Irreducible representations of the Witt–Jacobson Lie algebra of rank 2

2nd October 2005

Ph. D.-thesis

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1 Introduction

This thesis gathers the results I have obtained during my graduate studies at University of Aarhus. Through the last four years I have been working with problems in "representation theory of Lie algebras in prime characteristic". First, I will give a historical review of the subject.

E. Witt's discovery of a non classical simple Lie algebra (a Lie algebra not associated to a smooth algebraic group) is the starting point of the theory of modular Lie algebras. Subsequently, more non classical simple Lie algebras were found and a new type of simple restricted Lie algebras, simple restricted Lie algebras of Cartan type, were introduced. They fall into four categories [27, 4]: Witt–Jacobson Lie algebras $W(n)$, special Lie algebras $S(n)$, hamiltonian Lie algebras $H(2n)$ and contact Lie algebras $K(2n+1)$.

In 1966, A. Kostrikin and I. Shafarevic enunciated their famous conjecture asserting that any simple restricted finite dimensional Lie algebra is either classical or of Cartan type. This was proved by R. Block and R. Wilson [1] in 1988 if the characteristic of the ground field K is $p > 7$. Later, this was improved by A. Premet and H. Strade by managing the case $p = 7$. For $p > 5$ the simple restricted finite dimensional Lie algebras then fall into two categories: Classical Lie algebras and Lie algebras of Cartan type. For $p = 5$ one has to add the series of Melikyan algebras constructed in [18].

The representation theory of $U_\chi(\mathfrak{g})$, where $\mathfrak{g}$ is a classical Lie algebra, was first studied by Kac and Weisfeiler [15, 28], and further developed by Friedlander and Parshall [8, 9]. In 1995, Premet [21] proved a conjecture of Kac and Weisfeiler on the dimension of irreducible $U_\chi(\mathfrak{g})$–modules, where $\mathfrak{g} = \mathrm{Lie}(G)$ is the Lie algebra of a simple, connected algebraic group G such that $\mathfrak{g}$ admits a non-degenerate trace form. On the other hand, for restricted Lie algebras of Cartan type, Chang gave a classification of the irreducible $U_\chi(\mathfrak{g})$–modules, when $\mathfrak{g}$ is the smallest (rank 1) Witt–Jacobson Lie algebra [2]. Later, Strade [25] gave proofs of many of Chang's results in a different approach. N. Koreshkov [16] and T. Wichers [29] studied the next smallest (rank 2) Witt–Jacobson Lie algebra. R. Holmes [10] gave a uniform treatment for irreducible modules of small height (the height is an invariant attached to irreducible modules).

The main theme in the thesis concerns the classification of the irreducible $U_\chi(W)$–modules, where W denotes the next smallest Witt–Jacobson Lie algebra and χ is an arbitrary p–character. Already, N. Koreshkov [16] and T. Wichers [29] have studied irreducible modules for that algebra and this thesis contain improvements of the results obtained there together with new results and examples. During the thesis, I will compare the results obtained here with the results by Koreshkov and Wichers.

The first approach is to find a Lie p–subalgebra of W such that irreducible W–modules are induced from irreducible modules for that subalgebra. It is well known that W is a graduated restricted Lie algebra and it contains a Lie p–subalgebra of codimension 2 (the standard maximal subalgebra of W which I denote by $W_{\geq 0}$). We shall induce irreducible $W_{\geq 0}$–modules to W and try to decide whether the induced module is irreducible [i.e., if S is an irreducible $U_\chi(W_{\geq 0})$–module then the $U_\chi(W)$–module induced from S is $U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S$]. If we know that induction is a bijection between the set of irreducible $U_\chi(W_{\geq 0})$–modules and the set of irreducible $U_\chi(W)$–modules, then questions, such as dimension and number of irreducible $U_\chi(W)$–modules, are reduced to the same questions for irreducible $U_\chi(W_{\geq 0})$–modules. Often, irreducible $U_\chi(W_{\geq 0})$–modules are induced from irreducible modules for a supersolvable Lie p–subalgebra of codimension 1, so we will take a closer look at supersolvable Lie p–algebras also.

In some cases it will be convenient to induce from other Lie p–subalgebras than $W_{\geq 0}$. We shall define a Lie p–subalgebra $\mathfrak{g}$ of W of codimension p and induce irreducible $\mathfrak{g}$–modules to W. It often turns out to be best way when considering so–called exceptional characters defined in Section 11.

1.1 Notation

In this paper p will always denote a prime number, K will always denote an algebraically closed field of characteristic p and W will denote the Witt–Jacobson Lie algebra of rank 2. The term *K–algebra* will mean *associative K–algebra with a unit.*

The height of a character $\chi \in W^*$ is the unique integer r with $-1 \leq r \leq 2p-2$ such that $\chi(W_{\geq r}) = 0$ but $\chi(W_{r-1}) \neq 0$. If the height is $r = 2p-2$ we say that χ has maximal height. The height was used implicit in Chang's work [2], but explicitly defined for $W(1)$, the smallest Witt–Jacobson Lie algebra, in [25] by Strade. One can also find the height introduced by Rudakov in [22].

We shall several times use the tensor product when studying irreducible representations. Unless otherwise specified, it should be clear from the context what we are tensoring over.

Whenever we use the notation :=, we define what is on the left hand side to be equal to what is on the right hand side. For example, $W := W(2)$.

The following section contains a summary of the main themes in the thesis and the main results obtained. For details on the statements one should read the respective sections.

1.2 Summary

The paper is organized in 14 sections. I include four appendices: In the first, we compute the action of several matrices on W. In the second, we consider Jacobson's formula for $p = 3$ and prove results about the $[p]$–mapping on elements in W. In Appendix C we consider characters of height at most 1. The main source is [10]. The thesis will therefore mainly concern irreducible W–modules with p–character χ of height > 1. The final appendix gathers questions which I have not been able to answer due to lack of time as well as my mathematical limitations.

Below, I give a short review over all sections.

In Section 2, I settle the notation and recall well–known facts about Witt–Jacobson Lie algebras. I restrict myself to the next smallest Witt–Jacobson Lie algebra in Section 3. The main sources for sections 2–3 are [4], [12], [16] and [27].

In Section 4 we prove that each W_{s-1}, for $s \neq p-1$, can be written as $W_{s-1} = U \bigoplus V$, where U and V are irreducible $GL_2(K)$–submodules of W_{s-1}. Next, we identify each dual space U^* and V^* with homogeneous polynomials of appropriate degree. The results from Section 4 give us representatives for $\chi \in W^*$ with respect to the $GL_2(K)$–action on W^*. This is the subject for Section 5.

In Section 6 we give general criteria for irreducibility. The general setup is: We let $(\mathfrak{g}, [p])$ be a finite dimensional restricted Lie algebra over an algebraically closed field K of characteristic $p > 0$ and $\mathfrak{h} \subset \mathfrak{g}$ is a Lie p–subalgebra. If N is an irreducible $U_\chi(\mathfrak{h})$–module, then we give criteria for the induced $\mathfrak{g}$–module $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$ to be irreducible. The first criterion [27, 5, 5.7] requires the existence of an ideal $\mathfrak{a} \subset \mathfrak{g}$ with $\chi([\mathfrak{a}, \mathfrak{a}]) = 0$ and $\mathfrak{h}$ is defined via $\mathfrak{a}$. The sources are [25] and [26]. The second criterion [25] requires the existence of a unipotent p–ideal $\mathfrak{a} \subset \mathfrak{h}$ with $\chi(\mathfrak{a}) = 0$. The third criterion is used intensively in sections 11–14 if none of criteria 1–2 can be used.

We apply the theory from the first criterion to $\mathfrak{g} = W_{\geq 0}$, the standard maximal Lie p–subalgebra of W, in Section 7. We denote by W_{012} a supersolvable Lie p–subalgebra in $W_{\geq 0}$ of codimension 1. We consider a p–character χ of height $r > 1$ and prove, except for a single type of characters, that there exists $g \in GL_2(K)$ such that induction induces a bijection between the isomorphism classes of irreducible $U_{\chi^g}(W_{012})$–modules and the isomorphism classes of irreducible $U_{\chi^g}(W_{\geq 0})$–modules. As a consequence, one can use the theory for supersolvable Lie p–algebras and show that there exists a polarization $P \subset W_{\geq 0}$ of some $\lambda \in W_{\geq 0}^*$ such that induction induces a bijection between the isomorphism classes of irreducible $U_\chi(P)$–modules and the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules. The type of characters excluded are referred to as characters of height $r = 2p-3$ and Type II.a (the notation comes from Section 5). The results obtained are an improvement of Koreshkov's results in [16] and Wichers' results in [29]; they prove that induction induces a surjection. In fact, Koreshkov claims that the result is true for characters of height $2p-3$ and Type II.a also, but the example given in Section 13.13 shows a quite different behavior than the one described above for those type of characters when $p = 3$.

In Section 8, we apply the theory from the second criterion to $\mathfrak{g} = W$ and $\mathfrak{h} = W_{\geq 0}$. If χ is a p–character of height $r > 1$, we prove that induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and the isomorphism classes of irreducible $U_\chi(W)$–modules if $\mathfrak{st}(\chi, W_{\geq r}) = \{x \in W \mid \chi([x,y]) = 0 \ \forall y \in W_{\geq r}\} = W_{\geq 0}$. The results obtained are again a slightly an improvement of Koreshkov's results in [16] and Wichers' results in [29]; they prove that induction induces a surjection. In fact, Koreshkov claims that we can remove the additional assumption $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$, but computations in sections 12–13 show that he is wrong at that point.

If we consider a p–character χ of height $r > 1$ such that $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$ and if χ is not of Type II.a when $r = 2p-3$, then the results from sections 7–8 say that questions, such as dimension and number of irreducible $U_\chi(W)$–modules, are reduced to the same questions for irreducible modules for a supersolvable Lie p–algebra. Thus we take a closer look at supersolvable Lie p–algebras in Section 9. If L is a supersolvable Lie p–algebra and $\chi \in L^*$, then irreducible $U_\chi(L)$–modules are induced from one dimensional modules over some restricted Lie p–subalgebra of L. We describe a condition that tells us how to find restricted Lie subalgebras P of L and one dimensional $U_\chi(P)$–modules such that the induced $U_\chi(L)$–module is irreducible. I follow [6] and [13], where the description of the theory is given.

Section 10 is very technical but we end up with some of the main results in this thesis. After having introduced Vergne polarizations and compatible polarizations in Section 9, we ask ourselves the following question (we now consider the supersolvable Lie p–subalgebra W_{012} of W and Vergne polarizations are computed with respect to an appropriate chain of ideals in W_{012}): Given a p–character $\chi \in W^*$. Is it possible to find $\lambda \in W_{012}^*$ such that the Vergne polarization $\mathfrak{p}_\lambda$ of λ is compatible with χ and such that $\mathfrak{p}_\lambda = \mathfrak{p}_\chi$? The answer is yes except possibly for a single type of characters when $\mathfrak{p}_\chi$ is unipotent. The existence of λ with that property now has the following application: If $\chi \in W^*$ is a p–character of height $r > 1$ but $r \leq 2p-3$ with $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$ such that χ does not have Type II.a if $r = 2p-3$, then we can prove the following:

1) The dimension of any irreducible $U_\chi(W)$–module is $p^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$, where $\mathfrak{c}_W(\chi)$ is the stabiliser of χ in W. We have $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$.

2) The number of irreducible $U_\chi(W)$–modules (up to isomorphism) is $p^{\operatorname{rk} \mathfrak{c}_W(\chi)}$, where rk $\mathfrak{c}_W(\chi)$ is the dimension of any maximal torus in $\mathfrak{c}_W(\chi)$. We have rk $\mathfrak{c}_W(\chi) \in \{0, 1\}$.

The number of irreducible $U_\chi(W)$–modules and the dimension of all irreducibles are thus completely determined by $\mathfrak{c}_W(\chi)$ (If we assume that $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$ and that χ does not have Type II.a if $r = 2p-3$). Note that $\operatorname{codim}_W \mathfrak{c}_W(\chi)$ is even, since $\mathfrak{c}_W(\chi)$ is the radical of the bilinear, antisymmetric form $(x,y) \longmapsto \chi([x,y])$.

In Section 11, we take a closer look at the exceptional characters (i.e., characters $\chi \in W^*$ of height $r > 1$ with $\mathfrak{st}(\chi, W_{\geq r}) \neq W_{\geq 0}$; one can check that $r > p-2$ for such χ). We prove that any exceptional character is conjugate under $\operatorname{Aut}(W)$ to exactly one of two types of exceptional characters (referred to as Type A– and Type B characters) and we can easily tell which one. We study Type A– and Type B characters and prove, among other things, similar results to 1), 2) above under additional assumptions. [For Type B characters we do not find the explicit dimension formula in 1) since the main theorem in Section 10.1 has not been proved for all characters and it has not been possible for me to improve the result in that sense.]

The subject for Section 12 is characters of rank 2 (i.e., $\chi \in W^*$ with $\operatorname{rk} \mathfrak{c}_W(\chi) = 2$). We find that the only nonzero χ with that property has height 1, height $p-1$, height p or maximal height. For χ of height 1 and $\operatorname{rk} \mathfrak{c}_W(\chi) = 2$, we can apply the results in Appendix C. For height $p-1$ and $\operatorname{rk} \mathfrak{c}_W(\chi) = 2$, we classify the irreducible $U_\chi(W)$–modules and we see in fact a quite different behavior as in 1),2) above. We apply results from the representation theory of $W(1)$, the smallest Witt–Jacobson Lie algebra. For χ of height p with $\operatorname{rk} \mathfrak{c}_W(\chi) = 2$, I have no ideas what happens; computations in Section 13.12 for $p = 3$ indicate that no methods from the height $p-1$ case can be used. Finally, we consider characters of maximal height with $\dim_K \mathfrak{c}_W(\chi) = 2$ and $\mathfrak{c}_W(\chi) \cap W_{\geq 0} = 0$. We prove that the dimension of any irreducible $U_\chi(W)$–module is maximal in that case (i.e., equal to p^{p^2-1} by Mil'ner's result [19]). In particular, we can apply the result above to some characters χ of maximal height and $\operatorname{rk} \mathfrak{c}_W(\chi) = 2$.

Section 13 contains the main examples in this thesis. We shall see that none of the assumptions on χ (i.e., χ has height $r > 1$ with $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$ but $r \neq 2p-3$ if χ has Type II.a) can be removed in order to obtain 1),2) above. We give a classification of the irreducible $U_\chi(W)$–modules if χ has height $r = 2$ or $r = 3$. If $p > 3$ we have $r < 2p-3$ and $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$, so the dimension formula in 1) and the formula for the number of irreducibles in 2) can be applied. A complete list of the possibilities are given for $r = 2$ and $r = 3$ when $p > 3$. The interesting part in Section 13 occurs when $p = 3$ and $\mathfrak{st}(\chi, W_{\geq r}) \neq W_{\geq 0}$ or χ has Type II.a and $r = 3$.

If $\mathfrak{st}(\chi, W_{\geq r}) \neq W_{\geq 0}$, then we observe in some cases a behavior quite different from the one described in 1),2) above. I will try to sketch the differences (for $p = 3$) in the following items:

- If $\operatorname{rk} \mathfrak{c}_W(\chi) = 0$, one can have > 1 isomorphism classes of irreducible $U_\chi(W)$–modules.
- If $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$, one can have > 3 isomorphism classes of irreducible $U_\chi(W)$–modules.
- It is possible to have $\operatorname{rk} \mathfrak{c}_W(\chi) \not\subset W_{\geq 0}$ and it is possible to have $\operatorname{rk} \mathfrak{c}_W(\chi) = 2$.
- If $\operatorname{rk} \mathfrak{c}_W(\chi) = 2$, one can have 3^2 isomorphism classes of irreducible $U_\chi(W)$–modules.
- All irreducible $U_\chi(W)$–modules do not have the same dimension.
- The dimension of an irreducible $U_\chi(W)$–module is not always divisible by 3.

As a consequence, induction does not always take irreducible $U_\chi(W_{\geq 0})$–modules to irreducible $U_\chi(W)$–modules if $\mathfrak{st}(\chi, W_{\geq r}) \neq W_{\geq 0}$.

If χ has Type II.a and $r = 3 = 2p - 3$, then we also observe a behavior different from the one described in 1),2) above. The differences (for $p = 3$) are:

- The number of isomorphism classes of irreducible $U_\chi(W)$–modules is not always divisible by 3.
- All irreducible $U_\chi(W)$–modules do not have the same dimension.
- The dimension of an irreducible $U_\chi(W)$–module is not always divisible by 3.

Finally, we consider characters of maximal height in Section 14. The representation theory here is not very well understood. Koreshkov [16] claim that we end up in three possible cases, but his proof is very mysterious. In [16] and [29] there are examples where one construct χ such that all irreducible $W_{\geq 0}$–modules with p–character χ have maximal dimension equal to p^{p^2-1} and hence they all extend to W. It is however not clear how to compute the number of irreducibles.

We try to get a better understanding by looking at the case where $p = 3$. The most interesting observation says that we can find χ of maximal height and irreducible $U_\chi(W)$–modules of non–maximal dimension (without a classification of these).

Even for $p = 3$ it has not been possible to classify the set of irreducible W–modules (of maximal height) and general statements on the irreducibles (arbitrary p) are therefore far away (for me).

1.3 Acknowledgement

I would like to express my gratitude to my advisor Jens Carsten Jantzen, who has been of great inspiration during my project. His immense knowledge and new ideas within this field have been of invaluable support. I feel privileged to have had his direction and guidance.

During the fall of 2004, I visited the University of Manchester, and I hereby take the opportunity to acknowledge the support I received at the mathematical department. A special thank you to Professor Alexander Premet for informative discussions during my stay.

My sincere appreciation to Ulf Raben-Pedersen, my Ph.D. colleague and very good friend, whose high spirits have been contagious. I am very grateful for his mental guidance and deeply appreciate our intensive mathematical discussions throughout the years.

Last but not least, I thank my family for their patience and support.

Peter Johannes Steffensen
Aarhus, July 2005

2 The Witt–Jacobson Lie algebras

Let K be an algebraically closed field of characteristic $p > 0$. In [27, 4, §2], Strade and Farnsteiner define the generalized Witt–Jacobson Lie algebras over K. Here we will focus on the restricted generalized Witt–Jacobson Lie algebras (or just Witt–Jacobson Lie algebras). By [27, 4, Lemma 2.1 (3) and Theorem 2.4] they can be realized in the following way: For any positive integer n set

$$B_n = K[X_1, X_2, \dots, X_n]/(X_1^p, X_2^p, \dots, X_n^p) \tag{2.1}$$

where $K[X_1, X_2, \dots, X_n]$ is the polynomial ring in n indeterminates $X_1, X_2, \dots, X_n$. The image of X_i in B_n will be denoted by x_i. For each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$ set

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}. \tag{2.2}$$

Set

$$I(n) = \{\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n \mid 0 \le \alpha_i < p \text{ for all } i\}.$$

Then all x^α with $\alpha \in I(n)$ form a K–basis for B_n; so $\dim_K B_n = p^n$.

Set $W(n)$ equal to the set of all K–linear derivations of B_n. Then $W(n)$ is a restricted Lie algebra over K (the n'th Witt–Jacobson algebra) where the Lie bracket is the usual commutator and the p–mapping is given as p times composition; for $D \in W(n)$ we have $D^{[p]} = D \circ D \circ \cdots \circ D$ (p times). It is easy to see that $W(n)$ is a B_n–submodule of $\mathrm{End}_K(B_n)$.

For $i = 1, 2, \dots, n$ set $\partial_i = \frac{\partial}{\partial x_i}$ (the partial derivative). Each ∂_i is a derivation of $K[X_1, X_2, \dots, X_n]$ with $\partial_i(X_j^p) = 0$ for all j; hence it preserves the ideal generated by all X_j^p and induces then a derivation on B_n as in (2.1). Denote (again) by ∂_i the induced derivation on B_n with $\partial_i(x_j) = \delta_{ij}$ for all j. For any derivation $D \in W(n)$ we have $D(1) = 0$ and D is uniquely determined by the values $D(x_1), D(x_2), \dots, D(x_n)$. This implies that

$$D = \sum_{i=1}^{n} D(x_i)\partial_i. \tag{2.3}$$

So $W(n)$ is free as a B_2–module with basis $\partial_1, \partial_2, \dots, \partial_n$; hence $\dim_K W(n) = np^n$.

Each $W(n)$ is in fact a graded Lie algebra. Let me first discuss the natural grading on B_n: The K–algebra B_n has a grading $B_n = \bigoplus_{i \ge 0} (B_n)_i$ such that each x_j is homogeneous of degree 1. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in I(n)$ we define $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Then all x^α with $\alpha \in I(n)$ and $|\alpha| = i$ form a K–basis for $(B_n)_i$. It is easy to see that $(B_n)_i = 0$ for $i > n(p-1)$. The grading on B_n now induce a grading on $W(n)$ in the following way: For all $i \in \mathbb{Z}$ set

$$W(n)_i = \{D \in W(n) \mid D((B_n)_j) \subset (B_n)_{i+j} \text{ for all } j\}.$$

Then $W(n)_i$ is a subspace of $W(n)$ and the sum of the $W(n)_i$ is direct. We also have $[W(n)_i, W(n)_j] \subset W(n)_{i+j}$ for all i, j and the graduation is restricted: If $D \in W(n)_i$ then $D^{[p]} \in W(n)_{pi}$. The partial derivative ∂_i belongs to $W(n)_{-1}$ since $\partial_i(x^\alpha) = \alpha_i x^{\alpha - \varepsilon_i}$, where $\varepsilon_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the i'th position [define $x^\gamma = 0$ if $\gamma \notin \mathbb{N}^n$]. It follows that $x^\alpha \partial_i \in W(n)_{|\alpha|-1}$ for all i and all $\alpha \in I(n)$; in fact all $x^\alpha \partial_i$ form a K–basis for $W(n)_{|\alpha|-1}$ and we have that

$$W(n) = \bigoplus_{i=-1}^{n(p-1)-1} W(n)_i$$

is a graded restricted Lie algebra. For all integer $s \ge -1$, set $W(n)_{\ge s} = \bigoplus_{i \ge s} W(n)_i$. For $s = 0$ we get the *standard maximal* subalgebra $W(n)_{\ge 0}$ of $W(n)$.

We have formulas for the commutator from [27, 4,2.1]: If $f_1, f_2 \in B_n$ and $D_1, D_2 \in W(n)$ then we have

$$[f_1D_1, f_2D_2] = f_1D_1(f_2)D_2 - f_2D_2(f_1)D_1 + f_1f_2[D_1, D_2].$$

In particular,

$$[x^\alpha \partial_i, x^\beta \partial_j] = \beta_i x^{\alpha+\beta-\varepsilon_i}\partial_j - \alpha_j x^{\alpha+\beta-\varepsilon_j}\partial_i. \tag{2.4}$$

Now it is easy to verify that $[W(n)_{-1}, W(n)_i] = W(n)_{i-1}$ for all i (check the definitions). [This is proved in the Kreknin paper [17] in a more general setup.] The $[p]$–mapping operates on our basis elements via

$$(x^\alpha \partial_i)^{[p]} = \begin{cases} x_i\partial_i & \text{if } \alpha = \varepsilon_i, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

In order to see this, note that $(x^\alpha\partial_k)^{[p]}(x_l) = (x^\alpha\partial_k)^p(x_l) = 0$ for all $l \neq k$. So we get that $(x^\alpha\partial_k)^{[p]} = (x^\alpha\partial_k)^p(x_k)\partial_k$ since any derivation $D \in W(n)$ is uniquely determined by the values $D(x_1), D(x_2), \ldots, D(x_n)$ (see (2.3)). If $\alpha = \varepsilon_i$ then $(x^\alpha\partial_i)(x_i) = x_i\partial_i(x_i) = x_i$ and so $(x^\alpha\partial_i)^p(x_i) = x_i$ also. If $\alpha = 0$ then already $(x^\alpha\partial_i)^2(x_i) = \partial_i(1) = 0$. If $\alpha_i > 0$, then any $(x^\alpha\partial_i)^r(x_i)$ is by induction a multiple of $x^{r\alpha-(r-1)\varepsilon_i}$. If $\alpha_j \neq 0$ for some $j \neq i$, then $(x^\alpha\partial_i)^p(x_i)$ is a multiple of $x_j^{p\alpha_j} = 0$; if $\alpha_i > 1$, then we get a multiple of $x_i^{p(\alpha_i-1)+1} = 0$.

2.1 Simplicity

The Witt–Jacobson Lie algebra $W(n)$ is simple unless $p = 2$ and $n > 1$. The proof can be found in [12, Thm. 1] or [27, 4, Thm. 2.4 (1)]. If $p = 2$ and $n = 1$ then e_0, e_1 form a K–basis for $W(1)$ and $[e_0, e_1] = e_1$. Therefore, Ke_1 is a proper nonzero ideal in $W(1)$.

The centre of $W(n)$ is equal to 0 [for $(p, n) \neq (2, 1)$ this follows from the simplicity of $W(n)$ and it is easy to check for $(p, n) = (2, 1)$].

2.2 Automorphisms

Let $g \in \mathrm{Aut}(W(n))$. The centre of $W(n)$ is 0 so g is a restricted automorphism of the Lie algebra $W(n)$, i.e., $g(D^{[p]}) = g(D)^{[p]}$ for all $D \in W(n)$: First, the adjoint representation of $W(n)$ is injective and for any $D \in W(n)$ the element $D^{[p]}$ is uniquely determined by the condition that $\mathrm{ad}(D^{[p]}) = \mathrm{ad}(D)^p$. Moreover, we have $\mathrm{ad}(g(D)) = g\circ\,\mathrm{ad}(D) \circ g^{-1}$; hence

$$\mathrm{ad}(g(D))^p = (g \circ \mathrm{ad}(D) \circ g^{-1})^p = g \circ \mathrm{ad}(D)^p \circ g^{-1} = g \circ \mathrm{ad}(D^{[p]}) \circ g^{-1} = \mathrm{ad}(g(D^{[p]})).$$

This implies that $g(D^{[p]}) = g(D)^{[p]}$ for any automorphism g of the Lie algebra $W(n)$ and for any derivation $D \in W(n)$. Any element $\varphi \in \mathrm{Aut}_{K-\mathrm{alg}}\, B_n$ induces an automorphism σ_φ of the Lie algebra $W(n)$ in the following way:

$$\sigma_\varphi(D) := \varphi \circ D \circ \varphi^{-1} \quad \forall D \in W(n) = \mathrm{Der}_K(B_n). \tag{2.6}$$

We have to check that $\sigma_\varphi(D) \in W(n)$ for $D \in W(n)$. The linearity is clear and given any $f_1, f_2 \in B_n$ we get

$$\begin{aligned} \big(\sigma_\varphi(D)\big)(f_1f_2) &= \varphi \circ D(\varphi^{-1}(f_1)\varphi^{-1}(f_2)) \\ &= \varphi\big(D(\varphi^{-1}(f_1))\varphi^{-1}(f_2)\big) + \varphi\big(\varphi^{-1}(f_1)D(\varphi^{-1}(f_2))\big) \\ &= \big(\sigma_\varphi\big)(f_1)f_2 + f_1\big(\sigma_\varphi\big)(f_2) \end{aligned}$$

which shows that $\sigma_\varphi(D)$ is a derivation — i.e., $\sigma_\varphi(D) \in \mathrm{Der}_K(B_n) = W(n)$. Note that the second equality follows since D is a derivation. Therefore σ_φ is a linear isomorphism on $W(n)$ (with inverse $\sigma_{\varphi^{-1}}$). In order to show that $\sigma_\varphi \in \mathrm{Aut}(W(n))$, we need to know that the action of σ_φ respects the commutator:

$$\begin{aligned}\sigma_\varphi\big([D_1, D_2]\big) &= \varphi \circ (D_1 \circ D_2 - D_2 \circ D_1) \circ \varphi^{-1} \\ &= (\varphi \circ D_1 \circ \varphi^{-1}) \circ (\varphi \circ D_2 \circ \varphi^{-1}) - (\varphi \circ D_2 \circ \varphi^{-1}) \circ (\varphi \circ D_1 \circ \varphi^{-1}) \\ &= \big[\sigma_\varphi(D_1), \sigma_\varphi(D_2)\big].\end{aligned}$$

The map $\varphi \longmapsto \sigma_\varphi$ is therefore a homomorphism between the two automorphism groups.

Theorem 2.2.1. *Suppose that $p \geq 5$. Then $\varphi \longmapsto \sigma_\varphi$ is a group isomorphism from $\mathrm{Aut}_{K-\mathrm{alg}}\, B_n$ onto $\mathrm{Aut}(W(n))$.*

Proof. See [12, Thm.6] □

Remark 2.2.2. The theorem above is also true for $p = 3$ and $n > 1$. Indeed, consider $h_i = \partial_i + x_i\partial_i$ for $i = 1, 2, \ldots, n$ and let $\mathfrak{h} = \bigoplus_i K h_i$. Then

$$[h_i, h_j] = 0 \quad \text{and} \quad h_i^{[p]} = h_i \tag{2.7}$$

for all i, j. For the second claim use that $h_i^{[p]}(x_i) = h_i^p(x_i) = 1 + x_i = h_i(x_i)$ and that $h_i^{[p]}(x_j) = h_i^p(x_j) = 0 = h_i(x_j)$ for $j \neq i$. Now, suppose that g is an automorphism on $W(n)$. Define $D_i := g(h_i)$ and note that all D_i satisfy (2.7) also, since g is a restricted automorphism. Since $W(n)_{\geq 0} \cap \mathfrak{h} = 0$, we have $W(n)_{\geq 0} \cap g(\mathfrak{h}) = 0$ also. Indeed, for any automorphism g of the Lie algebra $W(n)$, we have $g(W(n)_{\geq 0}) = W(n)_{\geq 0}$ (see [17] or use formula (2.14) in Section 2.3). Now, apply [4, Thm. 1] and find $\varphi \in \mathrm{Aut}_{K-\mathrm{alg}}\, B_n$ such that $\sigma_\varphi \circ g(h_i) = h_i$ for all i; hence $\sigma_\varphi \circ g = \mathrm{Id}_{|W(n)}$ by [12, Thm. 5]. Therefore $g = \sigma_\varphi^{-1} = \sigma_{\varphi^{-1}}$ and so the map in Theorem 2.2.1 is surjective. For the injectivity, use [12, Thm.5].

Proposition 2.2.3. *Let $D = \sum_{i=1}^n f_i\partial_i \in W(n)$ for some $f_i \in B_n$ and let $\varphi \in \mathrm{Aut}_{K-\mathrm{alg}} B_n$. With the action in (2.6) we have*

$$\sigma_\varphi(D) = \sum_{i=1}^n \sum_{l=1}^n f_l(\varphi(x_1), \ldots, \varphi(x_n)) \frac{\partial \varphi^{-1}(x_i)}{\partial x_l}\big(\varphi(x_1), \ldots, \varphi(x_n)\big)\partial_i. \tag{2.8}$$

Proof. It is enough to find $\sigma_\varphi(D)(x_i)$ for $i = 1, 2, \ldots, n$ by (2.3). We use (2.6) and get

$$\begin{aligned}\sigma_\varphi(D)(x_i) &= \varphi \circ D \circ \varphi^{-1}(x_i) \\ &= \varphi\big(\textstyle\sum_{l=1}^n f_l \frac{\partial \varphi^{-1}(x_i)}{\partial x_l}\big)\end{aligned}$$

which implies that

$$\sigma_\varphi(D)(x_i) = \sum_{l=1}^n f_l(\varphi(x_1), \ldots, \varphi(x_n)) \frac{\partial \varphi^{-1}(x_i)}{\partial x_l}\big(\varphi(x_1), \ldots, \varphi(x_n)\big).$$

Now apply (2.3) and obtain:

$$\sigma_\varphi(D) = \sum_{i=1}^n \sum_{l=1}^n f_l(\varphi(x_1), \ldots, \varphi(x_n)) \frac{\partial \varphi^{-1}(x_i)}{\partial x_l}(\varphi(x_1), \ldots, \varphi(x_n))\partial_i.$$

The proof is completed. □

2.3 Decomposition

Set $G = \mathrm{Aut}_{K-\mathrm{alg}}\, B_n$ and $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$ the unique maximal ideal in B_n. Recall that B_n has a grading $B_n = \bigoplus_{i\geq 0}(B_n)_i$ such that each x_j is homogeneous of degree 1. For any $\varphi \in G$, we have $\varphi(\mathfrak{m}) = \mathfrak{m}$; thus we can write

$$\varphi(x_i) = \sum_{j=1}^{n} a_{ji} x_j + f_i \quad \text{for } i = 1, 2, \ldots, n \tag{2.9}$$

where $a_{ji} \in K$ and $f_i \in (B_n)_{\geq 2}$ for all i, j. One checks easily that the map taking φ to the matrix (a_{ji}) is a group homomorphism from G to $GL_n(K)$. On the other hand, given a matrix (a_{ji}) in $GL_n(K)$ and elements $f_i \in (B_n)_{\geq 2}$, then there exists $\varphi \in G$ such that (2.9) is satisfied. So we get a short exact sequence of groups

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow GL_n(K) \longrightarrow 1 \tag{2.10}$$

where $G_1 := \{\varphi \in G \mid \varphi(x_i) - x_i \in (B_n)_{\geq 2} \text{ for all } i\}$. Note that the short exact sequence in (2.10) splits: Associate to any $(a_{ji}) \in GL_n(K)$ the automorphism as in (2.9) with all $f_i = 0$. We identify $GL_n(K)$ with a subgroup of G via

$$GL_n(K) = \{\varphi \in G \mid \varphi\big((B_n)_i\big) = (B_n)_i \text{ for all } i\}.$$

From the splitting in (2.10), we have a decomposition of G: $G = GL_n(K) \ltimes G_1$. Now apply Theorem 2.2.1 and Remark 2.2.2 and obtain a decomposition of $\mathrm{Aut}(W(n))$:

$$\mathrm{Aut}(W(n)) = GL_n(K) \ltimes \mathrm{Aut}^*(W(n)) \tag{2.11}$$

where

$$GL_n(K) := \{g \in \mathrm{Aut}(W(n)) \mid g(W(n)_i) = W(n)_i \text{ for all } i\} \tag{2.12}$$

is a subgroup of $\mathrm{Aut}(W(n))$ via Theorem 2.2.1. Moreover,

$$\mathrm{Aut}^*(W(n)) = \{g \in \mathrm{Aut}(W(n)) \mid g(D) - D \in W(n)_{\geq i+1} \text{ for all } D \in W(n)_i \text{ and all } i\}. \tag{2.13}$$

From (2.11), (2.12) and (2.13), we get also

$$g(W(n)_{\geq i}) = W_{\geq i} \quad \text{for all } i \text{ and all } g \in \mathrm{Aut}(W(n)). \tag{2.14}$$

Remark 2.3.1. In view of (2.14), the height function

$$\mathrm{ht} : W(n)^* \longrightarrow \{-1, 0, 1, \ldots, n(p-1) - 1\}$$

given by $\mathrm{ht}\ \chi = \min\{s \geq -1 \mid \chi(W(n)_{\geq s}) = 0\}$, for $\chi \in W(n)^*$, is an invariant of the $\mathrm{Aut}(W(n))$–orbits of $W(n)^*$.

2.4 Maximal torus subalgebras

It has been proved in [4, §3] that there are $n+1$ conjugacy classes under $\mathrm{Aut}(W(n))$ of maximal torus subalgebras (and Cartan subalgebras). Representatives are given as:

$$\begin{array}{rcl} T_0 & = & \sum_{i=1}^{n} K x_i \partial_i, \\ T_1 & = & \sum_{i=1}^{n-1} K x_i \partial_i \oplus K(1 + x_n)\partial_n, \\ & \vdots & \\ T_n & = & \sum_{i=1}^{n} K(1 + x_i)\partial_i. \end{array}$$

Thus any two maximal torus subalgebras (or Cartan subalgebras) whose intersection with $W(n)_{\geq 0}$ have the same dimension are conjugate by elements of $\mathrm{Aut}(W(n))$. The *rank* of $W(n)$ is n which is the dimension of any maximal torus in $W(n)$. In the thesis we will focus on the Witt–Jacobson Lie algebra of rank 2.

3 The Witt–Jacobson Lie algebra $W(2)$

Let K be an algebraically closed field of characteristic $p > 0$. From now we will concentrate on the restricted Witt–Jacobson Lie algebra $W(2)$ of rank 2 defined (in a more general setup) in the previous section. We have $\dim_K W(2) = 2p^2$. In the following, set

$$W := W(2).$$

It will be convenient to define basis elements

$$\boxed{e_{ijk} := x_1^i x_2^j \cdot \partial_k}$$

if i and j are integers with $0 \le i, j < p$ and $k = 1, 2$ and we define $e_{ijk} = 0$ otherwise. If we use (2.4) we get the following:

$$\big[e_{rs1}, e_{ij1}\big] = (i - r)e_{i+r-1,j+s,1}, \tag{3.1a}$$
$$\big[e_{rs1}, e_{ij2}\big] = -se_{i+r,j+s-1,1} + ie_{i+r-1,j+s,2}, \tag{3.1b}$$
$$\big[e_{rs2}, e_{ij1}\big] = je_{i+r,j+s-1,1} - re_{i+r-1,j+s,2}, \tag{3.1c}$$
$$\big[e_{rs2}, e_{ij2}\big] = (j - s)e_{i+r,j+s-1,2}. \tag{3.1d}$$

Actually (3.1c) follows from (3.1b) and the antisymmetry of the commutator $[\ ,\ ]$. The p–mapping is p–fold composition and from (2.5) we have:

$$e_{ijk}^{[p]} = \begin{cases} e_{012} & \text{if } (i,j,k) = (0,1,2), \\ e_{101} & \text{if } (i,j,k) = (1,0,1), \\ 0 & \text{otherwise.} \end{cases}$$

For $s \ge 0$ and $s \le 2p - 2$ we have

$$W_{s-1} = \sum_{i+j=s} \sum_{k=1,2} Ke_{ijk}.$$

3.1 Ordering

Following Koreshkov's paper [16] we introduce an order relation on the set of indices of basis elements. Let $(i', j', k') \preceq (i, j, k)$ if $(i', j', k') = (i, j, k)$ or

1) $i' + j' < i + j$ or

2) if $k < k'$ and $i' + j' = i + j$ or

3) if $i < i'$ and $k' = k$ and $i' + j' = i + j$.

The ordering of indices looks like

$$002 \preceq 001 \preceq 102 \preceq 012 \preceq 101 \preceq 011 \preceq 202 \preceq \cdots \preceq (p-1, p-1, 1).$$

For any triple (ijk) with $(002) \preceq (ijk) \preceq (p-1, p-1, 1)$, set

$$W_{ijk} := \sum_{(i,j,k) \preceq (\alpha\beta\gamma)} Ke_{\alpha\beta\gamma}.$$

Note that W_{002} is just W and W_{102} is the standard maximal Lie subalgebra $W_{\ge 0}$ of W. The ordering of indices induces a chain of subspaces:

$$W = W_{002} \supset W_{001} \supset W_{102} \supset W_{012} \supset W_{101} \supset W_{011} \supset W_{202} \supset \cdots \supset W_{(p-1,p-1,1)}.$$

They are all subspaces of W, by construction, and the next lemma says that most of them are in fact Lie p–subalgebras of W.

Lemma 3.1.1. *All W_{ijk} with $(ijk) \succeq (102)$ are Lie p–subalgebras of W.*

Proof. Since all basis elements $e_{\alpha\beta\gamma}$ of W satisfy $e_{\alpha\beta\gamma}^{[p]} = 0$ or $e_{\alpha\beta\gamma}^{[p]} = e_{\alpha\beta\gamma}$ we only need to check that all W_{ijk} are Lie subalgebras of W in order to show that all W_{ijk} are Lie p–subalgebras. Let $(ijk) \succeq (102)$ and consider $(\alpha\beta\gamma) \succeq (ijk)$ and $(\alpha'\beta'\gamma') \succeq (ijk)$. Then it follows from (3.1a),(3.1b),(3.1d) that

$$[e_{\alpha\beta\gamma}, e_{\alpha'\beta'\gamma'}] = \begin{cases} (\alpha'-\alpha)e_{\alpha+\alpha'-1,\beta+\beta',1} & \gamma=\gamma'=1, \\ (\beta'-\beta)e_{\alpha+\alpha',\beta+\beta'-1,2} & \gamma=\gamma'=2, \\ -\beta e_{\alpha+\alpha',\beta+\beta'-1,1} + \alpha' e_{\alpha+\alpha'-1,\beta+\beta',2} & \gamma=1, \gamma'=2. \end{cases}$$

Consider the case $\gamma = \gamma' = 1$. Assume $\alpha \neq \alpha'$ [otherwise the commutator is zero]. If $\alpha+\alpha'-1+\beta+\beta' > \alpha+\beta$ or $\alpha+\alpha'-1+\beta+\beta' > \alpha'+\beta'$ we are done since $(\alpha\beta\gamma) \succeq (ijk)$ and $(\alpha'\beta'\gamma') \succeq (ijk)$. Since $\alpha+\alpha'-1+\beta+\beta' \geq \alpha+\beta$ and $\alpha+\alpha'-1+\beta+\beta' \geq \alpha'+\beta'$ for $(\alpha\beta\gamma), (\alpha'\beta'\gamma') \succeq (102)$ we may assume that $\alpha+\alpha'-1+\beta+\beta' = \alpha+\beta = \alpha'+\beta'$. This implies that $\alpha+\beta = 1 = \alpha'+\beta'$ and hence $\alpha = 1, \alpha' = 0$ or $\alpha = 0, \alpha' = 1$. If $\alpha = 1, \alpha' = 0$ we have $(\alpha+\alpha'-1, \beta+\beta', 1) \succeq (\alpha, \beta, 1) \succeq (ijk)$. If $\alpha = 0, \alpha' = 1$ we have $(\alpha+\alpha'-1, \beta+\beta', 1) \succeq (\alpha', \beta', 1) \succeq (ijk)$. Hence the commutator lies in W_{ijk}.

Consider the case $\gamma = \gamma' = 2$. Assume $\beta \neq \beta'$ [otherwise the commutator is zero]. If $\alpha+\alpha'+\beta+\beta'-1 > \alpha+\beta$ or $\alpha+\alpha'+\beta+\beta'-1 > \alpha'+\beta'$ we are done since $(\alpha\beta\gamma) \succeq (ijk)$ and $(\alpha'\beta'\gamma') \succeq (ijk)$. As above we may assume that $\alpha+\alpha'+\beta+\beta'-1 = \alpha+\beta = \alpha'+\beta'$. This implies that $\alpha+\beta = 1 = \alpha'+\beta'$ and hence $\alpha = 1, \alpha' = 0$ or $\alpha = 0, \alpha' = 1$. If $\alpha = 1, \alpha' = 0$ we have $(\alpha+\alpha', \beta+\beta'-1, 2) = (\alpha, \beta, 2) \succeq (ijk)$. If $\alpha = 0, \alpha' = 1$ we have $(\alpha+\alpha', \beta+\beta'-1, 2) = (\alpha', \beta', 2) \succeq (ijk)$. Hence the commutator lies in W_{ijk}.

Finally, let $\gamma = 1, \gamma' = 2$. From the ordering we see easily that $(\alpha+\alpha', \beta+\beta'-1, 1) \succeq (\alpha', \beta', 2) \succeq (ijk)$ and hence the first term lies in W_{ijk}. For the second term note we may assume that $\alpha+\beta = 1 = \alpha'+\beta'$ [apply similar arguments as above]. If $\alpha = \alpha' = 1$ then $(\alpha+\alpha'-1, \beta+\beta', 2) = (\alpha, \beta, 2) \succeq (ijk)$. If $\alpha \neq \alpha'$ then $\alpha+\alpha'-1 < \alpha$ or $\alpha+\alpha'-1 < \alpha'$ and hence [recall the ordering] the second term lies in W_{ijk} also. □

3.2 Certain automorphisms

Let φ be a K–algebra automorphism on $K[X_1, X_2]/(X_1^p, X_2^p)$ given by

$$\varphi(x_1) = x_1 + \sum_{i=0}^{s} a_i x_1^i x_2^{s-i} \quad \text{and} \quad \varphi(x_2) = x_2 + \sum_{i=0}^{s} b_i x_1^i x_2^{s-i}$$

for some $a_i, b_i \in K$ and $s \geq 2$. Note that

$$\varphi^{-1}(x_1) = x_1 - \sum_{i=0}^{s} a_i x_1^i x_2^{s-i} + (\text{degree } \geq\ s+1)$$

$$\varphi^{-1}(x_2) = x_2 - \sum_{i=0}^{s} b_i x_1^i x_2^{s-i} + (\text{degree } \geq\ s+1).$$

Set $x = \sum_{i=0}^{s} a_i e_{i,s-i,1} + \sum_{i=0}^{s} b_i e_{i,s-i,2} \in W_{s-1}$. I claim that the automorphism σ_φ induced by φ satisfies

$$\sigma_\varphi(y) \equiv y + [x, y] \pmod{W_{\geq r+s}} \quad \text{for } y \in W_r. \tag{3.2}$$

Because of linearity, it is enough to check this for $y = e_{km1}$ and $y = e_{km2}$. First we calculate

$$\begin{aligned}\varphi(x_1)^k\varphi(x_2)^m &\equiv x_1^k x_2^m + \textstyle\sum_{i=0}^s a_i k x_1^{i+k-1} x_2^{m+s-i} \\ &\quad + \textstyle\sum_{i=0}^s b_i m x_1^{k+i} x_2^{m+s-i-1} \pmod{\text{degree } \geq k+m+s}\end{aligned}$$

and

$$\begin{aligned}\tfrac{\partial\varphi^{-1}(x_1)}{\partial x_1}(\varphi(x_1),\varphi(x_2)) &\equiv 1 - \textstyle\sum_{i=0}^s a_i i x_1^{i-1} x_2^{s-i} \pmod{\text{degree } \geq s} \\ \tfrac{\partial\varphi^{-1}(x_2)}{\partial x_1}(\varphi(x_1),\varphi(x_2)) &\equiv -\textstyle\sum_{i=0}^s b_i i x_1^{i-1} x_2^{s-i} \pmod{\text{degree } \geq s}.\end{aligned}$$

If we use Proposition 2.3, we get

$$\begin{aligned}\sigma_\varphi(e_{km1}) &\equiv e_{km1} + \textstyle\sum_{i=0}^s (a_i k - a_i i) e_{k+i-1,m+s-i,1} \\ &\quad + \textstyle\sum_{i=0}^s m b_i e_{k+i,m+s-i-1,1} \\ &\quad - \textstyle\sum_{i=0}^s i b_i e_{k+i-1,m+s-i,2} \pmod{W_{k+m+s-1}}\end{aligned}$$

which is exactly $e_{km1} + [x, e_{km1}]$ with x defined above. In the same way, one can prove that $\sigma_\varphi(e_{km2})$ has the desired form.

3.3 $GL_2(K)$–action

Note that we have an inclusion $\mathrm{Aut}_{K-\mathrm{alg}}\, B_2 \supset GL_2(K)$ in the following way:

$$\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{ccc} x_1 & \longmapsto & ax_1 + cx_2 \\ x_2 & \longmapsto & bx_1 + dx_2 \end{array} \tag{3.3}$$

where $ad - bc \neq 0$. For any $\varphi \in GL_2(K)$, the automorphism σ_φ is determined by (2.8) and (3.3). In Appendix A we have listed formulas for the $GL_2(K)$–action on basis elements. For $\varphi \in GL_2(K)$ we will write $\varphi(w)$ instead of $\sigma_\varphi(w)$ whenever $w \in W$.

Lemma 3.3.1. *Suppose that $\varphi \in GL_2(K) \subset \mathrm{Aut}_{K-\mathrm{alg}}\, B_2$. If $\varphi(W_{011}) = W_{011}$, then φ is a diagonal matrix composed with some lower triangular matrix with 1 at the diagonal. In particular, $\varphi(W_{012}) = W_{012}$.*

Proof. Otherwise, φ is given by $\varphi_1 \circ \Theta \circ T \circ \varphi_1'$, where φ_1, φ_1' are lower triangular matrices and T is a diagonal matrix and Θ is the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Use the Bruhat decomposition of $GL_2(K)$ described in [23, 8]. Now apply the relations in Appendix A.1, A.2, A.4 and get $\varphi(e_{011}) \equiv (\text{nonzero constant}) \cdot e_{102} \pmod{W_{012}}$. This is however a contradiction. □

4 Decomposition

Let $r \geq 0$ but $r \leq 2p-2$. If $r \neq p-1$, then W_{r-1} has a K–basis given by $x_0^{(r)}, x_1^{(r)}, \ldots, x_{\text{top}}^{(r)}$ and $y_1^{(r)}, y_2^{(r)}, \ldots, y_{\text{top}}^{(r)}$ defined below. The top index for $\{x_i^{(r)}\}_{i=0}^{\text{top}}$ is $r+1$ when $r \leq p-2$ and $2p-r-1$ when $r \geq p$ and the top index for $\{y_i^{(r)}\}_{i=1}^{\text{top}}$ is r when $r \leq p-2$ and $2p-r-2$ when $r \geq p$. In all cases, I will refer to the top index as "top" (this allow us to treat the cases $r \leq p-2$ and $r \geq p$ at the same time).

The new basis elements are given by:

$$\begin{array}{lrcll} \underline{r \leq p-2}: & x_i^{(r)} & = & (r+1-i)e_{r-i,i,2} - ie_{r+1-i,i-1,1} & \text{for } 0 \leq i \leq r+1, \\ & y_i^{(r)} & = & e_{r+1-i,i-1,1} + e_{r-i,i,2} & \text{for } 1 \leq i \leq r \end{array}$$

$$\begin{array}{lrcll} \underline{r \geq p}: & x_i^{(r)} & = & e_{p-1-i,r+1+i-p,2} + e_{p-i,r+i-p,1} & \text{for } 0 \leq i \leq 2p-r-1, \\ & y_i^{(r)} & = & ie_{p-1-i,r+i+1-p,2} + (r+i+1-p)e_{p-i,r+i-p,1} & \text{for } 1 \leq i \leq 2p-r-2. \end{array}$$

4.1 The case that $r \leq p-2$

Apply (3.1a),(3.1b),(3.1c),(3.1d) and find:

$$\begin{array}{rcll} [e_{011}, x_i^{(r)}] & = & (r+1-i)x_{i+1}^{(r)} & \text{for } 0 \leq i \leq r+1, \\ [e_{012}, x_i^{(r)}] & = & (i-1)x_i^{(r)} & \text{for } 0 \leq i \leq r+1, \\ [e_{101}, x_i^{(r)}] & = & (r-i)x_i^{(r)} & \text{for } 0 \leq i \leq r+1, \\ [e_{102}, x_i^{(r)}] & = & ix_{i-1}^{(r)} & \text{for } 0 \leq i \leq r+1, \end{array}$$

$$\begin{array}{rcll} [e_{011}, y_i^{(r)}] & = & (r-i)y_{i+1}^{(r)} & \text{for } 1 \leq i \leq r, \\ [e_{012}, y_i^{(r)}] & = & (i-1)y_i^{(r)} & \text{for } 1 \leq i \leq r, \\ [e_{101}, y_i^{(r)}] & = & (r-i)y_i^{(r)} & \text{for } 1 \leq i \leq r, \\ [e_{102}, y_i^{(r)}] & = & (i-1)y_{i-1}^{(r)} & \text{for } 1 \leq i \leq r. \end{array}$$

We define $x_{i+1}^{(r)} = 0$ for $i = r+1$ and $x_{i-1}^{(r)} = 0$ for $i = 0$ and $y_{i+1}^{(r)} = 0$ for $i = r$ and $y_{i-1}^{(r)} = 0$ for $i = 1$.

4.2 The case that $r \geq p$

Apply (3.1a),(3.1b),(3.1c),(3.1d) and find:

$$\begin{array}{rcll} [e_{011}, x_i^{(r)}] & = & -(i+1)x_{i+1}^{(r)} & \text{for } 0 \leq i \leq 2p-r-1, \\ [e_{012}, x_i^{(r)}] & = & (r+i)x_i^{(r)} & \text{for } 0 \leq i \leq 2p-r-1, \\ [e_{101}, x_i^{(r)}] & = & -(i+1)x_i^{(r)} & \text{for } 0 \leq i \leq 2p-r-1, \\ [e_{102}, x_i^{(r)}] & = & (r+i)x_{i-1}^{(r)} & \text{for } 0 \leq i \leq 2p-r-1, \end{array}$$

$$\begin{array}{rcll} [e_{011}, y_i^{(r)}] & = & -iy_{i+1}^{(r)} & \text{for } 1 \leq i \leq 2p-r-2, \\ [e_{012}, y_i^{(r)}] & = & (r+i)y_i^{(r)} & \text{for } 1 \leq i \leq 2p-r-2, \\ [e_{101}, y_i^{(r)}] & = & -(i+1)y_i^{(r)} & \text{for } 1 \leq i \leq 2p-r-2, \\ [e_{102}, y_i^{(r)}] & = & (r+i+1)y_{i-1}^{(r)} & \text{for } 1 \leq i \leq 2p-r-2. \end{array}$$

We define $x_{i+1}^{(r)} = 0$ for $i = 2p-r-1$ and $x_{i-1}^{(r)} = 0$ for $i = 0$ and $y_{i+1}^{(r)} = 0$ for $i = 2p-r-2$ and $y_{i-1}^{(r)} = 0$ for $i = 1$.

4.3 Decomposition of W_{r-1} when $r \neq p-1$

Proposition 4.3.1. *Suppose that $r \neq p-1$. Then $U := \sum_{j=0}^{\text{top}} Kx_j^{(r)}$ and $V := \sum_{j=1}^{\text{top}} Ky_j^{(r)}$ are irreducible $GL_2(K)$-submodules of W_{r-1} and $W_{r-1} = U \bigoplus V$.*

Proof. For $r \leq p-2$ the $GL_2(K)$–action on U is given by the following formulas (use the relations in Appendix A and the definition of $x_i^{(r)}$):

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot x_i^{(r)} = -x_{r+1-i}^{(r)},$$

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \cdot x_i^{(r)} = \sum_{s=0}^{r+1-i} \binom{r+1-i}{s} \alpha^s x_{i+s}^{(r)},$$

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \cdot x_i^{(r)} = \sum_{s=0}^{i} \binom{i}{s} \alpha^s x_{i-s}^{(r)},$$

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot x_i^{(r)} = t^{r+1-2i} x_i^{(r)},$$

$$\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \cdot x_i^{(r)} = t_1^{r-i} t_2^{i-1} x_i^{(r)}.$$

Note that $U = \bigoplus_{i=0}^{r+1} U_{\mu_i}$ (the top index is $r+1$) where $U_{\mu_i} = Kx_i^{(r)}$ is the weight space of U for the weight $\mu_i : T \longrightarrow K^*$, where T is the subgroup of all diagonal matrices in $SL_2(K)$. The weight μ_i is given by

$$\mu_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = t^{r+1-2i}.$$

If $N \neq 0$ is a $GL_2(K)$–submodule in U, then it is well known that N is a direct sum of N_{μ_j}, where $N_{\mu_j} = Kx_j^{(r)}$ and where $j \in I$ for a nonempty subset I of $\{0, 1, \dots, r+1\}$. Since N is a $GL_2(K)$–module, we must have $I = \{0, 1, \dots, m+1\}$ (apply the $GL_2(K)$–action above); hence $N = U$. Similar arguments as those applied to U can be used to show that V is an irreducible $GL_2(K)$–module also. Finally, observe that $x_0^{(r)}, x_1^{(r)}, \dots, x_{\text{top}}^{(r)}$ and $y_1^{(r)}, y_2^{(r)}, \dots, y_{\text{top}}^{(r)}$ are linear independent. Therefore $U \bigoplus V$ is a $GL_2(K)$–submodule of W_{r-1} of dimension $2r+2 = \dim_K W_{r-1}$; hence $W_{r-1} = U \bigoplus V$.

If $r \geq p$ the $GL_2(K)$–action on U is given by the following formulas:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot x_i^{(r)} = x_{2p-r-1-i}^{(r)},$$

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \cdot x_i^{(r)} = \sum_{s=0}^{2p-r-1-i} \binom{p-1-i}{s} \alpha^s x_{i+s}^{(r)},$$

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \cdot x_i^{(r)} = \sum_{s=0}^{i} \binom{r+i-p}{s} \alpha^s x_{i-s}^{(r)},$$

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot x_i^{(r)} = t^{2p-2i-r-1} x_i^{(r)},$$

$$\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \cdot x_i^{(r)} = t_1^{p-1-i} t_2^{r+i-p} x_i^{(r)}.$$

Again we have $U = \bigoplus_{i=0}^{r+1} U_{\mu_i}$ where $U_{\mu_i} = Kx_i^{(r)}$ is the weight space of U for the weight $\mu_i : T \longrightarrow K^*$ and where T is the subgroup all diagonal matrices in $SL_2(K)$. The weight μ_i is given by

$$\mu_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = t^{r+1-2i}.$$

We can now show that U is an irreducible $GL_2(K)$–module (apply arguments similar to those for $r \leq p-2$) and that V is an irreducible $GL_2(K)$–module. Finally, we obtain $W_{r-1} = U \bigoplus V$. $\square$

4.4 $SL_2(K)$–action on polynomials

We still assume that $r \neq p-1$ and write $W_{r-1} = U \oplus V$ as in Proposition 4.3.1. Then $W_{r-1}^* = U^* \oplus V^*$, where $U^* = \sum_j K\chi_j$ and $V^* = \sum_j K\chi_j'$ and where χ_j and χ_j' are defined via:

$$\chi_j : \ \chi_j(x_i^{(r)}) = \delta_{ij} \quad \text{and} \quad \chi_j(y_i^{(r)}) = 0 \ \ \forall i,$$

$$\chi_j' : \ \chi_j'(y_i^{(r)}) = \delta_{ij} \quad \text{and} \quad \chi_j'(x_i^{(r)}) = 0 \ \ \forall i.$$

Let $K[X,Y]$ be the polynomial ring in two variables. Each element in $SL_2(K)$ (and $GL_2(K)$) defines an automorphisms on $K[X,Y]$ in the following way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(X,Y) = f(aX+cY, bX+dY).$$

All $L_n := \{ f \in K[X,Y] \mid f \text{ is homogeneous of degree } n\}$ are $GL_2(K)$–submodules of $K[X,Y]$ with dimension $n+1$.

Lemma 4.4.1. *For $0 \leq n \leq p-1$ all L_n are irreducible $GL_2(K)$–modules.*

Proof. Note that $X^n, XY^{n-1}, \ldots, Y^n$ form a K–basis for L_n. We have

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot X^{n-i}Y^i = t^{n-2i}X^{n-i}Y^i \quad \text{for } 0 \leq i \leq n.$$

Let T denote the subgroup of all diagonal matrices in $SL_2(K)$. Then $L_n = \bigoplus_i (L_n)_{\mu_i}$, where $\mu_i : T \longrightarrow K^*$ is given by

$$\mu_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = t^{n-2i}$$

and

$$(L_n)_{\mu_i} = \{ f \in L_n \mid t \cdot f = \mu_i(t) f \ \ \forall t \in T\} = KX^{n-i}Y^i.$$

Any $GL_2(K)$–submodule $M_n \neq 0$ of L_n is a direct sum $\bigoplus_{i \in I} KX^{n-i}Y^i$ for a nonempty subset I of $\{0,1,\ldots,n\}$. Since M_n is a $GL_2(K)$–module and $n < p$, we have $I = \{0,1,\ldots,n\}$ (use the $GL_2(K)$–action on L_n); hence $M_n = U$. Indeed, for all $0 \leq i \leq n$, we have:

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \cdot X^{n-i}Y^i = \sum_{s=0}^{i} \binom{i}{s} \alpha^{i-s} X^{n-s}Y^s,$$

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \cdot X^{n-i}Y^i = \sum_{s=0}^{n-i} \binom{n-i}{s} \alpha^{n-i-s} X^{s}Y^{n-s}.$$

The proof is completed. $\square$

4.5 Identification

We are now in position to identify U^* and V^* with appropriate L_n.

Theorem 4.5.1. *As $SL_2(K)$–modules we have isomorphisms*

$$U^* = \bigoplus_{j=0}^{\text{top}} K\chi_j \simeq L_{\dim_K U-1} \qquad V^* = \bigoplus_{j=1}^{\text{top}} K\chi'_j \simeq L_{\dim_K V-1}.$$

Proof. Let me concentrate on the case involving U^* (one can use similar computations for V^*). The Bruhat decomposition of $GL_2(K)$ shows that any matrix in $SL_2(K)$ belongs to $\{D_t \circ \Phi_1\} \cup \{\Phi'_1 \circ \Theta \circ d_t \circ \Phi_1\}$, where

$$\Phi_1 = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \quad \text{and} \quad \Phi'_1 = \begin{pmatrix} 1 & 0 \\ \alpha' & 1 \end{pmatrix}$$

denote matrices in the subgroup of all lower triangular matrices in $SL_2(K)$ with 1 at the diagonal and D_t, d_t, Θ are the matrices

$$D_t = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}, \qquad d_t = \begin{pmatrix} -t^{-1} & 0 \\ 0 & t \end{pmatrix}, \qquad \Theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The $SL_2(K)$–action on U^* in the case where $r \leq p-2$ is given via (use the formulas for the action on U in the proof of Proposition 4.3.1):

$$\begin{aligned}
\Phi_1 \cdot \chi_j &= \textstyle\sum_{s=0}^{j} \binom{r+1-s}{j-s} (-\alpha)^{j-s} \chi_s \\
D_t \cdot \chi_j &= t^{r+1-2j} \chi_j \\
d_t \cdot \chi_j &= (-1)^{r-j} t^{r+1-2j} \chi_j \\
\Theta \cdot \chi_j &= -\chi_{r+1-j}.
\end{aligned}$$

We now define a K-linear map $\psi : U^* \longrightarrow L_{\dim_K U-1}$ by $\psi(\chi_j) = (-1)^j \binom{r+1}{j} X^j Y^{r+1-j}$ for $j = 0, 1, \ldots, r+1$. Since ψ is a bijective and

$$\begin{aligned}
\psi(\Phi_1 \cdot \chi_j) &= \Phi_1 \cdot \psi(\chi_j) \\
\psi(\Theta \circ d_t \cdot \chi_j) &= \Theta \circ d_t \cdot \psi(\chi_j) \\
\psi(D_t \cdot \chi_j) &= D_t \cdot \psi(\chi_j)
\end{aligned}$$

for all j, we conclude that ψ is a $SL_2(K)$–isomorphism.

The $SL_2(K)$–action on U^* in the case where $r \geq p$ is given via (use the formulas for the action on U in the proof of Proposition 4.3.1):

$$\begin{aligned}
\Phi_1 \cdot \chi_j &= \textstyle\sum_{s=0}^{j} \binom{p-1-s}{j-s} (-\alpha)^{j-s} \chi_s \\
D_t \cdot \chi_j &= t^{2p-2j-r-1} \chi_j \\
d_t \cdot \chi_j &= (-1)^{j} t^{2p-2j-r-1} \chi_j \\
\Theta \cdot \chi_j &= \chi_{2p-r-1-j}.
\end{aligned}$$

We now define a K-linear map $\phi : U^* \longrightarrow L_{\dim_K U-1}$ by $\phi(\chi_j) = X^i Y^{2p-r-1-j}$ for $j = 0, 1, \ldots, 2p-r-1$. Since ϕ is a bijection and

$$\begin{aligned}
\phi(\Phi_1 \cdot \chi_j) &= \Phi_1 \cdot \phi(\chi_j) \\
\phi(\Theta \circ d_t \cdot \chi_j) &= \Theta \circ d_t \cdot \phi(\chi_j) \\
\phi(D_t \cdot \chi_j) &= D_t \cdot \phi(\chi_j)
\end{aligned}$$

for all j, then ϕ is a $SL_2(K)$–isomorphism ☐

4.6 $SL_2(K)$–orbits in L_n

Let f be a homogeneous polynomial in two variables X, Y. We can write f as a product of polynomials of degree 1: Assume that

$$f = \sum_{i=0}^{n} c_i X^i Y^{n-i} = Y^n \sum_{i=0}^{n} c_i \Big(\frac{X}{Y}\Big)^i \quad \text{for some } c_i \in K.$$

Then $\sum_{i=0}^{n} c_i \Big(\frac{X}{Y}\Big)^i \in K[\frac{X}{Y}]$ can be written in linear factors (since K is algebraically closed); we then obtain

$$f = Y^n \prod_{i=1}^{n} (a_i \frac{X}{Y} + b_i) = \prod_{i=1}^{n} (a_i X + b_i Y).$$

Eventually, one has to put $b_i = 1, a_i = 0$ for $i \geq m$ and some $m \leq n$. We can now write $f = a f_1^{m_1} f_2^{m_2} \cdots f_s^{m_s}$, where $a \in K^*$ and $f_1, f_2, \ldots, f_s$ are linear, non–proportional polynomials. For $g \in SL_2(K)$ we have $g \cdot f = a(g \cdot f_1)^{m_1} (g \cdot f_2)^{m_2} \cdots (g \cdot f_s)^{m_s}$, which again is the factorization of $g \cdot f$ in linear, non–proportional polynomials. Therefore: If two polynomials l, h with factorizations $l = a_l l_1^{n_1} l_2^{n_2} \cdots l_t^{n_t}$ and $h = a_h h_1^{k_1} h_2^{k_2} \cdots h_s^{k_s}$ lie in the same $SL_2(K)$–orbit, then $s = t$. Moreover, $(n_1, n_2, \ldots, n_t) \in S_t \cdot (k_1, k_2, \ldots, k_t)$ where S_t is the permutations group on t elements.

In the next sections we consider a polynomial $f = a f_1^{m_1} f_2^{m_2} \cdots f_s^{m_s}$, where $a \in K^*$ and $f_1, \ldots, f_s$ are linear, non–proportional polynomials. We say that s is the *length* of f.

4.7 Length 1

Then $f = a(f_1)^n$ for some $a \in K^*$ and some $n \in \mathbb{N}$. There exists $g \in SL_2(K)$ such that $g \cdot f_1 = bY$ for some $b \in K^*$; hence f and Y^n lie in the same $SL_2(K)$–orbit. Moreover,

$$\mathrm{Stab}_{GL_2(K)}\Big(Y^n\Big) = \Big\{ \begin{pmatrix} a & 0 \\ c & \xi \end{pmatrix} \,\big|\, \xi^n = 1, a \in K^*, c \in K \Big\}$$

is a linear algebraic group of dimension 2.

4.8 Length 2

Then $f = a(f_1)^{m_1}(f_2)^{m_2}$, where $a \in K^*$ and f_1, f_2 are non–proportional and $m_1, m_2 > 0$. Take $g \in SL_2(K)$ such that $g \cdot f_1 = X$ and $g \cdot f_2 = bY$ for some $b \in K^*$. Now, observe that all $aX^{n-i}Y^i$, where i is an integer with $0 < i \leq \frac{n}{2}$ and $a \in K^*$, are representatives for the $SL_2(K)$–orbit of f. This follows since $aX^{n-i}Y^i \in SL_2(K) \cdot aX^iY^{n-i}$ for i with $\frac{n}{2} < i < n$: Use $\Theta \circ \begin{pmatrix} -\alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in SL_2(K)$ where $\alpha \in K^*$ such that $(-1)^{n-i}\alpha^{n-2i} = 1$.

The stabilizer is a linear algebraic group of dimension 1 given by:

$$\mathrm{Stab}_{GL_2(K)}\Big(aX^{n-i}Y^i\Big) = \begin{cases} \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \,\Big|\, t_1^{n-i}t_2^i = 1 \right. & 0 < i < \frac{n}{2}, \\ \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,\Big|\, t_1^{n-i}t_2^i = 1 \right. & i = \frac{n}{2}. \end{cases}$$

4.9 Length ≥ 3

We have $f = a(f_1)^{m_1}(f_2)^{m_2}(f_3)^{m_3}\cdots$ with $a \in K^*$ and $m_1, m_2, m_3 > 0$ and $f_1, f_2, f_3, \ldots$ are non–proportional polynomials. As in the case $s = 2$ we find $g \in SL_2(K)$ such that $g \cdot f_1 = X$ and $g \cdot f_2 = bY$ for some $b \in K^*$. Now the $SL_2(K)$–orbit of f includes a polynomial, which is divisible by both X and Y and involves at least 2 monomials (else f is equal to $a(f_1)^{n-j}(f_2)^j$ for some $j \geq 0$ which is a contradiction to our assumption $s \geq 3$). Therefore the $SL_2(K)$–orbit of f has a representative given by

$$\sum_{i=1}^{n-1} a_i Y^i X^{n-i} \qquad \text{where } \exists\, i_1 \neq i_2 :\ a_{i_1} \neq 0 \neq a_{i_2}.$$

The stabilizer is finite.

5 Representatives of characters

In this section we shall use the results obtained in the previous section and find certain representatives for $\chi \in W^*$ with respect to the $GL_2(K)$–action on W^*. Suppose that $\chi \in W^*$ has height r (i.e., $\chi(W_{\geq r}) = 0$ but $\chi(W_{r-1}) \neq 0$). We assume that $r \neq p - 1$. Write $W_{r-1} = U \bigoplus V$ as in Proposition 4.3.1.

5.1 Characters of Type I

Suppose that $\chi(V) = 0$ and $\chi(U) \neq 0$ (or equivalent $\chi \in U^*$). The isomorphism in Theorem 4.5.1 identifies $\chi_{|W_{r-1}}$ with a homogeneous polynomial $f = af_1^{m_1}f_2^{m_2}\cdots f_s^{m_s}$ for some $a \in K^*$ and linear, non–proportional factors $f_1, f_2, \ldots, f_s$. I will treat the cases $s = 1, s = 2$ and $s \geq 3$ separately. As in Section 4, I will refer to the top index for $\{x_i^{(r)}\}_{i=0}^{\mathrm{top}}$ as top.

a) Combine the case $s = 1$ discussed in Section 4.7 and the isomorphism from Theorem 4.5.1 to find a representative for the orbit of χ as (abuse of notation):

$$\chi :\ \chi(x_0^{(r)}) \neq 0,\ \chi(x_i^{(r)}) = 0 \text{ for } i > 0 \quad \text{and} \quad \chi(y_i^{(r)}) = 0\,\forall\, i.$$

b) Use the strategy in a) with $s = 2$ to find a representative for the orbit of χ as:

$$\chi :\ \exists\, j \neq 0, \mathrm{top} \text{ with } \chi(x_j^{(r)}) \neq 0,\ \chi(x_i^{(r)}) = 0 \text{ for } i \neq j \text{ and } \chi(y_i^{(r)}) = 0\,\forall\, i.$$

c) Use the strategy in a) with $s = 3$ to find a representative for the orbit of χ as:

$$\chi :\ \exists\, j_1 \neq j_2 \text{ with } \chi(x_{j_1}^{(r)}) \neq 0 \neq \chi(x_{j_2}^{(r)}) \text{ for } j_1, j_2 \neq 0, \mathrm{top} \text{ and}$$

$$\chi(x_0^{(r)}) = \chi(x_{\mathrm{top}}^{(r)}) = 0 \quad \text{and} \quad \chi(y_i^{(r)}) = 0\,\forall\, i.$$

Characters as in a),b) or c) above will be referred to as characters of type I.a, I.b or I.c.

5.2 Characters of Type II

Assume $\chi(U) = 0$ and $\chi(V) \neq 0$ which means $\chi \in V^*$. As above, I will refer to the top index for $\{y_i^{(r)}\}_{i=0}^{\text{top}}$ as top.

a) Combine the case $s = 1$ discussed in Section 4.7 and the isomorphism from Theorem 4.5.1 to find a representative for the orbit of χ as (abuse of notation):

$$\chi \,:\, \chi(y_1^{(r)}) \neq 0,\ \chi(y_i^{(r)}) = 0 \ \text{ for } i > 1 \quad \text{ and } \quad \chi(x_i^{(r)}) = 0 \,\forall\, i.$$

b) Use the strategy in a) with $s = 2$ to find a representative for the orbit of χ as:

$$\chi \,:\, \exists\, j \neq 1, \text{top} \ \text{ with } \ \chi(y_j^{(r)}) \neq 0,\ \chi(y_i^{(r)}) = 0 \ \text{ for } i \neq j \ \text{ and } \ \chi(x_i^{(r)}) = 0 \,\forall\, i.$$

c) Use the strategy in a) with $s = 3$ to find a representative for the orbit of χ as:

$$\chi \,:\, \exists\, j_1 \neq j_2 \ \text{ with } \ \chi(y_{j_1}^{(r)}) \neq 0 \neq \chi(y_{j_2}^{(r)}) \ \text{ for } j_1, j_2 \neq 1, \text{top} \quad \text{ and }$$
$$\chi(y_1^{(r)}) = \chi(y_{\text{top}}^{(r)}) = 0 \quad \text{ and } \quad \chi(x_i^{(r)}) = 0 \,\forall\, i.$$

Characters as in a),b) or c) above will be referred to as characters of type II.a, II.b or II.c.

5.3 Characters of Type III

Assume $\chi(U) \neq 0 \neq \chi(V)$. If $\big(a_i\big)_{i \in I}$ denote representatives for the $SL_2(K)$–orbit on U^* and $(a, b) \in W_{r-1}^* = U^* \bigoplus V^*$, then there exists $i \in I$ such that $g \cdot (a, b) = (a_i, b')$ and b' can only be changed by using $\mathrm{Stab}_{\mathrm{Aut}(W)}(a_i)$. A representative τ for the orbit of χ can be chosen such that $\tau_{|U^*}$ is a representative in U^* and τ is arbitrary on V. Characters τ as above, where the restriction to U^* defines a character of type I and $\tau(V) \neq 0$ will be referred to as characters of type III.

5.4 A lemma

From the description of the representatives above, we have:

Lemma 5.4.1. *Suppose that $\chi \in W^*$ with height $r \geq 0$ but $r \neq p - 1$. Then there exists $g \in \mathrm{Aut}(W)$ and $x \in W_{r-1}$ such that $\chi^g([x, e_{102}]) \neq 0 = \chi^g([x, W_{012}])$ except for the case where $r = 2p - 3$ and χ has type* II.a *as in* 5.2.

Proof. The computations above say that we can find $g \in GL_2(K)$ such that χ^g has the following properties: Either we have $0 \leq t := \max\{0 \leq j \leq \text{top} \mid \chi^g(x_j^{(r)}) \neq 0\} < \text{top}$ or $1 \leq s := \max\{1 \leq j \leq \text{top} \mid \chi^g(y_j^{(r)}) \neq 0\} < \text{top}$. Now, set $x = x_{t+1}^{(r)}$ or $x = y_{s+1}^{(r)}$ and apply the relations in Section 4.1, 4.2. It follows that $\chi^g([x, e_{102}]) \neq 0 = \chi^g([x, W_{012}])$ as required. $\square$

6 Criteria for irreducibility

Let K be an algebraically closed field of characteristic $p > 0$ and $(\mathfrak{g}, [p])$ be a finite dimensional restricted Lie algebra over K. Every irreducible $\mathfrak{g}$–module is finite dimensional and therefore admits a p–character $\chi \in \mathfrak{g}^*$ (see [14, 2.4]). Conversely, for any linear form $\chi \in \mathfrak{g}^*$, there exists a finite dimensional associative algebra $U_\chi(\mathfrak{g})$ which is a quotient of the universal enveloping algebra $U(\mathfrak{g})$ and whose irreducible modules are exactly the irreducible $\mathfrak{g}$–modules with p–character χ (see [14, 2.7]). Hence the algebras $U_\chi(\mathfrak{g})$, where $\chi \in \mathfrak{g}^*$, play a major role in studying the representations of $\mathfrak{g}$. If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie p–subalgebra, we will use the notation $U_\chi(\mathfrak{h})$ for the reduced enveloping algebra $U_{\chi_{|\mathfrak{h}}}(\mathfrak{h})$.

6.1 Setup

Let $\chi \in \mathfrak{g}^*$ and let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie p–subalgebra. We shall define the *stabilizer* of χ in $\mathfrak{h}$ as

$$\mathfrak{st}(\chi, \mathfrak{h}) := \{y \in \mathfrak{g} \mid \chi([y, x]) = 0 \text{ for all } x \in \mathfrak{h}\}. \tag{6.1}$$

It is easy to verify that $\mathfrak{st}(\chi, \mathfrak{h})$ is a Lie p–subalgebra of $\mathfrak{g}$. In the following N will denote an irreducible $U_\chi(\mathfrak{h})$–module. We will give criteria for the induced $\mathfrak{g}$–module

$$U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$$

to be irreducible. The first criterion is described in [27, 5, 5.7] and the second in [25, I]. The third criterion has been made because none of the first two criteria could be applied in examples for $p = 3$ in Section 13.

Let $e_1, \dots, e_n$ be a basis for a complement to $\mathfrak{h}$ in $\mathfrak{g}$. For each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$ set $e^\alpha = e_1^{\alpha_1} e_2^{\alpha_2} \cdots e_n^{\alpha_n}$ and $|\alpha| = \sum_{i=1}^n \alpha_i$. Define $\varepsilon_j = (0, 0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^n$, where 1 occurs at the j'th place. Let I denote the set of all $\alpha \in \mathbb{N}^n$ with $\alpha_i < p$ for all i. From [14, 4.1] we have a direct sum decomposition as a vector space

$$U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N = \bigoplus_{\alpha \in I} e^\alpha \otimes N. \tag{6.2}$$

For all $u \in U(\mathfrak{g})$ and all $\alpha \in I$, we have

$$ue^\alpha = \sum_\beta \binom{\alpha}{\beta} e^\beta \mathrm{ad}'(e_n)^{\alpha_n - \beta_n} \circ \cdots \circ \mathrm{ad}'(e_1)^{\alpha_1 - \beta_1}(u) \tag{6.3}$$

where we sum over all $\beta \in I$ with $\beta_i \leq \alpha_i$ for all i and where $\mathrm{ad}'(x)(y) = [y, x]$ for all $x, y \in \mathfrak{g}$. The binomial coefficient is given by

$$\binom{\alpha}{\beta} = \prod_{i=1}^n \binom{\alpha_i}{\beta_i}.$$

6.2 Criterion 1

Recall the further setup from [27, 5, 5.7]: Let $\mathfrak{a} \subset \mathfrak{g}$ be an ideal such that $\chi([\mathfrak{a}, \mathfrak{a}]) = 0$ and set $\mathfrak{h} := \mathfrak{st}(\chi, \mathfrak{a})$ where $\mathfrak{st}(\chi, \mathfrak{a}) = \{y \in \mathfrak{g} \mid \chi([y, x]) = 0 \text{ for all } x \in \mathfrak{a}\}$. Since $\mathfrak{a} \subset \mathfrak{g}$ is an ideal, $\mathfrak{h}$ is a Lie p–subalgebra of $\mathfrak{g}$ and moreover $\mathfrak{h}$ contains $\mathfrak{a}$ because $\chi([\mathfrak{a}, \mathfrak{a}]) = 0$. If $u \in U(\mathfrak{a})$ the factor after e^β in (6.3) belongs to $U(\mathfrak{a})$. We get then for all $\alpha \in I$ and all $v \in N$

$$u(e^\alpha \otimes v) \in e^\alpha \otimes uv - \sum_{j=1}^n \alpha_j e^{\alpha - \varepsilon_j} \otimes [e_j, u]v + \sum_{|\beta| \leq |\alpha| - 2} e^\beta \otimes N. \tag{6.4}$$

We need one further assumption: We shall assume that N is an irreducible $U_\chi(\mathfrak{h})$–module such that $y \cdot v = \chi(y)v$ for all $y \in \mathfrak{a}$ and all $v \in N$.

Lemma 6.2.1.

1) *There exist $y_1, y_2, \dots, y_n \in \mathfrak{a}$ such that $\chi([y_i, e_j]) = \delta_{ij}$.*

2) *For all $v \in N$ we have*

$$(y_i - \chi(y_i)) \cdot e^\alpha \otimes v - \alpha_i e^{\alpha - \varepsilon_i} \otimes v \in \sum_{|\beta| \leq |\alpha| - 2} K e^\beta \otimes N.$$

Proof. 1) Set $U = \sum_{i=1}^n Ke_i$. The bilinear form $B_\chi(x,y) := \chi([x,y])$ defines a linear mapping

$$\varphi : U \longrightarrow \mathfrak{a}^*, \quad x \longmapsto \chi([-,x]).$$

The definition of $\mathfrak{h}$ and the assumption on $e_1, e_2, \ldots, e_n$ as a cobasis for $\mathfrak{h}$ in $\mathfrak{g}$ imply that φ is injective and the linear functionals $\varphi(e_1), \varphi(e_2) \ldots, \varphi(e_n)$ are linear independent. Hence there are $y_1, y_2, \ldots, y_n \in \mathfrak{a}$ with $\varphi(e_i)(y_j) = \delta_{ij}$, $1 \leq i, j \leq n$.

2) It follows from (6.4) and 1) that

$$(y_i - \chi(y_i)) \cdot e^\alpha \otimes v \in e^\alpha \otimes (y_i - \chi(y_i))v + \sum_{j=1}^n \delta_{ij}\alpha_j e^{\alpha-\varepsilon_j} \otimes v + \sum_{|\beta| \leq |\alpha|-2} Ke^\beta \otimes N.$$

Now use that each $y \in \mathfrak{a}$ acts as multiplication by $\chi(y)$ on N and obtain

$$(y_i - \chi(y_i)) \cdot e^\alpha \otimes v - \alpha_i e^{\alpha-\varepsilon_i} \otimes v \in \sum_{|\beta| \leq |\alpha|-2} Ke^\beta \otimes N.$$

The proof is completed. □

Remark 6.2.2. The previous lemma works for all finite dimensional $U_\chi(\mathfrak{h})$–modules N such that $y \cdot v = \chi(y)v$ for all $y \in \mathfrak{a}$ and all $v \in N$. The assumption on irreducibility is not needed anywhere.

Proposition 6.2.3. *Let M be a $\mathfrak{g}$–submodule of $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$. Then there exists a $\mathfrak{h}$–submodule X of N such that $M \cap (1 \otimes N) = 1 \otimes X$ and such that $M \simeq U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} X$.*

Proof. Set

$$X = \{v \in N \mid 1 \otimes v \in M\}.$$

Clearly, X is a $\mathfrak{h}$–submodule of N. Since we have a direct sum decomposition of $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$ as in (6.2), we also obtain $M \cap (1 \otimes N) = 1 \otimes X$. For the isomorphism, note that we have a canonical embedding

$$\phi : U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} X \hookrightarrow U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$$

with image $\sum_{\alpha \in I} Ke^\alpha \otimes X$ inside $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$. Clearly, the image of ϕ is contained in M since $1 \otimes X \subset M$ and M is a $\mathfrak{g}$–submodule of $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$. I claim that the image of ϕ is all M. This will follow if we prove that

$$M \cap \sum_{|\alpha| \leq j} Ke^\alpha \otimes N = \sum_{|\alpha| \leq j} e^\alpha \otimes X \quad \text{for all } j \geq 0.$$

I will use induction on $j \geq 0$. The case $j = 0$ follows directly from the first part of the lemma. So let $j > 0$ and suppose that $x \in M \subset U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$ such that

$$x \equiv \sum_{k=1}^l \sum_{|\alpha| \leq j} \gamma(k,\alpha) e^\alpha \otimes v_k \pmod{\sum_{|\alpha| \leq j} Ke^\alpha \otimes X} \tag{6.5}$$

where $\gamma(k,\alpha) \in K$ and $v_1, v_2, \ldots, v_l \in N$ are chosen such that $N = X \oplus \oplus_{k=1}^l Kv_k$. We shall prove that $\gamma(k,\alpha) = 0$ for all $\alpha \in I$ with $|\alpha| = j$.

