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On Frobenius Splittings of the Wonderful Compactification of the Unipotent Variety

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Contents

Introduction

This thesis consists of the results that I have obtained during my graduate studies at University of Aarhus. The following section contains a summary as well as a historical review of the three main themes: the wonderful compactification, the unipotent variety, and Frobenius splittings. The results of this thesis concerns with the relation of these themes.

Summary

The paper is organized in 5 chapters. In the first chapter, I settle the notation and recall several well-known facts of linear algebraic groups and representations of these.

In chapter 2, I review the construction of the wonderful compactification of a connected semi-simple adjoint linear algebraic group G_{ad} over an algebraically closed field of positive characteristic and show the properties of it. The main ideas go back to the original paper [DC-P] by De Concini and Procesi in 1983. They construct the wonderful compactification of any symmetric variety in characteristic zero. Note that a symmetric variety is the quotient of G_{ad} with the fixpoint set under an involution of G_{ad}. The wonderful compactification X is a smooth, irreducible compactification containing the symmetric variety as an open subset. The boundary is a finite union of smooth G-stable divisors with normal crossings. Further, the compactification has finitely many G_{ad}-orbits. Also, De Concini and Procesi proved that the Picard group $\mathrm{Pic}(X)$ of the wonderful compactification is a sublattice of the $\Lambda \times \Lambda$ where Λ is the weight space.

Consider the involution $(x, y) \mapsto (y, x)$ of $G_{ad} \times G_{ad}$. Then the fixpoint set is the diagonal $\Delta(G_{ad}) = \{(g, g) \,|\, g \in G_{ad}\}$ and hence the symmetric variety is $G_{ad} \times G_{ad} / \Delta(G_{ad}) \simeq G_{ad}$. Thus, G_{ad} is a symmetric variety. The wonderful compactification of G_{ad} is called the *group compactification*. In her paper [Str] from 1987, Strickland constructs the group compactification in positive characteristics and proves the properties mentioned above. Further, she proves that the wonderful compactification is Frobenius split and also gives an explicit description of Pic(X): It is the elements on the form $(-w_o\lambda, \lambda)$ for a weight $\lambda \in \Lambda$. Hence, in the group compactification, the Picard group is isomorphic to the weight space Λ.

In 1999 in their paper [DC-S], De Concini and Springer generalized the results of [DC-P] to all positive characteristics. Also, they proved that the wonderful compactification of any symmetric variety is Frobenius split and the global sections of any ample line bundle on X has a good filtration.

The third chapter covers the unipotent variety $\mathcal{U}$, i.e. the subset of all unipotent elements of G. Especially, it is proven that $\mathcal{U}$ is a complete intersection and that it is regular in codimension 1 which combined implies that $\mathcal{U}$ is normal.

The proof that $\mathcal{U}$ is a complete intersection, goes back to the paper [St1] from 1965. Herein, the Steinberg map and the cross-section $\mathcal{C}$ are introduced which are essential parts of the proof. Also, a algebraic geometric result by Kostant [K] (1963) is used. The proof of normality relies also on the paper [St1] where Steinberg obtained several criteria for regularity. Actually, he extended the definition of regularity by Chevalley ([Chev] p. 7-03) such that an element of a semi-simple linear algebraic group G is regular (in the sense of Steinberg; see [St1] §1) when the dimension of its centralizer equals the rank of G. In particular, Steinberg proves in [St1] Theorem 6.11 that the regular elements in $\mathcal{U}$ form a open set and the complement has at least codimension 2 in $\mathcal{U}$. Since he also proved in [St1] Theorem 3.3 that the regular unipotent elements are all conjugate, the regular unipotent elements are non-singular in $\mathcal{U}$ (cf. [St2] Theorem 3.10.7). Thus, $\mathcal{U}$ is normal by Serre's normality criterion (for a locally complete intersection; see e.g. [Ha1] Proposition II.8.23(b)). However, the approach in this thesis uses the 'shortcut' that there are finitely many unipotent conjugacy classes, to prove the existence of regular unipotent element. This well-known, but non-trivial result is due to Richardson in characteristic zero ([Ri]) and Lusztig in positive characteristics ([Lu]). Chapter 3 ends with a result that shows that the unipotent variety of G maps bijectively onto the unipotent variety of G_{ad} under a central isogeny $\pi : G \to G_{ad}$.

The unipotent variety $\mathcal{U}$ of a connected semi-simple linear algebraic group is studied in many details. However the closure of $\mathcal{U}$ in the group compactification has not been studied as closely. A natural question (at least for an algebraic geometer) is if the closure $\bar{\mathcal{U}}$ is Frobenius split. Here, we answer affirmative in certain cases. In fact, this is the main result of this thesis and appears in chapter 4 where we find a explicit Frobenius splitting of the group compactification X such that the boundary divisors and the closure of $\mathcal{U}$ in X is compatibly Frobenius split. This is however under the very restrictive assumption that the fundamental characters[1] map the identity element of G to zero in $\Bbbk$. See Remark 4.3.5 on page 48 for the explicit cases. Further, the Frobenius splitting constructed is B-canonical.

We begin the chapter by reviewing the general theory of Frobenius splittings as defined by Mehta and Ramanathan in their fundamental paper [M-R] from 1985. The (absolute) Frobenius map on a scheme Z is the identity on points and the p'th power map on the level of the functions $\mathcal{O}_Z \to F_*\mathcal{O}_Z$. Then Z is Frobenius split if $\mathcal{O}_Z \to F_*\mathcal{O}_Z$ splits. The main reason that the method of Frobenius splittings turn out so powerful is that the pull back $F_*\mathcal{L}$ of a line bundle on a Frobenius split scheme equals $\mathcal{L}^p$. This implies among other things the vanishing of all the higher cohomology groups of ample line bundles on the Frobenius split projective scheme.

As Mehta and Ramanathan discovered in [M-R] then the flag variety ${}^{G}/_{B}$ is Frobenius split compatibly splitting the Schubert varieties in ${}^{G}/_{B}$. Here B is a Borel subgroup of a semi-simple algebraic group G. Actually, flag varieties and Schubert varieties are one of main classes of examples of Frobenius split varieties. From a Frobenius splitting of a flag variety, Strickland showed how to construct a Frobenius splitting of the group compactification X compatibly splitting all orbit-closures in X. A crucial ingredient of her proof is that ${}^{G}/_{B} \times {}^{G}/_{B}$ is the unique closed orbit in X. In section 4.2, we look closely at this relation between Frobenius splittings of X and Frobenius splitting of ${}^{G}/_{B} \times {}^{G}/_{B}$. Further, we give a criterion for Frobenius splittings of ${}^{G}/_{B} \times {}^{G}/_{B}$ due to Lauritzen and Thomsen [L-T]. Other important examples (from the view of this thesis) of Frobenius split varieties are the wonderful compactifications of arbitrary symmetric varieties (cf. [DC-S]) and the large Schubert varieties (cf. [B-P2]), i.e. the closure in the group compactification of double cosets $B\dot{w}B$ of G. Here the $\dot{w}$ is a representative in the normalizer of a maximal torus T (contained in the Borel subgroup B) of G where w is an element of the Weyl group of G with respect to T. Actually, the latter paper (i.e. [B-P2]) has been of great inspiration in our attempt to prove that the wonderful compactification $\bar{\mathcal{U}}$ of the unipotent variety $\mathcal{U}$ is Frobenius split. Here, $\bar{\mathcal{U}}$ is the closure in the group compactification of the unipotent variety $\mathcal{U}$.

Section 4.3 contains our main result. It is joint work with my advisor Jesper Funch Thomsen. We construct a global sections ϕ_i of certain line bundles $\mathcal{L}_i$ on the group compactification X such that $\phi_{i|G_{ad}} \circ \pi$ equals the fundamental characters where $\pi : G \to G_{ad}$ is a simply connected covering. The common zero subset of fundamental characters is related to

[1] A fundamental character is the composition of the trace map and the representation associated with the fundamental weight

the unipotent variety giving us the result together with the results developed in section 4.2.

After a short crash course on B-canonical endomorphisms, we can prove that the Frobenius splitting constructed in the previous section (section 4.3) is B-canonical. The notion of B-canonical endomorphism is due to Mathieu in his paper [Mat] from 1990. Here, he proves what is the motivation for definition (and for us); namely, that if a G-scheme B-canonical Frobenius split then the global sections of a G-linearized line bundle allow a good filtration. We use a criterion for a smooth projective variety to be B-canonical Frobenius split due to van der Kallen in [vdK2]. With this, we prove that the Frobenius splitting of X which compatibly splits $\mathcal{U}$ is in fact also B-canincal. This implies that $\bar{\mathcal{U}}$ is B-canonical F-split.

The last chapter provides some geometric applications of the main result in chapter 4. Using that the wonderful compactification $\bar{\mathcal{U}}$ of the unipotent variety is Frobenius split in certain cases, we prove that $\bar{\mathcal{U}}$ is a locally complete intersection and is regular in codimension 1 in these cases. Thus, $\mathcal{U}$ is normal by [Ha1] Proposition II.8.23(b). We also give a partial result on the Picard group of $\mathcal{U}$ using that $\mathcal{U}$ is a locally complete intersection (in certain cases). The map $\mathrm{Pic}(X) \to \mathrm{Pic}(\bar{\mathcal{U}})$ induced by the inclusion map $\bar{\mathcal{U}} \to X$ is injective.

I include two appendices. In the first, we find the restrictions on G_{ad} coming from the condition that all fundamental characters χ_i map the identity element e of G to zero. Observing that $\chi_i(e)$ is the dimension of the representation associated to the fundamental weight ω_i. When answering one question, more questions arise. In appendix B, I have gathered some of these questions that I have not been able to answer; this may be ascribed lack of time as well as my mathematical limitations.

I should note that general references are [Spr], [Jan], [Ha1], [E], and [Hum]. In chapter 2 and 4, I have referred to [B-K] and in chapter 3, the main reference is [Hum2]. I hope this proves to be of some convenience for the reader. I have tried to include historical comments to give the reader a feeling of the development of the theories.

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During my graduate studies, I visited University of North Carolina, Chapel Hill during the fall 2003. I take here the opportunity to acknowledge the support I received at the mathematical department. Especially, I like to thank Shrawan Kumar for his and his wife's help and hospitality toward me and my family during our stay in Chapel Hill. Also, I am indebted to Shrawan Kumar for letting me read the at that time unpublished book [B-K] by him and Michel Brion. I benefited a lot gaining much insight in the methods of Frobenius splittings.

Last but not the least, I thank my family for patience and support.

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Chapter 1

Prerequisites

This chapter fixes notation not already defined and reviews some known results which shall be used in the subsequent chapters. General references are [Spr], [Jan], [Ha1], [E], and [Hum].

1.1 Linear Algebraic Groups

Let G denote a connected reductive linear algebraic group over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. Let T (respectively $B \supseteq T$) be a maximal torus (respectively a Borel subgroup) of G. Let B^- be the opposite Borel subgroup, i.e. $B \cap B^- = T$. The dimension of T is called the rank of G and is denoted ℓ.

Let Φ denote the roots of G wrt. T (see e.g. [Spr] §7.4.3) and we denote the positive roots wrt. B with Φ^+ ([Spr] Proposition 7.4.6). Then the characters $X^*(T)$ of T is a sublattice of the weight lattice Λ. Let Δ be the simple roots, i.e. linearly independent positive roots such that any other positive root is a linear combination of these roots with non-negative coefficients cf. [Spr] Theorem 8.2.8(iii). Indexing the simple roots $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ let ω_i be the corresponding fundamental weight of α_i (see [Spr] Exercise 8.2.11(3)). The fundamental weights form a basis of the weights Λ cf. [Hum] §13.1. Let $\rho = \sum_{i=1}^{\ell} \omega_i = {}^1\!/_2 \sum_{\alpha \in \Phi^+} \alpha$. Note that a character of B gives a character of T. Conversely, by [Spr] Theorem 6.3.5(iv) $B = TB_u$ where B_u is the unipotent radical of B and hence we get a map $\gamma : B \to {}^B\!/_{B_u} \to T$. Thus, if $\chi \in X^*(T)$ then the composition $\chi \circ \gamma$ a character of B.

The finite group $W = {}^{N_G(T)}\!/_T$ is called the Weyl group ([Spr] §7.1.4). By [Spr] Theorem 8.2.8(i) W is generated by the simple reflections $s_i = s_{\alpha_i}$, i.e. the reflections associated to the simple roots $\alpha_i \in \Delta$. Let w_o denote the unique element of W such that $w_o(\Phi^+) = -\Phi^+$ cf. [Spr] Proposition 8.2.4(ii). Note that $-\Phi^+$ is the positive roots in Φ wrt. B^-. Further, w_o is the longest element in W by [Spr] Lemma 8.3.2(ii). Note that $-w_o\rho = \rho$ by definition of w_o (i.e. $w_o(\Phi^+) = -\Phi^+$).

For each root $\alpha \in \Phi$, we have a homomorphism of groups $u_\alpha : \mathbb{G}_a \to G$ which is an isomorphism on its image U_α by [Spr] Proposition 8.1.1. Thus, U_α is a unique closed subgroup which we call the *root subgroup* associated with α. Further, $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for $t \in T, x \in \Bbbk$. By Chevalley's commutator formula ([Spr] Proposition 8.2.3) we have that $U_w := \prod_{\alpha\in\Phi(w)} U_\alpha$ is a subgroup of B_u for any ordering of the roots and any $w \in W$ where $\Phi(w) = \{\alpha \in \Phi^+ | w(\alpha) \in -\Phi^+\}$ (see also [Spr] Lemma 8.3.5). In particular, $B_u = U_{w_o} = \prod_{\alpha\in\Phi^+} U_\alpha$. Hence, U_w is an affine space of dimension $|\Phi(w)| = l(w)$ where the length $l(w)$ is defined to be the smallest integer such that w is a product of $l(w)$ simple reflections. By [Spr] Exercise 8.1.12(2) there exist non-zero constants $c_{w,\alpha}$ such that $\dot{w}u_\alpha(x)\dot{w}^{-1} = u_{w(\alpha)}(c_{w,\alpha}x)$ for all $x \in \Bbbk$, $w \in W$, $\alpha \in \Phi$. Therefore, $B^- = w_oBw_o^{-1}$.

For a set of chosen representatives $\dot{w} \in N_G(T)$ of $w \in W$ we consider the $B \times B$-orbits $B\dot{w}B$ $w \in W$. Then we have the Bruhat's Lemma ([Spr] Theorem 8.3.8): G is a disjoint union of $B\dot{w}B$ for $w \in W$. This implies the Bruhat decomposition ([Spr] Corollary 8.3.9), i.e. that an element $g \in G$ can be written uniquely in the form $u\dot{w}b$ with $w \in W$, $u \in U_{w^{-1}}$, $b \in B$. Observe that $B\dot{w}_oB$ is the unique[1] open double coset in G.

For the rest of the thesis, we let G_{ad} denote a connected semisimple adjoint linear algebraic group over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. Let T_{ad} (respectively $B_{ad} \supseteq T_{ad}$) be a maximal torus (respectively a Borel subgroup) of G_{ad}. Let B_{ad}^- be the opposite Borel subgroup, i.e. $B_{ad} \cap B_{ad}^- = T_{ad}$.

Consider the simply connected covering $\pi : G \to G_{ad}$, i.e. a surjective homomorphism of algebraic groups from the connected, semisimple, simply connected linear algebraic group G to G_{ad} where the kernel lies in the center $Z(G)$ of G (cf. [Spr] Exercise 10.1.4(1)). Let T, B, and B^- be a maximal torus, respectively Borel subgroups of G such that $\pi(T) = T_{ad}$, $\pi(B) = B_{ad}$, and $\pi(B^-) = B_{ad}^-$. By [Spr] §8.1.11 $X^*(T_{ad})$ is the root lattice and $X^*(T)$ the weight lattice. Unless explicitly stated otherwise, G denote the simply connected covering of G_{ad}.

Example 1.1.1. This is *the* standard example in this thesis which we will refer to again and again.

[1]it is unique among the the cosets on the form $B\dot{w}B$ for $w \in W$

Let $p = 2$ and $G_{ad} = PSl_2$. The simply connected covering is $G = Sl_2$. Note that the kernel of $\pi : G \to G_{ad}$ is $\{\pm \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\}$. Let T be the diagonal matrices, B the upper triangular matrices, and B^- the lower triangular matrices in G. Then $\alpha : T \to \Bbbk$ given by $\left[\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right] \mapsto t^2$ is the only positive root. Further, $\rho = \omega_\alpha$ is the weight $\left[\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right] \mapsto t$.

1.2 Representations

A *(rational) representation* of a (general) algebraic group H in a finite dimensional vector space over $\Bbbk$ is a homomorphism of algebraic group $r : H \to Gl(V)$. We use the words '(rational) representation of G' and 'G-module' interchangably. Note that if V is a H-module then $\mathbb{P}(V)$ is a G-variety, i.e. a variety Z with an action of G such that $G \times Z \to Z$ is a morphism of varieties.

A very important G-module is $\mathrm{H}^0({}^G\!/\!_B, \mathcal{L}_{G/B})$ with the line bundle on ${}^G\!/\!_B$. Let $\Bbbk_\lambda$ denote B-module $\Bbbk$ where the action of B is given by $b.x = \lambda(b)^{-1}x$ for $x \in \Bbbk$, $b \in B$. Then $G \times^B \Bbbk_\lambda$ is a G-equivariant line bundle on ${}^G\!/\!_B$ defined by λ (see e.g. [Spr] §8.5.7). Let $\mathcal{L}_{G/B}(\lambda)$ denote the corresponding locally free sheaf of rank 1 cf. [Ha1] Exercise II.5.18. Then $\mathrm{H}^0(\lambda) := \mathrm{H}^0({}^G\!/\!_B, \mathcal{L}_{G/B}(\lambda))$ is finite dimensional vector space by [Spr] 8.5.8. Further, $\mathrm{H}^0(\lambda) \neq 0$ if and only if $\lambda \in \Lambda^+$, the dominant weights. By [Spr] §8.5.7, we can regard $\mathrm{H}^0(\lambda)$ as the following subset of the coordinate ring $\Bbbk[G]$

$$\{f \in \Bbbk[G] | f(gb) = \lambda(b)f(g)\,,\, \forall b \in B,\, g \in G\}$$

where $\lambda \in X^*(T)$ is considered as a character of B. Furthermore, $\mathrm{H}^0(\lambda)$ is a G-module. From the proof of [Spr] Theorem 8.5.8, any weight χ of $\mathrm{H}^0(\lambda)$ satisfies $-\lambda \leq \chi \leq -w_o\lambda$ where the ordering $\leq$ of the weights is given by $\mu \leq \nu$ if $\nu - \mu$ is a non-negative linear combination of the simple roots. This ordering is called the *dominant ordering* on Λ. A dominant weight is called *regular* if when written as a linear combination of the fundamental weights all coefficients are positive.

Let $V(\lambda) = \mathrm{H}^0(\lambda)^*$ and hence $V(\lambda)$ is a G-module with highest weight λ and lowest weight $w_o\lambda$. It is called the Weyl module. Another important G-module is the Steinberg module $\mathrm{H}^0((p-1)\rho)$. It is denoted St. By [Jan] §II.3.18 and Corollary II.2.5, St is selfdual and irreducible since $-w_o\rho = \rho$.

Having H-modules (for any group H) M, N we consider $M \boxtimes N$ as a $H \times H$-module via the action $(g,h).m \otimes n = g.m \otimes h.n$, $m \in M, n \in N, g, h \in H$. It is called the *external tensor product*.

