Torsion Subgroups of Jacobians Acting on Moduli Spaces of Vector Bundles

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Electronic versions of this thesis and associated code will be available at `http://www.girlz.dk/frank/phd/` for as long as possible.

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Preface

This thesis presents the outcome of work I have done under the supervision of Jørgen Ellegaard Andersen at the University of Aarhus during the years 2000–2005. During this period I have had the privilege of working alongside an impressive range of mathematicians, while being constantly supported and encouraged by the Department of Mathematical Sciences. Furthermore, I have had opportunities to visit University of California, Berkeley as well as Consejo Superior de Investigaciones Científicas, Madrid for extended periods of time.

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Abstract

Dansk resumé.
Denne afhandling handler om aspekter af den matematiske disciplin kaldet topologisk kvantefeltteori *(TQFT). Udviklingen heraf blev startet i 1989–1990 af Fieldsmedaljevinderen, Ed Witten og Abelprisvinderen, Michael Atiyah.*
Ideen bag TQFT er affødt af fysik, eller mere præcist, fænomenet geometrisk kvantisering. *– Man starter med et vanskeligt forståeligt geometrisk objekt, og ved at tænke på det som faserum for et fysisk system, kan man imitere kvantiseringsprocessen af sådanne, hvorved der fremkommer et meget mere velforstået objekt, et vektorrum, indeholdende information om strukturen af det oprindelige objekt.*
Afhandlingen præsenterer en konstruktion af en struktur på TQFT-vektorrummene som repræsentation af nogle bestemte endelige grupper.

Abstract in English

This thesis is associated to the area of mathematics known as *topological quantum field theory*.

In ultra-brief, the purpose of topological quantum field theory is to build invariants of three manifolds by constructing certain functors from a cobordism category of three manifolds to vector spaces. A well-known way to do this is to construct so-called *modular functors*, associating vector spaces to closed, oriented surfaces.

The thesis is concerned with an aspect of the *gauge theoretic* approach to the construction of modular functors. -More specifically, with a natural structure of the TQFT vector spaces as representations of certain finite groups.

However, whilst being motivated and inspired by both gauge theory and topological quantum field theory, the main body of work in the thesis is algebraic geometric by nature.

In order to define the representations, a thorough understanding is needed of the natural action of torsion subgroups of the Jacobian variety of a Riemann surface on the moduli spaces of semistable vector bundles on that surface.

In particular, the geometry of the fixed point varieties of the action is studied, with emphasis on intersection properties. The outcome of this study is used to define certain groups of *lifts*, acting on the determinant line bundles on the moduli spaces.

1 Introduction

1.1 Motivation and perspectives

Let Σ denote a complete, non-singular algebraic curve over $\mathbb{C}$ (in other words, a compact Riemann surface) of genus $g \geq 2$. Let $n \geq 2$ be an integer and let $d \in \{0, 1, \ldots, n-1\}$.

The moduli space, $M(n, \Delta)$, of semistable holomorphic bundles on Σ of rank n and a fixed determinant Δ of degree d is a projective algebraic variety, of dimension $(n^2-1)(g-1)$. According to work of Narasimhan and Seshadri [9], its underlying topological space is identified with $\mathrm{Hom}(\pi_1(\Sigma), SU(n))/SU(n)$. As explained by Drezet and Narasimhan in [8], the Picard group of $M(n, \Delta)$ is isomorphic to $\mathbb{Z}$ with a canonical ample generator, $\mathcal{L}$. [1]

For $k \geq 1$, the complex vector spaces $Z_k(n, \Delta)$ of algebraic sections of $\mathcal{L}^{\otimes k}$ have been subject to great interest, because they appear as an important ingredient in the gauge theoretic approach to $2+1$ dimensional topological quantum field theory (TQFT), as originally suggested by Witten [19]. In brief, the outlines of the gauge theoretic approach are as follows:

Given a compact, oriented surface S of genus g, the spaces $Z_k(n, \Delta)$ (for fixed n, Δ and k) constitute a vector bundle over the Teichmüller space, T_S. It is becoming popular to refer to this bundle as the *Verlinde bundle* ([3]).

Hitchin and Axelrod, Della Pietra and Witten have shown that the Verlinde bundle allows a projectively flat connection ([16] and [18]), inducing a projective representation of the mapping class group on the space of covariantly constant projective sections.

[1] We will use slightly different notation for this in the thesis.

It has long been expected that the projective ambiguity in the above can be circumvented by changing the bundle on Teichmüller space slightly, allowing the action of the mapping class group to be replaced by an action of a central extension. This way, a genuinely *flat* connection would be possible, enabling a modular functor to be defined as the space of covariantly constant sections in the new, modified Verlinde bundle.

The task of completing the above construction entirely within the gauge theoretic setting has not yet been accomplished. There is, however, a closely related approach arising from conformal field theory (CFT). In this theory, another vector bundle on Teichmüller space and a projectively flat connection have been developed [22]. Laszlo has shown that this construction is equivalent to the gauge theoretic one ([20]). Recent work by Andersen and Ueno performs the modification mentioned above within the setting of CFT and proves that the construction yields a modular functor (a notion originally invented within CFT by Segal [23]). Since any modular functor can be extended to a full TQFT (this is due to Walker – see also [15]), the work of Andersen and Ueno completes the program of constructing TQFT's from the spaces $Z_k(n, \Delta)$.

In a completely different approach, Reshetikhin and Turaev have described in the early 90's how TQFT's can be constructed from quantum groups by constructing modular tensor categories. This has led to the celebrated Reshetikhin-Turaev quantum invariants for 3-manifolds [21].

Blanchet, Habegger, Masbaum and Vogel have shown how the modular tensor category corresponding to $sl_2(\mathbb{C})$ can be constructed in a simple way, using the Kauffman bracket. In their paper, certain groups of involutions on the TQFT vector spaces arose and were used to decompose the vector space into direct summands [14].

In the paper [1], Andersen and Masbaum pursued the idea that the involutions from [14] should have an analogy in the gauge theoretic setting. They found that involutions do exist on $Z_k(2, \Delta)$ as well. They are induced by certain lifts to $\mathcal{L}(n, \Delta)$ of the natural action of $J^{(2)}(\Sigma)$ on $M(2, \Delta)$, and they constitute central extensions of $J^{(2)}(\Sigma)$. Andersen and Masbaum were able to identify the involutions with the ones from Blanchet et al. Furthermore, they determined the characters of the representations, leading to the conclusion that $Z_k(2, \Delta)$ are isomorphic, as representations of the groups of involutions, to the TQFT vector spaces constructed in [14].

According to Blanchet, constructions similar to those of [14] exist for $sl_n(\mathbb{C})$ ($n \in \mathbb{N}$). The TQFT functors associated to the resulting modular tensor categories (denoted by Blanchet as $H^{SU(n,k)}$) also come with certain unipotent group actions and corresponding splittings into direct summands. (See e.g. [11].)

It has long been believed, and is currently becoming known ([6]), that the TQFT vector spaces arising from $sl_n(\mathbb{C})$ are the same as the gauge theoretic ones. For instance, the dimension of the (complexified) vector space which the Reshetikhin-Turaev TQFT associates to S at level k is known to be equal to the dimension of $Z_k(n, \mathcal{O})$. Both are given by the amazing Verlinde formula:

$$d_{n,k}(g) = ((n+k)^{n-1}n)^{g-1} \sum_{\lambda \in \Gamma_{n,k}} \prod_{1 \le i < j \le n} (2\sin(\lambda_i - i - \lambda_j + j)\frac{\pi}{n+k})^{2-2g}$$

– where $\Gamma_{n,k} = \{\lambda = (\lambda_1, \ldots, \lambda_n), k \ge \lambda_1 \ge \ldots \ge \lambda_{n-1} \ge \lambda_n = 0\}$. (See e.g. [11] for the topological version and [13] for a version in algebraic geometry.)

This thesis generalises the idea of Andersen and Masbaum to ranks greater than 2, aiming to "rediscover" the group representations on the Verlinde spaces in the gauge theoretic setting. In the process, a wealth of structure on the moduli spaces becomes apparent.

1.2 Notational conventions

Throughout this thesis, a "bundle" on a Riemann surface will mean a holomorphic vector bundle. Similarly, a "bundle" on a complex algebraic variety will mean an algebraic vector bundle. By a "bundle map" I will mean a holomorphic (resp. algebraic), fibrewisely linear map between the total spaces of two bundles, which does *not* necessarily induce the identity on the base space. The terms "homomorphism", "endomorphism" and "automorphism" will be reserved for basepoint preserving maps. (I.e. sections in the derived bundles Hom, End and Aut respectively.)

Furthermore, the following symbols will be used without further introduction.

SET	:	The category of sets
VAR	:	The category of (not necessarily irreducible) $\mathbb{C}$-varieties
μ_n	:	The group of n'th roots of unity in $\mathbb{C}$
ζ_n	:	$e^{2\pi i/n}$ (Or the reader's favourite generator of μ_n)
$A \overset{\circ}{\subseteq} B$	:	"A is an open subset of B"

1.3 Outline and main results

The thesis is organised as follows:

Chapter 1

-Is this introduction to the thesis.

Chapter 2

The second chapter gathers some fundamental technical results needed in the thesis. In particular, in lack of proper reference, a detailed account is given on the correspondence between the Weil pairing and the intersection pairing on a compact Riemann surface. (Proposition 2.16). Furthermore, a very general technique is introduced for determining when maps between moduli spaces are actually morphisms. (Proposition 2.31).

Chapter 3

The third chapter is a study of the action, and notably the fixed points, of a single torsion element, α in the Jacobian $J(\Sigma)$ on the moduli space, $M(n,\Delta_d)$ of semistable bundles with fixed rank n and determinant Δ_d of degree d.

The exposition involves the extension and application of some theory of Narasimhan and Ramanan, leading to the following main result (theorem 3.33 in the thesis):

Theorem 1. *Given a primitive $\alpha \in J^{(n)}$, the fixed point variety $|M(n,\Delta_d)|_\alpha$ of the action of α on $M(n,\Delta_d)$ has $r = (n,d)$ connected components, each of which is isomorphic to the quotient of the Prym variety P_α under the action of μ_q, $q = \frac{n}{(n,d)}$.*

If n is odd, the set of connected components is canonically identified with μ_r.

If n is even, this set is canonically identified with $(\mu_r \times \frac{\alpha}{2})/\sim$.

Here $\frac{\alpha}{2}$ denotes the set of elements $a \in J^{(2n)}$ with $2a = \alpha$, and $(\zeta_1, a_1) \sim (\zeta_2, a_2)$ if and only if $\zeta_1 = \lambda^q \zeta_2$, where $\lambda = \lambda_{2n}(a_1, a_2) \in \{\pm 1\}$ is the Weil pairing.

In both cases, the identification maps are given by definitions 3.36 and 3.37.

Finally, a generalisation to the case of non-primitive elements (theorem 3.44) is presented.

Chapter 4

The fourth chapter contains an examination of the relative positions of the fixed point varieties corresponding to different torsion elements. The main results include a criterion for when and how the fixed point varieties of independent, primitive torsion elements intersect (corollary 4.7 and proposition 4.9 in the thesis):

Theorem 2. *Given primitive, independent elements α and β in $J^{(n)}$, the fixed point varieties $|M(n,\Delta_d)|_\alpha$ and $|M(n,\Delta_d)|_\beta$ intersect if and only if $\lambda_n(\alpha,\beta) \in \mu_n$ is of order $q = \frac{n}{(n,d)}$.*

When they do intersect, the intersection is evenly divided into a finite number of so-called "layers" (i.e. orbits of the action of $J^{(n)}/\langle\alpha,\beta\rangle$). In each of these layers, every component of $|M(n,\Delta_d)|_\alpha$ intersects each component of $|M(n,\Delta_d)|_\beta$ in finitely many points, giving rise to a picture similar to the one in figure 4.1 (and the front page).

Furthermore, in the case of odd ranks, a complete description is given of how individual components of the fixed point varieties for three torsion elements with certain relations intersect (theorem 4.17):

Theorem 3. *Assume n is odd. Let $r = (n,d)$ and $q = \frac{n}{r}$. Suppose $\alpha,\beta \in J^{(n)}$ are primitive elements with $\langle\alpha\rangle \cap \langle\beta\rangle = 0$ and $\mathrm{ord}(\lambda_n(\alpha,\beta)) = q$. Let $\gamma = \alpha+\beta$. For each triple $\zeta,\zeta',\zeta'' \in \mu_r$ we have:*

$$|M(n,\Delta_d)|^{\zeta,\zeta',\zeta''}_{\alpha,\beta,\gamma} \neq \emptyset \Leftrightarrow \frac{\zeta''}{\zeta\zeta'} = 1.$$

–Here, $|M(n,\Delta_d)|^{\zeta,\zeta',\zeta''}_{\alpha,\beta,\gamma}$ denotes the intersection of the fixed point components of α, β and γ, corresponding to the roots of unity ζ, ζ' and ζ'', respectively, under the identification mentioned in theorem 1.

Finally, two partial results are presented in the case of even ranks, including the following (theorem 4.28):

Theorem 4. *Assume $n = 2\tilde{n}$, where $\tilde{n}$ is odd. Suppose $\alpha,\beta \in J^{(n)}$ are primitive elements with $\langle\alpha\rangle \cap \langle\beta\rangle = 0$ and $\lambda_n(\alpha,\beta) = 1$. Let $a \in \frac{\alpha}{2}$, $b \in \frac{\beta}{2}$, and define $\gamma = \alpha+\beta$ and $c = a+b$. For each triple $\zeta,\zeta',\zeta'' \in \mu_n$, we have:*

$$|M(n,\mathcal{O})|^{\zeta,\zeta',\zeta''}_{a,b,c} \neq \emptyset \Leftrightarrow \left(\frac{\zeta''}{\zeta\zeta'}\right) = \lambda_{2n}(a,b)^{\frac{n}{2}}.$$

–Here $|M(n,\mathcal{O})|^{\zeta,\zeta',\zeta''}_{a,b,c}$ denotes the intersection of the fixed point components of α, β and γ, corresponding to the pairs (ζ,a), (ζ',b) and (ζ'',c), respectively, under the identification mentioned in theorem 1.

Chapter 5

The fifth chapter recollects the notion of elementary modification and introduces the Hecke correspondence in degrees zero and one. Furthermore, some auxiliary results for use in chapter 6 are given.

Chapter 6

The sixth chapter introduces certain lifts of the action to the determinant line bundle on the moduli spaces.

In the case of n being odd, the group of lifts is generated by elements: $\rho_{\alpha,d}$, ($\alpha \in J^{(n)}$), uniquely determined by the demand that $\rho_{\alpha,d}$ act trivially on fibres above the connected component of $|M(n,\Delta_d)|_\alpha$ corresponding to $1 \in \mu_r$ ($r = (n,d)$). (In the case $d = 0$, it is the component containing the bundle: $\bigoplus_{i=1}^n L_\alpha^{\otimes i}$, where L_α is the line bundle corresponding to α.)

In the case of n being even, the group of lifts is generated by elements: $\rho_{a,d}$, ($a \in J^{(2n)}$), uniquely determined by the demand that $\rho_{a,d}$ act trivially on fibres above the connected component of $|M(n,\Delta_d)|_\alpha$ corresponding to $(1,a) \in \mu_r \times \frac{\alpha}{2}$. ($\alpha = 2a$ and $r = (n,d)$). (In the case $d = 0$, it is the component containing the bundle: $\bigoplus_{i=1}^n L_a^{\otimes(2i-1)}$, where L_a is the line bundle corresponding to a.)

In both cases, the group of lifts constitute a central extension of $J^{(n)}$.

The main results of the chapter include a determination of the bilinear form of the central extension induced by the group of lifts in degree zero (proposition 6.13 in the thesis):

Theorem 5. *When $d = 0$, the alternating form of the central extension consisting of the lifts is the Weil pairing. In other words:*

For $\alpha, \beta \in J^{(n)}$, we have when n is odd,

$$\rho_{\alpha,0}\rho_{\beta,0}(\rho_{\alpha,0})^{-1}(\rho_{\beta,0})^{-1} = \lambda_n(\alpha,\beta).$$

For $a, b \in J^{(2n)}$, $\alpha = 2a$, $\beta = 2b$, we have when n is even,

$$\rho_{a,0}\rho_{b,0}(\rho_{a,0})^{-1}(\rho_{b,0})^{-1} = \lambda_n(\alpha,\beta).$$

Finally, a result about the complete presentation of the group of lifts, in terms of generators and relations, is given in the special case where n is an odd prime (conjecture 6.14). It is "almost" a theorem, in the sense that a proof existed until an error was located in one of the technical lemmas, during the final hours of writing. Hence the result has been downgraded to a conjecture, giving the partial proof and stressing what remains for the proof to be complete. It is strongly believed that the problems can be sorted out. In case they are solved, an erratum will be made available as soon as possible, turning the conjecture into a theorem.

Conjecture 6. *Assume n is an odd prime. We have for all $\alpha, \beta \in J^{(n)}$:*

$$\rho_{\alpha,d}\rho_{\beta,d} = \rho_{\alpha+\beta,d} \quad , \quad d \neq 0$$

and

$$\rho_{\alpha,0}\rho_{\beta,0} = \lambda_n(\alpha,\beta)^{\frac{n+1}{2}} \cdot \rho_{\alpha+\beta,0}$$

Chapter 7

The seventh and final chapter puts the whole work of the thesis into the broader perspective of topological quantum field theory. The chapter contains no new material, but serves mainly as a survey over the construction of $(2+1)$-dimensional TQFT from the gauge theoretic approach, and how the project of the thesis relates to this construction.

2
Preliminaries

This chapter covers some of the background material needed in the rest of the thesis. Most of the contents are already treated very well elsewhere, and are listed mainly for the sake of easy reference. Further details can be found in [31], [30], [28], [24], and [27]. Also included are proofs of some "well known" facts for which I have not been able to find suitable references, as well as some fundamental technical results concerning moduli spaces. Finally, the section also serves to fix notation.

Throughout the chapter, Σ will denote a compact Riemann surface of genus $g \geq 2$.

2.1 The Picard group, divisors and the Jacobian

The set of isomorphism classes of holomorphic line bundles on Σ constitute a group, $\mathrm{Pic}(\Sigma)$ with tensoring and dualisation as its operations. Once in a while, working with an element $L \in \mathrm{Pic}(\Sigma)$ requires picking a bundle representing it. I will often use the same symbol to denote both the bundle and its isomorphism class, sometimes not even mentioning the choice of representative. Of course, this is only done when the conclusions are independent of the choices made.

The group of divisors (i.e. finitely supported maps $\Sigma \to \mathbb{Z}$) is denoted $\mathrm{Div}(\Sigma)$. The degree of a divisor is the sum of its values, and the subgroup of degree zero divisors is denoted $\mathrm{Div}_0(\Sigma)$. Any nonzero meromorphic function g on Σ defines a divisor (g), whose value at $x \in \Sigma$ is the order of the first non-zero term in a Laurant series expansion of g around x. Divisors arising in this way are called *principal* and the subgroup of principal divisors is denoted $\mathrm{Div}_{\mathrm{pr}}(\Sigma)$.

Notice that principal divisors are always of degree zero by the residue theorem.

The Jacobian variety of Σ is defined by: $J(\Sigma) = H^1(\Sigma, \mathcal{O})/H^1(\Sigma, \mathbb{Z})$, where $\mathcal{O}$ is the sheaf of holomorphic functions on Σ. By Serre duality, it is isomorphic to $H^0(\Sigma, \Omega)^*/H_1(\Sigma, \mathbb{Z})$ (where the inclusion of $H_1(\Sigma, \mathbb{Z})$ in $H^0(\Sigma, \Omega)^*$ is given by integration along cycles). It is a complex torus of dimension equal to the genus of Σ.

There are several identifications between the groups introduced above, all of which will be used frequently throughout the thesis. I shall summarise them below for the sake of easy reference.

Lemma 2.1. $\mathrm{Pic}(\Sigma) \cong H^1(\Sigma, \mathcal{O}^*)$.

Proof. Given a line bundle L on Σ, choose a covering $(U_i)_{i\in I}$ of Σ with local trivialisations of L. On each intersection, $U_i \cap U_j$, the coordinate-change function h_{ij} is holomorphic and non-zero. Clearly, (h_{ij}) is a cocycle. The induced cohomology class in $H^1(\Sigma, \mathcal{O}^*)$ depends only on the isomorphism class of L and is independent of the choice of trivialisations. Conversely, the line bundle may be reconstructed (up to isomorphism) by gluing trivial bundles on each U_i together, using the non-zero holomorphic functions as gluing functions on overlaps. □

Lemma 2.2. $\mathrm{Div}(\Sigma) \cong H^0(\Sigma, \mathcal{M}^*/\mathcal{O}^*)$.

Proof. The map going to the right is given by choosing small neighbourhoods U_x around each $x \in \mathrm{Supp}(D)$ and meromorphic functions g_x on U_x with $(g_x) = D|_{U_x}$. These, along with the constant function 1 on $\Sigma \setminus \mathrm{Supp}(D)$, generate an element in $H^0(\Sigma, \mathcal{M}^*/\mathcal{O}^*)$. The map is clearly an isomorphism. □

Notice that the principal divisors correspond, under this identification, to the image of $H^0(\Sigma, \mathcal{M}^*) \to H^0(\Sigma, \mathcal{M}^*/\mathcal{O}^*)$ in the long exact sequence induced by $0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0$.

Lemma 2.3. $\mathrm{Div}(\Sigma)/\mathrm{Div}_{\mathrm{pr}}(\Sigma) \cong H^1(\Sigma, \mathcal{O}^*)$.

Proof. This follows from 2.2 and the long exact sequence induced by the short exact sequence: $0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0$. The map is induced by the degree zero Bockstein map. It is surjective, because $H^1(\Sigma, \mathcal{M}^*) = 0$. (See for instance p.215-216 in [25]). □

Lemma 2.4. $\mathrm{Div}(\Sigma)/\mathrm{Div}_{\mathrm{pr}}(\Sigma) \cong \mathrm{Pic}(\Sigma)$.

Proof. This is simply composing 2.1 with 2.3. □

Remark 2.5. Given a divisor D on Σ the corresponding element in $\mathrm{Pic}(\Sigma)$ is denoted by $[D]$. It is given explicitly as follows: Choose a covering U_i of Σ and meromorphic functions g_i on U_i such that $(g_i) = D|_{U_i}$. Now, for each i, j, $\frac{g_j}{g_i} \in \mathcal{O}^*(U_i \cap U_j)$, and hence they define an element of $H^1(\Sigma, \mathcal{O}^*)$. Gluing trivial bundles on the U_i together with the $\frac{g_j}{g_i}$ as coordinate-change functions gives $[D]$. Notice that the g_i piece together to give a meromorphic section in $[D]$ with divisor D. This section gives an isomorphism between the sheaf of holomorphic sections of $[D]$ and the sheaf $\mathcal{O}_D$ of meromorphic functions g with $(g) \geq -D$.

For an element $L = [D] \in \mathrm{Pic}(\Sigma)$, we define the *degree* of L as the degree of D. The subgroup of degree zero elements in $\mathrm{Pic}(\Sigma)$ is denoted $\mathrm{Pic}_0(\Sigma)$. This is of course isomorphic to $\mathrm{Div}_0(\Sigma)/\mathrm{Div}_{\mathrm{pr}}(\Sigma)$.

Lemma 2.6. $J(\Sigma) \cong \mathrm{Pic}_0(\Sigma)$.

Proof. The long exact sequence induced by the exponential short exact sequence: $0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$ yields (since the induced map $H^0(\Sigma, \mathcal{O}) \xrightarrow{\exp} H^0(\Sigma, \mathcal{O}^*)$ is surjective):

$$0 \to H^1(\Sigma, \mathbb{Z}) \to H^1(\Sigma, \mathcal{O}) \to H^1(\Sigma, \mathcal{O}^*) \to H^2(\Sigma, \mathbb{Z}).$$

The kernel of the rightmost map corresponds to degree zero divisors under the map from Lemma 2.3, and hence to $\mathrm{Pic}_0(\Sigma)$ under the map from Lemma 2.1. In fact, the map associates to a line bundle its first Chern class, and it can be shown that for a divisor D on Σ: $c_1([D]) = \deg(D)$, when using the identification: $H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}$, given by the orientation of Σ. (See section 3.4 in [31]). □

Remark 2.7. Given an element $\alpha \in J(\Sigma)$, the corresponding element in $\mathrm{Pic}_0(\Sigma)$ is denoted by L_α.

Given $n \in \mathbb{N}$, the subgroups of n-torsion elements in $J(\Sigma)$ is denoted by $J^{(n)}(\Sigma)$, or sometimes simply $J^{(n)}$. It is abstractly isomorphic to $\mathbb{Z}_n^{2g}$. In fact, we have canonically:

$$J^{(n)}(\Sigma) \cong H^1(\Sigma, \mathbb{Z})/(n \cdot H^1(\Sigma, \mathbb{Z})) \cong H^1(\Sigma, \mathbb{Z}_n) \cong \mathrm{Hom}(H_1(\Sigma, \mathbb{Z}), \mathbb{Z}_n) \quad (2.1)$$

For later use, I note the following explicit isomorphism: Let $\alpha \in J^{(n)}(\Sigma)$. Let L_α be the corresponding element in $\mathrm{Pic}_0(\Sigma)$ and $D_\alpha \in \mathrm{Div}(\Sigma)$ such that $[D_\alpha] = L_\alpha$. The fact that α is n-torsion means that nD_α is principal, so choose $f_\alpha \in \mathcal{M}(\Sigma)$ such that $(f_\alpha) = nD_\alpha$.

Lemma 2.8. *The map:* $\phi : J^{(n)}(\Sigma) \to \mathrm{Hom}(H_1(\Sigma, \mathbb{Z}), \mu_n)$*, sending* α *to* $(\gamma \mapsto \exp(\frac{1}{n}\int_\gamma \frac{df_\alpha}{f_\alpha}))$*, is a well defined isomorphism.*

Proof. First of all, notice that $\exp(\frac{1}{n}\int_\gamma \frac{df_\alpha}{f_\alpha})$ is an n'th root of unity for every $\gamma \in H_1(\Sigma, \mathbb{Z})$. This is because locally and away from the poles and zeros of f_α, we have: $\frac{df_\alpha}{f_\alpha} = d\log f_\alpha$ for any choice of logarithm. So, integrating is just a matter of summing differences between local logarithms, i.e. integer multiples of $2\pi i$.

Furthermore, f_α is determined by α up to multiplication by an n'th power of a non-zero meromorphic function, say h. But this simply adds $n \cdot \frac{dh}{h}$ to $\frac{df_\alpha}{f_\alpha}$, and hence an integer multiple of $n \cdot 2\pi i$ to the integral. This shows that the map is well defined.

Since ϕ is clearly a group homomorphism, it remains only to show that the kernel is trivial. Suppose $\alpha \in \mathrm{Ker}(\phi)$. Choose a point $p_0 \in \Sigma$ and define $g \in \mathcal{M}(\Sigma)$ by:

$$g(p) = \exp\left(\frac{1}{n}\int_{p_0}^{p} \frac{df_\alpha}{f_\alpha}\right).$$

The assumption assures that this is independent of the path chosen for the integration. Locally (in a neighbourhood around $\tilde{p}$ where f_α has a logarithm), $g(p) = g(\tilde{p}) \cdot \exp(\frac{1}{n}(\log f(p) - \log f(\tilde{p})))$. Hence g is meromorphic, and g^n is a scalar multiple of f_α. Thus, $nD_\alpha = (f_\alpha) = n(g)$. This shows that D_α is principal, i.e. $\alpha = 0$. □

Remark 2.9. Since $J(\Sigma)$ is a complex torus, it possible, for any $\alpha \in J(\Sigma)$ and any $k \in \mathbb{N} \setminus 0$ to find an $\tilde{\alpha} \in J(\Sigma)$ with $k\tilde{\alpha} = \alpha$. $J^{(k)}(\Sigma)$ then acts freely and transitively on the set of such $\tilde{\alpha}$.

Notation 2.10. For each $\alpha \in J(\Sigma)$, and every $k \in \mathbb{N} \setminus 0$ denote:

$$\frac{\alpha}{k} = \{\tilde{\alpha} \in J(\Sigma) \mid k\tilde{\alpha} = \alpha\}.$$

2.2 The Weil pairing

The Weil pairing plays an important role in the project. This section recollects the definition and some basic properties.

Let f be a non-zero meromorphic function on Σ. Let $D = \sum n_j x_j$ be a divisor on Σ with $n_j = 0$ whenever x_j is a pole or a zero of f. We define:

$$f(D) = \prod f(x_j)^{n_j} \in \mathbb{C} \setminus \{0\}.$$

Let $f, g \in \mathcal{M}(\Sigma)$ and D, D' be divisors on Σ.

Lemma 2.11. *Let $f, g \in \mathcal{M}(\Sigma)$, $D, D' \in \mathrm{Div}(\Sigma)$, and ϕ an automorphism of Σ. We have the following, whenever both sides of the equations are defined:*

- $fg(D) = f(D)g(D)$
- $f(D + D') = f(D)f(D')$
- $f(\phi^*(D)) = (\phi^{-1})^*(f)(D)$

Weil reciprocity is the statement that whenever f and g are meromorphic functions on Σ and the set of poles and zeros of f is disjoint from the one of g, then $f((g)) = g((f))$. (Recall that (f) denotes the divisor, whose value at x is the order of the first non-zero term in a Laurant series expansion of f around x.)

Remark 2.12. Weil reciprocity is easily derived on $\mathbb{CP}^1$: Suppose $f, g \in \mathcal{M}(\mathbb{CP}^1)$ have divisors with disjoint support. Chose a homogeneous chart, such that both f and g are defined and non-zero at infinity. Hence f and g induce meromorphic functions on $\mathbb{C}$ which are bounded and non-zero near infinity. By compactness of $\mathbb{CP}^1$, f and g have only finitely many zeros and poles. Denote these $\{z_1, z_2 \ldots, z_k\}$ for f and $\{w_1, w_2, \ldots, w_l\}$ for g. Let $a_i = \mathrm{ord}_{z_i}(f)$ and $b_i = \mathrm{ord}_{w_i}(g)$. Since f and g both have as many poles as zeros (counted with multiplicities), $h_1(z) = \prod(z - z_i)^{a_i} f(z)$ and $h_2(z) = \prod(z - w_i)^{b_i} g(z)$ are holomorphic and bounded on $\mathbb{C}$ and thus constant. Using this, we may write: $f(z) = K_1 / \prod_i (z - z_i)^{a_i}$ and $g(z) = K_2 / \prod_i (z - w_i)^{b_i}$, $(K_1, K_2 \in \mathbb{C} \setminus \{0\})$. Now:

$$f((g)) = \frac{K_1^{\sum b_j}}{\prod_{i,j}(w_j - z_i)^{a_i b_j}} = \frac{1}{\prod_{i,j}(w_j - z_i)^{a_i b_j}}$$

$$g((f)) = \frac{K_2^{\sum a_j}}{\prod_{i,j}(z_j - w_i)^{b_i a_j}} = \frac{1}{\prod_{i,j}(z_j - w_i)^{b_i a_j}}$$

These differ only by a factor $\prod_{i,j}(-1)^{a_i b_j} = \prod_i ((-1)^{a_i})^{\sum b_j} = 1$. This proves the special case of Weil reciprocity. In fact, the general case can now be derived, using some easy results of section 2.3 (see remark 2.21).

Now let $n \in \mathbb{N}$. The order n Weil pairing on a compact Riemann surface, Σ, can be defined as follows[1]: Suppose $\alpha, \beta \in J(\Sigma) \cong \mathrm{Div}_0(\Sigma)/\mathrm{Div}_{\mathrm{pr}}(\Sigma)$ are of order n. Choose generators D_α and D_β in $\mathrm{Div}_0(\Sigma)$. The fact that α and β are of

[1] See lemma 2.14, however.

order n means that nD_α and nD_β are principal. Pick $f_\alpha, f_\beta \in \mathcal{M}(\Sigma)$ such that $(f_\alpha) = nD_\alpha$ and $(f_\beta) = nD_\beta$.

Definition 2.13. The Weil pairing of α and β is $\lambda_n(\alpha, \beta) = \frac{f_\alpha(D_\beta)}{f_\beta(D_\alpha)} \in \mu_n$

Lemma 2.14. *The above is independent of all choices made.*

Proof. First of all one has to argue that D_α and D_β can be chosen with disjoint support. This follows essentially from the Riemann-Roch theorem and corollary 17.16 in [30]:

1. $\dim H^0(\Sigma, \mathcal{O}_D) - \dim H^1(\Sigma, \mathcal{O}_D) = 1 - g + \deg(D)$ for all $D \in \mathrm{Div}(\Sigma)$.
2. $\dim H^1(\Sigma, \mathcal{O}_D) = 0$, whenever $\deg(D) > 2g - 2$.

If $D_\alpha = \sum_{i=1}^N a_i x_i$ represents α, then choosing $x \in \Sigma \setminus \{x_1, \ldots, x_N\}$ and putting $B = (2g-1)x$ and $C_i = (2g-1)x + 1x_i$, we may choose elements: $f_i \in H^0(\Sigma, \mathcal{O}_{C_i}) \setminus H^0(\Sigma, \mathcal{O}_B)$ (the dimensions of the two spaces being $g+1$ resp. g). In other words, f_i has a simple pole in x_i, and no other poles, except possibly in x. By adding constants to the f_i, ensure that $\mathrm{ord}_{x_j}(f_i) = \delta_{i,j}$. (I.e. that f_i is non-zero at x_j for $i \neq j$.) Hence, by adding or subtracting (f_i) to D_β the desired result follows.

Finally one must show that $\lambda_n(\alpha, \beta)$ is an n'th root of unity and that it is independent of the choices made; i.e. that its value does not change when f_α and f_β are multiplied by holomorphic functions (i.e. constants) and when principal divisors are added to D_α and D_β (at least as long as their supports remain disjoint). These properties all follow from the Weil reciprocity law and the calculational rules stated earlier. ☐

Lemma 2.15. *The following properties of the Weil pairing are all consequences of the definition and the calculational rules mentioned above.*

- $\lambda_n(\alpha + \beta, \gamma) = \lambda_n(\alpha, \gamma)\lambda_n(\beta, \gamma)$ *for all* $\alpha, \beta, \gamma \in J^{(n)}(\Sigma)$.
- $\lambda_n(\alpha, \beta) = \lambda_n(\beta, \alpha)^{-1}$ *for all* $\alpha, \beta \in J^{(n)}(\Sigma)$.
- $\lambda_{kn}(\alpha, \beta)^k = \lambda_n(k\alpha, k\beta)$ *for all* $\alpha, \beta \in J^{(kn)}(\Sigma)$, $k \in \mathbb{N}$.
- $\lambda_{kn}(\alpha, \beta) = \lambda_n(\alpha, \beta)^k$ *for all* $\alpha, \beta \in J^{(n)}(\Sigma)$, $k \in \mathbb{N}$.

I conclude the section with a proof of the fundamental property of the Weil pairing.

Proposition 2.16. *The Weil pairing is a perfect pairing. In other words, the homomorphism: $J^{(n)}(\Sigma) \to \mathrm{Hom}(J^{(n)}(\Sigma), \mu_n)$, given by $\delta \mapsto \lambda_n(-, \delta)$ is an isomorphism.*

More explicitly, we show that under the isomorphism, ϕ, from lemma 2.8, we get for $\alpha, \beta \in J^{(n)}(\Sigma)$:

$$\lambda_n(\alpha, \beta) = [\phi(\alpha), \phi(\beta)]$$

–Where $[-,-]$ is the perfect pairing, induced on $\mathrm{Hom}(H_1(\Sigma, \mathbb{Z}), \mu_n)$ by the intersection pairing.

Proof. First choose a canonical basis $\gamma_1, \gamma_2, \ldots, \gamma_{2g}$ for $H_1(\Sigma, \mathbb{Z})$. The intersection pairing, given by: $\gamma_i \cap \gamma_j = \delta_{i+g,j}$ when $i \leq g$ and $\gamma_i \cap \gamma_j = -\delta_{i-g,j}$ when $i > g$, induces a perfect pairing on $\mathrm{Hom}(H_1(\Sigma, \mathbb{Z}), \mathbb{Z})$ defined by:

$$\{\psi_1, \psi_2\} = \sum_{i=1}^{g} \psi_1(\gamma_i)\psi_2(\gamma_{i+g}) - \psi_1(\gamma_{i+g})\psi_2(\gamma_i),$$

and this, in turn, defines a perfect μ_n-valued pairing on $\mathrm{Hom}(H_1(\Sigma, \mathbb{Z}), \mu_n)$ as follows: Given $\phi_1, \phi_2 \in \mathrm{Hom}(H_1(\Sigma, \mathbb{Z}), \mu_n)$, we may choose lifts: $\psi_1, \psi_2 \in \mathrm{Hom}(H_1(\Sigma, \mathbb{Z}), \mathbb{Z})$, such that $\phi_i = \exp(\frac{2\pi i}{n}\psi_i)$. We define (independently of that choice):

$$[\phi_1, \phi_2] = \exp(\frac{2\pi i}{n}\{\psi_1, \psi_2\}).$$

Now, let $\alpha, \beta \in J^{(n)}(\Sigma)$. Choose $D_\alpha, D_\beta, f_\alpha$ and f_β be as in lemma 2.8. Clearly, $\psi_\alpha : (\gamma \mapsto \frac{1}{2\pi i}\int_\gamma \frac{df_\alpha}{f_\alpha})$ and $\psi_\beta : (\gamma \mapsto \frac{1}{2\pi i}\int_\gamma \frac{df_\beta}{f_\beta})$ are lifts, respectively, of $\phi(\alpha)$ and $\phi(\beta)$ through $\exp(\frac{2\pi i}{n})$. (Depending of course on the choice of f_α and f_β). Hence,

$$\begin{aligned} [\phi(\alpha), \phi(\beta)] &= \exp\left(\frac{2\pi i}{n}\{\psi_\alpha, \psi_\beta\}\right) && (2.2) \\ &= \exp\left(\frac{1}{n2\pi i}\sum_{i=1}^{g}\left(\int_{\gamma_i}\frac{df_\alpha}{f_\alpha}\int_{\gamma_{i+g}}\frac{df_\beta}{f_\beta} - \int_{\gamma_{i+g}}\frac{df_\alpha}{f_\alpha}\int_{\gamma_i}\frac{df_\beta}{f_\beta}\right)\right). \end{aligned}$$

To see that this is in fact the Weil pairing of α and β, we imitate an argument made in [29]: Consider the topological representation of Σ as the quotient of an open polygon Δ with $4g$ edges identified with $\gamma_1, \gamma_2, \gamma_1^{-1}, \gamma_2^{-1}, \ldots, \gamma_{2g}$. Furthermore, let x_0 be the common base point of the γ_i, and choose disjoint curves α_i, connecting x_0 to each of the zeros and poles p_i of f_α.

Now, let $\Delta' = \Delta \setminus \bigcup_i \alpha_i$. Since Δ' is simply connected and f_α has neither zeros nor poles in Δ', we may choose a holomorphic logarithm, $\log f_\alpha$, in Δ'. Define: $\phi = \log f_\alpha \frac{df_\beta}{f_\beta}$. This is a meromorphic 1-form on Δ' which has a simple pole at each pole and zero, q, of f_β, with residue given by: $\mathrm{ord}_q(f_\beta) \cdot \log f_\alpha(q)$. Consequently, by the residue theorem:

$$\int_{\partial\Delta'} \phi = 2\pi i \cdot \sum_{q \in \mathrm{supp} D_\beta} \mathrm{ord}_q(f_\beta) \cdot \log f_\alpha(q)$$

(Here, $\int_{\partial\Delta'}$ means integration along a curve in Δ' running very close alongside the boundary.)

On the other hand, we may calculate the integral piece by piece as follows. For points, $p \in \gamma_i$, $p' \in \gamma_i^{-1}$ on $\partial\Delta'$ identified on Σ,

$$\log f_\alpha(p') - \log f_\alpha(p) = \int_{\gamma_{i+g}} d \log f_\alpha = \int_{\gamma_{i+g}} \frac{df_\alpha}{f_\alpha},$$

whereas by continuity, $\frac{df_\beta}{f_\beta}(p') = \frac{df_\beta}{f_\beta}(p)$. Hence,

$$\int_{\gamma_i + \gamma_i^{-1}} \phi = \left(\int_{\gamma_i} \frac{df_\beta}{f_\beta} \right) \left(- \int_{\gamma_{i+g}} \frac{df_\alpha}{f_\alpha} \right).$$

Similarly:

$$\int_{\gamma_{i+g} + \gamma_{i+g}^{-1}} \phi = \left(\int_{\gamma_{i+g}} \frac{df_\beta}{f_\beta} \right) \left(\int_{\gamma_i} \frac{df_\alpha}{f_\alpha} \right).$$

Furthermore, for points $p \in \alpha_i$, $p' \in \alpha_i^{-1}$ on $\partial\Delta'$ identified on Σ,

$$\log f_\alpha(p) - \log f_\alpha(p') = \mathrm{Res}(d \log f_\alpha, p_i) = \mathrm{ord}_{p_i}(f_\alpha),$$

whereas, again, $\frac{df_\beta}{f_\beta}(p') = \frac{df_\beta}{f_\beta}(p)$. Hence,

$$\int_{\alpha_i + \alpha_i^{-1}} \phi = 2\pi i \cdot \mathrm{ord}_{p_i}(f_\alpha) \cdot \int_{\alpha_i} \frac{df_\beta}{f_\beta}.$$

Choosing logarithms of f_β on neighbourhoods of each α_i, making sure that they agree in x_0, we get:

$$\sum_i \int_{\alpha_i+\alpha_i^{-1}} \phi \quad = 2\pi i \sum_i \mathrm{ord}_{p_i}(f_\alpha)(\log f_\beta(p_i) - \log f_\beta(x_0))$$
$$= 2\pi i \sum_{p\in \mathrm{supp} D_\alpha} \mathrm{ord}_p(f_\alpha)\cdot \log f_\beta(p)$$

Thus, equating the two expressions for $\int_{\partial\Delta'}\phi$ shows that the sum in (2.2) above is equal to:

$$2\pi i \Bigg(\sum_{q\in \mathrm{Supp} D_\beta} \mathrm{ord}_q(f_\beta)\cdot \log f_\alpha(q) - \sum_{p\in \mathrm{Supp} D_\alpha} \mathrm{ord}_p(f_\alpha)\cdot \log f_\beta(p) \Bigg).$$

Hence, exponentiating:

$$[\phi(\alpha),\phi(\beta)] = \prod_{q\in \mathrm{supp} D_\beta} f_\alpha(q)^{\frac{1}{n}\mathrm{ord}_q(f_\beta)} \Bigg/ \prod_{p\in \mathrm{supp} D_\alpha} f_\beta(p)^{\frac{1}{n}\mathrm{ord}_p(f_\alpha)} = \lambda_n(\alpha,\beta)$$

□

2.3 Holomorphic coverings and the norm map

Suppose that $\pi : \Sigma' \to \Sigma$ is a (branched) covering of compact Riemann surfaces. There are several maps, called the *Norm* or *Albanese* map, associated to π:

Definition 2.17. $\mathrm{Nm} : \mathcal{M}(\Sigma') \to \mathcal{M}(\Sigma)$ is defined by

$$(\mathrm{Nm}(f))(x) = \prod_{y_i\in\pi^{-1}(x)} f(y_i)^{\nu(y_i)} \qquad (f\in\mathcal{M}(\Sigma')),$$

where $\nu(y_i)$ is the multiplicity with which π takes the value x at y_i.

