Estimates and are adjusted according to Resurvey Loss Estimates. The Preliminary Loss Estimate of anticipated insured property losses is typically prepared and released within several days to two weeks after occurrence of a PCS Identified Catastrophe. If a catastrophe is large enough, PCS will continue to survey loss information to determine whether its preliminary estimate should be adjusted. PCS generally resurveys PCS Identified Catastrophes that, based upon its Preliminary Estimate, appear to have caused more than \$250 million of insured property damage. PCS usually releases the initial Resurvey Estimate to subscribers approximately 60 days after the Preliminary Estimate is issued. PCS may continue the resurvey process and publish additional Resurvey Estimates approximately every 60 days after the previous Preliminary Estimate or Resurvey Estimate, until it believes that the industry insured property loss has been reasonably approximated. This means that the insured losses due to certain catastrophes may continue to develop after PCS-option settlement. PCS compiles its estimates of insured property damage using a combination of procedures, including a general survey of insurers, its National Risk Profile, and where appropriate, its own on-the-ground survey. PCS will report the PCS loss indices on each CBoT trading day. But the indices are only changed when a new Preliminary Loss Estimate is released or a Resurvey Loss Estimate is released.

Let us now return to the problems of the CAT-futures and how the PCS options solve them. Neither American Insurance Services Group nor any person employed by American Insurance Services Group will disclose any estimate of total insured losses following a catastrophe to any person prior to its official publication. This means that all investors receive the same information at the same time. Thereby the problem of asymmetric information is eliminated.

When PCS estimates the loss indices, they conduct surveys of the market. These surveys are confidential and they are not used directly in the estimation of the indices. So it is extremely difficult for insurance companies to affect the indices, and thereby the Moral Hazard Problem is eliminated.

The construction of the PCS-option also eliminates the problem by late occured catastrophes. The PCS index does not directly depend on a number of reported claims and the time from the end of the event period to the time the index is settled is also longer for the PCS-option than it was for the CAT-future.

That these problems were solved, was probably the main reason for the PCS-options higher trading activity compared to the CAT-future. But the fact that the new product was more logically constructed than the old one could also have had an effect. Hereby we mean that a construction using options instead of futures and “options on futures”, seems more logical, when all the trading activities are in options.

Next will be explained, as mentioned in section 2, that the call spreads work much like aggregate excess-of-loss reinsurance agreements.

4. Hedging with PCS-options

This Subsection will describe how to hedge against catastrophe losses using PCS-option. Only the PCS-option spreads will be considered since most of the trading activity occurs in these products.

The buyers of PCS-options spreads are mainly large insurance companies and reinsurance companies. The sellers could be investors, as for instance companies earning money in relation to catastrophes (such as building supply firms or construction companies). To illustrate how PCS-option spreads work, lets now consider a hypothetical insurance company. The Safe-place Insurance Company is a property/casualty insurance company with a book of business heavily concentrated in the Eastern region.

Assume that:

(a) Safe-Place Insurance Company has a 0.2% market share (measured in written premium)
(b) Safe-Place's book is less exposed on average to hurricane risk than that of the industry. More specifically, assume that Safe-Place anticipates its losses to be 80% of the industry on average.
(c) Safe-Place wants to hedge catastrophe losses in the hurricane season (the third quarter) by buying a layer of protection of \$ 6 million in excess of \$ 4 million.

The question is now, what kind of September Eastern option spreads should Safe-Place buy (the September Eastern contract track third-quarter losses for the eastern region of the United states)? Based on the given assumption, Safe-Place calculates the appropriate amount of protection by relating its attachment point to the industrys attachment points as follows:

$$\text{Strike value} = \text{Comp. loss} \times \frac{1}{\text{comp. market share}} \times \frac{1}{\text{loss experience}}$$

At the \$4 million attachment point:

$$\text{Strike value} = \$\ 4 \text{ million} \times \frac{1}{0.2\ \%} \times \frac{1}{80\ \%} = \$\ 2.5 \text{ billion or } 25 \text{ p.}$$

At the \$10 million attachment point(\$ 6 million excess of \$ 4 million):

$$\text{Strike value} = \$\ 10 \text{ million} \times \frac{1}{0.2\ \%} \times \frac{1}{80\ \%} = \$\ 6.25 \text{ billion or } 62.5 \text{ p.}$$

Safe-Place's \$ 6 million in excess of \$ 4 million level of protection is now approximated by a 25/65 call spread (PCS-option contract specifications call only for five-point strike intervals). This translates to an industry loss range of \$ 2.5 billion to \$ 6.5 billion.

Safe-Place calculates the proper number of spreads as follows:

$$\text{Number of spreads} = \frac{\text{ammount of protection needed}}{\text{amount of protection offered by each layer}}$$

$$\text{Number of spreads} = \frac{\$\ 6 \text{ million}}{\$((65-25)\times\$200)} = 750 \text{ spreads}$$

They therefore decides to buy 750 25/65 September Eastern call spreads (the September Eastern contract track third-quarter losses for the eastern region of the United states). In other words they would buy 750 call at strike value 25 and simultaneously sell 750 call at strike value 65. The value of a 25/65 call spread can be illustrated by figure 3. The two dotted lines illustrate the payoff from selling a call 65 and from buying a call 25. The full-drawn line is the the total payoff from buying a call spread 25/65.

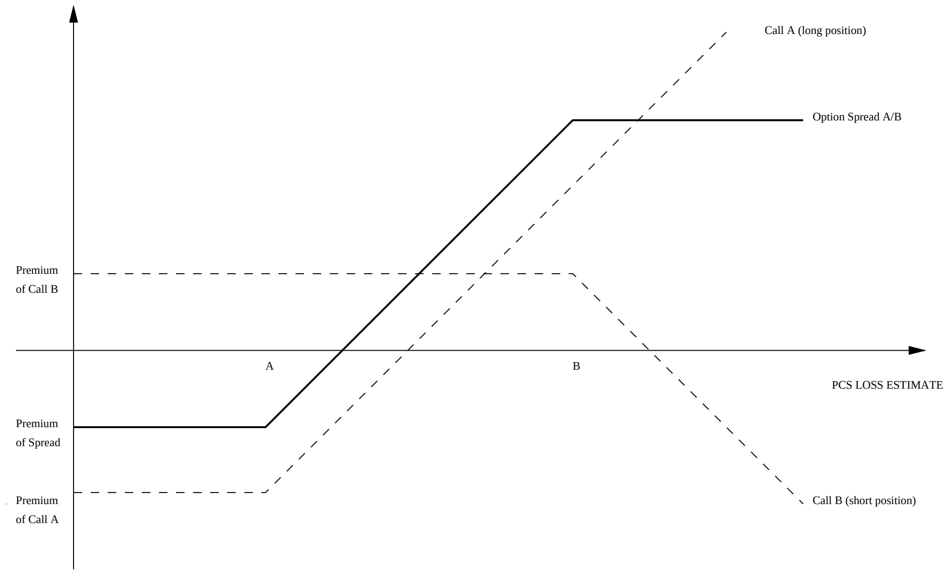

FIGURE 3. A 25/65 call option spread.

If the losses have been estimated below \$2.5 billion, the 25/65 call spread has no value. If the losses have been estimated above \$65 billion Safe-Place receives a full protection payment, that is, $40 \times \$200 \times 750$, or \$6 million which is the amount of protection originally desired by the firm. If the losses have been estimated between the 25/65 attachment points, Safe-Place is compensated for the difference between the lower attachment point (25) and the actual settlement value. For instance, assume that the losses have been estimated to \$4 billion. This industry loss amount corresponds to a \$6.4 million aggregate loss for Safe-Place, or a \$2.4 million excess of the \$4 million retained by the company. On settlement day Safe-Place receives compensation equal to $(40 - 25) \times$ $\$200 \times 750$ spreads, or \$2.25 million. This amount helps to offset the companys original loss.

This example shows the PCS call spreads work much like layers of aggregate excess-of-loss reinsurance, but as we shall see in the next section they are not perfect substitutes.

5. PCS-options versus Reinsurance contracts

As mentioned in section 4, the PCS-options are similar to the structures of typical stop loss reinsurance contracts, but there are important differences between reinsurance and financial contracts.

The buyers and sellers of a PCS-option are anonymous to each other and the price is determined through an auction market. For the reinsurance contract it is different, here the contract is negotiated between the buyer and the seller, and the price is determined through a negotiation process. These conditions make the buyers of an actively traded PCS-options more assured of receiving an arbitrage-free price than the buyers of a reinsurance contract. Even though the buyer negotiated with several reinsurers before making a decision.

A main problem of the PCS-option is that the loss, estimated by the PCS, is not necessarily perfectly correlated to the buyer's losses. This is not a problem in reinsurance because it is the buyer's own loss experiences that are covered. But that the PCS-options are standardized to all buyers and sellers does have some advantages. If the general risk exposure suddenly changes then the people on the option market have the possibility of closing out a position by taking the opposite position. The portfolio can also be adjusted using the experience of losses included in the index. For instance if a company has an upper layer, A say, and there had been already many catastrophes, so that A is likely to be exceeded, the company can adjust the portfolio by buying AB-spreads for $B > A$, giving them a new upper layer B. This is usually not possible for people on the reinsurance market, because of the buyer specific reinsurance contracts.

Another advantage of the PCS-options is the time factor. The PCS-options can be bought and sold in a second, assuming the buyer has been accepted by the clearinghouse. Unlike the reinsurance contract, where it often takes quite a while before the contract is negotiated and underwritten by the reinsurer. And in the reinsurance market there is no clearinghouse to assure the buyers and sellers about the credit worthiness of the other party.

PCS-options and reinsurance can both be used in hedging underwritten risk, but as seen above they are not perfect substitutes. It is therefore likely that we will see both type of contracts in the future.

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(C. Vorm Christensen) Department of Theoretical Statistics and Operations Research, University of Aarhus, Ny Munkegade 116, 8000 Aarhus C, Denmark

E-mail address: vorm@imf.au.dk

A NEW MODEL FOR PRICING CATASTROPHE INSURANCE DERIVATIVES

CLAUS VORM CHRISTENSEN

Abstract. We want to price catastrophe insurance derivatives and we are therefore facing two main problems. The first problem is that under a realistic model for the underlying loss process the market is incomplete and there exist many equivalent martingale measures. Hence there are several arbitrage free prices of the product. The other problem is that we prefer a heavy tail for the underlying loss index, but heavy tails often give computational problems. In this note we will present a model which in some sense takes care of both the problems. We will in particular consider the PCS option, but the approach can also be used for pricing other securities relying on a catastrophe loss index.

1. Introduction

After Hurricane Andrew in 1992 there followed a reinsurance capacity shortage and a huge increase in property catastrophe reinsurance premiums. Both phenomena where reinforced by the occurrence of the Northridge Earthquake in 1994. The reinsurance industry therefore needed new capital. This capital should be found in the financial market and the way to obtain it was to create the securitization market.

One of the first products on this market was the CAT future introduced by the CBOT (Chicago Board of Trade) in 1992. And ever since the market has tried to fulfil the investors requirements, but the market is still not well launched. J. A. Tilley [12] is mentioning the following four reasons why this market is emerging so slowly. First, since 1994 there has been a generally favourable catastrophe loss experience and as a result of this the reinsurance prices have decreased. This becomes a problem because many cedents of risk both primary writers and reinsurers have considered securitization as an alternative to reinsurance rather than complementary to reinsurance. Second, insurers are unwilling to be pioneers, because of the high development cost. Third, the fact that the products are uncorrelated to other financial products is not a good enough selling story for investors. Investors

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want to understand the nature of the risk, and this takes time. And finally there still remains unanswered questions about what form and structure of insurance linked securities and derivatives will be viewed most favourable by investors.

But even though the market have not been well launched and the products has not been standardized, academics has tried to model the prices of such products see [1], [2], [5], [6] and [8]. And this is also the aim of this paper. We aim to find a model that solves two of the main problems related to pricing. The first problem is that if we choose a realistic model for the underlying loss process the market will be incomplete and there will exist many equivalent martingale measures. Hence there exists a large set of arbitrage free prices of the product. The next problem is that we like a heavy tail for the underlying loss index, but heavy tails often give computational problems. The model presented takes in some sense care of both these problems and is to our knowledge the first one to do so. To derive the price of the securities we use results from Gerber and Shiu [10]. In [10] it is shown that the Esscher transform is an unique and transparent technique for valuing derivative securities if the logarithms of the underlying process are governed by a certain stochastic process with stationary and independent increments (a Levy process). We propose here such a model and by way of example we calculate prices for one the most standardized products on the market namely the PCS option.

In the remaining part of this introduction we give a short description of the PCS option, and the keywords that specify the PCS option are given. In Section 2 we present the model of the underlying loss index for the PCS option, and show how the option price is determined within this model. In Section 3, we show how the results from [10] can be used in our context to find the risk neutral Esscher measure. In Section 4 we show how to compute the risk neutral Esscher measure for the loss period and the development period. In Section 5 we then calculate the price for the PCS option based on the results obtained in the previous sections. And finally there are some concluding remarks.

1.1. **Specification of the PCS option.** In this section the definitions of the keywords that specify the PCS options are given. For a more detailed description of the PCS option see [3] or [4].

The PCS options are traded by Chicago Board of Trade and are regional contracts whose value is tied to the so called PCS Index. The PCS index tracks PCS estimates for insured industry losses resulting from catastrophic events (as identified by PCS) in the area and loss period covered. The options are traded as capped contracts, i.e. the cap limit the amount of losses that can be included under each contract.The value of a PCS call option at expiration day T, with exercise price A

and cap value K is given by

$$C(T, L_T) = \min(\max(L_T - A, 0), K - A)$$

where L_T is the value of the PCS index at time T.

PCS options can be traded as European calls, puts, or spreads. Most of the trading activity occurs in call spreads, since they essentially work like aggregate excess-of-loss reinsurance agreements see [4] for further explanations.

The option contract includes both a loss period and a development period. The **loss period** is the time during which a catastrophic event must occur in order for resulting losses to be included in a particular index. During the loss period, PCS provides loss estimates as catastrophes occur. The **development period** is the time after the loss period during which PCS continues to estimate and reestimate losses from catastrophes occurred during the loss period. The reestimations may result (and have resulted historically) in adjustments upwards and downwards. PCS option users can choose either a six-month or twelve-month development period. The settlement value for each index represents the sum of then-current PCS insured loss estimates provided and revised over the loss and development periods.

2. The underlying model

The most natural way to model the underlying loss index is to model it by a marked point process with a heavy tailed distribution function for the marks. But as previous papers has shown it is hard to price derivative securities in such a model see [1], [2], [5] and [8]. The idea in this paper is therefore to search for another model than the marked point process. This model may not be perfect, but on the other hand we hope to find a model that has a heavy tail, allows for fluctuation and gives the possibility to express the price in a closed form.

The model we present below is inspired by Gerber and Shiu [10]. In [10] they show how one can obtain a risk neutral measure in an unique and transparent way if the logarithms of the value of the underlying security is a Levy process. The idea is now to choose such a model.

Let now L_t be the underlying loss index for a catastrophe insurance derivative, $[0, T_1]$ be the loss period and $[T_1, T_2]$ be the development period. We then assume that L_t for all $t \in [0, T_2]$ is described by

$$L_t = L^0 \exp(X_t)$$

where X_t is a levy process in the development period and in the loss period and $L^0 \in \mathbb{R}^+$. We model X_t differently in the loss period and the development period, for a similar model see [11]. The question is then how to model X_t for the loss period and for the development period.

For $t \in [0, T_1]$ we will model X_t by a compound Poisson process

$$X_t = \sum_{i=1}^{N_t} Y_i \qquad \forall t \in [0, T_1]$$

where N_t is a Poisson process with a fixed parameter λ_1, and Y_i is exponentially distributed with parameter α. We hereby obtain one of the desired properties namely as mentioned above the heavy tail for L_t, e.g. when $X_t \sim$Exp(α) then $L_t - L^0$ is Pa(α, L^0) distributed. But as mentioned above this model is not chosen because it is the most obvious one but because it has a heavy tail, allows for fluctuation and gives the possibility to express the price in a closed form. The model therefore also has some disadvantages compared to a more natural model, firstly late catastrophes become more severe than earlier ones and secondly $L_0 = L^0 > 0$. For this reason this model should only be used as a first "crude" approximation to the real world. We have tried to work out these problems, but this seems to be hard.

It should be mentioned that it is possible to make a model which allows for more fluctuation, e.g. we can also express the price in a closed form with $Y_i = \sum_{j=1}^{M_i} Y_{ij}$ where M_i is negative binomial distributed and Y_{ij} is exponentially distributed, but to keep the model tractable we consider the simple model. Also in order to keep the model simple we assume that all the adjustments are done in the development period.

We now have to choose a model for X_t for the development period. We know that the adjustments are done both upwards and downwards we will therefore again describe X_t as a compound Poisson process for $t \in [T_1, T_2]$

$$X_t = X_{T_1} + \sum_{i=1}^{\tilde{N}_{t-T_1}} \tilde{Y}_i$$

where $\tilde{N}_t$ is a Poisson process with a fixed parameter λ_2 and $\tilde{Y}_i$ is normally distributed (N(μ, σ)), where the most natural choice of μ is $\mu = 0$ (unbiased previous estimates). In order to use the results from Gerber and Shiu [10] we need to assume that the process X_t for $t \in [0, T_1]$ is independent of the process $X_t - X_{T_1}$ for $t \in [T_1, T_2]$. In the real world one will expect some dependence but the assumption is invariable in order to use the results form [10].

