Contents

Preface

This document is my Ph.D. thesis. It consist of four parts

(1) An introduction.
(2) Friis, Peter de Place, *d'Alembert's and Wilson's equations on Lie groups.* Submitted. Preprint Series 2000 No 8, Matematisk Institut, Aarhus University, Denmark. pp 1-12.
(3) Friis, Peter de Place, Stetkær, H. *On the Cosine-Sine Functional Equation on Groups.* Submitted. Preprint Series 2000 No 10, Matematisk Institut, Aarhus University, Denmark. pp 1-17.
(4) Friis, Peter de Place, *The Sine and Cosine addition functional equations on non-abelian groups.* Manuscript pp 1-21.

The two preprints have been submitted to Aequationes Mathematicae. The second preprint is, as indicated, a joint work with my supervisor Henrik Stetkær. This was my introduction to the field of functional equations and I wish to express my gratitude for this well considered starting project. The manuscript is an almost preprint meaning that I consider it to be in its final form, but it have not formally been made into a preprint at the time where I handed in my thesis. Finally thanks to Søren Fournais and Kåre Nielsen for many enjoyable conversations over the years on mathematical subject and other matters. Further thanks are due to Kåre for his technical assistance with LaTeX. Lastly thanks to my supervisor Henrik Stetkær for accepting me as a ph.d.-student in the first place and for a good introduction into the interesting field of functional equations.

CHAPTER 1

Introduction

As the title of my thesis indicates, the subject is trigonometric functional equations on groups. These are partially motivated by a generalization of the addition formulas for Sine and Cosine from the real line to an arbitrary group. A related source of inspiration is the study of spherical functions which is a classical field of interest. Roughly speaking, the spherical functions play the role in harmonic analysis on homogeneous spaces that the trigonometric polynomials play in Fourier analysis. The spherical functions can be characterized as the solutions of certain functional equations. Let us be more precise. Let G be a topological Hausdorff group, K a compact transformation group acting on G, let $k\cdot$ denote the action of $k \in K$, and dk the normalized Haar measure on K. A K-spherical function is a solution $f \in C(G)$, $f \neq 0$, to the functional equation

$$\int_K f(xk \cdot y)dk = f(x)f(y), \;\; x, y \in G. \tag{1}$$

For general G little is known. If G is a connected abelian Lie group or a compact abelian group then all solutions are given in [**17**]. If G is abelian and locally compact then all essentially bounded solutions are given in [**3**] and [**2**]. For G abelian with the usual $\mathbb{Z}_2$-action, i.e.,

$$f(x+y) + f(x-y) = 2f(x)f(y), \;\; x, y \in G, \tag{2}$$

the general solution was found by Kannappan as late as in 1968 (see [**13**]). Corovei has found all solutions when G is a nilpotent group where all elements have odd order (see [**5**]). This was my inspiration for showing that the form of the solutions are the same on connected nilpotent Lie groups as in the abelian case (see [**8**]). All solutions on metabelian groups are given by [**7**] and [**20**].

When studying (1) on Lie groups one is led to consider a generalisation of (1) (see [**12**] Proposition IV.2.4 page 402), namely

$$\int_K f(xk \cdot y)dk = f(x)\phi(y), \;\; x, y \in G. \tag{3}$$

On abelian groups with the usual $\mathbb{Z}_2$-action the form of the solutions of (3) are known. In [**6**] Corovei gives all solutions when G is a nilpotent group where all elements are of odd order and ϕ is not identically 1. This was my inspiration for proving that in the case where ϕ is not identically 1 the solutions of (3) on nilpotent connected Lie groups are the same as in the abelian case (see [**8**]). This of course leaves the case where $\phi \equiv 1$. That is we are considering Jensen's equation

$$f(xy) + f(xy^{-1}) = 2f(x), \;\; x, y \in G. \tag{4}$$

If G is abelian or $f(xy) = f(yx)$ then it follows from [**1**] that f is an additive function. Ng has solved Jensen's equation for all free groups and for $GL_n(\mathbb{Z})$

for $n \geq 3$ (see [**14**]). I solve the equation in the case $\phi \equiv 1$ on a semidirect product of two abelian groups. Given that the solutions to Wilson's equation on nilpotent connected Lie groups when $\phi \not\equiv 1$ where the same as in the abelian case, the following is a bit surprising: I have shown that on the Heisenberg group the solutions differ from the solutions in the abelian case. The Heisenberg group is after all the simplest connected nilpotent Lie group which is not abelian.

A natural generalization of (1) and (3) is

$$\int_K f(xk \cdot y)dk = \sum_{i=1}^{n} g_i(x)h_i(y), \ x, y \in G. \tag{5}$$

This has been studied in [**15**], [**16**], [**17**], and [**18**] for example. In a joint work with my supervisor Stetkær [**9**] we solve the following special case of (5)

$$(f(x+y) + f(x+\sigma y))/2 = f(x)g(y) + g(x)f(y) + h(x)h(y), x, y \in G, \tag{6}$$

where σ is a continuous involution of G. The solutions are certain exponential polynomials. This is an extension of the results in [**4**], where the authors consider the case $\sigma(x) = x, \ x \in G$. We also solve the signed equation

$$\frac{f(x+y) - f(x+\sigma y)}{2} = f(x)g(y) + g(x)f(y) + h(x)h(y), \ x, y \in G. \tag{7}$$

This turns out to be surprisingly simpler to solve.

Stetkær has found a necessary condition on the functions $g_1, \ldots, g_n$, $h_1, \ldots, h_n$ in (2) when K acts on G by homomorphisms (see [**18**]). My most recent work [**10**] is an extention of that result to the case where for each $k \in K$ the action of k is either a homomorphism or an antihomomorphism, where G of course need not be abelian. The functions $g_1, \ldots, g_n, h_1, \ldots, h_n \in C(G)$ has to satisfy the equation

$$\begin{aligned} &\sum_{i=1}^{n} g_i(x) \int_K [h_i(yk \cdot z) + h_i((k \cdot z)y)]dk \\ = &\sum_{i=1}^{n} \left[\int_K g_i(xk \cdot y)dk h_i(z) + \int_K g_i(xk \cdot z)dk h_i(y) \right], \ x, y, z \in G, \end{aligned} \tag{8}$$

in order that a solution can exist. This necessary condition is applied to the equations

$$\int_K f(xk \cdot y)dk = f(x)g(y) + g(x)f(y), \ x, y \in G, \tag{9}$$

and

$$\int_K f(xk \cdot y)dk = f(x)f(y) + g(x)g(y), \ x, y \in G, \tag{10}$$

and I show how to reduce the problem of solving these equations to the problem of solving simpler equations. This is an extension of results in [**19**]. The conclusion is the same but the assumptions are weaker. I do not assume that G is abelian, and K acts by homomorphisms and possibly also antihomomorphisms, and not exclusively by homomorphisms. It is perhaps a bit unexpected that the conclusion is the same.

Further developments

While obviously a number of functional equations have been solved this has often been done using ad hoc methods. What I am trying to find is more necessary conditions for existence of solutions for different left hand sides of (5) for non-abelian G. Any such result would be a valuable contribution due to the scarcity of general methods. The reason for looking at different left hand sides of (2) is primarily that it is not obvious how to generalize a functional equation from the abelian case to the non-abelian case. The obvious generalization is of course not to change anything but other possibilities occur. Perhaps the left hand side should be changed to $\int_K f(xk \cdot y)dk + \int_K f(k \cdot yx)dk$, which is a generalization of a left hand side occuring in [**6**].

Furthermore the methods used are most often algebraic in nature. I think another perspective might prove fruitful. Applying abstract harmonic analysis and representation theory for groups to provide information about and existence of the solutions might be useful. Of course some work has already been done in this direction, see [**11**], [**15**], [**16**], [**18**], and [**21**].

Bibliography

[1] Aczél, J., Chung, J. K., and Ng, C. T. *Symmetric second differences in product form on groups*, in: Th. M. Rasssias (Ed.), *Topics in Mathematical Analysis*, pp. 1-22. Ser. Pure Math., 11, World Scientific Publ. Co., Teaneck, NJ, 1989.

[2] Badora, R., *On a joint generalization of Cauchy's and d'Alembert's functional equations.* Aequationes Math. 43 (1992), 72-89.

[3] Chojnacki, W. *Fonctions cosinus hilbertiennes bornée dans les groupes commutatifs localement compacts.* Compositio Math. 57 (1986), no. 1, 15-60.

[4] Chung, J. K., Kannappan, PL. and Ng, C. T., *A Generalization of the Cosine-Sine Functional Equation on Groups.* Linear Algebra Appl. **66** (1985), 259-277.

[5] Corovei, I, *The cosine functional equation for nilpotent groups*, Aequationes Math. 15 (1977), 99-106.

[6] Corovei, I., *The functional equation* $f(xy)+f(xy^{-1}) = 2f(x)g(y)$ *for nilpotent groups*, Mathematica (Cluj) 22 (45) (1980), 33-41.

[7] Corovei, I., *The d'Alembert functional equation on metabelian groups*, Aequationes Math. 57 (1999), 201-205.

[8] Friis, Peter de Place, *d'Alembert's and Wilson's equations on Lie groups.* Submitted. Preprint Series 2000 No 8, Matematisk Institut, Aarhus University, Denmark. pp 1-12.

[9] Friis, Peter de Place, Stetkær, H. *On the Cosine-Sine Functional Equation on Groups.* Submitted. Preprint Series 2000 No 10, Matematisk Institut, Aarhus University, Denmark. pp 1-17.

[10] Friis, Peter de Place, *The Sine and Cosine addition functional equations on non-abelian groups.* Manuscript.

[11] Gajda, Z., *Unitary Representations of Topological Groups and Functional Equations.* J. Math. Anal. Appl. 152 (1990), 6-19.

[12] Helgason, S., *Groups and Geometric Analysis; Integral Geometry, Invariant Differential Operators and Spherical Functions.* Academic Press, New York, 1984.

[13] Pl. Kannappan, *The functional equation* $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ *for groups.* Proc. Amer. Math. Soc. **19** (1968), 69-74.

[14] C. T. Ng, *Jensen's functional equation on groups, II*, Aequationes Math. 58 (1999), 311-320.

[15] Penney, R. C., Rukhin, A. L., *D'Alembert's functional equation on groups*, Proc. Amer. Math. Soc. 77 (1979), 73-80.

[16] Rukhin, A. L., *The solution of the functional equation of d'Alembert's type for commutative groups.* Internat. J. Math. Math. Sci. 5 (1982), 315-335.

[17] Stetkær, H., *Functional Equations and Spherical Functions.* Preprint Series 1994 No **18**, Matematisk Institut, Aarhus University, Denmark. pp. 1-28.

[18] Stetkær, H., *Functional equations on abelian groups with involution.* Aequationes Math. 54 (1997), 144-172.

[19] Stetkaer, H., *Trigonometric functional equations of rectangular type.* Aequationes Math. **56** (1998) 1-19.

[20] Stetkær, H. *D'Alembert's funtional equations on metabelian groups*, Aequationes Math. 59 (2000), 306-320.

[21] Székelyhidi, L., Convolution Type Functional Equations on Topological Abelian Groups, World Scientific. Singapore, New Jersey, London, Hong Kong. 1991.

CHAPTER 2

D'Alembert's and Wilson's Equations on Lie Groups

Peter de Place Friis

1. Introduction

In this paper we discuss certain topics in the theory of functional equations on non-abelian groups. Our first aim is to study d'Alembert's equation

$$g(xy) + g(xy^{-1}) = 2g(x)g(y), \ x, y \in G, \tag{11}$$

where G is a group and g is a complex valued function on G, and the following generalization that comes out naturally of the study of Wilson's functional equation (see Corovei [**3**])

$$g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4g(x)g(y), \ x, y \in G. \tag{12}$$

Secondly we will study Wilson's equation

$$f(xy) + f(xy^{-1}) = 2f(x)g(y), \ x, y \in G. \tag{13}$$

Finally we will solve Jensen's equation

$$f(xy) + f(xy^{-1}) = 2f(x), \ x, y \in G, \tag{14}$$

on a semidirect product of groups.

Notation: The following notation will be used throughout the article. G denotes a group with e as neutral element and $Z(G)$ its centre. $\mathbb{C}^*$ denotes the multiplicative group of non-zero complex numbers. If $m : G \mapsto \mathbb{C}^*$ is a homomorphism, then $\check{m} : G \mapsto \mathbb{C}^*$ is the homomorphism given by $\check{m}(x) = m(x^{-1})$, $x \in G$. A group G is said to be 2-divisible, if for any $x \in G$ there exists $y \in G$ such that $y^2 = x$. y is not assumed to be unique. An involution τ of G is a map $\tau : G \mapsto G$ such that $\tau(xy) = \tau(y)\tau(x)$, $\forall x, y \in G$ and $\tau(\tau(x)) = x$ for all $x \in G$.

If $m : G \to \mathbb{C}^*$ is a homomorphism, then

$$g(x) = \frac{m + \check{m}}{2}(x), \ x \in G, \tag{15}$$

is a solution to (11). No restrictions on the group are needed for that statement. For abelian groups the converse is true: Any nonzero solution of (11) has this form (Kannappan [**6**]). This is true for certain other groups as well (Corovei [**2**] and Stetkær [**10**]). We are going to show it for still another class of groups.

Our main results are the following:

(1) We show that any solution g to (11) and (12) is of the form (15) when G is a connected nilpotent Lie group (see Theorem 2.6 and Corollary 2.8).

(2) We give all solutions to Wilson's equation on connected nilpotent Lie groups, provided that it is not the degenerate version of Wilson's equation where $g \equiv 1$, i.e. Jensen's equation (see Theorem 3.4).
(3) We give the solution to Jensen's equation on a semidirect product of two groups, where we suppose that the normalized solutions to Jensen's equation on the groups, which enter in the formation of the semidirect product, are homomorphisms. This is the case if they are abelian (see Theorem 4.1).

Ng has solved Jensen's equation for all free groups and $GL_n(\mathbb{Z})$ for $n \geq 3$ (see Ng [**8**]). Ng has also studied the following version of Jensen's equation

$$f(xy) + f(y^{-1}x) = 2f(x),\ \forall x \in G. \tag{16}$$

We are not going to pursue this, but we will compare our results to his on the Heisenberg group (see Example 4.2). The difference is somewhat surprising.

The parts of the present paper concerned with d'Alembert's and Wilson's functional equations are closely related to and inspired by Corovei [**2**] and [**3**]. However there is a shift of emphasis from insisting on that all elements have odd order to looking at 2-divisibility as we do. Apart from the trivial group, connected Lie groups contain elements of infinite order, so it is essentially a phenomenon for discrete groups that all elements have odd order. We manage to treat the connected nilpotent Lie groups which play an important role in Analysis. This is one reason that the results are interesting. These groups are 2-divisible. Since 2-divisibility was what made Corovei's proofs work, some of his proofs are copied with only modest changes. But our results are more general (see Remark 2.7 and Remark 3.5).

From Lemma 1 in [**1**] we know that a solution $f : G \mapsto \mathbb{C}$ of Jensen's equation with $f(e) = 0$ and $f(xy) = f(yx)$ for all $x, y \in G$ is a homomorphism. But not all solutions f with $f(e) = 0$ are homomorphisms. We have a counterexample when G is the Heisenberg group (see Example 4.2), the simplest connected nilpotent Lie group which is not abelian.

2. d'Alembert's equation on nilpotent connected Lie groups

In this section we will solve (12). It is a generalization of (11), because $g(xy) = g(yx)$ for any solution g of (11) (see Remark V.2 in [**10**]) so that any solution of (11) is also a solution of (12).

LEMMA 2.1. *Let $g : G \to \mathbb{C}$ be a non-zero solution of the equation*

$$g(xy) + g(yx) + g(x\tau(y)) + g(\tau(y)x) = 4g(x)g(y),\ x, y \in G, \tag{17}$$

where τ is an involution of G. Then $g(e) = 1$, $g \circ \tau = g$, and

$$g(x^2) + \frac{g(x\tau(x)) + g(\tau(x)x)}{2} = 2g(x)^2,\ x \in G. \tag{18}$$

PROOF. See Lemma III.1 of [**10**]. □

THEOREM 2.2. *Let g be a solution of the following extension of d'Alembert's functional equation*

$$g(xy) + g(yx) + g(x\tau(y)) + g(\tau(y)x) = 4g(x)g(y),\ \forall x, y \in G, \tag{19}$$

where $\tau : G \mapsto G$ is an involution and $Z(g) = \{u \in G : g(xuy) = g(xyu),\ \forall x, y \in G\}$.

a: *If there exists $u \in Z(g)$ such that $g(u)^2 \neq g(u\tau(u))$ then g has the form*

$$g = \frac{m + m \circ \tau}{2}, \tag{20}$$

where $m : G \to \mathbb{C}^*$ is a homomorphism.

b: *If $g(u)^2 = g(u\tau(u))$ for all $u \in Z(g)$ then*

$$g(xu) = g(x)g(u), \ \forall x \in G, \ \forall u \in Z(g). \tag{21}$$

PROOF. See Theorem III.2 of [**10**]. □

LEMMA 2.3. *Let H be a 2-divisible subgroup of G. Let g be a solution of*

$$g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4g(x)g(y), \ x, y \in G. \tag{22}$$

If $g^2 \equiv 1$ on H then $g \equiv 1$ on H.

PROOF. For any $x \in H$ there exists $y \in H$ such that $y^2 = x$. Using Lemma 2.1 we get $g(x) = g(y^2) = 2g(y)^2 - g(e) = 1$. □

LEMMA 2.4. *If $g : G \to \mathbb{C}$ is a solution to*

$$g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4g(x)g(y), \ x, y \in G, \tag{23}$$

and $g(u) = 1, \ \ \forall u \in Z(G)$, then one can define $F : G/Z(G) \to \mathbb{C}$ by $F(\bar{x}) = g(x), \ \forall x \in G$, where $\bar{x} = xZ(G)$. Furthermore F satisfies the equation

$$F(\bar{x}\bar{y}) + F(\bar{y}\bar{x}) + F(\bar{x}\bar{y}^{-1}) + F(\bar{y}^{-1}\bar{x}) = 4F(\bar{x})F(\bar{y}), \ \bar{x}, \bar{y} \in G/Z(G). \tag{24}$$

PROOF. See Lemma 4 of [**2**]. □

LEMMA 2.5. *If G is a connected nilpotent Lie group, then G is 2-divisible. Furthermore $Z(G)$ and $G/Z(G)$ are connected nilpotent Lie groups and hence also 2-divisible.*

PROOF. Let $\mathcal{G}$ be the Lie algebra of G. Since G is connected and nilpotent, the exponential map $\exp : \mathcal{G} \to G$ is onto (see Corollary VI 4.4 of [**5**] (p. 269)). Let $x \in G$, there exists $X \in \mathcal{G}$ such that $\exp(X) = x$. $\mathcal{G}$ being a vector space, put $y = \exp(\frac{1}{2}X)$, then

$$y^2 = \exp(\frac{1}{2}X)\exp(\frac{1}{2}X) = \exp(X) = x. \tag{25}$$

So G is 2-divisible. Z(G) is a closed subgroup of G and hence a Lie group in it's own right. Furthermore Z(G) is connected (see Corollary 3.6.4 of [**11**]). Being nilpotent Z(G) is 2-divisible. Since Z(G) is a closed normal subgroup of G, it follows that $G/Z(G)$ is a Lie group (see Theorem 2.9.6 of [**11**]). The natural map $\pi : G \to G/Z(G)$ given by $\pi(g) = gZ(G)$ is continuous, so $G/Z(G)$ is connected. Hence $G/Z(G)$ is a connected nilpotent Lie group. □

For Lie groups the following theorem extends Proposition V.5 of [**10**].

THEOREM 2.6. *If G is a nilpotent connected Lie group, then $g : G \to \mathbb{C}$ is a non-zero solution to*

$$g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4g(x)g(y), \ x, y \in G, \tag{26}$$

if and only if g has the form

$$g = \frac{m + \check{m}}{2}, \tag{27}$$

where $m : G \to \mathbb{C}^*$ is a homomorphism.

PROOF. Standard technique. Let $\{e\} = Z_0 < \cdots < Z_n = G$ be an ascending central series for G, with $Z_{i+1}/Z_i = Z(G/Z_i)$. We will prove the result by induction on n. If $n = 0, 1$ then $Z(G) = G$, hence G is abelian, and g therefore satisfies the equation

$$g(xy) + g(xy^{-1}) = 2g(x)g(y), \ x, y \in G. \tag{28}$$

That is d'Alembert's equation on an abelian group, where it is known that g has the stated form. Let $n \in \mathbb{N}$ and assume that the result is true for all nilpotent connected Lie groups with ascending central series of length n. Let G be a nilpotent connected Lie group with ascending central series $\{e\} = Z_0 < Z_1 < \cdots < Z_n < Z_{n+1} = G$, where $Z_{i+1}/Z_i = Z(G/Z_i)$. If there exists $u \in Z(G)$ such that $g(u)^2 \neq 1$, then it follows from the previous theorem that g has the stated form. So we can assume that $g(u)^2 = 1, \quad \forall u \in Z(G)$. Since Z(G) is 2-divisible it follows that $g(u) = 1, \ \forall u \in Z(G)$. By the previous lemma $G/Z(G)$ is a nilpotent connected Lie group. Furthermore $Z_1/Z_1 < \cdots < Z_{n+1}/Z_1 = G/Z_1$ is an ascending central series for $G/Z_1 = G/Z(G)$ with $(Z_{i+1}/Z_1)/(Z_i/Z_1) = Z((G/Z_1)/(Z_i/Z_1))$. We have shown above that we can define $F : G/Z_1 \to \mathbb{C}$ by $F(\bar{x}) = g(x)$, where $\bar{x} = xZ_1$, and furthermore $F : G/Z_1 \to \mathbb{C}$ is a solution to

$$F(\bar{x}\bar{y}) + F(\bar{y}\bar{x}) + F(\bar{x}\bar{y}^{-1}) + F(\bar{y}^{-1}\bar{x}) = 4F(\bar{x})F(\bar{y}), \ \bar{x}, \bar{y} \in G/Z(G). \tag{29}$$

By assumption, there exists a homomorphism $M : G/Z_1 \to \mathbb{C}^*$ such that $F = (M + \check{M})/2$. Define a homomorphism $m : G \to \mathbb{C}^*$ by $m(x) = M(\bar{x})$, then $\check{m}(x) = \check{M}(\bar{x})$. So we have that

$$g(x) = F(\bar{x}) = \frac{M + \check{M}}{2}(\bar{x}) = \frac{m + \check{m}}{2}(x), \ x \in G. \tag{30}$$

The theorem now follows by induction on n. □

REMARK 2.7. (a) Instead of $g : G \to \mathbb{C}$ we could consider $g : G \to K$ where K is any quadratically closed field with characteristic different from 2.