Use Lemma 6.2.1 to get

$$(y_i - \chi(y_i)) \cdot \sum_{k=1}^{l} \sum_{|\alpha| \le j} \gamma(k,\alpha) e^{\alpha} \otimes v_k - \sum_{k=1}^{l} \sum_{|\alpha| = j} \gamma(k,\alpha) \alpha_i e^{\alpha - \varepsilon_i} \otimes v_k \in \sum_{|\beta| \le j-2} K e^{\beta} \otimes N$$

for $i = 1, 2, \ldots, n$ [here $y_1, y_2, \ldots, y_n$ are chosen such that $\chi([y_i, e_j]) = \delta_{ij}$ for $1 \le i, j \le n$]. Now use induction to obtain

$$(y_i - \chi(y_i)) \cdot x \in M \cap \sum_{|\beta| \le j-1} K e^{\beta} \otimes N = \sum_{|\beta| \le j-1} K e^{\beta} \otimes X. \tag{6.6}$$

Since $y \cdot v_k = \chi(y) v_k$ for $k = 1, 2, \ldots, l$ and all $y \in \mathfrak{a}$ we have:

$$(y_i - \chi(y_i)) \cdot x \in \sum_{k=1}^{l} \sum_{|\alpha| = j} \gamma(k,\alpha) \alpha_i e^{\alpha - \varepsilon_i} \otimes v_k + \sum_{|\beta| \le j-2} K e^{\beta} \otimes N + \sum_{|\alpha| \le j-1} K e^{\alpha} \otimes X \tag{6.7}$$

for $i = 1, 2, \ldots, n$. Combine (6.6) and (6.7) and obtain $\alpha_i \gamma(k,\alpha) = 0$ for all α with $|\alpha| = j$ and all $k = 1, 2, \ldots, l$. Since $j > 0$ there exists i such that $\alpha_i > 0$; hence $\gamma(k,\alpha) = 0$ for all $k = 1, 2, \ldots, l$ and all $\alpha \in \mathbb{N}^n$ with $|\alpha| = j$. Now induction induces that

$$x \in M \cap \sum_{|\beta| \le j-1} K e^{\beta} \otimes N = \sum_{|\beta| \le j-1} K e^{\beta} \otimes X.$$

The proof is completed. ☐

Corollary 6.2.4. *Let N be an irreducible $U_\chi(\mathfrak{h})$–module such that $y \cdot v = \chi(y) v$ for all $y \in \mathfrak{a}$ and all $v \in N$. Then $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$ is irreducible if and only if N is irreducible.*

Proof. If $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$ is an irreducible $\mathfrak{g}$–module, it is clear that N is an irreducible $\mathfrak{h}$–module. Now suppose that N is an irreducible $\mathfrak{h}$–module and let M be a $\mathfrak{g}$–submodule $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$. By virtue of Proposition 6.2.3, there exists a $\mathfrak{h}$–submodule X of N such that $M \simeq U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} X$. Since N is an irreducible $\mathfrak{h}$–module, we have $M = 0$ or $M = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$ accordingly as $X = 0$ or $X = N$. ☐

For any $\mathfrak{g}$–module M set $M^\chi := \{m \in M \mid x \cdot m = \chi(x) m \ \forall x \in \mathfrak{a}\}$. Note that M^χ is a $\mathfrak{h}$–submodule of M.

Corollary 6.2.5. *If M is an irreducible $U_\chi(\mathfrak{g})$–module with $M^\chi \neq 0$, then M^χ is an irreducible $U_\chi(\mathfrak{h})$–module and $M \simeq U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} M^\chi$.*

Proof. There is a homomorphism of $\mathfrak{g}$–modules $\phi : U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} M^\chi \longrightarrow M$ given by $\phi(x \otimes v) = x \cdot v$ for $x \in U_\chi(\mathfrak{g})$ and $v \in M^\chi$. Since $\phi \neq 0$ and M is irreducible, ϕ is surjective. The kernel of ϕ is a $\mathfrak{g}$–module of $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} M^\chi$ and intersects $1 \otimes M^\chi$ trivially. Now use Proposition 6.2.3 to get $\mathrm{Ker}(\phi) = 0$. Hence ϕ is an isomorphism and M^χ is now irreducible by Corollary 6.2.4. ☐

Definition 6.2.6. If M is a $\mathfrak{g}$–module, we say that χ is an eigenvalue function for M if $M^\chi = \{m \in M \mid x \cdot m = \chi(x) m \ \forall x \in \mathfrak{a}\} \neq 0$. In a similar way, we say that χ is an eigenvalue function for a $\mathfrak{h}$–module N if $N^\chi = \{v \in N \mid x \cdot v = \chi(x) v \ \forall x \in \mathfrak{a}\} \neq 0$.

Remark 6.2.7. If N is an irreducible $\mathfrak{h}$–module with eigenvalue function χ then $N^\chi = N$: Indeed, since $\mathfrak{a} \subset \mathfrak{h}$ is an ideal and $\chi([\mathfrak{h}, \mathfrak{a}]) = 0$, it follows that N^χ is a (nonzero) $U_\chi(\mathfrak{h})$–submodule of N. Therefore, $N^\chi = N$ by irreducibility.

Theorem 6.2.8. *The maps $M \longmapsto M^\chi$ and $N \longmapsto U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$ induce inverse bijections between the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules with eigenvalue function χ and isomorphism classes of irreducible $U_\chi(\mathfrak{h})$–modules with eigenvalue function χ.*

Proof. Suppose that M is an irreducible $U_\chi(\mathfrak{g})$–module with eigenvalue function χ. Then $M^\chi \neq 0$ and by Corollary 6.2.5 then M^χ is an irreducible $U_\chi(\mathfrak{h})$–module with eigenvalue function χ. Moreover $M \simeq U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} M^\chi$. If N is an irreducible $U_\chi(\mathfrak{h})$–module with eigenvalue function χ set $M = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$. Then M is an irreducible $U_\chi(\mathfrak{g})$–module (apply Corollary 6.2.4) and by Remark 6.2.7, we have $1 \otimes N \subset M^\chi$. Since $M \simeq U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} M^\chi$ by Corollary 6.2.5, we have $\dim_K 1 \otimes N = \dim_K M^\chi$ and so $M^\chi = 1 \otimes N \simeq N$. □

Remark 6.2.9. Let $\mathfrak{g}'$ be a restricted Lie subalgebra of $\mathfrak{g}$ with $\mathfrak{g}' \supset \mathfrak{h}$. It follows that $\mathfrak{g}' \supset \mathfrak{a}$, that $\mathfrak{a}$ is an ideal in $\mathfrak{g}'$, and that $\mathfrak{h} = \{y \in \mathfrak{g}' \mid \chi([y, x]) = 0 \ \forall x \in \mathfrak{a}\}$. We can therefore apply everything above to $\mathfrak{g}'$ instead of $\mathfrak{g}$. So the isomorphism classes of irreducible $U_\chi(\mathfrak{g}')$–modules with eigenvalue function χ are in bijection with the same isomorphism classes of irreducible $U_\chi(\mathfrak{h})$–modules with eigenvalue function χ as in the theorem. Combining this with the theorem we see that we have a bijection from the isomorphism classes of irreducible $U_\chi(\mathfrak{g}')$–modules with eigenvalue function χ to the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules with eigenvalue function χ. The isomorphism of tensor products

$$U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N \simeq U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}')} \left(U_\chi(\mathfrak{g}') \otimes_{U_\chi(\mathfrak{h})} N\right)$$

shows that the bijection is induced by $M' \longmapsto U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}')} M'$. Clearly, the inverse map is given by $M \longmapsto U_\chi(\mathfrak{g}')M^\chi$.

6.3 Criterion 2

Fix the notation from Section 6.1. Recall that $e_1, e_2, \ldots, e_n$ form a basis for a complement to $\mathfrak{h}$ in $\mathfrak{g}$, where $\mathfrak{h} \subset \mathfrak{g}$ is a Lie p–subalgebra. Let N be an irreducible $U_\chi(\mathfrak{h})$–module and denote by $\sigma : \mathfrak{h} \longrightarrow \mathfrak{gl}(N)$ the corresponding representation. In the previous section $\mathfrak{a}$ was an ideal in $\mathfrak{g}$ with $\chi([\mathfrak{a}, \mathfrak{a}]) = 0$. Now we will change our definition of $\mathfrak{a}$. In this section $\mathfrak{a} \subset \mathfrak{h}$ denotes a unipotent p–ideal with $\chi(\mathfrak{a}) = 0$. This implies that

$$\mathfrak{a} \cdot N = 0 \quad \text{or} \quad \sigma(\mathfrak{a}) = 0. \tag{6.8}$$

This follows from:

Lemma 6.3.1. *Let $(\mathfrak{h}, [p])$ be a restricted Lie algebra and $\mathfrak{a} \lhd \mathfrak{h}$ a p–ideal which is unipotent. If V is an irreducible $\mathfrak{h}$–module with p–character χ and $\chi(\mathfrak{a}) = 0$, then $\mathfrak{a} \cdot V = 0$.*

Proof. Set

$$V^{\mathfrak{a}} = \{x \in V \mid x \cdot v = 0 \ \ \forall x \in \mathfrak{a}\}.$$

Note that $V^{\mathfrak{a}} \subset V$ is a $\mathfrak{h}$–submodule, since $\mathfrak{a} \lhd \mathfrak{h}$ is an ideal. I claim that $V^{\mathfrak{a}}$ is nonzero. Indeed, since V is finite dimensional, it contains an irreducible restricted ($\chi(\mathfrak{a}) = 0$) $\mathfrak{a}$–module M. Since $\mathfrak{a}$ is unipotent M is isomorphic to the trivial module (see [14, 3.2]) and therefore contained $V^{\mathfrak{a}}$. So $V^{\mathfrak{a}}$ is a nonzero $U_\chi(\mathfrak{h})$–submodule of V; hence $V = V^{\mathfrak{a}}$ by irreducibility. □

In the following, set $\mathfrak{l} := [\mathfrak{g}, \mathfrak{a}]$.

Remark 6.3.2. If $\mathfrak{l} \subset \mathfrak{h}$, then $\mathfrak{l} \subset \mathfrak{h}$ and $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{h}$ are ideals. This follows from the definition of $\mathfrak{l}$ and the fact that $\mathfrak{a} \subset \mathfrak{h}$ is an ideal.

Now we have to make our assumptions:

Theorem 6.3.3. *Let N be an irreducible $U_\chi(\mathfrak{h})$–module. Suppose that $\mathfrak{l} = [\mathfrak{g}, \mathfrak{a}]$ has a basis $l_1, l_2, \ldots, l_k$ with $l_i^{[p]} = 0$ for all i and that the following conditions are satisfied:*

1) $\mathfrak{l}, [\mathfrak{g}, \mathfrak{l}] \subset \mathfrak{h}$,

2) $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{h}$ *is a unipotent p–ideal with* $\chi([\mathfrak{l}, \mathfrak{l}]) = 0$,

3) $\mathfrak{st}(\chi, \mathfrak{a}) = \mathfrak{h}$.

Then

$$1 \otimes N = \{w \in U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N \mid \mathfrak{a} \cdot w = 0\}. \tag{6.9}$$

Before I start with the proof, let me make some remarks.

Remark 6.3.4. Remark 6.3.2 and 1) show that $\mathfrak{l}, [\mathfrak{l}, \mathfrak{l}]$ are ideals inside $\mathfrak{h}$. If $\mathfrak{l}$ has a basis as in the theorem, then we can apply *Jacobson's formula* and show that $\mathfrak{l}, [\mathfrak{l}, \mathfrak{l}]$ are p–ideals inside $\mathfrak{h}$. Indeed, consider $x = a_1 l_1 + a_2 l_2 + \cdots + a_k l_k \in \mathfrak{l}$ and obtain $x^{[p]} \in [\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{l}$. By 2), we obtain that $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{h}$ is a unipotent p–ideal with $\chi([\mathfrak{l}, \mathfrak{l}]) = 0$. It follows that $[\mathfrak{l}, \mathfrak{l}] \cdot N = 0$, hence $\sigma([\mathfrak{l}, \mathfrak{l}]) = 0$.

Remark 6.3.5. For $1 \le i, j \le n$, each $[e_i, f_j]^{[p]}$ acts trivially on any irreducible $U_\chi(\mathfrak{h})$–module N. Indeed, $[e_i, f_j] \in \mathfrak{l}$ and can be written as $b_1 l_1 + b_2 l_2 + \cdots + b_k l_k$ for some $b_1, b_2, \ldots, b_k \in K$. The assumption on the basis elements of $\mathfrak{l}$ implies that $[e_i, f_j]^{[p]} \in [\mathfrak{l}, \mathfrak{l}]$. Now apply Remark 6.3.4.

Remark 6.3.6. We have defined $e_1, e_2, \ldots, e_n$ such that $\mathfrak{g} = (\oplus_{i=1}^n K e_i) \oplus \mathfrak{h}$. It now follows that $\mathfrak{st}(\chi, \mathfrak{a}) = \mathfrak{h}$ if and only if

$$\forall\, \varphi \in (\sum_{i=1}^n K e_i)^* \; \exists\, f \in \mathfrak{a} \,:\, \varphi(z) = \chi([z, f]) \quad \forall\, z \in \sum_{i=1}^n K e_i.$$

For the "if" part: Consider $\varphi_i \in (\sum_{i=1}^n K e_i)^*$ with $\varphi_i(e_j) = \delta_{ij}$ and choose $f_i \in \mathfrak{a}$ such that $\varphi_i(z) = \chi([z, f_i])$ for all $z \in \sum_{i=1}^n K e_i$. It follows that there exist $f_1, f_2, \ldots, f_n \in \mathfrak{a}$ such that $\chi([e_i, f_j]) = \delta_{ij}$ for $1 \le i, j \le n$. Now, let $y = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n + h \in \mathfrak{st}(\chi, \mathfrak{a})$ where $a_1, a_2, \ldots, a_n \in K$ and $h \in \mathfrak{h}$. Since $f_1, f_2, \ldots, f_n \in \mathfrak{a}$ and $\mathfrak{a} \lhd \mathfrak{h}$ with $\chi(\mathfrak{a}) = 0$, the relations $\chi([y, f_1]) = \chi([y, f_2]) = \cdots = \chi([y, f_n]) = 0$ imply that $a_1 = a_2 = \cdots = a_n = 0$; hence $y = h \in \mathfrak{h}$. So the stabilizer of χ in $\mathfrak{a}$ is contained in $\mathfrak{h}$. The other inclusion is obvious.

If $\mathfrak{st}(\chi, \mathfrak{a}) = \mathfrak{h}$, then consider the linear mapping:

$$\psi : K e_1 \oplus K e_2 \oplus \cdots \oplus K e_n \longrightarrow \mathfrak{a}^*, \quad x \longmapsto \chi([-, x]).$$

The assumption $\mathfrak{st}(\chi, \mathfrak{a}) = \mathfrak{h}$ says that ψ is injective and that $\psi(e_1), \psi(e_2), \ldots, \psi(e_n)$ are linearly independent. Hence there are $f_1, f_2, \ldots, f_n \in \mathfrak{a}$ with $\psi(e_i)(f_j) = \chi([e_i, f_j]) = \delta_{ij}$ for $1 \le i, j \le n$. Now consider any $\varphi \in (\sum_{i=1}^n K e_i)^*$ and assume that $\varphi(e_i) = r_i$ for some $r_i \in K$ and all $i = 1, 2, \ldots, n$. If $f = \sum_{i=1}^n r_i f_i \in \mathfrak{a}$ we have $\varphi(z) = \chi([z, f])$ for all $z \in \sum_{i=1}^n K e_i$ as required.

It follows that $\mathfrak{st}(\chi, \mathfrak{a}) = \mathfrak{h}$ if and only if there are $f_1, f_2, \ldots, f_n \in \mathfrak{a}$ with $\chi([e_i, f_j]) = \delta_{ij}$ for $1 \le i, j \le n$. So the existence of $f_1, f_2, \ldots, f_n \in \mathfrak{a}$ with $\chi([e_i, f_j]) = \delta_{ij}$ for $1 \le i, j \le n$ is equivalent to statement 3) in Theorem 6.3.3.

Now to the proof of Theorem 6.3.3.

Proof. Set $\mathfrak{b} := \mathfrak{a} + \{x \in \mathfrak{l} \mid \chi(x) = 0\}$. Note that $\mathfrak{b} \subset \mathfrak{h}$ is a Lie p–subalgebra. In order to show that $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{b}$, we only need to consider $x, y \in \mathfrak{l}$ with $\chi(x) = 0 = \chi(y)$: Apply Remark 6.3.2 and 2) to get $[x, y] \in \mathrm{Ker}(\chi) \cap \mathfrak{l}$. Next, consider $z = a_1 l_1 + a_2 l_2 + \cdots + a_k l_k \in \mathfrak{b}$ and use the assumption that $l_i^{[p]} = 0$ for all i to obtain $z^{[p]} \in [\mathfrak{l}, \mathfrak{l}] \subset \mathrm{Ker}(\chi) \cap \mathfrak{l} \subset \mathfrak{b}$. Observe that $\mathfrak{b}$ is unipotent also since $z^{[p]} \in \mathfrak{a} + [\mathfrak{l}, \mathfrak{l}]$ for all $z \in \mathfrak{b}$. Now, we can use 2) to get $z^{[p^r]} = 0$ for some $r > 0$. Therefore, $\chi(\mathfrak{b}) = 0$ implies:

$$\text{The only irreducible } U_\chi(\mathfrak{b})\text{–module is the trivial one dimensional module } K. \tag{6.10}$$

Let now J denote the subspace of $\mathrm{End}_K(N)$ spanned by all $\sigma(x_1)\sigma(x_2)\cdots\sigma(x_s)$ with $s \geq 1$ and all $x_i \in \mathfrak{b}$. Then J is obviously closed under multiplication; it is the associative algebra "without 1" generated by $\sigma(\mathfrak{b})$. Denote by J^m the span of all $u_1 u_2 \cdots u_m$ with all $u_i \in J$.

Lemma 6.3.7. *We have $J^m = 0$ for all $m \geq \dim(N)$.*

Proof. Choose a composition series

$$N = N_k \supset N_{k-1} \supset N_{k-2} \supset \cdots \supset N_1 \supset N_0 = 0$$

of N considered as a $U_\chi(\mathfrak{b})$–module. Now (6.10) implies that $\dim_K N_j / N_{j-1} = 1$ for all j (hence $k = \dim_K N$) and $\sigma(\mathfrak{b}) N_j \subset N_{j-1}$. It follows that $u N_j \subset N_{j-1}$ for all $u \in J$ and hence $J^m N_j \subset N_{j-m}$ for all j where we write $N_l = 0$ for $l < 0$. We get in particular, $J^k N = J^k N_k \subset N_0 = 0$, hence the claim. □

Set A equal to the associative algebra with 1 generated by $\sigma(\mathfrak{a} + \mathfrak{l})$. So this is the subspace of $\mathrm{End}_K(N)$ spanned by all $\sigma(y_1)\sigma(y_2)\cdots\sigma(y_s)$ with $s \geq 0$ and $y_i \in \mathfrak{a} + \mathfrak{l}$ (for $s = 0$ we pick up the identity). We have clearly $J \subset A$.

Lemma 6.3.8. *The algebra A is commutative. The ideal AJ in A satisfies $(AJ)^m = 0$ for all $m \geq \dim(N)$.*

Proof. We have $[\sigma(y_1), \sigma(y_2)] = \sigma([y_1, y_2])$ for all $y_1, y_2 \in \mathfrak{a} + \mathfrak{l}$. Since $[y_1, y_2] \in \mathfrak{a} + [\mathfrak{l}, \mathfrak{l}]$ we use (6.8) and Remark 6.3.4 to obtain $[\sigma(y_1), \sigma(y_2)] = \sigma([y_1, y_2]) = 0$. So all generators $\sigma(y)$ with $y \in \mathfrak{a} + \mathfrak{l}$ commute with each other and A is commutative. Now AJ is the span of all zu with $z \in A$ and $u \in J$. The commutativity of A implies that AJ is an ideal in A (a priori it is only a left ideal) and that $(AJ)^m = AJ^m$ for all m. Now apply Lemma 6.3.7. □

Now we are in position to finish the proof. We only have to prove the inclusion " $\supset$ " in (6.9) because of (6.8). Let w be a nonzero element from the right hand side in (6.9). Write $w = \sum_{\alpha \in I} e^\alpha \otimes w_\alpha$ with $w_\alpha \in N$. Let $q(w)$ denote the maximum of all $|\alpha|$ with $w_\alpha \neq 0$. The claim says that $q(w) = 0$ so let us assume that $q(w) > 0$ and get a contradiction. We will use Remark 6.3.6 in order to use assumption 3) in the theorem: So let $f_1, f_2, \ldots, f_n \in \mathfrak{a}$ such that $\chi([e_i, f_j]) = \delta_{ij}$ for $1 \leq i, j \leq n$. Now apply (6.3) in order to evaluate $f_i \cdot w$ for $i = 1, 2, \ldots, n$. Since $f_i \cdot w = 0$ and $f_i \cdot w_\alpha \in \mathfrak{a} \cdot N = 0$, we get from assumption 1):

$$0 \in \sum_{|\alpha| = q(w)} \sum_{j=1}^{n} \alpha_j e^{\alpha - \varepsilon_j} \otimes [e_j, f_i] w_\alpha + \sum_{|\beta| \leq q(w) - 2} e^\beta \otimes N \ \text{ for all } i = 1, 2, \ldots, n. \tag{6.11}$$

For each β with $|\beta| = q(w) - 1$ we have (use the direct sum in equation (6.2) together with (6.11))

$$\sum_{j=1}^{n} (\beta_j + 1)[e_j, f_i] w_{\beta+\varepsilon_j} = 0 \quad \text{for } i = 1, 2, \ldots, n. \tag{6.12}$$

The aim is to show that $w_{\beta+\varepsilon_j} = 0$ for all β and j with $\beta + \varepsilon_j \in I$. Then we get $w_\alpha = 0$ for all α with $|\alpha| = q(w)$, a contradiction to our definition of $q(w)$. Consider $\beta \in I$ with $|\beta| = q(w) - 1$ and define $I(\beta) := \{1 \le t \le n \mid \beta + \varepsilon_t \in I\}$. We need to show that $w_{\beta+\varepsilon_l} = 0$ for $l \in I(\beta)$. By Remark 6.3.5 each $[e_i, f_i]^p$ acts as multiplication on N by $\chi([e_i, f_i])^p$. Therefore the action of $z_i = \chi([e_i, f_i])^{-p} \sigma([e_i, f_i])^{p-1} \in A$ is inverse to that of $[e_i, f_i]$. Now (6.12) gives

$$w_{\beta+\varepsilon_i} = \sum_{j \ne i, j \in I(\beta)} (\beta_j + 1) z_i [e_j, f_i](w_{\beta+\varepsilon_j}) \qquad (*)$$

for all $i \in I(\beta)$. Consider a numbering of the elements $i_1 < i_2 < \cdots < i_s$ in $I(\beta)$. If we apply $(*)$ successive we get for all r with $1 \le r \le s$:

$$w_{\beta+\varepsilon_{i_r}} \in AJ(w_{\beta+\varepsilon_{i_r}}) + \sum_{j>r} AJ(w_{\beta+\varepsilon_{i_j}}). \tag{6.13}$$

In particular, we have $w_{\beta+\varepsilon_{i_s}} = u(w_{\beta+\varepsilon_{i_s}})$ for some $u \in AJ$. Hence $w_{\beta+\varepsilon_{i_s}} = u^m(w_{\beta+\varepsilon_{i_s}})$ for all $m > 0$. Now apply Lemma 6.3.8 and get $w_{\beta+\varepsilon_{i_s}} = 0$. Therefore $w_{\beta+\varepsilon_{i_{s-1}}} \in AJ(w_{\beta+\varepsilon_{i_{s-1}}})$ also and a similar calculation as before shows that $w_{\beta+\varepsilon_{i_{s-1}}} = 0$. Now, continue this process and get $w_{\beta+\varepsilon_l} = 0$ for all $l \in I(\beta)$. $\square$

Remark 6.3.9. The theorem shows that the isomorphism class of the $\mathfrak{g}$–module $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$ determines the isomorphism class of the $\mathfrak{h}$–module N. On the other hand, we get from (6.9) the simplicity of $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$: Any nonzero $\mathfrak{g}$–submodule M contains an irreducible $U_\chi(\mathfrak{a})$–module V. We get then $\mathfrak{a} \cdot V = 0$, hence $V \subset 1 \otimes N$ by (6.9), hence $M \cap (1 \otimes N) \ne 0$. This intersection is an $\mathfrak{h}$–submodule of $1 \otimes N$. So the simplicity of $1 \otimes N \simeq N$ implies that $1 \otimes N = M \cap (1 \otimes N) \subset M$, hence $U_\chi(\mathfrak{g})(1 \otimes N) \subset M$ and $M = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$.

For each $\mathfrak{g}$–module M set $M^{\mathfrak{a}} := \{w \in M \mid \mathfrak{a} \cdot w = 0\}$. It is easy to see that $M^{\mathfrak{a}}$ is a $U_\chi(\mathfrak{h})$–module since $\mathfrak{a} \lhd \mathfrak{h}$ with $\chi(\mathfrak{a}) = 0$. One could hope that the functors F and G defined by

$$G : \{U_\chi(\mathfrak{g}) - \text{modules}\} \longrightarrow \{U_\chi(\mathfrak{h}) - \text{modules}\}, \quad M \longmapsto M^{\mathfrak{a}}$$

and

$$F : \{U_\chi(\mathfrak{h}) - \text{modules}\} \longrightarrow \{U_\chi(\mathfrak{g}) - \text{modules}\}, \quad V \longmapsto U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} V$$

are inverse equivalence of categories. The hope turns out to be false: Take the regular $U_\chi(\mathfrak{h})$–module, then this is a free module over the local algebra $U_\chi(\mathfrak{a}) = U_0(\mathfrak{a})$, so that $\dim_K U_\chi(\mathfrak{h}) = \dim_K U_\chi(\mathfrak{h})^{\mathfrak{a}} \cdot p^{\dim_K \mathfrak{a}}$, whence $\mathfrak{a} = (0)$ and $\mathfrak{h} = \mathfrak{g}$. [In order to get $\dim_K U_\chi(\mathfrak{h}) = \dim_K U_\chi(\mathfrak{h})^{\mathfrak{a}} \cdot p^{\dim_K \mathfrak{a}}$, use that $U_0(\mathfrak{a})$ has a simple socle, hence $\dim_K U_0(\mathfrak{a})^{\mathfrak{a}} = 1$ since $\mathfrak{a}$ is unipotent. Now $U_\chi(\mathfrak{h}) \simeq U_0(\mathfrak{a})^c$ implies that $\dim_K U_\chi(\mathfrak{h})^{\mathfrak{a}} = c$.]

One could perhaps hope the following: Abusing notation, we denote by χ the linear form on $\mathfrak{h}/\mathfrak{a}$ induced by $\chi \in \mathfrak{g}^*$. Then G actually takes values in $U_\chi(\mathfrak{h}/\mathfrak{a})$, and defines a functor

$$G' : \{U_\chi(\mathfrak{g}) - \text{modules}\} \longrightarrow \{U_\chi(\mathfrak{h}/\mathfrak{a}) - \text{modules}\}.$$

By composing F with the pull-back along the projection $U_\chi(\mathfrak{h}) \twoheadrightarrow U_\chi(\mathfrak{h}/\mathfrak{a})$, we obtain a functor

$$F' : \{U_\chi(\mathfrak{h}/\mathfrak{a}) - \text{modules}\} \longrightarrow \{U_\chi(\mathfrak{g}) - \text{modules}\}.$$

Frobenius reciprocity implies that (F', G') is an adjoint pair, so that the front and rear adjunctions are candidates for the desired equivalences. In fact, if F' and G' are exact, then (6.3.3) and induction would yield such an assertion. Since F' is exact, the problem resides in the exactness of G', which is usually only left exact.

In fact, this hope turns out to be false also: Let $\mathfrak{g} := \mathfrak{sl}_2(K) \oplus L(1)$ be the semidirect product of $\mathfrak{sl}_2(K)$ and its two dimensional standard module $L(1)$. [Here $L(1)$ is the abelian restricted Lie algebra with bracket and p–mapping being zero.] Let $\{v_1, v_2\}$ be the standard basis of $L(1)$, $\{e, h, f\}$ the standard basis of $\mathfrak{sl}_2(K)$. Define $\chi \in \mathfrak{g}^*$ via $\chi(\mathfrak{sl}_2(K) = 0 = \chi(v_1)$ and $\chi(v_2) = 1$, so that $\mathfrak{a} := Kv_1$ and $\mathfrak{h} := \mathfrak{st}(\chi, \mathfrak{a}) = Kh \oplus Ke \oplus L(1)$. Then the conditions in Theorem 6.3.3 are fulfilled (with $\mathfrak{l} = L(1)$), and one would now hope for an equivalence of $U_\chi(\mathfrak{g})$–mod. and $U_\chi(Kh \oplus Ke \oplus Kv_2)$–mod.

However, take a look at the regular $U_\chi(\mathfrak{g})$–module $U_\chi(\mathfrak{g})$. We have

$$\dim_K U_\chi(\mathfrak{g})^{\mathfrak{a}} = p^{\dim_K \mathfrak{g}/\mathfrak{a}} = p^4$$

and if there is an equivalence, we should also have

$$U_\chi(\mathfrak{g}) \simeq U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} U_\chi(\mathfrak{g})^{\mathfrak{a}}. \tag{6.14}$$

It is not hard to see that

$$U_\chi(\mathfrak{g})^{\mathfrak{a}} = \bigoplus_{ijkl} K v_1^{p-1} v_2^i e^j h^k f^l.$$

Consider the module on the right hand side of (6.14): I claim that

$$\dim_K \big(U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} U_\chi(\mathfrak{g})^{\mathfrak{a}}\big)^{\mathfrak{a}} > \dim_K U_\chi(\mathfrak{g})^{\mathfrak{a}} \tag{6.15}$$

and thus an isomorphism as in (6.14) is impossible. In order to prove (6.15), take

$$x = f \otimes v_1^{p-1} v_2^{p-1}$$

and note that $v_1 \cdot x = f \otimes v_1^p v_2^{p-1} - 1 \otimes v_1^{p-1} v_2^p = 0$. It follows that

$$1 \otimes U_\chi(\mathfrak{g})^{\mathfrak{a}} \oplus Kx \subset \big(U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} U_\chi(\mathfrak{g})^{\mathfrak{a}}\big)^{\mathfrak{a}}$$

such that we have (6.15).

6.4 Criterion 3

Keep the notation from Section 6.1 but assume now that N is a finite dimensional $U_\chi(\mathfrak{h})$–module (we do not require irreducibility). For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$ set

$$e^\alpha = e_1^{\alpha_1} e_2^{\alpha_2} \cdots e_n^{\alpha_n}.$$

Let I denote the set of all $\alpha \in \mathbb{N}^n$ with $\alpha_i < p$ for all i. For each integer l with $1 \le l \le n+1$, set $I_l = \{\alpha \in I \mid \alpha_1 = \alpha_2 = \cdots = \alpha_{l-1} = 0\}$. It is easy to see that we have inclusions: $I_{n+1} = 0 \subset I_n \subset \cdots \subset I_2 \subset I_1 := I$. If N is a $U_\chi(\mathfrak{h})$–module we define

$$N_l = \bigoplus_{\alpha \in I_l} K e^\alpha \otimes N \subset U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N. \tag{6.16}$$

Proposition 6.4.1. *Suppose that there exists $l \le n$ and $f \in \mathfrak{g}$ such that either $[f, e_l]$ acts bijectively on $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$ and $(\mathrm{ad}\, e_l)^i(f) \cdot N_{l+1} \subset N_{l+1}$ for all i or $[f, e_l] = \mu e_l$ for some $\mu \in K^*$. If there exists $\lambda \in K$ such that $f \cdot y = \lambda y$ for all $y \in N_{l+1}$, then*

$$\{x \in N_l \mid f \cdot x = \lambda x\} = N_{l+1}.$$

Proof. The inclusion $\supset$ is clear by our assumption. Choose $x \in N_l$ such that $f \cdot x = \lambda x$. If $x \notin N_{l+1}$ we can find m with $0 < m < p$ such that

$$x = \sum_{i=0}^{m} e_l^i \cdot v_i \quad \text{where } v_i \in N_{l+1} \text{ for all } i \text{ and } v_m \neq 0. \quad (*)$$

First, suppose that $[f, e_l]$ acts bijectively on $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$ and that $(\mathrm{ad}\, e_l)^i(f) \cdot N_{l+1} \subset N_{l+1}$ for all i. Since $f \cdot x = \lambda x$ and $f \cdot v_i = \lambda v_i$ for all i we get:

$$\lambda x \in \lambda e_l^m \cdot v_m + \lambda e_l^{m-1} \cdot v_{m-1} + m e_l^{m-1} \cdot [f, e_l] \cdot v_m + \sum_{i \le m-2} e_l^i \cdot N_{l+1}.$$

Here we use our assumption that $(\mathrm{ad}\, e_l)^i(f) \cdot N_{l+1} \subset N_{l+1}$ for all i. It follows from the direct sum in (6.16) that $m e_l^{m-1} \cdot [f, e_l] \cdot v_m = 0$. This is a contradiction since $m \neq 0$ and $v_m \neq 0$ and since $[f, e_l]$ acts bijectively on $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N \supset N_{l+1}$.

Next, suppose that $[f, e_l] = \mu e_l$ for some $\mu \neq 0$. Consider x as in $(*)$ with $f \cdot x = \lambda x$: Since $f \cdot v_i = \lambda v_i$ for all i we get:

$$\lambda x \in (\lambda + \mu + m) e_l^m \cdot v_m + (\lambda + \mu + m - 1) e_l^{m-1} \cdot v_{m-1} + \sum_{i \le m-2} e_l^i \cdot N_{l+1}.$$

We conclude, by (6.16), that $\lambda = \lambda + \mu + m$ but also $\lambda = \lambda + \mu + m - 1$. This is a contradiction.

The proof is completed. ☐

Remark 6.4.2. Note that $[f, e_l]$ acts bijectively on $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$ if $\chi([f, e_l]) \neq 0$ and $[f, e_l]^{[p]} = [f, e_l]$ or $[f, e_l]^{[p]} = 0$: Suppose that $v \in U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$ such that $[f, e_l] \cdot v = 0$. Then $[f, e_l]^{[p]} \cdot v = 0$ also (since $[f, e_l]^{[p]} = [f, e_l]$ or $[f, e_l]^{[p]} = 0$). We conclude that $\chi([f, e_l])^p v = 0$; hence $v = 0$ since $\chi([f, e_l]) \neq 0$.

Corollary 6.4.3. *Suppose that there exist $f_1, f_2, \dots, f_n \in \mathfrak{g}$ such that each $[f_l, e_l]$ either acts bijectively on $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$ and $(\mathrm{ad}\, e_l)^i(f_l) \cdot N_{l+1} \subset N_{l+1}$ for all i or $[f_l, e_l] = \mu_l e_l$ for some $\mu_l \in K^*$. If, for each l, there exists $\lambda_l \in K$ such that $f_l \cdot y = \lambda_l y$ for all $y \in N_{l+1}$, then*

$$\{x \in U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N \mid f_i \cdot x = \lambda_i x \ \ \forall\, i\} = N_{n+1} = 1 \otimes N.$$

Proof. Apply Proposition 6.4.1 with our assumptions (n times). ☐

7 Induction from W_{012} to $W_{\geq 0}$

Now consider the second restricted Witt–Jacobson algebra $W = W(2)$ over an algebraically closed field K of characteristic $p > 0$. We want to apply the theory in Section 6.2 to $\mathfrak{g} = W_{\geq 0}$. Let $\chi \in W^*$ of height r. Recall the Lie p–subalgeba $W_{012} \subset W$ of codimension 3 defined in Section 3.1. We shall prove that there exists an automorphism g such that induction induces a bijection between the isomorphism classes of irreducible $U_{\chi^g}(W_{012})$–modules and the isomorphism classes of irreducible $U_{\chi^g}(W_{\geq 0})$–modules except possibly the case where $r = 2p - 3$ and χ has Type II.a as in Section 5.2.

Suppose that $r > 1$. Then $\mathfrak{a} = W_{\geq r-1}$ is an unipotent ideal in $\mathfrak{g}$ with $\chi([\mathfrak{a}, \mathfrak{a}]) \subset \chi(W_{\geq r}) = 0$. So χ defines a one–dimensional $\mathfrak{a}$–module K_χ. This is actually a $U_\chi(\mathfrak{a})$–module since at least all basis elements e_{ijk} of $\mathfrak{a}$ satisfy $e_{ijk}^{[p]} = 0$. It is in fact the only irreducible $U_\chi(\mathfrak{a})$–module since $\mathfrak{a}$ is unipotent (see [14, 3.3]). Finally, $\mathfrak{h}$ is the Lie p–subalgebra of $\mathfrak{g} = W_{\geq 0}$ given by $\{y \in W_{\geq 0} \mid \chi([y, x]) = 0 \text{ for all } x \in W_{\geq r-1}\}$.

7.1 The case that $r \neq p - 1$

Keep the notation from above. Let M be an irreducible $U_\chi(\mathfrak{g})$–module. Then M contains an irreducible $U_\chi(\mathfrak{a})$–module, which is a copy of K_χ. Hence $M^\chi = \{m \in M \mid x \cdot m = \chi(x)m \ \forall x \in \mathfrak{a}\}$ is nonzero and by Corollary 6.2.5 an irreducible $U_\chi(\mathfrak{h})$–module. Hence all irreducible $U_\chi(\mathfrak{g})$–modules have eigenvalue function χ. In a similar way (since $\mathfrak{a} \subset \mathfrak{h}$) we could prove that all irreducible $U_\chi(\mathfrak{h})$–modules have eigenvalue function χ; hence, by Theorem 6.2.8, the map $M \longmapsto M^\chi$ induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules and isomorphism classes of irreducible $U_\chi(\mathfrak{h})$–modules; the inverse is given by induction.

The definition of $\mathfrak{h}$ implies that

$$\mathfrak{h} = W_{\geq 1} \oplus \{y \in W_0 \mid \chi([y, W_{r-1}]) = 0\}. \tag{7.1}$$

Indeed, any $y \in W_{\geq 1}$ satisfies $[y, \mathfrak{a}] \subset W_{\geq r} \subset \mathrm{Ker}(\chi)$; if $y \in W_0$ then $\chi([y, W_{\geq r-1}]) = \chi([y, W_{r-1}])$ since $\chi([y, W_{\geq r}]) = 0$. If there exists $x \in W_{r-1}$ such that $\chi([x, e_{102}]) \neq 0 = \chi([x, W_{012}])$, then (7.1) shows that $\mathfrak{h} \subset W_{012}$. Therefore we have (in fact for arbitrary $r > 1$):

Lemma 7.1.1. *Suppose that $r > 1$. If there exists $x \in W_{r-1}$ such that $\chi([x, e_{102}]) \neq 0 = \chi([x, W_{012}])$, then induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{012})$–modules and the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules.*

Proof. Follows immediately from Remark 6.2.9 with $\mathfrak{g}' = W_{012}$ and $\mathfrak{g} = W_{\geq 0}$. □

It now follows from Lemma 5.4.1 that:

Proposition 7.1.2. *If $\chi \in W^*$ of height $r > 1$ and $r \neq p-1$, then there exists $g \in \mathrm{Aut}(W)$ such that induction induces a bijection between the isomorphism classes of irreducible $U_{\chi^g}(W_{012})$–modules and isomorphism classes of irreducible $U_{\chi^g}(W_{\geq 0})$–modules except possibly the case where $r = 2p - 3$ and χ has* Type II.a *as in* 5.2.

7.2 The case that $r = p - 1$

Keep the notation from above. We shall prove the statement in Proposition 7.1.2 in the case where $r = p - 1$. Note that $g(\mathfrak{h}) = \{y \in W_{\geq 0} \mid \chi^{g^{-1}}([y, W_{\geq r-1}]) = 0\}$ for $\mathfrak{h}$ is defined as in (7.1). We thus have to investigate the question: When does there exists $g \in \mathrm{Aut}(W)$ such that

$$g\big(\{y \in W_0 \mid \chi([y, W_{r-1}]) = 0\}\big) \subset W_{012}.$$

If such a g exists, then we can apply Remark 6.2.9 with $\chi^{g^{-1}}$ instead of χ, $g(\mathfrak{h})$ instead of $\mathfrak{h}$ and $\mathfrak{g}' = W_{012}$ and $\mathfrak{a} = W_{\geq r-1}$. The proof is then completed.

Clearly it is enough to consider $g \in GL_2(K)$. We have an isomorphism $W_0 \simeq \mathfrak{gl}_2(K)$:

$$ae_{101} + be_{102} + ce_{011} + de_{012} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{7.2}$$

for $a, b, c, d \in K$. Since the $GL_2(K)$–action is compatible with this isomorphism we arrive the question: For which Lie subalgebras $\mathfrak{s}$ of $\mathfrak{gl}_2(K)$ does there exists $g \in GL_2(K)$ such that

$$g \cdot \mathfrak{s} \cdot g^{-1} \subset \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}. \qquad (\sharp)$$

We will apply our discussion on $\mathfrak{s} = \{y \in W_0 \mid \chi([y, W_{r-1}]) = 0\}$. If $\mathfrak{s}$ is solvable it follows from [24, Satz 3] that the dimension of each irreducible $\mathfrak{s}$–module is a power of p. Therefore each irreducible submodule of the tautological representation on K^2 has dimension 1 (recall that $p > 2$ since $r = p - 1 > 1$). Pick such an irreducible submodule Kx_2 and extend to a basis x_1, x_2 for K^2. Then each $y \in \mathfrak{s}$ has a lower triangular matrix with respect to x_1, x_2. With $g =$ right base change, we have $(\sharp)$.[1]

This leaves us with the case where $\mathfrak{s}$ is not solvable. In particular, we have $\dim_K \mathfrak{s} \geq 3$. If $\dim_K \mathfrak{s} = 4$, then $\mathfrak{s} = \mathfrak{gl}_2(K)$ and nothing can be done [but this case is not of our interest since $\mathfrak{s} = \mathfrak{gl}_2(K)$ corresponds to $W_0 = \{y \in W_0 \mid \chi([y, W_{r-1}]) = 0\}$ via the isomorphism in (7.2) – contradiction since $[W_0, W_{r-1}] = W_{r-1}$].

If $\dim_K \mathfrak{s} = 3$ and $\dim_K([\mathfrak{s}, \mathfrak{s}]) < 3$ then $[\mathfrak{s}, \mathfrak{s}]$ is solvable, hence so is $\mathfrak{s}$ — contradiction. We can thus assume that $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}]$ and that $\dim_K \mathfrak{s} = 3$. By dimension comparison this yields $\mathfrak{s} = \mathfrak{sl}_2(K)$. So altogether there exists $g \in GL_2(K)$ such that $(\sharp)$ above is satisfied unless $\mathfrak{s} \supset \mathfrak{sl}_2(K)$. Going back to W_0, we see that the bad case occurs when

$$\mathfrak{g}_1 := Ke_{102} + K(e_{012} - e_{101}) + Ke_{011} \subset \{y \in W_0 \mid \chi([y, W_{r-1}]) = 0\}. \tag{7.3}$$

Set $W'_{r-1} = W_{r-1} \cap \mathrm{Ker}(\chi)$. This is a one codimensional subspace in W_{r-1} and the inclusion in (7.3) is equivalent to $[y, W_{r-1}] \subset W'_{r-1}$ for all $y \in \mathfrak{g}_1$. If we regard W_{r-1} as a module for $\mathfrak{g}_1 \simeq \mathfrak{sl}_2(K)$ via ad, then W'_{r-1} has to be a submodule and W_{r-1}/W'_{r-1} a trivial one dimensional module. Now, let $h = e_{012} - e_{101}$. Then $\mathrm{ad}(h)$ acts diagonalisably on W_{r-1}, hence also on W'_{r-1}. Since $\mathrm{ad}(h)$ acts trivially on W_{r-1}/W'_{r-1}, we see that W'_{r-1} contains all eigenspaces for $\mathrm{ad}(h)$ in W_{r-1} corresponding to nonzero eigenvalues. In the case $r = p - 1$ this means that W'_{r-1} contains all e_{ijk} with $i + j = p - 1$ except possibly for $e_{0,p-1,2}$ and $e_{0,p-1,1}$. But now $[e_{011}, e_{1,p-2,1}] = e_{0,p-1,1}$ and $[e_{102}, e_{p-1,0,1}] = -e_{p-1,0,2}$ show that $W'_{r-1} = W_{r-1}$ — contradiction. So the inclusion in (7.3) is impossible.

Lemma 7.2.1. *Suppose that $\chi \in W^*$ has height $r = p - 1$. Then there exists $g \in \mathrm{Aut}(W)$ and $x \in W_{r-1}$ such that $\chi^g([x, e_{102}]) \neq 0 = \chi^g([x, W_{012}])$.*

Proof. The discussion above says that we can find an automorphism g such that

$$\{y \in W_{\geq 0} \mid \chi^g([y, W_{\geq r-1}]) = 0\} \subset W_{012}.$$

[1] Since Lie's Theorem [11, p. 16] holds whenever the dimension of the relevant module is $< p$, one can also use this result to obtain the desired presentation.

Set $\mathfrak{h} := \{y \in W_{\geq 0} \mid \chi^g([y, W_{\geq r-1}]) = 0\}$. Let $e_1, e_2, \dots, e_n$ be a cobasis for $\mathfrak{h}$ in $W_{\geq 0}$ such that $e_1 = e_{102}$ and $e_i \in W_{012}$ for $i > 1$. The bilinear form $B_{\chi^g}(x, y) := \chi^g([x, y])$ defines a linear mapping

$$\varphi : U := \sum_{i=1}^n Ke_i \quad \longrightarrow \quad W^*_{\geq r-1}$$

given by $\varphi(z) = \chi^g([z, -])$. The assumption on $e_1, e_2, \dots, e_n$ as a cobasis for $\mathfrak{h}$ in $W_{\geq 0}$ implies that φ is injective and the linear functionals $\varphi(e_1), \varphi(e_2) \dots, \varphi(e_n)$ are linear independent. Hence there are $f_1, f_2, \dots, f_n \in W_{\geq r-1}$ with $\varphi(e_i)(f_j) = \delta_{ij}$, $1 \leq i, j \leq n$. It follows that $\chi^g([e_{102}, f_1]) = 1$ $(e_1 = e_{102})$. Moreover, any $z \in W_{012}$ can written as $z = \sum_{i>1} a_i e_i + h$ for some $a_2, a_3, \dots, a_n \in K$ and some $h \in \mathfrak{h}$ (note that $W_{012} = \sum_{i>1} Ke_i \oplus \mathfrak{h}$). Therefore, $\chi^g([f_1, z]) = 0$ since $\chi^g([e_i, f_1]) = 0$ for $i > 1$ and $\chi^g([\mathfrak{h}, f_1]) = 0$. The proof is completed by setting $x = f_1$. □

Proposition 7.2.2. *Suppose that $\chi \in W^*$ has height $r = p - 1$. Then there exists $g \in$* Aut(W) *such that induction induces a bijection between the isomorphism classes of irreducible $U_{\chi^g}(W_{012})$–modules and the isomorphism classes of irreducible $U_{\chi^g}(W_{\geq 0})$–modules.*

Proof. Follows immediately from Lemma 7.1.1 and Lemma 7.2.1. □

7.3 Arbitrary r

If we combine the results in Lemma 5.4.1, 7.2.1 and Proposition 7.1.2, 7.2.2, we obtain:

Lemma 7.3.1. *Let $\chi \in W^*$ of height $r \geq 0$. Then there exists $g \in$* Aut(W) *and $x \in W_{r-1}$ such that $\chi^g([x, e_{102}]) \neq 0 = \chi^g([x, W_{012}])$ except the case where $r = 2p - 3$ and χ has* Type II.a *as in* 5.2.

Theorem 7.3.2. *Suppose that $\chi \in W^*$ with height $r > 1$. Then there exists $g \in$* Aut(W) *such that induction induces a bijection between the isomorphism classes of irreducible $U_{\chi^g}(W_{012})$–modules and the isomorphism classes of irreducible $U_{\chi^g}(W_{\geq 0})$–modules except possibly the case where $r = 2p - 3$ and χ has* Type II.a *as in* 5.2.

Remark 7.3.3. Let $\chi \in W^*$ be a character of height $r > 1$ such that $r \neq 2p - 3$ if χ has Type II.a as in 5.2. Let g be as above. Then one can show that each irreducible $U_{\chi^g}(W_{012})$–module is induced from a one dimensional $U_{\chi^g}(P)$–module where $P \subset W_{\geq 0}$ is a polarization of some $\lambda \in W^*_{\geq 0}$ and the number of irreducible $U_{\chi^g}(W_{012})$–modules is equal to the number of irreducible $U_{\chi^g}(P)$–modules. Here we apply a result proved in Lemma 9.4.1 saying that W_{012} is supersolvable. [In Section 9 we take a closer look at supersolvable Lie algebras and one can check that the statements above follow from Proposition 9.3.5 and Lemma 9.3.2, 9.3.7.] It follows that induction induces a bijection between the isomorphism classes of irreducible $U_{\chi^g}(P)$–modules and the isomorphism classes of irreducible $U_{\chi^g}(W_{\geq 0})$–modules. This implies that induction induces a bijection between the irreducible $U_\chi(g(P))$–modules and the irreducible $U_\chi(W_{\geq 0})$–modules and $g(P) \subset W_{\geq 0}$ is a polarization of $\lambda^{g^{-1}} \in W^*_{\geq 0}$. This result is obtained in [29, 10.16] also but the statement there only says that induction induces a surjection.

Remark 7.3.4. The bad case for induction from W_{012} to $W_{\geq 0}$ is the situation where $r = 2p-3$ and χ has Type II.a as in 5.2. The hope was that Theorem 7.3.2 could be improved in the following way: If $\chi \in W^*$ of Type II.a as in 5.2 with height $r = 2p-3$ then there exists an automorphism g such that induction is a bijection between the isomorphism classes of irreducible $U_{\chi^g}(W_{012})$–modules and the isomorphism classes of irreducible $U_{\chi^g}(W_{\geq 0})$–modules. But the example in Section 13.13 shows that it turns out to be a false hope. This is indicated in [29, p. 80] also but some of the arguments leading to [29, Satz 10.16] are suspicious. We will discuss an example for $p = 3$ in Section 13.13.

8 Induction from $W_{\geq 0}$ to W

In this section we will apply the results in Section 6.3 to $\mathfrak{g} = W$ and $\mathfrak{h} = W_{\geq 0}$. Let $\chi \in W^*$ be a character of height $r > 1$ but $r \leq 2p - 3$. Clearly, $\mathfrak{a} := W_{\geq r} \lhd \mathfrak{h}$ is a unipotent p–ideal with $\chi(W_{\geq r}) = 0$; hence $W_{\geq r}$ acts trivially on every irreducible $U_\chi(\mathfrak{h})$–module. Note that $\mathfrak{l} := [W, W_{\geq r}] = W_{\geq r-1} \subset \mathfrak{h}$ has a basis $l_1, l_2, \ldots, l_k$ such that $l_i^{[p]} = 0$ for all i and $[W, \mathfrak{l}] = W_{\geq r-2} \subset \mathfrak{h}$. Moreover, $[\mathfrak{l}, \mathfrak{l}] \subset W_{\geq 2r-2} \subset W_{\geq r} \subset \mathfrak{h}$ is a unipotent p–subalgebra with $\chi([\mathfrak{l}, \mathfrak{l}]) = 0$. If $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$, then we can apply Theorem 6.3.3 and Corollary **??**: Induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and isomorphism classes of irreducible $U_\chi(W)$–modules.

8.1 Good induction

We have shown:

Theorem 8.1.1. *Let $\chi \in W^*$ be a character of height $r > 1$. If $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$, then induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and the isomorphism classes of irreducible $U_\chi(W)$–modules.*

In order to use Theorem 8.1.1 the following remark is important:

Remark 8.1.2. The result in Theorem 8.1.1 is true for χ if and only if it is true for χ^g, where $g \in \mathrm{Aut}(W)$. This follows from the fact that $g(W_{\geq 0}) = W_{\geq 0}$.

Lemma 8.1.3. *Let $\chi \in W^*$ with height $1 < r \leq p - 2$. Then the induction functor induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and the isomorphism classes of irreducible $U_\chi(W)$–modules.*

Proof. By Theorem 8.1.1 it is enough to prove that $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$. The assumption $r \leq p-2$ implies that $\chi(e_{\alpha\beta\gamma}) = 0$ for $\alpha + \beta \geq p - 1$. So there exists an index (i, j, k) with $i < p - 1$ maximal such that $\chi(e_{ijk}) \neq 0$ and an index (i', j', k') with $j' < p - 1$ maximal such that $\chi(e_{i'j'k'}) \neq 0$. Now define $e_i = e_{00i}$, $f_1 = e_{i+1,j,k}$ and $f_2 = e_{i',j'+1,k'}$ and observe:

$$\begin{aligned} \chi([e_1, f_1]) = (i+1)\chi(e_{ijk}) \quad &\neq \quad 0 = i'\chi(e_{i'-1,j'+1,k'}) = \chi([e_1, f_2]), \\ \chi([e_2, f_2]) = (j'+1)\chi(e_{i',j',k'}) \quad &\neq \quad 0 = j\chi(e_{i+1,j-1,k}) = \chi([e_2, f_1]). \end{aligned}$$

It follows that $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$. $\square$

Remark 8.1.4. The results obtained in Theorem 8.1.1 and Lemma 8.1.3 are also proved in [29, 11.1], but the proof there only says that induction induces a surjection in the case where $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$ (T.Wichers uses the notation $W_{\geq r}^{\perp}$ for $\mathfrak{st}(\chi, W_{\geq r})$).

If $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$, then the classification and dimension formulas for the irreducible $U_\chi(W)$–modules are given in terms of the corresponding data for the irreducible $U_\chi(W_{\geq 0})$–modules (see Theorem 8.1.1). Except for an exceptional case (see Theorem 7.3.2), there exists $g \in GL_2(K)$ such that induction induces a bijection between the isomorphism classes of irreducible $U_{\chi^g}(W_{012})$–modules and isomorphism classes of irreducible $U_{\chi^g}(W_{\geq 0})$–modules. Therefore: The classification and dimension formulas for the irreducible $U_\chi(W)$–modules are now reduced to the study of W_{012}–modules with p–character χ^g for a suitable $g \in GL_2(K)$. Since W_{012} is a supersolvable Lie p–subalgebra of W this reduction turns out to be very useful. Let me summarize: If $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$ and we exclude the case where $r = 2p - 3$ and χ has Type II.a, then there exists $g \in GL_2(K)$ such that induction induces a bijection between the isomorphism classes of irreducible $U_{\chi^g}(W_{012})$–modules and isomorphism classes of irreducible $U_{\chi^g}(W)$–modules.

Thus: We will study supersolvable Lie algebras a little closer.

9 Supersolvable Lie-algebras

A finite dimensional Lie algebra L over K is called *supersolvable* if there exists a chain

$$0 = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_n = L \tag{9.1}$$

of ideals in L such that the factor algebras L_j/L_{j-1} are one–dimensional for integers j with $1 \leq j \leq n$. It is clear that subalgebras and factor algebras of supersolvable Lie algebras are again supersolvable.

9.1 Restriction

In the rest of this section we restrict ourselves to supersolvable restricted Lie algebras L over K such that L is a direct sum of a torus T (i.e., a commutative Lie p-subalgebra with basis $h_1, h_2, \ldots, h_l$ such that $h_i^{[p]} = h_i$ for all i) and a p–nilpotent ideal U in L (i.e., $\forall x \in U \; \exists \, s > 0 : x^{[p^s]} = 0$): We assume that $L = T \oplus U$. It follows that any restricted Lie subalgebra of L can decomposed in that way also.

Lemma 9.1.1. *Let $L' \subset L$ be a restricted Lie subalgebra. Then there exists a (maximal) torus $T' \subset L'$ such that $L' = T' \oplus (L' \cap U)$.*

Proof. First, note that we have an isomorphism $L'/(L' \cap U) \simeq (L' + U)/U$ of restricted Lie algebras and an inclusion $(L' + U)/U \subset L/U$. Since $L/U \simeq T$ is a torus and any restricted subalgebra of a torus is again a torus, we conclude that $L'/(L' \cap U)$ is a torus. Now apply [27, 2, Lemma 4.4 (2)] to find torus $T' \subset L'$ such that $L' = T' + (L' \cap U)$ (one should check that the definition of a torus given in [27, 2] is equivalent to the definition given just before the lemma, see [27, 2, Theorem 3.6]). Clearly $T' \cap (L' \cap U) = 0$ and thus we have $L' = T' \oplus (L' \cap U)$ as required. □

For any restricted L–module $V \neq 0$ the subspace

$$V^U = \{v \in V \mid x \cdot v = 0 \; \forall \, x \in U\}$$

of fixed points of U is nonzero. Indeed, since V is finite dimensional, it contains an irreducible restricted U–module M. Since U is unipotent, by assumption, it follows from [14, 3.2] that M is isomorphic to the trivial module and therefore contained in V^U. Moreover the set of fixed points of U is a L–submodule of V since U is an ideal in L. If V is irreducible we then conclude that $V = V^U$. Therefore, if V is irreducible, then V is an irreducible module for $L/U \simeq T$. Since T is commutative, this implies $\dim_K V = 1$. We have shown:

Lemma 9.1.2. *All restricted irreducible L–modules are one dimensional.*

Let me now consider $U_\chi(L)$–modules for an arbitrary $\chi \in L^*$. Each linear form $\lambda \in L^*$ with $\lambda(U) = 0$ defines a one dimensional L–module K_λ where each $x \in T$ acts as multiplication with $\lambda(x)$ and where each $y \in U$ acts as $\lambda(y) = 0$. Note that K_λ is restricted if and only if $\lambda(h^{[p]}) = \lambda(h)^p$ for all $h \in T$. We do not have to worry about U, since it is a p–ideal.

Lemma 9.1.3. *Let E be any irreducible $U_\chi(L)$–module. For $\lambda \in L^*$ with $\lambda(U) = 0$ and $\lambda(h^{[p]}) = \lambda(h)^p \; \forall \, h \in T$, each $E \otimes_K K_\lambda$ is an irreducible $U_\chi(L)$–module. Any irreducible $U_\chi(L)$–module is isomorphic to one of these $E \otimes_K K_\lambda$.*

Proof. Let now V_1, V_2 be irreducible $U_\chi(L)$–modules. Define the dual module V_1^* via $(x.f)(m) = -f(x.m)$. It follows that V_1^* has p–character $-\chi$. Moreover $V_1^* \otimes_K V_2$ becomes an L–module via $x.(f \otimes v_2) = x.f \otimes v_2 + f \otimes x.v_2$ with p–character $-\chi + \chi = 0$. Then $\mathrm{Hom}_K(V_1, V_2) \simeq V_1^* \otimes_K V_2$ is a non-zero $U_0(L)$–module. Since $\mathrm{Hom}_K(V_1, V_2)$ is finite dimensional, it contains an irreducible p–representation isomorphic to some K_λ, where $\lambda(h^{[p]}) = \lambda(h)^p$ for $h \in T$ and $\lambda(x) = 0$ for $x \in U$ (since U is unipotent it acts nilpotently on each restricted U–module). Then the trivial one-dimensional L–module isomorphic to $K_\lambda^* \otimes K_\lambda$ is contained in

$$K_\lambda^* \otimes_K V_1^* \otimes_K V_2 \simeq \mathrm{Hom}_K(V_1 \otimes_K K_\lambda, V_2).$$

In other words, we have $\mathrm{Hom}_K(V_1 \otimes_K K_\lambda, V_2) \neq 0$. Observe that $V_1 \otimes_K K_\lambda$ is an irreducible $U_\chi(L)$–module (V_1 is irreducible). By Schur's Lemma the proof is completed. □

Let E be an irreducible $U_\chi(L)$–module. Set

$$L^*_{\mathbb{F}_p} = \{\lambda \in L^* \mid \lambda(U) = 0, \lambda(h^{[p]}) = \lambda(h)^p \ \forall\, h \in T\}.$$

If $l := \dim_K T$ then T has a basis $h_1, h_2, \ldots, h_l$ with $h_i^{[p]} = h_i$ for all i. Let $\lambda \in L^*$ and note that $\lambda \in L^*_{\mathbb{F}_p}$ if and only if $\lambda(U) = 0$ and $\lambda(h_i) \in \mathbb{F}_p$ for all $i = 1, 2, \ldots, l$. Hence $|L^*_{\mathbb{F}_p}| = p^l$. The set of $\lambda \in L^*_{\mathbb{F}_p}$ with $E \otimes_K K_\lambda \simeq E$ form an $\mathbb{F}_p$–subspace of $L^*_{\mathbb{F}_p}$. If this subspace has dimension m then there are p^{l-m} isomorphism classes of irreducible $U_\chi(L)$–modules.

Remark 9.1.4. If L is supersolvable and all irreducible $U_\chi(L)$–modules are one dimensional, then there are p^l isomorphism classes of irreducible $U_\chi(L)$–modules, where l is the dimension of any maximal torus in L. Indeed, use the observations just above with $E = K_\mu$ for some $\mu \in L^*$ to get $m = 0$ (m is the dimension of the subspace of $L^*_{\mathbb{F}_p}$ consisting of all $\lambda \in L^*_{\mathbb{F}_p}$ with $K_\mu \otimes_K K_\lambda \simeq K_\mu$ and it is easy to see that only $\lambda = 0$ has that property). In fact, if there exists an irreducible $U_\chi(L)$–module of dimension one, then all irreducible $U_\chi(L)$–modules are one dimensional by Lemma 9.1.3.

For any $U_\chi(L)$–module V denote by $P(V)$ its projective cover in the category of $U_\chi(L)$–modules. We have for E and λ as above

$$P(E \otimes_K K_\lambda) \simeq P(E) \otimes_K K_\lambda.$$

See [6, Lemma 1]. Each $P(E \otimes_K K_\lambda)$ occurs $\dim_K E \otimes_K K_\lambda = \dim_K E$ times in a direct sum decomposition of $U_\chi(L)$ into indecomposables. This induces

$$p^{l-m} \dim_K E \cdot \dim_K P(E) = p^{\dim_K U_\chi(L)}. \tag{9.2}$$

In particular, both $\dim_K E$ and $\dim_K P(E)$ are powers of p.

Lemma 9.1.5. *Let $\chi \in L^*$, let $H \subset L$ be a Lie p–subalgebra of codimension 1.*

a) If M is an irreducible $U_\chi(H)$–module, then either M can be extended to a $U_\chi(L)$–module or $U_\chi(L) \otimes_{U_\chi(H)} M$ is an irreducible $U_\chi(L)$–module.

b) If V is an irreducible $U_\chi(L)$–module and $M \subset V$ is an irreducible $U_\chi(H)$–submodule, then either $M = V$ or V is isomorphic to $U_\chi(L) \otimes_{U_\chi(H)} M$.

Proof. a) There exists an irreducible $U_\chi(L)$–module V with a surjective homomorphism

$$\pi : U_\chi(L) \otimes_{U_\chi(H)} M \longrightarrow V$$

of $U_\chi(L)$–modules. This map is nonzero on $1 \otimes M$ since this subspace generates the induced module over $U_\chi(L)$. Since $1 \otimes M \simeq M$ is an irreducible $U_\chi(H)$–module it has to be mapped injectively into V. This implies that

$$\dim_K M \leq \dim_K V \leq \dim_K U_\chi(L) \otimes_{U_\chi(H)} M = p\dim_K M.$$

It follows from (9.2) that both $\dim_K V$ and $\dim_K M$ are powers of p. So we have either $\dim_K V = \dim_K M$ or $\dim_K V = p \cdot \dim_K M$. In the second case the induced module is irreducible. In the first case V is isomorphic to M as a $U_\chi(H)$–module. In that case we can extend M to a $U_\chi(L)$–module.

b) The inclusion of M into V induces a homomorphism of L–modules

$$U_\chi(L) \otimes_{U_\chi(H)} M \longrightarrow V \tag{9.3}$$

that has to be surjective. If $M \neq V$ then $\dim_K V > \dim_K M$. Since $\dim_K V$ and $\dim_K M$ are powers of p, by (9.2), this implies that $\dim_K V \geq p \cdot \dim_K M$. In that case (9.3) is an isomorphism. If $M = V$, then (9.3) shows that the induced module is not irreducible. □

Let $P \subset L$ be a Lie subalgebra of L. Then any $\lambda \in L^*$ with $\lambda([P, P]) = 0$ defines a one dimensional P–module K_λ where each $x \in P$ acts as multiplication with $\lambda(x)$. Let $\chi \in L^*$ be a linear form such that K_λ is a $U_\chi(P)$–module, i.e, with

$$\chi(x)^p = \lambda(x)^p - \lambda(x^{[p]}) \tag{9.4}$$

We can then define the induced module

$$U_\chi(L) \otimes_{U_\chi(P)} K_\lambda. \tag{9.5}$$

The annihilator of the module in (9.5) is an ideal in L contained in P, in fact in the kernel of $\lambda_{|P} \in P^*$. Indeed, if $x \in L$ with $x \notin P$, then $x(1 \otimes 1) = x \otimes 1$ is a nonzero element in the induced module [choose a basis for a complement to P in L containing x and apply the PBW–theorem for reduced enveloping algebras]. If $x \in P$ then $x(1 \otimes 1) = 1 \otimes \lambda(x)$ is zero if and only if $\lambda(x) = 0$. On the other hand we also have:

Lemma 9.1.6. *Let $A \subset P$ be an ideal in L with $\lambda(A) = 0$. Then A annihilates the module in* (9.5).

Proof. The set of v in the module with $Av = 0$ is a L–submodule, since A is an ideal in L. It contains $1 \otimes 1$ since $\lambda(A) = 0$. Thus it contains $U_\chi(L)(1 \otimes 1)$ which is the entire module. □

Consider the set B of all $x \in L$ that act on the module in (9.5) as a scalar. This set is an ideal in L. Consider $x \in B$ and look at $x(1 \otimes 1)$. If $x \notin P$ then $x(1 \otimes 1) = x \otimes 1 \notin K(1 \otimes 1)$; Thus $B \subset P$. Then $x(1 \otimes 1) = \lambda(x)(1 \otimes 1)$ and it follows that each $x \in B$ acts as the scalar $\lambda(x)$. So for all $y \in L$ and $x \in B$ the actions of x and y on the module commute. Therefore the commutator $[y, x]$ annihilates the module. This implies $\lambda([L, B]) = 0$ as we have seen above. On the other hand:

Lemma 9.1.7. *Let $B \subset P$ be an ideal in L with $\lambda([L, B]) = 0$. Then each $x \in B$ acts as multiplication by $\lambda(x)$ on the module in* (9.5).

Proof. Denote that module by V and set $A = B \cap \mathrm{Ker}(\lambda)$. We have $[L, B] \subset A$. It follows that $A \subset P$ is an ideal in L with $\lambda(A) = 0$. By Lemma 9.1.6 we get $A \cdot V = 0$. Set

$$V^\lambda = \{v \in V \mid xv = \lambda(x)v \text{ for all } x \in B\}.$$

Clearly, $1 \otimes 1 \in V^\lambda$. Consider $v \in V^\lambda$ and $x \in L$. Then we get for all $y \in B$:

$$y(xv) = [y, x]v + x(yv) = 0 + \lambda(y)xv$$

since $[y, x] \in A$ and $A \cdot V = 0$. Thus V^λ is a L–submodule of V and it contains $1 \otimes 1$; hence $V^\lambda = V$. $\square$

9.2 Polarizations

The further development requires the notion of a polarization. We will do that in a general setup; so let $\mathfrak{g}$ be a Lie algebra (not necessarily supersolvable) defined over an arbitrary field F. If $\lambda \in \mathfrak{g}^*$, then a Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called a polarization of λ, if $\mathfrak{p}$ is a maximal totally isotropic subspace with respect to the alternating form b_λ on $\mathfrak{g}$ given by $b_\lambda(x, y) = \lambda([x, y])$ for $x, y \in \mathfrak{g}$. As a consequence we have

$$\dim_F \mathfrak{p} = \frac{\dim_F \mathfrak{g} + \dim_F \mathfrak{c}_\mathfrak{g}(\lambda)}{2} \tag{9.6}$$

where $\mathfrak{c}_\mathfrak{g}(\lambda) = \{x \in \mathfrak{g} \mid \lambda([x, y]) = 0 \text{ for all } y \in \mathfrak{g}\}$ denotes the stabilizer of λ in $\mathfrak{g}$. Note that $\mathfrak{c}_\mathfrak{g}(\lambda)$ is the radical of the skew–symmetric bilinear form $(x, y) \longmapsto \lambda([x, y])$ on $\mathfrak{g}$; hence $\mathfrak{c}_\mathfrak{g}(\lambda)$ is a Lie p–subalgebra of $\mathfrak{g}$ and its codimension in $\mathfrak{g}$ is even.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra of $\mathfrak{g}$ of codimension 1. Set $\lambda' = \lambda_{|\mathfrak{h}}$ and define $\mathfrak{c}_\mathfrak{h}(\lambda') = \{x \in \mathfrak{h} \mid \lambda'([x, y]) = 0 \text{ for all } y \in \mathfrak{h}\}$. We are now in one of the two following cases:

1) We have $\mathfrak{c}_\mathfrak{g}(\lambda) \subset \mathfrak{h}$. Then $\mathfrak{c}_\mathfrak{g}(\lambda)$ is a subspace of codimension 1 in $\mathfrak{c}_\mathfrak{h}(\lambda')$. Each polarization $\mathfrak{p} \subset \mathfrak{h}$ of λ' is also a polarization of λ.

2) We have $\mathfrak{c}_\mathfrak{g}(\lambda) \not\subset \mathfrak{h}$. Then $\mathfrak{c}_\mathfrak{h}(\lambda') = \mathfrak{c}_\mathfrak{g}(\lambda) \cap \mathfrak{h}$. If $\mathfrak{p} \subset \mathfrak{h}$ is a polarization of λ', then we can find a maximal totally isotropic subspace for b_λ that contains $\mathfrak{p}$ as a subspace of codimension 1.

This is proved in [5, 1.12.2]. The general characteristic zero assumption of that book is not needed here.

Finally: Suppose that $\mathfrak{g}$ is a Lie p–algebra and that $\mathfrak{p}$ is a polarization of some $\lambda \in \mathfrak{g}^*$. Then $\mathfrak{p}$ is a Lie p–subalgebra of $\mathfrak{g}$. [Let $x \in \mathfrak{p}$. We have for all $y \in \mathfrak{p}$:

$$B_\lambda(x^{[p]}, y) = \lambda\big((\operatorname{ad} x^{[p]})(y)\big) = \lambda\big((\operatorname{ad} x)^p(y)\big) = B_\lambda\big(x, (\operatorname{ad} x)^{p-1}(y)\big) = 0$$

since $(\operatorname{ad} x)^{p-1}(y) \in \mathfrak{p}$. Therefore the subspace $\mathfrak{p} + Kx^{[p]}$ of $\mathfrak{g}$ is totally isotropic. By maximality, $x^{[p]} \in \mathfrak{p}$].

9.3 Vergne Polarization

Let us again consider supersolvable restricted Lie algebras over K (K is an algebraically closed field of characteristic $p > 0$): We assume that L is supersolvable Lie algebra such that L is a direct sum of a torus T and a p–nilpotent ideal U in L.

Definition 9.3.1. Let $\chi \in L^*$ and $\lambda \in L^*$. A polarization P of λ is said to be *compatible with* χ if $\lambda(x)^p - \lambda(x^{[p]}) = \chi(x)^p$ for all $x \in P$.

Lemma 9.3.2. *Let $\chi \in L^*$. If P is a polarization of $\lambda \in L^*$ which is compatible with χ, then all irreducible $U_\chi(P)$–modules are one dimensional and the number isomorphism classes of irreducible $U_\chi(P)$–modules is p^l where l is the dimension of any maximal torus in P.*

Proof. It follows from Lemma 9.1.1 that P, as a restricted Lie subalgebra of L, is a direct sum of a torus and a p–nilpotent ideal in P. If there exists a one dimensional (and hence irreducible) $U_\chi(P)$–module we can apply Lemma 9.1.3 and Remark 9.1.4 to complete the proof. But $\lambda \in L^*$ defines a one dimensional P–module K_λ where each $x \in P$ acts as multiplication with $\lambda(x)$ [note that $\lambda([P, P]) = 0$ since P is a polarization]. Since $\lambda(x)^p - \lambda(x^{[p]}) = \chi(x)^p$ for all $x \in P$ we have that K_λ is an irreducible $U_\chi(P)$–module. □

Let $\lambda \in L^*$ and consider now a chain of ideals as in (9.1). Set for all i

$$\mathfrak{s}_i{}^\lambda = \{x \in L_i \mid \lambda([x, y]) = 0 \ \ \forall y \in L_i\}. \tag{9.7}$$

Then $\mathfrak{p}_\lambda = \mathfrak{s}_1^\lambda + \cdots + \mathfrak{s}_n^\lambda$ is a polarization of λ with $\mathfrak{p}_{\lambda_{|L_i}} = \mathfrak{p}_\lambda \cap L_i$ [here we define $\mathfrak{p}_{\lambda_{|L_i}} = \mathfrak{s}_1^\lambda + \cdots + \mathfrak{s}_i^\lambda$]. See [5, 1.12.3 and 1.12.10]. We shall call a polarization constructed thus a Vergne polarization of λ with respect to the chain (9.1). It also follows that $\mathfrak{p}_{\lambda_{|L_i}}$ is a polarization of $\lambda_{|L_i}$.

Remark 9.3.3. The annihilator $\{x \in \mathfrak{g} \ : \ x.v = 0\}$ of an element $v \in V$ of a restricted $\mathfrak{g}$–module V is a p–subalgebra of $\mathfrak{g}$: It is easy to see that the annihilator is a subspace of $\mathfrak{g}$. If $x.v = 0 = y.v$, then

$$[x, y].v = x.(y.v) - y.(x.v) = 0$$

and

$$x^{[p]}.v = x^p.v = 0.$$

Hence $\{x \in \mathfrak{g} \ : \ x.v = 0\}$ is a p–subalgebra of $\mathfrak{g}$.

In the following, we assume that all L_i are p–ideals.

Lemma 9.3.4.

a) All $\mathfrak{s}_i^\lambda$ with $1 \le i \le n$ are Lie p–subalgebras of L.

b) If $\mathfrak{s}_i^\lambda \not\subset L_{i-1}$ for some $1 \le i \le n$, then there exists a nonzero $x \in \mathfrak{s}_i^\lambda$ such that $\mathfrak{s}_i^\lambda = Kx \oplus (\mathfrak{s}_i^\lambda \cap L_{i-1})$.

Proof. a) Apply Remark 9.3.3 with $\mathfrak{g} := L_i$, $V := L_i^*$ and $v := \lambda_{|L_i}$.

For the proof of b) let $x \in \mathfrak{s}_i^\lambda$ but $x \notin L_{i-1}$. I claim that $\mathfrak{s}_i^\lambda = Kx \oplus (\mathfrak{s}_i^\lambda \cap L_{i-1})$. So let $y \in \mathfrak{s}_i^\lambda$: If $y \in L_{i-1}$, then clearly $y \in \mathfrak{s}_i^\lambda \cap L_{i-1}$. If $y \notin L_{i-1}$, then there exists $a \in K$ such that $y - ax \in L_{i-1}$ since L_i/L_{i-1} is one–dimensional. Moreover, $y - ax \in \mathfrak{s}_i^\lambda$ and the proof is then completed. □

Proposition 9.3.5. *Let $\lambda \in L^*$, let $\mathfrak{p}_\lambda$ be the Vergne polarization of λ constructed via (9.1). Let $\chi \in L^*$ such that $\mathfrak{p}_\lambda$ is compatible with χ; i.e., $\chi(x)^p = \lambda(x)^p - \lambda(x^{[p]})$ for all $x \in \mathfrak{p}_\lambda$. Then $U_\chi(L) \otimes_{U_\chi(\mathfrak{p}_\lambda)} K_\lambda$ is an irreducible L–module.*

Proof. The condition on χ ensures that K_λ is a $U_\chi(\mathfrak{p}_\lambda)$–module and thus we can construct the module $U_\chi(L) \otimes_{U_\chi(\mathfrak{p}_\lambda)} K_\lambda$. We will use induction on $\dim_K L$. If $\mathfrak{c}_L(\lambda) = L$ we have $\mathfrak{p}_\lambda = L$ and so the induced module has dimension 1. In that case the claim is trivial [this takes care of $\dim_K L \leq 1$]. Assume from now on that $\mathfrak{c}_L(\lambda) \neq L$. In that case there exists $j > 0$ such that $L_j \not\subset \mathfrak{c}_L(\lambda)$ and $L_{j-1} \subset \mathfrak{c}_L(\lambda)$. Set

$$H = \{x \in L \mid \lambda([x,y]) = 0 \text{ for all } y \in L_j\}. \quad (*)$$

It follows from Remark 9.3.3 that H is a Lie p–subalgebra of L. Choose $y_0 \in L_j$ with $L_j = Ky_0 \oplus L_{j-1}$ and observe that

$$H = \{x \in L \mid \lambda([x,y_0]) = 0\}$$

[since $L_{j-1} \subset \mathfrak{c}_L(\lambda)$ by assumption]. Thus H is the kernel of the nonzero linear form on L given by $x \longmapsto \lambda([x,y_0])$ and hence H has codimension 1 in L. The chain

$$0 = L_0 \cap H \subset L_1 \cap H \subset L_2 \cap H \subset \cdots \subset L_n \cap H = H \quad (**)$$

is a chain as in (9.1). But there has to occur one repetition in $(**)$. We still can use it to construct a Vergne polarization of $\mathfrak{p}_{\lambda_{|H}}$ of $\lambda_{|H}$ as

$$\mathfrak{p}_{\lambda_{|H}} = {\mathfrak{s}'_1}^{\lambda} + {\mathfrak{s}'_2}^{\lambda} + \cdots + {\mathfrak{s}'_n}^{\lambda}$$

where

$${\mathfrak{s}'_i}^{\lambda} = \{x \in L_i \cap H \mid \lambda([x,y]) = 0 \text{ for all } y \in L_i \cap H\}.$$

I claim that $\mathfrak{p}_{\lambda_{|H}} = \mathfrak{p}_\lambda$. Our choice of j says for all $i < j$ that $L_i \subset \mathfrak{c}_L(\lambda)$, hence $\mathfrak{s}_i^\lambda = L_i$ and $L_i \subset H$, which then implies that $\mathfrak{s}_i^\lambda = L_i = {\mathfrak{s}'_i}^{\lambda}$. On the other hand, for $i \geq j$ any $x \in \mathfrak{s}_i^\lambda$ satisfies $\lambda([x,y]) = 0$ for all $y \in L_i$. We have in particular $\lambda([x,y_0]) = 0$ since $y_0 \in L_j \subset L_i$. We get $x \in H$ and therefore $x \in {\mathfrak{s}'_i}^{\lambda}$. Hence $\mathfrak{s}_i^\lambda \subset {\mathfrak{s}'_i}^{\lambda}$. This implies $\mathfrak{p}_\lambda \subset \mathfrak{p}_{\lambda_{|H}}$. Since $\mathfrak{c}_L(\lambda) \subset H$, any polarization of $\lambda_{|H}$ is also a polarization of λ [see statement 1) in Section 9.2]. Apply this to $\mathfrak{p}_{\lambda_{|H}}$ and get that $\mathfrak{p}_\lambda$ and $\mathfrak{p}_{\lambda_{|H}}$ are both polarizations of λ. Hence they have the same dimension. We get $\mathfrak{p}_\lambda = \mathfrak{p}_{\lambda_{|H}}$ as claimed. That claim implies that $\mathfrak{p}_\lambda$ is a polarization of $\lambda_{|H}$. Since $\dim_K H < \dim_K L$ we may apply induction and get that

$$V = U_\chi(H) \otimes_{U_\chi(\mathfrak{p}_\lambda)} K_\lambda$$

is an irreducible H–module. If we apply Lemma 9.1.7 to H instead of L and to the ideal $B = L_j$ we see that each $x \in L_j$ acts as scalar multiplication by $\lambda(x)$ on V [note that $(*)$ says that $\lambda([H,B]) = 0$ and that $L_j \subset \mathfrak{p}_\lambda$]. We have by transitivity of induction that

$$U_\chi(L) \otimes_{U_\chi(\mathfrak{p}_\lambda)} K_\lambda \simeq U_\chi(L) \otimes_{U_\chi(H)} V.$$

If the right hand side is not irreducible it follows from Lemma 9.1.5 that we can extend V to a $U_\chi(L)$–module. Since any $x \in L_j$ acts on V as scalar multiplication by $\lambda(x)$, it commutes then with the action of each $y \in L$. So $[y,x]$ acts as 0 on V. But $[y,x] \in L_j$, hence acts as $\lambda([y,x])$ on V. We get thus $\lambda([y,x]) = 0$ for all $y \in L$ and all $x \in L_j$, hence $L_j \subset \mathfrak{c}_L(\lambda)$ – a contradiction to the choice of L_j. □

Let $\lambda \in L^*$ and let $\chi \in L^*$ such that the Vergne polarization $\mathfrak{p}_\lambda$ of λ is compatible with χ. If P is any polarization of λ, then $\lambda([P,P]) = 0$ and so λ defines a one–dimensional P–module K_λ. Since P is a polarization of λ we have

$$\dim_K P = \dim_K \mathfrak{p}_\lambda = \frac{\dim_K L + \dim_K \mathfrak{c}_L(\lambda)}{2}$$

where $\mathfrak{c}_L(\lambda) = \{x \in L \mid \lambda([x,y]) = 0 \ \ \forall\, y \in L\}$ denotes the stabiliser of λ in L.

Lemma 9.3.6. *Let $\lambda \in L^*$ and let $\chi \in L^*$ such that $\mathfrak{p}_\lambda$ is compatible with χ. If P is any polarization of λ compatible with χ, then K_λ is a $U_\chi(P)$–module and the induced module*

$$U_\chi(L) \otimes_{U_\chi(P)} K_\lambda \tag{9.8}$$

is an irreducible $U_\chi(L)$–module.

Proof. The first claim is obvious. For irreducibility of the induced module, apply Lemma 9.1.3, 9.3.5 and find that all irreducible $U_\chi(L)$–modules have dimension

$$p^{\dim_K L - \dim_K \mathfrak{p}_\lambda}.$$

But this is the dimension of the module in (9.8) since $\dim_K P = \dim_K \mathfrak{p}_\lambda$. □

Lemma 9.3.7. *Let $\lambda \in L^*$ and let $\chi \in L^*$ such that $\mathfrak{p}_\lambda$ is compatible with χ. If P is any polarization of λ compatible with χ, then the number of isomorphism classes of irreducible $U_\chi(L)$–modules is p^l, where l is the dimension of any maximal torus in P. In particular, if P is unipotent then the number of isomorphism classes of irreducible $U_\chi(L)$–modules is 1.*

Proof. We shall use a result by Feldvoss in [6, Theorem 5]. The result is proved for supersolvable Lie algebras such that $[L, L]$ is p–nilpotent. For such type of Lie algebras we have that every irreducible restricted L–module is one–dimensional. Feldvoss mentions that in the first two lines of the proof and the rest of the proof can be applied to any supersolvable Lie algebra. Thus: If we know that every irreducible restricted L–module is one–dimensional we can use the result in [6, Theorem 5]. But this is proved in Lemma 9.1.2 (for L with $L = T \oplus U$ which is our assumption, T is a torus and U is a p–nilpotent ideal in L). The result in [6, Theorem 5] says then: If S is any irreducible module isomorphic to some induced module

$$U_\chi(L) \otimes_{U_\chi(\widetilde{P})} K_\mu$$

where $\widetilde{P}$ is a polarization of some $\mu \in L^*$ such that $\mu(x)^p - \mu(x^{[p]}) = \chi(x)^p$ for all $x \in \widetilde{P}$, then the number of isomorphism classes of irreducible $U_\chi(L)$–modules is p^l where l is the dimension of any maximal torus in $\widetilde{P}$. But we may apply this to $\mu = \lambda$ and $\widetilde{P} = P$ and let S be the induced module in (9.8). □

Remark 9.3.8. Let $\chi \in L^*$ and suppose that $\lambda \in L^*$ such that $\mathfrak{p}_\lambda$ is compatible with χ. If P is any polarization of λ compatible with χ, then the dimension l_1 of any maximal torus in $\mathfrak{p}_\lambda$ and the dimension l_2 of any maximal torus in P are equal. Indeed, we apply Lemma 9.3.7 to $\mathfrak{p}_\lambda$ and P and get that the number of isomorphism classes of irreducible $U_\chi(L)$–modules is p^{l_1} and p^{l_2}; hence $l_1 = l_2$. In particular, we cannot have $\mathfrak{p}_\lambda$ unipotent and P non unipotent or P unipotent and $\mathfrak{p}_\lambda$ non unipotent.