Now, the character group $X^*(T \times T)$ of $T \times T$ is isomorphic to $X^*(T) \times X^*(T)$. Hence, let $\mathcal{L}_{G/B \times G/B}(\lambda, \mu)$ be the corresponding $G \times G$-equivariant line bundle on ${}^{G}/_{B} \times {}^{G}/_{B}$ defined as before (use $G \times G$, $B \times B$ in stead of G, B). Similarly, $\mathrm{H}^0(\lambda, \mu) := \mathrm{H}^0({}^{G}/_{B} \times {}^{G}/_{B}, \mathcal{L}_{G/B \times G/B}(\lambda, \mu))$ is a $G \times G$-module. Furthermore, we have an isomorphism $\mathrm{H}^0(\lambda, \mu) \simeq \mathrm{H}^0(\lambda) \boxtimes \mathrm{H}^0(\mu)$ as $G \times G$-modules.

Note that a G_{ad}-module is also a G-module via $\pi : G \to G_{ad}$. The following Lemma is the criterion on a G-module V for the action of G on $\mathbb{P}(V)$ to factorize through the action of G_{ad}. It seems to be a well known result but we have included a proof since we were not able to find a reference.

Lemma 1.2.1. *Let $V = \oplus_{\lambda \in \Lambda} V_\lambda$ be a G-module. Then the action of G on $\mathbb{P}(V)$ factors through the action of G_{ad} if the difference of any two weights of V is in the root lattice.*

Proof. Assume that any difference of two weights of the G-module V is in the root lattice $X^*(T_{ad})$. For $x \in V$ write $x = (x_{\lambda_1}, \ldots, x_{\lambda_N})$ with $N = \dim(V)$. Let $[x] = [(x_{\lambda_1}, \ldots, x_{\lambda_N})]$ denote the image of x in $\mathbb{P}(V)$. Let $\mathbb{P}(V)_{\lambda_i}$ denote the standard open subset of $\mathbb{P}(V)$, e.g. the subset of $x \in \mathbb{P}(V)$ such that $x_{\lambda_i} \neq 0$. Observe that $\mathbb{P}(V)_{\lambda_i} \simeq \mathbb{A}^{N-1}$ by [Ha1] Proposition I.2.2.

Note that the center of G is $Z(G) = \cap_{\alpha \in \Phi} \ker(\alpha)$ by [Spr] Proposition 8.1.8(i) and $Z(G) \subseteq T$ by [Spr] Corollary 7.6.4(iii). Therefore for $g \in Z(G)$ and $x \in \mathbb{P}(V)_{\lambda_1}$ we have

$$[g.x] = [(\lambda_1(g)x_{\lambda_1}, \ldots, \lambda_N(g)x_{\lambda_N})] = [(x_{\lambda_1}, (\lambda_2 - \lambda_1)(g)x_{\lambda_2}, \ldots, (\lambda_N - \lambda_1)(g)x_{\lambda_N})]$$

Since $\lambda_i - \lambda_1 \in X^*(T_{ad})$ for $i \geq 2$ and $g \in Z(G) = \cap_{\alpha \in \Phi} \ker(\alpha)$ we have that $(\lambda_i - \lambda_1)(g) = 1$ (again for $i \geq 2$). Thus, $[g.x] = [x]$.

Therefore, we can define an action (on the level of points) of G_{ad} on $\mathbb{P}(V)_{\lambda_i}$ by $g.[x] = [g.x]$ for each i. It remains to prove that $G_{ad} \times \mathbb{P}(V)_{\lambda_i} \to \mathbb{P}(V)_{\lambda_i}$ is a morphism for all i.

Consider the following diagram:

$$\begin{tikzcd}
T \times \mathbb{P}(V)_{\lambda_i} \arrow[r, "i \times id"] \arrow[d, "\pi_{|T} \times id"] & G \times \mathbb{P}(V)_{\lambda_i} \arrow[r, "\phi"] \arrow[d, "\pi \times id"] & \mathbb{P}(V)_i \\
T_{ad} \times \mathbb{P}(V)_{\lambda_i} \arrow[r, "i_{ad} \times id"] & G_{ad} \times \mathbb{P}(V)_{\lambda_i} \arrow[ur, "\phi_{ad}"'] &
\end{tikzcd}$$

where $i : t \to G$ and $i_{ad} : T_{ad} \to G_{ad}$ are the inclusion maps. First we prove that the map $\phi_{ad|T_{ad}} : T_{ad} \times \mathbb{P}(V)_{\lambda_i} \to \mathbb{P}(V)_{\lambda_i}$ is a morphism of varieties.

Identify $\mathbb{P}(V)_{\lambda_i}$ with the affine space $\mathbb{A}^{N-1}$ then the action of G on $\mathbb{A}^{N-1}$ is $t.a = (\mu_1(t)a_1, \ldots, \mu_{N-1}(t)a_{N-1})$ where $a = (a_1, \ldots, a_{N-1}) \in \mathbb{A}^{N-1}$ and μ_i lies in the root lattice for all i. Since $Z(G) \subseteq T$ we can define similarly (at least on the level of points) the action of T_{ad} on $\mathbb{A}^{N-1}$ for some $\tilde{\mu}_i$ in the root lattice. By [Spr] §9.6.1, π defines a bijection of the roots Φ. Further, π induces a injective map $\pi^* : X^*(T_{ad}) \to X^*(T)$. As μ_i lies in the root lattice, μ_i lies in the image of π^*. Thus, $\mu_i = \pi^*(\tilde{\mu}_i)$ for $\tilde{\mu}_i$ and all i. Therefore, the morphism $\phi_{|T} : T \times \mathbb{P}(V)_{\lambda_i} \to \mathbb{P}(V)_{\lambda_i}$ factors through π and $\phi_{ad|T_{ad}} : T_{ad} \times \mathbb{P}(V)_{\lambda_i} \to \mathbb{P}(V)_{\lambda_i}$. Thus, we conclude that $\phi_{ad|T_{ad}}$ is a morphism.

We know that $B_u \times B_u^- \times T$ (respectively $(B_{ad})_u \times (B_{ad}^-)_u \times T_{ad}$) is an open dense subset of G (respectively of G_{ad}). Again using that π is a central isogeny we get a bijection of the roots in T_{ad} onto the roots of T and furthermore the root subgroups $U_\alpha \simeq U_{\beta,ad}$ are isomorphic under π where β is the image of α under this bijection. Therefore, we deduce that $B_u \simeq (B_{ad})_u$ and $B_u^- \simeq (B_{ad}^-)_u$. Thus ϕ_{ad} is a morphism of varieties when restricted to the open dense set $(B_{ad})_u \times (B_{ad}^-)_u \times T_{ad}$ and therefore ϕ_{ad} is a morphism of varieties. □

In the following chapter, we need the next Lemma. Again, we have included a proof since we could not find a reference.

Lemma 1.2.2. *Let $\phi_i \in H^0(-w_o\omega_i)_{\omega_i}$. Then ϕ_i is $B \times B$-eigenvector of weight $(\omega_i, -w_o\omega_i)$ and the zero-subset $\mathcal{V}_G(\phi_i)$ is the closure in G of Bs_iw_oB.*

Proof. We know that ϕ_i satisfies $\phi_i(gb) = (-w_o\omega_i)(b)\phi_i(g)$ and $b.\phi_i = \omega_i(b)\phi_i$ for all $b \in B$, $g \in G$. Since the action of $G \times G$ on G is given by $(g,h).x = gxh^{-1}$ for all $g,h,x \in G$ we also have that $G \times G$ acts on the coordinate ring $\Bbbk[G]$ of G by $((g,h).f)(x) = f((g^{-1},h^{-1}).x) = f(g^{-1}xh)$ for $g,h,x \in G, f \in \Bbbk[G]$. Thus,

$$((b,b').\phi_i)(x) = \phi_i(b^{-1}xb') = \omega_i(b)(-w_o\omega_i)(b')\phi_i(x)$$

for $b, b' \in B$, $x \in G$. This shows that ϕ_i is a $B \times B$-eigenvector of weight $(\omega_i, -w_o\omega_i)$.

To prove the second assertion, we consider $(t.\phi_i)(\dot{w})$ where $t \in T$ and $\dot{w}$ a representant of $w \in W = N_G(T)/T$. First observe that $(t.\phi_i)(\dot{w}) = \omega_i(t)\phi_i(\dot{w})$ by definition of ϕ_i. Now, this can be calculated differently:

$$(t.\phi_i)(\dot{w}) = \phi_i(t^{-1}\dot{w}) = \phi_i(\dot{w}\dot{w}^{-1}t^{-1}\dot{w}) = (-w_o\omega_i)(\dot{w}^{-1}t^{-1}\dot{w})\phi_i(\dot{w})$$

since $\dot{w} \in N_G(T)$. Now, $(-w_o\omega_i)(\dot{w}^{-1}t^{-1}\dot{w})\phi_i(\dot{w}) = (ww_o\omega_i)(t)\phi_i(\dot{w})$. This implies that either $\phi_i(\dot{w}) = 0$ or $ww_o\omega_i = \omega_i$. Put otherwise; if $ww_o\omega_i \neq \omega_i$ then $\phi_i(\dot{w}) = 0$.

Consider $w = s_iw_o$. Then $ww_o\omega_i = s_i\omega_i = \omega_i - \alpha_i \neq \omega_i$. Therefore $\phi_i(\dot{w}) = 0$. Since ϕ_i is $B \times B$-eigenvector, we have that the closure of $Bs_i\dot{w}_oB$ in G is contained in the zero subset $\mathcal{V}_G(\phi_i)$.

Now, $\Bbbk[G]$ is a unique factorisation domain as G is simply connected[2]. Hence, we can write $\phi_i = \prod_j \phi_{ij}$ where ϕ_{ij} are prime elements in $\Bbbk[G]$. Since $\phi(gb) = \omega_i(b)\phi(g)$ for all $b \in B$ and $g \in G$ also $\phi_{ij}(gb) = \chi_{ij}(b)\phi_{ij}(g)$ where $\sum_j \chi_{ij} = \omega_i$.

As the fundamental weights form a basis for the weight space Λ, we conclude that only one of the χ_{ij} is non-zero, i.e. this non-zero weight is ω_i. Hence, ϕ is irreducible and it implies that $\mathcal{V}_G(\phi)$ is irreducible. Therefore, $\mathcal{V}_G(\phi)$ is the closure in G of Bs_iw_oB by [Spr] Proposition 1.8.2 since both sets are closed, irreducible, and of the same dimension (i.e. codimension 1). □

Example 1.2.3. We continue our example (see the previous example 1.1.1).

We identify the Steinberg module $St = \mathrm{H}^0((p-1)\rho)$ with $\Bbbk^2$. Let the G-action on $\Bbbk$ be the obvious, namely $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} ax+by \\ cx+dy \end{smallmatrix}\right)$.

Let $m_\rho, m_{-\rho}$ denote the basis vectors $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, respectively $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Notice that m_ρ is a highest weight vector in St and $m_{-\rho}$ is a lowest weight vector.

1.3 Filtrations and Tilting Modules

It turns out that tilting modules are important for the construction of the wonderful compactification of G_{ad} as we will see in the next chapter. Here we introduce tilting modules and their properties.

Definition 1.3.1. Let M be a G-module.
(i) M is said to have a *good filtration* if there exists a filtration $(0) = F_0 \subseteq F_1 \subseteq \ldots$ of G-submodules such that $\cup_i F_i = M$ and for any $i \geq 1$ we have ${}^{F_i}/_{F_{i-1}} \simeq \mathrm{H}^0(\lambda_i)$ with $\lambda_i \in \Lambda^+$.
(ii) M is said to have a *Weyl filtration* if there exists a filtration $(0) = F_0 \subseteq F_1 \subseteq \ldots$ of G-submodules such that $\cup_i F_i = M$ and for any $i \geq 1$ we have ${}^{F_i}/_{F_{i-1}} \simeq \mathrm{H}^0(\lambda_i)^*$ with $\lambda_i \in \Lambda^+$.
(iii) Assume furthermore that M is finite dimensional. Then M is called a *tilting module* if M has both a Weyl filtration and a good filtration.

Remark 1.3.2. *(i) A Weyl filtration and a good filtration are usually distinct in characteristic $p > 0$ since $H^0(\lambda)^*$ is not always isomorphic to $H^0(\mu)$ for some $\mu \in \Gamma$.*
(ii) It is clear that M has a Weyl filtration if and only if M^* has a good filtration. Thus, M is a tilting module if both M and M^* have a good filtration.

[2] It follows by [Iv] Theorem 2.7 and [Ha1] Proposition II.6.2 and Corollary II.6.16

In characteristic zero, it is easy to see that tilting modules exist because $\mathrm{H}^0(\lambda)$ is a tilting module for each $\lambda \in \Lambda^+$. More generally, if $\mathrm{H}^0(\lambda)$ is irreducible then $\mathrm{H}^0(-w_o\lambda)$ is a tilting module since $\mathrm{H}^0(-w_o\lambda)^* \simeq \mathrm{H}^0(\lambda)$. But in positive characteristics, the existence of tilting modules is not trivial. Fortunately, we have:

Theorem 1.3.3. ([Jan] Lemma II.E.3 + Lemma II.E.5)
(i) For any $\lambda \in \Lambda^+$, there is a unique indecomposable tilting module $T(\lambda)$ such that $\dim T(\lambda)_\lambda = 1$ *and any weight μ of $T(\lambda)$ satisfies $\mu \leq \lambda$ where $\leq$ is the dominant order on Λ.*
(ii) $T(\mu) \simeq T(\lambda)$ if and only if $\mu = \lambda$. Especially, $T(\lambda)^* = T(-w_o\lambda)$.

Chapter 2

The wonderful compactification of $\mathbf{G_{ad}}$

For the construction of the wonderful compactification X of G_{ad} as well as some of the properties of X, we follow [B-K] chapter 6. The main ideas come from the original paper on the wonderful compactifications of symmetric varieties [DC-P] by De Concini and Procesi. The results were obtained in characteristic zero. Strickland in [Str] generalized the theory for compactification of the adjoint group. Later in [DC-S], De Concini and Springer extended the results of [DC-P] to positive characteristics (except characteristic 2).

2.1 Construction

Let G_{ad} be a connected, semisimple, adjoint linear algebraic group over $\Bbbk$. Consider the simply connected covering $\pi : G \to G_{ad}$, i.e. a surjective homomorphism of algebraic groups from the connected, semisimple, simply connected linear algebraic group G to G_{ad} where the kernel lies in the center $Z(G)$ of G (cf. [Spr] 10.1.4(1)). Then we will in this section construct a smooth $G_{ad} \times G_{ad}$-equivariant compactification X of G_{ad} (here we extend the $G_{ad} \times G_{ad}$ action on G_{ad} given by left and right multiplication to X).

The main idea is to embed G_{ad} into the projective space $\mathbb{P}(\mathrm{End}_{\Bbbk}(M))$ of endomorphism of some suitable G-module M. It can be done $G \times G$-equivariantly. Therefore, the closure of G_{ad} in $\mathbb{P}(\mathrm{End}_{\Bbbk}(M))$ is an equivariant compactification of G_{ad}.

Next, we summarize some of the properties needed for the G-module M:

Lemma 2.1.1. *([B-K] Lemma 6.1.1)*
Let $\lambda \in \Lambda$ be a regular dominant weight. Then there exists a finite dimensional G-module M with the following properties:
(i) The T-eigenspace M_λ has dimension 1, and all other weights of M are $\lneqq \lambda$
(ii) For $\alpha \in \Phi^+$ we have $\mathfrak{g}_{-\alpha} M_\lambda \neq 0$
(iii) $\mathfrak{g}_\alpha M^*_{-\lambda} \neq 0$ for all $\alpha \in \Phi^+$ where $M^*_{-\lambda}$ is the T-eigenspace of weight $-\lambda$ in M^*

Proof. Let M be the indecomposable tilting module $T(\lambda)$ cf. [Jan] Proposition II.E.6. From this proposition (i) is clear (see also Theorem 1.3.3 on page 7).

To prove (ii) observe that $\mathfrak{g}_\alpha M_\lambda = 0$ since λ is the highest weight of M. Therefore for $\alpha \in \Phi^+$, $X_\alpha \in \mathfrak{g}_\alpha$, $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$, and $m \in M_\lambda$ gives

$$X_\alpha X_{-\alpha}.m = [X_\alpha, X_{-\alpha}].m = \langle \lambda, \alpha^\vee \rangle m$$

It suffices to prove that $\langle \lambda, \alpha^\vee \rangle$ differs from zero when considered as an element of $\Bbbk$.

We have that $V(\lambda)_\lambda \simeq M_\lambda$ since $\dim(M_\lambda) = 1$. Hence $V(\lambda) \subseteq M$ because M is indecomposable and $V(\lambda)$ is generated by the B-stable line of weight λ by [Jan] Lemma II.2.13(b). Therefore (ii) follows from [Jan] Lemma II.8.4 since λ is a regular dominant weight.

Since $T(\lambda)^* \simeq T(-w_o\lambda)$, the highest weight line wrt. B in M^* also satisfies (ii) (i.e. $\mathfrak{g}_{-\alpha} M^*_{-w_o\lambda} \neq 0$ for all simple roots α). As $Bw_o = w_o B^-$ we get by multiplying with w_o that $w_o M^*_{-w_o\lambda}$ is line consisting of B^--eigenvectors. Hence we have that $\mathfrak{g}_\alpha M^*_{-\lambda} \neq 0$ for all simple roots α showing (iii). $\square$

Remark 2.1.2. *In characteristic zero, M can be taken as the simple G-module with highest weight λ (λ is still a regular dominant weight), e.g. $H^0(\lambda)$. When char($\Bbbk$) $= p > 0$ the Steinberg module $St = H^0((p-1)\rho)$ can be chosen (i.e. $\lambda = (p-1)\rho$).*

Another remark - although stated here as a Lemma - shows how to construct a G-module satisfying the properties (i) to (iii) of Lemma 2.1.1 above.

Lemma 2.1.3. *(a) Let $M = M_1 \otimes \cdots \otimes M_s$ where M_i is a tilting module with highest weight λ_i such that $\sum_{i=1}^s \lambda_i$ is a regular dominant weight. Then M satisfies (i)–(iii) in Lemma 2.1.1.*
(b) Let M', M'' be G-modules that satisfies (i)–(iii) of Lemma 2.1.1. Then $M = M' \otimes M''$ also satisfies (i)–(iii) of Lemma 2.1.1. More generally, M'' only has to satisfy (i) in Lemma 2.1.1.

Proof. Note first that M is a G-module with the action $g.(v_1 \otimes \cdots \otimes v_s) = g.v_1 \otimes \ldots g.v_s$ for $g \in G$ and $v_i \in M_i$.

Since M_i is a finitely dimensional module with highest weight λ_i, M is a finitely dimensional module with highest weight $\sum_{i=1}^{s} \lambda_i$. Also, the weights of M_i are less than λ_i in the dominant ordering. Hence M satisfies condition (i).

Observe that the Lie algebra action of $\mathfrak{g}$ is

$$A.(v_1 \otimes \cdots \otimes v_s) = \sum_{i=1}^{s} v_1 \otimes \cdots \otimes A.v_i \otimes \cdots \otimes v_s$$

for $A \in \mathfrak{g}$. Therefore to prove condition (ii), we note that for each $\alpha \in \Delta$ there exists $i \in \{1, \ldots, s\}$ such that $\langle \lambda_i, \alpha^\vee \rangle \neq 0$ since $\sum_{i=1}^{s} \lambda_i$ is regular. By the calculations in the proof of condition (ii) in Lemma 2.1.1 on page 9, we deduce that condition (ii) holds for M.

To prove the last condition, notice that M^* is a tilting module with highest weight $\sum_{i=1}^{s} -w_o\lambda_i$. This is again a regular dominant weight. So argueing similarly as for condition (ii) proves condition (iii) of Lemma 2.1.1.