The value of the PCS call option at expiration day T_2, with exercise price A and cap value K is given by

$$C(T_2, L_{T_2}) = \min(\max(L_{T_2} - A, 0), K - A).$$

Let $\mathcal{F}_t$ be the information available at time t. We will then make the following assumption

- $\mathcal{F}_t$ is the smallest right continuous complete filtration, such that the aggregate amount of reported losses at time t (L_t) is ($\mathcal{F}_t$)-adapted.

The value of the option at time t is then

$$C(t, L_t) = \exp(-r(T_2 - t))E^*[C(T_2, L_{T_2})|\mathcal{F}_t]$$

where r is the risk free interest rate and E^* is the mean value according to a risk neutral measure. Before we can proceed further in the calculation of the option price we will have to choose a risk neutral measure. This is done in the next sections.

3. The computation of the risk neutral measure

This section describe how to compute a risk neutral measure using the Esscher Transform. The theory was introduced by Gerber and Shiu [10]. But some adjustments have to be made in order to use their results in our context.

3.1. **The computation of the risk neutral measure.** Let L_t be the value of the PCS index at time t.

$$L_t = L^0 \exp(X_t), \qquad \forall t \in [0, T_1], \tag{3.1}$$

where X_t is a Levy process. To keep it simple we only consider the loss period, we extend the results to the development period later. Let $M(z,t)$ be the moment generating function defined by:

$$M(z,t) := E[\exp(zX_t)] = \int_{-\infty}^{\infty} \exp(zx)F(dx,t) \tag{3.2}$$

provided the integral is finite, where F denotes the distribution function for X_t. Because of the independent stationary increments we then have, (see [9], section IX.5), that

$$M(z,t) = (M(z,1))^t \tag{3.3}$$

For any $h \in \mathbb{R}$ the Esscher-Transformation $F(dx, t; h)$ is defined as:

$$F(dx,t;h) = \frac{\exp(hx)F(dx,t)}{M(h,t)}. \tag{3.4}$$

From this transformed density we define the Esscher-transformed moment generating function as:

$$M(z,t;h) = \int_{-\infty}^{\infty} \exp(zx)F(dx,t;h) = \frac{M(z+h,t)}{M(h,t)}. \tag{3.5}$$

Then it follows from (3.3) and (3.5) that

$$M(z,t;h) = (M(z,1;h))^t \tag{3.6}$$

The idea of Gerber and Shiu [10] is to choose $h = h^*$ such that the discounted underlying process here $\{e^{-rt}L_t\}$ becomes a martingale under the Esscher transformed measure. But absence of arbitrage arguments do not apply because the underlying process L_t is a loss index and not a price process, i.e. it gives no meaning to derive the risk neutral measure under the conditions that $\{e^{-rt}L_t\}$ should be a martingale. So we have to consider another process. Let P_t be the deterministic premium paid till time t to receive the value L_t at time t and assume that the index $\{L_t/P_t\}$ is a traded asset. We then use the idea of Gerber and Shiu by choosing $h = h^*$ such that the process $\{e^{-rt}L_t/P_t\}$ is a martingale under the Esscher transformed measure.

The question now is how to model P_t. We have to consider the loss period and the development period separately. We therefore first consider the loss period. Insurance markets are competitive and certainly with the introduction of securitization which offer an alternative to reinsurance, we argue that it is reasonable to assume that the insurance markets creates no arbitrage possibilities. We will therefore calculate the premium according to the adjusted parameter principle suggested by Venter [13]. See the latter paper for a description of the premium principle and a discussion of why this premium principle is arbitrage free. Let now $\tilde{\lambda}_1$ and $\tilde{\alpha}$ be the adjusted parameters and let $\tilde{X}_t$ be the adjusted process, i.e. $\tilde{X}_t$ is a compound Poisson process with Poisson parameter $\tilde{\lambda}_1$ and with marks that are exponentially distributed with parameter $\tilde{\alpha}$. . The premium is then:

$$\begin{aligned} P_t &= E_P[\tilde{L}_t] \\ &= E_P[L^0 \exp(\tilde{X}_t)] \\ &= L^0 \exp(\frac{\tilde{\lambda}_1 t}{(\tilde{\alpha}-1)}) \end{aligned}$$

Motivated by this we will use the following model for P_t

$$P_t = L^0 \exp(\beta_1 t)$$

We are now ready to find the parameter h_l^* and thereby derive the risk neutral measure in the loss period. h_l^* is chosen such that the process $\{e^{-rt}L_t/P_t\}$ is a martingale under the Esscher transformed measure

$$\begin{aligned} & E^*[\exp(-rt)L_t/P_t] = 1 \\ \Rightarrow \quad & \exp((r+\beta_1)t) = E^*[\exp(X_t)] \\ \Rightarrow \quad & \exp((r+\beta_1)t) = \int_{-\infty}^{\infty} \exp(x) F(dx, t; h_l^*) \\ \Rightarrow \quad & \exp((r+\beta_1)t) = M(1, t; h_l^*) \end{aligned} \tag{3.7}$$

By (3.6) it follows that the condition for h_l^* in the loss period is:

$$\exp(r+\beta_1) = M(1,1,h_l^*) \tag{3.8}$$

For the development period the situation is similar, the model for X_t is just different. Let now $\tilde{\lambda_2}$, $\tilde{\mu}$ and $\tilde{\sigma}$ be the adjusted parameters corresponding to the development period. For $t \in [T_1, T_2]$ the premium is:

$$\begin{aligned} P_t = E_P[L^0 \exp(\tilde{X}_t)] \\ &= E_P[L^0 \exp(\tilde{X}_{T_1}) \exp(\tilde{X}_t - \tilde{X}_{T_1})] \\ &= L^0 \exp(\beta_1 T_1) \exp(\tilde{\lambda_2}(t-T_1)(e^{\frac{\tilde{\sigma}^2}{2}+\tilde{\mu}} - 1)) \\ &= L^0 \exp(\beta_1 T_1) \exp(\beta_2(t-T_1)). \end{aligned}$$

And as before it follows that the condition for h_d^* in the development period is given by:

$$\begin{aligned} & E^*[\exp(-rt)L_t/P_t] = 1 \\ \Rightarrow\quad & e^{(r+\beta_1)T_1} e^{(r+\beta_2)(t-T_1)} = E^*[\exp(X_{T_1})\exp(X_t - X_{T_1})] \\ \Rightarrow\quad & e^{(r+\beta_1)T_1} e^{(r+\beta_2)(t-T_1)} = M_l(1,T_1,h_l^*) M_d(1, t-T_1, h_d^*) \end{aligned}$$

where $M_l(1,t,h)$ and $M_d(1,t,h)$ denotes the Esscher-transformed moment generating function for X_t for t in the loss period and development period respectively. It hereby follows that the condition for h_d^* in the development period is:

$$\exp(r+\beta_2) = M(1,1,h_d^*) \tag{3.9}$$

The Radon-Nikodym derivative for the risk neutral Esscher measure on the σ-algebra $\mathcal{F}_t$ can now be characterized

$$\frac{dQ}{dP}|\mathcal{F}_t = \begin{cases} \frac{e^{h_l^* X_t}}{M_l(h_l^*,t)} & t \in [0,T_1] \\ \frac{e^{h_l^* X_{T_1}}}{M_l(h_l^*,T_1)} \frac{e^{h_d^*(X_t - X_{T_1})}}{M_d(h_d^*, t-T_1)} & t \in [T_1, T_2] \end{cases}$$

where h_l^* and h_d^* are given by equation 3.8 and equation 3.9 respectively.

Above we derive the risk neutral measure Q under the assumption that $\{L_t/P_t\}$ is a traded index, but it should be mentioned that this assumption is not strictly needed. It is basically the information in P_t that determines the Q measure, i.e. the measure Q is derived such that it evaluates the risk in the same way as P_t does. So instead of using that $\{L_t/P_t\}$ is a traded index, the measure Q can be derived directly from the equation $e^{-rt}E_Q[\exp(\sum_{i=1}^{N_t} X_i)] = \exp(\beta_1 t)$.

Although there is more than one equivalent measure, the risk neutral Esscher measure provides a unique and transparent answer. Motivated by [10] we now try to justify the choice of the Esscher measure by looking at a representative investor maximizing his expected utility.

3.2. **The representative investor with power utility function.** Consider a market with only the traded index (L_t/P_t) and a risk free bond and their derivative securities. To keep it simple we only consider the loss period. Assume there exists a representative investor who has m shares of of the traded index L_t/P_t and who bases his decisions on a risk-averse utility function $u(x)$. Consider then a derivative security that provides a payment π_t at time t, $t > 0$; π_t is a function of the index process (L_t/P_t) until time t. What is the investors price for the derivative security, such that it is optimal for him not to buy or sell any multiple of it? Let V_0 denote this price. Mathematically, this leads to the function

$$\psi(\eta) = E[u(mL_t/P_t + \eta(\pi_t - e^{rt}V_0))]$$

is maximal for $\eta = 0$. From

$$\psi'(0) = 0$$

we obtain

$$V_0 = e^{-rt}\frac{E[\pi_t u'(mL_t/P_t)]}{E[u'(mL_t/P_t)]}$$

(as a necessary and sufficient condition, since $\psi''(\eta) < 0$ if $u''(x) < 0$). In the particular case of a power utility function with parameter $c > 0$,

$$u(x) = \begin{cases} \frac{x^{1-c}}{1-c} & \text{if } c \neq 1 \\ \ln(x) & \text{if } c = 1 \end{cases}$$

We have $u'(x) = x^{-c}$, and

$$V_0 = e^{-rt}\frac{E[\pi_t(m(L_t/P_t))^{-c}]}{E[(m(L_t/P_t))^{-c}]} = e^{-rt}\frac{E[\pi_t(L_t/P_t)^{-c}]}{E[(L_t/P_t)^{-c}]} \tag{3.10}$$

Formula (3.10) must hold for all derivative securities. For $\pi_t = L_t/P_t$ and therefore $V_0 = 1$, (3.10) becomes

$$1 = e^{-rt}\frac{E[(L_t/P_t)^{1-c}]}{E[(L_t/P_t)^{-c}]}$$

or

$$\exp((r + \beta_1)t) = \frac{M(1-c, t)}{M(-c, t)}. \tag{3.11}$$

Comparing (3.11) with (3.7) we see that the value of the parameter $-c$ is h_l^*. Hence P_t is indeed the expectation of the losses L_t calculated with respect to the risk neutral measure.

The results from this section will now be used in the next section, where the concrete risk neutral Esscher measures for both the loss period and the development period will be computed.

4. The risk neutral Esscher measures

In this section we compute the risk neutral Esscher measures for both the loss period and the development period.

4.1. **The risk neutral Esscher measure for the loss period.**

In this section we will compute the risk neutral Esscher measure for the compound Poisson process where the marks are exponentially distributed.

$$X_t = \sum_{i=1}^{N_t} Y_i$$

where $N_t \sim \text{Po}(\lambda_1 t)$ and $Y_i \sim$Exp(α).

The moment generating function of X_t is

$$\begin{aligned} M(z,t) &= E[\exp(z(\sum_{i=1}^{N_t} Y_i))] \\ &= \exp\left(\lambda_1 t\left(\frac{\alpha}{\alpha - z} - 1\right)\right). \end{aligned} \tag{4.1}$$

The Esscher-transformed moment generating function is then computed according to (3.5)

$$\begin{aligned} M(z,t;h) = \exp\left(\lambda_1 t\Big(\frac{\alpha}{\alpha-(z+h)} - \frac{\alpha}{\alpha-h}\Big)\right) \\ = \exp\left(\lambda_1 \frac{\alpha}{\alpha-h} t\Big(\frac{(\alpha-h)}{(\alpha-h)-z} - 1\Big)\right). \end{aligned} \tag{4.2}$$

From (4.1) and (4.2) it follows that the Esscher transformed process is again a process of the same type as the original one, provided $h < \alpha$. Let $\tilde{X}_t$ denote the Esscher transformed process, then

$$\tilde{X}_t = \sum_{i=1}^{\tilde{N}_t} \tilde{Y}_i$$

where we now have that $\tilde{N}_t \sim \text{Po}(\lambda_1 \frac{\alpha}{\alpha-h} t)$ and $\tilde{Y}_i \sim$Exp$(\alpha - h)$.

We are now ready to calculate the parameter h_l^* which determines the risk neutral Esscher measure.

We use (3.8) to find h_l^*

$$\begin{aligned} & M(1,1,h_l^*) = \exp(r+\beta_1) \\ & \Rightarrow \exp\left(\lambda_1 \frac{\alpha}{\alpha - h_l^*}(\frac{(\alpha - h_l^*)}{(\alpha - h_l^*) - 1} - 1)\right) = \exp(r+\beta_1) \\ & \Rightarrow (h_l^*)^2 - (2\alpha - 1)h_l^* + \alpha(\alpha - 1 + \frac{\lambda_1}{r+\beta_1}) = 0 \end{aligned} \tag{4.3}$$

(4.3) is a second order equation where only one of the solutions fulfills the restriction $h_l^* < \alpha$.

Finally we will compute $P^*(X_t \leq x)$

$$
\begin{aligned}
P^*(X_t \leq x) &= P(\tilde{X}_t \leq x) \\
&= \sum_{n=0}^{\infty} P(N_t = n) P(\sum_{i=0}^{n} \tilde{Y}_i \leq x) \\
&= \sum_{n=0}^{\infty} P(N_t = n) \int_0^x (\alpha - h_l^*)^n \Gamma(n)^{-1} z^{n-1} e^{-(\alpha - h_l^*)z} dz \\
&= \sum_{n=0}^{\infty} e^{-\lambda_1^* t} \frac{(\lambda_1^* t)^n}{n!} \Gamma^*(n, \alpha, x) \qquad (4.4)
\end{aligned}
$$

where $\lambda_1^* = \lambda_1 \frac{\alpha}{\alpha - h_l^*}$ and $\Gamma^*(n, \alpha, x) = \int_0^x (\alpha - h_l^*)^n \Gamma(n)^{-1} z^{n-1} e^{-(\alpha - h_l^*)z} dz$

4.2. **The risk neutral Esscher measure for the development period.** In this section we will compute the risk neutral Esscher measure for the compound Poisson process where the claim sizes are normally distributed.

$$X_t = \sum_{i=1}^{N_t} Y_i$$

where $N_t \sim \mathrm{Po}(\lambda_2 t)$, and $Y_i \sim N(\mu, \sigma)$.

The moment generating function of X_t is

$$
\begin{aligned}
M(z,t) &= E[\exp(z(\sum_{i=1}^{N_t} Y_i))] \\
&= \exp\left(\lambda_2 t \left(e^{\frac{\sigma^2}{2}z^2 + \mu z} - 1\right)\right) \qquad (4.5)
\end{aligned}
$$

The Esscher-transformed moment generating function is then computed according to (3.5).

$$
\begin{aligned}
M(z,t;h) &= \exp\left(\lambda_2 t \left(e^{\frac{\sigma^2}{2}(z+h)^2 + \mu(z+h)} - e^{\frac{\sigma^2}{2}h^2 + \mu h}\right)\right) \\
&= \exp\left(\lambda_2 t e^{\frac{\sigma^2}{2}h^2 + \mu h} \left(e^{\frac{\sigma^2}{2}z^2 + (\mu + \sigma^2 h)z} - 1\right)\right) \qquad (4.6)
\end{aligned}
$$

From (4.5) and (4.6) it follows that the Esscher transformed process is again a process of the same type as the original one. Let $\tilde{X}_t$ denote the Esscher transformed process then.

$$\tilde{X}_t = \sum_{i=1}^{\tilde{N}_t} \tilde{Y}_i$$

where we now have $\tilde{N}_t \sim \mathrm{Po}(\lambda_2 e^{\frac{\sigma^2}{2}h^2 + \mu h} t)$ and $\tilde{Y}_i \sim N(\sigma^2 h + \mu, \sigma^2)$.

We are now ready to calculate the parameter h_d^* which determines the risk neutral Esscher measure.

We use (3.9) to find h_d^*

$$M(1,1,h^*) = \exp(r+\beta_2)$$
$$\Rightarrow \exp\Big(\lambda_2 1\Big(e^{\frac{\sigma^2}{2}(1+h_d^*)^2+\mu(1+h_d^*)} - e^{\frac{\sigma^2}{2}h_d^{*2}+\mu h_d^*}\Big)\Big) = \exp(r+\beta_2)$$
$$\Rightarrow e^{\frac{\sigma^2}{2}(1+h_d^*)^2+\mu(1+h_d^*)} - e^{\frac{\sigma^2}{2}h_d^{*2}+\mu h_d^*} = \frac{r+\beta_2}{\lambda_2} \tag{4.7}$$

the solution to (4.7) has to be found numerically.