Lemma 9.3.9. *Let $\chi \in L^*$, let $\lambda_i \in L_i^*$ such that $\lambda_i(x)^p - \lambda_i(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_{\lambda_i}$. Then there exists an extension $\lambda \in L^*$ such that the Vergne polarization $\mathfrak{p}_\lambda$ of λ constructed via* (9.1) *is compatible with χ.*

Proof. We can assume that $L_i \neq L$ (otherwise let $\lambda = \lambda_i$). We use induction on $\dim_K L$. If $\dim_K L = 1$ we necessarily have $\mathfrak{p}_\lambda = L$. If x is a basis for L then there exists $a \in K$ such that $x^{[p]} = ax$. Since K is algebraically closed we can find $b \in K$ such that $b^p - ab = \chi(x)^p$. We define $\lambda \in L^*$ by $\lambda(x) = b$.

Suppose now that $\dim_K L > 1$. Denote the last but one term in (9.1) by $L' = L_{n-1}$. So we have $\dim_K L' = \dim_K L - 1$ and we can apply induction to L', working again with the chain (9.1), just with the last term removed. So there is by induction an extension

$\lambda' \in (L')^*$ of λ_i such that the Vergne polarization $\mathfrak{p}_{\lambda'}$ of λ' with respect to that chain satisfies $\lambda'(x)^p - \lambda'(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_{\lambda'}$. We want to construct λ as an extension of λ' (and so of λ_i). Since $[L, L] \subset L'$ the Vergne polarization of *any* extension λ of λ' to L is equal to $\mathfrak{p}_\lambda = \mathfrak{p}_{\lambda'} + \mathfrak{s}$ where

$$\mathfrak{s} = \{x \in L \mid \lambda'([x, L]) = 0\}.$$

We have now two possibilities: If $\mathfrak{p}_\lambda = \mathfrak{p}_{\lambda'}$, then we take an arbitrary extension of λ' to L, and the claim holds.

Assume now the other possibility holds, i.e., that $\mathfrak{p}_\lambda \neq \mathfrak{p}_{\lambda'}$. Then $\mathfrak{s} \not\subset \mathfrak{p}_{\lambda'}$, equivalently, $\mathfrak{s} \not\subset L'$. So there exists $y \in L$ with $y \notin L'$ and $y \in \mathfrak{p}_\lambda$. We have then $L = Ky \oplus L'$ and $\mathfrak{p}_\lambda = Ky \oplus \mathfrak{p}_{\lambda'}$. We can find linear form $\lambda \in L^*$ such that $\lambda_{|L'} = \lambda'$ and $\lambda(y)^p - \lambda(y^{[p]}) = \chi(y)^p$. [We can write $y^{[p]} = ay + y'$ with $a \in K$ and $y' \in L'$. Then $\lambda(y)$ can be chosen as any element in K with $\lambda(y)^p - a\lambda(y) - \lambda'(y') = \chi(y)^p$.] Now $\mathfrak{p}_\lambda$ is the Vergne polarization of λ with respect to our chain. We have to show that $\lambda(x)^p - \lambda(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_\lambda$. We can write $x = by + z$ with $b \in K$ and $z \in \mathfrak{p}_{\lambda'}$. Note that $(by + z)^{[p]} - (by)^{[p]} - z^{[p]}$ is a linear combination of terms

$$[x_1, [x_2, \ldots, [x_{p-1}, x_p] \ldots]]$$

where each x_i is either by or z. Now each of these terms is in $[\mathfrak{p}_\lambda, \mathfrak{p}_\lambda]$. So they are all in the kernel of λ, since $\mathfrak{p}_\lambda$ is a polarization of λ. Our assumptions of λ' and the choice of $\lambda(y)$ give:

$$\begin{aligned} \lambda(x^{[p]}) &= \lambda((by)^{[p]}) + \lambda(z^{[p]}) \\ &= b^p \lambda(y^{[p]}) + \lambda'(z^{[p]}) \\ &= b^p \lambda(y)^p - b^p \chi(y)^p + \lambda'(z^{[p]}). \end{aligned}$$

Therefore we obtain:

$$\begin{aligned} \lambda(x)^p - \lambda(x^{[p]}) &= b^p \lambda(y)^p + \lambda(z)^p - b^p \lambda(y)^p + b^p \chi(y)^p - \lambda'(z^{[p]}) \\ &= \lambda'(z)^p - \lambda'(z^{[p]}) + b^p \chi(y)^p = \chi(z)^p + \chi(by)^p = \chi(x)^p. \end{aligned}$$

The proof is completed. □

Proposition 9.3.10. *Let $\chi \in L^*$, let E be an irreducible $U_\chi(L)$–module. Then there exists a linear form $\lambda \in L^*$ such that the Vergne polarization $\mathfrak{p}_\lambda$ of λ constructed via* (9.1) *is compatible with χ and $E \simeq U_\chi(L) \otimes_{U_\chi(\mathfrak{p}_\lambda)} K_\lambda$.*

Proof. There exists by Lemma 9.3.9 a linear form $\lambda \in L^*$ such that the Vergne polarization constructed via (9.1) satisfies $\lambda(x)^p - \lambda(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_\lambda$. Proposition 9.3.5 says that the L–module $E' = U_\chi(L) \otimes_{U_\chi(\mathfrak{p}_\lambda)} K_\lambda$ is irreducible. By Lemma 9.1.3 there exists $\mu \in L^*$ with $\mu(h^{[p]}) = \mu(h)^p$ for all $h \in T$ and $\mu(U) = 0$ such that $E \simeq E' \otimes K_\mu$. We get then $E \simeq U_\chi(L) \otimes_{U_\chi(\mathfrak{p}_\lambda)} K_{\lambda+\mu}$. Clearly, $\mathfrak{p}_\lambda$ is also the Vergne polarization of $\lambda + \mu$ constructed via (9.1) since μ vanishes on U and hence on $[L, L]$. So the claim follows. □

9.4 A supersolvable subalgebra of W

Recall the ordering on page 13. In this section we consider the subspace W_{012}. It is a Lie subalgebra of W [see Lemma 3.1.1] of codimension 3 and can be written as a direct sum of a torus $Ke_{012} \oplus Ke_{101}$ and a p–nilpotent ideal W_{011} in W_{012}. In fact we have:

Lemma 9.4.1. *The subspace W_{012} is a supersolvable restricted Lie algebra.*

Proof. First notice, that each W_{ijk} with $(i,j,k) \geq (0,1,2)$ is an ideal in W_{012} and form a chain

$$W_{012} \supset W_{101} \supset W_{011} \supset W_{202} \supset \cdots \supset W_{(p-1,p-1,1)} \supset 0 \tag{9.9}$$

such that $\dim_K W_{ijk}/W_{i'j'k'} = 1$, where (i',j',k') is a successor for (i,j,k). Indeed it is clear that $W_{ijk}/W_{i'j'k'}$ is spanned by the coset of e_{ijk}. For the ideal property, i.e., for $\Big[W_{012}, W_{ijk}\Big] \subset W_{ijk}$, one has to use the equations (3.1a), (3.1b), (3.1c), (3.1d) and the ordering of indices. Furthermore W_{012} can be written as $T \oplus U$, where $T = Ke_{012} \oplus Ke_{101}$ is a torus and $U = W_{011}$ is a p–ideal that is unipotent. □

All Vergne polarizations of linear forms on W_{012} are constructed with respect to the chain

$$W_{012} \supset W_{101} \supset W_{011} \supset W_{202} \supset \cdots \supset W_{p-1,p-1,1} \supset 0. \tag{9.10}$$

Let $\lambda \in W_{012}^*$. The Vergne polarization of λ with respect to the chain above is defined as

$$\mathfrak{p}_\lambda = \mathfrak{s}_{012}^\lambda + \mathfrak{s}_{101}^\lambda + \cdots + \mathfrak{s}_{p-1,p-1,2}^\lambda + \mathfrak{s}_{p-1,p-1,1}^\lambda \tag{9.11}$$

where

$$\mathfrak{s}_{ijk}^\lambda = \big\{x \in W_{ijk} \,|\, \lambda\big([x,y]\big) = 0 \text{ for all } y \in W_{ijk}\big\}. \tag{9.12}$$

Remark 9.4.2. Assume that $r > 1$ and $\lambda(W_{\geq r}) = 0$. Set $s = [r/2]$. So s is $r/2$ if r is even and $(r+1)/2$ if r is *odd*. It follows that $W_{\geq s} \subset \mathfrak{p}_\lambda$. This follows by observing that $W_{\geq s} = \mathfrak{s}_{s+1,0,2}^\lambda$. Indeed, since $W_{s+1,0,2} = W_{\geq s}$ (recall the ordering), and $\lambda([W_{\geq s}, W_{\geq s}]) \subset \lambda(W_{\geq 2s}) \subset \lambda(W_{\geq r}) = 0$ the claim follows.

Lemma 9.4.3. *Let $\lambda \in W_{012}^*$ such that $\mathfrak{p}_\lambda$ is non unipotent. Then $\mathfrak{p}_\lambda$ is a direct sum of a torus and a p–nilpotent ideal in $\mathfrak{p}_\lambda$. If $\lambda_{|W_{011}} \neq 0$ there exists a nonzero toral element $h \in \mathfrak{p}_\lambda$ such that $\mathfrak{p}_\lambda = Kh \oplus \mathfrak{p}_\lambda \cap W_{011}$ and $\lambda([h, W_{011}]) = 0$.*

Proof. Let $\lambda \in W_{012}^*$ such that $\mathfrak{p}_\lambda$ non unipotent. It follows directly from Lemma 9.1.1 that $\mathfrak{p}_\lambda$, as a Lie p–subalgebra of W_{012}, can be written as a direct sum of a torus and a p–nilpotent ideal $\mathfrak{p}_\lambda \cap W_{011}$ in $\mathfrak{p}_\lambda$. If $\lambda_{|W_{011}} = 0$ it is clear that $\mathfrak{p}_\lambda = W_{012}$ and so written as a direct sum of a torus $Ke_{012} \oplus Ke_{101}$ and a p–nilpotent ideal W_{011} in W_{012}.

Suppose that $\lambda_{|W_{011}} \neq 0$. Since $\mathfrak{p}_\lambda$ is non unipotent we have $\mathfrak{s}_{101}^\lambda \not\subset W_{011}$ or $\mathfrak{s}_{012}^\lambda \not\subset W_{011}$. Moreover, $\mathfrak{s}_{101}^\lambda \subset W_{011}$ or $\mathfrak{s}_{012}^\lambda \subset \mathfrak{s}_{101}^\lambda$. Indeed, let (ijk) be the maximal triple [with respect to the ordering on page 13] such that $\lambda(e_{ijk}) \neq 0$. If $k = 1$ we have $[W_{011}, e_{ijk}] \subset Ke_{i-1,j+1,1} \oplus W_{\geq i+j}$ such that $\lambda([W_{011}, e_{ijk}]) = 0$ [recall the ordering]. If $k = 2$ then $[W_{011}, e_{ijk}] \subset Ke_{ij1} \oplus Ke_{i-1,j+1,2} \oplus W_{\geq i+j}$ and so $\lambda([W_{011}, e_{ijk}]) = 0$. Hence $\lambda([W_{011}, e_{ijk}]) = 0$.

Suppose that $e_{101} + z \in \mathfrak{s}_{101}^\lambda$ for some $z \in W_{011}$. From the relations $\lambda([e_{101} + z, e_{ijk}]) = 0$ and $\lambda([z, W_{ijk}]) = 0$ we get $i = 1$ if $k = 1$ and $i = 0$ if $k = 2$. Next, consider $ae_{012} + be_{101} + z \in \mathfrak{s}_{012}^\lambda$ for some $z \in W_{011}$ and obtain from the relation $\lambda([ae_{012} + be_{101} + z, e_{ijk}]) = 0$ that $aj + b(i-1) = 0$ if $k = 1$ and $a(j-1) + bi = 0$ if $k = 2$. Since $(i,j,k) \succeq (0,1,1)$ we conclude that $a = 0$. This implies that $\mathfrak{s}_{012}^\lambda \subset \mathfrak{s}_{101}^\lambda$ if $\mathfrak{s}_{101}^\lambda \not\subset W_{011}$. Now apply Lemma 9.3.4.b to find nonzero element $h \in \mathfrak{s}_{012}^\lambda \cup \mathfrak{s}_{101}^\lambda$ such that $\mathfrak{s}_{101}^\lambda = Kh \oplus \mathfrak{s}_{101}^\lambda \cap W_{011}$ or $\mathfrak{s}_{012}^\lambda = Kh \oplus \mathfrak{s}_{012}^\lambda \cap W_{101}$. By Lemma 9.3.4.a and Lemma 9.1.1 we may assume that h is toral. It follows that $\mathfrak{p}_\lambda = Kh \oplus \mathfrak{p}_\lambda \cap W_{011}$ for some nonzero toral element $h \in \mathfrak{p}_\lambda$ as required. Finally, $\lambda([h, W_{011}]) = 0$ since $h \in \mathfrak{s}_{012}^\lambda \cup \mathfrak{s}_{101}^\lambda$. □

Suppose that $\mathfrak{p}_\lambda$ is non unipotent and let $0 \neq h \in \mathfrak{p}_\lambda$ be a toral element. There exists $g \in \mathrm{Aut}(W)$ such that $g(h) \in Ke_{012} \oplus Ke_{101}$ (see [4, Thm.1]). The next lemma shows that the $GL_2(K)$–part of g is a lower triangular matrix; in particular, $g(W_{012}) = W_{012}$ and $g(W_{011}) = W_{011}$. Each $g \in \mathrm{Aut}(W)$ can be written as $g = g_1 \circ g_2$, where $g_1 \in GL_2(K)$ and $g_2 \in \mathrm{Aut}^*(W)$. The $GL_2(K)$–part of g will be defined as g_1. This is well defined since $\mathrm{Aut}(W)$ is a semidirect product of $GL_2(K)$ and $\mathrm{Aut}^*(W)$.

Lemma 9.4.4. *Let $0 \neq h \in W_{012}$ with $h^{[p]} = h$. Then there exists $g \in \mathrm{Aut}(W)$ such that $g(h) \in Ke_{012} \oplus Ke_{101}$ and $g(W_{012}) = W_{012}$ and $g(W_{011}) = W_{011}$.*

Proof. Let $h = ae_{012} + be_{101} + ce_{011} + v$ where $v \in W_{\geq 1}$. If there exists a lower triangular matrix g with $g(h) \in Ke_{012} \oplus Ke_{101} + W_{\geq 1}$ we are done. Indeed, the proof of [4, Thm.1] says that there exists $g' \in \mathrm{Aut}^*(W)$ such that $(g' \circ g)(h) \in Ke_{012} \oplus Ke_{101}$. Now set $g := g' \circ g$. Otherwise, let g be an automorphism on W such that $g(h) \in Ke_{012} \oplus Ke_{101}$. There are two possibilities for the $GL_2(K)$–part g_1 of g: Either $g_1 = D \circ \Phi_1$ or $g_1 = \Phi_1' \circ \Theta \circ D \circ \Phi_1$, where D is a diagonal matrix and Φ_1, Φ_1' are lower triangular matrices with 1 at the diagonal and Θ is the matrix defined in Appendix A.4. We may assume that $g_1 = \Phi_1' \circ \Theta \circ D \circ \Phi_1$ (since D and Φ_1 preserves W_{012}). Moreover, assume that the coefficient of e_{011} in $D \circ \Phi_1(h)$ is nonzero [otherwise we are in the situation discussed in the beginning of the proof]. If the coefficient of e_{011} in $D \circ \Phi_1(h)$ is nonzero then, by the relations in Appendix A.2, the coefficient of e_{102} in $\Phi_1(h)$ is nonzero – contradiction. $\square$

Lemma 9.4.5. *Suppose that $\lambda \in W_{012}^*$ and let $g \in \mathrm{Aut}(W)$ with $g(W_{012}) = W_{012}$. Then $\mathfrak{p}_{\lambda^g} = g^{-1}\big(\mathfrak{p}_\lambda\big)$. In particular, $\mathfrak{p}_\lambda \not\subset W_{011}$ if and only if $\mathfrak{p}_{\lambda^g} \not\subset W_{011}$.*

Proof. Since g preserves W_{012}, the $GL_2(K)$ part of g must be a diagonal matrix composed with some lower triangular matrix with 1 at the diagonal. For such a g we have $\mathfrak{s}_{ijk}^{\lambda^g} = g^{-1}\big(\mathfrak{s}_{ijk}^{\lambda}\big)$, since $g^{-1}(W_{ijk}) = W_{ijk}$ (recall the ordering on the set of indices of basis elements and the action of g on basis elements). In order to finish the proof, we just have to use $s_{ijk}^{\lambda^g} = g^{-1}\big(\mathfrak{s}_{ijk}^{\lambda}\big)$ for all $(ijk) \succeq (012)$ with (9.11): It follows that $\mathfrak{p}_{\lambda^g} = g^{-1}\big(\mathfrak{p}_\lambda\big)$. The proof is completed. $\square$

Lemma 9.4.6. *Assume that $\lambda \in W_{012}^*$. Then $\mathfrak{p}_\lambda \not\subset W_{011}$ if and only if there exists $g \in \mathrm{Aut}(W)$ with $g(W_{012}) = W_{012}$ such that $ae_{012} + be_{101} \in \mathfrak{p}_{\lambda^g}$ for $a, b \in \mathbb{F}_p$ with $a \neq 0$ or $b \neq 0$ and $\lambda^g([ae_{012} + be_{101}, W_{012}]) = 0$.*

Proof. Apply Lemma 9.4.3 and find nonzero $h \in \mathfrak{p}_\lambda$ with $h^{[p]} = h$ and $\lambda([h, W_{011}]) = 0$. Let g^{-1} be an automorphism on W preserving W_{012} such that $g^{-1}(h) \in Ke_{012} \oplus Ke_{101}$ (see Lemma 9.4.4). Since $g(W_{011}) = W_{011}$ also, we have $\lambda^g\big(\big[g^{-1}(h), W_{011}\big]\big) = 0$. The fact that $g^{-1}(h) \in Ke_{012} \oplus Ke_{101}$ implies that $\lambda^g\big(\big[g^{-1}(h), W_{012}\big]\big) = 0$, which implies that $g^{-1}(h) \in \mathfrak{p}_{\lambda^g}$. The other implication follows immediately from Lemma 9.4.5. Since g^{-1} is a restricted automorphism, we have $ae_{012} + be_{101} = g^{-1}(h) = g^{-1}(h^{[p]}) = g^{-1}(h)^{[p]} = a^p e_{012} + b^b e_{101}$, which implies that $a, b \in \mathbb{F}_p$. $\square$

Remark 9.4.7. Return to the case where $\chi \in W^*$ is a p–character and let $g \in \mathrm{Aut}(W)$ with $g(W_{012}) = W_{012}$. If $\lambda \in W_{012}^*$ satisfies that $\lambda(x)^p - \lambda(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_\lambda$ it follows from Lemma 9.4.5 that $\lambda^g(x)^p - \lambda^g(x^{[p]}) = \chi^g(x)^p$ for all $x \in \mathfrak{p}_{\lambda^g}$. It can also be formulated as (Definition 9.3.1): If $\mathfrak{p}_\lambda$ is compatible with χ then $\mathfrak{p}_{\lambda^g}$ is compatible with χ^g.

10 Compatible polarizations

Let $\chi \in W^*$ be a character of height $r > 1$. Note that we have defined the stabilizer of χ in $W_{\geq r}$ as $\mathfrak{st}(\chi, W_{\geq r}) = \{x \in W \mid \chi([x,y]) = 0 \, \forall \, y \in W_{\geq r}\}$.

10.1 Existence

In the next two sections we will prove:

Theorem 10.1.1. *There exists a linear form $\lambda \in W^*_{012}$ such that the Vergne polarization $\mathfrak{p}_\lambda$ of λ constructed via* (9.10) *is compatible with χ and such that*

a) $\mathfrak{p}_\lambda = \mathfrak{p}_\chi$ if $\mathfrak{p}_\chi$ is non unipotent.

b) $\mathfrak{p}_\lambda = \mathfrak{p}_\chi$ if $\mathfrak{p}_\chi$ is unipotent and $r \leq p$ or $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$ or $\chi([e_{011}, W_{r-1}]) \neq 0$.

The existence of $\lambda \in W^*_{012}$ such that the Vergne polarization $\mathfrak{p}_\lambda$ of λ is compatible with χ (i.e., that $\lambda(x)^p - \lambda(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_\lambda$) follows from the construction in Lemma 9.3.9. The proof of Theorem 10.1.1 will be divided into two parts. In Section 10.2 we consider χ such that $\mathfrak{p}_\chi$ is non unipotent and Section 10.3 deals with χ such that $\mathfrak{p}_\chi$ is unipotent. We will several times use the construction given in the proof of Lemma 9.3.9.

For $\lambda \in W^*_{012}$ and $\chi \in W^*$, it will be convenient to define

$$\lambda_{|W_{\alpha\beta\gamma}} := \lambda_{\alpha\beta\gamma} \quad \text{and} \quad \chi_{|W_{\alpha\beta\gamma}} := \chi_{\alpha\beta\gamma} \quad \text{for any triple } (\alpha\beta\gamma) \succeq (012).$$

For any $(\alpha\beta\gamma) \succeq (012)$ we define $\mathfrak{p}_{\lambda_{\alpha\beta\gamma}}$ with respect to the chain

$$W_{\alpha\beta\gamma} \supset W_{\alpha'\beta'\gamma'} \supset \cdots \subset W_{p-1,p-1,1} \supset 0.$$

where $(\alpha'\beta'\gamma')$ is the successor for $(\alpha\beta\gamma)$ with respect to the ordering $\preceq$ on page 13.

Lemma 10.1.2. *We can choose $\lambda \in W^*_{012}$ such that $\mathfrak{p}_\lambda$ is compatible with χ and*

a) If $r > p$ then $\lambda_{|W_{\geq 2}} = \chi_{|W_{\geq 2}}$.

b) If $r \leq p$ then $\lambda_{|W_{\geq 1}} = \chi_{|W_{\geq 1}}$.

Proof. a) Set $\lambda' := \chi_{|W_{\geq 2}}$. Then we have $\lambda'(x)^p - \lambda'(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_{\lambda'}$ since $x^{[p]} = 0$ for any $x \in W_{\geq 2}$. Now, by Lemma 9.3.9, let $\lambda \in W^*_{012}$ be an extension of λ' such that $\mathfrak{p}_\lambda$ is compatible with χ.

b) In this case we let $\lambda' := \chi_{|W_{\geq 1}}$. Note $\lambda'(x)^p - \lambda'(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_{\lambda'}$ since $x^{[p]} \in W_{\geq p}$ for any $x \in W_{\geq 1}$ and hence $\lambda'(x^{[p]}) = \chi(x^{[p]}) = 0$ $(r \leq p)$. Now, by Lemma 9.3.9, let $\lambda \in W^*_{012}$ be an extension of λ' such that $\mathfrak{p}_\lambda$ is compatible with χ. $\square$

10.2 The non unipotent case

Assume that the Vergne polarization $\mathfrak{p}_\chi$ of χ is non unipotent. Then there exists a nonzero toral element h such that $\mathfrak{p}_\chi = Kh \oplus \mathfrak{p}_\chi \cap W_{011}$ and $\chi([h, W_{011}]) = 0$ [see Lemma 9.4.3]. In order to show that $\mathfrak{p}_\lambda = \mathfrak{p}_\chi$ we may assume that $h \in Ke_{012} \oplus Ke_{101}$. Indeed, use Lemma 9.4.4 to find $g \in \mathrm{Aut}(W)$ with $g(W_{012}) = W_{012}$ such that $g(h) \in Ke_{012} \oplus Ke_{101}$. Therefore, by Lemma 9.4.5, we have $g(h) \in \mathfrak{p}_{\chi^{g^{-1}}}$. If we can find $\lambda^{g^{-1}} \in W^*_{012}$ such that $\mathfrak{p}_{\lambda^{g^{-1}}}$ is a polarization of $\lambda^{g^{-1}}$ compatible with $\chi^{g^{-1}}$ such that $\mathfrak{p}_{\lambda^{g^{-1}}} = \mathfrak{p}_{\chi^{g^{-1}}}$ then, since

$\mathfrak{p}_{\lambda^{g^{-1}}} = g(\mathfrak{p}_\lambda)$ and $\mathfrak{p}_{\chi^{g^{-1}}} = g(\mathfrak{p}_\chi)$, we have $\mathfrak{p}_\lambda = \mathfrak{p}_\chi$ and, by Remark 9.4.7, also that $\mathfrak{p}_\lambda$ is compatible with χ [here λ is the linear form $(\lambda^{g^{-1}})^g$]. So assume $h = ae_{012} + be_{101}$.

The proof of Theorem 10.1.1.a will be a consequence of several lemmas. Define (i, j, k) as the maximal index with respect to the ordering $\preceq$ on page 13 such that $\chi(e_{ijk}) \neq 0$. The assumption $r > 1$ implies that $i + j > 1$. Define, for $0 \leq \alpha, \beta < p$ and $\gamma = 1, 2$, $a_{\alpha\beta\gamma} \in K$ via the formula $[h, e_{\alpha\beta\gamma}] = a_{\alpha\beta\gamma} e_{\alpha\beta\gamma}$. It is easy to check, by (3.1a),(3.1b),(3.1c),(3.1d), that

$$a_{\alpha\beta\gamma} = \begin{cases} a - b & (\alpha\beta\gamma) = (011), \\ 2b - a & (\alpha\beta\gamma) = (202), \\ b & (\alpha\beta\gamma) = (112), \\ a & (\alpha\beta\gamma) = (022), \\ b & (\alpha\beta\gamma) = (201), \\ a & (\alpha\beta\gamma) = (111), \\ 2a - b & (\alpha\beta\gamma) = (021). \end{cases}$$

Lemma 10.2.1. *Suppose that $a_{021} = 0$. Set $\lambda := \chi_{021} = \chi_{|W_{021}}$. Then $\lambda(x)^p - \lambda(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_\lambda$.*

Proof. We can assume that $r > p$ [see Lemma 10.1.2.b]. This implies in particular that $p > 3$ since $r \leq 2p - 3$ also. Set $\lambda' := \chi_{|W_{\geq 2}}$. Then $\lambda'(x)^p - \lambda'(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_{\lambda'}$ since $x^{[p]} = 0$ for any $x \in W_{\geq 2}$. Let $\lambda \in W_{021}^*$ be an extension of $\lambda_{|W_{\geq 2}}$ such that $\lambda(x)^p - \lambda(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_\lambda$. See Lemma 9.3.9. The claim says that we can choose λ such that $\lambda(e_{021}) = \chi(e_{021})$. This will follow if we can prove that $\mathfrak{s}_{021}^\chi \subset W_{\geq 2}$. Indeed, for any extension λ we have $\mathfrak{s}_{021}^\chi = \mathfrak{s}_{021}^\lambda$ [since χ and λ are equal on $W_{\geq 2}$ by assumption]. Therefore the construction in the proof of Lemma 9.3.9 shows that we can choose $\lambda(e_{021})$ arbitrarily.

If there exists $y \in W_{r-2}$ such that $\chi([e_{021}, y]) \neq 0$, then we can choose $\lambda(e_{021})$ arbitrarily. Indeed, it follows that $\chi([W_{\geq 2}, y]) \subset \chi(W_{\geq r}) = 0$ such that $\mathfrak{s}_{021}^\chi \subset W_{\geq 2}$.

Let (i, j, k) be the maximal index such that $\chi(e_{ijk}) \neq 0$ [note that $i + j = r$]. If $k = 2$ and $i < p - 1$ we have $[e_{021}, e_{i+1,r-i-2,2}] = -2e_{i+1,r-i-1,1} + (i+1)e_{i,r-i,2}$ such that $\chi([e_{021}, e_{i+1,r-i-2,2}]) \neq 0$. So we can assume that $k = 2$ and $i = p - 1$ or $k = 1$ [otherwise we can choose $\lambda(e_{021}) = \chi(e_{021})$]. If $k = 1$ and $i < p - 1$ we have $\chi([e_{021}, e_{i+1,r-i-2,1}]) = (i+1)\chi(e_{i,r-i,1}) \neq 0$.

Thus: In order to prove that we can choose λ with $\lambda(e_{021}) = \chi(e_{021})$ we can assume that $i = p - 1$. Next, use that $0 \neq h = ae_{012} + be_{101} \in \mathfrak{p}_\chi$ for some $a, b \in \mathbb{F}_p$. Since $W_{r-1} \subset \mathfrak{p}_\chi$, by Remark 9.4.2, and $\chi([\mathfrak{p}_\chi, \mathfrak{p}_\chi]) = 0$ we have in particular $\chi([h, W_{r-1}]) = 0$. We then have

$$\chi([h, e_{p-1,r+1-p,k}]) = 0 \Longrightarrow \begin{cases} a(r+1) - 2b = 0 & \text{if } k = 1, \\ ar - b = 0 & \text{if } k = 2. \end{cases}$$

Since $a_{021} = 0$ we also have $2a - b = 0$. Putting all this together we get that

$$\begin{cases} r = p + 3 & \text{if } k = 1, \\ r = p + 2 & \text{if } k = 2. \end{cases}$$

Suppose that $\mathfrak{s}_{021}^\chi \not\subset W_{\geq 2}$. Then there exists $y \in W_{021}$ such that $\mathfrak{s}_{021}^\chi = Ky \oplus \mathfrak{s}_{021}^\chi \cap W_{\geq 2}$ and $\lambda(y)$ is defined via

$$\lambda(y)^p - \lambda(y^{[p]}) = \chi(y)^p. \tag{10.1}$$

Write $y = e_{021} + z_2 + z_3$ with $z_2 \in W_2$ and $z_3 \in W_{\geq 3}$. Now $y^{[p]} - (e_{021} + z_2)^{[p]} - z_3^{[p]}$ is a linear combination of terms

$$[x_1, [x_2, \ldots, [x_{p-1}, x_p] \ldots]] \quad (*)$$

where each x_i is either $e_{021} + z_2$ or z_3. If z_3 appears s times in $(*)$ then

$$[x_1, [x_2, \ldots, [x_{p-1}, x_p] \ldots]] \in W_{\geq p+2s}.$$

We may assume that $x_1 = z_3$ in order to prove that λ vanishes on terms in $(*)$. Indeed, if x_1 is $e_{021} + z_2$ note that

$$\lambda([e_{021} + z_2 + z_3, [x_2, \ldots, [x_{p-1}, x_p] \ldots]]) = 0$$

since $e_{021} + z_2 + z_3 \in \mathfrak{s}_{021}^{\chi}$. Therefore

$$\lambda([e_{021} + z_2, [x_2, \ldots, [x_{p-1}, x_p] \ldots]]) = -\lambda([z_3, [x_2, \ldots, [x_{p-1}, x_p] \ldots]]).$$

That is; we may assume that $x_1 = z_3$. This implies that $s > 1$ [since we can assume $x_p \neq x_{p-1}$] such that all terms in $(*)$ belong to $W_{\geq p+4}$. Hence λ (which is equal to χ on $W_{\geq 2}$ and so has height r also) vanishes on all terms in $(*)$ since $r \leq p+3$. Since $z_3^{[p]} = 0$ we therefore obtain

$$\lambda(y^{[p]}) = \lambda\big((e_{021} + z_2)^{[p]}\big).$$

Now $(e_{021} + z_2)^{[p]} - e_{021}^{[p]} - z_2^{[p]}$ is a linear combination of terms

$$[x_1, [x_2, \ldots, [x_{p-1}, x_p] \ldots]] \quad (**)$$

where each x_i is either e_{021} or z_2. If z_2 appears $s > 0$ (we can assume that $x_p \neq x_{p-1}$) times in $(**)$ then

$$[x_1, [x_2, \ldots, [x_{p-1}, x_p] \ldots]] \in W_{\geq p+s}.$$

We want to prove that λ vanishes on all terms as in $(**)$. If so; then, since $z_2^{[p]} = e_{021}^{[p]} = 0$, we have $\lambda(y^{[p]}) = 0$. Hence, by (10.1), we have $\lambda(y)^p = \chi(y)^p$. This implies that $\lambda(y) = \chi(y)$ and therefore $\lambda(e_{021}) = \chi(e_{021})$ [recall our assumption $\lambda_{|W_{\geq 2}} = \chi_{|W_{\geq 2}}$]. If $s > 2$ we see that λ vanishes on all terms in $(**)$ since $r \leq p+3$. So we need to handle the cases where $s = 1$ or $s = 2$.

First, write

$$z_2 = \sum_{t=0}^{3} a_t e_{t,3-t,1} + \sum_{t=0}^{3} b_t e_{t,3-t,2} \in W_2$$

for some $a_t, b_t \in K$.

Note that $\mathrm{ad}(e_{021})(z_2) \in \sum_{t=0}^{3} K e_{t,4-t,1} + \sum_{t=0}^{2} K e_{t,4-t,2}$ from (1.2a) and (1.2b). It now follows that

$$\begin{aligned}
(\mathrm{ad}\, e_{021})^2(z_2) &\in \textstyle\sum_{t=0}^{2} K e_{t,5-t,1} + \sum_{t=0}^{1} K e_{t,5-t,2} \\
(\mathrm{ad}\, e_{021})^3(z_2) &\in \textstyle\sum_{t=0}^{1} K e_{t,6-t,1} + K e_{062} \\
(\mathrm{ad}\, e_{021})^4(z_2) &\in K e_{071}, \\
(\mathrm{ad}\, e_{021})^5(z_2) &= 0.
\end{aligned}$$

If $s = 1$ we can assume that $x_p = z_2$ such that (**) is equal to $\mathrm{ad}^{p-1}e_{021}(z_2)$. For $p > 5$ this implies that $\mathrm{ad}^{p-1}e_{021}(z_2) = 0$. If $p = 5$ we have $\mathrm{ad}^4 e_{021}(z_2) \in Ke_{071}$ which is zero also (we have defined $e_{rst} = x_1^r x_2^s \frac{\partial}{\partial x_t}$ for $0 \leq r, s < p$ and $t = 1, 2$ and equal to zero otherwise. If $p = 5$ then $e_{071} = 0$ since $7 > 5$). So λ vanishes on all terms in (**) if $s = 1$.

Suppose that $s = 2$. If $k = 2$ we have $r = p + 2$ and hence λ vanishes on all terms in (**). So assume that $k = 1$ and then $r = p + 3$. If $x_1 = e_{021}$ then $y := [x_2, [x_3, \ldots, [x_{p-1}, x_p] \ldots]] \in W_{p+1} = W_{r-2}$. In that case we can assume that (**) is zero. Otherwise we have $\chi([e_{021}, y]) \neq 0$ and in that case we can choose $\lambda(e_{021})$ arbitrarily [recall the arguments in the beginning of the proof]. Therefore we only need to show that λ vanishes on all terms in (**) with $x_1 = z_2$. We can also assume that $x_p = z_2$. This implies that

$$[x_2, [x_3, \ldots, [x_{p-1}, x_p] \ldots]] = \mathrm{ad}^{p-2}(e_{021})(z_2).$$

If $p > 5$ then $\mathrm{ad}^{p-2}(e_{021})(z_2) = 0$. If $p = 5$ we have $\mathrm{ad}^3(e_{021})(z_2) = 0$ since $e_{061} = e_{062} = e_{151} = 0$ for $p = 5$. So λ vanishes on all terms in (**) if $s = 2$. The proof is completed. $\square$

Lemma 10.2.2. *There exists linear form $\lambda \in W_{012}^*$ such that $\mathfrak{p}_\lambda$ is compatible with χ and:*

a) If $a_{202} \neq 0$ then $\lambda_{|W_{\geq 1}} = \chi_{|W_{\geq 1}}$,

b) If $a_{202} = 0$ then $\lambda_{112} = \chi_{112}$.

Proof. We may assume that $r > p$ and that $\lambda_{|W_{\geq 2}} = \chi_{|W_{\geq 2}}$ by Lemma 10.1.2. Since $W_{r-1} \subset \mathfrak{p}_\chi$ and $0 \neq h = ae_{012} + be_{101} \in \mathfrak{p}_\chi$ by assumption, we then have $\chi([h, W_{r-1}]) = 0$. It follows that (remember that (i, j, k) denotes the maximal index with respect to the ordering $\preceq$ on page 13 such that $\chi(e_{ijk}) \neq 0$):

$$\chi([h, e_{ijk}]) = 0 \Longrightarrow \begin{cases} aj + b(i-1) = 0 & \text{if } k = 1, \\ a(j-1) + bi & \text{if } k = 2. \end{cases}$$

Since $r = i + j > p$ we have $i > 1$ and $j > 1$ and therefore $a \neq 0$ and $b \neq 0$. We conclude that $a_{\alpha\beta\gamma} \neq 0$ for all $(112) \preceq (\alpha\beta\gamma) \preceq (111)$ [use the relations on page 47 with $a \neq 0 \neq b$]. Therefore $\chi([h, e_{\alpha\beta\gamma}]) = a_{\alpha\beta\gamma}\chi(e_{\alpha\beta\gamma}) = 0$ implies that $\chi(e_{\alpha\beta\gamma}) = 0$ for $(112) \preceq (\alpha\beta\gamma) \preceq (111)$. If $a_{202} \neq 0$ we get also $\chi(e_{202}) = 0$ and if $a_{021} \neq 0$ we get $\chi(e_{021}) = 0$.

Now let $(202) \preceq (\alpha\beta\gamma) \preceq (021)$ and $(\alpha'\beta'\gamma')$ be the successor of $(\alpha\beta\gamma)$ with respect to the ordering $\preceq$ on page 13. We will only consider $(\alpha\beta\gamma) = (202)$ in the case where $a_{202} \neq 0$ and we will only consider $(\alpha\beta\gamma) = (021)$ if $a_{021} \neq 0$. If $a_{021} = 0$ it follows from Lemma 10.2.1 that $\lambda' = \chi_{021}$ satisfies that $\lambda'(x)^p - \lambda'(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_{\lambda'}$. So for the successor $(\alpha'\beta'\gamma')$ we may assume, by induction, that $\lambda_{\alpha'\beta'\gamma'} = \chi_{\alpha'\beta'\gamma'}$ and $\lambda_{\alpha'\beta'\gamma'}(x)^p - \lambda_{\alpha'\beta'\gamma'}(x^{[p]}) = \chi_{\alpha'\beta'\gamma'}(x)^p$ for all $x \in \mathfrak{p}_{\lambda_{\alpha'\beta'\gamma'}}$. There are now two possibilities:

1) If $\mathfrak{s}_{\alpha\beta\gamma}^{\lambda_{\alpha\beta\gamma}} \subset \mathfrak{s}_{\alpha'\beta'\gamma'}^{\lambda_{\alpha'\beta'\gamma'}}$ we can choose $\lambda_{\alpha\beta\gamma}(e_{\alpha\beta\gamma})$ arbitrarily. In that case set

$$\lambda_{\alpha\beta\gamma}(e_{\alpha\beta\gamma}) = \chi(e_{\alpha\beta\gamma}).$$

2) Suppose that there exists $y := e_{\alpha\beta\gamma} - y' \in W_{\alpha\beta\gamma}$ with $y' \in W_{\alpha'\beta'\gamma'}$ such that

$$\mathfrak{s}_{\alpha\beta\gamma}^{\lambda_{\alpha\beta\gamma}} = Ky \oplus \mathfrak{s}_{\alpha\beta\gamma}^{\lambda_{\alpha\beta\gamma}} \cap W_{\alpha'\beta'\gamma'}.$$

The construction given in the proof of Lemma 9.3.9 shows that we shall define $\lambda_{\alpha\beta\gamma}(y)$ from the relation

$$\lambda_{\alpha\beta\gamma}(y)^p - \lambda_{\alpha\beta\gamma}(y^{[p]}) = \chi(y)^p.$$

Since $\chi(e_{\alpha\beta\gamma}) = 0$ and $\lambda_{\alpha\beta\gamma}(y') = \chi(y')$ we get that

$$\lambda_{\alpha\beta\gamma}(e_{\alpha\beta\gamma})^p = \lambda_{\alpha\beta\gamma}(y^{[p]}).$$

If $\lambda_{\alpha\beta\gamma}(y^{[p]}) = 0$ we have $\lambda_{\alpha\beta\gamma}(e_{\alpha\beta\gamma}) = 0 = \chi(e_{\alpha\beta\gamma})$ as required. So assume that $\lambda_{\alpha\beta\gamma}(y^{[p]}) \neq 0$ and hence $\lambda_{\alpha\beta\gamma}(e_{\alpha\beta\gamma}) \neq 0$. Consider the Lie p–subalgebra of W_{012} given by

$$W' = Kh \oplus W_{\alpha\beta\gamma}.$$

Any extension of $\lambda_{\alpha\beta\gamma}$ to a character $\lambda' \in W'$ satisfies that $\mathfrak{p}_{\lambda'}$ is unipotent. Here $\mathfrak{p}_{\lambda'}$ is constructed via the chain

$$W' \supset W_{\alpha\beta\gamma} \supset W_{\alpha'\beta'\gamma'} \supset \cdots \supset W_{p-1,p-1,1} \supset 0.$$

Otherwise there exists $z \in W_{\alpha\beta\gamma}$ such that $\lambda'([h+z, W']) = 0$. Since $y = e_{\alpha\beta\gamma} - y' \in \mathfrak{s}_{\alpha\beta\gamma}^{\lambda_{\alpha\beta\gamma}}$ we get

$$\begin{aligned} 0 &= \lambda'([h,y]) + \lambda'([z,y]) \\ &= \lambda'([h,e_{\alpha\beta\gamma}]) - \lambda'([h,y']) + \lambda'([z,y]) \\ &= a_{\alpha\beta\gamma}\lambda(e_{\alpha\beta\gamma}) - \chi([h,y']) + \chi([z,y]) \\ &= a_{\alpha\beta\gamma}\lambda(e_{\alpha\beta\gamma}). \end{aligned}$$

Since $\lambda(e_{\alpha\beta\gamma}) \neq 0$ by assumption and $a_{\alpha\beta\gamma} \neq 0$ this is impossible [we only consider $(\alpha\beta\gamma) = (202)$ if $a_{202} \neq 0$]. That is; we have $\mathfrak{p}_{\lambda'} \subset W_{\alpha\beta\gamma}$ and can then choose $\lambda'(h)$ arbitrarily. Suppose that $\lambda'(h)^p - \lambda'(h^{[p]}) = \chi(h)^p$.

Next, consider $x \in \mathfrak{p}_{\lambda'} \cap W_{\alpha'\beta'\gamma'}$. I claim that $[h,x] \in \mathfrak{p}_{\lambda'} \cap W_{\alpha'\beta'\gamma'}$ also. Indeed, since $\lambda'_{\alpha'\beta'\gamma'} = \chi_{\alpha'\beta'\gamma'}$ by assumption we have $\mathfrak{p}_{\lambda'} \cap W_{\alpha'\beta'\gamma'} = \mathfrak{p}_{\chi} \cap W_{\alpha'\beta'\gamma'}$. Now use that $h \in \mathfrak{p}_{\chi}$ to get $[h,x] \in \mathfrak{p}_{\chi} \cap W_{\alpha'\beta'\gamma'} = \mathfrak{p}_{\lambda'} \cap W_{\alpha'\beta'\gamma'}$. It follows that

$$P = Kh \oplus \mathfrak{p}_{\lambda'} \cap W_{\alpha'\beta'\gamma'}$$

is a Lie subalgebra of W'. In fact, it is restricted since $h^{[p]} = h$ and $\mathfrak{p}_{\lambda'} \cap W_{\alpha'\beta'\gamma'}$ is restricted. Since $\lambda'_{\alpha'\beta'\gamma'} = \chi_{\alpha'\beta'\gamma'}$ and $\chi([h, W_{012}]) = 0$ we then get:

$$\lambda'([P,P]) \subset \lambda'([\mathfrak{p}_{\lambda'} \cap W_{\alpha'\beta'\gamma'}, \mathfrak{p}_{\lambda'} \cap W_{\alpha'\beta'\gamma'}]) + \lambda'([h, \mathfrak{p}_{\lambda'} \cap W_{\alpha'\beta'\gamma'}]) = 0 + 0 = 0.$$

Moreover, $\dim_K P = \dim_K \mathfrak{p}_{\lambda'}$ such that P is actually a polarization of λ'. We also have $\lambda'(x)^p - \lambda'(x^{[p]}) = \chi(x)^p$ for all $x \in P$ [it is true for all basis elements!]. So P is a non unipotent polarization of λ' compatible with χ and $\mathfrak{p}_{\lambda'}$ is a unipotent polarization of λ' compatible with χ. Now get a contradiction via Remark 9.3.8.

So if $a_{202} \neq 0$ then $\lambda' := \chi_{|W_{\geq 1}}$ satisfies that $\lambda'(x)^p - \lambda'(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_{\lambda'}$. Now, by Lemma 9.3.9, let $\lambda \in W_{012}^*$ be an extension of λ' such that $\mathfrak{p}_{\lambda}$ is compatible with χ. If $a_{202} = 0$ then $\lambda' := \chi_{112}$ satisfies that $\lambda'(x)^p - \lambda'(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_{\lambda'}$ and then, by Lemma 9.3.9, let $\lambda \in W_{012}^*$ be an extension of λ' such that $\mathfrak{p}_{\lambda}$ is compatible with χ. The proof is completed. □

Proof of Theorem 10.1.1 *for non unipotent* $\mathfrak{p}_\chi$: It follows from Lemma 10.2.2 that we can choose $\lambda \in W^*_{012}$ such that $\mathfrak{p}_\lambda$ is compatible with χ and $\lambda_{|W_{\geq 1}} = \chi_{|W_{\geq 1}}$ if $a_{202} \neq 0$ or $1 < r \leq p$ and $\lambda_{112} = \chi_{112}$ if $a_{202} = 0$ and $r > p$. If $\mathfrak{s}^\lambda_{011} \subset \mathfrak{s}^\lambda_{202}$, then we can choose $\lambda(e_{011})$ arbitrarily; in this case we choose $\lambda(e_{011}) = 0$.

Note that we always have

$$\mathfrak{s}^\lambda_{011} = \mathfrak{s}^\chi_{011} \quad \text{since } [W_{011}, W_{011}] \subset W_{112}.$$

Therefore $\mathfrak{p}_\chi \cap W_{011} = \mathfrak{p}_\lambda \cap W_{011}$. If $\mathfrak{p}_\lambda$ is non unipotent, then Lemma 9.4.3 implies that there exists a nonzero toral element $h' \in \mathfrak{p}_\lambda$ such that $\mathfrak{p}_\lambda = Kh' \oplus \mathfrak{p}_\chi \cap W_{011}$. If $\mathfrak{p}_\lambda$ is unipotent, then $\mathfrak{p}_\lambda = \mathfrak{p}_\chi \cap W_{011}$.

We have $\lambda([h, W_{\geq 1}]) = 0$: If $a_{202} \neq 0$, then $\lambda([h, W_{\geq 1}]) = \chi([h, W_{\geq 1}]) = 0$. If $a_{202} = 0$, then $\lambda([h, e_{202}]) = \lambda(0) = 0$ and $\lambda([h, W_{112}]) = 0$.

If now $\lambda([h, e_{011}]) = 0$, then $\lambda([h, W_{012}]) = 0$ since $[h, e_{012}] = 0 = [h, e_{101}]$. It then follows that $h \in \mathfrak{s}^\lambda_{012} \subset \mathfrak{p}_\lambda$ and so we have $\mathfrak{p}_\lambda = Kh \oplus \mathfrak{p}_\chi \cap W_{011} = \mathfrak{p}_\chi$.

So assume that $0 \neq \lambda([h, e_{011}]) = a_{011}\lambda(e_{011})$. It follows that $\lambda(e_{011}) \neq 0$, hence $\mathfrak{s}^\lambda_{011} \not\subset \mathfrak{s}^\lambda_{202}$ by our choice.

Consider first the case that $\mathfrak{p}_\lambda$ is non unipotent. There exists $y = e_{011} - y' \in \mathfrak{s}^\lambda_{011}$ for some $y' \in W_{\geq 1}$. From Lemma 9.4.3 there exists a nonzero toral element $h' \in \mathfrak{p}_\lambda$ such that $\lambda([h', W_{011}]) = 0$. Write $h' = a'e_{012} + b'e_{101} + z'$ for some $z' \in W_{011}$ and $a', b' \in K$. Note that $a' \neq 0$ or $b' \neq 0$ since h' is a toral element. Recall the definition of (i, j, k) as the maximal index with respect to the ordering $\preceq$ defined on page 13 such that $\chi(e_{ijk}) \neq 0$. Then we have $\lambda([z', e_{ijk}]) = 0 = \chi([z', e_{ijk}])$ [use the ordering and that $z' \in W_{011}$; for details see the proof of Lemma 9.4.3]. Therefore the relation $\lambda([h', e_{ijk}]) = 0$ implies that

$$\begin{cases} a'j + b'(i-1) & k = 1, \\ a'(j-1) + b'i & k = 2. \end{cases}$$

Since $\chi([h, e_{ijk}]) = 0$ we also have

$$\begin{cases} aj + b(i-1) & k = 1, \\ a(j-1) + bi & k = 2. \end{cases}$$

This shows that if we evaluate the matrix

$$\begin{pmatrix} a' & b' \\ a & b \end{pmatrix}$$

at

$$\begin{pmatrix} j \\ i-1 \end{pmatrix} \text{ if } k = 1 \text{ and } \begin{pmatrix} j-1 \\ i \end{pmatrix} \text{ if } k = 2$$

we get zero. Since $i + j > 1$ these vectors are nonzero and therefore the matrix is singular. In other words, there exists $c \in K^*$ such that $a' = ca$ and $b' = cb$. Hence $h' = ch + z'$. Since $y = e_{011} - y' \in \mathfrak{s}^\lambda_{011}$ we get

$$\begin{aligned} 0 &= \lambda([h', y]) \\ &= c\lambda([h, e_{011}]) + c\lambda([h, -y']) + \lambda([z', y]) \\ &= c\lambda([h, e_{011}]) + c\lambda([h, -y']) + 0. \end{aligned}$$

If $a_{202} \neq 0$ we have $\lambda_{|W_{\geq 1}} = \chi_{|W_{\geq 1}}$ and hence $\lambda([h,-y']) = \chi([h,-y']) = 0$. If $a_{202} = 0$ we have $[h,-y'] \in W_{112}$ and again, since $\lambda_{112} = \chi_{112}$ in that case, we get $\lambda([h,-y']) = \chi([h,-y']) = 0$. It follows that $\lambda([h,e_{011}]) = 0$ – contradiction.

Consider now the case that $\mathfrak{p}_\lambda$ is unipotent. Then the proof of Lemma 9.3.9, shows that we can take an arbitrary extension of λ_{011} to W_{012}. Define $\lambda(h)$ via $\lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p$. Set $P := Kh \oplus \mathfrak{p}_\chi \cap W_{\geq 1}$. Note that $P \subset W_{012}$ is a Lie subalgebra of $\mathfrak{p}_\chi$ since $[h, \mathfrak{p}_\chi \cap W_{\geq 1}] \subset \mathfrak{p}_\chi \cap W_{\geq 1}$. In fact, it is restricted since $h^{[p]} = h$ and $\mathfrak{p}_\chi \cap W_{\geq 1}$ is restricted. Since $\mathfrak{p}_\chi \cap W_{\geq 1} = \mathfrak{p}_\lambda \cap W_{\geq 1}$ we also have

$$[P,P] \subset [h, W_{\geq 1}] + [\mathfrak{p}_\lambda, \mathfrak{p}_\lambda]$$

and hence $\lambda([P,P]) = 0$ since $\lambda([h,W_{\geq 1}]) = 0 = \lambda([\mathfrak{p}_\lambda, \mathfrak{p}_\lambda])$ (the statement $\lambda([h,W_{\geq 1}]) = 0$ follows from a remark we made in the beginning of this proof). Finally, $\dim_K P = \dim_K \mathfrak{p}_\lambda$ and therefore P is a polarization of λ.

We also have $\lambda(x)^p - \lambda(x^{[p]}) = \chi(x)^p$ for all $x \in P$ [true for all basis elements!]. So P is a non unipotent polarization of λ compatible with χ and $\mathfrak{p}_\lambda$ is a unipotent polarization of λ compatible with χ. Now get a contradiction by Remark 9.3.8. $\square$

Corollary 10.2.3. *If $\xi \in W^*$ of height $r > 1$ with $\mathfrak{p}_\xi = Kh \oplus \mathfrak{p}_\xi \cap W_{011}$ for some nonzero toral element $h \in Ke_{012} \oplus Ke_{101}$ with $\xi([h, W_{011}]) = 0$. Then $\xi(y^{[p]}) = 0$ for all $y \in \mathfrak{s}^\xi_{\alpha\beta\gamma}$ with $(\alpha\beta\gamma) \succeq (112)$. Moreover,*

1) *If $a_{202} \neq 0$ then $\xi(y^{[p]}) = 0$ for all $y \in \mathfrak{s}^\xi_{202}$.*

2) *If $a_{011} \neq 0$ then $\xi(e_{011}) = 0$.*

Proof. 1) Use Lemma 10.2.1, 10.2.2 to find $\lambda \in W^*_{012}$ with $\lambda_{|W_{112}} = \xi_{|W_{112}}$ and $\lambda_{|W_{\geq 1}} = \xi_{|W_{\geq 1}}$ if $a_{202} \neq 0$ such that $\lambda(x)^p - \lambda(x^{[p]}) = \xi(x)^p$ for all $x \in \mathfrak{p}_\lambda$. So for $(\alpha\beta\gamma) \succeq (112)$ and $(\alpha\beta\gamma) \succeq (202)$ for $a_{202} \neq 0$ consider $y \in \mathfrak{s}^\xi_{\alpha\beta\gamma} = \mathfrak{s}^\lambda_{\alpha\beta\gamma}$. Then we get

$$\lambda(y)^p - \lambda(y^{[p]}) = \xi(y)^p \Longrightarrow \xi(y^{[p]}) = \lambda(y^{[p]}) = 0.$$

2) This is obvious since $\xi([h,e_{011}]) = a_{011}\xi(e_{011}) = 0$ and $a_{011} \neq 0$ by assumption. $\square$

10.3 The unipotent case

Let $\chi \in W^*$ of height $r > 1$ with unipotent $\mathfrak{p}_\chi$. We now prove that there exists a linear form $\lambda \in W^*_{012}$ such that the Vergne polarization of λ is compatible with χ and unipotent also.

Let $\lambda \in W^*_{012}$ such that $\mathfrak{p}_\lambda$ is compatible with χ. We will assume that $\lambda(e_{\alpha\beta\gamma}) = \chi(e_{\alpha\beta\gamma})$ whenever $\mathfrak{s}^{\lambda_{\alpha\beta\gamma}}_{\alpha\beta\gamma} \subset \mathfrak{s}^{\lambda_{\alpha'\beta'\gamma'}}_{\alpha'\beta'\gamma'}$ where $(\alpha'\beta'\gamma')$ is the successor for $(\alpha\beta\gamma)$ with respect to the ordering $\preceq$. It is possible to construct λ in this way such that $\mathfrak{p}_\lambda$ is compatible with χ. See the proof of Lemma 9.3.9.

Suppose that $\mathfrak{p}_\lambda$ is non unipotent. It follows, by Lemma 9.4.3, that there exists a nonzero toral element $h \in \mathfrak{p}_\lambda$ such that $\mathfrak{p}_\lambda = Kh \oplus \mathfrak{p}_\lambda \cap W_{011}$ and $\lambda([h, W_{011}]) = 0$. The aim is to prove that $\mathfrak{p}_\chi$ is non unipotent [and so to get a contradiction]. In order to prove that we may replace χ by any χ^g where g is an automorphism on W such that $g(W_{012}) = W_{012}$. See Lemma 9.4.5.

Let, by Lemma 9.4.4, g^{-1} be an automorphism on W such that $g^{-1}(W_{012}) = W_{012}$ and $g^{-1}(h) \in Ke_{012} \oplus Ke_{101}$. Then apply Lemma 9.4.5 and Remark 9.4.7 to get $\mathfrak{p}_{\lambda^g} = Kg^{-1}(h) \oplus \mathfrak{p}_{\lambda^g} \cap W_{011}$ and $\lambda^g(x)^p - \lambda^g(x^{[p]}) = \chi^g(x)^p$ for all $x \in \mathfrak{p}_{\lambda^g}$. Therefore we may assume that $h \in Ke_{012} \oplus Ke_{101}$ and $\lambda([h, W_{011}]) = 0$ since $g(W_{011}) = W_{011}$ by Lemma 9.4.4.

By Corollary 10.2.3 it follows that $\lambda(y^{[p]}) = 0$ for all $y \in \mathfrak{s}^{\lambda}_{\alpha\beta\gamma}$ with $(\alpha\beta\gamma) \succeq (112)$ and $(\alpha\beta\gamma) \succeq (202)$ if $a_{202} \neq 0$. I claim that this implies that $\lambda_{112} = \chi_{112}$ and $\lambda_{|W_{\geq 1}} = \chi_{|W_{\geq 1}}$ if $a_{202} \neq 0$. Indeed, let $(\alpha\beta\gamma) \succeq (112)$ [or $(\alpha\beta\gamma) \succeq (202)$ if $a_{202} \neq 0$] with successor $(\alpha'\beta'\gamma')$ for $(\alpha\beta\gamma)$ with respect to the ordering $\preceq$. We may assume $\lambda_{\alpha'\beta'\gamma'} = \chi_{\alpha'\beta'\gamma'}$ by induction. Now, for any extension $\lambda_{\alpha\beta\gamma}$ of $\lambda_{\alpha'\beta'\gamma'}$ we either have

$$\mathfrak{s}^{\lambda_{\alpha\beta\gamma}}_{\alpha\beta\gamma} \subset \mathfrak{s}^{\lambda_{\alpha'\beta'\gamma'}}_{\alpha'\beta'\gamma'} \quad \text{or} \quad \mathfrak{s}^{\lambda_{\alpha\beta\gamma}}_{\alpha\beta\gamma} \not\subset \mathfrak{s}^{\lambda_{\alpha'\beta'\gamma'}}_{\alpha'\beta'\gamma'}.$$

In the first case we have $\lambda_{\alpha\beta\gamma}(e_{\alpha\beta\gamma}) = \chi(e_{\alpha\beta\gamma})$ by our choice. In the second case there exists $y \in \mathfrak{s}^{\lambda_{\alpha\beta\gamma}}_{\alpha\beta\gamma}$ with $y \notin W_{\alpha'\beta'\gamma'}$ and the assumption on λ says that

$$\lambda(y)^p - \lambda(y^{[p]}) = \chi(y)^p.$$

But we have $\lambda(y^{[p]}) = 0$ and hence $\lambda_{\alpha\beta\gamma} = \chi_{\alpha\beta\gamma}$. So $\lambda \in W^*_{012}$ defined in the beginning of this section satisfies that $\lambda_{112} = \chi_{112}$ and $\lambda_{W_{\geq 1}} = \chi_{W_{\geq 1}}$ if $a_{202} \neq 0$. Therefore $\mathfrak{s}^{\chi}_{011} = \mathfrak{s}^{\lambda}_{011}$ since $[W_{011}, W_{011}] \subset W_{112}$.

We have $\chi([h, W_{\geq 1}]) = 0$: If $a_{202} \neq 0$, then $\chi([h, W_{\geq 1}]) = \lambda([h, W_{\geq 1}]) = 0$. If $a_{202} = 0$, then $\chi([h, e_{202}]) = \chi(0) = 0$ and $\chi([h, W_{112}]) = 0$.

If now $\chi([h, e_{011}]) = 0$, then $\chi([h, W_{012}]) = 0$ since $[h, e_{012}] = 0 = [h, e_{101}]$. It then follows that $h \in \mathfrak{s}^{\chi}_{012} \subset \mathfrak{p}_{\chi}$ – contradiction.

So assume that $0 \neq \chi([h, e_{011}]) = a_{011}\chi(e_{011})$. In particular, $\chi(e_{011}) \neq 0 \neq a_{011}$. Since $\lambda([h, e_{011}]) = 0$ we then have $\lambda(e_{011}) = 0$. So $\lambda(e_{011}) \neq \chi(e_{011})$ and hence $\mathfrak{s}^{\chi}_{011} \not\subset \mathfrak{s}^{\chi}_{202}$ by our choice. Then there exists $y = e_{011} - y' \in \mathfrak{s}^{\chi}_{011} = \mathfrak{s}^{\lambda}_{011}$ with $y' \in W_{\geq 1}$. It follows that

$$\mathfrak{p}_{\lambda} = Kh \oplus Ky \oplus \mathfrak{p}_{\lambda} \cap W_{\geq 1} \quad \text{and} \quad \mathfrak{p}_{\chi} = Ky \oplus \mathfrak{p}_{\chi} \cap W_{\geq 1}.$$

Set $P := Kh \oplus \mathfrak{p}_{\chi} \cap W_{\geq 1}$. Note that $P \subset W_{012}$ is a Lie subalgebra of $\mathfrak{p}_{\chi}$ since $[h, \mathfrak{p}_{\chi} \cap W_{\geq 1}] \subset \mathfrak{p}_{\chi} \cap W_{\geq 1}$. In fact, it is restricted since $h^{[p]} = h$ and $\mathfrak{p}_{\chi} \cap W_{\geq 1}$ is restricted. Note that

$$[P, P] \subset [h, W_{\geq 1}] + [\mathfrak{p}_{\chi}, \mathfrak{p}_{\chi}]$$

such that $\chi([P, P]) = 0$ since $\chi([h, W_{\geq 1}]) = 0 = \chi([\mathfrak{p}_{\chi}, \mathfrak{p}_{\chi}])$ (the statement $\chi([h, W_{\geq 1}]) = 0$ follows from a remark we made in the beginning of this proof). Finally, $\dim_K P = \dim_K \mathfrak{p}_{\chi}$ and therefore P is a polarization of χ. Now we can find $\tau \in W^*_{012}$ such that

$$\tau(x)^p = \chi(x)^p - \chi(x^{[p]}) \quad \forall\, x \in P \cup \mathfrak{p}_{\chi}.$$

This formula defines one linear form on $\mathfrak{p}_{\chi}$ and one on P. These coincide on $\mathfrak{p}_{\chi} \cap P$. So one can find a common extension to W_{012}. Now, by Remark 9.3.8, we get a contradiction if we consider the number of isomorphism classes of irreducible $U_{\tau}(W_{012})$–modules: On one hand we have $\chi(x)^p - \chi(x^{[p]}) = \tau(x)^p\ \forall\, x \in \mathfrak{p}_{\chi}$ such that the number of isomorphism classes of irreducible $U_{\tau}(W_{012})$–modules is 1. On the other hand we have $\chi(x)^p - \chi(x^{[p]}) = \tau(x)^p\ \forall\, x \in P$ for a polarization P of χ, where any maximal torus has dimension 1. This shows that the number of isomorphism classes of irreducible $U_{\tau}(W_{012})$–modules is p. We have a contradiction.

We have thus shown:

Lemma 10.3.1. *If $\mathfrak{p}_{\chi}$ is unipotent then there exists $\lambda \in W^*_{012}$ such that $\mathfrak{p}_{\lambda}$ is compatible with χ and such that $\mathfrak{p}_{\lambda}$ is unipotent.*

Lemma 10.3.2. *If $r \le p$ and $\mathfrak{p}_\chi$ is unipotent then there exists $\lambda \in W_{012}^*$ such that $\mathfrak{p}_\lambda$ is compatible with χ and $\mathfrak{p}_\lambda = \mathfrak{p}_\chi$.*

Proof. If $r \le p$, then there exists by Lemma 10.1.2.b a linear form $\lambda \in W_{012}^*$ such that $\mathfrak{p}_\lambda$ is compatible with χ and $\lambda_{|W_{\ge 1}} = \chi_{|W_{\ge 1}}$; in particular, $\mathfrak{s}^\chi_{\alpha\beta\gamma} = \mathfrak{s}^\lambda_{\alpha\beta\gamma}$ for all $(\alpha\beta\gamma) \succeq (011)$. So we have $\mathfrak{p}_\chi \cap W_{011} = \mathfrak{p}_\lambda \cap W_{011}$ and by the part of the claim already proved we have that $\mathfrak{p}_\lambda$ is unipotent also. Hence $\mathfrak{p}_\lambda = \mathfrak{p}_\lambda \cap W_{011} = \mathfrak{p}_\chi \cap W_{011} = \mathfrak{p}_\chi$. □

Suppose in the rest of this section that $r > p$. Let $\lambda \in W_{012}^*$ such that $\mathfrak{p}_\lambda$ is compatible with χ. We can assume that $\lambda_{|W_{\ge 2}} = \chi_{|W_{\ge 2}}$, by Lemma 10.1.2.a. Therefore $\mathfrak{p}_{\lambda_{|W_{\ge 1}}} = \mathfrak{p}_{\chi_{|W_{\ge 1}}}$ since $\mathfrak{s}^\lambda_{\alpha\beta\gamma} = \mathfrak{s}^\chi_{\alpha\beta\gamma}$ for all $(\alpha\beta\gamma) \succeq (202)$. Now write

$$\mathfrak{p}_\lambda = \mathfrak{s}^\lambda_{011} + \mathfrak{s} \quad \text{and} \quad \mathfrak{p}_\chi = \mathfrak{s}^\chi_{011} + \mathfrak{s} \ \text{ for } \ \mathfrak{s} := \mathfrak{p}_{\lambda_{|W_{\ge 1}}} = \mathfrak{p}_{\chi_{|W_{\ge 1}}}. \tag{10.2}$$

Lemma 10.3.3. *If $\chi([e_{011}, W_{r-1}]) \neq 0$ then $\mathfrak{p}_\lambda = \mathfrak{s} = \mathfrak{p}_\chi$.*

Proof. The assumption says that there exists $y \in W_{r-1}$ such that $\chi([e_{011}, y]) \neq 0$. This implies that $\mathfrak{s}^\chi_{011} \subset \mathfrak{s}^\chi_{202} \subset \mathfrak{s}$. Indeed, consider $u = ae_{011} + z \in \mathfrak{s}^\chi_{011}$ for some $a \in K$ and $z \in W_{\ge 1}$. Then $\chi([u, y]) = a\chi([e_{011}, y]) + \chi([z, y]) = 0$. But $[z, y] \in W_{\ge r}$ so we have $\chi([z, y]) = 0$ and therefore $a = 0$ since $\chi([e_{011}, y]) \neq 0$ by assumption. The assumption on λ says in particular that $\lambda_{|W_{\ge r-1}} = \chi_{|W_{\ge r-1}}$. This implies that $\lambda([e_{011}, y]) \neq 0$ and hence (apply the same arguments as before) we get $\mathfrak{s}^\lambda_{011} \subset \mathfrak{s}$ also; therefore $\mathfrak{p}_\lambda = \mathfrak{s} = \mathfrak{p}_\chi$. □

So assume that $\chi([e_{011}, W_{r-1}]) = 0$. We shall recall the basis for W_{r-1} given in Section 4.2: There exist basis elements $x_0^{(r)}, x_1^{(r)}, \dots, x_{2p-r-1}^{(r)}$ and $y_1^{(r)}, y_2^{(r)}, \dots, y_{2p-r-2}^{(r)}$ with the following properties:

$$\left[e_{011}, x_i^{(r)}\right] = -(i+1)x_{i+1}^{(r)} \quad \text{and} \quad \left[e_{011}, y_i^{(r)}\right] = -iy_{i+1}^{(r)}.$$

Since $\chi([e_{011}, W_{r-1}]) = 0$ it follows from the relations above that $\chi(x_i^{(r)}) = 0$ for all $i > 0$ and $\chi(y_j^{(r)}) = 0$ for all $j > 1$. But $\chi(W_{r-1}) \neq 0$, so we also have $\chi(x_0^{(r)}) \neq 0$ or $\chi(y_1^{(r)}) \neq 0$. This implies that $\chi(e_{p-1,r+1-p,2}) \neq 0$ or $\chi(e_{p-2,r+2-p,2}) + (r+2)\chi(e_{p-1,r+1-p,1}) \neq 0$. We also have $\chi(e_{\alpha\beta 1}) = 0$ for all α, β with $0 \le \alpha, \beta < p$ and $\alpha + \beta = r$ and $\alpha < p-1$. Moreover, $\chi(e_{\alpha\beta 2}) = 0$ for all α, β with $0 \le \alpha, \beta < p$ and $\alpha + \beta = r$ and $\alpha < p-2$. See Section 4.2 and use the assumptions on χ just obtained.

Lemma 10.3.4. *If $\mathfrak{st}(\chi, W_{\ge r}) = W_{\ge 0}$ and $\chi(x_1^{(r)}) = 0$, then*

$$\chi(e_{p-2,r+2-p,2}) = -\chi(e_{p-1,r+1-p,1}) \neq 0.$$

Proof. If $\chi(e_{p-2,r+2-p,2}) = 0$, then we have $e_{001} \in \mathfrak{st}(\chi, W_{\ge r})$ – contradiction. So we have $\chi(e_{p-2,r+2-p,2}) \neq 0$. Now, since $\chi(x_1^{(r)}) = 0$ we find that

$$0 = \chi([e_{011}, e_{p-1,r+1-p,2}]) = -(\chi(e_{p-2,r+2-p,2}) + \chi(e_{p-1,r+1-p,1})).$$

The proof is completed. □

Lemma 10.3.5. *Suppose that $\mathfrak{st}(\chi, W_{\ge r}) = W_{\ge 0}$ and $\chi([e_{011}, W_{r-1}]) = 0$. Then:*

a) We have inclusions $\mathfrak{s}^\chi_{201} \subset \mathfrak{s}^\chi_{111} \subset \mathfrak{s}^\chi_{021} \subset W_{\ge 2}$ and $\mathfrak{s}^\chi_{202} \subset \mathfrak{s}^\chi_{112}$.

b) We can choose $\lambda \in W_{012}^*$ such that $\mathfrak{p}_\lambda$ is compatible with χ and such that $\lambda_{201} = \chi_{201}$ and $\lambda(e_{202}) = \chi(e_{202})$.

Proof. 1) Set $y_{021} = e_{p-1,r-p,2}$ and $y_{111} = e_{p-1,r-p,1}$ and $y_{201} = e_{p-2,r-p+1,1}$. Then, for $(201) \preceq (\alpha\beta\gamma) \preceq (021)$, we have $\chi([e_{\alpha\beta\gamma}, y_{\alpha\beta\gamma}]) \neq 0$ but $\chi([W_{\alpha'\beta'\gamma'}, y_{\alpha\beta\gamma}]) = 0$, where $(\alpha'\beta'\gamma')$ is the successor for $(\alpha\beta\gamma)$ with respect to $\preceq$. The existence of $y_{\alpha\beta\gamma}$ implies that $\mathfrak{s}^\chi_{\alpha\beta\gamma} \subset W_{\alpha'\beta'\gamma'}$; hence $\mathfrak{s}^\chi_{\alpha\beta\gamma} \subset \mathfrak{s}^\chi_{\alpha'\beta'\gamma'}$.

If $r \neq 2p-4$ we set $y_{202} = e_{p-3,r+2-p,1}$ and find that $[e_{202}, y_{202}] = (r+2)e_{p-1,r+1-p,1} - 2e_{p-2,r+2-p,2}$; hence, by Lemma 10.3.4, we get $\chi([e_{202}, y_{202}]) = (r+4)\chi(e_{p-1,r+1-p,1}) \neq 0$. But also $\chi([W_{112}, y_{202}]) = 0$ from the assumptions on χ (see the remarks done just before Lemma 10.3.4). Hence $\mathfrak{s}^\chi_{202} \subset \mathfrak{s}^\chi_{112}$ for $r \neq 2p-4$.

If $r = 2p-4$, set $y_{202} = e_{p-4,r+3-p,2}$ (since $r = 2p-4 > p$ we have in particular, $p > 3$ and $r+3-p = p-1$; so $y_{202} \neq 0$). By Lemma 10.3.4, we get $\chi([e_{202}, y_{202}]) = (r+3-p)\chi(e_{p-2,r+2-p,2}) \neq 0$ but also $\chi([W_{112}, y_{202}]) = 0$ from the assumptions on χ. Therefore $\mathfrak{s}^\chi_{202} \subset \mathfrak{s}^\chi_{112}$ for $r = 2p-4$ also.

2) Apply the construction in the proof of Lemma 9.3.9 with Lemma 10.1.2.a and the inclusions obtained in 1) and choose then λ such that $\lambda(e_{\alpha\beta\gamma}) = \chi(e_{\alpha\beta\gamma})$ for $(\alpha\beta\gamma) \succeq (201)$ and $(\alpha\beta\gamma) = (202)$. $\square$

Lemma 10.3.6. *Suppose that $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$ and $\chi([e_{011}, W_{r-1}]) = 0$. If $\mathfrak{s}^\chi_{022} \subset \mathfrak{s}^\chi_{201}$ or there exists $x \in \mathfrak{s}^\chi_{022}$ with $\mathfrak{s}^\chi_{022} = Kx \oplus \mathfrak{s}^\chi_{022} \cap W_{201}$ and $\chi(x^{[p]}) = 0$, then there exists $\lambda \in W^*_{012}$ such that $\mathfrak{p}_\lambda$ is compatible with χ and such that $\mathfrak{p}_\lambda = \mathfrak{p}_\chi$.*

Proof. First, by the construction in the proof of Lemma 9.3.9 and Lemma 10.3.5, choose $\lambda_{201} \in W^*_{201}$ such that $\lambda_{201} = \chi_{201}$ and such that $\lambda_{201}(z)^p - \lambda_{201}(z^{[p]}) = \chi(z)^p$ for all $z \in \mathfrak{p}_{\lambda_{201}}$. Let λ_{022} be an extension of λ_{201} such that $\lambda_{022}(z)^p - \lambda_{022}(z^{[p]}) = \chi(z)^p$ for all $z \in \mathfrak{p}_{\lambda_{022}}$. If $\mathfrak{s}^\chi_{022} \subset \mathfrak{s}^\chi_{201}$, then also $\mathfrak{s}^{\lambda_{022}}_{022} \subset \mathfrak{s}^{\lambda_{201}}_{201}$ and then choose $\lambda_{022}(e_{022}) = \chi(e_{022})$. Otherwise, there exists x such that $\mathfrak{s}^\chi_{022} = Kx \oplus \mathfrak{s}^\chi_{022} \cap W_{201}$ and $\chi(x^{[p]}) = 0$. It follows that $\lambda_{022}(x)^p - \lambda_{022}(x^{[p]}) = \chi(x)^p$. Since $\lambda_{022}(x^{[p]}) = \chi(x^{[p]}) = 0$ we have $\lambda_{022}(x) = \chi(x)$; hence $\lambda_{022}(e_{022}) = \chi(e_{022})$. So it is possible, by Lemma 10.3.5.a and the construction in the proof of Lemma 9.3.9, to find an extension $\lambda \in W^*_{012}$ of $\lambda_{022} = \chi_{022}$ such that $\mathfrak{p}_\lambda$ is compatible with χ. Choose $\lambda \in W^*_{012}$ in that way.

Note that Lemma 10.3.5 implies that $\mathfrak{p}_\chi \cap W_{\geq 1} = \mathfrak{p}_\chi \cap W_{112}$. Therefore

$$[\mathfrak{s}^\lambda_{011}, \mathfrak{p}_\chi] \subset [\mathfrak{s}^\lambda_{011}, \mathfrak{s}^\chi_{011}] + [\mathfrak{s}^\lambda_{011}, \mathfrak{p}_\chi \cap W_{112}] \subset \mathrm{Ker}(\chi)$$

since $\chi([\mathfrak{s}^\lambda_{011}, \mathfrak{s}^\chi_{011}]) \subset \chi([W_{011}, \mathfrak{s}^\chi_{011}]) = 0$ and $\chi([\mathfrak{s}^\lambda_{011}, \mathfrak{p}_\chi \cap W_{112}]) = \lambda([\mathfrak{s}^\lambda_{011}, \mathfrak{p}_\chi \cap W_{112}]) \subset \lambda([\mathfrak{s}^\lambda_{011}, W_{011}]) = 0$ (here we use that $[\mathfrak{s}^\lambda_{011}, \mathfrak{p}_\chi \cap W_{112}] \subset W_{022}$ and that $\lambda_{022} = \chi_{022}$ by our choice). It follows that $\mathfrak{s}^\lambda_{011} + \mathfrak{p}_\chi$ is a totally isotropic subspace with respect to χ; hence, by maximality of $\mathfrak{p}_\chi$, we have $\mathfrak{s}^\lambda_{011} \subset \mathfrak{p}_\chi$. This implies that $\mathfrak{p}_\lambda \subset \mathfrak{p}_\chi$. By symmetry, we can also prove that $\mathfrak{p}_\chi \subset \mathfrak{p}_\lambda$. $\square$

So assume from now that:

1) $\chi([e_{011}, W_{r-1}]) = 0$,

2) $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$,

3) $\mathfrak{s}^\chi_{022} \not\subset \mathfrak{s}^\chi_{201}$,

4) $\mathfrak{s}^\chi_{011} \not\subset \mathfrak{s}^\chi_{022}$ or $\mathfrak{s}^\lambda_{011} \not\subset \mathfrak{s}^\lambda_{022}$.

Assumption 3) implies that there exists $z \in W_{201}$ such that $x := e_{022} + z \in \mathfrak{s}^\chi_{022}$. Write

$$z = ae_{201} + be_{111} + ce_{021} + z' \quad \text{for some } z' \in W_{\geq 2}.$$

Since $\chi([e_{022}, e_{p-2,r+1-p,1}]) = 0 = \chi([z', e_{p-2,r+1-p,1}])$ we get

$$\chi([ae_{201} + be_{111} + ce_{021}, e_{p-2,r+1-p,1}]) = 0$$

and hence $a = 0$ from the assumptions on χ. Moreover, $\chi([e_{022} + z, e_{p-1,r-p,1}]) = 0$ implies that $r = 2b$.

Let $\lambda \in W_{012}^*$ with $\lambda_{|W_{\geq 2}} = \chi_{|W_{\geq 2}}$ such that $\mathfrak{p}_\lambda$ is compatible with χ. We can assume that $\mathfrak{s}_{011}^\chi \not\subset \mathfrak{s}_{022}^\chi$ or $\mathfrak{s}_{011}^\lambda \not\subset \mathfrak{s}_{022}^\lambda$ by assumption 4) above.

Let $\lambda \in W_{012}^*$ with $\lambda_{|W_{\geq 2}} = \chi_{|W_{\geq 2}}$ such that $\mathfrak{p}_\lambda$ is compatible with χ. If there exists $y = e_{011} + y' \in \mathfrak{s}_{011}^\chi$ with $y' \in W_{\geq 1}$, then $[y, e_{022} + z] \in \mathfrak{p}_\chi$. If there exists $y = e_{011} + y' \in \mathfrak{s}_{011}^\lambda$ with $y' \in W_{\geq 1}$, then $[y, e_{022} + z] \in \mathfrak{p}_\lambda$ since $e_{022} + z \in \mathfrak{s}_{022}^\chi$ and $\mathfrak{s}_{022}^\chi = \mathfrak{s}_{022}^\lambda$. It is easy to obtain

$$[e_{011} + y', e_{022} + z] = (b-1)e_{021} + w$$

for some $w \in W_{\geq 2}$. If $e_{011} + y' \in \mathfrak{s}_{011}^\chi$ then $(b-1)e_{021} + w \in \mathfrak{p}_\chi \cap W_{021}$. If $e_{011} + y' \in \mathfrak{s}_{011}^\lambda$ then $(b-1)e_{021} + w \in \mathfrak{p}_\lambda \cap W_{021} = \mathfrak{p}_\chi \cap W_{021}$. But $\mathfrak{p}_\chi \cap W_{021} \subset W_{\geq 2}$ by Lemma 10.3.5.a; hence $b = 1$. Since $r = 2b$ and $p < r \leq 2p - 3$ we get $r = p + 2$.

So we only need to handle $r = p + 2$. Write

$$x = e_{022} + e_{111} + ce_{021} + z' \quad \text{where } c \in K \text{ and } z' \in W_{\geq 2}.$$

The next lemma proves that $\chi(x^{[p]}) = 0$ and hence $\mathfrak{p}_\lambda = \mathfrak{p}_\chi$ by Lemma 10.3.6.

Lemma 10.3.7. *If $r = p + 2$, then $\chi(x^{[p]}) = 0$.*

Proof. First, note that $(z')^{[p]} = 0$ since $z' \in W_{\geq 2}$. Next, I claim that

$$(e_{022} + e_{111} + ce_{021})^{[p]} = 0.$$

Indeed, set $z_1 := e_{022} + e_{111}$ and $z_2 := e_{021}$. Then $(e_{022} + e_{111} + ce_{021})^{[p]} - z_1^{[p]} - z_2^{[p]}$ is a linear combination of terms

$$[x_1, [x_2, \ldots, [x_{p-1}, x_p] \ldots]] \qquad (*)$$

where each x_i is either z_1 or z_2. Set $x_{p-1} = z_1$ and $x_p = z_2$. Note that $[z_1, e_{0j1}] = (j-1)e_{0,j+1,1}$ and $[e_{021}, e_{0j1}] = 0$ for all j. Hence $(\text{ad } z_1)^k(z_2) \in Ke_{0,k+2,1}$. If there exists $1 \leq i < p-1$ such that $x_i = z_2$, then the term in $(*)$ is zero [let $i < p-1$ be maximal and get:

$$[x_1, [x_2, \ldots, [x_{p-1}, x_p] \ldots]] = [x_1, [x_2, \ldots, x_{i-1}, [x_i, (\text{ad } z_2)^{p-i-1}(z_1)] \ldots]].$$

Since $(\text{ad } z_2)^{p-i-1}(z_1) \in Ke_{0,p+1-i,1}$ and $[x_i, e_{0,p+1-i,1}] = 0$, we have that terms in $(*)$ with $x_i = z_2$ for some $1 \leq i < p-1$ are zero].

Suppose that $x_1 = x_2 = \cdots = x_{p-1} = z_1$. Then

$$[x_1, [x_2, \ldots, [x_{p-1}, x_p] \ldots]] = (\text{ad } z_2)^{p-1}(z_1) \in Ke_{0,p+1,1} = 0$$

since $e_{0,p+1,1} = 0$. This implies that $(e_{022} + \frac{r}{2}e_{111} + ce_{021})^{[p]} - z_1^{[p]} - z_2^{[p]} = 0$. But $z_2^{[p]} = 0$ since $e_{021}^{[p]} = 0$.

Next, consider $z_1 = e_{022} + e_{111}$. Then $z_1^{[p]} - e_{022}^{[p]} - e_{111}^{[p]}$ is a linear combination of terms

$$[x_1, [x_2, \ldots, [x_{p-1}, x_p] \ldots]] \qquad (**)$$

where each x_i is either e_{022} or e_{111}. We may assume that $x_{p-1} = e_{022}$ and $x_p = e_{111}$. Note that $[e_{022}, e_{1j1}] = je_{1,j+1,1}$ and $[e_{111}, e_{1j1}] = 0$ for all j. Hence $(\text{ad } e_{022})^k(e_{111}) \in Ke_{1,k+1,1}$.

If there exists $1 \leq i < p-1$ such that $x_i = e_{111}$, then the term in $(**)$ is zero [let $i < p-1$ be maximal and get:

$$[x_1, [x_2, \dots, [x_{p-1}, x_p] \dots]] = [x_1, [x_2, \dots, x_{i-1}, [x_i, (\text{ad } e_{022})^{p-i-1}(e_{111})] \dots]].$$

Since $(\text{ad } e_{022})^k(e_{111}) \in Ke_{1,k+1,1}$ and $[x_i, e_{1,p-i,1}] = 0$, we have that terms in $(**)$ with $x_i = e_{111}$ for some $1 \leq i < p-1$ are zero].