(b) Instead of nilpotent connected Lie groups, we could consider any class $\mathcal{C}$ of nilpotent groups G, for which $G \in \mathcal{C}$ implies $Z(G)$ is 2-divisible and $G/Z(G) \in \mathcal{C}$. Note that if we take $\mathcal{C}$ to be all nilpotent groups where the order of all elements are odd, then $\mathcal{C}$ fulfils the requirement. So if we formulate the theorem for classes $\mathcal{C}$ with the above mentioned properties, instead of for connected nilpotent Lie groups, then it contains as a special case Theorem 2 in [**2**].

COROLLARY 2.8. *If G is a connected nilpotent Lie group, and K be a quadratically closed field with characteristic different from 2. Then $g : G \to K$ is a nonzero solution of d'Alembert's equation*

$$g(xy) + g(xy^{-1}) = 2g(x)g(y), \ x, y \in G, \tag{31}$$

if and only if g has the form $g = (m+\check{m})/2$, where $m : G \to K^*$ is a homomorphism. Furthermore suppose $K = \mathbb{C}$, then g is continuous if and only if m is continuous.

PROOF. Let $g : G \to K$ be a non-zero solution of d'Alembert's equation. Then $g(xy) = g(yx) \ \forall x, y \in G$. Hence g satisfies the equation

$$g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4g(x)g(y), \ x, y \in G. \tag{32}$$

Hence by the previous theorem g has the form $g = (m+\check{m})/2$, where $m : G \to K^*$ is a homomorphism. The converse result is trivial. Suppose $K = \mathbb{C}$. If m is continous then obviously so is g. If g is continuous then it follows from Theorem 1 in [**6**] or Proposition V.7 of [**10**] that m is continuous. □

3. Wilson's equation on connected nilpotent Lie groups

The following lemma is a slight extension of Lemma 1 of [**3**] in that the group inversion has been replaced by a general involution.

LEMMA 3.1. *Let the pair $f, g : G \to \mathbb{C}$ be a solution of Wilson's equation*

$$f(xy) + f(x\tau(y)) = 2f(x)g(y),\ x, y \in G. \tag{33}$$

where $\tau : G \mapsto G$ is an involution. If f is not identically zero then g satisfies the following equation

$$g(xy) + g(yx) + g(x\tau(y)) + g(\tau(y)x) = 4g(x)g(y),\ x, y \in G. \tag{34}$$

PROOF.

$$\begin{aligned} 8f(x)g(y)g(z) &= 4f(x)g(y)g(z) + 4f(x)g(z)g(y) \quad (35)\\ &= 2f(xy)g(z) + 2f(x\tau(y))g(z) + 2f(xz)g(y) + 2f(x\tau(z))g(y)\\ &= f(xyz) + f(xy\tau(z)) + f(x\tau(y)z) + f(x\tau(y)\tau(z))\\ &\quad + f(xzy) + f(xz\tau(y)) + f(x\tau(z)y) + f(x\tau(z)\tau(y))\\ &= 2f(x)[g(yz) + g(zy) + g(y\tau(z)) + g(\tau(z)y)],\ x, y, z \in G \end{aligned}$$

Since f is assumed not to be identically zero, the result follows. □

The following theorem is a slight extension of Theorem 1 in [**3**], again because the group inversion has been replaced by a general involution τ.

THEOREM 3.2. *Let G be a group. Suppose that the pair f and g is a solution to Wilson's equation,*

$$f(xy) + f(x\tau(y)) = 2f(x)g(y),\ x, y \in G, \tag{36}$$

where τ is an involution and f is nonzero. Suppose furthermore that there exists $u \in Z(G)$ such that $g(u)^2 \neq g(u\tau(u))$, then f and g has the form,

$$f = A\frac{m + m\circ\tau}{2} + B\frac{m - m\circ\tau}{2},\quad g = \frac{m + m\circ\tau}{2}, \tag{37}$$

where m is a homomorphism of G into $\mathbb{C}^*$, and $A, B \in \mathbb{C}$ are constants.

PROOF. Since f is non-zero it follows from the previous lemma that g satisfies the following equation

$$g(xy) + g(yx) + g(x\tau(y)) + g(\tau(y)x) = 4g(x)g(y),\ x, y \in G. \tag{38}$$

Since we assume that there exists $u_0 \in Z(G)$ such that $g(u_0)^2 \neq g(u_0\tau(u_0))$ then it follows from Theorem 2.2 that $g = (m + m\circ\tau)/2$. Now $g(u_0)^2 \neq g(u_0\tau(u_0))$ implies that $m(u_0) \neq m(\tau(u_0))$. Now fix $x_0 \in G$ for the moment and consider the smallest abelian subgroup G_{x_0} of G which contains $Z(G)$ and x_0 ($G_{x_0} = \{x_0^n z : n \in \mathbb{Z}, z \in Z(G)\}$). We obviously have

$$f(xy) + f(x\tau(y)) = 2f(x)g(y),\ x, y \in G_{x_0}, \tag{39}$$

It follows from Theorem III.4 in [**9**] that f's restriction to G_{x_0} has the form

$$f(x) = c_1(x_0)\frac{m + m\circ\tau}{2}(x) + c_2(x_0)\frac{m - m\circ\tau}{2}(x),\ x \in G_{x_0}, \tag{40}$$

where $c_1(x_0), c_2(x_0) \in \mathbb{C}$ are constants. Putting $x = e$ and $x = u_0$ we find that

$$c_1(x_0) = f(e), \tag{41}$$

and

$$c_2(x_0) = \frac{2}{m(u_0) - m(\tau(u_0))}(f(u_0) - f(e)g(u_0)). \tag{42}$$

So the constants $A = c_1(x_0)$, $B = c_2(x_0) \in \mathbb{C}$ do not depend on our particular choice of x_0. So for arbitrary $x_0 \in G$ we have

$$f(x_0) = A\frac{m + m \circ \tau}{2} + B\frac{m - m \circ \tau}{2}. \tag{43}$$

□

LEMMA 3.3. *Let G be a 2-divisible group. Let the pair $f, g : G \to \mathbb{C}$ be a solution to Wilson's equation*

$$f(xy) + f(xy^{-1}) = 2f(x)g(y), \ \forall x, y \in G, \tag{44}$$

where f is non-zero. Suppose that $\forall u \in Z(G) : g(u) = 1$ and $\exists x \in G : g(x) \neq 1$. Then the functions $F_1, G_1 : G/Z(G) \to \mathbb{C}$ can be defined by $F_1(\bar{x}) = f(x), \ \ \forall x \in G$,and $G_1(\bar{x}) = g(x), \ \ \forall x \in G$, where $\bar{x} = xZ(G)$. The functions F_1 and G_1 fulfil the equation

$$F_1(\bar{x}\bar{y}) + F_1(\bar{x}\bar{y}^{-1}) = 2F_1(\bar{x})G_1(\bar{y}), \ \bar{x}, \bar{y} \in G/Z(G). \tag{45}$$

PROOF. This is like the proof of Lemma 2 of [**3**] with minor modifications. We already know from Lemma 2.4 that $G_1(\bar{x}) = g(x), \ \ \forall x \in G$ is a valid definition, since $g(xu) = g(x), \ \forall x \in G, \ \forall u \in Z(G)$.

We split f into its even and odd parts $f(x) = f_1(x) + f_2(x)$ where $f_1(x^{-1}) = f_1(x)$ and $f_2(x^{-1}) = -f_2(x)$.

$$2f(e)g(y) = f(y) + f(y^{-1}) = 2f_1(y). \tag{46}$$

So $f_1(x) = Ag(x)$ where $A = f(e)$. Now

$$f_1(xy) + f_2(xy) + f_1(xy^{-1}) + f_2(xy^{-1}) = 2[f_1(x) + f_2(x)]g(y) \tag{47}$$

implies that

$$f_2(xy) + f_2(xy^{-1}) = 2[Ag(x) + f_2(x)]g(y) - A[g(xy) + g(xy^{-1})]. \tag{48}$$

Exchange x and y in this equation.

$$f_2(yx) + f_2(yx^{-1}) = 2[Ag(y) + f_2(y)]g(x) - A[g(yx) + g(yx^{-1})]. \tag{49}$$

Note that $f_2(yx^{-1}) = -f_2((yx^{-1})^{-1}) = -f_2(xy^{-1})$, and $g(xy^{-1}) = g((xy^{-1})^{-1}) = g(yx^{-1})$. Adding the previous two equations, and using these two facts we get

$$f_2(xy) + f_2(yx) = 2f_2(x)g(y) + 2f_2(y)g(x) + A[g(y^{-1}x) - g(xy^{-1})]. \tag{50}$$

Taking $y = x$ in the following identity

$$f(xyu) + f(xy^{-1}u^{-1}) = 2f(x)g(uy) = 2f(x)g(y) = f(xy) + f(xy^{-1}), \tag{51}$$

we find that

$$f(x^2u) + f(u^{-1}) = f(x^2) + A, \ \ \forall x \in G, \ \ \forall u \in Z(G). \tag{52}$$

Since G is 2-divisible we get

$$f(xu) + f(u^{-1}) = f(x) + A, \ \ \forall x \in G, \ \ \forall u \in Z(G). \tag{53}$$

We know that

$$f_1(xu) = Ag(xu) = Ag(x) = f_1(x), \ \ \forall x \in G, \ \ \forall u \in Z(G). \tag{54}$$

So we get

$$\begin{aligned} f_2(xu) &= f(xu) - f_1(xu) = f(xu) - f(x) + f_2(x) \\ &= f_2(x) + A - Ag(u^{-1}) - f_2(u^{-1}) = f_2(x) + f_2(u). \end{aligned} \tag{55}$$

From (50)) we have that

$$\begin{aligned} f_2(xu) + f_2(ux) &= 2f_2(x)g(u) + 2g(x)f_2(u) + A[g(u^{-1}x) - g(xu^{-1})] \\ &= 2f_2(x) + 2g(x)f_2(u). \end{aligned} \tag{56}$$

Hence

$$f_2(x) + f_2(u) = f_2(xu) = f_2(x) + g(x)f_2(u). \tag{57}$$

So we get

$$0 = f_2(u)[g(x) - 1], \ \ \forall x \in G, \ \ \forall u \in Z(G). \tag{58}$$

Since there exists $x \in G$ such that $g(x) \neq 1$ we deduce that $f_2(u) = 0, \ \ \forall u \in Z(G)$. So

$$f_2(xu) = f_2(x), \ \forall x \in G, \ \forall u \in Z(G). \tag{59}$$

Hence

$$f(xu) = f_1(xu) + f_2(xu) = f_1(x) + f_2(x) = f(x), \ \ \forall x \in G, \ \ \forall u \in Z(G). \tag{60}$$

So $F_1(\bar{x}) = f(x)$ is a valid definition. It is trivial to check that

$$F_1(\bar{x}\bar{y}) + F_1(\bar{x}\bar{y}^{-1}) = 2F_1(\bar{x})G_1(\bar{y}), \ \bar{x}, \bar{y} \in G/Z(G). \tag{61}$$

□

THEOREM 3.4. *Let G be a connected nilpotent Lie group. Let $f, g : G \to \mathbb{C}$ be a solution to Wilson's equation, where f is non-zero. Suppose that there exists $x \in G$ such that $g(x) \neq 1$. Then f and g have the form:*

$$f = A\frac{m + \check{m}}{2} + B\frac{m - \check{m}}{2}, \quad g = \frac{m + \check{m}}{2} \tag{62}$$

where $A, B \in \mathbb{C}$ are constants, and $m : G \to \mathbb{C}^*$ is a homomorphism. Conversely if f and g have this form where A, B are arbitrary constants, then the pair f, g is a solution to Wilsons equation.

PROOF. The last claim is a trivial calculation. The proof of the fact that the solutions must have this form is standard technique. Let $\{e\} = Z_0 < \cdots < Z_n = G$ be an ascending central series for G, with $Z_{i+1}/Z_i = Z(G/Z_i)$. We will prove the result by induction on n. If $n = 1$ then $Z(G) = G$ and it follows by the previous theorem that f and g has the stated form. Let $n \in \mathbb{N}$ and suppose that the result is true for any connected nilpotent Lie group with an ascending central series af length n. Let G be a connected nilpotent Lie group with an ascending central series of length $n+1$, $\{e\} = Z_0 < \cdots < Z_n < Z_{n+1} = G$, with $Z_{i+1}/Z_i = Z(G/Z_i)$, $Z_1 = Z(G)$. If there exists $u \in Z(G)$ such that $g(u) \neq 1$, then the result follows by the Theorem 3.2. So suppose $g(u) = 1, \ \ \forall u \in Z(G)$. By the previous lemma we can define $F_1, G_1 : G/Z_1 \to \mathbb{C}$ by $F_1(\bar{x}) = f(x)$ and $G_1(\bar{x}) = g(x)$, furthermore F_1 and G_1 satisfy the equation

$$F_1(\bar{x}\bar{y}) + F_1(\bar{x}\bar{y}^{-1}) = 2F_1(\bar{x})G_1(\bar{y}), \ \bar{x}, \bar{y} \in G/Z_1. \tag{63}$$

G/Z_1 is a connected nilpotent Lie group with an ascending central series $Z_1/Z_1 < \cdots < Z_{n+1}/Z_1 = G/Z_1$ with $(Z_{i+1}/Z_1)/(Z_i/Z_1) = Z((G/Z_1)/(Z_i/Z_1))$. By assumption there exist a homomorphism $M : G/Z_1 \to \mathbb{C}^*$ and constants $A, B \in \mathbb{C}$ such that

$$F_1 = A\frac{M + \check{M}}{2} + B\frac{M - \check{M}}{2}, \quad G_1 = \frac{M + \check{M}}{2}. \tag{64}$$

Define $m : G \to \mathbb{C}^*$ by $m(x) = M(\bar{x})$, m is a homomorphism and $\check{m}(x) = \check{M}(\bar{x})$.

$$f(x) = F_1(\bar{x}) = A\frac{M + \check{M}}{2}(\bar{x}) + B\frac{M - \check{M}}{2}(\bar{x}) = A\frac{m + \check{m}}{2}(x) + B\frac{m - \check{m}}{2}(x), \tag{65}$$

and

$$g(x) = G_1(\bar{x}) = \frac{M + \check{M}}{2}(\bar{x}) = \frac{m + \check{m}}{2}(x). \tag{66}$$

The theorem follows by induction on n. □

REMARK 3.5. (a) Instead of $f, g : G \to \mathbb{C}$ we could consider $f, g : G \to K$ where K is any quadratically closed field with characteristic different from 2.

(b) Instead of nilpotent connected Lie groups, we could consider a class $\mathcal{C}$ of nilpotent groups G, for which $G \in \mathcal{C}$ implies G and $Z(G)$ are 2-divisible and $G/Z(G) \in \mathcal{C}$. Note that if we take $\mathcal{C}$ to be all nilpotent groups where all elements are of odd order then $\mathcal{C}$ fulfils the requirement. So if we formulate the theorem in terms of classes $\mathcal{C}$ with the above mentioned properties, instead of for connected nilpotent Lie groups, then it contains as a special case Theorem 2 of [**3**].

4. Jensen's equation on a semidirect product of two groups

Let G be a semidirect product of G_1 and G_2. So we assume that G_1 is a transformation group of G_2 acting by homomorphisms, that is $a \cdot (xy) = (a \cdot x)(a \cdot y)$, $\forall a \in G_1$, $\forall x, y \in G_2$, and that the group operation in $G = G_2 \times G_1$, is given by

$$(x, a)(y, b) = (x(a \cdot y), ab), \ \forall (x, a), (y, b) \in G_2 \times G_1. \tag{67}$$

We let e_i denote the neutral element of G_i for $i = 1, 2$. Then $e = (e_1, e_2)$. Note that $(x, a)^{-1} = (a^{-1} \cdot x^{-1}, a^{-1})$, $\forall (x, a) \in G_2 \times G_1$. The idea is to reduce the study of functional equations on G to the study of functional equations on the subgroups G_1 and G_2. Clearly constant functions on G and homomorphisms of G into $\mathbb{C}^*$ are solutions to Jensen's equation on G. If f is a solution to Jensen's equation on G, then so is $f - f(e)$, so we may assume that $f(e) = 0$. If G is abelian and f is a solution of Jensen's equation on G such that $f(e) = 0$, then f is a homomorphism of G into $\mathbb{C}^*$ (see Lemma 1 in [**1**]).

THEOREM 4.1. *Assume that G_1 and G_2 satisfy the following. If $f_i : G_i \mapsto \mathbb{C}$ satisfies*

$$f_i(cd) + f_i(cd^{-1}) = 2f_i(c) \ \forall c, d \in G_i \ \text{ and } \ f_i(e_i) = 0, \tag{68}$$

then $f_i \in Hom(G_i, \mathbb{C})$ $i = 1, 2$. Then $f : G \mapsto \mathbb{C}$ is a solution to Jensens equation on G

$$f((x, a)(y, b)) + f((x, a)(y, b)^{-1}) = 2f(x, a), \ \forall (x, a), (y, b) \in G, \tag{69}$$

such that $f(e) = 0$ if and only if

$$f(x, a) = A_1(a) + A_2(x) + A_2(a^{-1} \cdot x), \ \forall (x, a) \in G, \tag{70}$$

where $A_i \in Hom(G_i, \mathbb{C})$ $, i = 1, 2$ *and*

$$A_2((ab) \cdot x) = A_2(a \cdot x) + A_2(b \cdot x) - A_2(x), \ \forall (x, a) \in G_2 \times G_1. \tag{71}$$

Suppose f *is of this form. Then* $f \in Hom(G, \mathbb{C})$ *if and only if* $A_2(a \cdot x) = A_2(x), \ \forall (x, a) \in G$.

Proof. Assume that $f : G \mapsto \mathbb{C}$ is a solution to Jensen's equation with $f(e) = 0$. Then

$$2f(x, a) = f(x(a \cdot y), ab) + f(x(ab^{-1} \cdot y^{-1}), ab^{-1}), \ \forall (x, a), (y, b) \in G. \tag{72}$$

Putting $b = e_1$ and fixing $a \in G_1$ in (72) we have

$$2f_a(x) = 2f(x, a) = f_a(x(a \cdot y)) + f_a(x(a \cdot y)^{-1}), \ \forall x, y \in G_2. \tag{73}$$

Since $a\cdot : G_2 \mapsto G_2$ is a bijection, it follows that $f_a : G_2 \mapsto \mathbb{C}$ is a solution to Jensen's equation on G_2. Hence

$$f(x, a) = f_a(x) = A_a(x) + f_a(e_2) = A_a(x) + f(e_2, a), \ \forall (x, a) \in G, \tag{74}$$

where $A_a : G_2 \mapsto \mathbb{C}$ is additive. Put $y = e_2$ and fix $x \in G_2$ in (72)

$$2f^x(a) = 2f(x, a) = f^x(ab) + f^x(ab^{-1}), \ \forall a, b \in G_1. \tag{75}$$

Hence $f^x : G_1 \mapsto \mathbb{C}$ is a solution to Jensen's equation on G_1.

$$f^x(a) = A^x(a) + f^x(e_1) = A^x(a) + f(x, e_1), \ \forall (x, a) \in G, \tag{76}$$

where $A^x : G_1 \mapsto \mathbb{C}$ is additive.

$$f(x, e_1) = A_{e_1}(x) + f(e_2, e_1) = A_{e_1}(x), \ \forall x \in G_2, \tag{77}$$

and

$$f(e_2, a) = f^{e_2}(a) = A^{e_2}(a) + f(e_2, e_1) = A^{e_2}(a), \ \forall a \in G_1. \tag{78}$$

Hence we have

$$f(x, a) = A_a(x) + A^{e_2}(a) = A^x(a) + A_{e_1}(x), \ \forall (x, a) \in G. \tag{79}$$

Note that

$$A^{xy}(a) = A_a(xy) + A^{e_2}(a) - A_{e_1}(xy) = A^x(a) + A^y(a) - A^{e_2}(a). \tag{80}$$

Now

$$\begin{aligned}
2f(x, a) &= f((x, a)(x, a)) + f((x, a)(x, a)^{-1}) = f(x(a \cdot x), a^2) \\
&= A^{x(a \cdot x)}(a^2) + A_{e_1}(x(a \cdot x)) \\
&= 2(A^x(a) + A_{e_1}(x)) - A_{e_1}(x) + 2(A^{a \cdot x}(a) - A^{e_2}(a)) + A_{e_1}(a \cdot x) \\
&= 2f(x, a) - A_{e_1}(x) + 2A_a(a \cdot x) - A_{e_1}(a \cdot x), \ \forall (x, a) \in G.
\end{aligned}$$

So

$$A_a(a \cdot x) = \frac{1}{2}(A_{e_1}(x) + A_{e_1}(a \cdot x)), \ \forall (x, a) \in G. \tag{81}$$

Substitute $a^{-1} \cdot x$ for x, that gives us

$$A_a(x) = \frac{1}{2}(A_{e_1}(x) + A_{e_1}(a^{-1} \cdot x)), \ \forall (x, a) \in G. \tag{82}$$

So

$$f(x, a) = A_a(x) + A^{e_2}(a) = A^{e_2}(a) + B(x) + B(a^{-1} \cdot x), \forall (x, a) \in G, \tag{83}$$

where $B = \frac{1}{2}A_{e_1}$ is an additive function on G_2. $f(x,a) = A^{e_2}(a)+B(x)+B(a^{-1}\cdot x)$ is a solution to Jensen's equation if and only if $g(x,a) = B(x) + B(a^{-1}\cdot x)$ is a solution to Jensen's equation. The computation

$$\begin{aligned} 2B(x) + 2B(a^{-1}\cdot x) &= 2g(x,a) = g(x(a\cdot y), ab) + g(x(ab^{-1}\cdot y^{-1}), ab^{-1}) \\ &= B(x(a\cdot y)) + B((ab)^{-1}\cdot(x(a\cdot y))) \\ &\quad +B(x(ab^{-1}\cdot y^{-1})) + B((ab^{-1})^{-1}\cdot(x(ab^{-1}\cdot y^{-1}))) \\ &= 2B(x) + B(a\cdot y) + B(b^{-1}a^{-1}\cdot x) + B(b^{-1}\cdot y) \\ &\quad +B(ab^{-1}\cdot y^{-1}) + B(ba^{-1}\cdot x) + B(y^{-1}), \end{aligned} \tag{84}$$

shows that g is a solution to Jensen's equation if and only if

$$\begin{aligned} 2B(a^{-1}\cdot x) &= B(a\cdot y) + B(b^{-1}a^{-1}\cdot x) + B(b^{-1}\cdot y) \\ &\quad +B(ab^{-1}\cdot y^{-1}) + B(ba^{-1}\cdot x) - B(y),\ \forall(x,a),(y,b)\in G. \end{aligned} \tag{85}$$

Put $x = e_2$ in (85) to get

$$0 = B(a\cdot y) + B(b^{-1}\cdot y) - B(ab^{-1}\cdot y) - B(y). \tag{86}$$

Put $a = b$ in (86) to get

$$2B(y) = B(a\cdot y) + B(a^{-1}\cdot y). \tag{87}$$

In particular we have

$$2B(a^{-1}\cdot x) = B(ba^{-1}\cdot x) + B(b^{-1}a^{-1}\cdot x). \tag{88}$$

It is now obvious that if conversely (86) holds, i.e. if

$$0 = B(a\cdot y) + B(b^{-1}\cdot y) - B(ab^{-1}\cdot y) - B(y),\ \ \forall a,b\in G_1,\ \ \forall y\in G_2, \tag{89}$$

then g is a solution to Jensen's equation. The condition (86) is equivalent to

$$0 = B(a\cdot y) + B(b\cdot y) - B(ab\cdot y) - B(y),\ \ \forall a,b\in G_1,\ \ \forall y\in G_2. \tag{90}$$

Now all that remains is to determine when f is additive. This is the case if and only if g is additive.

$$\begin{aligned} g((x,a)(y,b)) - g(x,a) - g(y,b) &= B(x(a\cdot y)) + B((ab)^{-1}\cdot(x(a\cdot y))) \qquad (91) \\ &\quad -B(x) - B(a^{-1}\cdot x) - B(y) - B(b^{-1}\cdot y) \\ &= B(b^{-1}a^{-1}\cdot x) - B(a^{-1}\cdot x) + B(a\cdot y) - B(y). \end{aligned}$$

Suppose that g is additive

$$0 = B(b^{-1}a^{-1}\cdot x) - B(a^{-1}\cdot x) + B(a\cdot y) - B(y). \tag{92}$$

Put $x = e_2$

$$B(a\cdot y) = B(y),\ \forall a\in G_1,\ \forall y\in G_2. \tag{93}$$

Conversely if this condition is fulfilled then g is additive.