Finally we will compute $P^*(X_t \le x)$

$$\begin{aligned} P^*(X_t \le x) &\\ = \;& P(\tilde{X}_t \le x)\\ = \;& \sum_{n=0}^{\infty} \exp(-\lambda_2 t e^{\frac{\sigma^2}{2}h_d^{*2}+\mu h_d^*}) \frac{(\lambda_2 t e^{\frac{\sigma^2}{2}h_d^{*2}+\mu h_d^*})^n}{n!} P(\sum_{i=1}^{n}\tilde{Y}_i \le x)\\ = \;& \sum_{n=0}^{\infty} \exp(-\lambda_2 t e^{\frac{\sigma^2}{2}h_d^{*2}+\mu h_d^*}) \frac{(\lambda_2 t e^{\frac{\sigma^2}{2}h_d^{*2}+\mu h_d^*})^n}{n!} \Phi(\frac{x-n(\sigma^2 h_d^*+\mu)}{\sqrt{n\sigma^2}})\\ = \;& \sum_{n=0}^{\infty} \exp(-\lambda_2^* t) \frac{(\lambda_2^* t)^n}{n!} \Phi(\frac{x-n\mu^*}{\sqrt{n\sigma^2}}) \qquad (4.8) \end{aligned}$$

where $\lambda_2^* = \lambda_2 t e^{\frac{\sigma^2}{2}h_d^{*2}+\mu h_d^*}$ and $\mu^* = \sigma^2 h_d^* + \mu$. For later use we also define $\lambda_2^{*+1} = \lambda_2 t e^{\frac{\sigma^2}{2}(h_d^*+1)^2+\mu(h_d^*+1)}$ and $\mu^{*+1} = \sigma^2(h_d^*+1)+\mu$.

5. Calculation of the PCS option price

Now we have calculated the risk neutral Esscher measure for both the loss period and the development period. We are therefore ready to calculate the PCS option price.

Let us now consider the PCS call option with exercise price A, cap K and expiring date T_2. Let $v_1(t) := \ln(A/(L^0\exp X_t))$ and $v_2(t) = \ln(K/(L^0\exp X_t))$. The value of the option at time $t \in [T_1, T_2]$ is:

$$\begin{aligned} C(t, L_t) &\\ = \;& E^*[\exp(-r(T_2-t))\min(\max(L_{T_2}-A,0),K-A)|\mathcal{F}_t]\\ = \;& E^*[\exp(-r(T_2-t)\min(\max(L^0\exp(X_t)\exp(X_{T_2}-X_t)-A,0),K-A)|\mathcal{F}_t]\\ = \;& \exp(-r(T_2-t))\Big(\int_{v_1(t)}^{v_2(t)}(L_t\exp(x)-A)F(dx,T_2-t,h_d^*)\\ & +(K-A)(1-F(v_2(t),T_2-t,h_d^*))\Big)\\ = \;& \exp(-r(T_2-t))\Big(L_t\int_{v_1(t)}^{v_2(t)}\exp(x)F(dx,T_2-t,h_d^*)\\ & +K(1-F(v_2(t),T_2-t,h_d^*)) - A(1-F(v_1(t),T_2-t,h_d^*))\Big). \end{aligned}$$

Using (3.4), (3.5) and (3.7) the integrand can be reduced further:

$$\begin{aligned}
\exp(x)F(dx, T_2 - t, h_d^*) &= \frac{\exp((h_d^*+1)x)F(dx, T_2-t)}{M(h_d^*, T_2-t)} \\
&= \frac{M(h_d^*+1, T_2-t)}{M(h_d^*, T_2-t)}F(dx, T_2-t, h_d^*+1) \\
&= M(1, T_2-t, h_d^*)F(dx, T_2-t, h_d^*+1) \\
&= \exp(\beta_2(T_2-t))F(dx, T_2-t, h_d^*+1)
\end{aligned}$$

We hereby obtain:

$$\begin{aligned}
&C(t, L_t) = \exp(-r(T_2-t))\Big(L_t \exp(\beta_2(T_2-t))\Big(F(v_2(t), T_2-t, h_d^*+1) \\
&-F(v_1(t), T_2-t, h_d^*+1)\Big) + K(1 - F(v_2(t), T_2-t, h_d^*)) - \\
&A(1 - F(v_1(t), T_2-t, h_d^*))\Big) \qquad (5.1)
\end{aligned}$$

And if we use the results from the Section 4.2 we obtain

$$\begin{aligned}
&C(t, L_t) \\
&= e^{(\beta_2-r)(T_2-t)}L_t \sum_{n=0}^{\infty} \exp(-\lambda_2^{*+1}(T_2-t))\frac{(\lambda_2^{*+1}(T_2-t))^n}{n!} \\
&\quad \Big(\Phi(\frac{v_2(t)-n\mu^{*+1}}{\sqrt{n\sigma^2}}) - \Phi(\frac{v_1(t)-n\mu^{*+1}}{\sqrt{n\sigma^2}})\Big) \\
&\quad + e^{-r(T_2-t)}\sum_{n=0}^{\infty} \exp(-\lambda_2^*(T_2-t))\frac{(\lambda_2^*(T_2-t))^n}{n!} \\
&\quad \big(K(1-\Phi(\frac{v_2(t)-n\mu^*}{\sqrt{n\sigma^2}})) \\
&\qquad -A(1-\Phi(\frac{v_1(t)-n\mu^*}{\sqrt{n\sigma^2}}))\big) \qquad (5.2)
\end{aligned}$$

If $t \in [0, T_1]$ the value of the call option at time t is given by

$$\begin{aligned}
C(t, L_t) &= e^{-r(T_2-t)}E^*[C(T_2, L_{T_2})|\mathcal{F}_t] \\
&= e^{-r(T_2-t)}E^*[E^*[C(T_2, L_{T_2})|\mathcal{F}_{T_1}]|\mathcal{F}_t]
\end{aligned}$$

the value of $E^*[C(T_2, L_{T_2})|\mathcal{F}_{T_1}]$ is known from (5.2). At time $t < T_1$ the values of $L_{T_1}, v_1(T_1)$ and $v_2(T_1)$ are stochastic. The conditional mean value at time t is therefore obtained by integrating the expression with respect to the distribution function for the risk neutral Esscher transformed process in the loss period which we calculated in section 4.1.

The values of L_{T_1}, $v_1(T_1)$ and $v_2(T_1)$ can all be expressed by the value of an independent copy $(\tilde{X}_t)$ of X_t.

- $L_{T_1} = L^0 \exp(X_t)\exp(X_{T_1} - X_t) \stackrel{\mathrm{d}}{=} L(t)\exp \tilde{X}_{(T_1-t)}$
- $v_1(T_1) = v_1(t) - X_{T_1} + X_t \stackrel{\mathrm{d}}{=} v_1(t) - \tilde{X}_{(T_1-t)}$
- $v_2(T_1) = v_2(t) - X_{T_1} + X_t \stackrel{\mathrm{d}}{=} v_2(t) - \tilde{X}_{(T_1-t)}$

Let now $\tilde{F}(dx, T_1 - t, h_l^*)$ denote the Esscher transformed distribution for $\tilde{X}_{(T_1-t)}$, which we found in section 4.1 (h_l^* denote the parameter which determines the risk neutral Esscher measure in the loss period). The value of the option price at time $t \in [0, T_1]$ is then

$$\begin{aligned}
C(t, L_t) &= e^{(\beta_2 - r)(T_1 - t)} L^0 \exp(X_t) \\
&\int_{-\infty}^{\infty} \exp(x) \sum_{n=0}^{\infty} \exp(-\lambda_2^{*+1}(T_2 - T_1)) \frac{(\lambda_2^{*+1}(T_2 - T_1))^n}{n!} \\
&\Big(\Phi(\frac{v_2(t) - x - n(\sigma^2(h_d^* + 1) + \mu)}{\sqrt{n\sigma^2}}) \\
&\qquad -\Phi(\frac{v_1(t) - x - n(\sigma^2(h_d^* + 1) + \mu)}{\sqrt{n\sigma^2}})\Big) \tilde{F}(dx, (T_1 - t), h_l^*) \\
&+ e^{-r(T_2 - t)} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \exp(-\lambda_2^*(T_2 - T_1)) \frac{(\lambda_2^*(T_2 - T_1))^n}{n!} \\
&\Big(K(1 - \Phi(\frac{v_2(t) - x - n\mu^*}{\sqrt{n\sigma^2}})) \\
&\qquad -A(1 - \Phi(\frac{v_1(t) - x - n\mu^*}{\sqrt{n\sigma^2}}))\Big) \tilde{F}(dx, (T_1 - t), h_l^*)
\end{aligned} \tag{5.3}$$

where $\tilde{F}(dx, (T_1 - t), h_l^*)$ is given by (4.4)

$$\begin{aligned}
&\tilde{F}(dx, (T_1 - t), h_l^*) \\
&= \sum_{n=0}^{\infty} \exp(-\lambda_1^*(T_1 - t)) \frac{(\lambda_1^*(T_1 - t))^n}{n!} \Gamma^*(n, \alpha, x)
\end{aligned}$$

From the above results we can now state the following theorem.

Theorem 5.4. *Let L_t be the value of the PCS index at time t;*

$$L_t = L^0 \exp(X_t) \qquad \forall t \geq 0$$

For $t \in [0, T_1]$ (the loss period) assume that

$$X_t = \sum_{i=1}^{N_t} Y_i$$

where $N_t \sim Po(\lambda_1 t)$ and $Y_i \sim Exp(\alpha)$.

For $t \in [T_1, T_2]$ (the development period) assume that

$$X_t = X_{T_1} + \sum_{i=1}^{\tilde{N}_{t-T_1}} \tilde{Y}_i$$

where $\tilde{N}_t \sim Po(\lambda_2 t)$, and $Y_i \sim N(\mu, \sigma)$.

Then the price of the PCS call option for $t \in [T_1, T_2]$ is given by (5.2) and for $t \in [0, T_1]$ it is given by (5.3).

6. Conclusion

The purpose of this note was to derive a model for pricing insurance derivatives which allows for heavy tails and also provide a unique pricing measure. We succeeded in finding such a model by modelling the logarithms of the loss process as a compound Poisson process with exponential distributed marks in the loss period and with normal distributed marks in the development period. The price was then found by evaluating the future payout of the insurance derivative under the risk neutral measure derived by the Esscher approach. We then calculated the exact price in the case of the PCS option.

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(C. Vorm Christensen) Department of Theoretical Statistics and Operations Research, University of Aarhus, Ny Munkegade 116, 8000 Aarhus C, Denmark

E-mail address: `vorm@imf.au.dk`

IMPLIED LOSS DISTRIBUTIONS FOR CATASTROPHE INSURANCE DERIVATIVES

CLAUS VORM CHRISTENSEN

Abstract. We analyse prices for catastrophe insurance derivatives in the same way as Lane and Movchan [10] considering the "implied loss distributions" embedded in the traded prices. There are two main problems in this analysis. First, what kind of distribution should be chosen for the implied losses and, second how should the involved parameters be estimated? In this paper we give answers to these two questions.

1. Introduction

Since the introduction of the insurance derivatives in 1992, there have been a problem pricing these products and several attempts has been made, see [1], [2], [5], [6], [7], [9] and [12]. It has not been possible to find a unique model like the Black Scholes model because the underlying cannot be described by a distribution as simple as the log normal and furthermore, the underlying is not traded. The underlying (the aggregate catastrophe losses, which we in the following will denote L_T) would instead most naturally be described by a marked point process with heavy tail distributed marks. But the problem of such a model for the underlying is that the market becomes incomplete and it is then an open question how the pricing measure should be determined. Furthermore, the heavy tailed distribution often gives computationally problems, e.g. if the pricing measure is determined by a representative agent with an exponential utility function, the marks must have an exponentially decreasing tail. So, estimating parameters for the marked point process and calculating consistent prices using a closed form pricing model is just now not a workable plan.

We therefore lead our analysis in another direction. We follow a procedure familiar to the conventional option market which also is suggested by Lane and Movchan in [10], namely rather than estimating volatilities and calculate consistent prices using, say the Black Scholes model, take the traded prices and extract the volatilities consistent with those prices, i.e. find the implied volatility. We cannot use exact the same procedure on the insurance derivative market, since as

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mentioned above, we are not able to characterize the price by a single parameter. But we can do something similar. We can choose a model for the implied loss distribution and then estimate the implied parameters from observed prices.

This analysis can be used to evaluate cheapness and dearness among different prices and different insurance derivative products. We simply calculate implied prices from the implied loss distributions and compare them to the observed prices. This analysis is very relevant seen in relation to the recent trading success observed at the Chicago Board of Trade (CBoT) competitors namely by The Bermuda Commodities Exchange (BCOE) and The Catastrophe Risk Exchange (CATEX).

There are two main problems related to this analysis. First what kind of distribution should be used for the implied loss distribution and second, how the involved parameters should be estimated. We are going to answer these two questions in this paper. The data material used for this analysis are the prices for the National PCS call spreads announced by the CBoT on January 1st 1999. For a description of the PCS-option see [4].

The paper will proceed as follows. In section 2 we will present the different models for the implied loss distributions, in section 3 we will describe the procedure for estimating the parameters, in section 4 we present the data and estimate the parameters, in section 5 we evaluate the different models and finally, there are some concluding remarks.

2. The implied loss distribution

In this section we present six different models for the implied losses. First, we will give a general description of the price for a PCS call spread expressed by the implied loss distribution.

Consider now a PCS call spread expiring at time T with upper and lower strike K_u and K_l respectively. Let $\widetilde{F}_{L_T}$ and $\widetilde{f}_{L_T}$ be the implied distribution function and the implied density function for the aggregate PCS loss index (L_T) at time T. The value of the PCS call spread at time 0 is then given by

$$\begin{aligned} P_{K_u,K_l}(L_0,0) &= \widetilde{E}[\min(\max(L_T - K_l, 0), K_u - K_l)] \\ &= \int_{K_l}^{K_u} (x-K_l)\widetilde{f}_{L_T}(x)dx + (K_u - K_l)(1-\widetilde{F}_{L_T}(K_u)) \end{aligned}$$

The question is now, how is this implied loss distribution of the PCS index related to the real statistical distribution, i.e. the distribution under the P-measure? Before we try to answer this question we consider an example.

Example 2.1. Let the statistical distribution for the aggregate losses be described by a compound Poisson process, i.e.

$$L_T = \sum_{i=1}^{N_T} Y_i$$

where $N_T \sim \text{Pois}(\lambda T)$, $Y_i \sim F_Y$ and Y_i are iid and independent of N_T. We will now price the PCS call spread by the approach of Embrechts and Meister [9]. There the general equilibrium approach is used, where all the utility functions of the agents are of exponential type. More precisely, let at time t the price of the PCS call spread be given by $P_{K_u,K_l}(L_t, t)$, the value of the PCS index be given by L_t and the information be given by $\mathcal{F}_t$, the price at time t see [9], is

$$P_{K_u,K_l}(L_t, t) = \frac{E_P[\exp(\alpha L_T)\min(\max(L_T - K_l, 0), K_u - K_l) \mid \mathcal{F}_t]}{E_P[\exp(\alpha L_T) \mid \mathcal{F}_t]}$$

where α is the risk aversion coefficient.

The term $\exp(\alpha L_T)/E_P[\exp(\alpha L_T) \mid \mathcal{F}_t]$ is strictly positive and integrates to one. Thus it is the Radon-Nikodym derivative dQ/dP of an equivalent measure. We can therefore express the price $P_{K_u,K_l}(L_t, t)$ as

$$P_{K_u,K_l}(L_t, t) = E_Q[\min(\max(L_T - K_l, 0), K_u - K_l) \mid \mathcal{F}_t]$$

where $dQ/dP = \exp(\alpha L_T)/E_P[\exp(\alpha L_T) \mid \mathcal{F}_t]$. If $P_{K_u,K_l}(L_t, t)$ is the correct price, the distribution of the PCS index under the risk neutral measure Q should coincide with the implied loss distribution. Therefore, if we are able to find the distribution of the PCS index under the risk neutral measure, we are also able to say something about the implied loss distribution.

Let us now try to find this distribution. For an introduction to change of measure methods, we refer to [11]. Let $M_Y(\alpha)$ denote the moment generating function for Y. By the above definition of the Q-measure, it follows that $1 \leq i \leq n$

$$\begin{aligned}
&Q(N_T = n, Y_i \in C_i) \\
&= \frac{1}{E_P[\exp(\alpha L_T)]} E_P[\exp(\alpha L_T)\mathbf{1}_{\{N_T=n\}}\mathbf{1}_{\{Y_i \in C_i\}}] \\
&= \frac{1}{E_P[\exp(\alpha L_T)]} E_P[\exp(\alpha \sum_{i=1}^{n} Y_i)\mathbf{1}_{\{N_T=n\}}\mathbf{1}_{\{Y_i \in C_i\}}] \\
&= \frac{1}{\exp(\lambda(M_Y(\alpha) - 1))} M_Y(\alpha)^n P(N_T = n) \prod_{i=1}^{n} \frac{E_P[\exp(\alpha Y_i)\mathbf{1}_{\{Y_i \in C_i\}}]}{M_Y(\alpha)} \\
&= \frac{1}{\exp(\lambda(M_Y(\alpha) - 1))} M_Y(\alpha)^n \frac{(\lambda)^n}{n!} e^{-\lambda} \prod_{i=1}^{n} \frac{E_P[\exp(\alpha Y_1)\mathbf{1}_{\{Y_i \in C_i\}}]}{M_Y(\alpha)} \\
&= e^{-\lambda M_Y(\alpha)} \frac{(\lambda M_Y(\alpha))^n}{n!} \prod_{i=1}^{n} \frac{E_P[\exp(\alpha Y_1)\mathbf{1}_{\{Y_i \in C_i\}}]}{M_Y(\alpha)}
\end{aligned}$$

Hereby it follows that the process L_T is under the new measure Q a process of the same type but with different parameters as under P. Under Q, N_T is a Poisson process with rate $\lambda M_Y(\alpha)$ and the individual claims have the distribution function $F_Y^Q(x) = \int_0^x e^{\alpha y} dF(y)/M_Y(\alpha)$ (e.g. if $Y \sim \Gamma(\beta, \gamma)$ then $Y \sim \Gamma(\beta - \alpha, \gamma)$ under Q). Similar results for another model can be found in [5].