Suppose that $x_1 = x_2 = \cdots = x_{p-1} = e_{022}$. Then

$$[x_1, [x_2, \dots, [x_{p-1}, x_p] \dots]] = (\text{ad } e_{022})^{p-1}(e_{111}) \in Ke_{1,p,1} = 0$$

since $e_{1,p,1} = 0$. It follows, since $e_{022}^{[p]} = 0 = e_{111}^{[p]}$, that

$$(e_{022} + e_{111} + ce_{021})^{[p]} = 0$$

as claimed in the beginning.

Since $x = e_{022} + e_{111} + ce_{021} + z'$ and $(z')^{[p]} = 0$ also, we get that $x^{[p]}$ is a linear combination of terms

$$[x_1, [x_2, \dots, [x_{p-1}, x_p] \dots]] \qquad (***)$$

where each x_i is either $e_{022} + e_{111} + ce_{021}$ or z'. If z' occurs s times in $(***)$ then $[x_1, [x_2, \dots, [x_{p-1}, x_p] \dots]] \in W_{\geq p+s}$. We can assume that $x_1 = z'$ in order to prove that χ vanishes on all terms in $(***)$. Indeed, if $x_1 = e_{022} + e_{111} + ce_{021}$ note that

$$\chi([e_{022} + e_{111} + ce_{021} + z', [x_2, \dots, [x_{p-1}, x_p] \dots]]) = 0$$

since $e_{022} + e_{111} + ce_{021} + z' \in \mathfrak{s}_{022}^{\chi}$. Therefore

$$\chi([e_{022} + e_{111} + ce_{021}, [x_2, \dots, [x_{p-1}, x_p] \dots]]) = -\chi([z', [x_2, \dots, [x_{p-1}, x_p] \dots]]).$$

That is; we may assume that $x_1 = z'$. So we have $s > 1$ [since we can assume $x_p \neq x_{p-1}$] and hence χ of height $p+2$ vanishes on all terms in $(***)$; therefore $\chi(x^{[p]}) = 0$ also. $\square$

Corollary 10.3.8. *Suppose that $\mathfrak{p}_\chi$ is unipotent and $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$. Then there exists $\lambda \in W_{012}^*$ such that $\mathfrak{p}_\lambda$ is compatible with χ and $\mathfrak{p}_\lambda = \mathfrak{p}_\chi$.*

10.4 Applications

Let $\chi \in W^*$. The stabilisers of χ in W and $W_{\geq 0}$ and W_{012} are defined as (see Section 9.2):

$$\begin{aligned} \mathfrak{c}_W(\chi) &= \{y \in W \mid \chi([y,x]) = 0 \text{ for all } x \in W\}, \\ \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) &= \{y \in W_{\geq 0} \mid \chi([y,x]) = 0 \text{ for all } x \in W_{\geq 0}\}, \\ \mathfrak{c}_{W_{012}}(\chi_{|W_{012}}) &= \{y \in W_{012} \mid \chi([y,x]) = 0 \text{ for all } x \in W_{012}\}. \end{aligned}$$

Some general observations: If $\mathfrak{g}$ is a Lie algebra over an arbitrary field and $\lambda \in \mathfrak{g}^*$, then it is well known that $\mathfrak{c}_\mathfrak{g}(\lambda)$ is a Lie subalgebra of $\mathfrak{g}$ (in fact a Lie p–subalgebra if $\mathfrak{g}$ is restricted) and its codimension in $\mathfrak{g}$ is even.

Suppose that $\mathfrak{g}$ is restricted. Then we define the *rank* of $\mathfrak{c}_\mathfrak{g}(\lambda)$ as the maximal dimension of all tori in $\mathfrak{c}_\mathfrak{g}(\lambda)$. We will write rk $\mathfrak{c}_\mathfrak{g}(\lambda)$ for the rank of $\mathfrak{c}_\mathfrak{g}(\lambda)$. Clearly, we have rk $\mathfrak{c}_\mathfrak{g}(\lambda) = $ rk $\mathfrak{c}_\mathfrak{g}(\lambda^g)$ for any $g \in \text{Aut}(\mathfrak{g})$ such that $g(x^{[p]}) = g(x)^{[p]}$ for all $x \in \mathfrak{g}$ (i.e., g is a restricted automorphism on $\mathfrak{g}$).

Going back to W we get rk $\mathfrak{c}_W(\chi) = $ rk $\mathfrak{c}_W(\chi^g)$ for all $g \in \text{Aut}(W)$ (since automorphisms on W are restricted). It follows by (2.14) that $g(W_{\geq 0}) = W_{\geq 0}$; therefore we have rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = $ rk $\mathfrak{c}_{W_{\geq 0}}(\chi^g_{|W_{\geq 0}})$ for $g \in \text{Aut}(W)$.

Lemma 10.4.1. *Let $\chi \in W^*$ of height $r > 1$ such that $\chi([W_{012}, x]) = 0 \neq \chi([e_{102}, x])$ for some $x \in W_{r-1}$. Then $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ is a subspace of $\mathfrak{c}_{W_{012}}(\chi_{|W_{012}})$ of codimension 1 and $\mathfrak{p}_\chi$ is a polarization of $\chi_{|W_{\geq 0}}$.*

Proof. Let $y = ae_{102} + be_{012} + ce_{101} + de_{011} + v \in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ for some $v \in W_{\geq 1}$ and get $a = 0$ from $0 = \chi([x, y]) = a\chi([e_{102}, x])$; thus we have $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) \subset W_{012}$. Now conclude by 1) in Section 9.2. $\square$

Lemma 10.4.2. *Let $\chi \in W^*$ of height $r > 1$. Then we have* rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) \in \{0, 1\}$.

Proof. If $\mathfrak{h} := \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ has rank 2, then $\mathfrak{h}$ contains a maximal torus of $W(2)$. Using Demushkin's result [4, Thm. 1] in conjunction with $\mathfrak{h}$ being contained in $W_{\geq 0}$, we can assume that $T_0 := Ke_{012} \oplus Ke_{101} \subset \mathfrak{h}$. Since T_0 is self–centralizing, it follows that $W_{\geq 1} \subset \operatorname{Ker}(\chi)$, whence $r \leq 1$ — contradiction. $\square$

Lemma 10.4.3. *Let $\chi \in W^*$ of height $r > 1$ such that χ does not have* Type II.a *as in* 5.2 *if $r = 2p-3$. We have* rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 1$ *if and only if all polarizations of $\chi_{|W_{\geq 0}}$ are non unipotent.*

Proof. First, note that both claims in the lemma are $\operatorname{Aut}(W)$–stable [let $g \in \operatorname{Aut}(W)$: Then rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 1$ if and only if rk $\mathfrak{c}_{W_{\geq 0}}(\chi^g_{|W_{\geq 0}}) = 1$ and all polarizations of $\chi_{|W_{\geq 0}}$ are non unipotent if and only if all polarizations of $\chi^g_{|W_{\geq 0}}$ are non unipotent: For the last statement note that the map $P \longmapsto g^{-1}(P)$ induces a bijection between the set of polarizations of $\chi_{|W_{\geq 0}}$ and the set of polarizations of $\chi^g_{|W_{\geq 0}}$].

Suppose that rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 1$ and let $h \in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ be a nonzero toral element. If $P \subset W_{\geq 0}$ is any polarization of $\chi_{|W_{\geq 0}}$ then $\chi_{|W_{\geq 0}}([P + Kh, P + Kh]) = 0$ since $h \in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$; hence $h \in P$ by maximality. In particular, P is non unipotent.

Suppose that any polarization of $\chi_{|W_{\geq 0}}$ is non unipotent. The remark in the beginning of the proof together with Lemma 7.3.1 say that we can assume $\chi([x, e_{102}]) \neq 0 = \chi([x, W_{012}])$ for some $x \in W_{r-1}$. In particular, $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) \subset W_{012}$ and by 1) in Section 9.2 then: Any polarization $\mathfrak{p} \subset W_{012}$ of $\chi_{|W_{012}}$ is a polarization of $\chi_{|W_{\geq 0}}$ also. In particular, the Vergne polarization $\mathfrak{p}_\chi$ is a polarization of $\chi_{|W_{\geq 0}}$ and therefore non unipotent. Write $\mathfrak{p}_\chi = Kh \oplus \mathfrak{p}_\chi \cap W_{011}$ for some h with $h^{[p]} = h$.

Let g be an automorphism on W with $g(W_{012}) = W_{012}$ such that $0 \neq h \in \mathfrak{p}_{\chi^g}$ and $h^{[p]} = h$ and $\chi^g([h, W_{012}]) = 0$ (see Lemma 9.4.6). Therefore, $g(h) \in \mathfrak{c}_{W_{012}}(\chi_{|W_{012}})$.

I claim that we can find $c \in K$ such that $h' = g(h) + cx \in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$. Indeed, define $c \in K$ such that $\chi([h', e_{102}]) = 0$ (here we use that $\chi([x, e_{102}]) \neq 0$). Therefore $\chi([h', e_{102}]) = 0$. But we also have $\chi([h', W_{012}]) = 0$, since $\chi([x, W_{012}]) = 0$ and $\chi([g(h), W_{012}]) = \chi^g([h, W_{012}]) = 0$; it follows that $h' \in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$. Since $g(h)$ is a toral element, we get rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 1$ (see Lemma B.1.3). $\square$

Remark 10.4.4. The proof of Lemma 10.4.3 shows: If $\chi([e_{102}, x]) \neq 0 = \chi([W_{012}, x])$ for some $x \in W_{r-1}$, then $\mathfrak{p}_\chi$ is non unipotent if and only if rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 1$. Indeed, the assumption on x says that $\mathfrak{p}_\chi$ is a polarization of $\chi_{|W_{\geq 0}}$. Therefore, rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 1$ implies that $\mathfrak{p}_\chi$ is non unipotent by Lemma 10.4.3. The other implication follows in a similar way as the last part of the proof above.

Theorem 10.4.5. *Let χ be a character of height $r > 1$ such that χ does not have* Type II.a *as in* 5.2 *if $r = 2p - 3$. Then there are p isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules if and only if* rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 1$. *If* rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 0$ *and $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$ or $r \leq p$ or $\chi([e_{011}, W_{r-1}]) \neq 0$, then there is 1 isomorphism class of irreducible $U_\chi(W_{\geq 0})$–modules.*

Proof. First, note that both conditions in the theorem are $\mathrm{Aut}(W)$–stable: Indeed, the number of isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules is equal to the number isomorphism classes of irreducible $U_{\chi^g}(W_{\geq 0})$–modules for any $g \in \mathrm{Aut}(W)$, since we have an isomorphism $U_\chi(W_{\geq 0}) \simeq U_{\chi^g}(W_{\geq 0})$ of K–algebras. We also have $\mathrm{rk}\ \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = \mathrm{rk}\ \mathfrak{c}_{W_{\geq 0}}(\chi^g_{|W_{\geq 0}})$ for any $g \in \mathrm{Aut}(W)$.

Therefore we can assume, by Lemma 7.3.1, that $\chi([e_{102}, x]) \neq 0 = \chi([W_{012}, x])$ for some $x \in W_{r-1}$ and by Theorem 7.3.2 then that induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{012})$–modules and isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules.

To the proof: If $\mathrm{rk}\ \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 1$ then all polarizations of $\chi_{|W_{\geq 0}}$ are non unipotent by Lemma 10.4.3. It follows from Lemma 10.4.1 that $\mathfrak{p}_\chi$ is a polarization of $\chi_{|W_{\geq 0}}$ and therefore non unipotent. Choose $\lambda \in W_{012}^*$ such that $\lambda(x)^p - \lambda(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_\lambda$ and such that $\mathfrak{p}_\lambda = \mathfrak{p}_\chi$. See Theorem 10.1.1.a. Any maximal torus in $\mathfrak{p}_\lambda = \mathfrak{p}_\chi$ has dimension 1 if $\mathfrak{p}_\chi$ is non unipotent [see Lemma 9.4.3], and so there are, by Lemma 9.3.7 with $L = W_{012}$, exactly p isomorphism classes of irreducible $U_\chi(W_{012})$–modules and since induction is a bijection (by assumption), we conclude that there are p isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules.

Suppose that there are p isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules. Since induction is a bijection (by assumption), there exist p isomorphism classes of irreducible $U_\chi(W_{012})$–modules or equivalently, by Lemma 9.3.7, $\mathfrak{p}_\lambda$ is non unipotent [for any $\lambda \in W_{012}^*$ such that $\lambda(x)^p - \lambda(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_\lambda$]. Now apply Theorem 10.1.1 to get that $\mathfrak{p}_\chi$ is non unipotent also. Finally conclude via Remark 10.4.4 (recall our assumption on χ in the beginning of the proof).

If $\mathrm{rk}\ \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 0$ then $\mathfrak{p}_\chi$ is unipotent by Remark 10.4.4. Now apply Theorem 10.1.1.b (recall our assumption in this case) and Lemma 9.3.7 with our assumption that induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{012})$–modules and isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules: It follows that there is 1 isomorphism class of irreducible $U_\chi(W_{\geq 0})$–modules. ☐

Theorem 10.4.6. *Let χ be a character of height $1 < r \leq 2p - 3$ such that χ does not have* Type II.a *as in* 5.2 *if $r = 2p - 3$. If $\mathfrak{p}_\chi$ is non unipotent or $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$ or $r \leq p$ or $\chi([e_{011}, W_{r-1}]) \neq 0$, then each irreducible $U_\chi(W_{\geq 0})$–module has dimension* $p^{\mathrm{codim}_{W_{\geq 0}} \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})/2}$.

Proof. First, apply Lemma 7.3.1 and find $g \in \mathrm{Aut}(W)$ such that $\chi^g([W_{012}, x]) = 0 \neq \chi^g([e_{102}, x])$ for some $x \in W_{r-1}$. But $g(W_{\geq 0}) = W_{\geq 0}$ so g induces an isomorphism $\mathfrak{c}_{W_{\geq 0}}(\chi^g_{|W_{\geq 0}}) \simeq \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$. Therefore, $\mathrm{codim}_{W_{\geq 0}} \mathfrak{c}_{W_{\geq 0}}(\chi^g_{|W_{\geq 0}}) = \mathrm{codim}_{W_{\geq 0}} \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ and we can thus assume that $\chi([W_{012}, x]) = 0 \neq \chi([e_{102}, x])$ for some $x \in W_{r-1}$ (irreducible $U_\chi(W_{\geq 0})$–modules and irreducible $U_{\chi^g}(W_{\geq 0})$–modules have the same dimension since $U_\chi(W_{\geq 0}) \simeq U_{\chi^g}(W_{\geq 0})$ as K–algebras).

Now apply Lemma 10.4.1 and get $\dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = \dim_K \mathfrak{c}_{W_{012}}(\chi_{|W_{012}}) - 1$. Moreover, the existence of x says that induction induces a bijection between isomorphism classes of irreducible $U_\chi(W_{012})$–modules and isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules. See Lemma 7.1.1.

Pick $\lambda \in W_{012}^*$ such that $\mathfrak{p}_\lambda$ is compatible with χ (i.e., $\lambda(x)^p - \lambda(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_\lambda$) and such that $\mathfrak{p}_\lambda = \mathfrak{p}_\chi$. This can be done by Theorem 10.1.1 (recall our assumptions). Now the dimension of each irreducible $W_{\geq 0}$–module with p–character χ

(use that $\dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = \dim_K \mathfrak{c}_{W_{012}}(\chi_{|W_{012}}) - 1$):

$$p \cdot p^{\operatorname{codim}_{W_{012}} \mathfrak{p}_\chi} = p^{\operatorname{codim}_{W_{\geq 0}} \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})/2}.$$

The proof is completed. □

Suppose that $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$. Then it is easy to see that $\mathfrak{c}_W(\chi) \subset \mathfrak{st}(\chi, W_{\geq r})$ from the definitions; hence $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$. This implies that

$$\mathfrak{c}_W(\chi) \subset \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}). \tag{10.3}$$

The next lemma says that $\mathfrak{c}_W(\chi)$ is a subalgebra of $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ of codimension 2 if (10.3) holds. In the proof we only use (10.3) and not the assumption $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$.

Lemma 10.4.7. *If $\mathfrak{c}_W(\chi) \subset \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$, then we have $\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) -$ 2. In particular; any polarization $P \subset W_{\geq 0}$ of $\chi_{|W_{\geq 0}}$ is polarization of χ also.*

Proof. In general, if V is a vector space and $f : V \times V \longrightarrow K$ is a bilinear, antisymmetric form, then $\operatorname{codim}_V \mathfrak{c}_V(f)$ is even, where $\mathfrak{c}_V(f) = \{v \in V \mid f(v, V) = 0\}$. Suppose that $\operatorname{codim}_V \mathfrak{c}_V(f) = 2m_V$; then m_V is the maximal dimension of an isotropic subspace in $V/\mathfrak{c}_V(f)$ and therefore $m_V + \dim_K \mathfrak{c}_V(f)$ is the maximal dimension of an isotropic subspace in V. Apply these observations to $V = W$ (resp. $V = W_{\geq 0}$) and $f = \chi([\ ,\ ])$ and use that any isotropic subspace in $W_{\geq 0}$ is also an isotropic subspace in W: We obtain that $\dim_K \mathfrak{c}_W(\chi) \geq \dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) - 2$.

We also have $\dim_K \mathfrak{c}_W(\chi) - \dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) + 2 \in 2\mathbb{Z}$ since $\mathfrak{c}_{W'}(\chi) \subset W'$ is a subspace of even codimension for $W' = W$ or $W' = W_{\geq 0}$. Together with the inclusion $\mathfrak{c}_W(\chi) \subset \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$, this leave us with two possibilities: Either $\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) - 2$ or $\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$. The assumption on the height of χ says that $W_{\geq r} \subset \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$. But we have $W_{\geq r} \not\subset \mathfrak{c}_W(\chi)$ since $[W_{-1}, W_r] = W_{r-1}$ and $\chi(W_{r-1}) \neq 0$; hence we cannot have $\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$. We get $\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) - 2$ as required.

The final statement follows since $P \subset W$ is a Lie p–subalgebra with $\chi([P, P]) = 0$ and the dimension formula given in (9.6) follows by the part of the claim already proved. □

In the next lemma we will only use the inclusion in (10.3) and not the assumption $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$.

Lemma 10.4.8. *Let $\chi \in W^*$ of height $r > 1$ with $\mathfrak{c}_W(\chi) \subset \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$. Then we have* $\operatorname{rk} \mathfrak{c}_W(\chi) = 0$ *or* $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$.

Proof. Follows immmediately from Lemma 10.4.2 and $\mathfrak{c}_W(\chi) \subset \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$. □

Lemma 10.4.9. *Let $\chi \in W^*$ of height $r > 1$ with $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$ such that χ does not have* Type II.a *as in* 5.2 *if $r = 2p - 3$. We have* $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$ *if and only if all polarizations of χ are non unipotent.*

Proof. First, note that both claims in the lemma are $\operatorname{Aut}(W)$–stable [let $g \in \operatorname{Aut}(W)$: Then $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$ if and only if $\operatorname{rk} \mathfrak{c}_W(\chi^g) = 1$ and all polarizations of χ are non unipotent if and only if all polarizations of χ^g are non unipotent: For the last statement note that the map $P \longmapsto g^{-1}(P)$ induces a bijection between the set of polarizations of χ and the set of polarizations of χ^g].

Suppose that $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$ and let $h \in \mathfrak{c}_W(\chi)$ be a nonzero toral element. If P is any polarization of χ then $\chi([P + Kh, P + Kh]) = 0$ since $h \in \mathfrak{c}_W(\chi)$; hence $h \in P$ by maximality. In particular, P is non unipotent.

Suppose that any polarization of χ is non unipotent. The remark in the beginning of the proof together with Lemma 7.3.1 say that we can assume $\chi([x, e_{102}]) \neq 0 = \chi([x, W_{012}])$ for some $x \in W_{r-1}$. In particular, $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) \subset W_{012}$ and by Lemma 10.4.1 then: Any polarization $\mathfrak{p} \subset W_{012}$ of $\chi_{|W_{012}}$ is a polarization of $\chi_{|W_{\geq 0}}$ and so a polarization of χ by Lemma 10.4.7; hence non unipotent. In particular, the Vergne polarization $\mathfrak{p}_\chi$ is non unipotent. Write $\mathfrak{p}_\chi = Kh \oplus \mathfrak{p}_\chi \cap W_{011}$ for some h with $h^{[p]} = h$.

Let g be an automorphism g on W with $g(W_{012}) = W_{012}$ such that $0 \neq h \in \mathfrak{p}_{\chi^g}$ and $h^{[p]} = h$ and $\chi^g([h, W_{012}]) = 0$ (see Lemma 9.4.6). It follows that $g(h) \in \mathfrak{c}_{W_{012}}(\chi_{|W_{012}})$.

I claim that we can find $f \in W_r$ and $c \in K$ such that $h' = g(h) + f + cx \in \mathfrak{c}_W(\chi)$ for $x \in W_{r-1}$ defined such that $\chi([x, e_{102}]) \neq 0 = \chi([x, W_{012}])$. Indeed, define $c \in K$ such that $\chi([h', e_{102}]) = 0$ (here we use that $\chi([x, e_{102}]) \neq 0$). I claim that there exists $f \in W_r$ such that $\chi([f, e_{001}]) = -\chi([g(h) + cx, e_{001}])$ and $\chi([f, e_{002}]) = -\chi([g(h) + cx, e_{002}])$: Indeed, use that $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$ and apply Remark 6.3.6 with $\mathfrak{a} = W_{\geq r}$ and $\mathfrak{h} = W_{\geq 0}$ to $\varphi(z) = -\chi([g(h) + cx, z])$ for $z \in W_{-1}$. Therefore $\chi([h', W_{-1}]) = 0$. We have defined $c \in K$ such that $\chi([h', e_{102}]) = 0$ and moreover $\chi([h', W_{012}]) = 0$, since $\chi([x, W_{012}]) = 0 = \chi([f, W_{012}])$ and $\chi([g(h), W_{012}]) = \chi^g([h, W_{012}]) = 0$. It follows that $h' \in \mathfrak{c}_W(\chi)$ and therefore $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$ since $g(h)$ is toral (apply Lemma B.1.3 in Appendix B). The proof is completed. $\square$

Lemma 10.4.10. *Suppose that $\chi \in W^*$ of height $1 < r \leq 2p - 3$ with $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$. Then $\operatorname{rk} \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 1$ if and only if $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$.*

Proof. The "if" part is easy by Lemma 10.4.2 and since we have $\mathfrak{c}_W(\chi) \subset \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ by (10.3). Next, suppose that $\operatorname{rk} \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 1$ and let $h \in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ be a nonzero toral element. I claim that we can find $f \in W_r$ such that $h' = h + f \in \mathfrak{c}_W(\chi)$. Indeed, apply Remark 6.3.6 with $\mathfrak{a} = W_{\geq r}$ and $\mathfrak{h} = W_{\geq 0}$ to $\varphi(z) = -\chi([h, z])$ for $z \in W_{-1}$: There exists $f \in W_r$ such that $\chi([h + f, W_{-1}]) = 0$. It follows that $h' = h + f \in \mathfrak{c}_W(\chi)$ since $h \in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ and $\chi([f, W_{\geq 0}]) = 0$. Therefore we have $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$ since h is toral (see Lemma B.1.3) and $\operatorname{rk} \mathfrak{c}_W(\chi) \leq 1$ by Lemma 10.4.8. $\square$

If $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$ then, by Theorem 8.1.1, induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and the isomorphism classes of irreducible $U_\chi(W)$–modules. Now, apply Theorem 10.4.5 and Lemma 10.4.10 and find:

Theorem 10.4.11. *Let χ be a character of height $1 < r \leq 2p - 3$ such that $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$ and such that χ does not have* Type II.a *as in* 5.2 *if $r = 2p - 3$. Then there are p isomorphism classes of irreducible $U_\chi(W)$–modules if and only if $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$. If $\operatorname{rk} \mathfrak{c}_W(\chi) = 0$, then there is 1 isomorphism class of irreducible $U_\chi(W)$–modules.*

From Theorem 10.4.6 and Lemma 10.4.7 we get:

Theorem 10.4.12. *Let χ be a character of height $1 < r \leq 2p-3$ such that χ does not have* Type II.a *as in* 5.2 *if $r = 2p-3$. If $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$, then each irreducible $U_\chi(W)$–module has dimension $p^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$.*

11 Exceptional characters

In this section we will study characters where we cannot apply Theorem 8.1.1 in order to study induction from $W_{\geq 0}$ to W. First, let us introduce those characters.

11.1 Definition

In the following we consider $\chi \in W^*$ of height r with $p-2 < r \leq 2p-3$. We say that χ is *exceptional* if the stabilizer of χ in $W_{\geq r}$ intersects W_{-1} non trivial or equivalent:

$$\mathfrak{st}(\chi, W_{\geq r}) \neq W_{\geq 0}.$$

Note that $\mathfrak{st}(\chi, W_{\geq r}) = \{w \in W \mid \chi([w,x]) = 0 \, \forall x \in W_{\geq r}\}$. It is clear that there exist characters with that property: Indeed, consider χ of height r and let $0 \leq j \leq p-2$ such that $r = p-1+j$. Suppose that $\chi(e_{\alpha\beta\gamma}) = 0$ for all $\gamma = 1,2$ and all $0 \leq \alpha, \beta \leq p-1$ with $\alpha + \beta = r$ and $\alpha < p-1$. Then $e_{001} \in \mathfrak{st}(\chi, W_{\geq r})$ since we have $[e_{001}, W_{\geq r}] \subset \sum_{0 \leq \alpha,\beta \leq p-1_{\alpha < p-1}} \sum_{\gamma=1,2} Ke_{\alpha\beta\gamma} \oplus W_{\geq r}$ and hence $\chi([e_{001}, W_{\geq r}]) = 0$.

So $\mathfrak{st}(\chi, W_{\geq r}) = Ke_{001} \oplus W_{\geq 0}$. In fact, we have:

Lemma 11.1.1. *Let $\chi \in W^*$ of height r. Then $\mathfrak{st}(\chi, W_{\geq r}) = Ke_{001} \oplus W_{\geq 0}$ if and only if $\chi(e_{\alpha\beta\gamma}) = 0$ for all $\gamma = 1,2$ and all $0 \leq \alpha, \beta \leq p-1$ with $\alpha + \beta = r$ and $\alpha < p-1$.*

Proof. The "if" part follows from the remarks we made just before the lemma. Now we assume that $\mathfrak{st}(\chi, W_{\geq r}) = Ke_{001} \oplus W_{\geq 0}$. Then consider $\gamma = 1,2$ and $0 \leq \alpha, \beta \leq p-1$ such that $\alpha + \beta = r$ and $\alpha < p-1$. We find that $e_{\alpha\beta\gamma} = (\alpha+1)^{-1}[e_{001}, e_{\alpha+1,\beta,\gamma}]$ and so the assumption $\mathfrak{st}(\chi, W_{\geq r}) = Ke_{001} \oplus W_{\geq 0}$ implies that $\chi(e_{\alpha\beta\gamma}) = 0$. □

11.2 Conjugation

Let $\chi \in W^*$ be an exceptional character and let $j \in \mathbb{N} \cup \{0\}$ with $j \leq p-2$ and $r = p-1+j$. In the following lemma we will often use the relations (A.4)–(A.11) in Appendix A.

Lemma 11.2.1. *There exists $g \in GL_2(K)$ such that $\mathfrak{st}(\chi^g, W_{\geq r}) = Ke_{001} \oplus W_{\geq 0}$. Moreover, we can choose g such that $\chi^g(e_{p-1,j,1}) = 1$ and $\chi^g(e_{p-1,j,2}) = 0$ or $\chi^g(e_{p-1,j,2}) = 1$ and $\chi^g(e_{p-1,j,1}) = 0$ also.*

Proof. Since $\mathfrak{st}(\chi, W_{\geq r}) \neq W_{\geq 0}$ there exists $\pi \in W_{-1}$ such that $\mathfrak{st}(\chi, W_{\geq r}) = K\pi \oplus W_{\geq 0}$. Let $\pi = ae_{001} + be_{002}$ for some $a, b \in K$ and let $g_1^{-1} \in GL_2(K)$ be defined via

$$g_1^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ if } a = 0 \quad \text{and} \quad g_1^{-1} = \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \text{ if } a \neq 0.$$

It follows that $0 \neq g_1^{-1}(\pi) \in Ke_{001}$ and $\mathfrak{st}(\chi^{g_1}, W_{\geq r}) = g_1^{-1}(\mathfrak{st}(\chi, W_{\geq r})) = Ke_{001} \oplus W_{\geq 0}$. Therefore $\chi^{g_1}(e_{p-1,j,1}) \neq 0$ or $\chi^{g_1}(e_{p-1,j,2}) \neq 0$ by Lemma 11.1.1 and the fact that χ^{g_1} has height r also by [10, 1.2 (1)].

If $\chi^{g_1}(e_{p-1,j,1}) \neq 0$ let g_2 be the lower triangular matrix given by

$$g_2 = \begin{pmatrix} \delta_1 & 0 \\ \chi^{g_1}(e_{p-1,j,2})\chi^{g_1}(e_{p-1,j,1})^{-1}\delta_1 & 1 \end{pmatrix}$$

for some $\delta_1 \in K$ with $\delta_1^{p-2} = \chi^{g_1}(e_{p-1,j,1})^{-1}$. Then we have $\chi^{g_1 \circ g_2}(e_{p-1,j,1}) = 1$ and $\chi^{g_1 \circ g_2}(e_{p-1,j,2}) = 0$ and $\mathfrak{st}(\chi^{g_1 \circ g_2}, W_{\geq r}) = Ke_{001} \oplus W_{\geq 0}$ since $g_2^{-1}(e_{001}) = \delta_1^{-1}e_{001}$. Now set $g = g_1 \circ g_2$.

If $\chi^{g_1}(e_{p-1,j,2}) \neq 0 = \chi^{g_1}(e_{p-1,j,1})$ let $\delta_2 \in K$ such that $\delta_2^{j-1} = \chi^{g_1}(e_{p-1,j,2})^{-1}$. Then the diagonal matrix

$$g_2 = \begin{pmatrix} 1 & 0 \\ 0 & \delta_2 \end{pmatrix}$$

satisfies that $\chi^{g_1 \circ g_2}(e_{p-1,j,2}) = 1$ and $\chi^{g_1 \circ g_2}(e_{p-1,j,1}) = 0$. We still have $\mathfrak{st}(\chi^{g_1 \circ g_2}, W_{\geq r}) = Ke_{001} \oplus W_{\geq 0}$ since $g_2^{-1}(e_{001}) = e_{001}$. The proof is now completed if we set $g = g_1 \circ g_2$. $\square$

11.3 The case that $\chi([e_{001}, W_{\geq r}]) = 0$

Let $\chi \in W^*$ of height r with $p-2 < r \leq 2p-3$ such that $\mathfrak{st}(\chi, W_{\geq r}) = Ke_{001} \oplus W_{\geq 0}$ and let j be given such that $0 \leq j \leq p-2$ and $r = p-1+j$. Let $\mathfrak{p} := \mathfrak{p}_\lambda$ be a polarization of $\lambda \in W_{012}^*$ such that $\lambda(z)^p - \lambda(z^{[p]}) = \chi(z)^p$ for all $z \in \mathfrak{p}$ [the existence follows from Lemma 9.3.9]. If we use Proposition 9.3.5 and Lemma 9.3.7 we see that induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{p})$–modules and the isomorphism classes of irreducible $U_\chi(W_{012})$–modules. There exists $x \in W_{r-1}$ $(= e_{p-2,j+1,k})$ such that $\chi([x, e_{102}]) \neq 0 = \chi([x, W_{012}])$: Clearly, $\chi([e_{p-2,j+1,k}, e_{102}]) \neq 0$ by assumption and since

$$[e_{p-2,j+1,k}, W_{012}] \subset \bigoplus_{0 \leq \alpha < p-1} \bigoplus_{\alpha+\beta=r} Ke_{\alpha\beta\gamma} \oplus W_{\geq r}$$

we also have $\chi([e_{p-2,j+1,k}, W_{012}]) = 0$. Now apply Lemma 7.1.1 and get that induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{012})$–modules and the isomorphism classes irreducible $U_\chi(W_{\geq 0})$–modules also. Hence induction is a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{p})$–modules and the isomorphism classes irreducible $U_\chi(W_{\geq 0})$–modules.

Let K_λ be the one dimensional $U_\chi(\mathfrak{p})$–module where each $z \in \mathfrak{p}$ acts as multiplication with $\lambda(z)$. Set $S_\lambda := U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{p})} K_\lambda$. Note that we have $1 \otimes_\mathfrak{p} 1 \in \mathrm{Soc}_\mathfrak{p} S_\lambda$ with $K \cdot 1 \otimes_\mathfrak{p} 1 \simeq_\mathfrak{p} K_\lambda$ (here $\otimes_\mathfrak{p}$ is a short notation for $\otimes_{U_\chi(\mathfrak{p})}$).

I claim that $\mathrm{Soc}_\mathfrak{p} S_\lambda = K \cdot 1 \otimes_\mathfrak{p} 1$. Otherwise there exists a nonzero element $w \notin K \cdot 1 \otimes_\mathfrak{p} 1$ in $\mathrm{Soc}_\mathfrak{p} S_\lambda$ such that Kw is an irreducible $\mathfrak{p}$–submodule of $\mathrm{Soc}_\mathfrak{p} S_\lambda$ [see Lemma 9.3.2] and one of the following cases will occur:

If $Kw \simeq K_\mu$ for some $\mu \neq \lambda$ [$\mu \in W_{012}^*$ such that $\mu(z)^p - \mu(z^{[p]}) = \chi(z)^p$ for all $z \in \mathfrak{p}$] we obtain from 'Frobenius reciprocity' that $S_\mu \simeq S_\lambda$ which is a contradiction, since K_λ and K_μ are non isomorphic and induction is a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{p})$–modules and the isomorphism classes irreducible $U_\chi(W_{\geq 0})$–modules.

If $Kw \simeq K_\lambda$ we apply 'Frobenius reciprocity' again to get a $U_\chi(W_{\geq 0})$–endomorphism $S_\lambda \longrightarrow S_\lambda$ given by $1 \otimes_\mathfrak{p} 1 \longmapsto w$ and so not proportional to the identity map – contradiction since S_λ is irreducible. We have thus shown:

Lemma 11.3.1. *Let $\chi \in W^*$ of height r such that $\mathfrak{st}(\chi, W_{\geq r}) = Ke_{001} \oplus W_{\geq 0}$. Let $\lambda \in W_{012}^*$ such that the Vergne polarization $\mathfrak{p}$ of λ satisfies $\lambda(x)^p - \lambda(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}$. Then induction is a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{p})$–modules and the isomorphism classes irreducible $U_\chi(W_{\geq 0})$–modules. We have* $\mathrm{Soc}_\mathfrak{p} S_\lambda \simeq_\mathfrak{p} K_\lambda$.

11.4 Socle elements

Keep the assumptions from the previous section. Moreover, assume that $\chi(e_{p-1,j,1}) = 0$ or $\chi(e_{p-1,j,2}) = 0$ [any exceptional character is conjugate to a character of that particular type, see Lemma 11.2.1]. Define $k \in \{1, 2\}$ be defined via $\chi(e_{p-1,j,k}) = 1$.

Since $\mathfrak{st}(\chi, W_{\geq r}) = Ke_{001} \oplus W_{\geq 0}$ it follows from Lemma 11.1.1 that $\chi(e_{\alpha\beta\gamma}) = 0$ for all $0 \leq \alpha, \beta \leq p-1$ with $\alpha + \beta = r$ and $\alpha < p-1$ [and $\gamma = 1, 2$].

There exists $y := e_{p-1,j+1,k} \in W_r$ such that $\chi([y, e_{002}]) \neq 0 = \chi([y, e_{001}])$. Set $x := [y, e_{001}]$. We will use that notation in the proof of the following lemma.

We can choose $\lambda \in W_{012}^*$ such that the Vergne polarization $\mathfrak{p}$ of λ is compatible with χ (i.e., $\lambda(z)^p - \lambda(z^{[p]}) = \chi(z)^p$ for all $z \in \mathfrak{p}$) and such that $\lambda_{|W_{\geq 2}} = \chi_{|W_{\geq 2}}$. See Lemma 10.1.2. Since $[x, e_{001}]^{[p]} = 0$, it follows from the construction in the proof of Lemma 9.3.9 that we can choose $\lambda([x, e_{001}]) = \chi([x, e_{001}])$. Let $M_\lambda := U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S_\lambda$ be the W–module induced from $S_\lambda = U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{p})} K_\lambda$.

Lemma 11.4.1. *If Kw is an irreducible $\mathfrak{p}$–submodule of $\mathrm{Soc}_{\mathfrak{p}} M_\lambda$, then there exists a nonzero $v \in \mathrm{Soc}_{\mathfrak{p}} 1 \otimes S_\lambda$ and $0 \leq b \leq p-1$ such that*

$$w \in e_{001}^b \cdot v + e_{001}^{b-1} \cdot 1 \otimes S_\lambda + \sum_{k<b-1} \sum_{k+m \leq b-1} e_{001}^k e_{002}^m \cdot 1 \otimes S_\lambda.$$

Proof. Write

$$w = \sum_{k+m \leq b} e_{001}^k e_{002}^m \cdot w_{km}$$

for some $w_{km} \in 1 \otimes S_\lambda$ and $0 \leq k, m \leq p-1$ and suppose that b is chosen such that $0 \leq b \leq 2p-2$ and such that $w_{km} \neq 0$ for some k, m with $0 \leq k, m \leq p-1$ and $k + m = b$.

Assume $b > 0$. For $0 \leq k, m \leq p-1$ with $k + m = b$ we consider components in $e_{001}^k e_{002}^m \cdot 1 \otimes S_\lambda$ from $x \cdot w = 0$ and obtain $x \cdot w_{km} = 0$ for all $0 \leq k, m \leq p-1$ with $k + m = b$. This follows since

$$x \cdot w \in \sum_{k+m=b} e_{001}^k e_{002}^m \cdot x \cdot w_{km} + \sum_{k+m<b} e_{001}^k e_{002}^m \cdot 1 \otimes S_\lambda.$$

[If

$$x \cdot w = \sum_{st} e_{001}^s e_{002}^s \otimes v_{st}, \qquad v_{st} \in S_\lambda$$

then we can use the PBW theorem and the assumption $x \cdot w = 0$ to get $v_{st} = 0$ for all s, t. In particular, $v_{km} = 0$ for all k, m with $k + m = b$. So the phrase "*consider components in* $e_{001}^k e_{002}^m \otimes S_\lambda$ *from* $x \cdot w = 0$" means that all $v_{km} = 0$ when $k + m = b$.]

For $i > 0$, consider components in $e_{001}^{b-i} e_{002}^{i-1} \cdot 1 \otimes S_\lambda$ from $y \cdot w = 0$ and get

$$0 = i[y, e_{002}] \cdot w_{b-i,i} + (b+1-i) x \cdot w_{b+1-i,i-1} + y \cdot w_{b-i,i-1}$$

and hence $i[y, e_{002}] \cdot w_{b-i,i} = 0$ since $x \cdot w_{b+1-i,i-1} = 0 = y \cdot w_{b-i,i-1}$ (note that $y \in W_{\geq r}$ annihilates $1 \otimes S_\lambda$ by Lemma 6.3.1 with $\mathfrak{h} = W_{\geq 0}$ and $\mathfrak{a} = W_{\geq r}$). But $\chi([y, e_{002}]) \neq 0$ and therefore, since $[y, e_{002}]^{[p]} = 0$, it follows that $[y, e_{002}]$ acts bijectively on M_λ; hence $w_{b-i,i} = 0$. We conclude that $w_{b-i,i} = 0$ for all i with $0 < i \leq b$.

So we can write

$$w = e_{001}^b \cdot w_{b,0} + \sum_{k+m \leq b-1} e_{001}^k e_{002}^m \cdot w_{km}. \tag{11.1}$$

Let us show that $w_{b,0} \in \mathrm{Soc}_{\mathfrak{p}} 1 \otimes S_\lambda$. For $z \in \mathfrak{p} \cap W_{011}$ we have $[e_{001}, z] \in W_{\geq 0}$ since $[e_{001}, e_{011}] = 0$. This implies that

$$z e_{001}^b = e_{001}^b z + \sum_{k+m \leq b-1} e_{001}^k e_{002}^m z_{km} \quad \text{for some } z_{km} \in W_{\geq 0}.$$

We also have
$$z \cdot \sum_{k+m \le b-1} e_{001}^k e_{002}^m \in \sum_{k+m \le b-1} e_{001}^k e_{002}^m \cdot W_{\ge 0}.$$
If we use the previous remarks with (11.1) and the assumption $z \cdot w = \lambda(z)w$ we get
$$\lambda(z)w = e_{001}^b \cdot z \cdot w_{b,0} + \sum_{k+m \le b-1} e_{001}^k e_{002}^m \cdot w'_{km}$$
for some $w'_{km} \in 1 \otimes S_\lambda$. This equality shows [if we consider components in $e_{001}^b \cdot 1 \otimes S_\lambda$] that $z \cdot w_{b,0} = \lambda(z) w_{b,0}$.

If $\mathfrak{p}$ is non unipotent let $0 \ne h \in \mathfrak{p}$ given by $h = \alpha e_{012} + \beta e_{101} + z$ for some $z \in W_{011}$. Choose $\tau \in W_{012}^*$ such that $Kw \simeq_{\mathfrak{p}} K_\tau$. This implies that
$$\tau(h)w = e_{001}^b \cdot (h - b\beta) w_{b,0} + \sum_{k+m \le b-1} e_{001}^k e_{002}^m \cdot w'_{km}$$
for some $w'_{km} \in 1 \otimes S_\lambda$ and so $h \cdot w_{b,0} = (\tau(h) + b\beta) w_{b,0}$; hence $w_{b,0}$ lies in $\mathrm{Soc}_{\mathfrak{p}}(1 \otimes S_\lambda)$ as required. □

Corollary 11.4.2. *If $\mathfrak{p} \subset Ke_{012} \oplus W_{011}$, then any irreducible $\mathfrak{p}$–submodule of $\mathrm{Soc}_{\mathfrak{p}} M_\lambda$ is isomorphic to K_λ.*

Proof. Let Kw be an irreducible $\mathfrak{p}$–submodule of $\mathrm{Soc}_{\mathfrak{p}} M_\lambda$. It follows from Lemma 11.4.1 that there exists b with $0 \le b \le p-1$ such that
$$w = e_{001}^b \cdot v + e_{001}^{b-1} \cdot v_{b-1,0} + \sum_{k<b-1} \sum_{k+m \le b-1} e_{001}^k e_{002}^m \cdot v_{km}$$
where $0 \ne v \in \mathrm{Soc}_{\mathfrak{p}} 1 \otimes S_\lambda$ and $v_{b-1,0}, v_{km} \in 1 \otimes S_\lambda$. If $\mathfrak{p}$ is unipotent then it follows that any irreducible $U_\chi(\mathfrak{p})$–module is isomorphic to K_λ (apply Lemma 9.1.3 with $T = 0$). So we are left with the case where $\mathfrak{p} = Kh \oplus \mathfrak{p} \cap W_{011}$ for some nonzero $h = e_{012} + z$ with $z \in W_{011}$ and $h^{[p]} = h$ [see Lemma 9.4.3]. There exists, by Lemma 9.1.3, a linear form $\mu \in W_{012}^*$ with $\mu_{|W_{011}} = \lambda_{|W_{011}}$ and $\mu(x)^p - \mu(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}$ such that $Kw \simeq_{\mathfrak{p}} K_\mu$. I claim that $\mu(h) = \lambda(h)$ such that $Kw \simeq_{\mathfrak{p}} K_\mu \simeq_{\mathfrak{p}} K_\lambda$. Since $[e_{012}, e_{001}] = 0$ and $ze_{001} \equiv e_{001} z \pmod{W_{\ge 0}}$ for $z \in W_{011}$ we get
$$hw = \lambda(h) e_{001}^b \cdot v + \sum_{k<b} \sum_{k+m \le b-1} e_{001}^k e_{002}^m \cdot v'_{km}$$
for some $v'_{km} \in 1 \otimes S_\lambda$. But $hw = \mu(h)w$ and hence
$$(\lambda(h) - \mu(h)) e_{001}^b \cdot v \in \sum_{k+m \le b-1} e_{001}^k e_{002}^m \cdot 1 \otimes S_\lambda$$
which is a contradiction unless $\lambda(h) = \mu(h)$. □

As a consequence of Lemma 11.4.1 we get an upper bound for the dimension of $\mathrm{End}_W(M_\lambda)$.

Corollary 11.4.3. *We have $\dim_K \mathrm{End}_W(M_\lambda) \le p$.*

Proof. Since induction satisfies 'Frobenius reciprocity' we have functorial isomorphisms

$$\mathrm{End}_W(M_\lambda) \simeq \mathrm{Hom}_{W_{\geq 0}}(S_\lambda, M_\lambda) \simeq \mathrm{Hom}_{\mathfrak{p}}(K_\lambda, M_\lambda).$$

Define $V_\lambda := \{w \in M_\lambda \mid Kw \simeq_{\mathfrak{p}} K_\lambda\} \cup \{0\}$. We have an isomorphism $V_\lambda \simeq \mathrm{Hom}_{\mathfrak{p}}(K_\lambda, M_\lambda)$ as vector spaces. Let $\mathrm{Soc}_{\mathfrak{p}} 1 \otimes S_\lambda = Kv$ and define for $b = 0, 1, \ldots, p$ the following subspaces:

$$V_\lambda^b := \mathrm{sp}_K\{e_{001}^b \cdot v + \sum_{k+m \leq b-1} e_{002}^k e_{001}^m \cdot v_{km} \in V_\lambda \mid v_{km} \in 1 \otimes S_\lambda\}.$$

I claim that $V_\lambda = \sum_{b=0}^{p-1} V_\lambda^b$. Indeed, we know from from Lemma 11.4.1 that any $w \in V_\lambda$ can be written as

$$w = e_{001}^b \cdot v + \sum_{k+m \leq b-1} e_{002}^k e_{001}^m \cdot v_{km}$$

for some $0 \leq b < p$ and some $v_{km} \in 1 \otimes S_\lambda$. We choose $0 < b_1 < b_2 < \cdots < b_r < p$ such that $V_\lambda^{b_j} \neq 0$ for all $j \in \{1, 2, \ldots, r\}$ and $V_\lambda^b = 0$ for nonzero $b \notin \{b_1, b_2, \ldots, b_r\}$. Take nonzero elements w_{b_j} inside $V_\lambda^{b_j}$ for $j = 1, 2, \ldots, r$. Then

$$V_\lambda = \bigoplus_{j=1}^{r} K w_{b_j} \oplus Kv. \tag{11.2}$$

To see that let $w \in V_\lambda \setminus \{0\}$. If $w \in 1 \otimes S_\lambda$ we have $w \in Kv$. Otherwise $w \in V_\lambda^{b_k}$ for some $k = 1, 2, \ldots, r$ and it follows from Lemma 11.3.1 that there exists $a \in K$ such that $w - a w_{b_k} \in Kv$ or $w - a w_{b_k} \in V_\lambda^{b_j}$ for some $0 < j < k$. In the last case we may use induction on the set $\{1, 2, \ldots, r\}$ to see that $w - a w_{b_k} \in \oplus_{j=1}^{k-1} K w_{b_j} \oplus Kv$. Therefore (11.2) holds and consequently $\dim_K \mathrm{End}_W(M_\lambda) = \dim_K V_\lambda = r + 1 \leq p$. The proof is completed. □

Remark 11.4.4. If $\mathfrak{p} \subset K e_{012} \oplus W_{011}$ and $q \in U_\chi(W_{\geq 0})$ such that $(e_{001} + q) \cdot v \in \mathrm{Soc}_{\mathfrak{p}} M_\lambda$ for some $v \in \mathrm{Soc}_{\mathfrak{p}} 1 \otimes S_\lambda$, then we have $\dim_K \mathrm{End}_W(M_\lambda) = p$. Indeed, consider the W–endomorphism $\psi : M_\lambda \longrightarrow M_\lambda$ given by $\psi(w_{\lambda,0}) = (e_{001} + q) \cdot w_{\lambda,0}$. I claim that $\psi^0 := \mathrm{Id}_{|M_\lambda}, \psi, \psi^2, \ldots, \psi^{p-1}$ are linear independent. Otherwise there exists a dependence relation

$$\sum_{k=0}^{p-1} a_k \psi^k = 0 \tag{11.3}$$

where $a_0, a_1, \ldots, a_{p-1} \in K$ [in (11.3) the zero on the right hand side is the zero map on M_λ]. For each $i = 0, 1, \ldots, p-1$ we can write

$$\psi^i(w_{\lambda,0}) = e_{001}^i \cdot w_{\lambda,0} + \sum_{k+m \leq i-1} e_{002}^k e_{001}^m \cdot v_{kmi} \tag{11.4}$$

for some $v_{kmi} \in 1 \otimes S_\lambda$. Let $i > 0$ be maximal such that $a_i \neq 0$ in (11.3). Now apply (11.3) and (11.4) and get a relation $a_i e_{001}^i \cdot w_{\lambda,0} + \sum_{k+m \leq i-1} e_{001}^k e_{002}^m \cdot v'_{kmi} = 0$ for some $v'_{kmi} \in 1 \otimes S_\lambda$. Since $a_i \neq 0$ this is in contradiction with the PBW–theorem for reduced enveloping algebras. We conclude that $\psi^0 := \mathrm{Id}_{|M_\lambda}, \psi, \psi^2, \ldots, \psi^{p-1}$ are linear independent. On the other hand; By Lemma 11.4.3 it follows that $\dim_K \mathrm{End}_W M_\lambda \leq p$ and therefore $\psi^0 := \mathrm{Id}_{|M_\lambda}, \psi, \psi^2, \ldots, \psi^{p-1}$ form a basis for $\mathrm{End}_W M_\lambda$.

11.5 Two types of characters

Let $\chi \in W^*$ of height r with $p-2 < r \leq 2p-3$ and let j be defined via $r = p-1+j$. It follows that $0 \leq j < p-1$. We will assume that χ is an exceptional character; i.e., a character with $\mathfrak{st}(\chi, W_{\geq r}) \neq W_{\geq 0}$.

Define the following subsets of W:

$$A := \sum_{1 \leq a+b \leq r-1} \sum_{0<b\leq j+1} Ke_{ab1} + \sum_{a=1}^{p-2} Ke_{aj1} \oplus \sum_{0\leq a+b\leq r-1} \sum_{0\leq b\leq j} Ke_{ab2}, \tag{11.5}$$

$$B := \sum_{2\leq a+b\leq r-1} \sum_{a>0} \sum_{0\leq b\leq j} \sum_{c=1,2} Ke_{abc} \oplus K(e_{012} + je_{101}) \oplus Ke_{102} \oplus Ke_{002}. \tag{11.6}$$

We shall consider two types of characters:

Type A : $\tau \in W^*$ of height r with $\tau(e_{p-1,j,1}) = 1$ and $\tau(e_{p-1,j,2}) = 0 = \tau(A)$ and

$$\mathfrak{st}(\tau, W_{\geq r}) = Ke_{001} \oplus W_{\geq 0}.$$

Type B : $\tau \in W^*$ of height r with $\tau(e_{p-1,j,2}) = 1$ and $\tau(e_{p-1,j,1}) = 0 = \tau(B)$ and

$$\mathfrak{st}(\tau, W_{\geq r}) = Ke_{001} \oplus W_{\geq 0}.$$

For any r with $p-2 < r \leq 2p-3$ we define characters of Type A and Type B in the way above.

Lemma 11.5.1. *If $\tau \in W^*$ has* Type A *or* Type B*, then induction induces a bijection between the isomorphism classes of irreducible $U_\tau(W_{012})$–modules and the isomorphism classes of irreducible $U_\tau(W_{\geq 0})$–modules.*

Proof. If τ has Type A, then apply Lemma 7.1.1 with $x = e_{p-2,j+1,1}$ and if τ has Type B then apply Lemma 7.1.1 with $x = e_{p-2,j+1,2}$. ☐

We say that two characters χ and χ' are *conjugate* under $\mathrm{Aut}(W)$ if there exists an automorphism $g \in \mathrm{Aut}(W)$ such that $\chi^g = \chi'$.

Proposition 11.5.2. *If $\chi \in W^*$ of height r and $\mathfrak{st}(\chi, W_{\geq r}) \neq W_{\geq 0}$, then χ is conjugate under* $\mathrm{Aut}(W)$ *to a character of* Type A *or* Type B*. Moreover, no characters of* Type A *and* Type B *are conjugate.*

Proof. By Lemma 11.2.1 we can assume that either

1) $\chi(e_{p-1,j,1}) = 1$ and $\chi(e_{p-1,j,2}) = 0$ and $\mathfrak{st}(\chi, W_{\geq r}) = Ke_{001} \oplus W_{\geq 0}$ or

2) $\chi(e_{p-1,j,2}) = 1$ and $\chi(e_{p-1,j,1}) = 0$ and $\mathfrak{st}(\chi, W_{\geq r}) = Ke_{001} \oplus W_{\geq 0}$.

Note that we have a decomposition of $\mathrm{Aut}(W)$: $\mathrm{Aut}(W) = GL_2(K) \ltimes \mathrm{Aut}^*(W)$, where $\mathrm{Aut}^*(W) = \{g \in \mathrm{Aut}(W) \mid g(D) - D \in W_{\geq i+1}$ for all $D \in W_i$ and all $i\}$. See Section 2.3. The idea is to find $g \in \mathrm{Aut}^*(W)$ such that $\chi^g(A) = 0$ for χ as in 1) and $\chi^g(B) = 0$ for χ as in 2). The construction of g will complete the proof.

If we for each character χ as in 1) and each n with $-1 \leq n \leq r-2$ can find an automorphism $g_n \in \mathrm{Aut}^*(W)$ with $(g_n - \mathrm{Id}_{|W})(W_n) \subset W_{\geq r-1}$ and $(g_n - \mathrm{Id}_{|W})(W_m) \subset W_{\geq r}$ for $m > n$ such that $\chi^{g_n}(A \cap W_n) = 0$, then g can be constructed. Indeed; we construct inductively $g_{r-2}, g_{r-3}, \ldots$ such that for each n the character $\chi^{g_{r-2}\circ\cdots\circ g_n}$ satisfies that $\chi^{g_{r-2}\circ\cdots\circ g_n}(A \cap W_n) = 0$. Now $g := g_{r-2} \circ g_{r-3} \circ \cdots \circ g_{-1}$ works: To see that, let $x \in A \cap W_n$ for some n with $-1 \leq n \leq r-2$. Since $g_{n-1} \circ \cdots \circ g_{-1}(x) \equiv x \pmod{W_{\geq r}}$, it follows that

$$\chi^g(x) = \chi^{g_{r-2}\circ\cdots\circ g_n}(x) = 0.$$

In a similar way, we can prove: If we for each character χ as in 2) and each n with $-1 \leq n \leq r-2$ can find an automorphism $g_n \in \mathrm{Aut}^*(W)$ with $(g_n - \mathrm{Id}_{|W})(W_n) \subset W_{\geq r-1}$ and $(g_n - \mathrm{Id}_{|W})(W_m) \subset W_{\geq r}$ for $m > n$ such that $\chi^{g_n}(B \cap W_n) = 0$, then g can be constructed.

So we only need to find an automorphism g_n with the properties described above. Consider an automorphism on W induced by a K–algebra automorphism φ on $A(2) = K[X_1, X_2]/(X_1^p, X_2^p)$ given by (x_i is the image of X_i in $A(2)$)

$$\begin{aligned}\varphi(x_1) &= x_1 + \textstyle\sum_{k+l=r-n} \sum_{0\leq k,l<p} a_{kl} x_1^k x_2^l,\\ \varphi(x_2) &= x_2 + \textstyle\sum_{k+l=r-n} \sum_{0\leq k,l<p} b_{kl} x_1^k x_2^l.\end{aligned}$$

Set

$$x_n := \sum_{k+l=r-n} \sum_{0\leq k,l<p} a_{kl} e_{kl1} + \sum_{k+l=r-n} \sum_{0\leq k,l<p} b_{kl} e_{kl2} \in W_{r-1-n}.$$

The automorphism g_n satisfies (see (3.2))

$$g_n(y) \equiv y + [x_n, y] \pmod{W_{r-n+s}}$$

for each $y \in W_s$. In particular, we have $g_n \in \mathrm{Aut}^*(W)$ with $(g_n - \mathrm{Id}_{|W})(W_n) \subset W_{\geq r-1}$ and $(g_n - \mathrm{Id}_{|W})(W_m) \subset W_{\geq r}$ for $m > n$.

a) If χ is a character as in 1) above and if $e_{ab1} \in A \cap W_n$ and $e_{ab2} \in A \cap W_n$, then it follows from the formulas

$$\begin{aligned}\chi^{g_n}(e_{ab1}) &= \chi(e_{ab1}) + 2a \cdot a_{p-a,j-b} + b \cdot b_{p-1-a,j+1-b},\\ \chi^{g_n}(e_{ab2}) &= \chi(e_{ab2}) - a_{p-1-a,j+1-b}(j+1-b)\end{aligned}$$

that we can choose appropriate $a_{p-1-a,j+1-b}, b_{p-1-a,j+1-b} \in K$ such that $\chi^{g_n}(e_{ab1}) = 0 = \chi^{g_n}(e_{ab2})$: For each a, b with $a + b = n + 1$ and $b \leq j$ and $a \leq p-1$ choose $a_{p-1-a,j+1-b} \in K$ such that $\chi^{g_n}(e_{ab2}) = 0$ and for each a, b with $a + b = n + 1$ and $0 < b \leq j + 1$ and $a \leq p - 1$ choose $b_{p-1-a,j+1-b} \in K$ such that $\chi^{g_n}(e_{ab1}) = 0$. If $j = 0$ (and hence $r = p - 1$) we can choose $a_{p-(n+1),0,1} \in K$ (for $1 \leq n + 1 \leq p - 2$) such that $\chi^{g_n}(e_{n+1,0,1}) = 0$. It follows that there exists an automorphism g_n such that $\chi^{g_n}(A \cap W_n) = 0$.

2) If χ is a character as in 2) above and if $e_{ab1} \in B \cap W_n$ and $e_{ab2} \in B \cap W_n$, then it follows from the formulas

$$\begin{aligned}\chi^{g_n}(e_{ab1}) &= \chi(e_{ab1}) + a \cdot b_{p-a,j-b},\\ \chi^{g_n}(e_{ab2}) &= \chi(e_{ab2}) + a \cdot a_{p-a,j-b} - b_{p-1-a,j+1-b}(j+1-2b)\end{aligned}$$

that we can choose $a_{p-a,j-b}, b_{p-a,j-b} \in K$ such that $\chi^{g_n}(e_{ab1}) = 0 = \chi^{g_n}(e_{ab2})$. For $n \geq 1$ and each a, b with $a + b = n + 1$ and $a > 0$ choose $a_{p-a,j-b}, b_{p-a,j-b} \in K$ such that $\chi^{g_n}(e_{ab1}) = 0 = \chi^{g_n}(e_{ab2})$. For a, b with $a + b = 1$ choose $b_{p-1,j}, b_{p-2,j+1} \in K$ such that $\chi^{g_n}(e_{102}) = 0$ and $\chi^{g_n}(e_{012} + je_{101}) = 0$. For $n = -1$ choose $b_{p-1,j+1} \in K$ such that $\chi^{g_n}(e_{002}) = 0$. It follows that $\chi^{g_n}(B \cap W_n) = 0$.

For the final remark apply Lemma 11.6.1 below. The proof is completed. □

11.6 A $GL_2(K)$–submodule of W_{r-1}

For χ of height r, where r is given by $r = p-1+j$, we define

$$V_{r-1} := \sum_{a=j+1}^{p-1} K\big((r+1-a)e_{a,r-a,1} - ae_{a-1,r+1-a,2}\big). \tag{11.7}$$

Note that V_{r-1} is a $GL_2(K)$–submodule of W_{r-1}: To see this, define for each a with $j+1 \le a \le p-1$ elements $v_a = (r+1-a)e_{a,r-a,1} - ae_{a-1,r+1-a,2}$ and apply the relations in Appendix A to get:

$$\begin{aligned}
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot v_a &= -v_{r+1-a}, \\
\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \cdot v_a &= t_1^{a-1}t_2^{r-a}v_a \ \text{ for } t_1, t_2 \in K^*, \\
\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \cdot v_a &= \textstyle\sum_{s=0}^{a} \binom{a}{s}\alpha^s \cdot v_{a-s} \quad \alpha \in K.
\end{aligned}$$

But any $GL_2(K)$–matrix can be written as a diagonal matrix composed with a lower triangular matrix with 1 at the diagonal or a composition of a diagonal matrix, lower triangular matrices with 1 at the diagonal and the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Use the Bruhat decomposition of $GL_2(K)$ in [23, 8]. From the relations above, we conclude that V_{r-1} is a $GL_2(K)$–submodule of W_{r-1}.

Lemma 11.6.1. *Suppose that* $\mathfrak{st}(\chi, W_{\ge r}) \neq W_{\ge 0}$ *and let* V_{r-1} *be defined as in* (11.7)*. Then* χ *is conjugate under* $\mathrm{Aut}(W)$ *to a character of type* Type A *if and only if* $\chi(V_{r-1}) \neq 0$.

Proof. We know that V_{r-1} is a $GL_2(K)$–submodule of W_{r-1}. Therefore $g(V_{r-1}) \subset V_{r-1} + W_{\ge r}$ (use the decomposition of $\mathrm{Aut}(W)$). It follows that $\chi(V_{r-1}) \neq 0$ if and only if $\chi^g(V_{r-1}) \neq 0$ for all $g \in \mathrm{Aut}(W)$. Therefore: If χ is conjugate to a character of Type A, then we can use (11.7) and the definition of Type A characters to get $\chi(V_{r-1}) \neq 0$. In a similar way; if χ is conjugate to a character of Type B then $\chi(V_{r-1}) = 0$. □

11.7 Restricted subalgebras

In this section we will introduce two restricted Lie subalgebras of W. The subalgebras defined below will be of great importance when we shall describe the set of irreducible $U_\chi(W)$–modules for $p = 3$. First, define

$$\mathfrak{g} := \bigoplus_{0 \le i < p} \bigoplus_{0 < j < p} Ke_{ij2} \oplus \bigoplus_{0 \le i < p} \bigoplus_{0 \le j < p} Ke_{ij1}. \tag{11.8}$$

We can think of $\mathfrak{g}$ as W except all e_{i02} for $i = 0, 1, \ldots, p-1$. Let us check that $\mathfrak{g}$ is in fact a restricted Lie subalgebra of W. Consider two basis elements $e_{abc}, e_{\alpha\beta\gamma} \in \mathfrak{g}$. If we apply the commutator relations (3.1a), (3.1b) and (3.1d) we get:

$$[e_{abc}, e_{\alpha\beta\gamma}] = \begin{cases} (\alpha - a)e_{a+\alpha-1,b+\beta,1} & \text{if } c = \gamma = 1, \\ -be_{a+\alpha,b+\beta-1,1} + \alpha e_{a+\alpha-1,b+\beta,2} & \text{if } c = 1 \text{ and } \gamma = 2, \\ (\beta - b)e_{a+\alpha,b+\beta-1,2} & \text{if } c = \gamma = 2. \end{cases}$$

If $c = \gamma = 1$ we clearly have $[e_{ab1}, e_{\alpha\beta 1}] \in \mathfrak{g}$. If $c = 1$ and $\gamma = 2$ then $\beta > 0$ and hence $b + \beta > 0$ also; it follows that $[e_{ab1}, e_{\alpha\beta 2}] \in \mathfrak{g}$. Finally, $[e_{ab2}, e_{\alpha\beta 2}] \in \mathfrak{g}$ since $b + \beta - 1 > 0$ when $b > 0$ and $\beta > 0$. Moreover, $\mathfrak{g}$ is restricted since $e_{abc}^{[p]} \in \mathfrak{g}$ for all basis elements e_{abc} of $\mathfrak{g}$ (note that $e_{abc}^{[p]} = e_{abc}$ or $e_{abc}^{[p]} = 0$ by the properties of the $[p]$–mapping).

In particular, $\mathfrak{g} \cap W_{\geq 0}$ is a restricted Lie subalgebra of $W_{\geq 0}$: In fact it is a restricted Lie subalgebra of W_{012} hence supersolvable. We have $\operatorname{codim}_{\mathfrak{g}} \mathfrak{g} \cap W_{\geq 0} = 1$ and $\operatorname{codim}_W \mathfrak{g} = p$.

We define

$$\mathfrak{h} := \mathfrak{g} \cap W_{\geq 0} = \bigoplus_{0 \leq i < p} \bigoplus_{0 < j < p} K e_{ij2} \oplus \bigoplus_{0 < j < p} K e_{0j1} \oplus \bigoplus_{0 < i < p} \bigoplus_{0 \leq j < p} K e_{ij1}. \tag{11.9}$$

If we intersect the chain from (9.10) with $\mathfrak{h}$, then we get a chain

$$\mathfrak{h} \supset \mathfrak{h} \cap W_{101} \supset \mathfrak{h} \cap W_{011} \supset \cdots \supset 0 \tag{11.10}$$

that we can use to construct Vergne polarizations (after moving repetitions).

Define

$$\mathfrak{a} := \sum_{k=0}^{p-1} (\operatorname{ad} e_{001})^k (W_{\geq r}) \tag{11.11}$$

where r is the height of the exceptional character χ introduced in the beginning of this section. Note that $\mathfrak{a} \subset W_{\geq r+1-p} \subset W_{\geq 0}$. Let j be the integer with $0 \leq j < p - 1$ defined by $r = p - 1 + j$.

Lemma 11.7.1. *Let $\mathfrak{a}$ be defined as in* (11.11)*. Then $\mathfrak{a}$ is a p–ideal of $\mathfrak{g}$ and for all $s \geq 0$ we have that $\mathfrak{a} \cap W_{\geq s}$ is a p–ideal of $\mathfrak{h}$.*

Proof. Note that all elements e_{abc} with $b \geq j + 1$ and $c = 1, 2$ form a basis for $\mathfrak{a}$.

Consider $e_{uv1} \in \mathfrak{g}$. If $e_{ab1} \in \mathfrak{a}$, then $[e_{uv1}, e_{ab1}] = (a - u) e_{u+a-1, v+b, 1} \in \mathfrak{a}$ (clearly, $v + b \geq j + 1$ when $b \geq j + 1$) and $[e_{uv1}, e_{ab2}] = -v e_{u+a, v+b-1, 1} + a e_{u+a-1, v+b, 2}$ [if $v > 0$ this element lies in $\mathfrak{a}$ since $v + b, v + b - 1 \geq j + 1$ if $b \geq j + 1$. If $v = 0$ the first term is zero and $e_{u+a-1, b, 2} \in \mathfrak{a}$ since $b \geq j + 1$]. Hence $[e_{uv1}, \mathfrak{a}] \subset \mathfrak{a}$ for $e_{uv1} \in \mathfrak{g}$.

Next, let $e_{uv2} \in \mathfrak{g}$ with $v > 0$. From the relations $[e_{uv2}, e_{ab1}] = b e_{u+a, v+b-1, 1} - u e_{u+a-1, v+b, 2}$ and $[e_{uv2}, e_{ab2}] = (b - v) e_{u+a, v+b-1, 2}$ it follows that $[e_{uv2}, \mathfrak{a}] \subset \mathfrak{a}$ since $v + b - 1, v + b \geq j + 1$ for $v > 0$ and $b \geq j + 1$. We have thus shown that $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. All basis elements e_{abc} of $\mathfrak{a}$ satisfy $e_{abc}^{[p]} = e_{abc}$ or $e_{abc}^{[p]} = 0$; hence $\mathfrak{a}$ is a p–ideal of $\mathfrak{g}$. The final statement follows since $\mathfrak{h} \subset \mathfrak{g}$ and $[\mathfrak{h}, W_{\geq s}] \subset W_{\geq s}$ for $s \geq 0$. $\square$

Proposition 11.7.2. *If $\chi(\mathfrak{a}) \neq 0$ then induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{h})$–modules and the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules.*

Proof. Let $s > 0$ be defined such that $\chi(\mathfrak{a} \cap W_{s-1}) \neq 0$ but $\chi(\mathfrak{a} \cap W_{\geq s}) = 0$. Let N be an irreducible $U_\chi(\mathfrak{h})$–module. If $f \in \mathfrak{a} \cap W_{\geq s}$ such that $\chi([e_{001}, f]) \neq 0$ then

$$\{x \in U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N \mid f \cdot x = 0\} = 1 \otimes N. \tag{11.12}$$

To see this, adopt the notation from Section 6.4 with $G = \mathfrak{g}$ and $H = \mathfrak{h}$: Note that $f \cdot y = 0$ for all $y \in 1 \otimes N$ since $\mathfrak{a} \cap W_{\geq s}$ annihilates N by Lemma 6.3.1. Moreover, $(\operatorname{ad} e_{001})^i (f) \in \mathfrak{a} \subset \mathfrak{h}$ for all i we also have $(\operatorname{ad} e_{001})^i (f) \cdot 1 \otimes N \subset 1 \otimes N$. We can assume that $f = e_{abc}$ for appropriate a, b, c. It follows that $[e_{001}, f]$ acts bijectively on $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$ since $\chi([e_{001}, f]) \neq 0$ and $[e_{001}, f]^{[p]} = 0$ or $[e_{001}, f]^{[p]} = [e_{001}, f]$ (see Remark 6.4.2). Now apply Proposition 6.4.1 with $n = 1$ and $e_1 = e_{001}$ to get (11.12).

This implies that $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N$ is irreducible: Any irreducible W–submodule M has a nonzero intersection with $1 \otimes N$ (take an eigenvector for f considered as linear map on M and use that $f^{[p]} = 0$ and $\chi(f) = 0$). Therefore $M \cap (1 \otimes N) \neq 0$. But $M \cap (1 \otimes N)$ is a nonzero $U_\chi(\mathfrak{h})$–submodule of $1 \otimes N$ and therefore, by irreducibility, $M \cap (1 \otimes N) = 1 \otimes N$. In particular, we have $M \supset 1 \otimes N$ and hence M is the entire induced module.

If N_1, N_2 are irreducible $U_\chi(\mathfrak{h})$–modules such that

$$\varphi : U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N_1 \simeq U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} N_2$$

is an isomorphism, then φ induces a $U_\chi(\mathfrak{h})$–isomorphism $\overline{\varphi} : N_1 \simeq N_2$. Indeed, there exists a nonzero $x_1 \in N_1$ such that $\varphi(1 \otimes x_1) = 1 \otimes x_2 \in 1 \otimes N_2$ (look at φ applied to any eigenvector for f and use (11.12)). It follows that $\varphi(1 \otimes N_1) = \varphi(U_\chi(\mathfrak{h}) \cdot 1 \otimes x_1) = U_\chi(\mathfrak{h}) \cdot 1 \otimes x_2 = 1 \otimes N_2$; hence $N_1 \simeq N_2$. □

We will return to the two types of characters introduced in Section 11.5.

11.8 Type A characters

Keep the notation from the previous section. In this section we will take a closer look at characters of Type A. So let $\chi \in W^*$ be a character of Type A defined in Section 11.5. Let $\mathfrak{c}_W(\chi)$ be the stabilizer of χ in W defined in Section 10.4. We also define rk $\mathfrak{c}_W(\chi)$ as the dimension of any maximal torus in $\mathfrak{c}_W(\chi)$.

Theorem 11.8.1. *Let χ be a character of height $r > p$ and* Type A *such that $\chi(\mathfrak{a}) \neq 0$ with $\mathfrak{a}$ as in* (11.11)*. Then any irreducible $U_\chi(W)$–module has dimension $p^{\mathrm{codim_W}\mathfrak{c}_\mathrm{W}(\chi)/2}$ and the number of isomorphism classes of irreducible $U_\chi(W)$–modules is given by:*

$$\begin{cases} p & \textit{if } \mathrm{rk}\ \mathfrak{c}_W(\chi) = 1, \\ 1 & \textit{if } \mathrm{rk}\ \mathfrak{c}_W(\chi) = 0. \end{cases}$$

Proof. We proceed in several steps:

1) Let ρ be the integer with $1 < \rho \leq p-1$ such that $\chi(\mathfrak{a} \cap W_{\geq r+1-\rho}) = 0 \neq \chi(\mathfrak{a} \cap W_{\geq r-\rho})$. Such an integer exists since $\chi(\mathfrak{a}) \neq 0$ and $\mathfrak{a} \subset W_{\geq r+1-p}$.

2 For any ρ with $1 < \rho \leq p-1$, set

$$\mathfrak{h}_\rho := \mathfrak{h} \oplus \bigoplus_{a=\rho}^{p-1} K e_{a02}.$$

It is easy to check that any $\mathfrak{h}_\rho$ is a subalgebra of W (apply commutator relations). Moreover, $\mathfrak{h}_\rho$ is stable under the p–mapping (true for all basis elements) such that $\mathfrak{h}_\rho$ is a Lie p–subalgebra of W. It is supersolvable as a Lie p–subalgebra of W_{012}.

The idea is to use Theorem 6.3.3 and get for ρ as in 1): Induction is a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{h}_\rho)$–modules and the isomorphism classes irreducible $U_\chi(W)$–modules.

3) Let ρ be the integer from 1). Note that $\mathfrak{a} \cap W_{\geq r+1-\rho}$ has a basis consisting of all elements e_{abc} with $b \geq j+1$ and $a + b \geq r + 2 - \rho$ and $c = 1, 2$. Moreover, since $r + 1 - \rho \geq 3$ for $r > p$ and $\rho \leq p-1$ we have $\mathfrak{a} \cap W_{\geq r+1-\rho} \subset W_{\geq 3}$. By Lemma 11.7.1 we have that $\mathfrak{a} \cap W_{\geq r+1-\rho} \lhd \mathfrak{h}$ is a p–ideal. Moreover, $[e_{a02}, \mathfrak{a} \cap W_{\geq r+1-\rho}] \subset W_{\geq r} \subset \mathfrak{a} \cap W_{\geq r+1-\rho}$ for $a \geq \rho$. We conclude that $\mathfrak{a} \cap W_{\geq r+1-\rho}$ is a unipotent p–ideal in $\mathfrak{h}_\rho$ with $\chi(\mathfrak{a} \cap W_{\geq r+1-\rho}) = 0$.

4) Set $\mathfrak{l} := [W, \mathfrak{a} \cap W_{\geq r+1-\rho}]$. Then $\mathfrak{l} \subset W_{\geq 2}$ and is generated by all e_{abc} with $b \geq j$ and $a + b \geq r + 1 - \rho$ and $c = 1, 2$. It follows that $\mathfrak{l}$ has a basis $l_1, l_2, \dots, l_n$ such that $l_i^{[p]} = 0$ for all $i = 1, 2, \dots, n$. Moreover $[\mathfrak{l}, \mathfrak{l}] \subset W_{\geq 4}$ is unipotent. Since $\mathfrak{l} \subset W_{\geq r-\rho}$ we conclude that $[\mathfrak{l}, \mathfrak{l}] \subset W_{2r-2\rho}$ and it follows from commutator relations that a subset of $\{e_{abc} \mid b \geq 2j-1 \text{ and } a+b \geq 2r-2\rho+1\}$ form a basis for $[\mathfrak{l}, \mathfrak{l}]$. But $2j-1 \geq j+1$ (since $j > 1$ when $r = p-1+j > p$) and $a + b \geq 2r - 2\rho + 1 \geq r + 2 - \rho$ so $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{a} \cap W_{\geq r+1-\rho}$; hence $\chi([\mathfrak{l}, \mathfrak{l}]) = 0$.

5) We have already observed that $\mathfrak{l}$ is contained in $W_{\geq 2}$ and generated by elements e_{abc} with $b \geq j > 0$. Therefore $[W, \mathfrak{l}]$ is contained in $W_{\geq 1}$ and generated by elements e_{abc} with $b \geq j - 1 > 0$; hence $\mathfrak{l}$ and $[W, \mathfrak{l}]$ are contained in $\mathfrak{h} \subset \mathfrak{h}_\rho$.

6) We have $\mathfrak{st}(\chi, \mathfrak{a} \cap W_{\geq r+1-\rho}) = \{x \in W \mid \chi([x, y]) = 0 \ \forall y \in \mathfrak{a} \cap W_{\geq r+1-\rho}\} = \mathfrak{h}_\rho$. To see this note that we clearly have $\mathfrak{h}_\rho \subset \mathfrak{st}(\chi, \mathfrak{a} \cap W_{\geq r+1-\rho})$ by 3). On the other hand, consider an element

$$x = a_0 e_{002} + a_1 e_{102} + \cdots + a_{\rho-1} e_{\rho-1,0,2} + c e_{001}$$

in $\mathfrak{st}(\chi, \mathfrak{a} \cap W_{\geq r+1-\rho})$ for some $a_0, a_1, \dots, a_{\rho-1}, c \in K$.

We now define $f_i := (\text{ad } e_{001})^i (e_{p-1,j+1,1}) \in \mathfrak{a} \cap W_{\geq r+1-\rho}$ for $i = 0, 1, 2, \dots, \rho - 1$. By assumption on χ (Type A) we have

$$\chi([e_{001}, (\text{ad } e_{001})^i (e_{p-1,j+1,1})]) = 0$$

for all $i = 0, 1, \dots, \rho - 1$ since χ vanishes on all $e_{a,j+1,1}$ with $a \geq 0$. Moreover, $\chi([e_{i02}, f_i]) \neq 0$ for all $i = 0, 1, \dots, \rho - 1$. Thus we get:

$$\begin{aligned} \chi([x, f_0]) = 0 &\implies a_0 = 0 \\ \chi([x, f_1]) = 0 &\implies a_1 = 0 \\ &\vdots \\ \chi([x, f_{\rho-1}]) = 0 &\implies a_{\rho-1} = 0. \end{aligned}$$

Moreover, the assumption on χ implies that there exists $f \in \mathfrak{a} \cap W_{\geq r+1-\rho}$ such that $\chi([e_{001}, f]) \neq 0$. Now use that $\chi([x, f]) = 0$ and $a_0 = a_1 = \cdots = a_{\rho-1} = 0$ to get $c = 0$ also. Hence $\mathfrak{st}(\chi, \mathfrak{a} \cap W_{\geq r+1-\rho}) = \mathfrak{h}_\rho$.

7) Set $e_i = e_{i02}$ for $i = 0, 1, \dots, \rho - 1$ and $e_\rho = e_{001}$. Then $e_0, e_1, \dots, e_\rho$ is a basis for a complement to $\mathfrak{h}_\rho$ in W. We will apply Theorem 6.3.3: Adopt the notation from Section 6.3 and set $\mathfrak{g} = W$ and $\mathfrak{h} = \mathfrak{h}_\rho$ and $\mathfrak{a} = \mathfrak{a} \cap W_{\geq r+1-\rho}$. Then use step 6) and Remark 6.3.9 to find $f_0', f_1', \dots, f_{\rho-1}', f_\rho' \in \mathfrak{a} \cap W_{\geq r+1-\rho}$ such that $\chi([e_i, f_j']) = \delta_{ij}$. We are now in position to apply Theorem 6.3.3: Induction is a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{h}_\rho)$–modules and the isomorphism classes irreducible $U_\chi(W)$–modules.

Before we finish our proof we need two remarks:

Remark 11.8.2. Since induction is a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{h}_\rho)$–modules and the isomorphism classes irreducible $U_\chi(W)$–modules and since $\mathfrak{h}_\rho \subset W_{\geq 0}$ it follows that induction is a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and the isomorphism classes irreducible $U_\chi(W)$–modules

Remark 11.8.3. The proof of $\mathfrak{st}(\chi, \mathfrak{a} \cap W_{\geq r+1-\rho}) = \mathfrak{h}_\rho$ has an important application: Use Remark 6.3.6 and find $f \in \mathfrak{a} \cap W_{\geq r+1-\rho}$ such that $\chi([e_{001}, f]) \neq 0$ but $\chi([f, e_{l02}]) = 0$ for $l \leq \rho - 1$. Since $\chi([f, \mathfrak{h}_\rho]) = 0$ also we get: $\chi([f, e_{001}]) \neq 0 = \chi([f, e_{002}]) = \chi([f, W_{\geq 0}])$.

Now we are in position to prove the dimension formula for irreducible $U_\chi(W)$–modules where χ is exceptional (under the additional assumption $\chi(\mathfrak{a}) \neq 0$): We will use results from Section 10.4. But many results in that section may not be true if $r = 2p - 3$ and if χ has type II.a as in 5.2. For our χ this case does not occur so we do not have to worry about that point!

Take $\lambda \in W_{012}^*$ such that the Vergne polarization $\mathfrak{p}_\lambda$ of λ constructed via (9.10) is compatible with χ and equal to $\mathfrak{p}_\chi$. The existence follows from Theorem 10.1.1. Thus it follows from Theorem 10.4.6 that any irreducible $U_\chi(W_{\geq 0})$–module has dimension

$$p^{\operatorname{codim}_{W_{\geq 0}} \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})/2}.$$

Finally, use that $\mathfrak{c}_W(\chi) \subset \mathfrak{st}(\chi, \mathfrak{a} \cap W_{\geq r+1-\rho}) = \mathfrak{h}_\rho \subset W_{\geq 0}$ with Lemma 10.4.7 to get $\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) - 2$. Since induction takes irreducible $U_\chi(W_{\geq 0})$–modules to irreducible $U_\chi(W)$ we obtain the dimension formula: Any irreducible $U_\chi(W)$–module has dimension $p^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$.

By Theorem 10.4.5 the number of irreducible $U_\chi(W_{\geq 0})$–modules is p if rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 1$ and 1 otherwise (i.e., if rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 0$ by Lemma 10.4.2).

If rk $\mathfrak{c}_W(\chi) = 1$ then rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 1$ (apply Lemma 10.4.2 together with the inclusion $\mathfrak{c}_W(\chi) \subset \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$. Since induction from $W_{\geq 0}$ to W is a bijection the number of irreducible $U_\chi(W)$–modules is p as claimed.

If rk $\mathfrak{c}_W(\chi) = 0$ then I claim rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 0$ [Otherwise there exists a nonzero toral element $h \in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$. By Remark 11.8.3 above we can easily find $a, b \in K$ such that $h' := h + af + be_{p-1,j+1,1} \in \mathfrak{c}_W(\chi)$. But $(h')^{[p]} \in h + W_{011}$. Therefore rk $\mathfrak{c}_W(\chi) \neq 0$ which is a contradiction]. Now use that induction from $W_{\geq 0}$ to W is a bijection to get that the number of irreducible $U_\chi(W)$–modules is 1.

The proof is completed. □

Remark 11.8.4. For $r = p - 1$ or $r = p$ we cannot use the proof above. Whether one can extend the proof above to $r = p - 1, p$, I don't know. But examples for $p = 3$ (see Theorem 13.3.2.a and Theorem 13.11.6.a) show that the theorem might extend to $r = p$ and $r = p - 1$.

Theorem 11.8.5. *If* $\chi(\mathfrak{a} \cap W_{0,r+2-p,1}) = 0$, *then induction induces a bijection between the isomorphism classes of irreducible* $U_\chi(\mathfrak{g})$*–modules and the isomorphism classes of irreducible* $U_\chi(W)$*–modules with* $\mathfrak{g}$ *as in* (11.8).