□

EXAMPLE 4.2. The Heisenberg-group $H_3 = \mathbb{R}^2 \times_s \mathbb{R}$, where the action of $\mathbb{R}$ on $\mathbb{R}^2$ is given by $x\cdot(y,z) = (y, z+xy)$. Here $e = ((0,0),0)$.

PROPOSITION 4.3. *f is a solution to Jensen's equation on H_3 with $f(e)=0$ if and only if and only if*

$$f((y,z),x) = A_1(x) + A_2(y) + 2A_3(z - \frac{1}{2}xy),\ \forall x,y,z \in \mathbb{R}, \tag{94}$$

where $A_i \in Hom(\mathbb{R},\mathbb{C})$ are arbitrary, $i=1,2,3$.

Suppose f is a solution to Jensen's equation. Then $f \in Hom(H_3,\mathbb{C})$ if and only if $A_3 \equiv 0$.

PROOF. Let $B : \mathbb{R}^2 \mapsto \mathbb{C}$ be any additive function on $\mathbb{R}^2$. It is a simple calculation to check that

$$B((x_1+x_2)\cdot(y,z)) + B(y,z) = B(x_1\cdot(y,z)) + B(x_2\cdot(y,z)),\ \forall x_1,x_2,y,z \in \mathbb{R}. \tag{95}$$

So it follows immediately from Theorem 4.1 that the solutions to Jensen's equation on H_3 are of the form

$$f((y,z),x) = A_1(x) + B(y,z) + B(y,z-xy), \forall x,y,z \in \mathbb{R}, \tag{96}$$

where $B \in Hom(\mathbb{R}^2,\mathbb{C})$ is arbitrary. For $B \in Hom(\mathbb{R}^2,\mathbb{C})$ there exist $A_2, A_3 \in Hom(\mathbb{R},\mathbb{C})$ such that $B(y,z) = \frac{1}{2}A_2(y) + A_3(z),\ \forall y,z \in \mathbb{R}$. When is f additive? We know from Theorem 4.1 that it is the case if and only if

$$\frac{1}{2}A_2(y)+A_3(z+xy) = B(x\cdot(y,z)) = B(y,z) = \frac{1}{2}A_2(y)+A_3(z),\ \forall x,y,z \in \mathbb{R}, \tag{97}$$

that is if and only if $A_3 \equiv 0$. □

Note in particular that $f((y,z)x) = 2z - xy$ is a solution to Jensen's equation on the Heisenberg group with $f(e)=0$ which is not a homomorphism. So this example show that genuine differences occur, from the abelian case, when we attempt to solve Jensen's equation on non-abelian groups. Furthermore H_3 is a connected nilpotent Lie group, so the example also shows that contrary to what Theorem 3.4 might lead one to suspect, the solutions to the degenerate Wilson's equation, i.e. Jensen's equation, need not be of the classical form, even on connected nilpotent Lie groups. By the term the classical form we mean homomorphisms, which is the form of the solutions in the case where G is abelian. In contrast Ng has shown that all solutions to (16) with $f(e)=0$ are homomorphisms for certain groups including the Heisenberg group (private communication and presented in his talk at the 37th ISFE).

EXAMPLE 4.4. The (ax+b)-group $G = \mathbb{R} \times_s \mathbb{R}_+$ where the action of $\mathbb{R}_+$ on $\mathbb{R}$ is given by $a\cdot y = ay$. Here $e=(0,1)$.

PROPOSITION 4.5. *$f : G \mapsto \mathbb{C}$ is a solution to Jensen's equation with $f(e)=0$ if and only if $f(x,a) = A(a)$, $\forall a \in \mathbb{R}_+$, $\forall x \in \mathbb{R}$, where $A : \mathbb{R}_+ \mapsto \mathbb{C}$ is additive.*

PROOF. Assume that $B : \mathbb{R} \mapsto \mathbb{C}$ is additive and

$$B((ab)\cdot x) + B(x) = B(a\cdot x) + B(b\cdot x),\ \forall x \in \mathbb{R},\ \forall a,b \in \mathbb{R}_+. \tag{98}$$

Put $a=b=2$

$$B(4x) = B(2\cdot x) + B(2\cdot x) = B(4x) + B(x),\ \forall x \in \mathbb{R}. \tag{99}$$

Hence $B(x)=0$, $\forall x \in \mathbb{R}$. The proposition now follows immediately from Theorem 4.1. □

Bibliography

[1] J. Aczél, J. K. Chung, and C. T. Ng, *Symmetric second differences in product form on groups*, in: Th. M. Rasssias (Ed.), *Topics in Mathematical Analysis*, pp. 1-22. Ser. Pure Math., 11, World Scientific Publ. Co., Teaneck, NJ, 1989.

[2] I. Corovei, *The functional* $f(xy) + f(yx) + f(xy^{-1}) + f(y^{-1}x) = 4f(x)f(y)$ *for nilpotent groups*, (Romanian, English summary) Bul. Ştiinţ. Inst. Politech. Cluj-Napoca Ser. Mat.-Fiz.-Mec. Apl. 20 (1977), pp. 25-28.

[3] I. Corovei, *The functional equation* $f(xy)+f(xy^{-1}) = 2f(x)g(y)$ *for nilpotent groups.* Mathematica (Cluj) 22 (45) (1980), 33-41.

[4] I. Corovei, *The cosine functional equation for nilpotent groups*, Aequationes Math. 15 (1977), 99-106.

[5] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces.* Academic Press. New York 1978.

[6] Pl. Kannappan, *The functional equation* $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ *for groups.* Proc. Amer. Math. Soc. **19** (1968), 69-74.

[7] C. T. Ng, *Jensen's functional equation on groups*, Aequationes Math. 39 (1990), 85-99.

[8] C. T. Ng, *Jensen's functional equation on groups, II*, Aequationes Math. 58 (1999), 311-320.

[9] H. Stetkær, *Functional equations on abelian groups with involution.*, Aequationes Math. 54 (1997), 144-172.

[10] H. Stetkær, *d'Alembert's functional equations on metabelian groups*, Aequationes Math. 59 (2000) 306-320.

[11] V. S. Varadarajan, *Lie Groups, Lie Algebras, and their Representations*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey 1974.

Peter de Place Friis
Department of Mathematics
University of Aarhus
Ny Munkegade
DK 8000 Aarhus C
Denmark
E-mail: pfriis@imf.au.dk

CHAPTER 3

On the Cosine-Sine Functional Equations on Groups

Peter de Place Friis and Henrik Stetkær

1. Introduction

An ambitious project is to obtain the general solution $f, g, h \in C(G)$ of the functional equation

$$\int_K f(xk \cdot y)dk = f(x)g(y) + g(x)f(y) + h(x)h(y),\ x, y \in G, \tag{1}$$

where G is a topological group, and $C(G)$ denotes the algebra of all continuous, complex valued functions on G. Furthermore K is a compact, transformation group, acting by automorphisms on G, and $k \cdot x$ denotes the action of $k \in K$ on $x \in G$. In particular the map $(k, x) \mapsto k \cdot x$ of $K \times G$ into G is continuous. Finally dk is the normalized Haar measure on K. This notation will be used throughout the paper.

We give the complete continuous solution to (1) for $K = \mathbf{Z}_2$ acting on a topological abelian group G. That is we solve the functional equation

$$\frac{f(x+y) + f(x+\sigma y)}{2} = f(x)g(y) + g(x)f(y) + h(x)h(y),\ x, y \in G, \tag{2}$$

where $\sigma : G \to G$ denotes a continuous involutive automorphism of G. Obvious examples of such automorphisms are $\sigma = I$ and $\sigma = -I$, where I denotes the identity operator. Letting σ be a reflection in a hyperplane in $G = \mathbf{R}^n$ we get an example for which $\sigma \neq \pm I$. It turns out that the solutions are certain exponential polynomials. Chung, Kanappan and Ng's paper [**4**] deals with the functional equation

$$f(xy) = f(x)g(y) + g(x)f(y) + h(x)h(y),\ x, y \in G, \tag{3}$$

that can be viewed as the case of $\sigma = I$ in (2). Our results encompass those of [**4**] (see Remark 3.4). For $G = \mathbf{R}$ the functional equation (2) describes involved addition formulas for trigonometric and related functions. See also [**7**].

The classical example $\sigma = -I$ of the equation (2) has been studied extensively for d'Alembert's functional equation ($g = f - h = 0$), the trigonometric functional equations in [**5**] ($h = 0$ and $g = f = ih$) and the quadratic equation ($h = g - 1 = 0$). The special case of $h = g$ turns up as part of a system of 2 functional equations in ([**5**]; Formula (3.6)) and in ([**11**]; Lemma V.3). The case of $g = 1$ is Swiataks equation (see [**3**] and [**13**]).

The general form of the solution sets for functional equations of d'Alembert's type, i.e., with the left hand side $(f(x + y) + f(x - y))/2$, can be found in Rukhin [**10**].

The new of the present paper is that we:

i: Produce the explicit solution formulas for the special functional equations (2) in question.
ii: Do it for any involutive automorphism σ, not just for $\sigma = \pm I$.
iii: Take continuity into account.

We reveal part of the underlying structure in the set up by discussing the general equation (1). The results of the present paper can be compared with the ones of [**4**] because we formulate them in the same way. It is intriguing to see that many of the methods of [**4**] carry over to the more general situations (1) and (2). However, our formulas for the solutions of (2) contain certain types of functions that are absent in [**4**], because they vanish for $\sigma = I$. For example the 4'th order term in Proposition 3.2. So new phenomena show up.

With more than one term on the left hand side the possibility of varying signs exists. We give the complete solution of the functional equation

$$\frac{f(x+y) - f(x+\sigma y)}{2} = f(x)g(y) + g(x)f(y) + h(x)h(y), \ x, y \in G, \tag{4}$$

in Section 6.

2. Main result

The following notation will be used throughout the paper unless explicitly stated otherwise.

<u>**Notation**</u> $(G, +)$ is an abelian topological group, 0 its identity element. We let $\sigma : G \to G$ be an continuous automorphism of order 2. $\mathcal{A}(G)$ is the vector space of all continuous additive maps from G to $\mathbf{C}$ and $\mathcal{A}^{\pm}(G) := \{A \in \mathcal{A}(G) : A \circ \sigma = \pm A\}$. Furthermore $\mathcal{S}^{-}(G)$ denotes the vector space of all continuous, biadditive, symmetric maps $S : G \times G \to \mathbf{C}$ for which $S(\sigma x, y) = -S(x, y)$ for all $x, y \in G$. If $S^- \in \mathcal{S}^-(G)$ we let for brevity S^- also denote the function $S^-(x) := S^-(x, x), x \in G$. With $K = \mathbb{Z}_2 = (\pm 1, \cdot)$ equipped with the discrete topology, the action of K on G given by $1 \cdot x = x, \forall x \in G$ and $-1 \cdot x = \sigma x, \forall x \in G$. A *K-spherical function* is a function $\phi \in C(G)$ such that $\phi \neq 0$ and ϕ satifies $\int_K \phi(xk \cdot y)dk = \phi(x)\phi(y)$ for all $x, y \in G$, in the case $K = \mathbb{Z}_2$ the K-spherical functions are given by theorem III.1 in [**12**]. If f is a function on G and $k \in K$ we define the function $k \cdot f$ by $(k \cdot f)(x) := f(k^{-1} \cdot x)$ for $x \in G$. We let $\mathbf{C}^*$ denote the multiplicative group of nonzero complex numbers.

Our main result is

THEOREM 2.1. *Let (f, g, h) be a continuous solution of*

$$\frac{f(x+y) + f(x+\sigma y)}{2} = f(x)g(y) + g(x)f(y) + h(x)h(y), x, y \in G. \tag{5}$$

Then f,g and h have one of the following six forms, and conversely.

(A): $f = h = 0$ *and* $g \in C(G)$.
(B):

$$\left\{ \begin{array}{c} f \\ g \\ h \end{array} \right\} = \left\{ \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right\} \left\{ \begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \end{array} \right\}, \tag{6}$$

where ϕ_1,ϕ_2 and ϕ_3 are K-spherical functions on G and $a_i, b_i, c_i, i = 1, 2, 3$ are complex constants satisfying the matrix equation

$$\left\{ \begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} \right\} \left\{ \begin{array}{ccc} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{array} \right\} = \left\{ \begin{array}{ccc} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{array} \right\}. \tag{7}$$

(C):

$$\left\{ \begin{array}{c} f \\ g \\ h \end{array} \right\} = \left\{ \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right\} \left\{ \begin{array}{c} \phi_1 \\ \phi_2 \\ (m_2 A + (m_2 \circ \sigma)(A \circ \sigma))/2 \end{array} \right\}, \tag{8}$$

where ϕ_1 is a K-spherical function on G, $m_2 : G \to \mathbf{C}^*$ is a continuous homomorphism for which $m_2 \neq m_2 \circ \sigma$, ϕ_2 is the corresponding K-spherical function $\phi_2 = \int_K k \cdot m_2 dk$, $A \in \mathcal{A}(G)$, and $a_i, b_i, c_i, i = 1, 2, 3$ are complex constants satisfying the matrix equation

$$\left\{ \begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} \right\} \left\{ \begin{array}{ccc} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{array} \right\} = \left\{ \begin{array}{ccc} a_1 & 0 & 0 \\ 0 & a_2 & a_3 \\ 0 & a_3 & 0 \end{array} \right\}. \tag{9}$$

Furthermore $a_3 = 1$. If f, g and h are linearly independent it may also be assumed that $a_1 = -a_2$.

(D):

$$\left\{ \begin{array}{c} f \\ g \\ h \end{array} \right\} = \left\{ \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right\} \left\{ \begin{array}{c} \phi_1 \\ m_2 \\ m_2 q \end{array} \right\}, \tag{10}$$

where ϕ_1 is a K-spherical function on G, $m_2 : G \to \mathbf{C}^*$ is a continuous homomorphism for which $m_2 = m_2 \circ \sigma$, $q = A^+ + S^-$ where $A^+ \in \mathcal{A}^+(G)$ and $S^- \in \mathcal{S}^-(G)$ and $a_i, b_i, c_i, i = 1, 2, 3$ are complex constants satisfying the matrix equation (9). It may be assumed that $a_3 = 1$. If f,g and h are linearly independent it may also be assumed that $a_1 = -a_2$.

(E):

$$\left\{ \begin{array}{c} f \\ g \\ h \end{array} \right\} = \left\{ \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{array} \right\} \left\{ \begin{array}{c} (mA_1 + (m \circ \sigma)(A_1 \circ \sigma))/2 \\ (m + m \circ \sigma)/2 \\ (mA + (m \circ \sigma)(A \circ \sigma))/2 \\ (mA^2 + (m \circ \sigma)(A^2 \circ \sigma))/2 \end{array} \right\}, \tag{11}$$

where $m : G \to \mathbf{C}^*$ is a continuous homomorphism for which $m \neq m \circ \sigma$, $A, A_1 \in \mathcal{A}(G)$, and $a_i, b_i, c_i, i = 1, 2, 3, 4$ are complex constants satisfying the matrix equation

$$\left\{ \begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{array} \right\} \left\{ \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \\ a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \end{array} \right\} = \left\{ \begin{array}{cccc} 0 & a_1 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & a_3 & 2a_4 & 0 \\ 0 & a_4 & 0 & 0 \end{array} \right\}. \tag{12}$$

Futhermore $a_1 = 1$ and $a_3 = 0$. If f,g and h are linearly independent then it may also be assumed that $a_2 = 0$ and that $a_4 = 1/2$.

(F):

$$\left\{ \begin{array}{c} f \\ g \\ h \end{array} \right\} = \left\{ \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right\} \left\{ \begin{array}{c} mF \\ m \\ mq \end{array} \right\}, \tag{13}$$

where $a_i, b_i, c_i, i = 1,2,3$ are complex constants satisfying the matrix equation

$$\left\{\begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right\}\left\{\begin{array}{ccc} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{array}\right\} = \left\{\begin{array}{ccc} 0 & a_1 & 0 \\ a_1 & a_2 & a_3 \\ 0 & a_3 & a_1 \end{array}\right\}, \tag{14}$$

$m : G \to \mathbf{C}^*$ is a continuous homomorphism for which $m = m \circ \sigma$, and

$$F = \frac{1}{2}(A^+)^2 + \frac{1}{6}(A^-)^4 + A^+(A^-)^2 + A_1^+ + S^-, \tag{15}$$

$$q = A^+ + (A^-)^2, \tag{16}$$

where $A^+, A_1^+ \in \mathcal{A}^+(G)$, $A^- \in \mathcal{A}^-(G)$ and $S^- \in \mathcal{S}^-(G)$. We may even assume that

$$\left\{\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right\} = \left\{\begin{array}{ccc} 1 & 0 & 0 \\ -z^2/2 & 1 & z \\ -z & 0 & 1 \end{array}\right\} \text{ for some } z \in \mathbf{C}. \tag{17}$$

PROOF. It is elementary to check that the functions listed are solutions. Thus what is left is to show that each continuous solution (f, g, h) of (5) occurs in the list. Apart from the last section the rest of the paper is dedicated to this. □

REMARK 2.2. (a) Theorem 2.1 above yields for $\sigma = I$ the main result of [**4**]. In checking that this is so it is advisable to take the discussion on p. 276 of [**4**] into account or the remark after our Lemma 3.3.

(b) The matrix equations (7), (9) and (12) occur in [3].

3. Technicalities

PROPOSITION 3.1. *The solutions $q \in C(G)$ of the quadratic equation*

$$\frac{q(x+y) + q(x+\sigma y)}{2} = q(x) + q(y),\ x, y \in G, \tag{18}$$

are the functions of the form $q = A^+ + S^-$, where $A^+ \in \mathcal{A}^+(G)$ and $S^- \in \mathcal{S}^-(G)$.

PROOF. This is Corollary III.8 of [**12**]. □

PROPOSITION 3.2. *The solutions $F, q \in C(G)$ of the system of functional equations*

$$\begin{aligned} \frac{F(x+y) + F(x+\sigma y)}{2} &= F(x) + F(y) + q(x)q(y),\ x, y \in G, \\ \frac{q(x+y) + q(x+\sigma y)}{2} &= q(x) + q(y),\ x, y \in G, \end{aligned} \tag{19}$$

are

$$\begin{aligned} F &= \frac{1}{2}(A^+)^2 + \frac{1}{6}(A^-)^4 + A^+(A^-)^2 + q', \\ q &= A^+ + (A^-)^2, \end{aligned} \tag{20}$$

where $A^\pm \in \mathcal{A}^\pm(G)$ and where $q' \in C(G)$ is a solution of the quadratic equation (18).

PROOF. It suffices to prove that q has the stated form. Indeed, if so then $f_0 := (A^+)^2/2 + A^+(A^-)^2 + (A^-)^4/6$ is a particular solution of the first equation of (19). Its complete solution is $F = f_0 + q'$ where q' ranges over the solutions of the corresponding homogeneous equation, i.e. of the quadratic equation.

If the function $x \mapsto q(x+t) - q(x)$ is a constant, say $c(t)$, for any $t \in G$ then $c(t) = q(t) - q(0) = q(t)$, implying that q is additive. Substituting σy for y in (18) shows that any solution of the quadratic equation is invariant under σ, hence $q \in \mathcal{A}^+(G)$. So from now on we may assume that there exists a $t \in G$ such that the function $x \mapsto q(x+t) - q(x)$ is not constant. The function $F_t(x) := F(x+t) - F(x) - F(t)$, $x \in G$ satisfies

$$\frac{F_t(x+y) + F_t(x+\sigma y)}{2} = F_t(x) + [q(x+t) - q(x)]q(y), \ x, y \in G, \tag{21}$$

which is a version of the functional equation of symmetric differences. It is solved by theorem IV.1 of [**12**] according to which there are five cases (a)-(e) to take into account: The cases (a) and (b) do not apply under our assumptions here. (c) gives $q = A^+ + (A^-)^2$. In each of the two remaining cases q has the form $q = c(\phi - 1)$ where $c \in \mathbf{C}$ and ϕ is a $\mathbf{Z}_2$-spherical function. But if a function q of this form satisfies the second equation of (19) then it is 0. $\square$

The general result that makes thing work is the following technical lemma. It says that if $f, g, h \in C(G)$ constitute a solution of (22) then g and h satisfy functional equations of the same nature as f in (22). The corresponding result of [**4**] is there expressed by (3.7) and (3.8).