It is easy to see that $\mathrm{Nm}(f)$ is in fact a well-defined meromorphic function on Σ: Locally (around a point $y\in\Sigma'$ and $x=\pi(y)$), π is simply the function: $\{z\in\mathbb{C} \mid |z|<1\} \to \{z\in\mathbb{C} \mid |z|<1\}$ given by $z\mapsto z^k$ where k is the multiplicity of π at y. Hence, it is sufficient to show that for any meromorphic function f on the disc, the function $h=\prod_{\zeta\in\mu_k} f(\zeta z)$ factors through $z\mapsto z^k$. But this is indeed the case, since a Laurant series expansion of h can only contain non-zero terms in degrees given by integer multiples of k. (The function $z \mapsto h(z) - h(\zeta_n z)$ is zero.)

The following result can be obtained, using the fact that $\mathcal{M}(\Sigma)$ is "quasi-algebraically closed." See [28] for a sketched proof.

Proposition 2.18. $\mathrm{Nm} : \mathcal{M}(\Sigma') \to \mathcal{M}(\Sigma)$ *is surjective.*

Definition 2.19. $\mathrm{Nm} : \mathrm{Div}(\Sigma') \to \mathrm{Div}(\Sigma)$ is defined by

$$(\mathrm{Nm}(D))(x) = \sum_{y_i \in \pi^{-1}(x)} \nu(x_i) D(y_i) \qquad (D \in \mathrm{Div}(\Sigma')),$$

where $\nu(y_i)$ is the multiplicity with which π takes the value x at y_i.

Notice that for $f \in \mathcal{M}(\Sigma')$, $\mathrm{Nm}((f))$ is equal to $(\mathrm{Nm}(f))$. The following lemma states that the norm map works well together with pull-backs. (Recall that evaluation of meromorphic functions on divisors was defined in 2.2.)

Lemma 2.20. *For every $f \in \mathcal{M}(\Sigma), g \in \mathcal{M}(\Sigma'), D \in \mathrm{Div}(\Sigma), E \in \mathrm{Div}(\Sigma')$, we have the following calculational rules:*

- $\mathrm{Nm}(\pi^*(f)) = f^n$
- $\mathrm{Nm}(\pi^*(D)) = nD$
- $\pi^*(f)(E) = f(\mathrm{Nm}(E))$
- $g(\pi^*(D)) = \mathrm{Nm}(g)(D)$

Remark 2.21. We can now show the general case of Weil reciprocity: Let $f, g \in \mathcal{M}(\Sigma)$. Write $f = \tilde{f}^*(h)$ where $\tilde{f} : \Sigma \to \mathbb{CP}^1$ is the induced branched covering and $h \in \mathcal{M}(\mathbb{CP}^1)$ is the meromorphic function corresponding to $1_{\mathbb{CP}^1}$. By Weil reciprocity on $\mathbb{CP}^1$: $f((g)) = \tilde{f}^*(h)((g)) = h((\mathrm{Nm}(g))) = \mathrm{Nm}(g)((h)) = g((\tilde{f}^*(h))) = g((f))$.

Except for the remark above, all the coverings appearing in this thesis will be unbranched Galois coverings. Assume therefore henceforth that π is unbranched and Galois with finite Galois group, G. In this case, $\mathrm{Nm}(f)$ for $f \in \mathcal{M}(\Sigma')$ is simply the function on $\Sigma'/G = \Sigma$ induced by the G-invariant function $\prod_{g \in G} g^* f$ on Σ'. Likewise for divisors.

Notice that Nm restricts to a homomorphism between the multiplicative groups $\mathcal{M}(\Sigma) \setminus \{0\}$ and $\mathcal{M}(\Sigma') \setminus \{0\}$. In fact, this is how it got its name; Theorems 14.13, 8.3 and 8.12 in [30] show that $\pi^* : \mathcal{M}(\Sigma) \to \mathcal{M}(\Sigma')$ is a Galois field extension of degree n, with Galois group G (acting by pull-back on $\mathcal{M}(\Sigma')$). Obviously, Nm is equal to the norm map defined in Galois theory. For later use, I state the following result on cyclic Galois extensions:

Theorem 2.22 (Hilbert's theorem 90). *Let $k \to K$ be a cyclic Galois extension with Galois group $\langle g \rangle$. For any element $x \in K \setminus \{0\}$, $\mathrm{Nm}(x) = 1$ if and only if there exists an element $y \in K \setminus \{0\}$ such that $g(y) = x^{-1}y$.*

Proof. See [39]. □

Finally, since Nm takes principal divisors to principal divisors, it induces a map: $\mathrm{Pic}(\Sigma') \to \mathrm{Pic}(\Sigma)$. Given $L \in \mathrm{Pic}(\Sigma')$, $\mathrm{Nm}(L)$ can be constructed explicitly using "descent" of the equivariant bundle $\bigotimes_{g \in G} g^*L$, as described in section 3.3.

2.4 Stable and semistable bundles

As always, let Σ be a compact Riemann surface of genus $g \geq 2$. The degree of a vector bundle E of rank n on Σ is simply the degree of its determinant bundle, $\Lambda^n(E)$. For every line bundle L on Σ, $\Lambda^n(L \otimes E) \cong \Lambda^n(E) \otimes L^{\otimes n}$. This implies that the degree is unchanged under tensoring with line bundles of degree zero. Notice also that tensoring with a line bundle of degree one increases the degree by n.

We can now introduce the notion of stable and semistable bundles. Whilst originally arising from Mumford's geometric invariant theory, the development of this notion is mainly due to Narasimhan and Seshadri (see [9]).

Definition 2.23. Let E be any holomorphic vector bundle on Σ of positive rank. The *slope* of E is:

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}.$$

E is said to be *stable* if for every proper, non-zero sub-bundle F of E, we have: $\mu(F) < \mu(E)$. E is said to be *semistable* if for every proper, non-zero sub-bundle F of E, we have: $\mu(F) \leq \mu(E)$.

Proposition 2.24. *We have the following properties:*

- *A bundle E with $\deg(E)$ and $\mathrm{rk}(E)$ coprime is stable if and only if it is semistable.*
- *Line bundles are stable.*

- *Every stable bundle E is simple. (I.e. $H^0(\Sigma, \mathrm{End}(E)) = \mathbb{C}$). In particular, every stable bundle is indecomposable.*

- *If E is (semi-)stable and L is a line bundle, then $E \otimes L$ is (semi-)stable.*

Proof. The first two properties are trivial. The latter two are proven in section 4 of [9]. □

The semistable bundles with fixed slope, $\mu \in \mathbb{R}$, constitute an abelian category, which is noetherian and artinian. The simple objects in this category are the stable bundles.

Hence, according to the theorem of Jordan-Hölder, every semistable bundle E has a filtration:

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_s = E$$

–where the quotients $D_i = E_i/E_{i-1}$ are all stable with slopes $\mu(D_i) = \mu(E)$. Furthermore, the bundle: $Gr(E) = \bigoplus_{i=1}^{s} D_i$ is uniquely determined by E up to isomorphism, even though the filtration is not. It is called the *graded object* or *graded bundle* of E.

Definition 2.25. Two semistable bundles E and E' are said to be *S-equivalent* if $Gr(E) \cong Gr(E')$.

Due to the the uniqueness of the graded bundle, any direct sum of stable bundles with equal slopes is isomorphic to its own graded bundle. In particular, S-equivalence restricts to isomorphism on stable bundles. It also follows that every semistable bundle is S-equivalent to its own graded bundle. Therefore, every S-equivalence class has a representative which is the direct sum of stable bundles with equal slopes, and is unique up to isomorphism. This is called the *graded* representative of the S-equivalence class.

The construction of $Gr(E)$ shows that $Gr(E) \cong Gr(E')$ whenever $E \cong E'$. (I.e. S-equivalence is weaker than isomorphism). Furthermore, since $\mathrm{rk}(E) = \mathrm{rk}(Gr(E))$ and $\deg(E) = \deg(Gr(E))$, both rank and degree are discrete invariants of semistable bundles up to s-equivalence.

Finally, given a line bundle L and a semistable bundle E, $Gr(E \otimes L) \cong Gr(E) \otimes L$. Hence, tensoring with line bundles is well defined on S-equivalence classes of semistable bundles.

2.5 Moduli spaces, general theory

In this section, we recollect the abstract notion of moduli spaces and go on proving a simple category theoretical result, which will be useful for determining when maps between moduli spaces are in fact morphisms.

A *moduli problem* consists of the following data:

- A category, A, whose objects we wish to parametrise up to some equivalence. Usually one restricts to fixed values of every discrete invariant of that equivalence, in order to ensure connectedness of the moduli space.
- The notion of a *family* of elements in A parametrised by a variety (sometimes more generally; a scheme or a stack) S. For a one-point variety, $\{x\}$, a family of elements in A parametrised by $\{x\}$ must be, canonically, a single object of A.

  Furthermore, any morphism of varieties, $f : S \to S'$ must give rise to a pull-back of families parametrised by S' to families parametrised by S. In particular, the inclusion of a point $s \in S$ induces the notion of "evaluating" a family parametrised by S at s.
- A notion of *equivalence of families*, such that the equivalence classes of families parametrised by a variety S constitute a set. The equivalence must be compatible with pull-backs. In particular we get an equivalence, $\sim$ on A, and $A/\sim$ is a set.
- The contravariant functor $\mathcal{J} : \mathbf{VAR} \to \mathbf{SET}$, assigning to a variety S the set of equivalence classes of families parametrised by S, and to an arrow $f : S \to S'$ the pull back of families by f. This functor is referred to as the "family functor" of the moduli problem.

Every moduli problem is eventually described by a family functor $\mathcal{J}$. The aim of moduli theory is to find, in some sense, a *universal* variety for $\mathcal{J}$. This is made precise in the following definition:

Definition 2.26. A *fine moduli space* for a moduli problem with family functor $\mathcal{J}$ consists of a variety M together with a natural isomorphism $\Phi : \mathcal{J} \dot{\to} \mathrm{Hom}(-, M)$.

Proposition 2.27. *The above definition is equivalent to saying that there exists a* universal family *parametrised by M. That is, a family $\mathcal{U}$ representing an element of*

$\mathcal{J}(M)$, such that for each variety S and each $\mathcal{F} \in \mathcal{J}(S)$ there exists a unique morphism $f : S \to M$ satisfying that $\mathcal{F} \sim f^*\mathcal{U}$.

Proof. If M is a fine moduli space, simply let $\mathcal{U}$ be a representative of $\Phi^{-1}(1_M)$. Then, given S and $\mathcal{F}$, the desired morphism is simply $f = \Phi(\mathcal{F})$. Both uniqueness of f and the fact that $\mathcal{F} \sim f^*\mathcal{U}$ follows from chasing $\mathcal{U}$ around the diagram below.

$$\begin{tikzcd} \mathcal{J}(M) \arrow[r, "\Phi"] \arrow[d, "f^*"] & \mathrm{Hom}(M, M) \arrow[d, "f^*"] \\ \mathcal{J}(S) \arrow[r, "\Phi"] & \mathrm{Hom}(S, M) \end{tikzcd}$$

Conversely, if $\mathcal{U}$ is a universal family, this induces a transformation $\Phi : \mathcal{J} \to \mathrm{Hom}(-, M)$. It is seen to be bijective by the same diagram as before. (Only, now the top vertical map sends $\mathcal{U}$ to 1_M because of the uniqueness property.) To see that it is natural, suppose $g : S_1 \to S_2$ is a morphism and consider the diagram:

$$\begin{tikzcd} \mathcal{J}(S_2) \arrow[r, "\Phi_{S_2}"] \arrow[d, "g^*"] & \mathrm{Hom}(S_2, M) \arrow[d, "g^*"] \\ \mathcal{J}(S_1) \arrow[r, "\Phi_{S_1}"] & \mathrm{Hom}(S_1, M) \end{tikzcd}$$

Let $\mathcal{F}_2 \in \mathcal{J}(S_2)$ be arbitrary. Let $f_2 = \Phi_{S_2}(\mathcal{F}_2)$, $f_1 = f_2 \circ g$ and $\mathcal{F}_1 = \Phi_{S_1}^{-1}(f_1) = f_1^*(\mathcal{U})$. Then: $g^*(\Phi_{S_2}(\mathcal{F}_2)) = g^*(f_2) = f_1$, whereas $\Phi_{S_1}(g^*(\mathcal{F}_2)) = \Phi_{S_1}(g^*(f_2^*(\mathcal{U}))) = \Phi_{S_1}(f_1^*(\mathcal{U})) = \Phi_{S_1}(\mathcal{F}_1) = f_1$.

□

However, sometimes fine moduli spaces do not exist. In those cases one may look for a weaker notion:

Definition 2.28. A *coarse moduli space* for a moduli problem with family functor $\mathcal{J}$ consists of a variety M together with a natural transformation $\Phi : \mathcal{J} \stackrel{\cdot}{\to} \mathrm{Hom}(-, M)$, which satisfies the following two conditions:

- $\Phi(\{x\}) : \mathcal{J}(\{x\}) \to \mathrm{Hom}(\{x\}, M)$ is bijective for each one-point variety $\{x\}$.
- (M, Φ) is *universal* in the sense that for every other variety N and every natural transformation $\Psi : \mathcal{J} \stackrel{\cdot}{\to} \mathrm{Hom}(-, N)$, there exists a unique natural

transformation: $\chi : \mathrm{Hom}(-, N) \dot{\to} \mathrm{Hom}(-, M)$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{J} \arrow[r, "\Psi"] \arrow[dr, "\Phi"'] & \mathrm{Hom}(-, N) \\ & \mathrm{Hom}(-, M) \arrow[u, dotted, "\chi"'] \end{tikzcd} \tag{2.3}$$

Clearly, any fine moduli space is also a coarse moduli space. Furthermore, both types are unique up to isomorphism of M (together with composition of Φ with the induced natural isomorphism of $\mathrm{Hom}(-, M)$). Indeed, if (M, Φ) and (M', Φ') are coarse moduli spaces for $\mathcal{J}$, we get, by the universality, a unique natural isomorphism: $F : \mathrm{Hom}(-, M) \dot{\to} \mathrm{Hom}(-, M')$, satisfying that $\Phi = F \circ \Phi'$. To get the desired isomorphism from M to M', we only need to apply the following two lemmas:

Lemma 2.29 (Yoneda). *For every contravariant functor K : $\mathbf{VAR} \to \mathbf{SET}$ and every variety S, the set of natural transformations $\mathrm{Hom}(-, S) \dot{\to} K$ is isomorphic to $K(S)$. The map is given by $T \mapsto T(1_S)$.*

Proof. See section 3.2 in [38] including exercise 2. □

Lemma 2.30. *In the special case where $K = \mathrm{Hom}(-, S')$ for a variety S', the mapping $f = T(1_S) \in \mathrm{Hom}(S, S')$ has the property that the induced transformation $f_* : \mathrm{Hom}(-, S) \dot{\to} \mathrm{Hom}(-, S')$, given by $(g : V \to S) \mapsto f \circ g$, is equal to T.*

Proof. Let $g \in \mathrm{Hom}(V, S)$. The naturality of T implies that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}(S, S) \arrow[r, "T"] \arrow[d, "g^*"] & \mathrm{Hom}(S, S') \arrow[d, "g^*"] \\ \mathrm{Hom}(V, S) \arrow[r, "T"] & \mathrm{Hom}(V, S') \end{tikzcd} \tag{2.4}$$

Therefore $f_*(g) = g^*(f) = g^*(T(1_S)) = T(g^*(1_S)) = T(g)$. □

Both fine and coarse moduli spaces have the property that:

$$M \cong \mathrm{Hom}(\{x\}, M) \cong J(\{x\}) \cong A/\sim$$

–Canonically, as point sets. Furthermore, the definitions ensure that all ”naturally” constructed maps into and out of $J(\{x\})$ become *morphisms* into and out of M. This statement is made precise in the following proposition:

Proposition 2.31. *Let $\mathcal{J}_1, \mathcal{J}_2 : \mathbf{VAR} \to \mathbf{SET}$ be family functors for two moduli problems. Suppose (M_ν, Φ_ν) are coarse moduli spaces for $\mathcal{J}_\nu$ ($\nu = 1, 2$). Then any natural transformation from $\mathcal{J}_1$ to $\mathcal{J}_2$ induces a canonical morphism from M_1 to M_2. Moreover, the morphism associated to $T : \mathcal{J}_1 \dot{\to} \mathcal{J}_2$ is given by the following mapping of sets:*

$$M_1 = \mathcal{J}_1(\{x\}) \xrightarrow{T(\{x\})} \mathcal{J}_2(\{x\}) = M_2 \tag{2.5}$$

Proof. Let $T : \mathcal{J}_1 \to \mathcal{J}_2$ be a natural transformation. Define $\widetilde{T}$ by the following diagram of natural transformations (using the universality of Φ_1):

$$\begin{tikzcd} \mathcal{J}_1 \arrow[r, "T"] \arrow[drr, "\Phi_1"] & \mathcal{J}_2 \arrow[r, "\Phi_2"] & \mathrm{Hom}(-, M_2) \\ & & \mathrm{Hom}(-, M_1) \arrow[u, dotted, "\widetilde{T}"] \end{tikzcd} \tag{2.6}$$

The naturality of Φ_1 and Φ_2 actually shows that $T \mapsto \widetilde{T}$ defines a natural transformation: $\mathrm{Nat}(\mathcal{J}_1, \mathcal{J}_2) \to \mathrm{Nat}(\mathrm{Hom}(-, M_1), \mathrm{Hom}(-, M_2))$. By the Yoneda lemma:

$$\begin{array}{rcl} \mathrm{Nat}(\mathrm{Hom}(-, M_1), \mathrm{Hom}(-, M_2)) & \cong & \mathrm{Hom}(M_1, M_2) \\ S & \mapsto & S(1_{M_1}) \end{array}$$

This shows the first part of the proposition. To see that the induced morphism agrees with $T(\{x\})$ on sets, one has simply to chase 1_{M_1} around the following diagram for every element $x \in M_1$:

$$\begin{tikzcd} \mathrm{Hom}(M_1, M_1) \arrow[d, "\mathrm{Ev}_x"] \arrow[rrr, bend left, "\widetilde{T}(M_1)"] & \mathcal{J}_1(M_1) \arrow[l, "\Phi_1(M_1)"'] \arrow[r, "T(M_1)"'] \arrow[d, "i^*"] & \mathcal{J}_2(M_1) \arrow[r, "\Phi_2(M_1)"'] \arrow[d, "i^*"] & \mathrm{Hom}(M_1, M_2) \arrow[d, "\mathrm{Ev}_x"] \\ M_1 & \mathcal{J}_1(\{x\}) \arrow[l, "\cong"] \arrow[r, "T(\{x\})"] & \mathcal{J}_2(\{x\}) \arrow[r, "\cong"] & M_2 \end{tikzcd} \tag{2.7}$$

□

2.6 Moduli spaces of vector bundles

2.6.1 Stable bundles

Let $A(n,d)$ denote the category of stable, holomorphic vector bundles on Σ with fixed rank, n, and degree, d. The moduli problem of $A(n,d)$ is defined as follows:

- A family of elements in $A(n,d)$, parametrised by a variety S, is an algebraic vector bundle F on $S \times \Sigma$ with the property that its restriction to $\{s\} \times \Sigma$ is stable for every $s \in S$. For a morphism $f : S \to S'$, and a family, F, parametrised by S', we define the pull-back of F by f to be simply the pull-back of vector bundles through $f \times 1_\Sigma$.
- Two families F and F', parametrised by a variety S are said to be equivalent if their restrictions to every point $s \in S$ are isomorphic. Obviously, the induced equivalence relation on $A(n,d)$ is isomorphism of bundles. The demand is equivalent to saying that F, as a bundle on $S \times \Sigma$, be isomorphic to $F' \otimes p_S^*(L)$ for some line bundle L on S. (P_S being the projection: $S \times \Sigma \to S$.) (See lemma 5.10 in [24]).
- The induced family functor $\mathcal{J}_{n,d}$ assigning to each variety S the set of equivalence classes of families parametrised by S.

For the moduli problem of stable vector bundles on Σ, we have the following results, all of which can be found in [27]. (See also later in this section for an outline of the construction.)

Theorem 2.32 (Narasimhan-Seshadri). *There exists a coarse moduli space for $\mathcal{J}_{n,d}$ for each value of n and d. It is denoted $M_s(n,d)$. When n and d are coprime, $M_s(n,d)$ is actually a fine moduli space. Generally, $M_s(n,d)$ is a smooth, irreducible, quasi-projective algebraic variety of dimension $n^2(g-1)+1$. If n and d are coprime, it is projective and normal. In particular, $M_s(1,0)$ is isomorphic to the Jacobian $J(\Sigma)$.*

Remark 2.33. In the case where n and d are coprime, the universal family $\mathcal{U}$ promised by proposition 2.27, is called the *Poincaré bundle* on $M_s(n,d) \times \Sigma$. It is defined up to isomorphism, and has the property that for every bundle E on Σ representing a point $[E] \in M_s(n,d)$, the restriction of $\mathcal{U}$ to $\{[E]\} \times \Sigma$ is isomorphic to E.

Using the results from the previous section, we now see that the following maps are in fact morphisms, because they extend to natural maps of family functors.

Proposition 2.34. *The following maps define morphisms of moduli spaces:*

- *Pull backs by endomorphisms of* $\Sigma : M_s(n,d) \to M_s(n,d)$.
- *Dualisation:* $M_s(n,d) \to M_s(n,-d)$.
- *The tensor product:* $M_s(1,d') \times M_s(n,d) \to M_s(n, n'd + nd')$
- *In particular, the isomorphisms:* $M_s(n,d) \to M_s(n, d+n)$, *given by tensoring with a fixed line bundle of degree* 1.
- *The determinant:* $M_s(n,d) \to M_s(1,d) \cong M_s(1,0) \cong J(\Sigma)$.

Proof. It is straightforward to check that all the maps extend to natural transformations of family functors, and hence by proposition 2.31 they define morphisms between the corresponding moduli spaces.

The most complicated case is the one concerning the tensor product: Use the fact that $\mathrm{Hom}(-, M \times M')$ is naturally isomorphic to $\mathrm{Hom}(-, M) \times \mathrm{Hom}(-, M')$ to show that $\mathcal{J}_{1,d'} \times \mathcal{J}_{n,d}$ is a family functor with $M_s(1,d') \times M_s(n,d)$ as a coarse moduli space, and consider the natural transformation: $\mathcal{J}_{1,d'} \times \mathcal{J}_{n,d} \dot{\to} \mathcal{J}_{n,d+nd'}$ which takes a pair of vector bundles on $\Sigma \times S$ to their tensor product.

□

2.6.2 Semistable bundles

The story about semistable bundles is slightly more complicated. There does *not* exist a "moduli space of semistable bundles" in the sense of the previous section. It is not even clear what should be the notion of a family of semistable bundles, and less so what should be the notion of equivalence between such families. (Thanks to P.E. Newstead for clearing up this point for me.)

However, the semistable bundles turn interesting in the case where n and d are not coprime. The reason is that there exists, in this case, a canonical compactification of the moduli space of stable bundles, whose points parametrise exactly s-equivalence classes of semistable bundles. It is customary to refer to this compactification as "the moduli space of semistable bundles", $M_{ss}(n,d)$.

To see that the morphisms defined on $M_s(n,d)$ extend to $M_{ss}(n,d)$, one needs to understand a bit about the way the two are constructed. In outline, it goes as follows: (Details and references may be found in [27])

Definition 2.35. Let $n \geq 2$ and $d > n(2g-1)$ be integers. Let $p = d - n(2g-1)$ and $P(T) = p + nT$. Let $\mathcal{O}^p$ denote the trivial bundle of rank p on Σ.

Let $A(P)$ denote the category of pairs $(\mathcal{F}, q)$ where $\mathcal{F}$ is a coherent sheaf on Σ with Hilbert polynomial P, and q is a surjective morphism: $\mathcal{O}^p \to \mathcal{F}$. The moduli problem of $A(P)$ is defined by the following data:

- A family of elements in $A(P)$, parametrised by a noetherian $\mathbb{C}$-scheme S is a coherent sheaf $\mathcal{F}$ on $S \times_{\mathbb{C}} \Sigma$, which is flat over Y, together with a surjective morphism: $p_\Sigma^*(\mathcal{O}^p) \to \mathcal{F}$, such that for each point $s \in S$, the Hilbert polynomial of $\mathcal{F}_s$ is P. For a morphism $f : S \to S'$ and a family $\mathcal{F}$, parametrised by S', the pull-back of $\mathcal{F}$ by f is simply pull-back under $f \times 1_\Sigma$.

- Two families, $\mathcal{F}$ and $\mathcal{F}'$, parametrised by a $\mathbb{C}$-scheme S, are said to be equivalent if there exists an isomorphism of sheaves: $g : \mathcal{F} \to \mathcal{F}'$, such that the following diagram commutes:

$$\begin{tikzcd} p_X^*(E) \arrow[r] \arrow[d, "Id"] & \mathcal{F} \arrow[d, "g"] \\ p_X^*(E) \arrow[r] & \mathcal{F}' \end{tikzcd}$$

- The induced family functor $\mathcal{J}_P$ assigning to each $\mathbb{C}$-scheme, S, the set of equivalence classes of families parametrised by S.

A deep result by Grothendieck states that there exists a fine moduli space for this moduli problem for each value of n and d as above. It is a projective $\mathbb{C}$-scheme, denoted by: $\mathrm{Quot}^P_{E/\Sigma/\mathbb{C}}$, or simply Q.

There is an action of $PGL(p)$ on Q. It is given, on the underlying set, simply as composing the quotient map q of a pair $(\mathcal{F}, q)$ with the natural action of a matrix $A \in GL(p)$ on E. Clearly, this is compatible with the equivalence relation defined above. The action of $\mathbb{C}^* I \subset GL(p)$ is trivial on equivalence classes, making the action of $PGL(p)$ well defined.

Furthermore, there is an open subset, R, of Q, which is invariant under the $PGL(p)$-action. It it defined, as a set, to be the points $[(\mathcal{F}, q)] \in Q$, for which $\mathcal{F}$ is locally free, and q induces an isomorphism: $H^0(\Sigma, \mathcal{O}^p) \to H^0(\Sigma, \mathcal{F})$. It can be shown that R is a quasi-projective, irreducible and smooth variety.

Also, one can show that every vector bundle of rank n and degree d is a quotient of $\mathcal{O}^p$ (i.e. it is the underlying sheaf of an element in R), and two such quotients yield isomorphic vector bundles if and only if they are equivalent under the $PGL(p)$-action. (Lemmas 20 and 22 in [27]).

Geometric invariant theory gives the existence of a good quotient of the stable (resp. semistable) points in R under the action of $PGL(p)$. These quotients are by definition $M_s(n,d)$, resp. $M_{ss}(n,d)$. As suggested by the name, stable (resp. semistable) points correspond exactly to stable (resp. semistable) bundles. Two semistable points become identified in the good quotient precisely if they are S-equivalent.

The properties of $M_{ss}(n,d)$ are summarised below:

Theorem 2.36 (Narasimhan-Seshadri). *$M_{ss}(n,d)$ is a complete, normal, projective variety. In the special case, $g=2, n=2, d=0$, it is smooth. In all other cases, its singular points are exactly the ones corresponding to non-stable bundles. Moreover, these points consist of strata of codimension at least 2, leaving $M_s(n,d)$ as a dense, open subset consisting of the smooth points.*

One may check that all the maps in proposition 2.34 above extend to natural transformations of the functors $\mathcal{J}_P$, and hence define morphisms between the appropriate quot-schemes.

Furthermore, it is straightforward to check that all the maps are invariant under the action of $PGL(p)$, up to equivalence under the $PGL(p)$-action. Hence they descent to the good quotient. We summarise this in a proposition:

Proposition 2.37. *The following maps define morphisms of moduli spaces:*

- *Pull backs by endomorphisms of Σ: $M_{ss}(n,d) \to M_{ss}(n,d)$.*
- *Dualisation: $M_{ss}(n,d) \to M_{ss}(n,-d)$.*
- *The tensor product: $M_{ss}(1,d') \times M_{ss}(n,d) \to M_{ss}(n,d+nd')$*
- *In particular, the morphisms: $M_{ss}(n,d) \to M_{ss}(n,d+nd')$, given by tensoring with a fixed element of $M_{ss}(1,d')$.*
- *The determinant: $M_{ss}(n,d) \to M_{ss}(1,d) \cong M_{ss}(1,0) \cong J(\Sigma)$.*

There is one final subtlety that needs to be taken care of. Notice that the quot-schemes and thus $M_{ss}(n,d)$ were constructed only for sufficiently large values of d. (Namely, $d > n(g-1)$.)

However, for $d \leq n(g-1)$, denoting by $M_{ss}(n,d)$ simply the set of S-equivalence classes of semistable bundles of rank n and degree d, one may use the bijections of sets: $M_{ss}(n,d) \to M_{ss}(n,d+nd')$, given by tensoring with a line bundle of degree d', to define variety structures on $M_{ss}(n,d)$.

This way the morphisms in proposition 2.37 all extend to morphisms in the remaining degrees. However, one needs to show that the variety structure is independent of the choice of large d'. This follows from the following results:

Lemma 2.38. *When $d > n(g-1)$ and $d' > g-1$, the bijection of sets: $M_{ss}(n,d) \to M_{ss}(n,d+nd')$ given by tensoring with a line bundle of degree d' is an isomorphism.*

Proof. By (a version of) Zariski's main theorem (see e.g. Thm. 5.2.8. in [36]), the map needs only be birational, i.e. an isomorphism between a non-empty open subsets of $M_{ss}(n,d)$ and $M_{ss}(n,d+nd')$. But according to proposition 2.34 the stable parts $M_s(n,d)$ and $M_s(n,d+nd')$ constitute exactly such non-empty open sets. □

Corollary 2.39. *The bijection of sets, $M_{ss}(n,d) \to M_{ss}(n,d+n)$ given by tensoring with a line bundle of degree one is an isomorphism.*

Notation 2.40. Throughout the thesis, I will denote $M_{ss}(n,d)$ simply by $M(n,d)$. I will refer to the image of $M_s(n,d)$ under the inclusion into $M(n,d)$ as the *stable part* of $M(n,d)$. For every $\Delta \in \mathrm{Pic}_d(\Sigma)$ I denote by $M(n,\Delta)$, the closed subvariety of $M(n,d)$ given by $\det^{-1}(\Delta)$.

Remark 2.41. As with the Picard groups, it will be convenient to adopt a bit of notational abuse, using the same notation, sometimes, for a point in the moduli space (i.e. an S-equivalence class of semistable bundles), and some chosen generator of that class. I.e. sometimes writing $E \in M(n,d)$ for a given vector bundle E.

3
The fixed point varieties

I will now turn to the action of $J^{(n)}(\Sigma)$ on $M(n,d)$.

Definition 3.1. Let $n \in \mathbb{N}$ and $d \in \{0, 1, \ldots, n-1\}$. $J^{(n)}(\Sigma)$ acts on $M(n,d)$ by tensoring with the associated line bundles. By slight abuse of notation, the automorphism of $M(n,d)$ induced by $\alpha \in J^{(n)}$ is denoted simply by:

$$\alpha : \begin{cases} M(n,d) \to M(n,d) \\ [E] \mapsto [E \otimes L_\alpha] \end{cases}$$

This chapter will be an investigation of the points fixed by an element $\alpha \in J^{(n)} = J^{(n)}(\Sigma)$. It generalises section 5 in [1], and the methods used are quite similar. As in [1], the main idea for describing the fixed points is due to Narasimhan and Ramanan ([7]). However, in some sense the results of Narasimhan and Ramanan are more general than needed here, allowing me to do a few things more explicitly. On the other hand, [7] focuses mainly on *stable* bundles. This calls for some extra trouble of generalising to semistable bundles.

For the entire chapter, fix an element $\alpha \in J^{(n)}$. Let $|M(n,d)|_\alpha$ denote the set of points in $M(n,d)$ fixed by the action of α. The investigation of $|M(n,d)|_\alpha$ begins with a few easy simplifications:

3.1 Simplifications

The first lemma explains why it suffices to consider degrees $d \in \{0, 1, \ldots, n-1\}$.

Lemma 3.2. *The isomorphism $M(n,d) \to M(n,d+n)$ given by tensoring with a line bundle of degree 1 commutes with the action of α. In particular, it induces an isomorphism of the fixed point varieties.*

Proof. This is obvious because of the commutativity of tensor products and the fact that isomorphic bundles are S-equivalent. $\square$

Lemma 3.3. *For each $\Delta \in \mathrm{Pic}_d(\Sigma)$, the closed subvariety $M(n,\Delta) \subseteq M(n,d)$ is invariant under the $J^{(n)}$-action. Furthermore, given $\Delta_1, \Delta_2 \in \mathrm{Pic}_d(\Sigma)$, $M(n,\Delta_1)$ and $M(n,\Delta_2)$ are isomorphic and the isomorphism commutes with the action of $J^{(n)}$.*

Proof. The first claim is true because $\det(E \otimes L) \cong \det(E) \otimes L^{\otimes n}$ for any E representing a point in $M(n,d)$ and any line bundle L. For the second claim, observe that $\Delta_1^{-1} \otimes \Delta_2 \in \mathrm{Pic}_0(\Sigma) \cong J(\Sigma)$. Since $J(\Sigma)$ is a complex torus, we may find an element $L \in \mathrm{Pic}_0(\Sigma)$ such that $L^{\otimes n} \cong \Delta_1^{-1} \otimes \Delta_2$. Now, tensoring with L gives an automorphism of $M(n,d)$, which by the above formula restricts to an isomorphism between $M(n,\Delta_1)$ and $M(n,\Delta_2)$. It commutes with the action of α for the same reason as in the previous lemma. $\square$

The above lemma shows that the action of α can be studied inside each of the $M(n,\Delta)$ and that it suffices to consider one Δ for each degree. We therefore introduce the following:

Notation 3.4. Pick a point $p \in \Sigma$ and define for each $d \in \{1,2,\dots,n-1\}$, $\Delta_d = [d \cdot p]$. Denote by $|M(n,\Delta_d)|_\alpha$ the intersection $|M(n,d)|_\alpha \cap M(n,\Delta_d)$.

The rest of the chapter is devoted to describing $|M(n,\Delta_d)|_\alpha$.

3.2 Primitive torsion points, associated coverings

Assume that $\alpha \in J^{(n)}$ is primitive, i.e. $\mathrm{ord}(\alpha) = n$, postponing the issue of non-primitive elements until section 3.7.

There is a cyclic, n-sheeted Galois covering, $\pi_\alpha : \Sigma^\alpha \to \Sigma$ associated to each primitive element $\alpha \in J^{(n)}$. I will construct it explicitly, and then discuss how it relates to the classification of finite Galois covers.

Recall that L_α denotes the element in $\mathrm{Pic}(\Sigma)$ corresponding to α, as well as a (chosen) line bundle representing that element. We now take the latter point of view, and fix an isomorphism: $L_\alpha^{\otimes n} \cong \mathcal{O}_\Sigma$. This will be treated as an equality. Define (topologically for now):

$$\begin{aligned}\Sigma^\alpha &= \{\xi \in L_\alpha \mid \xi^{\otimes n} = (\pi(\xi), 1) \in L_\alpha^{\otimes n} = \mathcal{O}_\Sigma\} \\ \pi_\alpha &= \pi|_{\Sigma^\alpha}\end{aligned}$$

–Where π denotes the projection in L_α. Let $\phi : L_\alpha|_U \cong U \times \mathbb{C}$ be a local trivialisation. This induces $\phi \otimes \ldots \otimes \phi : L_\alpha^{\otimes n}|_U \cong (U \times \mathbb{C})^{\otimes n} \cong U \times \mathbb{C}$ and the following diagram commutes:

$$\begin{tikzcd} & L_\alpha|_U \arrow[r, "\phi"] \arrow[d, "\xi \mapsto \xi^{\otimes n}"'] & U \times \mathbb{C} \arrow[d, "(x,\lambda)\mapsto(x,\lambda^n)"] \\ \mathcal{O}_\Sigma|_U \arrow[r, "="] & L_\alpha^{\otimes n}|_U \arrow[r, "\phi\otimes\ldots\otimes\phi"'] & U \times \mathbb{C} \end{tikzcd} \tag{3.1}$$

–Denote by s, the non-zero holomorphic section induced in the lower right $U \times \mathbb{C}$ by the constant section 1 in $\mathcal{O}_\Sigma|_U$. Choosing U small enough, we may assume that s has an n'th root — i.e. a section s_0 in the upper right $U \times \mathbb{C}$ with $s_0^n = s$. Composing ϕ with division by s_0 induces a homeomorphism: $\pi_\alpha^{-1}(U) \to U \times \mu_n$. (Recall that μ_n denotes the n'th roots of unity in $\mathbb{C}$.) Since π_α corresponds to projection on U under this homomorphism, it is clearly an n-sheeted covering.

Furthermore, the action of μ_n by multiplication in the fibres of L_α, restricts to deck transformations on Σ^α. In the local description above, the action of μ_n on $\pi_\alpha^{-1}(U) \cong U \times \mu_n$ is simply given by multiplication in μ_n. Clearly, the action is transitive on each fibre of π_α. Hence, the covering is Galois.

In fact, μ_n is the entire Galois group. This follows from uniqueness of lifts (of 1_Σ) through connected coverings, as soon as we show that Σ^α is connected.

Lemma 3.5. *Σ^α is path-connected.*

Proof. Assume that $V \subseteq \Sigma^\alpha$ is a non-empty path connected component (thus closed and open). It is easy to see that $r_x = \#(V \cap \pi_\alpha^{-1}(x))$ is independent of $x \in \Sigma$. (Σ being path-connected, the curve lifting property assures that $r_x > 0$. Finding one path between two fibres $\pi_\alpha^{-1}(x_1)$ and $\pi_\alpha^{-1}(x_2)$ and moving it with the deck-transformations gives a one-to-one correspondence between $\pi_\alpha^{-1}(x_1)$ and $\pi_\alpha^{-1}(x_2)$, taking $V \cap \pi_\alpha^{-1}(x_1)$ to $V \cap \pi_\alpha^{-1}(x_2)$.) Let r denote its constant value.

Define a section: $\Sigma \to L_\alpha^{\otimes r}$ by assigning to each $x \in \Sigma$ the value $\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_r$ where $\{\xi_1, \xi_2, \ldots, \xi_r\} = V \cap \pi_\alpha^{-1}(x)$. This is well-defined (i.e. independent of the ordering of the ξ_i's) because L_α is a line bundle. (Interchanging the ξ_i's simply amounts to moving around scalars in the tensor product).

In the local description, $\pi_\alpha^{-1}(U) \cong U \times \mu_n$. Being open and closed herein, $V \cap \pi_\alpha^{-1}(U)$ must correspond to $U \times \{\zeta_1, \ldots, \zeta_r\}$ for some $\zeta_1, \ldots, \zeta_r \in \mu_n$.

Using the induced trivialisation; $L_\alpha^{\otimes r}|_U \cong (U \times \mathbb{C})^{\otimes r} \cong U \times \mathbb{C}$, the section defined above corresponds to the constant, non-zero section $\zeta_1 \cdots \zeta_r$, which is holomorphic.

Having defined a global, holomorphic and nowhere zero section in $L_\alpha^{\otimes r}$ we see that $r = n$, since L_α was of order n. Thus, $V = \Sigma^\alpha$. □

Finally, equip Σ^α with the unique complex structure that makes π_α a local biholomorphism. (see [30] p.22).

Notice that $\pi_\alpha^*(L_\alpha)$ is trivial. Indeed, a global, non-zero section is given by mapping each $\xi \in \Sigma^\alpha$ (say, $\pi_\alpha(\xi) = x \in \Sigma$) to itself – now seen as an element of $(L_\alpha)_x = (\pi_\alpha^*(L_\alpha))_\xi$. In fact, this property determines Σ^α completely in the following sense:

Lemma 3.6. *Consider α as an element of $H^1(\Sigma, \mathbb{Z}_n) \approx \mathrm{Hom}(H_1(\Sigma), \mathbb{Z}_n)$. We then have:*

$$\pi_{\alpha*}(H_1(\Sigma^\alpha)) = \mathrm{Ker}(\alpha).$$

Any other cyclic, n-sheeted, path-connected Galois cover: $\tilde{\pi} : \tilde{\Sigma} \to \Sigma$ with $\tilde{\pi}^*(\alpha) = 0$ is (holomorphically) equivalent to $\pi_\alpha : \Sigma^\alpha \to \Sigma$.