In example 2.1 we find that the implied loss distribution and the statistical distribution are of the same type only with different parameters. It would be convenient if this were true in general. When modelling the implied losses, we would only have to look among the models which reasonably could be used to describe the real losses. But is this true in general?

As mentioned earlier the most natural way to describe the real losses is by a marked point process with positive marks. Let us now recall the definition of a marked point process. A marked point process with positive marks is a sequence $(T_n, Y_n)_{n \geq 1}$ of stochastic pairs, where $T_1, T_2, \ldots$ are non-negative and represent time of occurrence of some phenomena represented by the non-negative elements $Y_1, Y_2, \ldots$ referred to as the marks of the process. By this definition it follows that if (T_n, Y_n) is a marked point process with positive marks under P, then (T_n, Y_n) is a marked point process with positive marks under Q, where Q is an equivalent measure. So, if the real losses are described by a marked point process the implied losses should also be described by a marked point process.

The distribution for the implied losses and the distribution for the real losses will in general not be the same. But by the discussion above, we will only use models which could reasonably be used to describe the real losses, when we now start to model the implied losses. We now present six models for the implied losses.

2.1. **Model 1.** The first model we will use in our analysis is the same model as the one suggested by Lane and Movchan [10], namely a compound Poisson model with gamma distributed claims, i.e.

$$\widetilde{L}_T = \sum_{i=1}^{N_T} Y_i$$

where $\widetilde{L}_T$ are the implied losses, $N_T \sim \text{Pois}(\lambda T)$ and $Y_i \sim \Gamma(\gamma, \alpha)$. This model is also suggested in [1], here a closed form pricing model are derived in the framework of general economic equilibrium theory under uncertainty.

The nice thing about this model is that we know the nth convolution of the Y's ($Y_1 + \ldots + Y_n \sim \Gamma(n\gamma, \alpha)$). This fact makes the computations very simple. A disadvantage of the model is that the claims are light tailed, whereas data give evidence that the distribution tail of the aggregate claims is heavy tailed. In this model we can only approximate a heavy tail by choosing low values of γ and α.

It is important to have this model in the analysis in order to see how the result from this model differs from the following and more complicated models. The value of the PCS call spread with strikes K_l and K_u at time 0 is given by

$$\begin{aligned}
&P_{K_u,K_l}(L_0,0)\\
&= \int_{K_l}^{K_u} (x-K_l)\widetilde{f}_{L_T}(x)dx + (K_u-K_l)(1-\widetilde{F}_{L_T}(K_u))\\
&= \sum_{n=1}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\Big(\int_{K_l}^{K_u} \alpha^{n\gamma}\Gamma(n\gamma)^{-1}x^{n\gamma}e^{-\alpha x}dx\\
&\quad + K_u\int_{K_u}^{\infty} \alpha^{n\gamma}\Gamma(n\gamma)^{-1}x^{n\gamma-1}e^{-\alpha x}dx\\
&\quad - K_l\int_{K_l}^{\infty} \alpha^{n\gamma}\Gamma(n\gamma)^{-1}x^{n\gamma-1}e^{-\alpha x}dx\Big)
\end{aligned}$$

where $L_0 = 0$.

2.2. **Model 2.** Looking at the listed call spreads we see, that the one with the lowest strikes is the 40/60 call spread. The bid and ask for this call spread is 12 and 15 respectively, which are relatively large values for a product that has a maximal pay-out of 20. These facts could therefore indicate that the market expects that the loss index will be above a given threshold K_0 for sure. If this is true and $K_0 > 40$, then there is no market for a 20/40 call spread, because the market will expect the call spread to be worth 20 for sure. Based on these indications we extend model 1,

$$\widetilde{L}_T = K_0 + \sum_{i=1}^{N_T} Y_i$$

where $\widetilde{L}_T$ are the implied losses, K_0 is a constant indicating the threshold the market expects the losses to be above for sure, $N_T \sim \text{Pois}(\lambda T)$ and $Y_i \sim \Gamma(\gamma,\alpha)$. According to the bid of the 40/60 call spread (12), we will not allow K_0 to be above 52 (40+12). A possible interpretation of this model is to think of K_0 as the mean value of the "normal" claims and of the compound Poisson process as a model of the excesses.

The computations are still very simple. The value of the PCS call spread with strikes K_l and K_u at time 0 is here given by

$$
\begin{aligned}
&P_{K_u,K_l}(L_0,0)\\
&= \int_{K_l}^{K_u}(x-K_l)\widetilde{f}_{L_T}(x)dx+(K_u-K_l)(1-\widetilde{F}_{L_T}(K_u))\\
&= \sum_{n=1}^{\infty}e^{-\lambda}\frac{\lambda^n}{n!}\Big(\int_{(K_l-K_0)^+}^{K_u-K_0}\alpha^{n\gamma}\Gamma(n\gamma)^{-1}x^{n\gamma}e^{-\alpha x}dx\\
&\quad+(K_u-K_0)\int_{K_u-K_0}^{\infty}\alpha^{n\gamma}\Gamma(n\gamma)^{-1}x^{n\gamma-1}e^{-\alpha x}dx\\
&\quad-(K_l-K_0)\int_{(K_l-K_0)^+}^{\infty}\alpha^{n\gamma}\Gamma(n\gamma)^{-1}x^{n\gamma-1}e^{-\alpha x}dx\Big)
\end{aligned}
$$

recall that we require $K_0 < 52$ so the term $(K_0-K_l)^+$ is only included in the price of the 40/60 call spread.

2.3. **Model 3.** The next model will also rely on a light tail distribution but we will now put more fluctuation into the model.

The PCS index can be viewed as the sum of losses from the individual catastrophes, and the losses from the individual catastrophe can be viewed as the sum of the individual claims corresponding to this single catastrophe. We could therefore model the PCS index L_T as

$$L_T=\sum_{i=1}^{N_T}\sum_{j=1}^{M_i}Y_{ij}$$

where N_T is the number of catastrophes, M_i the number of claims from the ith catastrophe and Y_{ij} is claim size number j from catastrophe number i. A similar model is also suggested in [6], here a closed form pricing model are derived and it is shown how the above model can be used to incorporate the reporting times of the claims.

The number of claims from a catastrophe is very large, so by the strong law of the large numbers it follows that $\sum_{j=1}^{M_i}Y_{ij}\approx M_iE[Y_{ij}]$. If the approximation should be good we will also need the $\mathrm{Var}(Y_{ij})$ to be small. If we use this approximation we could describe L_T as $L_T\approx\sum_{i=1}^{N_T}M_iY$ where $Y=E[Y_{ij}]$. Motivated by this, we now model the implied PCS index as

$$\widetilde{L}_T=\sum_{i=1}^{N_T}M_iY$$

where $N_T\sim\mathrm{Pois}(\lambda T)$, $M_i\sim\mathrm{NB}(\gamma,\mathrm{p})$ or more precisely by a mixed Poisson distribution with a Γ mixing parameter $\lambda_i\sim\Gamma(\gamma,\alpha)$ and Y is a constant. We now have a model allowing for more fluctuation but we also have four parameters to estimate. The value of the PCS call spread with strikes K_l and K_u at time 0 is here given by

$$\begin{aligned}
&P_{K_u,K_l}(L_0,0)\\
&= \sum_{n=1}^{\infty} P(N_T = n) \sum_{m=1}^{\infty} P(M_1 + \ldots + M_n = m) g(m)\\
&= \sum_{n=1}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!} \sum_{m=1}^{\infty} E[P(M_1 + \ldots + M_n = m \mid \lambda_1, \ldots, \lambda_n)] g(m)\\
&= \sum_{n=1}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!} \sum_{m=1}^{\infty} E[e^{-(\lambda_1+\ldots+\lambda_n)}\frac{(\lambda_1 + \ldots + \lambda_n)^m}{m!}] g(m)\\
&= \sum_{n=1}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!} \sum_{m=1}^{\infty} \Big(\int_0^{\infty} e^{-x}\frac{(x)^m}{m!}\alpha^{n\gamma}\Gamma(n\gamma)^{-1}x^{n\gamma-1}e^{-\alpha x}dx\Big) g(m)\\
&= \sum_{n=1}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!} \sum_{m=1}^{\infty} \frac{\alpha^{n\gamma}\Gamma(n\gamma + m)g(m)}{(\alpha+1)^{n\gamma+m}\Gamma(n\gamma)m!}
\end{aligned}$$

where $g(m) = (mY - K_l)^+ - (mY - K_u)^+$.

2.4. **Model 4.** We now construct a model allowing heavy tails. We simply use the same model as model 1, but we now choose a Pareto distribution for the Y's, i.e. $L_T = \sum_{i=1}^{N_T} Y_i$ where $N_T \sim \text{Pois}(\lambda T)$ and $Y_i \sim \text{Pa}(\alpha, \beta)$ ($f_Y(x) = \alpha\beta^{\alpha}(\beta + x)^{-\alpha}$). However, there is no closed form formula for the nth convolution of the Ys. We solve this problem by the following approximation. The value of the PCS call spread with strikes K_l and K_u at time 0 is given by

$$\begin{aligned}
&P_{K_u,K_l}(L_0,0)\\
&= \sum_{n=1}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\Big(\int_{K_l}^{K_u}(x - K_l)f^{*n}(x)dx + (K_u - K_l)\int_{K_u}^{\infty} f^{*n}(x)dx\Big)\\
&\approx \sum_{n=1}^{4} e^{-\lambda}\frac{\lambda^n}{n!}\Big(\int_{K_l}^{K_u}(x - K_l)f^{*n}(x)dx + (K_u - K_l)\int_{K_u}^{\infty} f^{*n}(x)dx\Big)
\end{aligned}$$

where $f^{*n}(x)$ denotes the density for the nth convolution of the Pareto distribution.

The first 4 convolutions are then found by the well-known general formula for the Lebesgue convolution ($f^{*2}(x) = \int_0^x f(x-y)f(y)dy$, $f^{*3}(x) = \int_0^x f(x-y)(\int_0^y f(y-z)f(z)dz)dy, \ldots$). By taking the first 4 convolutions in the sum only it should be possible for a computer to calculate the expression. And if λ (the average number of catastrophes) is small, the approximation is good because the term $\sum_5^{\infty} e^{-\lambda}\lambda^n/n!$ will be small.

2.5. **Model 5.** This model is inspired by the volatility surface models which try to explain the volatility smile, i.e. models where the volatility depends on the strike of the option. We construct a similar model where the most explanatory parameter in the implied loss distribution is dependent on the strike. We assume that the implied loss index is Pareto distributed, i.e.

$$\widetilde{L_T} \sim \mathrm{Pa}(\alpha, \beta)$$

β being just a scale parameter, the most explanatory parameter in this loss distribution is the α parameter. We therefore choose α to be the strike dependent parameter, i.e. we assume that α is a function of the strike ($\alpha(k) = f(K)$). The estimation of the parameters is done in four steps, because we have to chose the function f first. The four steps are the following

1. We assume that $\alpha = \alpha_0$, i.e. independent of the strike. We then let $\widetilde{L}_T \sim Pa(\alpha_0, \beta_0)$ and estimate the parameters α_0 and β_0.
2. We now keep β fixed as β_0 and then for each PCS call spread with strikes K_l^i and K_u^i, we estimate an α_i from the traded price or the bid/ask spread dependent of what is available. These values are then plotted. A possible picture could be the one given by figure 2.1.

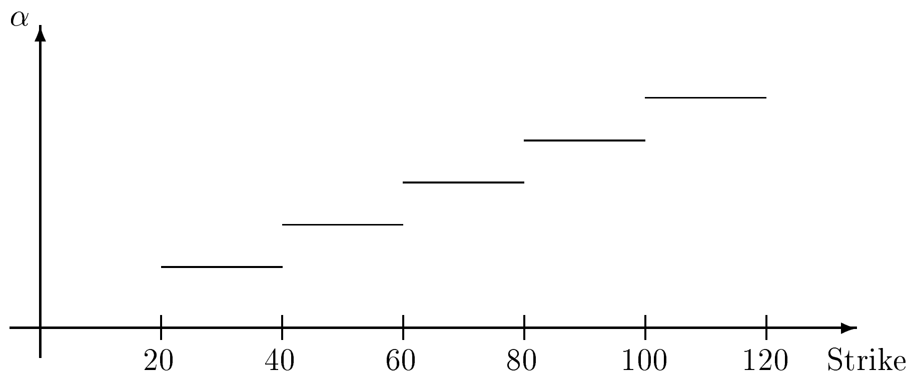

FIGURE 2.1. The α values.

3. From this plot we choose a function to describe α, i.e if we choose a function f with three parameters a, b and c, $a, b, c \in \mathbb{R}$. We can now describe α by $\alpha(K) = f(a, b, c, K)$. The implied loss distribution for a PCS call option with strike K is therefore given by

$$\widetilde{L}_T \sim \mathrm{Pa}(\alpha(K), \beta)$$

where $\alpha(K) = f(a, b, c, K)$.
4. Let now $P_K(0, L_0)$ denote the value of a PCS call option at time 0, i.e. $P_K(0, L_0) = \int_K^\infty (x - K)\widetilde{f}_L(x)dx$. The value of the PCS call

spread with strikes K_l and K_u at time 0 is here given by

$$\begin{aligned} P_{K_u,K_l}(L_0,0) &= P_{K_l}(0,L_0) - P_{K_u}(0,L_0) \\ &= \int_{K_l}^{\infty}(x-K_l)\widetilde{f}_L(x;K_l)dx - \int_{K_u}^{\infty}(x-K_u)\widetilde{f}_L(x;K_u)dx \\ &= \int_{K_l}^{\infty}(x-K_l)\alpha(K_l)\beta^{\alpha(K_l)}(\beta+x)^{-\alpha(K_l)-1}dx \\ &\quad - \int_{K_u}^{\infty}(x-K_u)\alpha(K_u)\beta^{\alpha(K_u)}(\beta+x)^{-\alpha(K_u)-1}dx \end{aligned}$$

by use of this expression the parameters a, b c and β can now be estimated.

2.6. **Model 6.** The models 3, 4 and 5 could also be extended by including a threshold as it was done for model 1 in model 2. But we will desist from doing this, as model 3 and 5 will be over parameterized and model 4 will be computationally too heavy. We will return to this discussion later. The last model we consider is a very simple model, which we expect to be computationally very fast. It will be interesting to compare the results of this model with the results of the other more complicated models. We again include a threshold as we did for model 2 and then model L_T by

$$\widetilde{L}_T = K_0 + Y_T$$

where $\widetilde{L}_T$ are the implied losses, K_0 is a constant indicating the threshold the market expects the losses to be above almost surely, $Y_T \sim \mathrm{Pa}(\alpha,\beta)$. Again we will not allow K_0 to be above 52. The value of the PCS call spread with strikes K_l and K_u at time 0 is here given by

$$\begin{aligned} &P_{K_u,K_l}(L_0,0) \\ &= \int_{K_l}^{K_u}(x-K_l)\widetilde{f}_{L_T}(x)dx \\ &\quad + (K_u-K_l)(1-\widetilde{F}_{L_T}(K_u)) \\ &= \int_{(K_l-K_0)^+}^{(K_u-K_0)}(x+K_0-K_l)\alpha\beta^{\alpha}(\beta+x)^{-\alpha-1}dx \\ &\quad + (K_u-K_l)(\alpha\beta^{\alpha}(\beta+(K_u-K_0))^{-\alpha}) \end{aligned}$$

The above six models are the models we are going to test on our data. The next step in our analysis is to describe the objective function from which the parameters should be found. This objective function is described in the next section.

3. The objective function

Lane and Movchan [10] estimate the parameters for the implied loss distribution by the following procedure. *"The parameters are chosen* *such that they generate prices that are (i) lower than known offers; (ii) higher than known bids, and (iii) closest to actual traded prices. The optimization is two-tier. First, get inside the bid-offer spread. Second, get closest to actual traded prices. The two-tier effect is achieved by attaching (ideally non-Archimedean) weights to each of the two objective function. "Closest" is defined as the absolute value of the difference between the actual traded price and the theoretical (or fitted) prices".*

We agree that it is desirable that the parameters are chosen such that the prices fulfill (i) and (ii), but we do not think that the requirements should be invariable because, if the spreads are very small, it could be a problem to find a solution. And if the theoretical prices appear to be far away from the spread, it could be used to indicate that the chosen model may be wrong. We also agree on point (iii), i.e. if our data contain only traded prices, the parameters should be found by a least square fit. But the data primarily consist of spreads and single bids or asks, we therefore suggest the following objective function.