LEMMA 3.3. *In this Lemma G need not be abelian, so we use the multiplicative way of writing the group composition. If $f, g, h \in C(G)$ constitute a solution of*

$$\int_K f(xk \cdot y)dk = f(x)g(y) + g(x)f(y) + h(x)h(y), \ x, y \in G, \tag{22}$$

and $f \neq 0$ then there exists constants $\alpha, \beta, \gamma, \delta \in \mathbf{C}$ such that $\gamma^2 = \alpha + \beta\delta$ and

$$\begin{aligned} &\int_K g(xk \cdot y)dk - g(x)g(y) \\ &\quad = \alpha f(x)f(y) + \beta[f(x)h(y) + h(x)f(y)] + \gamma h(x)h(y), \ x, y \in G, \end{aligned} \tag{23}$$

$$\begin{aligned} &\int_K h(xk \cdot y)dk - g(x)h(y) - h(x)g(y) = \\ &\quad \beta f(x)f(y) + \gamma[f(x)h(y) + h(x)f(y)] + \delta h(x)h(y), \ x, y \in G. \end{aligned} \tag{24}$$

PROOF. This is Propostition II.5 of [**9**]. We have included the proof for the readers convinience as [**9**] is not readily available.

Case A : f and h are linearly independent. Lemma II.2 in [**12**] implies here that

$$G(x,y)f(z) + H(x,y)h(z) = G(y,z)f(x) + H(y,z)h(x), \ x, y, z \in G, \tag{25}$$

where

$$\begin{aligned} G(x,y) &= \int_K g(xk \cdot y)dk - g(x)g(y), \ x, y \in G, \\ H(x,y) &= \int_K h(xk \cdot y)dk - g(x)h(y) - h(x)g(y), \ x, y \in G. \end{aligned} \tag{26}$$

By (25) we have for any $z_1, z_2 \in G$ that

$$\begin{pmatrix} f(z_1) & h(z_1) \\ f(z_2) & h(z_2) \end{pmatrix} \begin{pmatrix} G(x,y) \\ H(x,y) \end{pmatrix} = \begin{pmatrix} G(y,z_1)f(x) + H(y,z_1)h(x) \\ G(y,z_2)f(x) + H(y,z_2)h(x) \end{pmatrix}. \tag{27}$$

Since f and h are linearly independent there exists $z_1, z_2 \in G$ such that the matrix on the left is invertible (see Lemma 14.1 in [**1**]), and so

$$\begin{aligned} G(x,y) &= \phi_1(y)f(x) + \psi_1(y)h(x), \\ H(x,y) &= \phi_2(y)f(x) + \psi_2(y)h(x), \end{aligned} \tag{28}$$

for some functions $\phi_1, \phi_2, \psi_1, \psi_2 \in C(G)$. When we substitute this back into (25) we get by the linear independence of f and h that

$$\begin{aligned} \phi_1(y)f(z) + \phi_2(y)h(z) &= \phi_1(z)f(y) + \psi_1(z)h(y),\ y,z \in G, \\ \psi_1(y)f(z) + \psi_2(y)h(z) &= \phi_2(z)f(y) + \psi_2(z)h(y),\ y,z \in G. \end{aligned} \tag{29}$$

Using the linear independence of f and h once more we get that there exist constants $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{C}$ such that $\phi_i = a_i f + b_i h$ and $\psi_i = c_i f + d_i h$ for $i = 1, 2$. Substituting this back into (29) we find that $b_1 = c_1$, $a_2 = b_1$, $b_2 = d_1$, $c_1 = a_2$, $c_2 = b_2$, $d_1 = c_2$, so that $\phi_1 = \alpha f + \beta h$, $\phi_2 = \psi_1 = \beta f + \gamma h$, and $\psi_2 = \gamma f + \delta h$, where $\alpha = a_1$, $\beta = b_1$, $\gamma = b_2$, and $\delta = d_2$. This means by (28) that G and H have the forms stated in (23) and (24). That $\gamma^2 = \alpha + \beta\delta$ follows from applying Lemma II.2 in [**12**] to any of the two identities (23) and (24).

Case B: f and h are linearly dependent. Since $f \neq 0$ there exists a constant $c \in \mathbb{C}$ such that $h = \sqrt{2}cf$. The identity (22) then becomes

$$\begin{aligned} \int_K f(xk \cdot y)dk &= f(x)g(y) + g(x)f(y) + 2c^2 f(x)f(y) \\ &= f(x)[g + c^2 f](y) + [g + c^2 f](x)f(y),\ x,y \in G. \end{aligned} \tag{30}$$

From Lemma V.1 in [**11**] it follows after elementary computations that there exists a constant $\kappa \in \mathbb{C}$ such that

$$\int_K g(xk \cdot y)dk - g(x)g(y) = \kappa^2 f(x)f(y),\ x,y \in G. \tag{31}$$

From (22) we get that

$$\int_K h(xk \cdot y)dk - g(x)h(y) - h(x)g(y) = \sqrt{2}ch(x)h(y),\ x,y \in G, \tag{32}$$

So all that remains to be show is that there exist $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $\gamma^2 = \alpha + \beta\delta$, $\alpha + 2\sqrt{2}c\beta + 2c^2\gamma = \kappa^2$, and $\beta + 2\sqrt{2}c\gamma + 2c^2\delta = \sqrt{2}c$. But this is obvious. $\square$

REMARK 3.4. Let K be the trivial group as in [**4**] and let (f, g, h) be a solution of (22) with $f \neq 0$. Then it follows from (22),(23) and (24) that each of the functions f, g and h satisfies Kannappans condition. Hence we may assume without loss of generality that G is abelian in this case. Unfortunately the argument does not generalize to $K = \mathbf{Z}_2$ so here we assume that G is Abelian.

PROPOSITION 3.5. *Let each of the functions ϕ_1, ϕ_2, ϕ_3 be a K-spherical function on G or the zero function. For $a_i, b_i, c_i \in \mathbf{C}, i = 1, 2, 3$, we define the functions $f := \sum_{i=1}^3 a_i\phi_i$, $g := \sum_{i=1}^3 b_i\phi_i$ and $h := \sum_{i=1}^3 c_i\phi_i$. If the coefficients $a_i, b_i, c_i \in \mathbf{C}, i = 1, 2, 3$ satisfy the matrix equation (7) then the triple (f,g,h) constitutes a solution of the functional equation (22).*

Conversely if the triple (f,g,h) solves (22) and if f,g and h are linearly independent then the coefficients satisfy the matrix equation (7).

PROOF. To prove the converse result note that f, g, and h linearly independent implies that ϕ_1, ϕ_2, ϕ_3 are linearly independent and the result follows by direct computations. $\square$

PROPOSITION 3.6. *Let $m_1, m_2 : G \to \mathbf{C}^*$ be continuous homomorphisms and let $A \in \mathcal{A}(G)$. For $a_i, b_i, c_i \in \mathbf{C}, i = 1, 2, 3$ we define the functions*

$$\left\{ \begin{array}{c} f \\ g \\ h \end{array} \right\} = \left\{ \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right\} \left\{ \begin{array}{c} \phi_1 \\ \phi_2 \\ \int_K k \cdot (m_2 A) dk \end{array} \right\}, \tag{33}$$

where $\phi_i, i = 1, 2$ denotes the K-spherical function $\phi_i := \int_K k \cdot m_i dk$.

If the coefficients satisfy the matrix equation (9) then the triple (f,g,h) constitutes a solution of the functional equation (22).

Conversely if the triple $(f, g, h) \in C(G)$ solves (22) and if f, g and h are linearly independent then the coefficients satisfy the matrix equation (9).

4. The case of linear independence

Let us assume that the triple (f, g, h) solves the functional equation (22) and that $f \neq 0$. Explicit calculations based on the identities (22), (23) and (24) reveal that

$$\int_K \left\{ \begin{array}{c} f \\ g \\ h \end{array} \right\} (xk \cdot y) dk = \Phi(y)^t \left\{ \begin{array}{c} f \\ g \\ h \end{array} \right\} (x),\ x, y \in G, \tag{34}$$

where Φ is defined by

$$\Phi = gI + f \left\{ \begin{array}{ccc} 0 & \alpha & \beta \\ 1 & 0 & 0 \\ 0 & \beta & \gamma \end{array} \right\} + h \left\{ \begin{array}{ccc} 0 & \beta & \gamma \\ 0 & 0 & 1 \\ 1 & \gamma & \delta \end{array} \right\}. \tag{35}$$

Elementary computations based on the definition of Φ and the identities (22), (23) and (24) where $\gamma^2 = \alpha + \beta\delta$ show that Φ satisfies the spherical equation

$$\int_K \Phi(xk \cdot y) dk = \Phi(x)\Phi(y) \text{ for all } x, y \in G. \tag{36}$$

Since the right hand sides of (22), (23) and (24) are symmetric in x and y it follows that

$$\int_K F(xk \cdot y) dk = \int_K F(yk \cdot x) dk,\ \ \forall x, y \in G,\ F \in \{f, g, h\}. \tag{37}$$

Hence

$$\Phi(x)\Phi(y) = \int_K \Phi(xk \cdot y) dk = \int_K \Phi(yk \cdot x) dk = \Phi(y)\Phi(x),\ \forall x, y \in G. \tag{38}$$

By linear Algebra this ensures the existence of a 3×3 complex matrix A such that $A^{-1}\Phi(x)A$ is upper triangular for all $x \in G$. Below we find such an A explicitly. If we put $y = e$ in (22) we get that

$$(g(e) - 1)f + f(e)g + h(e)h = 0. \tag{39}$$

If f, g and h are linearly independent this means that $g(e) = 1$ and $f(e) = h(e) = 0$. In particular we find in this case that $\Phi(e) = I$ so Φ is a matrix valued K-spherical function. In the remaining part of this section we shall assume that the triple (f,g,h) constitutes a solution of (22) and that f, g and h are linearly independent.

CASE 1: $\beta = 0$ but $\alpha = \gamma^2 \neq 0$. Here Φ takes the form

$$\Phi = gI + f\left\{\begin{array}{ccc} 0 & \gamma^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \gamma \end{array}\right\} + h\left\{\begin{array}{ccc} 0 & 0 & \gamma \\ 0 & 0 & 1 \\ 1 & \gamma & \delta \end{array}\right\}. \tag{40}$$

Let

$$\lambda_\pm := \frac{\delta}{2} \pm \sqrt{\left(\frac{\delta}{2}\right)^2 + 2\gamma}. \tag{41}$$

CASE 1.A: $\lambda_+ \neq \lambda_-$. Here

$$\left\{\begin{array}{ccc} -\gamma & \gamma & \gamma \\ 1 & 1 & 1 \\ 0 & \lambda_+ & \lambda_- \end{array}\right\}^{-1} \Phi \left\{\begin{array}{ccc} -\gamma & \gamma & \gamma \\ 1 & 1 & 1 \\ 0 & \lambda_+ & \lambda_- \end{array}\right\} = \tag{42}$$

$$\left\{\begin{array}{ccc} g - \gamma f & 0 & 0 \\ 0 & g + \gamma f + \lambda_+ h & 0 \\ 0 & 0 & g + \gamma f + \lambda_- h \end{array}\right\}.$$

None of the three functions in the diagonal are zero, because f, g, and h are linearly independent. From (36) we read that $\phi_1 := g - \gamma f$, $\phi_2 := g + \gamma f + \lambda_+ h$, and $\phi_3 := g + \gamma f + \lambda_- h$ are K-spherical functions on G. We find that there exist constants $a_i, b_i, c_i \in \mathbf{C}$ for $i = 1,2,3$ such that $f = \sum_{i=1}^3 a_i\phi_i$, $g = \sum_{i=1}^3 b_i\phi_i$, and $h = \sum_{i=1}^3 c_i\phi_i$. It follows from Proposition 3.5 that the coefficients satisfy the matrix equation (7). The solution occurs in (B) of the list of Theorem 2.1.

CASE 1.B: $\lambda_+ = \lambda_-$. This means that $\gamma = -\delta^2/8$. In particular $\delta \neq 0$ since $\gamma \neq 0$. Here we find that

$$\left\{\begin{array}{ccc} \frac{\delta^2}{8} & -\frac{\delta^2}{8} & 0 \\ 1 & 1 & 0 \\ 0 & \frac{\delta}{2} & 1 \end{array}\right\}^{-1} \Phi \left\{\begin{array}{ccc} \frac{\delta^2}{8} & -\frac{\delta^2}{8} & 0 \\ 1 & 1 & 0 \\ 0 & \frac{\delta}{2} & 1 \end{array}\right\} = \tag{43}$$

$$\left\{\begin{array}{ccc} g + \frac{\delta^2}{8}f & 0 & 0 \\ 0 & g - \frac{\delta^2}{8}f + \frac{\delta}{2}h & h \\ 0 & 0 & g - \frac{\delta^2}{8}f + \frac{\delta}{2}h \end{array}\right\}.$$

We see from (36) that $\phi_1 := g + \delta^2 f/8$ and $\phi_2 := g - \delta^2 f/8 + \delta h/2$ are K-spherical functions on G. Furthermore

$$f = \left(\frac{2}{\delta}\right)^2 (\phi_1 - \phi_2 + \frac{\delta}{2}h) \text{ and } g = \frac{1}{2}(\phi_1 + \phi_2 - \frac{\delta}{2}h), \tag{44}$$

and h is a non-zero solution of the functional equation

$$\int_K h(xk \cdot y)dk = \phi_2(x)h(y) + h(x)\phi_2(y),\ x, y \in G. \tag{45}$$

We specialize to $\mathbf{Z}_2$ for a moment. By Theorem III.1 of [**12**] (or Theorem 3 of [**2**]) there exists a continuous homomorphism $m_2 : G \to \mathbf{C}^*$ such that $\phi_2 = (m_2 + m_2 \circ \sigma)/2$. By Theorem V.1 of [**13**] there are two possibilities for h :

CASE 1.B.1: $m_2 \neq m_2 \circ \sigma$. Here

$$h = \frac{m_2 + m_2 \circ \sigma}{2}A^+ + \frac{m_2 - m_2 \circ \sigma}{2}A^- = \frac{m_2 A + (m_2 \circ \sigma)(A \circ \sigma)}{2}, \tag{46}$$

where $A^{\pm} \in \mathcal{A}^{\pm}(G)$, $A \in \mathcal{A}(G)$. It follows from Proposition 3.6 that the coefficients satisfy the matrix equation (9). The solution occurs in (C) of the list of Theorem 2.1.

<u>CASE 1.B.2:</u> $m_2 = m_2 \circ \sigma$. Here $h = m_2 q$ where q is a solution of the quadratic equation (18). We find that

$$\left\{ \begin{array}{c} f \\ g \\ h \end{array} \right\} = \left\{ \begin{array}{ccc} (\frac{2}{\delta})^2 & -(\frac{2}{\delta})^2 & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{\delta^2}{8} \\ 0 & 0 & \frac{\delta}{2} \end{array} \right\} \left\{ \begin{array}{c} \phi_1 \\ m_2 \\ m_2 q \end{array} \right\}, \tag{47}$$

so that the solution occurs in (D) of the list of Theorem 2.1.

<u>CASE 2:</u> $\alpha = \beta = \gamma = 0$ but $\delta \neq 0$. We find that

$$\left\{ \begin{array}{ccc} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & \delta & -1/\delta \end{array} \right\}^{-1} \Phi \left\{ \begin{array}{ccc} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & \delta & -1/\delta \end{array} \right\} = \left\{ \begin{array}{ccc} g & 0 & f - h/\delta \\ 0 & g + \delta h & 0 \\ 0 & 0 & g \end{array} \right\}. \tag{48}$$

From here we proceed exactly as in Case 1.B above. The solutions occur in (C) and (D) of the list of Theorem 2.1.

<u>CASE 3:</u> $\alpha = \beta = \gamma = \delta = 0$. We get

$$\left\{ \begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right\}^{-1} \Phi \left\{ \begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right\} = \left\{ \begin{array}{ccc} g & h & f \\ 0 & g & h \\ 0 & 0 & g \end{array} \right\}, \tag{49}$$

so that g is a K-spherical function. We know from Theorem III.1 of [**12**] (or Theorem 3 of [**2**]) that there exists a continuous homomorphism $m : G \to \mathbf{C}^*$ such that $g = (m + m \circ \sigma)/2$.

<u>CASE 3.A:</u> $m \neq m \circ \sigma$. By Proposition III.6 of [**12**] there exists a continuous homomorphism $M : G \to GL(3, \mathbf{C})$ of the form

$$M = \left\{ \begin{array}{ccc} m & \psi & \phi \\ 0 & m & \psi \\ 0 & 0 & m \end{array} \right\} \text{ such that } \left\{ \begin{array}{ccc} g & h & f \\ 0 & g & h \\ 0 & 0 & g \end{array} \right\} = \frac{1}{2}(M + M \circ \sigma). \tag{50}$$

The homomorphism property of M means that

$$\begin{aligned} \psi(x+y) &= m(x)\psi(y) + \psi(x)m(y), \\ \phi(x+y) &= \phi(x)m(y) + m(x)\phi(y) + \psi(x)\psi(y). \end{aligned} \tag{51}$$

Dividing by $m(x+y) = m(x)m(y)$ in the above identities we get that

$$\frac{\psi}{m}(x+y) = \frac{\psi}{m}(x) + \frac{\psi}{m}(y), \; x, y \in G, \tag{52}$$

so that $\psi = mA$ where $A \in \mathcal{A}(G)$, and

$$\frac{\phi}{m}(x+y) = \frac{\phi}{m}(x) + \frac{\phi}{m}(y) + A(x)A(y), \; x, y \in G. \tag{53}$$

A particular solution of this inhomogeneous equation is $\frac{\phi}{m} = A^2/2$ so its complete solution is $\frac{\phi}{m} = A_1 + A^2/2$ where $A_1 \in \mathcal{A}(G)$. Now

$$M = m \left\{ \begin{array}{ccc} 1 & A & A_1 + \frac{1}{2}A^2 \\ 0 & 1 & A \\ 0 & 0 & 1 \end{array} \right\}, \tag{54}$$

from which we find that

$$\left\{\begin{array}{c} f \\ g \\ h \end{array}\right\} = \left\{\begin{array}{cccc} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right\} \left\{\begin{array}{c} (mA_1 + (m\circ\sigma)(A_1\circ\sigma))/2 \\ (m + m\circ\sigma)/2 \\ (mA + (m\circ\sigma)(A\circ\sigma))/2 \\ (mA^2 + (m\circ\sigma)(A^2\circ\sigma))/2 \end{array}\right\}. \tag{55}$$

The solution occurs in (E) in the list of Theorem 2.1.

CASE 3.B: $m = m\circ\sigma$. In this case $g = m = m\circ\sigma$. The original functional equation (22) and the one for h from Lemma 3.3 are in this case:

$$\frac{f(x+y)+f(x+\sigma y)}{2} = f(x)m(y) + m(x)f(y) + h(x)h(y),\ x,y\in G, \tag{56}$$

$$\frac{h(x+y)+h(x+\sigma y)}{2} = h(x)m(y) + m(x)h(y),\ x,\in G. \tag{57}$$

We find that

$$\left\{\begin{array}{c} f \\ g \\ h \end{array}\right\} = \left\{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right\} \left\{\begin{array}{c} mF \\ m \\ mq \end{array}\right\}, \tag{58}$$

where the functions $F := f/m$ and $q := h/m$ satisfy the equations

$$\frac{F(x+y)+F(x+\sigma y)}{2} = F(x) + F(y) + q(x)q(y),\ x,y\in G, \tag{59}$$

$$\frac{q(x+y)+q(x+\sigma y)}{2} = q(x) + q(y),\ x,y\in G. \tag{60}$$

Proposition 3.2 shows that the solution occurs in (F) of the list Theorem 2.1.

CASE 4: $\beta \neq 0$. For any $z\in \mathbf{C}$ we put

$$G := g - \frac{z^2}{2}f - zh \ \text{ and } H := h + zf. \tag{61}$$

Note that

$$\left\{\begin{array}{ccc} 1 & 0 & 0 \\ \frac{-z^2}{2} & 1 & -z \\ z & 0 & 1 \end{array}\right\}^{-1} = \left\{\begin{array}{ccc} 1 & 0 & 0 \\ \frac{-z^2}{2} & 1 & z \\ -z & 0 & 1 \end{array}\right\}. \tag{62}$$

Now f, G and H are linearly independent because f, g and h are so. Brute force calculations show that

$$\int_K f(xk\cdot y)dk = f(x)G(y) + G(x)f(y) + H(x)H(y), \tag{63}$$

$$\int_K G(xk\cdot y)dk - G(x)G(y) = \tag{64}$$
$$Af(x)f(y) + B[f(x)H(y) + H(x)f(y)] + CH(x)H(y),$$

$$\int_K H(xk\cdot y)dk - G(x)H(y) - H(x)G(y) = \tag{65}$$
$$Bf(x)f(y) + C[f(x)H(y) + H(x)f(y)] + DH(x)H(y),$$

where

$$\begin{aligned} A &= -\frac{3}{4}z^4 - \delta z^3 + 3\gamma z^2 - 3\beta z + \alpha, && (66) \\ B &= z^3 + \delta z^2 - 2\gamma z + \beta, && (67) \\ C &= -\frac{3}{2}z^2 - \delta z + \gamma, && (68) \\ D &= \delta + 3z. && (69) \end{aligned}$$

We note that $C^2 = A + BD$ corresponding to the earlier identity $\gamma^2 = \alpha + \beta\delta$. Choosing $z \in \mathbf{C}$ such that $B = 0$ we can apply the earlier results to the new set of functions $\{f, G, H\}$ replacing $\{\alpha, \beta, \gamma, \delta\}$ by $\{A, B, C, D\}$. Thus

$$\left\{ \begin{array}{c} f \\ G \\ H \end{array} \right\} = \left\{ \begin{array}{ccc} a_1 & \dots & a_n \\ b_1 & \dots & b_n \\ c_1 & \dots & c_n \end{array} \right\} \left\{ \begin{array}{c} \psi_1 \\ \vdots \\ \psi_n \end{array} \right\}, \tag{70}$$

corresponding to the various cases above. Now

$$\left\{ \begin{array}{c} f \\ g \\ h \end{array} \right\} = \left\{ \begin{array}{ccc} 1 & 0 & 0 \\ \frac{-z^2}{2} & 1 & z \\ -z & 0 & 1 \end{array} \right\} \left\{ \begin{array}{c} f \\ G \\ H \end{array} \right\} = \left\{ \begin{array}{ccc} a_1 & \cdots & a_n \\ b_1(z) & \cdots & b_n(z) \\ c_1(z) & \cdots & c_n(z) \end{array} \right\} \left\{ \begin{array}{c} \psi_1 \\ \vdots \\ \psi_n \end{array} \right\}, \tag{71}$$

where $b_i(z) := b_i - \frac{1}{2}z^2 a_i + z c_i$ and $c_i(z) := c_i - z a_i$ for $z \in \mathbf{C}$ and $i = 1, \dots, n$. We are through by the following matrix identity:

$$\left\{ \begin{array}{ccc} a_1 & b_1(z) & c_1(z) \\ \vdots & \vdots & \vdots \\ a_n & b_n(z) & c_n(z) \end{array} \right\} \left\{ \begin{array}{ccc} b_1(z) & \cdots & b_n(z) \\ a_1 & \cdots & a_n \\ c_1(z) & \cdots & c_n(z) \end{array} \right\} = \tag{72}$$

$$\left\{ \begin{array}{ccc} a_1 & b_1 & c_1 \\ \vdots & \vdots & \vdots \\ a_n & b_n & c_n \end{array} \right\} \left\{ \begin{array}{ccc} b_1 & \cdots & b_n \\ a_1 & \cdots & a_n \\ c_1 & \cdots & c_n \end{array} \right\}.$$

Indeed, in all the cases (B)-(F) of Theorem 2.1 the matrix equation contains only the a_i entries on the right hand side and they are independent of z.