To prove (b), we can argue similarly to prove (i). Now, observe that the Lie algebra action is factorwise. Thus, since M' satisfy (ii) and (iii) in Lemma 2.1.1 so does M. $\square$

Fix a choice for a weight $\lambda \in \Lambda$ and a G-module M satisfying conditions (i) to (iii) in Lemma 2.1.1 on page 9. Consider the $G \times G$-module $\mathrm{End}_{\Bbbk} M \simeq M^* \otimes M$. Let $h \in \mathrm{End}_{\Bbbk}(M)$ denote the identity whereas $[h]$ denotes its image in $\mathbb{P}(\mathrm{End}_{\Bbbk}(M))$. We will need the following observations later:

Remark 2.1.4. *([B-K] Lemma 6.1.5)*
We can write $h = \sum_\mu h_\mu \in End_{\Bbbk}(M)$ where h_μ is a $1 \times T$-eigenvector of weight μ. Choose a basis (m_i) of T-eigenvectors of M and let (m_i^*) denote the dual basis of M^*. Since $End_{\Bbbk}(M) \simeq M^* \otimes M$ we get that $h_\mu = \sum m_i^* \otimes m_i$ where the sum is over those i such that m_i has weight μ. From Lemma 2.1.1 (i) and (ii), the properties below of h_μ are easily deduced:
(i) $h_\lambda = m_\lambda^* \otimes m_\lambda$
(ii) $h_{\lambda-\alpha} \neq 0$ for all positive roots $\alpha \in \Phi^+$
(iii) If $h_\mu \neq 0$ then $\mu \leq \lambda$

Consider the closure in $\mathbb{P}(\mathrm{End}_{\Bbbk}(M))$ of $(G \times G).[h]$. We will denote this closure by X. We claim that X is in fact the wonderful compactification of G_{ad}:

Lemma 2.1.5. *([B-K] Lemma 6.1.3)*
The orbit $(G \times G).[h] \subseteq \mathbb{P}(End_{\Bbbk}(M))$ is isomorphic to the homogeneous space G_{ad}.

Proof. Observe that by Lemma 1.2.1 on page 4 $G_{ad} \times G_{ad}$ acts on $\mathbb{P}(\mathrm{End}_{\Bbbk}(M))$ since M satisfis condition (i)–(iii) in Lemma 2.1.1 on page 9. Therefore, the orbit $(G_{ad} \times G_{ad}).[h]$ equals the orbit $(G \times G).[h]$ since G_{ad} is isomorphic to ${}^{G}/_{Z(G)}$ as abstract groups.

Consider the isotropy group $(G_{ad} \times G_{ad})_{[h]}$. It consists of the pairs (g_1, g_2) such that $g_1 g_2^{-1}$ acts on M by a scalar; thus $g_1 g_2^{-1} \in Z(G_{ad})$ implies that $g_1 = g_2$.

In the same way, the isotropy Lie algebra $(\mathfrak{g} \times \mathfrak{g})_{[h]}$ consists of the pairs (x, x). So we can conclude that we have the above isomorphism by [Spr] Theorem 5.3.2(c) since ${}^{G_{ad} \times G_{ad}}/_{\Delta(G_{ad})} \simeq G_{ad}$. □

Notice that by Lemma 2.1.5 above, X is a $G_{ad} \times G_{ad}$-equivariant compactification of G_{ad} such that G_{ad} is an open subset of X (cf. [Spr] Lemma 2.3.3(i)). The next section will show that X satisfies the rest of the properties of the wonderful compactification, e.g. that X is smooth and has ℓ smooth boundary divisors which cross normally.

Example 2.1.6. We continue our example (see the previous examples - examples 1.1.1, 1.2.3).

We choose $\lambda = \rho$ and $M = St = \Bbbk^2$ as described in Example 1.2.3 on page 6. Let $m_\rho^*, m_{-\rho}^*$ denote the dual basis of $m_\rho, m_{-\rho}$. Then $h = m_\rho^* \otimes m_\rho + m_{-\rho}^* \otimes m_{-\rho} \in St^* \otimes St$. This is G-invariant as follows:

$$\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right).h = \left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right) + \left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) = h$$

in characteristic $p = 2$. Since $St^* \otimes St \simeq \mathrm{Mat}_2(\Bbbk)$, we get that $\mathbb{P}(\mathrm{End}_{\Bbbk}(St)) \simeq \mathbb{P}^3$. As $G_{ad} = PSl_2$ has dimension 3, we conclude that the wonderful compactification of G_{ad} is $X = \mathbb{P}^3$. Let the isomorphism $St^* \otimes St \simeq \Bbbk^4$ be given by $m_\rho^* \otimes m_\rho \mapsto \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $m_\rho^* \otimes m_{-\rho} \mapsto \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, $m_{-\rho}^* \otimes m_\rho \mapsto \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, and $m_{-\rho}^* \otimes m_{-\rho} \mapsto \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

Observe that

$$\left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right).h = a \cdot m_\rho^* \otimes m_\rho + b \cdot m_{-\rho}^* \otimes m_\rho + c \cdot m_\rho^* \otimes m_{-\rho} + d \cdot m_{-\rho}^* \otimes m_{-\rho}$$

which maps to $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ under the isomorphism $St^* \otimes St \simeq \mathrm{Mat}_2(\Bbbk)$ as defined above. Therefore, the $G \times G$-action on $\mathrm{Mat}_2(\Bbbk)$ is $(g_1, g_2).A = g_1^{-1} A g_2$. Hence, we get a $G_{ad} \times G_{ad}$-action on $\mathbb{P}^3$ when points in $\mathbb{P}^3$ are considered as matrices by the above isomorphism.

2.2 Properties of X

We have constructed a $G_{ad} \times G_{ad}$-equivariant compactification of G_{ad}, namely X. We will in this subsection show that the constructed X is actually the celebrated wonderful compactification cf. [DC-P], [Str], and [DC-S].

To be more precise, we will show the following theorem:

Theorem 2.2.1. *([B-K] Theorem 6.1.8)*
(i) X is an irreducible, smooth, projective $G_{ad} \times G_{ad}$-equivariant variety containing G_{ad} as an open subset.
(ii) The boundary $\partial X := X \setminus G_{ad}$ is the union of ℓ smooth prime divisors $X_1, \ldots, X_\ell$ with normal crossings.
(iii) For each subset $I \subseteq \{1, \ldots, \ell\}$, the intersection $X_I := \cap_{i \in I} X_i$ is the closure Y_I of a unique $G_{ad} \times G_{ad}$-orbit. Conversely, any closure of a $G_{ad} \times G_{ad}$-orbit in X equals Y_I for a unique I.
(iv) The unique closed $G_{ad} \times G_{ad}$-orbit in X is $Y := Y_{\{1,\ldots,\ell\}} = \cap_{i=1}^{\ell} X_i$ which is $G \times G$-isomorphic to ${}^{G}/_{B} \times {}^{G}/_{B}$.

The outline of the proof is as follows: We will first prove that T_{ad} can be embedded in the affine space $\mathbb{A}^\ell$ such that $\mathbb{A}^\ell \setminus T_{ad}$ is the union of ℓ smooth hyperplanes of $\mathbb{A}^\ell$ that have normal crossings. Furthermore, the closure of T_{ad}-orbits are in 1-1 correspondence with subsets of $\{1, \ldots, \ell\}$ similar to (iii) of the Theorem 2.2.1 above. To relate these results to X, we find an open affine subset X_o of X such that $\overline{T_{ad}} \cap X_o \simeq \mathbb{A}^\ell$. Further, we prove that $X = (G \times G).X_o$ and, most significantly, that $X_o = (B_u \times B_u^-).\mathbb{A}^\ell$.

Following the idea outlined above we consider T_{ad}. By Lemma 2.1.5 we have that T_{ad} can be identified with the orbit $(T \times T).[h]$. Let $\overline{T_{ad}}$ denote the closure in X of T_{ad} (viewed as the orbit $(T \times T).[h]$). Consider the affine subset $\mathbb{P}_o$ in $\mathbb{P}(\mathrm{End}(M))$ defined to be the elements where the coefficient to $m_\lambda^* \otimes m_\lambda$ equals 1. Note that $\mathbb{P}_o$ is $B \times B^-$-stable. Define $\overline{T_{ad,o}} = \overline{T_{ad}} \cap \mathbb{P}_o$. This is an affine open $T \times T$-invariant subset of $\overline{T_{ad}}$.

Consider $\mathbb{A}^\ell$ as a T-variety with T-action given by

$$t.(x_1, \ldots, x_\ell) = (\alpha_1(t^{-1})x_1, \ldots, \alpha_\ell(t^{-1})x_\ell)$$

Then $\mathbb{A}^\ell$ is an embedding of T_{ad} (i.e. $t \mapsto t.(1, \ldots, 1)$).

The next result shows that there is an isomorphism of $\mathbb{A}^\ell$ onto $\overline{T_{ad,o}}$. Further, $\mathbb{A}^\ell \setminus T_{ad}$ has the required properties.

Lemma 2.2.2. *([B-K] Lemma 6.1.6)*
(i) Define $\gamma : \mathbb{A}^\ell \to \mathbb{P}(End_{\Bbbk}(M))$ by

$$(x_1, \ldots, x_\ell) \longmapsto \Big[\sum_{(n_1,\ldots,n_\ell) \in \mathbb{N}^\ell} x_1^{n_1} \cdots x_\ell^{n_\ell} \, h_{\lambda - n_1\alpha_1 - \cdots - n_\ell\alpha_\ell} \Big] \tag{2.1}$$

Then γ is isomorphism onto its image and $Im(\gamma) = \overline{T_{ad,o}}$.
(ii) The Weyl group W of G with respect to T acts on $\overline{T_{ad,o}}$ via the diagonal

action[1] and $\overline{T_{ad}} = \cup_{w\in W}(w,w).\overline{T_{ad,o}}$
(iii) The boundary $\mathbb{A}^\ell \setminus T_{ad}$ is the union of ℓ hyperplanes which have normal crossings. Further, the T-orbit closures are in 1-1 correspondance to subsets of $\{1,\dots,\ell\}$.

Proof. The coefficient in front of h_λ in the expression (2.1) is 1 and hence $\mathrm{Im}(\gamma) \subseteq \mathbb{P}_o$. Further, the coefficient in front of h_μ (for $\mu \leq \lambda$ in the dominant order) is a polynomial expression of the coefficients in front of $h_{\lambda-\alpha_i}$ for $i = 1,\dots,\ell$. These are non-zero (by Remark 2.1.4 on page 11) and linearly independent as the simple roots are. Especially, γ is an isomorphism onto its image.

To find $\mathrm{Im}(\gamma)$, consider the $T\times T$-action on h. By Remark 2.1.4, we get:

$$\begin{aligned}(1,t).h &= \sum_\mu \mu(t)h_\mu \\ &= \sum_{(n_1,\dots,n_\ell)\in\mathbb{N}^\ell} (\lambda - n_1\alpha_1 - \cdots - n_\ell\alpha_\ell)(t)h_{\lambda-n_1\alpha_1-\cdots-n_\ell\alpha_\ell}\end{aligned}$$

This shows that the map $T_{ad} \to (T_{ad}\times T_{ad}).[h]$, $t \mapsto (1,t).[h]$ factors through γ and the map $T_{ad} \to \mathbb{A}^\ell$, $t \mapsto t.(1,\dots,1)$.

Now, $\overline{T_{ad,o}}$ is closed in $\mathbb{P}_o$ and hence $\gamma^{-1}(\overline{T_{ad,o}})$ is closed in $\mathbb{A}^\ell$. We also have that $\gamma^{-1}(\overline{T_{ad,o}})$ contains the open dense subset T_{ad} of $\mathbb{A}^\ell$. Thus, $\gamma^{-1}(\overline{T_{ad,o}}) = \mathbb{A}^\ell$. Therefore $\mathrm{Im}(\gamma) \subseteq \overline{T_{ad,o}}$. But the calculation above shows that they must be equal since T_{ad} is an open subset of $\overline{T_{ad,o}}$.

To prove (ii), observe that $N_G(T)$ acts on T_{ad} by $(n,n).t = ntn^{-1}$ for $t \in T_{ad}$ and $n \in N_G(T)$. Thus $N_G(T)$ acts on the closure $\overline{T_{ad}}$. Hence T acts on $\overline{T_{ad}}$ and furthermore, T acts trivially on T_{ad}. Consequently, $W = {}^{N_G(T)}/_{T}$ acts on $\overline{T_{ad}}$.

Since $\overline{T_{ad,o}}$ is an open subset of $\overline{T_{ad}}$ then $Z := \bigcup_{w\in W}(w,w).\overline{T_{ad,o}}$ is also an open subset of $\overline{T_{ad}}$. We will prove that Z is complete. Hence $Z = \overline{Z} = \overline{T_{ad}}$ since the latter is irreducible. By (i), $\overline{T_{ad,o}}$ is the toric variety associated with the negative Weyl chamber. Thus Z is the toric variety associated with all Weyl Chambers and hence Z is complete by [Fult] Proposition 2.4 finishing the proof of (ii).

For the last assertion, observe that $\mathbb{A}^\ell \setminus T_{ad}$ is the closed subvariety of $\mathbb{A}^\ell$ where at least one coordinate is zero. Hence $\mathbb{A}^\ell \setminus T_{ad}$ is the union of the coordinate hyperplanes H_i in $\mathbb{A}^\ell$, i.e. H_i is the subvariety of $\mathbb{A}^\ell$ where the i-th coordinate is zero. Thus, $H_i \simeq \mathbb{A}^{\ell-1}$. So we have that the hyperplanes

[1] The action of $w \in W$ on $x \in \overline{T_{ad}}$ is the element $(w,w).x$ (the action here being that of $G\times G$)

H_i are closed smooth subvarieties of $\mathbb{A}^\ell$ of codimension 1. Furthermore they cross normally (cf. [Ha1] page 391, line 2) since $\cap_{i\in I} H_i$ is the subvariety where the i'th coordinate is zero for all $i \in I$. Thus, $\cap_{i\in I} H_i \simeq \mathbb{A}^{\ell-|I|}$.

Define $[h_I] \in \mathbb{A}^\ell$ such that the i-th coordinate is 0 if $i \in I$ and 1 otherwise. Notice that the i'th coordinate of $t.[h_I]$ is zero if $i \in I$. Therefore,

$$\mathbb{A}^\ell \setminus T_{ad} = \bigcup_{\emptyset \neq I \subseteq \{1,\ldots,\ell\}} T.[h_I]$$

The union is disjoint.

This shows that every T-orbit equals one of the above, i.e. there exists a subset $I \subseteq \{1,\ldots,\ell\}$ such that the orbit equals $T.[h_I]$.

Note that since the closure of $T_{ad}.[h_I]$ with $I = \{i\}$ is the hyperplane H_i we get that for an arbitrary $I \subseteq \{1,\ldots,\ell\}$, the closure of $T_{ad}.[h_I]$ is the intersection of the hyperplanes H_i with $i \in I$. □

To follow the outlined idea, we need to construct an affine subset X_o of X with the mentioned properties. It is not hard to define X_o. Let $X_o := X \cap \mathbb{P}_o$. Observe that X_o is an affine open $B \times B^-$-stable subset of X. Notice that $X_o \cap \overline{T_{ad}} = \overline{T_{ad,o}} \simeq \mathbb{A}^\ell$ by Lemma 2.2.2 on page 13.

We can give an explicit description of the $B \times B^-$-structure on X_o. Consider map $\Gamma : B_u \times B_u^{-1} \times \mathbb{A}^\ell \to X$ by $(u,v,x) \mapsto (u,v).\gamma(x)$. Notice that $\mathrm{Im}(\Gamma) \subseteq X_o$ since $\mathrm{Im}(\gamma) = \overline{T_{ad,o}}$ and X_o is $B \times B^-$-stable. Hence, Γ is a $B_u \times B_u^-$-equivariant morphism where $B_u \times B_u^-$ acts on $B_u \times B_u^- \times \mathbb{A}^\ell$ via multiplication componentwise on the first two factors. Our crucial proposition is

Proposition 2.2.3. *([B-K] Proposition 6.1.7)*
$\Gamma : B_u \times B_u^- \times \mathbb{A}^\ell \to X_o$ is an isomorphism.

Proof. We will construct a $B_u \times B_u^-$-equivariant morphism $\beta : X_o \to B_u \times B_u^-$ such that the restriction of $\beta \circ \Gamma$ to $B_u \times B_u^- \times T_{ad}$ is the map $(u,v,t) \mapsto (u,v)$. Hence β is surjective onto $B_u \times B_u^-$. Let $\beta^{-1}(1,1)$ denote the fibre of β over $(1,1) \in B_u \times B_u^-$. Hence, we can define the map $\Gamma' : B_u \times B_u^- \times \beta^{-1}(1,1) \to X_o$ by $(u,v,x) \mapsto (u,v).x$.

Using the $B_u \times B_u^-$-equivariance of β we find that $\Psi : X_o \to B_u \times B_u^- \times \beta^{-1}(1,1)$ given by $x \mapsto (\beta(x), \beta(x)^{-1}.x)$ is a $B_u \times B_u^-$-equivariant morphism such that $\Gamma' \circ \Psi = Id_{X_o}$ and $\Psi \circ \Gamma' = Id_{B_u \times B_u^- \times \beta^{-1}(1,1)}$ by the construction of Γ' and Ψ. Thus, Γ' is an isomorphism. In particular, the scheme-theoretic fiber $\beta^{-1}(1,1)$ is a irreducible variety since X_o is. Furthermore, $\dim(\beta^{-1}(1,1)) = \dim(X_o) - \dim(B_u) - \dim(B_u^{-1}) = \ell$. Observe that $\Gamma(1 \times 1 \times T_{ad}) \subseteq \beta^{-1}(1,1)$ by the construction of β. Thus

$\beta^{-1}(1,1) \supseteq \Gamma(1 \times 1 \times \mathbb{A}^\ell) = \gamma(\mathbb{A}^\ell) = \overline{T_{ad,o}}$ and $\overline{T_{ad,o}}$ is closed in X_o hence in $\beta^{-1}(1,1)$. Therefore, $\beta^{-1}(1,1) = \overline{T_{ad,o}}$ by [Spr] Proposition 1.8.2 as they are closed, irreducible varieties of the same dimension.

It remains to construct the $B \times B^-$-equivariant map β. Consider $\mathbb{P}(\mathrm{End}(M))$ as a space of rational self-maps of $\mathbb{P}(M)$ and consider the basis (m_i) of T-eigenvectors of M and its dual basis (m_i^*) cf. Remark 2.1.4 on page 11. Every morphism in $\mathrm{End}_{\Bbbk}(M) \simeq M^* \otimes M$ can then be written as $\sum_{\mu,\nu \le \lambda} a_{\mu,\nu} m_\mu^* \otimes m_\nu$. $\mathbb{P}_o$ is then the rational maps in $\mathbb{P}(\mathrm{End}(M))$ that are defined in $[m_\lambda]$ and map $[m_\lambda]$ to a point of $\mathbb{P}(M)_o$, the subset of $\mathbb{P}(M)$ with non-zero coefficient for m_λ because if $[\phi] \in \mathbb{P}_o$ then $a_{\lambda,\lambda} = 1$ where $\phi = \sum_{\mu,\nu \le \lambda} a_{\mu,\nu} m_\mu^* \otimes m_\nu$ as above.

Observe that $((g_1, g_2).h)(m) = g_1.h(g_2^{-1}.m) = g_1 g_2^{-1}.m$ since h is the identity map in $\mathrm{End}_{\Bbbk}(M)$. Hence elements in $(G \times G).[h]$ map $G.[m_\lambda]$ to $G.[m_\lambda]$.