3.1. **The objective function.** The objective function O that we propose be minimized in order to find the parameters is the following

$$\begin{aligned} O \;=\; & \underbrace{\sum_{\text{bids}}\Big(\big(\frac{P_i^{\text{bid}}-P_i^{\text{th}}}{P_i^{\text{bid}}}\big)^+\Big)^2}_{\text{term 1}} + \underbrace{\sum_{\text{asks}}\Big(\big(\frac{P_i^{\text{th}}-P_i^{\text{ask}}}{P_i^{\text{ask}}}\big)^+\Big)^2}_{\text{term 2}} \\ & + \underbrace{\delta_1\Big(\frac{1}{\#\text{spreads}}\sum_{\text{spreads}}\frac{(P_i^{\text{bid}}-P_i^{\text{ask}})}{(P_i^{\text{bid}}+P_i^{\text{ask}})/2}\Big)}_{\text{term 3}} \\ & \underbrace{\sum_{\text{spreads}}\Big(\frac{P_i^{\text{th}}-(P_i^{\text{bid}}+P_i^{\text{ask}})/2}{P_i^{\text{bid}}-P_i^{\text{ask}}}\Big)^2 \wedge \frac{1}{4}}_{\text{term4}} \\ & + \underbrace{\delta_2\sum_{\text{single bids}}\Big(\big(\frac{P_i^{\text{th}}-2P_i^{\text{bid}}}{P_i^{\text{bid}}}\big)^+\Big)^2}_{\text{term 5}} \\ & + \underbrace{\delta_2\sum_{\text{single asks}}\Big(\big(\frac{0.5P_i^{\text{ask}}-P_i^{\text{th}}}{P_i^{\text{ask}}}\big)^+\Big)^2\Big)}_{\text{term 6}} \end{aligned}$$

where the P_i^{th}'s are the theoretical prices, the P_i^{bid}'s are the observed bids and the P_i^{ask}'s are the observed asks. δ_1 and δ_2 are both constants. When P_i is a traded price, P_i is considered as both a bid and an ask where $P_i^{\text{bid}} = P_i^{\text{ask}}$.

Term 1 and 2 are included because as mentioned above we prefer the theoretical prices to be above the observed bids and below the observed asks. And as for the optimal case where we have only traded prices and no bid/ask spreads, these two terms alone will give us the commonly used least square fit.

As long as the average length of the spreads is small we are close to the optimal case where we only have traded prices and term 1 and 2 will probably be sufficient to find a solution. But if the average length of the spread is large there is less information about the prices and we will probably be unable to find a unique solution. We therefore add term 4, in this term we value the information from the bid and the ask equally, i.e. we prefer the theoretical price to be in the middle of the bid/ask spread. We cap the single terms in the sum at 1/4, because if $P_i^{\text{th}} = P_i^{\text{bid}}$ or $P_i^{\text{th}} = P_i^{\text{ask}}$ the single term in the sum is equal to 1/4, and if $P_i^{\text{th}} > P_i^{\text{bid}}$ or $P_i^{\text{th}} < P_i^{\text{ask}}$ then it is punished in term 1 or 2. How much this fourth term should be valued compared to term 1 and 2 is then adjusted by term 3. Term 3 is a constant δ_1 and a term denoting the average length of the spread. In agreement with the comments above, we thereby obtain, that if the average length of the spreads is small, we weight P_i^{th} being in the middle less than if the average length of the spreads is large.

The term 5 and 6 are included in order to secure that the theoretical prices do not get too far away from the single bid's or ask's. By too far away we mean that a theoretical price is punished if it is lower than 50 % of a single ask or higher than 200 % of a single bid. By the term δ_2 we are able to adjust how much the fifth term should be valued compared to the other terms.

If we also had information about the volumes that are bid and asked, it could be argued that the bid and ask prices should be weighted by the corresponding volumes. This is due to the fact that if the volume is large, the traders are more concerned about the price, and therefore the price should be more accurate. The problem with this argument is that there is an opposite effect, namely if the asks are much higher than the "true" price, we also expect the volumes to be large and if the bids are much lower than the "true" price, we expect the volume to be large. We therefore do not take the volumes into the objective function. We are now ready to estimate the parameters for the models described in section 2 by the above objective function. This estimation is the subject of the next section.

4. The parameter estimation

In this section we estimate the parameters and evaluate the six models described in section 2. Before we start to estimate we first present the data.

4.1. The data. The data material that we are going to use for this analysis are the prices for the National PCS call spreads announced by the CBoT on January 7th 1999.

The first change in the underlying PCS index was made January 19th, where the index increased from 0 to 7.6. We have chosen the data from January 7th because the last changes in the bids and asks before January 19th were made here. If we take data from dates after January 19th we have to take the value of the index into account. If we consider data from a time point t where the PCS index is greater than 0, some adjustments have to be done. The implied losses at expiration time T can, at time t, be written as $\widetilde{L}_T = (\widetilde{L}_T - L_t) + L_t$, where L_t is a constant and $\widetilde{L}_T - L_t$ is the implied losses in the period from t to T. $\widetilde{L}_T - L_t$ can then be described by the same models as we used to describe $\widetilde{L}_T$, but the parameters will probably be changed. Even though we are looking at a model where $\widetilde{L}_T$ is a stationary process, we cannot expect the same parameters since the PCS index is influenced by some large seasonal effects.

The National PCS call spreads announced by the CBoT on January 7th 1999 is given by table 4.1.

Call Spreads	K_l/K_U	bid	ask
National	40/60	12.0	15.0
National	60/80	6.0	12.0
National	80/100	4.0	8.0
National	100/120	2.8	4.0
National	150/200	4.3	6.0
National	200/250	2.8	4.0
National	250/300		3.5
National	300/350		3.0

TABLE 4.1. The National PCS call spread prices.

4.2. **The estimation.** The parameters are found by minimizing the objective function, with $\delta_1 = 0.001$ and $\delta_2 = 0.1$. We return to the discussion of these parameters later. The objective function is a function depending on a higher dimensional variable (the dimension is given by the number of parameters in the model). We therefore choose to minimize it by using a modification of the method of steepest decent described by Broyden see [3] and [8].

When we evaluate the models it is important to consider the computer time used in order to find the optimum. This time is of course dependent on the chosen initial parameter values, it is therefore not possible to compare the computer times directly. But after running the programs a couple of times, one gets an indication of how fast or slow the different models are. The table below gives a brief indication of how computationally heavy the models are. The figures in this table

Model	Computer time used
1	hours
2	hours
3	days
4	days
5	hours
6	seconds

TABLE 4.2. The computer time used.

are very brief. But from the table it is very clear that model 3 and 4 seems to be computationally heavy compared to model 1, 2, 5 and 6. We will keep this in mind for the evaluation of the models. It is not possible to disqualify any of the models, because the computer times could be reduced with a faster computer, or by using method tailored for the problem to solve.

4.3. **The theoretical prices.** We are now ready to estimate the parameters. The estimation of the parameters for model 1, 2, 3, 4 and 6 is straightforward. We simply write a computer program that calculates the theoretical prices according to the above formulas and minimize the objective function according to the description in [3] and [8]. For model 5 we have to do the estimation in four steps as described in section 2. In step 1 we find $\alpha = 7.66$ and $\beta = 508$. In step 2 we then set $\beta = 508$ and estimate the different α's, the α's are shown in figure 4.1.

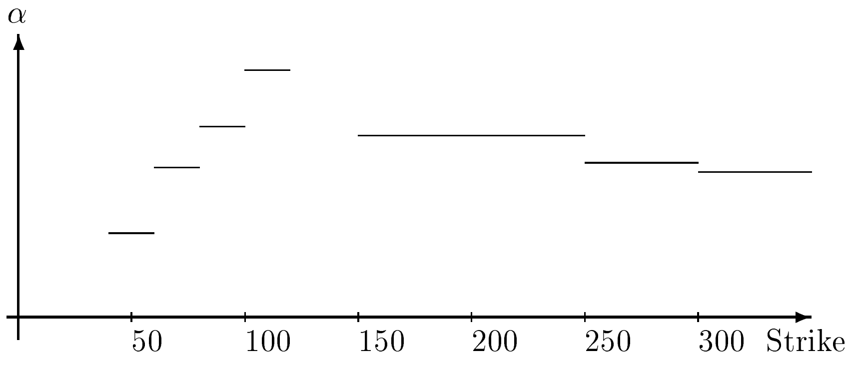

FIGURE 4.1. The α values.

In step 3 we then have to choose a function to describe the dependence between the α-parameter and the strike value. Figure 4.1 shows that we need at least three parameters to describe this dependence. Another thing we have to keep in mind is that the α-parameter has to be positive. We therefore choose the following function $\alpha(K) = \exp(aK^{-2} + bK^{-1} + c)$ to describe the dependence between the α-parameter and the strike value. With this function we run the computer program.

The parameters found by minimizing the objective function, the corresponding mean values and variances for the implied losses and the theoretical prices are listed in table 4.3, table 4.4 and table 4.5 respectively.

Model	par. 1	par. 2	par. 3	par. 4	value
1	$\lambda = 70$	$\alpha = 0.0123$	$\gamma = 0.0129$		0.058
2	$\lambda = 55$	$\alpha = 0.0050$	$\gamma = 0.0039$	$x = 47.2$	0.00015
3	$\lambda = 36$	$\alpha = 0.00019$	$\gamma = 0.0266$	$Y = 0.015$	0.086
4	$\lambda = 2.6$	$\alpha = 3.50$	$\beta = 90.7$		0.060
5	$\beta = 58$	$a = -0.117$	$b = -4.082$	$c = 0.596$	0.013
6	$\beta = 24$	$\alpha = 1.25$	$x = 40.0$		0.00010

TABLE 4.3. The estimated parameters.

	M1	M2	M3	M4	M5	M6
Mean value	74	90	77	96	(73;91)	139
Variance	6096	8453	6178	11652	∞	∞

TABLE 4.4. Mean value and variance.

K_l/K_U	bid	M1	M2	M3	M4	M5	M6	ask
40/60	**12.0**	9.87	13.56	10.02	9.33	13.57	13.57	**15.0**
60/80	**6.0**	7.61	6.55	7.72	7.27	8.00	7.48	**12.0**
80/100	**4.0**	5.88	4.82	5.96	5.63	5.49	5.03	**8.0**
100/120	**2.8**	4.55	3.78	4.60	4.36	4.07	3.73	**4.0**
150/200	**4.3**	5.07	5.07	5.06	4.92	5.08	4.88	**6.0**
200/250	**2.8**	2.71	3.35	2.67	2.78	3.41	3.45	**4.0**
250/300		1.45	2.29	1.41	1.67	2.46	2.64	**3.5**
300/350		0.78	1.60	0.74	1.06	1.87	2.11	**3.0**

TABLE 4.5. The theoretical prices.

5. EVALUATION OF THE MODELS

In this section we evaluate the results of the estimations from the previous section.

5.1. **Model 1.** From table 4.5 we see that model 1 is unable to generate prices that get into the bid/ask spread of the 40/60 and 200/250 call spreads and we also see that it produces very low prices for the 250/300 and 300/350 call spreads. This indicates that model 1 is a bad description of the implied losses. But recall that this is the model that was successfully suggested by Lane and Movchan in [10] so why now this difference? In [10] they consider market prices midyear 1998 where the PCS index was nearly 40 and this apparently makes a difference.

We also tried to model the midyear 1998 prices with model 1 and our objective function. The results are shown in table 5.1 (The α parameter from [10] has been adjusted to correspond to the index value and not the Billion \$ value).

K_l/K_U	bid	LM	CVC	ask
40/60	**11.0**	11.0	12.0	
60/80	**6.0**	7.5	8.2	**10.0**
80/100		5.7	6.1	**8.0**
100/120	**3.5**	4.4	4.6	**6.0**
100/150		9.4	9.5	**12.0**
120/140	**1.0**	3.5	3.5	**6.0**
250/300	**0.5**	1.9	1.4	**2.5**
100/200		14.7	14.4	**20.0**
150/200	**4.0**	5.4	4.9	**7.5**
180/200	**0.4**	1.8	1.6	**1.8**
λ		2.23	2.17	
γ		0.1887	0.2645	
α		0.0089	0.0124	

TABLE 5.1. The data from [10] contra our data.

From table 5.1 we see that model 1 in our objective function also generates reasonable results for the midyear 1998 prices. We therefore conclude that the reason for the bad fit of the 1999 prices is the model and not the objective function. Another important thing to note from table 5.1 is that there are remarkable differences in the prices obtained by Lane and Movchan and the prices we obtain. We thereby see that the valuation of the bid's and ask's is highly dependent on the choice of the objective function.

5.2. **Model 2.** Looking at the results from model 2 we see a remarkably better fit. All the prices are now in the bid/ask spreads and we also have reasonable prices for the 250/300 and 300/350 call spreads. So the shift of the distribution to $x = 47.2$ apparently has a large effect. This is in agreement with the results from [10]. If we compare the 1999 prices with the midyear 1998 prices we see that they are quite similar, see table 5.2

Looking at the figures in table 5.2 and recalling that the compound Poisson distribution fits the midyear 1998 prices (where the index value was 40), it is not suprising that the shifted compound Poisson distribution fits the 1999 prices. Model 1 is a special case of model 2 ($x = 0$), so model 2 will also be able to fit the midyear 1998 prices.

K_l/K_U	bid99	ask99	bid98	ask98
40/60	11.0		12.0	15.0
60/80	6.0	10.0	6.0	12.0
80/100		8.0	4.0	8.0
100/120	3.5	6.0	2.8	4.0
150/200	4.0	7.5	4.3	6.0
250/300	0.5	2.5		3.5

TABLE 5.2. The 99 prices contra the midyear 98 prices

5.3. **Model 3.** From table 4.5 we see that model 3 generates prices very similar to model 1, so we do not obtain much by including the extra parameter. And as for model 1 we conclude that model 3 is a bad description of the implied losses. As done for model 1 we could also extend model 3 by shifting it, but we will desist from doing this because we will then have five parameters, which we consider too many. How many parameters one will allow in a model is of course individual, but we set the limit by four. We discuss this further in the dicussion of model 5.

5.4. **Model 4.** A very important thing to note about this model is that the computation of the convolutions is very time consuming and this is also why we only include the first four convolutions in the model. The prices we obtain by including only the first four convolutions are very similar to the prices from model 1 and model 3, and as for model 1 and model 3 it is not possible to get into the 40/60 and 200/250 call spreads. The prices for the 250/300 and the 300/350 call spreads seem a little bit better than those for model 1 and 3. The model was justified by assuming that the term $\sum_5^\infty e^{-\lambda}\lambda^n/n!$ should be small. Here the term is 0.12, which can hardly be considered small. So, with only four convolutions, we consider the model as a bad describtion of the implied losses.

If we had included more convolutions, 10 say, we would have expected a remarkably better fit, but this would have been way to timecomsuming. The time factor is also the reason why we desist from shifting the distribution.

5.5. **Model 5.** The first thing to note from model 5 is the number of data being very small, which therefore makes it hard to really gain anything from figure 4.1 (the relation between the α-parameter and the strike). The function we choose based on figure 4.1 is therefore also very general. The three parameters included in the function are many compared to the number of data. It is therefore not surprising that we get a better fit than for model 1, 3 and 4, and if we had increased the number of parameters even more, say to four or five, we would probably also obtain a better fit than we did in model 2. But increasing the number of parameters does not make the model any better in relation of describing the implied losses. After the estimation of the prices for model 5 it is our general impression that model 5 is a bad description of the implied losses. We thereby do not conclude that there is no strike dependency on the inplied parameters, but if there is, it has to be modeled in another way.

5.6. **Model 6.** It is very fast to compute the prices corresponding to model 6 and the fit we obtain is surprisingly good. Suprisingly because it is not a compound process, but only a shifted Pareto distribution, and it generates prices that are better than the prices from model 2. But even though the model generates a good fit, the prices are very different from the prices obtained by model 2. In general it derives prices that are higher than the prices from model 2. The differences in prices are not surprising if we compare the mean values and the variances from model 1 with the ones from model 6, see table 4.4.

We now have two models both generating reasonable prices and having nearly the same value of the objective function. But this is not the same as saying that the two models are equally good. Based on the discussion in section 2, model 2 seems to be the best theoretically founded model. But if we look at the model 2 prices for the 60/80 and the 80/100 call spread, one could get the impression that the implied distribution from model 2 generates too little risk, which indicates that model 2 is not a perfect model.

After this discussion we find that model 2 is the best model to use even though it is not perfect. But we also find that one should use model 6 simultaneously because it is very fast and it could be used to support the evaluation of the prices.

5.7. **The δ_1 and δ_2 parameter.** How the parameters δ_1 and δ_2 should be set depends on the data set. If the data set consist of only bid/ask spreads and our model generates prices that are inside these spreads, then the value of δ_1 has no effect on the results. But if we have also traded prices in our data set, the value of δ_1 gets more important. In this case we believe that one should estimate prices for different values of δ_1 and then set the parameter based on an evaluation of these results. Model 2 and model 6 both generate results that are inside the bid/ask spread and the value of δ_1 therefore becomes unimportant for the data set considered in this paper.

The value of δ_2 is set to 0.1, i.e. if the prices are more than 100% above a single bid or 50% below a single ask we punish it only 10% as hard as when the prices are below or above a bid or an ask. We find this value reasonable but again we could for a given data set try with different values and based on these estimations set the parameter. But again for model 2 and model 6, the value of δ_2 does not affect the results in this paper.

6. Conclusion

We analyse prices for catastrophe insurance derivatives by looking at the "implied loss distributions" embedded in the traded prices. As mentioned in the introduction there are two main problems in this analysis. First, what kind of distribution should be chosen for the implied losses and second, how should the involved parameters be estimated?

In relation to how the parameters should be estimated we find that an improvement of the procedure from [10] was necessary because of the following two reasons. First we agree that it is desirable that the parameters are chosen such that the prices are lower than known offers and higher than known bids, but we do not think that the requirements should be invariable because, if the spreads are very small, it could be a problem to find a solution. And if the theoretical prices appear to be far away from the spread, it could be used to indicate that the chosen model may be wrong. Second we agree on point that the parameters should be chosen such that the prices gets closest to the actual traded prices, i.e. if our data contain only traded prices, the parameters should be found by a least square fit. But because the data primarily consists of spreads and single bids or asks, we find that this should be incorporated in the objective function. No matter what objective function one uses, it is clear from the discussion of model 1 that the choice of the objective function has a large effect on the derived prices, and it should therefore be chosen carefully.