Proof. Since the identifications $J^{(n)} \approx H^1(\Sigma, \mathbb{Z}_n) \approx \mathrm{Hom}(H_1(\Sigma), \mathbb{Z}_n)$ are natural, we have for every $x \in H_1(\Sigma^\alpha)$: $0 = \pi_\alpha^*(\alpha)(x) = \alpha(\pi_{\alpha*}(x))$, whence $\pi_{\alpha*}(H_1(\Sigma^\alpha)) \subseteq \mathrm{Ker}(\alpha)$. Conversely, consider the diagram:

$$\begin{array}{ccc} \pi_1(\Sigma^\alpha) & \xrightarrow{h} & H_1(\Sigma^\alpha) \\ \downarrow{\scriptstyle \pi_{\alpha*}} & & \downarrow{\scriptstyle \pi_{\alpha*}} \\ \pi_1(\Sigma) & \xrightarrow{h} & H_1(\Sigma) \end{array} \tag{3.2}$$

–where the horizontal maps are the natural, surjective ones with kernel the commutator subgroup of $\pi_1(\Sigma)$ resp. $\pi_1(\Sigma^\alpha)$. Proposition 1.39 in [37] states that $\pi_{\alpha*}(\pi_1(\Sigma^\alpha)) \subseteq \pi_1(\Sigma)$ is a normal subgroup with cyclic quotient of order n. In particular, it must contain the commutator of $\pi_1(\Sigma)$, and hence $\pi_{\alpha*}(H_1(\Sigma^\alpha)) = h(\pi_{\alpha*}(\pi_1(\Sigma^\alpha)))$ must also have cyclic quotient of order n. Since Ker(α) has cyclic quotient of order n, the inclusion $\pi_{\alpha*}(H_1(\Sigma^\alpha)) \subseteq \mathrm{Ker}(\alpha)$ must be an equality.

In the above diagram, h sets up a one-to-one correspondence between subgroups of $H_1(\Sigma)$ and normal subgroups of $\pi_1(\Sigma)$ containing $[\pi_1(\Sigma), \pi_1(\Sigma)]$ (i.e.

those with abelian quotient). Thus, the second claim of the lemma follows by the classification of Galois covers. (proposition 1.37 in [37]). ☐

3.3 Equivariant bundles, direct images

The following is not at all limited to the case of holomorphic or cyclic coverings. However, since I will not need further generality, I will use the terminology of the present, rather special case:

An *equivariant* bundle on Σ^α is a vector bundle E together with an action of μ_n on the total space of E (acting as bundle maps), such that for all $\zeta \in \mu_n$ the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\zeta"] \arrow[d, "\pi_E"'] & E \arrow[d, "\pi_E"] \\ \Sigma^\alpha \arrow[r, "\zeta"] & \Sigma^\alpha \end{tikzcd} \tag{3.3}$$

An equivariant bundle, E on Σ^α defines by "descent" a bundle on Σ. The total space of the descended bundle is simply the quotient of the total space of E under the action of μ_n. It is easy to see that this becomes a vector bundle with projection induced by $\pi_\alpha \circ \pi_E$.

Remark 3.7. For every bundle, F on Σ, the pull-back $\pi_\alpha^*(F)$ is equivariant. The action of $\zeta \in \mu_n$ is defined by $(x, \xi) \mapsto (\zeta(x), \xi)$ whenever $x \in \Sigma^\alpha, \xi \in F_{\pi_\alpha(x)} = F_{\pi_\alpha(\zeta(x))}$. The descent of $\pi_\alpha^*(F)$ is canonically isomorphic to F.

Two equivariant bundles E and E' on Σ^α are called isomorphic (understood "as equivariant bundles" or "equivariantly") if there is an isomorphism $\phi : E \cong E'$ such that $\phi(\zeta(\xi)) = \zeta(\phi(\xi))$ for every $\xi \in E$ and $\zeta \in \mu_n$.

Remark 3.8. Equivariant bundles define isomorphic bundles by descent if and only if they are equivariantly isomorphic. Hence, in view of the above, two bundles on Σ are isomorphic if and only if their pull-backs to Σ^α are equivariantly isomorphic.

Lemma 3.9. *Let E be a simple bundle on Σ^α. If $\zeta_n^*(E) \cong E$, then there exists a bundle F on Σ, such that $\pi_\alpha^*(F) \cong E$.*

Proof. Let F be the bundle arising from making E into a μ_n-equivariant bundle and descending it to Σ. The action of ζ_n on E is given as follows: Let ϕ be the composition of an isomorphism $E \cong \zeta_n^*(E)$ with the canonical bundle map (inducing ζ_n on the base): $\zeta_n^*(E) \to E$. Since E is simple, and ϕ^n is an automorphism of E, we may assume, by scaling, that $\phi^n = 1$. Thus ϕ generates a group action of μ_n. The quotient map $E \to F$ then defines an isomorphism: $E \cong \pi_\alpha^*(F)$. □

For any holomorphic bundle E on Σ^α we define its *push-down* or *direct image* $\pi_{\alpha*}(E)$ as the descent of the equivariant bundle $\oplus_{\zeta' \in \mu_n} \zeta'^*(E)$. The action of $\zeta \in \mu_n$ on this bundle is given by shifting summands, i.e. $(\xi_{\zeta'})_{\zeta' \in \mu_n} \mapsto (\xi_{\zeta \cdot \zeta'})_{\zeta' \in \mu_n}$ whenever the $\xi_{\zeta'}$ lie in $E_{\zeta'(x)}$ for some $x \in \Sigma^\alpha$.

Remark 3.10. Another way of describing the push-down $\pi_{\alpha*}(E)$, which will be useful in the proof of theorem 6.14 is the following: Let $\lambda \in \mu_n$ be any generator. Let $A : \lambda^*(E) \to E$ be the canonical map covering λ on Σ^α. Then,

$$\pi_{\alpha*}(E) = \left(\bigoplus_{i=0}^{n-1} \lambda^{i*} E \right) \Big/ \left\langle \begin{bmatrix} 0 & A & & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & A \\ A & 0 & \dots & 0 \end{bmatrix} \right\rangle$$

Remark 3.11. $\pi_{\alpha*}E$ depends only (up to isomorphism) on the isomorphism class of E.

Remark 3.12. Normally one constructs the push down of a bundle E by identifying E with its (locally free) sheaf of holomorphic sections. One then defines the sheaf $\pi_{\alpha*}(E)$ by: $\pi_{\alpha*}(E)(U) = E(\pi_\alpha^{-1}(U))$. This is again locally free, i.e. a vector bundle. It is easy to see that the two constructions agree.

Lemma 3.13. *We have the following properties:*

- *For every bundle E on Σ^α: $\pi_\alpha^*(\pi_{\alpha*}(E)) \cong \bigoplus_{\zeta' \in \mu_n} \zeta'^* E$ equivariantly.*
- *Given bundles $E_1, E_2, \dots, E_k$ on Σ^α: $\pi_{\alpha*} \bigoplus_{i=1}^{k} E_i \cong \bigoplus_{i=1}^{k} \pi_{\alpha*} E_i$.*
- *If E, E' are indecomposable bundles on Σ^α, then $\pi_{\alpha*}(E) \cong \pi_{\alpha*}(E')$ if and only if $E' \cong \zeta^*(E)$ for some $\zeta \in \mu_n$.*
- *For every bundle E on Σ^α and any line bundle L on Σ: $\pi_{\alpha*}(E) \otimes L \cong \pi_{\alpha*}(E \otimes \pi_\alpha^* L)$.*

Proof. The first two statements are easy consequences of the definition. The third follows from the first by the theorem of Krull-Remak-Schmidt. The last statement is known as the *pull-push formula*. It follows by pulling both sides back to Σ_α and noticing that they are equivariantly isomorphic by tracing the action of $\zeta \in \mu_n$ through the sequence of isomorphisms:

$$\begin{aligned}
\pi_\alpha^*(\pi_{\alpha *}(E) \otimes L) &\cong \pi_\alpha^*(\pi_{\alpha *}(E)) \otimes \pi_\alpha^*(L) \\
&\cong \Big(\bigoplus_{\zeta' \in \mu_n} \zeta'^* E\Big) \otimes \pi_\alpha^*(L) \\
&\cong \bigoplus_{\zeta' \in \mu_n} \zeta'^*(E \otimes \pi_\alpha^*(L)) \\
&\cong \pi_\alpha^*(\pi_{\alpha *}(E \otimes \pi_\alpha^* L)).
\end{aligned}$$

□

As mentioned before, all of the above is valid for any finite, cyclic Galois covering. Now I will start imposing the special properties of Σ^α. It has already been shown that the pull-back of L_α to Σ^α is trivial. However, as an equivariant bundle it is not trivial:

Lemma 3.14. *$\pi_\alpha^*(L_\alpha)$ is isomorphic (as an equivariant bundle) to $\mathcal{O}_{\Sigma^\alpha}^{(-1)}$. I.e. the trivial bundle with $\zeta \in \mu_n$ acting by $(x, \lambda) \mapsto (\zeta(x), \zeta^{-1}\lambda)$.*

Proof. Recall that $\pi_\alpha^*(L_\alpha)$ has a non-zero holomorphic section, s, given by mapping each $x \in \Sigma^\alpha$ to itself – now seen as an element of $(L_\alpha)_{\pi_\alpha(x)}$. This induces an isomorphism: $\pi_\alpha^*(L_\alpha) \cong \mathcal{O}_{\Sigma^\alpha}$ given by $(x, \xi) \mapsto (x, \lambda)$ where $\xi = \lambda s(x)$. Since clearly, $s(\zeta(x)) = \zeta \cdot s(x)$, it is easily verified that the claimed action of μ_n on $\mathcal{O}_{\Sigma^\alpha}$ makes the following diagram commute:

$$\begin{tikzcd}
\pi_\alpha^*(L_\alpha) \arrow[r, "(x,\xi)\mapsto(\zeta(x),\xi)"] \arrow[d, "\cong"] & \pi_\alpha^*(L_\alpha) \arrow[d, "\cong"] \\
\mathcal{O}_{\Sigma^\alpha} \arrow[r, "(x,\lambda)\mapsto(\zeta(x),\zeta^{-1}\lambda)"] & \mathcal{O}_{\Sigma^\alpha}
\end{tikzcd} \qquad (3.4)$$

□

Using this, one can determine exactly which line bundles have trivial pull-backs:

Lemma 3.15. *The kernel of* $\pi_\alpha^* : \mathrm{Pic}(\Sigma) \to \mathrm{Pic}(\Sigma^\alpha)$ *is equal to* $\langle L_\alpha \rangle$.

Proof. There are only n ways of giving $\mathcal{O}_{\Sigma^\alpha}$ the structure of a μ_n-equivariant bundle, because the action of the generator $\zeta_n \in \mu_n$ on $\mathcal{O}_{\Sigma^\alpha}$ is given by multiplication with a holomorphic (hence constant) function f on Σ^α with $1 = \mathrm{Nm}_\alpha(f) = f^n$. Since for any $k \in \mathbb{Z}$, $\pi_\alpha^*(L_\alpha^{\otimes k}) \cong (\mathcal{O}_{\Sigma^\alpha}^{-1})^{\otimes k} \cong \mathcal{O}_{\Sigma^\alpha}^{(-k)}$ (I.e. the trivial bundle with $\zeta \in \mu_n$ acting by $(x, \lambda) \mapsto (\zeta(x), \zeta^{-k}\lambda)$, all the structures are occupied and therefore, any $L \in \mathrm{Pic}(\Sigma)$ with $\pi_\alpha^*(L) \cong \mathcal{O}_{\Sigma^\alpha}$ must have $\pi_\alpha^*(L) \cong \pi_\alpha^*(L_\alpha^{\otimes k})$ as equivariant bundles for some k, and by descent, $L \cong L_\alpha^{\otimes k}$. ☐

Lemma 3.16. *For any line bundle* L *on* Σ *we have:*

$$\pi_{\alpha *}(\pi_\alpha^* L) \cong \bigoplus_{i=0}^{n-1} L \otimes L_\alpha^{\otimes i}$$

Proof. By lemma 3.13, $\pi_{\alpha *}(\pi_\alpha^* L) \cong \pi_{\alpha *}(\pi_\alpha^*(L) \otimes \mathcal{O}_{\Sigma^\alpha}) \cong L \otimes \pi_{\alpha *}(\mathcal{O}_{\Sigma^\alpha})$. Hence, it suffices to prove that $\pi_{\alpha *}(\mathcal{O}_{\Sigma^\alpha}) \cong \bigoplus_{i=0}^{n-1} {L_\alpha}^{\otimes i}$. Pulling both sides back to Σ^α yields $\mathcal{O}_{\Sigma^\alpha}^{\oplus n}$. Furthermore, the canonical equivariant actions of $\zeta_n \in \mu_n$ are given respectively by:

$$\zeta_n : (x, (\xi_1, \ldots, \xi_n)) \mapsto (\zeta_n . x, A(\xi_1, \ldots, \xi_n))$$

$$\zeta_n : (x, (\xi_1, \ldots, \xi_n)) \mapsto (\zeta_n . x, B(\xi_1, \ldots, \xi_n))$$

Where ...

$$A = \begin{bmatrix} 0 & \ldots & 0 & 1 \\ 1 & \ddots & \vdots & 0 \\ & \ddots & 0 & \vdots \\ 0 & & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & & & 0 \\ & \zeta_n^{-1} & & \\ & & \ddots & \\ 0 & & & \zeta_n^{-n+1} \end{bmatrix}$$

Since the characteristic polynomial of A is $(-T)^n + (-1)^{n-1}$, diagonalising A yields B. The invertible matrix C that gives: $C^{-1}AC = B$ induces an automorphism C of $\mathcal{O}_{\Sigma^\alpha}^{\oplus n}$ such that the following diagram commutes:

$$\begin{tikzcd}
\pi_\alpha^*(\pi_{\alpha *}(\mathcal{O}_{\Sigma^\alpha})) \arrow[r, "\cong"] \arrow[d, "\zeta_n"] & \mathcal{O}_{\Sigma^\alpha}^{\oplus n} \arrow[r, "C"] \arrow[d, "A"] & \mathcal{O}_{\Sigma^\alpha}^{\oplus n} \arrow[r, "\cong"] \arrow[d, "B"] & \pi_\alpha^* \bigoplus_{i=0}^{n-1} {L_\alpha}^{\otimes i} \arrow[d, "\zeta_n"] \\
\pi_\alpha^*(\pi_{\alpha *}(\mathcal{O}_{\Sigma^\alpha})) \arrow[r, "\cong"] & \mathcal{O}_{\Sigma^\alpha}^{\oplus n} \arrow[r, "C"] & \mathcal{O}_{\Sigma^\alpha}^{\oplus n} \arrow[r, "\cong"] & \pi_\alpha^* \bigoplus_{i=1}^{n-1} {L_\alpha}^{\otimes i}
\end{tikzcd} \tag{3.5}$$

This shows that the two pull-backs are equivariantly isomorphic and hence by descent: $\pi_{\alpha *}(\mathcal{O}_{\Sigma^\alpha}) \cong \bigoplus_{i=0}^{n-1} {L_\alpha}^{\otimes i}$ as desired. □

An immediate consequence of the pull-push formula (lemma 3.13, last item) is that every direct image of a line bundle on Σ^α is invariant under tensoring with L_α. This is the fundamental idea of Narasimhan and Ramanan [7], as described in the next section. For later use, I note the following explicit isomorphism between $\pi_{\alpha *}(L)$ and $\pi_{\alpha *}(L) \otimes L_\alpha$.

Lemma 3.17. *The map:* $\bigoplus_{j=0}^{n-1} \zeta_n^{j*}(L) \to \bigoplus_{j=0}^{n-1} \zeta_n^{j*}(L)$ *given by multiplication with the matrix*

$$B = \begin{bmatrix} 1 & & & 0 \\ & \zeta_n & & \\ & & \ddots & \\ 0 & & & \zeta_n^{n-1} \end{bmatrix}$$

defines an equivariant isomorphism: $\tilde{\psi} : \pi_\alpha^*(\pi_{\alpha *}(L)) \cong \pi_\alpha^*(\pi_{\alpha *}(L) \otimes L_\alpha)$. *Hence, it descends to an isomorphism:* $\psi : \pi_{\alpha *}(L) \to \pi_{\alpha *}(L) \otimes L_\alpha$.

Proof. By lemma 3.13 and 3.14, $\pi_\alpha^*(\pi_{\alpha *}(L) \otimes L_\alpha)$ is equivariantly isomorphic to $\bigoplus_{j=0}^{n-1} \zeta_n^{j*}(L)$ with $\zeta \in \mu_n$ acting by the usual shifting of summands composed with multiplication by ζ^{-1}. Hence, the following diagram commutes (where each of the vertical maps is the canonical action of $\zeta \in \mu_n$).

$$\begin{tikzcd}
\pi_\alpha^*(\pi_{\alpha *}(L)) \arrow[r, "\cong"] \arrow[d] & \bigoplus_{j=0}^{n-1} \zeta_n^{j*}(L) \arrow[r, "B"] \arrow[d] & \bigoplus_{j=0}^{n-1} \zeta_n^{j*}(L) \arrow[r, "\cong"] \arrow[d] & \pi_\alpha^*(\pi_{\alpha *}(L) \otimes L_\alpha) \arrow[d] \\
\pi_\alpha^*(\pi_{\alpha *}(L)) \arrow[r, "\cong"] & \bigoplus_{j=0}^{n-1} \zeta_n^{j*}(L) \arrow[r, "B"] & \bigoplus_{j=0}^{n-1} \zeta_n^{j*}(L) \arrow[r, "\cong"] & \pi_\alpha^*(\pi_{\alpha *}(L) \otimes L_\alpha)
\end{tikzcd} \tag{3.6}$$

From this, the lemma follows easily. □

3.4 Fixed points as direct images

We now aim to apply the ideas of Narasimhan and Ramanan ([7]), describing the fixed point as direct images under π_α. First we need to adapt the notation of

[7].

Denoting by $\widehat{\mu}_n$ the 1-dimensional characters on μ_n (i.e. the group of homomorphisms $\mu_n \to \mathbb{C}^*$), every element $\chi \in \widehat{\mu}_n$ gives rise to a line bundle on Σ: $L_\chi = (\Sigma^\alpha \times \mathbb{C})/_\sim$ where $(\zeta(x), \lambda) \sim (x, \chi(\zeta)\lambda)$ for all $\zeta \in \mu_n, x \in \Sigma^\alpha, \lambda \in \mathbb{C}$. This is, in fact a very general construction, which uses the fact that Σ^α is a *principal bundle* on Σ with its structure group, μ_n acting on $\mathbb{C}$ via χ. For details about principal bundles, see [32].

Now $L_{\chi_1} \otimes L_{\chi_2} \cong L_{\chi_1\chi_2}$ for all $\chi_1, \chi_2 \in \widehat{\mu}_n$. Denoting elements in L_χ by $[x, \lambda]_\chi$, an isomorphism is given by $[x, \lambda]_{\chi_1} \otimes [x, \gamma]_{\chi_2} \mapsto [x, \lambda\gamma]_{\chi_1\chi_2})$ for all $x \in \Sigma^\alpha, \lambda, \gamma \in \mathbb{C})$. This implies that the L_χ are always n-torsion points.

Tensoring with L_χ gives an action of $\widehat{\mu}_n$ on $M(n, d)$. This is the terminology throughout [7]. Since $L_{\chi_0} \cong L_\alpha$ for the canonical generator χ_0 of $\widehat{\mu}_n$ (i.e. the inclusion $\mu_n \hookrightarrow \mathbb{C}^*$) by mapping $[x, \lambda]_{\chi_0} \mapsto \lambda x$, this action is exactly the same as that of $\langle L_\alpha \rangle \subseteq J^{(n)}$.

For use in proposition 3.23, I state the following:

Lemma 3.18. *Let ψ_{reg} denote the regular representation over $\mathbb{C}$ of μ_n ($\zeta \in \mu_n$ acting on $\mathbb{C}[\mu_n]$ by multiplication on the group elements). Composing ψ_{reg} with the determinant yields a 1-dimensional character χ_{reg}. The associated line bundle $L_{\chi_{\text{reg}}}$ is trivial when n is odd and isomorphic to $L_\alpha^{\otimes n/2}$ when n is even.*

Proof. Choosing the ordered basis $(1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1})$ for $\mathbb{C}[\mu_n]$, we see that

$$\chi_{\text{reg}}(\zeta_n) = \det\left(\begin{bmatrix} 0 & \ldots & 0 & 1 \\ 1 & \ddots & \vdots & 0 \\ & \ddots & 0 & \vdots \\ 0 & & 1 & 0 \end{bmatrix}\right) = (-1)^{n-1}.$$

Thus, for n odd, χ_{reg} and hence $L_{\chi_{\text{reg}}}$ is trivial. For n even, $\chi_{\text{reg}} = \chi_0^{n/2}$ proving that $L_{\chi_{\text{reg}}} \cong L_\alpha^{\otimes n/2}$. □

Now, importing some results from [7], yields:

Proposition 3.19. *Let E be a bundle on Σ^α.*

- $\deg(\pi_{\alpha*}(E)) = \deg(E)$ *and* $\operatorname{rk}(\pi_{\alpha*}(E)) = n \cdot \operatorname{rk}(E)$.
- $\pi_{\alpha*}(E)$ *is semistable if and only if E is semistable.*

- $\pi_{\alpha*}(E)$ *is stable if and only if* E *is both stable* and *no two of the bundles* $\zeta^*(E)$ $(\zeta \in \mu_n)$ *are isomorphic.*

Proof. See [7] □

Since $\pi_{\alpha*}$ extends naturally to family functors, the previous proposition combined with the methods in 2.5 and lemma 3.13 shows that $\pi_{\alpha*}$ induces a morphism: $\mathrm{Pic}_d(\Sigma^\alpha) \to |M(n,d)|_\alpha$. The key result is the following:

Proposition 3.20. $\pi_{\alpha*} : \mathrm{Pic}_d(\Sigma^\alpha) \to |M(n,d)|_\alpha$ *is surjective.*

Proof. It is shown in [7] that every simple bundle E of rank n on Σ with $E \otimes L_\alpha \cong E$ is isomorphic to the direct image of a line bundle. In particular, every stable point of $|M(n,d)|_\alpha$ lies within the image of $\pi_{\alpha*}$.

Assume that $E = \oplus_{j=0}^{s} E_j$ is the graded representative of any semistable, but not stable, fixed point ($s \geq 1$). The assumption that E is a fixed point implies that $E \cong \oplus_{j=0}^{s} E_j \otimes L_\alpha$ (both being their own graded bundles).

Define $r = \mathrm{rk}(E_0)$ and $q = \min\{q' \in \mathbb{N}\setminus 0 \mid E_0 \otimes L_\alpha^{\otimes q'} \cong E_0\}$. Since $\det(E_0) \cong \det(E_0 \otimes L_\alpha^{\otimes q}) \cong \det(E_0) \otimes L_\alpha^{\otimes qr}$, we see that $L_\alpha^{\otimes qr} \cong \mathcal{O}_\Sigma$, whence $n \mid qr$. On the other hand, the assumption $E \cong E \otimes L_\alpha \cong \ldots \cong L_\alpha^{\otimes q-1} \otimes E$ combined with the fact that $E_i \otimes L_\alpha^{\otimes j}$ is stable and hence indecomposable for every i, j shows by the theorem of Krull-Remak-Schmidt that for each $j \in \{1, \ldots, q-1\}$, $E_0 \otimes L_\alpha^{\otimes j} \cong E_i$ for some i. Since by minimality of q, $E_0, E_0 \otimes L_\alpha, \ldots, E_0 \otimes L_\alpha^{\otimes q-1}$ are pairwise non-isomorphic, we may rearrange the E_i's to get: $E_i \cong E_0 \otimes L_\alpha^{\otimes i}$ for $i \in \{1, \ldots, q-1\}$. Considering ranks, we get $qr \leq n$, which combined with the above gives $qr = n$. All in all, we have:

$$E \cong \bigoplus_{j=0}^{q-1} E_0 \otimes L_\alpha^{\otimes j}$$

Case 1: $r = 1$

In this case E_0 is a line bundle, and by lemma 3.16: $E \cong \bigoplus_{j=0}^{q-1} E_0 \otimes L_\alpha^{\otimes j} \cong \pi_{\alpha*}(\pi_\alpha^*(E_0))$. Thus, $E \in \mathrm{Im}(\pi_{\alpha*})$.

Case 2: $r > 1$

In this case, notice that $E_0 \in M(r, \deg(E_0))$ is fixed by $\tilde{\alpha} = q\alpha \in J^{(r)}$ of order r. We may use section 3.2 on $\tilde{\alpha}$ instead of α, to get $\pi_{\tilde{\alpha}} : \Sigma^{\tilde{\alpha}} \to \Sigma$ with Galois group

μ_r. For simplicity, choose $L_\alpha^{\otimes q}$ to be the line bundle representing $\tilde\alpha$ in $\mathrm{Pic}(\Sigma)$ and use the identification $L_{\tilde\alpha}^{\otimes r} = L_\alpha^{\otimes n} = \mathcal{O}_\Sigma$ when constructing $\Sigma^{\tilde\alpha}$. There is a map $\pi : \Sigma^\alpha \to \Sigma^{\tilde\alpha}$ taking $\xi \in \Sigma^\alpha \subseteq L_\alpha$ into $\xi^{\otimes q} \in \Sigma^{\tilde\alpha} \subseteq L_{\tilde\alpha} = L_\alpha^{\otimes q}$. Clearly, the following diagram commutes:

$$\begin{tikzcd} \Sigma^\alpha \arrow[rr, "\pi"] \arrow[dr, "\pi_\alpha"'] & & \Sigma^{\tilde\alpha} \arrow[dl, "\pi_{\tilde\alpha}"] \\ & \Sigma & \end{tikzcd} \tag{3.7}$$

Locally (L_α being trivial over U), Σ^α and $\Sigma^{\tilde\alpha}$ are homeomorphic to $U \times \mu_n$ and $U \times \mu_r$, respectively. It is easy to check that under these homeomorphisms, π corresponds to the mapping $(x, \zeta) \mapsto (x, \zeta^q)$. This shows that π is in fact a Galois covering with Galois group $\mu_q \subset \mu_n$. Since locally, π is a composition of biholomorphisms, it is holomorphic. Since $\pi^*(\pi_{\tilde\alpha}^* L_\alpha)$ is trivial, lemma 3.6 shows that π is in fact the covering that corresponds to $\pi_{\tilde\alpha}^*(L_\alpha)$. Furthermore, since the action of $\zeta \in \mu_n$ on Σ^α covers the action of $\zeta^q \in \mu_r$ on $\Sigma^{\tilde\alpha}$, it is easy to show that $\pi_{\alpha *} = \pi_{\tilde\alpha *} \circ \pi_*$.

Since E_0 is stable, [7] shows that there exists a line bundle L on $\Sigma^{\tilde\alpha}$ with $\pi_{\tilde\alpha *}(L) = E_0$. Hence, using lemma 3.16 for the covering π corresponding to $\pi_{\tilde\alpha}(L_\alpha)$ and lemma 3.13 for $\pi_{\tilde\alpha}$, we get:

$$\begin{aligned} \pi_{\alpha *}(\pi^*(L)) \quad \cong \quad \pi_{\tilde\alpha *}\pi_*(\pi^*(L)) \quad &\cong \quad \pi_{\tilde\alpha *}\bigoplus_{j=0}^{q-1} L \otimes \pi_{\tilde\alpha}^*(L_\alpha)^{\otimes j} \\ &\cong \quad \bigoplus_{j=0}^{q-1} \pi_{\alpha *}(L) \otimes L_\alpha^{\otimes j} \\ &\cong \quad E \end{aligned}$$

□

The calculations in the above proof also show another important fact:

Addendum 3.21. For any two line bundles L, L' on Σ^α, $\pi_{\alpha *}(L)$ is S-equivalent to $\pi_{\alpha *}(L')$ if and only if $\pi_{\alpha *}(L) \cong \pi_{\alpha *}(L')$. In particular, μ_n acts *transitively* (by pull back) on every fibre of $\pi_{\alpha *} : \mathrm{Pic}_d(\Sigma^\alpha) \to |M(n,d)|_\alpha$.

Proof. I claim that for any line bundle L on Σ^α, $\pi_{\alpha *}(L) \cong Gr(\pi_{\alpha *}(L))$. Clearly, this will prove the first part of the addendum.

Let L be a line bundle on Σ^α. Let $r = \min\{r' \in \mathbb{N} \setminus 0 \mid \zeta_n^{r'*}(L) \cong L\}$. If $r = n$, then by proposition 3.19, $\pi_{\alpha *}(L)$ is stable and hence isomorphic to its graded bundle. If $r \neq n$, then by minimality, $r \mid n$ (say, $n = qr$). In this case, L is isomorphic to the pull back of a line bundle L' on $\Sigma^{q\alpha}$, according to lemma 3.9.

By the minimality of r and the fact that $\zeta^*\pi^*(L') = \pi^*\zeta^{q*}(L')$ for each $\zeta \in \mu_n$, we see that $L', \zeta_n^{q*}(L'), \ldots, \zeta_n^{q(r-1)*}(L')$ are non-isomorphic and hence, by proposition 3.19, $\pi_{q\alpha *}(L')$ is stable.

The calculations in the proof of proposition 3.20 then show that

$$\pi_{\alpha *}(L) \cong \bigoplus_{i=0}^{q-1} \pi_{q\alpha *}(L') \otimes {L_\alpha}^{\otimes i}$$

which is isomorphic to its graded bundle.

Finally, the second statement follows by proposition 3.13. □

The next obvious question is: How does $\pi_{\alpha *}$ behave with respect to the partition into the closed subvarieties $|M(n, \Delta_d)|_\alpha$? To answer this, we need the following:

Definition 3.22. For $L \in \mathrm{Pic}(\Sigma^\alpha)$, define $\vartheta_\alpha(L) = \det(\pi_{\alpha *}(L))$.

Proposition 3.23.

$$\vartheta_\alpha(L) = \begin{cases} \mathrm{Nm}_\alpha(L) & , \quad n \text{ odd} \\ \mathrm{Nm}_\alpha(L) \otimes L_\alpha^{\otimes n/2} & , \quad n \text{ even} \end{cases} \tag{3.8}$$

Proof. Since the determinant is a fibrewise construction, $\det(\pi_{\alpha *}(L))$ can be calculated as the descent of the equivariant bundle:

$$\det\Big(\bigoplus_{\zeta' \in \mu_n} \zeta'^*(L)\Big) \cong \bigotimes_{\zeta' \in \mu_n} \zeta'^*(L) \cong \Big(\bigotimes_{\zeta' \in \mu_n} \zeta'^*(L)\Big) \otimes \mathcal{O}_{\Sigma^\alpha}. \tag{3.9}$$

Identifying the fibres of $\bigoplus_{\zeta' \in \mu_n} \zeta'^*(L)$ with $\mathbb{C}[\mu_n]$, notice that the action of μ_n is simply the regular representation, together with the moving of fibres. Hence, the action on the left hand side of (3.9) multiplies with the determinant of the regular representation.

Therefore, the action on the right hand side of (3.9) that makes the isomorphism equivariant is the natural one on $\bigotimes_{\zeta' \in \mu_n} \zeta'^*(L)$ tensored with the action on $\mathcal{O}_{\Sigma^\alpha}$ induced by the regular character, χ_{reg}.

Descent commutes with (equivariant) tensor products. Hence $\det(\pi_{\alpha *}(L))$ is given by the descent of $\bigotimes_{\zeta' \in \mu_n} \zeta'^*(L)$ (with the natural action) tensored with the descent of $\Sigma^\alpha \times \mathbb{C}$ (with the action induced by the regular character). The first yields $\mathrm{Nm}_\alpha(L)$. The latter yields $L_{\chi_{\mathrm{reg}}}$. (Following the notation introduced in the beginning of section 3.4.) The proposition follows by lemma 3.18. ☐

Corollary 3.24. *For each $d \in \{0, 1, \ldots, n-1\}$, the subvariety $\vartheta_\alpha^{-1}(\Delta_d) \subseteq \mathrm{Pic}_d(\Sigma^\alpha)$ is isomorphic to $\mathrm{Nm}_\alpha^{-1}(\mathcal{O}_\Sigma)$. An isomorphism is obtained by choosing an element $L_0 \in \vartheta_\alpha^{-1}(\Delta_d)$ (this will be done explicitly in section 3.6) and restricting the morphism $\mathrm{Pic}_0(\Sigma^\alpha) \to \mathrm{Pic}_d(\Sigma^\alpha) : L \mapsto L \otimes L_0$ to $\mathrm{Nm}_\alpha^{-1}(\mathcal{O}_\Sigma)$.*

Proof. One only has to check that $\mathrm{Nm}_\alpha^{-1}(\mathcal{O}_\Sigma) \otimes L_0 = \vartheta_\alpha^{-1}(\Delta_d)$. Both inclusions follow directly from proposition 3.23. ☐

3.5 The kernel of Nm_α

By now, it is clear that a detailed description of $\mathrm{Nm}_\alpha^{-1}(\mathcal{O}_\Sigma)$ is necessary. The case $n = 2$ is treated in a sequence of exercises in appendix B of [28]. The general case is "well known", but I have been unable to find a reference for it in the literature.

The description resembles the one given by Hilbert's theorem 90 of the kernel of $\mathrm{Nm}_\alpha : \mathcal{M}(\Sigma^\alpha) \to \mathcal{M}(\Sigma)$. Not surprisingly, it starts with an application of that theorem.

Proposition 3.25. *There exists a meromorphic function $g_\alpha \in \mathcal{M}(\Sigma^\alpha) \setminus \{0\}$ such that $\zeta_n^*(g_\alpha) = \zeta_n^{-1} \cdot g_\alpha$. In particular, its divisor (g_α) is invariant under pull-back with ζ_n. Furthermore, $(g_\alpha) = \pi_\alpha^*(C)$ for some $C \in \mathrm{Div}(\Sigma)$. This divisor satisfies $[C] = L_\alpha \in \mathrm{Pic}(\Sigma)$.*

Proof. Apply theorem 2.22 to the constant function $\zeta_n \in \mathcal{M}(\Sigma^\alpha)$ to get g_α. It follows that (g_α) is invariant and hence given as $\pi_\alpha^*(C)$ for some $C \in \mathrm{Div}(\Sigma)$. It remains to show that $[C] = L_\alpha$. Let L be a line bundle representing $[C]$. According to section 2.1 L has a meromorphic section with divisor equal to C. Pulling this back to Σ^α gives a meromorphic section s in $\pi_\alpha^*([L])$ with divisor equal to (g_α) and with $s(\zeta(x)) = s(x) \in L_{\pi_\alpha(x)}$ for every $x \in \Sigma^\alpha$ and $\zeta \in \mu_n$. Notice that s/g_α is a nonzero holomorphic section, inducing an isomorphism: $\phi : \pi_\alpha^*(L) \to \mathcal{O}_{\Sigma^\alpha}$ by mapping an element (x, ξ) into (x, λ) where $\lambda \in \mathbb{C}$ satisfies $\xi = \lambda \cdot s(x)/g_\alpha(x)$.

It is easy to see that the equivariant action of $\zeta \in \mu_n$ on $\mathcal{O}_{\Sigma^\alpha}$ that makes ϕ an equivariant isomorphism is $(x, \lambda) \mapsto (\zeta(x), \zeta^{-1}\lambda)$. Hence, both $\pi_\alpha^*(L_\alpha)$ and $\pi_\alpha^*(L)$ are equivariantly isomorphic to $\mathcal{O}_{\Sigma^\alpha}^{(-1)}$, and by descent: $L \cong L_\alpha$. □

Definition 3.26. For each $k \in \mathbb{Z}$, the functions $\Phi_\alpha^k : \mathrm{Pic}_k(\Sigma^\alpha) \to \mathrm{Pic}_0(\Sigma^\alpha)$ are defined by $\Phi_\alpha^k(L) = \zeta_n^*(L) \otimes L^{-1}$.

Lemma 3.27. *For every $k \in \mathbb{Z}$, $\mathrm{Im}\Phi_\alpha^k = \mathrm{Im}\Phi_\alpha^{k+n}$.*

Proof. Pick a point $x \in \Sigma$. Tensoring with $L_0 = \pi_\alpha^*[x] \in \mathrm{Pic}_n(\Sigma^\alpha)$ gives a bijection between $\mathrm{Pic}_k(\Sigma^\alpha)$ and $\mathrm{Pic}_{n+k}(\Sigma^\alpha)$ satisfying: $\Phi_\alpha^{k+n}(L \otimes L_0) = \Phi_\alpha^k(L)$ for every $L \in \mathrm{Pic}_k(\Sigma^\alpha)$ □

Proposition 3.28. *The kernel of $\mathrm{Nm}_\alpha : \mathrm{Pic}(\Sigma^\alpha) \to \mathrm{Pic}(\Sigma)$ has the following composition into connected components:*

$$\mathrm{Nm}_\alpha^{-1}(\mathcal{O}_\Sigma) = \bigcup_{k=0}^{n-1} \mathrm{Im}\Phi_\alpha^k \tag{3.10}$$

Proof. Identifying $\mathrm{Pic}_k(\Sigma^\alpha) = \mathrm{Pic}_0(\Sigma^\alpha) = J(\Sigma^\alpha)$ as complex varieties, the results in 2.6 ensures that the Φ_α^k are morphisms. $J(\Sigma^\alpha)$ being complete and connected, it follows that the $\mathrm{Im}\Phi_\alpha^k$ are connected and closed. This leaves only to show that the union in (3.10) is disjoint and yields $\mathrm{Nm}_\alpha^{-1}(\mathcal{O}_\Sigma)$.

For the inclusion to the right, assume that $L \in \mathrm{Nm}_\alpha^{-1}(\mathcal{O}_\Sigma)$ is represented by a divisor $D \in \mathrm{Div}(\Sigma^\alpha)$. Denote by $\{x_1, x_2, \ldots, x_m\}$ the image of the support of D under π_α and denote $\pi_\alpha^{-1}(x_i) = \{x_{i1}, x_{i2}, \ldots, x_{in}\}$ for each $i \in \{1, 2, \ldots, m\}$. Choose the numbering such that $\zeta_n(x_{ij}) = x_{i(j+1)}$. [1]

Write $D = \sum_{i,j} a_{ij} x_{ij}$ $(a_{ij} \in \mathbb{Z})$. The assumption means that $\mathrm{Nm}_\alpha(D) = \sum_{i,j} a_{ij} x_i$ is principal. Pick $f \in \mathcal{M}(\Sigma)$ such that $\sum_{i,j} a_{ij} x_i = (f)$. Using proposition 2.18, pick $\tilde{f} \in \mathcal{M}(\Sigma^\alpha)$ such that $\mathrm{Nm}_\alpha(\tilde{f}) = f$. Subtracting the divisor of $\tilde{f}$ from D gives another representative for L, $\tilde{D} = \sum_{i,j} \tilde{a}_{ij} x_{ij}$, satisfying $\mathrm{Nm}_\alpha(D') = 0$. This implies that for each i, $\sum_j \tilde{a}_{ij} = 0$, which is a sufficient condition for solving the equations $b_{i(j+1)} - b_{ij} = \tilde{a}_{ij}$, $(b_{ij} \in \mathbb{Z})$. Doing that, $\bar{L} = [\sum_{i,j} b_{ij} x_{ij}] \in \mathrm{Pic}(\Sigma^\alpha)$ satisfies $\Phi_\alpha^k(\bar{L}) = \zeta_n^*(\bar{L}) \otimes \bar{L}^{-1} = [\sum_{i,j}(b_{i(j+1)} - b_{ij}) x_{ij}] = L$. It follows from lemma 3.27 that $L \in \mathrm{Im}\Phi_\alpha^k$ for some $k \in \{0, 1, \ldots, n-1\}$.

[1] $(j+1)$ is to be calculated modulo n.

The inclusion to the left being trivial, it remains only to show that the union is disjoint. Assume therefore that $\Phi_\alpha^{k_1}(L_1) = \Phi_\alpha^{k_2}(L_2)$ for $k_\nu \in \mathbb{Z}$ and $L_\nu \in \mathrm{Pic}_{k_\nu}(\Sigma^\alpha)$ $(\nu = 1, 2)$. We need to show that $k_1 \equiv k_2 \mod n$. Rewriting the assumption gives: $\zeta_n^*(L_1 \otimes L_2^{-1}) \otimes (L_1 \otimes L_2^{-1})^{-1} \cong \mathcal{O}_{\Sigma^\alpha}$.

Let $D \in \mathrm{Div}_{k_1-k_2}(\Sigma^\alpha)$ be such that $[D] = L_1 \otimes L_2^{-1}$. The assumption then implies that there exists an $f \in \mathcal{M}(\Sigma^\alpha)$ such that $\zeta_n^*(D) - D = (f)$. Let g_α and C be as in proposition 3.25. Adjusting D with a principal divisor as in lemma 2.14, we may assume that D and (g_α) have disjoint support. Then, since (g_α) is invariant under ζ_n, its support is also disjoint from the support of $\zeta_n^*(D)$ and hence the one of $\zeta_n^*(D) - D = (f)$. Weil reciprocity gives:

$$1 = \frac{f((g_\alpha))}{g_\alpha((f))} = \frac{f((g_\alpha))g_\alpha(D)}{g_\alpha(\zeta_n^*(D))} \tag{3.11}$$

Now, $(\mathrm{Nm}_\alpha(f)) = \mathrm{Nm}_\alpha(\zeta_n^*(D) - D) = 0$ means that $\mathrm{Nm}_\alpha(f)$ is holomorphic and hence constant. Since also $n \cdot \deg(C) = \deg((g_\alpha)) = 0$, $f((g_\alpha)) = f(\pi_\alpha^*(C)) = \mathrm{Nm}_\alpha(f)(C) = z^{\deg(C)} = 1$ (where $z \neq 0$ is the constant value of $\mathrm{Nm}_\alpha(f)$).