5. The case of linear dependence

This section deals with the remaining case of f, g and h linearly dependent. We divide it into three subcases (A), (B) and (C).

(A) f and h linearly independent, so that $g = \lambda f + \mu h$ for some $\lambda, \mu \in \mathbf{C}$. Substituting this expression for g into the functional equation (1) and introducing $H := h + \mu f$ instead of h we get

$$\int_K f(xk \cdot y)dk = (2\lambda - \mu^2)f(x)f(y) + H(x)H(y), \; x, y \in G. \tag{73}$$

If $2\lambda = \mu^2$ then taking $y = e$ in (73) we find that $f = H(e)H = H(e)h + H(e)\mu f$, contradicting that f and h are linearly independent. So $2\lambda - \mu^2 \neq 0$. Letting $\rho \in \mathbf{C}$ be a square root of $2\lambda - \mu^2$ the equation (73) becomes

$$\int_K F(xk \cdot y) = F(x)F(y) + G(x)G(y), \; x, y \in G, \tag{74}$$

where $F := \rho^2 f$ and $G := \rho H$. The solutions of (74) are written down as Theorem V.5 of [**13**] for $K = \mathbf{Z}_2$ with n=1. The theorem states that there are only the following possibilities (a)-(e):

(a) $F = G = 0$. This implies here that $f = 0$. However that possibility must be excluded since f and h are assumed to be linearly independent.

(b) There exists a continuous homomorphism $m : G \to \mathbf{C}^*$ and a $c \in \mathbf{C} \setminus \{\pm i\}$ such that

$$\rho H = \frac{c}{1+c^2}\frac{m + m\circ\sigma}{2} \text{ and } \rho^2 f = \frac{1}{1+c^2}\frac{m + m\circ\sigma}{2}. \tag{75}$$

It follows that $c\rho f = h + \mu f$. But f and h are assumed to be linearly independent so the possibility (b) must also be excluded.

(c) There exist continuous homomorphisms $m_1, m_2 : G \to \mathbf{C}^*$ and a $c \in \mathbf{C}\setminus\{\pm i\}$ such that

$$\rho H = \frac{c}{1+c^2}\left(\frac{m_1 + m_1\circ\sigma}{2} - \frac{m_2 + m_2\circ\sigma}{2}\right), \tag{76}$$

$$\rho^2 f = \frac{1}{1+c^2}\frac{m_1 + m_1\circ\sigma}{2} + \frac{c^2}{1+c^2}\frac{m_2 + m_2\circ\sigma}{2}. \tag{77}$$

Letting $\phi_1 := (m_1 + m_1\circ\sigma)/2$ and $\phi_2 := (m_2 + m_2\circ\sigma)/2$ we find that

$$\left\{\begin{array}{c} f \\ g \\ h \end{array}\right\} = \frac{1}{\rho^2}\frac{1}{1+c^2}\left\{\begin{array}{cc} 1 & c^2 \\ \lambda + \mu\rho c - \mu^2 & c(\lambda c - \mu\rho - c\mu^2) \\ \rho c - \mu & -c(\rho + \mu c) \end{array}\right\}\left\{\begin{array}{c} \phi_1 \\ \phi_2 \end{array}\right\}. \tag{78}$$

A calculation reveals that this fits into case (B) of Theorem 2.1 when we take

$$\left\{\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right\} = \frac{1}{\rho^2}\frac{1}{1+c^2}\left\{\begin{array}{ccc} 1 & c^2 & 0 \\ \lambda + \mu\rho c - \mu^2 & c(\lambda c - \mu\rho - c\mu^2) & 0 \\ \rho c - \mu & -c(\rho + \mu c) & 0 \end{array}\right\}. \tag{79}$$

(d) There exists a continuous homomorphism $m : G \to \mathbf{C}^*$ for which $m \neq m\circ\sigma$, $A^+ \in \mathcal{A}^+(G)$ and $A^- \in \mathcal{A}(G)^-$ such that

$$\rho H = \frac{m + m\circ\sigma}{2}A^+ + \frac{m - m\circ\sigma}{2}A^-, \tag{80}$$

$$\rho^2 f = \frac{m + m\circ\sigma}{2} \pm i\left[\frac{m + m\circ\sigma}{2}A^+ + \frac{m - m\circ\sigma}{2}A^-\right]. \tag{81}$$

With $A := \pm i\rho^{-2}(A^+ + A^-)$ we find that

$$\left\{\begin{array}{c} f \\ g \\ h \end{array}\right\} = \left\{\begin{array}{ccc} 0 & \rho^{-2} & 1 \\ 0 & (\rho^2 - \mu^2)\rho^{-2}/2 & (\rho \mp i\mu)^2/2 \\ 0 & -\mu\rho^{-2} & \mp i(\rho \mp i\mu) \end{array}\right\}\left\{\begin{array}{c} (m + m\circ\sigma)/2 \\ (m + m\circ\sigma)/2 \\ (mA + (mA)\circ\sigma)/2 \end{array}\right\}. \tag{82}$$

The solution fits into case (C) of Theorem 2.1.

(e) There exists a continuous homomorphism $m : G \to \mathbf{C}^*$ for which $m = m\circ\sigma$ and a solution $q \in C(G)$ of the quadratic equation (18) such that $\rho H = mq$ and $\rho^2 f = m \pm imq$. Here we find that

$$\left\{\begin{array}{c} f \\ g \\ h \end{array}\right\} = \left\{\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right\}\left\{\begin{array}{c} m \\ m \\ mq \end{array}\right\}, \tag{83}$$

where

$$\left\{\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right\} = \left\{\begin{array}{ccc} 0 & \rho^{-2} & 1 \\ 0 & \frac{1}{2}(\rho^2-\mu^2)\rho^{-2} & -\frac{1}{2}(\mp i\rho-\mu)^2 \\ 0 & -\mu\rho^{-2} & \mp i\rho-\mu \end{array}\right\}. \tag{84}$$

A calculation reveals that this fits into case (D) of Theorem 2.1.

(B) $f=0$. Here $h=0$ and $g\in C(G)$, which is the trivial case (A) of Theorem 2.1.

(C) $f\neq 0$ and h are linearly dependent, so that $h=\alpha f$ for some $\alpha\in\mathbf{C}$. Here the functional equation (5) reduces to

$$\frac{f(x+y)+f(x+\sigma y)}{2} = f(x)\left[g(y)+\frac{\alpha^2}{2}f(y)\right]+\left[g(x)+\frac{\alpha^2}{2}f(x)\right]f(y), x,y\in G, \tag{85}$$

which is once again a well known functional equation. The solutions of the equation (85) are written down as Theorem V.4 of [**13**] for $K=\mathbf{Z}_2$ with n=1. The theorem state that there are only the following possibilities (a)-(e):

(a) $f=0$. This possibility is excluded by assumption.

(b) There exists a continuous homomorphism $m: G\to\mathbf{C}^*$ and a constant $c\in\mathbf{C}$ such that $g+\alpha^2 f/2=(m+m\circ\sigma)/4$ and $f=c(m+m\circ\sigma)$.

Letting $\phi_1=\phi_2=\phi_3=(m+m\circ\sigma)/2$ we get

$$\left\{\begin{array}{c} f \\ g \\ h \end{array}\right\} = \left\{\begin{array}{ccc} 2c & 0 & 0 \\ \frac{1}{2}-\alpha^2 c & 0 & 0 \\ 2\alpha c & 0 & 0 \end{array}\right\}\left\{\begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \end{array}\right\}, \tag{86}$$

from which a small calculation reveals that we this is case (B) of Theorem 2.1.

(c) There exist continuous homomorphisms $m_1,m_2: G\to\mathbf{C}^*$ and a constant $c\in\mathbf{C}$ such that $g+\alpha^2 f/2=(m_1+m_1\circ\sigma+m_2+m_2\circ\sigma)/4$ and $f=c[m_1+m_1\circ\sigma-(m_2+m_2\circ\sigma)]$.

Letting $\phi_1:=(m_1+m_1\circ\sigma)/2$ and $\phi_2=\phi_3:=(m_2+m_2\circ\sigma)/2$ we find that

$$\left\{\begin{array}{c} f \\ g \\ h \end{array}\right\} = \left\{\begin{array}{ccc} 2c & -2c & 0 \\ \frac{1}{2}-\alpha^2 c & \frac{1}{2}+\alpha^2 c & 0 \\ 2\alpha c & -2\alpha c & 0 \end{array}\right\}\left\{\begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \end{array}\right\}, \tag{87}$$

from which a small calculation reveals that this is case (B) of Theorem 2.1.

(d) There exists a continuous homomorphism $m: G\to\mathbf{C}^*$ for which $m\neq m\circ\sigma$, $A^+\in\mathcal{A}^+(G)$ and $A^-\in\mathcal{A}^-(G)$ such that

$$g+\frac{1}{2}\alpha^2 f=\frac{m+m\circ\sigma}{2} \text{ and } f=\frac{m+m\circ\sigma}{2}A^+ + \frac{m-m\circ\sigma}{2}A^-. \tag{88}$$

We find with $A:=A^+ + A^-$ that

$$\left\{\begin{array}{c} f \\ g \\ h \end{array}\right\} = \left\{\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & -\alpha^2/2 \\ 0 & 0 & \alpha \end{array}\right\}\left\{\begin{array}{c} (m+m\circ\sigma)/2 \\ (m+m\circ\sigma)/2 \\ (mA+(mA)\circ\sigma)/2 \end{array}\right\}, \tag{89}$$

from which a small calculation reveals that this is case (C) of Theorem 2.1.

(e) There exists a continuous homomorphism $m: G\to\mathbf{C}^*$ for which $m=m\circ\sigma$, and a solution $q\in C(G)$ of the quadratic equation (18) such that $g+\alpha^2 f/2=m$

and $f = mq$. We find that

$$\left\{\begin{array}{c} f \\ g \\ h \end{array}\right\} = \left\{\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & -\alpha^2/2 \\ 0 & 0 & \alpha \end{array}\right\} \left\{\begin{array}{c} m \\ m \\ mq \end{array}\right\}, \tag{90}$$

from which a small calculation reveals that this is case (D) of Theorem 2.1.

6. The signed equation

PROPOSITION 6.1. *$f, g, h \in C(G)$ is a solution to*

$$\int_K f(x + k \cdot y)\overline{\chi(k)}dk = f(x)g(y) + g(x)f(y) + h(x)h(y),\ x, y \in G, \tag{91}$$

where χ is a continuous homomorphism from K into the circle group $\{z \in \mathbb{C} : |z| = 1\}$ and $\chi \neq 1$, if and only if one of the following three conditions holds:

- **a):** *$f = h = 0$, and $g \in C(G)$.*
- **b):** *$f = \nu$, $g = -\alpha^2\nu/2$, and $h = \alpha\nu$ where $\alpha \in \mathbb{C}$ and $\nu \in C(G)$ is a solution to*

$$\int_K \nu(x + k \cdot y)\overline{\chi(k)}dk = 0,\ x, y \in G. \tag{92}$$

- **c):** *$g = -\mu^2 f/2 + \mu H$, $h = H - \mu f$ where $\mu \in \mathbb{C}$, and $f, H \in C(G)$ is a solution to*

$$\int_K f(x + k \cdot y)\overline{\chi(k)}dk = H(x)H(y),\ x, y \in G. \tag{93}$$

PROOF. That the conditions are sufficient is verified by trivial calculations. Now suppose that the triplet f, g, and h is a solution. Suppose $f(k \cdot x) = \chi(k)f(x)$ for all $x \in G$ and for all $k \in K$. Then

$$\begin{aligned} \int_K f(y + k \cdot x)dk &= \int_K f(k^{-1} \cdot x + y)dk = \int_K f(k^{-1} \cdot (x + k \cdot y))dk \\ &= \int_K f(x + k \cdot y)\chi(k^{-1})dk = \int_K f(x + k \cdot y)\overline{\chi(k)}dk \\ &= f(y)g(x) + g(y)f(x) + h(y)h(x) \\ &= \int_K f(y + k \cdot x)\overline{\chi(k)}dk, \end{aligned} \tag{94}$$

where we have used that K is unimodular since it is compact (see Theorem 15.13 and Theorem 15.14 in [**6**]). Taking $x = e$ we get

$$f(y) = \int_K f(y + k \cdot e)dk = \int_K f(y + k \cdot e)\overline{\chi(k)}dk = f(y)\int_K \overline{\chi(k)}dk = 0, \tag{95}$$

since $\overline{\chi} \neq 1$ (see Lemma 23.19 in [**6**]). So if $f \neq 0$ we can not have $f(k \cdot x) = \chi(k)f(x)$ for all $x \in G$ and for all $k \in K$. We will use this observation to exclude a number of cases and thereby prove that the only possible solutions are those given by the proposition.

Suppose that f, g, and h are linearly independent. it follows immediately from Theorem II.2 in [**11**] that $f(k \cdot x) = \chi(k)f(x)$, $\forall x \in G$, $\forall k \in K$. But this is impossible since $f \neq 0$. So f, g, and h have to be linearly dependent.

Case A: f and h are linearly independent. Then $g = \lambda f + \mu h$ for some $\lambda, \mu \in \mathbb{C}$. Define $H = h + \mu f$, then we have

$$\int_K f(x + k \cdot y)\overline{\chi(k)}dk = (2\lambda - \mu^2)f(x)f(y) + H(x)H(y), \ x, y \in G. \tag{96}$$

Take $\rho \in \mathbb{C}$ such that $\rho^2 = 2\lambda - \mu^2$. Suppose $\rho \neq 0$ and define $F = \rho f$. We have

$$\int_K f(x + k \cdot y)\overline{\chi(k)}dk = F(x)F(y) + H(x)H(y), \ x, y \in G. \tag{97}$$

$F = \rho f$ and $H = h + \mu f$ are linearly independent since f and h are linearly independent. Again it follows from Theorem II.2 in [**11**] that $F(k \cdot x) = \chi(k)F(x)$ and hence $f(k \cdot x) = \chi(k)f(x), \ \forall x \in G, \ \forall k \in K$. So $f = 0$, but this is impossible so $\rho = 0$. Hence

$$\int_K f(x + k \cdot y)\overline{\chi(k)}dk = H(x)H(y), \ x, y \in G. \tag{98}$$

This is case c in Proposition 6.1.

Case B: f and h are linearly dependent.

Case B1: $f \equiv 0$, then $h \equiv 0$ and $g \in C(G)$ can be arbitrary. This is case a in Proposition 6.1.

Case B2: $f \not\equiv 0$, so $h = \alpha f$ and we have

$$\int_K f(x + k \cdot y)\overline{\chi(k)}dk = f(x)[\frac{\alpha^2}{2}f + g](y) + [\frac{\alpha^2}{2}f + g](x)f(y), \ x, y \in G. \tag{99}$$

Suppose f and $\alpha^2 f/2 + g$ are linearly independent then again it follows from Theorem II.2 in [**11**] that $f(k \cdot x) = \chi(k)f(x), \ \forall x \in G, \ \forall k \in K$. So $f \equiv 0$. But this is impossible so f and $\alpha^2 f/2 + g$ have to be linearly dependent. So $\alpha^2 f/2 + g = \lambda^2 f/2$ for some $\lambda \in \mathbb{C}$. Hence we have

$$\int_K f(x + k \cdot y)\overline{\chi(k)}dk = (\lambda f)(x)(\lambda f)(y), \ x, y \in G. \tag{100}$$

Suppose $\lambda \neq 0$ then, using Theorem II.2 in [**11**] it follows that $f(k \cdot x) = \chi(k)f(x), \ \forall x \in G, \ \forall k \in K$ so $f = 0$ and this is impossible. So $\lambda = 0$ and we have

$$\int_K f(x + k \cdot y)\overline{\chi(k)}dk = 0, \ x, y \in G. \tag{101}$$

This is case b in Proposition 6.1 This proves the proposition. □

Proposition 6.2. *$f, g, h \in C(G)$ is a solution to*

$$\frac{f(x + y) - f(x + \sigma y)}{2} = f(x)g(y) + g(x)f(y) + h(x)h(y), \ x, y \in G, \tag{102}$$

if and only if one of the following conditions holds:

- **a):** *$f = h = 0$ and $g \in C(G)$.*
- **b):** *$f = \nu$, $g = -\alpha^2\nu/2$, $h = \alpha\nu$, where $\alpha \in \mathbb{C}$ and $\nu(x + y) = \nu(x + \sigma y), \ \forall x, y \in G$.*
- **c):** *$g = -\mu^2 f/2 + \mu H$, $h = H - \mu f$, where $\mu \in \mathbb{C}$, and where $f = c^2(m + m \circ \sigma)/2 + \nu$ and $H = c(m - m \circ \sigma)/2$ where $c \in \mathbb{C}$ and $\nu(x + y) = \nu(x + \sigma y), \ \forall x, y \in G$ and $m : G \to \mathbb{C}^*$ is a continuous homomorphism such that $m \neq m \circ \sigma$.*

d): *$g = -\mu^2 f/2 + \mu H$, $h = H - \mu f$, where $\mu \in \mathbb{C}$, and $f = m(A^-)^2/2 + \nu$ and $H = mA^-$, where $A \in \mathcal{A}^-(G)$, and $m : G \to C^*$ is a continuous homomorphism such that $m = m \circ \sigma$, and $\nu(x+y) = \nu(x+\sigma y)$, $\forall x, y \in G$.*

Proof. To check that anything on the list is a solution is trivial. Now suppose that f, g, and h is a solution. We let $K = \mathbb{Z}_2$ act in the usual way on G. We define $\chi : K \mapsto \mathbb{C}^*$ by $\chi(1) = 1$ and $\chi(-1) = -1$. Then we have

$$\begin{aligned} \int_K f(x + k \cdot y)\overline{\chi(k)}dk &= \frac{f(x+y) - f(x+\sigma y)}{2} \\ &= f(x)g(y) + g(x)f(y) + h(x)h(y), \ x, y \in G. \quad (103) \end{aligned}$$

This equation was treated in Proposition 6.1.

If we are in case a in Proposition 6.1 then we are in case a in Proposition 6.2. If we are in case b in Proposition 6.1 then we are in case b in Proposition 6.2.

If we are in case c in Proposition 6.1. Then for some $\mu \in \mathbb{C}$ we have $g = -\mu^2 f/2 + \mu H$ and $h = H - \mu f$ where $f, H \in C(G)$ is a solution to

$$\frac{f(x+y) - f(x+\sigma y)}{2} = H(x)H(y), \ x, y \in G. \quad (104)$$

The equation (104) has been solved in Corollary III.5 in [**12**].

If we are in case 1 or 2 in Corollary III.5 in [**12**] then $H = 0$ and $f(x+y) = f(x+\sigma y)$, $x, y \in G$. This is case b of Proposition 6.2.

If we are in case 3 in Corollary III.5 in [**12**] then there exist a continuous homomorphism $m : G \to \mathbb{C}^*$ for which $m \neq m \circ \sigma$, $c \in \mathbb{C} \setminus \{0\}$, $c_1, c_2 \in \mathbb{C}$, and $\nu \in C(G)$ for which $\nu(x+y) = \nu(x+\sigma y)$, $x, y \in G$ such that

$$c_1 \frac{m + m \circ \sigma}{2} + c_2 \frac{m - m \circ \sigma}{2} = H = c \frac{m - m \circ \sigma}{2}, \quad (105)$$

and

$$f = cc_2 \frac{m + m \circ \sigma}{2} + cc_1 \frac{m - m \circ \sigma}{2} + \nu. \quad (106)$$

From (105) it follows that $c_1 = H(e) = 0$, and since $m \neq m \circ \sigma$ it follows from (105) that $c_2 = c$, and we are in case c of Proposition 6.2.

If we are in case 4 of Corollary III.5 in [**12**] then there exist a continuous homomorphism $m : G \to \mathbb{C}^*$ for which $m = m \circ \sigma$, $c, c_1 \in \mathbb{C}$, $A^- \in \mathcal{A}(G)$, and $\nu \in C(G)$ for which $\nu(x+y) = \nu(x+\sigma y)$, $x, y \in G$, such that

$$cm + c_1 mA^- = H = mA^-, \quad (107)$$

and

$$f = cmA^- + \frac{1}{2} c_1 m (A^-)^2 + \nu. \quad (108)$$

From equation (107) it follows that $c = H(e) = mA^-(e) = 0$, if $A^- \equiv 0$ we can take c_1 to be 1, if $A^- \neq 0$ then c_1 has to be 1. We are in case d in Proposition 6.2. □

Remark 6.3. Note that if G is 2-divisible and $\sigma = -I$ then the condition $\nu(x+y) = \nu(x+\sigma y)$, $x, y \in G$ implies that ν is constant. 2-divisible means that for any $x \in G$ there is a $y \in G$ such that $y^2 = x$.

Bibliography

[1] Aczél, J. andDhombres, J., *Functional equations in several variables.* Cambridge University Press. Cambridge, New York, New Rochelle, Melbourne, Sydney 1989.

[2] Baker, J. A., *The stability of the cosine equation.* Proc. Amer. Math. Soc. **80** (1980), 411-416.

[3] Chung, J. K., Ebanks, B. R., Ng, C. T. and Sahoo, P. K., *On a quadratic trigonometric functional equation and some applications.* Trans. Amer. Math. Soc. **347** (1995), 1131-1161.

[4] Chung, J. K., Kannappan, PL. and Ng, C. T., *A Generalization of the Cosine-Sine Functional Equation on Groups.* Linear Algebra Appl. **66** (1985), 259-277.