Proof. Set $e_l = e_{i02}$ for $l = 0, 1, \ldots, p - 1$. Then $e_0, e_1, \ldots, e_{p-1}$ form a basis for a complement to $\mathfrak{g}$ in W. Set

$$f_l = \begin{cases} e_{p-1-l,j+1,1} + (l+1)(j+2)^{-1} e_{p-2-l,j+2,2} & \text{if } r < 2p-3, \\ e_{p-1-l,p-1,1} & \text{if } r = 2p-3 \end{cases}$$

for $l = 0, 1, 2, \ldots, p - 1$ (note that $r < 2p - 3$ implies that $j + 2 \neq 0$). It is easy to see that $\chi([e_l, f_l]) \neq 0$. Moreover, we have $[e_l, f_l] \in Ke_{p-1,j,1} \oplus Ke_{p-2,j+1,2}$; hence $[e_l, f_l]^{[p]} = 0$ since

$$\begin{aligned} e_{p-1,j,1}^{[p]} &= 0, \\ e_{p-2,j+1,2}^{[p]} &= 0, \\ \big[e_{p-1,j,1}, [e_{p-1,j,1}, e_{p-2,j+1,2}]\big] &= 0, \\ \big[e_{p-2,j+1,2}, [e_{p-2,j+1,2}, e_{p-1,j,1}]\big] &= 0. \end{aligned}$$

Now adopt the notation from Section 6.4: Let N be an irreducible $U_\chi(\mathfrak{g})$–module and define for $l = 0, 1, \ldots, p-1$:

$$N_l = \bigoplus_{0 \le i_l, \ldots, i_{p-1} < p} e_l^{i_l} \cdots e_{p-1}^{i_{p-1}} \otimes N \subset U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} N.$$

Interpret N_p as $1 \otimes N$. Note that each N_l with $l > 0$ is a W_{l02}–module (see page 13 for notation and recall that W_{l02} is a Lie p–subalgebra for $l > 0$ by Lemma 3.1.1): Indeed, the PBW theorem says that any element $y \in U_\chi(W_{l02})$, where $0 < l \le p-1$, can be written as

$$y = \sum_{0 \le i_l, \ldots, i_{p-1} < p} e_l^{i_l} \cdots e_{p-1}^{i_{p-1}} \cdot u_{i_l, \ldots, i_{p-1}} \quad \text{for } \; u_{i_l, \ldots, i_{p-1}} \in \mathfrak{g}. \qquad (*)$$

If $x \in W_{l02}$ then $x \cdot e_l^{i_l} \cdots e_{p-1}^{i_{p-1}} \in U_\chi(W_{l02})$ and we can use $(*)$ to get $x \cdot N_l \subset N_l$ for $l > 0$.

I claim that $f_l \cdot N_{l+1} = 0$. For $r < 2p-3$ we have $[e_k, f_l] = 0$ for $k > l$ and hence $f_l \cdot N_{l+1} = 0$ if $f_l \cdot N = 0$. If $r = 2p-3$ we have $f_l e_{l+1}^{i_{l+1}} = e_{l+1}^{i_{l+1}} f_l + (l+1) i_{l+1} e_{l+1}^{i_{l+1}-1} e_{p-1,p-1,1}$ and since $[f_l, e_k] = 0 = [e_{p-1,p-1,1}, e_k]$ for k with $l+1 < k \le p-1$ we have $f_l \cdot N_{l+1} = 0$ if $f_l \cdot N = 0 = e_{p-1,p-1,1} \cdot N = 0$. But N is an irreducible $U_\chi(\mathfrak{g})$–module and therefore a homomorphic image of $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{h})} S$, where S is an irreducible $U_\chi(\mathfrak{h})$–module. Now, observe that $f_l \in \mathfrak{a} \cap W_{0,r+2-p,1}$ and that $(\mathrm{ad}\, e_{001})(f_l) = -(l+1) f_{l+1}$ for all l; thus we have $(\mathrm{ad}\, e_{001})^m (f_l) \in \mathfrak{a} \cap W_{0,r+2-p,1}$ for all m, l. Moreover, $(\mathrm{ad}\, e_{001})^m (e_{p-1,p-1,1}) \in \mathfrak{a} \cap W_{0,r+2-p,1}$ for all m. So we only need to show that $\mathfrak{a} \cap W_{0,r+2-p,1}$ annihilates S in order to show that $f_l \cdot N_{l+1} = 0$. But $\mathfrak{a} \cap W_{0,r+2-p,1}$ is a unipotent ideal in $\mathfrak{h}$ since $[\mathfrak{h}, \mathfrak{a}] \subset \mathfrak{a}$ and $[\mathfrak{h}, W_{0,r+2-p,1}] \subset W_{0,r+2-p,1}$ and $\chi(\mathfrak{a} \cap W_{0,r+2-p,1}) = 0$; hence $(\mathfrak{a} \cap W_{0,r+2-p,1}) \cdot S = 0$ by Lemma 6.3.1.

In order to apply Corollary 6.4.3 we only need that $(\mathrm{ad}\, e_l)^k (f_l) \cdot N_{l+1} \subset N_{l+1}$ for all k. Since $(\mathrm{ad}\, e_l)^k (f_l) \in W_{l+1,0,2}$ for all k we are done since each N_l is a W_{l02}–module for $l > 0$.

Thus we have by Corollary 6.4.3:

$$\{x \in U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} N \mid f_l \cdot x = 0 \;\text{ for } l = 0, 1, \ldots, p-1\} = 1 \otimes N.$$

Therefore,

$$\{x \in U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} N \mid (\mathfrak{a} \cap W_{0,r+2-p,1}) \cdot x = 0\} \subset 1 \otimes N.$$

This implies that $U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} N$ is irreducible: Any irreducible W–submodule M has a nonzero intersection with $1 \otimes N$. [Since $\mathfrak{a} \cap W_{0,r+2-p,1}$ is unipotent with $\chi(\mathfrak{a} \cap W_{0,r+2-p1}) = 0$ the trivial $\mathfrak{a} \cap W_{0,r+2-p,1}$–module K is the only irreducible $U_\chi(\mathfrak{a} \cap W_{0,r+2-p,1})$–module (up to isomorphism) by [14, 3.2] so an irreducible $U_\chi(\mathfrak{a} \cap W_{0,r+2-p,1})$–submodule of M has a nonzero intersection with $1 \otimes N$.] But $M \cap (1 \otimes N)$ is a nonzero $U_\chi(\mathfrak{g})$–submodule of $1 \otimes N$ and therefore, by irreducibility, $M \cap (1 \otimes N) = 1 \otimes N$. In particular, we have $M \supset 1 \otimes N$ and hence M is the entire induced module.

If N_1, N_2 are irreducible $U_\chi(\mathfrak{g})$–modules such that

$$\varphi : U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} N_1 \simeq U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} N_2$$

is an isomorphism, then φ induces a $U_\chi(\mathfrak{g})$–isomorphism $\overline{\varphi} : N_1 \simeq N_2$. Indeed, we have $\varphi(1 \otimes N_1) \cap (1 \otimes N_2) \neq 0$. (Look at the elements annihilated by $\mathfrak{a} \cap W_{0,r+2-p,1}$.) Since $\varphi(1 \otimes N_1)$ and $1 \otimes N_2$ are irreducible $U_\chi(\mathfrak{g})$–modules, we get $\varphi(1 \otimes N_1) = 1 \otimes N_2$; hence $N_1 \simeq N_2$. $\square$

11.9 Type B characters

In this section we will take a closer look at characters of Type B. So let $\chi \in W^*$ be a character of Type B defined in Section 11.5.

Theorem 11.9.1. *Let χ be a character of height $r > p$ and* Type B *such that $\chi(\mathfrak{a}) \neq 0$ with $\mathfrak{a}$ as in* (11.11)*. If* rk $\mathfrak{c}_W(\chi) = 1$ *then there exist up to isomorphism p irreducible $U_\chi(W)$–module of dimension $p^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$. If* rk $\mathfrak{c}_W(\chi) = 0$ *then there exists up to isomorphism 1 irreducible $U_\chi(W)$–module and the dimension is $p^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2-1}$ or $p^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$ or $p^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2+1}$.*

Proof. We proceed in several steps:

1) Define integer ρ with $1 < \rho \leq p-1$ such that $\chi(\mathfrak{a} \cap W_{\geq r+1-\rho}) = 0 \neq \chi(\mathfrak{a} \cap W_{\geq r-\rho})$. Such an integer exists since $\chi(\mathfrak{a}) \neq 0$ and $\mathfrak{a} \subset W_{\geq r+1-p}$.

2) For ρ with $1 < \rho \leq p-1$, set

$$\mathfrak{h}_\rho := \mathfrak{h} \oplus \bigoplus_{a=\rho}^{p-1} K e_{a02}$$

as in the proof of Theorem 11.8.1. It is supersolvable as a Lie p–subalgebra of W_{012}.

The idea is to use Theorem 6.3.3 and get for ρ as in 1): Induction is a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{h}_\rho)$–modules and the isomorphism classes irreducible $U_\chi(W)$–modules.

3) Let ρ be the integer from 1). Set $\mathfrak{a}_\rho := \mathfrak{a} \cap W_{\geq r+1-\rho} + \sum_{k=0}^{\rho-2} (\text{ad } e_{001})^k (e_{p-1,j,1}) \subset W_{\geq r+1-\rho}$. Note that $\mathfrak{a}_\rho$ has a basis consisting of all elements e_{ab2} with $b \geq j+1$ and $a+b \geq r+2-\rho$ and e_{ab1} with $b \geq j$ for $a+b \geq r+2-\rho$.

Consider a basis element $e_{rs1} \in \mathfrak{h}_\rho$. Then $[e_{rs1}, e_{ab1}] = (a-r)e_{r+a-1,s+b,1} \in \mathfrak{a}_\rho$ for all $e_{ab1} \in \mathfrak{a}_\rho$ (clearly, $s+b \geq j$ when $b \geq j$). Next, consider $[e_{rs1}, e_{ab2}] = -se_{r+a,s+b-1,1} + ae_{r+a-1,s+b,2}$. If $s > 0$ this element lies clearly in $\mathfrak{a}_\rho$, since $s+b, s+b-1 \geq j+1$ if $b \geq j+1$. If $s = 0$ the first term is zero and $e_{r+a-1,b,2} \in \mathfrak{a}_\rho$ since $b \geq j+1$. Hence $[e_{rs1}, \mathfrak{a}_\rho] \subset \mathfrak{a}_\rho$ for $e_{rs1} \in \mathfrak{h}_\rho$.

Consider e_{rs2} with $r+s \geq \rho$ and observe that $[e_{rs2}, \mathfrak{a}_\rho] \subset W_{\geq r} \subset \mathfrak{a}_\rho$. Finally, let $e_{rs2} \in \mathfrak{h}_\rho$ with $s > 0$. From the relations $[e_{rs2}, e_{ab1}] = be_{r+a,s+b-1,1} - re_{r+a-1,s+b,2}$ and $[e_{rs2}, e_{ab2}] = (b-s)e_{r+a,s+b-1,2}$ it follows that $[e_{rs2}, \mathfrak{a}_\rho] \subset \mathfrak{a}_\rho$ since $s+b-1 \geq j$ for $s > 0$ and $b \geq j$ and $s+b, s+b-1 \geq j+1$ for $b \geq j+1$. Since $\mathfrak{a}_\rho \subset W_{\geq 3}$ ($r > p$ and $\rho \leq p-1$), we conclude that $\mathfrak{a}_\rho \lhd \mathfrak{h}_\rho$ is a unipotent p–ideal with $\chi(\mathfrak{a}_\rho) = 0$ (since $\chi(\mathfrak{a} \cap W_{\geq r+1-\rho}) = 0$ and since $\chi(e_{aj1}) = 0$ for all $a > 0$ – Type B).

4) Set $\mathfrak{l} := [W, \mathfrak{a}_\rho]$. Then $\mathfrak{l} \subset W_{\geq 2}$ and is generated by all e_{ab2} with $b \geq j$ and $a+b \geq r+1-\rho$ and $c = 1,2$ and all e_{ab1} with $b \geq j-1$ and $a+b \geq r+1-\rho$. It follows that $\mathfrak{l}$ has a basis $l_1, l_2, \ldots, l_n$ such that $l_i^{[p]} = 0$ for all $i = 1, 2, \ldots, n$. Moreover $[\mathfrak{l}, \mathfrak{l}] \subset W_{\geq 4}$ is unipotent. Since $\mathfrak{l} \subset W_{\geq r-\rho}$ we conclude that $[\mathfrak{l}, \mathfrak{l}] \subset W_{2r-2\rho}$ and it follows from commutator relations that a subset of

$$\{e_{ab1} \mid b \geq 2j-2 \text{ and } a+b \geq 2r-2\rho+1\} \cup \{e_{ab2} \mid b \geq 2j-1 \text{ and } a+b \geq 2r-2\rho+1\}$$

form a basis for $[\mathfrak{l}, \mathfrak{l}]$. But $2j-2 \geq j$ and $2j-1 \geq j+1$ (since $j > 1$ when $r = p-1+j > p$) and $a+b \geq 2r-2\rho+1 \geq r+2-\rho$ so $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{a}_\rho$; hence $\chi([\mathfrak{l}, \mathfrak{l}]) = 0$.

5) We have already observed that $\mathfrak{l}$ is contained in $W_{\geq 2}$ and generated by elements e_{ab2} with $b \geq j > 0$ and elements e_{ab1} with $b \geq j-1 > 0$. Therefore $[W, \mathfrak{l}]$ is contained in $W_{\geq 1}$ and generated by elements e_{ab2} with $b \geq j-1 > 0$ and e_{ab1} with $b \geq j-2 \geq 0$; hence $\mathfrak{l}$ and $[W, \mathfrak{l}]$ are contained in $\mathfrak{h} \subset \mathfrak{h}_\rho$.

6) We have $\mathfrak{st}(\chi, \mathfrak{a}_\rho) = \{x \in W \mid \chi([x,y]) = 0 \ \forall y \in \mathfrak{a}_\rho\} = \mathfrak{h}_\rho$. To see this note that we clearly have $\mathfrak{h}_\rho \subset \mathfrak{st}(\chi, \mathfrak{a}_\rho)$ by 3). On the other hand, consider an element

$$x = a_0 e_{002} + a_1 e_{102} + \cdots + a_{\rho-1} e_{\rho-1,0,2} + c e_{001}$$

in $\mathfrak{st}(\chi, \mathfrak{a}_\rho)$ for some $a_0, a_1, \ldots, a_{\rho-1}, c \in K$.

Set $f_0 = e_{p-1,j+1,2}$ and $f_i := (\text{ad } e_{001})^{i-1}(e_{p-1,j,1}) \in \mathfrak{a}_\rho$ for $i = 1, 2, \ldots, \rho-1$. By assumption on χ (Type B) we have

$$\chi([e_{001}, (\text{ad } e_{001})^{i-1}(e_{p-1,j,1})]) = 0$$

for all $i = 1, \ldots, \rho-1$ since χ vanishes on all e_{aj1} with $a \geq 0$. Moreover, we have $\chi([e_{i02}, f_i]) \neq 0$. Thus we get:

$$\begin{aligned} \chi([x, f_0]) = 0 &\implies a_0 = 0 \\ \chi([x, f_1]) = 0 &\implies a_1 = 0 \\ &\vdots \\ \chi([x, f_{\rho-1}]) = 0 &\implies a_{\rho-1} = 0. \end{aligned}$$

The assumption on χ implies that there exists $f \in \mathfrak{a}_\rho$ such that $\chi([e_{001}, f]) \neq 0$. Now use that $\chi([x,f]) = 0$ and $a_0 = a_1 = \cdots = a_{\rho-1} = 0$ to get $c = 0$ also. Hence $\mathfrak{st}(\chi, \mathfrak{a}_\rho) = \mathfrak{h}_\rho$.

7) Set $e_i = e_{i02}$ for $i = 0, 1, \ldots, \rho-1$ and $e_\rho = e_{001}$. Then $e_0, e_1, \ldots, e_\rho$ is a basis for a complement to $\mathfrak{h}_\rho$ in W. We will apply Theorem 6.3.3: Adopt the notation from Section 6.3 and set $\mathfrak{g} = W$ and $\mathfrak{h} = \mathfrak{h}_\rho$ and $\mathfrak{a} = \mathfrak{a}_\rho$. Then use step 6) and Remark 6.3.9 to find $f_0', f_1', \ldots, f_{\rho-1}', f_\rho' \in \mathfrak{a}_\rho$ such that $\chi([e_i, f_j']) = \delta_{ij}$. We are now in position to apply Theorem 6.3.3: Induction is a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{h}_\rho)$–modules and the isomorphism classes irreducible $U_\chi(W)$–modules.

Before we finish our proof we need two remarks:

Remark 11.9.2. Since induction is a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{h}_\rho)$–modules and the isomorphism classes irreducible $U_\chi(W)$–modules and since $\mathfrak{h}_\rho \subset W_{\geq 0}$ it follows that induction is a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and the isomorphism classes irreducible $U_\chi(W)$–modules

Remark 11.9.3. The proof of $\mathfrak{st}(\chi, \mathfrak{a}_\rho) = \mathfrak{h}_\rho$ has an important application: Use Remark 6.3.6 and find $f \in \mathfrak{a}_\rho$ such that $\chi([e_{001}, f]) \neq 0$ but $\chi([f, e_{l02}]) = 0$ for $l \leq \rho - 1$. Since $\chi([f, \mathfrak{h}_\rho]) = 0$ also we get: $\chi([f, e_{001}]) \neq 0 = \chi([f, e_{002}]) = \chi([f, W_{\geq 0}])$.

Now we are in position to prove the claims for the dimension of irreducible $U_\chi(W)$–modules where χ is exceptional (under the additional assumption $\chi(\mathfrak{a}) \neq 0$): We will use results from Section 10.4. But many results in that section may not be true if $r = 2p-3$ and if χ has type II.a as in 5.2. For our χ this case does not occur so we do not have to worry about that point!

Take $\lambda \in W_{012}^*$ such that the Vergne polarization $\mathfrak{p}_\lambda$ of λ constructed via (9.10) is compatible with χ and equal to $\mathfrak{p}_\chi$. This time we cannot be sure that $\mathfrak{p}_\lambda = \mathfrak{p}_\chi$. [If $\mathfrak{p}_\chi$ is non unipotent we can choose λ that way by Theorem 10.1.1.a but for unipotent $\mathfrak{p}_\chi$ the computations in Section 10.3 require $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$ or $r \leq p$ or $\chi([e_{001}, W_{r-1}]) \neq 0$, but none of these conditions are satisfied for Type B characters of height $> p$; in the unipotent case we can therefore only say that $\mathfrak{p}_\lambda$ is unipotent and that $\dim_K \mathfrak{p}_\lambda = \dim_K \mathfrak{p}_\chi$ or $\dim_K \mathfrak{p}_\lambda = \dim_K \mathfrak{p}_\chi - 1$ or $\dim_K \mathfrak{p}_\lambda = \dim_K \mathfrak{p}_\chi + 1$.]

If rk $\mathfrak{c}_W(\chi) = 1$ then rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 1$ (apply Lemma 10.4.2 together with the inclusion $\mathfrak{c}_W(\chi) \subset \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$). Therefore $\mathfrak{p}_\chi$ is non unipotent [since $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ is a subspace of W_{012} (use Lemma 10.4.1 with $x = e_{p-2,j+1,2}$) and therefore contained in $\mathfrak{p}_\chi$!] Since induction from $W_{\geq 0}$ to W is a bijection the number of irreducible $U_\chi(W)$–modules is p as claimed. The dimension is given by $p^{\mathrm{codim}_W \mathfrak{c}_W(\chi)/2}$ (in the non unipotent case we can choose λ such that $\mathfrak{p}_\lambda = \mathfrak{p}_\chi$; now conclude as in the proof of Theorem 11.8.1).

If rk $\mathfrak{c}_W(\chi) = 0$ then I claim rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 0$ (otherwise there exists a nonzero toral element $h \in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$. By Remark 11.9.3 above we can easily find $a, b \in K$ such that $h' := h + af + be_{p-1,j+1,1} \in \mathfrak{c}_W(\chi)$. But $(h')^{[p]} \in h + W_{011}$. Therefore rk $\mathfrak{c}_W(\chi) \neq 0$ which is a contradiction). Now use that induction from $W_{\geq 0}$ to W is a bijection to get that the number of irreducible $U_\chi(W)$–modules is 1. But now we can either choose λ such that $\dim_K \mathfrak{p}_\lambda = \dim_K \mathfrak{p}_\chi$ or $\dim_K \mathfrak{p}_\lambda = \dim_K \mathfrak{p}_\chi - 1$ or $\dim_K \mathfrak{p}_\lambda = \dim_K \mathfrak{p}_\chi + 1$. Now the claim on the dimension follows.

The proof is completed. □

Remark 11.9.4. For $r = p - 1$ or $r = p$ we cannot use the proof above. Whether one can extend the proof above to $r = p - 1, p$, I don't know. But examples for $p = 3$ (see Theorem 13.4.5.a and Theorem 13.12.6.a) show that the theorem might extend to $r = p$ and $r = p - 1$.

We now seek for an analogous result for Type B characters to the one proved for Type A characters in Theorem 11.8.5. We will have to require $\chi(\mathfrak{a}) = 0$ instead of just $\chi(\mathfrak{a} \cap W_{0,r+2-p,1}) = 0$.

Before we prove that result we need a lemma.

Lemma 11.9.5. *Let $\mathfrak{g}$ be as in* (11.8) *and let $\mathfrak{a}$ be as in* (11.11). *If M is a $U_\chi(W)$–module and $M \neq 0$, then*

$$\{x \in M \mid \mathfrak{a} \cdot x = 0\} \neq 0$$

and there exists an irreducible $U_\chi(\mathfrak{g})$–submodule $X \subset M$ with $\mathfrak{a} \cdot X = 0$.

Proof. This is clear for $r > p - 1$ since $\mathfrak{a}$ is unipotent in that case; thus we can take an irreducible $U_\chi(\mathfrak{a})$–submodule of M which, by [14, 3.2], is isomorphic to the trivial module K. So there exists a nonzero $x \in M$ with $\mathfrak{a} \cdot x = 0$. Consider the case that $r = p - 1$: Set $\mathfrak{b} = \mathfrak{a} \oplus K e_{p-1,0,1}$. Since $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{b} \cap W_{011}$ there exists a $U_\chi(\mathfrak{b})$–module K_l as being equal to K as a vector space and where the module structure is given by: $e \cdot 1 = 0$ for $e \in \mathfrak{b} \cap W_{011}$ and $e_{012} \cdot 1 = l$ (since $e_{012} \in \mathfrak{a}$ with $\chi(e_{012}) = 0$ we have $l \in \mathbb{F}_p$). Since $\mathfrak{b} \subset W_{012}$ is supersolvable any irreducible $U_\chi(\mathfrak{b})$–module is isomorphic to some K_l with $l \in \mathbb{F}_p$ by Lemma 9.1.3. So there exists a nonzero $x \in M$ with $(\mathfrak{b} \cap W_{011}) \cdot x = 0$ and $e_{012} \cdot x = lx$ for some $l \in \mathbb{F}_p$. If $l > 0$, set $y := e^l_{p-1,0,2} \cdot x \in M$. Then $y \neq 0$ since $\chi(e_{p-1,0,2}) \neq 0$. Moreover, we have $e_{012} \cdot y = e^l_{p-1,0,2}(e_{012} - l) \cdot x = 0$ and $(\mathfrak{b} \cap W_{011}) \cdot y = 0$ since $[e_{p-1,0,2}, \mathfrak{b} \cap W_{011}] \subset \mathfrak{b} \cap W_{011}$. We conclude that that $\mathfrak{a} \cdot x = 0$ if $l = 0$ and $\mathfrak{a} \cdot y = 0$ if $l > 0$; hence $\{x \in M \mid \mathfrak{a} \cdot x = 0\} \neq 0$.

The final statement in the lemma is now clear: Take nonzero $x \in M$ such that $\mathfrak{a} \cdot x = 0$. Then $U_\chi(\mathfrak{g}) \cdot x$ is a $U_\chi(\mathfrak{g})$–submodule of M annihilated by $\mathfrak{a}$ (since $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ and $\mathfrak{a} \cdot x = 0$). Thus it contains an irreducible $U_\chi(\mathfrak{g})$–submodule X such that there exists an irreducible $U_\chi(\mathfrak{g})$–submodule $X \subset M$ with $\mathfrak{a} \cdot X = 0$. □

Theorem 11.9.6. *If $\chi(\mathfrak{a}) = 0$, then induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules annihilated by $\mathfrak{a}$ and the isomorphism classes of irreducible $U_\chi(W)$–modules.*

Proof. Set $e_l = e_{l02}$ for $l = 0, 1, \ldots, p-1$. Then $e_0, e_1, \ldots, e_{p-1}$ form a basis for a complement to $\mathfrak{g}$ in W. Set

$$f_l = e_{p-1-l,j+1,2} \quad \text{for} \quad l = 0, 1, 2, \ldots, p-1.$$

It follows that $[e_l, f_l] \in K e_{p-1,j,2}$; hence $[e_l, f_l]^{[p]} = 0$. It is easy to see that $\chi([e_l, f_l]) \neq 0$ and that $[e_k, f_l] = 0$ for k with $l < k \leq p-1$.

Now adopt the notation from Section 6.4: Let N be an irreducible $U_\chi(\mathfrak{g})$–module with $\mathfrak{a} \cdot N = 0$ and define for $l = 0, 1, \ldots, p-1$:

$$N_l = \bigoplus_{0 \leq i_l, \ldots, i_{p-1} < p} e_l^{i_l} \cdots e_{p-1}^{i_{p-1}} \otimes N \subset U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} N.$$

Interpret N_p as $1 \otimes N$. Note that each N_l with $l > 0$ is a W_{l02}–module (see the proof of Proposition 11.8.5). For $k > l$ we have $[e_k, f_l] = 0$; hence $f_l \cdot N_{l+1} = 0$ since $f_l \in \mathfrak{a}$ with $f_l \cdot N = 0$ by assumption.

In order to apply Corollary 6.4.3 we only need to prove that $(\operatorname{ad} e_l)^k(f_l) \cdot N_{l+1} \subset N_{l+1}$ for all k. Since $(\operatorname{ad} e_l)^k(f_l) \in W_{l+1,0,2}$ for all k we can use that N_{l+1} is a $W_{l+1,0,2}$–module; hence $(\operatorname{ad} e_l)^k(f_l) \cdot N_{l+1} \subset N_{l+1}$ for all k. Thus we have by Corollary 6.4.3:

$$\{x \in U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} N \mid \mathfrak{a} \cdot x = 0\} = 1 \otimes N.$$

This implies that $U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} N$ is irreducible: Any irreducible W–submodule M has a nonzero intersection with $1 \otimes N$ by Lemma 11.9.5. Therefore $M \cap (1 \otimes N)$ is a nonzero $U_\chi(\mathfrak{g})$–submodule of $1 \otimes N$ and, by irreducibility, $M \cap (1 \otimes N) = 1 \otimes N$. In particular, we have $M \supset 1 \otimes N$ and hence M is the entire induced module.

If N_1, N_2 are irreducible $U_\chi(\mathfrak{g})$–modules annihilated by $\mathfrak{a}$ such that we have an isomorphism $\varphi : U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} N_1 \simeq U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} N_2$, then φ induces a $U_\chi(\mathfrak{g})$–isomorphism $\overline{\varphi} : N_1 \simeq N_2$ since $\varphi(1 \otimes N_1) \cap (1 \otimes N_2) \neq 0$ (look at elements annihilated by $\mathfrak{a}$). But $\varphi(1 \otimes N_1)$ and $1 \otimes N_2$ are irreducible $U_\chi(\mathfrak{g})$–modules so we get $\varphi(1 \otimes N_1) = 1 \otimes N_2$; hence $N_1 \simeq N_2$.

We have thus shown: Induction induces an injection from the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules annihilated by $\mathfrak{a}$ into the isomorphism classes of irreducible $U_\chi(W)$–modules.

Now, let Y be an arbitrary irreducible $U_\chi(\mathfrak{g})$–module. I claim that we can find an irreducible $U_\chi(\mathfrak{g})$–module X with $\mathfrak{a} \cdot X = 0$ and

$$U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} X \longrightarrow U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} Y.$$

First, apply Lemma 11.9.5 to find an irreducible $U_\chi(\mathfrak{g})$–submodule $X \subset U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} Y$ with $\mathfrak{a} \cdot X = 0$; thus we have inclusion maps:

$$X \hookrightarrow U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} Y.$$

Now apply 'Frobenius reciprocity' on the inclusion $X \hookrightarrow U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} Y$ and obtain a (nonzero) $U_\chi(W)$–homomorphism:

$$U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} X \longrightarrow U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} Y. \tag{11.13}$$

This implies that every $U_\chi(W)$–module is induced from a $U_\chi(\mathfrak{g})$–module annihilated by $\mathfrak{a}$: Indeed, any irreducible $U_\chi(W)$–module V contains an irreducible $U_\chi(\mathfrak{g})$–module Y; hence, by 'Frobenius reciprocity', V is a homomorphic image of $U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} Y$ and by (11.13) then also a homomorphic image of $U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} X$ for some irreducible $U_\chi(\mathfrak{g})$–module X with $\mathfrak{a} \cdot X = 0$. By the part of the claim already proved we therefore have $V \simeq U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} X$. The proof is completed. □

Remark 11.9.7. If $r > p-1$ then $\mathfrak{a} \subset W_{\geq 1}$ and therefore $\mathfrak{a}$ is a unipotent ideal in $\mathfrak{g}$ with $\chi(\mathfrak{a}) = 0$; by Lemma 6.3.1 all irreducible $U_\chi(\mathfrak{g})$–modules are annihilated by $\mathfrak{a}$. So Proposition 11.9.6 says for $r > p-1$: If $\chi(\mathfrak{a}) = 0$, then induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules and the isomorphism classes of irreducible $U_\chi(W)$–modules.

12 Rank 2

Let $p > 2$ and let χ be a character of height r such that rk $\mathfrak{c}_W(\chi) = 2$. Define

$$\begin{array}{rcl} T_0 &:=& Ke_{012} \oplus Ke_{101}, \\ T_1 &:=& K(e_{001} - e_{p-1,01}) \oplus Ke_{012}, \\ T_2 &:=& K(e_{001} + e_{101}) \oplus K(e_{002} + e_{012}). \end{array}$$

It is easy to check that T_0 and T_2 are maximal tori. I claim that T_1 is a maximal torus also: Since $[e_{012}, e_{001} - e_{p-1,0,1}] = 0$ we only need to prove that $(e_{001} - e_{p-1,0,1})^{[p]} = e_{001} - e_{p-1,0,1}$. This will be a consequence of the following: Let $b \in K^*$ and let $D_b = e_{001} + be_{p-1,0,1}$ be the derivation on $B_2 = K[X_1, X_2]/(X_1^p, X_2^p)$ given by

$$D_b = \frac{\partial}{\partial x_1} + bx_1^{p-1} \frac{\partial}{\partial x_1} \tag{12.1}$$

where x_1 is the image of X_1 in B_2. Any derivation on B_2 is determined (uniquely) by its values on x_1 and x_2. If we evaluate D_b on x_1 we get $1+bx_1^{p-1}$. For $s = 2, 3, \ldots, p-1$ we easily find $D_b^s(x_1) = b \prod_{i=p-s+1}^{p-1} ix_1^{p-s}$. In particular, $D_b^{p-1}(x_1) = -bx_1$ since $1 \cdot 2 \cdots (p-1) \equiv -1 \pmod{p}$. Now we get

$$D_b^{[p]}(x_1) = (e_{001} + be_{p-1,0,1})^p(x_1) = -b(1 + bx_1^{p-1}).$$

Finally, $D_b^{[p]}(x_2) = 0$ so we have $(e_{001} + be_{p-1,0,1})^{[p]}(x_1) = \big(b(e_{001} + be_{p-1,0,1})\big)(x_1)$ and $(e_{001} + be_{p-1,0,1})^{[p]}(x_2) = (e_{001} + be_{p-1,0,1})(x_2)$. Therefore we have $(e_{001} + be_{p-1,0,1})^{[p]} = -b(e_{001} + be_{p-1,0,1})$ and so D_{-1} is a toral element. Moreover, KD_b is a torus since cD_b is toral for some $c \in K^*$: In fact, we shall choose $c \in K$ such that $c^{p-1}b = -1$.

Note that none of T_0, T_1, T_2 are conjugate under Aut(W). In [4, Thm. 1], Demushkin proves that any maximal torus in W is conjugate under Aut(W) to exactly one of the maximal tori T_0, T_1 or T_2.

So we can find $g \in$ Aut(W) such that $\mathfrak{c}_W(\chi^g)$ contains one of T_0, T_1 or T_2. It is well known that the representation theory of $U_\chi(W)$ depends only on the Aut(W)–orbit of χ. Thus: We will in the following assume that $\chi \neq 0$ and that $\mathfrak{c}_W(\chi)$ contains T_0 or T_1 or T_2.

Note: There exists a nonzero character χ such that $\mathfrak{c}_W(\chi)$ contains T_0 or T_1 or T_2. In fact, if T is any torus then $T = \bigoplus_{i=1}^n Kh_i$ for $n \leq 2$ and $h_i^{[p]} = h_i$ for all i. It follows that $\mathrm{ad}(h_i)^p = \mathrm{ad}(h_i^{[p]}) = \mathrm{ad}(h_i)$ for all i so each $\mathrm{ad}(h_i)$ acts diagonalisably on W. But all h_i commute so $\mathrm{ad}_W T$ is simultaneously diagonalizable. Now T is a $\mathrm{ad}_W T$–submodule of W so, by [3, 3.12], there exists a $\mathrm{ad}_W T$–submodule V of W such that $W = T \oplus V$. Now define $\chi \in W^*$ such that $\chi(V) = 0$ and $\chi(T) \neq 0$. Then $\chi \neq 0$ and $\mathfrak{c}_W(\chi) \supset T$.

12.1 Stabilizers of rank 2

Lemma 12.1.1. *If $\mathfrak{c}_W(\chi) \supset T_0$ then $\chi = 0$ or $\chi(T_0) \neq 0 = \chi(W_{-1} \oplus Ke_{102} \oplus W_{011}) = 0$.*

Proof. Note that

$$[e_{012}, e_{ijk}] = \begin{cases} je_{ijk} & k = 1, \\ (j-1)e_{ijk} & k = 2. \end{cases} \quad \text{and} \quad [e_{101}, e_{ijk}] = \begin{cases} (i-1)e_{ijk} & k = 1, \\ ie_{ijk} & k = 2. \end{cases}$$

It follows that $\chi(e_{ijk}) = 0$ unless $(ijk) = (012)$ or $(ijk) = (101)$. □

Lemma 12.1.1 says that χ has height -1 (or equivalent: $\chi = 0$) or χ has height 1 if $\mathfrak{c}_W(\chi) \supset T_0$. The representation theory of χ can be obtained from our computations in the "height at most one" case. See Appendix C. If $\chi = 0$ we require $p > 3$ in order to use [10].

Lemma 12.1.2. *If $\chi \neq 0$ and $\mathfrak{c}_W(\chi) \supset T_1$ then $\chi(e_{ij1}) = 0$ if $j \neq 0$ and $\chi(e_{ij2}) = 0$ if $j \neq 1$. We have $\chi(e_{001}) = 2\chi(e_{p-1,0,1})$ and $\chi(e_{i01}) = 0$ for $1 \leq i \leq p-2$ and we have $\chi(e_{012}) = \chi(e_{p-1,1,2})$ and $\chi(e_{i12}) = 0$ for $1 \leq i \leq p-2$.*

Proof. Since $e_{012} \in \mathfrak{c}_W(\chi)$ we have $\chi(e_{ij1}) = 0$ unless $j = 0$ and $\chi(e_{ij2}) = 0$ unless $j = 1$. This follows from

$$[e_{012}, e_{ijk}] = \begin{cases} je_{ijk} & k = 1, \\ (j-1)e_{ijk} & k = 2. \end{cases}$$

The assumption $e_{001} - e_{p-1,0,1} \in \mathfrak{c}_W(\chi)$ implies that:

$$i\chi(e_{i-1,0,1}) - (i+1)\chi(e_{p-2+i,0,1}) = 0, \tag{12.2}$$

$$i\chi(e_{i-1,1,2}) - i\chi(e_{p-2+i,1,2}) = 0, \tag{12.3}$$

for all i with $0 \leq i \leq p-1$ [we define $e_{i-1,0,1} = 0 = e_{i-1,0,2}$ for $i = 0$ and we define $e_{p-2+i,0,1} = 0 = e_{p-2+i,1,2}$ for $i \geq 2$]. The relations (12.2), (12.3) follow from our assumption $e_{001} - e_{p-1,0,1} \in \mathfrak{c}_W(\chi)$ and $\chi([e_{001} - e_{p-1,0,1}, e_{i01}]) = 0 = \chi([e_{001} - e_{p-1,0,1}, e_{i12}])$.

For all $i \geq 2$ we have $\chi(e_{i-1,0,1}) = 0$ by (12.2). For $i = 1$ we get $\chi(e_{001}) = 2\chi(e_{p-1,0,1})$ and $\chi(e_{p-2,0,1}) = 0$ is just (12.2) with $i = 0$.

For all $i \geq 2$ we have $\chi(e_{i-1,1,2}) = 0$ by (12.3). For $i = 1$ we get $\chi(e_{012}) = \chi(e_{p-1,1,2})$ and $\chi(e_{p-2,1,2}) = 0$ is just (12.3) with $i = 0$. □

The lemma above says that any nonzero χ with $\mathfrak{c}_W(\chi) \supset T_1$ has height $p-1$ or p. In Section 12.2 we give a complete description of the irreducible $U_\chi(W)$–modules for χ of height $p-1$ and $\mathfrak{c}_W(\chi) \supset T_1$.

Lemma 12.1.3. *If $\chi \neq 0$ and $\mathfrak{c}_W(\chi) \supset T_2$ then $\chi(W_{2p-3}) \neq 0$; i.e., χ has maximal height.*

Proof. Otherwise we can find $x \in W_r$, where $r = \text{ht}\ \chi$, such that $\chi([e_{001}, x]) \neq 0$ or $\chi([e_{002}, x]) \neq 0$ (use that $[W_r, W_{-1}] = W_{r-1}$). Therefore

$$\chi([e_{001} + e_{101}, x]) \neq 0 \quad \text{or} \quad \chi([e_{002} + e_{012}, x]) \neq 0$$

since $\chi([e_{101}, x]) = \chi([e_{012}, x]) = 0$. We have a contradiction with $\mathfrak{c}_W(\chi) \supset T_2$. □

Lemma 12.1.3 says that χ has maximal height if $\mathfrak{c}_W(\chi) \supset T_2$. The representation theory of χ with maximal height is not very well understood. However, we will study χ with maximal height and rank 2 a little closer in Section 12.3.

12.2 Some characters of height $p-1$

Let $a \in K$. We shall consider a character χ_a of height $p-1$ such that $\chi_a(e_{p-1,0,1}) \neq 0$ and $\chi_a(e_{001}) = a$. Moreover, $\chi_a(e_{ij2}) = 0$ for all i,j with $(ij2) \neq (012)$ and $\chi_a(e_{ij1}) = 0$ for all $j > 0$ and all i. Finally, $\chi_a(e_{i01}) = 0$ for i with $1 \leq i \leq p-2$. Note that each χ_a is a character of Type A where Type A–characters are defined in Section 11.5.

If $a = 2\chi_a(e_{p-1,0,1})$ and $\chi_a(e_{012}) = 0$, then $\mathfrak{c}_W(\chi_a) \supset T_1$ by Lemma 12.1.2 so the computations below contain the case, where χ_a has height $p-1$ and rk $\mathfrak{c}_W(\chi_a) = 2$.

Define $\mathfrak{g}$ as in (11.8):

$$\mathfrak{g} = \bigoplus_{0\leq i<p} \bigoplus_{0<j<p} Ke_{ij2} \oplus \bigoplus_{0\leq i<p} \bigoplus_{0\leq j<p} Ke_{ij1}.$$

We can think of $\mathfrak{g}$ as W except all e_{i02} for $i = 0,1,\ldots,p-1$. Inside $\mathfrak{g}$ we have a p–ideal given by

$$\mathfrak{a} = \sum_{i=0}^{p-1} (\mathrm{ad}\, e_{001})^i (W_{\geq p-1}) = \sum_{k=1}^{2} \sum_{j\geq 1} \sum_{i=0}^{p-1} Ke_{ijk}.$$

See (11.11) and Lemma 11.7.1.

It is easy to see that each χ_a vanishes on $\mathfrak{a}$; hence induction induces a bijection between the isomorphism classes of irreducible $U_{\chi_a}(\mathfrak{g})$–modules and the isomorphism classes of irreducible $U_{\chi_a}(W)$–modules by Theorem 11.8.5.

So we concentrate on the irreducible $U_{\chi_a}(\mathfrak{g})$–modules from now. Let $\mathfrak{h} = \mathfrak{g} \cap W_{\geq 0}$ as in (11.9). Since $\mathfrak{h}$ is supersolvable we can construct Vergne polarizations with respect to the chain (11.10). Let $\mathfrak{p}_a$ denote the Vergne polarization of $(\chi_a)_{|\mathfrak{h}}$ with respect to the chain (11.10). Since χ_a has height $p-1$, it follows that $W_{\geq \frac{p+1}{2}} \cap \mathfrak{h} \subset \mathfrak{p}_a$ since $\chi_a([W_{\geq s}, W_{\geq s}]) \subset \chi_a(W_{\geq p}) = 0$. In order to compute $\mathfrak{p}_a$ recall the definition given in (9.7):

a) $e_{i-t,t,2} \in \mathfrak{s}^{\chi_a}_{i-t,t,2}$ for $0 < i \leq \frac{p+1}{2}$ and $0 < t \leq i$: This follows immediately from the definition of χ_a and the inclusion:

$$[e_{i-t,t,2}, W_{i-t,t,2} \cap \mathfrak{h}] \subset \sum_{c=1}^{2} \sum_{b>0} \sum_{a+b\geq i} Ke_{abc}.$$

b) $\mathfrak{s}^{\chi_a}_{i01} \subset \mathfrak{s}^{\chi_a}_{i-1,1,1}$ for $0 < i \leq \frac{p-1}{2}$: Use that $e_{p-i,0,1} \in W_{i01} \cap \mathfrak{h}$ and that

$$\chi_a([e_{i01}, e_{p-i,0,1}]) \neq 0 = \chi_a([W_{i-1,1,1} \cap \mathfrak{h}, e_{p-i,0,1}]).$$

c) $e_{i-t,t,1} \in \mathfrak{s}^{\chi_a}_{i-t,t,1}$ for $0 < i \leq \frac{p+1}{2}$ and $0 < t \leq i$: It follows immediately from the inclusion:

$$[e_{i-t,t,1}, W_{i-t,t,1} \cap \mathfrak{h}] \subset \sum_{c=0}^{2} \sum_{b>0} \sum_{a+b\geq i} Ke_{abc}$$

and the definition of χ_a.

d) $e_{\frac{p+1}{2},0,1} \in \mathfrak{s}^{\chi_a}_{\frac{p+1}{2},0,1}$ since $\chi_a([e_{\frac{p+1}{2},0,1}, W_{\frac{p+1}{2},0,1} \cap \mathfrak{h}]) = 0$.

The observations in a)–d) and the definition of $\mathfrak{p}_a$ implies that

$$\mathfrak{p}_a = \bigoplus_{n=1}^{\frac{p-1}{2}} \bigoplus_{i=0}^{n} Ke_{i,n+1-i,1} \oplus Ke_{\frac{p+1}{2},0,1} \oplus \bigoplus_{n=0}^{\frac{p-1}{2}} \bigoplus_{i=0}^{n} Ke_{i,n+1-i,2} \oplus W_{\geq \frac{p+1}{2}}. \tag{12.4}$$

It follows that $\mathfrak{p}_a$ has rank one ($\mathfrak{p}_a = Ke_{012} \oplus \mathfrak{p}_a \cap W_{011}$) and $\dim_K \mathfrak{g} - \dim_K \mathfrak{p}_a = \frac{p+1}{2}$. Define $\nu \in K$ such that $\nu^3 - \nu = \chi_a(e_{012})^3$. Let K_ν the one dimensional $\mathfrak{p}_a$–module, where $x \in \mathfrak{p}_a \cap W_{011}$ acts as multiplication by $\chi_a(x)$ and e_{012} acts as multiplication by ν. Since $\chi_a(x^{[p]}) = 0$ for all $x \in \mathfrak{p}_a \cap W_{011}$ (true for all basis elements!), it follows that K_ν is an irreducible $U_{\chi_a}(\mathfrak{p}_a)$–module.

There exists $\lambda \in \mathfrak{h}^*$ such $\mathfrak{p}_a$ is the Vergne polarization of λ and compatible with χ_a (choose λ such that $\lambda_{|\mathfrak{h}\cap W_{101}} = (\chi_a)_{|\mathfrak{h}\cap W_{101}}$ and $\lambda(e_{012})^3 - \lambda(e_{012}) = \chi_a(e_{012})^3$). Therefore. by Proposition 9.3.5 and Lemma 9.3.7, there exist up to isomorphism p irreducible $U_{\chi_a}(\mathfrak{h})$–modules represented by $N_\nu := U_{\chi_a}(\mathfrak{h}) \otimes_{U_{\chi_a}(\mathfrak{p}_a)} K_\nu$, where $\nu \in K$ such that $\nu^3 - \nu = \chi_a(e_{012})^3$. The set

$$\{e_{101}^{i_1} e_{201}^{i_2} \cdots e_{\frac{p-1}{2},0,1}^{i_{\frac{p-1}{2}}} \otimes 1 \mid 0 \leq i_j \leq p-1 \text{ for } j = 1, 2, \ldots, \frac{p-1}{2}\}$$

form a basis for N_ν. Since $[e_{012}, e_{k01}] = 0$ for all k it follows that e_{012} acts as multiplication by ν on N_ν. We define the $U_{\chi_a}(\mathfrak{g})$–module induced from N_ν by

$$S_\nu := U_{\chi_a}(\mathfrak{g}) \otimes_{U_{\chi_a}(\mathfrak{h})} N_\nu. \tag{12.5}$$

If $\{v_j\}_{j\in J}$ form a basis for N_ν then the set $\{e_{001}^i \otimes_{U_{\chi_a}(\mathfrak{h})} v_j \mid i = 0, 1, \ldots, p-1, \ \ j \in J\}$ form a basis for S_ν. Since $[e_{012}, e_{001}] = 0$ and since e_{012} acts as multiplication by ν on N_ν it follows that e_{012} acts as multiplication by ν on S_ν also.

Lemma 12.2.1. *If $\nu \neq 0$ then S_ν is irreducible and if $\nu, \mu \neq 0$ and $S_\nu \simeq S_\mu$ then $\nu = \mu$.*

Proof. Let $\nu \in K$ with $\nu^3 - \nu = \chi_a(e_{012})^3$. Then

$$\{v \in S_\nu \mid e_{112} \cdot v = 0\} = 1 \otimes N_\nu. \tag{12.6}$$

Note that $e_{112} \cdot 1 \otimes N_\nu = 0$ (use that $e_{112} \in \mathfrak{a} \cap W_{\geq 1}$ and that $\mathfrak{a} \cap W_{\geq 1} \lhd \mathfrak{h}$ is unipotent with $\chi_a(\mathfrak{a} \cap W_{\geq 1}) = 0$; hence $\mathfrak{a} \cap W_{\geq 1}$ annihilates all irreducible $U_{\chi_a}(\mathfrak{h})$–modules by Lemma 6.3.1).

Suppose that there exists $m > 0$ and a nonzero element $v_m \in N_\nu$ such that

$$v \in e_{001}^m \otimes v_m + \sum_{k=0}^{m-1} e_{001}^k \otimes N_\nu$$

is annihilated by e_{112}. Since $e_{112} \cdot N_\nu = 0$ we get: $0 \in e_{001}^{m-1} \otimes e_{012} v_m + \sum_{k=0}^{m-2} e_{001}^k \otimes N_\nu$. Now apply the PBW theorem for reduced enveloping algebras and obtain $e_{012} \cdot v_m = 0$; hence $v_m = 0$ since $\nu \neq 0$ by assumption and since e_{012} acts as multiplication by ν on S_ν. We have a contradiction. It follows that (12.6) holds.

Therefore each S_ν is irreducible and if $S_\nu \simeq S_\mu$ then $\nu = \mu$: Indeed, let X be a $\mathfrak{g}$–submodule of S_ν. Take a nonzero $x \in X$ such that $e_{112} \cdot x = 0$ (for instance, take $x \in \mathrm{Soc}_{\mathfrak{p}_a} X$). Then $X \cap (1 \otimes N_\nu) \neq 0$. But then $X \cap (1 \otimes N_\nu) = 1 \otimes N_\nu$ since $X \cap (1 \otimes N_\nu)$ is a $U_{\chi_a}(\mathfrak{h})$–submodule of $1 \otimes N_\nu$ and since $1 \otimes N_\nu$ is an irreducible $U_{\chi_a}(\mathfrak{h})$–module; hence $X \supset 1 \otimes N_\nu$ and therefore also $X \supset S_\nu$.

Since e_{012} acts on each S_ν as multiplication by ν it follows that $\nu = \mu$ if $S_\nu \simeq S_\mu$. ☐

Proposition 12.2.2. *If $\chi_a(e_{012}) \neq 0$ then there exist up to isomorphism p irreducible $U_{\chi_a}(W)$–modules of dimension $p^{\frac{3p+1}{2}}$.*

Proof. If $\chi_a(e_{012}) \neq 0$ then all $\nu \in K$ with $\nu^3 - \nu = \chi_a(e_{012})^3$ are nonzero and so, by the lemma above, induction induces in that case a bijection between the isomorphism classes of irreducible $U_{\chi_a}(\mathfrak{h})$–modules and the isomorphism classes of irreducible $U_{\chi_a}(\mathfrak{g})$–modules [we could obtain this from Lemma 11.7.2 also]. Moreover, induction induces a bijection between the isomorphism classes of irreducible $U_{\chi_a}(\mathfrak{g})$–modules and the isomorphism classes of irreducible $U_{\chi_a}(W)$–modules by Theorem 11.8.5. There are up to isomorphism p irreducible $U_{\chi_a}(\mathfrak{h})$–modules of dimension $p^{\frac{p-1}{2}}$ and $\dim_K W - \dim_K \mathfrak{h} = p+1$. The proof is completed. □

Proposition 12.2.3. *Assume that $\chi_a(e_{012}) = 0$. Then there exist up to isomorphism*

$$\begin{cases} 2p-1 & \text{irreducible } U_{\chi_a}(W) - \text{modules} \quad \text{if } a \neq 0, \\ 2p-2 & \text{irreducible } U_{\chi_a}(W) - \text{modules} \quad \text{if } a = 0. \end{cases}$$

There exist $p-1$ representatives of dimension $p^{\frac{3p+1}{2}}$ and

$$\begin{cases} p & \text{representatives of dimension } p^{\frac{3p-1}{2}} \quad \text{if } a \neq 0, \\ p-1 & \text{representatives of dimension } p^{\frac{3p-1}{2}} \quad \text{if } a = 0. \end{cases}$$

Proof. If $\chi_a(e_{012}) = 0$ then $S_1, S_2, \ldots, S_{p-1}$ are irreducible $U_{\chi_a}(\mathfrak{g})$–modules and non–isomorphic by Lemma 12.2.1. Note that any irreducible $U_{\chi_a}(\mathfrak{g})$–module X contains an irreducible $U_{\chi_a}(\mathfrak{p}_a)$–module which is a copy of some K_ν. By 'Frobenius reciprocity' X is isomorphic to some S_ν with $\nu \neq 0$ or a homomorphic image of S_0. If X is a homomorphic image of S_0 then $\mathfrak{a} \cdot X = 0$ since $\mathfrak{a} \cdot S_0 = 0$ [note that $\mathfrak{a} \subset \mathfrak{h}$ with $\mathfrak{a} \cdot N_0 = 0$ since $e_{012} \cdot N_0 = 0$ and since $\mathfrak{a} \cap W_{011} \triangleright \mathfrak{h}$ is a unipotent p–ideal with $\chi_a(\mathfrak{a} \cap W_{011}) = 0$ and therefore annihilates all irreducible $U_{\chi_a}(\mathfrak{h})$–modules by Lemma 6.3.1]. It follows that representatives for irreducible $U_{\chi_a}(\mathfrak{g})$–modules are $S_1, S_2, \ldots, S_{p-1}$ and representatives for irreducible $U_{\chi_a}(\mathfrak{g})$–modules annihilated by $\mathfrak{a}$.

We take a closer look at irreducible $U_{\chi_a}(\mathfrak{g})$–modules annihilated by $\mathfrak{a}$. We can write

$$\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{i=0}^{p-1} K e_{i01}.$$

We now observe that $\bigoplus_{i=0}^{p-1} K e_{i01}$ is isomorphic to the Witt–Jacobson algebra $W(1)$ of rank 1 defined in [7]. It has a K–basis e_i where $-1 \leq i \leq p-2$ and the Lie bracket and the p–mapping are given by $[e_i, e_j] = (j-i)e_{i+j}$ for all $-1 \leq i, j \leq p-2$ where $e_{i+j} := 0$ if $i + j \notin \{-1, 0, \ldots, p-1\}$ and $e_i^{[p]} := \delta_{i0} e_{i0}$ for any i with $-1 \leq i \leq p-2$. Now the map $e_{i01} \longmapsto e_i$ is an isomorphism $\bigoplus_{i=0}^{p-1} K e_{i01} \simeq W(1)$ of restricted Lie algebras.

It is well known that irreducible $U_{\chi_a}(\mathfrak{g})$–modules annihilated by $\mathfrak{a}$ are in one to one correspondence with irreducible $U_{\chi_a}(\mathfrak{g}/\mathfrak{a}) \simeq U_{\chi_a}\big(W(1)\big)$–modules. [Any irreducible $U_{\chi_a}\big(W(1)\big)$–module X extends to $\mathfrak{g}$ if we define $\mathfrak{a} \cdot X = 0$. On the other hand: Any irreducible $U_{\chi_a}(\mathfrak{g})$–module is an irreducible $U_{\chi_a}\big(W(1)\big)$–module. Therefore, we can think of irreducible $U_{\chi_a}(\mathfrak{g})$–modules annihilated by $\mathfrak{a}$ as irreducible $U_{\chi_a}\big(W(1)\big)$–modules extended to $\mathfrak{g}$ with trivial $\mathfrak{a}$–action.]

Since $\chi_a(e_{p-1,0,1}) \neq 0$ we have $r(\chi_a) = p-1$ with the definition of $r(\chi_a)$ defined in [7, p. 448]. So the irreducible $U_{\chi_a}\big(W(1)\big)$–modules are described in [7, Theorem C].

If $a \neq 0$ then $\mathfrak{c}_{W(1)}(\chi_a) = K(e_{001} + \frac{p-1}{2} a \chi_a(e_{p-1,0,1})^{-1} e_{p-1,0,1})$ is a torus and so we can apply [7, Theorem C (i)] (in the beginning of this section we proved that KD_b with D_b as in (12.1) is a torus). If $a = 0$ then $\mathfrak{c}_{W(1)}(\chi_a) = Ke_{001}$ is unipotent and we can then apply [7, Theorem C (ii)]. □

Remark 12.2.4. Any character τ of height $p-1$ with rk $\mathfrak{c}_W(\tau) = 2$ is conjugate under $\mathrm{Aut}(W)$ to some χ_a with $\chi_a(e_{012}) = 0$ for some $a \neq 0$: Since τ^g has height $p-1$ for any automorphism g we can find an automorphism g such that $\mathfrak{c}_W(\tau^g) \supset T_1$; hence $\tau^g = \chi_a$ and $\chi_a(e_{012}) = 0$ for some $a \in K$ by Lemma 12.1.2.

If $a \neq 0$ and $\chi_a(e_{012}) = 0$ then rk $\mathfrak{c}_W(\chi_a) = 2$: Take a diagonal matrix T with entries t_1 and 1 (in that order) such that $a = \chi_a(e_{001}) = 2t_1^{p-1}\chi_a(e_{p-1,0,1})$. Then $\chi_a^T(e_{ijk}) = 0$ unless $(ijk) = (001)$ and $(ijk) = (p-1, 0, 1)$ and we have $\chi_a^T(e_{001}) = 2\chi_a^T(e_{p-1,0,1})$. Therefore, $T^{-1}(\mathfrak{c}_W(\chi)) = \mathfrak{c}_W(\chi_a^T) \supset T_1$. So irreducible modules for characters χ with rk $\mathfrak{c}_W(\chi) = 2$ and height $p-1$ are described in Proposition 12.2.3.

Finally, if $a \neq 0$ then χ_a and χ_0 with $\chi_a(e_{012}) = 0 = \chi_0(e_{012})$ are not conjugate under $\mathrm{Aut}(W)$ because of Proposition 12.2.3. In particular, rk $\mathfrak{c}_W(\chi_0) = 1$ since already $e_{012} \in \mathfrak{c}_W(\chi_0)$.

12.3 Some characters of maximal height

First, we introduce another basis for $B_2 = K[X_1, X_2]/(X_1^p, X_2^p)$. Let x_i be the image of X_i in B_2. Set $y_i := 1 + x_i \in B_2$. Note that each y_i is a unit in B_2 since $y_i^p = x_i^p + 1 = 1$. Thus we can define y^α for any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$:

$$y^\alpha = y_1^{\alpha_1} y_2^{\alpha_2}. \tag{12.7}$$

Let $I(2)$ be the set of $\alpha \in \mathbb{Z}^2$ with $0 \leq \alpha_1 < p$ and $0 \leq \alpha_2 < p$. It is easy to see that all y^α with $\alpha \in I(2)$ are a basis for B_2.

Define

$$e_\alpha^{(i)} = y_i y^\alpha \frac{\partial}{\partial x_i} \quad \text{for } i = 1,2 \text{ and } \alpha \in \mathbb{Z}^2. \tag{12.8}$$

The y^α with $\alpha \in I(2)$ form a basis for B_2; so do the $y_i y^\alpha$ with $\alpha \in I(2)$ since y_i is a unit in B_2. Therefore the $e_\alpha^{(i)}$ with $i = 1,2$ and $\alpha \in I(2)$ defined above for a basis of W (recall that W is a free B_2–module with basis $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$).

The commutator of any two basis elements as in (12.8) is given by (apply the Lie bracket in W introduced in Section 3):

$$[e_\alpha^{(i)}, e_\beta^{(j)}] = \beta_i e_{\alpha+\beta}^{(j)} - \alpha_j e_{\alpha+\beta}^{(i)}. \tag{12.9}$$

The p–mapping is given as follows:

$$(e_\alpha^{(i)})^{[p]} = \begin{cases} e_0^{(i)} & \text{if } \alpha_i \equiv 0 \pmod{p}, \\ 0 & \text{else.} \end{cases} \tag{12.10}$$

In order to obtain (12.10), note that $(e_\alpha^{(i)})^{[p]}(x_i) = (e_\alpha^{(i)})^p(x_i)\frac{\partial}{\partial x_i}$. Then use induction over t to show that

$$(e_\alpha^{(i)})^t(x_i) = \prod_{j=1}^{t-1}(j\alpha_i + 1)y^{t\alpha+\varepsilon_i}, \qquad \text{hence} \quad (e_\alpha^{(i)})^p(x_i) = \prod_{j=1}^{p-1}(j\alpha_i + 1)y_i$$

where $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the i–th position. In case $\alpha_i \equiv 0 \pmod{p}$ the product in the last statement is equal to 1; if $\alpha_i \not\equiv 0 \pmod{p}$, then there exists j with $0 < j < p$ such that $j\alpha_i \equiv p-1 \pmod{p-1}$ and so the product is 0.

Set

$$h_i = e_0^{(i)} \text{ for } i = 1,2 \quad \text{and} \quad \mathfrak{h} = \sum_{i=1}^{2} K h_i. \tag{12.11}$$

We have $h_1 = e_{001} + e_{101}$ and $h_2 = e_{002} + e_{012}$; hence $\mathfrak{h} = T_2$.

Lemma 12.3.1. *If $\mathfrak{c}_W(\chi) \supset T_2$ and $\chi(h_1) \neq 0 \neq \chi(h_2)$ then $\mathfrak{c}_W(\chi) = T_2$.*

Proof. First, note that $\chi(e_\alpha^{(i)}) = 0$ for all $\alpha \neq 0$ and $i = 1, 2$ since $h_1, h_2 \in \mathfrak{c}_W(\chi)$ with $[h_1, e_\alpha^{(i)}] = \alpha_1 e_\alpha^{(i)}$ and $[h_2, e_\alpha^{(i)}] = \alpha_2 e_\alpha^{(i)}$. In order to prove our claim suppose that

$$y = \sum_{i=1}^{2} \sum_{\alpha \neq 0} a_{\alpha,i} e_\alpha^{(i)} \in \mathfrak{c}_W(\chi) \quad \text{for some } a_{\alpha,i} \in K.$$

Given $\alpha = (\alpha_1, \alpha_2) \in I(2)$ such that $\alpha \neq 0$ and $a_{\alpha,i} \neq 0$ for some $i = 1, 2$. We can assume that $a_{\alpha,1} \neq 0$ for some $\alpha \neq 0$ (apply the interchanging automorphism on W introduced in Appendix A). Set $\alpha' = (p - \alpha_1, p - \alpha_2)$. Now use the relations

$$[e_\alpha^{(1)}, e_{\alpha'}^{(1)}] = -\,2\alpha_1 h_1, \tag{12.12}$$

$$[e_\alpha^{(1)}, e_{\alpha'}^{(2)}] = -\,\alpha_2 h_1 - \alpha_1 h_2, \tag{12.13}$$

$$[e_\beta^{(r)}, e_{\alpha'}^{(s)}] \in \sum_{i=1}^{2} \sum_{\gamma \neq 0} K e_\gamma^{(i)} \quad \text{for } r, s \in \{1, 2\} \text{ and for } \beta \in I(2) \text{ with } \beta \neq \alpha. \tag{12.14}$$

It follows from (12.12),(12.13) and (12.14) that we get a contradiction if $a_{\alpha,1} \neq 0$ for some $\alpha \neq 0$. □

Remark 12.3.2. If $\chi(h_1) = 0$ or $\chi(h_2) = 0$ we do not have $\mathfrak{c}_W(\chi) = T_2$. Suppose that $\chi(h_1) = 0 \neq \chi(h_2)$: Then we can apply (12.12),(12.13) and (12.14) above to get:

$$\mathfrak{c}_W(\chi) = T_2 \oplus \sum_{j=1}^{p-1} e_{(0,j)}^{(1)}.$$

If $\chi(h_2) \neq 0 = \chi(h_2)$ one gets a similar result.

In the following we will take a closer look at the situation where χ has maximal height and $\dim_K \mathfrak{c}_W(\chi) = 2$ and $\mathfrak{c}_W(\chi) \cap W_{\geq 0} = 0$. The computations below (of course) include the case where

$$\mathfrak{c}_W(\chi) = T_2 = K(e_{001} + e_{101}) \oplus K(e_{002} + e_{012}).$$

We shall prove that all irreducible $U_\chi(W)$–modules have maximal dimension; i.e., any irreducible $U_\chi(W)$–module has dimension equal to p^{p^2-1}. See [20, 6.4, Remark 1]. First, we need a reduction:

Lemma 12.3.3. *If $\chi \in W^*$ of maximal height, then there exists $g \in \mathrm{Aut}(W)$ such that $\chi^g(e_{p-1,p-1,2}) = 1$ and $\chi^g(e_{p-1,p-1,1}) = 0$ and $\chi^g(W_{p-1} \oplus W_p \oplus \cdots \oplus W_{2p-4}) = 0$.*

Proof. If we use a suitable automorphism in $GL_2(K)$ we can assume that $\chi(e_{p-1,p-1,2}) = 1$ and $\chi(e_{p-1,p-1,1}) = 0$ (the final part of the proof of Lemma 11.2.1 does not use the assumption on the height in that section and can then be applied here). Let m be an integer with $p - 1 \leq m \leq 2p - 4$ and define

$$x_m = \sum_{r+s=2p-2-m} a_{rs} e_{rs1} + \sum_{r+s=2p-2-m} b_{rs} e_{rs2} \in W_{2p-3-m}.$$

From the formulas $(i + j = m + 1)$

$$\begin{aligned} \chi([x_m, e_{ij1}]) &= i b_{p-i,p-1-j} \\ \chi([x_m, e_{ij2}]) &= i a_{p-i,p-1-j} + 2j b_{p-1-i,p-j} \end{aligned}$$

it follows from Section 3.2 that we can find automorphism g_m on W induced by x_m defined as above (for suitable $a_{rs}, b_{rs} \in K$) such that $\chi^{g_m}(W_m) = 0$. For each g_m constructed in this way note that $\chi^{g_m}_{|W_j} = \chi_{|W_j}$ for $j > m$ (this follows since $[x_m, W_j] \subset W_{\geq 2p-2} = 0$).

We are now in position to construct $g \in \mathrm{Aut}(W)$ with $\chi^g(W_{p-1} \oplus W_p \oplus \cdots \oplus W_{2p-4}) = 0$ and $\chi^g(e_{p-1,p-1,2}) = 1$ and $\chi^g(e_{p-1,p-1,1}) = 0$. Set $g = g_{2p-4} \circ \cdots \circ g_p \circ g_{p-1}$ and suppose, for $p-1 \leq i \leq 2p-4$, that g_i is chosen such that $\chi^{g_{2p-4} \circ \cdots \circ g_{i-1} \circ g_i}(W_i) = 0$ (this can be done by the calculations above). It follows that $\chi^g(W_{p-1} \oplus W_p \oplus \cdots \oplus W_{2p-4}) = 0$. $\square$

In order to prove our claim (that all irreducible $U_\chi(W)$–modules have maximal dimension if $\dim_K \mathfrak{c}_W(\chi) = 2$ and $\mathfrak{c}_W(\chi) \cap W_{\geq 0} = 0$) we can assume that $\chi(e_{p-1,p-1,2}) = 1$ and $\chi(e_{p-1,p-1,1}) = 0$ and $\chi(W_{p-1} \oplus W_p \oplus \cdots \oplus W_{2p-4}) = 0$. Let that be our assumption from now.

I claim that $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 0$: First, use the assumption on $\mathfrak{c}_W(\chi)$ and find $y_1, y_2 \in W_{\geq 0}$ such that $e_{001} + y_1, e_{002} + y_2$ form a basis for $\mathfrak{c}_W(\chi)$. If $y \in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$, then $\chi([y, e_{001} + y_1]) = 0 = \chi([y, e_{002} + y_2])$ implies that $\chi([y, e_{001}]) = 0 = \chi([y, e_{002}])$ since $\chi([y, y_1]) = 0 = \chi([y, y_2])$. It follows that $y \in \mathfrak{c}_W(\chi) \cap W_{\geq 0} = 0$.

Since $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 0$, we can apply 1) in Section 9.2 and get $\dim_K \mathfrak{c}_{W_{012}}(\chi_{|W_{012}}) = 1$. Therefore, any polarization of χ has dimension $\frac{2p^2-3+1}{2} = p^2 - 1$ by (9.6). I claim that the Vergne polarization of χ is given as

$$\mathfrak{p}_\chi = W_{\geq p-1} \oplus \bigoplus_{\beta = \frac{p+1}{2}}^{p-1} \bigoplus_{\gamma=1}^{2} K h_{0\beta\gamma} \tag{12.15}$$

where $h_{0\beta 1} = e_{0\beta 1}$ and $h_{0\beta 2} = e_{0\beta 2} - 2\beta e_{1,\beta-1,1}$. First, note that $W_{\geq p-1} \subset \mathfrak{p}_\chi$ since $[W_{\geq p-1}, W_{\geq p-1}] \subset W_{\geq 2p-2} = 0$ and therefore $W_{\geq p-1} = \mathfrak{s}^\chi_{p-1,1,2} \subset \mathfrak{p}_\chi$ (see (9.11) and (9.12) in Section 9.4). For $\beta \geq \frac{p+1}{2}$ and $\beta \leq p-1$ we have $e_{0\beta 1} \in \mathfrak{s}^\chi_{0\beta 1}$ since our assumption on χ says that

$$[e_{0\beta 1}, W_{0\beta 1}] \subset \sum_{a+b \geq p} K e_{ab1} + \sum_{a+b \geq p} \sum_{a < p-1} K e_{ab2} \subset \mathrm{Ker}(\chi).$$

For $\beta \geq \frac{p+1}{2}$ and $\beta \leq p-1$ we also have $h_{0\beta 2} \in \mathfrak{s}^\chi_{0\beta 2}$: First, observe that

$$[e_{0\beta 2} - 2\beta e_{1,\beta-1,1}, e_{ij1}] \subset \sum_{a+b \geq p} K e_{ab1} \sum_{a+b \geq p} \sum_{a < p-1} K e_{ab2} \subset \mathrm{Ker}(\chi) \text{ for } i + j \geq \frac{p+1}{2}.$$

If $i + j \geq \frac{p+1}{2}$ but $i \neq p-1$ we have

$$[h_{0\beta 2}, e_{ij2}] \subset \sum_{a+b \geq p} K e_{ab1} + \sum_{a+b \geq p} \sum_{a < p-1} K e_{ab2} \subset \mathrm{Ker}(\chi).$$

If $i + j \geq \frac{p+1}{2}$ and $i = p-1$ but $j \neq p - \beta$ we have

$$[h_{0\beta 2}, e_{ij2}] \subset \sum_{a+b \geq p} K e_{ab1} + \sum_{a+b \geq p} \sum_{b < p-1} K e_{ab2} \subset \mathrm{Ker}(\chi).$$

Finally, $[h_{0\beta 2}, e_{p-1,p-\beta,2}] = 0$ by construction of $h_{0\beta 2}$. It follows that $h_{0\beta 2} \in \mathfrak{s}^\chi_{0\beta 2}$. We have thus shown that the Vergne polarization of χ contains

$$P := W_{\geq p-1} \oplus \bigoplus_{\beta = \frac{p+1}{2}}^{p-1} \bigoplus_{\gamma=1}^{2} K h_{0\beta\gamma}.$$

But we easily get

$$\dim_K P = 2\big((p-1)+(p-2)+\cdots+2+1\big) = p^2 - p + 2\big(p-1-\frac{p-1}{2}\big) = p^2-1$$

which is the dimension of $\mathfrak{p}_\chi$; hence $\mathfrak{p}_\chi = P$ and so (12.15) holds. In fact, $\mathfrak{p}_\chi$ is a polarization of χ compatible with χ [i.e., $\chi(x)^p - \chi(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_\chi$ since

$$x \in W_{\geq p-1} \implies x^{[p]} = 0,$$

$$x = h_{0\beta 1} \implies x^{[p]} = 0,$$

$$x = h_{0\beta 2} \implies x^{[p]} \in Ke_{1,(p-1)\beta,1} = 0].$$

Now we can define the induced module

$$U_\chi(W_{012}) \otimes_{U_\chi(\mathfrak{p}_\chi)} K_\chi \tag{12.16}$$

where K_χ is the one dimensional $\mathfrak{p}_\chi$–module where each $x \in \mathfrak{p}_\chi$ acts as multiplication with $\chi(x)$ (since $\mathfrak{p}_\chi$ is a polarization of χ compatible with χ it follows that K_χ is a $U_\chi(\mathfrak{p}_\chi)$–module).

Next, apply Proposition 9.3.5 and Lemma 9.3.7 with $L = W_{012}$ and $\lambda = \chi$ and $P = \mathfrak{p}_\chi$ to get: There exists (up to isomorphism) one irreducible $U_\chi(W_{012})$–module of dimension p^{p^2-2}. Since induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{012})$–modules and the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules (use Lemma 7.1.1 with $x = e_{p-1,p-1,1}$) and since p^{p^2-1} is the maximal dimension for irreducible $U_\chi(W)$–modules we thus get:

Theorem 12.3.4. *Suppose that $\chi \in W^*$ has maximal height and $\dim_K \mathfrak{c}_W(\chi) = 2$ and $\mathfrak{c}_W(\chi) \cap W_{\geq 0} = 0$. Then there exists (up to isomorphism) one irreducible $U_\chi(W_{\geq 0})$–module of dimension p^{p^2-1}. For any irreducible $U_\chi(W_{\geq 0})$–module S there exists a W–module structure on S which extends the given $W_{\geq 0}$–module structure. In particular, all irreducible $U_\chi(W)$–modules have dimension*

$$p^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2} = p^{p^2-1}.$$

13 Characters of height 2 and 3

Let K be an algebraically closed field of characteristic $p > 2$ and let W denote the second Witt–Jacobson algebra over K. We will consider $\chi \in W^*$ of height 2 and 3.

13.1 The good case

We assume that $\chi \in W^*$ is a character of height r, where $r = 2$ or $r = 3$, such that $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$ and such that χ does not have Type II.a as in 5.2 if $r = 3$ and $p = 3$. In that case we can apply the main theorems in Section 10.4: The dimension of any irreducible $U_\chi(W)$–module is, by Theorem 10.4.12, equal to $p^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$, where $\mathfrak{c}_W(\chi)$ denotes the stabilizer of χ in W. Theorem 10.4.11 says that the number of isomorphism classes of irreducible $U_\chi(W)$–modules is p if $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$; otherwise (i.e., $\operatorname{rk} \mathfrak{c}_W(\chi) = 0$ by Lemma 10.4.8) the number of isomorphism classes of irreducible $U_\chi(W)$–modules is 1.

Below we will describe the possible dimension for irreducible $U_\chi(W)$–modules and the number of isomorphism classes (denoted by $|\operatorname{Irr}(W, \chi)|$) for characters as above. The representation theory of $U_\chi(W)$ depends only on the $\operatorname{Aut}(W)$–orbit of χ, so we can assume that there exists $x \in W_{r-1}$ with $\chi([x, e_{102}]) \neq 0 = \chi([x, W_{012}])$ by Lemma 7.3.1. Now use Lemma 10.4.1 to get $\dim_K \mathfrak{c}_{W_{012}}(\chi_{|W_{012}}) = \dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) + 1$. Since $\mathfrak{st}(\chi, W_{\geq 2}) = W_{\geq 0}$ we also have $\mathfrak{c}_W(\chi) \subset \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ by (10.3) and the arguments before that; hence

$$\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) - 2$$

by Lemma 10.4.7. We conclude that $\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_{W_{012}}(\chi_{|W_{012}}) - 3$ or, by (9.6),

$$\operatorname{codim}_W \mathfrak{c}_W(\chi)/2 = \dim_K W - \dim_K \mathfrak{p}_\chi \tag{13.1}$$

where $\mathfrak{p}_\chi$ denotes the Vergne polarization of χ. Now (13.1) allows us to find the possible dimension for irreducible $U_\chi(W)$–modules. Note that either $\mathfrak{p}_\chi$ is unipotent or there exists a nonzero toral element $h \in \mathfrak{p}_\chi$ such that $\mathfrak{p}_\chi = Kh \oplus \mathfrak{p}_\chi \cap W_{011}$ (see Lemma 9.4.3).

For characters of height 2 we have $W_{\geq 1} \subset \mathfrak{p}_\chi$ by Remark 9.4.2. We get the following possibilities:

Characters of height 2 with $\mathfrak{st}(\chi, W_{\geq 2}) = W_{\geq 0}$:

$\lvert\operatorname{Irr}(W, \chi)\rvert$	Possible dimension
1	p^5 or p^6
p	p^4 or p^5

For characters of height 3 we have $W_{\geq 2} \subset \mathfrak{p}_\chi$. In order to find $\mathfrak{p}_\chi$, we now have to compute all $\mathfrak{s}_{ijk}^\chi$ for (ijk) with $(012) \preceq (ijk) \preceq (021)$. We only consider representatives from Section 5 and only characters of height 3 and of Type II.a as in 5.2 if $p > 3$. One can obtain the following scheme:

Characters of height 3 with $\mathfrak{st}(\chi, W_{\geq 3}) = W_{\geq 0}$:

Type	$\lvert\operatorname{Irr}(W, \chi)\rvert$	Possible dimension
Type I	1	p^8 or p^9
	p	p^7 or p^8
Type II	1	p^7 or p^8
	p	p^6 or p^7
Type III	1	p^8 or p^9
	p	p^6 or p^7 or p^8

As we shall see in the next sections the situation is much more complicated if we consider $\chi \in W^*$ of height r, where $r = 2$ or $r = 3$, such that $\mathfrak{st}(\chi, W_{\geq r}) \neq W_{\geq 0}$ (χ is an exceptional character) or $p = 3$ and χ is a character of Type II.a as in 5.2 with height $r = 3$.

13.2 Exceptional characters of height 2

Suppose that $p = 3$ and let $\chi \in W^*$ be a character of height 2 with $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$ (for $p > 3$ we have $\mathfrak{st}(\chi, W_{\geq 2}) = W_{\geq 0}$ by Lemma 8.1.3 and its proof). We shall see that the situation is much more complicated when $p = 3$ and $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$: Induction does not always take irreducible $U_\chi(W_{\geq 0})$–modules to irreducible $U_\chi(W)$–modules and we shall see that not all irreducible $U_\chi(W)$–modules have the same dimension and the number of irreducible $U_\chi(W)$–modules is not always a power of p. This is a quite different pattern than we saw in the previous section.

We will study two types of characters (see Section 11.5):

Type A : $\tau \in W^*$ of height 2 with $\tau(e_{201}) = 1$ and $\tau(e_{202}) = \tau(e_{102}) = \tau(e_{002})$ and $\tau(e_{101}) = 0 = \tau(e_{011})$ and $\mathfrak{st}(\tau, W_{\geq 2}) = Ke_{001} \oplus W_{\geq 0}$.

Type B : $\tau \in W^*$ of height 2 with $\tau(e_{202}) = 1$ and $\tau(e_{201}) = \tau(e_{012}) = 0$ and $\tau(e_{102}) = \tau(e_{002}) = 0$ and $\mathfrak{st}(\tau, W_{\geq 2}) = Ke_{001} \oplus W_{\geq 0}$.

13.3 Type A characters of height 2

Consider $\chi \in W^*$ be a character of height 2 and Type A. Recall the characters defined in Section 12.2: There we consider arbitrary $p > 2$ and for $a \in K$ we define $\chi_a \in W^*$ via $\chi_a(e_{ijk}) = 0$ unless $(ijk) = (012)$ or $(ijk) = (001)$ or $(ijk) = (p-1, 0, 1)$. We have $\chi_a(e_{001}) = a$ and $\chi_a(e_{p-1,0,1}) \neq 0$. The irreducible $U_{\chi_a}(W)$–modules are described in Proposition 12.2.2, 12.2.3.

In our situation, $\chi = \chi_a$ for $a = \chi(e_{001})$. Before we write down we need information on the stabilizer $\mathfrak{c}_W(\chi)$ of χ.

Lemma 13.3.1. *Let $\chi \in W^*$ be a character of height 2 and* Type A. *Then we have*

$$\dim_K \mathfrak{c}_W(\chi) = \begin{cases} 8 & \text{if } \chi(e_{012}) \neq 0, \\ 10 & \text{if } \chi(e_{012}) = 0. \end{cases}$$

Moreover,

$$\operatorname{rk} \mathfrak{c}_W(\chi) = \begin{cases} 1 & \text{if } \chi(e_{012}) \neq 0 \text{ or } \chi(e_{012}) = 0 = \chi(e_{001}), \\ 2 & \text{if } \chi(e_{012}) = 0 \neq \chi(e_{001}), \end{cases}$$

where $\operatorname{rk} \mathfrak{c}_W(\chi)$ *is the dimension of any maximal torus in $\mathfrak{c}_W(\chi)$. Finally, $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$ if and only if $\chi(e_{012}) \neq 0$.*

Proof. First, note that $e_{221}, e_{222}, e_{121}, e_{122}, e_{212} \in \mathfrak{c}_W(\chi)$. Since $\chi(e_{011}) = \chi(e_{021}) = 0$ and $\chi(e_{111}) = \chi(e_{112}) = 0$ we also have $e_{021} \in \mathfrak{c}_W(\chi)$. Finally, $e_{012} \in \mathfrak{c}_W(\chi)$ since $\chi(e_{002}) = 0 = \chi(e_{102})$ and since $\chi([e_{012}, W_{012}]) = 0$.

Therefore we consider $x \in \mathfrak{c}_W(\chi)$ written as

$$x = \sum_{(ijk)} a_{ijk} e_{ijk} \tag{13.2}$$

where $a_{012} = a_{021} = a_{121} = a_{122} = a_{212} = a_{221} = a_{222} = 0$. Since $\mathfrak{c}_W(\chi) \subset \mathfrak{st}(\chi, W_{\geq 2}) = Ke_{001} \oplus W_{\geq 0}$ we also have $a_{002} = 0$.

The relations $\chi([x, e_{111}]) = 0$ and $\chi([x, e_{201}]) = 0$ say that $a_{102} = a_{101} = 0$ since $\chi(e_{101}) = 0 = \chi(e_{011})$ and $a_{202} = 0$ because $\chi([x, e_{011}]) = 0$. Next, observe that $a_{011} = 0 = a_{111}$: First use that $\chi([x, e_{202}]) = 0 = \chi(e_{102}) = 0$ to get $a_{011} = 0$ and then $\chi([x, e_{102}]) = 0 = \chi(e_{002})$ to get $a_{111} = 0$.

In order to determine x we now only have to consider the following relations:

1) $\chi([x, e_{112}]) = 0 \Longrightarrow a_{001}\chi(e_{012}) = 0$,

2) $\chi([x, e_{101}]) = 0 \Longrightarrow a_{201} = a_{001}\chi(e_{001})$,

3) $\chi([x, e_{001}]) = 0 \Longrightarrow a_{112}\chi(e_{012}) = 0$,

4) $\chi([x, e_{002}]) = 0 \Longrightarrow a_{211} = a_{022}\chi(e_{012})$.

If $\chi(e_{012}) \neq 0$ it follows from 1)–3) that $a_{001} = a_{201} = a_{112} = 0$ and by 4) we have $e_{022} + \chi(e_{012})e_{211} \in \mathfrak{c}_W(\chi)$. Moreover, x as in (13.2) is unique (up to multiplication with elements from K). We conclude that $\dim_K \mathfrak{c}_W(\chi) = 8$. Finally, Ke_{012} is a maximal torus in $\mathfrak{c}_W(\chi)$.

Next, suppose that $\chi(e_{012}) = 0$. We find that e_{112}, e_{022} and $e_{001} + \chi(e_{001})e_{201}$ belong to $\mathfrak{c}_W(\chi)$. Moreover, x as in (13.2) is a linear combination of these. We conclude that $\dim_K \mathfrak{c}_W(\chi) = 10$ in that case.

Since $\chi = \chi_a$ where $a = \chi(e_{001})$ and χ_a is a character as in Section 12.2 with $p = 3$ we can apply Remark 12.2.4 in order to find $\operatorname{rk} \mathfrak{c}_W(\chi)$: If $\chi(e_{001}) \neq 0$ then $\operatorname{rk} \mathfrak{c}_W(\chi) = 2$ and if $\chi(e_{001}) = 0$ then $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$ as required. □

We are now in position to describe the irreducible $U_\chi(W)$–modules for Type A characters of height 2. We will formulate the results in terms of the ideal introduced in (11.11):

$$\mathfrak{a} = \bigoplus_{k=1}^{2} Ke_{01k} \oplus \bigoplus_{k=1}^{2} Ke_{11k} \oplus \bigoplus_{k=1}^{2} Ke_{02k} \oplus W_{\geq 2}. \tag{13.3}$$

Note that $\chi(\mathfrak{a}) = 0$ if and only if $\chi(e_{012}) = 0$.

Theorem 13.3.2. *Let $\chi \in W^*$ be a character of height 2 and* Type A *and let $\mathfrak{a}$ be as in* (13.3).

a) If $\chi(\mathfrak{a}) \neq 0$ then there exist up to isomorphism 3 irreducible $U_\chi(W)$–modules all of dimension $3^5 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$. Moreover, $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$ with $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$.

b) If $\chi(\mathfrak{a}) = 0 = \chi(e_{001})$ then there exist up to isomorphism 4 irreducible $U_\chi(W)$–modules. Two representatives have dimension $3^5 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2+1}$ and two representatives have dimension $3^4 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$. Moreover, $\mathfrak{c}_W(\chi) \not\subset W_{\geq 0}$ with $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$.

c) If $\chi(\mathfrak{a}) = 0 \neq \chi(e_{001})$ then there exist up to isomorphism 5 irreducible $U_\chi(W)$–modules. Two representatives have dimension $3^5 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2+1}$ and three representatives have dimension $3^4 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$. Moreover, $\mathfrak{c}_W(\chi) \not\subset W_{\geq 0}$ with $\operatorname{rk} \mathfrak{c}_W(\chi) = 2$.