At the same time:

$$\begin{aligned} g_\alpha(D) &= \prod_{D(x')\neq 0} g_\alpha(x')^{D(x')} \\ &= \prod_{D(\zeta_n(x'))\neq 0} g_\alpha(\zeta_n(x'))^{D(\zeta_n(x'))} \\ &= \prod_{\zeta_n^*(D)(x')\neq 0} (\zeta_n^{-1} g_\alpha(x'))^{\zeta_n^*(D)(x')} = \zeta_n^{-\deg(D)} g_\alpha(\zeta_n^*(D)). \end{aligned}$$

Putting everything into (3.11) yields $1 = \zeta_n^{-\deg(D)} = \zeta_n^{k_2-k_1}$ proving that $k_1 \equiv k_2 \mod n$. □

The component of $\mathrm{Nm}_\alpha^{-1}(\mathcal{O}_\Sigma)$ containing $\mathcal{O}_{\Sigma^\alpha}$ (i.e. $\mathrm{Im}\Phi_\alpha^0$) is called the *generalised Prym variety* P_α associated to α. Any subgroup of μ_n acts on P_α by pullback. The quotients are called *generalised Kummer varieties*.

The remaining components of $\mathrm{Nm}_\alpha^{-1}(\mathcal{O}_\Sigma)$ are simply translations of P_α. Indeed, for any $L \in \mathrm{Im}\Phi_\alpha^k$ and any $k' \in \mathbb{Z}$, we have: $L \otimes \mathrm{Im}\Phi_\alpha^{k'} = \mathrm{Im}\Phi_\alpha^{k+k'}$ The enumeration by $k \in \{0, 1, \ldots, n-1\}$ depends, however, on the choice of ζ_n. The following gives a more natural characterisation of the components.

Lemma 3.29. *For any $L \in \mathrm{Pic}_{k'}(\Sigma^\alpha)$, the bundle $\zeta_n^{k*} L \otimes L^{-1}$ lies in $\mathrm{Im}\Phi_\alpha^{kk'}$.*

Proof. We have: $\zeta_n^{k*} L \otimes L^{-1} \cong \Phi_\alpha^{kk'}(\zeta_n^{(k-1)*} L \otimes \zeta_n^{(k-2)*} L \otimes \ldots \otimes L)$. □

Definition 3.30. For each $\zeta \in \mu_n$ let P_α^ζ be the component of $\mathrm{Nm}_\alpha^{-1}(\mathcal{O}_\Sigma)$ containing $\zeta^*(L) \otimes L^{-1}$ for each $L \in \mathrm{Pic}_1(\Sigma^\alpha)$.

Proposition 3.31. *We have the following properties:*

- $P_\alpha^{\zeta_n^k} = \mathrm{Im}\Phi_\alpha^k$.
- *The connected components of* $\mathrm{Nm}_\alpha^{-1}(\mathcal{O}_\Sigma)$ *are* P_α^ζ, $(\zeta \in \mu_n)$.
- *For* $L \in P_\alpha^\zeta$ *and* $\zeta' \in \mu_n$, *we have* $L \otimes P_\alpha^{\zeta'} = P_\alpha^{\zeta\zeta'}$.
- *For* $L \in \mathrm{Pic}_k(\Sigma^\alpha)$ *and* $\zeta \in \mu_n$, *we have* $\zeta^*(L) \otimes L^{-1} \in P_\alpha^{\zeta^k}$
- *Conversely, whenever* $\zeta \in \mu_n$ *is primitive and* $L \in P_\alpha^{\zeta^k}$ *for some* k, *there exist a* $K \in \mathrm{Pic}_k(\Sigma^\alpha)$ *such that* $L \cong \zeta^*(K) \otimes K^{-1}$.

Proof. This is just a reformulation of the above, save for the last claim. To prove this, assume that $\zeta = \zeta_n^i$ and $\zeta_n = \zeta^j$. Then for $L \in P_\alpha^{\zeta^k} = P_\alpha^{\zeta_n^{ik}} = \mathrm{Im}\Phi_\alpha^{ik}$, there exists a $\widetilde{K} \in \mathrm{Pic}_{ik}(\Sigma^\alpha)$ with $L \cong \zeta_n^*(\widetilde{K}) \otimes \widetilde{K}^{-1} \cong (\zeta^j)^*(\widetilde{K}) \otimes \widetilde{K}^{-1} \cong \zeta^*(K) \otimes K^{-1}$, where $K = \widetilde{K} \otimes \zeta^*(\widetilde{K}) \otimes \ldots \otimes \zeta^{(j-1)*}(\widetilde{K}) \in \mathrm{Pic}_{jik}(\Sigma^\alpha)$. Now, $jik \equiv k \mod n$. Hence, as in the proof of lemma 3.27 we may adjust K with a pull-back bundle from Σ to achieve that $\deg(K) = k$. □

Given another element $\beta \in J^{(n)}$, then $\mathrm{Nm}_\alpha(\pi_\alpha^*(L_\beta)) \cong L_\beta^{\otimes n} \cong \mathcal{O}_\Sigma$. This means that $\pi_\alpha^*(L_\beta) \in \mathrm{Nm}_\alpha^{-1}(\mathcal{O}_\Sigma)$. One may ask in which component it lies. This is where the Weil pairing enters the scene.

Proposition 3.32. *Let* $\beta \in J^{(n)}$. *We then have:*

$$\pi_\alpha^*(L_\beta) \in P_\alpha^{\lambda_n(\alpha,\beta)}$$

Proof. By proposition 3.28, $\pi_\alpha^*(L_\beta) \in \mathrm{Im}\Phi_\alpha^k = P_\alpha^{\zeta_n^k}$, for some k. We need to show that $\zeta_n^k = \lambda_n(\alpha, \beta)$.

Let C be the divisor on Σ with $\pi_\alpha^*(C) = (g_\alpha)$ and hence $nC = (\mathrm{Nm}_\alpha(g_\alpha))$ and $[C] = L_\alpha$ (see proposition 3.25).

Pick $B \in \mathrm{Div}(\Sigma)$ such that $[B] = L_\beta$ and adjust it with principal divisors until B and C have disjoint support.

Choose $L \in \mathrm{Pic}_k(\Sigma^\alpha)$ such that $\zeta_n^*(L) \otimes L^{-1} \cong \pi_\alpha^*(L_\beta)$. Pick $D \in \mathrm{Div}_k(\Sigma^\alpha)$ such that $[D] = L$ and adjust it until D and (g_α) have disjoint support. Since (g_α) is invariant, this implies that $\zeta_n^*(D) - D$ and (g_α) have disjoint support. Now,

$\pi_\alpha^*(B)$ and $\zeta_n^*(D) - D$ represent the same element in $\mathrm{Pic}(\Sigma^\alpha)$, so they differ only by a principal divisor. Pick $h \in \mathcal{M}(\Sigma^\alpha)$ such that $\pi_\alpha^*(B) = \zeta_n^*(D) - D + (h)$. Taking norms on both sides gives: $nB = (\mathrm{Nm}_\alpha(h))$.

The Weil pairing can now be calculated as follows:

$$\lambda_n(\alpha, \beta) = \frac{\mathrm{Nm}_\alpha(g_\alpha)(B)}{\mathrm{Nm}_\alpha(h)(C)} = \frac{g_\alpha(\pi_\alpha^*(B))}{h(\pi_\alpha^*(C))} = g_\alpha(\zeta_n^*(D) - D)\frac{g_\alpha((h))}{h((g_\alpha))}$$

As in the proof of proposition 3.28, $g_\alpha(\zeta_n^*(D) - D) = \zeta_n^{\deg(D)} = \zeta_n^k$. Finally, by Weil reciprocity: $\lambda_n(\alpha, \beta) = \zeta_n^k$. ☐

3.6 Description of the fixed point varieties

The aim of this section is to prove the following theorem:

Theorem 3.33. *Given a primitive $\alpha \in J^{(n)}$, the fixed point variety $|M(n, \Delta_d)|_\alpha$ of the action of α on $M(n, \Delta_d)$ has exactly $r = (n, d)$ connected components, each of which is isomorphic to the quotient of the Prym variety P_α under the action of μ_q, $q = \frac{n}{(n,d)}$.*

If n is odd, the set of connected components is canonically identified with μ_r.

If n is even, this set is canonically identified with $(\mu_r \times \frac{\alpha}{2})/\sim$ where $\frac{\alpha}{2}$ denotes the set of elements $a \in J^{(2n)}$ with $2a = \alpha$, and $(\zeta_1, a_1) \sim (\zeta_2, a_2)$ if and only if $\zeta_1 = \lambda^q \zeta_2$, where $\lambda = \lambda_{2n}(a_1, a_2) \in \{\pm 1\}$.

In both cases, the identification maps are given by definitions 3.36 and 3.37.

The starting point is the following:

Proposition 3.34.

$$\vartheta_\alpha^{-1}(\Delta_d)/\mu_n \cong |M(n, \Delta_d)|_\alpha$$

Proof. By proposition 3.20 and addendum 3.21, $\pi_{\alpha *} : \vartheta_\alpha^{-1}(\Delta_d) \to |M(n, \Delta_d)|_\alpha$ is surjective, and μ_n acts transitively on its fibres.

By (a very simply case of) geometric invariant theory, the action of μ_n on $\vartheta_\alpha^{-1}(\Delta_d)$ gives rise to a good quotient. By definition of good quotients, $\pi_{\alpha *}$ descends to an isomorphism. ☐

Definition 3.35. For each $d \in \{0, 1, \ldots, n-1\}$ let $r = (n, d)$ and $q = \frac{n}{r}$. Recall from section 3.1 that $\Delta_d = [d \cdot p]$ for some $p \in \Sigma$. Now pick a point $p_\alpha \in \Sigma^\alpha$ with $\pi_\alpha(p_\alpha) = p$, and define:

$$D_d^\alpha = \sum_{\zeta \in \mu_r} \frac{d}{r} \cdot \zeta(p_\alpha) \in \mathrm{Div}_d(\Sigma^\alpha).$$

$$\Delta_d^\alpha = [D_d^\alpha] \in \mathrm{Pic}_d(\Sigma^\alpha).$$

Notice that $\mathrm{Nm}_\alpha(\Delta_d^\alpha) \cong \Delta_d$ and that $\zeta^* \Delta_d^\alpha \cong \Delta_d^\alpha$ for each $\zeta \in \mu_r \subseteq \mu_n$. Furthermore, if n is even, choose an element $a \in \frac{\alpha}{2}$.

By propositions 3.23 and 3.28, we have the following partition into connected components:

$$\vartheta_\alpha^{-1}(\Delta_d) = \begin{cases} \bigcup\limits_{\zeta \in \mu_n} \Delta_d^\alpha \otimes \pi_\alpha^*(L_a) \otimes P_\alpha^\zeta & , \quad n \quad \text{even} \\ \bigcup\limits_{\zeta \in \mu_n} \Delta_d^\alpha \otimes P_\alpha^\zeta & , \quad n \quad \text{odd} \end{cases} \tag{3.12}$$

Next choose $\beta \in J^{(n)}$ with $\lambda_n(\alpha, \beta) = \zeta_n$. By proposition 3.32, $\pi_\alpha^*(L_{k\beta}) \in P_\alpha^{\zeta_n^k}$ for every $k \in \{0, 1, \ldots, n-1\}$. Furthermore, by proposition 3.31, for every $\zeta \in \mu_n$: $\zeta^*(\Delta_d^\alpha) \otimes (\Delta_d^\alpha)^{-1} \in P_\alpha^{\zeta^d}$. Hence: $\zeta_n^{j*}(\Delta_d^\alpha) \otimes \pi_\alpha^*(L_{k\beta}) \otimes P_\alpha^1 = \Delta_d^\alpha \otimes P_\alpha^{\zeta_n^{jd+k}}$. This gives another description:

$$\vartheta_\alpha^{-1}(\Delta_d) = \begin{cases} \bigcup\limits_{k=0}^{r-1} \bigcup\limits_{j=0}^{q-1} \zeta_n^{j*} \Delta_d^\alpha \otimes \pi_\alpha^*(L_{a+k\beta}) \otimes P_\alpha^1 & , \quad n \quad \text{even} \\ \bigcup\limits_{k=0}^{r-1} \bigcup\limits_{j=0}^{q-1} \zeta_n^{j*} \Delta_d^\alpha \otimes \pi_\alpha^*(L_{k\beta}) \otimes P_\alpha^1 & , \quad n \quad \text{odd} \end{cases} \tag{3.13}$$

From (3.13), it is obvious how μ_n acts on $\vartheta_\alpha^{-1}(\Delta_d)$. Indeed,

$$\vartheta_\alpha^{-1}(\Delta_d)/\mu_n \cong \begin{cases} \bigcup\limits_{k=0}^{r-1} \Delta_d^\alpha \otimes \pi_\alpha^*(L_{a+k\beta}) \otimes (P_\alpha^1/\mu_q) & , \quad n \quad \text{even} \\ \bigcup\limits_{k=0}^{r-1} \Delta_d^\alpha \otimes \pi_\alpha^*(L_{k\beta}) \otimes (P_\alpha^1/\mu_q) & , \quad n \quad \text{odd} \end{cases} \tag{3.14}$$

In particular, $|M(n, \Delta_d)|_\alpha$ has r components, each of which is isomorphic to the generalised Kummer variety P_α/μ_q.

When n is odd, the components are canonically indexed by $\mu_n/\mu_q \cong \mu_r$. Namely, for each $\zeta \in \mu_r$, by picking a q'th root $\zeta' \in \mu_n$ of ζ (or equivalently picking a generator for the class in μ_n/μ_q) and associating to ζ the component containing the image under $\pi_{\alpha *}$ of $\Delta_d^\alpha \otimes P_\alpha^{\zeta'}$. (This is independent of the choice of ζ', since any other choice is given by $\xi^d\zeta'$, ($\xi \in \mu_n$), and then: $\Delta_d^\alpha \otimes P_\alpha^{\xi^d\zeta'} = \xi^*(\Delta_d^\alpha \otimes P_\alpha^{\zeta'})$.) In other words, to $\zeta_n^{kq} \in \mu_r$ we associate the component corresponding to k in the description (3.14).

When n is even, the description depends on the choice of $a \in J^{(2n)}$. But having chosen this, we may still associate to $\zeta_n^{kq} \in \mu_r$ the component corresponding to k in (3.14). I.e. the one containing $\pi_{\alpha *}(\Delta_d^\alpha \otimes \pi_\alpha^*(L_a) \otimes P_\alpha^{\zeta'})$ for any (and hence all) $\zeta' \in \mu_n$ with $(\zeta')^q = \zeta$. To sum up:

Definition 3.36. When n is odd, for every $\zeta \in \mu_r$ denote by $|M(n,\Delta_d)|_\alpha^\zeta$ the component of $|M(n,\Delta_d)|_\alpha$ given by $\pi_{\alpha *}(\Delta_d^\alpha \otimes P_\alpha^{\zeta'})$ for any $\zeta' \in \mu_n$ with $(\zeta')^q = \zeta$. We then have:

$$|M(n,\Delta_d)|_\alpha = \bigcup_{\zeta \in \mu_r} |M(n,\Delta_d)|_\alpha^\zeta$$

Definition 3.37. When n is even, for every $\zeta \in \mu_r$ and every $a \in \frac{\alpha}{2}$ denote by $|M(n,\Delta_d)|_a^\zeta$ the component of $|M(n,\Delta_d)|_\alpha$ given by $\pi_{\alpha *}(\Delta_d^\alpha \otimes \pi_\alpha^*(L_a) \otimes P_\alpha^{\zeta'})$ for any $\zeta' \in \mu_n$ with $(\zeta')^q = \zeta$. We then have for each $a \in \frac{\alpha}{2}$:

$$|M(n,\Delta_d)|_\alpha = \bigcup_{\zeta \in \mu_r} |M(n,\Delta_d)|_a^\zeta$$

Remark 3.38. Notice that the construction of Δ_d^α depended on a choice of $p_\alpha \in \Sigma^\alpha$ with $\pi_\alpha(p_\alpha) = p$, making Δ_d^α only canonically defined up to pull-back by $\xi \in \mu_n$. This ambiguity does not affect the above definitions, though: According to proposition 3.31, $\xi^*(\Delta_d^\alpha) \otimes (\Delta_d^\alpha)^{-1} \in P_\alpha^{\xi^d}$, and hence for each $\zeta' \in \mu_n$:

$$\xi^*(\Delta_d^\alpha) \otimes P_\alpha^{\zeta'} = \Delta_d^\alpha \otimes \xi^*(\Delta_d^\alpha) \otimes (\Delta_d^\alpha)^{-1} \otimes P_\alpha^{\zeta'} = \Delta_d^\alpha \otimes P_\alpha^{\zeta' \cdot \xi^d}$$

(Again using proposition 3.31 in the last equality.) But $(\zeta' \cdot \xi^d)^q = (\zeta')^q$, and hence definitions 3.36 and 3.37 remain the same when Δ_d^α is replaced by $\xi^*(\Delta_d^\alpha)$.

In the case when n is even, we also need to know how the definitions depend on the choice of $a \in \frac{\alpha}{2}$. First we need a tiny calculation, which is surprisingly tricky to show directly from the definitions, because of the assumption that divisors need to be disjoint in order to calculate the Weil pairing.

Lemma 3.39. *Assume n is even. Let $a_1, a_2 \in J^{(2n)}$. If $2a_1 = 2a_2 = \alpha$, then:*

$$\lambda_n(\alpha, a_1 - a_2) = \lambda_{2n}(a_1, a_2).$$

Proof. Let $b = a_1 - a_2 \in J^{(2)}$. Pick $\tilde{b} \in J^{(4)} \subseteq J^{(2n)}$, such that $2\tilde{b} = b$. We then have, using lemma 2.15:

$$\lambda_{2n}(a_1, a_2) = \lambda_{2n}(b, a_2) = \lambda_{2n}(\tilde{b}, a_2)^2 = \lambda_n(2\tilde{b}, 2a_2) = \lambda_n(\alpha, a_1 - a_2).$$

–Where the last equality is due to the fact that $\lambda_n(a_1 - a_2, \alpha) \in \{\pm 1\}$. □

Proposition 3.40. *Assume that n is even and let $a_1, a_2 \in \frac{\alpha}{2}$. Let $r = (n, d)$, $q = \frac{n}{r}$ and $\lambda = \lambda_{2n}(a_1, a_2)$. Then for each $\zeta \in \mu_r$:*

$$|M(n, \Delta_d)|^{\zeta}_{a_1} = |M(n, \Delta_d)|^{\lambda^q \zeta}_{a_2}$$

Proof. Let $\zeta \in \mu_r$. Choose $\zeta' \in \mu_n$ with $(\zeta')^q = \zeta$. Since $a_1 - a_2 \in \mu_2 \subseteq \mu_n$, proposition 3.32 and lemma 3.31 show that:

$$\begin{aligned} \Delta_d^{\alpha} \otimes \pi_{\alpha}^*(L_{a_1}) \otimes P_{\alpha}^{\zeta'} &= \Delta_d^{\alpha} \otimes \pi_{\alpha}^*(L_{a_2}) \otimes \pi_{\alpha}^*(L_{a_1 - a_2}) \otimes P_{\alpha}^{\zeta'} \\ &= \Delta_d^{\alpha} \otimes \pi_{\alpha}^*(L_{a_2}) \otimes P_{\alpha}^{\lambda \zeta'}. \end{aligned}$$

–Where $\lambda = \lambda_n(\alpha, a_1 - a_2) = \lambda_{2n}(a_1, a_2)$. By definition 3.37, the image under $\pi_{\alpha *}$ of the first is equal to $|M(n, \Delta_d)|^{\zeta}_{a_1}$ and the image of the latter is equal to $|M(n, \Delta_d)|^{\lambda^q \zeta}_{a_2}$. □

Corollary 3.41. *Assume that n is even. Let $q = \frac{n}{(n,d)}$. If q is even (for instance, whenever d and hence r is odd), the component $|M(n, \Delta_d)|^{\zeta}_{a}$ is independent of the choice of a. If q is odd, two choices a_1 and a_2 yield the same component if and only if $\lambda_{2n}(a_1, a_2) = 1$.*

Proof. This is obvious, since $\lambda_{2n}(a_1, a_2) = \pm 1$. □

Finally, we address the question of how the action of the remaining elements in $J^{(n)}$ behaves on the fixed point set for α.

Proposition 3.42. *Let $\delta \in J^{(n)}$. The action of δ on $M(n, \Delta_d)$ induces a permutation on the set of connected components in $|M(n, \Delta_d)|_{\alpha}$. Denote $r = (n, d)$ and $q = \frac{n}{r}$. When n is odd, we have for $\zeta \in \mu_r$:*

$$L_{\delta} \otimes |M(n, \Delta_d)|^{\zeta}_{\alpha} = |M(n, \Delta_d)|^{\lambda_n(\alpha,\delta)^q \zeta}_{\alpha}$$

When n is even, we have for $\zeta \in \mu_r$ and $a \in \frac{\alpha}{2}$:

$$L_{\delta} \otimes |M(n, \Delta_d)|^{\zeta}_{a} = |M(n, \Delta_d)|^{\lambda_n(\alpha,\delta)^q \zeta}_{a}$$

Proof. Since $J(\Sigma)$ is commutative, the action of δ on $M(n, \Delta_d)$ restricts to an automorphism of $|M(n, \Delta_d)|_\alpha$, hence permuting its components. By the pull-push formula, the automorphism lifts through $\pi_{\alpha *}$ to tensoring with $\pi_\alpha^*(L_\delta)$ on $\vartheta_\alpha^{-1}(\Delta_d)$.[2] As in the previous proof, by proposition 3.32 and lemma 3.31, we get for any $\zeta = (\zeta')^q \in \mu_r$ when n is odd:

$$\pi_\alpha^*(L_\delta) \otimes \Delta_d^\alpha \otimes P_\alpha^{\zeta'} = \Delta_d^\alpha \otimes P_\alpha^{\lambda_n(\alpha,\delta)\zeta'}$$

And similarly when n is even. □

The above result has a nice corollary that will be useful later:

Corollary 3.43. *Suppose $k \in \{1, 2, \dots, n-1\}$ has $(k, n) = 1$. Then $k\alpha$ is primitive and has the same fixed points as α. When n is odd, we have for all $\zeta \in \mu_r$:*

$$|M(n, \Delta_d)|_\alpha^\zeta = |M(n, \Delta_d)|_{k\alpha}^{\zeta^k}$$

When n is even, and $a \in \frac{\alpha}{2}$, we have for all $\zeta \in \mu_r$:

$$|M(n, \Delta_d)|_a^\zeta = |M(n, \Delta_d)|_{ka}^{\zeta^k}$$

Proof. Assume n is odd. Clearly, since Σ^α is equivalent to $\Sigma^{k\alpha}$ by lemma 3.6 (only the actions of μ_n are different), we may assume that $\pi_{\alpha *}(\Delta_d^\alpha) \cong \pi_{k\alpha *}(\Delta_d^{k\alpha})$. This shows that

$$|M(n, \Delta_d)|_\alpha^1 = |M(n, \Delta_d)|_{k\alpha}^1$$

When n is even, notice that since k must be odd, $L_\alpha^{\frac{n}{2}} \cong L_{k\alpha}^{\frac{n}{2}}$, and hence:

$$\pi_{\alpha *}(\pi_\alpha^*(L_a) \otimes \Delta_d^\alpha) \cong L_\alpha^{\frac{n}{2}} \otimes \pi_{\alpha *}(\Delta_d^\alpha) \cong \pi_{k\alpha *}(\pi_{k\alpha}^*(L_{ka}) \otimes \Delta_d^{k\alpha}),$$

showing that:

$$|M(n, \Delta_d)|_a^1 = |M(n, \Delta_d)|_{ka}^1$$

In both cases, for each $\zeta \in \mu_n$, choose $\delta \in J^{(n)}$ with $\lambda_n(\alpha, \delta) = \zeta$. Then, $\lambda_n(k\alpha, \delta) = \zeta^k$, and the corollary follows from proposition 3.42. □

[2] This will be the fundamental observation in the next chapter.

3.7 Non-primitive torsion points

When α is of order $m < n$, the geometry of the fixed point varieties in $M(n, \Delta_d)$, under the action of α, becomes much more complicated.

However, we may prove the following generalisation of theorem 3.33).

Theorem 3.44. *Given an element $\alpha \in J^{(n)}$ of order $m \mid n$, the fixed point variety, $|M(n, \Delta_d)|_\alpha$ has exactly $\tilde{r} = (m, \tilde{d})$ connected components, where $\tilde{d} = d/(d, \frac{n}{m})$.*

If n is odd, the set of connected components is canonically identified with $\mu_{\tilde{r}}$. If n is even, the set of connected components is canonically identified with $(\mu_{\tilde{r}} \times \frac{\alpha}{2})/\sim$. Here, $\frac{\alpha}{2} = \{a \in J^{(2m)} \mid 2a = \alpha\}$, and $(\zeta_1, a_1) \sim (\zeta_2, a_2)$ if and only if $\zeta_1 = \lambda^{s\tilde{q}}\zeta_2$, where $\lambda = \lambda_{2m}(a_1, a_2) \in \{\pm 1\}$, $s = \frac{n}{m}$ and $\tilde{q} = \frac{m}{\tilde{r}}$.

Remark 3.45. Notice that whenever $s\tilde{q}$ is even (including the case when m divides $\frac{n}{2}$ — in particular when m is odd, and the case when $\tilde{q}$ is even — in particular when m is even and $\tilde{d}$ is odd, the relation $\sim$ is simply equality in the roots of unity, so that $(\mu_{\tilde{r}} \times \frac{\alpha}{2})/\sim$ is simply $\mu_{\tilde{r}}$ in this case.

Most of the proof is very similar to the one given in the primitive case. Of course, we may construct the m-sheeted cyclic Galois covering $\pi_\alpha : \Sigma^\alpha \to \Sigma$, with $\ker(\pi_\alpha^* : J(\Sigma) \to J(\Sigma^\alpha)) = \langle L_\alpha \rangle$, and construct fixed points as push down of vector bundles on Σ^α of rank $\frac{n}{m}$. In other words, according to proposition 3.19, we get a map: $\pi_{\alpha *} : M_\alpha(\frac{n}{m}, d) \to |M(n, d)|_\alpha$. (Where $M_\alpha(\frac{n}{m}, d)$ denotes the moduli space of semistable vector bundles on Σ^α of rank $\frac{n}{m}$ and degree d.)

Narasimhan and Ramanan ([7]) show that every fixed point in $M(n, \Delta_d)$, represented by a *simple* bundle is isomorphic to the direct image of a bundle on Σ^α. As before, it is possible to extend that result to the following:

Proposition 3.46. *$\pi_{\alpha *} : M_\alpha(\frac{n}{m}, d) \to |M(n, d)|_\alpha$ is surjective.*

Proof. The technique required is the same as in proposition 3.20. However, graded representatives of semistable fixed points are now generally of the form:

$$E = \bigoplus_{i=0}^{s-1} \bigoplus_{j=0}^{q_i - 1} E_i \otimes L_\alpha^{\otimes j}$$

where each E_i is stable of rank r_i and q_i is minimal such that $L_\alpha^{q_i} \otimes E_i \cong E_i$. By minimality, $q_i \mid m$. Now, for each $i \in \{0, \ldots s-1\}$, E_i is stable and fixed by $q_i \cdot \alpha$, and therefore the direct image of a bundle on $\Sigma^{q_i \cdot \alpha}$. Pulling this further back to Σ^α (using the diagram (3.7)) gives a stable bundle F_i, with $\pi_{\alpha *}(F_i) \cong \bigoplus_{j=0}^{q_i - 1} L_\alpha^j \otimes E_i$, and hence $E \cong \pi_{\alpha *}(\bigoplus_i F_i)$. □

As with proposition 3.20, the proof shows another important fact:

Addendum 3.47. Any indecomposable bundle E on Σ^α has $\pi_{\alpha *}(E) \cong Gr(\pi_{\alpha *}E)$. In particular, for any two indecomposable bundles E, E' on Σ^α, $\pi_{\alpha *}(E)$ is S-equivalent to $\pi_{\alpha *}(E')$ if and only if $\pi_{\alpha *}(E) \cong \pi_{\alpha *}(E')$.

Proof. One may repeat the proof of addendum 3.21, word by word, with L replaced by E, and n replaced by m. □

Corollary 3.48. *If E is the graded representative of a point in $M(\frac{n}{m}, d)$, then $\pi_{\alpha *}(E)$ is isomorphic to its graded bundle. In particular, if E and E' are graded representatives of points in $M(\frac{n}{m}, d)$ and $\pi_{\alpha *}(E)$ is S-equivalent to $\pi_{\alpha *}(E')$, then $\pi_{\alpha *}(E) \cong \pi_{\alpha *}(E')$.*

Proof. Suppose $E = \bigoplus_{i=1}^s E_i$, where E_i are stable with slope $\mu(E)$. This means that $\pi_{\alpha *}(E) \cong \bigoplus_{i=1}^s \pi_{\alpha *}(E_i) \cong \bigoplus_{i=1}^s Gr(\pi_{\alpha *}(E_i))$, which is isomorphic to its graded bundle. □

We may define:

Definition 3.49. For $E \in M_\alpha(\frac{n}{m}, d)$, let $\vartheta_\alpha(E) = \det(\pi_{\alpha *}(E))$.

We then have the following generalisation of proposition 3.23, at the same time correcting a minor imprecision in lemma 3.4 of [7]. [3]

Proposition 3.50.

$$\vartheta_\alpha(E) = \begin{cases} \mathrm{Nm}_\alpha(\det(E)) & , \quad m \ \text{ odd} \\ \mathrm{Nm}_\alpha(\det(E)) \otimes L_\alpha^{\otimes n/2} & , \quad m \ \text{ even} \end{cases} \tag{3.15}$$

Proof. Again, the proof is more or less the same as the one for 3.23, only substituting the calculation:

$$\det\Big(\bigoplus_{\zeta' \in \mu_m} \zeta'^*(E)\Big) \cong \bigotimes_{\zeta' \in \mu_m} \zeta'^*(\det(E)) \cong \Big(\bigotimes_{\zeta' \in \mu_m} \zeta'^*(\det(E))\Big) \otimes \mathcal{O}_{\Sigma^\alpha}.$$

Identifying the fibres of $\bigoplus_{\zeta' \in \mu_m} \zeta'^*(E)$ with $\mathbb{C}^{\frac{n}{m}}[\mu_m] = \mathbb{C}[\mu_m]^{\oplus \frac{n}{m}}$, notice that the action of μ_m is simply $\frac{n}{m}$ copies of the regular representation, together with the moving of fibres. Hence, the action on the left hand side multiplies with the $\frac{n}{m}$'th power of the determinant of the regular representation.

[3] Whether or not there is a mistake in [7] depends on how "regular representation" is interpreted. It might seem that the authors forget the power $\frac{n}{m}$ arising in the present proof.

And so the action on the right hand side that makes the isomorphism equivariant, is the natural one on $\bigotimes_{\zeta' \in \mu_m} \zeta'^*(\det E)$ tensored with the action on $\mathcal{O}_{\Sigma^\alpha}$ induced by $(\chi_{\mathrm{reg}})^{\frac{n}{m}}$.

Consequently, $\det(\pi_{\alpha *}(E))$ is given by the descent of $\bigotimes_{\zeta' \in \mu_m} \zeta'^*(\det E)$ (with the natural action) tensored with the descent of $\Sigma^\alpha \times \mathbb{C}$ (with the action induced by $\chi_{\mathrm{reg}}^{\frac{n}{m}}$). The former yields $\mathrm{Nm}_\alpha(\det E)$. The latter yields $L_{\chi_{\mathrm{reg}}}^{\otimes \frac{n}{m}}$. The proposition follows by lemma 3.18. □

Remark 3.51. Notice that whenever m divides $\frac{n}{2}$, $L_\alpha^{\otimes n/2}$ is trivial, and hence some of the precautions taken in the following are in fact completely vacuous. However, for the sake of simplicity, I will not distinguish this situation from the general one.

Proposition 3.46 shows that: $|M(n, \Delta_d)|_\alpha = \pi_{\alpha *}(\vartheta_\alpha^{-1}(\Delta_d))$. Introducing proposition 3.50, together with the description of the kernel of Nm_α in proposition 3.31, we have, choosing $a \in \frac{\alpha}{2}$ when m is even, the following disjoint union:

$$\vartheta_\alpha^{-1}(\Delta_d) = \begin{cases} \det{}^{-1}\Big(\bigcup\limits_{\zeta \in \mu_m} \Delta_d^\alpha \otimes P_\alpha^\zeta\Big) & , \quad m \ \text{odd} \\ \det{}^{-1}\Big(\bigcup\limits_{\zeta \in \mu_m} \Delta_d^\alpha \otimes \pi_\alpha^*(L_a)^{\otimes \frac{n}{m}} \otimes P_\alpha^\zeta\Big) & , \quad m \ \text{even} \end{cases} \tag{3.16}$$

where $\det$ denotes the determinant map: $M_\alpha(\frac{n}{m}, d) \to \mathrm{Pic}_d(\Sigma^\alpha)$.

The following general result implies that the above is actually a composition into connected components.

Lemma 3.52. *Let* $\det : M_\alpha(\frac{n}{m}, d) \to \mathrm{Pic}_d(\Sigma^\alpha)$ *be the determinant map. For any path-connected subset* $U \subseteq \mathrm{Pic}_d(\Sigma^\alpha)$*, the inverse image,* $\det^{-1}(U)$ *is path-connected.*

Proof. The key element in the proof is the fact that the map: $J(\Sigma^\alpha) \to J(\Sigma^\alpha)$ given by $L \mapsto L^{\otimes k}$ is a branched covering for any $k \in \mathbb{N}$. This implies that any curve $\gamma : [0,1] \to J(\Sigma^\alpha)$ can be lifted to a curve $\tilde{\gamma} : [0,1] \to J(\Sigma^\alpha)$ with $\tilde{\gamma}(t)^{\otimes k} = \gamma(t)$.

First we show that the fibres $\det^{-1}(\Delta)$ are connected. Given E_0 and E_1 in $\det^{-1}(\Delta)$, choose a curve $\gamma : [0,1] \to M_\alpha(\frac{n}{m}, d)$ with $\gamma(\nu) = E_\nu$, $\nu = 0, 1$. Consider the curve $\bar{\gamma} : [0,1] \to J(\Sigma^\alpha)$ given by $\bar{\gamma}(t) = \det(\gamma(t))^{-1} \otimes \Delta$, and let $\tilde{\gamma}$ be a lift of $\bar{\gamma}$ as described above, with $k = \frac{n}{m}$. Now $\delta : [0,1] \to M_\alpha(\frac{n}{m}, d)$ defined by $\delta(t) = \gamma(t) \otimes \tilde{\gamma}(t)$ is a curve in $\det^{-1}(\Delta)$, connecting E_0 and E_1.

Finally, given E_0 and E_1 in $\det^{-1}(U)$, choose a curve $\gamma : [0,1] \to U$ with $\gamma(\nu) = \det(E_\nu)$. Again, let $\tilde{\gamma} : [0,1] \to J(\Sigma^\alpha)$ be such that $\tilde{\gamma}(t)^{\otimes \frac{n}{m}} = \gamma(t) \otimes \gamma(0)^{-1}$. Then, $\delta(t) = E_0 \otimes \tilde{\gamma}(t)$ is a curve in $\det^{-1}(U)$, connecting E_0 to a point with the same determinant as E_1. So, by the above, we connect this point to E_1 with another curve running inside $\det^{-1}(\det(E_1))$. □

Definition 3.53. When n is odd, for each $\zeta \in \mu_m$, denote:

$$B^\zeta_\alpha = \det^{-1}\left(\Delta^\alpha_d \otimes P^\zeta_\alpha\right).$$

When m is even, for each $\zeta \in \mu_m$ and each $a \in \frac{\alpha}{2}$, denote:

$$B^\zeta_a = \det^{-1}\left(\Delta^\alpha_d \otimes \pi^*_\alpha(L_a)^{\otimes \frac{n}{m}} \otimes P^\zeta_\alpha\right).$$

Remark 3.54. Notice the sudden change of division into cases. In view of the previous, it would seem more obvious to distinguish between whether or not m is odd. However, if n is odd, so is m. And if n is even, and m happens to be odd, m divides $\frac{n}{2}$, and we have: $\pi^*_\alpha(L_a)^{\frac{n}{m}} \cong \mathcal{O}_{\Sigma^\alpha}$.

So, in all cases we see by the previous results that when n is odd:

$$\pi_{\alpha *}(\bigcup_{\zeta \in \mu_m} B^\zeta_\alpha)$$

and when n is even and $a \in \frac{\alpha}{2}$:

$$\pi_{\alpha *}(\bigcup_{\zeta \in \mu_m} B^\zeta_a)$$

Remark 3.55. The only obstacle left towards identifying the components of the fixed point variety lies in the fact that μ_m no longer acts transitively on the fibres of $\pi_{\alpha *}$. For instance, if $n = 4$ and $m = 2$, we may have line bundles L_1 and L_2 on Σ^α with equal degrees $d \in \{0,1\}$, $\pi_{\alpha *}(L_1)$ and $\pi_{\alpha *}(L_2)$ stable, but $\zeta^* L_1$ being non-isomorphic to L_2 for any $\zeta \in \mu_r$. Then $\pi_{\alpha *}(L_1 \oplus L_2) \cong \pi_{\alpha *}(\zeta^* L_1 \oplus L_2) \in |M(4, 2d)|_\alpha$.

This phenomenon sometimes causes the images of the B^ζ_a to intersect, even if one is not the pull-back of the other. In the particular case mentioned above, $\det(\zeta^* L_1 \oplus L_2) = \det(L_1 \oplus L_2) \otimes \zeta^* L_1 \otimes L_1^{-1}$. If $d = 0$, $\zeta^* L_1 \otimes L_1^{-1}$ lies in P^1_α, whence $\zeta^* L_1 \oplus L_2$ lies in the same component of $\det^{-1}(\mathrm{Nm}^{-1}_\alpha(\mathcal{O}_{\Sigma^\alpha}))$ as $L_1 \oplus L_2$. Hence $\pi_{\alpha *}(L_1 \oplus L_2)$ does not give rise to an intersection between otherwise disjoint components. On the other hand, if $d = 1$, $\zeta^* L_1 \otimes L_1^{-1}$ lies in P^ζ_α, causing the images of all the B^ζ_a to intersect in $\pi_{\alpha *}(L_1 \oplus L_2)$.

The above remark actually concludes the proof of theorem 3.44 in the case $n = 4$ upon observing that the case $m = 1$ is trivial. For general values of n, one needs to consider all possible gradings of a semistable bundle in $\vartheta_\alpha^{-1}(\Delta_d)$.

Proposition 3.56. *Let $\tilde{d} = \frac{d}{(d, \frac{n}{m})}$. Let $\tilde{r} = (m, \tilde{d})$ and $\tilde{q} = \frac{m}{\tilde{r}}$.*
When n is odd, for $\zeta_1, \zeta_2 \in \mu_m$:

$$\pi_{\alpha *}(B_\alpha^{\zeta_1}) \cap \pi_{\alpha *}(B_\alpha^{\zeta_2}) \neq \emptyset \Leftrightarrow \frac{\zeta_1}{\zeta_2} \in \mu_{\tilde{q}}$$

When n is even, for $a \in \frac{\alpha}{2}$ and $\zeta_1, \zeta_2 \in \mu_m$:

$$\pi_{\alpha *}(B_a^{\zeta_1}) \cap \pi_{\alpha *}(B_a^{\zeta_2}) \neq \emptyset \Leftrightarrow \frac{\zeta_1}{\zeta_2} \in \mu_{\tilde{q}}$$

Proof. The proof is exactly the same in both cases, except for the difference in notation. We will do it for n odd.

Suppose first that $E_1 \in B_\alpha^{\zeta_1}$, $E_2 \in B_\alpha^{\zeta_2}$ and $\pi_{\alpha *}(E_1)$ is S-equivalent to $\pi_{\alpha *}(E_2)$. We may assume without loss of generality that E_1 and E_2 are isomorphic to their graded bundles, and hence by corollary 3.48, $\pi_{\alpha *}(E_1) \cong \pi_{\alpha *}(E_2)$.

Suppose $E_i = \bigoplus_{j=1}^{s_i} E_{i,j}$, where $E_{i,j}$ is stable and has the same slope as E_i.

Since $\pi_{\alpha *}(E_1) \cong \pi_{\alpha *}(E_2)$, we get by pulling back to Σ^α that:

$$\bigoplus_{\zeta \in \mu_m} \bigoplus_{j=1}^{s_1} \zeta^*(E_{1,j}) \cong \bigoplus_{\zeta \in \mu_m} \bigoplus_{j=1}^{s_2} \zeta^*(E_{2,j})$$

Consequently, by the theorem of Krull-Remak-Schmidt, each of the $E_{1,j}$ is isomorphic to a pull back of one of the $E_{2,j}$. We may assume (handling the general situation recursively) that $E_{1,1} \cong \zeta^*(E_{2,1})$ and $E_{1,j} \cong E_{2,j}$ for $j > 1$. Now,

$$\det(E_1) = \bigotimes_{j=1}^{s} \det(E_{1,j}) \cong \zeta^*(\det(E_{2,1})) \otimes \det(E_{2,1})^{-1} \otimes \det(E_2)$$

Suppose $\deg(E_{2,1}) = d_1$, $\mathrm{rk}(E_{2,1}) = r_1$. Since $E_{2,1}$ has the same slope as E_2, we have:

$$d_1 = r_1 \frac{d}{\frac{n}{m}} = r_1 \left(\frac{\frac{n}{m}}{(d, \frac{n}{m})} \right)^{-1} \frac{d}{(d, \frac{n}{m})}$$

which can be any integer multiple of $\tilde{d} = \frac{d}{(d, \frac{n}{m})}$.