[5] Chung, J. K., Kanappan, PL. and Ng, C. T., *On two trigonometric functional equations.* Mathematics Reports Toyama University **11** (1988), 153-165.

[6] Hewitt, E. and Ross, K. A., *Abstract Harmonic Analysis I.* Springer-Verlag, Berlin-Göttingen-Heidelberg 1963.

[7] Horinuchi, S. and Kannappan, PL., *On the System of Functional Equations* $f(x+y) = f(x)+f(y)$ *and* $f(xy) = p(x)f(y)+q(y)f(x)$ Aequationes Math. **6** (1973), 195-201.

[8] Kannappan, PL., *The functional equation* $f(xy)+f(xy^{-1}) = 2f(x)f(y)$ *for groups.* Proc. Amer. Math. Soc. **19** (1968), 69-74.

[9] Poulsen, Thomas A. and Stetkær, H., *On the addition and subtraction formulas.* Manuscript of August 15, 1995. 15 pp.

[10] Rukhin, A. L., *The solution of the functional equation of d'Alembert's type for commutative groups.* Internat. J. Math. Math. Sci. **5** (1982), 315-335.

[11] Stetkær, H., *Functional Equations and Spherical Functions.* Preprint Series 1994 No **18**, Matematisk Institut, Aarhus University, Denmark. pp. 1-28.

[12] Stetkær, H., *Functional equations on abelian groups with involution.* Aequationes Math. **54** (1997), 144-172.

[13] Stetkær, H., *Trigonometric functional equations of rectangular type.* Aequationes Math. **56** (1998) 1-19.

[14] Swiatak, H., *On two equations connected with the equation* $\phi(x+y)+\phi(x-y) = 2\phi(x)+2\phi(y)$. Aequationes Math. **5** (1970), 3-9.

Peter de Place Friis and Henrik Stetkær
Department of Mathematics
Aarhus University
Ny Munkegade
DK-8000 Aarhus C
Denmark
e-mail: pfriis@imf.au.dk and stetkaer@imf.au.dk

CHAPTER 4

The Sine and Cosine Addition Functional Equations on Nonabelian Groups

Peter de Place Friis

1. Introduction

Let K be a compact transformation group acting on a topological group G by homomorphisms. Suppose $f, g_1, \ldots, g_n, h_1, \ldots, h_n \in C(G)$ is a solution to the following functional equation

$$\int_K f(xk \cdot y)dk = \sum_{i=1}^{n} g_i(x)h_i(y),\ x, y \in G, \tag{1}$$

then $g_1, \ldots, g_n, h_1, \ldots, h_n$ is a solution to

$$\sum_{i=1}^{n} \int_K g_i(xk \cdot y)dk h_i(z) = \sum_{i=1}^{n} g_i(x) \int_K h_i(yk \cdot z)dk,\ x, y, z \in G. \tag{2}$$

See Lemma II.2 in [**14**]. This necessary condition is very useful as can be seen from [**14**] for example. Usually functional equations on groups are solved by ad hoc methods. There is very little theory to our disposal. Therefore any necessary condition is useful. When G is not abelian group inversion is not a homomorphism but an antihomomorphism. So the standard action of $\mathbb{Z}_2$ on a nonabelian group does not fit into the general set-up in [**14**]. Hence we can not use the necessary condition (2) on the equations

$$f(xy) + f(xy^{-1}) = 2f(x)g(y) + 2g(x)f(y),\ x, y \in G, \tag{3}$$

and

$$f(xy) + f(xy^{-1}) = 2f(x)f(y) + 2g(x)g(y),\ x, y \in G. \tag{4}$$

So it would be of interest to find a necessary condition in the case where K acts by both homomorphisms and antihomomorphism. Such a condition is given in the section General Theory and we deduce some additional properties in this case. Then we apply the necessary condition to the functional equations

$$\int_K f(xk \cdot y)dk = f(x)g(y) + g(x)f(y),\ x, y \in G, \tag{5}$$

and

$$\int_K f(xk \cdot y)dk = f(x)f(y) + g(x)g(y),\ x, y \in G \tag{6}$$

and we show how the problem of solving these equations can be reduced to the problem of solving some simpler equations. This is an extension of result in [**15**]. The conclusion is the same but the assumptions are weaker. G need not be abelian and K is not assumed to act by homomorphisms only, but by homomorphisms and possibly also antihomomorphisms. That the conclusion is unaltered is perhaps somewhat surprising.

The main new results of the present paper is that we:

(1) Find a necessary condition that $g_1, \dots, g_n, h_1, \dots, h_n \in C(G)$ must satisfy in order that a solution to

$$\int_K f(xk \cdot y)dk = \sum_{i=1}^{n} g_i(x)h_i(y), \; x, y \in G, \tag{7}$$

can exist in the case where K acts by homomorphisms and possibly also antihomomorphisms (Proposition 2.4).

(2) Reduce the problem of solving the equations

$$\int_K f(xk \cdot y)dk = f(x)g(y) + g(x)f(y), \; x, y \in G, \tag{8}$$

and

$$\int_K f(xk \cdot y)dk = f(x)f(y) + g(x)g(y), \; x, y \in G, \tag{9}$$

to the problem of solving simpler equations when G is not necessarily abelian and K acts by homomorphisms and possibly also antihomomorphisms (Theorem 4.3 and Theorem 5.3).

2. General Theory

We recall the concept of a topological transformation group as defined in, e.g., [**9**] Ch. II §3.

Definition 2.1. Let K be a group with neutral element e, and let X be a nonempty set. We say that K acts on X if there is given an action of K on X, i.e. a map $(k, x) \mapsto k \cdot x$ of $K \times X$ into X such that

(1) $(k_1 k_2) \cdot x = k_1 \cdot (k_2 \cdot x)$ for $k_1, k_2 \in K$ and $x \in X$.
(2) $e \cdot x = x$ for $x \in X$.

If X is a group we say that the action is by automorphisms if the bijection $x \mapsto k \cdot x$ of X onto X is an automorphism for each $k \in K$. Let a topological group K act on a topological space X. We say that K is a topological transformation group of X if the map $(k, x) \mapsto k \cdot x$ is a continuous mapping of the product space $K \times X$ onto X.

We shall throughout the paper work in the following set-up.

General Set-up and Notation K is a compact Hausdorff group. We denote by dk its normalized Haar measure. Furthermore K is also a topological transformation group acting on G, where G is a topological group. The action of $k \in K$ on $x \in G$ is denoted by $k \cdot x$. Furthermore we assume that for every $k \in K$, $k \cdot = \phi_k : G \to G$ is either a homomorphism or an antihomomorphism. The neutral element of G is denoted e. As usual $C(G)$ is the algebra of complex valued continuous functions on G. A function f on G is said to be K-invariant if $f(k \cdot x) = f(x)$ for all $k \in K$ and $x \in G$. For any function $f : G \to \mathbb{C}$ the function $\check{f} : G \to \mathbb{C}$ is defined by $\check{f}(x) = f(x^{-1})$, $x \in G$. We let $M_2(\mathbb{C})$ denote the algebra of complex 2×2 matrices. The group $\mathbb{Z}_2$ will be viewed as the multiplicative group $\mathbb{Z}_2 = \{\pm 1, \cdot\}$. It acts on any group G by $(+1) \cdot x = x$ and $(-1) \cdot x = x^{-1}$. We will call this action for the standard action by $\mathbb{Z}_2$ on G. Finally we let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ be the multiplicative group of the nonzero complex numbers.

DEFINITION 2.2. We will call a non-zero function $\phi \in C(G)$ a K-spherical function if it satisfies d'Alembert's functional equation

$$\int_K \phi(xk \cdot y)dk = \phi(x)\phi(y), \;\; x, y \in G. \tag{10}$$

REMARK 2.3. Note that this is an extension of the usual definition, as found in [**2**] Definition 8.1 for example, in that K is not assumed to act by homomorphisms only but by homomorphisms and possibly also antihomomorphisms. If K acts on G by homomorphisms there is a connection with the classical theory of spherical functions. We define the group $\tilde{G} = G \times_s K$ as a semidirect product of G and K where the topology is the product topology and the group operation is given by $(g_1, k_1)(g_2, k_2) = (g_1 k_1 \cdot g_2, k_1 k_2)$. $K_s = \{e_G\} \times K$ is a closed compact subgroup of $\tilde{G}$. Consider $\tilde{f} : \tilde{G} \to \mathbb{C}$ then $\tilde{f}((g, k)(e_G, k_0)) = \tilde{f}(g, kk_0)$ so a K_s-rightinvariant function on $\tilde{G}$ is a function which only depends on the first variable. So we can identify functions on $\tilde{G}/K_s$ and functions on G. Furthermore $\tilde{f}((e_G, k_0)(g, e_K)) = \tilde{f}(k_0 \cdot g, k_0)$, so K_s-biinvariance on $\tilde{G}$ corresponds to K-invariance on G. A K_s-spherical function on the group $\tilde{G}$ continuous K_s-biinvariant function with satisfies

$$\int_{K_s} f(xky)dk = f(x)f(y), \;\; x, y \in \tilde{G}. \tag{11}$$

We can-as we saw above-think of $\tilde{f}$ as defined on $\tilde{G}/K_s$ that is on G as a topological K space. So if K acts on G with homomorphisms what we call K-spherical functions correspond to K_s-spherical functions on $\tilde{G}$. For more information on this point of view see [**13**]. However, when K acts by both homomophisms and antihomomorphisms we can not form the semidirect product $G \times_s K$ so in this case there is no obvious connection with the classical theory of spherical functions. The standard Z_2 action serves as motivation for also studying the case where K acts both homomorphisms and antihomomorphisms.

PROPOSITION 2.4. *If $f, g_1, \ldots, g_n, h_1, \ldots, h_n \in C(G)$ is a solution to*

$$\int_K f(xk \cdot y)dk = \sum_{i=1}^{n} g_i(x)h_i(y), \;\; x, y \in G, \tag{12}$$

then

$$\begin{aligned} &\sum_{i=1}^{n} g_i(x) \int_K [h_i(yk \cdot z) + h_i((k \cdot z)y)]dk \\ &= \sum_{i=1}^{n} \left[\int_K g_i(xk \cdot y)dk h_i(z) + \int_K g_i(xk \cdot z)dk h_i(y) \right], \;\; x, y, z \in G. \end{aligned} \tag{13}$$

PROOF.

$$
\begin{aligned}
&\sum_{i=1}^n g_i(x)\int_K [h_i(yk_1\cdot z)+h_i((k_1\cdot z)y)]dk_1 \qquad (14)\\
&= \int_K\int_K [f(xk\cdot(yk_1\cdot z))+f(xk\cdot((k_1\cdot z)y))]dkdk_1\\
&= \int_K\int_K [f(x(k\cdot y)(kk_1\cdot z))+f(x(kk_1\cdot z)(k\cdot y))]dkdk_1\\
&= \int_K\int_K [f(x(k\cdot y)(kk_1\cdot z))+f(x(kk_1\cdot z)(k\cdot y))]dk_1dk\\
&= \int_K\int_K [f(x(k\cdot y)(k_1\cdot z))+f(x(k_1\cdot z)(k\cdot y))]dk_1dk\\
&= \int_K\sum_{i=1}^n g_i(xk\cdot y)h_i(z)dk+\int_K\int_K f(x(k_1\cdot z)(k\cdot y))]dkdk_1\\
&= \int_K\sum_{i=1}^n g_i(xk\cdot y)h_i(z)dk+\int_K\sum_{i=1}^n g_i(xk_1\cdot z)h_i(y)dk_1,\ x,y,z\in G.
\end{aligned}
$$

□

REMARK 2.5. Suppose that K acts by homomorphisms. Furthermore assume that $f, g_1, \dots, g_n, h_1, \dots, h_n \in C(G)$ is a solution to

$$\int_K f(xk\cdot y)dk=\sum_{i=1}^n g_i(x)h_i(y),\ x,y\in G. \qquad (15)$$

As we will prove in Proposition 2.8 this implies that

$$\sum_{i=1}^n g_i(x)h_i(k\cdot y)=\sum_{i=1}^n g_i(x)h_i(y),\ x,y\in G,\ k\in K. \qquad (16)$$

So using the fact that K is compact and hence unimodular (see Theorem 15.13 and Theorem 15.14 in [**10**]) we find that

$$
\begin{aligned}
&\sum_{i=1}^n g_i(x)\int_K h_i((k\cdot z)y)dk=\int_K\sum_{i=1}^n g_i(x)h_i(k\cdot(zk^{-1}\cdot y))dk\\
&=\int_K\sum_{i=1}^n g_i(x)h_i(zk^{-1}\cdot y)dk=\sum_{i=1}^n g_i(x)\int_K h_i(zk\cdot y)dk,\ x,y,z\in G.
\end{aligned} \qquad (17)
$$

Stetkær's necessary condition (2) gives

$$\sum_{i=1}^n\int_K g_i(xk\cdot y)dkh_i(z)=\sum_{i=1}^n g_i(x)\int_K h_i(yk\cdot z)dk,\ x,y,z\in G. \qquad (18)$$

Exchanging y and z gives

$$\sum_{i=1}^n\int_K g_i(xk\cdot z)dkh_i(y)=\sum_{i=1}^n g_i(x)\int_K h_i(zk\cdot y)dk,\ x,y,z\in G. \qquad (19)$$

Adding these two equations using (17) yields

$$\begin{aligned}&\sum_{i=1}^{n} g_i(x)\int_K [h_i(yk\cdot z)+h_i((k\cdot z)y)]dk\\&\quad=\sum_{i=1}^{n}\int_K g_i(xk\cdot y)dkh_i(z)+\sum_{i=1}^{n}\int_K g_i(xk\cdot z)dkh_i(y),\ x,y,z\in G.\end{aligned}\tag{20}$$

This is precisely our necessary condition (13). So while our necessary condition can be applied in the more general case where K acts by both homomorphism and antihomomorphisms, it does not reduce to Stetkær's condition in the case where K acts by homomorphisms only. So it is in some sense weaker. Also, the fact that our necessary condition (13) is symmetric in y and z would make it likely that it is a bit harder to use it to obtain results than using Stetkær's necessary condition; the use of our necessary condition on the Sine and the Cosine addition equations seems to confirm this. However, it is applicable in the more general situation where K acts by both homomorphisms and antihomomorphisms so some additional complications are to be expected.

Corollary 2.6. *If $f, g_1, \ldots, g_n, h_1, \ldots, h_n \in C(G)$ is a solution to*

$$\int_K f((k\cdot y)x)dk=\sum_{i=1}^{n} g_i(x)h_i(y),\ x,y\in G,\tag{21}$$

then

$$\begin{aligned}&\sum_{i=1}^{n} g_i(x)\int_K [h_i(yk\cdot z)+h_i((k\cdot z)y)]dk\\&\quad=\sum_{i=1}^{n}\left[\int_K g_i((k\cdot y)x)dkh_i(z)+\int_K g_i((k\cdot z)x)dkh_i(y)\right],\ x,y,z\in G.\end{aligned}\tag{22}$$

Proof. Since group inversion is continuous it follows that $\check{f}, \check{g}_1, \ldots, \check{g}_n, \check{h}_1, \ldots, \check{h}_n \in C(G)$. Since for any $k \in K$ we have that $\phi_k = k\cdot : G \to G$ is a homomorphism or an antihomomorphism, we have that $\phi_k(x^{-1}) = \phi_k(x)^{-1}$, $x \in G$. So we get

$$\begin{aligned}\int_K \check{f}(xk\cdot y)dk &= \int_K f((xk\cdot y)^{-1})dk=\int_K f((k\cdot y^{-1})x^{-1})dk\\&=\sum_{i=1}^{n} g_i(x^{-1})h_i(y^{-1})=\sum_{i=1}^{n}\check{g}_i(x)\check{h}_i(y),\ x,y\in G.\end{aligned}\tag{23}$$

Using (13) gives

$$\begin{aligned}&\sum_{i=1}^{n} g_i(x^{-1})[\int_K h_i((k\cdot z^{-1})y^{-1})dk+\int_K h_i(y^{-1}k\cdot z^{-1})dk]\\&=\sum_{i=1}^{n}\check{g}_i(x)[\int_K \check{h}_i(yk\cdot z)dk+\int_K \check{h}_i((k\cdot z)y)dk]\\&=\sum_{i=1}^{n}[\int_K \check{g}_i(xk\cdot y)dk\check{h}_i(z)+\int_K \check{g}_i(xk\cdot z)dk\check{h}_i(y)]\\&=\sum_{i=1}^{n}[\int_K g_i((k\cdot y^{-1})x^{-1})dkh_i(z^{-1})+\int_K g_i((k\cdot z^{-1})x^{-1})dkh_i(y^{-1}),\ x,y,z\in G.\end{aligned}\tag{24}$$

□

REMARK 2.7. Let $f, g_1, \dots, g_n, h_1, \dots h_n \in C(G)$. Note that we have proved that if

$$\int_K f((k\cdot y)x)dk = \sum_{i=1}^{n} g_i(x)h_i(y),\ x, y \in G, \tag{25}$$

then

$$\int_K \check{f}(xk\cdot y)dk = \sum_{i=1}^{n} \check{g}_i(x)\check{h}_i(y),\ x, y \in G. \tag{26}$$

The converse result is proved in the same manner. So if we have a result for the equation

$$\int_K f(xk\cdot y)dk = \sum_{i=1}^{n} g_i(x)h_i(y),\ x, y \in G, \tag{27}$$

we automatically get a corresponding result for the equation

$$\int_K f((k\cdot y)x)dk = \sum_{i=1}^{n} g_i(x)h_i(y),\ x, y \in G, \tag{28}$$

just by translating. So while it is essentially only necessary to treat one of the equations it is convenient for our later work to have the results written down explicitly for both equations.

PROPOSITION 2.8. *If $f, g_1, \dots, g_n, h_1, \dots, h_n \in C(G)$ is a solution to*

$$\int_K f(xk\cdot y)dk = \sum_{i=1}^{n} g_i(x)h_i(y),\ x, y \in G, \tag{29}$$

then

$$\sum_{i=1}^{n} g_i(x)h_i(k\cdot y) = \sum_{i=1}^{n} g_i(x)h_i(y),\ x, y \in G. \tag{30}$$

If in particular $g_1, \dots, g_n$ are linearly independent then $h_i(k\cdot x) = h_i(x)$, $x \in G$, $k \in K$.

PROOF. Since K is compact it is unimodular (see Theorem 15.13 and Theorem 15.14 in [**10**]). So we get

$$\begin{aligned} \sum_{i=1}^{n} g_i(x)h_i(k_1\cdot y) &= \int_K f(xk\cdot(k_1\cdot y))dk = \int_K f(x(kk_1\cdot y))dk \\ &= \int_K f(xk\cdot y)dk = \sum_{i=1}^{n} g_i(x)h_i(y),\ x, y \in G. \end{aligned} \tag{31}$$

□

PROPOSITION 2.9. *If $f, g_1, \dots, g_n, h_1, \dots, h_n \in C(G)$ is a solution to*

$$\int_K f((k\cdot y)x)dk = \sum_{i=1}^{n} g_i(x)h_i(y),\ x, y \in G, \tag{32}$$

then

$$\sum_{i=1}^{n} g_i(x)h_i(k\cdot y) = \sum_{i=1}^{n} g_i(x)h_i(y),\ x, y \in G. \tag{33}$$

If in particular $g_1, \ldots, g_n$ *are linearly independent then* $h_i(k\cdot x) = h_i(x)$, $x \in G, k \in K$.

PROPOSITION 2.10. *If* $f, g_1, \ldots, g_n, h_1, \ldots, h_n \in C(G)$ *is a solution to*

$$\int_K f(xk \cdot y)dk = \sum_{i=1}^{n} g_i(x)h_i(y) = \sum_{i=1}^{n} g_i(y)h_i(x), \;\; x, y \in G, \tag{34}$$

then $f(k \cdot x) = f(x)$, $x \in G$, *and*

$$\int_K f(xk \cdot y)dk = \int_K f((k \cdot y)x)dk, \;\; x, y \in G, \tag{35}$$

$$\sum_{i=1}^{n} g_i(k \cdot x)h_i(y) = \sum_{i=1}^{n} g_i(x)h_i(y), \;\; x, y \in G. \tag{36}$$

In particular if $h_1, \ldots, h_n$ *are linearly independent then* $g_i(k \cdot x) = g_i(x)$, $x \in G$.

PROOF. First note that

$$f(x) = \int_K f(xk \cdot e)dk = \sum_{i=1}^{n} g_i(e)h_i(x), \;\; x \in G. \tag{37}$$

From Proposition 2.8 we have that

$$f(k \cdot x) = \sum_{i=1}^{n} g_i(e)h_i(k \cdot x) = \sum_{i=1}^{n} g_i(e)h_i(x) = f(x), \;\; x \in G, \;\; k \in K. \tag{38}$$

Hence we have that

$$f(x) = \int_K f(x)dk = \int_K f(k \cdot x)dk, \;\; x \in G. \tag{39}$$

Using this expression for f gives us

$$\begin{aligned}
&\int_K f(xk \cdot y)dk + \int_K f((k \cdot y)x)dk \qquad (40)\\
&= \int_K \int_K f(k_1 \cdot (xk \cdot y))dk_1 dk + \int_K \int_K f(k_1 \cdot ((k \cdot y)x))dk_1 dk\\
&= \int_K \int_K [f((k_1 \cdot x)(k_1 k \cdot y)) + f((k_1 k \cdot y)(k_1 \cdot x))]dk_1 dk\\
&= \int_K \int_K [f((k_1 \cdot x)(k_1 k \cdot y)) + f((k_1 k \cdot y)(k_1 \cdot x))]dk dk_1\\
&= \int_K \int_K [f((k_1 \cdot x)k \cdot y) + f((k \cdot y)k_1 \cdot x)]dk dk_1\\
&= \int_K \sum_{i=1}^{n} g_i(k_1 \cdot x)h_i(y)dk_1 + \int_K \int_K f((k \cdot y)(k_1 \cdot x))dk_1 dk\\
&= \int_K \sum_{i=1}^{n} g_i(x)h_i(y)dk_1 + \int_K \sum_{i=1}^{n} g_i(k \cdot y)h_i(x)dk\\
&= \sum_{i=1}^{n} g_i(x)h_i(y) + \sum_{i=1}^{n} g_i(y)h_i(x) = 2\sum_{i=1}^{n} g_i(x)h_i(y) = 2\int_K f(xk \cdot y)dk.
\end{aligned}$$

Hence

$$\int_K f((k \cdot y)x)dk = \int_K f(xk \cdot y)dk, \;\; x, y \in G. \tag{41}$$

□

COROLLARY 2.11. *If $f, g_1, \dots, g_n, h_1, \dots, h_n \in C(G)$ is a solution to*

$$\int_K f((k \cdot y)x)dk = \sum_{i=1}^{n} g_i(x)h_i(y) = \sum_{i=1}^{n} g_i(y)h_i(x),\ x, y \in G, \tag{42}$$

then $f(k \cdot x) = f(x)$, $x \in G$, and

$$\int_K f(xk \cdot y)dk = \int_K f((k \cdot y)x)dk,\ x, y \in G, \tag{43}$$

$$\sum_{i=1}^{n} g_i(k \cdot x)h_i(y) = \sum_{i=1}^{n} g_i(x)h_i(y),\ x, y \in G. \tag{44}$$

In particular if $h_1, \dots, h_n$ are linearly independent then $g_i(k \cdot x) = g_i(x)$, $x \in G$.