Consider the subset $A := \{\phi \in \mathrm{End}_{\Bbbk}(M) \,|\, \phi(G.m_\lambda) \subseteq G.m_\lambda\}$. First notice that $A = \bigcap_{g \in G} \{\phi \in \mathrm{End}_{\Bbbk}(M) \,|\, \phi(g.m_\lambda) \subseteq G.m_\lambda\}$. Since the map $G \times^B M_\lambda \to M$ is a closed map then $\{\phi \in \mathrm{End}_{\Bbbk}(M) \,|\, \phi(g.m_\lambda) \subseteq G.m_\lambda\}$ is closed in $\mathrm{End}_{\Bbbk}(M)$. Thus, we deduce that each element of X_o is a rational self-map of $G.[m_\lambda]$ because elements of X_o are contained in $\{[\phi] \,|\, \phi \in A\}$.

From Bruhat's Lemma (cf. [Spr] Theorem 8.3.8), we find that $G = \cup_{w \in W} B\dot{w}B \simeq \cup_{w \in W} B^-\dot{w}B$ since $w_o B w_o^{-1} = B^-$ and $\dot{w}_o W = W$ (the $\dot{w}$'s are chosen representatives for $w \in W$). Hence, $G.[m_\lambda] = \cup_{w \in W} B^-\dot{w}.[m_\lambda]$ since $B.[m_\lambda] = [m_\lambda]$ by Lemma 2.1.1 on page 9. If $\dot{w}.[m_\lambda] \neq [m_\lambda]$ then $\dot{w}.[m_\lambda]$ is a weight vector of weight $w^{-1}.\lambda \lneqq \lambda$ hence $B^-\dot{w}.[m_\lambda] \cap \mathbb{P}(M)_o = \emptyset$. So we conclude that $G.[m_\lambda] \cap \mathbb{P}(M)_o = B_u^-.[m_\lambda]$.

Consider the obvious map $B_u^- \to B_u^-.[m_\lambda]$ (i.e. $u \mapsto u.[m_\lambda]$). It is an isomorphism (of homogeneous spaces for B_u^-) by [Spr] Theorem 5.3.2(iii) and by Lemma 2.1.1 (ii) on page 9. Therefore, we have a morphism $X_o \to B_u^-$ given by $\phi \mapsto v_\phi$ where $v_\phi \in B_u^-$ is the unique (from Bruhat decomposition cf. [Spr] Corollary 8.2.9) element such that $\phi([m_\lambda]) = v_\phi.[m_\lambda]$.

Regarding $\mathbb{P}(\mathrm{End}(M)) \simeq \mathbb{P}(\mathrm{End}(M^*))$ as a space of rational self-maps of $\mathbb{P}(M^*)$, we find by the arguments as above a morphism $X_o \to B_u$ mapping ψ to u_ψ where $\psi([m_\lambda^*]) = u_\psi.[m_\lambda^*]$. Now, we can define $\beta : X_o \to B_u \times B_u^-$ as the product morphism, i.e. the morphism given by $\phi \mapsto (u_\phi, v_\phi)$ in the above notation. Clearly, β sends any $\phi \in T_{ad}$ to $(1,1)$. Furthermore, β is $B_u \times B_u^-$-equivariant by construction. Thus, $(\beta \circ \Gamma)(u, v, t) = (u, v)$ for all $(u, v, t) \in B_u \times B_u^- \times T_{ad}$ is satisfied as required and therefore the proof is completed. □

Hence X_o is an affine space. Thus X_o is smooth. So we only need to prove the last assertion for X_o, namely that X is the union of $G \times G$-

translates of X_o. This follows from a more general result:

Lemma 2.2.4. *If Y is the unique closed orbit in a G-variety X and if V is a open subset of X satisfying $V \cap Y \neq \emptyset$ then $X = \bigcup_{g \in G} g.V$*

Proof. Obviously the inclusion '$\supseteq$' holds. So consider an element $x \in X$, but not in the union $\bigcup_{g \in G} g.V$. Hence, $g.x \notin V$ for all $g \in G$ and therefore $G.x \subseteq V^c$. Since V is open we get $\overline{G.x} \subseteq V^c$. Thus, $\overline{G.x} \cap V = \emptyset$.

As $\overline{G.x}$ is closed and G-stable, $\overline{G.x}$ contains a closed G-orbit. But a closed G-orbit in $\overline{G.x}$ is also a closed G-orbit of X. Therefore $Y \subseteq \overline{G.x}$. Hence $Y \cap V = \emptyset$ contradicting the assumption of the Lemma. □

Hence, it suffices to prove that X contains a unique closed $G \times G$-orbit intersecting X_o non-trivially. First observe that X contains a closed $G \times G$-orbit since X is closed and $G \times G$-stable by definition.

Lemma 2.2.5. *X contains a unique closed $G \times G$-orbit, namely $Y := (G \times G).[h_\lambda]$. And $Y \cap X_o \neq \emptyset$.*

Proof. Since a closed orbit in X is also a closed orbit in $\mathbb{P}(\mathrm{End}(M))$, we only need to prove that the latter has a unique closed orbit. Then this closed orbit actually lies in X.

Let $(G \times G).[m]$ be a closed orbit in $\mathbb{P}(\mathrm{End}(M))$ where $m \in \mathrm{End}(M)$ is non- zero. Then $(G \times G).[m] \simeq {}^{G\times G}/_{(G\times G)_{[m]}}$ is projective and hence $(G \times G)_{[m]}$ is parabolic. Thus, it contains some Borel subgroup of $G \times G$ by [Spr] Theorem 6.2.7(i) and therefore we have $x \in G \times G$ such that $B \times B \subseteq x(G \times G)_{[m]}x^{-1}$. Notice that for $y \in (G \times G)_{[m]}$ we have $(xyx^{-1}).(x.m) = x.(y.m) \in \Bbbk(x.m)$ which implies that $x.m$ is a $B \times B$-eigenvector. Hence $x.m \in M^*_{-w_o\lambda} \otimes M_\lambda$. Now, as M_λ and $M^*_{-w_o\lambda}$ are 1-dimensional, $x.m \in \Bbbk h_\lambda$. Now, therefore $[h_\lambda] = [x.m] = x.[m]$ and thus, $(G \times G).[m] = (G \times G).[h_\lambda]$.

This proves that $Y := (G \times G).[h_\lambda]$ is closed and unique. Furthermore that $(G \times G).[h_\lambda] \simeq {}^{G\times G}/_{B\times B}$ by Lemma 2.1.1 on page 9. Now, $Y \cap X_o = Y \cap \mathbb{P}_o$ which clearly contains the element $[h_\lambda]$. Therefore, $Y \cap X_o$ is non-empty. □

This description of X_o finally enables us to prove Theorem 2.2.1 on page 13:

Proof of Theorem 2.2.1. To prove (i), the only thing we need to show is the smoothness of X. We know by Lemma 2.2.5 and 2.2.4 that $X = (G \times G).X_o$. Since X_o is smooth so is X.

For (ii) and (iii), recall the definition of $[h_I]$ in the proof of Lemma 2.2.2 on page 13 for a subset I of $\{1, \ldots, \ell\}$. Observe that if $\lambda - \mu \in \sum_{i \notin I} \mathbb{N}\alpha_i$

then the coefficient in front of h_μ in $\gamma([h_I])$ is 1 cf. definition of γ in Lemma 2.2.2 on page 13. Therefore $[h_I] \in \mathbb{P}(\mathrm{End}(M))$ is the projection to the sum of those weight subspaces M_μ with $\lambda - \mu \in \sum_{i \notin I} \mathbb{N}\alpha_i$. As this is never M (because $I \neq \emptyset$) we find in particularly that $[h_I] \notin G_{ad}$. Therefore together with Proposition 2.2.3 on page 15, we deduce that Γ restricts to an isomorphism $B_u \times B_u^- \times (\mathbb{A}^\ell \setminus T_{ad}) \to X_o \setminus G_{ad} = X_o \cap \partial X$. Together with X is $(G \times G)$-translates of $\overline{T_{ad,o}} \simeq \mathbb{A}^\ell$, this implies that ∂X is a union of ℓ smooth prime divisors X_i with normal crossings proving (ii).

This also proves (iii) by the isomorphism $B_u \times B_u^- \times (\mathbb{A}^\ell \setminus T_{ad}) \to X_o \cap \partial X$ and Lemma 2.2.2 on page 13.

The unique closed orbit is $(G \times G).[h_\lambda] \simeq {}^{G}/_{B} \times {}^{G}/_{B}$ as found in the proof of Lemma 2.2.5 on the previous page. Now, $[h_\lambda] = [h_{\{1,\ldots,\ell\}}]$ by Lemma 2.2.2 on page 13. Thus, the unique closed orbit is $\cap_{i=1}^{\ell} X_i$. $\square$

So far the construction of X depends on a choice of a regular dominant weight and a choice for the G-module $M(\lambda)$ that has the properties (i)–(iii) of Lemma 2.1.1 on page 9. We will prove independence of both of these choices. Before that we need the following 'observation':

Lemma 2.2.6. *([DC-S] Proposition 3.15)*
Let M_i be G-modules with highest weight λ_i for $i = 1, \ldots, N$. Assume that the tensorproduct $\otimes_{i=1}^N M_i$ satisfies the conditions (i)–(iii) of Lemma 2.1.1 on page 9. Let $h_i \in End_{\Bbbk}(M_i)$ be the identity element and let $[h_i]$ denote its image in $\mathbb{P}(End_{\Bbbk}(M_i))$. Then X is isomorphic to the closure of the $G \times G$-orbit of the following element:

$$([h_1], \ldots, [h_\ell]) \in \prod_{i=1}^{N} \mathbb{P}(End_{\Bbbk}(M_i))$$

Proof. We prove the case where $N = 2$ because this case together with induction will take care of the general case.

It is clear that the element $h = h_1 \otimes h_2 \in M$ is the identity map in $\mathrm{End}_{\Bbbk}(M)$ where $M = M_1 \otimes M_2$. Thus, the closure of the $G \times G$-orbit of $[h] \in \mathbb{P}(\mathrm{End}_{\Bbbk}(M))$ is isomorphic to the variety X constructed above.

Consider the Segre embedding:

$$s : \mathbb{P}(\mathrm{End}_{\Bbbk}(M_1)) \times \mathbb{P}(\mathrm{End}_{\Bbbk}(M_1)) \hookrightarrow \mathbb{P}(\mathrm{End}_{\Bbbk}(M))$$

It is $G \times G$-equivariant since $\mathrm{End}_{\Bbbk}(M_i)$ is a $G \times G$-module for $i = 1, 2$. Furthermore, the Segre embedding s is a closed map. Hence, the closure of the $G \times G$-orbit of the element $([h_1], [h_2])$ maps isomorphically onto the closure of the $G \times G$-orbit of $[h]$ which is the variety X. $\square$

We are ready to prove independence of the choice of λ and of a G-module that satisfies properties (i)–(iii) of 2.1.1.

Proposition 2.2.7. *X is independent of the choices of the regular dominant weight λ and the associated G-module $M(\lambda)$ that satisfies the properties (i)–(iii) in Lemma 2.1.1 on page 9.*

Proof. Consider two compactifications X', X'' of G_{ad} associated with two different choices of regular dominant weights λ', λ'' and G-modules $M' = M(\lambda'), M'' = M(\lambda'')$ having the properties (i)–(iii) of Lemma 2.1.1 on page 9.

Now, we can embed G_{ad} diagonally in $X' \times X''$. From Lemma 2.2.6 above we get that $X' \times X'' \subseteq \mathbb{P}(End_{\Bbbk}(M' \otimes M''))$. Therefore we can define X to be the closure of $\Delta(G_{ad})$ in $X' \times X''$. Then X is $G_{ad} \times G_{ad}$-equivariant compactification of G_{ad} with two equivariant projections $\pi' : X \to X'$ and $\pi'' : X \to X''$. The closures of T_{ad} in X' and X'' are the same by Lemma 2.2.2 on page 13 such that the closure of T_{ad} in $X' \times X''$ is mapped isomorphically to these via the projections π', π''.

Define X_o to be the preimage of X'_o under π'. Then $X_o \simeq B_u \times B_u^- \times \overline{T_{ad,o}}$ by Proposition 2.2.3 on page 15. Therefore, π' restricted to X_o is an isomorphism $\pi' : X_o \to X'_o$. Because all $G_{ad} \times G_{ad}$-orbits meet X_o and π' is equivariant, it is an isomorphism everywhere, i.e. $X \simeq X'$ proving that X is independent of the choices made when constructing X. $\square$

Example 2.2.8. We continue our example (see the previous examples - examples 1.1.1, 1.2.3 , 2.1.6).

Notice that G_{ad} is the subset

$$\{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mid ad - bc \neq 0\} \subseteq \mathbb{P}^3$$

Thus, ∂X is the closed set $\{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mid ad - bc = 0\}$. This is also $Y = {}^{G}/_{B} \times {}^{G}/_{B} = \mathbb{P}^1 \times \mathbb{P}^1$.

2.3 The Picard group of X

In this section, we determine the Picard group of the wonderful compactification X of G_{ad}. Since X is a smooth variety, we have that the Picard group $\mathrm{Pic}(X)$ is isomorphic to the divisor class group $\mathrm{Cl}(X)$ (see e.g. [Ha1] Corollary II.6.16).

The first result due to M. Brion (see [DC-S] Proposition 4.4) describes the divisor class group of X:

Lemma 2.3.1. *([B-K] Proposition 6.1.9)*
The irreducible components of $X \setminus X_o$ are the prime divisors $\overline{B\dot{s}_iB^-}$ where s_i is the simple reflection in W associated with α_i. The equivalence classes of these prime divisors generate the abelian group $Cl(X)$ freely.

Proof. All irreducible components of $X \setminus X_o$ have codimension 1 by [Ha2] Proposition II.3.1 as X_o is affine (cf. Proposition 2.2.3 on page 15). Further, X_o has the property (proved by Lemma 2.2.4 on page 17 and Lemma 2.2.5 on page 17) that it meets all $G_{ad} \times G_{ad}$-orbits. Thus $X \setminus X_o$ does not contain any $G_{ad} \times G_{ad}$-orbits.

We claim that this implies that $G_{ad} \setminus X_o$ is dense in $X \setminus X_o$. If not there exists an irreducible component C of $X \setminus X_o$ such that $C \subsetneqq \overline{G_{ad} \setminus X_o}$. Consider the open subset $G_{ad} \cap C$ of C. We have that $G_{ad} \cap C \subseteq C \cap \overline{G_{ad} \setminus X_o}$ per choice of C. This implies that the closure in C of $G_{ad} \cap C$ is the empty set since C is irreducible. Thus, $C \subseteq X \setminus G_{ad}$. But then C is one of the divisors X_i since it has codimension 1 in X by [Ha2] Proposition II.3.1. But these are $G \times G$-stable contradicting that $X \setminus X_o$ does not contain any $G_{ad} \times G_{ad}$-orbits. Therefore, $G_{ad} \setminus X_o$ is dense in $X \setminus X_o$.

Now, observe that $G_{ad} \cap X_o \simeq B_u \times B_u^- \times T_{ad}$ which is nothing else but the big cell of G_{ad}. Therefore the Bruhat decomposition ([Spr] Theorem 8.3.8) reveals that $G_{ad} \setminus X_o = \bigcup_{w \neq id} B\dot{w}B^-$. Being a dense subset of $X \setminus X_o$ we find when taking closures in X that $X \setminus X_o = \bigcup_{w \neq id} \overline{B\dot{w}B^-}$. By [Spr] Proposition 8.5.5, this can be refined:

$$X \setminus X_o = \bigcup_{i=1}^{\ell} \overline{B\dot{s}_iB^-}$$

proving the first assertion of the Lemma.

To prove the second assertion, let D be a divisor of X. As X_o is an affine space, $\mathrm{Cl}(X_o)$ is trivial cf. [Ha1] Proposition II.6.2. Hence, $\mathrm{Cl}(X)$ is generated by the irreducible components of $X \setminus X_o$ which are the $\overline{Bs_iB^-}$'s $(i = 1, \ldots, \ell)$ according to the first assertion.

In order to finish the proof, we need to prove that the $\overline{B\dot{s}_iB^-}$'s are linearly independent in $\mathrm{Cl}(X)$. Assume for contradiction that there exists a trivial linear combination of the $\overline{B\dot{s}_iB^-}$'s. Hence there exists a non-constant rational function on X which has zeros and poles along divisors contained in $X \setminus X_o$. But then the restriction $f_{|X_o}$ is a non-constant invertible function of the affine space X_o giving a contradiction. Thus, the second assertion is proved. □

As mentioned above $\mathrm{Pic}(G) \simeq \mathrm{Cl}(X)$ since X is smooth. This Lemma then gives one description of $\mathrm{Pic}(X)$. The definition of $\mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles on X. Since G is semisimple and simply connected and X is normal we have that any line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ admits a unique $G \times G$-linearization. The restriction of $\mathcal{L}$ to the unique closed $G_{ad} \times G_{ad}$-orbit $Y \simeq {}^{G}/_{B} \times {}^{G}/_{B}$ is then also a $G \times G$-linearized line bundle on Y.

It is a well-known result that $\mathrm{Pic}({}^{G}/_{B}) \simeq X^*(T) = \Lambda$ (see e.g. [Jan] §4.2) and thus $\mathrm{Pic}(Y) \simeq \Lambda \times \Lambda$. Define $\mathcal{L}_Y(\lambda, \mu) := \mathcal{L}(\lambda) \boxtimes \mathcal{L}(\mu)$. Now, as a $G \times G$-linerized invertible sheaf on Y the restriction of $\mathcal{L} \in \mathrm{Pic}(X)$ is isomorphic to $\mathcal{L}(\lambda) \boxtimes \mathcal{L}(\mu)$ for uniquely determined weights $\lambda, \mu \in \Lambda$.

Since X is a smooth variety, there is a line bundle $\mathcal{L}_X(D)$ associated to each divisor D on X ([Ha1] Proposition II.6.11 and Prosition II.6.13). Define $D_i = \overline{Bs_i w_o B}$ where the closure is taken in X. Then D_i is a prime divisor on X. Let τ_i denote the unique (up to a scalar) section of $\mathcal{L}_X(D_i)$ such that the zero subset of τ_i is D_i. Observe that the classes of D_i form a basis of $\mathrm{Cl}(X)$ since $D_i = (1, \dot{w}_o)\overline{Bs_i B^-}$ is linear equivalent to $\overline{Bs_i B^-}$.

Further, let σ_i denote the unique (up to a scalar) section of the line bundle $\mathcal{L}_X(X_i)$ such that the zero subset of σ_i is X_i. For the action of $G \times G$ on $\mathrm{H}^0(X, \mathcal{L}_X(D_i))$ we have that τ_i is a $B \times B$-eigenvector. Similarly, σ_i is $G \times G$-invariant.

Now, we determine $\mathrm{Pic}(X)$ as an explicit subset of $\mathrm{Pic}(Y)$:

Proposition 2.3.2. *([B-K] Proposition 6.1.11)*
The restriction map $\imath^* : Pic(X) \to Pic(Y)$ is injective where $\imath : Y \to X$ is the inclusion map. Further, the image of $\imath^*$ consists of $\mathcal{L}_Y(-w_o\lambda, \lambda)$ with $\lambda \in \Lambda$.

Proof. Note that there is a unique (up to scalar) $B \times B$-eigenvector in $\Bbbk[G]$ of weight $(\omega_i, -w_o\omega_i)$ such that the zero subscheme of this eigenvector is the closure in G of $Bs_i w_o B$ cf. Lemma 1.2.2 on page 5. Hence also τ_i has weight $(\omega_i, -w_o\omega_i)$ since $V_X(\tau_i) \cap G$ is the closure in G of $Bs_i w_o B$. The restriction of τ_i to Y is contained in $\mathrm{H}^0(\lambda) \boxtimes \mathrm{H}^0(\mu)$ for unique $\lambda, \mu \in \Lambda$. As τ_i is a $B \times B$-eigenvector of weight $(\omega_i, -w_o\omega_i)$ we must have that $\lambda = -w_o\omega_i$ and $\mu = \omega_i$. Thus, $\imath^*\mathcal{L}_X(D_i) = \mathcal{L}_Y(-w_o\omega_i, \omega_i)$.