By proposition 3.31, $\zeta^*(\det(E_{2,1})) \otimes \det(E_{2,1})^{-1} \in P_\alpha^{\zeta^{d_1}}$, and hence: $\zeta_1 = \zeta^{d_1}\zeta_2$. It remains only to realize that $\zeta^{d_1} \in (\mu_m)^{\tilde{d}} = \mu_{\tilde{q}}$.

Conversely, using the considerations above, given $\zeta_1, \zeta_2 \in \mu_m$ with $\frac{\zeta_1}{\zeta_2} \in \mu_{\tilde{q}}$, it is straightforward to construct one's own bundles $E_i \in B_\alpha^{\zeta_i}$ with $\pi_{\alpha*}(E_1) \cong \pi_{\alpha*}(E_2)$, by applying the considerations above. □

Definition 3.57. Let $\zeta \in \mu_{\tilde{r}}$. When n is odd, denote by $|M(n,\Delta_d)|_\alpha^\zeta$ the component of $|M(n,\Delta_d)|_\alpha$ that contains $\pi_{\alpha*}(B_\alpha^{\zeta'})$ for any (and hence all) $\zeta' \in \mu_m$ with $\zeta'^{\tilde{q}} = \zeta$. Similarly, when n is even, denote by $|M(n,\Delta_d)|_a^\zeta$ the component of $|M(n,\Delta_d)|_\alpha$ that contains $\pi_{\alpha*}(B_a^{\zeta'})$ for any (and hence all) $\zeta' \in \mu_m$ with $\zeta'^{\tilde{q}} = \zeta$.

To prove theorem 3.44, it remains only to examine how the definitions depend on the choice of a, when m is even.

Proposition 3.58. *Assume that n is even and let $a_1, a_2 \in \frac{\alpha}{2}$. Let $s = \frac{n}{m}$, $\tilde{r} = (m, \tilde{d})$ and $\tilde{q} = \frac{m}{\tilde{r}}$, where $\tilde{d} = \frac{d}{(d,\frac{n}{m})}$. Furthermore, let $\lambda = \lambda_{2m}(a_1, a_2) \in \mu_2$.*

Then for each $\zeta \in \mu_{\tilde{r}}$:

$$|M(n,\Delta_d)|_{a_1}^\zeta = |M(n,\Delta_d)|_{a_2}^{\lambda^{s\tilde{q}}\zeta}.$$

Proof. By proposition 3.32, $\pi_\alpha^*(L_{a_1-a_2}^{\otimes s}) \in P_\alpha^{\lambda^s}$, where $\lambda = \lambda_m(\alpha, a_1 - a_2) = \lambda_{2m}(a_1, a_2)$. This shows that for $\zeta' \in \mu_m$: $B_{a_1}^{\zeta'} = B_{a_2}^{\lambda^s\zeta'}$. □

Finally, most applications, including the way in which the remaining elements of $J^{(n)}$ permute the components, become clear from the following observation.

Proposition 3.59. *Let $\alpha \in J^{(n)}$ be primitive and $k \in \{0, 1, \ldots, n-1\}$. Let $r = (n, d)$, $q = \frac{n}{r}$. Let $m = \mathrm{ord}(k\alpha)$, and $\tilde{d}, \tilde{r}, \tilde{q}$ as before. Finally, let $\tilde{k} = \frac{\tilde{q}k}{q}$.*

If n is odd, we have for all $\zeta \in \mu_r$:

$$|M(n,\Delta_d)|_\alpha^\zeta \subseteq |M(n,\Delta_d)|_{k\alpha}^{\zeta^{\tilde{k}}}.$$

If n is even, we have for all $\zeta \in \mu_r$ and all $a \in \frac{\alpha}{2}$:

$$|M(n,\Delta_d)|_a^\zeta \subseteq |M(n,\Delta_d)|_{ka}^{\zeta^{\tilde{k}}}.$$

Proof. First of all, we may assume that k divides n. (Indeed, in the general case, $k = \frac{k}{(n,k)}(n,k)$, and $\frac{k}{(n,k)}$ being prime to n, corollary 3.43 reduces the statement of the proposition to the one with α replaced by $\frac{k}{(n,k)}\alpha$ and k replaced by (n,k).)

Let $\tilde{\alpha} = k\alpha$, which is of order $\frac{n}{k}$. Consider the diagram of coverings:

$$\begin{tikzcd} \Sigma^{\alpha} \arrow[rr, "\pi"] \arrow[dr, "\pi_{\alpha}"'] & & \Sigma^{\tilde{\alpha}} \arrow[dl, "\pi_{\tilde{\alpha}}"] \\ & \Sigma & \end{tikzcd} \tag{3.17}$$

where π is the k-sheeted covering associated to $\pi^*_{\tilde{\alpha}}(L_\alpha)$. Recall from the discussion in the proof of proposition 3.20 that the action of $\zeta \in \mu_n$ on Σ^α covers the action of ζ^k on $\Sigma^{\tilde{\alpha}}$.This implies that $\mathrm{Nm}(P^{\zeta}_{\alpha}) = P^{\zeta^k}_{\tilde{\alpha}}$ where Nm is the norm map associated to π. Furthermore, we may arrange that $\mathrm{Nm}(\Delta^{\alpha}_{d}) \cong \Delta^{\tilde{\alpha}}_{d}$.

Thus, in the case where n is odd, given $\zeta \in \mu_n$, for any $L \in \Delta^{\alpha}_{d} \otimes P^{\zeta}_{\alpha}$ we have:

$$\pi_{\alpha *}(L) \in |M(n, \Delta_d)|^{\zeta^q}_{\alpha}$$

where $q = \frac{n}{(n,d)}$. But at the same time, $\pi_{\alpha *}(L) \cong \pi_{\tilde{\alpha} *}(\pi_*(L))$, where

$$\det(\pi_*(L)) \cong \mathrm{Nm}(L) \in \Delta^{\tilde{\alpha}}_{d} \otimes P^{\zeta^k}_{\tilde{\alpha}}.$$

This shows that $\pi_{\alpha *}(L) \in |M(n, \Delta_d)|^{\zeta^{\tilde{q}k}}_{k\alpha} = |M(n, \Delta_d)|^{(\zeta^q)^{\tilde{k}}}_{k\alpha}$.

In the case where n is even, and $a \in \frac{\alpha}{2}$, we may do almost the same: Given $\zeta \in \mu_n$, for any $L \in \pi^*_{\alpha}(L_a) \otimes \Delta^{\alpha}_{d} \otimes P^{\zeta}_{\alpha}$ we have:

$$\pi_{\alpha *}(L) \in |M(n, \Delta_d)|^{\zeta^q}_{a}$$

where $q = \frac{n}{(n,d)}$. But at the same time, $\pi_{\alpha *}(L) \cong \pi_{\tilde{\alpha} *}(\pi_*(L))$, where

$$\det(\pi_*(L)) \cong \mathrm{Nm}(L) \in \pi^*_{\tilde{\alpha}}(L_a)^k \otimes \Delta^{\tilde{\alpha}}_{d} \otimes P^{\zeta^k}_{\tilde{\alpha}},$$

if k is odd, and

$$\det(\pi_*(L)) \cong \mathrm{Nm}(L) \otimes \pi^*_{\tilde{\alpha}}(L_\alpha)^{\frac{k}{2}} \in \pi^*_{\tilde{\alpha}}(L_a)^{2k} \otimes \Delta^{\tilde{\alpha}}_{d} \otimes P^{\zeta^k}_{\tilde{\alpha}},$$

if k is even.

Since $\pi^*_{\tilde{\alpha}}(L_a)^k \cong \pi^*_{\tilde{\alpha}}(L_{ka})^k$, when k is odd, and $\pi^*_{\tilde{\alpha}}(L_a)^{2k} \cong \mathcal{O}_{\Sigma^{\tilde{\alpha}}} \cong \pi^*_{\tilde{\alpha}}(L_{ka})^k$, when k is even, we get in both cases that

$$\pi_{\alpha *}(L) \in |M(n,\Delta_d)|^{\zeta^{\tilde{q}k}}_{ka} = |M(n,\Delta_d)|^{(\zeta^q)^{\tilde{k}}}_{ka}.$$

□

Corollary 3.60. *Let $\tilde{\alpha}, \delta \in J^{(n)}$, such that* $\operatorname{ord}(\tilde{\alpha}) = m$. *The action of δ on $M(n,\Delta_d)$ induces a permutation on the set of connected components in $|M(n,\Delta_d)|_{\tilde{\alpha}}$. Denote $\tilde{r} = (m,d)$ and $\tilde{q} = \frac{m}{\tilde{r}}$. When n is odd, we have for $\zeta \in \mu_{\tilde{r}}$:*

$$L_\delta \otimes |M(n,\Delta_d)|^{\zeta}_{\tilde{\alpha}} = |M(n,\Delta_d)|^{\lambda_n(\tilde{\alpha},\delta)^{\tilde{q}}\zeta}_{\tilde{\alpha}}$$

When n is even, we have for $\zeta \in \mu_r$ and $a \in \frac{\tilde{\alpha}}{2}$:

$$L_\delta \otimes |M(n,\Delta_d)|^{\zeta}_{a} = |M(n,\Delta_d)|^{\lambda_n(\tilde{\alpha},\delta)^{\tilde{q}}\zeta}_{a}$$

Proof. Choose $\alpha \in J^{(n)}$ primitive, such that $\tilde{\alpha} = k\alpha$ for $k = \frac{n}{m}$.

Now, for $E \in |M(n,\Delta_d)|^{\zeta}_{\alpha} \subseteq |M(n,\Delta_d)|^{\zeta^{\tilde{k}}}_{k\alpha}$, we have by proposition 3.42 that $L_\delta \otimes E$ lies in $|M(n,\Delta_d)|^{\lambda^q\zeta}_{\alpha} \subseteq |M(n,\Delta_d)|^{(\lambda^q\zeta)^{\tilde{k}}}_{k\alpha} = |M(n,\Delta_d)|^{\lambda^{k\tilde{q}}\zeta^{\tilde{k}}}_{k\alpha}$, where $\lambda = \lambda_n(\alpha,\delta)$, and hence $\lambda^k = \lambda_n(\tilde{\alpha},\delta)$. □

Finally, as an apology for the notation, I supply an overview of the values of the different variables in the some particular cases. In each case, assuming $\tilde{\alpha} = k\alpha$, where $\alpha \in J^{(n)}$ is primitive, and applying the results of the section to $\tilde{\alpha}$, which is of order $m = \frac{n}{(n,k)}$.

Remark 3.61. We have in the following cases:

$m = n$:	$\tilde{d} = d$, $\tilde{r} = r$, $\tilde{q} = q$, and $\tilde{k} = k$.
$d = 0$:	$\tilde{d} = 0$, $\tilde{r} = m$, $\tilde{q} = 1 = q$, and $\tilde{k} = k$.
$(n,d) = 1$:	$\tilde{d} = d$, $\tilde{r} = 1 = r$, $\tilde{q} = m$, and $\tilde{k} = \frac{mk}{n}$.
$n = 4$, $d = 2$, $m = 2$:	$\tilde{d} = 1$, $\tilde{r} = 1$, $\tilde{q} = 2$, $\tilde{k} = k$.
$n = 6$, $d = 4$, $m = 3$, $(2 \mid k)$:	$\tilde{d} = 2$, $\tilde{r} = 1$, $\tilde{q} = 3$, $\tilde{k} = k$.
$n = 6$, $d = 4$, $m = 2$, $(3 \mid k)$:	$\tilde{d} = 4$, $\tilde{r} = 2$, $\tilde{q} = 1$, $\tilde{k} = \frac{k}{3}$.

4
Intersections of fixed point varieties

Given different elements in $J^{(n)}(\Sigma)$, how do their fixed point varieties intersect? In the cases where the fixed point varieties are not connected, how do the individual components intersect? These questions are addressed below. We immediately restrict our attention to primitive torsion elements. The results generalise the ones of section 6 in [1].

4.1 Pairwise intersections

Suppose $\alpha, \beta \in J^{(n)}$ are both of order n. Assume furthermore, for simplicity, that $\langle\alpha\rangle \cap \langle\beta\rangle = \{0\}$.

Notation 4.1. Define: $|M(n,\Delta_d)|_{\alpha,\beta} = |M(n,\Delta_d)|_\alpha \cap |M(n,\Delta_d)|_\beta$

Notice that the action of β preserves $|M(n,\Delta_d)|_\alpha$. In fact, the pull-push formula states that the following diagram is commutative:

$$\begin{tikzcd} \vartheta_\alpha^{-1}(\Delta_d) \arrow[r, "\otimes\pi_\alpha^*(L_\beta)"] \arrow[d, "\pi_{\alpha*}"] & \vartheta_\alpha^{-1}(\Delta_d) \arrow[d, "\pi_{\alpha*}"] \\ |M(n,\Delta_d)|_\alpha \arrow[r, "\otimes L_\beta"] & |M(n,\Delta_d)|_\alpha \end{tikzcd} \tag{4.1}$$

This, combined with proposition 3.20 and its addendum gives the following description:

$$|M(n,\Delta_d)|_{\alpha,\beta} \quad = \quad \pi_{\alpha *}\big(\bigcup_{\zeta\in\mu_n}\{L\in\vartheta_\alpha^{-1}(\Delta_d)\mid L\otimes\pi_\alpha^*(L_\beta)\cong\zeta^*(L)\}\big) \tag{4.2}$$

Remark 4.2. Notice that $|M(n,\Delta_d)|_{\alpha,\beta}$ is contained in the stable part of $M(n,d)$. Indeed, suppose $L\otimes\pi_\alpha^*(L_\beta)\cong\zeta^*(L)$ for some $L\in\vartheta_\alpha^{-1}(\Delta_d)$ and $\zeta\in\mu_n$. Then for every $j\in\{0,1,\ldots,n-1\}$, we have $\zeta^{j*}(L)\cong L\otimes\pi_\alpha^*(L_\beta)^{\otimes j}$. Since the right-hand sides of these equations are non-isomorphic (by lemma 3.15 and the assumption that $\langle\alpha\rangle\cap\langle\beta\rangle=0$), ζ must be of order n and proposition 3.19 shows that $\pi_{\alpha *}(L)$ is stable.

Definition 4.3. Define:

$$A(\alpha,\beta,d)=\{\zeta\in\mu_n\mid\langle\zeta\rangle=\mu_n,\zeta^d=\lambda_n(\alpha,\beta)\}.$$
$$a(\alpha,\beta,d)=\#A(\alpha,\beta,d).$$

Remark 4.4. Notice the following properties:

- $a(\alpha,\beta,n-d)=a(\alpha,\beta,d)$ for $d\in\{1,\ldots,n-1\}$
- $\displaystyle\sum_{d=0}^{n-1}a(\alpha,\beta,d)=\varphi(n)$, where φ is Euler's phi-function.
- $a(\alpha,\beta,0)=\begin{cases}\varphi(n) & \text{if}\quad \lambda_n(\alpha,\beta)=1\\ 0 & \text{if}\quad \lambda_n(\alpha,\beta)\neq 1\end{cases}$
- $a(\alpha,\beta,1)=\begin{cases}1 & \text{if}\quad \lambda_n(\alpha,\beta)\text{ generates }\mu_n.\\ 0 & \text{if}\quad \lambda_n(\alpha,\beta)\text{ does not generate }\mu_n.\end{cases}$

Definition 4.5. For each $\zeta\in\mu_n$, define:

$$I_\zeta=\{L\in\vartheta_\alpha^{-1}(\Delta_d)\mid L\otimes\pi_\alpha^*(L_\beta)\cong\zeta^*(L)\}.$$

Proposition 4.6. *The intersection $|M(n,\Delta_d)|_{\alpha,\beta}$ consists of $a(\alpha,\beta,d)$ disjoint sets, henceforth called* "layers", *namely:*

$$|M(n,\Delta_d)|_{\alpha,\beta}=\bigcup_{\zeta\in A(\alpha,\beta,d)}\pi_{\alpha *}(I_\zeta)$$

On each of these layers, the quotient group, $J^{(n)}/\langle\alpha,\beta\rangle$ acts freely and transitively. In particular, $|M(n,\Delta_d)|_{\alpha,\beta}$ has $a(\alpha,\beta,d)\cdot n^{2g-2}$ elements.

Proof. Clearly, the I_ζ are disjoint (according to the discussion in remark 4.2 and invariant under pull-back by any $\zeta' \in \mu_n$. Hence, their images under $\pi_{\alpha *}$ are disjoint. By (4.2) these images constitute a partition of $|M(n,\Delta_d)|_{\alpha,\beta}$ into "layers". We need only show that $I_\zeta \neq \emptyset$ if and only if $\zeta \in A(\alpha,\beta,d)$.

Assume first that $L \in I_\zeta$ for some $L \in \vartheta_\alpha^{-1}(\Delta_d)$ and $\zeta \in \mu_n$. By remark 4.2, this implies that ζ is of order n. Combining propositions 3.31 and 3.32, $\pi_\alpha^*(L_\beta) \cong \zeta_n^*(L) \otimes L^{-1}$ implies that $P_\alpha^{\lambda_n(\alpha,\beta)} = P_\alpha^{\zeta^d}$ and thus, $\zeta^d = \lambda_n(\alpha,\beta)$. All in all we get, $\zeta \in A(\alpha,\beta,d)$.

Conversely, assume that $\zeta \in A(\alpha,\beta,d)$. Propositions 3.31 and 3.32 immediately give an $L \in \mathrm{Pic}_d(\Sigma^\alpha)$ with $\zeta^*(L) \otimes L^{-1} \cong \pi_\alpha^*(L_\beta)$. Pick an $\widetilde{L} \in \mathrm{Pic}_0(\Sigma)$ with

$$\widetilde{L}^{\otimes n} \cong \Delta_d \otimes \vartheta_\alpha(L)^{-1}$$

Now, by proposition 3.23, $\pi_\alpha^*(\widetilde{L}) \otimes L$ lies in $\vartheta_\alpha^{-1}(\Delta_d)$, and it has the same transformation property as L. All in all, $\pi_\alpha^*(\widetilde{L}) \otimes L \in I_\zeta$.

Finally, we prove that $J^{(n)}/\langle\alpha\rangle$ acts freely and transitively on each I_ζ, ($\zeta \in A(\alpha,\beta,d)$). (The class generated by $\gamma \in J^{(n)}$ acts by tensoring with $\pi_\alpha^*(L_\gamma)$.) Indeed, the action is free by lemma 3.15. It is transitive because if $L_1, L_2 \in I_\zeta$, then $L_1 \otimes L_2^{-1} = \pi_\alpha^*(L)$ for some $L \in \mathrm{Pic}(\Sigma)$. Proposition 3.23 gives: $L^{\otimes n} \cong \mathrm{Nm}_\alpha(L_1 \otimes L_2^{-1}) \cong \vartheta_\alpha(L_1) \otimes \vartheta_\alpha(L_2)^{-1} \cong \mathcal{O}_\Sigma$. Thus, $L = L_\gamma$ for some $\gamma \in J^{(n)}$. Since the action of β is transitive on the fibres of $\pi_{\alpha *}$, $J^{(n)}/\langle\alpha,\beta\rangle$ acts freely and transitively on each layer. ☐

Corollary 4.7. *$|M(n,\Delta_d)|_{\alpha,\beta}$ is non-void if and only if $\lambda_n(\alpha,\beta) = \zeta^d$ for some generator ζ of μ_n. This criterion is, in turn, equivalent to $\lambda_n(\alpha,\beta)$ being of order $q = \frac{n}{(n,d)}$.*

Next, we will examine how the individual components of the fixed point varieties intersect.

Notation 4.8. When n is odd and $\zeta, \zeta' \in \mu_r$, denote by $|M(n,\Delta_d)|_{\alpha,\beta}^{\zeta,\zeta'}$ the intersection $|M(n,\Delta_d)|_\alpha^\zeta \cap |M(n,\Delta_d)|_\beta^{\zeta'}$.

When n is even, $a \in \frac{\alpha}{2}, b \in \frac{\beta}{2}$ and $\zeta, \zeta' \in \mu_r$, denote by $|M(n,\Delta_d)|_{a,b}^{\zeta,\zeta'}$ the intersection $|M(n,\Delta_d)|_a^\zeta \cap |M(n,\Delta_d)|_b^{\zeta'}$.

Proposition 4.9. *Assume that $|M(n,\Delta_d)|_{\alpha,\beta} \neq \emptyset$. If n is odd, $|M(n,\Delta_d)|_{\alpha,\beta}$ is the disjoint union of the component-intersections $|M(n,\Delta_d)|_{\alpha,\beta}^{\zeta,\zeta'}$ where $\zeta, \zeta' \in \mu_r$. If n is even, and $a \in \frac{\alpha}{2}, b \in \frac{\beta}{2}$, $|M(n,\Delta_d)|_{\alpha,\beta}$ is the disjoint union of the component-*

intersections $|M(n, \Delta_d)|^{\zeta,\zeta'}_{a,b}$ *where* $\zeta, \zeta' \in \mu_r$. *In both cases, each of the component-intersections has* $\frac{n^{2g-2}}{r^2}$ *elements in each of the* $a(\alpha, \beta, d)$ *layers.*

Proof. Only the last statement is non-trivial. For this let ζ and ζ' be any two elements of μ_r. Choose $\delta, \delta' \in J^{(n)}$ with $\lambda_n(\alpha, \delta) = \zeta$, $\lambda_n(\beta, \delta') = \zeta'$ and $\lambda_n(\alpha, \delta') = \lambda_n(\beta, \delta) = 1$. (This uses proposition 2.16.) By proposition 3.42, tensoring with $L_{\delta+\delta'}$ defines a bijection between $|M(n, \Delta_d)|^{1,1}_{\alpha,\beta}$ and $|M(n, \Delta_d)|^{\zeta,\zeta'}_{\alpha,\beta}$ (resp. $|M(n, \Delta_d)|^{1,1}_{a,b}$ and $|M(n, \Delta_d)|^{\zeta,\zeta'}_{a,b}$). Since the bijection respects the partition into layers, the claim follows from proposition 4.6. $\square$

Example 4.10. One should have the picture in figure 4.1 in mind. It illustrates the case $n = 4, d = 2, \lambda_n(\alpha, \beta) = (\zeta_4)^2 = -1$. In this case, the fixed point varieties have $r = 2$ components each, and the number of layers is equal to 2.

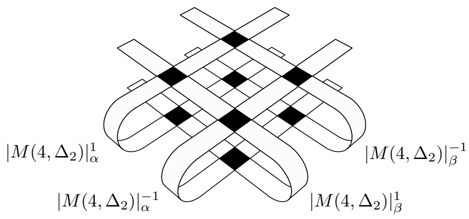

Figure 4.1: Structure of the component intersections.

4.2 The covering associated to a pair

Before studying triple intersections, we need to introduce some auxiliary definitions and a couple of technical results.

Given primitive elements, $\alpha, \beta \in J^{(n)}$, with $\langle\alpha\rangle \cap \langle\beta\rangle = \{0\}$, there is an associated Galois covering, $\tilde{\pi} : \tilde{\Sigma} \to \Sigma$, which is determined by the demand that the kernel of $\tilde{\pi}^* : J(\Sigma) \to J(\tilde{\Sigma})$ be equal to $\langle\alpha, \beta\rangle$. As in section 3.2, I will give an explicit construction in terms of L_α and L_β. Define:

$$\tilde{\Sigma} = \{(\xi, \eta) \in L_\alpha \oplus L_\beta \mid \xi^{\otimes n} = \eta^{\otimes n} = 1 \in \mathcal{O}_\Sigma\}$$

$$\tilde{\pi} = \pi|_{\tilde{\Sigma}}$$

where π is the projection in $L_\alpha \oplus L_\beta$.

By the same method as presented in lemma 3.5, one may show that this yields a connected, holomorphic Galois covering with Galois group $\mu_n \times \mu_n$. The projection maps from $L_\alpha \oplus L_\beta$ to L_α, L_β and $L_\alpha \otimes L_\beta \cong L_{\alpha+\beta}$ induce maps π^α, π^β and π^γ such that the following diagram commutes:

$$\begin{tikzcd}
& \tilde{\Sigma} \arrow[dl, "\pi^\alpha"'] \arrow[d, "\pi^\beta"] \arrow[dr, "\pi^\gamma"] & \\
\Sigma^\alpha \arrow[dr, "\pi_\alpha"'] & \Sigma^\beta \arrow[d, "\pi_\beta"] & \Sigma^\gamma \arrow[dl, "\pi_\gamma"] \\
& \Sigma &
\end{tikzcd}$$

-where γ denotes $\alpha + \beta$. (Notice that γ is primitive, because of the assumption that $\langle\alpha\rangle \cap \langle\beta\rangle = \{0\}$.)

In fact, π^α is isomorphic (as a covering of Σ^α) to the cyclic, Galois covering associated to $\pi_\alpha^*(L_\beta)$, as described in section 3.2. Similarly for π^β and π^γ. This follows from the uniqueness property in lemma 3.6.

Since by now μ_n is the Galois group of at least six different coverings, we need to introduce more precise notation for the actions of μ_n on the different covering spaces.

Notation 4.11. For $\zeta \in \mu_n$, let ζ_α, ζ_β and ζ_γ denote the deck-transformations of Σ_α, Σ_β and Σ_γ, respectively, given by multiplication with ζ in the the fibres of L_α, L_β and L_γ. Furthermore, let ζ^α, ζ^β and ζ^γ denote the three maps: $\tilde{\Sigma} \to \tilde{\Sigma}$ given by restriction of the following three automorphisms of $L_\alpha \oplus L_\beta$:

$$\begin{array}{lcll}
\zeta^\alpha & : & (\xi, \eta) \mapsto (\xi, \zeta\eta) & \\
\zeta^\beta & : & (\xi, \eta) \mapsto (\zeta\xi, \eta) & \\
\zeta^\gamma & : & (\xi, \eta) \mapsto (\zeta^{-1}\xi, \zeta\eta) & = \quad \zeta^\alpha \circ (\zeta^\beta)^{-1}
\end{array}$$

Notice that ζ^α, ζ^β and ζ^γ generate the Galois groups of the coverings π^α, π^β and π^γ respectively. Furthermore, notice that the following diagrams are commutative:

$$
\begin{tikzcd}
\tilde{\Sigma} \arrow[r, "\zeta^{\beta},(\zeta^{\gamma})^{-1}"] \arrow[d, "\pi^{\alpha}"] & \tilde{\Sigma} \arrow[d, "\pi^{\alpha}"] \\
\Sigma^{\alpha} \arrow[r, "\zeta_{\alpha}"] & \Sigma^{\alpha}
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{\Sigma} \arrow[r, "\zeta^{\alpha},\zeta^{\gamma}"] \arrow[d, "\pi^{\beta}"] & \tilde{\Sigma} \arrow[d, "\pi^{\beta}"] \\
\Sigma^{\beta} \arrow[r, "\zeta_{\beta}"] & \Sigma^{\beta}
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{\Sigma} \arrow[r, "\zeta^{\alpha},\zeta^{\beta}"] \arrow[d, "\pi^{\gamma}"] & \tilde{\Sigma} \arrow[d, "\pi^{\gamma}"] \\
\Sigma^{\gamma} \arrow[r, "\zeta_{\gamma}"] & \Sigma^{\gamma}
\end{tikzcd}
$$

Lemma 4.12. *Suppose L_1, L_2 and L_3 are line bundles on Σ^{α}, Σ^{β} and Σ^{γ}, respectively, such that $\pi_{\alpha *}(L_1) \cong \pi_{\beta *}(L_2) \cong \pi_{\gamma *}(L_3)$. Then $\pi^{\alpha *}(L_1) \cong \pi^{\beta *}(L_2) \cong \pi^{\gamma *}(L_3)$.*

Proof. Let $E \in |M(n, \Delta_d)|_{\alpha,\beta,\gamma}$ be the point represented by $\pi_{\alpha *}(L_1) \cong \pi_*(L_2) \cong \pi_*(L_3)$. By the description of the pairwise intersections in section 4.1, ${\zeta_{\alpha}}^*(L_1) \cong L_1 \otimes \pi_{\alpha}^*(L_{\beta})$ for some generator $\zeta \in \mu_n$. Hence,

$$\tilde{\pi}^*(E) \cong \pi_{\alpha}^*(\bigoplus_{i=0}^{n-1} \zeta^{i*} L_1) \cong \pi_{\alpha}^*(\bigoplus_{i=0}^{n-1} L_1 \otimes \pi_{\alpha}^*(L_{\beta})^{\otimes i}) \cong \pi^{\alpha *}(L_1)^{\oplus n}.$$

Similarly, $\tilde{\pi}^*(E) \cong \pi^{\beta *}(L_2)^{\oplus n} \cong \pi^{\gamma *}(L_3)^{\oplus n}$. Since line bundles are simple, the desired result follows but the theorem of Krull-Remak-Schmidt. □

Denoting by Nm^{α}, Nm^{β} and Nm^{γ} the norm maps corresponding to π^{α}, π^{β} and π^{γ}, recall definition 3.35 of D_d^{α}. We may construct, similarly, D_d^{β} and D_d^{α}. Notice that

$$\mathrm{Nm}^{\gamma}(\pi^{\beta *}(D_d^{\beta})) \cong \pi_{\gamma}^*(\mathrm{Nm}_{\beta}(D_d^{\beta})) \cong \pi_{\gamma}^*(d \cdot p) \cong \pi_{\gamma}^*(\mathrm{Nm}_{\alpha}(D_d^{\alpha})) \cong \mathrm{Nm}^{\gamma}(\pi^{\alpha *}(D_d^{\alpha})).$$

That means:

$$\pi^{\beta *}(D_d^{\beta}) \otimes \pi^{\alpha *}(D_d^{\alpha})^{-1} \in \mathrm{Ker}(\mathrm{Nm}^{\gamma}).$$

-And similarly for any permutation of (α, β, γ). Recall that ζ_n denotes a fixed, chosen generator of μ_n and let $r = (n, d)$ and $q = \frac{n}{r}$.

Lemma 4.13. *With suitable choices of the points p_{α}, p_{β} and p_{γ} in definition 3.35, there exists a divisor, F of degree $\frac{dn(q-1)}{2}$ on $\tilde{\Sigma}$, such that:*

$$(\zeta_n)^{\gamma *} F - F = \pi^{\beta *}(D_d^{\beta}) - \pi^{\alpha *}(D_d^{\alpha})$$

and

$$(\zeta_n)^{\alpha *} F - F = \pi^{\beta *}(D_d^{\beta}) - \pi^{\gamma *}(D_d^{\gamma}).$$

Proof. The proof consists of a construction. To avoid obscuring the argument with unnecessary generality, it is made in the particular case $n = 6, d = 2$, (hence, $r = 2, q = 3$), where the general situation protrudes clearly.

Choosing a point $\tilde{p} \in \tilde{\Sigma}$ with $\tilde{\pi}(\tilde{p}) = p$, we may describe the values of a divisor on $\tilde{\Sigma}$ in the fibre $\tilde{\pi}^{-1}(p)$ by an $n \times n$ matrix, the (i,j)'th entry of which being the value of the divisor at $(\zeta_n^i)^\alpha(\zeta_n^j)^\beta(\tilde{p})$. When stating that a divisor equals an $n \times n$ matrix, we adopt the convention that it is understood to be zero outside $\tilde{\pi}^{-1}(p)$.

Choosing $p_\alpha = \pi^\alpha(\tilde{p})$, $p_\beta = \pi^\beta(\tilde{p})$ and $p_\gamma = \pi^\gamma((\zeta_n^{-1})^\beta(\tilde{p})$, we have:

$$\pi^{\beta *}(D_d^\beta) - \pi^{\alpha *}(D_d^\alpha) = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ \\ 0 & 1 & 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\pi^{\beta *}(D_d^\beta) - \pi^{\gamma *}(D_d^\gamma) = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

-Furthermore, pull-back by ζ_n^α and ζ_n^β, respectively, corresponds to shifting rows upward and columns to the left. Hence, one may check easily that the following matrix describes a divisor F with the desired properties. Clearly: $\deg(F) = r^2 \cdot \frac{d}{r} \cdot \frac{q(q-1)}{2} = \frac{dn(q-1)}{2}$.

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$\square$

Addendum 4.14. Divisor F from lemma 4.13 furthermore satisfies:

$$(\zeta_n^{-1})^{\beta *}(\zeta_n^{\alpha *}F) - (\zeta_n^{\alpha *}F) = \pi^{\gamma *}(D_d^{\gamma}) - \pi^{\alpha *}(D_d^{\alpha}).$$

With different choices of p_α, p_β and p_γ, there exists a (different) divisor, F on $\tilde{\Sigma}$ of degree $\frac{dn(q-1)}{2}$ on $\tilde{\Sigma}$, such that:

$$(\zeta_n)^{\alpha *}F - F = \pi^{\beta *}(D_d^{\beta}) - \pi^{\gamma *}(D_d^{\gamma})$$

and

$$(\zeta_n)^{\beta *}F - F = \pi^{\alpha *}(D_d^{\alpha}) - \pi^{\gamma *}(D_d^{\gamma}).$$

Proof. For the first claim, simply observe that:

$$\begin{aligned} \pi^{\gamma *}(D_d^{\gamma}) - \pi^{\alpha *}(D_d^{\alpha}) &= \pi^{\beta *}(D_d^{\beta}) - \pi^{\alpha *}(D_d^{\alpha}) - (\pi^{\beta *}(D_d^{\beta}) - \pi^{\gamma *}(D_d^{\gamma})) \\ &= \zeta_n^{\gamma *}F - F - \zeta_n^{\alpha *}F + F \\ &= (\zeta_n^{-1})^{\beta *}(\zeta_n^{\alpha *}F) - (\zeta_n^{\alpha *}F) \end{aligned}$$

For the second claim, proceed in the proof of lemma 4.13, only this time letting $p_\alpha = \pi^{\alpha}(\tilde{p})$, $p_\beta = \pi^{\beta}(\tilde{p})$ and $p_\gamma = \pi^{\gamma}(\tilde{p})$. Then put

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

□

4.3 Triple intersections

Assume for the entire section that $\alpha, \beta \in J^{(n)}$ are primitive with $\langle \alpha \rangle \cap \langle \beta \rangle = 0$.

Notation 4.15. Let $\gamma \in J^{(n)}$. Denote:

$$|M(n, \Delta_d)|_{\alpha,\beta,\gamma} = |M(n, \Delta_d)|_{\alpha} \cap |M(n, \Delta_d)|_{\beta} \cap |M(n, \Delta_d)|_{\gamma}$$

For n odd and for $\zeta, \zeta', \zeta'' \in \mu_r$, $r = (n, d)$, denote:

$$|M(n, \Delta_d)|_{\alpha,\beta,\gamma}^{\zeta,\zeta',\zeta''} = |M(n, \Delta_d)|_{\alpha}^{\zeta} \cap |M(n, \Delta_d)|_{\beta}^{\zeta'} \cap |M(n, \Delta_d)|_{\gamma}^{\zeta''}$$

For n even, $a \in \frac{\alpha}{2}$, $b \in \frac{\beta}{2}$ and $c \in \frac{\gamma}{2}$, and for $\zeta, \zeta', \zeta'' \in \mu_r$, $r = (n, d)$, denote:

$$|M(n,\Delta_d)|^{\zeta,\zeta',\zeta''}_{a,b,c} = |M(n,\Delta_d)|^{\zeta}_{a} \cap |M(n,\Delta_d)|^{\zeta'}_{b} \cap |M(n,\Delta_d)|^{\zeta''}_{c}$$

Lemma 4.16. *Assume that $|M(n,\Delta_d)|_{\alpha,\beta} \neq \emptyset$. Let $\gamma \in J^{(n)}$. We then have:*

$$|M(n,\Delta_d)|_{\alpha,\beta,\gamma} \neq \emptyset \Leftrightarrow \gamma \in \langle \alpha, \beta \rangle.$$

In the affirmative case, $|M(n,\Delta_d)|_{\alpha,\beta,\gamma} = |M(n,\Delta_d)|_{\alpha,\beta}$.

Proof. Suppose that $|M(n,\Delta_d)|_{\alpha} \cap |M(n,\Delta_d)|_{\beta} \cap |M(n,\Delta_d)|_{\gamma} \neq \emptyset$. The first claim follows because $J^{(n)}/\langle \alpha, \beta \rangle$ acts freely on $|M(n,\Delta_d)|_{\alpha,\beta}$ according to proposition 4.6. The second claim is trivial. □

The next obvious question is: When $\gamma = \alpha + \beta$, how do the components of $|M(n,\Delta_d)|_{\alpha}$, $|M(n,\Delta_d)|_{\beta}$ and $|M(n,\Delta_d)|_{\gamma}$ intersect?

Notice that the results about $|M(n,\Delta_d)|_{\alpha}$ apply to $|M(n,\Delta_d)|_{\beta}$ and $|M(n,\Delta_d)|_{\gamma}$ as well. Moreover, $\lambda_n(\alpha,\beta) = \lambda_n(\alpha,\gamma) = \lambda_n(\gamma,\beta)$. Hence, the above description of the pairwise intersections applies to each of the three pairs.

4.3.1 Component intersections, odd rank

When n is odd, there is a complete answer to the question of which triple component intersection are non-empty.

Theorem 4.17. *Assume n is odd. Let $r = (n, d)$ and $q = \frac{n}{r}$. Suppose $\alpha, \beta \in J^{(n)}$ are primitive elements with $\langle \alpha \rangle \cap \langle \beta \rangle = 0$ and $\mathrm{ord}(\lambda_n(\alpha,\beta)) = q$. Let $\gamma = \alpha + \beta$. For each triple $\zeta, \zeta', \zeta'' \in \mu_r$ we have:*

$$|M(n,\Delta_d)|^{\zeta,\zeta',\zeta''}_{\alpha,\beta,\gamma} \neq \emptyset \Leftrightarrow \frac{\zeta''}{\zeta\zeta'} = 1.$$

Proof. Given $\zeta, \zeta', \zeta'' \in \mu_r$, we may choose $\delta \in J^{(n)}$ such that $\lambda_n(\alpha,\delta)^q = \zeta^{-1}$ and $\lambda_n(\beta,\delta)^q = (\zeta')^{-1}$. Then, by proposition 3.42 we have:

$$E \in |M(n,\Delta_d)|^{\zeta,\zeta',\zeta''}_{\alpha,\beta,\gamma} \Leftrightarrow L_\delta \otimes E \in |M(n,\Delta_d)|^{1,1,\frac{\zeta''}{\zeta\zeta'}}_{\alpha,\beta,\gamma}.$$

Hence it suffices to prove the seemingly weaker statement that for each $\zeta \in \mu_r$,

$$|M(n,\Delta_d)|^{1,1,\zeta}_{\alpha,\beta,\gamma} \neq \emptyset \Leftrightarrow \zeta = 1.$$

Furthermore, since the assumptions ensure that $|M(n,\Delta_d)|^{1,1}_{\alpha,\beta} \neq \emptyset$, it is enough to show that any $E \in |M(n,\Delta_d)|^{1,1}_{\alpha,\beta}$ must lie in $|M(n,\Delta_d)|^{1}_{\gamma}$.

Let $E \in |M(n,\Delta_d)|^{1,1}_{\alpha,\beta}$. Suppose $E \in |M(n,\Delta_d)|^{\zeta}_{\gamma}$. According to section 3.6 we may find line bundles L_1, L_2 and L_3 on Σ^α,Σ^β and Σ^γ respectively, such that $E \cong \pi_{\alpha *}(L_1) \cong \pi_{\beta_*}(L_2) \cong \pi_{\gamma_*}(L_3)$. By proposition 3.31 and definition 3.36, there exist line bundles K_1 on Σ^α and K_2 on Σ^β, both of degree zero, such that:

$$L_1 \cong \Delta_d^\alpha \otimes (\zeta_n^{-1})^*_\alpha(K_1) \otimes K_1^{-1}$$
$$L_2 \cong \Delta_d^\beta \otimes (\zeta_n)^*_\beta(K_2) \otimes K_2^{-1}$$

Now define:

$$\widetilde{K}_3 = \pi^{\alpha *}(K_1)^{-1} \otimes \pi^{\beta *}(K_2) \otimes [F]$$

where F is the divisor from lemma 4.13. Notice that:

$$\begin{aligned}
(\zeta_n)^{\gamma *}(\widetilde{K}_3) &\cong \pi^{\alpha *}((\zeta_n^{-1})^*_\alpha(K_1))^{-1} \otimes \pi^{\beta *}((\zeta_n)^*_\beta(K_2)) \otimes [(\zeta_n)^{\gamma *}F] \\
&\cong \widetilde{K}_3 \otimes \pi^{\alpha *}(L_1^{-1} \otimes \Delta_d^\alpha) \otimes \pi^{\beta *}(L_2 \otimes (\Delta_d^\beta)^{-1}) \otimes [(\zeta_n)^{\gamma *}F - F] \\
&\cong \widetilde{K}_3
\end{aligned}$$

–Where the last step used lemma 4.12 as well as the first property of F.

The above calculation implies that $\widetilde{K}_3 \cong \pi^{\gamma *}(K_3)$ for some line bundle K_3 on Σ^γ. Notice that $\deg(K_3) = \frac{1}{n}\deg(\widetilde{K}_3) = \frac{1}{n}\deg(F) = \frac{d(q-1)}{2}$.