PROOF. We note that $\check{f}, \check{g}_1, \dots, \check{g}_n, \check{h}_1, \dots, \check{h}_n \in C(G)$ are a solution to

$$\begin{aligned}\int_K \check{f}(xk \cdot y)dk &= \sum_{i=1}^{n} \check{g}_i(x)\check{h}_i(y) = \sum_{i=1}^{n} g_i(x^{-1})h_i(y^{-1}) \\ &= \sum_{i=1}^{n} g_i(y^{-1})h_i(x^{-1}) = \sum_{i=1}^{n} \check{g}_i(y)\check{h}_i(x),\ x, y \in G.\end{aligned} \tag{45}$$

Hence by the Proposition 2.10

$$\int_K f((k \cdot y^{-1})x^{-1})dk = \int_K \check{f}(xk \cdot y)dk = \int_K \check{f}((k \cdot y)x)dk = \int_K f(x^{-1}k \cdot y^{-1})dk. \tag{46}$$

Thus

$$\int_K f(xk \cdot y)dk = \int_K f((k \cdot y)x)dk = \sum_{i=1}^{n} g_i(x)h_i(y),\ x, y \in G, \tag{47}$$

so by the previous proposition this proves the corollary. □

PROPOSITION 2.12. *If $\mathbb{Z}_2 = \{1, -1\}$ acts on G by $1 \cdot x = x$, $x \in G$ and $(-1) \cdot x = \sigma x$, $x \in G$, where σ is an continuous antihomomorphism of order 2 and*

$$\frac{f(xy) + f(x\sigma(y))}{2} = \sum_{i=1}^{n} g_i(x)h_i(y) = \sum_{i=1}^{n} g_i(y)h_i(x),\ x, y \in G, \tag{48}$$

then $f(xy) = f(yx)$, $x, y \in G$.

PROOF. We note first how it is made to fit into the general framework. Endow $\mathbb{Z}_2$ with the discrete topology. Then $\mathbb{Z}_2$ is compact Hausdorff group. It is trivial to check that $\mathbb{Z}_2$ acts on G. So we only need to check that the map $(k, x) \mapsto k \cdot x$ is continuous but that is true since σ is continuous. From the previous proposition we have

$$\begin{aligned}\frac{f(yx) + f(\sigma(y)x)}{2} &= \int_{\mathbb{Z}_2} f((k \cdot y)x)dk = \int_{\mathbb{Z}_2} f(xk \cdot y)dk \\ &= \int_{\mathbb{Z}_2} f(yk \cdot x)dk = \frac{f(yx) + f(y\sigma x)}{2},\ x, y \in G,\end{aligned} \tag{49}$$

so $f(\sigma(y)x) = f(y\sigma x) = f(\sigma(y\sigma x)) = f(x\sigma y)$, $x, y \in G$. Hence $f(xy) = f(yx)$, $x, y \in G$. □

3. Technicalities

This section contains some lemmas we will need later on.

LEMMA 3.1. *Let X be a nonempty set. Suppose $f : X \to \mathbb{C}$ and $D : X \times X \to \mathbb{C}$ are a solution to*

$$f(z)D(x,y) = -f(y)D(x,z), \; x,y,z \in X, \tag{50}$$

and $f \not\equiv 0$, then $D \equiv 0$.

PROOF. There exists $z_0 \in X$ such that $f(z_0) \neq 0$. So

$$D(x,y) = -f(y)D(x,z_0)/f(z_0) = d(x)f(y), \; x,y \in X, \tag{51}$$

where $d(x) = -D(x,z_0)/f(z_0)$. Insert this expression for D in (50)

$$f(z)d(x)f(y) = -f(y)d(x)f(z), \; x,y,z \in X. \tag{52}$$

Since $f \not\equiv 0$ it follows that $d \equiv 0$ and hence $D \equiv 0$. □

LEMMA 3.2. *Let X be a nonempty set. Suppose $f : X \to \mathbb{C}$ and $\Phi, \Psi : X \times X \to \mathbb{C}$ are a solution to*

$$f(x)[\Phi(y,z) + \Psi(y,z)] = \Phi(x,y)f(z) + \Phi(x,z)f(y), \; x,y,z \in X, \tag{53}$$

and $f \not\equiv 0$. Then there exists a function $\psi : X \to \mathbb{C}$ such that $\Phi(x,y) = f(x)\psi(y)$, $x,y \in X$ and $\Psi(x,y) = \psi(x)f(y)$, $x,y \in X$.

If X is a topological space and Φ is continuous then so is ψ.

PROOF. There exists $x_0 \in X$ such that $f(x_0) \neq 0$. So

$$\Phi(y,z) + \Psi(y,z) = \frac{1}{f(x_0)}\Phi(x_0,y)f(z) + \frac{1}{f(x_0)}\Phi(x_0,z)f(y) = \psi(y)f(z) + \psi(z)f(y), \tag{54}$$

where $\psi(x) = \Phi(x_0,x)/f(x_0)$. Hence

$$f(x)[\psi(y)f(z) + \psi(z)f(y)] = \Phi(x,y)f(z) + \Phi(x,z)f(y), \; x,y,z \in X. \tag{55}$$

Rearranging terms gives

$$[\Phi(x,y) - f(x)\psi(y)]f(z) = -f(y)[\Phi(x,z) - f(x)\psi(z)], \; x,y,z \in X. \tag{56}$$

Define $D(x,y) = \Phi(x,y) - f(x)\psi(y)$. Then

$$D(x,y)f(z) = -D(x,z)f(y) \tag{57}$$

By Lemma 3.1 $D \equiv 0$. So

$$\Phi(x,y) = f(x)\psi(y), \; x,y \in X, \tag{58}$$

and

$$\Psi(x,y) = \psi(x)f(y), \; x,y \in X. \tag{59}$$

That the continuity of Φ implies the continuity of ψ is trivial by the definition of ψ. □

LEMMA 3.3. *Let X and Y be nonempty topological spaces. Assume that Y is compact, $F : X \times Y \to \mathbb{C}$ is continuous, and μ is a finite measure on a σ-algebra $\mathcal{Y}$ in Y which contains all Borel sets in Y, then $\Phi : X \to \mathbb{C}$ given by*

$$\Phi(x) = \int_Y F(x,y)d\mu(y), \; x \in X, \tag{60}$$

is continuous.

PROOF. We will show that Φ is continuous at the arbitrary point $x_0 \in X$, so let ϵ be given. For every $y \in Y$ there exist open neighbourhoods V_y of x_0 and W_y of y such that $(x, z) \in V_y \times W_y$ implies $|F(x_0, y) - F(x, z)| < \epsilon/2$. Since $\{W_y : y \in Y\}$ is an open cover of Y and since Y is compact there exist a finite subcover $\{W_{y_i} : i = 1, \ldots, n\}$ of Y. Put $V = \cap_{i=1}^{n} V_{y_i}$. For any $x \in V$ and any $y \in Y$ there exists i_0 such that $y \in W_{y_{i_0}}$, $x \in V_{y_{i_0}}$, $|F(x, y) - F(x_0, y)| \leq |F(x, y) - F(x_0, y_{i_0})| + |F(x_0, y_{i_0}) - F(x_0, y)| < \epsilon$. Thus for any $x \in V$ we have that

$$|\int_Y F(x, y) d\mu(y) - \int_Y F(x_0, y) d\mu(y)| \leq \int_Y |F(x, y) - F(x_0, y)| d\mu(y) \leq \epsilon\mu(Y). \quad (61)$$

$\square$

COROLLARY 3.4. *If $f \in C(G)$ then $\Phi : G \times G \to \mathbb{C}$ given by*

$$\Phi(x, y) = \int_K f(xk \cdot y) dk, \; x, y \in G, \quad (62)$$

is continuous.

PROOF. $F : (G \times G) \times K \to \mathbb{C}$ given by $F((x, y), k) = f(xk \cdot y)$ is continuous, so the corollary follows from the previous lemma. $\square$

4. The Sine Addition Equation

In this section we will study the equation

$$\int_K f(xk \cdot y) dk = f(x)g(y) + g(x)f(y), \; x, y \in G. \quad (63)$$

We will show how reduce the problem of solving this equation to the problem of solving simpler equations. Note that if $K = \{e\}$ is the trivial group, then (63) reduces to

$$f(xy) = f(x)g(y) + g(x)f(y), \; x, y \in G. \quad (64)$$

If $G = (\mathbb{R}, +)$ then $f = sin$ and $g = cos$ is a solution. For a thorough discussion of trigonometric functional equations on $\mathbb{R}$ see Chapter 13 in [**1**].

LEMMA 4.1. *If $f, g \in C(G)$ are a solution to*

$$\int_K f(xk \cdot y) dk = f(x)g(y) + g(x)f(y), \; x, y \in G, \quad (65)$$

and $f \not\equiv 0$, then there exists a function $\psi \in C(G)$ such that

$$\int_K g(xk \cdot y) dk = g(x)g(y) + f(x)\psi(y), \; x, y \in G, \quad (66)$$

and

$$\int_K g((k \cdot y)x) dk = g(x)g(y) + \psi(x)f(y), \; x, y \in G. \quad (67)$$

PROOF. Apply the necessary condition (13) to (65). This gives

$$\begin{aligned} &f(x)[\int_K g(yk\cdot z)dk + \int_K g((k\cdot z)y)dk] + g(x)[\int_K f(yk\cdot z)dk + \int_K f((k\cdot z)y)dk] \\ &= \int_K f(xk\cdot y)dkg(z) + \int_K g(xk\cdot y)dkf(z) + \int_K f(xk\cdot z)dkg(y) \\ &+ \int_K g(xk\cdot z)dkf(y),\ x,y,z\in G. \end{aligned} \tag{68}$$

Using the fact that

$$\int_K f((k\cdot y)x)dk = \int_K f(xk\cdot y)dk = f(x)g(y) + g(x)f(y),\ x,y\in G, \tag{69}$$

which follows from Proposition 2.10, we get that

$$\begin{aligned} &f(x)[\int_K g(yk\cdot z)dk + \int_K g((k\cdot z)y)dk] + 2g(x)[f(y)g(z) + g(y)f(z)] \qquad (70) \\ &= [f(x)g(y) + g(x)f(y)]g(z) + \int_K g(xk\cdot y)dkf(z) + [f(x)g(z) + g(x)f(z)]g(y) \\ &\quad + \int_K g(xk\cdot z)dkf(y),\ x,y,z\in G. \end{aligned}$$

Cancelling and rearranging terms we get

$$\begin{aligned} &f(x)[\int_K g(yk\cdot z)dk - g(y)g(z) + \int_K g((k\cdot z)y)dk - g(y)g(z)] \qquad (71) \\ &= [\int_K g(xk\cdot y)dk - g(x)g(y)]f(z) + [\int_K g(xk\cdot z)dk - g(x)g(z)]f(y),\ x,y,z\in G. \end{aligned}$$

Define $\Phi, \Psi : G\times G \to \mathbb{C}$ by

$$\Phi(x,y) = \int_K g(xk\cdot y)dk - g(x)g(y),\ x,y\in G, \tag{72}$$

$$\Psi(x,y) = \int_K g((k\cdot y)x)dk - g(x)g(y),\ x,y\in G. \tag{73}$$

It follows from Corollary 3.4 that Φ is continuous. Using the definition of Φ and Ψ in (71) we get

$$f(x)[\Phi(y,z) + \Psi(y,z)] = \Phi(x,y)f(z) + \Phi(x,z)f(y),\ x,y\in G. \tag{74}$$

By Lemma 3.2 there exists $\psi\in C(G)$ such that $\Phi(x,y) = f(x)\psi(y)$, $x,y\in G$ and $\Psi(x,y) = \psi(x)f(y),\ x,y\in G$. □

PROPOSITION 4.2. *If $f,g\in C(G)$ is a solution to*

$$\int_K f(xk\cdot y)dk = f(x)g(y) + g(x)f(y),\ x,y\in G, \tag{75}$$

and $f\not\equiv 0$, then there exists $\lambda\in\mathbb{C}$ such that

$$\int_K g(xk\cdot y)dk = g(x)g(y) + \lambda^2 f(x)f(y),\ x,y\in G. \tag{76}$$

PROOF. By the Lemma 4.1 there exists $\psi \in C(G)$ such that

$$\int_K g(xk \cdot y)dk = g(x)g(y) + f(x)\psi(y), \;\; x, y \in G, \tag{77}$$

and

$$\int_K g((k \cdot y)x)dk = g(x)g(y) + \psi(x)f(y), \;\; x, y \in G. \tag{78}$$

If $\psi \equiv 0$ then it follows from (77) that $\lambda = 0$ will do. So assume that $\psi \not\equiv 0$. Apply the necessary condition (13) to (77)

$$\begin{aligned} &g(x)\int_K [g(yk \cdot z) + g((k \cdot z)y)]dk + f(x)\int_K [\psi(yk \cdot z) + \psi((k \cdot z)y)]dk \\ &= \int_K g(xk \cdot y)dkg(z) + \int_K f(xk \cdot y)dk\psi(z) + \int_K g(xk \cdot z)dkg(y) \\ &+ \int_K f(xk \cdot z)dk\psi(y), \;\; x, y, z \in G. \end{aligned} \tag{79}$$

Using (75), (77), and (78) we get

$$\begin{aligned} g(x)[2g(y)g(z) + f(y)\psi(z) + \psi(y)f(z)] + f(x)\int_K [\psi(yk \cdot z) + \psi((k \cdot z)y)]dk \\ = \;\; [g(x)g(y) + f(x)\psi(y)]g(z) + [f(x)g(y) + g(x)f(y)]\psi(z) \\ +[g(x)g(z) + f(x)\psi(z)]g(y) + [f(x)g(z) + g(x)f(z)]\psi(y), \;\; x, y, z \in G. \end{aligned} \tag{80}$$

Cancelling terms we get

$$f(x)[\int_K \psi(yk \cdot z)dk + \int_K \psi((k \cdot z)y)dk] = 2f(x)g(y)\psi(z) + 2f(x)\psi(y)g(z). \tag{81}$$

Since $f \not\equiv 0$ this implies

$$\int_K \psi(xk \cdot y)dk + \int_K \psi((k \cdot y)x)dk = 2g(x)\psi(y) + 2\psi(x)g(y), \;\; x, y \in G. \tag{82}$$

Using the necessary condition (22) on (78) we get

$$\begin{aligned} g(x)\int_K [g(yk \cdot z) + g((k \cdot z)y)]dk + \psi(x)\int_K [f(yk \cdot z) + f((k \cdot z)y)]dk \\ = \;\; \int_K g((k \cdot y)x)dkg(z) + \int_K \psi((k \cdot y)x)f(z) + \int_K g((k \cdot z)x)dkg(y) \\ + \int_K \psi((k \cdot z)x)dkf(y), \;\; x, y, z \in G. \end{aligned} \tag{83}$$

Using (77), (78), and the fact that

$$\int_K f((k \cdot y)x)dk = \int_K f(xk \cdot y)dk = f(x)g(y) + g(x)f(y), \;\; x, y \in G, \tag{84}$$

which follows from Proposition 2.10, we get

$$\begin{aligned} &g(x)[2g(y)g(z) + f(y)\psi(z) + \psi(y)f(z)] + 2\psi(x)[f(y)g(z) + g(y)f(z)] \\ &= \;\; [g(x)g(y) + \psi(x)f(y)]g(z) + \int_K \psi((k \cdot y)x)dkf(z) + [g(x)g(z) + \psi(x)f(z)]g(y) \\ &+ \int_K \psi((k \cdot z)x)dkf(y), \;\; x, y, z \in G. \end{aligned} \tag{85}$$

Rearranging and cancelling terms we get

$$f(z)[\int_K \psi((k\cdot y)x)dk - \psi(x)g(y) - g(x)\psi(y)] \tag{86}$$
$$= \quad -f(y)[\int_K \psi((k\cdot z)x)dk - g(x)\psi(z) - \psi(x)g(z)],\ x,y,z \in G.$$

This is of the form

$$f(z)D(x,y) = -f(y)D(x,z),\ x,y,z \in G, \tag{87}$$

with

$$D(x,y) = \int_K \psi((k\cdot y)x)dk - \psi(x)g(y) - g(x)\psi(y),\ x,y \in G. \tag{88}$$

Hence $D \equiv 0$, see Lemma 3.1. So

$$\int_K \psi((k\cdot y)x)dk = \psi(x)g(y) + g(x)\psi(y),\ x,y \in G. \tag{89}$$

Subtract (89) from (82). This gives

$$\int_K \psi(xk\cdot y)dk = \psi(x)g(y) + g(x)\psi(y),\ x,y \in G. \tag{90}$$

It follows from Lemma 4.1 that there exists a function $\phi \in C(G)$ such that

$$\int_K g(xk\cdot y)dk = g(x)g(y) + \psi(x)\phi(y),\ x,y \in G. \tag{91}$$

Compare this with (77). It follows that

$$f(x)\psi(y) = \psi(x)\phi(y),\ x,y \in G. \tag{92}$$

If $\phi \equiv 0$ then $f(x)\psi(y) = 0$, $x,y \in G$. Since $f \not\equiv 0$ we get $\psi \equiv 0$. This is excluded by assumption so $\phi \not\equiv 0$. Hence there exists $y_0 \in G$ such that $\phi(y_0) \neq 0$ so $\psi = \lambda^2 f$ where $\lambda^2 = \psi(y_0)/\phi(y_0)$. So from (77) we get that

$$\int_K g(xk\cdot y)dk = g(x)g(y) + \lambda^2 f(x)f(y),\ x,y \in G. \tag{93}$$

□

The following theorem is an extension of Theorem V.2 in [**15**] in that G is not assumed to be abelian and K is not supposed to act by homomorphisms only.

THEOREM 4.3. *The solutions $f,g \in C(G)$ of the functional equation*

$$\int_K f(xk\cdot y)dk = f(x)g(y) + g(x)f(y),\ x,y \in G, \tag{94}$$

can be listed as follows

(1) *$f \equiv 0$ and g arbitrary in $C(G)$.*
(2) *There exists a K-spherical function $\phi \in C(G)$ and a constant $c \in \mathbb{C}\setminus\{0\}$ such that $f = c\phi$ and $g = \phi/2$.*
(3) *There exists two K-spherical functions $\phi_1,\phi_2 \in C(G)$, $\phi_1 \neq \phi_2$, and a constant $c \in \mathbb{C}\setminus\{0\}$ such that*
$$g = \frac{\phi_1+\phi_2}{2},\ f = c\frac{\phi_1-\phi_2}{2}. \tag{95}$$
(4) *g is a K-spherical function and f is a solution of the original functional equation (94).*

PROOF. It is easy to check that each ot the pairs described in Theorem 4.3 is a solution to equation (94). So assume that the pair $f, g \in C(G)$ is a solution to (94). We will show that it fits into one of the four cases. If $f \equiv 0$ then obviously g can be chosen arbitrarily in $C(G)$. This is case 1. So we will assume that $f \not\equiv 0$. Note that if $g \equiv 0$ then

$$f(x) = \int_K f(xk \cdot e)dk = f(x)g(e) + g(x)f(e) = 0, \tag{96}$$

which we have excluded, so $g \not\equiv 0$. If f and g are linearly dependent, then there exists $c \in \mathbb{C}\backslash\{0\}$ such that $g = cf$. It then follows from (94) that $2cf$ is K-spherical. This is case 2. Now assume that f and g are linearly independent. From

$$f(x) = \int_K f(xk \cdot e)dk = f(x)g(e) + g(x)f(e), \ x \in G, \tag{97}$$

we deduce that $f(e) = 0$ and $g(e) = 1$. It follows from Proposition 4.2 that there exists $\lambda \in \mathbb{C}$ such that

$$\int_K g(xk \cdot y)dk = g(x)g(y) + \lambda^2 f(x)f(y), \ x, y \in G. \tag{98}$$

If $\lambda = 0$ then g is a K-spherical function. This is case 4. So assume that $\lambda \neq 0$. We now proceed as on page 18 in [**12**]. Define $\Phi : G \to M_2(\mathbb{C})$ by

$$\Phi(x) = \left\{ \begin{array}{cc} g(x) & \lambda^2 f(x) \\ f(x) & g(x) \end{array} \right\}, \ x \in G. \tag{99}$$

It is easy to check that

$$\int_K \Phi(xk \cdot y)dk = \Phi(x)\Phi(y), \ x, y \in G. \tag{100}$$

Put

$$A = \left\{ \begin{array}{cc} \lambda & -\lambda \\ 1 & 1 \end{array} \right\}. \tag{101}$$

Define $\Psi : G \to M_2(\mathbb{C})$ by

$$\Psi = A^{-1}\Phi A = \left\{ \begin{array}{cc} g + \lambda f & 0 \\ 0 & g - \lambda f \end{array} \right\}. \tag{102}$$

Since

$$\int_K \Psi(xk \cdot y)dk = A^{-1} \int_K \Phi(xk \cdot y)dkA = A^{-1}\Phi(x)AA^{-1}\Phi(y)A = \Psi(x)\Psi(y), \tag{103}$$

and $(g + \lambda f)(e) = (g - \lambda f)(e) = 1$ we have that $\phi_1 = g + \lambda f \in C(G)$ and $\phi_2 = g - \lambda f \in C(G)$ are K-spherical functions. This is case 3 with $c = 1/\lambda$. □

REMARK 4.4. Notice that in the proof of this theorem we could use the same methods as Stetkær used in section V of [**12**] and in section V of [**15**] with the exception of the proof of Proposition 4.2. It was fairly simple to prove the similar result Lemma V.1 in [**12**] in the case where K acted by homomorphisms. But proving Proposition 4.2 when K does not act exclusively by homomorphisms was the difficult part of proving Theorem 4.3.