After the discussion in the previous section we find that model 1, suggested by Lane and Movchan [10], is unable to fit the PCS-option prices in general. Instead, we find that model 2 is a better model to use for the implied losses. However, it would be preferable to use model 6 also in order to support it. It is clear that none of the suggested models fit the implied losses perfectly, but we believe that model 2 supported by model 6 will be a good tool for investors analysing prices of catastrophe insurance derivatives.

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(C. Vorm Christensen) Department of Theoretical Statistics and Operations Research, University of Aarhus, Ny Munkegade, 8000 Aarhus C, Denmark

E-mail address: `vorm@imf.au.dk`

HOW TO HEDGE UNKNOWN RISK

CLAUS VORM CHRISTENSEN

ABSTRACT. In this paper we are considering risk with more than one prior estimate of the frequency, e.g. environmental health risk of new and little known epidemics, or risk induced by scientific uncertainty in predicting the frequency and severity of catastrophic events. It is not possible to hedge this kind of risk only using traditional insurance practice. A new method is called for. In this paper we show how to manage this unknown risk by using traditional insurance practice and by trading in the security market simultaneously.

1. INTRODUCTION

During the last years, the market for risk related to natural phenomena such as different catastrophes has witnessed important changes.

Such risk have traditionally been distributed through the insurance and reinsurance system. Insurance companies accumulate the risk of individual entities and redistribute the risk to the global reinsurance industry. But the volatility of weather, taken together with population movement to warm coastal areas and change of property prices has made catastrophic risk highly unpredictable. It is therefore no longer possible to diversify this risk using traditional insurance practices. A new way to manage such risk or unknown risk in general is called for.

When we talk about unknown risk, we refer to risk which frequency we do not know, i.e. there is more than one estimate of the frequency of the risk. Examples of unknown risk are environmental health risk of new and little known epidemics, or risk induced by scientific uncertainty in predicting the frequency and severity of catastrophic events.

As we will show in this paper, motivated by [2], the way to handle unknown risk is to use two different approaches of hedging risk simultaneously, namely the statistical approach known from the insurance industry and the economic approach known from the securities industry.

In this paper we consider a general model for an insurance company, where the company faces n states of the world. For each of these states

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the insurance company is able to estimate the frequency of the risk, but the risk related to the states is unknown. We then show how the company should handle this unknown risk. This is done by using the statistical approach to handle the known risk, i.e. the risk related to a given state. And by using the economic approach to handle the risk related to the different states.

In the next section we make the basic assumptions and present the general model underlying the theory. In section 3 we show how to handle the unknown risk in the case where the market is complete. We show how the statistical approach and the economic approach are used simultaneously. It is not always the case that the insurance company can charge any premium they want, it is therefore natural to consider the case with a restricted premium. This case we consider in section 4 where four different ways of choosing the n state premiums are suggested. These four different choices are then evaluated in section 5 and 6. In section 7 we consider the incomplete market case and finally there are some concluding remarks.

2. The Model

Let S denote the state of the world. We make the following assumptions:

- There are n states denoted by $\{s_1, \dots, s_n\}$, $\quad S \in \{s_1, \dots, s_n\}$.
- The probabilities corresponding to the n states are known
$$P(S = s_i) = p_i, \quad i = 1, \dots, n, \quad \sum_{i=1}^{n} p_i = 1$$
- F_i is known for all $i \in \{1, \dots, n\}$, where F_i denotes the distribution of the loss (L) of the insurance companies given the state is i ($L|\{S = s_i\} \sim F_i$). Let $L_i = L|\{S = s_i\}$.
- If the insurance company knows the state S then the statistical approach by adding a safety loading would work, i.e. if the insurance company knew that $S = s_i$ it would be reasonable to charge the premium P_i given by
$$P_i = E[L_i] + \delta_i$$
where δ_i is a safety loading calculated by a standard premium calculation principle.
- There exist n "state securities" traded on the n states. Security number j pays the amount c_{ij} if the state is i. Let c_i be the vector $c_i = (c_{i1}, \dots, c_{in})$ and let C be the matrix given by
$$C = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Let further $\tilde{c_j}$ be the jth column in C.
- The market is complete, i.e. the n columns in C are linearly independent.
- The market for these securities is arbitrage free and there exists an unique risk neutral measure. We denote the risk neutral probabilities by $q_1, \dots, q_n$, and let q be the vector given by $q = (q_1, \dots, q_n)$. From basic finance courses it is known that these risk neutral probabilities can be used to price the state securities, i.e. the price of state security number i is given by the discounted value of $q\tilde{c_j}$.
- There exists a risk free security and for simplicity we assume that the risk free interest rate is zero. This is no loss of generality because we can discount all securities.

We now have a model where the insurance company exactly knows how they should handle the insurance risk if the state of the world is known. But because of the uncertainty on the state of the world the general risk for the insurance company becomes unknown. In the next section we will show how the insurance company is able to handle this unknown risk.

3. How to handle unknown risk in a complete market

The expected loss for the insurance company is given by

$$E[L] = p_1 E[L_1] + \cdots + p_n E[L_n]$$

To cover these losses the insurance company has to charge a premium P, but charging a premium is not enough. Because if the insurance company charges a premium P we obtain a safety loading in state i given by $\tilde{\delta}_i = P - E[L_i]$. The problem by this is that we do not obtain the desired safety loading. For some i's we have that $\delta_i < \tilde{\delta}_i$ which means that the insurance company has been overcharging. And for some i's we have that $\delta_i > \tilde{\delta}_i$ which means that the insurance company has been undercharging, which could lead to a dangerous position. Before we solve this problem we make the two following definitions.

Definition 3.1. A **trading strategy** for the insurance company is defined as a vector $m = (m_1, \dots, m_n)^T$ where m_i denotes how many securities i the insurance company buys.

Definition 3.2. An **optimal trading strategy** for the insurance company is a costless trading strategy such that

$$P + c_i m - E[L_i] = \delta_i \qquad \forall i = 1, \cdots, n. \tag{3.3}$$

The question is now whether it is possible to obtain this optimal strategy and if it is, what premium should be charged in order to obtain it? These questions is answered in the following theorem.

Theorem 3.4. *An optimal trading strategy can be obtained if and only if*

$$P = q_1 P_1 + \cdots + q_n P_n.$$

In this case, the strategy m has to be chosen by 3.5.

Proof. We first show the if part. Assume that the insurance company charge the premium P given by $P = q_1 P_1 + \cdots + q_n P_n$. We then show that there exist a trading strategy $m = (m_1, \ldots, m_n)^T$ that solves condition (3.3) in definition (3.2). m if found by solving the following equation

$$\begin{bmatrix} P \\ \vdots \\ P \end{bmatrix} - \begin{bmatrix} E[L_1] \\ \vdots \\ E[L_n] \end{bmatrix} + \begin{bmatrix} c_{11} & \ldots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \ldots & c_{nn} \end{bmatrix} \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{bmatrix}.$$

The market is complete, C is therefore invertible. A solution exists and is given by

$$m = C^{-1} \begin{bmatrix} -P + E[L_1] + \delta_1 \\ \vdots \\ -P + E[L_n] + \delta_n \end{bmatrix}. \tag{3.5}$$

It now remains to show that this trading strategy is costless. The value of the ith security is the expected value of the payoff calculated under the risk neutral measure. Let v_i denote the value of the ith security

$$v_i = q\tilde{c_j}.$$

Let v be given by $v = (v_1, \ldots, v_n)$. The cost of the portfolio is then given by

$$\begin{aligned} \text{Cost of pf.} &= vm \\ &= qCC^{-1} \begin{bmatrix} -P + E[L_1] + \delta_1 \\ \vdots \\ -P + E[L_n] + \delta_n \end{bmatrix} \\ &= \sum_{i=1}^{n} q_i(-P + E[L_i] + \delta_i) \\ &= -P \sum_{i=1}^{n} q_i + \sum_{i=1}^{n} q_i P_i = 0. \end{aligned}$$

We now show the only if part. Assume therefore that there exist an optimal trading strategy. Equation (3.5) then holds and $vm = 0$. Using this and that $v = qC$ we obtain

$$\sum_{i=1}^{n} q_i(-P + E[L_i] + \delta_i) = 0$$

and rearranging the terms we obtain

$$P = q_1 P_1 + \cdots + q_n P_n$$

□

Remark 3.6. The problem can be simplified if we consider it in the following way. The insurance company wants to obtain the premiums $(P_1, \dots, P_n)$ corresponding to the n states. This could be obtained for all i if we for all i buy P_i of Arrow-Debreu (AD) security number i. AD security i is a security that pays 1 if the state is i and pays zero in all other states. These AD securities exist because the market is complete, and the price of AD security number i is given by q_i. The total price of this AD portfolio is therefore given by

$$\text{Total price} = \sum_{i=1}^{n} P_i q_i$$

So by charging a premium $P = \sum_{i=1}^{n} P_i q_i$ the insurance company can obtain the optimal strategy. This only works if the market is complete, we will return to the incomplete case later.

4. The restricted premium case

In the previous section we found the optimal premium to charge for the insurance company. But the insurance company may be unable to charge this premium because of some competitive reasons. We therefore now assume that the premium which the insurance company can charge is fixed at P_0.

The insurance company should therefore now choose a trading strategy which they find "optimal" under the restriction that the cost of the trading strategy equals P_0. What we mean by "optimal" is discussed later in this section. In this complete market case choosing a trading strategy m is equivalent to choosing premiums $(P_1, \dots, P_n)$. We have the following relation between $(P_1, \dots, P_n)$ and m

$$(P_1, \dots, P_n)^T = Cm.$$

The restriction can also be expressed in terms of the P_i's instead of m.

$$\begin{aligned} & vm = P_0 \\ \Rightarrow \quad & qCC^{-1}(P_1, \dots, P_n)^T = P_0 \\ \Rightarrow \quad & q_1 P_1 + \cdots + q_n P_n = P_0. \end{aligned}$$

These observations now allows us to reformulate the problem to a problem in terms of the premiums $(P_1, \dots, P_n)$ instead of a problem in term of the trading strategy m.

The problem in this fixed premium case is therefore to find the "optimal" choice of the P_i's subject to the constrain $P_0 = P_1 q_1 + \cdots + P_n q_n$. We will now consider four different ways of solving this optimal premium choice (OPC), i.e. defining "optimal". In the following we let

O_i denote the difference between the premium received in state i and the losses paid in state i, i.e. $O_i = P_i - L_i$.

4.1. **OPC1: Equal risk quantity in all states.** The goal here is to obtain the same risk quantity in all the states. To measure the risk quantity we will use the mean divided by the standard deviation, i.e. high value of the quantity corresponds to a low risk. The solution to OPC1 is found by solving the following n equations with n unknowns:

$$\frac{E[O_1]}{\sqrt{\mathrm{Var}(O_1)}} = \frac{E[O_i]}{\sqrt{\mathrm{Var}(O_i)}}, \qquad i = 2, \dots, n$$
$$P_0 = P_1 q_1 + \cdots + P_n q_n$$

Note that we here have $Var(O_i) = Var(L_i)$. OPC1 therefore ensures that expected gain $E[P_i - L_i]$ will be high in states where $Var(L_i)$ is large, i.e. we obtain a high gain in the risky states.

4.2. **OPC2: Equal ruin probabilities in all states.** The goal here is to obtain the same ruin probabilities in all the states. The ruin probabilities are usually calculated according to the initial capital, we therefore include the initial capital u in the OPC. The solution is found by solving the following n equations with n unknowns:

$$P(O_1 + u < 0) = P(O_i + u < 0), \qquad i = 2, \dots, n$$
$$P_0 = P_1 q_1 + \cdots + P_n q_n$$

4.3. **OPC3: Equal expected utility in all states.** The goal here is to obtain the same expected utility in all the states. In order to solve the problem we have to choose a utility function. Let the utility function be given by $v(x), x \in \mathbb{R}$, where we assume, $v(0) = 0$ (only for convenience), $v' > 0$ (less losses are preferred) and $v'' < 0$ (stronger weights for higher losses). The solution is found by solving the following n equations with n unknowns:

$$E[v(u + O_1)] = E[v(u + O_i)], \qquad i = 2, \dots, n$$
$$P_0 = P_1 q_1 + \cdots + P_n q_n$$

4.4. **OPC4: Maximal expected utility.** The goal here is to obtain the maximal expected utility. We have the same utility function as in OPC3, but we now have to solve the following maximization problem:

$$\max E[v(u + P_s - L_s)]$$
$$\text{st} \quad P_0 = P_1 q_1 + \cdots + P_n q_n$$

We have now stated four different ways of solving the OPC, but which one is the best? At first it seems most natural to use OPC4, i.e. maximize the expected utility. But a problem by this approach is that when we only consider the general expected utility we could end up with some very risky individual states. One could imagine a situation where the insurance company maximizes its expected utility and thereby obtains a very large ruin probability in one of the states. Such a premium choice could then, because of the high ruin probability, be refused by shareholders, authorities or the like. OPC4 is therefore not necessarily the best OPC to use in general. In the following we will analyse the OPC's further.

5. Solutions of the different OPC's

In this section we will try to evaluate and compare the different OPC's. This is done by first solving the different OPC's one by one, and then secondly compare them by an example where different premiums are calculated.

5.1. **Solution of OPC1.** Let now μ_i denote the mean value and σ_i denote the standard deviation of L_i. We then have $\sqrt{\mathrm{Var}(O_i)} = \sigma_i$ and OPC1 can now be written as

$$\begin{aligned} \frac{P_1 - \mu_1}{\sigma_1} &= \frac{P_i - \mu_i}{\sigma_i}, \qquad i = 2, \ldots, n \\ P_0 &= P_1 q_1 + \cdots + P_n q_n \end{aligned}$$

The general solution to these n equations with n unknowns is given by

$$\begin{aligned} P_i &= a_i P_1 + b_i, \qquad i = 2, \ldots, n \\ P_1 &= \frac{P_0 - q_2 b_2 - \cdots - q_n b_n}{q_1 + q_2 a_2 + \cdots + q_n a_n} \end{aligned}$$

where $a_i = \sigma_i / \sigma_1$ and $b_i = \mu_i - a_i \mu_1$.

OPC1 is easy to solve, and it only requires that the first two moments of all the loss distributions exists. From the solution we see that P_i is increasing in μ_i and if $P_0 > \sum_{i=1}^{n} q_i \mu_i$ also increasing in σ_i. This also seems intuitively clear. Here the risk measure must be the same for all the states, the premium for state i is therefore expected to increase if the mean value or the standard deviation corresponding to state i increases.

5.2. **Solution of OPC2.** Consider first

$$\begin{aligned} &\mathrm{P}(O_i + u < 0) \\ &= \mathrm{P}(P_i - L_i + u < 0) = \mathrm{P}(L_i > P_i + u) = 1 - F_i(P_i + u) \end{aligned}$$

OPC2 can therefore be written as

$$F_1(P_1+u)=F_i(P_i+u), \qquad i=2,\dots,n$$
$$P_0=P_1q_1+\cdots+P_nq_n$$

Plugging the $(n-1)$ equations into equation number n we obtain

$$\sum_{i=1}^{n} q_i F_i^{-1}(F_1(P_1+u))=P_0+u$$

Let then $G(x)$ be given by

$$G(x)=\sum_{i=1} q_i F_i^{-1}(F_1(x))$$

$G(x)$ is then increasing, hence invertible. In the cases where F_i is continuous the solution is given by

$$P_1=G^{-1}(P_0+u)-u$$
$$P_i=F_i^{-1}(F_1(P_1+u))-u$$

If F_i is not continuous, it may happen that no solution exists.

Remark 5.1. An alternative way of solve the problem, will be to solve the problem numerically by setting up the following minimization problem

$$\min \sum_{i=2}^{n}(F_1(P_1+u)-F_i(P_i+u))^2+(P_0-P_1q_1-\cdots-P_nq_n)^2$$

Note that the solution to the minimization problem is only a solution to the OPC2 problem if the optimal value is 0.

5.3. **Solution of OPC3.** We first solve OPC3 in general, i.e. where the only assumption we have on the utility function is that $v(0)=0$, $v'>0$ and $v''<0$. Let now $g_i(x)$ be given by

$$g_i(x)=E[v(x-L_i+u)]$$

$g_i(x)$ is then strictly increasing and continuous, hence the inverse exists, and is also strictly increasing and continuous. The n equations corresponding to OPC3 can therefore be written as

$$P_i=g_i^{-1}(g_1(P_1)), \qquad i=2,\dots,n$$
$$P_0=\sum_{i=1}^{n} q_i g_i^{-1}(g_1(P_1)).$$

Let now $G(x)$ be given by

$$G(x)=\sum_{i=1}^{n} q_i g_i^{-1}(g_i(x)).$$

$G(x)$ is then increasing, hence invertible and the solution is given by

$$P_1 = G^{-1}(P_0).$$

We now have a general solution of OPC3. In the next part of this section we consider OPC3 with a further assumption, namely that the P_i's has to be independent of u. How this assumption will restrict our choice of the utility function is shown by the following lemma.