Now let $L = \Delta_d^\gamma \otimes (\zeta_n^{-1})^*_\gamma(K_3) \otimes K_3^{-1}$. We have:

$$\begin{aligned}
\pi^{\gamma *}(L) &\cong \pi^{\gamma *}(\Delta_d^\gamma) \otimes (\zeta_n)^{\alpha *}(\widetilde{K}_3) \otimes \widetilde{K}_3^{-1} \\
&\cong \pi^{\gamma *}(\Delta_d^\gamma) \otimes (\zeta_n)^{\alpha *}\pi^{\beta *}(K_2) \otimes \pi^{\beta *}(K_2)^{-1} \otimes [(\zeta_n)^{\alpha *}F - F] \\
&\cong \pi^{\beta *}(\Delta_d^\beta \otimes (\zeta_n)^*_\beta(K_2) \otimes K_2^{-1}) \\
&\cong \pi^{\beta *}(L_2) \\
&\cong \pi^{\gamma *}(L_3)
\end{aligned}$$

–where we used the second property of F as well as lemma 4.12 in the last two steps.

Since $\mathrm{Ker}(\pi^{\gamma *}) = \langle \pi^*_\gamma(L_\beta) \rangle$, we get for some $j \in \{0, 1, \ldots n-1\}$:

$$L \cong L_3 \otimes \pi^*_\gamma(L_\beta)^{\otimes j}.$$

Hence, since $E \in |M(n,\Delta_d)|_{\alpha,\beta,\gamma}$:

$$\pi_{\gamma *}(L) \cong \pi_{\gamma *}(L_3 \otimes \pi_\gamma^*(L_\beta)^{\otimes j}) \cong E \otimes L_\beta^{\otimes j} \cong E.$$

But by proposition 3.31, $L \in \Delta_d^\gamma \otimes P_\gamma^{\zeta_n^{-\deg(K_3)}}$, and hence: $E \cong \pi_{\gamma *}(L) \in |M(n,\Delta_d)|_\gamma^\zeta$, where

$$\zeta = (\zeta_n^{-\deg(K_3)})^q = \zeta_n^{-\frac{dq(q-1)}{2}} = 1$$

– since n and thereby q is odd, and $n \mid dq$.

$\square$

4.3.2 Component intersections, even rank

When n is even, the situation is somewhat more complicated. In the general case we only show a partial result, which is complete in the case where $(n,d) = 2$, thus generalising the important case $n = 2, d = 0$ treated in [1].

Subsequently we propose a method for treating values of n that are twice an odd number. Unfortunately, it is not clear how to generalise to greater powers of 2, leaving the case $n = 4$ still a partial mystery.

Proposition 4.18. *Assume that both n and d are even. Let $r = (n,d)$ and $q = \frac{n}{r}$. Suppose $\alpha, \beta \in J^{(n)}$ are primitive elements with $\langle\alpha\rangle \cap \langle\beta\rangle = 0$ and $\operatorname{ord}(\lambda_n(\alpha,\beta)) = q$. Let $a \in \frac{\alpha}{2}$, $b \in \frac{\beta}{2}$, and define $\gamma = \alpha + \beta$ and $c = a + b$. For each triple $\zeta, \zeta', \zeta'' \in \mu_r$, we have:*

$$|M(n,\Delta_d)|_{a,b,c}^{\zeta,\zeta',\zeta''} \neq \emptyset \Rightarrow \left(\frac{\zeta''}{\zeta\zeta'}\right)^{\frac{r}{2}} = (-1)^{\frac{d(q-1)}{2}} \lambda_{2n}(a,b)^{\frac{n}{2}}.$$

Proof. Suppose $E \cong \pi_{\alpha *}(L_1) \cong \pi_{\beta *}(L_2) \cong \pi_{\gamma *}(L_3) \in |M(n,\Delta_d)|_{a,b,c}^{\zeta,\zeta',\zeta''}$. By proposition 3.31 and definition 3.37, we may pick line bundles K_1, K_2 and K_3 on Σ^α, Σ^β and Σ^γ respectively, such that:

$$L_1 \cong \Delta_d^\alpha \otimes \pi_\alpha^*(L_a) \otimes (\zeta_n)_\alpha^* K_1 \otimes K_1^{-1} \qquad , \qquad (\zeta_n)^{\deg(K_1)\cdot q} = \zeta$$

$$L_2 \cong \Delta_d^\beta \otimes \pi_\beta^*(L_b) \otimes (\zeta_n)_\beta^* K_2 \otimes K_2^{-1} \qquad , \qquad (\zeta_n)^{\deg(K_2)\cdot q} = \zeta'$$

$$L_3 \cong \Delta_d^\gamma \otimes \pi_\gamma^*(L_c) \otimes (\zeta_n)_\gamma^* K_3 \otimes K_3^{-1} \qquad , \qquad (\zeta_n)^{\deg(K_3)\cdot q} = \zeta''.$$

Pick divisors E_i with $[E_i] = K_i$, $(i = 1, 2, 3)$. Furthermore, pick divisors D_a, D_b and D_c on Σ, representing L_a, L_b and L_c, respectively. Define:

$$D_1 = D_d^\alpha + \pi_\alpha^*(D_a) + (\zeta_n)_\alpha^* E_1 - E_1$$

$$D_2 = D_d^\beta + \pi_\beta^*(D_b) + (\zeta_n)_\beta^* E_2 - E_2$$

$$D_3 = D_d^\gamma + \pi_\gamma^*(D_c) + (\zeta_n)_\gamma^* E_3 - E_3$$

I.e. $L_i \cong [D_i]$. Now, lemma 4.12 implies the existence of meromorphic functions h_1 and h_2 on $\tilde{\Sigma}$, such that:

$$\pi^{\alpha *}(D_1) + (h_1) = \pi^{\beta *}(D_2) + (h_2) = \pi^{\gamma *}(D_3).$$

For the sake of readability as well as later reference, the rest of the proof is divided into lemmas. □

Lemma 4.19. *Let g_α, g_β and g_γ denote the meromorphic functions on Σ^α, Σ^β and Σ^γ, given by proposition 3.25.*

We may choose D_a, D_b and D_c so that $\pi_\alpha^*(2D_a) = (g_\alpha)$, $\pi_\beta^*(2D_b) = (g_\beta)$ and $\pi_\gamma^*(2D_c) = (g_\gamma)$.

Proof. Let Σ^a denote the $2n$-sheeted covering associated to a. Let g_a be the meromorphic function, given by lemma 3.25 applied to a. Consider the diagram:

$$\begin{tikzcd} \Sigma^a \arrow[rr, "\pi"] \arrow[dr, "\pi_a"'] & & \Sigma^\alpha \arrow[dl, "\pi_\alpha"] \\ & \Sigma & \end{tikzcd}$$

Let Nm denote the norm map associated to π, and notice that for $\zeta \in \mu_{2n}$: $(\zeta^2)_\alpha^*(\mathrm{Nm}(g_a)) = \mathrm{Nm}(\zeta_a^*(g_a)) = \mathrm{Nm}(\zeta^{-1} \cdot g_a) = (\zeta^2)^{-1} \cdot \mathrm{Nm}(g_a)$. This shows that we may use $\mathrm{Nm}(g_a)$ as g_α. Thus, choosing D_a such that $\pi_a^*(D_a) = (g_a)$, we get: $\pi_\alpha^*(2D_a) = \mathrm{Nm}(\pi^*(\pi_\alpha^*(D_a))) = \mathrm{Nm}((g_a)) = (g_\alpha)$.

Similarly for b and c. □

Lemma 4.20. *We may assume without loss of generality that $\mathrm{Nm}^\alpha(h_2) = (g_\alpha)^{\frac{n}{2}}$, $\mathrm{Nm}^\beta(h_1) = (g_\beta)^{\frac{n}{2}}$ and $\mathrm{Nm}^\gamma(\frac{h_1}{h_2}) = (g_\gamma)^{\frac{n}{2}}$.*

Furthermore, for $\zeta \in \mu_n$ letting $k_A(\zeta) = \frac{\zeta^{\alpha *} h_1}{h_1}$, $k_B(\zeta) = \frac{\zeta^{\beta *} h_2}{h_2}$ and $k(\zeta) = \frac{k_A(\zeta)}{k_B(\zeta)}$, then $k(\zeta)$ is constant, and $k(\zeta)^n = \zeta^{-\frac{n}{2}} \in \{\pm 1\}$

Proof. For the first claim, notice that

$$\begin{aligned}
(\pi^{\alpha*}(\mathrm{Nm}^{\alpha}(h_2))) &= \pi^{\alpha*}(\mathrm{Nm}^{\alpha}(\pi^{\gamma*}(D_3) - \pi^{\beta*}(D_2)) \\
&= \tilde{\pi}^*(\mathrm{Nm}_{\gamma}(D_3) - \mathrm{Nm}_{\beta}(D_2)) \\
&= \tilde{\pi}^*(\mathrm{Nm}_{\gamma}(D_d^{\gamma}) + nD_c - \mathrm{Nm}_{\beta}(D_d^{\beta}) - nD_b) \\
&= \pi^{\alpha*}(\pi_{\alpha}^*(nD_a)) \\
&= \pi^{\alpha*}(\frac{n}{2}(g_{\alpha}))
\end{aligned}$$

-where we used lemma 4.19 in the last step. This shows that $\mathrm{Nm}^{\alpha}(h_2)$ and $(g_{\alpha})^{\frac{n}{2}}$ have the same divisors. Hence, after scaling h_2, we may assume that $\mathrm{Nm}^{\alpha}(h_2) = (g_{\alpha})^{\frac{n}{2}}$. Similarly in the other two cases.

For the final claim, we calculate:

$$\begin{aligned}
(k(\zeta)) &= (k_A(\zeta)) - (k_B(\zeta)) \\
&= \zeta^{\alpha*}(\pi^{\gamma*}(D_3) - \pi^{\alpha*}(D_1)) - (\pi^{\gamma*}(D_3) - \pi^{\alpha*}(D_1)) \\
&\quad -\zeta^{\beta*}(\pi^{\gamma*}(D_3) - \pi^{\beta*}(D_2)) + (\pi^{\gamma*}(D_3) - \pi^{\beta*}(D_2)) \\
&= \zeta^{\alpha*}(\pi^{\gamma*}(D_3)) - \zeta^{\beta^*}(\pi^{\gamma*}(D_3)) \\
&= \pi^{\gamma*}(\zeta_{\gamma}^*(D_3) - \zeta_{\gamma}^*(D_3)) \\
&= 0
\end{aligned}$$

This shows that $k(\zeta)$ is constant. To see that it is a root of unity:

$$k(\zeta)^n = \mathrm{Nm}^{\beta}(k(\zeta)) = \mathrm{Nm}^{\beta}(\frac{\zeta^{\alpha*}(h_1)}{h_1}) = \left(\frac{\zeta_{\beta}^*(g_{\beta})}{g_{\beta}}\right)^{\frac{n}{2}} = \zeta^{-\frac{n}{2}} = \pm 1$$

□

Lemma 4.21. $\lambda_{2n}(a,b)^{\frac{n}{2}} = (-1)^{\frac{d(q-1)}{2}} \left(\frac{\zeta''}{\zeta\zeta'}\right)^{\frac{r}{2}}$

Proof. Since by lemma 4.19, $2nD_a = (\mathrm{Nm}_{\alpha}(g_{\alpha}))$ and $2nD_b = (\mathrm{Nm}_{\beta}(g_{\beta}))$, we may calculate: (Using lemma 4.20 in the second equality.)

$$
\begin{aligned}
\lambda_{2n}(a,b)^{\frac{n}{2}} &= \left(\frac{\mathrm{Nm}_\alpha(g_\alpha)(D_b)}{\mathrm{Nm}_\beta(g_\beta)(D_a)}\right)^{\frac{n}{2}} \\
&= \frac{\widetilde{\mathrm{Nm}}(h_2)(D_b)}{\widetilde{\mathrm{Nm}}(h_1)(D_a)} \\
&= \frac{h_2(\tilde{\pi}^* D_b)}{h_1(\tilde{\pi}^* D_a)} \\
&= \frac{h_2(\tilde{\pi}^* D_c - \tilde{\pi}^* D_a)}{h_1(\tilde{\pi}^* D_c - \tilde{\pi}^* D_b)} \\
&= \frac{h_2(\pi^{\gamma *}(D_3 - D_d^\gamma - (\zeta_n)^*_\gamma E_3 + E_3) - \pi^{\alpha *}(D_1 - D_d^\alpha - (\zeta_n)^*_\alpha E_1 + E_1))}{h_1(\pi^{\gamma *}(D_3 - D_d^\gamma - (\zeta_n)^*_\gamma E_3 + E_3) - \pi^{\beta *}(D_2 - D_d^\beta - (\zeta_n)^*_\beta E_2 + E_2))} \\
&= X \cdot Y \cdot Z
\end{aligned}
$$

–Here (by Weil reciprocity):

$$
X = \frac{h_2(\pi^{\gamma *} D_3 - \pi^{\alpha *} D_1)}{h_1(\pi^{\gamma *} D_3 - \pi^{\beta *} D_2)} = \frac{h_2(h_1)}{h_1(h_2)} = 1.
$$

And:

$$
Y = \frac{h_2(\pi^{\alpha *}(D_d^\alpha) - \pi^{\gamma *} D_d^\gamma))}{h_1(\pi^{\beta *}(D_d^\alpha) - \pi^{\gamma *}(D_d^\gamma))} = \frac{h_2((\zeta_n)^{\beta *}(F) - F)}{h_1((\zeta_n)^{\alpha *}(F) - F)} = \frac{\frac{(\zeta_n^{-1})^{\beta *} h_2}{h_2}(F)}{\frac{(\zeta_n^{-1})^{\alpha *} h_1}{h_1}(F)}
$$

–where F is the divisor from addendum 4.14. So, by the last part of lemma 4.20, and using the assumption that d is even:

$$
Y = k(\zeta_n^{-1})(F) = k(\zeta_n^{-1})^{\deg(F)} = k(\zeta_n^{-1})^{n\frac{d(q-1)}{2}} = (-1)^{\frac{d(q-1)}{2}}.
$$

And finally:

$$
\begin{aligned}
Z &= \frac{h_2(\pi^{\alpha *}((\zeta_n)^*_\alpha E_1 - E_1) - \pi^{\gamma *}((\zeta_n)^*_\gamma E_3 - E_3))}{h_1(\pi^{\beta *}((\zeta_n)^*_\beta E_2 - E_2) - \pi^{\gamma *}((\zeta_n)^*_\gamma E_3 - E_3))} \\
&= \frac{\mathrm{Nm}^\alpha(h_2)((\zeta_n)^*_\alpha E_1 - E_1)}{\mathrm{Nm}^\beta(h_1)((\zeta_n)^*_\beta E_2 - E_2)} \mathrm{Nm}^\gamma(\frac{h_2}{h_1})((\zeta_n)^*_\gamma E_3 - E_3) \\
&= \left(\frac{g_\alpha((\zeta_n)^*_\alpha E_1 - E_1)}{g_\beta((\zeta_n)^*_\beta E_2 - E_2) \cdot g_\gamma((\zeta_n)^*_\gamma E_3 - E_3)} \right)^{\frac{n}{2}} \\
&= \left(\frac{\zeta_n^{\deg(E_1)}}{\zeta_n^{\deg(E_2)} \zeta_n^{\deg(E_3)}} \right)^{\frac{n}{2}} \\
&= \left(\frac{\zeta_n^{\deg(K_3)\cdot q}}{\zeta_n^{\deg(K_1)\cdot q} \zeta_n^{\deg(K_2 \cdot q)}} \right)^{\frac{n}{2q}} \\
&= \left(\frac{\zeta''}{\zeta\zeta'} \right)^{\frac{r}{2}}
\end{aligned}
$$

–Where we used the assumption that d is even, and hence $q \mid \frac{n}{2}$, again in the second but last equality.

□

Corollary 4.22. *With the assumptions in proposition 4.18, if $(n, d) = 2$, we have:*

$$|M(n, \Delta_d)|^{\zeta,\zeta',\zeta''}_{a,b,c} \neq \emptyset \Leftrightarrow \frac{\zeta''}{\zeta\zeta'} = (-1)^{\frac{d(q-1)}{2}} \lambda_{2n}(a, b)^{\frac{n}{2}}.$$

Proof. The arrow to the right comes from proposition 4.18. Conversely, given ζ, ζ', ζ'' such that the right hand statement is true, the assumptions ensure by proposition 4.9 and lemma 4.16 that

$$|M(n, \Delta_d)|^{\zeta,\zeta',\xi}_{a,b,c} \neq \emptyset$$

for some $\xi \in \mu_r$, but then the implication to the right gives that

$$\xi = \zeta\zeta'(-1)^{\frac{d(q-1)}{2}} \lambda_{2n}(a, b)^{\frac{n}{2}} = \zeta''$$

□

Finally, we seek to strengthen the result of proposition 4.18 in the special case where $d = 0$ and n is twice an odd number. Assume throughout the investigation that $n = 2\tilde{n}$, where $\tilde{n}$ is odd, let $\alpha \in J^{(n)}$ be primitive, and denote:

$$\alpha_1 = 2\alpha \in J^{(\tilde{n})}$$

$$\alpha_2 = \tilde{n}\alpha \in J^{(2)}$$

Lemma 4.23. *If $E_1 \in |M(\tilde{n}, \mathcal{O})|_{\alpha_1}$ and $E_2 \in |M(2, \mathcal{O})|_{\alpha_2}$, then:*

$$E_1 \otimes E_2 \in |M(n, \mathcal{O})|_\alpha$$

Proof. Observe that

$$L_\alpha \cong L_\alpha^{-\tilde{n}} \otimes (L_\alpha^2)^{\frac{\tilde{n}+1}{2}} \cong L_{\alpha_1}^{-1} \otimes L_{\alpha_2}^{\frac{\tilde{n}+1}{2}}.$$

Hence, if $E_1 \in |M(\tilde{n}, \mathcal{O})|_{\alpha_1}$ and $E_2 \in |M(2, \mathcal{O})|_{\alpha_2}$, then:

$$(E_1 \otimes E_2) \otimes L_\alpha \cong (E_1 \otimes L_{\alpha_1}^{-1}) \otimes (E_2 \otimes L_{\alpha_2}^{\frac{\tilde{n}+1}{2}}) \cong E_1 \otimes E_2.$$

□

Next, introduce the coverings Σ^{α_1}, Σ^{α_2} and Σ^α, and notice that we have the following commutative diagram:

$$\begin{tikzcd}
 & \Sigma^\alpha \arrow[dl, "\pi^{\alpha_1}"'] \arrow[dr, "\pi^{\alpha_2}"] \arrow[dd, "\pi_\alpha"] & \\
\Sigma^{\alpha_1} \arrow[dr, "\pi_{\alpha_1}"'] & & \Sigma^{\alpha_2} \arrow[dl, "\pi_{\alpha_2}"] \\
 & \Sigma &
\end{tikzcd}$$

–where π^{α_1} is the 2-sheeted covering associated to $\pi_{\alpha_1}^*(L_\alpha)$ and π^{α_2} is the $\tilde{n}$-sheeted covering associated to $\pi_{\alpha_2}^*(L_\alpha)$.

Assume that $\beta \in J^{(n)}$ is primitive, with $\langle\alpha\rangle \cap \langle\beta\rangle = 0$ and $\lambda_n(\alpha, \beta) = 1$. Denote:

$$\beta_1 = 2\beta \in J^{(\tilde{n})}$$

$$\beta_2 = \tilde{n}\beta \in J^{(2)}$$

Lemma 4.24. *Suppose that*

$$E_1 \cong \pi_{\alpha_1 *}(L_1) \in |M(\tilde{n}, \mathcal{O})|_{\alpha_1,\beta_1}$$

and

$$E_2 \cong \pi_{\alpha_2 *}(L_2) \in |M(2, \mathcal{O})|_{\alpha_2,\beta_2}$$

Then:

$$E_1 \otimes E_2 \cong \pi_{\alpha *}(\pi^{\alpha_1 *}(L_1) \otimes \pi^{\alpha_2 *}(L_2))$$

Proof. By lemma 4.23, $E_1 \otimes E_2 \in |M(n, \mathcal{O})|_{\alpha,\beta}$. Hence, by proposition 4.6, we may find some line bundle L on Σ^α with $\pi_{\alpha *}(L) \cong E_1 \otimes E_2$. Pulling $E_1 \otimes E_2$ back to Σ^α then yields:

$$\pi_\alpha^*(E_1 \otimes E_2) \cong \pi_\alpha^*(\pi_{\alpha *}(L)) \cong \bigoplus_{\zeta \in \mu_n} \zeta^* L \cong \bigoplus_{i=1}^{n} L \otimes \pi_\alpha^*(L_\beta)^{\otimes i} \tag{4.3}$$

Where the last isomorphism is because $L \in I_\zeta$ for some primitive $\zeta \in \mu_n$ by proposition 4.6.

On the other hand,

$$\begin{aligned}
\pi_\alpha^*(E_1 \otimes E_2) &\cong \pi_\alpha^*(E_1) \otimes \pi_\alpha^*(E_2) \\
&\cong \pi^{\alpha_1 *}(\bigoplus_{\zeta \in \mu_{\tilde{n}}} \zeta^*(L_1)) \otimes \pi^{\alpha_2 *}(\bigoplus_{\zeta \in \mu_2} \zeta^*(L_2)) \\
&\cong \pi^{\alpha_1 *}(\bigoplus_{i=1}^{\tilde{n}} L_1 \otimes \pi_{\alpha_1}^*(L_{\beta_1})^{\otimes i}) \otimes \pi^{\alpha_2 *}(\bigoplus_{i=1}^{2} L_2 \otimes \pi_{\alpha_2}^*(L_{\beta_2})^{\otimes i}) \\
&\cong \pi^{\alpha_1 *}(L_1) \otimes \pi^{\alpha_2 *}(L_2) \otimes \bigoplus_{i=1}^{\frac{n}{2}} \bigoplus_{j=1}^{2} \pi_\alpha^*(L_\beta)^{2i + \frac{n}{2} j} \\
&\cong (\pi^{\alpha_1 *}(L_1) \otimes \pi^{\alpha_2 *}(L_2)) \otimes \bigoplus_{i=1}^{n} \pi_\alpha^*(L_\beta)^i
\end{aligned} \tag{4.4}$$

Putting (4.3) and (4.4) together, the theorem of Krull-Remak-Schmidt gives that $\pi^{\alpha_1 *}(L_1) \otimes \pi^{\alpha_2 *}(L_2) \cong L \otimes \pi_\alpha^*(L_\beta)^{\otimes i}$ for some i, and hence:

$$\pi_{\alpha *}(\pi^{\alpha_1 *}(L_1) \otimes \pi^{\alpha_2 *}(L_2)) \cong \pi_{\alpha *}(L) \otimes L_\beta^{\otimes i} \cong E_1 \otimes E_2.$$

□

Remark 4.25. So far we have silently assumed that $E_1 \otimes E_2$ is semistable and has the right determinant. (Only through the choice of notation, though. -We have not used it otherwise.)

Notice that it now follows from the above lemma in our special case, where E_i is a fixed point under the action of α_i as well as β_i.

As explained in the proof of proposition 3.20, the action of $\zeta \in \mu_n$ on Σ^α covers the action of ζ^2 on Σ^{α_1} as well as the action of $\zeta^{\frac{n}{2}}$ on Σ^{α_2}, $\mu_2 \subseteq \mu_n$ and $\mu_{\tilde{n}} \subseteq \mu_n$ being the Galois groups for the coverings π^{α_1} and π^{α_2} respectively.

Lemma 4.26. *The map: $|M(\tilde{n}, \mathcal{O})|_{\alpha_1,\beta_1} \times |M(2, \mathcal{O})|_{\alpha_2,\beta_2} \to |M(n, \mathcal{O})|_{\alpha,\beta}$ given by $(E_1, E_2) \mapsto E_1 \otimes E_2$ is bijective.*

Proof. To prove surjectivity, assume that $E \in |M(n, \mathcal{O})|_{\alpha,\beta}$. By proposition 4.6, we may pick a line bundle L on Σ^α with $\pi_{\alpha *}(L) \cong E$ and $\zeta^*(L) \cong \pi_\alpha^*(L_\beta)$ for some primitive element $\zeta \in \mu_n$. Thus, for all $i \in \{1, 2, \ldots, n\}$:

$$(\zeta^i)^*(L) \cong \pi_\alpha^*(L_\beta)^{\otimes i}.$$

Now let $L_1 = (\mathrm{Nm}^{\alpha_1}(L) \otimes \pi_{\alpha_1}^*(L_\alpha))^{\otimes \frac{\tilde{n}+1}{2}}$ and $L_2 = \mathrm{Nm}^{\alpha_2}(L)^{-1}$.

First observe that:

$$\begin{aligned}
\pi^{\alpha_1 *}(L_1) \otimes \pi^{\alpha_2 *}(L_2) &\cong \left(\bigotimes_{i=1}^{2} (\zeta^{\tilde{n}i})^* L \right)^{\frac{\tilde{n}+1}{2}} \otimes \left(\bigotimes_{i=1}^{\tilde{n}} (\zeta^{2i})^* L \right)^{-1} \\
&\cong \left(\bigotimes_{i=1}^{2} L \otimes \pi_\alpha^*(L_\beta)^{\tilde{n}i} \right)^{\frac{\tilde{n}+1}{2}} \otimes \left(\bigotimes_{i=1}^{\tilde{n}} L \otimes \pi_\alpha^*(L_\beta)^{2i} \right)^{-1} \\
&\cong L^{2 \cdot \frac{\tilde{n}+1}{2}} \otimes L^{-\tilde{n}} \otimes \left(\bigotimes_{i=1}^{2} \pi_\alpha^*(L_\beta)^{\tilde{n}i} \right)^{\frac{\tilde{n}+1}{2}} \otimes \left(\bigotimes_{i=1}^{\tilde{n}} \pi_\alpha^*(L_\beta)^{2i} \right)^{-1} \\
&\cong L \otimes \pi_\alpha^*(L_\beta)^k \\
&\cong (\zeta^k)^*(L)
\end{aligned}$$

–for some $k \in \{1, 2, \ldots, n\}$. Consequently,

$$\pi_{\alpha_1 *}(L_1) \otimes \pi_{\alpha_2 *}(L_2) \cong \pi_{\alpha *}(\pi^{\alpha_1 *}(L_1) \otimes \pi^{\alpha_2 *}(L_2)) \cong \pi_{\alpha *}(L) \cong E.$$

One needs to check that $\pi_{\alpha_1 *}(L_1)$ and $\pi_{\alpha_2 *}(L_2)$ have the right determinants.

For this, observe that since $\mathcal{O}_\Sigma \cong \det(\pi_{\alpha *}(L)) \cong \mathrm{Nm}_\alpha(L) \otimes L_\alpha^{\otimes \tilde{n}}$ (by proposition 3.23), we get: (Using proposition 3.23 with the covering π_{α_1} in the first step)

$$\begin{aligned} \det(\pi_{\alpha_1 *}(L_1)) &\cong \mathrm{Nm}_{\alpha_1}(L_1) \\ &\cong \left(\mathrm{Nm}_\alpha(L) \otimes L_\alpha^{\tilde{n}}\right)^{\frac{\tilde{n}+1}{2}} \\ &\cong \left(L_\alpha^{\tilde{n}} \otimes L_\alpha^{\tilde{n}}\right)^{\frac{\tilde{n}+1}{2}} \\ &\cong \mathcal{O}_\Sigma \end{aligned}$$

And also: (Using proposition 3.23 with the covering π_{α_2}.)

$$\begin{aligned} \det(\pi_{\alpha_2 *}(L_2)) &\cong \mathrm{Nm}_{\alpha_2}(L_2) \otimes L_{\alpha_2} \\ &\cong \mathrm{Nm}_\alpha(L)^{-1} \otimes L_\alpha^{\tilde{n}} \\ &\cong \mathcal{O}_\Sigma \end{aligned}$$

This proves surjectivity. Finally, by proposition 4.6 the number of elements in $|M(n,\mathcal{O})|_{\alpha,\beta}$ is:

$$a(\alpha,\beta,0) \cdot n^{2g-2} = \phi(n) \cdot n^{2g-2}.$$

–Whereas $|M(\tilde{n},\mathcal{O})|_{\alpha_1,\beta_1}$ has $\phi(\tilde{n}) \cdot \tilde{n}^{2g-2}$ elements, and $|M(2,\mathcal{O})|_{\alpha_2,\beta_2}$ has 2^{2g-2} elements. Hence, the number of elements in $|M(\tilde{n},\mathcal{O})|_{\alpha_1,\beta_1} \times |M(2,\mathcal{O})|_{\alpha_2,\beta_2}$ is

$$\phi(\tilde{n}) \cdot \tilde{n}^{2g-2} \cdot 2^{2g-2} = \phi(n) \cdot n^{2g-2}.$$

□

Next, we ask how the map in lemma 4.26 relates the components of the fixed point varieties.

Lemma 4.27. *Suppose $a \in \frac{\alpha}{2}$. If $E_1 \in |M(\tilde{n},\mathcal{O})|_{\alpha_1}^{\zeta_1}$ for some $\zeta_1 \in \mu_{\tilde{n}}$ and $E_2 \in |M(2,\mathcal{O})|_{\tilde{n}a}^{\zeta_2}$ for some $\zeta_2 \in \mu_2$, then $E_1 \otimes E_2 \in |M(n,\mathcal{O})|_a^{\zeta_1 \cdot \zeta_2}$.*

Proof. Assume

$$E_1 \cong \pi_{\alpha_1 *}(L_1) \in |M(\tilde{n},\mathcal{O})|_{\alpha_1}^{\zeta_1}$$

$$E_2 \cong \pi_{\alpha_2 *}(L_2) \in |M(2,\mathcal{O})|_{\tilde{n}a}^{\zeta_2}.$$

This is to say that $L_1 \in P_{\alpha_1}^{\zeta_1}$ and $L_2 \in \pi_{\alpha_2}^*(L_{\tilde{n}a}) \otimes P_{\alpha_2}^{\zeta_2}$.

For all $\zeta \in \mu_n$, we have:

$$\zeta_\alpha^* \circ \pi^{\alpha_1 *} = \pi^{\alpha_1 *} \circ (\zeta^2)^*_{\alpha_1}$$

$$\zeta_\alpha^* \circ \pi^{\alpha_2 *} = \pi^{\alpha_2 *} \circ (\zeta^{\tilde{n}})^*_{\alpha_2}$$

Since pull-back takes $\mathrm{Pic}_k(\Sigma^{\alpha_1})$ to $\mathrm{Pic}_{2k}(\Sigma^{\alpha})$ and $\mathrm{Pic}_k(\Sigma^{\alpha_2})$ to $\mathrm{Pic}_{\tilde{n}k}(\Sigma^{\alpha})$, proposition 3.31 shows that:

$$\pi^{\alpha_1 *}(P^{\zeta_1}_{\alpha_1}) \subseteq P^{\zeta_1}_{\alpha}$$

$$\pi^{\alpha_2 *}(P^{\zeta_2}_{\alpha_2}) \subseteq P^{\zeta_2}_{\alpha}$$

Consequently, $\pi^{\alpha_1 *}(L_1) \in P^{\zeta_1}_\alpha$ and $\pi^{\alpha_2 *}(L_2) \in \pi^*_\alpha(L_{\tilde{n}a}) \otimes P^{\zeta_2}_\alpha$. But

$$\pi^*_\alpha(L_{\tilde{n}a}) \cong \pi^*_\alpha(L_a)^{\otimes \tilde{n}} \cong \pi^*_\alpha(L_a) \otimes \pi^*_\alpha(L_\alpha)^{\otimes \frac{\tilde{n}-1}{2}} \cong \pi^*_\alpha(L_a),$$

so $\pi^{\alpha_1 *}(L_1) \otimes \pi^{\alpha_2 *}(L_2) \in \pi^*_\alpha(L_a) \otimes P^{\zeta_1 \cdot \zeta_2}_\alpha$, and hence:

$$E_1 \otimes E_2 \in |M(n, \mathcal{O})|^{\zeta_1 \cdot \zeta_2}_a.$$

□

Theorem 4.28. *Assume $n = 2\tilde{n}$, where $\tilde{n}$ is odd. Suppose $\alpha, \beta \in J^{(n)}$ are primitive elements with $\langle\alpha\rangle \cap \langle\beta\rangle = 0$ and $\lambda_n(\alpha, \beta) = 1$. Let $a \in \frac{\alpha}{2}$, $b \in \frac{\beta}{2}$, and define $\gamma = \alpha + \beta$ and $c = a + b$. For each triple $\zeta, \zeta', \zeta'' \in \mu_n$, we have:*

$$|M(n, \mathcal{O})|^{\zeta, \zeta', \zeta''}_{a,b,c} \neq \emptyset \Leftrightarrow \left(\frac{\zeta''}{\zeta\zeta'}\right) = \lambda_{2n}(a, b)^{\frac{n}{2}}.$$

Proof. Suppose $E \in |M(n, \mathcal{O})|^{\zeta, \zeta', \zeta''}_{a,b,c}$ for some $\zeta, \zeta'\zeta'' \in \mu_n$.

Denote $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ as in the preceding lemmas. Let $a_2 = \tilde{n}a$, $b_2 = \tilde{n}b$ and $c_2 = \tilde{n}c$.

By lemma 4.26, $E \cong E_1 \otimes E_2$ for some bundles, $E_1 \in |M(\tilde{n}, \mathcal{O})|_{\alpha_1, \beta_1}$ and $E_2 \in |M(2, \mathcal{O})|_{\alpha_2, \beta_2}$.

Suppose that

$$E_1 \in |M(\tilde{n}, \mathcal{O})|^{\zeta_1, \zeta_1', \zeta_1''}_{\alpha_1, \beta_1, \gamma_1}$$

$$E_2 \in |M(2, \mathcal{O})|^{\zeta_2, \zeta_2', \zeta_2''}_{a_2, b_2, c_2}$$

– for certain $\zeta_1, \zeta_1', \zeta_1'' \in \mu_{\tilde{n}}$ and $\zeta_2, \zeta_2', \zeta_2'' \in \mu_2$. Then by lemma 4.27, $\zeta_1\zeta_2 = \zeta$, $\zeta_1'\zeta_2' = \zeta'$ and $\zeta_1''\zeta_2'' = \zeta''$. But by theorem 4.17,

$$\frac{\zeta_1''}{\zeta_1\zeta_1'} = 1$$

and by corollary 4.22 (or simply theorem 2.3 in [1]),

$$\frac{\zeta_2''}{\zeta_2\zeta_2'} = \lambda_4(a_2, b_2) = \lambda_{4\tilde{n}}(a, b)^{\tilde{n}} = \lambda_{2n}(a, b)^{\frac{n}{2}}.$$

The converse implication follows in precisely the same manner as in corollary 4.22.

□

5

The action of $J^{(n)}$ on the Hecke correspondence

So far, I have only treated one value of d at a time. There is an important interplay, known as the Hecke correspondence, between the moduli spaces corresponding to different values of d. I will only treat the case of degrees zero and one. This is done in order to avoid the topic of parabolic semistability. It will be sufficient as a tool in chapter 6 in most cases, including the ones where n is prime.

5.1 Elementary modification

Recall that $p \in \Sigma$ was chosen back in section 3.1. Suppose E is a vector bundle on Σ and $\mathcal{F} \subseteq E_p$ a codimension one subspace. Elementary modification of E at p in the direction of $\mathcal{F}$ yields a bundle E' on Σ. It is constructed as follows:

Definition 5.1. Let $\mathbb{C}_p$ denote the skyscraper sheaf on Σ with support at p. Choose an isomorphism: $E_p/\mathcal{F} \cong \mathbb{C}$. Denote by $\Gamma(E)$ the sheaf of holomorphic sections in E and define the sheaf morphism: $\lambda : \Gamma(E) \to \mathbb{C}_p$ as follows: Given $U \overset{\circ}{\subseteq} \Sigma$ and a section s on U:

$$\lambda_U(s) = \begin{cases} 0 & , \quad p \notin U \\ [s(p)] \in E_p/\mathcal{F} \cong \mathbb{C} & , \quad p \in U \end{cases}$$

The kernel of λ is clearly independent of the chosen isomorphism. According to the following lemma, it is a locally free sheaf of rank n, and hence that is is equal to $\Gamma(E')$ for some bundle E' on Σ of rank n. This bundle is the elementary modification of E in the direction of $\mathcal{F}$.

Lemma 5.2. *The kernel of λ in the definition above is a locally free sheaf of rank n.*

Proof. On a small neighbourhood U of p, we may choose a coordinate z centred at p and a holomorphic frame $e_1, e_2, \ldots, e_n$ of E such that $e_1(p) \notin \mathcal{F}$ and $e_j(p) \in \mathcal{F}$ for $j \neq 1$. This induces identifications: $\Gamma(E)(U) = \mathcal{O}(U)^{\oplus n}$ and $\mathrm{Ker}(\lambda)(U) = \{(f_1, \ldots, f_n) \in E|_U \mid f_1(p) = 0\}$. These are isomorphic (as $\mathcal{O}(U)$-modules) under multiplication in the first term by z. This shows that $\mathrm{Ker}(\lambda)$ is locally free. □

Lemma 5.3. *We have:*

$$\det(E') \cong \det(E) \otimes [-p].$$

Proof. The exact sequences below show that $\det(E') \cong \det(E) \otimes \det(\mathbb{C}_p)^{-1} \cong \det(E') \otimes \mathcal{O}_{[-p]}$.

$$0 \longrightarrow \Gamma(E') \longrightarrow \Gamma(E) \xrightarrow{\lambda} \mathbb{C}_p \longrightarrow 0 \tag{5.1}$$

$$0 \longrightarrow \mathcal{O}_{[-p]} \longrightarrow \mathcal{O} \xrightarrow{\mathrm{Ev}_p} \mathbb{C}_p \longrightarrow 0$$

□

Remark 5.4. Semistability is not preserved under elementary modification. If for example $E = L \oplus L'$ where L and L' are line bundles of degree 1 and $\mathcal{F} = L_p$, then $E' = L \oplus (L' \otimes [-p])$, which is not semistable. This is where the notion of parabolic (semi-)stability would normally enter naturally. However, if E is assumed to be of degree 1, then (semi-)stability of E ensures stability of E'.

Lemma 5.5. *In the above construction, if $\deg(E) = 1$ and E is stable, then E' is stable as well.*

Proof. Any sub-bundle F' of E' induces (via the sheaf inclusion $E' \to E$) a sub-bundle F of E.

If F_p is not contained in $\mathcal{F}$, then F' is given by elementary modification of F along $\mathcal{F} \cap F_p$. In particular, $\mathrm{rk}(F) = \mathrm{rk}(F')$ and $\deg(F) = \deg(F') + 1$. E being semistable implies:

$$\mu(F') = \frac{\deg(F) - 1}{\mathrm{rk}(F)} \leq \frac{\deg(E)}{\mathrm{rk}(E)} - \frac{1}{\mathrm{rk}(F)} < \frac{\deg(E) - 1}{\mathrm{rk}(E)} = \mu(E').$$

If F_p is contained in $\mathcal{F}$ then the sheaf inclusion $E' \to E$ restricts to an isomorphism between F' and F. In this case (this is where we need E to be of degree 1):

$$\deg(F')\mathrm{rk}(E') = \deg(F)\mathrm{rk}(E) < \deg(E)\mathrm{rk}(F) = 1 \cdot \mathrm{rk}(F) = \mathrm{rk}(F').$$

This can only be the case if $\deg(F')\mathrm{rk}(E') \leq 0 = \deg(E')\mathrm{rk}(F')$. □

5.2 The Hecke correspondence

The Hecke correspondence (in degrees zero and one) is a pair of morphisms:

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, "q_0"'] \arrow[dr, "q_1"] & \\ M(n, \Delta_0) & & M(n, \Delta_1) \end{tikzcd} \tag{5.2}$$

It is constructed in the following way: Let $\mathcal{U}$ be a Poincaré bundle (cf. remark 2.33) on $\Sigma \times M(n, \Delta_1)$. Denote by $\mathcal{U}_p$ the restriction of $\mathcal{U}$ to $\{p\} \times M(n, \Delta_1)$, considered as a bundle on $M(n, \Delta_1)$.

Definition 5.6. Define:

$$\mathcal{P} = \mathrm{Gr}_{n-1}(\mathcal{U}_p),$$

where Gr_{n-1} denote the Grassmann variety of dimension $n - 1$ subspaces. Let $q_1 : \mathcal{P} \to M(n, \Delta_1)$ be the projection induced from $\mathcal{U}_p$.

Remark 5.7. A point in $\mathcal{P}$ is given by an element $[E] \in M(n, \Delta_1)$ and an $(n-1)$-dimensional subspace $\mathcal{F}$ of $(U_p)|_{[E]} \cong E_p$. In other words, $\mathcal{P}$ consists of equivalence classes of pairs: $[(E, \mathcal{F})]$ with $\mathcal{F} \subseteq E_p$ a codimension one subspace. Two such pairs $(E_1, \mathcal{F}_1)$ and $(E_2, \mathcal{F}_2)$ are equivalent if there is an isomorphism $E_1 \cong E_2$ which takes $\mathcal{F}_1$ to $\mathcal{F}_2$.

Definition 5.8. Let $q_0 : \mathcal{P} \to M(n, \Delta_0)$ be the map that takes a class of pairs $[(E, \mathcal{F})]$ to the elementary modification of E in the direction of $\mathcal{F}$.

Remark 5.9. Notice that q_1 is a $\mathbb{CP}^{n-1}$-fibration. For later use we note the following explicit isomorphism:

Given $E \in M(n, \Delta_1)$, the fibre $q_1^{-1}(E)$ is canonically isomorphic to $\mathbb{P}(E_p^*)$. For $\omega \in E_p^* \setminus \{0\}$, the isomorphism takes $[\omega] \in \mathbb{P}(E_p^*)$ to $[(E, \mathrm{Ker}(\omega))] \in q_1^{-1}(E)$.