COROLLARY 4.5. *Assume that G is an abelian group, or G is an metabelian group generated by the squares x^2, $x \in G$, or that G is a semidirect product of two 2-divisible abelian groups, or G is a connected nilpotent Lie group. Then the solutions $f, g \in C(G)$ of the functional equation*

$$f(xy) + f(xy^{-1}) = 2f(x)g(y) + 2g(x)f(y), \ x, y \in G, \tag{104}$$

can be listed as follows

(1) $f \equiv 0$ *and* $g \in C(G)$ *arbitrary.*
(2) *There exists a continuous homomorphism* $m : G \to \mathbb{C}$ *and a constant* $c \in \mathbb{C} \setminus \{0\}$ *such that* $f = c(m + \check{m})/2$ *and* $g = (m + \check{m})/4$.
(3) *There exist two continuous homomorphisms* $m_1, m_2 : G \to \mathbb{C}^*$ *such that* $\phi_1 = (m_1 + \check{m}_1)/2 \neq \phi_2 = (m_2 + \check{m}_2)/2$, *and a constant* $c \in \mathbb{C} \setminus \{0\}$ *such that*
$$g = \frac{\phi_1 + \phi_2}{2}, \quad f = c\frac{\phi_1 - \phi_2}{2} \tag{105}$$
(4) *There exists a continuous homomorphism* $m : G \to \mathbb{C}^*$ *and* f *is a solution to the equation*

$$f(xy) + f(xy^{-1}) = f(x)(m + \check{m})(y) + (m + \check{m})(x)f(y), \ x, y \in G. \tag{106}$$

PROOF. First of all we need to determine the $\mathbb{Z}_2$-spherical functions. So assume that $\phi : G \to \mathbb{C}$ is a solution of

$$\phi(xy) + \phi(xy^{-1}) = 2\phi(x)\phi(y), \ x, y \in G. \tag{107}$$

Then there exists a homomorphism $m : G \to \mathbb{C}^*$ such that

$$\phi = \frac{m + \check{m}}{2}. \tag{108}$$

If G is abelian this follows from Theorem 2 in [**11**]. If G is a metabelian group generated by the squares x^2, $x \in G$, then it follows from Proposition V.5 in [**16**]. If G is a semidirect product of two abelian 2-divisible groups then it follows from Theorem 3.13 in [**8**]. If G is a connected nilpotent Lie group then it follows from Corollary 2.8 in [**7**]. It follows from Theorem 1 in [**11**] that ϕ is continuous if and only if m is. So the $\mathbb{Z}_2$-spherical functions on G are all functions ϕ of the form

$$\phi = \frac{m + \check{m}}{2}, \tag{109}$$

where $\phi \in C(G)$. The fact that any function of this form is a $\mathbb{Z}_2$-spherical function is trivial to check, and this is true for any group G. The corollary now follows from Theorem 4.3. $\square$

REMARK 4.6. If G is abelian then all solutions to the equation in case 4 are given by Theorem 1 in [**3**] and the corollary gives the solution formulas as found in Theorem 2 in [**3**] as noted by Stetkær in [**12**] p. 18. So the corollary only gives something new when G is not abelian. Suppose we have case 4 in the corollary and that $m = \check{m}$. Then $F = f/m$ is a solution to the quadratic equation

$$F(xy) + F(xy^{-1}) = 2F(x) + 2F(y), \ x, y \in G. \tag{110}$$

So we would like to solve the quadratic equation on any group. In [**5**] Corovei claims to have proved that all solutions to the quadratic equation have the form

$F(x) = B(x, x)$ where $B : G \times G \to \mathbb{C}$ is a symmetric biadditive map. Unfortunately the proof is faulty. It is based on the claim that a solution to the equation

$$a(xy) + a(yx) = 2a(x) + 2a(y),\ x, y \in G, \tag{111}$$

necessarily is a homomorphism. This is not true. That the claim is false can be seen in the following way. Let $h : G \to \mathbb{C}$ be a solution of Jensen's equation

$$h(xy) + h(xy^{-1}) = 2h(x),\ x, y \in G, \tag{112}$$

with $h(e) = 0$. Then $h(x^{-1}) = -h(x)$ and we get

$$2h(x) + 2h(y) = h(xy) + h(xy^{-1}) + h(yx) + h(yx^{-1}) = h(xy) + h(yx),\ x, y \in G. \tag{113}$$

So if the claim was true then any solution h to Jensen's equation with $h(e) = 0$ would necessarily be a homomorphism. This is however not the case. For a counterexample see Proposition 4.3 of [**7**]. So finding all solutions of the quadratic equation is still an open problem.

5. The Cosine Addition Equation

In this section we will study the equation

$$\int_K f(xk \cdot y)dk = f(x)f(y) + g(x)g(y),\ x, y \in G, \tag{114}$$

and show how to reduce the problem of solving this equation to the problem of solving simpler equations. Note that if $K = \{e\}$ is the trivial group then (114) reduces to

$$f(xy) = f(x)f(y) + g(x)g(y), x, y \in G. \tag{115}$$

If $G = (\mathbb{R}, +)$ then $f = \cos$ and $g = i \sin$ is a solution. Again see Chapter 13 in [**1**] for further information on trigonometric functional equations on $\mathbb{R}$.

Lemma 5.1. *If $f, g \in C(G)$ is a solution to*

$$\int_K f(xk \cdot y)dk = f(x)f(y) + g(x)g(y),\ x, y \in G, \tag{116}$$

then there exists $\psi \in C(G)$ such that

$$\int_K g(xk \cdot y)dk = f(x)g(y) + g(x)\psi(y),\ x, y \in G, \tag{117}$$

and

$$\int_K g((k \cdot y)x)dk = g(x)f(y) + \psi(x)g(y),\ x, y \in G. \tag{118}$$

Proof. If $g \equiv 0$ then the conclusion is trivially true. So we may assume that $g \not\equiv 0$. If we apply the necessary condition (13) to (116) we get

$$\begin{aligned} f(x)\int_K [f(yk \cdot z) + f((k \cdot z)y)]dk + g(x)[\int_K [g(yk \cdot z) + g((k \cdot z)y)]dk \\ = \int_K f(xk \cdot y)dkf(z) + \int_K g(xk \cdot y)dkg(z) + \int_K f(xk \cdot z)dkf(y) \\ + \int_K g(xk \cdot z)dkg(y),\ x, y, z \in G. \end{aligned} \tag{119}$$

Using the fact that

$$\int_K f((k \cdot y)x)dk = \int_K f(xk \cdot y)dk = f(x)f(y) + g(x)g(y),\ x, y \in G, \tag{120}$$

which follows from Proposition 2.10, we get

$$\begin{aligned} &2f(x)[f(y)f(z) + g(y)g(z)] + g(x)\int_K [g(yk \cdot z) + g((k \cdot z)y)]dk \qquad (121)\\ &= [f(x)f(y) + g(x)g(y)]f(z) + \int_K g(xk \cdot y)dkg(z) + [f(x)f(z) + g(x)g(z)]f(y)\\ &\quad + \int_K g(xk \cdot z)dkg(y),\ x, y, z \in G. \end{aligned}$$

Cancelling and rearranging terms we get

$$\begin{aligned} &g(x)[\int_K g(yk \cdot z)dk - f(y)g(z) + \int_K g((k \cdot z)y)dk - g(y)f(z)] \qquad (122)\\ &= [\int_K g(xk \cdot y)dk - f(x)g(y)]g(z) + [\int_K g(xk \cdot z)dk - f(x)g(z)]g(y). \end{aligned}$$

This is of the form

$$g(x)[\Phi(y, z) + \Psi(y, z)] = g(z)\Phi(x, y) + g(y)\Phi(x, z),\ x, y, z \in G, \tag{123}$$

with

$$\Phi(x, y) = \int_K g(xk \cdot y)dk - f(x)g(y),\ x, y \in G, \tag{124}$$

and

$$\Psi(x, y) = \int_K g((k \cdot y)x)dk - g(x)f(y),\ x, y \in G. \tag{125}$$

It follows from Corollary 3.4 that Φ is continuous. Hence by Lemma 3.2 there exists $\psi \in C(G)$ such that $\Phi(x, y) = g(x)\psi(y)$, $x, y \in G$, and $\Psi(x, y) = \psi(x)g(y)$, $x, y \in G$. $\square$

PROPOSITION 5.2. *If $f, g \in C(G)$ are a solution to*

$$\int_K f(xk \cdot y)dk = f(x)f(y) + g(x)g(y),\ x, y \in G, \tag{126}$$

then for some $\lambda \in \mathbb{C}$ we have

$$\int_K g(xk \cdot y)dk = f(x)g(y) + g(x)f(y) + \lambda g(x)g(y),\ x, y \in G. \tag{127}$$

PROOF. If $g \equiv 0$ then the conclusion is trivially true. So we may assume that $g \not\equiv 0$. By Lemma 5.1 there exists $\psi \in C(G)$ such that

$$\int_K g(xk \cdot y)dk = f(x)g(y) + g(x)\psi(y),\ x, y \in G, \tag{128}$$

and

$$\int_K g((k \cdot y)x)dk = g(x)f(y) + \psi(x)g(y),\ x, y \in G. \tag{129}$$

Apply the necessary condition (13) to (128)

$$\begin{aligned} f(x)\int_K [g(yk\cdot z)+g((k\cdot z)y)]dk + g(x)\int_K [\psi(yk\cdot z)+\psi((k\cdot z)y)]dk \quad (130)\\ = \int_K f(xk\cdot y)dkg(z) + \int_K g(xk\cdot y)dk\psi(z) + \int_K f(xk\cdot z)dkg(y)\\ + \int_K g(xk\cdot z)dk\psi(y),\ x,y,z\in G. \end{aligned}$$

Using (126), (128), and (129) we get

$$\begin{aligned} &f(x)[f(y)g(z)+g(y)f(z)+g(y)\psi(z)+\psi(y)g(z)]+g(x)\int_K[\psi(yk\cdot z)+\psi((k\cdot z)y)]dk\\ &=[f(x)f(y)+g(x)g(y)]g(z)+[f(x)g(y)+g(x)\psi(y)]\psi(z)\\ &\quad+[f(x)f(z)+g(x)g(z)]g(y)+[f(x)g(z)+g(x)\psi(z)]\psi(y),\ x,y,z\in G. \end{aligned} \tag{131}$$

Cancelling terms we get

$$g(x)[\int_K \psi(yk\cdot z)dk + \int_K \psi((k\cdot z)y)dk] = 2g(x)g(y)g(z)+2g(x)\psi(y)\psi(z),\ x,y,z\in G. \tag{132}$$

Since $g\not\equiv 0$ this implies that

$$\int_K \psi(xk\cdot y)dk + \int_K \psi((k\cdot y)x)dk = 2g(x)g(y)+2\psi(x)\psi(y),\ x,y\in G. \tag{133}$$

Apply the necessary condition (22) to (129)

$$\begin{aligned} g(x)\int_K [f(yk\cdot z)+f((k\cdot z)y)]dk + \psi(x)\int_K [g(yk\cdot z)+g((k\cdot z)y)]dk \quad (134)\\ = \int_K g((k\cdot y)x)dkf(z) + \int_K \psi((k\cdot y)x)dkg(z) + \int_K g((k\cdot z)x)dkf(y)\\ + \int_K \psi((k\cdot z)x)dkg(y),\ x,y,z\in G. \end{aligned}$$

Using (128), (129), and the fact that

$$\int_K f((k\cdot y)x)dk = \int_K f(xk\cdot y)dk = f(x)f(y)+g(x)g(y),\ x,y\in G, \tag{135}$$

which follows from Proposition 2.10, we get

$$\begin{aligned} 2g(x)[f(y)f(z)+g(y)g(z)]+\psi(x)[f(y)g(z)+g(y)f(z)+g(y)\psi(z)+\psi(y)g(z)] \quad (136)\\ = [g(x)f(y)+\psi(x)g(y)]f(z)+\int_K \psi((k\cdot y)x)dkg(z)+[g(x)f(z)+\psi(x)g(z)]f(y)\\ + \int_K \psi((k\cdot z)x)dkg(y),\ x,y,z\in G. \quad (137) \end{aligned}$$

Cancelling and rearranging terms we get

$$g(z)[\int_K \psi((k\cdot y)x)dk - \psi(x)\psi(y) - g(x)g(y)] \tag{138}$$

$$= -g(y)[\int_K \psi((k\cdot z)x)dk - \psi(x)\psi(z) - g(x)g(z)],\ x,y,z\in G. \tag{139}$$

This is of the form

$$g(z)D(x,y) = -g(y)D(x,z),\ x,y,z\in G, \tag{140}$$

with

$$D(x,y) = \int_K \psi((k\cdot y)x)dk - \psi(x)\psi(y) - g(x)g(y),\ x,y \in G. \tag{141}$$

Hence by Lemma 3.1 $D \equiv 0$. So

$$\int_K \psi((k\cdot y)x)dk = \psi(x)\psi(y) + g(x)g(y),\ x,y \in G. \tag{142}$$

Subtract (142) from (133) to get

$$\int_K \psi(xk\cdot y)dk = \psi(x)\psi(y) + g(x)g(y),\ x,y \in G. \tag{143}$$

By Lemma 5.1 there exists a function $\phi \in C(G)$ such that

$$f(x)g(y) + g(x)\psi(y) = \int_K g(xk\cdot y)dk = \psi(x)g(y) + g(x)\phi(y),\ x,y \in G, \tag{144}$$

where we have used (128). Hence

$$g(y)[\psi(x) - f(x)] = g(x)[\psi(y) - \phi(y)],\ x,y \in G. \tag{145}$$

Since $g \not\equiv 0$ there exists $y_0 \in G$ such that $g(y_0) \neq 0$. So $\psi(x) - f(x) = \lambda g(x)$ where $\lambda = (\psi(y_0) - \phi(y_0))/g(y_0)$. Insert $\psi = f + \lambda g$ into (128) and the proposition follows. □

The following theorem is an extension of Theorem V.3 in [**15**] in that G is not supposed to be abelian and K is not supposed to by acting by homomorphisms only.

THEOREM 5.3. *The solutions $f, g \in C(G)$ of the functional equation*

$$\int_K f(xk\cdot y)dk = f(x)f(y) + g(x)g(y),\ x,y \in G, \tag{146}$$

can be listed as follows:

(1) $f \equiv 0,\ g \equiv 0$.
(2) *There exists a K-spherical function $\phi \in C(G)$ and a constant $\lambda \in \mathbb{C}\backslash\{\pm i\}$ such that*

$$f = \frac{1}{1+\lambda^2}\phi \text{ and } g = \frac{\lambda}{1+\lambda^2}\phi. \tag{147}$$

(3) *There exist two K-spherical functions $\phi_1, \phi_2 \in C(G)$, $\phi_1 \neq \phi_2$, and a constant $\lambda \in \mathbb{C}\setminus\{0, i, -i\}$ such that*

$$f = \frac{\lambda\phi_1 + \lambda^{-1}\phi_2}{\lambda + \lambda^{-1}} \text{ and } g = \frac{\phi_1 - \phi_2}{\lambda + \lambda^{-1}}. \tag{148}$$

(4) *There exists a K-spherical function $\phi \in C(G)$ such that $g \in C(G)$ is a solution of*

$$\int_K g(xk\cdot y)dk = g(x)\phi(y) + \phi(x)g(y) \tag{149}$$

and $f = \phi + ig$ or $f = \phi - ig$.

PROOF. It is easy to see that each of the pairs f, g described in Theorem 5.3 is a solution of (146). So let us assume that the pair f, g is a solution to (146). We must show that it falls into one of the four cases listed above.

If $f \equiv 0$ then it obviously follows from (146) that $g \equiv 0$. This is case 1 above. So we may assume that $f \not\equiv 0$. If f and g are linearly dependent then there is a constant $\lambda \in \mathbb{C}$ such that $g = \lambda f$. This reduces the identity (146) to

$$\int_K f(xk \cdot y)dk = (1+\lambda^2)f(x)f(y),\ x,y \in G. \tag{150}$$

Here $1+\lambda^2 \neq 0$ because $1+\lambda^2 = 0$ in (150) would imply $f \equiv 0$ and this is excluded by assumption. We see from (150) that $(1+\lambda^2)f$ is a K-spherical function, this is case 2 above. So we may assume that f and g are linearly independent. Since we assume that f and g are linearly independent it follows from

$$f(x) = \int_K f(xk \cdot e)dk = f(x)f(e) + g(x)g(e),\ x \in G, \tag{151}$$

that $f(e) = 1$ and $g(e) = 0$. We now proceed along the same lines as on page 18 in [**12**]. It follows from 5.2 that there exists $\kappa \in \mathbb{C}$ such that

$$\int_K g(xk \cdot y)dk = f(x)g(y) + g(x)f(y) + 2\kappa g(x)g(y),\ x,y \in G. \tag{152}$$

Define $\Phi : G \to M_2(\mathbb{C})$ by

$$\Phi(z) = \left\{ \begin{array}{cc} f(z) & g(z) \\ g(z) & f(z)+2\kappa g(z) \end{array} \right\},\ z \in G. \tag{153}$$

Then a straightforward calculation shows that Φ fulfils

$$\int_K \Phi(xk \cdot y)dk = \Phi(x)\Phi(y),\ x,y \in G. \tag{154}$$

If $\kappa \neq \pm i$ then we put

$$A = \left\{ \begin{array}{cc} 1 & 1 \\ \kappa+\sqrt{1+\kappa^2} & \kappa-\sqrt{1+\kappa^2} \end{array} \right\}, \tag{155}$$

where $\sqrt{1+\kappa^2}$ is a square root of $1+\kappa^2$. Define $\Psi : G \to \mathbb{C}$ by

$$\Psi = A^{-1}\Phi A = \left\{ \begin{array}{cc} f+(\kappa+\sqrt{1+\kappa^2})g & 0 \\ 0 & f+(\kappa-\sqrt{1+\kappa^2})g \end{array} \right\}. \tag{156}$$

Put $\lambda = -\kappa-\sqrt{1+\kappa^2}$ and note that $-(\kappa+\sqrt{1+\kappa^2})(\kappa-\sqrt{1+\kappa^2}) = 1$ so $\lambda \neq 0$ and $\lambda^{-1} = \kappa-\sqrt{1+\kappa^2}$. From $\lambda+\lambda^{-1} = -2\sqrt{1+\kappa^2}$ it follows that $\kappa \neq \pm i$ implies that $\lambda \neq \pm i$. Since

$$\int_K \Psi(xk \cdot y)dk = A^{-1}\int_K \Phi(xk \cdot y)dkA = A^{-1}\Phi(x)AA^{-1}\Phi(y)A = \Psi(x)\Psi(y) \tag{157}$$

and $(f-\lambda g)(e) = (f+\lambda^{-1}g)(e) = 1$ we have that $\phi_2 = f-\lambda g \in C(G)$ and $\phi_1 = f+\lambda^{-1}g$ are K-spherical functions. We find that

$$f = \frac{\lambda\phi_1+\lambda^{-1}\phi_2}{\lambda+\lambda^{-1}},\ g = \frac{\phi_1-\phi_2}{\lambda+\lambda^{-1}}. \tag{158}$$

This is case 3.

Suppose $\kappa = \pm i$. Then we put

$$A = \left\{ \begin{array}{cc} 1 & \kappa \\ \kappa & 1 \end{array} \right\}, \tag{159}$$

and as above we define $\Psi : G \to M_2(\mathbb{C})$ by

$$\Psi = A^{-1}\Phi A = \left\{ \begin{array}{cc} f + \kappa g & 2g \\ 0 & f + \kappa g \end{array} \right\}. \tag{160}$$

Again

$$\int_K \Psi(xk \cdot y)dk = \Psi(x)\Psi(y),\ x, y \in G, \tag{161}$$

and since $(f + \kappa g)(e) = 1$, $\phi = f + \kappa g \in C(G)$ is K-spherical and

$$\int_K g(xk \cdot y)dk = g(x)f(y) + f(x)g(y) + 2\kappa g(x)g(y) = g(x)\phi(y) + \phi(x)g(y),\ x, y \in G. \tag{162}$$

This is case 4. □

Remark 5.4. A similar remark applies to this theorem as to Theorem 4.3. Also in this case we could use the same procedure as Stetkær with the exception of the proof of Proposition 5.2. Again the proof of this proposition turned out to be the difficult part of proving Theorem 5.3

Bibliography

[1] Aczél, J. andDhombres, J., *Functional equations in several variables.* Cambridge University Press. Cambridge, New York, New Rochelle, Melbourne, Sydney 1989.

[2] Benson, C., Jenkins, J., Ratcliff, G., *On Gelfand pairs associated with solvable Lie groups.* Trans. Amer. Math. Soc. 321 (1990), no. 1, 85-116.

[3] Chung, J. K., Kanappan, PL. and Ng, C. T., *On two trigonometric functional equations.* Mathematics Reports Toyama University **11** (1988), 153-165.

[4] Corovei, *The cosine functional equation for nilpotent groups*, Aequationes Math. 15 (1977), 99-106.

[5] Corovei, I., *Some functional equations connected with quadratic forms.* Anal. Numér. Théor. Approx. 19 (1990), no. 2, 123-127.

[6] Corovei, I., On some functional equations for the homomorphisms. Bul. Ştiinţ. Inst. Politehn. Cluj-Napoca Ser. Mat.-Fiz.-Mec. Apl. 22 (1979), 14-19.

[7] Friis, Peter de Place, *d'Alembert's and Wilson's equations on Lie groups.* Submitted. Preprint Series 2000 No 8, Matematisk Institut, Aarhus University, Denmark. pp 1-12.

[8] Friis, P., *Trigonometric Functional Equations on Groups*, Progress report, Department of Mathematics, University of Aarhus, December 1998.

[9] Helgason, S., *Differential Geometry, Lie Groups, and Symmetric Spaces.* Academic Press. New York 1978.

[10] Hewitt, E. and Ross, K. A., *Abstract Harmonic Analysis I.* Springer-Verlag, Berlin-Göttingen-Heidelberg 1963.

[11] Pl. Kannappan, *The functional equation $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ for groups.* Proc. Amer. Math. Soc. **19** (1968), 69-74.

[12] Stetkær, H., *Functional Equations and Spherical Functions.* Preprint Series 1994 No **18**, Matematisk Institut, Aarhus University, Denmark. pp. 1-28.

[13] Stetkaer, H., *Wilson's functional equations on groups.* Aequationes Math. 49, No. 3, 252-275 (1995).

[14] Stetkaer, H., *Functional equations on abelian groups with involution.* Aequationes Math. **54** (1997), 144-172.

[15] Stetkaer, H., *Trigonometric functional equations of rectangular type.* Aequationes Math. **56** (1998) 1-19.

[16] H. Stetkaer, *d'Alembert's functional equations on metabelian groups*, Aequationes Math. 59 (2000) 306-320.