Proof. Use that $\chi = \chi_a$ as in Section 12.2 with $a = \chi(e_{001})$ and $p = 3$. Then apply Proposition 12.2.2, 12.2.3 and Lemma 13.3.1. □

Remark 13.3.3. One can show that there are (up to isomorphism) 3 irreducible $U_\chi(W_{\geq 0})$–modules all of dimension 3^3. If $\chi(\mathfrak{a}) \neq 0$ then induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and the isomorphism classes of irreducible $U_\chi(W)$–modules. But Theorem 13.3.2 says that induction from $W_{\geq 0}$ to W does not always take irreducible $U_\chi(W_{\geq 0})$–modules to irreducible $U_\chi(W)$–modules. In fact, if $\chi(e_{012}) = 0$ then there are (up to isomorphism) 3 irreducible $U_\chi(W_{\geq 0})$–modules S_0, S_1, S_2 and one can prove that

$$\mathrm{End}_W\big(U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S_1\big) \simeq K[X]/\big(X^3 - X^2 - \chi(e_{001})^3\big).$$

Moreover: S_0, S_2 are non isomorphic irreducible $U_\chi(W_{\geq 0})$–modules and the induced W–modules $U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S_0$ and $U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S_2$ are irreducible and non isomorphic.

13.4 Type B characters of height 2

Let $\chi \in W^*$ of height 2 with and Type B. In particular, $\chi(e_{012}) = 0$ since $j = r+1-p = 0$ in this case and $\chi(e_{012} + je_{101}) = 0$. The Vergne polarization $\mathfrak{p}$ of χ constructed via the chain in (9.10) is given by

$$\mathfrak{p} = \begin{cases} W_{011} & \text{if } \chi(e_{011}) \neq 0, \\ K(e_{012} - e_{101}) \oplus W_{011} & \text{if } \chi(e_{011}) = 0. \end{cases}$$

We have $\chi([W_{011}, W_{011}]) = 0$ since $[W_{011}, W_{011}] \subset W_{112}$ and $\chi(W_{112}) = 0$; hence $W_{011} \subset \mathfrak{s}^\chi_{011} \subset \mathfrak{p}$. If $\chi(e_{011}) = 0$ we have $\chi([e_{012} - e_{101}, W_{012}]) = 0$ and hence $e_{012} - e_{101} \in \mathfrak{s}^\chi_{012}$. If $\chi(e_{011}) \neq 0$ it is easy to check that $\mathfrak{s}^\chi_{012} \subset \mathfrak{s}^\chi_{101} \subset W_{011}$.

Remark 13.4.1. We have $\mathrm{rk}\ \mathfrak{c}_W(\chi) \leq 1$ since any $\tau \in W^*$ of height 2 and $\mathrm{rk}\ \mathfrak{c}_W(\tau) = 2$ is conjugate under $\mathrm{Aut}(W)$ to a character of Type A by the results in Section 12.1. Moreover, Proposition 11.5.2 (or Lemma 11.6.1) says that no characters of Type A and Type B are conjugate. So $\mathrm{rk}\ \mathfrak{c}_W(\chi) = 2$ is impossible for χ of height 2 and Type B.

Lemma 13.4.2. *Let $h := e_{012} - e_{101}$. If $\chi \in W^*$ is a character of height 2 and* Type B*, then we have*

$$\dim_K \mathfrak{c}_W(\chi) = \begin{cases} 8 & \textit{if } \chi(e_{011}) \neq 0, \\ 10 & \textit{if } \chi(e_{011}) = 0 \ \textit{ and } \ \chi(h) \neq 0 \ \textit{ or } \ \chi(e_{001}) \neq 0, \\ 12 & \textit{if } \chi(e_{011}) = \chi(h) = \chi(e_{001}) = 0. \end{cases}$$

Moreover,

$$\mathrm{rk}\ \mathfrak{c}_W(\chi) = \begin{cases} 0 & \textit{if } \chi(e_{011}) \neq 0 \ \textit{or } \chi(h) = 0 \neq \chi(e_{001}), \\ 1 & \textit{otherwise.} \end{cases}$$

Finally; $\mathfrak{c}_W(\chi) \not\subset W_{\geq 0}$ *if and only if* $\chi(e_{011}) = \chi(h) = \chi(e_{001}) = 0$.

Proof. Our assumption on χ (Type B) says that $\chi(h) = 0$ if and only if $\chi(e_{101}) = 0$ since $\chi(e_{012}) = 0$. First, note that we always have $e_{221}, e_{222}, e_{121}, e_{122}, e_{211}, e_{021}, e_{022} \in \mathfrak{c}_W(\chi)$. Therefore we only consider $y \in \mathfrak{c}_W(\chi)$ written as

$$y = \sum_{(ijk)} a_{ijk} e_{ijk} \tag{13.4}$$

where $a_{ijk} = 0$ if $j = 2$ or $(ijk) = (211)$. Since $\mathfrak{c}_W(\chi) \subset \mathfrak{st}(\chi, W_{\geq 2}) = Ke_{001} \oplus W_{\geq 0}$ we can also assume that $a_{002} = 0$. I claim that $\mathfrak{c}_W(\chi) \subset W_{012}$ unless $\chi(e_{001}) = \chi(h) = \chi(e_{011}) = 0$. First, we prove that $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$ (or $a_{001} = 0$) unless $\chi(e_{001}) = \chi(h) = \chi(e_{011}) = 0$.

If $\chi(e_{011}) \neq 0$, then $a_{001} = 0$ since $\chi([y, e_{111}]) = 0$. If $\chi(h) \neq 0$, then we get $a_{001} = 0$ from $\chi([y, e_{112} + e_{201}]) = 0$. Finally; suppose that we have $\chi(h) = 0 = \chi(e_{011})$ but $\chi(e_{001}) \neq 0$. Then $a_{001} = 0$: Combine the relations $\chi([y, h]) = 0$ and $\chi([y, e_{201}]) = 0$ and get $a_{001}\chi(e_{001}) = 0$; hence $a_{001} = 0$ by our assumption.

If $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$, then we clearly have an inclusion $\mathfrak{c}_W(\chi) \subset \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$. Therefore $\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) - 2$ by Lemma 10.4.7. Next, apply Lemma 10.4.1 with $x = e_{112}$ and find that $\dim \mathfrak{c}_{W_{\geq 0}}(\chi) = \dim \mathfrak{c}_{W_{012}}(\chi) - 1$. We conclude by (9.6) that:

$$\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_{W_{012}}(\chi) - 3 = 2\dim_K \mathfrak{p} - \dim_K W_{012} - 3 = \begin{cases} 8 & \text{if } \chi(e_{011}) \neq 0, \\ 10 & \text{if } \chi(e_{011}) = 0. \end{cases}$$

If $\chi(e_{011}) \neq 0$ then $\mathrm{rk}(\mathfrak{c}_W(\chi)) = 0$: Indeed, we have $\mathfrak{c}_W(\chi) \subset W_{012}$ and therefore $\mathfrak{c}_W(\chi)$ is a subset of $\mathfrak{p}$ [for any $z \in \mathfrak{c}_W(\chi)$ we have $\chi([\mathfrak{p} + Kz, \mathfrak{p} + Kz]) = 0$ and hence $z \in \mathfrak{p}$ by maximality]. But $\mathfrak{p}$ is unipotent if $\chi(e_{011}) \neq 0$.

Next, suppose that $\chi(e_{011}) = 0$ but $\chi(h) \neq 0$ or $\chi(e_{001}) \neq 0$. We still have $y \in W_{012}$ where $y \in \mathfrak{c}_W(\chi)$ as in (13.4), but we can find further conditions on y from $\chi([y, W]) = 0$ by looking at $\chi([y, e_{\alpha\beta\gamma}]) = 0$ for all $(\alpha\beta\gamma)$. We find that $y \in \mathfrak{c}_W(\chi) \cap W_{012}$ if and only if $a_{202} = 0$ and that $a_{012}, a_{101}, a_{011}, a_{201}, a_{111}, a_{112}, e_{212} \in K$ satisfy the following relations:

1) $a_{012} + a_{101} = 0$,

2) $a_{011}\chi(h) + a_{201} - a_{112} = 0$,

3) $-a_{101}\chi(e_{001}) + a_{201}\chi(e_{101}) = 0$,

4) $-a_{011}\chi(e_{001}) - a_{111}\chi(e_{101}) - a_{212} = 0$.

It is easy to check from 1)–4) that $e_{011} - \chi(e_{101})e_{112} - \chi(e_{001})e_{212} \in \mathfrak{c}_W(\chi)$. We also have $e_{111} - \chi(e_{101})e_{212} \in \mathfrak{c}_W(\chi)$. Since $\dim_K \mathfrak{c}_W(\chi) = 10$ in this case (i.e., the case where $\chi(e_{011}) = 0$ but $\chi(h) \neq 0$ or $\chi(e_{001}) \neq 0$) we just have to find an element whose coefficients satisfy 1)–4) and which is not a linear combination of

i) $e_{221}, e_{222}, e_{121}, e_{122}, e_{211}, e_{021}, e_{022}$,

ii) $e_{111} - \chi(e_{101})e_{212}$,

iii) $e_{011} - \chi(e_{101})e_{112} - \chi(e_{001})e_{212}$.

If $\chi(h) = 0$ take $e_{201} + e_{112} \in \mathfrak{c}_W(\chi)$ (it is easy to check that 1)–4) above are satisfied for that element). In particular, $\mathfrak{c}_W(\chi)$ is unipotent.

If $\chi(h) \neq 0 = \chi(e_{001})$ take $h \in \mathfrak{c}_W(\chi)$ (it is easy to check that 1)–4) above are satisfied for that element). In particular, $\mathfrak{c}_W(\chi)$ has rank one since $\mathrm{rk}\ \mathfrak{c}_W(\chi) = 2$ is impossible by Remark 13.4.1.

If $\chi(h) \neq 0 \neq \chi(e_{001})$ take y as in (13.4) with $a_{112} = 1 = a_{201}$ and $a_{012} = \chi(h)\chi(e_{001})^{-1}$ and $a_{101} = -a_{012}$. Let $a_{\alpha\beta\gamma} = 0$ otherwise. Now it is straightforward to check that y defined that way satisfies 1)–4). By Lemma B.1.1 it follows that Ky is a torus and so $\mathfrak{c}_W(\chi)$ has rank one by Remark 13.4.1.

Finally, assume that $\chi(e_{001}) = \chi(h) = \chi(e_{011}) = 0$. Then $e_{111}, e_{011} \in \mathfrak{c}_W(\chi)$. Consider now $y \in \mathfrak{c}_W(\chi)$ written as in (13.4). Since $\mathfrak{c}_W(\chi) \subset \mathfrak{st}(\chi, W_{\geq 2}) = Ke_{001} \oplus W_{\geq 0}$ we have $a_{002} = 0$. The remarks just made show that we can assume that $a_{011} = a_{111} = 0$. We will find conditions on y from $\chi([y, W]) = 0$ by looking at $\chi([y, e_{\alpha\beta\gamma}]) = 0$ for all $(\alpha\beta\gamma)$. We get $a_{102} = a_{202} = a_{212} = 0$ and $a_{012} + a_{101} = 0$ and $a_{201} - a_{112} = 0$. Thus it follows that $e_{201} + e_{112} \in \mathfrak{c}_W(\chi)$ and that $h \in \mathfrak{c}_W(\chi)$, where $h = e_{012} - e_{101}$. Moreover, $e_{001} \in \mathfrak{c}_W(\chi)$. It follows that $\dim_K \mathfrak{c}_W(\chi) = 12$ if $\chi(e_{011}) = \chi(h) = \chi(e_{001}) = 0$. Since $h \in \mathfrak{c}_W(\chi)$ we have $\mathrm{rk}\ \mathfrak{c}_W(\chi) = 1$ by Remark 13.4.1. $\square$

First, we shall describe the irreducible $U_\chi(\mathfrak{g})$–modules in the situation where we have $\chi(e_{011}) \neq 0$. Let K_χ be the one dimensional $\mathfrak{p}$–module where each $x \in \mathfrak{p}$ acts as multiplication by $\chi(x)$. Actually, K_χ is a $U_\chi(\mathfrak{p})$–module since $\chi(x^{[3]}) = 0$ for all $x \in \mathfrak{p}$. Moreover, K_χ is the unique $U_\chi(\mathfrak{p})$–module since $\mathfrak{p}$ is unipotent. Set $S := U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{p})} K_\chi$ and note that S is irreducible with a basis given by $e_{102}^k e_{012}^l e_{101}^m \otimes 1$ where $0 \leq k, l, m < 3$ (the PBW theorem). Define $z_{klm} := e_{102}^k e_{012}^l e_{101}^m \otimes 1$ for $0 \leq k, l, m < 3$.

Let $M := U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S$ and let $w_0 = 1 \otimes z_{000} \in \mathrm{Soc}_\mathfrak{p} M$. Note that $w_0 \in \mathrm{Soc}_\mathfrak{p} 1 \otimes S$; thus it follows from Lemma 11.3.1 that $\mathrm{Soc}_\mathfrak{p} 1 \otimes S = K w_0$.

Proposition 13.4.3. *If $\chi(e_{011}) \neq 0$ then $\mathrm{Soc}_\mathfrak{p} M = K w_0$. In particular, there exist 1 isomorphism class of irreducible $U_\chi(W)$–modules of dimension 3^5 represented by M.*

Proof. We shall prove that $\mathrm{Soc}_\mathfrak{p} M = K w_0$; so suppose otherwise that $\mathrm{Soc}_\mathfrak{p} M \neq K w_0$. Let $w \in M$ such that Kw is an irreducible $\mathfrak{p}$–submodule of $\mathrm{Soc}_\mathfrak{p} M$; by Lemma 11.4.1 we have

$$w \in e_{001}^b \otimes z_{000} + e_{001}^{b-1} \otimes u + \sum_{k<b-1} \sum_{k+m \leq b-1} e_{001}^k e_{002}^m \otimes S$$

for some $b > 0$ and some $u \in S$. The assumption on w says that $x \cdot w - \chi(x) w = 0$ for all $x \in W_{011}$. For $x = e_{111}, e_{202}$ we have:

$$\chi(x) w \in \chi(x) e_{001}^b \otimes z_{000} + e_{001}^{b-1} \otimes (b[x, e_{001}] \cdot z_{000} + x \cdot u) + \sum_{k<b-1} \sum_{k+m \leq b-1} e_{001}^k e_{002}^m \otimes S \quad (*)$$

and therefore ($x = e_{111}$) in particular $b[e_{111}, e_{001}] \cdot z_{000} + e_{111} \cdot u = 0$ by the PBW theorem. It follows that $e_{111} \cdot u = b\chi(e_{011}) z_{000}$. Write $u = \sum_{klm} a_{klm} z_{klm}$. Note that $e_{111} \cdot z_{1lm} = 0 = e_{111} \cdot z_{01m}$ since $\chi(e_{112}) = \chi(e_{111}) = \chi(e_{201}) = 0$ (use the basis for S). Now the relation $e_{111} \cdot u = b\chi(e_{011}) z_{000}$ implies that

$$\sum_{lm} a_{2lm} e_{202} \cdot z_{0lm} - b\chi(e_{011}) z_{000} = 0. \quad (**)$$

It follows that $a_{2lm} = 0$ if $l > 0$ or $m > 0$ and $a_{200} = b\chi(e_{011}) \neq 0$; so we can write

$$u = a z_{200} + \sum_{lm} a_{0lm} z_{0lm} \quad \text{for some } a_{0lm} \in K \text{ and } a \in K^*. \quad (***)$$

Finally, use $(*)$ with $x = e_{202}$ and get $e_{202} \cdot u = u - b z_{100}$; next apply $(***)$ to get $e_{202} \cdot u \in a z_{200} + \sum_{lm} K z_{0lm}$; we have a contradiction and so we cannot have $\mathrm{Soc}_\mathfrak{p} M \neq K w_0$.

Therefore M is the only irreducible $U_\chi(W)$–module (up to isomorphism): For irreducibility note that any nonzero W–submodule of M contains w_0 and then $U_\chi(W) \cdot w_0 = M$. For uniqueness, let X be an irreducible $U_\chi(W)$–module. It contains a copy of S [note that S is the only irreducible $U_\chi(W_{\geq 0})$–module (up to isomorphism)]. Now use 'Frobenius reciprocity' to produce a nonzero $U_\chi(W)$–homomorphism $M \longrightarrow X$. Both modules are irreducible; hence $M \simeq X$. ∎

From now assume that $\chi(e_{011}) = 0$. Define $\mathfrak{L}$ as

$$\mathfrak{L} := K(-e_{001}) \oplus K(2e_{101} + e_{012}) \oplus K(e_{201} + e_{112}). \tag{13.5}$$

We have $[2e_{101} + e_{012}, -e_{001}] = -2(-e_{001})$ and $[2e_{101} + e_{012}, e_{201} + e_{112}] = 2(e_{201} + e_{112})$ and $[e_{201} + e_{112}, -e_{001}] = 2e_{101} + e_{012}$. Moreover,

$$(2e_{101} + e_{012})^{[3]} = 2e_{101} + e_{012}, \tag{13.6}$$

$$e_{001}^{[3]} = 0, \tag{13.7}$$

$$(e_{201} + e_{112})^{[3]} = 0. \tag{13.8}$$

Note that (13.6) holds since $[e_{012}, e_{101}] = 0$ and $e_{012}^{[3]} = e_{012}$ and $e_{101}^{[3]} = e_{101}$ and (13.7) follows from the properties of the $[p]$–mapping on W. Finally, (13.8) holds because $e_{112}^{[3]} = 0 = e_{201}^{[3]}$ and $\big[e_{201}, [e_{201}, e_{112}]\big] = 0 = \big[e_{112}, [e_{112}, e_{201}]\big]$ (see (B.2) in Appendix B). Therefore $\mathfrak{L} \simeq \mathfrak{sl}_2(K)$ as restricted Lie algebras.

Let $\mathfrak{a}$ be as in (13.3). By Lemma 11.7.1 we know that $\mathfrak{a} \triangleleft \mathfrak{g}$ is a p–ideal and the definition of $\mathfrak{L}$ above implies that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{L}$. Therefore $\mathfrak{g}/\mathfrak{a} \simeq \mathfrak{L}$ as restricted Lie algebras. It is well known that irreducible $U_\chi(\mathfrak{g})$–modules annihilated by $\mathfrak{a}$ are in one to one correspondence with irreducible $U_\chi(\mathfrak{g}/\mathfrak{a}) \simeq U_\chi(\mathfrak{L})$–modules. [Any irreducible $U_\chi(\mathfrak{L})$–module X extends to $\mathfrak{g}$ if we define $\mathfrak{a} \cdot X = 0$. On the other hand: Any irreducible $U_\chi(\mathfrak{g})$–module is an irreducible $U_\chi(\mathfrak{L})$–module. Therefore, we can think of irreducible $U_\chi(\mathfrak{g})$–modules annihilated by $\mathfrak{a}$ as irreducible $U_\chi(\mathfrak{L})$–modules extended to $\mathfrak{g}$ with trivial $\mathfrak{a}$–action.] Since $\mathfrak{L} \simeq \mathfrak{sl}_2(K)$ the irreducible $U_\chi(\mathfrak{L})$–modules are classified. We now prove (the first claim being obvious):

Lemma 13.4.4. *Suppose that $\chi(e_{011}) = 0$ or equivalently: $\chi(\mathfrak{a}) = 0$ for $\mathfrak{a}$ as in (13.3). Then irreducible $U_\chi(\mathfrak{g})$–modules annihilated by $\mathfrak{a}$ are in one to one correspondence with irreducible $U_\chi(\mathfrak{L})$–modules. The number of isomorphism classes and dimension formulas are given as follows:*

a) If $\chi(e_{001}) = 0 = \chi(e_{101})$ then there exist up to isomorphism 3 irreducible $U_\chi(\mathfrak{g})$–modules annihilated by $\mathfrak{a}$ of dimension $1, 2, 3$.

b) If $\chi(e_{001}) \neq 0 = \chi(e_{101})$ then there exist up to isomorphism 2 irreducible $U_\chi(\mathfrak{g})$–modules annihilated by $\mathfrak{a}$ all of dimension 3.

c) If $\chi(e_{101}) \neq 0$ then there exist up to isomorphism 3 irreducible $U_\chi(\mathfrak{g})$–modules annihilated by $\mathfrak{a}$ all of dimension 3.

Proof. In [27, 5, 5.2] the representation theory of $\mathfrak{sl}_2(K)$ is described. If we apply the description in [27] on $\mathfrak{L}$ we see that there are 3 isomorphism classes of irreducible $U_\chi(\mathfrak{L})$–modules if $\chi(\mathfrak{L}) = 0$ or $\chi(2e_{101} + e_{012})^2 - \chi(e_{001})\chi(e_{201} + e_{112}) \neq 0$ and 2 isomorphism classes of irreducible $U_\chi(\mathfrak{L})$–modules if $\chi(2e_{101} + e_{012})^2 - \chi(e_{001})\chi(e_{201} + e_{112}) = 0$.

Since $\chi(e_{201} + e_{112}) = 0 = \chi(e_{012})$ we have three situations: If $\chi(e_{001}) = 0 = \chi(e_{101})$ then there exist up to isomorphism 3 irreducible $U_\chi(\mathfrak{L})$–modules of dimension $1, 2, 3$ (in this case $\chi(\mathfrak{L}) = 0$). If $\chi(e_{001}) \neq 0 = \chi(e_{101})$ then there exist up to isomorphism 2 irreducible $U_\chi(\mathfrak{g})$–modules all of dimension 3. Finally, if $\chi(e_{101}) \neq 0$ then there exist up to isomorphism 3 irreducible $U_\chi(\mathfrak{g})$–modules all of dimension 3. The proof is completed. $\square$

Theorem 13.4.5. *Let $\chi \in W^*$ be a character of height 2 and* Type B *and let $\mathfrak{a}$ be as in* (13.3).

a) If $\chi(\mathfrak{a}) \neq 0$, then there exist up to isomorphism 1 irreducible $U_\chi(W)$–module of dimension $3^5 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$. Moreover, $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$ with $\operatorname{rk} \mathfrak{c}_W(\chi) = 0$.

b) If $\chi(\mathfrak{a}) = \chi(e_{101}) = \chi(e_{001}) = 0$ then there exist up to isomorphism 3 irreducible $U_\chi(W)$–modules of dimension $3^3 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$ and $2 \cdot 3^3 = 2 \cdot 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$ and $3^4 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2+1}$. Moreover, $\mathfrak{c}_W(\chi) \not\subset W_{\geq 0}$ with $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$.

c) If $\chi(\mathfrak{a}) = \chi(e_{101}) = 0 \neq \chi(e_{001})$ then there exist up to isomorphism 2 irreducible $U_\chi(W)$–modules all of dimension $3^4 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$. Moreover, $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$ with $\operatorname{rk} \mathfrak{c}_W(\chi) = 0$.

d) If $\chi(e_{101}) \neq 0$ then there exist up to isomorphism 3 irreducible $U_\chi(W)$–modules all of dimension $3^4 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$. Moreover, $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$ with $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$.

Proof. Use Lemma 13.4.2 and Proposition 13.4.3 for part a) and Theorem 11.9.6, Lemma 13.4.2 and Lemma 13.4.4 for b),c),d). ☐

Remark 13.4.6. If $\chi(e_{101}) \neq 0$ or $\chi(\mathfrak{a}) \neq 0$ then one can show that irreducible $U_\chi(W)$–modules are induced from irreducible $U_\chi(W_{\geq 0})$–modules. This is not the situation if $\chi(e_{101}) = \chi(\mathfrak{a}) = 0$: There exist (up to isomorphism) 3 irreducible $U_\chi(W_{\geq 0})$–modules S_0, S_1, S_2 all of dimension 3^2. Theorem 13.3.2 says that induction from $W_{\geq 0}$ to W does not always take irreducible $U_\chi(W_{\geq 0})$–modules to irreducible $U_\chi(W)$–modules. In fact, one can prove that there exist nonzero $U_\chi(W)$–homomorphisms

$$U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S_1 \ {}^{\psi}\!\rightleftarrows_{\varphi} U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S_2$$

such that $\varphi \circ \psi = \chi(e_{001})^3 \cdot \mathrm{Id}_1$ and $\psi \circ \varphi = \chi(e_{001})^3 \cdot \mathrm{Id}_2$ (Id_k is the identity map on the W–module induced by S_k for $k = 1, 2$). If $\chi(e_{001}) = 0$ then $\mathrm{Ker}(\psi)$ is a proper nonzero W–submodule of $U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S_1$ and $\mathrm{Ker}(\varphi)$ is a proper nonzero W–submodule of $U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S_2$. Moreover, $U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S_0$ is irreducible.

13.5 Original χ

Consider an arbitrary character $\chi \in W^*$ of height 2 with $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$. So there exists a nonzero element $\pi \in W_{-1}$ such that $\pi \in \mathfrak{st}(\chi, W_{\geq 2})$. Suppose that $\pi = ae_{001} + be_{002}$ for some $a, b \in K$. We cannot have $W_{-1} \subset \mathfrak{st}(\chi, W_{\geq 2})$ since $[W_{-1}, W_2] = W_1$ and $\chi(W_1) \neq 0$ by assumption. We conclude that

$$\mathfrak{st}(\chi, W_{\geq 2}) = K\pi \oplus W_{\geq 0}.$$

We will classify the set of irreducible $U_\chi(W)$–modules. For any automorphism g on W, the algebras $U_\chi(W)$ and $U_{\chi^g}(W)$ are isomorphic. Thus: If we know the number of isomorphism classes of irreducible $U_{\chi^g}(W)$–modules and the dimension of all representatives, then we know the number of isomorphism classes of irreducible $U_\chi(W)$–modules and the dimension of all representatives.

It follows from Proposition 11.5.2 that $\chi \in W^*$ of height 2 with $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$ is conjugate to a character of Type A or Type B (defined in Section 11.5). Moreover, Lemma 11.6.1 says exactly when χ is conjugate under $\mathrm{Aut}(W)$ to a character of Type A. It turns out that χ is conjugate to a character of Type A if and only if $\chi(V) \neq 0$, where

$$V = K(e_{201} + e_{112}) \oplus K(e_{111} + e_{022}). \tag{13.9}$$

In the next sections we will use the following properties whenever we consider two conjugate characters: Suppose that $\chi \sim \chi'$. Then we have $\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_W(\chi')$ and $\mathrm{rk}\, \mathfrak{c}_W(\chi) = \mathrm{rk}\, \mathfrak{c}_W(\chi')$ and $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$ if and only if $\mathfrak{c}_W(\chi') \subset W_{\geq 0}$. If induction induces a bijection between the isomorphism classes of irreducible $U_{\chi'}(W_{\geq 0})$–modules and the isomorphism classes of irreducible $U_{\chi'}(W)$–modules, then induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and the isomorphism classes of irreducible $U_\chi(W)$–modules.

By Lemma 11.2.1 there exists $g \in GL_2(K)$ such that $\mathfrak{st}(\chi^g, W_{\geq 2}) = Ke_{001} \oplus W_{\geq 0}$ and either $\chi^g(e_{201}) = 1, \chi^g(e_{202}) = 0$ or $\chi^g(e_{202}) = 1, \chi^g(e_{201}) = 0$ and Proposition 11.5.2 says that we can find automorphism g^* such that $\chi^{g \circ g^*}$ has Type A or Type B. For $\chi \sim$ Type A–character we will find equivalent conditions for $\chi^{g\circ g^*}(e_{012}) = 0$ and $\chi^{g\circ g^*}(e_{001}) = 0$. For $\chi \sim$ Type B–character we will find equivalent conditions for $\chi^{g\circ g^*}(e_{011}) = 0$ and $\chi^{g\circ g^*}(h) = 0$ ($h = e_{012} - e_{101}$) and $\chi^{g\circ g^*}(e_{001}) = 0$.

First, we will determine the action of g^* on appropriate basis elements. We will consider the two types of characters separately.

Lemma 13.5.1. *Let $\chi \in W^*$ of height 2 be an exceptional character and let $g \in GL_2(K)$ such that $\mathfrak{st}(\chi^g, W_{\geq 2}) = Ke_{001} \oplus W_{\geq 0}$ and $\chi^g(e_{201}) = 1, \chi^g(e_{202}) = 0$. Then there exists g^* such that $\chi^{g \circ g^*}$ has* Type A *and we have the following properties:*

$$\begin{aligned} g^*(e_{012}) &\in e_{012} + [e_{012}, W_{\geq 1}] + W_{\geq 2}, \\ g^*(e_{001}) &\in e_{001} - \chi(e_{101})e_{101} + \chi(e_{011})\chi(e_{102})e_{201} - \chi(e_{102})e_{011} - \chi(e_{011})e_{102} + \mathrm{Ker}(\chi). \end{aligned}$$

Proof. If we look at the proof of Proposition 11.5.2 it follows that $g^* = g_{-1} \circ g_0$ where g_{-1}, g_0 are automorphisms on W induced by K–algebra automorphisms φ_{-1}, φ_0 on $A(2) = K[X_1, X_2]/(X_1^p, X_2^p)$ given by (x_i is the image of X_i in $A(2)$):

$$\begin{aligned} \varphi_{-1}(x_1) &= x_1 + \chi(e_{002})x_1^2x_2, \\ \varphi_{-1}(x_2) &= x_2, \\ \\ \varphi_0(x_1) &= x_1 + \chi(e_{102})x_1x_2 + \chi(e_{101})x_1^2, \\ \varphi_0(x_2) &= x_2 - \chi(e_{011})x_1^2 + \chi(e_{101})x_1^2. \end{aligned}$$

For the explicit formulas one has to go through step 1) in the proof of Proposition 11.5.2. The inverses satisfy:

$$\begin{aligned} \varphi_{-1}^{-1}(x_1) &\in x_1 - \chi(e_{002})x_1^2x_2 + Kx_1^2x_2^2, \\ \varphi_{-1}^{-1}(x_2) &\in x_2 + Kx_1^2x_2^2, \\ \\ \varphi_0^{-1}(x_1) &\equiv x_1 - \chi(e_{102})x_1x_2 - \chi(e_{101})x_1^2 + \textstyle\sum_{i+j\geq 3} Kx_1^ix_2^j, \\ \varphi_0^{-1}(x_2) &\equiv x_2 + \chi(e_{011})x_1^2 - \chi(e_{101})x_1^2 + \textstyle\sum_{i+j\geq 3} Kx_1^ix_2^j. \end{aligned}$$

The formula for $g^*(e_{012}) = (g_{-1} \circ g_0)(e_{012})$ follows from (3.2): We get

$$\begin{aligned} (g_{-1} \circ g_0)(e_{012}) &\in g_{-1}(e_{012} + [e_{012}, W_1] + W_{\geq 2}) \\ &\in e_{012} + [e_{012}, W_{\geq 1}] + W_{\geq 2} \end{aligned}$$

as required.

Finally, we shall use Proposition 2.2.3 to get a formula for $(g_{-1} \circ g_0)(e_{001})$. First, we obtain:

$$\begin{aligned} g_0(e_{001}) \in\ & e_{001} - \chi(e_{101})e_{101} + \chi(e_{011})\chi(e_{102})e_{201} - \chi(e_{102})e_{011} \\ & -\chi(e_{011})e_{102} + \textstyle\sum_{k=1,2}\sum_{i+j\geq 2}\sum_{i<2} Ke_{ijk}. \end{aligned}$$

Now use the action of g_{-1} (see (3.2)) and get (note that all e_{ijk} with $i + j \geq 2$ but $i < 2$ are contained in $\mathrm{Ker}(\chi)$):

$$g^*(e_{001}) \in e_{001} - \chi(e_{101})e_{101} + \chi(e_{011})\chi(e_{102})e_{201} - \chi(e_{102})e_{011} - \chi(e_{011})e_{102} + \mathrm{Ker}(\chi).$$

The proof is completed. $\square$

Lemma 13.5.2. *Let $\chi \in W^*$ of height 2 be an exceptional character and let $g \in GL_2(K)$ such that $\mathfrak{st}(\chi^g, W_{\geq 2}) = Ke_{001} \oplus W_{\geq 0}$ and $\chi^g(e_{202}) = 1, \chi^g(e_{201}) = 0$. Set $h := e_{012} - e_{101}$. Then there exists g^* such that $\chi^{g \circ g^*}$ has* Type B *and we have the following properties:*

$$\begin{array}{rcl} g^*(e_{011}) & \in & e_{011} + [e_{011}, W_{\geq 1}] + W_{\geq 2}, \\ g^*(h) & \in & h + [h, W_{\geq 1}], \\ g^*(e_{001}) & \in & e_{001} + \chi(e_{012})e_{102} + \chi(e_{102})\chi(e_{012})e_{202} + \mathrm{Ker}(\chi). \end{array}$$

Proof. If we look at the proof of Proposition 11.5.2 it follows that $g^* = g_{-1} \circ g_0$ where g_{-1}, g_0 are automorphisms on W induced by K–algebra automorphisms φ_{-1}, φ_0 on $A(2) = K[X_1, X_2]/(X_1^p, X_2^p)$ given by (x_i is the image of X_i in $A(2)$):

$$\begin{array}{rcl} \varphi_{-1}(x_1) & = & x_1 + \chi(e_{002})x_1^2 x_2, \\ \varphi_{-1}(x_2) & = & x_2, \\ & & \\ \varphi_0(x_1) & = & x_1, \\ \varphi_0(x_2) & = & x_2 + \chi(e_{102})x_1 x_2 - \chi(e_{012})x_1^2. \end{array}$$

For the explicit formulas one has to go through step 2) in the proof of Proposition 11.5.2. The inverses satisfy:

$$\begin{array}{rcl} \varphi_{-1}^{-1}(x_1) & \in & x_1 - \chi(e_{002})x_1^2 x_2 + Kx_1^2 x_2^2, \\ \varphi_{-1}^{-1}(x_2) & \in & x_2 + Kx_1^2 x_2^2, \\ & & \\ \varphi_0^{-1}(x_1) & \equiv & x_1 + \sum_{i+j \geq 3} Kx_1^i x_2^j, \\ \varphi_0^{-1}(x_2) & \equiv & x_2 - \chi(e_{102})x_1 x_2 + \chi(e_{012})x_1^2 + \sum_{i+j \geq 3} Kx_1^i x_2^j. \end{array}$$

The formula for $g^*(e_{011}) = (g_{-1} \circ g_0)(e_{011})$ and $g^*(h) = (g_{-1} \circ g_0)(h)$ follows from (3.2): We get for all $y \in W_0$ (in particular; $y = e_{011}$ and $y = h$)

$$\begin{array}{rcl} (g_{-1} \circ g_0)(y) & \in & g_{-1}(y + [y, W_1] + W_{\geq 2}) \\ & \in & y + [y, W_{\geq 1}] + W_{\geq 2} \end{array}$$

as required.

Finally, we shall use Proposition 2.2.3 to get a formula for $(g_{-1} \circ g_0)(e_{001})$. First, we obtain:

$$g_0(e_{001}) \quad \in \quad e_{001} + \chi(e_{012})e_{102} + \chi(e_{102})\chi(e_{012})e_{202} + \sum_{k=1,2} \sum_{i+j \geq 2} \sum_{i<2} Ke_{ijk}.$$

Now use the action of g_{-1} (see (3.2)) and get (note that all e_{ijk} with $i + j \geq 2$ but $i < 2$ are contained in $\mathrm{Ker}(\chi)$):

$$g^*(e_{001}) \in e_{001} + \chi(e_{012})e_{102} + \chi(e_{102})\chi(e_{012})e_{202} + \mathrm{Ker}(\chi).$$

The proof is completed. □

Now we are in position to find equivalent conditions for $\chi^{g \circ g^*}(e_{012}) = 0$ and $\chi^{g \circ g^*}(e_{001}) = 0$ if $\chi \sim$ Type A–character and equivalent conditions for $\chi^{g \circ g^*}(e_{011}) = 0$ and $\chi^{g \circ g^*}(h) = 0$ ($h = e_{012} - e_{101}$) and $\chi^{g \circ g^*}(e_{001}) = 0$ if $\chi \sim$ Type B–character. Here and in the rest of this section, $g \in GL_2(K)$ is an automorphism such that $\mathfrak{st}(\chi^g, W_{\geq 2}) = Ke_{001} \oplus W_{\geq 0}$ and either $\chi^g(e_{201}) = 1, \chi^g(e_{202}) = 0$ [if $\chi^{g \circ g^*}$ has Type A] or $\chi^g(e_{202}) = 1, \chi^g(e_{201}) = 0$ [if $\chi^{g \circ g^*}$ has Type B]. The explicit formula for g can be found in the proof of Lemma 11.2.1.

Proposition 13.5.3. *Let $\chi \in W^*$ of height 2 with $\mathfrak{st}(\chi, W_{\geq 2}) = K\pi \oplus W_{\geq 0}$ (where $\pi = ae_{001} + be_{002}$) such that $\chi(V) \neq 0$ and let $h := e_{012} - e_{101}$. Let g, g^* be automorphisms on W such that $\chi^{g\circ g^*}$ has* Type A. *Then:*

a) $\chi^{g\circ g^*}(e_{012}) = 0$ if and only if

a1) $\chi(e_{022})\chi(ae_{011} - be_{101}) - \chi(e_{021})\chi(ae_{012} - be_{102}) = 0$ and

a2) $\chi(ae_{201} + be_{202})\chi(ae_{012} - be_{102}) - \chi(e_{202})\chi(a^2e_{011} + abh - b^2e_{102}) = 0.$

b) $\chi^{g\circ g^*}(e_{001}) = 0$ if and only if

b1) $\chi(ae_{201} + be_{202})\chi(ae_{001} + be_{002}) - a^2\big(\chi(h)^2 + \chi(e_{102})\chi(e_{011})\big) = 0$ and

b2) $\chi(ae_{021} + be_{022})\chi(ae_{001} + be_{002}) - b^2\big(\chi(h)^2 + \chi(e_{102})\chi(e_{011})\big) = 0.$

Proof. a) First suppose that $a = 0$. Then $\chi(e_{201}) = 0 = \chi(e_{202})$ since $e_{201} = [e_{002}, e_{211}]$ and $e_{202} = [e_{002}, e_{212}]$ and $e_{002} \in \mathfrak{st}(\chi, W_{\geq 2})$. Our statement in a) then says: $\chi^{g\circ g^*}(e_{012}) = 0$ if and only if $\chi(e_{022})\chi(e_{101}) - \chi(e_{021})\chi(e_{102}) = 0$. Let g, g^* be automorphisms on W such that $\chi^{g\circ g^*}$ has Type A. It follows that

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \chi(e_{022})^{-1} & 0 \\ \chi(e_{021})\chi(e_{022})^{-2} & 1 \end{pmatrix}$$

and g^* is the automorphism on W from Proposition 11.5.2. Note that $\chi^g([e_{012}, W_{\geq 1}]) = 0$ since $\chi^g([e_{201}, e_{012}]) = 0$. Therefore, $\chi^{g\circ g^*}(e_{012}) = \chi^g(e_{012})$ by Lemma 13.5.1 and so $\chi^{g\circ g^*}(e_{012}) = 0$ if and only if $\chi^g(e_{012}) = 0$.

Since $g(e_{012}) = e_{101} - \chi(e_{021})\chi(e_{022})^{-1}e_{102}$ we see that $\chi^g(e_{012}) = 0$ if and only if $\chi(e_{022})\chi(e_{101}) - \chi(e_{021})\chi(e_{102}) = 0$ as required.

Next, suppose that $a \neq 0$. Now g is given by

$$g = \begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \chi(e_{201} + a^{-1}be_{202})^{-1} & 0 \\ \chi(e_{202})\chi(e_{201} + a^{-1}be_{202})^{-2} & 1 \end{pmatrix}.$$

Since

$$g(e_{012}) = e_{012} - a^{-1}be_{102} - \chi(e_{202})\chi(e_{201} + a^{-1}be_{202})^{-1}(e_{011} + a^{-1}bh - a^{-2}b^2e_{102})$$

we have $\chi^g(e_{012}) = 0$ iff $\chi(ae_{201}+be_{202})\chi(ae_{012}-be_{102})-\chi(e_{202})\chi(a^2e_{011}+abh-b^2e_{102}) = 0$ or equivalent:

$$\chi(e_{201})\chi(ae_{012} - be_{102}) - \chi(e_{202})\chi(ae_{011} - be_{101}) = 0. \quad (*)$$

If $b = 0$ we have $\chi(e_{022}) = 0 = \chi(e_{021})$ since $e_{021} = [e_{001}, e_{121}]$ and $e_{022} = [e_{001}, e_{122}]$ and $e_{001} \in \mathfrak{st}(\chi, W_{\geq 2})$. Our statement in a) then says: $\chi^{g\circ g^*}(e_{012}) = 0$ if and only if $\chi(e_{201})\chi(e_{012}) - \chi(e_{202})\chi(e_{011}) = 0$ equivalent to $(*)$ for $b = 0$. If $b \neq 0$ then, since $ae_{001} + be_{002} \in \mathfrak{st}(\chi, W_{\geq 2})$, we have

$$\begin{aligned} a\chi(e_{022}) &= b\chi(e_{112}), \\ a\chi(e_{112}) &= b\chi(e_{202}), \\ a\chi(e_{021}) &= b\chi(e_{111}), \\ a\chi(e_{111}) &= b\chi(e_{201}) \end{aligned}$$

and so $(*)$ is equivalent to $\chi(e_{021})\chi(ae_{012} - be_{102}) - \chi(e_{022})\chi(ae_{011} - be_{101}) = 0$. But if $a \neq 0 \neq b$ we have $\chi(e_{021})\chi(ae_{012} - be_{102}) - \chi(e_{022})\chi(ae_{011} - be_{101}) = 0$ if and only if $\chi(ae_{201}+be_{202})\chi(ae_{012}-be_{102})-\chi(e_{202})\chi(a^2e_{011}+abh-b^2e_{102})=0$ because of the relations listed just above. Therefore the conditions a1) and a2) in the proposition are equivalent and equivalent to $(*)$. The proof of a) is completed.

b) Let g, g^* be automorphisms on W such that $\chi^{g\circ g^*}$ has Type A. By Lemma 13.5.1 we have

$$\chi^{g\circ g^*}(e_{001})=\chi^g(e_{001})-\chi^g(e_{101})^2-\chi^g(e_{011})\chi^g(e_{102}). \tag{13.10}$$

First suppose that $a=0$. Then $\chi(e_{201})=0=\chi(e_{202})$ since $e_{201}=[e_{002},e_{211}]$ and $e_{202}=[e_{002},e_{212}]$ and $e_{002}\in\mathfrak{st}(\chi,W_{\geq 2})$. Our statement in b) then says: $\chi^{g\circ g^*}(e_{001})=0$ if and only if $\chi(e_{022})\chi(e_{002})-\chi(h)^2-\chi(e_{102})\chi(e_{011})=0$ (note that $h=e_{012}-e_{101}$). It follows that

$$g=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\chi(e_{022})^{-1}&0\\\chi(e_{021})\chi(e_{022})^{-2}&1\end{pmatrix}.$$

If we apply (13.10) with the relations in Appendix A we get $\chi^{g\circ g^*}(e_{001})=\chi(e_{022})\chi(e_{002})-\chi(h)^2-\chi(e_{102})\chi(e_{011})$ as required.

Next, suppose that $a\neq 0$. Then

$$g=\begin{pmatrix}1&-a^{-1}b\\0&1\end{pmatrix}\begin{pmatrix}\chi(e_{201}+a^{-1}be_{202})^{-1}&0\\\chi(e_{202})\chi(e_{201}+a^{-1}be_{202})^{-2}&1\end{pmatrix}$$

and if we apply (13.10) with the formulas in Appendix A we find $\chi^{g\circ g^*}(e_{001})=0$ if and only if

$$\chi(ae_{201}+be_{202})\chi(ae_{001}+be_{002})-a^2\big(\chi(h)^2+\chi(e_{102})\chi(e_{011})\big)=0. \quad (*)$$

If $b=0$ we have $\chi(e_{022})=0=\chi(e_{021})$ since $e_{021}=[e_{001},e_{121}]$ and $e_{022}=[e_{001},e_{122}]$ and $e_{001}\in\mathfrak{st}(\chi,W_{\geq 2})$. Our statement in b) then says: $\chi^{g\circ g^*}(e_{001})=0$ if and only if $\chi(e_{201})\chi(e_{001})-\chi(h)^2-\chi(e_{102})\chi(e_{011})=0$ equivalent to $(*)$ for $b=0$.

If $b\neq 0$ then, since $ae_{001}+be_{002}\in\mathfrak{st}(\chi,W_{\geq 2})$, we have

$$\begin{aligned}a\chi(e_{022})&=b\chi(e_{112}),\\a\chi(e_{112})&=b\chi(e_{202}),\\a\chi(e_{021})&=b\chi(e_{111}),\\a\chi(e_{111})&=b\chi(e_{201})\end{aligned}$$

and therefore we get $\chi(ae_{201}+be_{202})\chi(ae_{001}+be_{002})-a^2\big(\chi(h)^2+\chi(e_{102})\chi(e_{011})\big)=0$ if and only if $\chi(ae_{021}+be_{022})\chi(ae_{001}+be_{002})-b^2\big(\chi(h)^2+\chi(e_{102})\chi(e_{011})\big)=0$. That is: Both conditions in the proposition are equivalent and equivalent to $(*)$. The proof is completed. □

Proposition 13.5.4. *Let $\chi\in W^*$ of height 2 with $\mathfrak{st}(\chi,W_{\geq 2})=K\pi\oplus W_{\geq 0}$ (where $\pi=ae_{001}+be_{002}$) such that $\chi(V)=0$. Let g,g^* be automorphisms on W such that $\chi^{g\circ g^*}$ has* Type B. *Set $h:=e_{012}-e_{101}$. Then:*

a) $\chi^{g\circ g^*}(e_{011})=0$ if and only if $\chi(a^2e_{011}+abh-b^2e_{102})=0$.

b) If $\chi^{g\circ g^*}(e_{011})=0$ then $\chi^{g\circ g^*}(h)=0$ if and only if $\chi(ae_{011}-bh)=0=\chi(ah+be_{102})$.

c) If $\chi^{g\circ g^*}(e_{011})=\chi^{g\circ g^*}(h)=0$ then $\chi^{g\circ g^*}(e_{001})=0$ if and only if

c1) $\chi(e_{021})\chi(ae_{001}+be_{002})-\chi(e_{011})\chi(ae_{011}+be_{012})=0$ and

c2) $\chi(e_{202})\chi(ae_{001}+be_{002})-\chi(e_{102})\chi(ae_{012}+be_{102})=0$.

Proof. a) Let g, g^* be an automorphisms on W such that $\chi^{g\circ g^*}$ has Type B. It follows from Lemma 11.2.1 and its proof that

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \chi(e_{021}) \end{pmatrix} \text{ if } a = 0 \quad \text{and } g = \begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \chi(e_{202}) \end{pmatrix} \text{ if } a \neq 0.$$

Moreover, $\chi^{g\circ g^*}(e_{011}) = \chi^g(e_{011})$, since $g^*(e_{011}) \equiv e_{011} \pmod{\mathrm{Ker}(\chi^g)}$ by Lemma 13.5.2.

If $a = 0$, then $g(e_{011}) = \chi(e_{021})e_{102}$ such that $\chi^g(e_{011}) = 0$ if and only if $\chi(e_{102}) = 0$ if and only if $\chi(a^2e_{011} + abh - b^2e_{102}) = 0$ (since $a = 0$).

If $a \neq 0$, then $g(e_{011}) = \chi(e_{021})(e_{011} + a^{-1}bh - a^{-2}b^2e_{102})$ from the $GL_2(K)$–action in Appendix A. We conclude that $\chi^g(e_{011}) = 0$ if and only if $\chi(a^2e_{011} + abh - b^2e_{102}) = 0$. The proof of a) is completed.

b) Note that $g^*(h) \in h + [h, W_{\geq 1}] + W_{\geq 2}$ by Lemma 13.5.2. Since $[h, e_{202}] = 0$ we have $[h, W_{\geq 1}] \subset W_{112} \subset \mathrm{Ker}(\chi)$. Therefore $\chi^{g\circ g^*}(h) = 0$ if and only if $\chi^g(h) = 0$.

First suppose that $a = 0$. Let g be as above (in the $a = 0$ case). Since $\chi^g(e_{011}) = 0$, by assumption, we have $\chi(e_{102}) = 0$ by a); hence $\chi(ah + be_{102}) = 0$ for $a = 0$. We also have $\chi^g(h) = -\chi(h)$. Therefore $\chi^g(h) = 0$ if and only if $\chi(ae_{011} - bh) = 0$ for $a = 0$.

Next, suppose that $a \neq 0$. Let g be as above (in the $a \neq 0$ case). Now, use the $GL_2(K)$–action in Appendix A and obtain $\chi^g(h) = \chi(h) + a^{-1}b\chi(e_{102})$. Since $\chi^g(e_{011}) = 0$ by assumption we have $\chi(a^2e_{011} + abh - b^2e_{102}) = 0$ by a). Now it is easy to get $\chi^g(h) = 0$ if and only if $\chi(ae_{011} - bh) = 0$ and $\chi(ah + be_{102}) = 0$. This completes the proof of b).

c) First, apply Lemma 13.5.2 and get

$$\chi^{g\circ g^*}(e_{001}) = \chi^g(e_{001}) - \chi^g(e_{012})\chi^g(e_{102}). \quad (*)$$

Again we treat $a = 0$ and $a \neq 0$ separately. First suppose that $a = 0$. Let g be as above (in the $a = 0$ case). Now use $(*)$ to get $\chi^{g\circ g^*}(e_{001}) = 0$ if and only if $\chi(e_{002})\chi(e_{021}) - \chi(e_{011})\chi(e_{101}) = 0$ (use the $GL_2(K)$ action in Appendix A). But $\chi^{g\circ g^*}(h) = -\chi(h) = 0$ implies that $\chi(e_{012}) = \chi(e_{101})$; therefore $\chi^{g\circ g^*}(e_{001}) = 0$ if and only if $\chi(e_{002})\chi(e_{021}) - \chi(e_{011})\chi(e_{012}) = 0$. But $\chi(e_{102}) = 0$ (apply b)) and since $e_{002} \in \mathfrak{st}(\chi, W_{\geq 2})$ we also have $\chi(e_{202}) = 0$ from the relation $e_{202} = [e_{002}, e_{212}]$. It follows that the condition in c2) is always true and moreover, the condition in c1) is just $\chi(e_{002})\chi(e_{021}) - \chi(e_{011})\chi(e_{012}) = 0$ for $a = 0$.

Let $a \neq 0$ and let g be as above (in the $a \neq 0$ case). Now we have $\chi^{g\circ g^*}(e_{001}) = 0$ if and only if $\chi(e_{202})\chi(ae_{001} + be_{002}) - \chi(e_{102})\chi(ae_{012} + be_{102}) = 0$.

If $b = 0$ then $\chi(e_{021}) = 0$ since $e_{001} \in \mathfrak{st}(\chi, W_{\geq 2})$ and $e_{021} = [e_{001}, e_{121}]$. It follows that the condition in c1) is always true and the condition c2) is equivalent to $\chi^{g\circ g^*}(e_{001}) = 0$.

Finally, suppose that $a \neq 0 \neq b$. Then c1) and c2) are equivalent and so equivalent to $\chi(e_{202})\chi(ae_{001} + be_{002}) - \chi(e_{102})\chi(ae_{012} + be_{102}) = 0$ as required. In order to obtain the equivalence of c1) and c2) use that $ae_{001} + be_{002} \in \mathfrak{st}(\chi, W_{\geq 2})$ and find the following relations:

$$\begin{aligned} a\chi(e_{022}) &= b\chi(e_{112}), \\ a\chi(e_{112}) &= b\chi(e_{202}), \\ a\chi(e_{021}) &= b\chi(e_{111}), \\ a\chi(e_{111}) &= b\chi(e_{201}). \end{aligned}$$

Moreover, $\chi^g(e_{201}) = 0$ since χ^g has Type B; hence $a\chi(e_{201}) + b\chi(e_{202}) = 0$. It follows that $\chi(e_{202}) = -a^3b^{-3}\chi(e_{021})$. By assumption we have $\chi^{g\circ g^*}(h) = 0$ and hence, by a), $\chi(ae_{011} - bh) = 0 = \chi(ah + be_{102})$; it follows that $a\chi(e_{011}) = b\chi(h)$ and $-b\chi(e_{102}) = a\chi(h)$. With these relations in mind it is easy to check that c1) is equivalent with c2). □

13.6 Stabilizer of dimension 8

Theorem 13.6.1. *Let $\chi \in W^*$ of height 2. If $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$ and if $\dim_K \mathfrak{c}_W(\chi) = 8$ then any irreducible $U_\chi(W)$–module has dimension $3^5 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$ and induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and the isomorphism classes of irreducible $U_\chi(W)$–modules. The number of isomorphism classes of irreducible $U_\chi(W)$–modules is 1 if $\chi(V) = 0$ and 3 if $\chi(V) \neq 0$ with V as in (13.9). Finally; $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$ and $\operatorname{rk} \mathfrak{c}_W(\chi) = 0$ if $\chi(V) = 0$ and $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$ if $\chi(V) \neq 0$.*

Proof. If $\chi(V) \neq 0$, then χ is conjugate to a character of Type A by Lemma 11.6.1. Now apply Lemma 13.3.1 and Theorem 13.3.2.a: It follows that there exist 3 isomorphism classes of irreducible $U_\chi(W)$–modules; each representative has dimension $3^5 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$ (note that $\dim_K W = 18$ for $p = 3$).

If $\chi(V) = 0$, then χ is conjugate to a character of Type B by Lemma 11.6.1. Now apply Lemma 13.4.2 and Theorem 13.4.5: It follows that there exist 1 isomorphism class of irreducible $U_\chi(W)$–modules; any representative has dimension $3^5 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$.

Moreover, by Remark 13.3.3 (if $\chi(V) \neq 0$) and Remark 13.4.6 (if $\chi(V) = 0$) induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and the isomorphism classes of irreducible $U_\chi(W)$–modules.

The final statement follows from Lemma 13.3.1 and Lemma 13.4.2. □

The next lemmas say exactly when we have $\dim_K \mathfrak{c}_W(\chi) = 8$ for $\chi \in W^*$ of height 2 with $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$. We will discuss $\chi(V) = 0$ and $\chi(V) \neq 0$ separately. Recall that we have defined $a, b \in K$ such that $0 \neq \pi = ae_{001} + be_{002} \in \mathfrak{st}(\chi, W_{\geq 2})$. Set $h := e_{012} - e_{101}$.

Lemma 13.6.2. *Let $\chi \in W^*$ of height 2 with $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$ such that $\chi(V) = 0$. Then $\dim_K \mathfrak{c}_W(\chi) = 8$ if and only if $\chi(a^2 e_{011} + abh - b^2 e_{102}) \neq 0$.*

Proof. Follows immediately from Lemma 13.4.2 and Proposition 13.5.4.a. □

Lemma 13.6.3. *Let $\chi \in W^*$ of height 2 such that $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$ and $\chi(V) \neq 0$. Then $\dim_K \mathfrak{c}_W(\chi) = 8$ if and only if $\chi(e_{022})\chi(ae_{011} - be_{101}) - \chi(e_{021})\chi(ae_{012} - be_{102}) \neq 0$ or $\chi(ae_{201} + be_{202})\chi(ae_{012} - be_{102}) - \chi(e_{202})\chi(a^2 e_{011} + abh - b^2 e_{102}) \neq 0$.*

Proof. Follows immediately from Lemma 13.3.1 and Proposition 13.5.3.a. □

13.7 Stabilizer of dimension 10 and $\chi(V) = 0$

In this section we consider $\chi \in W^*$ of height 2 with $\dim_K \mathfrak{c}_W(\chi) = 10$ such that $\chi(V) = 0$ and $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$. We define $a, b \in K$ such that $0 \neq \pi = ae_{001} + be_{002} \in \mathfrak{st}(\chi, W_{\geq 2})$. Set $h := e_{012} - e_{101}$. Note that χ is conjugate to a character of Type B by Lemma 11.6.1. Since $\dim_K \mathfrak{c}_W(\chi) = 10$ we have, by Lemma 13.4.2, either $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$ or $\operatorname{rk} \mathfrak{c}_W(\chi) = 0$.

Lemma 13.7.1. *Let $\chi \in W^*$ of height 2 such that $\chi(V) = 0$ and $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$ and $\dim_K \mathfrak{c}_W(\chi) = 10$. Then $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$ if and only if $\chi(ae_{011} - bh) \neq 0$ or $\chi(ah + be_{102}) \neq 0$ and $\operatorname{rk} \mathfrak{c}_W(\chi) = 0$ if and only if $\chi(ae_{011} - bh) = 0 = \chi(ah + be_{102})$.*

Proof. Note that $\chi^{g \circ g^*}$ has Type B for some $g, g^* \in \operatorname{Aut}(W)$ by Lemma 11.6.1. The assumption $\dim_K \mathfrak{c}_W(\chi^{g \circ g^*}) = \dim_K \mathfrak{c}_W(\chi) = 10$ implies that $\chi^{g \circ g^*}(e_{011}) = 0$. Moreover, $\operatorname{rk} \mathfrak{c}_W(\chi^{g \circ g^*}) = 0$ if and only if $\chi^{g \circ g^*}(h) = 0$ and $\operatorname{rk} \mathfrak{c}_W(\chi^{g \circ g^*}) = 1$ if and only if $\chi^{g \circ g^*}(h) \neq 0$ by Lemma 13.4.2. Now conclude by Proposition 13.5.4.b. □

Lemma 13.7.2. *Let $\chi \in W^*$ of height 2 such that $\chi(V) = 0$ and $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$. Then we have $\dim_K \mathfrak{c}_W(\chi) = 10$ if and only if $\chi(a^2 e_{011} + abh - b^2 e_{102}) = 0$ and either 1) or 2) are satisfied:*

1) $\chi(ae_{011} - bh) \neq 0$ or $\chi(ah + be_{102}) \neq 0$ or

2) one of the following conditions are satisfied:

$$\begin{aligned} \chi(e_{021})\chi(ae_{001} + be_{002}) - \chi(e_{011})\chi(ae_{011} + be_{012}) &\neq 0 \quad \text{or} \\ \chi(e_{202})\chi(ae_{001} + be_{002}) - \chi(e_{102})\chi(ae_{012} + be_{102}) &\neq 0. \end{aligned}$$

Proof. Follows immediately from Lemma 13.4.2 and Proposition 13.5.4. □

Theorem 13.7.3. *Let $\chi \in W^*$ of height 2 such that $\chi(V) = 0$ and $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$ and $\dim_K \mathfrak{c}_W(\chi) = 10$. Then $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$ and any irreducible $U_\chi(W)$–module has dimension $3^4 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$. The number of isomorphism classes of irreducible $U_\chi(W)$–modules is 3 if $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$ and 2 if $\operatorname{rk} \mathfrak{c}_W(\chi) = 0$.*

Proof. Suppose that $\chi^{g \circ g^*}$ has Type B for automorphism g, g^*. Since $\dim_K \mathfrak{c}_W(\chi) = 10$ we have in particular $\chi^{g \circ g^*}(e_{011}) = 0$. Moreover, it follows from Lemma 13.4.2 that $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$. Our assumption $\dim_K \mathfrak{c}_W(\chi) = 10$ implies, again by Lemma 13.4.2, that $\operatorname{rk} \mathfrak{c}_W(\chi) = \operatorname{rk} \mathfrak{c}_W(\chi^{g \circ g^*}) = 1$ if and only if $\chi^{g \circ g^*}(h) \neq 0$. If $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$ we thus have $\chi^{g \circ g^*}(e_{101}) \neq 0$ since $\chi^{g \circ g^*}(e_{101}) = \chi^{g \circ g^*}(h)$; now apply Theorem 13.4.5.c on $\chi^{g \circ g^*}$. If $\operatorname{rk} \mathfrak{c}_W(\chi) = 0$ then $\chi^{g \circ g^*}(e_{101}) = \chi^{g \circ g^*}(h) = 0 \neq \chi^{g \circ g^*}(e_{001}) \neq 0$; now apply Theorem 13.4.5.b on $\chi^{g \circ g^*}$. The proof is completed. □

13.8 Stabilizer of dimension 10 and $\chi(V) \neq 0$

In this section we consider $\chi \in W^*$ of height 2 and $\dim_K \mathfrak{c}_W(\chi) = 10$ such that $\chi(V) \neq 0$ and $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$. We define $a, b \in K$ such that $0 \neq \pi = ae_{001} + be_{002} \in \mathfrak{st}(\chi, W_{\geq 2})$. Set $h := e_{012} - e_{101}$. First, apply Lemma 13.3.1 and Lemma 13.6.3 and get:

Lemma 13.8.1. *Let $\chi \in W^*$ of height 2 such that $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$ and $\chi(V) \neq 0$. Then $\dim_K \mathfrak{c}_W(\chi) = 10$ if and only if $\chi(e_{022})\chi(ae_{011} - be_{101}) - \chi(e_{021})\chi(ae_{012} - be_{102}) = 0$ and $\chi(ae_{201} + be_{202})\chi(ae_{012} - be_{102}) - \chi(e_{202})\chi(a^2 e_{011} + abh - b^2 e_{102}) = 0$.*

Lemma 13.8.2. *Let $\chi \in W^*$ of height 2 such that $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$ and $\chi(V) \neq 0$. Suppose that $\dim_K \mathfrak{c}_W(\chi) = 10$. Then $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$ or $\operatorname{rk} \mathfrak{c}_W(\chi) = 2$. Moreover, we have $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$ if and only if $\chi(ae_{201} + be_{202})\chi(ae_{001} + be_{002}) - a^2\big(\chi(h)^2 + \chi(e_{102})\chi(e_{011})\big) = 0$ and $\chi(ae_{021} + be_{022})\chi(ae_{001} + be_{002}) - b^2\big(\chi(h)^2 + \chi(e_{102})\chi(e_{011})\big) = 0$.*

Proof. Since $\chi(V) \neq 0$ it follows that χ is conjugate to a character of Type A. Therefore $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$ or $\operatorname{rk} \mathfrak{c}_W(\chi) = 2$ by Lemma 13.3.1. Let g, g^* be automorphisms such that $\chi^{g \circ g^*}$ has Type A. Since $\dim_K \mathfrak{c}_W(\chi) = 10$ it follows from Lemma 13.3.1 that $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$ if and only if $\chi^{g \circ g^*}(e_{001}) = 0$. Now conclude by Proposition 13.5.3.b □

Theorem 13.8.3. *Let $\chi \in W^*$ of height 2 such that $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$ and $\chi(V) \neq 0$. Suppose that $\dim_K \mathfrak{c}_W(\chi) = 10$. If $\operatorname{rk} \mathfrak{c}_W(\chi) = 2$, then there exist 5 isomorphism classes of irreducible $U_\chi(W)$–modules; two representatives have dimension $3^5 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2+1}$ and three representatives have dimension $3^4 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$. If $\operatorname{rk} \mathfrak{c}_W(\chi) = 1$, then there exist 4 isomorphism classes of irreducible $U_\chi(W)$–modules; two representatives have dimension $3^5 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2+1}$ and two representatives have dimension $3^4 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$. Finally; $\mathfrak{c}_W(\chi) \not\subset W_{\geq 0}$.*

Proof. Since $\chi(V) \neq 0$ it follows that $\chi^{g \circ g^*}$ for some automorphisms g, g^*. We also have $\dim_K \mathfrak{c}_W(\chi^{g \circ g^*}) = \dim_K \mathfrak{c}_W(\chi) = 10$ and therefore $\chi^{g \circ g^*}(e_{012}) = 0$ (see Lemma 13.3.1). Now apply Theorem 13.3.2.b,c and note that $\chi^{g \circ g^*}(e_{001}) = 0$ if and only if $\mathrm{rk}\, \mathfrak{c}_W(\chi) = 1$ by Lemma 13.3.1. Finally; $\mathfrak{c}_W(\chi) \not\subset W_{\geq 0}$ by Lemma 13.3.1 again. The proof is completed. $\square$

13.9 Stabilizer of dimension 12

Theorem 13.9.1. *Let $\chi \in W^*$ of height 2. If $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$ and if $\dim_K \mathfrak{c}_W(\chi) = 12$, then $\chi(V) = 0$ with V as in* (13.9) *and $\mathfrak{c}_W(\chi) \not\subset W_{\geq 0}$ with $\mathrm{rk}\ \mathfrak{c}_W(\chi) = 1$. There exist 3 isomorphism classes of irreducible $U_\chi(W)$–modules; one representative has dimension $3^3 = 3^{\mathrm{codim}_W \mathfrak{c}_W(\chi)/2}$, one representative has dimension $2 \cdot 3^3 = 2 \cdot 3^{\mathrm{codim}_W \mathfrak{c}_W(\chi)/2}$ and one representative has dimension $3^4 = 3^{\mathrm{codim}_W \mathfrak{c}_W(\chi)/2+1}$.*

Proof. It follows from Lemma 13.3.1 that χ isn't conjugate to a character of Type A; hence $\chi(V) = 0$ by Lemma 11.6.1. Next, apply Lemma 13.4.2 and Theorem 13.4.5.a and obtain the required result (note that $\dim_K W = 18$ for $p = 3$ such that $\mathrm{codim}_W \mathfrak{c}_W(\chi) = 6$). $\square$

The next lemma says exactly when we have $\dim_K \mathfrak{c}_W(\chi) = 12$ for $\chi \in W^*$ of height 2 with $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$. We have defined $a, b \in K$ such that $0 \neq \pi = ae_{001} + be_{002} \in \mathfrak{st}(\chi, W_{\geq 2})$. Set $h := e_{012} - e_{101}$.

Lemma 13.9.2. *Let $\chi \in W^*$ of height 2 such that $\mathfrak{st}(\chi, W_{\geq 2}) \neq W_{\geq 0}$. Then we have $\dim_K \mathfrak{c}_W(\chi) = 12$ if and only if $\chi(a^2 e_{011} + abh - b^2 e_{102}) = 0$ and $\chi(ae_{011} - bh) = 0 = \chi(ah + be_{102})$ and*

$$\begin{aligned} \chi(e_{021})\chi(ae_{001} + be_{002}) - \chi(e_{011})\chi(ae_{011} + be_{012}) &= 0 \quad \text{and} \\ \chi(e_{202})\chi(ae_{001} + be_{002}) - \chi(e_{102})\chi(ae_{012} + be_{102}) &= 0. \end{aligned}$$

Proof. Follows immediately from Proposition 13.5.4 and Lemma 13.4.2. $\square$

13.10 Exceptional characters of height 3

Let $p = 3$ and let $\chi \in W^*$ be a character of height 3 such that $\mathfrak{st}(\chi, W_{\geq 3}) \neq W_{\geq 0}$. We will study two types of characters introduced in Section 11.5 (and I will use the terminology from the height 2 situation):

Type A : $\tau \in W^*$ of height 3 with $\tau(e_{211}) = 1, \tau(e_{212}) = 0$ and $\tau(e_{002}) = 0$ and $\tau(e_{102}) = \tau(e_{012}) = \tau(e_{011}) = 0$ and $\tau(e_{202}) = \tau(e_{112}) = \tau(e_{111}) = \tau(e_{021}) = 0$ and $\mathfrak{st}(\tau, W_{\geq 3}) = Ke_{001} \oplus W_{\geq 0}$.

Type B : $\tau \in W^*$ of height 3 with $\tau(e_{212}) = 1, \tau(e_{211}) = 0$ and $\tau(e_{002}) = 0$ and $\tau(e_{102}) = \tau(e_{012} + e_{101}) = 0$ and $\tau(e_{202}) = \tau(e_{112}) = \tau(e_{201}) = \tau(e_{111}) = 0$ and $\mathfrak{st}(\tau, W_{\geq 3}) = Ke_{001} \oplus W_{\geq 0}$.

The definition of $\mathfrak{a}$ in (11.11) now reads:

$$\mathfrak{a} = \sum_{k=1}^{2} Ke_{02k} \oplus \sum_{k=1}^{2} Ke_{12k} \oplus W_{\geq 3}. \tag{13.11}$$

Note that $\chi(\mathfrak{a} \cap W_{021}) = 0$ for $\chi \in W^*$ of Type A. Thus we can apply Theorem 11.8.5 for χ of Type A.

13.11 Type A characters of height 3

Consider $\chi \in W^*$ of height 3 and Type A. We shall classify the irreducible $U_\chi(\mathfrak{g})$–modules (for $\mathfrak{g}$ as in (11.8) with $p = 3$) and then use Proposition 11.8.5 to get information on the irreducible $U_\chi(W)$–modules.

Proposition 13.11.1. *If $\chi(e_{022}) \neq 0$ then there exists up to isomorphism 1 irreducible $U_\chi(\mathfrak{g})$–module of dimension 3^5.*

Proof. Note that $\mathfrak{h} = \mathfrak{g} \cap W_{\geq 0}$ is supersolvable and the Vergne polarization $\mathfrak{p}$ of χ constructed with respect to the chain (11.10) is given by

$$\mathfrak{p} = K(e_{112} + e_{201}) \oplus Ke_{022} \oplus Ke_{111} \oplus Ke_{021} \oplus W_{\geq 2}.$$

Indeed: By Remark 9.4.2 we have $W_{\geq 2} \subset \mathfrak{p}_1$ and $e_{021} \in \mathfrak{s}_{021}^\chi$ and $e_{111} \in \mathfrak{s}_{111}^\chi$ follows immediately. Since $\chi([e_{201}, e_{111}]) \neq 0 = \chi([e_{201}, W_{021}])$ we have $\mathfrak{s}_{201}^\chi \subset \mathfrak{s}_{111}^\chi$. Moreover, $e_{022} \in \mathfrak{s}_{022}^\chi$ and $e_{112} + e_{201} \in \mathfrak{s}_{112}^\chi$ since $\chi([e_{022}, W_{022}]) = 0$ and $\chi([e_{112} + e_{201}, W_{112}]) = 0$. Next, $\mathfrak{s}_{011}^\chi \subset W_{\geq 1}$ since $\chi([e_{011}, e_{212}]) \neq 0 = \chi([W_{\geq 1}, e_{212}])$. We also have $\mathfrak{s}_{101}^\chi \subset \mathfrak{s}_{011}^\chi$ by observing that $\chi([e_{101}, e_{211}]) \neq 0 = \chi([W_{011}, e_{211}])$. Finally, $\mathfrak{s}_{012}^\chi \subset \mathfrak{s}_{101}^\chi$; otherwise there exists $z \in W_{101}$ such that $e_{012} + z \in \mathfrak{s}_{012}^\chi$ and hence $\chi([e_{012} + z, e_{022}]) = 0$. But this implies that $\chi([z, e_{022}]) = -\chi(e_{022}) \neq 0$ since $\chi(e_{022}) \neq 0$ – contradiction since $[e_{101}, e_{022}] = 0$ and $\chi([W_{011}, e_{022}]) = 0$.

Let $\lambda \in \mathfrak{h}^*$ with $\lambda = \chi_{|\mathfrak{h}}$: Then the Vergne polarization of λ is equal to $\mathfrak{p}$ and compatible with χ (i.e., $\lambda(x)^p - \lambda(x^{[p]}) = \chi(x)^p$ for all $x \in \mathfrak{p}_1$). Therefore, by Proposition 9.3.5 and Lemma 9.3.7, there exists up to isomorphism 1 irreducible $U_\chi(\mathfrak{h})$–module of dimension 3^4. Now apply Proposition 11.7.2 and the fact that $\chi(\mathfrak{a}) \neq 0$ if and only if $\chi(e_{022}) \neq 0$. □

The idea now is to describe the irreducible $U_\chi(\mathfrak{g})$–modules when $\chi(e_{022}) = 0$. We define (as in the height 2 situation) $\mathfrak{L} = K(-e_{001}) \oplus K(2e_{101} + e_{012}) \oplus K(e_{201} + e_{112})$. It is a restricted Lie algebra isomorphic to $\mathfrak{sl}_2(K)$. Set $\mathfrak{b} := Ke_{012} \oplus Ke_{011} \oplus Ke_{111}$ and define

$$\mathfrak{s} := \mathfrak{L} \oplus \mathfrak{a} \oplus \mathfrak{b}. \tag{13.12}$$

It is easy to verify that $\mathfrak{s}$ is a restricted Lie subalgebra of $\mathfrak{g}$ and that $\mathfrak{a} \oplus \mathfrak{b}$ is a p–ideal in $\mathfrak{s}$ (apply commutator relations). Moreover, $\chi(\mathfrak{a} \oplus \mathfrak{b}) = 0$ and $\mathfrak{s}/(\mathfrak{a} \oplus \mathfrak{b}) \simeq \mathfrak{L} \simeq \mathfrak{sl}_2(K)$ as restricted Lie algebras.

We shall prove that induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{s})$–modules annihilated by $\mathfrak{a} \oplus \mathfrak{b}$ and the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules. In order to prove this we need a lemma.

Lemma 13.11.2. *Suppose that $\chi(\mathfrak{a} \oplus \mathfrak{b}) = 0$. If M is a $U_\chi(\mathfrak{g})$–module and $M \neq 0$, then $\{x \in M \mid (\mathfrak{a} \oplus \mathfrak{b}) \cdot x = 0\} \neq 0$ and there exists an irreducible $U_\chi(\mathfrak{s})$–submodule $X \subset M$ with $(\mathfrak{a} \oplus \mathfrak{b}) \cdot X = 0$.*

Proof. Since $[\mathfrak{a} \oplus \mathfrak{b}, \mathfrak{a} \oplus \mathfrak{b}] \subset (\mathfrak{a} \oplus \mathfrak{b}) \cap W_{011}$ there exists a $U_\chi(\mathfrak{a} \oplus \mathfrak{b})$–module K_l as being equal to K as a vector space and where the module structure is given by: $e \cdot 1 = 0$ for $e \in (\mathfrak{a} \oplus \mathfrak{b}) \cap W_{011}$ and $e_{012} \cdot 1 = l$ (since $e_{012} \in \mathfrak{a}$ with $\chi(e_{012}) = 0$ we have $l \in \mathbb{F}_3$). But $\mathfrak{a} \oplus \mathfrak{b} \subset W_{012}$ is supersolvable so we can apply Lemma 9.1.3: It follows that any irreducible $U_\chi(\mathfrak{a} \oplus \mathfrak{b})$–module is isomorphic to some K_l with $l \in \mathbb{F}_3$. So there exists a nonzero $x \in M$ with $(\mathfrak{a} \oplus \mathfrak{b}) \cap W_{011} \cdot x = 0$ and $e_{012} \cdot x = lx$ for some $l \in \mathbb{F}_3$. If $l = 1$, set $y := e_{211}^2 \cdot x \in M$ and if $l = 2$, let $y := e_{211}^1 \cdot x \in M$. Then $(\mathfrak{a} \oplus \mathfrak{b}) \cap W_{011} \cdot y = 0$ since $[e_{211}, (\mathfrak{a} \oplus \mathfrak{b}) \cap W_{011}] \subset (\mathfrak{a} \oplus \mathfrak{b}) \cap W_{011}$ and $e_{012} \cdot y = 0$ by construction. We conclude that $\{x \in M \mid (\mathfrak{a} \oplus \mathfrak{b}) \cdot x = 0\} \neq 0$.

The final part of the lemma is now easy: Take a nonzero $x \in M$ with $(\mathfrak{a} \oplus \mathfrak{b}) \cdot x = 0$ and take an irreducible $U_\chi(\mathfrak{s})$–submodule X of $U_\chi(\mathfrak{s}) \cdot x$. Since $\mathfrak{a} \oplus \mathfrak{b}$ is an ideal of $\mathfrak{s}$ with $(\mathfrak{a} \oplus \mathfrak{b}) \cdot x = 0$ we have $(\mathfrak{a} \oplus \mathfrak{b}) \cdot U_\chi(\mathfrak{s}) \cdot x = 0$ and therefore $(\mathfrak{a} \oplus \mathfrak{b}) \cdot X = 0$ as required. $\square$

Proposition 13.11.3. *Suppose that $\chi(e_{022}) = 0$. Let $\mathfrak{s}$ be defined as in* (13.12). *Then induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{s})$–modules annihilated by $\mathfrak{a} \oplus \mathfrak{b}$ and the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules.*

Proof. The assumption $\chi(e_{022}) = 0$ implies that $\chi(\mathfrak{a} \oplus \mathfrak{b}) = 0$. Set $e_1 = e_{112}$ and $e_2 = e_{212}$ and $e_3 = e_{211}$. Then e_1, e_2, e_3 form a basis for a complement to $\mathfrak{s}$ in $\mathfrak{g}$. Let X be an irreducible $U_\chi(\mathfrak{s})$–module annihilated by $\mathfrak{a} \oplus \mathfrak{b}$. The idea is to prove that

$$\{x \in U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X \mid (\mathfrak{a} \oplus \mathfrak{b}) \cdot x = 0\} = 1 \otimes X. \tag{13.13}$$

In order to prove (13.13) we will apply Proposition 6.4.1 with $\mathfrak{h} = \mathfrak{s}$ and $N = X$. Adopt the notation from Section 6.4: We define

$$\begin{aligned} X_1 &= \bigoplus e_1^i e_2^j e_3^k \otimes X, \\ X_2 &= \bigoplus e_2^j e_3^k \otimes X, \\ X_3 &= \bigoplus e_3^k \otimes X, \end{aligned}$$

where all i, j, k run over $\{0, 1, 2\}$. Note that $(\mathfrak{s} \oplus Ke_3) \cdot X_3 \subset X_3$ since $e_3 \cdot X_3 \subset X_3$ and $[e_3, \mathfrak{s} \oplus Ke_3] \subset Ke_3 \oplus \mathfrak{s}$. We also have $(\mathfrak{s} \cap W_{\geq 0} \oplus Ke_2 \oplus Ke_3) \cdot X_2 \subset X_2$ since $e_2 \cdot X_2 \subset X_2 \supset e_3 \cdot X_2$ and $[e_2, \mathfrak{s} \cap W_{\geq 0} \oplus Ke_2 \oplus Ke_3] \subset \mathfrak{s} \cap W_{\geq 0} \oplus Ke_2 \oplus Ke_3$. Finally, observe that $(\mathfrak{a} \oplus \mathfrak{b}) \cap W_{011} \cdot X_3 = 0$ and $(\mathfrak{a} \oplus \mathfrak{b}) \cap W_{\geq 1} \cdot X_2 = 0$. We will use these observations in the following.

Our aim is to prove that (13.13) holds; i.e., that any $x \in U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X$ such that $(\mathfrak{a} \oplus \mathfrak{b}) \cdot x = 0$ lies in $X_4 := 1 \otimes X$. So let $x \in U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X$ denote an element with $(\mathfrak{a} \oplus \mathfrak{b}) \cdot x = 0$.

Set $f_1 = e_{111}$. Since $f_1 \in \mathfrak{a}$ it follows that $f_1 \cdot x = 0$. Moreover, $\chi([e_{112}, f_1]) \neq 0$ but $[e_{112}, f_1]^{[3]} = (e_{211} - e_{122})^{[3]} = 0$. We also have $(\mathrm{ad}\ e_{112})^i(f_1) \cdot X_2 \subset X_2$ since $(\mathrm{ad}\ e_{112})^i(f_1) \in \mathfrak{s} \oplus Ke_2 \oplus Ke_3$ for all i. Finally, $f_1 \cdot X_2 = 0$ since $f_1 \in (\mathfrak{a} \oplus \mathfrak{b}) \cap W_{\geq 1}$.

Next, set $f_2 = e_{011}$. Then $f_2 \cdot x = 0$ and $\chi([e_{212}, f_2]) \neq 0 = [e_{212}, f_2]^{[3]} = (e_{211} + e_{122})^{[3]}$. Since $(\mathrm{ad}\ e_{212})^i(f_2) \in Ke_3 \oplus \mathfrak{s}$ for all i we also have $(\mathrm{ad}\ e_{212})^i(f_2) \cdot X_3 \subset X_3$. Finally, $f_2 \cdot X_3 = 0$ since $f_2 \in (\mathfrak{a} \oplus \mathfrak{b}) \cap W_{011}$.

Finally, set $f_3 = e_{012}$. Then $[f_3, e_3] = e_3$. We are now in position to apply Corollary 6.4.3 (with e_1, e_2, e_3 and f_1, f_2, f_3 and G, H, N defined above): We find that

$$\{x \in U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X \mid (\mathfrak{a} \oplus \mathfrak{b}) \cdot x = 0\} \subset 1 \otimes X$$

and since $(\mathfrak{a} \oplus \mathfrak{b}) \cdot X = 0$ the other conclusion is clear. We conclude that (13.13) holds.

This implies that $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X$ is irreducible: Any irreducible $\mathfrak{g}$–submodule M has a nonzero intersection with $1 \otimes X$ [Apply Lemma 13.11.2]. Therefore $M \cap (1 \otimes X)$ is a nonzero $U_\chi(\mathfrak{s})$–submodule of $1 \otimes X$ and, by irreducibility, $M \cap (1 \otimes X) = 1 \otimes X$. In particular, we have $M \supset 1 \otimes X$ and hence M is the entire induced module.

If X_1, X_2 are irreducible $U_\chi(\mathfrak{g})$–modules such that $(\mathfrak{a} \oplus \mathfrak{b}) \cdot X_1 = 0 = (\mathfrak{a} \oplus \mathfrak{b}) \cdot X_2$ and

$$\varphi : U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X_1 \simeq U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X_2$$

is an isomorphism, then φ induces a $U_\chi(\mathfrak{s})$–isomorphism $\overline{\varphi} : X_1 \simeq X_2$. Indeed, we have $\varphi(1 \otimes X_1) \cap (1 \otimes X_2) \neq 0$. (Look at the elements annihilated by $\mathfrak{a} \oplus \mathfrak{b}$.) Since $\varphi(1 \otimes X_1)$ and $1 \otimes X_2$ are irreducible $U_\chi(\mathfrak{g})$–modules, we get $\varphi(1 \otimes X_1) = 1 \otimes X_2$; hence $X_1 \simeq X_2$.

We have thus shown: Induction induces an injection from the isomorphism classes of irreducible $U_\chi(\mathfrak{s})$–modules annihilated by $\mathfrak{a} \oplus \mathfrak{b}$ into the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules.

Now, let Y be an arbitrary irreducible $U_\chi(\mathfrak{s})$–module. I claim that we can find an irreducible $U_\chi(\mathfrak{s})$–module X with $(\mathfrak{a} \oplus \mathfrak{b}) \cdot X = 0$ and

$$U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X \longrightarrow U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} Y.$$

First, apply Lemma 13.11.2 to find an irreducible $U_\chi(\mathfrak{s})$–submodule $X \subset U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} Y$ with $(\mathfrak{a} \oplus \mathfrak{b}) \cdot X = 0$; thus we have inclusion maps: $X \hookrightarrow U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} Y$. Now apply 'Frobenius reciprocity' on the inclusion $X \hookrightarrow U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} Y$ to produce a (nonzero) $U_\chi(\mathfrak{g})$–homomorphism:

$$U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X \longrightarrow U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} Y. \tag{13.14}$$

This implies that every $U_\chi(\mathfrak{g})$–module is induced from a $U_\chi(\mathfrak{s})$–module annihilated by $\mathfrak{a} \oplus \mathfrak{b}$: Indeed, any irreducible $U_\chi(\mathfrak{g})$–module V contains an irreducible $U_\chi(\mathfrak{s})$–module Y; hence, by 'Frobenius reciprocity', V is a homomorphic image of $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} Y$ and by (13.14) then also a homomorphic image of $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X$ for some irreducible $U_\chi(\mathfrak{s})$–module X with $(\mathfrak{a} \oplus \mathfrak{b}) \cdot X = 0$. By the part of the claim already proved we therefore have $V \simeq U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X$. The proof is completed. □

It is well known that irreducible $U_\chi(\mathfrak{s})$–modules annihilated by $\mathfrak{a} \oplus \mathfrak{b}$ are in one to one correspondence with irreducible $U_\chi(\mathfrak{s}/(\mathfrak{a} \oplus \mathfrak{b})) \simeq U_\chi(\mathfrak{L})$–modules. [Any irreducible $U_\chi(\mathfrak{s})$–module X extends to $\mathfrak{g}$ if we define $(\mathfrak{a} \oplus \mathfrak{b}) \cdot X = 0$. On the other hand: Any irreducible $U_\chi(\mathfrak{g})$–module is an irreducible $U_\chi(\mathfrak{s})$–module. So we can think of irreducible $U_\chi(\mathfrak{g})$–modules annihilated by $\mathfrak{a} \oplus \mathfrak{b}$ as irreducible $U_\chi(\mathfrak{s})$–modules extended to $\mathfrak{g}$ with trivial $\mathfrak{a} \oplus \mathfrak{b}$–action.]