Lemma 5.2. *The P_i's that solves OPC3 are independent of u for all distributions of L_i and all choices of $q_1, \ldots, q_n$ if and only if $v(x) = A(1 - e^{-\alpha x})$ for some $A > 0$ and $\alpha > 0$.*

Proof. Assume that $v(x) = A(1 - e^{-\alpha x})$. We then have to solve the following problem

$$\begin{aligned} &E[A(1 - e^{-\alpha(P_1 - L_1 + u)})] = E[A(1 - e^{-\alpha(P_i - L_i + u)})], \qquad i = 2, \ldots, n \\ &P_0 = P_1 q_1 + \cdots + P_n q_n \end{aligned}$$

These equations are equivalent to

$$\begin{aligned} &E[A(1 - e^{-\alpha(P_1 - L_1)})] = E[A(1 - e^{-\alpha(P_i - L_i)})], \qquad i = 2, \ldots, n \\ &P_0 = P_1 q_1 + \cdots + P_n q_n \end{aligned} \tag{5.3}$$

We here see that the P_i's are independent of u.

Assume that the P_i's are independent of u. Let there be two states and two corresponding AD prices q_1 and q_2 ($q_1 + q_2 = 1$). Let $P(L_1 = 2) = 1 - P(L_1 = 0) = r$ and $P(L_2 = 1) = 1$. P_1 and P_2 are now dependent on r and we will therefore in the following denote them $P_1(r)$ and $P_2(r)$. The OPC3 can now be written as

$$\begin{aligned} &rv(u + P_1(r) - 2) + (1 - r)v(u + P_1(r)) = v(u + P_2(r) - 1) \\ &q_1 P_1(r) + q_2 P_2(r) = P_0 \end{aligned}$$

From the second equation we can express $P_2(r)$ in terms of $P_1(r)$. This expression is used in the first equation to obtain

$$\begin{aligned} &rv(u + P_1(r) - 2) + (1 - r)v(u + P_1(r)) = \\ &v(u + \frac{P_0 - q_1 P_1(r)}{q_2} - 1). \end{aligned} \tag{5.4}$$

Taking the derivative of (5.4) with respect to u yields

$$\begin{aligned} &rv'(u + P_1(r) - 2) + (1 - r)v'(u + P_1(r)) = \\ &v'(u + \frac{P_0 - q_1 P_1(r)}{q_2} - 1). \end{aligned} \tag{5.5}$$

The derivative of (5.4) with respect to r is

$$\begin{aligned} &v(u + P_1(r) - 2) - v(u + P_1(r)) + P_1'(r) r v'(u + P_1(r) - 2) + \\ &P_1'(r)(1 - r)v'(u + P_1(r)) = -\tfrac{q_1}{q_2} P_1'(r) v'(u + \tfrac{P_0 - q_1 P_1(r)}{q_2} - 1). \end{aligned} \tag{5.6}$$

Plugging in (5.5) in (5.6) yields

$$\begin{aligned} & v(u+P_1(r)-2)-v(u+P_1(r))+ \\ & \frac{1}{q_2}P_1'(r)v'(u+\frac{P_0-q_1P_1(r)}{q_2}-1)=0. \end{aligned} \tag{5.7}$$

Note that $P_1'(r)>0$ follows immediately. The derivative of (5.7) with respect to u is

$$\begin{aligned} & v'(u+P_1(r)-2)-v'(u+P_1(r))+ \\ & \frac{1}{q_2}P_1'(r)v''(u+\frac{P_0-q_1P_1(r)}{q_2}-1)=0. \end{aligned} \tag{5.8}$$

and from the derivative with respect to r it follows that

$$\begin{aligned} & -\frac{1}{q_2^2}P_1'(r)^2v''(u+\frac{P_0-q_1P_1(r)}{q_2}-1)+ \\ & \frac{1}{q_2}P_1''(r)v'(u+\frac{P_0-q_1P_1(r)}{q_2}-1)=0. \end{aligned}$$

Thus $P_1''(r)\le 0$ and if we substitute $x=u+\frac{P_0-q_1P_1(r)}{q_2}-1$ we get

$$q_2\frac{P_1''(r)}{P_1'(r)^2}=\frac{v''(x)}{v'(x)}.$$

The right hand side is independent of r and the left hand side is independent of x. The left hand side and the right side is therefore both independent of r and x. And from this we conclude

$$\frac{v''(x)}{v'(x)}=-\alpha$$

for some $\alpha\ge 0$ and the assertion follows. □

Example 5.9. If we now set $v(x)=A(1-e^{-\alpha x})$ we are able to find a closed solution to OPC3. Let $M_{L_i}(\alpha)$ denote the moment generating function of L_i at the point α. From (5.3) we then know that OPC3 can be written as

$$\begin{aligned} P_i &= P_1-\frac{1}{\alpha}\ln(\frac{M_{L_1}(\alpha)}{M_{L_i}(\alpha)}), \qquad i=2,\dots,n \\ P_0 &= P_1q_1+\cdots+P_nq_n \end{aligned}$$

We now have a problem equivalent to OPC1 and the general solution is therefore given by

$$\begin{aligned} P_1 &= P_0+\frac{1}{\alpha}(q_2\ln(a_2)+\cdots+q_n\ln(a_n)) \\ P_i &= P_1-\frac{1}{\alpha}\ln(a_i), \qquad i=2,\dots,n \end{aligned}$$

where $a_i=\frac{M_{L_1}(\alpha)}{M_{L_i}(\alpha)}$.

Remark 5.10. A problem of the exponential utility function is that it only allows for loss distributions where the moment generating function exists. This means that for heavy tailed loss distributions we need to choose another utility function, e.g. the power utility function. If we do so the OPC3 becomes dependent of u and also much harder to solve. In most of the cases the solution will have to be found numerically.

5.4. **Solution of OPC4.** As for OPC3 we first solve OPC4 in general.

$$\begin{array}{ll} \max & p_1 E[v(u+P_1-L_1)] + \cdots + p_n E[v(u+P_n-L_n)] \\ \text{st} & P_0 = P_1 q_1 + \cdots + P_n q_n \end{array}$$

In order to solve this problem we construct the Lagrange function

$$\begin{aligned} F(P_1,\ldots,P_n) = p_1 E[v(u+P_1-L_1)] + \cdots + p_n E[v(u+P_n-L_n)] \\ +\lambda(P_0 - P_1 q_1 - \cdots - P_n q_n) \end{aligned}$$

The next step is to construct the first and second order condition corresponding to the maximization problem. The first order condition:

$$\begin{aligned} &\frac{\partial F}{\partial P_1} = p_1 E[v'(u+P_1-L_1)] - \lambda q_1 = 0 \\ &\vdots \\ &\frac{\partial F}{\partial P_n} = p_n E[v'(u+P_n-L_n)] - \lambda q_n = 0 \\ &\frac{\partial F}{\partial \lambda} = P_0 - P_1 q_1 - \cdots - P_n q_n = 0 \end{aligned}$$

Let now $g_i(x) = E[v'(u+x-L_i)]$, then $g_i(x)$ is strictly decreasing and continuous, hence the inverse exists, which also is continuous and decreasing. From equation number 1 and i, $i=2,\ldots,n$ we get

$$P_i = g_i^{-1}\left(\frac{p_1 q_i}{p_i q_1} g_1(P_1)\right). \tag{5.11}$$

P_1 is found by plugging (5.11) into the constraint, i.e. from the equation

$$P_0 = q_1 P_1 + q_2 g_2^{-1}\left(\frac{p_1 q_2}{p_2 q_1} g_1(P_1)\right) + \cdots + q_n g_n^{-1}\left(\frac{p_1 q_n}{p_n q_1} g_1(P_1)\right).$$

Note that $q_1 P_1$ and $q_i g_i^{-1}(\frac{p_1 q_i}{p_i q_1} g_1(P_1))$ for all i are strictly increasing in P_1. It is therefore possible to find a solution to this equation.

We now have a solution. In order to check whether it is optimal we check the second order condition. To check the second order condition, for a minimization problem with an equality constraint, we have to construct the so called bordered determinants, see [3] (p. 382). They are obtained by bordering the principal minors of the Hessian determinant of second partial derivatives of the Lagrange function by a row and a column containing the first partial derivatives of the constraint. The element in the southeast corner of each of these arrays is zero.

The second order condition for the problem is then satisfied if these bordered determinants alternate in sign, starting with plus, i.e. the sign of the determinants below must taken from above be +,–,+, etc.

$$\begin{bmatrix} p_1E[v''(u+P_1-L_1)] & 0 & -q_1 \\ 0 & p_2E[v''(u+P_2-L_2)] & -q_2 \\ -q_1 & -q_2 & 0 \end{bmatrix}$$

$\vdots$

$$\begin{bmatrix} p_1E[v''(u+P_1-L_1)] & \dots & 0 & -q_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & p_nE[v''(u+P_n-L_n)] & -q_n \\ -q_1 & \dots & -q_n & 0 \end{bmatrix}$$

It follows easily from the bordering determinants above that the second order condition is fulfilled.

In the following example we solve the problem explicitly for the exponential utility function.

Example 5.12. As for OPC3 we now solve OPC4 explicitly for $v(x) = A(1-e^{-\alpha x})$. The OPC4 problem then takes the following form:

$$\begin{array}{ll} \max & 1 - p_1E[e^{-\alpha(P_1-L_1)}] - \cdots - p_nE[e^{-\alpha(P_n-L_n)}] \\ \text{st} & P_0 = P_1q_1 + \cdots + P_nq_n \end{array}$$

As before we first construct the Lagrange function

$$\begin{aligned} F(P_1,\dots,P_n) = 1 - p_1M_{L_1}(\alpha)e^{-\alpha P_1} - \cdots - p_nM_{L_n}(\alpha)e^{-\alpha P_n} \\ +\lambda(P_0 - P_1q_1 - \cdots - P_nq_n) \end{aligned}$$

The next step is then to construct the first and second order condition corresponding to the maximization problem. The first order condition as before:

$$\begin{aligned} \frac{\partial F}{\partial P_1} &= \alpha p_1M_{L_1}(\alpha)e^{-\alpha P_1} - \lambda q_1 = 0 \\ &\vdots \\ \frac{\partial F}{\partial P_n} &= \alpha p_nM_{L_n}(\alpha)e^{-\alpha P_n} - \lambda q_n = 0 \\ \frac{\partial F}{\partial \lambda} &= P_0 - P_1q_1 - \cdots - P_nq_n = 0 \end{aligned}$$

From equation number 1 and i, $i = 2,\dots,n$ we get

$$P_i = P_1 - \frac{1}{\alpha}\ln a_i, \quad \text{where } a_i = \frac{p_1q_iM_{L_1}(\alpha)}{p_iq_1M_{L_i}(\alpha)}. \tag{5.13}$$

Plugging (5.13) into the constraint we obtain

$$P_1 = P_0 + \frac{1}{\alpha}(q_2 \ln(a_2) + \cdots + q_n \ln(a_n))$$

We now have a solution and we know from the general solution that it is optimal. So again we obtain a nice solution for the exponential utility function. But as for OPC3 the solution only work for distribution where the moment generating function exist. Note that for general utility functions, solutions may have to be found numerically.

We are now ready to evaluate the four OPC's.

6. Evaluation of the OPS's

In this section we try to evaluate the different OPC's. This is done by constructing two different examples. In the first example we consider a model with three different states. The losses corresponding to these states follow an exponential distribution, and two different gamma distributions, respectively. These light tailed distributions allow us to use the exponential utility function in OPC3 and OPC4. We are therefore able to compare all the different OPC's in this example. In the other example we include a heavy tailed distribution. We here consider two states, the losses in state two follow a Pareto distribution whereas the losses in state one still follow a light tailed distribution namely the exponential. When we include a heavy tailed distribution we are no longer able to use the exponential utility function. We therefore only compare OPC1 and OPC2 in this example.

6.1. **Example 1.** We start this example by specifying the distribution of the state losses and their mean values, variances and moment generating functions. The possible premium (P_0) is set to $1.2 \times E_P[L]$ ($P_0 = 7.44$).

	L_1	L_2	L_3
Distribution	Exp(1)	Γ(5,1)	Γ(2,0.1)
Mean value	1	5	20
Variance	1	5	200
$M_{L_i}(\alpha)$	$(\frac{1}{1-\alpha})$	$(\frac{1}{1-\alpha})^5$	$(\frac{0.1}{0.1-\alpha})^2$

We further assume that the p probabilities are given by $p_1 = 0.45$, $p_2 = 0.35$ and $p_3 = 0.2$. But before we set the q probabilities recall the following. The q probabilities can be seen as the prices of the AD securities. The q probabilities are therefore set by the agents in the market. Some of the agents trading in this market will be people from the insurance market, i.e. people who will be needing money if the world end up in state 3. We therefore expect the price of AD security

3 (q_3) to be higher than p_3 and as a consequence of this q_1 and/or q_2 to be lower than p_1 and/or p_2. With this in mind we now assume that the preferences in the market determines the following q probabilities $q_1 = 0.044$, $q_2 = 0.0345$ and $q_3 = 0.0215$. We have now set all the parameter and are therefore ready to calculate the different OPC.

In table 1 the different OPC are stated. For OPC2 we have chosen different values for the initial capital, and similar for OPC3/4 we have chosen different values for the risk aversion coefficient α.

OPC number	special parameter	P_1	P_2	P_3
1		1.022	5.108	24.318
2	$u = 0$	1.118	5.701	23.168
2	$u = 1$	0.356	5.212	25.512
2	$u = 10$	-6.224	3.742	46.742
3	$\alpha = 0.09$	-4.782	-0.590	45.339
3	$\alpha = 0.05$	0.284	4.387	26.984
3	$\alpha = 0.03341$	1.022	5.090	24.346
3	$\alpha = 0.03075$	1.118	5.181	24.002
3	$\alpha = 0.03$	1.145	5.206	23.908
3	$\alpha = 0.01$	1.739	5.759	21.806
3	$\alpha = 0.001$	1.953	5.955	21.053
3	$\alpha = 0.0001$	1.973	5.973	20.983
4	$\alpha = 0.09$	-4.525	-0.423	44.543
4	$\alpha = 0.05$	0.747	4.689	25.551
4	$\alpha = 0.04549$	1.022	4.938	24.591
4	$\alpha = 0.04384$	1.118	5.024	24.255
4	$\alpha = 0.03$	1.917	5.709	21.520
4	$\alpha = 0.01$	4.056	7.267	14.643
4	$\alpha = 0.001$	25.122	21.040	-50.571

TABLE 1. The OPC's for example 1

From table 1 we see that the safety loadings corresponding to state 1,2 and 3 for OPC1 are set to 2%, 2% and 22% respectively. The safety loadings are the same for state 1 and state 2. This is just as we expect, because they have the same ratio between the mean value and the variance. The safety loading for state 3 is much larger, this means that OPC1, as preferred, takes the higher risk in state 3 into account. How OPC1 changes when the risk in state 1 becomes more and more heavy tailed is considered in example 2.

Consider now OPC2. For $u = 0$ the safety loadings are 12%, 14% and 16% for state 1, 2 and 3 respectively. This means that OPC2 for $u = 0$ do not specially compensate for the higher risk in state 3. But if we increase the initial capital, we see that the premium in state 3 increases. It is preferable to have a OPC that compensates for the higher risk in state 3. But the question is how large a compensation is desirable? We see that for high values of u OPC2 suggest that we start selling out of AD security 1 in order to buy more of AD security 3. We hereby obtain a low ruin probability in all states but also the probability of loosing money on the sold AD securities. For some it will probably be better to accept a higher ruin probability in state 3 and then not to sell as many of AD security 1. If this is the case one should choose the u carefully or go for another OPC. The choice of u could also lead to other problem for OPC2, which we return to in example 2.

We now turn to OPC3. When we chose the risk aversion parameter (α) we have to remember that α has to be below 0.1 otherwise the moment generating function for the loss in state 3 does not exist. Values of α close to 0.1 therefore corresponds to a high risk aversion. For $\alpha = 0.09$ we also see a very risk averse behaviour. Here the agent fears state 3 so much that he sells out of AD security 1 and 2 in order to buy more of AD security 3. For lower values of the risk aversion the safety loading in state 3 decreases. The values of $\alpha = 0.03341$ and $\alpha = 0.03075$ are included in the table in order to see how OPC 3 is related to OPC1 and OPC2. For the two values we obtain $P_1^{OPC3} = P_1^{OPC1}$ and $P_1^{OPC3} = P_1^{OPC2}$, but we also see that is not possible to find a α such that the all P_i's are equal, but for $\alpha = 0.03341$ OPC1 and OPC3 are very similar. For $\alpha \to 0$, i.e. if we moving towards a risk neutral behaviour we see that the P_i's are converging. It can be shown that the limits are given by

$$\begin{aligned} P_i &= E[L_i] + P_0 - E_Q[L] \\ &= E[L_i] + 0.975 \end{aligned}$$

This is also what we expect, an agent with a risk neutral behaviour allocates the same amount of safety loading in all the states.

Finally consider OPC4. When we consider OPC4 we have to remember that OPC4 both include the p and the q probabilities. This means that the premium choices for an agent using OPC4 becomes depended on how he evaluates the market prices, i.e. how his preferences are compared to the preferences of the representative agent. This means that if the agent consider the price of AD security i (q_i) to low compared to his own preferences he will be buying more of AD security i. But when he buys more of AD security i his portfolio changes and at some level he will consider the price to high. So when we now evaluate OPC4 we will have to remember that the agent maximize his utility and evaluate the prices (the q probabilities) at the same time.