Any divisor D on X is generated by the D_i's by Lemma 2.3.1 and hence the image of $\imath^*$ is as described since $\mathrm{Cl}(X) \simeq \mathrm{Pic}(X)$. Assume that a divisor D on X has trivial image under $\imath^*$. Write $D = \sum_{i=1}^{\ell} n_i D_i$. Hence, $\mathcal{L}_X(D) = \bigotimes_{i=1}^{\ell} \mathcal{L}_X(D_i)^{\otimes n_i}$. Then $\imath^*\mathcal{L}_X(D) = \otimes_{i=1}^{\ell}\mathcal{L}_Y(-w_o\omega_i, \omega_i)^{\otimes n_i} = \mathcal{L}_Y(-w_o\lambda, \lambda)$ where $\lambda = \sum_{i=1}^{\ell} \omega_i$. By our assumption $\imath^*\mathcal{L}_X(D)$ is trivial and hence $\lambda = 0$. Thus, $\mathcal{L}_X(D)$ is trivial proving that $\imath^*$ is injective. □

As $\mathrm{Cl}(X) \simeq \mathrm{Pic}(X)$ we will also call $\imath^*$ the restriction to Y and denote it with res_Y. Let $\mathcal{L}_X(\lambda)$ denote the line bundle on X with restriction to Y

equaling $\mathcal{L}_Y(-w_o\lambda, \lambda)$ which makes sense by Proposition 2.3.2 above. Thus, the map $\Lambda \to \operatorname{Pic}(X)$, $\lambda \mapsto \mathcal{L}_X(\lambda)$ is an isomorphism of (abstract) groups.

Remark 2.3.3. *([B-K] Proposition 6.1.11)*
The line bundle $\mathcal{L}_X(X_i)$ corresponding to the prime divisor X_i is the line bundle $\mathcal{L}_X(\alpha_i)$ with the description above. Further, the canonical sheaf of X is $\omega_X = \mathcal{L}_X(-2\rho - \alpha_i - \cdots - \alpha_\ell)$.

Proof. Consider the restriction of $\mathcal{L}_X(X_i)$ to X_o. Since $X_o \cap \partial X \simeq B_u \times B_u^- \times \mathbb{A}^\ell \setminus T_{ad}$ (cf. the proof of Theorem 2.2.1 on page 13 on page 17) then we have that $X_o \cap X_i = V_{X_o}(x_i)$ where $\Bbbk[\mathbb{A}^\ell] = \Bbbk[x_1, \dots, x_\ell]$ (i.e. x_i is the regular function of $\mathbb{A}^\ell$ picking out the coefficient in front of the i'th coordinate). By Proposition 2.2.3 on page 15, x_i is a regular function of X_o.

We already have a $B \times B^-$-action on X_o and therefore we can define a $B \times B$-action on X_o by conjugating the second faktor with $\dot{w}_o$ because $\dot{w}_o B \dot{w}_o = B^-$. If $t \in T_{ad}$ the $\Gamma(u, v, t) = utv^{-1}$ for $u \in B_u$, $v \in B_u^-$ (the product taken in G_{ad}). Thus, $x_i(\Gamma(u, v, t)) = \alpha_i(vt^{-1}u^{-1}) = \alpha_i(t^{-1})$. Let $b, c \in B$ then

$$\begin{aligned} ((b,c).x_i)(utv^{-1}) &= x_i((b^{-1}, \dot{w}_o c^{-1} \dot{w}_o^-).utv^{-1}) = x_i(b^{-1}utv^{-1}\dot{w}_o c \dot{w}_o^{-1}) \\ &= \alpha_i(\dot{w}_o c^{-1} \dot{w}_o^{-1} v t^{-1} u^{-1} b) = (-w_o.\alpha_i)(c) \cdot \alpha_i(b) \cdot \alpha_i(t^{-1}) \end{aligned}$$

Hence, the $B \times B$-weight of x_i is $(\alpha_i, -w_o.\alpha_i)$.

Therefore, the restriction of x_i to Y gives a $B \times B$-eigenvector of weight $(\alpha_i, -w_o.\alpha_i)$. Consequently, the restriction of x_i to Y is contained in $\mathrm{H}^0(Y, \mathcal{L}_Y(-w_o.\alpha_i, \alpha_i))$ showing that $\mathcal{L}_X(X_i) = \mathcal{L}_X(\alpha_i)$.

The idea is now to combine this with [Ha1] Proposition II.8.20 to prove the second assertion. Notice that the ideal sheaf $\mathcal{I}_{X_i} = \mathcal{L}_X(-X_i) = \mathcal{L}_X(-\alpha_i)$ by [Ha1] Proposition II.6.18. Since the boundary divisors X_i cross normally, $\sigma_1, \dots, \sigma_\ell$ are linear independent $(\operatorname{mod} \mathcal{M}_y^2)$ where $y \in Y$, $\mathcal{M}_y$ is the maximal ideal in the regular local ring $\mathcal{O}_{X,x}$ and σ_i define the divisors X_i. Thus, $\mathcal{I}_Y/\mathcal{I}_Y^2$ has a basis consisting of these σ_i's showing that $\mathcal{I}_Y/\mathcal{I}_Y^2 \simeq \mathcal{L}_X(-X_i) \otimes \cdots \otimes \mathcal{L}_X(-X_i)$. Therefore the normal sheaf $\mathcal{N}_{Y/X} \simeq \bigotimes_{i=1}^\ell \mathcal{L}_X(\alpha_i)$. Therefore by [Ha1] Proposition II.8.20 we have that the canonical sheaf ω_Y of Y is $\omega_Y = \omega_X \otimes \mathcal{L}_X(\alpha_1 + \cdots + \alpha_\ell)$ which proves the claim. $\square$

Example 2.3.4. We continue our example (see the previous examples - examples 1.1.1, 1.2.3 , 2.1.6, 2.2.8).

As $w_o = s_\alpha$ we get that $D = \overline{B s_\alpha w_o B} = \overline{B}$. As a subset of $\mathbb{P}^3$ (in matrix notation) we have

$$\overline{B} = \{ [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in \mathbb{P}^3 \mid c = 0 \}$$

By [Ha1] Corollary II.6.17, $\mathrm{Pic}(X) \simeq \mathbb{Z}$. We also have that $\mathrm{Pic}(X) \simeq \mathrm{Cl}(X)$ since X is smooth. Thus, the isomorphism $\mathbb{Z} \simeq \mathrm{Pic}(X)$ is given by $1 \mapsto \mathcal{O}_X(1) \simeq \mathcal{L}_X(\omega_\alpha) = \mathcal{L}_X(\overline{B})$. Observe that $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is the diagonal map since $\mathrm{Pic}(Y) \simeq \mathbb{Z} \times \mathbb{Z}$ and every weight λ equals $m\rho$ for some $m \in \mathbb{Z}$. Hence, $\mathcal{L}_X(m\rho) = \mathcal{L}_X(\rho)^{\otimes m} \simeq \mathcal{O}_X(m)$. Especially, $\omega_X^{-1} = \mathcal{O}_X(4)$ by Remark 2.3.3 on the preceding page.

Let $\Bbbk[Z_{11}, Z_{12}, Z_{21}, Z_{22}]$ be the homogeneous coordinate ring for $X = \mathbb{P}^3$ where Z_{ij} is the function $\left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right] \mapsto a_{ij}$ for $i, j \in \{1, 2\}$. Then $\partial X = \mathcal{V}_X(Z_{11}Z_{22} - Z_{12}Z_{21})$ and the unique (up to a scalar) global section σ of $\mathcal{L}_X(\alpha) = \mathcal{O}_X(2)$ is $Z_{11}Z_{22} - Z_{12}Z_{21}$. We also have $\overline{B} = \mathcal{V}_X(Z_{21})$ and hence the unique (up to a scalar) global section σ of $\mathcal{L}_X(\rho) = \mathcal{O}_X(1)$ is Z_{21}.

Chapter 3

Unipotent varieties

In this section we will study the unipotent variety $\mathcal{U}$, i.e. the subset of a linear algebraic group H consisting of all unipotent elements. We will prove that when G is connected, semi simple, and simply connected, $\mathcal{U}$ is a complete intersection and furthermore normal. We also prove that the central isogeny $\pi : G \to G_{ad}$ maps $\mathcal{U}$ bijectively onto the unipotent variety $\mathcal{U}_{ad}$ of the adjoint linear algebraic group G_{ad}.

3.1 The Unipotent Varieties

Let G denote a linear algebraic group. Then an element $x \in G$ is called *unipotent* if the image of x in some embedding of G as a closed subset of $Gl_n(\Bbbk)$ have all eigenvalues equal to 1. By Jordan decomposition (see e.g. [Spr] Theorem 2.4.8) this is independent of the choice of embedding. Let $\mathcal{U}$ denote the subset of all unipotent elements in G. The condition for an element in $Gl_n(\Bbbk)$ to be unipotent is that $x - I$ is nilpotent where I is the identity element in $Gl_n(\Bbbk)$. This is a polynomial condition and therefore $\mathcal{U}$ is a closed subset of G. But we can show more than that:

Theorem 3.1.1. *([Hum2] Theorem 4.2)*
The subset $\mathcal{U}$ of all unipotent elements in a connected reductive linear algebraic group G is a closed irreducible subvariety of G of dimension $\dim(G)-\ell$ *where $\ell = rank(G)$.*

Proof. Let $\eta : G\times G \to G$ be defined by $(g, x) \mapsto g^{-1}xg$. Since the unipotent radical B_u of B is closed in G the pullback $Z := \eta^{-1}(B_u)$ is closed in $G \times G$. Let Z' denote the image of Z under the map $pr_{G/B} \times \mathrm{id} : G \times G \to {}^{G}/_{B} \times G$. Then

$$Z' = \mathrm{Im}(pr_{G/B} \times \mathrm{id}) = \{(gB, x)|\, g^{-1}xg \in B_u\}$$

This description of Z' makes sense since $b^{-1}xb \in B_u$ for all $b \in B$ and $x \in B_u$. Now, $pr_{G/B} \times \mathrm{id}$ is an open map cf. the quotient topology. Therefore Z' is closed since $(pr_{G/B} \times \mathrm{id})((G \times G) \setminus Z) = (G/B \times G) \setminus Z'$ is open in $G/B \times G$.

By definition of Z' we have that the map $G \times B_u \to Z'$ given by $(g, x) \mapsto (gB, g^{-1}xg)$ is surjective. Hence Z' is irreducible as $G \times B_u$ is ([Spr] Lemma 1.2.3(ii)).

Consider the projection $G/B \times G \to G$. It is a closed map since G/B is complete (cf. [Spr] 6.1.1). Furthermore, Z' is mapped onto $\mathcal{U}$ via this projection since $\mathcal{U} = \bigcup_{g \in G} gB_u g^{-1}$. Thus, $\mathcal{U}$ is closed and irreducible. Therefore, we can regard $\mathcal{U}$ as a closed subvariety of G by putting the reduced structure on $\mathcal{U}$.

We need to find the dimension of $\mathcal{U}$, but first we find the dimension of Z'. Consider the other projection $G/B \times G \to G/B$. Under this projection Z' is sent to G/B with fibres isomorphic to $gB_u g^{-1}$. Therefore, the fibres all have dimension equal to $\dim(B_u)$. Thus,

$$\begin{aligned}\dim(Z') &= \dim(G/B) + \dim(B_u) \\ &= \dim(G) - \dim(B) + \dim(B_u) = \dim(G) - \operatorname{rank}(G)\end{aligned}$$

So to prove the last assertion we only need to check that there exists a finite fibre of the surjection $Z' \to \mathcal{U}$. We have to show that there exists a $x \in B_u$ that only is contained in $gB_u g^{-1}$ for finitely many $g \in G$. This follows from the Lemma:

Lemma 3.1.2. *(Contained in [Hum2] Proposition 4.1)*
There exists $x \in B_u$ lying in only finitely many Borel subgroups.

Proof. Consider the root subgroup maps $u_\alpha : \Bbbk \to U_\alpha$ cf. [Spr] Proposition 8.1.1. Choose an ordering $\alpha_1, \ldots, \alpha_s$ of the positive roots such that the simple roots are the first ℓ. We have $B_u = \prod_{i=1}^{s} U_\alpha$ by [Spr] Proposition 8.2.3.

Let $x \in B_u$ such that when writing $x = \prod_{i=1}^{s} u_{\alpha_i}(x_i)$ for $x_i \in \Bbbk$ then $u_{\alpha_i}(x_i) \neq e$. We will prove that x lies in a unique Borel subgroup.

Assume that x lies in some other Borel subgroup B'. Then we have unique $w \in W, v \in U_{w^{-1}}$, $b \in B$ such that $B' = v\dot{w}bBb^{-1}\dot{w}^{-1}v^{-1} = v\dot{w}B\dot{w}^{-1}v^{-1}$ by the Bruhat decomposition ([Spr] Corollary 8.3.9) and the fact that all Borel subgroups in G are conjugated ([Spr] Theorem 6.2.7). Hence $\dot{w}^{-1}v^{-1}xv\dot{w} \in B$. Notice that $U_{w^{-1}}$ is a subgroup of B_u by [Spr] Lemma 8.3.5.

From [Spr] Proposition 8.2.3 we deduce that if $y = \prod_{i=1}^{s} u_{\alpha_i}(y_i)$ is an element of the commutator subgroup (B_u, B_u) then $u_{\alpha_i}(y_i) = e$ for $i = 1, \ldots, \ell$.

Hence this is the case for $x^{-1}v^{-1}xv$. Therefore, if $v^{-1}xv = \prod_{i=1}^{s} u_{\alpha_i}(z_i)$ then $z_i = x_i \neq 0$ for $i = 1, \dots, \ell$.

Now, $\dot{w}^{-1}v^{-1}xv\dot{w} = \prod_{i=1}^{s} u_{w^{-1}.\alpha_i}(c_{w,\alpha}z_i)$ by [Spr] Exercise 8.1.12(2). Note that $c_{w,\alpha}z_i \neq 0$ for $1 \leq i \leq \ell$. Further, $\dot{w}^{-1}v^{-1}xv\dot{w} \in B_u$. From [Spr] Proposition 8.2.1 and Exercise 8.1.12(1) we deduce that $B \cap \dot{w}B\dot{w}^{-} = U_{w_o w^{-1}} \prod_{\alpha\in\Phi^+, w^{-1}.\alpha\in\Phi^+} U_\alpha$. Therefore, we find that $w^{-1}.\alpha_i \in \Phi^+$ for all simple roots $\alpha_1, \dots, \alpha_\ell$. Thus, $w^{-1} = 1$ implying that $\dot{w} \in B$. Thus, $B' = vBv^{-1} = B$ since $v \in U_{w^{-1}} \subseteq B_u$. This concludes the proof of the Lemma. $\square$

Therefore, since there is one finite fibre, $\dim(\mathcal{U}) = \dim(X) = \dim(G) - \ell$ by [Spr] Corollary 5.2.7. $\square$

The closed subvariety $\mathcal{U}$ of G is called the *unipotent variety.*

Example 3.1.3. Let the situation be as in Example 1.1.1 on page 2. Then the unipotent variety of $G = Sl_2$ is

$$\mathcal{U} = \{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in G \,|\, a + d = 0\}$$

(since we are in characteristic $p = 2$).

3.2 Properties of the Unipotent Varieties

In this section, we prove that the unipotent variety of a connected, semi simple, simply connected linear algebraic group is a complete intersection. Further, it is normal.

We will need the following Lemma from commutative algebra due to Kostant:

Lemma 3.2.1. *([K] Lemma 1.6.4)*
Let $f_1, \dots, f_r \in \Bbbk[G]$ and set

$$X := \mathcal{V}_G(f_1, \dots, f_r) \qquad\qquad \mathcal{P} = \mathcal{I}_X(X)$$

Assume that X is irreducible and that there exists $x \in X$ such that $(df_i)_x$ are linearly independent in $\Omega_{G/\Bbbk}$. Then P is generated by $f_1, \dots, f_r$ as an ideal.

Proof. Let I be the ideal generated by $f_1, \dots, f_r$ in $\Bbbk[G]$. Consider the local ring $\Bbbk[G]_x$ with unique maximal ideal M_x. By the definition of X we have that the f_i's belong to M_x.

Since $(df_i)_x$ are linearly independent we get that $f_i + M_x^2$ are linearly independent in ${}^{M_x}/_{M_x^2}$. Hence, they form part of a basis for ${}^{M_x}/_{M_x^2}$. By Nakayama's Lemma ([E] Corollary 4.8) $f_1, \dots, f_r$ are part of a generating set of M_x (also called a regular system of parameters at x cf. [E] page 242). Consider the local ring $R := {}^{\Bbbk[G]_x}/_{\Bbbk[G]_x I}$ and let $\mathcal{M}$ denote its maximal ideal. By the principal ideal theorem ([E] Theorem 10.2) $\dim_{R/\mathcal{M}}({}^{\mathcal{M}}/_{\mathcal{M}^2}) \geq \dim(R)$. We know that the vector space ${}^{\mathcal{M}}/_{\mathcal{M}^2}$ is a quotient of ${}^{M_x}/_{M_x^2}$. Therefore $\dim_{R/\mathcal{M}}({}^{\mathcal{M}}/_{\mathcal{M}^2}) \leq \dim(\Bbbk[G]_x) - r$. On the other hand, $\dim(R) \geq \dim(\Bbbk[G]_x) - r$. Thus, $\dim_{R/\mathcal{M}}({}^{\mathcal{M}}/_{\mathcal{M}^2}) = \dim(R) - r$ and therefore R is regular local ring. Thus, R is an integral domain by [E] Corollary 10.14 implying that $\Bbbk[G]_x I$ is a prime ideal in $\Bbbk[G]_x$.

By Hilbert's Nullstellensatz ([E] Theorem 1.6) we have that the radical $\sqrt{I} = \mathcal{P}$. Hence localizing $\sqrt{\Bbbk[G]_x I} = \Bbbk[G]_x\mathcal{P}$. In fact, $\Bbbk[G]_x I = \Bbbk[G]_x\mathcal{P}$ because $\Bbbk[G]_x I$ is a prime ideal. By [E] Proposition 2.2, $\mathcal{P} = \Bbbk[G] \cap \Bbbk[G]_x\mathcal{P}$ as $\mathcal{P}$ is a prime ideal (by the assumption that X is irreducible). Thus, $\mathcal{P} = \Bbbk[G] \cap \Bbbk[G]_x I$.

Take an irredundant primary decomposition $I = Q_1 \cap \cdots \cap Q_s$ (see cg. [E] Theorem 3.10). Notice that the radical ideals $\sqrt{Q_i}$ are actually prime ideals. Since $\mathcal{P} = \sqrt{I}$ is prime, $\mathcal{P} \subseteq \sqrt{Q_i}$ for all i. As I is generated by $\mathrm{codim}(I)$ elements, the unmixedness Theorem ([E] Corollary 18.14) implies that all the $\sqrt{Q_i}$ are minimal primes over I. Hence, they all equal $\mathcal{P}$ implying that I is a primary ideal. Thus, $I = \Bbbk[G] \cap \Bbbk[G]_x I$ by [E] Theorem 3.1(c). Therefore $I = \mathcal{P}$. □

Consider the *fundamental characters* $\chi_1, \dots, \chi_\ell$: For the fundamental weight ω_i corresponding to the simple root α_i we have a G-module $\mathrm{H}^0(-w_o\omega) = \mathrm{H}^0({}^G/_B, \mathcal{L}_{G/B}(-w_o\omega_i))$. The fundamental character is then defined to be the composition of the homomorphism of algebraic groups and the trace map, i.e. $\chi_i : G \to Gl(\mathrm{H}^0(-w_o\omega_i)) \to \Bbbk$. Especially, χ_i is a class function, i.e. $\chi_i(gxg^{-1}) = \chi_i(x)$ for all $g, x \in G$.