Thus we obtain from the proposition above:

Corollary 13.11.4. *Suppose that $\chi(e_{022}) = 0$. The number of isomorphism classes and dimension formulas for irreducible $U_\chi(\mathfrak{g})$–modules are given as follows:*

a) If $\chi(e_{101}) \neq 0 = \chi(e_{201})$ then there exist up to isomorphism 3 irreducible $U_\chi(\mathfrak{g})$–modules all of dimension 3^4.

b) If $\chi(e_{101}) = \chi(e_{201}) = 0 \neq \chi(e_{001})$ then there exist up to isomorphism 2 irreducible $U_\chi(\mathfrak{g})$–modules all of dimension 3^4.

c) If $\chi(e_{101}) = \chi(e_{201}) = \chi(e_{001}) = 0$ then there exist up to isomorphism 3 irreducible $U_\chi(\mathfrak{g})$–modules of dimension $3^3, 2 \cdot 3^3$ and 3^4.

d) If $\chi(e_{201}) \neq 0 = \chi(2e_{101} + e_{012})^2 - \chi(e_{001})\chi(e_{201} + e_{112})$ then there exist up to isomorphism 2 irreducible $U_\chi(\mathfrak{g})$–modules all of dimension 3^4.

e) If $\chi(e_{201}) \neq 0 \neq \chi(2e_{101} + e_{012})^2 - \chi(e_{001})\chi(e_{201} + e_{112})$ then there exist up to isomorphism 3 irreducible $U_\chi(\mathfrak{g})$–modules all of dimension 3^4.

Proof. In [27, 5, 5.2] the representation theory of $\mathfrak{sl}_2(K)$ is described. If we apply the description in [27] on $\mathfrak{L}$ we see that there are 3 isomorphism classes of irreducible $U_\chi(\mathfrak{L})$–modules if $\chi(\mathfrak{L}) = 0$ or $\chi(2e_{101} + e_{012})^2 - \chi(e_{001})\chi(e_{201} + e_{112}) \neq 0$ and 2 isomorphism classes of irreducible $U_\chi(\mathfrak{L})$–modules if $\chi(2e_{101} + e_{012})^2 - \chi(e_{001})\chi(e_{201} + e_{112}) = 0$. If $\chi(\mathfrak{L}) \neq 0$ then each irreducible $U_\chi(\mathfrak{L})$–module has dimension 3 and if $\chi(\mathfrak{L}) = 0$ then there exist 3 irreducible $U_\chi(\mathfrak{L})$–modules of dimension $1, 2, 3$. Now it is straightforward to verify a)–e) by using Proposition 13.11.3. □

Lemma 13.11.5. *Let $\chi \in W^*$ be a character of height 3 and* Type A. *Then we have* $\dim_K \mathfrak{c}_W(\chi) = 2$ *and* $\mathrm{rk}\ \mathfrak{c}_W(\chi) = 0$ *and* $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$ *if* $\chi(e_{022}) \neq 0$. *If* $\chi(e_{022}) = 0$ *we have the following:*

$$\dim_K \mathfrak{c}_W(\chi) = \begin{cases} 6 & \text{if } \chi(e_{201}) = \chi(e_{101}) = \chi(e_{001}) = 0, \\ 4 & \text{else} \end{cases}$$

and

$$\mathrm{rk}\ \mathfrak{c}_W(\chi) = \begin{cases} 0 & \text{if } \chi(e_{101}) = 0 = \chi(e_{201}) \text{ and } \chi(e_{001}) \neq 0, \\ 0 & \text{if } \chi(e_{201}) \neq 0 = \chi(2e_{101} + e_{012})^2 - \chi(e_{001})\chi(e_{201} + e_{111}), \\ 1 & \text{else}, \end{cases}$$

Finally, $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$ *unless* $\chi(e_{201}) \neq 0$ *or* $\chi(e_{101}) = \chi(e_{201}) = \chi(e_{001}) = 0$.

Proof. First, note that $\mathfrak{c}_W(\chi) \subset \mathfrak{g}$ [Let $y \in \mathfrak{c}_W(\chi)$ and use the relations $\chi([y, e_{221}]) = \chi([y, e_{121}]) = \chi([y, e_{021}]) = 0$]. This leaves two possibilities for $\dim_K \mathfrak{c}_W(\chi)$: Either $\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_{\mathfrak{g}}(\chi) - 1$ or $\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_{\mathfrak{g}}(\chi) - 3$ (use similar ideas as in the proof of Lemma 10.4.7). If $\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_{\mathfrak{g}}(\chi) - 1$ then there exists $x \in \mathfrak{c}_{\mathfrak{g}}(\chi)$ such that $\mathfrak{c}_{\mathfrak{g}}(\chi) = \mathfrak{c}_W(\chi) \oplus Kx$. But we easily check that $e_{121}, e_{021} \in \mathfrak{c}_{\mathfrak{g}}(\chi)$ and since $(Ke_{121} \oplus Ke_{021}) \cap \mathfrak{c}_W(\chi) = 0$ it follows that $\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_{\mathfrak{g}}(\chi) - 1$ is impossible. Let me summarize: $\mathfrak{c}_W(\chi) \subset \mathfrak{g}$ of codimension 3.

If $\chi(e_{022}) \neq 0$ then we have $\mathfrak{c}_{\mathfrak{g}}(\chi) \subset \mathfrak{c}_{\mathfrak{g}} \cap W_{\geq 0} = \mathfrak{h}$ since we have $\chi([e_{001}, e_{122}]) \neq 0$ but $\chi([\mathfrak{g} \cap W_{\geq 0}, e_{122}]) = 0$. Now use that $\mathfrak{h} \subset \mathfrak{g}$ of codimension 1 to get $\dim_K \mathfrak{c}_{\mathfrak{g}}(\chi) = \dim_K \mathfrak{c}_{\mathfrak{h}}(\chi) - 1$. But the dimension of $\dim_K \mathfrak{c}_{\mathfrak{h}}(\chi)$ can be determined by the Vergne polarization of $\chi_{|\mathfrak{h}}$ (computed in the proof of Proposition 13.11.1); if we use (9.6) in Section 9.2 we get:

$$\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_{\mathfrak{h}}(\chi) - 4 = 2 \cdot \dim_K \mathfrak{p} - \dim_K \mathfrak{h} - 4 = 2.$$

Since $\mathfrak{c}_W(\chi) \subset \mathfrak{h}$ we have $\mathfrak{c}_{\mathfrak{h}}(\chi) \subset \mathfrak{p}$; hence $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$ and $\mathrm{rk}\ \mathfrak{c}_W(\chi) = 0$.

From now suppose that $\chi(e_{022}) = 0$. Let

$$y = \sum_{ijk} a_{ijk} e_{ijk} \in \mathfrak{c}_W(\chi)$$

for some $a_{ijk} \in K$. Since $\mathfrak{c}_W(\chi) \subset \mathfrak{g}$ we have $a_{002} = a_{102} = a_{202} = 0$. Moreover, it is easy to check that $a_{221} = a_{121} = a_{211} = a_{212} = a_{021} = a_{111} = a_{011} = 0$ also [use $\chi([y, e_{abc}]) = 0$ for appropriate a, b, c]. The final relations give the following conditions on the coefficients in the expression of y:

$$\begin{aligned} a_{201} - a_{112} &= 0, \\ a_{012} + a_{101} &= 0, \\ a_{101}\chi(e_{001}) - a_{201}\chi(e_{101}) &= 0, \\ a_{001}\chi(e_{001}) - a_{201}\chi(e_{201}) &= 0, \\ a_{101}\chi(e_{201}) - a_{001}\chi(e_{101}) &= 0. \end{aligned}$$

It follows that $e_{222}, e_{122}, e_{022} \in \mathfrak{c}_W(\chi)$ and that $e_{001}, e_{101} - e_{012}, e_{201} \in \mathfrak{c}_W(\chi)$ if $\chi(e_{201}) = \chi(e_{101}) = \chi(e_{001}) = 0$ and $\chi(e_{201})e_{001} + \chi(e_{101})(e_{101} - e_{012}) + \chi(e_{001})e_{201} \in \mathfrak{c}_W(\chi)$ otherwise. Moreover, y is a linear combination of these elements.

The dimension formula for $\mathfrak{c}_W(\chi)$ now follows and $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$ unless $\chi(e_{201}) \neq 0$ or $\chi(e_{201}) = \chi(e_{101}) = \chi(e_{001}) = 0$.

If $\chi(e_{201}) = 0$ then

$$\text{rk } \mathfrak{c}_W(\chi) = \begin{cases} 0 & \text{if } \chi(e_{101}) = 0 \neq \chi(e_{011}), \\ 1 & \text{else.} \end{cases}$$

Clearly, rk $\mathfrak{c}_W(\chi) = 0$ if $\chi(e_{101}) = 0 \neq \chi(e_{011})$ since $\mathfrak{c}_W(\chi) \subset W_{011}$ in that case. Suppose that $\chi(e_{101}) = 0 = \chi(e_{011})$. Then $e_{012} - e_{101} \in \mathfrak{c}_W(\chi)$ is a toral element. If rk $\mathfrak{c}_W(\chi) = 2$ then χ is conjugate under $\mathrm{Aut}(W)$ to a character of Type B by the results in Section 12.1. But this is a contradiction since no characters of Type A and Type B are conjugate. So rk $\mathfrak{c}_W(\chi) = 2$ is impossible for χ of height 3 and Type A.

Finally, suppose that $\chi(e_{101}) \neq 0$. Then $K(\chi(e_{101})(e_{101} - e_{012}) + \chi(e_{001})e_{201})$ is a torus since any $e_{101} - e_{012} + ce_{201}$ for $c \in K$ is a toral element by Lemma B.1.1. In fact, it is a maximal torus also.

If $\chi(e_{201}) \neq 0$ set $\alpha := \chi(e_{201})e_{001} + \chi(e_{101})(e_{101} - e_{012}) + \chi(e_{001})e_{201}$ and use (B.2) in Appendix B to get:

$$\alpha^{[3]} = \big(\chi(2e_{101} + e_{012})^2 - \chi(e_{001})\chi(e_{201} + e_{111})\big)\alpha.$$

Therefore

$$\text{rk } \mathfrak{c}_W(\chi) = \begin{cases} 0 & \text{if } \chi(2e_{101} + e_{012})^2 - \chi(e_{001})\chi(e_{201} + e_{111}) = 0, \\ 1 & \text{else.} \end{cases}$$

If $\chi(2e_{101} + e_{012})^2 - \chi(e_{001})\chi(e_{201} + e_{111}) \neq 0$ then rk $\mathfrak{c}_W(\chi) = 1$ by Lemma B.1.2. Suppose that $\chi(2e_{101} + e_{012})^2 - \chi(e_{001})\chi(e_{201} + e_{111}) = 0$. If rk $\mathfrak{c}_W(\chi) > 0$ then there exists a nonzero toral element $h \in \mathfrak{c}_W(\chi)$. It is easy to see that we can write

$$h = \alpha + z$$

for some $z \in Ke_{022} \oplus Ke_{122} \oplus Ke_{222}$. But $h^{[3]} \in W_{\geq 0}$ by (B.2) in Appendix B (use that $\alpha^{[3]} = 0$); therefore $h^{[3]} = h$ is impossible.

The proof is completed. □

The irreducible $U_\chi(W)$–modules are now described by using Theorem 11.8.5, Proposition 13.11.1, Corollary 13.11.4 and Lemma 13.11.5 (note that $\chi(\mathfrak{a}) \neq 0$ if and only if $\chi(e_{022}) \neq 0$):

Theorem 13.11.6. *Let $\chi \in W^*$ be a character of height 3 and* Type A *and let $\mathfrak{a}$ be as in* (11.11) *with $r = 3$.*

a) If $\chi(\mathfrak{a}) \neq 0$ then $\dim_K \mathfrak{c}_W(\chi) = 2$ and there exists up to isomorphism 1 irreducible $U_\chi(W)$–module of dimension $3^8 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$. We have rk $\mathfrak{c}_W(\chi) = 0$ *and $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$.*

Suppose that $\chi(\mathfrak{a}) = 0$.

b) If $\chi(e_{101}) \neq 0 = \chi(e_{201})$ then $\dim_K \mathfrak{c}_W(\chi) = 4$ and there exist up to isomorphism 3 irreducible $U_\chi(W)$–modules all of dimension $3^7 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$. We have rk $\mathfrak{c}_W(\chi) = 1$ *and $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$.*

c) If $\chi(e_{101}) = \chi(e_{201}) = 0 \neq \chi(e_{001})$ then $\dim_K \mathfrak{c}_W(\chi) = 4$ and there exist up to isomorphism 2 irreducible $U_\chi(W)$–modules all of dimension $3^7 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$. We have rk $\mathfrak{c}_W(\chi) = 0$ *and $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$.*

d) If $\chi(e_{101}) = \chi(e_{201}) = \chi(e_{001}) = 0$ then $\dim_K \mathfrak{c}_W(\chi) = 6$ and there exist up to isomorphism 3 irreducible $U_\chi(W)$–modules of dimension $3^6 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$, $2 \cdot 3^6 = 2 \cdot 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$ and $3^7 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2+1}$. We have $\operatorname{rk}\ \mathfrak{c}_W(\chi) = 1$ and $\mathfrak{c}_W(\chi) \not\subset W_{\geq 0}$.

e) If $\chi(e_{201}) \neq 0 = \chi(2e_{101} + e_{012})^2 - \chi(e_{001})\chi(e_{201} + e_{112})$ then $\dim_K \mathfrak{c}_W(\chi) = 4$ and there exist up to isomorphism 2 irreducible $U_\chi(W)$–modules all of dimension $3^7 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$. We have $\operatorname{rk}\ \mathfrak{c}_W(\chi) = 0$ and $\mathfrak{c}_W(\chi) \not\subset W_{\geq 0}$.

f) If $\chi(e_{201}) \neq 0 \neq \chi(2e_{101} + e_{012})^2 - \chi(e_{001})\chi(e_{201} + e_{112})$ then $\dim_K \mathfrak{c}_W(\chi) = 4$ and there exist up to isomorphism 3 irreducible $U_\chi(W)$–modules all of dimension $3^7 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$. We have $\operatorname{rk}\ \mathfrak{c}_W(\chi) = 1$ and $\mathfrak{c}_W(\chi) \not\subset W_{\geq 0}$.

Remark 13.11.7. One can show that induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and the isomorphism classes of irreducible $U_\chi(W)$–modules if $\chi(\mathfrak{a}) \neq 0$ or $\chi(\mathfrak{a}) = \chi(e_{201}) = 0 \neq \chi(e_{101})$.

But Theorem 13.11.6 says that induction from $W_{\geq 0}$ to W does not always take irreducible $U_\chi(W_{\geq 0})$–modules to irreducible $U_\chi(W)$–modules.

In fact, if $\chi(\mathfrak{a}) = \chi(e_{101}) = \chi(e_{201}) = 0$ then one can prove that there exist 3 irreducible $U_\chi(W_{\geq 0})$–modules S_0, S_1, S_2 and nonzero $U_\chi(W)$–homomorphisms

$$U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S_0 \ {}^{\psi}\!\rightleftarrows_{\varphi} U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S_2$$

such that $\varphi \circ \psi = \chi(e_{001})^3 \cdot \mathrm{Id}_0$ and $\psi \circ \varphi = \chi(e_{001})^3 \cdot \mathrm{Id}_2$ (Id_k denotes the identity map on the W–module induced by S_k for $k = 0, 2$). If $\chi(e_{001}) = 0$ then $\operatorname{Ker}(\psi)$ is a proper nonzero W–submodule of $U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S_0$ and $\operatorname{Ker}(\varphi)$ is a proper nonzero W–submodule of $U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S_2$. Moreover, $U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S_1$ is irreducible.

If $\chi(\mathfrak{a}) = 0 \neq \chi(e_{201})$ one can prove that there exists one irreducible $U_\chi(W_{\geq 0})$–module S with

$$\operatorname{End}_W\big(U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S\big) \simeq K[X]/\big(X^3 - X^2 - \big(\chi(2e_{101} + e_{012})^2 - \chi(e_{001})\chi(e_{201} + e_{112})\big)^3\big).$$

13.12 Type B characters of height 3

Consider $\chi \in W^*$ of height 3 and Type B. The Vergne polarization of χ constructed via the chain (9.10) is given by

$$\mathfrak{p} = \begin{cases} Ke_{012} \oplus Ke_{011} \oplus W_{022} & \text{if } \chi(e_{011}) = \chi(e_{022}) = \chi(e_{021}) = 0, \\ K(e_{011} - \chi(e_{021})e_{202} - \chi(e_{022})e_{201}) \oplus W_{022} & \text{otherwise.} \end{cases} \tag{13.15}$$

In order to see this we have to consider all $\mathfrak{s}^\chi_{ijk}$ for $(ijk) \succeq (012)$. First, we observe that $e_{\alpha\beta\gamma} \in \mathfrak{s}^\chi_{\alpha\beta\gamma}$ for $(\alpha\beta\gamma) \succeq (022)$; hence $W_{022} \subset \mathfrak{p}$. Moreover, $\mathfrak{s}^\chi_{112} \subset \mathfrak{s}^\chi_{022}$ and $\mathfrak{s}^\chi_{202} \subset \mathfrak{s}^\chi_{112}$ since $\chi([e_{112}, e_{201}]) \neq 0 = \chi([W_{022}, e_{201}])$ and $\chi([e_{202}, e_{111}]) \neq 0 = \chi([W_{112}, e_{111}])$.

Next, observe that $y := e_{011} - \chi(e_{021})e_{202} - \chi(e_{022})e_{201} \in \mathfrak{s}^\chi_{011}$ (one has to check that $\chi([y, e_{022}]) = \chi([y, e_{111}]) = \chi([y, e_{021}]) = 0$).

Finally, we have to consider $\mathfrak{s}^\chi_{101}$ and $\mathfrak{s}^\chi_{012}$. Since $\chi([e_{101}, e_{212}]) \neq 0 = \chi([W_{011}, e_{212}])$ we have $\mathfrak{s}^\chi_{101} \subset \mathfrak{s}^\chi_{011}$. Finally, suppose that $h \in \mathfrak{s}^\chi_{012}$ but $h \notin W_{101}$: Since $\chi([e_{012}, e_{212}]) = 0$ it follows that $h = e_{012} + z$ for some $z \in W_{011}$. If $\chi(e_{021}) \neq 0$, then $\chi([e_{012} + z, e_{021}]) \neq 0$ implies that $\chi([z, e_{021}]) \neq 0$ – contradiction since $\chi([e_{021}, W_{011}]) = 0$ for χ of Type B. If $\chi(e_{022}) \neq 0$ but $\chi(e_{021}) = 0$, then $\chi([e_{012} + z, e_{022}]) \neq 0$ implies that $\chi([z, e_{022}]) \neq 0$ and therefore

$$z \in K^* e_{202} + \sum_{(\alpha\beta\gamma) \neq (202)} K e_{\alpha\beta\gamma}.$$

But $\chi([e_{012}+z, e_{111}]) = 0 = \chi(e_{111})$ implies that $\chi([e_{111}, z]) = 0$ – contradiction for z written as above.

Suppose that $\chi(e_{011}) \neq 0$ but $\chi(W_1) = 0$ (or equivalent: $\chi(e_{022}) = \chi(e_{021}) = 0$). Then $\chi([e_{011}, W_{011}]) = 0$ and so we get a contradiction from $\chi([e_{012}+z, e_{011}]) = \chi(e_{011})$. It follows that $\mathfrak{s}^{\chi}_{011} \subset \mathfrak{s}^{\chi}_{101} \subset \mathfrak{s}^{\chi}_{011}$ if $\chi(e_{011}) \neq 0$ or $\chi(e_{021}) \neq 0$ or $\chi(e_{022}) \neq 0$.

Finally, if $\chi(e_{011}) = \chi(W_1) = 0$ (or equivalent: $\chi(e_{011}) = \chi(e_{022}) = \chi(e_{021}) = 0$) then $e_{012} \in \mathfrak{s}^{\chi}_{012}$ since $\chi([e_{012}, e_{212}]) = 0$. Therefore, the formula for $\mathfrak{p}$ in (13.15) holds.

At this point it will be convenient to describe the centralizer of χ.

Lemma 13.12.1. *If $\chi \in W^*$ is a character of height 3 and* Type B*, then we have*

$$\dim_K \mathfrak{c}_W(\chi) = \begin{cases} 4 & \text{if } \chi(e_{011}) \neq 0 \text{ or } \chi(e_{021}) \neq 0 \text{ or } \chi(e_{022}) \neq 0, \\ 8 & \text{else.} \end{cases}$$

Moreover,

$$\mathrm{rk}\ \mathfrak{c}_W(\chi) = \begin{cases} 0 & \text{if } \chi(e_{011}) \neq 0 \text{ or } \chi(e_{021}) \neq 0 \text{ or } \chi(e_{022}) \neq 0, \\ 1 & \text{if } \chi(e_{012}) = 0 \text{ and } \chi(e_{011}) = \chi(e_{021}) = \chi(e_{022}) = 0, \\ 2 & \text{if } \chi(e_{012}) \neq 0 \text{ and } \chi(e_{011}) = \chi(e_{021}) = \chi(e_{022}) = 0. \end{cases}$$

Finally; $\mathfrak{c}_W(\chi) \not\subset W_{\geq 0}$ if and only if $\chi(e_{011}) = \chi(e_{021}) = \chi(e_{022}) = 0$.

Proof. Let

$$y = \sum_{(\alpha\beta\gamma)} a_{\alpha\beta\gamma} e_{\alpha\beta\gamma} \in \mathfrak{c}_W(\chi) \qquad (*)$$

for $a_{\alpha\beta\gamma} \in K$. First, let us show that $a_{002} = a_{102}$: Since $\mathfrak{c}_W(\chi) \subset \mathfrak{st}(\chi, W_{\geq 3}) = Ke_{001} \oplus W_{\geq 0}$ we have $a_{002} = 0$. Next, use the relations $\chi([y, e_{211}]) = 0 \neq \chi([e_{102}, e_{211}])$ and $\chi([e_{001}, e_{211}]) = \chi([W_{012}, e_{211}]) = 0$ to get $a_{102} = 0$. It follows that $y \equiv a_{001}e_{001} \pmod{W_{012}}$.

If $\chi(e_{021}) \neq 0$ then $\chi([y, e_{121}]) = 0$ implies that $a_{001} = 0$ since $\chi([e_{001}, e_{121}]) \neq 0$ and since $\chi([W_{012}, e_{121}]) = 0$.

If $\chi(e_{022}) \neq 0$ then $\chi([y, e_{122}]) = 0$ implies that $a_{001} = 0$ since $\chi([e_{001}, e_{122}]) \neq 0$ and since $\chi([W_{012}, e_{122}]) = 0$.

Finally, suppose that $\chi(e_{021}) = \chi(e_{022}) = 0$ but $\chi(e_{011}) \neq 0$. It follows that $a_{202} = 0$ since $0 = \chi([y, e_{022}]) = 2a_{202}\chi(e_{212})$. Now use that $\chi([y, e_{111}]) = 0$ to get $a_{001} = 0$.

Therefore, $\mathfrak{c}_W(\chi) \subset \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ if $\chi(e_{011}) \neq 0$ or $\chi(e_{021}) \neq 0$ or $\chi(e_{022}) \neq 0$. Since $a_{102} = 0$ for any y as in $(*)$ we also have $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) \subset \mathfrak{c}_{W_{012}}(\chi_{|W_{012}})$. Now apply Lemma 10.4.1 and Lemma 10.4.7 to get $\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_{W_{012}}(\chi_{|W_{012}}) - 3$. But $\dim_K \mathfrak{c}_{W_{012}}(\chi_{|W_{012}})$ can be obtained by combining the dimension for $\mathfrak{p}$ and (9.6). Thus we get (see the formula for $\mathfrak{p}$ in (13.15)):

$$\dim_K \mathfrak{c}_W(\chi) = 2 \cdot \dim_K \mathfrak{p} - \dim_K W_{012} - 3 = 4.$$

Moreover, rk $\mathfrak{c}_W(\chi) = 0$ since $\mathfrak{c}_W(\chi) \subset \mathfrak{p}$ and $\mathfrak{p}$ is unipotent.

Assume from now that $\chi(e_{011}) = \chi(e_{021}) = \chi(e_{022}) = 0$ and let

$$y = \sum_{(\alpha\beta\gamma)} a_{\alpha\beta\gamma} e_{\alpha\beta\gamma} \in \mathfrak{c}_W(\chi)$$

for $a_{\alpha\beta\gamma} \in K$. It follows from the calculations above that $a_{002} = a_{102} = a_{202} = 0$ (in order to get $a_{202} = 0$ we use that $\chi([y, e_{022}]) = 0$ and $\chi(e_{022}) = \chi(e_{021}) = 0$). It is easy to verify (use the assumptions on χ) that $e_{012}, e_{021}, e_{121}, e_{221} \in \mathfrak{c}_W(\chi)$. Therefore we can also assume that $a_{012} = a_{021} = a_{121} = a_{221} = 0$ in the expression for y. Moreover, we also have $a_{101} = 0$ since $0 = \chi([y, e_{212}]) = 2a_{101}\chi(e_{212})$ (here we use that $\chi(e_{112}) = 0$). The final relations (i.e., $\chi([y, e_{abc}]) = 0$ for appropriate a, b, c) give the following conditions on the coefficients in the expression of y:

$$\begin{aligned}
a_{022} - a_{111} &= 0,\\
a_{001}\chi(e_{012}) + a_{201} &= 0,\\
a_{001}\chi(e_{012}) - a_{112} &= 0,\\
a_{001}\chi(e_{001}) + a_{212} &= 0,\\
a_{211} + a_{122} - a_{011}\chi(e_{012}) &= 0,\\
(a_{022} + a_{111})\chi(e_{012}) - a_{011}\chi(e_{001}) + a_{222} &= 0,\\
(a_{112} + a_{201})\chi(e_{012}) &= 0.
\end{aligned}$$

If $\chi(e_{012}) = 0$ then

$$y \in K(e_{211} - e_{122}) \oplus K(e_{022} + e_{111}) \oplus K(e_{001} - \chi(e_{001})e_{212}) \oplus K(e_{011} + \chi(e_{001})e_{222})$$

and by the list of relations above we have

$$\begin{aligned}
e_{211} - e_{122} &\in \mathfrak{c}_W(\chi),\\
e_{022} + e_{111} &\in \mathfrak{c}_W(\chi),\\
e_{001} - \chi(e_{001})e_{212} &\in \mathfrak{c}_W(\chi),\\
e_{011} + \chi(e_{001})e_{222} &\in \mathfrak{c}_W(\chi).
\end{aligned}$$

It follows that $\dim_K \mathfrak{c}_W(\chi) = 8$. We also have $\operatorname{rk}\, \mathfrak{c}_W(\chi) \geq 1$ since $e_{012} \in \mathfrak{c}_W(\chi)$ is a toral element. If we have $\operatorname{rk}\, \mathfrak{c}_W(\chi) = 2$ then it is easy to that there exists a toral element $h \in \mathfrak{c}_W(\chi)$ given by $h = e_{001} + z$ for some $z \in Ke_{011} \oplus Ke_{022} \oplus W_{111}$. This implies, by (B.2) in Appendix B, that $h^{[3]} \in W_{\geq 0}$ and therefore $h^{[3]} \neq h$ – contradiction.

If $\chi(e_{012}) \neq 0$ then

$$\begin{aligned}
y \in\ & K(e_{211} - e_{122}) \oplus K(e_{022} + e_{111} + \chi(e_{012})e_{222}) \oplus K(e_{011} + \chi(e_{001})e_{222} + \chi(e_{012})e_{211})\\
& \oplus K(e_{001} - \chi(e_{001})e_{212} + \chi(e_{012})(e_{112} - e_{201}))
\end{aligned}$$

and by the list of relations above we also have

$$\begin{aligned}
e_{211} - e_{122} &\in \mathfrak{c}_W(\chi),\\
e_{022} + e_{111} + \chi(e_{012})e_{222} &\in \mathfrak{c}_W(\chi),\\
e_{001} - \chi(e_{001})e_{212} + \chi(e_{012})(e_{112} - e_{201}) &\in \mathfrak{c}_W(\chi),\\
e_{011} + \chi(e_{001})e_{222} + \chi(e_{012})e_{211} &\in \mathfrak{c}_W(\chi).
\end{aligned}$$

It follows that $\dim_K \mathfrak{c}_W(\chi) = 8$. In this case we have $\operatorname{rk}\, \mathfrak{c}_W(\chi) = 2$: To see this, set

$$h := e_{001} - \chi(e_{001})e_{212} + \chi(e_{012})(e_{112} - e_{201}).$$

If we apply (B.2) in in Appendix B we get $h^{[3]} = h + \chi(e_{001})e_{012}$. It follows that we can choose $\gamma \in K$ such that $h + \gamma e_{012}$ is toral [since $[h, e_{012}] = 0$ we shall choose $\gamma \in K$ such that $h^{[3]} + \gamma^3 e_{012} = h + \gamma e_{012}$ or equivalent: We shall choose $\gamma \in K$ such that $\gamma = \gamma^3 + \chi(e_{001})$]. We conclude that $Ke_{012} \oplus K(h + \gamma e_{012})$ is a (maximal) torus. □

Let us describe the irreducible $U_\chi(W)$–modules. First, suppose that $\chi(e_{011}) \neq 0$ or $\chi(e_{021}) \neq 0$ or $\chi(e_{022}) \neq 0$.

Let K_χ be the one dimensional $\mathfrak{p}$–module where each $x \in \mathfrak{p}$ acts as multiplication by $\chi(x)$. Actually, K_χ is a $U_\chi(\mathfrak{p})$–module since $\chi(x^{[3]}) = 0$ for all $x \in \mathfrak{p}$ [in order to get $\chi(y^{[3]}) = 0$ for $y = e_{011} - \chi(e_{021})e_{202} - \chi(e_{022})e_{201}$ one has to use (B.2) in Appendix B]. Moreover, K_χ is the unique $U_\chi(\mathfrak{p})$–module since $\mathfrak{p}_2$ is unipotent.

Set $S := U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{p})} K_\chi$ and note that S is irreducible with a basis given by $z_{stklm} := e^s_{102}e^t_{202}e^k_{012}e^l_{101}e^m_{112} \otimes 1$ for $0 \leq s, t, k, l, m < 3$ (the PBW theorem).

Let $M := U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S$ and let $w_0 = 1 \otimes z_{00000} \in \mathrm{Soc}_{\mathfrak{p}}M$. Note that $w_0 \in \mathrm{Soc}_{\mathfrak{p}}1 \otimes S$; thus it follows from Lemma 11.3.1 that $\mathrm{Soc}_{\mathfrak{p}}1 \otimes S = Kw_0$.

We shall obtain results similar to those in Theorem 10.4.11 and Theorem 10.4.12 [in Theorem 10.4.11 and 10.4.12 we consider χ of height r with $\mathfrak{st}(\chi, W_{\geq r}) = W_{\geq 0}$: Except for one type of characters of height $2p-3$ the dimension of all irreducible $U_\chi(W)$–modules is $p^{\mathrm{codim}_W \mathfrak{c}_W(\chi)/2}$ and the number of isomorphism classes is p if rk $\mathfrak{c}_W(\chi) = 1$ and 1 if rk $\mathfrak{c}_W(\chi) = 0$]. This is illustrated by the following result.

Proposition 13.12.2. *If $\chi(e_{011}) \neq 0$ or $\chi(e_{022}) \neq 0$ or $\chi(e_{021}) \neq 0$ then there exist one irreducible $U_\chi(W)$–module of dimension $3^7 = 3^{\mathrm{codim}_W \mathfrak{c}_W(\chi)/2}$.*

Proof. Keep the notation from above. The idea is to prove that $\mathrm{Soc}_{\mathfrak{p}}M = Kw_0$; so suppose otherwise that $\mathrm{Soc}_{\mathfrak{p}}M \neq Kw_0$. Then there exists $w \in M$ such that Kw is an irreducible $\mathfrak{p}$–submodule of $\mathrm{Soc}_{\mathfrak{p}}M$ and by Lemma 11.4.1 we have

$$w \in e^b_{001} \otimes z_{00000} + e^{b-1}_{001} \otimes u + \sum_{k<b-1} \sum_{k+m \leq b-1} e^k_{001}e^m_{002} \otimes S \tag{13.16}$$

for some $b > 0$ and some $u \in S$. The assumption on w says that $x \cdot w - \chi(x)w = 0$ for all $x \in W_{011}$. For $x \in \{e_{111}, e_{022}, e_{211}, e_{122}, e_{121}\}$ we have:

$$\chi(x)w \in \chi(x)e^b_{001} \otimes z_{00000} + e^{b-1}_{001} \otimes (b[x, e_{001}] \cdot z_{00000} + x \cdot u) + \sum_{k<b-1} \sum_{k+m \leq b-1} e^k_{001}e^m_{002} \otimes S. \tag{$*$}$$

In particular, $b[x, e_{001}] \cdot z_{00000} + x \cdot u = 0$ by the PBW theorem. Use that relation with $x = e_{211}$ and $x = e_{022}$ and get (note that $[e_{001}, e_{022}] = 0$ and that $[e_{001}, e_{211}] \cdot z_{00000} = 0$)

$$\begin{aligned} e_{211} \cdot u &= 0, \\ e_{022} \cdot u &= 0. \end{aligned}$$

This implies that $u \in \sum_{klm} Kz_{00klm}$ since

$$\{u \in S \mid e_{211} \cdot u = 0 = e_{022} \cdot u\} \subset \sum_{klm} Kz_{00klm}. \tag{13.17}$$

In order to prove (13.17) use Proposition 6.4.1 with $\mathfrak{g} = W_{\geq 0}$ and $H = \mathfrak{p}$ and $N = K_\chi$. The cobasis is given by $e_1 = e_{102}, e_2 = e_{202}, e_3 = e_{012}, e_4 = e_{101}, e_5 = e_{112}$. There exists $f_1 = e_{211}$ such that $[e_1, f_1]$ acts bijectively on S (since $\chi([e_1, f_1]) \neq 0 = [e_1, f_1]^{[3]}$) and $f_1 \cdot \sum_{tklm} Kz_{0tklm} = 0$ (note that N_2 in Proposition 6.4.1 with $N = K_\chi$ corresponds to $\sum_{tklm} Kz_{0tklm}$). Finally, (ad e_1)$(f_1) \cdot N_2 \subset N_2$ and thus we can use Proposition 6.4.1: It follows that $u \in \sum_{tklm} Kz_{0tklm}$ if $e_{211} \cdot u = 0$.

Next, we can use that $e_{022} \cdot u = 0$. Set $f_2 = e_{022}$ and note that $[e_2, f_2]$ acts bijectively on S (since $\chi([e_2, f_2]) \neq 0 = [e_2, f_2]^{[3]}$) and $f_2 \cdot \sum_{tklm} Kz_{00klm} = 0$ (note that N_3 in Proposition 6.4.1 with $N = K_\chi$ corresponds to $\sum_{klm} Kz_{00klm}$). Finally, (ad e_2)$(f_2) \cdot N_3 \subset N_3$ and thus we can use Proposition 6.4.1 again: It follows that $u \in \sum_{klm} Kz_{00klm}$ if $e_{211} \cdot u = 0 = e_{022} \cdot u$. Therefore, (13.17) holds.

So for any $w \in M$ but $w \notin Kw_0$ such that Kw is a $\mathfrak{p}_2$–submodule of M we have $u \in \sum_{klm} Kz_{00klm}$ for w written as in (13.16).

The assumption in the theorem implies that we can find $x \in \mathfrak{p}_2$ such that $[x, e_{001}] \cdot z_{00000} \neq 0$ [take $x = e_{121}$ if $\chi(e_{021}) \neq 0$ and $x = e_{122}$ if $\chi(e_{022}) \neq 0$ and $x = e_{111}$ if $\chi(e_{011}) \neq 0$]. It follows from $(*)$ that $x \cdot u \neq 0$ for $x = e_{121}$ or $x = e_{122}$ or $x = e_{111}$. But now we have a contradiction since $u \in \sum_{klm} Kz_{00klm}$ by (13.17) and since $x \cdot z_{00klm} = 0$ for all k, l, m if $x = e_{121}$ or $x = e_{122}$ or $x = e_{111}$.

We conclude that $\mathrm{Soc}_{\mathfrak{p}} M = Kw_0$. This implies that M is irreducible and in fact the only irreducible $U_\chi(W)$–module up to isomorphism: Indeed, any nonzero W–submodule X of M contains w_0 (X has $\mathfrak{p}$–socle inside $\mathrm{Soc}_{\mathfrak{p}} M = Kw_0$); therefore X is the entire module. Any irreducible module M' contains a copy of K_χ and so a copy of S (use 'Frobenius reciprocity'). Thus we have a nonzero W–homomorphism $M \longrightarrow M'$. Since both M and M' are irreducible we have $M \simeq M'$.

Note that $\dim_K M = 3^7$. Now the dimension formula (i.e., $3^7 = 3^{\mathrm{codim}_W \mathfrak{c}_W(\chi)/2}$) follows from Lemma 13.12.1 The proof is completed. □

Finally we shall consider the case where $\chi(e_{011}) = 0 = \chi(W_1)$. It follows from (13.11) that $\chi(\mathfrak{a}) = 0$. Set $\mathfrak{b} := Ke_{011} \oplus Ke_{111} \oplus Ke_{211}$. It is easy to check that $\mathfrak{a} \oplus \mathfrak{b}$ is a p–ideal in $\mathfrak{g}$ with $\chi(\mathfrak{a} \oplus \mathfrak{b}) = 0$; hence $\mathfrak{a} \oplus \mathfrak{b}$ annihilates all irreducible $U_\chi(\mathfrak{g})$–modules. Moreover, let $\mathfrak{L}$ be the restricted Lie algebra isomorphic to $\mathfrak{sl}_2(K)$ defined in (13.5) and define

$$\mathfrak{H} := \mathfrak{L} \oplus Ke_{012} \oplus Ke_{112} \oplus Ke_{212}. \tag{13.18}$$

It is a Lie p–subalgebra of $\mathfrak{g}$ with $\mathfrak{g} = (\mathfrak{a} \oplus \mathfrak{b}) \oplus \mathfrak{H}$; hence $\mathfrak{g}/(\mathfrak{a} \oplus \mathfrak{b}) \simeq \mathfrak{H}$. Note that $\mathfrak{a} \oplus \mathfrak{b}$ annihilates all irreducible $U_\chi(\mathfrak{g})$–modules and that $\mathfrak{g}/(\mathfrak{a} \oplus \mathfrak{b}) \simeq \mathfrak{H}$. Thus irreducible $U_\chi(\mathfrak{g})$–modules are in one to one correspondence with irreducible $U_\chi(\mathfrak{H})$–modules. [We can think of irreducible $U_\chi(\mathfrak{g})$–modules as irreducible $U_\chi(\mathfrak{H})$–modules where $\mathfrak{a} \oplus \mathfrak{b}$ acts trivially.] Moreover, induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules and the isomorphism classes of irreducible $U_\chi(W)$–modules by Theorem 11.9.6 and Remark 11.9.7. Thus it will be enough for us to describe the irreducible $U_\chi(\mathfrak{H})$–modules in detail. This is the subject for the next propositions.

First, we define

$$\mathfrak{H}_0 := \mathfrak{H} \cap W_{\geq 0} = Ke_{012} \oplus Ke_{101} \oplus Ke_{112} \oplus Ke_{201} \oplus Ke_{212}. \tag{13.19}$$

Note that $\mathfrak{H}_0$, as a Lie p–subalgebra of W_{012}, is supersolvable. If we intersect the chain from (9.10) with $\mathfrak{H}_0$, then we get a chain

$$\mathfrak{H}_0 \supset \mathfrak{H}_0 \cap W_{101} \supset \mathfrak{H}_0 \cap W_{011} \supset \cdots \supset 0 \tag{13.20}$$

that we can use to construct Vergne polarizations (after moving repetitions). It is easy to see that the Vergne polarization $\mathfrak{p}_0$ of $\chi_{|\mathfrak{H}_0}$ with respect to (13.20) is given by

$$\mathfrak{p}_0 = Ke_{012} \oplus Ke_{201} \oplus Ke_{212}.$$

For any $\nu \in K$ with $\nu^3 - \nu = \chi(e_{012})^3$ we can define an (irreducible) $U_\chi(\mathfrak{p}_0)$–module K_ν where each $x \in \mathfrak{p}_0 \cap W_{011}$ acts as mutiplication by $\chi(x)$ and e_{012} acts as multiplication by ν. Moreover, any $U_\chi(\mathfrak{p}_0)$–module is isomorphic to one of these K_ν. It follows from Proposition 9.3.10 that any irreducible $U_\chi(\mathfrak{H}_0)$–module is isomorphic to some $U_\chi(\mathfrak{H}_0) \otimes_{U_\chi(\mathfrak{p}_0)} K_\nu$ (where $\nu \in K$ with $\nu^3 - \nu = \chi(e_{012})^3$). Set $N_\nu := U_\chi(\mathfrak{H}_0) \otimes_{U_\chi(\mathfrak{p}_0)} K_\nu$. Then all $z_{ij} := e_{101}^i e_{112}^j \otimes 1$ with $0 \leq i, j < 3$ form a basis for N_ν. Set

$$M_\nu := U_\chi(\mathfrak{H}) \otimes_{U_\chi(\mathfrak{H}_0)} N_\nu$$

and define $w_{\nu,0} := 1 \otimes z_{00} \in \mathrm{Soc}_{\mathfrak{p}_0} M_\nu$.

Proposition 13.12.3. *For $\nu \in K$ with $\nu^3 - \nu = \chi(e_{012})^3$ we have*

$$(e_{001} - e_{112} - e_{101}e_{112}) \cdot w_{\nu,0} \in \mathrm{Soc}_{\mathfrak{p}_0} M_\nu.$$

Proof. First, note that $e_{012} \cdot (e_{001} - e_{112} - e_{101}e_{112}) \cdot w_{\nu,0} = \nu(e_{001} - e_{101}e_{112}) \cdot w_{\nu,0}$ since $[e_{001}, e_{012}] = [e_{101}, e_{012}] = [e_{112}, e_{012}] = 0$ and $e_{012} \cdot w_{\nu,0} = \nu w_{\nu,0}$. Next, $[e_{201}, e_{001}] \cdot w_{\nu,0} = e_{101} \cdot w_{\nu,0}$ and $e_{201} \cdot (e_{112} + e_{101}e_{112}) \cdot w_{\nu,0} = e_{101} \cdot w_{\nu,0}$. Therefore we have

$$e_{201} \cdot (e_{001} - e_{112} - e_{101}e_{112}) \cdot w_{\nu,0} = 0$$

as required. Finally, we get $e_{212} \cdot (e_{112} + e_{101}e_{112}) \cdot w_{\nu,0} = e_{101} \cdot w_{\nu,0} = [e_{212}, e_{001}] \cdot w_{\nu,0}$ implying that $e_{212} \cdot (e_{001} - e_{112} - e_{101}e_{112}) \cdot w_{\nu,0} = (e_{001} - e_{112} - e_{101}e_{112}) \cdot w_{\nu,0}$. □

Set $q_\nu = e_{001} - e_{112} - e_{101}e_{112}$ and $w_\nu = q_\nu \cdot w_{\nu,0}$. Then Kw_ν is a $\mathfrak{p}_0$–submodule of M_ν by Proposition 13.12.3 and so $\mathrm{Hom}_{\mathfrak{p}_0}(K_\nu, M_\nu) \neq 0$ (take $\psi : K_\nu \longrightarrow M_\nu$ defined by $\psi(1) = w_\nu$). Now apply 'Frobenius reciprocity' once to produce a $U_\chi(\mathfrak{H}_0)$–homomorphism $\psi_0 : N_\nu \longrightarrow M_\nu$ given by $\psi(z_{00}) = w_\nu$ and secondly use 'Frobenius resiprocity' to produce $U_\chi(\mathfrak{H})$–homomorphism $\psi : M_\nu \longrightarrow M_\nu$ given by $\psi(w_{\nu,0}) = w_\nu$. It is easy to see that $\mathrm{Id}_{M_\nu}, \psi$ and ψ^2 are linear independent $U_\chi(\mathfrak{H})$–homomorphisms [one can use quite similar arguments as in Remark 11.4.4; the setup in Remark 11.4.4 is a little different from here but the type of arguments are exactly the same]. It follows that $\dim_K \mathrm{End}_{\mathfrak{H}}(M_\nu) = 3$ [we have to argue as before: One can use similar arguments as in Corollary 11.4.3; the setup is a little different from here but the type of arguments are exactly the same].

Proposition 13.12.4. *We have an isomorphism as K–algebras*

$$K[X]/\big(X^3 - \nu X - \chi(e_{001})^3\big) \simeq \mathrm{End}_{\mathfrak{H}}(M_\nu) \;\; ; \;\; X + (X^3 - \nu X - \chi(e_{001})^3) \longmapsto \psi$$

where ψ is the $\mathfrak{H}$–endomorphism given by $\psi(w_{\nu,0}) = (e_{001} - e_{112} - e_{101}e_{112}) \cdot w_{\nu,0}$.

Proof. Define K–algebra homomorphism $K[X] \longrightarrow \mathrm{End}_{\mathfrak{H}}(M_\nu)$ sending X to ψ (ψ as in the proposition). If we can prove that $\psi^3 - \nu\psi - \chi(e_{001})^3 \mathrm{Id}_{|M_\nu} = 0$ (where 0 is the zero endomorphism on M_ν), then we are done (compare dimension). Clearly, it is enough to prove that $\psi^3(w_{\nu,0}) - \nu\psi(w_{\nu,0}) - \chi(e_{001})^3 \cdot w_{\nu,0} = 0$ or equivalent:

$$(e_{001} - e_{112} - e_{101}e_{112})^3 \cdot w_{\nu,0} - \nu(e_{001} - e_{112} - e_{101}e_{112}) \cdot w_{\nu,0} - \chi(e_{001})^3 \cdot w_{\nu,0} = 0. \quad (13.21)$$

First, observe that

$$\begin{aligned}(e_{001} - e_{112} - e_{101}e_{112})^3 &= \chi(e_{001})^3 - \big[e_{001}, [e_{001}, e_{112} + e_{101}e_{112}]\big] \\ &\quad + \big[[e_{001}, e_{112} + e_{101}e_{112}], e_{112} + e_{101}e_{112}\big] \\ &\quad - (e_{112} + e_{101}e_{112})^3.\end{aligned}$$

Now use [27, 1, Prop. 1.3 (2)] to get:

$$\begin{aligned}[e_{001}, e_{112} + e_{101}e_{112}] &= e_{012} + e_{001}e_{112} + e_{101}e_{012}, \\ [e_{001}, e_{012} + e_{001}e_{112} + e_{101}e_{012}] &= -e_{001}e_{012}, \\ [e_{012}, e_{112} + e_{101}e_{112}] &= 0, \\ [e_{001}e_{112} + e_{101}e_{012}, e_{112} + e_{101}e_{112}] &= -(e_{112} + e_{101}e_{112})e_{012}.\end{aligned}$$

It follows that

$$(e_{001} - e_{112} - e_{101}e_{112})^3 = \chi(e_{001})^3 + (e_{001} - e_{112} - e_{101}e_{112})e_{012} - (e_{112} + e_{101}e_{112})^3. \quad (*)$$

But $(e_{112} + e_{101}e_{112})^3 = (e_{112}(1 + e_{101}))^3 = 0$ since $e_{112}^3 = e_{112}^{[3]} + \chi(e_{112})^3 = 0$ and since $(e_{101} + a)e_{112} = e_{112}(e_{101} + a + 1)$ for any $a \in K$. Now use $(*)$ and the fact that $e_{012} \cdot w_{\nu,0} = \nu w_{\nu,0}$ to get (13.21). □

Proposition 13.12.5. *Suppose that* $\chi(e_{011}) = \chi(e_{022}) = \chi(e_{021}) = 0$.

a) *If* $\chi(e_{012}) \neq 0$ *then there exist up to isomorphism* 3^2 *irreducible* $U_\chi(\mathfrak{H})$*–modules all of dimension* 3^2.

b) *If* $\chi(e_{012}) = 0$ *then there exist up to isomorphism* $2 \cdot 3 + 1$ *irreducible* $U_\chi(\mathfrak{H})$*–modules all of dimension* 3^2.

Proof. We shall consider irreducible $\mathfrak{H}$–submodules of M_ν.

If $\nu \neq 0$, then $X^3 - \nu X - \chi(e_{001})^3$ has three (different) roots $a_0, a_1, a_2 \in K$ and therefore M_ν decomposes into its isotypic components $M_{\nu,i} := \mathrm{Ker}(\psi - a_i \cdot \mathrm{Id}_{|M_\nu})$ for $i = 0, 1, 2$ (apply Proposition 13.12.4). There results an embedding

$$\bigoplus_{i=0}^{2} \mathrm{End}_{\mathfrak{H}}(M_{\nu,i}) \longrightarrow \mathrm{End}_{\mathfrak{H}}(M_\nu) \;\; ; \;\; (f_0, f_1, f_2) \longmapsto f_0 \oplus f_1 \oplus f_2$$

of K–algebras, which, for dimension reasons, is also onto. Consequently, $\dim_K \mathrm{End}_{\mathfrak{H}}(M_{\nu,i}) = 1$ as well as $\mathrm{Hom}_{\mathfrak{H}}(M_{\nu,i}, M_{\nu,j}) = (0)$ for $i \neq j$, implying that the $M_{\nu,i}$ are pairwise non-isomorphic (recall that $\dim_K \mathrm{End}_{\mathfrak{H}}(M_\nu) = 3$). Each $M_{\nu,i}$ contains a simple $U_\chi(\mathfrak{H}_0)$-module and thus has dimension $\geq p^2$. In view of $3p^2 = p^3 = \dim_K M_\nu$, each $M_{\nu,i}$ is an irreducible $\mathfrak{H}_0$–module; thus we obtain the irreducibility of each $\mathfrak{H}$–module $M_{\nu,i}$. Finally, note that M_ν is semisimple for $\nu \neq 0$.

For $\nu = 0$, we have $\mathrm{End}_{\mathfrak{H}}(M_0) \simeq K[T]/(T^3)$. Let $t := T + (T^3)$ and consider the filtration

$$(0) \subsetneq t^2 M_0 \subsetneq t M_0 \subsetneq M_0$$

of $\mathfrak{H}$-modules. The foregoing dimension arguments imply that this is a composition series. Moreover, multiplication by t induces isomorphisms between the composition factors. It is easy to verify that M_0 is indecomposable.

We are now in position to finish the proof:

If $\chi(e_{012}) \neq 0$ then $\nu \neq 0$ for all $\nu \in K$ with $\nu^3 - \nu = \chi(e_{012})^3$. Let $\nu_0, \nu_1, \nu_2 \in K^*$ be the roots in $X^3 - X - \chi(e_{012})^3$. Then $\{M_{\nu_i,j} \mid 0 \leq i, j \leq 2\}$ is a set of representative of non–isomorphic irreducible $U_\chi(\mathfrak{H})$–modules.

If $\chi(e_{012}) = 0$ then $\{M_{i,j}, t^2 M_0 \mid i = 1, 2, 0 \leq j \leq 2\}$ is a set of representative of non–isomorphic irreducible $U_\chi(\mathfrak{H})$–modules. [If $\phi : M_{i,j} \simeq t^2 M_0$ take nonzero $x_0 \in t^2 M_0$ such that $K x_0$ is $\mathfrak{p}_0$–submodules of $t^2 M_0$ and $x_i \in M_{i,j}$ such that $K x_i$ is $\mathfrak{p}_0$–submodules of $M_{i,j}$; in particular, $K x_i \simeq K_i$ and $K x_0 \simeq K_0$. Then $\phi(x_i) \in t^2 M_0$ and $K\phi(x_i) \subset M_0$ is a $\mathfrak{p}_0$–submodule; hence isomorphic to K_0. We conclude that $K_i \simeq_{\mathfrak{p}_0} K_0$–contradiction.] □

Theorem 13.12.6. *Let* $\chi \in W^*$ *be a character of height 3 and* Type B.

a) *If* $\chi(e_{011}) \neq 0$ *or* $\chi(e_{022}) \neq 0$ *or* $\chi(e_{021}) \neq 0$ *then there exists up to isomorphism* 1 *irreducible* $U_\chi(W)$*–module of dimension* $3^7 = 3^{\mathrm{codim}_W \mathfrak{c}_W(\chi)/2}$. *Moreover,* $\mathfrak{c}_W(\chi) \subset W_{\geq 0}$ *and* $\mathrm{rk}\, \mathfrak{c}_W(\chi) = 0$.

Suppose that $\chi(e_{011}) = \chi(e_{021}) = \chi(e_{022}) = 0$.

b) *If* $\chi(e_{012}) \neq 0$ *then there are up to isomorphism* 3^2 *irreducible* $U_\chi(W)$*–modules all of dimension* $3^5 = 3^{\mathrm{codim}_W \mathfrak{c}_W(\chi)/2}$. *Moreover,* $\mathfrak{c}_W(\chi) \not\subset W_{\geq 0}$ *and* $\mathrm{rk}\, \mathfrak{c}_W(\chi) = 2$.

c) *If* $\chi(e_{012}) = 0$ *then there are up to isomorphism* $2 \cdot 3 + 1$ *irreducible* $U_\chi(W)$*–modules of dimension* $3^5 = 3^{\mathrm{codim}_W \mathfrak{c}_W(\chi)/2}$. *Moreover,* $\mathfrak{c}_W(\chi) \not\subset W_{\geq 0}$ *and* $\mathrm{rk}\, \mathfrak{c}_W(\chi) = 1$.

Proof. Apply Proposition 13.12.2 and Lemma 13.12.1 for part a). For b),c) apply Lemma 13.12.1 and and Proposition 13.12.5. □

13.13 Type II.a characters of height 3

In this section we consider $\chi \in W^*$ of height 3 such that χ has Type II.a as in Section 5.2. Since χ has Type II.a we have

$$\chi(e_{122}) = -\chi(e_{211}) \neq 0$$

and $\chi(e_{212}) = 0 = \chi(e_{121})$. The representation theory of $U_\chi(W)$ depends only on the $\mathrm{Aut}(W)$–orbit of χ, so we may replace χ with any χ^g, for an automorphism g, in order to describe the irreducible $U_\chi(W)$–modules. Thus the next result becomes useful.

Lemma 13.13.1. *There exists an automorphism $g \in \mathrm{Aut}(W)$ with $g(W_{012}) = W_{012}$ such that $\chi^g(W_1) = 0 = \chi^g(e_{012} + e_{101})$.*

Proof. Set $x = a_1e_{202}+a_2e_{112}+a_3e_{022}+b_1e_{201}+b_2e_{111}+b_3e_{021}$ and denote by g_1 the automorphism on W induced by x (see Section 3.2). It follows that $g_1(y) \equiv y+[x, y] \pmod{W_{\geq 3}}$ for all $y \in W_1$. The formulas

$$\begin{aligned}
\chi\big([x, e_{202}]\big) &= -4b_3\chi(e_{211}),\\
\chi\big([x, e_{112}]\big) &= (a_3 - 2b_2)\chi(e_{211}),\\
\chi\big([x, e_{022}]\big) &= -a_2\chi(e_{211}),\\
\chi\big([x, e_{201}]\big) &= b_2\chi(e_{211}),\\
\chi\big([x, e_{201}]\big) &= (2a_2 - b_1)\chi(e_{211}),\\
\chi\big([x, e_{021}]\big) &= 4a_1\chi(e_{211}),
\end{aligned}$$

say that we can find appropriate $a_1, a_2, a_3, b_1, b_2, b_3 \in K$ such that $\chi^{g_1}(W_1) = 0$ (note that $\chi(e_{211}) \neq 0$).

Finally, denote by g_2 the automorphism on W induced by $y = \alpha e_{211}$ (see Section 3.2). It follows that $g_2(z) \equiv z \pmod{W_{\geq 3}}$ for all $z \in W_{\geq 1}$; hence $\chi^{g_1\circ g_2}(W_1) = 0$ if we choose g_1 as above. Moreover, we can choose $\alpha \in K$ such that $\chi^{g_1\circ g_2}(e_{012} + e_{101}) = 0$ since

$$\chi^{g_1\circ g_2}([y, e_{012} + e_{101}]) = -2\alpha\chi(e_{211}).$$

We have $\chi^{g_1\circ g_2}(e_{012} + e_{101}) = \chi^{g_1}(e_{012} + e_{101}) - 2\alpha\chi^{g_1}(e_{211}) = 0$ for some $\alpha \in K$ and also $\chi^{g_1\circ g_2}(W_1) = 0$. The proof is completed. $\square$

The discussion before Lemma 13.13.1 says that we can assume that χ is a character of height 3 with $\chi(W_1) = 0 = \chi(e_{012} + e_{101})$. First, we will prove a result on the stabilizer of $\chi_{|W_{\geq 0}}$ in $W_{\geq 0}$. In fact, we only need the assumption $\chi(W_1) = 0$ in order to prove this result.

Lemma 13.13.2. *Let $\chi \in W^*$ be a character of height 3 and of* Type II.a *as in 5.2. Suppose that $\chi(W_1) = 0$. Set $h := e_{012} - e_{101}$. Then*

$$\dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = \begin{cases} 8 & \text{if } \chi(e_{102}) = \chi(h) = \chi(e_{011}) = 0,\\ 6 & \text{else.}\end{cases}$$

Moreover,

$$\mathrm{rk}\ \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = \begin{cases} 1 & \text{if } \chi(e_{102}) = \chi(h) = \chi(e_{011}) = 0,\\ 1 & \text{if } \chi(h)^2 + \chi(e_{102})\chi(e_{011}) \neq 0,\\ 0 & \text{else.}\end{cases}$$

Finally; $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) \not\subset W_{012}$ if and only if $\chi(e_{011}) \neq 0$ or $\chi(e_{102}) = \chi(h) = \chi(e_{011}) = 0$.

Proof. It is easy to verify that

$$\begin{aligned} e_{222} &\in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}), \\ e_{221} &\in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}), \\ e_{121} &\in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}), \\ e_{211}+e_{122} &\in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}), \\ e_{212} &\in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}). \end{aligned}$$

Let

$$y = \sum_{\alpha\beta\gamma} a_{\alpha\beta\gamma} e_{\alpha\beta\gamma} \in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) \qquad (*)$$

with $a_{222} = a_{221} = a_{121} = a_{212} = 0$ and $a_{211}+a_{122} = 0$. It follows from $\chi([y, e_{012}+e_{101}]) = 0$ that $a_{211} = a_{122} = 0$ since $\chi([e_{012}+e_{101}, W_0+W_1]) = 0$. Moreover, from the relations $\chi([y, e_{\alpha\beta\gamma}]) = 0$, where $(\alpha\beta\gamma)$ denote all triples with $\alpha+\beta = 2$ and $\gamma = 1, 2$, we obtain $a_{\alpha\beta\gamma} = 0$ for all $(\alpha\beta\gamma)$ with $\alpha+\beta = 2$ and $\gamma = 1, 2$. Thus we can assume that y written as in $(*)$ belongs to W_0. Since $\chi([y, e_{211}]) = 0$ we have $a_{012}+a_{101} = 0$.

If $\chi(e_{102}) = \chi(h) = \chi(e_{011}) = 0$ then $e_{102} \in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ and $h \in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ and $e_{011} \in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ and y is a linear combination of these elements. In particular, $\dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 8$. In this case we have $\operatorname{rk} \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 1$: Clearly, Kh is a torus inside $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$; if $\operatorname{rk} \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) > 1$, then it is easy to see that there exists a nonzero toral element $h' \in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ given by

$$h' = e_{102} + z \quad \text{for some } z \in W_{011}$$

such that $Kh \oplus Kh'$ is a (maximal) torus. But $[h, h'] \equiv e_{102} \pmod{W_{012}}$ is a contradiction. Finally, $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) \not\subset W_{012}$.

Suppose that $\chi(e_{102}) \neq 0$ or $\chi(h) \neq 0$ or $\chi(e_{011}) \neq 0$. Since y written as in $(*)$ with $a_{\alpha\beta\gamma} = 0$ for $\alpha+\beta \geq 2$ and $a_{012}+a_{101} = 0$ satisfies that $\chi([y, W_{\geq 1}]) = 0$, we have $y \in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ if and only if $\chi([y, W_0]) = 0$. We can assume that $y \in W_0$ is given by

$$y = ae_{102} + bh + ce_{011}$$

for some $a, b, c \in K$. We now get the following relations:

$$\begin{aligned} b\chi(e_{102}) + c\chi(h) &= 0, \\ a\chi(e_{102}) - c\chi(e_{011}) &= 0, \\ a\chi(h) + b\chi(e_{011}) &= 0. \end{aligned}$$

It is easy to see that a, b, c are determined uniquely by the relations above: We get $a = \chi(e_{011})$ and $b = -\chi(h)$ and $c = \chi(e_{102})$. In particular, $\dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 6$. Moreover, $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) \not\subset W_{012}$ if and only if $\chi(e_{011}) \neq 0$.

In order to determine the rank of $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ we can first use (B.2) in appendix B with commutator relations and get

$$y^{[3]} = (\chi(h)^2 + \chi(e_{102})\chi(e_{011}))y.$$

If $\chi(h)^2+\chi(e_{102})\chi(e_{011}) \neq 0$ then Ky is a torus by Lemma B.1.2; hence $\operatorname{rk} \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 1$ in that case.

If $\chi(h)^2 + \chi(e_{102})\chi(e_{011}) = 0$ then $\operatorname{rk} \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 0$: Otherwise there exists a nonzero toral element $h' \in \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$. It is easy to see that $h' \in K^*y + W_{\geq 2}$ and hence $(h')^{[3]} \in K^*y^{[3]} + W_{\geq 2} = W_{\geq 2}$ since $y^{[3]} = 0$. Therefore $(h')^{[3]} = h'$ is impossible. $\square$

Theorem 13.13.3. *Suppose that $p > 3$ and let $\chi \in W^*$ be a character of height 3 and of* Type II.a *as in* 5.2*. Then* rk $\mathfrak{c}_W(\chi) = 1$ *or* rk $\mathfrak{c}_W(\chi) = 0$ *and all irreducible $U_\chi(W)$–modules have dimension $p^{\mathrm{codim_W}\mathfrak{c}_W(\chi)/2}$. If* rk $\mathfrak{c}_W(\chi) = 1$*, then there exist up to isomorphism p irreducible $U_\chi(W)$–modules and if* rk $\mathfrak{c}_W(\chi) = 0$*, then there exists up to isomorphism* 1 *irreducible $U_\chi(W)$–module.*

Proof. The fact that rk $\mathfrak{c}_W(\chi) = 1$ or rk $\mathfrak{c}_W(\chi) = 0$ follows from Lemma 10.4.8 in Section 10.4 (here we use that $r = 3 < 2p - 3$ when $p > 3$ and that $\mathfrak{st}(\chi, W_{\geq 3}) = W_{\geq 0}$ since $\chi([e_{001}, e_{222}]) \neq 0 = \chi([e_{002}, e_{222}])$ and since $\chi([e_{002}, e_{221}]) \neq 0 = \chi([e_{001}, e_{221}])$).

For the final statements use Theorem 10.4.11, 10.4.12 in Section 10.4. ☐

Remark 13.13.4. Since $\dim_K \mathfrak{c}_W(\chi) = \dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) - 2$ and $\dim_K W = 2p^2$ we can use Lemma 13.13.2 and find the possible dimension of all irreducible $U_\chi(W)$–modules combined with the number of irreducible in the following scheme ($|\mathrm{Irr}(W, \chi)|$ denotes the number of irreducible):

Type II.a and height 3 and $p > 3$

$\lvert\mathrm{Irr}(W, \chi)\rvert$	Possible dimension
1	p^{p^2-2}
p	p^{p^2-3} or p^{p^2-2}

In the rest of this section we assume that $p = 3$. This is the critical situation when we consider χ of height 3 and of Type II.a as in 5.2 (remember the exceptions from Section 10.4). By Lemma 13.13.1 we may consider $\chi \in W^*$ of height 3 and of Type II.a as in 5.2 such that $\chi(W_1) = 0 = \chi(e_{012} + e_{101})$. In the following, set (as in Lemma 13.13.2)

$$h := e_{012} - e_{101}.$$

Now observe that $Ke_{102} \oplus K(e_{012} - e_{101}) \oplus Ke_{011}$ is a restricted Lie subalgebra of $W_{\geq 0}$ isomorphic to $\mathfrak{sl}_2(K)$. The isomorphism is given by

$$e_{102} \longmapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_{012} - e_{101} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_{011} \longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus we set

$$\mathfrak{sl}_2(K) := Ke_{102} \oplus Kh \oplus Ke_{011}. \tag{13.22}$$

Next, define elements in W_1:

$$x_n := e_{3-n,n-1,1} + e_{2-n,n,2} \quad \text{for } 0 \leq n < 3. \tag{13.23}$$

It is easy to verify the following relations:

$$\begin{aligned}
[e_{102}, x_n] &= (n-1)x_{n-1} \ \ (= 0 \text{ if } n \leq 1), \\
[e_{012}, x_n] &= (n-1)x_n, \\
[e_{101}, x_n] &= (2-n)x_n, \\
[e_{011}, x_n] &= -(n+1)x_{n+1} \ \ (= 0 \text{ if } n \geq 2), \\
[x_0, x_1] &= 0, \\
[x_0, x_2] &= 0, \\
[x_1, x_2] &= 0.
\end{aligned}$$

From the relations above it follows that $\mathfrak{s}$ defined by

$$\mathfrak{s} := \mathfrak{sl}_2(K) \oplus K(e_{012} + e_{101}) \oplus \bigoplus_{i=0}^{2} Kx_i \oplus K(e_{211} + e_{122}) \oplus Ke_{121} \oplus Ke_{212} \oplus W_{\geq 3} \quad (13.24)$$

is a restricted Lie subalgebra of $W_{\geq 0}$. Moreover, $\mathfrak{a}$ defined by

$$\mathfrak{a} := K(e_{012} + e_{101}) \oplus \bigoplus_{i=0}^{2} Kx_i \oplus K(e_{211} + e_{122}) \oplus Ke_{121} \oplus Ke_{212} \oplus W_{\geq 3} \quad (13.25)$$

is an ideal in $\mathfrak{s}$ with $\chi(\mathfrak{a}) = 0$ (in order to show the ideal property we use in particular that $[e_{102}, e_{012} + e_{101}] = 0 = [e_{011}, e_{012} + e_{101}]$).

It is well known that irreducible $U_\chi(\mathfrak{s})$–modules annihilated by $\mathfrak{a}$ are in one to one correspondence with irreducible $U_\chi(\mathfrak{s}/\mathfrak{a}) \simeq U_\chi(\mathfrak{sl}_2(K))$–modules. [Any irreducible $U_\chi(\mathfrak{sl}_2(K))$–module X extends to $\mathfrak{s}$ if we define $\mathfrak{a} \cdot X = 0$. On the other hand: Any irreducible $U_\chi(\mathfrak{s})$–module is an irreducible $U_\chi(\mathfrak{sl}_2(K))$–module. So we can think of irreducible $U_\chi(\mathfrak{s})$–modules annihilated by $\mathfrak{a}$ as irreducible $U_\chi(\mathfrak{sl}_2(K))$–modules extended to $\mathfrak{s}$ with trivial $\mathfrak{a}$–action.]

Now the next results are essential in the description of the irreducible $U_\chi(W_{\geq 0})$–modules.

Lemma 13.13.5. *If M is a $U_\chi(W_{\geq 0})$–module and $M \neq 0$, then*

$$\{x \in M \mid \mathfrak{a} \cdot x = 0\} \neq 0$$

and there exists an irreducible $U_\chi(\mathfrak{s})$–submodule $X \subset M$ with $\mathfrak{a} \cdot X = 0$.

Proof. Since $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a} \cap W_{011}$ there exists $U_\chi(\mathfrak{a})$–module K_l as being equal to K as a vector space and where the module structure is given by: $e \cdot 1 = 0$ for $e \in \mathfrak{a} \cap W_{011}$ and $(e_{012} + e_{101}) \cdot 1 = l$ (since $e_{012} + e_{101} \in \mathfrak{a}$ with $\chi(e_{012} + e_{101}) = 0$ we have $l \in \mathbb{F}_3$). But $\mathfrak{a} \subset W_{012}$ is supersolvable so we can apply Lemma 9.1.3: Any irreducible $U_\chi(\mathfrak{a})$–module is isomorphic to some K_l with $l \in \mathbb{F}_3$. It follows that there exists a nonzero $x \in M$ with $\mathfrak{a} \cap W_{011} \cdot x = 0$ and $(e_{012} + e_{101}) \cdot x = lx$ for some $l \in \mathbb{F}_3$. Let $y := e_{211}^l \cdot x \in M$. Then $\mathfrak{a} \cap W_{011} \cdot y = 0$ since $[e_{211}, \mathfrak{a} \cap W_{011}] \subset \mathfrak{a} \cap W_{011}$ and $(e_{012} + e_{101}) \cdot y = 0$ by construction. We conclude that $\{x \in M \mid \mathfrak{a} \cdot x = 0\} \neq 0$.

The final part of the lemma is now easy: Take a nonzero $x \in M$ with $\mathfrak{a} \cdot x = 0$ and take an irreducible $U_\chi(\mathfrak{s})$–submodule X of $U_\chi(\mathfrak{s}) \cdot x$. Since $\mathfrak{a}$ is an ideal of $\mathfrak{s}$ with $\mathfrak{a} \cdot x = 0$ we have $\mathfrak{a} \cdot U_\chi(\mathfrak{s}) \cdot x = 0$ and therefore $\mathfrak{a} \cdot X = 0$ as required. □

Proposition 13.13.6. *Let $\mathfrak{s}$ be defined as in* (13.24)*. Then induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{s})$–modules annihilated by $\mathfrak{a}$ and the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules.*

Proof. Set $e_1 = e_{201}$ and $e_2 = e_{111}$ and $e_3 = e_{021}$ and $e_4 = e_{211}$. Then e_1, e_2, e_3, e_4 form a basis for a complement to $\mathfrak{s}$ in $W_{\geq 0}$. Let X be an irreducible $U_\chi(\mathfrak{s})$–module annihilated by $\mathfrak{a}$. The idea is to prove that

$$\{x \in U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{s})} X \mid \mathfrak{a} \cdot x = 0\} = 1 \otimes X. \quad (13.26)$$

In order to prove (13.26) we will apply Proposition 6.4.1. Adopt the notation from Section 6.4 with $\mathfrak{g} = W_{\geq 0}$ and $\mathfrak{h} = \mathfrak{s}$ and $N = X$: We define

$$\begin{aligned} X_1 &= \bigoplus e_1^i e_2^j e_3^k e_4^l \otimes X, \\ X_2 &= \bigoplus e_2^j e_3^k e_4^l \otimes X, \\ X_3 &= \bigoplus e_3^k e_4^l \otimes X, \\ X_4 &= \bigoplus e_4^l \otimes X, \end{aligned}$$

where all i, j, k, l run over $\{0, 1, 2\}$. Note that $(\mathfrak{s} \oplus Ke_4) \cdot X_4 \subset X_4$ with $(\mathfrak{a} \cap W_{011}) \cdot X_4 = 0$. The first claim follows from $e_4 \cdot X_4 \subset X_4$ and $[e_4, \mathfrak{s} \oplus Ke_4] \subset Ke_4 \oplus \mathfrak{s}$ and the second claim follows from $(\mathfrak{a} \cap W_{011}) \cdot X = 0$ and $[e_4, \mathfrak{a} \cap W_{011}] \subset \mathfrak{a} \cap W_{011}$. We also have

$$(\mathfrak{s} \cap W_{012} \oplus Ke_i \oplus \cdots \oplus Ke_4) \cdot X_i \subset X_i$$

for all $i = 1, 2, 3, 4$. This follows from the fact that

$$\begin{aligned} [e_j, \mathfrak{s} \cap W_{012}] &\subset \mathfrak{s} \cap W_{012} \oplus Ke_j \oplus \cdots \oplus Ke_4 \quad \text{for any } j = 1, 2, 3, 4, \\ [e_j, e_k] &\subset \mathfrak{s} \cap W_{012} \oplus \textstyle\bigoplus_{l > j} Ke_l \quad \text{for any } k, l. \end{aligned}$$

Finally, observe that

$$\begin{aligned} (Kx_1 \oplus Kx_2 \oplus \mathfrak{a} \cap W_{\geq 2}) \cdot X_3 &= 0, \\ (Kx_2 \oplus \mathfrak{a} \cap W_{\geq 2}) \cdot X_2 &= 0, \\ \mathfrak{a} \cap W_{\geq 2} \cdot X_1 &= 0. \end{aligned}$$

We will use these observations in the following. Our aim is to prove that (13.26) holds; i.e., that any $x \in U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{s})} X$ with $\mathfrak{a} \cdot x = 0$ lies in $X_5 := 1 \otimes X$. So let $x \in U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{s})} X$ denote an element such that $\mathfrak{a} \cdot x = 0$.

Set $f_1 = x_2$ as in (13.23). Since $f_1 \in \mathfrak{a}$ it follows that $f_1 \cdot x = 0$. Moreover, $\chi([e_1, f_1]) \neq 0$ but $[e_1, f_1]^{[3]} = (-e_{211})^{[3]} = 0$. We also have $(\text{ad } e_1)^i(f_1) \cdot X_2 \subset X_2$ since $(\text{ad } e_1)^i(f_1) \in (\mathfrak{s} \cap W_{012}) \oplus Ke_4$ for all i. Finally, $f_1 \cdot X_2 = 0$.

Next, set $f_2 = x_1$. Then $f_2 \cdot x = 0$ and $\chi([e_2, f_2]) \neq 0 = [e_2, f_2]^{[3]} = e_{122}^{[3]}$. Since $(\text{ad } e_2)^i(f_2) \in (\mathfrak{s} \cap W_{012}) \oplus Ke_4$ for all i we also have $(\text{ad } e_2)^i(f_2) \cdot X_3 \subset X_3$. Finally, $f_2 \cdot X_3 = 0$.

Set $f_3 = x_0$. Then $f_3 \cdot x = 0$ and $\chi([e_3, f_3]) \neq 0 = [e_3, f_3]^{[3]} = (e_{211} - e_{122})^{[3]}$. Since $(\text{ad } e_3)^i(f_3) \in (\mathfrak{s} \cap W_{012}) \oplus Ke_4$ for all i we also have $(\text{ad } e_3)^i(f_3) \cdot X_4 \subset X_4$. Finally, $f_3 \cdot X_4 = 0$.

Finally, set $e_4 = e_{012} + e_{101}$ and note that $[e_4, f_4] = e_4$. We are now in position to apply Corollary 6.4.3 (with e_1, e_2, e_3, e_4 and f_1, f_2, f_3, f_4 and G, H, N defined above): We find that

$$\{x \in U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X \mid \mathfrak{a} \cdot x = 0\} \subset 1 \otimes X$$

and since $\mathfrak{a} \cdot X = 0$ the other conclusion is clear. We conclude that (13.26) holds.

This implies that $U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{s})} X$ is irreducible: Any irreducible $W_{\geq 0}$–submodule M has a nonzero intersection with $1 \otimes X$ [Apply Lemma 13.13.5]. Therefore $M \cap (1 \otimes X)$ is a nonzero $U_\chi(\mathfrak{s})$–submodule of $1 \otimes X$ and, by irreducibility, $M \cap (1 \otimes X) = 1 \otimes X$. In particular, we have $M \supset 1 \otimes X$ and hence M is the entire induced module.

If X_1, X_2 are irreducible $U_\chi(W_{\geq 0})$–modules such that $\mathfrak{a} \cdot X_1 = 0 = \mathfrak{a} \cdot X_2$ and

$$\varphi : U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{s})} X_1 \simeq U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{s})} X_2$$

is an isomorphism, then φ induces a $U_\chi(\mathfrak{s})$–isomorphism $\overline{\varphi} : X_1 \simeq X_2$. Indeed, we have $\varphi(1 \otimes X_1) \cap (1 \otimes X_2) \neq 0$. (Look at the elements annihilated by $\mathfrak{a}$.) Since $\varphi(1 \otimes X_1)$ and $1 \otimes X_2$ are irreducible $U_\chi(W_{\geq 0})$–modules, we get $\varphi(1 \otimes X_1) = 1 \otimes X_2$; hence $X_1 \simeq X_2$.

We have thus shown: Induction induces an injection from the isomorphism classes of irreducible $U_\chi(\mathfrak{s})$–modules annihilated by $\mathfrak{a}$ into the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules.

Now, let Y be an arbitrary irreducible $U_\chi(\mathfrak{s})$–module. I claim that we can find an irreducible $U_\chi(\mathfrak{s})$–module X with $\mathfrak{a} \cdot X = 0$ and

$$U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{s})} X \longrightarrow U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{s})} Y.$$

First, apply Lemma 13.13.5 to find an irreducible $U_\chi(\mathfrak{s})$–submodule $X \subset U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{s})} Y$ with $\mathfrak{a} \cdot X = 0$; thus we have inclusion maps: $X \hookrightarrow U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{s})} Y$. Now apply 'Frobenius reciprocity' on the inclusion $X \hookrightarrow U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{s})} Y$ to produce a (nonzero) $U_\chi(W_{\geq 0})$–homomorphism:

$$U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{s})} X \longrightarrow U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{s})} Y. \tag{13.27}$$

This implies that every $U_\chi(W_{\geq 0})$–module is induced from a $U_\chi(\mathfrak{s})$–module annihilated by $\mathfrak{a}$: Indeed, any irreducible $U_\chi(W_{\geq 0})$–module V contains an irreducible $U_\chi(\mathfrak{s})$–module Y; hence, by 'Frobenius reciprocity', V is a homomorphic image of $U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{s})} Y$ and by (13.27) then also a homomorphic image of $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X$ for some irreducible $U_\chi(\mathfrak{s})$–module X with $\mathfrak{a} \cdot X = 0$. By the part of the claim already proved we therefore have $V \simeq U_\chi(W_{\geq 0}) \otimes_{U_\chi(\mathfrak{s})} X$. The proof is completed. □

Theorem 13.13.7. *Let $\chi \in W^*$ be a character of height 3 and of* Type II.a *as in* 5.2. *Suppose that $\chi(W_1) = 0 = \chi(e_{012} + e_{101})$.*

a) If $\chi(h)^2 + \chi(e_{102})\chi(e_{011}) \neq 0$ then rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 1$ *and* $\dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 6$ *and there exist up to isomorphism 3 irreducible $U_\chi(W_{\geq 0})$–modules of dimension $3^5 = 3^{\operatorname{codim}_{W_{\geq 0}} \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})/2}$. Moreover, $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) \not\subset W_{012}$ if and only if $\chi(e_{011}) \neq 0$.*

Suppose that $\chi(h)^2 + \chi(e_{102})\chi(e_{011}) = 0$.

b) If $\chi(h) \neq 0$ or $\chi(e_{102}) \neq 0$ or $\chi(e_{011}) \neq 0$ then rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 0$ *and we have* $\dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 6$ *and there exist up to isomorphism 2 irreducible $U_\chi(W_{\geq 0})$–modules of dimension $3^5 = 3^{\operatorname{codim}_{W_{\geq 0}} \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})/2}$. Moreover, $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) \not\subset W_{012}$ if and only if $\chi(e_{011}) \neq 0$.*

c) If $\chi(h) = \chi(e_{102}) = \chi(e_{011}) = 0$ then rk $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 1$ *and* $\dim_K \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) = 8$ *and there exist up to isomorphism 3 irreducible $U_\chi(W_{\geq 0})$–modules of dimension $3^4 = 3^{\operatorname{codim}_{W_{\geq 0}} \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})/2}$ and $2 \cdot 3^4 = 2 \cdot 3^{\operatorname{codim}_{W_{\geq 0}} \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})/2}$ and finally $3^5 = 3^{\operatorname{codim}_{W_{\geq 0}} \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})/2+1}$. Moreover, $\mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}}) \not\subset W_{012}$.*

Proof. It follows from Proposition 13.13.6 that induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{s})$–modules annihilated by $\mathfrak{a}$ and the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules. But irreducible $U_\chi(\mathfrak{s})$–modules annihilated by $\mathfrak{a}$ are just irreducible $U_\chi(\mathfrak{s}/\mathfrak{a}) \simeq U_\chi(\mathfrak{sl}_2(K))$–modules extended to $\mathfrak{s}$ with trivial $\mathfrak{a}$–action.

If we apply the description in [27, 5, 5.2] on $\mathfrak{sl}_2(K)$ defined in (13.22), we see that there are 3 isomorphism classes of irreducible $U_\chi(\mathfrak{sl}_2(K))$–modules if $\chi(\mathfrak{sl}_2(K)) = 0$ or $\chi(h)^2 + \chi(e_{102})\chi(e_{011}) \neq 0$ and 2 isomorphism classes of irreducible $U_\chi(\mathfrak{sl}_2(K))$–modules if $\chi(h)^2 + \chi(e_{102})\chi(e_{011}) = 0$. If $\chi(\mathfrak{sl}_2(K)) \neq 0$ then each irreducible $U_\chi(\mathfrak{sl}_2(K))$–module has dimension 3 and if $\chi(\mathfrak{sl}_2(K)) = 0$ then there exist 3 irreducible $U_\chi(\mathfrak{sl}_2(K))$–modules of dimension $1, 2, 3$. Now it is straightforward to verify a)–c) by using Proposition 13.13.6 and Lemma 13.13.2. □

Note that $\mathfrak{st}(\chi, W_{\geq 3}) = W_{\geq 0}$ since $\chi([e_{001}, e_{222}]) \neq 0 = \chi([e_{002}, e_{222}])$ and since $\chi([e_{002}, e_{221}]) \neq 0 = \chi([e_{001}, e_{221}])$. It follows that induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and the isomorphism classes of irreducible $U_\chi(W)$–modules. See Theorem 8.1.1. Moreover, $\mathfrak{c}_W(\chi) \subset \mathfrak{c}_{W_{\geq 0}}(\chi_{|W_{\geq 0}})$ since $\mathfrak{c}_W(\chi) \subset \mathfrak{st}(\chi, W_{\geq 3})$. Now apply Lemma 10.4.7 and Theorem 13.13.7 and get:

Theorem 13.13.8. *Let $\chi \in W^*$ be a character of height 3 and of* Type II.a *as in* 5.2. *Suppose that $\chi(W_1) = 0 = \chi(e_{012} + e_{101})$.*

a) If $\chi(h)^2 + \chi(e_{102})\chi(e_{011}) \neq 0$ then $\dim_K \mathfrak{c}_W(\chi) = 4$ and there exist up to isomorphism 3 irreducible $U_\chi(W)$–modules of dimension $3^7 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$.

Suppose that $\chi(h)^2 + \chi(e_{102})\chi(e_{011}) = 0$.

b) If $\chi(h) \neq 0$ or $\chi(e_{102}) \neq 0$ or $\chi(e_{011}) \neq 0$ then $\dim_K \mathfrak{c}_W(\chi) = 4$ and there exist up to isomorphism 2 irreducible $U_\chi(W)$–modules of dimension $3^7 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$.

c) If $\chi(h) = \chi(e_{102}) = \chi(e_{011}) = 0$ then $\dim_K \mathfrak{c}_W(\chi) = 6$ and there exist up to isomorphism 3 irreducible $U_\chi(W)$–modules of dimension $3^6 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$ and $2 \cdot 3^6 = 2 \cdot 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$ and $3^7 = 3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2+1}$. Moreover, $\mathfrak{c}_W(\chi) \not\subset W_{012}$.