For a high risk aversion coefficient we see, as for OPC3, that the agent sells out of AD security 1 and 2 in order to buy more of AD security 3. We also see that for $\alpha = 0.09$ P_3 is lower for OPC4 than for OPC3. The reason for this is as mentioned above that when the agent using OPC4 obtain a premium $P_3 = 44.543$ in state 3 he starts considering the price of AD security 3 (q_3) to high.

The values of $\alpha = 0.04549$ and $\alpha = 0.04384$ are included in the table in order to see how OPC 4 is related to OPC1 and OPC2. For the two values we obtain $P_1^{OPC3} = P_1^{OPC1}$ and $P_1^{OPC3} = P_1^{OPC2}$. We here observe that OPC4 put more weight into state 3 that both OPC1 and OPC2.

Finally we see that for $\alpha \to 0$, i.e. we are moving towards a risk neutral behaviour the agent, as expected, considers the price of AD security 3 higher and higher. He therefore sets P_3 lower and lower, i.e. selling more and more of AD security 3.

6.2. **Example 2.** As mentioned above we here assume that the total losses in state 1 is exponentially distributed ($L_1 \sim \mathrm{Exp}(\alpha_1)$) and the total losses in state 2 is Pareto distributed ($L_2 \sim \mathrm{Pa}(\alpha_2, \beta_2)$). The possible premium (P_0) is here set to $1.2 \times E_Q[L]$ and the q probabilities are set to $q_1 = 0.75$ and $q_2 = 0.25$. In this example we consider three different values for the parameters. This is done in order to see how the two OPC's (OPC1 and OPC2) differ when the loss distribution in state 2 becomes more and more heavy tailed. The parameter values, the mean values and the variances for the L_i's, the OPC, the risk quantity's and the ruin probabilities are all given in the table below. OPC2 is solved for different values of the initial capital. The value of the u's are given in the table.

If we first consider OPC1, we see that the value of P_2 increases and the risk quantity decreases when L_2 becomes more and more heavy tailed. So again OPC1 works the way we want, i.e. we have a relatively large safety loading in the states where our risk is high.

We now consider OPC2. Here the picture is different. For $u = 0$ and $u = 10$ we surprisingly observe that the the value of P_2 and the ruin probability both decreases when L_2 becomes more and more heavy tailed. But for $u = 20$ we observe a more preferable behaviour, namely that P_2 and the ruin probability both increases when L_2 becomes more and more heavy tailed. The reason for the "bad" OPC for $u = 0$ and $u = 10$ must be found in the shape of the distribution functions. In figure 1 and figure 2 $P(L < P+u)$ (1-"the ruin probability") are shown for $u = 0$ and $u = 20$ respectively. The labels corresponds to the following distributions of L $\mathrm{Exp}(0.1)(0)$, $\mathrm{Pa}(6, 50)(1)$ and $\mathrm{Pa}(2.1, 11)(3)$. The horizontal lines are the levels of the survival probability (1-ruin probability) for OPC2.

If we start analysing figure 1, it becomes clear why we observe the decreasing P_2 when L_2 becomes heavier tailed. Because of the low values of the initial capital we have a relative low level of the distribution

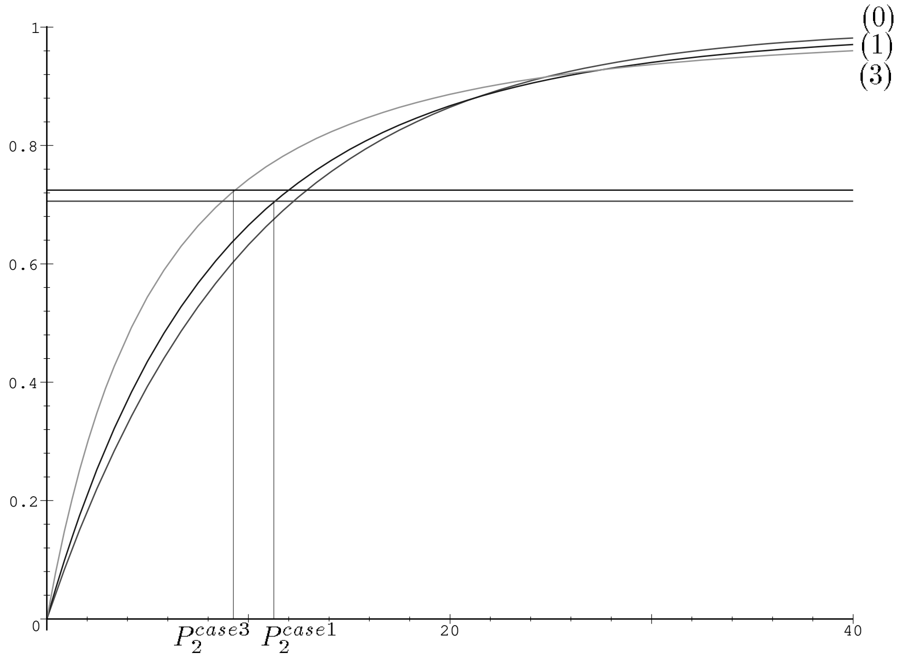

FIGURE 1. The shape of the distribution functions for $u = 0$.

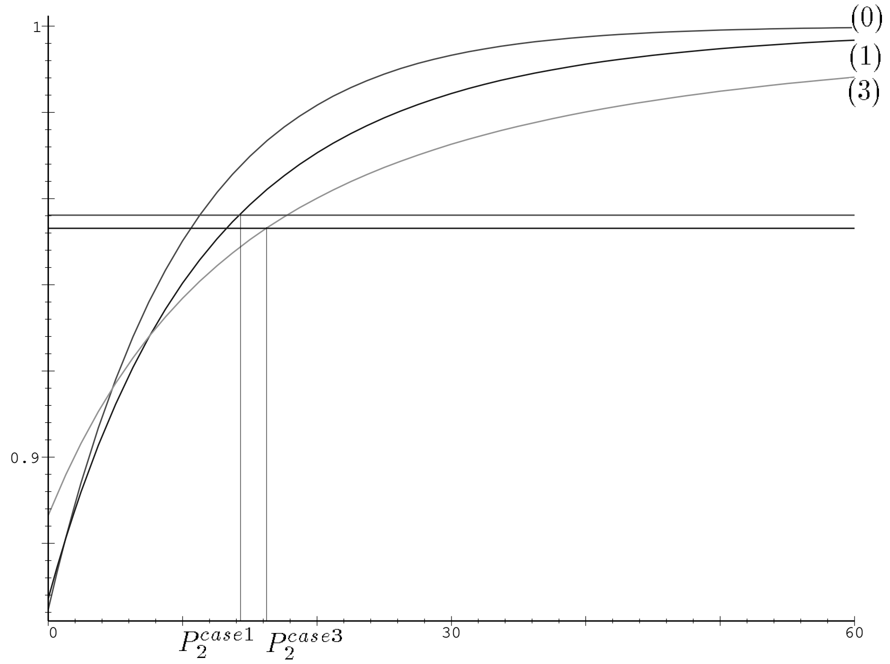

FIGURE 2. The shape of the distribution functions for $u = 20$.

	Case 1	Case 2	Case 3
α_1	0.1	0.1	0.1
α_2	6	3	2.1
β_2	50	20	11
$E[L_1]$	10	10	10
$Var(L_1)$	100	100	100
$E[L_2]$	10	10	10
$Var(L_2)$	150	300	2100
OPT1 P_1	**11.89**	**11.69**	**11.06**
OPC1 P_2	**12.32**	**12.93**	**14.83**
risk quantity	0.19	0.17	0.11
OPC2 P_1 $(u = 0)$	**12.23**	**12.54**	**12.89**
OPC2 P_2 $(u = 0)$	**11.31**	**10.38**	**9.32**
Ruin probability	0.294	0.285	0.276
OPC2 P_1 $(u = 10)$	**11.97**	**12.08**	**12.36**
OPC2 P_2 $(u = 10)$	**12.10**	**11.75**	**10.91**
Ruin probability	0.111	0.110	0.107
OPC2 P_1 $(u = 20)$	**11.27**	**10.75**	**10.59**
OPC2 P_2 $(u = 20)$	**14.20**	**15.75**	**16.22**
Ruin probability	0.044	0.046	0.047

TABLE 2. The OPC's for example 2

function in equilibrium. At this levels the curve of the case 3 Pareto distribution function is above the curve of the case 1 Pareto distribution function. We therefore observe the decreasing P_2's. But when we increase the initial capital and thereby increase the survival probability the picture changes. At this level the fat tail is taking over, and we get the desired effect namely that P_2 increases when the tail is getting fatter.

We now conclude that the simple OPC1 is easy to calculate, and it works reasonable. OPC2 is more complicated to calculate and we have to be more careful when we use it, because it is highly depended on the initial capital. But with a "suitable" level of the initial capital we have seen that it works well. It is not possible to say whether OPC1 or OPC2 is best in general.

6.3. **Example 3.** As seen above, it is not possible to rank the different OPC's. But in this example we will try to do it for an agent that weights all four OPC's equally and who has a known utility function.

Let the situation be as in example 1. We further assume that the considered agent has an exponential utility function with risk aversion coefficient $\alpha = 0.03$ and initial capital $u = 0$.

The state premiums are known from example 1.

OPC number	P_1	P_2	P_3
1	1.022	5.108	24.318
2	1.118	5.701	23.168
3	1.145	5.206	23.908
4	1.917	5.709	21.520

We will now evaluate how OPC1, OPC2, OPC3 and OPC4 are doing in relation of having equal risk quantity (RQ), equal ruin probability (RP), equal expected utility (EU) and maximal expected utility (MEU). This is done in the following way. Let RQ_{ij} denote the risk quantity in state j if P_j is determined from OPCi. We then introduce the variable RQ^i to denote how well OPCi is doing in relation of having equal risk quantities, let RQ^i be given by

$$\mathrm{RQ}^i = \frac{\sum_{j=1}^{3} p_j |\mathrm{RQ}_{1j} - \mathrm{RQ}_{ij}|}{\sum_{i=1}^{4}(\sum_{j=1}^{3} p_j |\mathrm{RQ}_{1j} - \mathrm{RQ}_{ij}|)}.$$

The numerator denotes how far the RQ is from the optimal RQ (the RQ obtained by OPC1) weighted by the state probabilities. The denominator is included in order to normalize the expression such that it can be compared with the three other cases. Note that RQ^i is 0 for $i = 1$.

A corresponding formula is used to calculate the variables that denotes how well the OPCi's are doing in relation of having equal ruin probabilities (the RP^i's) and equal expected utilities (the EU^i's). For the maximal expected utility we use the following formula

$$\mathrm{MEU}^i = \frac{|\mathrm{MEU}_4 - \mathrm{MEU}_i|}{\sum_{i=1}^{4}(|\mathrm{MEU}_4 - \mathrm{MEU}_i|)}$$

In the table below we have listed RQ^i, RP^i, EU^i and MEU^i for i=1,2,3 and 4.

OPC number	RQ	RP	EU	MEU	Σ
1	0	0.300	0.113	0.468	0.881
2	0.144	0	0.220	0.190	0.554
3	0.105	0.195	0	0.342	0.643
4	0.751	0.504	0.667	0	1.922

In the column to the right the sum of the figures are listed. From this column we can now rank the OPC's. It is seen that OPC2 is the best OPC to use for the agent considered in this example. But it is important to note that the analysis is highly depended on the preferences of the agent. If we set the risk aversion coefficient different we might obtain another ranking, e.g. if we set $\alpha = 0.04$ we will, based on table 1, expect OPC4 to do better.

7. The incomplete market case

In this section we consider the incomplete market case, i.e. we now consider a market where the number of states n is larger than the number of securities. Let now k denote the number of securities. Let again v_i be the price of state security number i and let $v = (v_1, \dots, v_k)$.

7.1. The unrestricted premium case.

Because of the incompleteness in this market we are no longer able to construct the n AD securities. We therefore cannot construct the optimal trading strategy and set the premium by $P = q_1 P_1 + \cdots + q_n P_n$. So instead of constructing the optimal trading strategy an alternative could be to choose the cheapest strategy which assures that the premium in state i is greater than or equal to $P_i = E[L_i] + \delta_i$ for all states, i.e. choose a trading strategy that solve the following problem

$$\min_m vm$$

$$\text{st} \quad \sum_{j=1}^{k} m_j c_{ij} \geq E[L_i] + \delta_i \qquad i = 2, \dots, n.$$

A problem of this strategy is that it could be very expensive. An alternative strategy is therefore to choose the premiums so that they get as close as possible to the optimal premiums $(P_1, \dots, P_n)$, i.e. choose the portfolio m that solves the following problem

$$\min_{m_1, \dots, m_k} \sum_{i=1}^{n} (\sum_{j=1}^{k} m_j c_{ij} - P_i)^2$$

or equivalent

$$\min_m ||Cm - \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix}||^2$$

where C now is a $n \times k$ matrix. This is a well known problem and it is solved by the least square solution which is given by, (see [1] p. 318),

$$m = (C^T C)^{-1} C^T \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix}$$

After these considerations we now make the following definition

Definition 7.1. A **least square strategy** is a trading strategy such that the insurance company gets as close as possible to the desired n

state premiums as possible in the least square sense, i.e. the **least square strategy** is obtained by the following portfolio of securities.

$$m = (C^T C)^{-1} C^T \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix}$$

The insurance company would of course prefer to follow the optimal trading strategy given by $P_i = E[L_i] + \delta_i$ but this is impossible in this market. But if it has been possible the insurance company would have been willing to pay more for the optimal strategy than for the least square strategy. Therefore if the insurance company follows the least square strategy they should charge a premium that is larger than the price of the least square strategy. They thereby get a compensation for not having the optimal strategy but only the least square strategy.

7.2. **The restricted premium case.** Let us now as in the complete case consider the situation where the insurance company is unable to charge the desired premium because of some competitive reasons. We again set the possible fixed premium that can be charged to P_0.

The problem now is that we want to set the P_i's according to OPC1, OPC2, OPC3 or OPC4 but the equation $P_0 = P_1 q_1 + \cdots + P_n q_n$ is no longer valid. We are no longer able to construct the n AD securities in this incomplete market. But instead of choosing the P_is according to OPC1, OPC2, OPC3 or OPC4 we could choose the corresponding least square solution. We then just have to replace the equation $P_0 = P_1 q_1 + \cdots + P_n q_n$ with an equation that makes sure that the price of the least square portfolio is equal to P_0. Before we solve this problem we make the following definition in relation to OPC1, OPC2 and OPC3. We return to OPC4 later.

Definition 7.2. The **incomplete optimal premium choice** (IOPC) is defined as a choice of premiums (P_i's) such that

- The P_i's solve the $n-1$ first equations in OPC1, OPC2 or OPC3.
- The price of the least square strategy corresponding to the P_i's is P_0.

The IOPC is then found by the following theorem

Theorem 7.3. *The solution to the IOPC corresponding to OPC1, OPC2 or OPC3 is found by solving the n equations with n unknowns from the OPC. But where the last equation in OPC1, OPC2 and OPC3 is replaced by*

$$P_0 = \tilde{q} \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix} = \tilde{q}_1 P_1 + \cdots + \tilde{q}_n P_n$$

where $\tilde{q}$ is given by

$$\tilde{q} = v(C^T C)^{-1} C^T$$

Proof. The least square strategy corresponding to $(P_1, \dots, P_n)$ is given by

$$m = (C^T C)^{-1} C^T \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix}$$

and the price of the state securities is given by $v = (v_1, \dots, v_k)$. The price of the least square strategy corresponding to $(P_1, \dots, P_n)$ is then given by

$$\begin{aligned} vm &= v(C^T C)^{-1} C^T \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix} \\ &= \tilde{q} \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix} = \tilde{q}_1 P_1 + \cdots + \tilde{q}_n P_n \qquad (7.4) \end{aligned}$$

where $\tilde{q}$ is given by

$$\tilde{q} = v(C^T C)^{-1} C^T$$

It then follows by definition 7.2 that if the last equation in OPC1, OPC2 or OPC3 is replaced by 7.4, we will obtain the corresponding IOPC. □

For OPC4 the situation is different. OPC4 is a maximization problem and we therefore just have to reformulate the problem to a problem of dimension k instead of a problem of dimension n. The IOPC for OPC4 therefore takes the following form

$$\begin{aligned} &\max_m E[v(u + c_i m - L_i)] \\ &\text{st} \quad vm = P_0. \end{aligned}$$

The maximization problem is solved in the same way as described in section 5.

8. Conclusion

In this paper we have been looking at risk with more than one prior estimate of the frequency. As mentioned in the introduction it is not possible to hedge this kind of risk using traditional insurance practice only, so a new method was called for. In this paper we present a model that is able to manage this kind of risk. The model works by using traditional insurance practice and trading in the security market simultaneously. The paper shows how this new method works both in complete and incomplete markets.

Further we consider the case where the premium the insurance company can charge is restricted. In this case the insurance company has to choose an allocation of the restricted premium corresponding to the states of the world. We propose four different methods of solving this problem. These four methods are then analysed and evaluated and by examples advantages and disadvantages are illustrated. We also show a way to rank the four methods in the case where we consider an agent that evaluates the four OPC equally and has a known utility function.

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(C. Vorm Christensen) Department of Theoretical Statistics and Operations Research, University of Aarhus, Ny Munkegade 116, 8000 Aarhus C, Denmark

E-mail address: vorm@imf.au.dk