Remark 5.10. In fact, $\mathcal{P}$ is isomorphic to a moduli space $M(\chi, a, d)$ of parabolic bundles of rank n, sequence of multiplicities $\chi = (1, (n-1))$, weights $a = (0, 0)$ and degree d. (See chapter 3 in [27].) Hence the methods of section 2.5 could be used to show that q_0 and q_1 are in fact morphisms.

Definition 5.11. There is a natural action of $J^{(n)}$ on $\mathcal{P}$. The action of $\alpha \in J^{(n)}$ maps a class $[(E, \mathcal{F})]$ into $[(E \otimes L_\alpha, \mathcal{F} \otimes L_\alpha)]$. By usual abuse of notation, this map will be denoted simply by α.

Lemma 5.12. *With the above definition, both q_0 and q_1 become $J^{(n)}$-equivariant.*

Proof. It is obvious that q_1 becomes equivariant.

As for q_0, Let $[(E, \mathcal{F})] \in \mathcal{P}$, and let $E' = q_0([(E, \mathcal{F})])$, such that $\Gamma(E') = \mathrm{Ker}(\lambda)$, with λ as in definition 5.1. Notice that elementary modification of $E \otimes L_\alpha$ in the direction of $\mathcal{F} \otimes L_\alpha$ is equal to $\mathrm{Ker}(\tilde{\lambda})$, where $\tilde{\lambda} : \Gamma(E \otimes L_\alpha) \to \mathbb{C}_p$ is given by

$$\tilde{\lambda} : s \in \Gamma_U(E \otimes L_\alpha) \mapsto \begin{cases} 0 & , \quad p \notin U \\ [s(p)] \in (E \otimes L_\alpha)_p / (\mathcal{F} \otimes L_\alpha) \cong \mathbb{C} & , \quad p \in U \end{cases}$$

Now, $\mathrm{Ker}(\tilde{\lambda}) = \mathrm{Ker}(\lambda) \otimes \Gamma(L_\alpha) = \Gamma(E' \otimes L_\alpha)$. This means that $E' \otimes L_\alpha \cong q_0([(E \otimes L_\alpha, \mathcal{F} \otimes L_\alpha)])$. ☐

Suppose that $E \in M(n, \Delta_1)$ is fixed by a primitive element, $\alpha \in J^{(n)}$. By the above, α acts on the fibre $q_1^{-1}(E) \cong \mathbb{P}(E_p^*)$. We will need an explicit description of this action.

Recall that the construction of Δ_1^α, back in section 3.6 involved picking a point $p_\alpha \in \pi_\alpha^{-1}(p) \subseteq (L_\alpha)_p$. Also recall that $\pi_{\alpha *} : \vartheta_\alpha^{-1}(\Delta_1) \to |M(n, \Delta_1)|_\alpha$ is a surjective morphism.

Lemma 5.13. *Suppose $E = \pi_{\alpha *}(L)$. Let $\psi : E \to E \otimes L_\alpha$ be the explicit isomorphism constructed in lemma 3.17.*

Define $A : E_p \to E_p$ by the following commutative diagram:

$$\begin{tikzcd} E_p \arrow[rr, "A"] \arrow[dr, "\psi_p"'] & & E_p \\ & E_p \otimes (L_\alpha)_p \arrow[ur, "\phi_{p_\alpha}"'] & \end{tikzcd} \tag{5.3}$$

– where ϕ_{p_α} is the map: $(x \otimes z \cdot p_\alpha) \mapsto z \cdot x$.

We then have the following commutative diagram:

$$\begin{tikzcd} \mathbb{P}(E_p^*) \arrow[r, "\mathbb{P}(A^T)"] \arrow[d, "\cong"] & \mathbb{P}(E_p^*) \arrow[d, "\cong"] \\ q^{-1}(E) \arrow[r, "\alpha"] & q^{-1}(E) \end{tikzcd}$$

–Where A^T is the dual map to A, and the vertical isomorphisms are the one from remark 5.9.

Proof. Let $\omega \in E_p^*$. Sending $[\omega]$ through the diagram, to the right, and then down, yields:

$$[(E, \mathrm{Ker}(\omega \circ A))].$$

Sending it down and then to the right yields:

$$[(E \otimes L_\alpha, \mathrm{Ker}(\omega) \otimes (L_\alpha)_p)].$$

But by (5.3), $\psi : E \to E \otimes L_\alpha$ takes $x \in \mathrm{Ker}(\omega \circ A)$ to $Ax \otimes p_\alpha$ which lies in $\mathrm{Ker}(\omega) \otimes (L_\alpha)_p$. This shows that the two classes are the same. $\square$

Again let α be a primitive element in $J^{(n)}$ and denote by $|\mathcal{P}|_\alpha$ the subvariety of $\mathcal{P}$ fixed by α. For every $L \in \vartheta_\alpha^{-1}(\Delta_1)$ the projection induces a canonical isomorphism:

$$(\bigoplus_{\zeta \in \mu_n} \zeta^* L)_{p_\alpha} \cong \pi_{\alpha *}(L)_p$$

and thereby specifies an $(n-1)$ dimensional subspace of $\pi_{\alpha *}(L)_p$, namely the one corresponding to $(\bigoplus_{\zeta \neq 1} \zeta^*(L))_{p_\alpha}$.

Proposition 5.14. *Let $\alpha \in J^{(n)}$ be primitive. There is a bijection: $j_{p_\alpha} : \vartheta_\alpha^{-1}(\Delta_1) \to |\mathcal{P}|_\alpha$ defined by $j_{p_\alpha}(L) = [(\pi_{\alpha *}(L), (\bigoplus_{\zeta \neq 1} \zeta^*(L))_{p_\alpha})]$, making the following diagram commutative:*

$$\begin{tikzcd} |\mathcal{P}|_\alpha \arrow[dr, "q_1"'] & & \vartheta_\alpha^{-1}(\Delta_1) \arrow[ll, "j_{p_\alpha}"'] \arrow[dl, "\pi_{\alpha*}"] \\ & |M(n,\Delta_1)|_\alpha & \end{tikzcd}$$

Proof. Let $L \in \vartheta_\alpha^{-1}(\Delta_1)$ and denote $\pi_{\alpha *}(L)$ by E and $(\bigoplus_{\zeta \neq 1} \zeta^*(L))_{p_\alpha}$ by $\mathcal{F}$. From lemma 3.17 we have an isomorphism $\psi : E \cong E \otimes L_\alpha$ given by descent of $\tilde{\psi}$, where $\tilde{\psi}$ is induced by the matrix $B = \mathrm{diag}(1, \zeta_n, \ldots, \zeta_n^{n-1})$ mapping $\bigoplus_{j=0}^{n-1} \zeta_n^{j*}(L)$ to itself. Since E is stable and therefore simple, any other such isomorphism is given by a non-zero complex scalar times ψ. Thus, $j_{p_\alpha}(L)$ is fixed by the action of α precisely if ψ takes $\mathcal{F}$ to $\mathcal{F} \otimes L_\alpha$. Clearly, this is always the case.

If $j_{p_\alpha}(L) = j_{p_\alpha}(L')$ for $L, L' \in \vartheta_\alpha^{-1}(\Delta_1)$, lemma 3.13 implies that $L' \cong \zeta'^*(L)$ for some $\zeta' \in \mu_n$. But then, $(\bigoplus_{\zeta \neq 1} \zeta^*(L'))_{p_\alpha} \cong (\bigoplus_{\zeta \neq \zeta'} \zeta^*(L))_{p_\alpha}$. Since this must be mapped to $(\bigoplus_{\zeta \neq 1} \zeta^*(L))_{p_\alpha}$ by an automorphism of E (again, those are all constant), we get that $\zeta' = 1$. This shows that j_{p_α} is injective.

As for surjectivity, let $[(E, \mathcal{F})] \in |\mathcal{P}|_\alpha$. Since $E \in |M(n, \Delta_d)|_\alpha$, pick $L \in \vartheta_\alpha^{-1}(\Delta_1)$ with $\pi_{\alpha *}(L) = E$. The fact that α fixes $[(E, \mathcal{F})]$ implies that $\mathcal{F}$ is induced by an $(n-1)$ dimensional subspace of $(\bigoplus_{j=0}^{n-1} \zeta_n^{j*}(L))_{p_\alpha}$, which is invariant under B. These are all given by $(\bigoplus_{\zeta \neq \zeta'} \zeta^*(L))_{p_\alpha})$ for some $\zeta' \in \mu_n$. Hence, $[(E', \mathcal{F})] = j_{p_\alpha}(\zeta'^*(L))$. ☐

Remark 5.15. Again, with a little more theory on parabolic bundles, the results of section 2.5 could be applied to show that j_{p_α} is an isomorphism of varieties. However, we will only need the fact that it is a bijection.

The map j_{p_α} fits nicely with the other half of the Heche diagram:

Proposition 5.16. *For any $L \in \vartheta_\alpha^{-1}(\Delta_1)$, we have:*

$$q_0(j_{p_\alpha}(L)) = \pi_{\alpha *}(L \otimes [p_\alpha]^{-1}).$$

Proof. Let $L \in \vartheta_\alpha^{-1}(\Delta_1)$. Let $E = \pi_{\alpha *}(L)$ and $\mathcal{F} \subseteq E_p$ the subspace such that $j_{p_\alpha}(L) = [(E, \mathcal{F})]$. Let $E' = q_0(j_{p_\alpha}(L))$.

Choose an isomorphism: $L_{p_\alpha} \cong \mathbb{C}$. Using the canonical isomorphism between E_p and $(\bigoplus_{\zeta\in\mu_n} \zeta^* L)_{p_\alpha}$, this gives an isomorphism:

$$E_p/\mathcal{F} \cong (\bigoplus_{\zeta\in\mu_n} \zeta^*(L))_{p_\alpha}/(\bigoplus_{\zeta\neq 1} \zeta^*(L))_{p_\alpha} \cong L_{p_\alpha} \cong \mathbb{C}$$

and hence a map $\lambda : \Gamma(E) \to \mathbb{C}_p$.

Let $\mathcal{E} = \Gamma(E)$, $\mathcal{E}' = \Gamma(E') = \mathrm{Ker}(\lambda)$, and let $\tilde{\mathcal{E}} = \Gamma(\pi_\alpha^*(E)) \cong \Gamma(\bigoplus_{\zeta\in\mu_n} \zeta^*(L))$. We have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{\mathcal{E}}' \arrow[r] & \tilde{\mathcal{E}} \arrow[r, "\tilde{\lambda}"] & \mathcal{S} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{E} \arrow[r, "\lambda"] \arrow[u, hook] & \mathbb{C}_p \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

–where $\mathcal{S} = \bigoplus_{\zeta\in\mu_n} \zeta^*(\mathbb{C}_{p_\alpha})$, the vertical maps are inclusions of invariant sections, and $\tilde{\lambda}$ is given by evaluation of sections in $\zeta^*(L)$ in $\zeta^{-1}(p_\alpha)$ (composed with the chosen isomorphism $L_{p_\alpha} \cong \mathbb{C}$). The kernel $\tilde{\mathcal{E}}'$ of $\tilde{\lambda}$ is simply the sheaf of sections in $\bigoplus_{\zeta\in\mu_n} \zeta^*(L \otimes [p_\alpha]^{-1})$.

The diagram shows that $\mathcal{E}'$ is the invariant part of $\tilde{\mathcal{E}}'$, and therefore, E' is isomorphic to the descent of $\bigoplus_{\zeta\in\mu_n} \zeta^*(L \otimes [p_\alpha]^{-1})$, i.e. to $\pi_{\alpha *}(L \otimes [p_\alpha]^{-1})$.

$\square$

6
Lifting the action

This chapter introduces certain groups of *lifts* of the action of $J^{(n)}$ on $M(n, \Delta_d)$ to the canonical ample generator of $\mathrm{Pic}(M(n, \Delta_d))$.

The groups constitute central extensions of $J^{(n)}$, and they turn out to be defined independently of the complex structure on Σ. In chapter 7 we shall further see how the lifts induce actions on the Verlinde bundles over Teichmüller space of the underlying surface of Σ, and thereby become represented on the Verlinde vector spaces in the construction of topological quantum field theories from gauge theory and conformal field theory.

During the investigation of the groups we gradually strengthen the assumptions, finally giving a complete presentation of the groups of lifts in the special case where n is an odd prime. (But *a priori* $n \geq 2$ is arbitrary.)

By ongoing abuse of notation, we will make no distinction between elements $\alpha \in J^{(n)}$ and the automorphisms of $M(n, \Delta_d)$ given by the action of α.

6.1 Definition of the lifts

A main result of [8] is that the Picard group of $M(n, d)$ is generated by pullbacks from $J(\Sigma)$ (under the determinant map) along with a canonical, ample[1] line bundle, $\mathcal{L}_{n,d}$. Furthermore, each subvariety $M(n, \Delta_d)$ have Picard group isomorphic to $\mathbb{Z}$ with the restriction of $\mathcal{L}_{n,d}$ as its generator. This restriction is of course still ample and will also be denoted by $\mathcal{L}_{n,d}$.

[1] For a brief introduction to ample line bundles, see [35] p.143-156.

Definition 6.1. By a *lift* of α to $\mathcal{L}_{n,d}$, we shall mean an invertible bundle map ρ from $\mathcal{L}_{n,d}$ to itself, inducing α on the base.

Lemma 6.2. *For each $\alpha \in J^{(n)}$, there exist lifts of α.*

Proof. Pull back by α induces an endomorphism α^* of $\mathrm{Pic}(M(n, \Delta_d)) \cong \mathbb{Z}$ with $(\alpha^*)^n$=1. Hence $\alpha^*(\mathcal{L}_{n,d}) = \mathcal{L}_{n,d}^{\otimes(\pm 1)}$. (The option "$-1$" being relevant only when n is even, of course.) But since $\mathcal{L}_{n,d}$ is ample, so is $\alpha^*(\mathcal{L}_{n,d})$ and by the Kodaira vanishing theorem, $\mathcal{L}_{n,d}^{-1}$ is not ample. Thus, $\alpha^*(\mathcal{L}_{n,d}) \cong \mathcal{L}_{n,d}$.

By choosing an isomorphism $\mathcal{L}_{n,d} \cong \alpha^*(\mathcal{L}_{n,d})$ and composing this with the canonical (invertible) bundle map $\alpha^*(\mathcal{L}_{n,d}) \to \mathcal{L}_{n,d}$ (inducing α on the base), we get a lift of α. □

Definition 6.3. Let $\mathcal{G}(J^{(n)}, \mathcal{L}_{n,d})$ denote the group consisting of all possible lifts of elements in $J^{(n)}$.

Lemma 6.4. *$\mathcal{G}(J^{(n)}, \mathcal{L}_{n,d})$ is a central extension:*

$$\{1\} \to \mathbb{C}^* \to \mathcal{G}(J^{(n)}, \mathcal{L}_{n,d}) \to J^{(n)} \to \{0\} \tag{6.1}$$

Consequently, a unique lift ρ of a given $\alpha \in J^{(n)}$ may be specified by demanding that ρ act by multiplication with a certain scalar on the fibre of $\mathcal{L}_{n,d}$ over a point $x \in M(n, \Delta_d)$ fixed by α.

The specified lift then acts by multiplication with the same scalar in every fibre above the connected component of $|M(n, \Delta_d)|_\alpha$ containing x.

Proof. A lift of $0 \in J^{(n)}$ is simply an algebraic function on $M(n, \Delta_d)$ and hence, $M(n, \Delta_d)$ being complete, a non-zero constant. This shows that the sequence (6.1) is a central extension.

Therefore, two different lifts of an element $\alpha \in J^{(n)}$ differ only by a scalar. And since a lift ρ of α multiplies by a non-zero scalar in a fibre of $\mathcal{L}_{n,d}$ over a point in $|M(n, \Delta_d)|_\alpha$, fixing this scalar determines ρ uniquely.

The final claim is true because ρ^n is a lift of the identity, and hence constant. Therefore, the action on fibres above fixed points can vary only by n'th roots of unity. By continuity, it must be constant on connected components. □

Definition 6.5. If n is odd, for each element $\alpha \in J^{(n)}$, define $\rho_{\alpha,d}$ to be the lift of α to $\mathcal{L}_{n,d}$, which acts as the identity on the fibre over each point in $|M(n, \Delta_d)|_\alpha^1$.

Denote by $\mathcal{E}(n, d)$ the subgroup of $\mathcal{G}(J^{(n)}, \mathcal{L}_{n,d})$ generated by $\{\rho_{\alpha,d} \mid \alpha \in J^{(n)}\}$.

Definition 6.6. If n is even, for each element $a \in J^{(2n)}$, define $\rho_{a,d}$ to be the lift of $\alpha = 2a$ to $\mathcal{L}_{n,d}$, which acts as the identity on the fibre over each point in $|M(n, \Delta_d)|^1_a$.

Denote by $\mathcal{E}(n, d)$ the subgroup of $\mathcal{G}(J^{(n)}, \mathcal{L}_{n,d})$ generated by $\{\rho_{a,d} \mid a \in J^{(2n)}\}$.

Remark 6.7. Notice that we suppress the dependence on n of $\rho_{\alpha,d}$ and $\rho_{a,d}$ in the notation. This should not cause confusion, since n will always be fixed when discussing the lifts.

Whenever appropriate, we will also suppress the dependence on d, writing: $\rho_{\alpha,d} = \rho_\alpha$ and $\rho_{a,d} = \rho_a$.

6.2 Investigation of the groups of lifts.

We now aim to give a description of the groups $\mathcal{E}(n, d)$ in terms of relations between the generators. We begin with a direct application of previous results.

Lemma 6.8. *For odd n, whenever $\alpha \in J^{(n)}$ and $k \in \{1, 2 \ldots, n\}$, we have:*

$$\rho_{k\alpha} = (\rho_\alpha)^k.$$

For even n, whenever $a \in J^{(2n)}$ and $k \in \{1, 2, \ldots n\}$, we have:

$$\rho_{ka} = (\rho_a)^k.$$

Proof. The proof is similar in the two cases, so we only do it for n odd.

We may assume without loss of generality that α is primitive. -Otherwise choose a primitive element $\beta \in J^{(n)}$ with $l\beta = \alpha$ for some l and use the statement for primitive elements twice:

$$\rho_{k\alpha} = \rho_{kl\beta} = (\rho_\beta)^{kl} = ((\rho_\beta)^l)^k = (\rho_\alpha)^k.$$

Notice that $\rho_{k\alpha}$ and $(\rho_\alpha)^k$ are both lifts of $k\alpha$, and thus scalar multiples of each other. But according to proposition 3.59:

$$|M(n, \Delta_d)|^1_\alpha \subseteq |M(n, \Delta_d)|^1_{k\alpha}.$$

This shows that in fibres of $\mathcal{L}_{n,d}$ over $|M(n, \Delta_d)|^1_\alpha$, both ρ_α (and hence $(\rho_\alpha)^k$) and $\rho_{k\alpha}$ act as the identity. Therefore, being scalar multiples of each other, they must agree everywhere. ∎

Using the tools we developed in chapter 5, a very useful property of the elements in $\mathcal{E}(n,0)$ arises.

Proposition 6.9. *Suppose n is odd. Let $\alpha \in J^{(n)}$ and $\rho = \rho_{\alpha,0} \in \mathcal{E}(n,0)$. Then for each $\zeta \in \mu_n$, ρ acts by multiplication with ζ in the fibres of $\mathcal{L}_{n,0}$ above the component $|M(n,\mathcal{O})|^{\zeta}_{\alpha}$.*

Suppose n is even. Let $a \in J^{(2n)}$ and $\rho = \rho_{a,0} \in \mathcal{E}(n,0)$. Then for each $\zeta \in \mu_n$, ρ acts by multiplication with ζ in the fibres of $\mathcal{L}_{n,0}$ above the component $|M(n,\mathcal{O})|^{\zeta}_{a}$.

Proof. We begin with some simplifications: First of all, the proof in the two cases differs only in notation, so we concentrate on the case where n is odd. Second, it is sufficient to show the theorem in the case where α is primitive. The general situation then follows by writing α as $k\tilde{\alpha}$ and $\rho = \tilde{\rho}^k$, where the claim holds for the pair $(\tilde{\alpha},\tilde{\rho})$. -Then applying lemma 6.8 and proposition 3.59.

Let E represent a point $[E]$ in $|M(n,\Delta_1)|_{\alpha}$. By proposition 5.14, the action of α on the fibre of q_1 above $[E]$ has exactly n fixed points. (Namely the ones corresponding to the fibre of $\pi_{\alpha *}$ under j_{p_α}.)

Let $L \in \vartheta_{\alpha}^{-1}(\Delta_1)$ be such that $E = \pi_{\alpha *}(L)$. According to proposition 3.34 and equation (3.13) (in this case, $r = 1$ and $q = n$), we may fix L uniquely (up to isomorphism) by demanding that $L \in \Delta_1^{\alpha} \otimes P^1_{\alpha}$. Then by proposition 3.31,

$$\zeta^*(L) = L \otimes (\zeta^*(L) \otimes L^{-1}) \in \Delta_1^{\alpha} \otimes P^{\zeta}_{\alpha}.$$

Notice that Δ_1^{α} is simply $[p_\alpha]$, so by proposition 5.16 and definition 3.36 (in degree zero):

$$q_0(j_{p_\alpha}(\zeta^*(L))) = \pi_{\alpha *}([p_\alpha]^{-1} \otimes L) \in |M(n,\mathcal{O})|^{\zeta}_{\alpha}.$$

One should have the following picture in mind:

$$\begin{tikzcd}
& q_1^{-1}(E) \cong \mathbb{P}(E_p^*) \arrow[dl, "q_0"'] \arrow[dr, "q_1"] & & \{\zeta^*(L) \mid \zeta \in \mu_n\} \arrow[ll, "j_{p_\alpha}"'] \arrow[dl, "\pi_{\alpha *}"] \\
|M(n,\mathcal{O})|_{\alpha} & & \{E\} &
\end{tikzcd}$$

The Poincaré bundle $\mathcal{U}$, used in the construction of $\mathcal{P}$, can be normalised in such a way that $q_0^*(\mathcal{L}_{n,0})|_{q_1^{-1}([E])}$ is isomorphic to the tautological line bundle $\mathcal{O}(1)$ over $q_1^{-1}([E]) \cong \mathbb{P}(E_p^*)$. (See Lemma 2.3. in [12].)

The action of ρ on $\mathcal{L}_{n,o}$ pulls back to a lift of the action of α to $q_0^*(\mathcal{L}_{n,0})|_{q_1^{-1}([E])} \cong \mathcal{O}(1)$ over $q_1^{-1}([E]) \cong \mathbb{P}(E_p^*)$.

The proof will be finished upon showing that this pull-back acts by multiplication with ζ in the fibre above $j_{p_\alpha}(\zeta^*(L))$.

In view of lemma 5.13, one possible lift of the action of α on $\mathbb{P}(E_p^*)$ is given by:

$$R_\alpha : \begin{cases} \mathcal{O}_{\mathbb{P}(E_p^*)}(1) & \to & \mathcal{O}_{\mathbb{P}(E_p^*)}(1) \\ (x,\omega) & \mapsto & (\mathbb{P}(A^T)(x), A^T(\omega)) \end{cases}$$

–where $x \in \mathbb{P}(E_p^*)$ and ω is a non-zero element in the line in E_p^* given by x.

This lift acts by multiplication with ζ in the the fibre above the point in $\mathbb{P}(E_p^*)$ corresponding to $j_{p_\alpha}(\zeta^*(L))$. (See lemma 6.10 for the calculation.)

Since $\mathbb{P}(E_p^*)$ is compact, any two lifts of the action of α to $q_0^*(\mathcal{L}_{n,0})|_{q_1^{-1}([E])} \cong \mathcal{O}(1)$ over $q_1^{-1}([E]) \cong \mathbb{P}(E_p^*)$ are scalar multiples of each other. Both R_α and $q_0^*(\rho_{\alpha,0})|_{q_1^{-1}([E])}$ being the identity on the fibre above $j_{p_\alpha}(L)$, they must be equal, and the claim follows. ☐

Lemma 6.10. *Under the identification $\mathbb{P}(E_p^*) \cong q_1^{-1}([E])$ from remark 5.9, the lift R_α, defined in the proof above, acts by multiplication with ζ in fibres of $\mathcal{O}_{\mathbb{P}(E_p^*)}(1)$ above the point corresponding to $j_{p_\alpha}(\zeta^*(L)) \in q_1^{-1}([E])$.*

Proof. Let $\zeta \in \mu_n$. Choose $\bar{\omega}_\zeta : (\bigoplus_{\zeta' \in \mu_n} \zeta'^*(L))_{p_\alpha} \to \mathbb{C}$, such that $\mathrm{Ker}(\bar{\omega}_\zeta) = (\bigoplus_{\zeta' \neq \zeta} \zeta'^*(L))_{p_\alpha}$. Under the canonical isomorphism $(\bigoplus_{\zeta' \in \mu_n} \zeta'^*(L))_{p_\alpha} \cong E_p$, $\bar{\omega}_\zeta$ induces an element $\omega_\zeta \in E_p^*$ with $[(E, \mathrm{Ker}(\omega_\zeta))] = j_{p_\alpha}(\zeta^*(L))$. In other words, ω_ζ corresponds to $j_{p_\alpha}(\zeta^*(L))$ under the identification from remark 5.9.

For the point $([\omega_\zeta], \omega_\zeta)$ in the fibre of $\mathcal{O}_{\mathbb{P}(E_p^*)}(1)$ above $[\omega_\zeta] \in \mathbb{P}(E_p^*)$, we have:

$$R_\alpha(([\omega_\zeta], \omega_\zeta)) = (\mathbb{P}(A^T)([\omega_\zeta]), A^T(\omega_\zeta)) = ([\omega_\zeta \circ A], \omega_\zeta \circ A)$$

Under the identification $E_p \cong (\bigoplus_{\zeta' \in \mu_n} \zeta'^*(L))_{p_\alpha}$ the map $A : E_p \to E_p$ from lemma 5.13 is given by multiplication with the matrix:

$$B = \begin{bmatrix} 1 & & & 0 \\ & \zeta_n & & \\ & & \ddots & \\ 0 & & & \zeta_n^{n-1} \end{bmatrix}$$

Therefore, for $\xi = (\xi_{\zeta'})_{\zeta' \in \mu_n} \in E_p \cong (\bigoplus_{\zeta' \in \mu_n} \zeta'^*(L))_{p_\alpha}$, we have:

$$\bar{\omega}_\zeta(A\xi) = \bar{\omega}_\zeta(\zeta \cdot \xi_\zeta) = \zeta \cdot \bar{\omega}_\zeta(\xi).$$

And hence $\omega_\zeta \circ A = \zeta \cdot \omega_\zeta$, proving that $R_\alpha(([\omega_\zeta], \omega_\zeta)) = ([\omega_\zeta], \zeta \cdot \omega_\zeta) = \zeta \cdot ([\omega_\zeta], \omega_\zeta)$.

□

Remark 6.11. The technique introduced in the proof of proposition 6.9, using a fixed point in degree 1 to construct a $\mathbb{CP}^{(n-1)}$-image inside $M(n, \mathcal{O})$, intersecting each of the fixed point components, will be used again in the proof of theorem 6.14.

Corollary 6.12. *If n is even, and $a_1, a_2 \in J^{(2n)}$ are such that $2a_1 = 2a_2 = \alpha$, then:*

$$\rho_{a_2,0} = \lambda_{2n}(a_1, a_2)\rho_{a_1,0}$$

Furthermore:

$$\rho_{a_1,0}\rho_{a_2,0} = \lambda_{2n}(a_1, a_2)\rho_{a_1+a_2,0}$$

Proof. Notice that $2\alpha = 2(a_1 + a_2) = 2(2a_1) \in J^{(\frac{n}{2})}$.

Since $\lambda = \lambda_{2\frac{n}{2}}(2a_1, a_1 + a_2) \in \{\pm 1\}$, we get from proposition 3.58 (with 2α playing the role as α, and hence $m = \frac{n}{2}$, $s = 2$, and $\tilde{q} = 1$):

$$|M(n, \mathcal{O})|^1_{2a_1} = |M(n, \mathcal{O})|^{\lambda^2}_{a_1+a_2} = |M(n, \mathcal{O})|^1_{a_1+a_2}.$$

Furthermore, by proposition 3.59,

$$|M(n, \mathcal{O})|^1_{a_1} \subseteq |M(n, \mathcal{O})|^1_{2a_1},$$

and by proposition 3.58,

$$|M(n, \mathcal{O})|^1_{a_1} = |M(n, \mathcal{O})|^{\lambda_{2n}(a_1,a_2)}_{a_2}.$$

Therefore, picking a point in $|M(n, \mathcal{O})|^1_{a_1}$, we see that in the fibre above, ρ_{a_1} acts as the identity, and so does $\rho_{a_1+a_2}$, whilst ρ_{a_2} acts by multiplication with $\lambda_{2n}(a_1, a_2)$, by proposition 6.9.

Consequently, since $\rho_{a_2}\rho_{a_1}^{-1}$ and $\rho_{a_1}\rho_{a_2}\rho_{a_1+a_2}^{-1}$ are both lifts of the identity, and hence constant, we get:

$$\rho_{a_2}\rho_{a_1}^{-1} = \lambda_{2n}(a_1, a_2),$$
$$\rho_{a_1}\rho_{a_2}\rho_{a_1+a_2}^{-1} = \lambda_{2n}(a_1, a_2).$$

□

Another immediate consequence is the following.

Proposition 6.13. *The alternating form of the central extension (6.1) is the Weil pairing when $d = 0$. In other words:*

For $\alpha, \beta \in J^{(n)}$, we have when n is odd,

$$\rho_{\alpha,0}\rho_{\beta,0}(\rho_{\alpha,0})^{-1}(\rho_{\beta,0})^{-1} = \lambda_n(\alpha, \beta).$$

For $a, b \in J^{(2n)}$, $\alpha = 2a$, $\beta = 2b$, we have when n is even,

$$\rho_{a,0}\rho_{b,0}(\rho_{a,0})^{-1}(\rho_{b,0})^{-1} = \lambda_n(\alpha, \beta).$$

Proof. The two cases differ only in notation, so we assume n is odd. Let $m = \mathrm{ord}(\alpha)$ and $\lambda = \lambda_n(\alpha, \beta) \in \mu_m$. Pick an element $E \in |M(n, \mathcal{O})|^{\lambda}_{\alpha}$. By proposition 3.60, we have: $L_\beta^{-1} \otimes E \in |M(n, \mathcal{O})|^{1}_{\alpha}$. Hence, for any ξ in the fibre of $\mathcal{L}_{n,0}$ above E, $(\rho_{\beta,0})^{-1}(\xi)$ lies in a fibre above $|M(n, \mathcal{O})|^{1}_{\alpha}$. Hence:

$$\rho_{\alpha,0}\rho_{\beta,0}(\rho_{\alpha,0})^{-1}(\rho_{\beta,0})^{-1}(\xi) = \rho_{\alpha,0}\rho_{\beta,0}(1 \cdot ((\rho_{\beta,0})^{-1}(\xi))) = \rho_{\alpha,0}(\xi) = \lambda \cdot \xi$$

Since $\rho_{\alpha,0}\rho_{\beta,0}(\rho_{\alpha,0})^{-1}(\rho_{\beta,0})^{-1}$ is a lift of the identity, and hence constant, we get the desired result. ☐

Finally, as promised, we give a complete presentation of the groups $\mathcal{E}(n, d)$ in the cases where n is an odd prime.

Theorem 6.14. *Assume n is an odd prime. We have for all $\alpha, \beta \in J^{(n)}$:*

$$\rho_{\alpha,d}\rho_{\beta,d} = \rho_{\alpha+\beta,d} \quad , \quad d \neq 0$$

and

$$\rho_{\alpha,0}\rho_{\beta,0} = \lambda_n(\alpha, \beta)^{\frac{n+1}{2}} \cdot \rho_{\alpha+\beta,0}$$

Proof. Denote for the entire proof: $\alpha + \beta = \gamma$. In both cases, if $\beta = k\alpha$, for some $k \in \{1, 2, \dots, n\}$, the claim follows by lemma 6.8:

$$\rho_\alpha\rho_\beta = \rho_\alpha(\rho_\alpha)^k = \rho_\alpha^{1+k} = \rho_{(1+k)\alpha} = \rho_{\alpha+\beta}.$$

Hence we may assume for the rest of the proof that $\langle\alpha\rangle \cap \langle\beta\rangle = 0$. We begin with the case $d \neq 0$:

If $\lambda_n(\alpha, \beta) \neq 1$, we have by corollary 4.7 and lemma 4.16 that $|M(n, \Delta_d)|_{\alpha,\beta,\gamma}$ is non-empty. Since each of the fixed point varieties is connected in this case (by theorem 3.33), we may choose a point in $|M(n,\Delta_d)|^{1,1,1}_{\alpha,\beta,\gamma}$. In the fibre of $\mathcal{L}_{n,d}$ above this point, we have: $\rho_\alpha = \rho_\beta = \rho_\gamma = 1$. Since $\rho_\alpha\rho_\beta(\rho_\gamma)^{-1}$ is a lift of $0 \in J^{(n)}$, and hence constant, we get: $\rho_\alpha\rho_\beta(\rho_\gamma)^{-1} = 1$.

If $\lambda_n(\alpha,\beta) = 1$, using proposition 2.16 (and the fact that $\langle\alpha\rangle \cap \langle\beta\rangle = 0$) , we may pick an element $\delta \in J^{(n)}$ with $\lambda_n(\alpha,\delta) = \lambda_n(\beta,\delta) = \zeta_n$.

Then, by the above:

$$\rho_\beta\rho_\delta = \rho_{\beta+\delta}.$$

And since $\lambda_n(\alpha,\beta+\delta) = \lambda_n(\alpha,\delta) = \zeta_n$:

$$\rho_\alpha\rho_{\beta+\delta} = \rho_{\alpha+\beta+\delta}.$$

And since $\lambda_n(\alpha+\beta+\delta,-\delta) = (\zeta_n)^{-2} \neq 1$:

$$\rho_{\alpha+\beta+\delta}\rho_{-\delta} = \rho_{\alpha+\beta+\delta-\delta} = \rho_{\alpha+\beta}.$$

Putting all this together:

$$\rho_{\alpha+\beta} = (\rho_\alpha(\rho_\beta\rho_\delta))\rho_{-\delta} = \rho_\alpha\rho_\beta.$$

This leaves the case $d = 0$.

If $\lambda_n(\alpha,\beta) = 1$, in this case, theorem 4.17 gives that $|M(n,\mathcal{O})|^{1,1,1}_{\alpha,\beta,\gamma} \neq \emptyset$, and hence evaluating in a fibre above a point in there, we get:

$$\rho_{\alpha,0}\rho_{\beta,0}(\rho_{\gamma,0})^{-1} = 1.$$

Finally, if $\lambda_n(\alpha,\beta) \neq 1$, a bit more effort is required. This case will take up the rest of the chapter.

We get from theorem 4.17 that $|M(n,\Delta_1)|^{1,1,1}_{\alpha,\beta,\gamma} \neq \emptyset$. Hence, we can find line bundles L_1, L_2 and L_3 on Σ^α, Σ^β and Σ^γ, respectively, such that:

$$\pi_{\alpha*}(L_1) \cong \pi_{\beta*}(L_2) \cong \pi_{\gamma*}(L_3)$$

represent an element $[E]$ in $|M(n,\Delta_1)|^{1,1,1}_{\alpha,\beta,\gamma}$. (Notice that we have not picked the representative E yet.)

Now consider the Hecke diagram. By pull-back under q_0, the lifts ρ_α, ρ_β and ρ_γ define maps:

$$q_0^*(\mathcal{L}_{n,0})|_{q_1^{-1}([E])} \to q_0^*(\mathcal{L}_{n,0})|_{q_1^{-1}([E])}$$

covering the action of α, β and γ.

As in the proof of proposition 6.9, the pull-back of ρ_α can be described explicitly by letting $E = \pi_{\alpha*}(L_1)$ and using the induced canonical identification from remark 5.9:

$$\begin{tikzcd} q_0^*(\mathcal{L}_{n,0})|_{q_1^{-1}([E])} \arrow[r, "\cong"] \arrow[d] & \mathcal{O}_{\mathbb{P}(E_p^*)}(1) \arrow[d] \\ q_1^{-1}([E]) \arrow[r, "\cong"] & \mathbb{P}(E_p^*) \end{tikzcd} \tag{6.2}$$

Indeed, fix L_1 uniquely (up to isomorphism) by demanding that $L_1 \in [p_\alpha] \otimes P^1_\alpha$, and choose a basis $(\omega_\zeta)_{\zeta\in\mu_n}$ for E_p^*, such that

$$\mathrm{Ker}(\omega_\zeta) = (\bigoplus_{\zeta'\neq\zeta} \zeta'^*(L_1))_{p_\alpha},$$

–under the canonical identification $E_p \cong (\bigoplus_{\zeta'\in\mu_n} \zeta'^*(L_1))_{p_\alpha}$. Then, by lemma 5.13 and the discussion in the proof of proposition 6.9, under the identification (6.2), the pull-back of ρ_α corresponds to the map:

$$R_\alpha : \begin{cases} \mathcal{O}_{\mathbb{P}(E_p^*)}(1) & \to & \mathcal{O}_{\mathbb{P}(E_p^*)}(1) \\ ([\omega],\omega) & \mapsto & ([(T_\alpha)(\omega)], T_\alpha(\omega)) \end{cases}$$

–where $T_\alpha : E_p^* \to E_p^*$ is the transpose of the map A defined in lemma 5.13. It is characterised by the fact that $T_\alpha(\omega_\zeta) = \zeta\cdot\omega_\zeta$. Completely similar remarks apply to β and γ.

The rest of the proof consists of finding explicit isomorphisms:

$$\pi_{\alpha*}(L_1) \cong \pi_{\beta*}(L_2) \cong \pi_{\gamma*}(L_3)$$

and showing that under these isomorphisms, the maps T_α, T_β and T_γ relate as follows:

$$T_\alpha \circ T_\beta = \lambda_n(\alpha,\beta)^{\frac{n+1}{2}} T_\gamma$$

This is done in the following sequence of lemmas, thus completing the proof. □

Remark 6.15. An error in the following argument was found during the last hours of writing. We emphasise in the following which part is missing by turning one lemma into a conjecture.

Hence, the final part of the theorem remains a conjecture at the writing. It is strongly believed that the arguments can be turned into a rigorous proof with a little extra effort.

Let L_1, L_2 and L_3 be line bundles on Σ^α, Σ^β and Σ^γ, chosen as described above. Let $\lambda = \lambda_n(\alpha, \beta)$, and recall that we are currently assuming that λ is primitive.

We begin with a simple observation.

Lemma 6.16.
$$\lambda_\alpha^*(L_1) \cong L_1 \otimes \pi_\alpha^*(L_\beta)$$
$$\lambda_\beta^*(L_2) \cong L_2 \otimes \pi_\beta^*(L_\alpha)^{-1}$$
$$\lambda_\gamma^*(L_3) \cong L_3 \otimes \pi_\gamma^*(L_\alpha)^{-1}$$

Proof. Notice that since $\lambda_n(\alpha, \beta) = \lambda_n(\beta, -\alpha) = \lambda_n(\gamma, -\alpha) = \lambda$, the description of the pairwise intersections in proposition 4.6 applies to each pair. In this case, $A(\alpha, \beta, 1) = A(\beta, -\alpha, 1) = A(\gamma, -\alpha, 1) = \{\lambda\}$. □

The next result is an analogue of lemma 4.20, only this time in odd rank.

Using definition 3.36 and proposition 3.31, choose line bundles K_1, K_2 and K_3 of degree zero on Σ^α, Σ^β and Σ^γ respectively, such that:

$$L_1 \cong \Delta_1^\alpha \otimes \lambda_\alpha^* K_1 \otimes K_1^{-1}$$
$$L_2 \cong \Delta_1^\beta \otimes \lambda_\beta^* K_2 \otimes K_2^{-1}$$
$$L_3 \cong \Delta_1^\gamma \otimes \lambda_\gamma^* K_3 \otimes K_3^{-1}$$

Pick divisors E_i with $[E_i] = K_i$, $(i = 1, 2, 3)$. Define:

$$D_1 = p_\alpha + \lambda_\alpha^* E_1 - E_1$$
$$D_2 = p_\beta + \lambda_\beta^* E_2 - E_2$$
$$D_3 = p_\gamma + \lambda_\gamma^* E_3 - E_3$$

I.e. $L_i \cong [D_i]$. Now introduce the covering $\tilde{\Sigma}$ from section 4.2. Lemma 4.12 implies the existence of meromorphic functions h_1 and h_2 on $\tilde{\Sigma}$, such that:

$$\pi^{\alpha *}(D_1) + (h_1) = \pi^{\beta *}(D_2) + (h_2) = \pi^{\gamma *}(D_3).$$

Lemma 6.17. *We may assume that* $\mathrm{Nm}^\alpha(h_2) = 1$, $\mathrm{Nm}^\beta(h_1) = 1$. *Furthermore, for* $\zeta \in \mu_n$*, letting* $k_A(\zeta) = \frac{\zeta^{\alpha*}(h_1)}{h_1}$, $k_B(\zeta) = \frac{\zeta^{\beta*}(h_2)}{h_2}$ *and* $k(\zeta) = \frac{k_A}{k_B}$*, then* $k(\zeta)$ *is constant, and* $k(\zeta) \in \mu_n$ *for all* $\zeta \in \mu_n$.