Remark 13.13.9. Note that Theorem 13.13.8 says in particular that Theorem 10.4.11, 10.4.12 in Section 7 cannot be improved to include characters of height $2p-3$ and of Type II.a as in 5.2 [characters of height $2p-3$ and of Type II.a as in Section 5.2 are excluded in most of the results in Section 10.4].

We also see that Theorem 7.3.2 does not hold for all characters of height $2p-3$ and of Type II.a as in 5.2: So there does not exists an automorphism g such that induction is a bijection between the isomorphism classes of irreducible $U_{\chi^g}(W_{012})$–modules and the isomorphism classes of irreducible $U_{\chi^g}(W_{\geq 0})$–modules. First, let us investigate what we can say about the irreducible $U_\chi(W_{\geq 0})$–modules if there exists an automorphism g with these properties. Since W_{012} is supersolvable it follows from Proposition 9.3.5 that the number of irreducible $U_{\chi^g}(W_{012})$–modules is p^l for some integer l (in fact we have $0 \leq l \leq 2$). By Lemma 9.3.7 the number of irreducible $U_{\chi^g}(W_{012})$–modules is p^m for some integer $m \geq 0$. Now use the assumption on g and the fact that $U_{\chi^g}(W_{\geq 0}) \simeq U_\chi(W_{\geq 0})$ to get:

1) There exists an integer l with $0 \leq l \leq 2$ such that the number of irreducible $U_\chi(W_{\geq 0})$–modules is p^l.

2) There exists an integer $m \geq 1$ such that the dimension of any irreducible $U_\chi(W_{\geq 0})$–modules is p^m.

We have seen that both 1) and 2) break down if we consider χ of Type II.a as in 5.2 of height $r = 2p-3$ (at least for $p = 3$).

14 Maximal height

In this section we consider $\chi \in W^*$ of maximal height (i.e., $\chi \in W^*$ with $\chi(W_{2p-3}) \neq 0$). The representation theory of χ with maximal height is not very well understood. So far, I have only seen examples in [29, §12], [16, p.45–46] and [20, 6.4], where one constructs χ such that irreducible $U_\chi(W_{\geq 0})$–modules have dimension p^{p^2-1} and then concludes that any irreducible $U_\chi(W_{\geq 0})$–module extends to W by Mil'ner's result [19, §7, Remark after Prop. 19]. In Section 14.4 we shall see an example (when $p=3$) where some irreducible $U_\chi(W)$–modules have dimension $< 3^{3^2-1}$. From now we will assume that $p=3$.

14.1 Representatives

First, we need to find certain representatives for the $\mathrm{Aut}(W)$–orbit of χ.

Lemma 14.1.1. *If χ has maximal height, then χ is conjugate under $\mathrm{Aut}(W)$ to a character χ' with $\chi'(e_{222}) = 1$ and $\chi'(e_{221}) = 0$ such that one of the following situations occur:*

(M1) $\chi'(W_2) = \chi'(e_{202}) = \chi'(e_{112}) = \chi'(e_{022}) = 0$ and $\chi'(e_{201}) = \chi'(e_{111}) = 0 \neq \chi'(e_{021})$ and $\chi'(e_{102}) = \chi'(e_{012}) = 0$.

(M2) $\chi'(W_2) = \chi'(e_{201}) = \chi'(e_{111}) = \chi'(e_{021}) = 0$ and $\chi'(e_{202}) = \chi'(e_{112}) = 0 \neq \chi'(e_{022})$ and $\chi'(e_{102}) = \chi'(e_{012}) = 0$.

(M3) $\chi'(W_2) = \chi'(W_1) = 0$ and $\chi'(e_{102}) = \chi'(e_{012}) = 0$.

Proof. We can assume that $\chi(e_{222}) = 1$ and $\chi(e_{221}) = 0 = \chi(W_2)$ by Lemma 12.3.3. If $\chi(e_{021}) \neq 0$, then we can apply a lower triangular matrix

$$\varphi_1 = \begin{pmatrix} 1 & 0 \\ \frac{\chi(e_{022})-\chi(e_{111})}{2\chi(e_{021})} & 1 \end{pmatrix}$$

and (A.6),(A.7) to get $\chi^{\varphi_1}(e_{022}) = \chi^{\varphi_1}(e_{111})$. We still have $\chi^{\varphi_1}(e_{222}) = 1$ and $\chi^{\varphi_1}(e_{221}) = 0 = \chi^{\varphi_1}(W_2)$. So we can assume that $\chi(e_{222}) = 1$ and $\chi(e_{221}) = 0 = \chi(W_2)$ and that $\chi(e_{111}) = \chi(e_{022})$ if $\chi(e_{021}) \neq 0$.

Now, let

$$x = a_1 e_{212} + a_2 e_{122} + b_1 e_{211} + b_2 e_{121}$$

and denote by g the automorphism on W induced by x (see Section 3.2). It follows that $g(y) = y + [x, y]$ for all $y \in W_2$ since $W_{\geq 4} = 0$. But $\chi^g\big([x, W_{\geq 2}]\big) = 0 = \chi^g\big([x, e_{021}]\big)$ and

$$\begin{aligned} \chi^g\big([x, e_{201}]\big) &= -a_2, \\ \chi^g\big([x, e_{111}]\big) &= a_1, \\ \chi^g\big([x, e_{202}]\big) &= -b_2, \\ \chi^g\big([x, e_{112}]\big) &= b_1 - a_2, \\ \chi^g\big([x, e_{022}]\big) &= a_1, \end{aligned}$$

so there exist appropriate $a_1, a_2, b_1, b_2 \in K$ such that $\chi^g(e_{222}) = 1$ and $\chi^g(e_{221}) = \chi^g(W_2) = 0$ and $\chi^g(e_{201}) = \chi^g(e_{111}) = \chi^g(e_{202}) = \chi^g(e_{112}) = 0$. Moreover, $\chi^g(e_{022}) = \chi^g(e_{111}) = 0$ if $\chi(e_{021}) \neq 0$. Finally, we apply a suitable automorphism g' on W induced by some $x = a e_{222} + b e_{221}$ (Section 3.2) such that $\chi^{g \circ g'}(e_{102}) = \chi^{g \circ g'}(e_{012}) = 0$; this can be done since

$$\begin{aligned} \chi^{g \circ g'}\big([x, e_{102}]\big) &= b, \\ \chi^{g \circ g'}\big([x, e_{012}]\big) &= -a. \end{aligned}$$

It follows that $\chi^{g \circ g'}(e_{222}) = 1$ and $\chi^{g \circ g'}(e_{221}) = 0$ and that $\chi^{g \circ g'}$ either satisfies (M1), (M2) or (M3). $\square$

14.2 Three types of characters

We consider three types of characters:

(M1) $\chi : \chi(e_{222}) = 1$ and $\chi(e_{221}) = 0$ and $\chi(W_2) = \chi(e_{202}) = \chi(e_{112}) = \chi(e_{022}) = 0$ and $\chi(e_{201}) = \chi(e_{111}) = 0 \neq \chi(e_{021})$ and $\chi(e_{102}) = \chi(e_{012}) = 0$.

(M2) $\chi : \chi(e_{222}) = 1$ and $\chi(e_{221}) = 0$ and $\chi(W_2) = \chi(e_{201}) = \chi(e_{111}) = \chi(e_{021}) = 0$ and $\chi(e_{202}) = \chi(e_{112}) = 0 \neq \chi(e_{022})$ and $\chi(e_{102}) = \chi(e_{012}) = 0$.

(M3) $\chi : \chi(e_{222}) = 1$ and $\chi(e_{221}) = 0$ and $\chi(W_2) = \chi(W_1) = 0$ and $\chi(e_{102}) = \chi(e_{012}) = 0$.

The result in the previous section says that any character of maximal height is conjugate under $\mathrm{Aut}(W)$ to a character of Type M1, Type M2 or Type M3.

So let χ be a character of Type M1, Type M2 or Type M3. We will find $\dim_K \mathfrak{c}_W(\chi)$ and rk $\mathfrak{c}_W(\chi)$ for $\chi \in W^*$ of maximal height. Let

$$y = \sum_{(\alpha\beta\gamma)} a_{\alpha\beta\gamma} e_{\alpha\beta\gamma} \in \mathfrak{c}_W(\chi). \tag{14.1}$$

The possible values for $a_{\alpha\beta\gamma}$ can be found from $\chi([y, e_{abc}]) = 0$, where (abc) runs over all valid triples (i.e., $(002) \preceq (abc) \preceq (221)$). We find:

1) $a_{102} = a_{112} = a_{121} = 0$.

2) $a_{012} = a_{101}$.

3) $a_{202} + a_{001}\chi(e_{021}) = 0$.

4) $a_{201} + a_{001}\chi(e_{022}) = 0$.

5) $a_{022} + a_{111} = 0$.

6) $a_{002}\chi(e_{011}) = a_{012}\chi(e_{021})$.

7) $a_{012}\chi(e_{022}) - a_{011}\chi(e_{021}) + a_{212} = 0$.

8) $a_{001}\chi(e_{011}) + a_{002}\chi(e_{101}) + a_{011}\chi(e_{021}) + a_{212} = 0$.

9) $a_{211} - a_{122} + a_{011}\chi(e_{022}) = 0$.

10) $a_{122} + a_{001}\chi(e_{101}) = 0$.

11) $a_{002}\chi(e_{001}) - a_{111}\chi(e_{021}) = 0$.

12) $a_{001}\chi(e_{001}) + a_{011}\chi(e_{011}) + a_{021}\chi(e_{021}) + a_{222} = 0$.

13) $a_{002}\chi(e_{002}) - a_{011}\chi(e_{011}) + a_{021}\chi(e_{021}) - a_{022}\chi(e_{022}) - a_{222} = 0$.

14) $a_{001}\chi(e_{002}) - a_{011}\chi(e_{101}) + a_{021}\chi(e_{022}) + a_{221} = 0$.

15) $a_{101}\chi(e_{001}) - a_{201}\chi(e_{101}) + a_{111}\chi(e_{011}) + a_{122}\chi(e_{022}) = 0$.

16) $a_{012}\chi(e_{002}) + a_{011}\chi(e_{001}) + a_{111}\chi(e_{101}) - a_{021}\chi(e_{011}) = 0$.

First, we will use the relations above to get results on $\mathfrak{c}_W(\chi) \cap W_{\geq 0}$ for χ of Type M1, Type M2 or Type M3.

Lemma 14.2.1. *If* χ *has* Type M1, *then* $\mathfrak{c}_W(\chi) \cap W_{\geq 0} = 0$.

Proof. Let y be as in (14.1) with $a_{002} = 0 = a_{001}$. Then $a_{201} = 0 = a_{202}$ by 3),4) and 2) and 6) imply that $a_{012} = 0 = a_{101}$. Next, use 5) and 7)–11) to get $a_{011} = a_{022} = a_{111} = a_{211} = a_{122} = a_{212} = 0$. Finally, 12)–14) imply that $a_{021} = a_{222} = a_{221} = 0$. It follows that $y = 0$ if $a_{002} = 0 = a_{001}$. □

Lemma 14.2.2. *If* χ *has* Type M2, *then*

$$\dim_K \mathfrak{c}_W(\chi) \cap W_{\geq 0} = \begin{cases} 2 & \textit{if } \chi(e_{001}) = 0 = \chi(e_{011}), \\ 1 & \textit{otherwise}, \end{cases}$$

and $\operatorname{rk} \mathfrak{c}_W(\chi) \cap W_{\geq 0} = 0$.

Proof. Let y be given as in (14.1) with $a_{002} = 0 = a_{001}$. Then $a_{202} = 0 = a_{201}$ by 3),4) and $a_{212} = a_{122} = a_{012} = a_{101} = 0$ by 2), 7), 8), 10). In particular, $\mathfrak{c}_W(\chi) \cap W_{\geq 0} \subset W_{011}$ and so $\operatorname{rk} \mathfrak{c}_W(\chi) \cap W_{\geq 0} = 0$. If we add 12) and 13) we get $a_{022} = 0$ since $\chi(e_{022}) \neq 0$; hence $a_{111} = 0$ by 5). The only conditions on the coefficients are now:

$$\begin{aligned} a_{011}\chi(e_{011}) + a_{222} &= 0, \\ a_{011}\chi(e_{101}) - a_{021}\chi(e_{022}) - a_{221} &= 0, \\ a_{011}\chi(e_{001}) - a_{021}\chi(e_{011}) &= 0, \\ a_{211} + a_{011}\chi(e_{022}) &= 0. \end{aligned}$$

The claim on the dimension follows. □

Lemma 14.2.3. *If* χ *has* Type M3, *then*

$$\dim_K \mathfrak{c}_W(\chi) \cap W_{\geq 0} = \begin{cases} 2 & \textit{if } \chi(e_{011}) \neq 0 \textit{ or } \chi(e_{011}) = 0 \neq \chi(e_{001}), \\ 3 & \textit{if } \chi(e_{011}) = \chi(e_{001}) = 0 \neq \chi(e_{101}), \\ 3 & \textit{if } \chi(e_{011}) = \chi(e_{001}) = \chi(e_{101}) = 0 \textit{ but } \chi(e_{002}) \neq 0, \\ 4 & \textit{if } \chi(e_{011}) = \chi(e_{001}) = \chi(e_{101}) = \chi(e_{002}) = 0, \end{cases}$$

and

$$\operatorname{rk} \mathfrak{c}_W(\chi) \cap W_{\geq 0} = \begin{cases} 1 & \textit{if } \chi(e_{011}) \neq 0, \\ 0 & \textit{if } \chi(e_{011}) = 0 \neq \chi(e_{001}), \\ 1 & \textit{if } \chi(e_{011}) = \chi(e_{001}) = 0 \neq \chi(e_{101}), \\ 0 & \textit{if } \chi(e_{011}) = \chi(e_{001}) = \chi(e_{101}) = 0 \textit{ but } \chi(e_{002}) \neq 0, \\ 1 & \textit{if } \chi(e_{011}) = \chi(e_{001}) = \chi(e_{101}) = \chi(e_{002}) = 0, \end{cases}$$

Proof. Let y be given as in (14.1) with $a_{002} = 0 = a_{001}$. Then we use 3),4) and 8)–10) to get $a_{202} = a_{201} = a_{212} = a_{122} = a_{211} = 0$. Then $a_{012} = a_{101}$ and:

$$\begin{aligned} a_{022} + a_{111} &= 0, \\ a_{011}\chi(e_{011}) + a_{222} &= 0, \\ a_{011}\chi(e_{101}) - a_{221} &= 0, \\ a_{101}\chi(e_{002}) + a_{011}\chi(e_{001}) - a_{021}\chi(e_{011}) + a_{111}\chi(e_{101}) &= 0, \\ a_{101}\chi(e_{001}) + a_{111}\chi(e_{011}) &= 0. \end{aligned}$$

If we compute the rank of the matrix determined by the system of equations the claim on the dimension of $\mathfrak{c}_W(\chi) \cap W_{\geq 0}$ follows immediately. It is easy to see that $a_{012} = a_{101} = 0$ if $\chi(e_{011}) = 0 \neq \chi(e_{001})$ or $\chi(e_{011}) = \chi(e_{001}) = \chi(e_{101}) = 0$ but $\chi(e_{002}) \neq 0$. If $\chi(e_{011}) = 0 \neq \chi(e_{001})$ or $\chi(e_{011}) = \chi(e_{001}) = \chi(e_{101}) = \chi(e_{002}) = 0$, then there exists nonzero toral element $h \in \mathfrak{c}_W(\chi) \cap W_{\geq 0}$; hence $\operatorname{rk} \mathfrak{c}_W(\chi) \cap W_{\geq 0} = 1$ since already $\operatorname{rk} \mathfrak{c}_W(\chi) \cap W_{\geq 0} \leq 1$. □

In order to compute $\mathfrak{c}_W(\chi)$ note that we have

$$\dim_K \mathfrak{c}_W(\chi) \le \dim_K \mathfrak{c}_W(\chi) \cap W_{\ge 0} + 2 \tag{14.2}$$

since $W_{\ge 0} \subset W$ is a subalgebra of codimension 2. It is well known that $\mathfrak{c}_W(\chi)$ is a Lie p–subalgebra of W and its codimension in W is even. We also have $\mathfrak{c}_W(\chi) \ne 0$: Indeed, there exists $\xi \in W^*$ such that $\mathfrak{c}_W(\xi)$ is a two dimensional torus and, by [20, 4.4], it then follows that $\dim_K \mathfrak{c}_W(\zeta) \ge 2$ for all $\zeta \in W^*$.

We will consider characters of Type M1 and prove that all irreducible $U_\chi(W)$–modules have maximal dimension (irreducible $W_{\ge 0}$–modules with p–character χ extend to W). Finally, we consider irreducible $U_\chi(W)$–modules for χ of Type M3. We cannot give a complete classification, but as a result of our computations we will see that: There exists χ of Type M3 such that irreducible $U_\chi(W)$–modules have non–maximal dimension (i.e., dimension $< 3^{3^2-1} = 3^8$).

14.3 Type M1 characters

First, let us compute $\mathfrak{c}_W(\chi)$.

Lemma 14.3.1. *If χ has* Type M1*, then $\mathfrak{c}_W(\chi)$ is 2–dimensional and $\mathfrak{c}_W(\chi) \cap W_{\ge 0} = 0$.*

Proof. Since $\mathfrak{c}_W(\chi)$ has even dimension > 0, we can apply (14.2) and find that $\mathfrak{c}_W(\chi)$ is 2–dimensional. The statement follows from Lemma 14.2.1. □

We can now apply Theorem 12.3.4:

Theorem 14.3.2. *If $\chi \in W^*$ has* Type M1*, then there exists (up to isomorphism) one irreducible $U_\chi(W_{\ge 0})$–module of dimension 3^8. For any irreducible $U_\chi(W_{\ge 0})$–module S there exists a W–module structure on S which extends the given $W_{\ge 0}$–module structure. In particular, all irreducible $U_\chi(W)$–modules have dimension*

$$3^{\mathrm{codim}_W \mathfrak{c}_W(\chi)/2} = 3^8.$$

Remark 14.3.3. I can't say anything about the number of irreducible $U_\chi(W)$–modules. Of course, the number is less than or equal to p^2 and equal to p^2 if and only if $U_\chi(W)$ is semisimple.

14.4 Type M3 characters

There are a number of cases to consider. First, let us see what we can say in general: If $y \in \mathfrak{c}_W(\chi)$ as in (14.1), then $a_{102} = a_{121} = a_{112} = 0$ and $a_{201} = 0 = a_{202}$ by the relations 1), 3) and 4) in Section 14.2 (since χ has Type M3 we have $\chi(e_{022}) = 0 = \chi(e_{021})$). Moreover, $a_{212} = 0$ and $a_{211} = a_{122}$ by 7) and 9).

a) If $\chi(e_{011}) \ne 0$, then $a_{001} = 0 = a_{002}$ by 6) and 8). Therefore, $\mathfrak{c}_W(\chi) = \mathfrak{c}_W(\chi) \cap W_{\ge 0}$. It follows that $\dim_K \mathfrak{c}_W(\chi) = 2$ and rk $\mathfrak{c}_W(\chi) = 1$ by Lemma 14.2.3.

b) If $\chi(e_{011}) = 0$ but $\chi(e_{001}) \ne 0$, then $a_{001} = 0 = a_{002}$ by 11)–13). Therefore, $\mathfrak{c}_W(\chi) = \mathfrak{c}_W(\chi) \cap W_{\ge 0}$. It follows that $\dim_K \mathfrak{c}_W(\chi) = 2$ and rk $\mathfrak{c}_W(\chi) = 0$ by Lemma 14.2.3.

c) If $\chi(e_{011}) = \chi(e_{001}) = 0 \neq \chi(e_{101})$, then $\dim_K \mathfrak{c}_W(\chi) \leq 5$ by Lemma 14.2.3 and (14.2). This implies that $\dim_K \mathfrak{c}_W(\chi) = 4$ and it is easy to see that

$$\mathfrak{c}_W(\chi) = K(e_{001} - \chi(e_{101})e_{211} - \chi(e_{101})e_{122} - \chi(e_{002})e_{221}) \oplus (\mathfrak{c}_W(\chi) \cap W_{\geq 0}).$$

It follows that rk $\mathfrak{c}_W(\chi) = 1$.

d) If $\chi(e_{001}) = \chi(e_{101}) = \chi(e_{011}) = 0$ but $\chi(e_{002}) \neq 0$, then $\dim_K \mathfrak{c}_W(\chi) \leq 5$ by Lemma 14.2.3 and (14.2). This implies that $\dim_K \mathfrak{c}_W(\chi) = 4$ and it is easy to see that $e_{001} - \chi(e_{002})e_{221}, e_{011}, e_{111} - e_{022}, e_{021}$ form a basis for $\mathfrak{c}_W(\chi)$. It follows that rk $\mathfrak{c}_W(\chi) = 0$.

e) If $\chi(e_{001}) = \chi(e_{101}) = \chi(e_{011}) = \chi(e_{002}) = 0$, then $\dim_K \mathfrak{c}_W(\chi) = 6$: It is easy to see that $e_{001}, e_{002}, e_{012} + e_{101}, e_{011}, e_{111} - e_{022}, e_{021}$ form a basis for $\mathfrak{c}_W(\chi)$. It follows that rk $\mathfrak{c}_W(\chi) = 1$.

We can now write down our observations:

Lemma 14.4.1. *If χ has* Type M3*, then*

$$\dim_K \mathfrak{c}_W(\chi) = \begin{cases} 2 & \text{if } \chi(e_{011}) \neq 0 \text{ or } \chi(e_{011}) = 0 \neq \chi(e_{001}), \\ 4 & \text{if } \chi(e_{011}) = \chi(e_{001}) = 0 \neq \chi(e_{101}), \\ 4 & \text{if } \chi(e_{011}) = \chi(e_{001}) = \chi(e_{101}) = 0 \text{ but } \chi(e_{002}) \neq 0, \\ 6 & \text{if } \chi(e_{011}) = \chi(e_{001}) = \chi(e_{101}) = \chi(e_{002}) = 0, \end{cases}$$

and

$$\text{rk } \mathfrak{c}_W(\chi) = \begin{cases} 1 & \text{if } \chi(e_{011}) \neq 0, \\ 0 & \text{if } \chi(e_{011}) = 0 \neq \chi(e_{001}), \\ 1 & \text{if } \chi(e_{011}) = \chi(e_{001}) = 0 \neq \chi(e_{101}), \\ 0 & \text{if } \chi(e_{011}) = \chi(e_{001}) = \chi(e_{101}) = 0 \text{ but } \chi(e_{002}) \neq 0, \\ 1 & \text{if } \chi(e_{011}) = \chi(e_{001}) = \chi(e_{101}) = \chi(e_{002}) = 0, \end{cases}$$

Theorem 14.4.2. *If χ has* Type M3 *and $\chi(e_{011}) \neq 0$, then there exist 3 irreducible $U_\chi(W)$–modules all of dimension*

$$3^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2} = 3^8.$$

Proof. Let $\mathfrak{g}$ be defined as in Section 11.8 (i.e., all e_{ijk} where $(ijk) \neq (002), (102), (202)$ form a basis for $\mathfrak{g}$). Let $\mathfrak{h} = \mathfrak{g} \cap W_{\geq 0}$.

First, we prove that induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{h})$–modules and the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules. We shall apply Theorem 6.2.8 in Section 6.2 with

$$\mathfrak{a} = Ke_{111} \oplus Ke_{021} \oplus Ke_{022} \oplus Ke_{121} \oplus Ke_{122} \oplus Ke_{211} \oplus Ke_{221} \oplus Ke_{222}.$$

All irreducible $U_\chi(\mathfrak{g})$–modules have eigenvalue function χ since $\mathfrak{a}$ is unipotent with $\chi(\mathfrak{a}) = 0$ (see Definition 6.2.6). It is easy to verify that $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ with $\chi([\mathfrak{a}, \mathfrak{a}]) = 0$. We also have $\mathfrak{st}(\chi, \mathfrak{a}) = \mathfrak{h}$: Indeed, consider $y \in \mathfrak{st}(\chi, \mathfrak{a})$ given by $y = e_{001} + x$ for some $x \in \mathfrak{h}$. Since $\chi([y, e_{022}]) = 0 = \chi([e_{001}, e_{022}])$ we have $\chi([e_{022}, x]) = 0$. Therefore,

$$x \in \mathfrak{h} \cap W_0 + \mathfrak{h} \cap W_1 + Ke_{211} + Ke_{121} + Ke_{122} + W_3.$$

This implies that $\chi([x, e_{111}]) = 0$, which is a contradiction since $\chi([e_{001}, e_{111}]) \neq 0$. We can now apply Theorem 6.2.8.

Since $\mathfrak{h}$ is supersolvable we can determine the irreducible $U_\chi(\mathfrak{h})$–modules from the Vergne polarization of χ with respect to the chain (11.10) in Section 11.7: One can show that

$$\mathfrak{p}_{\chi|\mathfrak{h}} = K(e_{012} + e_{101}) \oplus Ke_{011} \oplus K(e_{022} - e_{111}) \oplus Ke_{021} \oplus W_{\geq 2}$$

and from the construction in the proof of Lemma 9.3.9, there exists $\lambda \in \mathfrak{h}^*$ such that the Vergne polarization $\mathfrak{p}_\lambda$ of λ is compatible with χ and equal to $\mathfrak{p}_{\chi|\mathfrak{h}}$. Thus we obtain from Lemma 9.3.6, 9.3.7: There exist (up to isomorphism) 3 irreducible $U_\chi(\mathfrak{h})$–modules all of dimension 3^4. So, by the first part of the proof, there exist (up to isomorphism) 3 irreducible $U_\chi(\mathfrak{g})$–modules all of dimension 3^5.

Next, we will use Theorem 6.3.3 in Section 6.3 and study the induction functor from $\mathfrak{g}$ to W. Set $\mathfrak{b} := Ke_{021} \oplus Ke_{121} \oplus Ke_{221}$. It is easy to verify that $\mathfrak{b}$ is a unipotent p–ideal in $\mathfrak{g}$ with $\chi(\mathfrak{b}) = 0$. Moreover, we have $\mathfrak{st}(\chi, \mathfrak{b}) = \mathfrak{g}$ since $\chi([e_{021}, e_{002}]) \neq 0 = \chi([e_{021}, e_{102}]) = \chi([e_{021}, e_{202}])$ and $\chi([e_{221}, e_{102}]) \neq 0 = \chi([e_{221}, e_{202}])$ and $\chi([e_{121}, e_{202}]) \neq 0$.

Note that

$$[W, \mathfrak{b}] = \mathfrak{b} \oplus Ke_{011} \oplus Ke_{022} \oplus Ke_{122} \oplus Ke_{222}$$

has a basis $l_1, l_2, \dots, l_k$ with $l_i^{[p]} = 0$ for all i and that $\big[[W, \mathfrak{b}], [W, \mathfrak{b}]\big] \subset \mathfrak{h}$ is a unipotent p–ideal contained in $\mathrm{Ker}(\chi)$. Finally, we have $\big[W, [W, \mathfrak{b}]\big] \subset \mathfrak{g}$ and hence we can apply Theorem 6.3.3 in Section 6.3 with the results already obtained: There exist (up to isomorphism) 3 irreducible $U_\chi(W)$–modules all of dimension $3^8 = 3^{\mathrm{codim}_W \mathfrak{c}_W(\chi)/2}$. $\square$

Now suppose that $\chi(e_{011}) = 0$. We shall use the restricted Lie algebra $\mathfrak{g}$ defined in Section 11.7. The idea is to find a restricted Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ and a unipotent ideal $\mathfrak{a} \lhd \mathfrak{s}$ with $\chi(\mathfrak{a}) = 0$ such that the following conditions are satisfied:

1) Induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{s})$–modules and the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules.

2) $\mathfrak{s}/\mathfrak{a} \simeq \mathfrak{gl}_2(K)$ as restricted Lie algebras.

Note that $Ke_{012} \oplus Ke_{001} \oplus K(e_{012} - e_{101}) \oplus K(e_{201} + e_{112})$ is a restricted Lie algebra isomorphic to $\mathfrak{gl}_2(K)$ (The isomorphism sends $Ke_{001} \oplus K(e_{012} - e_{101}) \oplus K(e_{201} + e_{112})$ to $\mathfrak{sl}_2(K)$ and e_{012} to the identity matrix). Therefore, set

$$\begin{aligned} \mathfrak{gl}_2(K) &:= Ke_{012} \oplus Ke_{001} \oplus K(e_{012} - e_{101}) \oplus K(e_{201} + e_{112}), \\ \mathfrak{a} &:= Ke_{011} \oplus Ke_{111} \oplus Ke_{021} \oplus Ke_{022} \oplus Ke_{122} \oplus Ke_{121} \oplus Ke_{221}. \end{aligned}$$

It is clear that we now shall define

$$\mathfrak{s} := \mathfrak{gl}_2(K) \oplus \mathfrak{a}.$$

It is easy to verify that $\mathfrak{s}$ is a restricted Lie subalgebra of $\mathfrak{g}$ and that $\mathfrak{a}$ is a unipotent ideal $\mathfrak{a} \lhd \mathfrak{s}$ with $\chi(\mathfrak{a}) = 0$. In particular, $\mathfrak{a}$ acts trivially on any irreducible $U_\chi(\mathfrak{s})$–module (apply Lemma 6.3.1). Moreover, $\mathfrak{s}/\mathfrak{a} \simeq \mathfrak{gl}_2(K)$.

Lemma 14.4.3. *Let χ be a character of* Type M3 *with $\chi(e_{011}) = 0$. If M is a $U_\chi(\mathfrak{g})$–module and $M \neq 0$, then*

$$\{x \in M \mid \mathfrak{a} \cdot x = 0\} \neq 0$$

and there exists an irreducible $U_\chi(\mathfrak{s})$–submodule $X \subset M$ with $\mathfrak{a} \cdot X = 0$.

Proof. Since $[\mathfrak{a},\mathfrak{a}] \subset \mathfrak{a} \cap W_{011}$ there exists a $U_\chi(\mathfrak{a})$–module K_l as being equal to K as a vector space and where the module structure is given by: $e \cdot 1 = 0$ for $e \in \mathfrak{a} \cap W_{011}$ and $e_{012} \cdot 1 = l$ (since $e_{012} \in \mathfrak{a}$ with $\chi(e_{012}) = 0$ we have $l \in \mathbb{F}_p$). Since $\mathfrak{a} \subset W_{012}$ is supersolvable any irreducible $U_\chi(\mathfrak{a})$–module is isomorphic to some K_l with $l \in \mathbb{F}_p$ by Lemma 9.1.3. So there exists a nonzero $x \in M$ with $(\mathfrak{a} \cap W_{011}) \cdot x = 0$ and $e_{012} \cdot x = lx$ for some $l \in \mathbb{F}_p$. If $l > 0$, set $y := e_{222}^{3-l} \cdot x \in M$. Then $y \neq 0$ since $\chi(e_{222}) \neq 0$. Moreover, we have $e_{012} \cdot y = e_{222}^{3-l}(e_{012} + (3-l)) \cdot x = 0$ and $(\mathfrak{a} \cap W_{011}) \cdot y = 0$ since $[e_{222}, \mathfrak{a} \cap W_{011}] \subset \mathfrak{a} \cap W_{011}$. We conclude that that $\mathfrak{a} \cdot x = 0$ if $l = 0$ and $\mathfrak{a} \cdot y = 0$ if $l > 0$; hence $\{x \in M \mid \mathfrak{a} \cdot x = 0\} \neq 0$.

The final statement in the lemma is now clear: Take nonzero $x \in M$ such that $\mathfrak{a} \cdot x = 0$. Then $U_\chi(\mathfrak{g}) \cdot x$ is a $U_\chi(\mathfrak{g})$–submodule of M annihilated by $\mathfrak{a}$ (since $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ and $\mathfrak{a} \cdot x = 0$). Thus it contains an irreducible $U_\chi(\mathfrak{g})$–submodule X such that there exists an irreducible $U_\chi(\mathfrak{g})$–submodule $X \subset M$ with $\mathfrak{a} \cdot X = 0$. $\square$

Proposition 14.4.4. *Let χ be a character of* Type M3 *with $\chi(e_{011}) = 0$. Induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{s})$–modules annihilated by $\mathfrak{a}$ and the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules.*

Proof. We shall apply Corollary 6.4.3 in Section 6.4: Let e_1, e_2, e_3 be a basis for a complement to $\mathfrak{s}$ in $\mathfrak{g}$. We can choose

$$\begin{aligned} e_1 &= e_{112}, \\ e_2 &= e_{212}, \\ e_3 &= e_{222}. \end{aligned}$$

Let N be an irreducible $U_\chi(\mathfrak{s})$–module annihilated by $\mathfrak{a}$. Adopt the notation from Section 6.4:

$$\begin{aligned} N_1 &= \textstyle\bigoplus_{ijkl} K e_1^i e_2^j e_3^k \otimes N, \\ N_2 &= \textstyle\bigoplus_{jkl} K e_2^j e_3^k \otimes N, \\ N_3 &= \textstyle\bigoplus_{kl} K e_3^k \otimes N. \end{aligned}$$

The idea is to prove that

$$\{x \in U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} N \mid \mathfrak{a} \cdot x = 0\} = 1 \otimes N. \tag{14.3}$$

First, let $f_1 = e_{211}$. Observe that $\chi([f_1, e_1]) \neq 0$; hence $[f_1, e_1]$ acts bijectively on $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} N$. It is easy to verify that $(\mathrm{ad}\, e_1)^i(f_1) \cdot N_2 \subset N_2$ for all i. Moreover, $f_1 \cdot N_2 = 0$ since $e_{211} \in \mathfrak{a}$ and therefore $e_{211} \cdot N = 0$.

Next, let $f_2 = e_{111}$. Observe that $\chi([f_2, e_2]) \neq 0$; hence $[f_2, e_2]$ acts bijectively on $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} N$. It is easy to verify that $(\mathrm{ad}\, e_2)^j(f_2) \cdot N_3 \subset N_3$ for all j. Moreover, $f_2 \cdot N_3 = 0$ since $e_{111} \in \mathfrak{a}$ and therefore $e_{111} \cdot N = 0$.

Finally, set $f_3 = e_{012}$. Observe that $[f_3, e_3] = e_3$ and that $f_3 \cdot N_4 \in \mathfrak{a} \cdot N = 0$.

We are now in a position, where we can use Corollary 6.4.3 in Section 6.4 with the observations just made to show that (14.3) holds.

This implies that $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} N$ is irreducible: Any irreducible $\mathfrak{g}$–submodule M has a nonzero intersection with $1 \otimes N$ [Apply Lemma 14.4.3]. Therefore $M \cap (1 \otimes N)$ is a nonzero $U_\chi(\mathfrak{s})$–submodule of $1 \otimes N$ and, by irreducibility, $M \cap (1 \otimes N) = 1 \otimes N$. In particular, we have $M \supset 1 \otimes N$ and hence M is the entire induced module.

If X_1, X_2 are irreducible $U_\chi(\mathfrak{g})$–modules such that $\mathfrak{a} \cdot X_1 = 0 = \mathfrak{a} \cdot X_2$ and

$$\varphi : U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X_1 \simeq U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X_2$$

is an isomorphism, then φ induces a $U_\chi(\mathfrak{s})$–isomorphism $\overline{\varphi} : X_1 \simeq X_2$. Indeed, we have $\varphi(1 \otimes X_1) \cap (1 \otimes X_2) \neq 0$. (Look at the elements annihilated by $\mathfrak{a}$.) Since $\varphi(1 \otimes X_1)$ and $1 \otimes X_2$ are irreducible $U_\chi(\mathfrak{s})$–modules, we get $\varphi(1 \otimes X_1) = 1 \otimes X_2$; hence $X_1 \simeq X_2$.

We have thus shown: Induction induces an injection from the isomorphism classes of irreducible $U_\chi(\mathfrak{s})$–modules annihilated by $\mathfrak{a}$ into the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules.

Now, let Y be an arbitrary irreducible $U_\chi(\mathfrak{g})$–module. I claim that we can find an irreducible $U_\chi(\mathfrak{s})$–module X with $\mathfrak{a} \cdot X = 0$ and

$$U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X \longrightarrow U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} Y.$$

First, apply Lemma 14.4.3 to find an irreducible $U_\chi(\mathfrak{s})$–submodule $X \subset U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} Y$ with $\mathfrak{a} \cdot X = 0$; thus we have inclusion maps:

$$X \hookrightarrow U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} Y.$$

Now apply 'Frobenius reciprocity' on the inclusion $X \hookrightarrow U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} Y$ and obtain a (nonzero) $U_\chi(\mathfrak{g})$–homomorphism:

$$U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X \longrightarrow U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} Y. \tag{14.4}$$

This implies that every $U_\chi(\mathfrak{g})$–module is induced from a $U_\chi(\mathfrak{s})$–module annihilated by $\mathfrak{a}$: Indeed, any irreducible $U_\chi(\mathfrak{g})$–module V contains an irreducible $U_\chi(\mathfrak{s})$–module Y; hence, by 'Frobenius reciprocity', V is a homomorphic image of $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} Y$ and by (14.4) then also a homomorphic image of $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X$ for some irreducible $U_\chi(\mathfrak{s})$–module X with $\mathfrak{a} \cdot X = 0$. By the part of the claim already proved we therefore have $V \simeq U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} X$. The proof is completed. □

Corollary 14.4.5. *Let χ be a character of* Type M3 *with $\chi(e_{011}) = 0$. The dimension of irreducible $U_\chi(\mathfrak{g})$–modules and the number of irreducible $U_\chi(\mathfrak{g})$–modules (up to isomorphism) are given as:*

1) Each irreducible $U_\chi(\mathfrak{g})$–module has dimension 3^5 if $\chi(e_{001}) \neq 0$ or $\chi(e_{101}) \neq 0$. The number of irreducibles is 3^2 if $\chi(e_{101}) \neq 0$ and $2 \cdot 3$ if $\chi(e_{101}) = 0$.

2) If $\chi(e_{001}) = 0 = \chi(e_{101})$, then the number of irreducibles is 3^2 and there exist irreducible $U_\chi(\mathfrak{g})$–modules of dimension $3^4, 2 \cdot 3^4, 3^5$.

Proof. The result follows immediately from Proposition 14.4.4 and from the representation theory of $\mathfrak{gl}_2(K)$. By Proposition 14.4.4 induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{s})$–modules annihilated by $\mathfrak{a}$ and the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules. But $\mathfrak{s}/\mathfrak{a} \simeq \mathfrak{gl}_2(K)$ as restricted Lie algebras, so we can just think of irreducible $U_\chi(\mathfrak{s})$–modules annihilated by $\mathfrak{a}$ as irreducible $U_\chi(\mathfrak{gl}_2(K))$–modules. But $\mathfrak{gl}_2(K) = \mathfrak{sl}_2(K) \oplus Z(\mathfrak{gl}_2(K))$, where $Z(\mathfrak{gl}_2(K))$ denotes the centre of $\mathfrak{gl}_2(K))$, and $\mathfrak{sl}_2(K)$ and $Z(\mathfrak{gl}_2(K))$ commute. This implies that any irreducible $U_\chi(\mathfrak{gl}_2(K))$–module is isomorphic to $S_1 \otimes_K S_2$, where S_1 is an irreducible $U_\chi(\mathfrak{sl}_2(K))$–module and S_2 is an irreducible $U_\chi(Z(\mathfrak{gl}_2(K)))$–module (see [3, Thm. 10.38]). Since $Z(\mathfrak{gl}_2(K))$ is abelian, there exist (up to isomorphism) three irreducible $U_\chi(Z(\mathfrak{gl}_2(K)))$–modules of dimension 1 and the representation theory of $\mathfrak{sl}_2(K)$ is described in [27, 5, 5.2]. Now 1) and 2) are easy to obtain (we use that $\chi(e_{201} + e_{112}) = 0$ for χ of Type M3). □

Remark 14.4.6. If χ has Type M3 and $\chi(e_{001}) = \chi(e_{101}) = 0$, then it follows from Corollary 14.4.5 that there exist irreducible $U_\chi(W)$–modules of *non–maximal* dimension (i.e., of dimension strictly less than 3^8). Indeed, there exists an irreducible $U_\chi(W)$–module M such that M contains an irreducible $U_\chi(\mathfrak{g})$–module X of dimension 3^4 or $2 \cdot 3^4$ and by 'Frobenius reciprocity' there exists a (surjective) $U_\chi(W)$–homomorphism:

$$U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} X \twoheadrightarrow M.$$

In particular, $\dim_K M \leq 3^3 \cdot \dim_K X < 3^8$.

But I do not know whether $U_\chi(W) \otimes_{U_\chi(\mathfrak{g})} X$ is irreducible (when X is an irreducible $U_\chi(\mathfrak{g})$–module). If, for instance, induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules and the isomorphism classes of irreducible $U_\chi(W)$–modules, then we could apply the results in Corollary 14.4.5 and get a description of the irreducible $U_\chi(W)$–modules (when $\chi(e_{011}) = 0$). But all methods in Section 6 breaks down in order to show that result.

A $GL_2(K)$–action on basis elements

Any element $\varphi \in \mathrm{Aut}_{K-\mathrm{alg}}\, B_2$ induces an automorphism σ_φ of the Lie algebra W such that

$$\sigma_\varphi(D) := \varphi \circ D \circ \varphi^{-1} \quad \forall D \in W = \mathrm{Der}_K(B_2). \tag{A.1}$$

It is easy to see that $\sigma_\varphi^{-1} = \sigma_{\varphi^{-1}}$. So the map $\varphi \longmapsto \sigma_\varphi$ is a homomorphism between the two automorphism groups. In fact, by Theorem 2.2.1 and Remark 2.2.2, it is an isomorphism of groups for $p > 3$.

For $D = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} \in W$ and $\varphi \in \mathrm{Aut}_{K-\mathrm{alg}}\, B_2$ we have, with the action given in (A.1), that

$$\sigma_\varphi(D) = \sum_{l=1}^{2}\sum_{i=1}^{2} f_l(\varphi(x_1), \varphi(x_2)) \frac{\partial \varphi^{-1}(x_i)}{\partial x_l}\big(\varphi(x_1), \varphi(x_2)\big)\frac{\partial}{\partial x_i}. \tag{A.2}$$

See Proposition 2.2.3. Note that we have an inclusion $\mathrm{Aut}_{K-\mathrm{alg}}\, B_2 \supset GL_2(K)$ in the following way:

$$\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{ccc} x_1 & \longmapsto & ax_1 + cx_2 \\ x_2 & \longmapsto & bx_1 + dx_2 \end{array} \tag{A.3}$$

where $ad - bc \neq 0$. For any $\varphi \in GL_2(K)$, the automorphism σ_φ is determined by (A.2) and (A.3). For $\varphi \in GL_2(K)$ we will define $\sigma_\varphi(w) := \varphi(w)$ for $w \in W$.

A.1 Diagonal matrices

Let $t_1, t_2 \in K^*$ and define automorphism on B_2 given by

$$T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} : \begin{array}{ccc} x_1 & \longmapsto & t_1 x_1, \\ x_2 & \longmapsto & t_2 x_2. \end{array}$$

Now apply (A.2) and find:

$$T(e_{ij1}) = \sigma_T(e_{ij1}) = T \cdot e_{ij1} = t_1^{i-1} t_2^{j} e_{ij1}, \tag{A.4}$$

$$T(e_{ij2}) = \sigma_T(e_{ij2}) = T \cdot e_{ij2} = t_1^{i} t_2^{j-1} e_{ij2}. \tag{A.5}$$

A.2 Lower triangular matrices

Let $\alpha \in K$ and consider the automorphism on B_2 by

$$\varphi_1 = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} : \begin{array}{ccc} x_1 & \longmapsto & x_1 + \alpha \cdot x_2, \\ x_2 & \longmapsto & x_2. \end{array}$$

Now apply (A.2) and obtain:

$$\varphi_1(e_{ij1}) = \sigma_{\varphi_1}(e_{ij1}) = \sum_{s=0}^{i} \binom{i}{s} e_{i-s, j+s, 1} \cdot \alpha^s, \tag{A.6}$$

$$\varphi_1(e_{ij2}) = \sigma_{\varphi_1}(e_{ij2}) = \sum_{s=0}^{i} \binom{i}{s} \big(e_{i-s, j+s, 2} - \alpha \cdot e_{i-s, j+s, 1}\big) \cdot \alpha^s. \tag{A.7}$$

A.3 Upper triangular matrices

Let $\alpha \in K$ and consider the automorphism on B_2 by

$$\varphi_2 = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} : \begin{array}{ccc} x_1 & \longmapsto & x_1, \\ x_2 & \longmapsto & x_1 + \alpha \cdot x_2. \end{array}$$

Now apply (A.2) and obtain:

$$\varphi_2(e_{ij1}) = \sigma_{\varphi_2}(e_{ij1}) = \sum_{s=0}^{j} \binom{j}{s} \big(e_{i+s,j-s,1} - \alpha \cdot e_{i+s,j-s,2}\big) \cdot \alpha^s, \tag{A.8}$$

$$\varphi_2(e_{ij2}) = \sigma_{\varphi_2}(e_{ij2}) = \sum_{s=0}^{j} \binom{j}{s} e_{i+s,j-s,2} \cdot \alpha^s. \tag{A.9}$$

A.4 Interchanging

Finally consider the interchanging given by

$$\Theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \begin{array}{ccc} x_1 & \longmapsto & x_2, \\ x_2 & \longmapsto & x_1. \end{array}$$

Next, apply (A.2) and get:

$$\Theta(e_{ij1}) = \sigma_\Theta(e_{ij1}) = e_{ji2} \tag{A.10}$$

$$\Theta(e_{ij2}) = \sigma_\Theta(e_{ij2}) = e_{ji1}. \tag{A.11}$$

B Jacobson's formula for $p = 3$

Let K be an algebraically closed field of characteristic $p = 3$ and let W be the second Witt–Jacobson algebra over K. For $D_1, D_2 \in W$ we have

$$(D_1 + D_2)^{[3]} = D_1^{[3]} + D_2^{[3]} + \sum_{i=1}^{2} s_i(D_1, D_2)$$

where the $s_i(D_1, D_2)'$s are elements in W such that

$$\big(\mathrm{ad}(D_1 \otimes t + D_2 \otimes 1)\big)^2 (D_1 \otimes 1) = \sum_{i=1}^{2} i s_i(D_1, D_2) \otimes t^{i-1} \tag{B.1}$$

in $W \otimes_K K[t]$. The vector space $W \otimes_K K[t]$ obtains the structure of a Lie algebra via the commutator

$$[d \otimes_K f(t), d' \otimes_K g(t)] = [d, d'] \otimes_K f(t)g(t) \quad \text{for } \ d, d' \in W \ \text{ and } \ f, g \in K[t].$$

Now, apply (B.1) and get:

$$s_1(D_1, D_2) = \big[D_2, [D_2, D_1]\big] \quad \text{and} \quad s_2(D_1, D_2) = \big[D_1, [D_1, D_2]\big].$$

It follows that

$$(D_1 + D_2)^{[3]} = D_1^{[3]} + D_2^{[3]} + \big[D_2, [D_2, D_1]\big] + \big[D_1, [D_1, D_2]\big]. \tag{B.2}$$

B.1 Toral elements

Lemma B.1.1. *For all $c, c' \in K$ the element $e_{012} - e_{101} + ce_{201} + c'e_{112}$ is a toral element in W.*

Proof. Set $D_1 = e_{012} - e_{101}$ and $D_2 = ce_{201} + c'e_{112}$). Then $D_1^{[3]} = D_1$ and $D_2^{[3]} = 0$. Moreover, $\big[D_2, [D_2, D_1]\big] = 0$ and $\big[D_1, [D_1, D_2]\big] = D_2$. Now apply (B.2). $\square$

Lemma B.1.2. *If $D \in W$ and $0 \neq D^{[3]} \in KD$, then there exists $c \in K^*$ such that cD is a toral element.*

Proof. Suppose that $D^{[3]} = sD$ for some $s \in K^*$. Then c is determined by $(cD)^{[3]} = cD$ or equivalent $c^3 s = c$. Since $s \neq 0$ we shall choose $c \neq 0$ as a root in $X^3 - s^{-1}X = 0$. $\square$

Lemma B.1.3. *If $D \in W$ with $D^{[3]} \equiv cD \pmod{W_{011}}$ for some $c \in K^*$, then there exists $N > 0$ and $c' \in K^*$ such that $c'D^{[3^N]}$ is toral.*

Proof. Since W_{011} is unipotent there exists $N > 0$ such that $\big(D^{[3]} - cD\big)^{[3^N]} = 0$. Note that $D^{[3^j]}, D^{[3^{j+1}]}$ commute for all $j \geq 0$ since $[D^{[3^{j+1}]}, D^{[3^j]}] = (\mathrm{ad}\, D^{[3^j]})^3(D^{[3^j]}) = 0$. The choice of N now implies that that

$$D^{[3^{N+1}]} = c^N D^{[3^N]}.$$

Therefore $\big(D^{[3^N]}\big)^{[3]} \in KD^{[3^N]}$. Now apply Lemma B.1.2. $\square$

C Characters of height at most 1

Let K be an algebraically closed field of characteristic $p > 0$. I will consider the case where $\chi(W_{\geq 1}) = 0$; i.e., the height of χ is at most 1. [In [10] the height $\mathrm{ht}\, \chi$ of a character is defined as

$$\mathrm{ht}\, \chi = \min\big\{j \geq -1 \mid \chi(W_{\geq j}) = 0\big\}.]$$

Since $W_{\geq 1} \lhd W_{\geq 0}$, any W_0–module becomes a $W_{\geq 0}$–module via the canonical map $W_{\geq 0} \longrightarrow W_{\geq 0}/W_{\geq 1} \simeq W_0$. Since $\mathrm{ht}\, \chi \leq 1$ we have $\chi(W_{\geq 1}) = 0$, so any irreducible $U_\chi(W_0)$–module is an irreducible $U_\chi(W_{\geq 0})$–module via the canonical map (apply Lemma 6.3.1 in Section 6.3 with $\mathfrak{g} = W_{\geq 0}$ and $\mathfrak{h} = W_{\geq 1}$). On the other hand: Any irreducible $U_\chi(W_{\geq 0})$–module is an irreducible $U_\chi(W_0)$–module, since $W_{\geq 1}$ annihilates all irreducible $U_\chi(W_{\geq 0})$–modules and since $W_{\geq 0}/W_{\geq 1} \simeq W_0$. We can think of irreducible $U_\chi(W_{\geq 0})$–modules as irreducible $U_\chi(W_0)$–modules, where $W_{\geq 1}$ acts trivially.

Let S be an irreducible $U_\chi(W_0)$–module. If we extend S to $W_{\geq 0}$ in the way above, then S is an irreducible $U_\chi(W_{\geq 0})$–module and we can define the induced module

$$U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S.$$

We have a triangular decomposition $W_0 = Ke_{102} \oplus (Ke_{101} \oplus Ke_{012}) \oplus Ke_{011}$. Set $\mathfrak{t} = Ke_{101} \oplus Ke_{012}$ and $\mathfrak{n} = Ke_{011} \oplus W_{\geq 1}$. Let $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ and let M be a $\mathfrak{b}$–module. If $\lambda \in \mathfrak{t}^*$ we define $M_\lambda = \{m \in M \mid e_{101} \cdot m = \lambda(e_{101})m \text{ and } e_{012} \cdot m = \lambda(e_{012})m\}$. An element of M_λ is called a weight vector of weight λ. A nonzero element $m \in M_\lambda$ is a maximal vector (of weight λ) provided that $\mathfrak{n} \cdot m = 0$. If M has p–character χ and $0 \neq m \in M_\lambda$, then we have $\lambda(e_{101})^p m - \lambda(e_{101})m = \chi(e_{101})^p m$ and $\lambda(e_{012})^p m - \lambda(e_{012})m = \chi(e_{012})^p m$. This implies that $\lambda \in \Lambda_\chi = \big\{\lambda \in \mathfrak{t}^* \mid \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p \ \text{ for all } \ h \in \mathfrak{t}\, \big\}$.

C.1 Irreducible $W_{\geq 0}$–modules

Suppose that $p > 2$. Observe that W_0 can be written as $A_1 \oplus Z(W_0)$, where A_1 is the three dimensional Lie algebra with basis e_{102}, e_{011}, $e_{101} - e_{012}$ and $Z(W_0) = K(e_{101} + e_{012})$ is the center in W_0. It is easy to verify that A_1 is a restricted Lie subalgebra of W_0 isomorphic to $\mathfrak{sl}_2(K)$: The isomorphism is given by

$$e_{102} \longmapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_{012} - e_{101} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_{011} \longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since $W_0 = A_1 \oplus Z(W_0)$ and $[A_1, Z(W_0)] = 0$, it follows from [3, Thm.10.38] that each irreducible $U_\chi(W_0)$–module S is isomorphic to $S_1 \otimes_K S_2$, where S_1 is an irreducible $U_\chi(A_1)$–module and S_2 is an irreducible $U_\chi(Z(W_0))$–module. The number of irreducible $U_\chi(W_0)$–modules is now just the number of irreducible $U_\chi(A_1)$–modules times the number of irreducible $U_\chi(Z(W_0))$–modules. The irreducible $U_\chi(A_1)$–modules and irreducible $U_\chi(Z(W_0))$–modules are well described so we can use the observations just made to describe the irreducible $U_\chi(W_0)$–modules. Recall that we can think of irreducible $U_\chi(W_{\geq 0})$–modules as irreducible $U_\chi(W_0)$–modules, where $W_{\geq 1}$ acts trivially.

Denote by W_0' the subalgebra $A_1 \oplus W_{\geq 1}$.

Proposition C.1.1. *Suppose that $\chi(W_0') \neq 0$ but $\chi(e_{011}) = 0$. Then each irreducible $U_\chi(W_{\geq 0})$–module has dimension p and the number of irreducible $U_\chi(W_{\geq 0})$–modules are:*

$$\begin{cases} p^2 & \text{if } \chi(e_{012} - e_{101}) \neq 0, \\ p \cdot \frac{p+1}{2} & \text{if } \chi(e_{012} - e_{101}) = 0. \end{cases}$$

Proof. Since $\chi(W_0') \neq 0$ we have $\chi(A_1) \neq 0$; therefore irreducible $U_\chi(A_1)$–modules are of dimension p by [14, Prop.5.3]. Moreover, all irreducible $U_\chi(Z(W_0))$–modules have dimension 1 since $Z(W_0)$ is abelian. So any irreducible $U_\chi(W_0)$–module has dimension p and the number of irreducible $U_\chi(W_0)$–modules is p^2 if $\chi(e_{101} - e_{012}) \neq 0$ and $p \cdot \frac{p+1}{2}$ if $\chi(e_{101} - e_{012}) = 0$: The number of irreducible $U_\chi(A_1)$–modules is p if $\chi(e_{101} - e_{012}) \neq 0$ and $\frac{p+1}{2}$ if $\chi(e_{101} - e_{012}) = 0$ by [27, 5, 5.2] (here we use that $\chi(e_{011}) = 0$) and the number of irreducible $U_\chi(Z(W_0))$–modules is p. The proof is completed. □

Lemma C.1.2. *If $\chi(W_0') \neq 0$, then there exists an automorphism $\Phi \in \mathrm{Aut}(W)$ such that $\chi^\Phi(e_{011}) = 0$.*

Proof. If $\chi(e_{011}) \neq 0$ we can apply an automorphism $\Psi \in GL_2(K)$ given by $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$. We can now find an appropriate $\alpha \in K$ such that

$$\chi^\Psi(e_{102}) = -\chi(e_{011}) \cdot \alpha^2 - (\chi(e_{101}) - \chi(e_{012})) \cdot \alpha + \chi(e_{102}) \neq 0.$$

If we apply an automorphism $\Gamma \in GL_2(K)$ given by $\begin{pmatrix} 1 & \alpha' \\ 0 & 1 \end{pmatrix}$ we get for an appropriate $\alpha' \in K$, that

$$\chi^{\Psi \circ \Gamma}(e_{011}) = -\chi^\Psi(e_{102}) \cdot (\alpha')^2 + (\chi^\Psi(e_{101}) - \chi^\Psi(e_{012})) \cdot \alpha' + \chi^\Psi(e_{011}) = 0.$$

The formulas for $\Psi(e_{011})$ and $\Gamma(e_{102})$ follows from Appendix A. Set $\Phi = \Psi \circ \Gamma$. □

Suppose now that $\chi(W_0') = 0$. As in the proof of Proposition C.1.1, irreducible $U_\chi(W_{\geq 0})$–modules are determined by irreducible $U_\chi(A_1)$–modules and irreducible $U_\chi(Z(W_0))$–modules. Since $Z(W_0)$ is abelian, each irreducible $U_\chi(Z(W_0))$–module is given by some K_μ, where $e_{101} + e_{012}$ acts as multiplication by $\mu \in K$ and $\mu^p - \mu = \chi(e_{101} + e_{012})^p$. The irreducible modules for $A_1 \simeq \mathfrak{sl}_2(K)$ are described in [14, 5.2] and [11, 7.2]: For any irreducible $U_\chi(A_1)$–module S_1 there exist $0 \leq n < p$ such that S_1 has dimension $n + 1$. Given a basis $v_0, \ldots, v_n$ for S_1 the A_1-action is given by

$$\begin{aligned} e_{102} \cdot v_i &= \begin{cases} (n-i+1)v_{i-1} & i > 0, \\ 0 & i = 0 \end{cases} \\ e_{011} \cdot v_i &= \begin{cases} (i+1)v_{i+1} & i < n, \\ 0 & i = n \end{cases} \\ \big(e_{101} - e_{012}\big) \cdot v_i &= (n-2i)v_i. \end{aligned}$$

In this way any irreducible $U_\chi(W_{\geq 0})$–module is uniquely determined by a pair (μ, n) where $0 \leq n < p$ and $\mu \in K$ with $\mu^p - \mu = \chi(e_{101} + e_{012})^p$. In particular, there are p^2 irreducible $U_\chi(W_{\geq 0})$–modules.

Proposition C.1.3. *If $\chi(W_0') = 0$, then there are p^2 irreducible $U_\chi(W_{\geq 0})$–modules. For any integer n with $0 \leq n < p$ there exist p irreducible $U_\chi(W_{\geq 0})$–modules of dimension $n+1$. Any irreducible $U_\chi(W_{\geq 0})$–module is isomorphic to a $W_{\geq 0}$–submodule in $U_\chi(W_{\geq 0}) \otimes_{U_\chi(W_{012})} Kv$ of dimension $n + 1$, where Kv is a one-dimensional $U_\chi(W_{012})$–module with the action of e_{101}, e_{012} given by multiplication with $\frac{1}{2}(\mu + n + 2), \frac{1}{2}(\mu - n - 2)$ for some $\mu \in K$ with $\mu^p - \mu = \chi(e_{101} + e_{012})^p$ and some $0 \leq n < p$.*

Proof. From the descriptions above, any irreducible $U_\chi(W_{\geq 0})$–module S has a basis $v_0, \ldots, v_n$, where the W_0-action is given by

$$\begin{aligned} e_{102} \cdot v_i &= \begin{cases} (n-i+1)v_{i-1} & i > 0, \\ 0 & i = 0. \end{cases} \\ e_{011} \cdot v_i &= \begin{cases} (i+1)v_{i+1} & i < n, \\ 0 & i = n. \end{cases} \\ e_{101} \cdot v_i &= \tfrac{1}{2}(\mu + n - 2i)v_i, \\ e_{012} \cdot v_i &= \tfrac{1}{2}(\mu - n + 2i)v_i \end{aligned}$$

and where $W_{\geq 1}$ acts trivially. The formulas for $e_{101} \cdot v_i$ and $e_{012} \cdot v_i$ is a consequence of $\big(e_{101} - e_{012}\big) \cdot v_i = (n-2i)v_i$ and $\big(e_{101} + e_{012}\big) \cdot v_i = \mu v_i$. One can show that S is isomorphic to a submodule in $U_\chi(W_{\geq 0}) \otimes_{U_\chi(W_{012})} Kv$ where the action on the one–dimensional $U_\chi(W_{012})$–module Kv is given by

$$e_{101} \cdot v = \frac{1}{2}(\mu + n + 2) \quad e_{012} \cdot v = \frac{1}{2}(\mu - n - 2) \quad e_{rst} \cdot v = 0 \quad \text{for all } (rst) \succ (101).$$

More explicitly, the isomorphism of $U_\chi(W_{\geq 0})$–modules is given by

$$\begin{aligned} \phi : S &\xrightarrow{\sim} \textstyle\sum_{j=p-1-n}^{p-1} K e_{102}^j \otimes v \subseteq U_\chi(W_{\geq 0}) \otimes_{U_\chi(W_{012})} Kv \\ v_0 &\longmapsto e_{102}^{p-1} \otimes v. \end{aligned}$$

The proof is completed. □

C.2 Irreducible W–modules

Suppose that $p > 2$. If ht $\chi = 1$ it turns out that induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and isomorphism classes of irreducible $U_\chi(W)$–modules. For ht $\chi \leq 0$ we assume that $p > 3$ in order to use results proved in [10].

Following [10], we introduce the exceptional weights: The exceptional weights $\omega_0, \omega_1, \omega_2$ are elements in $\mathfrak{t}^*$ defined via

$$\begin{array}{ll} \omega_0(e_{101}) = -1 & \omega_0(e_{012}) = -1, \\ \omega_1(e_{101}) = 0 & \omega_1(e_{012}) = -1, \\ \omega_2(e_{101}) = 0 & \omega_2(e_{012}) = 0. \end{array}$$

Proposition C.2.1. *Suppose that $p > 2$. If $\chi \in W^*$ with* ht $\chi = 1$*, then induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and isomorphism classes of irreducible $U_\chi(W)$–modules.*

Proof. If $\chi(W_0') \neq 0$, we may assume that $\chi(e_{102}) \neq 0$. Indeed, apply an automorphism Ψ induced by a lower triangular matrix in $GL_2(K)$ given by $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$. Then we can find an appropriate $\alpha \in K$ such that $\chi^\Psi(e_{102}) = -\chi(e_{011}) \cdot \alpha^2 - (\chi(e_{101}) - \chi(e_{012})) \cdot \alpha + \chi(e_{102}) \neq 0$.

In order to prove that induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and isomorphism classes of irreducible $U_\chi(W)$–modules, we may replace χ with any χ^Ψ.

If $\chi(e_{102}) \neq 0$ and $p > 3$, then the result is proved in [10, 2.4]: R. Holmes prove, for an irreducible $U_\chi(W_{\geq 0})$–module S, that any maximal vector $v \in U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S$ of weight λ has the form $1 \otimes s$ for some maximal vector $s \in S$. Thus, $U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S$ is irreducible and if $U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S_1 \simeq U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S_2$ for irreducible $U_\chi(W_{\geq 0})$–modules S_1, S_2, then $S_1 \simeq S_2$.

For $p = 3$ let S be an irreducible $U_\chi(W_{\geq 0})$–module and let $v \in U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S$ be a maximal vector of weight λ given by

$$v = \sum_{0 \leq k,m < 3} e_{002}^k e_{001}^m \otimes x_{km}. \tag{C.1}$$

We shall prove that $x_{km} = 0$ unless $k = m = 0$. Note that $e_{202} \cdot v = 0$ and $e_{202} \cdot x_{km} = 0$ for all k, m. Now use that $[e_{202}, e_{002}] = 0$ and [27, 1, 1.3(4)] to get:

$$0 = e_{202} \cdot v = -2 \sum_{0 \leq k,m < 3} m e_{002}^k e_{001}^{m-1} \otimes e_{102} \cdot x_{km} + 2 \sum_{0 \leq k,m < 3} \binom{m}{2} e_{002}^{k+1} e_{001}^{m-2} \otimes x_{km}. \tag{C.2}$$

This implies that $e_{102} \cdot x_{22} = e_{102} \cdot x_{12} = e_{102} \cdot x_{02} = e_{102} \cdot x_{01} = 0$; hence $x_{22} = x_{12} = x_{02} = x_{01} = 0$ since $\chi(e_{102}) \neq 0$. Since $x_{12} = 0$ we have $e_{102} \cdot x_{21} = 0$ also and then $x_{21} = 0$. Since $x_{02} = 0$ we also obtain $e_{102} \cdot x_{11} = 0$ and then $x_{11} = 0$. Next, use that $e_{021} \cdot v = 0$ and $e_{021} \cdot x_{km} = 0$ for all k, m to get:

$$0 = e_{021} \cdot v = -2 \sum_{0 \leq k,m < 3} k e_{002}^{k-1} e_{001}^{m} \otimes e_{011} \cdot x_{km} + 2 \sum_{0 \leq k,m < 3} \binom{k}{2} e_{002}^{k-2} e_{001}^{m+1} \otimes x_{km}. \tag{C.3}$$

This implies that $x_{20} = 0$ since $x_{11} = 0$. In order to prove that $x_{10} = 0$, we use that $e_{112} \cdot v = 0$ and that $e_{112} \cdot x_{km} = 0$ for all k, m to get: $0 = -e_{002} \otimes e_{102} \cdot x_{10}$. Therefore $e_{102} \cdot x_{10} = 0$ and then $x_{10} = 0$. We conclude that $x_{km} = 0$ unless $k = m = 0$ and, since v is a maximal vector so is $x_{00} \in S$.

If $\chi(W_0')=0$ we have $\chi(e_{101}+e_{012})\neq 0$ and $\chi(e_{101})=\chi(e_{012})\neq 0$. If v is a maximal vector in $U_\chi(W)\otimes_{U_\chi(W_{\geq 0})} S$ (where S is an irreducible $U_\chi(W_{\geq 0})$–module) of weight λ, then there exist $\lambda_1,\lambda_2\in K$ such that $e_{101}\cdot v=\lambda_1 v$ and $e_{012}\cdot v=\lambda_2 v$. It follows that $\lambda_1^p-\lambda_1=\chi(e_{101})^p=\chi(e_{012})^p=\lambda_2^p-\lambda_2$. Hence $\lambda_1,\lambda_2\notin\mathbb{F}_p$. Therefore, $\lambda\in\mathfrak{t}^*$ given by $\lambda(e_{101})=\lambda_1$ and $\lambda(e_{012})=\lambda_2$ is not an exceptional weight (as introduced just before the proposition). In particular, S has no maximal vectors of weights w_0,w_1 or w_2 and it follows from [10, 2.4] when $p>3$ that $v=1\otimes s$ for some maximal vector $s\in S$. For $p=3$ consider v as in (C.1). The equations

$$\sum_{0\leq k,m<3} e_{002}^k e_{001}^m\otimes\lambda_1 x_{km}=\lambda_1 v=e_{101}\cdot v=\sum_{0\leq k,m<3} e_{002}^k e_{001}^m\otimes(e_{101}\cdot x_{km}-m x_{km})$$

$$\sum_{0\leq k,m<3} e_{002}^k e_{001}^m\otimes\lambda_2 x_{km}=\lambda_2 v=e_{012}\cdot v=\sum_{0\leq k,m<3} e_{002}^k e_{001}^m\otimes(e_{012}\cdot x_{km}-k x_{km})$$

imply that $e_{101}\cdot x_{km}=(\lambda_1+m)x_{km}$ and $e_{012}\cdot x_{km}=(\lambda_2+k)x_{km}$. Now, since $W_{011}\cdot v=0$ and $W_{\geq 1}\cdot S=0$ we get (apply [27, 1, 1.3(4)]):

$$0=e_{022}\cdot v=\sum_{0\leq k,m<3} 2\Big(\binom{k}{2}-k(\lambda_2+k)\Big)e_{002}^{k-1}e_{001}^m\otimes x_{km}, \tag{C.4}$$

$$0=e_{201}\cdot v=\sum_{0\leq k,m<3} 2\Big(\binom{m}{2}-m(\lambda_1+m)\Big)e_{002}^{k}e_{001}^{m-1}\otimes x_{km}. \tag{C.5}$$

Thus, if $x_{km}\neq 0$ then $0\leq k,m<3$ and

a) If $m\neq 0$ then $\lambda_1+m=\frac{m-1}{2}$.

b) If $k\neq 0$ then $\lambda_2+k=\frac{k-1}{2}$.

We conclude that $x_{km}=0$ unless $k=m=0$: Indeed, if $x_{km}\neq 0$ we get from either a) or b) that either $\lambda_1\in\mathbb{F}_p$ or $\lambda_2\in\mathbb{F}_p$ — contradiction. Hence $v=1\otimes s$ for some $s\in S$.

It follows that induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and isomorphism classes of irreducible $U_\chi(W)$–modules. □

Theorem C.2.2. *Suppose that $p>2$ and let $\chi\in W^*$ be a character of height 1. Then there are p^2 irreducible $U_\chi(W)$–modules and the dimension of any irreducible is p if $\chi(W_0')\neq 0$. If $\chi(W_0')=0$, then, for any integer n with $0\leq n<p$, there exist p irreducible $U_\chi(W)$–modules of dimension $n+1$.*

Proof. Use Lemma C.1.2 and assume that $\chi(e_{011})=0$ if $\chi(W_0')\neq 0$. Then combine Proposition C.1.1 and Proposition C.2.1. If $\chi(W_0')=0$ we combine Proposition C.1.3 and Proposition C.2.1. The proof is completed. □

Suppose that that $p>3$ and that $\chi\in W^*$ is a character of height 0. Let S be an irreducible $U_\chi(W_{\geq 0})$–module. The proof of Proposition C.1.3 says that there exists n with $0\leq n<p$ and $\mu\in K$ with $\mu^p-\mu=0$ such that S has a basis $v_0,v_1,\dots,v_n$, where the

W_0-action is given by

$$\begin{aligned}
e_{102} \cdot v_i &= \begin{cases} (n-i+1)v_{i-1} & i > 0, \\ 0 & i = 0. \end{cases} \\
e_{011} \cdot v_i &= \begin{cases} (i+1)v_{i+1} & i < n, \\ 0 & i = n. \end{cases}
\end{aligned}$$

$$\begin{aligned}
e_{101} \cdot v_i &= \tfrac{1}{2}(\mu + n - 2i)v_i, \\
e_{012} \cdot v_i &= \tfrac{1}{2}(\mu - n + 2i)v_i.
\end{aligned}$$

There are p^2 irreducible $U_\chi(W_{\geq 0})$–modules represented by all $S(\mu, n)$ for n with $0 \leq n < p$ and $\mu \in K$ such that $\mu^p - \mu = 0$; each $S(\mu, n)$ is an irreducible $U_\chi(W_{\geq 0})$–module with a basis $v_0, v_1, \dots, v_n$ where the W_0–action is given as above and $W_{\geq 1}$ acts trivially. Note that $S(\mu, n)$ has maximal vector v_n of weight $\tau \in \mathfrak{t}^*$ given by $\tau(e_{101}) = \frac{1}{2}(\mu - n)$ and $\tau(e_{012}) = \frac{1}{2}(\mu + n)$. We can now describe the exceptional weights $\omega_0, \omega_1, \omega_2$ in terms of μ, n in the following way:

$$\begin{aligned}
\omega_0 &\longleftrightarrow \mu = -2 \text{ and } n = 0, \\
\omega_1 &\longleftrightarrow \mu = -1 \text{ and } n = 1, \\
\omega_2 &\longleftrightarrow \mu = 0 \text{ and } n = 0.
\end{aligned}$$

Proposition C.2.3. *Suppose that $p > 3$ and let $\chi \in W^*$ with* ht $\chi = 0$. *Then induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules with $(\mu, n) \neq (-1, 1)$ and the isomorphism classes of irreducible $U_\chi(W)$–modules. In particular, there are $p^2 - 1$ irreducible $U_\chi(W)$–modules.*

Proof. This follows from [10, 4.3 (1), (2)] and description of the exceptional weights in terms of μ, n just above. ∎

Remark C.2.4. Consider the case where $(\mu, n) = (-1, 1)$. Then the induced $U_\chi(W)$–module $U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S(-1, 1)$ contains a unique W–submodule $M(-1, 1)$ of dimension p^2 (see [10, 3.10 (3), 3.12]) determined by the kernel of the W–homomorphism δ_1^χ introduced in [10, 3.8]. Since $n = 1$ we have $\dim_K S(-1, 1) = 2$. From [10, 4.3 (1)], it follows that the quotient $\big(U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S(-1, 1)\big)/M(-1, 1)$ is an irreducible $U_\chi(W)$–module isomorphic to $U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S(0, 0)$.

Proposition C.2.5. *Suppose that $p > 3$ and that* ht $\chi = -1$ *or equivalent $\chi(W) = 0$. Then there are p^2 isomorphism classes of irreducible W–modules with p–character χ represented by:*

1) *$U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S(\mu, n)$ if $(\mu, n) \neq (-2, 0), (-1, 1), (0, 0)$,*

2) *$L(\mu, n) := \big(U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S(\mu, n)\big)/M(\mu, n)$ if $(\mu, n) \in \{(-2, 0), (-1, 1), (0, 0)\}$ where $M(\mu, n)$ is the unique maximal W–submodule of $U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S(\mu, n)$.*

   If $(\mu, n) = (-2, 0)$ the W–submodule $M(-2, 0)$ is one-dimensional and is equal to $Ke_{002}^{p-1} e_{001}^{p-1} \otimes w$ where $S(-2, 0) = Kw$. If $(\mu, n) = (-1, 1)$ we have $\dim_K M(-1, 1) = p^2 + 1$ *and* $\dim_K S(-1, 1) = 2$ *when $(\mu, n) = (-1, 1)$. Finally, if $(\mu, n) = (0, 0)$ then $L(0, 0)$ is the one–dimensional trivial module equal to $S(0, 0)$ as a $W_{\geq 0}$–module. For the dimensions we have:*

$$\dim_K L(\mu, n) = \begin{cases} p^2 - 1 & (\mu, n) = (-2, 0), (-1, 1), \\ 1 & (\mu, n) = (0, 0). \end{cases}$$

Proof. The statement on the number of isomorphism classes of irreducible $U_\chi(W)$–modules follows from [10, 4.2 (1)] and the proof of 1), 2) is a consequence of [10, 4.1 and 4.2 (1), (2)]. If (μ, n) equals $(-2,0), (-1,1)$ or $(0,0)$ the W–module $U_\chi(W) \otimes_{U_\chi(W_{\geq 0})} S(\mu, n)$ contains (see [10, 4.1]) a unique maximal W–submodule $M(\mu, n)$ determined by the kernel of the W–homomorphism δ_1^χ introduced in [10, 3.8].

If $(\mu, n) = (-2, 0)$ we have $\dim_K S(-2,0) = 1$ and $L(-2,0)$ has dimension $p^2 - 1$ (see [10, 4.2 (3)]). Here $M(-2,0)$ is the unique maximal submodule of dimension 1 equal to $Ke_{002}^{p-1} e_{001}^{p-1} \otimes w$ where $S(-2,0) = Kw$. Indeed, this is the only choice since every W–submodule must contain $e_{002}^{p-1} e_{001}^{p-1} \otimes w$ after a suitable multiply by e_{002}, e_{001}.

Suppose that $(\mu, n) = (-1, 1)$. Then $\dim_K S(-1,1) = 2$ and $L(-1,1)$ has dimension $p^2 - 1$, which also shows that $\dim_K M(-1,1) = p^2 + 1$ (see [10, 4.2 (3)]).

Finally, if $(\mu, n) = (0,0)$ we use [10, 4.2 (3)] to conclude that $L(0,0)$ is the one–dimensional trivial module equal to $S(0,0)$ as a $W_{\geq 0}$–module. □

D Comments

Here, I will comment on the problems which are left open in this thesis. I have gathered questions which I have not been able to answer due to lack of time as well as my mathematical limitations.

Question1 : How can we classify the set of irreducible $U_\chi(W)$–modules in the case where $\mathfrak{st}(\chi, W_{\geq r}) \neq W_{\geq 0}$ (r denotes the height of χ and we assume that $r > 1$ but $r \leq 2p - 3$).

The assumption $\mathfrak{st}(\chi, W_{\geq r}) \neq W_{\geq 0}$ implies that $p - 2 < r \leq 2p - 3$. We have only been able to answer Question 1 in the situation where $p = 3$ and $\chi \in W^*$ of height $2, 3$ or χ has height $p - 1$ and $\operatorname{rk} \mathfrak{c}_W(\chi) = 2$. For simplicity, we only consider χ of Type A or Type B as defined in Section 11.5.

Strictly speaking, the idea is the following: Let $\mathfrak{g}$ be the Lie p–subalgebra of W such that all e_{ijk} with $(ijk) \notin \{(002), (102), \ldots, (p-1, 0, 2)\}$ form a basis for $\mathfrak{g}$. Then $\mathfrak{h} = \mathfrak{g} \cap W_{\geq 0}$ is a supersolvable Lie p–subalgebra of $\mathfrak{g}$. We define an ideal $\mathfrak{a}$ in $\mathfrak{g}$ via

$$\mathfrak{a} = \sum_{j=0}^{p-1} \operatorname{ad}^j(e_{001})(W_{\geq r}).$$

<u>TypeA – characters</u> : For Type A–characters we have, so far, seen the following results:

A1) If $\chi(\mathfrak{a}) \neq 0$, then each irreducible $U_\chi(W)$–module has dimension $p^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$ and the number of irreducibles is $p^{\operatorname{rk} \mathfrak{c}_W(\chi)}$. This result has been proved for general p if $r > p$ and we find the same results when $r = p - 1, p$ and $p = 3$. Can one extend the given proof to hold for $r = p - 1, p$ and arbitrary p ?

A2) If $\chi(\mathfrak{a}) = 0$, then induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules and the isomorphism classes of irreducible $U_\chi(W)$–modules; hence the classification of the irreducible $U_\chi(W)$–modules are reduced to the classification of irreducible $U_\chi(\mathfrak{g})$–modules.

- a1) If $r = 2$ and $p = 3$ or $r = p - 1$ for arbitrary p but $\operatorname{rk} \mathfrak{c}_W(\chi) = 2$, then irreducible $U_\chi(\mathfrak{g})$–modules are a (disjoint) union of $p - 1$ irreducible $U_\chi(\mathfrak{g})$–modules all induced from irreducible $U_\chi(\mathfrak{h})$–modules and the set of irreducible $U_\chi(\mathfrak{g})$–modules annihilated by $\mathfrak{a}$. Since $\mathfrak{h}$ is supersolvable, we can classify the $p - 1$

irreducible $U_\chi(\mathfrak{g})$–modules induced from irreducible $U_\chi(\mathfrak{h})$–modules and the irreducible $U_\chi(\mathfrak{g})$–modules annihilated by $\mathfrak{a}$ correspond to irreducible $U_\chi(W(1))$–modules, where $W(1)$ is the smallest Witt–Jacobson Lie algebra. We do not observe the same pattern as in the A1)–case.

a2) For $r = 3 = p$ we find a Lie p–subalgebra $\mathfrak{s} \subset \mathfrak{g}$ with the property that induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{s})$–modules and the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules and such that $\mathfrak{s}/\mathfrak{a} \simeq \mathfrak{sl}_2(K)$. It follows that the classification of irreducible $U_\chi(W)$–modules are reduced to the well–known classification of $\mathfrak{sl}_2(K)$ (or $W(1)$ since $W(1) \simeq \mathfrak{sl}_2(K)$ as restricted Lie algebras when $p = 3$). We do not observe the same pattern as in the A1)–case.

The reason for the difference when $r = p - 1$ and $r = p$ is, that $\mathfrak{a}$ is unipotent for $r > p - 1$ and hence $\chi(\mathfrak{a}) = 0$ automatically implies that $\mathfrak{a}$ annihilates all irreducible $U_\chi(\mathfrak{g})$–modules. This is not the case when $r = p - 1$ and we have to add extra irreducibles. I suggest that one, for arbitrary p, should prove similar statements as those in the a1)–case for $r = p - 1$ and similar statements as those in the a2)–case for $r > p - 1$. I don't know whether it is possible to find $\mathfrak{s}$ such that $\mathfrak{s}/\mathfrak{a} \simeq W(1)$ or $\mathfrak{s}/\mathfrak{a} \simeq \mathfrak{sl}_2(K)$, but the representation theory of $\mathfrak{s}/\mathfrak{a}$ should at least be well–described.

$\underline{\text{TypeB} - \text{characters}}$: For Type B–characters we have, so far, seen the following results:

B1) If $\chi(\mathfrak{a}) \neq 0$, then each irreducible $U_\chi(W)$–module has dimension $p^{\operatorname{codim}_W \mathfrak{c}_W(\chi)/2}$ and the number of irreducibles is $p^{\operatorname{rk} \mathfrak{c}_W(\chi)}$. This result has almost been proved for general p if $r > p$ and we find the same results when $r = p - 1, p$ and $p = 3$. Can one extend the given proof to hold for $r = p - 1, p$ and arbitrary p ?

B2) If $\chi(\mathfrak{a}) = 0$, then induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules annihilated by $\mathfrak{a}$ and the isomorphism classes of irreducible $U_\chi(W)$–modules; hence the classification of the irreducible $U_\chi(W)$–modules are reduced to the classification of irreducible $U_\chi(\mathfrak{g}/\mathfrak{a})$–modules.

b1) If $r = 2$ and $p = 3$ we have $\mathfrak{g}/\mathfrak{a} \simeq \mathfrak{sl}_2(K)$ and so irreducible $U_\chi(\mathfrak{g}/\mathfrak{a})$–modules are just irreducible $U_\chi(\mathfrak{sl}_2(K))$–modules. We do not observe the same pattern as in the B1)–case.

b2) For $r = 3 = p$ the situation is very complicated and requires computations, which are impossible to carry out for arbitrary p.

At the moment, I have no suggestions of what to do in the b1) and b2)–case (for arbitrary p). Maybe, one should try to find a Lie p–subalgebra $\mathfrak{s}$ inside $\mathfrak{g}$ such that induction induces a bijection between the isomorphism classes of irreducible $U_\chi(\mathfrak{s})$–modules and the isomorphism classes of irreducible $U_\chi(\mathfrak{g})$–modules annihilated by $\mathfrak{a}$. Moreover, one should know the representation theory of $\mathfrak{s}/\mathfrak{a}$.

Question 2 : How can we classify the set of irreducible $U_\chi(W)$–modules in the case where χ has height $r = 2p - 3$ and Type II.a.

It is enough to consider irreducible $U_\chi(W_{\geq 0})$–modules since induction induces a bijection between the isomorphism classes of irreducible $U_\chi(W_{\geq 0})$–modules and the isomorphism classes of irreducible $U_\chi(W)$–modules. The computations for $p = 3$ say that we can find a Lie p–subalgebra $\mathfrak{s}$ of $W_{\geq 0}$ and a p–ideal $\mathfrak{a} \lhd \mathfrak{s}$ such that irreducible $U_\chi(W_{\geq 0})$–modules are induced from irreducible $U_\chi(\mathfrak{s}/\mathfrak{a})$–modules. Now we are done since irreducible $U_\chi(\mathfrak{s}/\mathfrak{a})$–modules are well described ($\mathfrak{s}/\mathfrak{a} \simeq \mathfrak{sl}_2(K)$). But for $p = 3$, already, the computations are difficult.

Is it possible, for arbitrary p, to find $\mathfrak{s}$ and $\mathfrak{a}$ with the properties as above? Maybe one cannot prove $\mathfrak{s}/\mathfrak{a} \simeq \mathfrak{sl}_2(K)$ but as long the representation theory of $\mathfrak{s}/\mathfrak{a}$ is well known the reduction is useful.

Question 3 : How can we classify the set of irreducible $U_\chi(W)$–modules in the case where χ has maximal height.

The representation theory of $U_\chi(W)$ when χ has maximal height is not very well understood. In the examples I have seen so far, one constructs χ such that $W_{\geq 0}$–modules with p–character χ has maximal dimension p^{p^2-1}. Therefore, they all extend to W. It is however not clear how to compute the number of irreducibles in those cases.

I have considered the case where $p = 3$ but a complete understanding in this *simple* case is still far away (for me). The computations for $p = 3$ show that there exists χ of maximal height and an irreducible $U_\chi(W)$–module of non–maximal dimension (without a classification of these); this is in fact the most interesting observation.

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