Notice that in the embedding $G \to Gl(\mathrm{H}^0(-w_o\omega_i))$, the unipotent elements of G are map to elements with all eigenvalues equaling 1. Thus, $\chi_i(g) = \chi_i(e)$ for all i if g is unipotent.

In [Hum] Theorem 23.1 and the appendix to section 23, it is proven that the G-invariant polynomial functions on a semi simple Lie algebra L are isomorphic to the W-invariant polynomial functions on a Cartan subalgebra H of L. The arguments can be extended to show that the set of class functions $\Bbbk[G]^G$ in G (the action of G is conjugation) is isomorphic to $\Bbbk[T]^W$, the algebra of regular functions on T which are constant on W-orbits. Further, since G is simply connected, $\Bbbk[G]^G$ is freely generated by the fundamental characters [1]. This implies that two semisimple elements s, t in G are conjugate if and only if $\chi_i(s) = \chi_i(t)$ for all i.

Thus, $\Bbbk[T]^W$ is the coordinate ring of an affine space $\mathbb{A}^\ell$ when G is simply connected. This gives a surjective morphism $\kappa : G \to \mathbb{A}^\ell$ given by $g \mapsto (\chi_1(g), \dots, \chi_\ell(g))$; It is named the Steinberg map .

Consider the closed subvariety $\mathcal{C} := U_{\alpha_1}\dot{s}_1 \dots U_{\alpha_\ell}\dot{s}_\ell$ where s_i are the simple reflections, i.e. $s_i \in W$ is the reflection corresponding to the simple root α_i. Investigating κ, we find that the restriction of κ to $\mathcal{C}$ is isomorphic onto $\mathbb{A}^\ell$:

Theorem 3.2.2. *([Hum2] Theorem 4.17) The restriction of the Steinberg map κ to $\mathcal{C}$ induces a isomorphism of varieties $\mathcal{C} \simeq \mathbb{A}^\ell$. In particular, the $d\chi_i$'s are linearly independent in each point $x \in \mathcal{C}$.*

Sketch of proof. From [Spr] Lemma 8.3.6(i) we have the following isomorphism $\mathbb{A}^\ell \to \mathcal{C}$ given by $(c_1, \dots, c_\ell) \longmapsto u_{\alpha_1}(c_1)\dot{s}_1 \dots u_{\alpha_\ell}(c_\ell)\dot{s}_\ell$. Hence, we can regard $\kappa_{\mathcal{C}} : \mathcal{C} \to \mathbb{A}^\ell$ as a map from $\mathbb{A}^\ell$. Therefore it suffices to prove that to each c_i we can find a polynomial P in ℓ indeterminals such that $c_i = P(\chi_1(g), \dots, \chi_\ell(g))$ where $g = u_{\alpha_1}(c_1)\dot{s}_1 \dots u_{\alpha_\ell}(c_\ell)\dot{s}_\ell$.

This requires a somewhat complicated, but not too difficult book-keeping with roots and weights. See the proof of [Hum2] Theorem 4.17 for details. □

Now we have what is needed to prove that $\mathcal{U}$ is a complete intersection.

Proposition 3.2.3. *([Hum2] Theorem 4.24(a))*
The ideal in $\Bbbk[G]$ vanishing on $\mathcal{U}$ is generated by $\chi_i - \chi_i(e)$ for $1 \le i \le \ell$. Thus, $\mathcal{U}$ is a complete intersection.

Proof. We will make use of Lemma 3.2.1 above. First, we claim that if $\chi_i(g) = \chi_i(e)$ for all i then $g \in G$ is unipotent. Actually, we prove that $\chi_i(g) = \chi_i(g_s)$ where g_s is the semisimple part of g in Jordan decomposition (see e.g. [Spr] Theorem 2.4.8(i)).

To prove the claim, let $g \in G$. Then g lies in some Borel subgroup ([Spr] Theorem 6.4.5). Hence a conjugate of g lies in $B = TB_u$ with $g_s \in T$, $g_u \in B_u$. Write $g_u = \prod_{\alpha \in \Phi^+} u_\alpha(c_\alpha)$. For a non-zero element $c \in \Bbbk$, we can find $t \in T$ such that $\alpha_i(t) = c$ for all simple roots. Therefore,

$$tgt^{-1} = g_s(tg_ut^{-1}) = g_s u_c, \qquad \text{where } u_c := \prod_{\alpha \in \Phi^+} u_\alpha(c^{m_\alpha} c_\alpha)$$

Here $m_\alpha = \sum_{i=1}^{\ell} n_i$ with $\alpha = \sum_{i=1}^{\ell} n_i \alpha_i$.

[1] These results are contained in [Hum2] chapter 3 in full detail

Now, let C denote the conjugacy class in G of g then $u_c \in g_s^{-1}C$ for all non-zero $c \in \Bbbk$. For any $f \in \Bbbk[G]$ vanishing on $g_s^{-1}C$, the map $c \mapsto f(u_c)$ vanishes on $\Bbbk \setminus \{0\}$. Hence, it vanishes on $\Bbbk$. In particular, $f(e) = 0$ as $u_0 = e$. Thus, $e \in \overline{g_s^{-1}C}$. So we conclude that $g_s \in \overline{C}$ which implies that $\chi_i(g) = \chi_i(g_s)$ for all i.

If $\chi_i(g) = \chi_i(e)$ for all i then by the claim we have that $\chi_i(g_s) = \chi_i(e)$ for all i. Thus, g_s and e are conjugated. Therefore $g_s = e$ and hence g is unipotent.

Since $\mathfrak{U}$ is the common zero subset of $\chi_1 - \chi_1(e), \dots, \chi_\ell - \chi_\ell(e)$ and $\mathfrak{U}$ is irreducible by Theorem 3.1.1 on page 25 we only need to find a point $x \in \mathfrak{U}$ such that $d\chi_i$ are linearly independent in x.

From Theorem 3.2.2 above the element $x := \kappa_{|\mathfrak{C}}^{-1}(\chi_1(e), \dots, \chi_\ell(e))$ is unipotent and the $d\chi_i$'s are linearly independent in x. This concludes the proof. $\square$

We will also prove normality of the unipotent variety $\mathfrak{U}$. Since $\mathfrak{U}$ is a complete intersection by the Proposition 3.2.3 above, it suffices to prove that $\mathfrak{U}$ is regular in codimension 1 cf. [Ha1] Proposition 8.23(ii). In order to do so, we introduce the notion of regular elements in G:

Definition 3.2.4. An element g in G is *regular* if the dimension of the centralizer $C_G(g)$ equals the rank of G.

This is the smallest possible dimension by the following proposition:

Proposition 3.2.5. *([Hum2] Proposition 1.6)*
Let G be a connected reductive linear algebraic group. For all $g \in G$, $\dim(C_G(g)) \geq$ *rank*$(G) = \ell$

Proof. Consider a Borel subgroup B of G containing g (exists by [Spr] Theorem 6.4.5(i)). The commutator subgroup (B, B) is contained in the unipotent radical B_u of B. Thus, $(B, g) \subseteq B_u$ implying that $C := \{bgb^{-1} | b \in B\} \subseteq \{ug | u \in B_u\}$. The latter has dimension at most $\dim(B_u)$. We have that $\dim(C_B(g)) = \operatorname{codim}(C, B) \geq \operatorname{codim}(B_u, B) = \ell$. Now, $C_B(g) \subseteq C_G(g)$ finishing the proof. $\square$

Next we show that there exist regular unipotent elements in G. In doing so, we will make use of the following non-trivial, but well-known result without proof:

Theorem 3.2.6. *([Hum2] Theorem 3.9 and Theorem 3.11)*
Let G be a connected reductive linear algebraic group. Then G has finitely many unipotent conjugacy classes.

By using this Theorem, we are able to prove the existence of regular unipotent elements:

Proposition 3.2.7. *([Hum2] Section 4.3)*
Let G be a connected reductive linear algebraic group. Then regular unipotent elements exist in G and form a unique conjugacy class which is an open dense set of $\mathfrak{U}$. Further, the regular unipotent elements are non-singular elements in $\mathfrak{U}$.

Proof. By Theorem 3.2.6 above, G has only finitely many orbits and therefore there exists at least one conjugacy class C with $\dim(\bar{C}) = \dim(\mathfrak{U})$. For an element $c \in C$, we have that $\dim(C_G(c)) = \dim({}^{G}/_{\bar{c}}) = \dim(G) - \dim(\mathfrak{U}) = \ell$ by Theorem 3.1.1 on page 25. Thus, c is regular (by definition). As $\mathfrak{U}$ is irreducible, $\bar{C} = \mathfrak{U}$. And therefore C is open and hence dense in $\mathfrak{U}$ by [Spr] Lemma 2.3.3(i). Therefore, $\mathfrak{U} \setminus C$ is a proper closed subset and any unipotent conjugacy class $C' \neq C$ must have strictly less dimension than C. Hence, elements not in C are not regular. Therefore, the only regular unipotent elements are the elements in C.

To prove the last assertion, recall that the set of non-singular points of any variety form an open dense subset of that variety (see eg. [Ha1] Corollary 8.16). Thus, there is a non-singular point in C showing that C consists of non-singular points since C is a G-orbit. □

We found that if $x \in G$ is a unipotent, but *not* regular then the conjugacy class of x has dimension strictly less than $\dim(C)$. We can be even more precise:

Proposition 3.2.8. *([Hum2] Proposition 4.1 and Theorem 4.6)*
Let G be a connected reductive linear algebraic group and let $x \in G$ be a unipotent, but not regular element. Then $\dim(C_G(x)) \geq \ell + 2$.

Proof. We claim that $u \in G$ is conjugated to an element of the form

$$\prod_{\alpha \in \Phi^+} u_\alpha(x_\alpha)\,, \qquad x_{\alpha_i} \neq 0 \text{ for } i = 1, \dots, \ell \tag{3.1}$$

if and only if u is a regular unipotent element. Here, we have predetermined an order of the positive roots like in the proof of Lemma 3.1.2 on page 26. While proving this claim we actually prove the assertion of the proposition.

Assume that u is a regular unipotent element that is not conjugated to an element as in equation (3.1). We can find $g \in G$ such that $u' = gug^{-1} \in B_u$ since any two Borel subgroups are conjugated ([Spr] Theorem 6.2.7(iii)). Note that u' is also regular. When writing $u' = \prod_{\alpha \in \Phi^+} u_\alpha(x_\alpha)$ (wrt. the ordering chosen) then by assumption u' does not equal any element as in equation (3.1). Hence there exists j such that $x_{\alpha_j} = 0$. Therefore, $u' \in R_u(P_j)$, the unipotent radical of the minimal parabolic subgroup $P_j = B \cup B\dot{s}_j B$ cf. [Spr] Theorem 8.4.3.

Let C' denote the conjugacy class of u' in P_j. We have $C' \subseteq R_u(P_j)$. From [Spr] Theorem 8.4.3(iii) we find that $\dim(P_j) - \dim(R_u(P_j)) = \dim(L_j)$ where L_j is the Levi subgroup of P_j. It is a connected reductive group of semisimple rank 1 hence $\dim(L_j) = \ell + 2$ by [Spr] §7.3.2. Therefore we conclude that $\dim(C_{P_j}(u')) = \dim(P_j) - \dim(C') \geq \dim(P_j) - \dim(R_u(P_j)) = \ell + 2$. But this contradicts the regularity of u' since $C_{P_j}(u') \subseteq C_G(u')$.

Assume now that u is conjugate to an element v of equation (3.1). It suffices to prove that u is conjugated to a unipotent regular element. Let x be a regular unipotent element. By what we have proved so far x is conjugated to an element y of equation (3.1). We will prove that y and v are conjugate.

Write $v = \prod_{\alpha\in\Phi^+} u_\alpha(v_\alpha)$ and $y = \prod_{\alpha\in\Phi^+} u_\alpha(y_\alpha)$ (wrt. to the chosen ordering of the positive roots) with $v_{\alpha_i} \neq 0$ and $y_{\alpha_i} \neq 0$ for $i = 1, \ldots, \ell$. Observe that for $t \in T$ we get

$$tvt^{-1} = \prod_{\alpha\in\Phi^+} u_\alpha(\alpha(t)v_\alpha)$$

by [Spr] Proposition 8.1.1. Since the simple roots are linearly independent (see cf. [Spr] 8.2.8(iii)) we can choose $t \in T$ such that $\alpha_i(t)v_{\alpha_i} = y_{\alpha_i}$ for $i = 1, \ldots, \ell$.

Consider the set $V := \{zyz^{-1}y^{-1} | z \in B_u\}$. We have that V is isomorphic (as varieties) to the conjugacy class $V' = \{zyz^{-1} | z \in B_u\}$. Further, $V \subseteq H$ where $H = \{\prod_{\alpha\in\Phi^+} u_\alpha(z_\alpha) \,|\, z_{\alpha_i} = 0,\ 1 \leq i \leq \ell\}$. Note that $\operatorname{codim}(H, B_u) = \ell$. Also, we have that V is closed in B_u since every conjugacy class in B_u is closed (cf. [Spr] 2.4.14). Thus, $\operatorname{codim}(V, B_u) = \operatorname{codim}(V', B_u)$. As y is regular, $\operatorname{codim}(V', B_u) = \dim(C_{B_u}(y)) \leq \dim(C_G(y)) = \ell$. Hence $V \subseteq H$ is closed, irreducible, and of the same dimension as H and therefore have to be equal by [Spr] Proposition 1.8.2. Observe that $V \subseteq (B_u, B_u) \subseteq H$ and so the arguments above prove that $V = (B_u, B_u)$. Therefore, $vy^{-1} = zyz^{-1}y^{-1}$ for some $z \in B_u$ and hence $v = zyz^{-1}$. Therefore v (and hence u) is regular. This ends the proof of the claim.

Now, observe that in the first part of the proof, we proved that $\dim(C_G(x)) \geq \ell + 2$ for x that is not conjugated to an element of (3.1). We now know that this is equivalent to requiring that x is a unipotent, but not regular element. □

Finally, we can prove that $\mathfrak{U}$ is normal:

Proposition 3.2.9. *([Hum2] Theorem 4.24)*
The unipotent variety $\mathcal{U}$ in G is regular in codimension 1 and hence normal.

Proof. By Theorem 3.2.6 on page 30 we have that the regular unipotent elements form a unique conjugacy class C in $\mathcal{U}$ and all other unipotent conjugacy classes have dimension strictly less than $\dim(C)$. From Proposition 3.2.8 on page 31 we find that if $x \notin C$ then $\dim(C_G(x)) \geq \ell + 2$ and therefore $\mathrm{codim}(C', G) \geq \ell + 2$ where C' is the conjugacy class containing x. Since $\mathrm{codim}(\mathcal{U}, G) = \ell$ by Theorem 3.1.1 on page 25 we conclude that $\mathrm{codim}(C', \mathcal{U}) \geq 2$ for all unipotent conjugacy classes $C' \neq C$. Hence $\mathrm{codim}(\mathcal{U} \setminus C, \mathcal{U})$ cannot be less than 2. Therefore $\mathcal{U}$ is regular in codimension 1 since all regular unipotent elements are non-singular points of $\mathcal{U}$. As $\mathcal{U}$ is a complete intersection, $\mathcal{U}$ is hence normal by [Ha1] Proposition 8.23(b). □

Unless explicitly stated otherwise, G denotes a connected semisimple simply-connected linear algebraic group. Not all the above results were here proven to hold more generally. The following result relates the unipotent conjugacy classes of G and G_{ad}.

Proposition 3.2.10. *([Hum2] Proposition 1.8)*
The surjective homomorphism of linear algebraic groups $\pi : G \to G_{ad}$ restricts to bijective morphism of $\mathcal{U}$ and the unipotent variety $\mathcal{U}_{ad}$ of G_{ad}. Further, it induces a bijection of the unipotent conjugacy classes of G and the unipotent conjugacy classes of G_{ad}.

Proof. Let $g \in G$ be unipotent. Then by [Spr] Theorem 2.4.8(ii) $\pi(g) = \pi(g_s)\pi(g_u)$ where g_s is the semi simple part of g and g_u the unipotent part of g in Jordan decomposition. Hence $\pi(g)$ is unipotent. So π restricts to a map that maps unipotent elements of G to unipotent elements of G_{ad}.

To prove that this map is injective we assume that $g_1, g_2 \in G$ are two unipotent elements such that $\pi(g_1) = \pi(g_2)$. Hence, $g_2 \in \mathrm{Ker}(\pi)g_1$. Now, the kernel of π is central, i.e. $g_2 = zg_1$ for some $z \in Z(G)$. Hence z is semisimple and commutes with g_2. Thus by the uniqueness of the Jordan decomposition ([Spr] Theorem 2.4.8(i)) $z = 1$ and $g_1 = g_2$.

Next, we prove that it is surjective. Let $g' \in G_{ad}$ be unipotent. By surjectivity of π we can find $g \in G$ such that $g' = \pi(g)$. Let $g = su$ be the Jordan decomposition ([Spr] Theorem 2.4.8(i)) of g. Then $\pi(g) = \pi(s)\pi(u)$ is the Jordan decomposition of g' by [Spr] Theorem 2.4.8(ii). Thus $\pi(s) = e$ and $\pi(u) = g'$.

Therefore, we have a bijective map from the unipotent variety in G to the unipotent variety of G_{ad}. Furthermore, if the unipotent elements g_1, g_2

in G are conjugate then clearly $\pi(g_1), \pi(g_2)$ are also conjugate unipotent elements of G_{ad}. Hence, we have a bijection of the unipotent conjugacy classes of G and of G_{ad}. $\square$

Consider the unipotent variety $\mathcal{U}_{ad}$ in G_{ad} and let $\bar{\mathcal{U}}$ denote its closure in the wonderful compactification X of G_{ad} as described in chapter 2. We call $\bar{\mathcal{U}}$ the wonderful compactification of $\mathcal{U}_{ad}$.

Example 3.2.11. We continue our example cf. examples 1.1.1, 1.2.3, and 3.1.3.

The fundamental character $\chi_\alpha : G \to Gl(St) \to \Bbbk$ is the trace map $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \to a + d$. This is also the Steinberg map $\kappa : G \to \mathbb{A}^1$.

The closed subset $\mathcal{C}$ is the subset

$$\{\left[\begin{smallmatrix} -a & 1 \\ -1 & 0 \end{smallmatrix}\right] \mid a \in \Bbbk\}$$

since $\dot{s}_\alpha = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ by [Spr] Lemma 8.1.4(a). Then $\mathcal{U} \cap \mathcal{C} = \{\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]\}$. Notice that the centralizer of $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ is $SO_2 = \{\left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] \in G \mid a, b \in \Bbbk\}$. Thus, $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ is regular since $\dim(SO_2) = 1$.

We also have that $\mathcal{U}_{ad} = \{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in G_{ad} \mid (a+d)^2 = 0\}$. We have to take squares by the description of the coordinate ring $\Bbbk[G_{ad}]$ in [Spr] Exercise 2.1.5(3). Therefore, $\bar{\mathcal{U}}$ is the following subset of $\mathbb{P}^3$:

$$\{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathbb{P}^3 \mid a + d = 0\}$$

Chapter 4

Frobenius Splittings

Having an algebraic variety, it is natural (at least for an algebraic geometer) to ask whether it is Frobenius split or not. In this chapter we will prove that $\bar{\mathfrak{U}}$ is Frobenius split when the fundamental characters χ_i considered in Chapter 3 all equal 0 at the identity e of G. We use a result due to Strickland ([Str]) which enables us to construct Frobenius splittings of X by extending a Frobenius splitting of the unique closed orbit $Y = {}^{G}\!/_{B} \times {}^{G}\!/_{B}$. See Theorem 4.2.1 on page 40 for a more precise statement. Now, Frobenius splittings are closely related to global sections of the $p-1$'st power of the canonical sheaf and hence we find such a section which determines a Frobenius splitting and whose image under first restriction to G_{ad} composed with the map $\pi^*\colon \Bbbk[G_{ad}] \to \Bbbk[G]$ is divisable by a product of $p-1$'st powers of the fundamental characters χ_i's since they determine the unipotent variety $\tilde{\mathfrak{U}}$ by Proposition 3.2.3 on page 29.