Proof. Since

$$\begin{aligned}(\mathrm{Nm}^\alpha(h_2)) &= \mathrm{Nm}^\alpha(\pi^{\gamma*}(D_3) - \pi^{\beta*}(D_2)) \\ &= \pi_\alpha^*(\mathrm{Nm}_\gamma(D_3)) - \pi_\alpha^*(\mathrm{Nm}_\beta(D_2)) \\ &= \pi_\alpha^*(p-p) = 0,\end{aligned}$$

–we see that $\mathrm{Nm}^\alpha(h_2)$ is a non-zero constant. Hence, by scaling h_2, we may assume $\mathrm{Nm}^\alpha(h_2) = 1$. Similarly for $\mathrm{Nm}^\beta(h_1)$.

For the last claim, by the exact same argument as in the proof of lemma 4.20, we see that $(k(\zeta)) = 0$, and hence k must be constant. We may then calculate:

$$k(\zeta)^n = \mathrm{Nm}^\beta(k(\zeta)) = \mathrm{Nm}^\beta(k_A(\zeta)) = \frac{\zeta_\beta^*(\mathrm{Nm}^\beta(h_1))}{\mathrm{Nm}^\beta(h_1)} = 1$$

□

Conjecture 1. *With the above, we have:* $k(\lambda) = k(\lambda^{-1})$

Remark 6.18. The conjecture will not be used until the very last sentences of the proof.

Now let L denote $\pi^{\gamma*}(L_3)$. We make an auxiliary definition:

Definition 6.19. For $\zeta \in \mu_n$, denote by $C(\zeta)$ the canonical bundle map: $L \to L$, covering the deck transformation, ζ^γ, of π^γ. In mnemonics:

$$\begin{array}{ccc} L & \xrightarrow{C(\zeta)} & L \\ \downarrow & & \downarrow \\ \tilde{\Sigma} & \xrightarrow{\zeta^\gamma} & \tilde{\Sigma} \end{array}$$

Furthermore, choose isomorphisms: $\pi^{\alpha*}(L_1) \cong L$ and $\pi^{\beta*}(L_2) \cong L$, and define for $\zeta \in \mu_n$ the bundle maps $A(\zeta), B(\zeta) : L \to L$ by the following commutative diagrams:

$$\begin{tikzcd}
L \arrow[r, "\cong"] \arrow[d] \arrow[rrr, bend left, "A(\zeta)"] & \pi^{\alpha*}(L_1) \arrow[r] \arrow[d] & \pi^{\alpha*}(L_1) \arrow[r, "\cong"] \arrow[d] & L \arrow[d] \\
\tilde{\Sigma} \arrow[r, "\mathrm{Id}"] & \tilde{\Sigma} \arrow[r, "\zeta^{\alpha}"] & \tilde{\Sigma} \arrow[r, "\mathrm{Id}"] & \tilde{\Sigma}
\end{tikzcd}$$

$$\begin{tikzcd}
L \arrow[r, "\cong"] \arrow[d] \arrow[rrr, bend left, "B(\zeta)"] & \pi^{\beta*}(L_2) \arrow[r] \arrow[d] & \pi^{\beta*}(L_2) \arrow[r, "\cong"] \arrow[d] & L \arrow[d] \\
\tilde{\Sigma} \arrow[r, "\mathrm{Id}"] & \tilde{\Sigma} \arrow[r, "\zeta^{\beta}"] & \tilde{\Sigma} \arrow[r, "\mathrm{Id}"] & \tilde{\Sigma}
\end{tikzcd}$$

–where the upper vertical maps are, respectively, the chosen isomorphisms ($\cong$) and the canonical maps covering ζ^{β} and ζ^{α}. These definitions do not depend on the choice of isomorphism, since any two isomorphisms are scalar multiples of each other.

Lemma 6.20. *The actions defined above are related as follows:*

$$A(\zeta) \circ B(\zeta)^{-1} = k(\zeta)^{-1} \cdot C(\zeta)$$

$$B(\zeta)^{-1} \circ A(\zeta) = k(\zeta^{-1}) \cdot C(\zeta)$$

In particular:

$$A(\zeta) \circ B(\zeta) = \kappa(\zeta) \cdot B(\zeta) \circ A(\zeta)$$

–where $\kappa(\zeta) = k(\zeta^{-1}) \cdot k(\zeta) \in \mu_n$

Proof. Consider the corresponding maps on the sheaves of holomorphic sections in $\pi^{\alpha*}(L_1)$ and $\pi^{\beta*}(L_2)$ and L. I.e. the sheaves: $\mathcal{O}_{\pi^{\alpha*}(D_1)}$, $\mathcal{O}_{\pi^{\beta*}(D_2)}$ and $\mathcal{O}_{\pi^{\gamma*}(D_3)}$ of meromorphic functions on $\tilde{\Sigma}$ with certain restrictions on their pole orders (cf. remark 2.5).

On these, the canonical maps covering the deck transformations on $\tilde{\Sigma}$ are given as follows: For $\zeta \in \mu_n$, the map covering $\zeta^{\alpha} : \tilde{\Sigma} \to \tilde{\Sigma}$ takes $g \in \mathcal{O}_{\pi^{\alpha*}(D_1)}(U)$ into $(\zeta^{-1})^{\alpha*}(g) \in \mathcal{O}_{\pi^{\alpha*}(D_1)}(\zeta^{\alpha}(U))$. (And likewise for ζ^{β} and ζ^{γ}).

Furthermore, there are canonical isomorphisms: $\mathcal{O}_{\pi^{\gamma*}(D_3)} \to \mathcal{O}_{\pi^{\alpha*}(D_1)}$ and $\mathcal{O}_{\pi^{\gamma*}(D_3)} \to \mathcal{O}_{\pi^{\beta*}(D_2)}$ given by multiplication with h_1 and h_2 respectively.

Thus, for $g \in \mathcal{O}_{\pi^{\gamma*}(D_3)}$:

$$\begin{aligned}
A(\zeta)(B(\zeta)^{-1}(g)) &= h_1^{-1} \cdot ((\zeta^{-1})^{\alpha*}(h_1 \cdot (h_2^{-1} \cdot (\zeta^{\beta*}(h_2 \cdot g))))) \\
&= (\zeta^{-1})^{\alpha*}(k_A(\zeta)^{-1} \cdot k_B(\zeta) \cdot \zeta^{\beta*}(g)) \\
&= k(\zeta)^{-1} \cdot (\zeta^{-1})^{\gamma*}(g) \\
&= k(\zeta)^{-1} \cdot C(\zeta)(g)
\end{aligned}$$

–where we used the fact that $k(\zeta) = \frac{k_A(\zeta)}{k_B(\zeta)}$ is constant in the third step.

And similarly:

$$\begin{aligned}
B(\zeta)^{-1}(A(\zeta)(g)) &= h_2^{-1} \cdot (\zeta^{\beta*}(h_2 \cdot (h_1^{-1} \cdot ((\zeta^{-1})^{\alpha*}(h_1 \cdot g))))) \\
&= k_B(\zeta) \cdot \zeta^{\beta*}((\zeta^{-1})^{\alpha*}(k_A(\zeta)^{-1} g)) \\
&= \frac{k_B(\zeta)}{(\zeta^{-1})^{\gamma*}(k_A(\zeta))} \cdot (\zeta^{-1})^{\gamma*}(g) \\
&= \zeta^{\beta*}\frac{(\zeta^{-1})^{\beta*}(k_B(\zeta))}{(\zeta^{-1})^{\alpha*}(k_A(\zeta))} \cdot C(g) \\
&= \zeta^{\beta*}\frac{k_A(\zeta^{-1})}{k_B(\zeta^{-1})} \cdot C(g) \\
&= k(\zeta^{-1}) \cdot C(g)
\end{aligned}$$

□

Notice that $\pi_\alpha^*(L_1)$ is equivariantly isomorphic to L with equivariant action on L defined by $A(\zeta)$, $(\zeta \in \mu_n)$. Hence, since $\langle\lambda\rangle = \mu_n$, $L_1 \cong L/\langle A(\lambda)\rangle$. Similarly, $L_2 \cong L/\langle B(\lambda)\rangle$, and of course, $L_3 \cong L/\langle C(\lambda)\rangle$.

Furthermore, by lemma 6.16,

$$\pi^{\alpha*}(\lambda_\alpha^*(L_1)) \cong \pi^{\alpha*}(L_1 \otimes \pi_\alpha^*(L_\beta)) \cong \pi^{\alpha*}(L_1) \otimes \tilde{\pi}^*(L_\beta).$$

Now, $\tilde{\pi}^*(L_\beta) \cong \pi^{\beta*}(\pi_\beta^*(L_\beta))$ is equivariantly isomorphic to $\pi^{\beta*}(\mathcal{O}_{\Sigma^\beta}^{(-1)}) \cong \mathcal{O}_{\tilde{\Sigma}}^{(-1),(0)}$, i.e. the trivial bundle on $\tilde{\Sigma}$ with equivariant structure covering Σ^α given by

$$\zeta \in \mu_n : (x, \xi) \mapsto (\zeta^\alpha(x), \zeta^{-1} \cdot \xi)$$

and equivariant structure covering Σ^β given by

$$\zeta \in \mu_n : (x, \xi) \mapsto (\zeta^\beta(x), \xi).$$

This shows that $\lambda_\alpha^*(L_1) \cong L/\langle \lambda^{-1} \cdot A(\lambda)\rangle$. More generally we have:

Lemma 6.21.
$$(\lambda^i)_\alpha^*(L_1) \cong L/\langle \lambda^{-i} \cdot A(\lambda)\rangle \tag{6.3}$$
Similarly:
$$(\lambda^i)_\beta^*(L_2) \cong L/\langle \lambda^{i} \cdot B(\lambda)\rangle \tag{6.4}$$
and
$$(\lambda^i)_\gamma^*(L_3) \cong L/\langle \lambda^{-i} \cdot C(\lambda)\rangle \tag{6.5}$$

Proof. The general formula (6.3) follows by the same arguments, using lemma 6.16 recursively to show that $(\lambda^i)_\alpha^*(L_1) \cong L_1 \otimes \pi_\alpha^*(L_\beta)^{\otimes i}$. Formula (6.4) and (6.5) follow in the same way. Indeed, by lemma 6.16,

$$\pi^{\beta *}(\lambda_\beta^*(L_2)) \cong \pi^{\beta *}(L_2) \otimes \tilde{\pi}^*(L_\alpha)^{-1}$$

and
$$\pi^{\gamma *}(\lambda_\gamma^*(L_3)) \cong \pi^{\gamma *}(L_3) \otimes \tilde{\pi}^*(L_\alpha)^{-1}.$$

Furthermore,
$$\tilde{\pi}^*(L_\alpha)^{-1} \cong \mathcal{O}_{\tilde{\Sigma}}^{(0),(+1)}$$

–with equivariant action over Σ^β given by:

$$\zeta \in \mu_n : (x, \xi) \mapsto (\zeta^\beta(x), \zeta \cdot \xi)$$

and equivariant action covering Σ^γ given by:

$$\zeta \in \mu_n : (x, \xi) \mapsto (\zeta^\gamma(x), \zeta^{-1} \cdot \xi).$$

□

The canonical maps: $\lambda_\alpha^*(L_1) \to L_1$, $\lambda_\beta^*(L_2) \to L_2$ and $\lambda_\gamma^*(L_3) \to L_3$ may be described explicitly as follows.

Lemma 6.22. *Let $\kappa = \kappa(\lambda)$. The map $B(\lambda) : L \to L$ defines an equivariant bundle map:*
$$\pi^{\alpha *}((\kappa^{-1})_\alpha^*(L_1)) \to \pi^{\alpha *}(L_1)$$
covering λ^β on $\tilde{\Sigma}$. In particular, $\kappa^{-1} = \lambda$, and $B(\lambda)$ descends to the canonical map

$$\lambda_\alpha^*(L_1) \to L_1$$

covering λ_α on Σ^α. Furthermore, $A(\lambda) : L \to L$ descends to the canonical bundle maps

$$\lambda_\beta^*(L_2) \to L_2$$

and

$$\lambda_\gamma^*(L_3) \to L_3$$

– covering λ_β and λ_γ, respectively.

Proof. By lemma 6.20, we have the following commutative diagram:

$$\begin{tikzcd} L \arrow[r, "B(\lambda)"] \arrow[d, "\kappa\cdot A(\lambda)"'] & L \arrow[d, "A(\lambda)"] \\ L \arrow[r, "B(\lambda)"] & L \end{tikzcd} \tag{6.6}$$

This shows that $B(\lambda)$ descends to a bundle map:

$$L/\langle\kappa \cdot A(\lambda)\rangle \to L/\langle A(\lambda)\rangle$$

covering λ_α on Σ^α. By lemma 6.17 we may write $\kappa = \lambda^{-i}$ for some i. We then get by (6.3) a map: $(\lambda^i)_\alpha^*(L_1) \to L_1$, covering λ_α on Σ^α, and since $\pi_{\alpha*}(L_1)$ is stable, pull-backs of L_1 are non-isomorphic, hence $i = 1$.

Finally, such a map is unique up to a non-zero scalar, so by choosing the isomorphisms (6.3) right, we may assure that it corresponds to the canonical one.

The claims for L_2 and L_3 follow in the same way. For L_2 one needs only turn (6.6) upside down:

$$\begin{tikzcd} L \arrow[r, "A(\lambda)"] \arrow[d, "\kappa^{-1}\cdot B(\lambda)"'] & L \arrow[d, "B(\lambda)"] \\ L \arrow[r, "A(\lambda)"] & L \end{tikzcd} \tag{6.7}$$

For L_3, one calculates:

$$\begin{aligned} C(\lambda) \circ A(\lambda) &= k(\lambda)A(\lambda) \circ B(\lambda)^{-1} \circ A(\lambda) \\ &= k(\lambda)k(\lambda^{-1})A(\lambda) \circ C(\lambda) \\ &= \kappa(\lambda)A(\lambda) \circ C(\lambda) \end{aligned}$$

to get:

$$\begin{tikzcd} L \arrow[r, "A(\lambda)"] \arrow[d, "\kappa\cdot C(\lambda)"'] & L \arrow[d, "C(\lambda)"] \\ L \arrow[r, "A(\lambda)"] & L \end{tikzcd} \tag{6.8}$$

□

We emphasise the fact that arose during the proof:

Addendum 6.23. $\kappa(\lambda) = \lambda^{-1}$

Now, by the above we may construct $\pi_{\alpha*}(L_1)$, $\pi_{\beta*}(L_2)$ and $\pi_{\gamma*}(L_3)$ explicitly as follows: (Keeping in mind remark 3.10.)

$$\pi_{\alpha*}(L_1) \cong \left(\bigoplus_{i=0}^{n-1} L\right) \Big/ \langle X_1, Y_1 \rangle \tag{6.9}$$

–where:

$$X_1 = \begin{bmatrix} A(\lambda) & & & \\ & \lambda^{-1}\cdot A(\lambda) & & \\ & & \ddots & \\ & & & \lambda^{-(n-1)}\cdot A(\lambda) \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 0 & B(\lambda) & & 0 \\ \vdots & \ddots & \ddots & \\ 0 & & \ddots & B(\lambda) \\ B(\lambda) & 0 & \dots & 0 \end{bmatrix}$$

Similarly,

$$\pi_{\beta*}(L_2) \cong \left(\bigoplus_{i=0}^{n-1} L\right) \Big/ \langle X_2, Y_2 \rangle \tag{6.10}$$

– where:

$$X_2 = \begin{bmatrix} B(\lambda) & & & \\ & \lambda\cdot B(\lambda) & & \\ & & \ddots & \\ & & & \lambda^{(n-1)}\cdot B(\lambda) \end{bmatrix}$$

$$Y_2 = \begin{bmatrix} 0 & A(\lambda) & & 0 \\ \vdots & \ddots & \ddots & \\ & & & \\ 0 & & \ddots & A(\lambda) \\ A(\lambda) & 0 & \dots & 0 \end{bmatrix}$$

and

$$\pi_{\gamma *}(L_3) \cong \left(\bigoplus_{i=0}^{n-1} L \right) \Big/ \langle X_3, Y_3 \rangle \tag{6.11}$$

–where

$$X_3 = \begin{bmatrix} C(\lambda) & & & \\ & \lambda^{-1} \cdot C(\lambda) & & \\ & & \ddots & \\ & & & \lambda^{-(n-1)} \cdot C(\lambda) \end{bmatrix}$$

$$Y_3 = \begin{bmatrix} 0 & A(\lambda) & & 0 \\ \vdots & \ddots & \ddots & \\ & & & \\ 0 & & \ddots & A(\lambda) \\ A(\lambda) & 0 & \dots & 0 \end{bmatrix}$$

Remark 6.24. Notice that for each i, X_i and Y_i commute, so that the quotients (6.9), (6.10) and (6.11) are indeed well defined bundles on $\Sigma = \tilde{\Sigma} \big/ \langle \lambda^\alpha, \lambda^\beta \rangle$.

Lemma 6.25. *The automorphisms of $\bigoplus_{i=0}^{n-1} L$ given by the matrices M and N below descend through the identifications (6.9), (6.10) and (6.11) to isomorphisms:*

$$\Phi : \pi_{\beta *}(L_2) \to \pi_{\alpha *}(L_1)$$

and

$$\Psi : \pi_{\gamma *}(L_3) \to \pi_{\alpha *}(L_1)$$

We have:

$$M = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \lambda & \lambda^2 & \dots & \lambda^{n-1} \\ 1 & \lambda^2 & \lambda^4 & \dots & \lambda^{2n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda^{n-1} & \lambda^{n-2} & \dots & \lambda \end{bmatrix}$$

(In other words, the $(i, j)'th$ entry in M is $\lambda^{(i-1)(j-1)}$.)

Finally, we have: $N = DM$*, where*

$$D = \mathrm{diag}(d_1, d_2, d_3, \ldots, d_n)$$

–with $d_i = k(\lambda^{-1})^{i-1} \cdot \lambda^{-(1+2+\cdots+(i-2))}$.

Proof. The proof is simply a matter of checking that

$$MX_2 = Y_1M \quad , \quad MY_2 = X_1M$$

and

$$NY_3 = X_1N \quad , \quad Y_1N = N(Y_3X_3^{-1})$$

The fist two identities are straightforward computations, using that the rows and columns in M are "eigenvectors" of Y_1 and Y_2. [2]

For the third identity, notice that X_1 and D commute, so $NY_3 = D(MY_3) = D(MY_2) = D(X_1M) = X_1(DM)$. (Not using anything about D, except that it is diagonal.)

For the last identity we will argue that $D^{-1}Y_1DM = MY_3X_3^{-1}$. Let

$$\Lambda = \mathrm{diag}(1, \lambda, \lambda, \ldots, \lambda^{n-1}).$$

Furthermore, denote by $A(\lambda)$ the map given by $\mathrm{diag}(A(\lambda), A(\lambda), \ldots, A(\lambda))$, and similarly for $B(\lambda)$ and $C(\lambda)$. Notice that $X_1 = A(\lambda)\Lambda^{-1}$, $X_2 = B(\lambda)\Lambda$ and $X_3 = C(\lambda)\Lambda^{-1}$. Notice also that D has been constructed so that

$$\frac{d_{i+1}}{d_i} = k(\lambda^{-1}) \cdot \lambda^{-(i-1)}$$

(and in particular, $\frac{d_1}{d_n} = k(\lambda^{-1}) \cdot \lambda$). This implies that $D^{-1}Y_1D = k(\lambda^{-1})\Lambda^{-1}Y_1$.

We may now calculate:

$$\begin{aligned} D^{-1}Y_1DM &= k(\lambda^{-1})\Lambda^{-1}Y_1M \\ &= k(\lambda^{-1})\Lambda^{-1}MX_2 \\ &= k(\lambda^{-1})B(\lambda)\Lambda^{-1}M\Lambda \end{aligned}$$

And on the other hand:

$$\begin{aligned} MY_3X_3^{-1} &= MY_2X_3^{-1} \\ &= X_1MX_3^{-1} \\ &= A(\lambda)\Lambda^{-1}MC(\lambda)^{-1}\Lambda \\ &= k(\lambda^{-1})B(\lambda)\Lambda^{-1}M\Lambda \end{aligned}$$

[2] "Eigenvectors" is to be understood in a broad sense, since Y_1 and Y_2 are not really vector space automorphisms, because of the moving of fibres involved.

–where we used the commutator relations from lemma 6.20 in the final step. □

Lemma 6.26. *Let $E = \pi_{\alpha *}(L_1)$. The endomorphisms T_α, T_β and T_γ defined on E_p^* in the beginning of the proof are related as follows:*

$$T_\alpha \circ T_\beta = k(\lambda)^{-1} T_\gamma$$

If conjecture 1 is true, this concludes the proof of theorem 6.14.

Proof. Let $\tilde{p} \in \tilde{\Sigma}$ be the point with $\pi^\alpha(\tilde{p}) = p_\alpha$, $\pi^\beta(\tilde{p}) = p_\beta$ and $\pi^\gamma(\tilde{p}) = p_\gamma$. (Notice that this requires a certain choice of p_γ, namely $p_\gamma = p_\alpha \otimes p_\beta$ — considering the points as elements in L_α, L_β and L_γ respectively. Since no other restrictions have been made on the choice of p_γ, it is indeed possible.)

Choosing a basis for $(\bigoplus_{i=1}^n L)_{\tilde{p}}$ which is consistent with the direct sum, the dual basis corresponds to the basis $(\omega_i^\alpha) = (\omega_{\lambda^i}^\alpha)$ for $(\pi_{\alpha *}(L_1))_p^*$ introduced in lemma 6.10. (In lemma 6.10 it was used without the ordering and without the α in the notation, though.) Hence, T_α corresponds, in this basis, to multiplication with the matrix Λ introduced in the proof of lemma 6.25 above.

Similar remarks apply to β and γ, giving bases (ω_i^β) and ω_i^γ for $(\pi_{\beta *}(L_2))_p^*$ and $(\pi_{\gamma *}(L_3))_p^*$.

Furthermore, in these bases, $\Phi_p^T : (\pi_{\alpha *}(L_1))_p^* \to (\pi_{\beta *}(L_2))_p^*$ and $\Psi_p^T : (\pi_{\alpha *}(L_1))_p^* \to (\pi_{\gamma *}(L_3))_p^*$ are given by the matrices M^T and N^T from lemma 6.25.

Hence, under the isomorphisms Φ and Ψ, T_β and T_γ induce endomorphisms of $(\pi_{\alpha *}(L_1))_p^*)$ which, along with T_α are given in the basis (ω_i^α) as follows:

$$T_\alpha = \Lambda$$
$$T_\beta = (M^T)^{-1} \Lambda M^T$$
$$T_\gamma = (N^T)^{-1} \Lambda N^T$$

Since M is symmetric, the latter two may be calculated easily as follows. Notice that multiplying M with Λ from the left is the same as shifting its columns to the left, i.e. multiplying from the right with the matrix:

$$B = \begin{bmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

Hence,

$$T_\beta = M^{-1}\Lambda M = M^{-1}MB = B$$

Notice that

$$T_\alpha \circ T_\beta = \Lambda B = \begin{bmatrix} 0 & & & & 1 \\ \lambda & 0 & & & \\ & \lambda^2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \lambda^{n-1} & 0 \end{bmatrix}$$

Furthermore,

$$\begin{aligned} T_\gamma = D^{-1}M^{-1}\Lambda M D = D^{-1}BD &= \begin{bmatrix} 0 & & & & \frac{d_n}{d_1} \\ \frac{d_1}{d_2} & 0 & & & \\ & \frac{d_2}{d_3} & \ddots & & \\ & & \ddots & \ddots & \\ & & & \frac{d_{n-1}}{d_n} & 0 \end{bmatrix} \\ &= \frac{1}{k(\lambda^{-1})} \begin{bmatrix} 0 & & & & \lambda^{-1} \\ 1 & 0 & & & \\ & \lambda & \ddots & & \\ & & \ddots & \ddots & \\ & & & \lambda^{n-1} & 0 \end{bmatrix} \\ &= k(\lambda) \begin{bmatrix} 0 & & & & 1 \\ \lambda & 0 & & & \\ & \lambda^2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \lambda^{n-1} & 0 \end{bmatrix} \end{aligned}$$

–where we used addendum 6.23 in the last step. This proves the claim of the lemma.

Finally, provided conjecture 1 is true, addendum 6.23 shows that $k(\lambda)^{-2} = \lambda$. Since by lemma 6.17, $k(\lambda) \in \mu_n$, hence:

$$\lambda^{\frac{n+1}{2}} = k(\lambda)^{-2\frac{n+1}{2}} = k(\lambda)^{-1}$$

□

Remark 6.27. The groups $\mathcal{E}(n,d)$ act on the spaces $Z_k(n,\Delta_d)$ of algebraic sections in $\mathcal{L}_{n,d}^{\otimes k}$ as follows: An element $\rho \in \mathcal{E}(n,d)$ takes a section $s : M(n,\Delta_d) \to \mathcal{L}_{n,d}^{\otimes k}$ into the section: $\rho^{\otimes k} \circ s \circ \alpha^{-1}$.

The spaces $Z_k(n,\Delta_d)$ are isomorphic (although not canonically!) to the Verlinde Vector spaces of the underlying closed, oriented surface S of Σ, in the gauge theoretic construction of topological quantum field theories (corresponding to $SU(n)$, in dimension $2+1$). In the next chapter we shall see that it is in fact possible to extend the construction into a natural action on the Verlinde vector spaces.

7
The action on TQFT vector spaces

In this chapter we discuss the relevance of the thesis with respect to the gauge theoretic construction of topological quantum field theories.

We begin by presenting the moduli spaces from the gauge theoretic viewpoint and reviewing, briefly, the elements of Alexrod, Della Pietra and Witten's [18] and Hitchin's [16] construction of a projectively flat connection on the Verlinde vector bundles over Teichmüller space.

Next, we discuss the group actions on the moduli spaces from the gauge theoretic viewpoint and how the groups $\mathcal{E}(n,d)$ act on the Verlinde bundles.

Eventually, we outline how the action goes through the final stages of the construction of modular functors, hence inducing a structure on the Verlinde vector spaces as representations of the groups of lifts.

7.1 Differential geometry of the moduli spaces

I will take a slightly simpler, and more geometric approach to Hitchin's construction than originally proposed in [18] and [16]. This approach is due to Andersen. (See e.g. [2]).

Let S be a closed, oriented surface of genus $g \geq 2$, and fix $p \in S$. Let $n \geq 2$ and d be integers, and let M denote the set:

$$M = \mathrm{Hom}_d(\pi_1(\Sigma \setminus p), SU(n))^{\mathrm{irr}}/SU(n),$$

consisting of conjugacy classes of representations $\phi : \pi_1(\Sigma \backslash p) \to SU(n)$ sending a small, closed loop around p to $\exp(2\pi i \frac{d}{n}) I$ [1], and satisfying the irreducibility criterion that $\mathrm{Im}(\phi)$ has finite centraliser in $SU(n)$.

M has a natural structure as a smooth manifold of real dimension $(g-1)(n^2-1)$. Furthermore, using holonomy representations of the fundamental group (See chapter 2.9 in [32]), the elements of M can be identified with gauge equivalence classes of irreducible, flat connections in the trivial $SU(n)$ principal bundle $P = \Sigma \times SU(n)$.[2]

From the above identification M inherits the structure of a symplectic manifold. This structure can be described as follows: Fix an invariant inner product $\{\cdot, \cdot\}$ on $\mathfrak{su}(n)$, the Lie Algebra of $SU(n)$. It can be normalised by demanding that $\frac{1}{6}\{\vartheta \wedge [\vartheta \wedge \vartheta]\}$ yields a generator of the image of the integer cohomology inside real cohomology of degree 3 of $SU(n)$. (Here ϑ is the $\mathfrak{su}(n)$-valued Maurer Cartan 1-form on $SU(n)$, and the symbols $\{\cdot \wedge \cdot\}$ and $[\cdot \wedge \cdot]$ denote the maps:

$$\Omega^p(SU(n), \mathfrak{su}(n)) \otimes_{\Omega^0(SU(n)), \mathfrak{su}(n))} \Omega^q(SU(n), \mathfrak{su}(n)) \to \Omega^{p+q}(SU(n), \mathfrak{su}(n))$$

given by wedging forms on $SU(n)$ and taking inner product resp. bracket on coefficients.)

Then, given A a flat connection in P, representing a point $[A]$ in M, the tangent space $T_{[A]}M$ can be canonically identified with $H^1(\Sigma, d_A)$, where d_A is the connection and corresponding higher differential induced by A on the adjoint bundle $\mathrm{ad}P = P \times_{SU(n)} \mathfrak{su}(n)$. Hence, for a pair of tangent vectors, represented by d_A-closed 1-forms ϕ_1, ϕ_2 with values in $\mathrm{ad}P$, we may define:

$$\omega(\phi_1, \phi_2) = \int_\Sigma \{\phi_1 \wedge \phi_2\}.$$

Furthermore, Freed has constructed a Hermitian line bundle $\mathcal{L}$ on M with a connection ∇, compatible with the symplectic structure in the sense that the curvature of ∇ is equal to $\frac{i}{2\pi}\omega$. (See [2] for detailed references on this.) The induced connection on tensor powers of $\mathcal{L}$ will be denoted by ∇ as well.

Now, let $\mathcal{T}$ denote Teichmüller space of S. For each $\sigma \in \mathcal{T}$, S gets the structure of a Riemann Surface, which will be denoted Σ_σ.

This induces a complex structure I_σ on M as follows: The complex structure on Σ_σ defines a Hodge $*$-operator on 1-forms on S which, by Hodge theory,

[1] Notice that this is unchanged under conjugation by $SU(n)$ since $\exp(2\pi i \frac{d}{n})I$ is central.

[2] Since $\pi_1(SU(n)) = \pi_2(SU(n)) = 0$, any principal SU(n) bundle is in fact trivialisable and may be used.

gives a decomposition at each $[A] \in M$ (Again viewing M as the moduli space of flat connections, and using the same description of the tangent space as above):

$$T_{[A]}M = H^1(\Sigma, d_A) = \mathrm{Ker}(d_A) \cap \mathrm{Ker}(*d_A*).$$

Hence, the $*$-operator acts on $T_{[A]}M$, with $*^2 = -1$, and we may define a complex structure $I_\sigma = -*$ on M. Narasimhan and Seshadri ([10], see also section 2 in [16]) have shown that this in fact gives an integrable complex structure on M, hence making M into a Kähler manifold, $(M_\sigma, I_\sigma, \omega)$.

This also induces a structure on $\mathcal{L}$ as a holomorphic line bundle (denoted $\mathcal{L}_\sigma$), by formally letting $\nabla^{0,1}$ denote the differential operator:

$$\frac{1}{2}(1 + iI_\sigma)\nabla : C^\infty(M, L) \to \Omega^{0,1}(M, L)$$

and defining a local section s to be holomorphic if $\nabla^{0,1}s = 0$. (See [16] for details.)

Let $M_s^\sigma(n, \Delta_d)$ and $M_{ss}^\sigma(n, \Delta_d)$ denote the moduli spaces of stable resp. semistable bundles on Σ_σ.

Narasimhan and Seshadri ([9]) have shown that (M, I_σ, ω) can be identified with the underlying Kähler manifold of $M_s^\sigma(n, \Delta_d)$. Under this identification, the line bundle $\mathcal{L}_\sigma$ corresponds to $\mathcal{L}_{n,d}$, the restriction to the stable part of the ample generator of the Picard group of $M_{ss}^\sigma(n, \Delta_d)$.

Since $M_{ss}^\sigma(n, \Delta_d)$ has rational singularities, (a version of) Hartog's theorem ensures that every algebraic section in $\mathcal{L}_{n,d}^{\otimes k}$ over $M_s^\sigma(n, \Delta_d)$ can be extended across the singularities to $M_{ss}^\sigma(n, \Delta_d)$. Thus, the space Z_k of holomorphic sections in $\mathcal{L}_\sigma^{\otimes k}$ is identified with algebraic sections in $\mathcal{L}_{n,d}^{\otimes k}$ over $M_{ss}^\sigma(n, \Delta_d)$, on which our groups $\mathcal{E}(n, d)$ act naturally.

7.2 The projectively flat connection

Still, let $\mathcal{T}$ denote Teichmüller space of S. For $k \in \mathbb{N}$, let E_k be the trivial bundle: $\mathcal{T} \times C^\infty(M, \mathcal{L}^k)$. E_k contains the subbundle Z_k, whose fibre at each $\sigma \in \mathcal{T}$ consists of the sections that give holomorphic sections in $\mathcal{L}_\sigma^k$ over $(M_\sigma, I_\sigma, \omega)$. This is sometimes referred to as the *Verlinde bundle* of S ("At level k, corresponding to the given values of n and d.")

The idea is to construct a connection ∇^H in E_k which restricts to a connection in Z_k. I.e. such that for each section $s : \mathcal{T} \to H_k$ with $s(\sigma) \in H^0(M_\sigma, \mathcal{L}_\sigma^k)$ for each $\sigma \in \mathcal{T}$, and for each vector field X on $\mathcal{T}$, we have: $\nabla^{0,1}(\nabla_X^H(s)) = 0$.

The goal can be achieved by constructing a smooth map $u : T(\mathcal{T}) \to D(M, \mathcal{L}^k)$, where $D(M, \mathcal{L}^k)$ denotes the space of differential operators on $C^\infty(M, \mathcal{L}^k)$, such that for every section $s : \mathcal{T} \to E_k$ with $s(\sigma) \in H^0(M_\sigma, \mathcal{L}^k_\sigma)$ for all $\sigma \in \mathcal{T}$, and for every tangent vector field X on $\mathcal{T}$, we have:

$$\frac{i}{2} X[I] \nabla^{1,0}(s) + \nabla^{0,1}(u(X)(s)) = 0.$$

(Where X(I) means the derivative of I in the direction of X). Then letting

$$\nabla^H_v = \nabla^t_v - u(v),$$

where ∇^t is the trivial connection in E_k, gives a connection with the desired property. See [2] for the definition of u.

As mentioned before, Hitchin constructs the connection in a rather different way, using hypercohomology and Kodaira-Spencer deformation theory. However, he shows, eventually, that the resulting connection is given locally as described above (formulas (3.12) in [16]). Hence the two constructions agree, and we may import the following theorem:

Theorem 7.1 (Hitchin; Axelrod, Della Pietra, Witten). *The connection defined above in $Z_k(n, d)$ is projectively flat. I.e. it induces a flat connection in $\mathbb{P}(Z_k(n, d))$.*

Remark 7.2. Hitchin excludes the case $g = 2$ in his construction. However, Van Geemen and De Jong ([17]) have extended the construction to cover $g = 2$ as well.

7.3 The group actions from the topological viewpoint

For simplicity we restrict the attention to the case where n is an odd prime, and $d = 0$. In this case $M(S)$ consists of conjugacy classes of irreducible representations of $\pi_1(S)$.

We have for a compact Riemann surface Σ that $J^{(n)}(\Sigma) \cong \mathrm{Hom}(\pi_1(\Sigma), \mu_n)$. An explicit isomorphism, from right to left, is given by mapping a representation R to the line bundle given by:

$$\tilde{\Sigma} \times_{\pi_1(\Sigma)} \mathbb{C}$$

–where $\tilde{\Sigma}$ denotes the universal cover of Σ (not to be confused with $\tilde{\Sigma}$ from chapter 4) and $\pi_1(\Sigma)$ acts on $\mathbb{C}$ via R, and on $\tilde{\Sigma}$ as the group of deck transformations.

For a closed, oriented surface S, we may define the action of $\mathrm{Hom}(\pi_1(S), \mu_n)$ on M simply by multiplication. In other words, $R \in \mathrm{Hom}(\pi_1(S), \mu_n)$ takes a class $[\phi] \in M$ to $R \cdot \phi$. This is clearly independent of the choice of generator.

Furthermore, the Narasimhan-Seshadri map associates to a class $[\phi] \in M$ the vector bundle $E = \tilde{\Sigma} \times_{\pi_1(S)} \mathbb{C}^n$, where $\pi_1(S)$ acts on $\mathbb{C}^n$ via ϕ. Hence, whenever S is given a complex structure, the Narasimhan-Seshadri map takes the action of $\mathrm{Hom}(\pi_1(S), \mu_n)$ on M to the action of $J^{(n)}$ on $M(n, \mathcal{O})$.

It then makes sense to define the groups $\mathcal{E}(n, d)$, depending only on the closed, oriented surface S:

Let $R \in \mathrm{Hom}(\pi_1(S), \mu_n)$. Let $|M|^1_R$ be the connected component of M containing the element generated by $\mathrm{diag}(R, R^2, \ldots, R^n) : \pi_1(\Sigma) \to SU(n)$. We then define ρ_R to be the lift to $\mathcal{L}$ acting as the identity in fibres above $|M|^1_R$.

The groups $\mathcal{E}$ generated by such lifts acts naturally, by conjugation, on sections in $\mathcal{L}^{\otimes k}$. I.e. ρ_R taking a section $s \in C^\infty(M, \mathcal{L}^k)$ to the section $\rho_R^{\otimes k} \circ s \circ R^{-1}$.

For each point $\sigma \in \mathcal{T}$, the identification of M with $M_s^\sigma(n, \Delta_d)$ takes the lifts defined above to the ones defined in definition 6.5. This implies by remark 6.27 that the action of ρ_R preserves the holomorphic sections in $\mathcal{L}_\sigma$.

Remark 7.3. Of course, the presentation of the groups of lifts given in chapter 6 is far from apparent with this topological definition.

7.4 The action on the TQFT spaces

As described above, the groups $\mathcal{E}$ generated by the lifts ρ_R act on sections in $\mathcal{L}^{\otimes k}$, for each $\sigma \in \mathcal{T}$ preserving the holomorphic sections. Thus, they define a fibrewise (i.e. inducing the identity on the base) action on the Verlinde bundle Z_k over Teichüller space.

Work is in progress by the author to show that this action is compatible with Hitchin's connection, in the sense that for any section $s : \mathcal{T} \to E_k$, any vector field X on $\mathcal{T}$ and any lift ρ_R of an element $R \in \mathrm{Hom}(\pi_1(S), \mu_n)$, we have:

$$\rho_R(\nabla_X(\rho_R^{-1}(s))) = \nabla_X(s)$$

Remark 7.4. Since the trivial connection ∇^t is clearly invariant under the action (the latter being the identity on the base), it remains only to show that the map u is invariant.

The final steps of the construction of a modular functor (and hence a topological quantum field theory) go as follows. (For simplicity we fix a value of k and remove it from the notation wherever possible.)

One would like to construct a line bundle L over $\mathcal{T}$ with a connection ∇^{op} having the opposite curvature of ∇^H. This would allow a genuinely flat connection to be defined in the tensor product $Z_k \otimes L$.

There are obstructions towards doing this, however, in a way that preserves the action of the mapping class group. But allowing the passage to a central extension of the mapping class group, such obstructions disappear, and indeed, by transferring the entire situation to the realm of conformal field theory, Andersen and Ueno have recently completed the construction and showed that it yields a modular functor.

In outline, conformal field theory produces a vector bundle $V^\dagger$, the *bundle of conformal blocks*, over Teichmüller space, along with a projectively flat connection $\nabla^\dagger$. (See [22].) It has been shown by Laszlo ([20]) that this construction is equivalent to the Verlinde vector bundle endowed with Hitchin's connection.

Within the setting of conformal field theory, Andersen and Ueno have used a so-called opposite abelian theory to construct of a line bundle $L^\dagger_{\mathrm{ab}}$ over $\mathcal{T}$ with a connection $\nabla^\dagger_{\mathrm{ab}}$, whose curvature $\Omega_{\nabla_{\mathrm{ab}}}$ satisfies:

$$\Omega_{\nabla^\dagger} = \frac{c}{2}\Omega_{\nabla^\dagger_{\mathrm{ab}}}$$

–Where c is a rational number, called the *central charge* of the conformal field theory. It is given by: $c = (k \cdot \dim(SU(n)))/(k+n)$. The fibre of $L^\dagger_{\mathrm{ab}}$ over a point $\sigma \in \mathcal{T}$ is simply $\det(H^1(\Sigma_\sigma, \mathcal{O}))$.

Now, Teichmüller space being contractible, one may construct the fractional tensor power

$$(L^\dagger_{\mathrm{ab}})^{\otimes -c/2}$$

and endow the bundle:

$$V^\dagger \otimes (L^\dagger_{\mathrm{ab}})^{\otimes -c/2}$$

with the genuinely flat connection:

$$\nabla = \nabla^H \otimes 1 + 1 \otimes (\nabla^\dagger_{\mathrm{ab}})^{\otimes -c/2}$$

We may thus define $\mathcal{V}_k$ to be the space of covariantly constant sections in this bundle. Doing that, Andersen and Ueno ([5]) have shown, first of all, that the construction is compatible with the action of the mapping class group on Teichmüller spaces, ensuring that $\mathcal{V}_k$ becomes a representation of a central extension of the mapping class group.

Finally, Andersen and Ueno show that the association of $\mathcal{V}_k$ to the surface S satisfies all the axioms for a modular functor, and thus by the work of Walker and Grove ([15]) gives a topological quantum field theory in dimension $2+1$, with $\mathcal{V}$ as the TQFT vector spaces associated to S.

The action of the groups $\mathcal{E}(n,d)$ goes through this final step easily, simply tensoring with the identity action on the fractional power of $L^{\dagger}_{\mathrm{ab}}$. Supposing it preserves Hitchin's connection, it then preserves covariantly constant sections, thus inducing a representation on the TQFT vector spaces.

Remark 7.5. In a way, the representations on the TQFT vector spaces are not new. From the topological viewpoint (using modular tensor categories to construct Reshetikhin-Turaev modular functors), Blanchet and others have constructed similar representations and decompositions of the Verlinde vector spaces. Since the two constructions are currently being shown to be equivalent ([6]), it would be interesting to understand the topological realisation of the results in the thesis.

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