First, we introduce the general concept of Frobenius splittings in section 4.1 recapping some of the well-developed theory (see [M-R], [RR], and [R]). For the convenience of the reader and ourselves, we follow the book 'Frobenius Splittings Method in Geometry and Representation Theory' of [B-K]. In section 4.2 we look into extending Frobenius splittings of ${}^{G}\!/_{B} \times {}^{G}\!/_{B}$ to X. To gain control of the Frobenius splittings of X, we find an explicit criterion for Frobenius splittings of Y and furthermore, a $G \times G$-equivariant map $f : St \boxtimes St \to \mathrm{H}^0(X, \mathcal{L}_X((p-1)\rho))$ such that the composition with the restriction map to ${}^{G}\!/_{B} \times {}^{G}\!/_{B}$ is non-zero. In section 4.3, these results enable us to prove the main result: That $\bar{\mathfrak{U}}$ is Frobenius split when $\chi_i(e) = 0$ for $1 \leq i \leq \ell$. After a short crash-course on B-canonical split schemes we show that $\bar{\mathfrak{U}}$ is actually B-canonical split.

4.1 Basics on Frobenius splittings

This subsection is a review of the concept of Frobenius splittings. The notion 'Frobenius splittings' was defined in 1985 by Mehta and Ramanathan in [M-R] and many of the results in this section are due to them.

Let $p := \mathrm{char}(\Bbbk) > 0$. Having a scheme Z, the *absolute Frobenius morphism* $F : Z \to Z$ is the identity on point spaces and locally it raises the functions to the p'th power. Note that the absolute Frobenius morphism is not a morphism of schemes over $\Bbbk$. The starting point of Frobenius splittings is the following definition due to Mehta and Ramanathan ([M-R] Definition 2):

Definition 4.1.1. A scheme Z is called *Frobenius split* (or shorter *F-split*) if the map of $\mathcal{O}_X$-modules $F^\# : \mathcal{O}_Z \to F_*\mathcal{O}_Z$ defined by $\varphi \mapsto \varphi^p$ admits a splitting $s : F_*\mathcal{O}_Z \to \mathcal{O}_Z$ (i.e. $s \circ F^\# = \mathrm{id}$). Such a map is called a *F-splitting of* Z .

The second assertion of the following remark is due to Ramanathan ([R] Remark 1.3).

Remark 4.1.2. *([B-K] Remark 1.1.4 and Proposition 1.2.1)*
(i) A F-splitting of Z is an endomorphism ϕ of $\mathcal{O}_Z$ (considered as a sheaf of abelian groups on Z) satisfying $\phi(1) = 1$ and $\phi(f^p g) = f\phi(g)$ for $f, g \in \mathcal{O}_Z$
(ii) A F-split scheme Z is reduced.

Proof. Observe that for a sheaf $\mathcal{F}$ of $\mathcal{O}_Z$-modules, $F_*\mathcal{F}$ equals $\mathcal{F}$ as sheaves but the $\mathcal{O}_X$-module structure is twisted with F, i.e. $f \cdot s = f^p s$ for local sections $s \in \mathcal{F}, f \in \mathcal{O}_Z$. This gives immediately that $\phi \in \mathrm{Hom}_{\mathcal{O}_Z}(F_*\mathcal{O}_Z, \mathcal{O}_Z)$. Since $\phi(f^p) = f\phi(1)$, the composition $\phi \circ F^\#$ is multiplication with $\phi(1)$. Since $\phi \circ F^\#$ is a regular function on Z, ϕ is a F-splitting if and only if $\phi(1) = 1$ proving (i).

(ii) Let ϕ be a splitting of Z. Consider an affine open subset U of Z and a nilpotent element $f \in \mathrm{H}^0(Z, \mathcal{O}_Z)$, i.e. there exists an integer m such that $f^{p^m} = 0$ in $\mathrm{H}^0(Z, \mathcal{O}_Z)$. But

$$f^{p^{m-1}} = \phi(F^\#(f^{p^{m-1}})) = \phi(f^{p^m}) = 0$$

Hence by induction in m we get $f = 0$. Therefore Z is reduced. $\square$

Another important notion is 'compatibly F-split', also introduced in [M-R] Definition 3:

Definition 4.1.3. Let $V \subseteq Z$ be a closed subscheme with ideal sheaf $\mathcal{I}_V$. Then V is *compatibly F-split* in Z if we can choose a F-splitting s of Z such that $s(F_*\mathcal{I}_V) \subseteq \mathcal{I}_V$.

Observe that if V is compatibly F-split in Z then V itself is F-split. We have collected some useful results about F-splittings in the next Lemma which slightly generalize [M-R] Lemma 1 and [R] Proposition 1.9, respectively:

Lemma 4.1.4. *([B-K] Lemma 1.1.7 and Proposition 1.2.1)*
Let ϕ be a Frobenius splitting of a scheme Z.
(i) Let V be a closed irreducible subscheme of Z. If U is an open subset of Z such that $V \cap U \neq \emptyset$ then V is compatibly F-split in Z wrt. ϕ if and only if $U \cap V$ is compatibly F-split in U wrt. $\phi_{|U}$.
(ii) Assume furthermore that Z is noetherian. Let V_1, V_2 be closed subschemes of Z. If V_1 and V_2 are compatibly F-split in Z wrt. ϕ then $V_1 \cap V_2$, and $V_1 \cup V_2$ and any irreducible component of these are compatibly F-split in Z.

Proof. Note that $\phi(1) = 1$. Since ϕ restricts to a $\mathcal{O}_U$-linear map $F_*\mathcal{O}_U \to \mathcal{O}_U$ it is a F-splitting of U such that $U \cap Y$ is compatibly F-split as $\phi(F_*\mathcal{I}_V) \subseteq \mathcal{I}_V$ (by definition of compatibly F-split). This proves the 'only if' part in (i).

In order to prove the 'if' part of (i), we have to prove that $\phi(F_*\mathcal{I}_V) \subseteq \mathcal{I}_V$. Since $\phi(f^p) = f$ for any local section $f \in \mathcal{O}_Z$ (by definition of a F-splitting), we get that $\mathcal{I}_V \subseteq \phi(F_*\mathcal{I}_V)$. By [Ha1] Proposition II.5.7 and Proposition II.5.8 we have that $\phi(F_*\mathcal{I}_V)$ is quasi-coherent. It therefore determines a closed subscheme V' of Z by [Ha1] Proposition II.5.9 and further, V' is contained in V as $\phi(F_*\mathcal{I}_V) \supseteq \mathcal{I}_V$. As $\phi_{|U}$ is a compatibly F-splitting of $U \cap V$ in U we get $V' \cap U = V \cap U$. The ideal sheaf of V' (i.e $\phi(F_*\mathcal{I}_V)$) vanishes on the dense subset $U \cap V$ of V because $\mathcal{I}_V \subseteq \phi(F_*\mathcal{I}_V)$. Since V is reduced, the ideal sheaf $\phi(F_*\mathcal{I}_V)$ vanishes on V. Thus $V' = V$ proving that $\phi(F_*\mathcal{I}_V) = \mathcal{I}_V$. Hence V is compatibly F-split in Z.

Next, we prove the first assertion of (ii) by straightforward calculations:

$$\phi(F_*\mathcal{I}_{V_1\cap V_2}) = \phi(F_*(\mathcal{I}_{V_1} + \mathcal{I}_{V_2})) \subseteq \phi(F_*\mathcal{I}_{V_1}) + \phi(F_*\mathcal{I}_{V_2}) \subseteq \mathcal{I}_{V_1} + \mathcal{I}_{V_2} = \mathcal{I}_{V_1\cap V_2}$$

and similarly we find that $\phi(F_*\mathcal{I}_{V_1\cup V_2}) \subseteq \mathcal{I}_{V_1\cup V_2}$. Therefore, $V_1 \cup V_2$ and $V_1 \cap V_2$ are compatibly F-split wrt. ϕ.

To complete the proof, it suffices to prove that every irreducible component of Z is compatibly F-split in Z wrt. ϕ. Let C be an irreducible component of Z and let D be the union of all other irreducible components of Z. Again, we have that $\phi(F_*\mathcal{I}_C)$ is a coherent sheaf of ideals of $\mathcal{O}_Z$ containing $\mathcal{I}_C$. Further, both sheaves vanish on $X \setminus D$. And, $X \setminus D = C \setminus D$ which is dense in C. Since X is reduced, the ideals vanish on C and hence $\phi(F_*\mathcal{I}_C) = \mathcal{I}_C$. □

We now examine $\mathcal{H}om_{\mathcal{O}_Z}(F_*\mathcal{O}_Z, \mathcal{O}_Z)$ more closely in order to find a criterion for a smooth variety to be F-split.

For the rest of this section let Z denote a **smooth irreducible variety** of dimension N.

Let ω_Z denote the canonical sheaf. Let z be a point of Z. Then the local ring $\mathcal{O}_{Z,z}$ is regular, i.e. the maximal ideal $\mathcal{M}_z$ of $\Omega_{Z,z}$ satisfies $\dim_{\Bbbk}(\mathcal{M}_z/\mathcal{M}_z^2) = N$ $(= \dim(Z))$. Take elements in $\mathcal{M}_z$ such that their images is a basis for the vector space $\mathcal{M}_z/\mathcal{M}_z^2$. Then by Nakayama's Lemma (eg. [E] Corollary 4.8) these elements generate the maximal ideal $\mathcal{M}_z$. Such a set of minimal generators of $\mathcal{M}_z$ is called a *regular system of parameters*. By [E] Proposition 10.16 the completion of $\mathcal{O}_{Z,z}$ is isomorphic a power series ring $\Bbbk[[t_1, \dots, t_N]]$ where the variables $t_1, \dots, t_N$ can be chosen to be the elements in a regular system of parameters. Therefore we also call a regular system of parameters for *local coordinates* at the point z. Now, let $t_1, \dots, t_N$ be local coordinates at $z \in Z$ then $dt_1 \wedge \cdots \wedge dt_N$ generate the stalk $\omega_{Z,z}$ since $\omega_{Z,z}$ is locally free sheaf of rank 1 (see [Ha1] §II.5).

We adopt the multi-index notation: We will denote the monomial $t_1^{c_1} \cdots t_N^{c_N}$ by $\boldsymbol{t^c}$ where $\boldsymbol{c} = (c_1, \dots, c_N) \in \mathbb{N}^N$. It turns out that $(p-1, \dots, p-1)$ plays an important role and will be denoted $\boldsymbol{p-1}$. With this notation let $f \in \mathcal{O}_{Z,z}$ and consider the image of f in the completion $\Bbbk[[t_1, \dots, t_N]]$ for local coordinates at z. Write $f = \sum_{\boldsymbol{c}} f_{\boldsymbol{c}} \boldsymbol{t^c}$. Define

$$Tr : \Bbbk[[t_1, \dots, t_N]] \to \Bbbk[[t_1, \dots, t_N]], \qquad f \longmapsto \sum_{\boldsymbol{c}, \boldsymbol{c}=\boldsymbol{p-1}+p\boldsymbol{d}} f_{\boldsymbol{c}}^{1/p} \boldsymbol{t^d}$$

Having fixed the notation, we can give a local description of the evaluation map $\mathrm{Hom}_{\mathcal{O}_Z}(F_*\mathcal{O}_Z, \mathcal{O}_Z) \to \mathcal{O}_Z(Z)$ given by $\phi \mapsto \phi(1)$. Actually, we will consider it defined on sheaves:

$$\epsilon : \mathcal{H}om_{\mathcal{O}_Z}(F_*\mathcal{O}_Z, \mathcal{O}_Z) \to \mathcal{O}_Z, \qquad \phi \mapsto \phi(1)$$

From [Ha1] Exercise III.6.10 we have that $\mathcal{H}om_{\mathcal{O}_Z}(F_*\mathcal{O}_Z, \mathcal{O}_Z) = F_*(F^!\mathcal{O}_Z)$. The next theorem tells that $F^!\mathcal{O}_Z \simeq \omega_Z^{1-p}$ and furthermore gives a local description. It was proved in the original paper [M-R] §2 for smooth projective varieties by using Serre duality. The approach of Brion and Kumar in [B-K] is via duality for the Frobenius morphism following [vdK1]. An essential ingredient of the proof is the Cartier operator defined in [Cart] (see also [Katz]). We will not give a proof, but refer to [B-K].

Theorem 4.1.5. *([B-K] Theorem 1.3.8)*
The sheaf $F^!\mathcal{O}_Z$ is isomorphic to ω_Z^{1-p} identifying the evaluation map ϵ :

$F_*(F^!\mathcal{O}_Z) \to \mathcal{O}_Z$ to the map $\tau : F_*(\omega_Z^{1-p}) \to \mathcal{O}_Z$. Let $z \in Z$ and let $t_1, \dots, t_N$ be local coordinates in z then

$$\tau(f(t_1, \dots, t_N)(dt_1 \wedge \cdots \wedge dt_N)^{1-p}) = Tr(f)$$

Thus, Z is F-split if and only if there exists $\phi \in H^0(Z, \omega_Z^{1-p})$ such that $\tau(\phi) = 1$.
If, furthermore Z is projective, Z is F-split if and only if there exists $\phi \in H^0(Z, \omega_Z^{1-p})$ such that the monomial $\boldsymbol{t^{p-1}}$ occurs in the local expansion of ϕ at some closed point $z \in Z$.

Therefore, we call a global section of ω_Z^{1-p} for a *splitting section* of Z if its image under τ is a non-zero scalar, i.e. a splitting section determines a F-splitting up to a scalar.

Next, we prove that if Z is F-split wrt. $p-1$'st power of a global section σ of ω_Z^{-1} then σ^{p-1} gives compatibly F-splittings of the irreducible components of the subset of zeros of σ which is found as a remark after Proposition 8 in [M-R].

Proposition 4.1.6. *([B-K] Proposition 1.3.11)*
Assume that Z is projective. Let $\sigma \in H^0(Z, \omega_Z^{-1})$ such that σ^{p-1} is a splitting section of Z then any irreducible component of the zero subset $V_Z(\sigma)$ is compatibly F-split (wrt. the splitting section σ^{p-1}).

Proof. Choose a non-singular point x of the zero set of σ. Let Z' denote the unique irreducible component of that zero set such that $x \in Z'$. Note that Z' is a divisor. Since Z is non-singular, Z' corresponds to a line bundle $\mathcal{L}_{Z'}$ of Z. Choose an open subset U of Z such that the line bundles $\mathcal{L}_{Z'}$ and ω_Z restrict isomorphically to the structure sheaf $\mathcal{O}_U$ on U. Then there is a global section $f \in \mathcal{L}_{Z'|U}(U) \simeq \mathcal{O}_U(U)$ such that the zero subset $V_G(f) = Z' \cap U$. Find a regular system of parameters $t_1, \dots, t_N$ in $\mathcal{O}_{U,x}$ such that $t_1 = f$. Consider σ as an element of $\omega_{U,x} \simeq \mathcal{O}_{U,x}$ and hence as an element in the completion $\hat{\mathcal{O}}_{U,x}$. The completion $\hat{\mathcal{O}}_{U,x}$ is isomorphic to $\Bbbk[[t_1, \dots, t_n]]$ by [E] Theorem 7.7. That is, σ (as an element of $\hat{\mathcal{O}}_{U,x}$) can be written as

$$t_1^m g(t_1, \dots, t_N)(dt_1 \wedge \cdots \wedge dt_N)^{-1}$$

where $g(t_1, \dots, t_N)$ is not divisible by t_1. As σ vanishes on Z' we have that $m \geq 1$. Now, since σ^{p-1} is an F-splitting section of Z we have that the coefficient of $\boldsymbol{t^{p-1}}$ in $t_1^{m(p-1)} g(t_1, \dots, t_N)^{p-1}$ is non zero (cf. Theorem 4.1.5). Hence $m \leq 1$. Therefore $m = 1$ and Z' is compatibly F-split in Z wrt. σ^{p-1} at x. Thus, Z' is compatibly F-split in an open subset containing x. By Lemma 4.1.4 on page 37 Z' is compatibly F-split in Z. $\square$

4.2 Frobenius splittings of Y and of X

In Strickland's paper ([Str]) it is proved that X is F-split. In particular, she shows how to extend a F-splitting section on $Y = {}^{G}\!/_{B} \times {}^{G}\!/_{B}$ to a F-splitting section on X. In this section, we will explore the F-splitting section of Y to get good control on the F-splitting sections of X. Mehta and Ramanathan proved that ${}^{G}\!/_{B}$ is F-split in [M-R]. This can easily be generalized to ${}^{G}\!/_{B} \times {}^{G}\!/_{B}$. In [L-T], Lauritzen and Thomsen showed that the Steinberg module St plays a central role in the F-splitting of ${}^{G}\!/_{B}$ giving a very explicit criterion for F-splittings of ${}^{G}\!/_{B}$.

But first the starting point, namely Strickland's result[1]:

Theorem 4.2.1. *([Str] Theorem 3.1)*
Assume that $\phi \in H^0(2(p-1)(\rho,\rho))$ is a splitting section of Y. If $\psi \in H^0(X, \mathcal{L}_X(2(p-1)\rho))$ satisfies that $\psi_{|Y} = \phi$ then $\psi \prod_{i=1}^{\ell} \sigma_i^{p-1}$ is a splitting section of X where σ_i is the global section of $\mathcal{L}_X(X_i)$ having X_i as divisor.

Proof. Observe that $\omega_X = \mathcal{L}_X(2\rho + \sum_{i=1}^{p-1} \alpha_i)$ by Remark 2.3.3 on page 22. Thus $\psi \prod_{i=1}^{\ell} \sigma_i^{p-1} \in \mathrm{H}^0(X, \omega_X^{1-p})$.

Since $\phi \in \mathrm{H}^0(Y, \mathcal{L}_Y((p-1)(\rho,\rho)))$ is a splitting section it satisfies that there exists a point $y \in Y$ such that the term $(\prod_{i=1}^{d} t_i^{p-1})(dt_1 \wedge \cdots \wedge dt_d)^{1-p}$ has non-zero coefficient in the local expansion of ϕ at y (Theorem 4.1.5 on page 38). Here $t_1, \dots, t_d$ ($d = \dim(Y)$) is a regular system of parameters in $\mathcal{O}_{Y,y}$.

We have that $Y = \cap_{i=1}^{\ell} D_i$ and hence we get that the ideal sheaf $\mathcal{I}_Y$ of Y is locally generated by the elements $\sigma_1, \dots, \sigma_\ell$ because the D_i's have normal crossings (cf. Theorem 2.2.1 on page 13). Since X is smooth we find that $t_1', \dots, t_d', \sigma_1, \dots, \sigma_\ell$ form a regular sequence in $\mathcal{O}_{X,y}$ where $t_i' \in \mathcal{O}_{X,y}$ denotes a lift of t_i. Hence the local expansion of ψ is a polynomial in $t_1', \dots, t_d', \sigma_1, \dots, \sigma_\ell$.

Our assumption that $\psi_{|Y} = \phi$ implies that the coefficient of $(t_1' \dots t_d')^{p-1}$ in the local expansion of ψ is non-zero times $(dt_1', \dots, dt_d', d\sigma_1, \dots, d\sigma_\ell)^{1-p}$. Multiplying ψ with $\prod_{i=1}^{\ell} \sigma_i^{p-1}$ and using Theorem 4.1.5 on page 38 again gives that $\psi \prod_{i=1}^{\ell} \sigma_i^{p-1}$ is a F-splitting section of X. □

It would however be useless unless Y is F-split. This is fortunately the case which will be proven in Theorem 4.2.3 on page 42. Since our goal is to find an F-splitting of X such that $\bar{\mathcal{U}}$ is compatibly F-split wrt. it, we are interested in finding rather explicit descriptions of F-splittings of Y.

We will first prove that ${}^{G}\!/_{B}$ is F-split which in turn as a corollary gives that Y is F-split. Note that $\omega_{G/B} = \mathcal{L}_{G/B}(-2\rho)$ ([Jan] II.4.2.(5) and (6))

[1] This is, in fact, a weaker statement but we will not need more

and hence that $\mathrm{H}^0({}^G/_B, \omega_{G/B}^{1-p}) \simeq \mathrm{H}^0(2(p-1)\rho)$. Consider the multiplication map $m : \mathrm{H}^0((p-1)\rho) \otimes \mathrm{H}^0((p-1)\rho) \to \mathrm{H}^0(2(p-1)\rho)$. Hence, the Steinberg module $St = \mathrm{H}^0((p-1)\rho)$ becomes interesting. Which elements of $St \otimes St$ map to splitting sections of ${}^G/_B$?

Before answering that question we need an observation: Since St is selfdual ([Jan] §II.3.18 and Corollary II.2.5) we can choose an isomorphism $\gamma : St \to St^*$ which then gives a unique G-equivariant bilinear form $\chi : St \boxtimes St \to \Bbbk$ given by $\chi(u \otimes v) = \gamma(u)(v)$. Let $v_-, v_+ \in St$ denote respectively the lowest and highest weight vector in the Steinberg module St.

Proposition 4.2.2. *([B-K] Theorem 2.3.1 and Corollary 2.3.5) For any $u, v \in St$, $m(u \otimes v)$ is an F-splitting section (up to non-zero scalar) of ${}^G/_B$ if and only if $\chi(u \otimes v) \neq 0$.*

Proof. Equivalently, we have to prove that $\tau(m(u \otimes v)) \neq 1$ if and only if $\chi(u \otimes v) \neq 0$.

The multiplication map $m : St \otimes St \to \mathrm{H}^0(2(p-1)\rho)$ is G-equivariant by Frobenius reciprocity (see e.g. [Jan] Proposition I.3.4b). Now also τ is G-equivariant under the canonical G-structures of $\mathcal{O}_{G/B}$ and $F_*\mathcal{O}_{G/B}$ and by Theorem 4.1.5 on page 38.

Since the G-equivariant for $\chi : St \otimes St \to \Bbbk$ is unique (by Frobenius reciprocity), the composition $\tau \circ m$ is equal to χ up to a scalar $z \in \Bbbk$. To prove the proposition, it suffices to show that $z \neq 0$.

To prove this, we will show rather explicitly that $\tau(m(v_- \otimes v_+)) \neq 0$ showing that z must be non-zero.

We know from [Jan] Lemma 2.13(b) that $St^* \simeq V((p-1)\rho)$ is generated by a highest weight vector v^* (as a G-module). Viewing St as the double dual St^{**}, we have $m(v_- \otimes v_+)(g) = v_-(g.v^*)v_+(g.v^*)$. For $g \in B_u^-$, this expression simplifies to $m(v_- \otimes v_+)(g) = v_+(g.v^*)$ since $g.v^*$ then has weight strictly less than $(p-1)\rho$ (for $g \in B_u$).

Ordering the positive roots $\Phi^+ = \{\beta_1, \ldots, \beta_N\}$ then we have a kind of Taylor series[2] (cf. [Jan] §I.7.8 and §I.7.12)

$$u_{-\beta_i}(z).v^* = \sum_{j \geq 0} z^j (f_{\beta_i}^{(j)}.v^*)$$

where $f_\beta \in \mathfrak{g}_{-\beta}$ and $f_\beta^{(j)} = \frac{f_\beta^j}{j!} \otimes 1 \in \mathrm{Dist}(\mathbb{G}_a, \mathbb{Z}) \otimes_{\mathbb{Z}} \Bbbk$ as in [Jan] §I.7.6. By [Spr] Proposition 8.2.1, we find an isomorphism of varieties

$$\eta : \mathbb{G}_a^N \to B_u^- \quad , \qquad (z_1, \ldots, z_N) \longmapsto \prod_{i=1}^{N} u_{-\beta_i}(z_i)$$

[2] This is *not* the standard name but this is how I like to see it

Thus, for $(z_1, \dots, z_N) \in \mathbb{G}_a^N$

$$m(v_- \otimes v_+)(\eta(z_1, \dots, z_N)) = v_+\left(\sum_{(k_1,\dots,k_N)\in\mathbb{Z}_+^N} z_1^{k_1} \cdots z_N^{k_N} f_{\beta_1}^{(k_1)} \cdots f_{\beta_N}^{(k_N)}.v^* \right)$$

Now, St has a basis consisting of $f_{\beta_j}^{(k_j)}$ with $0 \le k_j < p$ for all roots $\beta \in \Phi^+$ by [Jan] §3.18 (2) and (4). Therefore, $f_{\beta_1}^{(p-1)} \cdots f_{\beta_N}^{(p-1)}.v^*$ is the only element in $V((p-1)\rho) \simeq St$ of lowest weight $-(p-1)\rho$; thus it is non-zero. Therefore, the coefficient of $z_1^{p-1} \cdots z_N^{p-1}$ is non-zero and hence, by Theorem 4.1.5 on page 38, $m(v_- \otimes v_+)$ is a splitting section (up to a scalar multiple). Thus, $\tau(m(v_- \otimes v_+)) \neq 0$ implying that $z \neq 0$ which proves the proposition. □

As a corollary we will generalize this result to $Y \simeq {}^{G}/_{B} \times {}^{G}/_{B}$:

Theorem 4.2.3. *([B-K] Theorem 2.3.8)*
Let $u_1 \otimes u_2, v_1 \otimes v_2 \in St \boxtimes St$ then $m^2(u_1 \otimes u_2 \otimes v_1 \otimes v_2)$ is a F-splitting section (up to a scalar) of Y if and only if $\langle u_1 \otimes u_2, v_1 \otimes v_2\rangle \neq 0$ where $\langle u_1 \otimes u_2, v_1 \otimes v_2\rangle = \chi(u_1 \otimes v_1)\chi(u_2 \otimes v_2)$, the unique (up to a scalar) $G \times G$-equivariant bilinear form on $(St \boxtimes St) \otimes (St \boxtimes St)$ and $m^2 : (St \boxtimes St)^{\otimes 2} \to H^0(2(p-1)\rho, 2(p-1)\rho)$ the standard multiplication map.

Proof. We have that $B \times B$ is a Borel subgroup of the product group $G \times G$. Hence the Steinberg module for $G \times G$ is $\mathrm{H}^0((p-1)\rho, (p-1)\rho) \simeq St \boxtimes St$. Note that $m^2((f_1 \otimes f_2) \otimes (g_1 \otimes g_2)) = m(f_1 \otimes g_1) \otimes m(f_2 \otimes g_2)$. Now apply Proposition 4.2.2 above on ${}^{G\times G}/_{B\times B} \simeq {}^{G}/_{B} \times {}^{G}/_{B}$. □

Consider $St \boxtimes St$. We can view $St \boxtimes St$ as a G-module via the action of the diagonal $\Delta(G)$ of G which we then denote $St \otimes St$. Since St is selfdual ([Jan] II.3.18(4)-(6) and Corollary II.2.5) we have $St \otimes St \simeq St^* \otimes St \simeq \mathrm{Hom}_{\Bbbk}(St, St)$. Whence by Frobenius reciprocity ([Jan] Proposition I.3.4b) $St \otimes St$ contains a unique (up to scalar) G-invariant element v.

Observe that the $\Delta(G)$-invariant element $v \in St \boxtimes St$ corresponds to the identity endomorphism in $\mathrm{End}(St) \simeq St^* \otimes St \simeq St \otimes St$. Thus, when expressing v in a basis consisting of weight vectors, the coefficient of $v_+ \otimes v_-$ is non-zero. Hence $\langle v_- \otimes v_+, v\rangle = \chi(v_- \otimes v_+) \cdot \chi(v_+ \otimes v_-) \neq 0$. This implies that $m^2((v_- \otimes v_+) \otimes v)$ is a F-splitting section of Y by Theorem 4.2.3 above.

To gain more control of the global sections of $\mathcal{L}_X((p-1)\rho)$ which restrict to F-splittings on Y we have the following crucial result due to Brion and Polo:

Lemma 4.2.4. *(Proof of [B-P2] Theorem 2)*
There exists a $G \times G$-homomorphism

$$f : St \boxtimes St \to H^0(X, \mathcal{L}_X((p-1)\rho))$$

of which the restriction to Y is non-zero (i.e. f splits).

Proof. In the proof of Proposition 2.3.2 on page 21 we have that the global section τ_i of $\mathcal{L}_X(D_i) \simeq \mathcal{L}_X(\omega_i)$ is a $B \times B$-eigenvector of weight $(\omega_i, -w_o\omega_i)$. Therefore, the $B \times B$-eigenvector $\tau = \prod_{i=1}^{\ell} \tau_i^{p-1}$ is contained in $\mathrm{H}^0(X, \mathcal{L}_X((p-1)\rho))$ as $\sum_{i=1}^{\ell} \omega_i = \rho$. Thus, we have a $B \times B$-homomorphism:

$$\Bbbk_{-(p-1)(\rho,\rho)} \to \mathrm{H}^0(X, \mathcal{L}_X((p-1)\rho))$$

given by $1 \mapsto \tau$. When first dualizing and then using Frobenius reciprocity ([Jan] Proposition I.3.4b) on this map, we get the $G \times G$-homomorphism

$$\mathrm{H}^0(X, \mathcal{L}_X((p-1)\rho))^* \to \mathrm{Ind}_{B\times B}^{G\times G}(\Bbbk_{(p-1)(\rho,\rho)}) \simeq St \boxtimes St$$

Hence by the selfduality of the St ([Jan] II.3.18(4)-(6) and Corollary II.2.5) we get the map

$$f : St \boxtimes St \to \mathrm{H}^0(X, \mathcal{L}_X((p-1)\rho))^{**} \simeq \mathrm{H}^0(X, \mathcal{L}_X((p-1)\rho))$$

Now, the eigenvector τ restricts to a non-zero element of $St \boxtimes St$ since Y is not contained in D_i for $i = 1, \dots, \ell$. $\square$

This provides good control on F-splitting sections of X constructed as in Theorem 4.2.1 on page 40 because $res_Y \circ f$ is the identity.

Example 4.2.5. We continue our example (see the previous examples - examples 1.1.1, 1.2.3 , 2.1.6, 2.2.8, 2.3.4).

We will also need the G-action of St^*. In general, $St^* \simeq St$ (as St is selfdual) and we let G act on St^* regarded as $\Bbbk^2$ in the same way as before (see Example 1.2.3 on page 6).

The isomorphism $\gamma : St \to St^*$ is in this example given by $m_\rho \mapsto m_\rho^* = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and $m_{-\rho} \mapsto m_{-\rho}^* = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ under γ. Hence, the G-invariant bilinear form $\chi : St \otimes St \to \Bbbk$ is given by $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto ay - bx$ since

$$\chi(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \otimes \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)) = \chi(\left(\begin{smallmatrix} ax+by \\ cx+dy \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} az+bw \\ cz+dw \end{smallmatrix}\right)) = xw - yz$$

using that $ad - bc = 1$.

Notice that the $\Delta(G)$-invariant element v of $St \otimes St$ is the element $m_{-\rho} \otimes m_\rho + m_\rho \otimes m_{-\rho}$. By Remark 4.3.1 on the following page, $f(v)_{|G_{ad}} \circ \pi$ is the character associated with St. Thus, $f(v)_{|G_{ad}} \circ \pi = \chi_\alpha$ considered in Example 3.2.11 on page 34. Observe that f equals the map g of Lemma 4.3.2 on page 45 which explains why we get this result.

We need an explicit description of the $G \times G$-equivariant map $f : St \otimes St \to \mathrm{H}^0(X, \mathcal{L}_X(\rho))$. As we saw in the construction of f (cf. Lemma 4.2.4 on page 42), we have the $B \times B$-equivariant map $\Bbbk_\rho \otimes \Bbbk_\rho \to \mathrm{H}^0(X, \mathcal{L}_X(\rho))$ given by $1 \otimes 1 \mapsto \tau = Z_{21}$. Thus, $f(m_\rho \otimes m_\rho) = Z_{21}$. Using that f is $G \times G$-equivariant, we get $f(g_1.m_\rho \otimes g_2.m_\rho) = (g_1, g_2).Z_{21}$. As $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, we get that f maps

$$m_\rho \otimes m_{-\rho} \overset{f}{\mapsto} -Z_{22}, \quad m_{-\rho} \otimes m_\rho \mapsto Z_{11}, \quad m_{-\rho} \otimes m_{-\rho} \mapsto -Z_{12}$$

Hence $f(v) = f(m_{-\rho} \otimes m_\rho + m_\rho \otimes m_{-\rho}) = Z_{11} - Z_{22}$ and $f(v_- \otimes v_+) = f(m_{-\rho} \otimes m_\rho) = Z_{11}$.

4.3 Frobenius splitting of $\bar{\mathcal{U}}$

In this section, we will prove one of our main results, namely that $\bar{\mathcal{U}}$ is F-split when $\chi_i(e) = 0$ in $\Bbbk$ for $1 \leq i \leq \ell$. Hence $\mathcal{U} = \mathcal{V}_G(\chi_1, \ldots, \chi_\ell)$. This turns out to be essential in what follows.

Now, we have seen that $m^2((v_- \otimes v_+) \otimes v)$ is a F-splitting section of Y by 4.2.3 since $\langle v_- \otimes v_+, v \rangle \neq 0$. Since $res_Y(f(v_- \otimes v_+)\, f(v)) = m^2((v_- \otimes v_+) \otimes v)$ by construction, we find that $f(v_- \otimes v_+)\, f(v) \prod_{i=1}^{\ell} \sigma_i^{p-1}$ is F-splitting section of X. Is $\bar{\mathcal{U}}$ compatibly F-split wrt. this F-splitting section?

Unfortunately, it is not easy to give an affirmative answer. What we could hope for is, that $f(v)_{|G_{ad}} \circ \pi$ is equal to a product of the fundamental characters to the $p-1$'st power. Since the common zero set in G of these fundamental characters is the unipotent variety $\mathcal{U}$, it would therefore give important imformation about the closure $\bar{\mathcal{U}}$. But as the following remark states, this is too much to hope for.

Remark 4.3.1. $f(v)_{|G_{ad}} \circ \pi \neq \prod_{i=1}^{\ell} \chi_i^{p-1}$.

Proof. We will construct a counterexample, but first we need to get a more explicit expression of $f(v)_{|G_{ad}} \circ \pi$. We claim that $(f(v)_{|G_{ad}} \circ \pi)(g) = \chi((e, g).v)$ where $v \in St \boxtimes St$ is the $\Delta(G)$-invariant element and $g \in G$ (e is the neutral element in G).

The coordinate ring $\Bbbk[G]$ allows a good filtration as a $G \times G$-module (the action is given by $(g_1, g_2).g = g_1 g g_2^{-1}$) c.f. [Jan] Proposition II.4.20 where one of the factors (which only occur with multiplicity 1) is $St \otimes St$. Therefore, since there is only one (up to a scalar) $G \times G$-equivariant map from $St \otimes St$ to itself it suffices to check that the claimed function in $\Bbbk[G]$ is $G \times G$-equivariant as a homomorphism $St \boxtimes St \to \Bbbk$, i.e. that $(g_1, g_2).\chi((e, g).v) = \chi((e, g)(g_1, g_2).v)$.

Notice that χ is $\Delta(G)$-invariant, i.e. $\chi(gv_1 \otimes gv_2) = \chi(v_1 \otimes v_2)$. Since $(e,g)(g^{-1},g^{-1}) = (g^{-1},e)$ we have that $\chi((e,g).v) = \chi((g^{-1},e).v)$. As $(g \mapsto \chi((e,g).v)) \in \Bbbk[G]$ we have that $(g_1,g_2)\chi((e,g).v) = \chi((e,(g_1^{-1},g_2^{-1}).g).v) = \chi((e,g_1^{-1}gg_2))$. Using the observation above we find the required proporty which proves the claim.

In fact, we can get an even more explicit formula for $f(v)_{|G_{ad}} \circ \pi$. Let $\{v_1,\ldots,v_d\}$ be a basis of St (where $d = \dim_\Bbbk(St)$) and $\{v_1^*,\ldots,v_d^*\}$ its dual basis. Identifying $St \otimes St \simeq St^* \times St \simeq \mathrm{End}(St)$ where the first map is $\gamma \otimes id$ and observing that the identity map is the $\Delta(G)$-invariant element in $\mathrm{End}(St)$ we find that $v = \sum_{i=1}^d \gamma^{-1}(v_i^*) \otimes v_i$. Hence $(f(v)_{|G_{ad}} \circ \pi)(g) = \chi((e,g).v) = \sum_{i=1}^d v_i^*(g.v_i)$ by the definition of χ. But $\sum_{i=1}^d v_i^*(g.v_i)$ is the trace of g's action on St, i.e. $f(v)_{|G_{ad}} \circ \pi$ is the character associated with St.

We are now ready to construct a counterexample. Consider $G = Sl_3$ in characteristic $p = 3$ and the element $t = \mathrm{diag}(a,b,c)$. Using that St is direct sum of its weight spaces we find:

$$\begin{aligned}\chi_{St}(t) \;=\; & a^4b^2 + a^4c^2 + a^3b^3 + a^3c^3 + a^3 + a^2b^4 + 2a^2b + a^2c^4 + 2a^2c \\ & +2ab^2 + 2ac^2 + b^4c^2 + b^3c^3 + b^3 + b^2c^4 + 2b^2c + 2bc^2 + c^3 + 3\end{aligned}$$

We have used that $abc = 1$ and that $\alpha_1(t) = ab^{-1}$ and $\alpha_2(t) = bc^{-1}$. Also $\rho = \alpha_1 + \alpha_2$ hence $\rho(t) = ac^{-1}$. Further calculations reveal that $\chi_{St}(t) = (a-b)^2(a-c)^2(b-c)^2$.

Now, $\chi_1(t) = a+b+c$ and $\chi_2 = bc+ac+ab$. Therefore we see that the coefficient of for example a^3 is 1 in $\chi_{St}(t)$ while 2 in $\chi_1^2(t)\chi_2^2(t)$ (in characteristic 3). Therefore the claim of the remark is proved. $\square$

Consequently, we are seeking global sections s_i of X such that $res_{G_{ad}} \circ \pi$ is the fundamental characters χ_i on G because of Proposition 3.2.3 on page 29. So we need the following map (as will become apparent later!):

Lemma 4.3.2. *There exists a $G \times G$-equivariant map*

$$g\colon Z := \bigotimes_{i=1}^{\ell} (H^0(-w_o\omega_i)^* \boxtimes H^0(-w_o\omega_i))^{\otimes(p-1)} \to H^0(X, \mathcal{L}_X((p-1)\rho))$$

satisfying $res_Y \circ g \neq 0$

Proof. Let M be a G-module that satisfies conditions (i)–(iii) in Lemma 2.1.1 on page 9. Then $M \otimes \bigotimes_{i=1}^{\ell} \mathrm{H}^0(-w_o\omega_i)^*$ also satisfies (i)–(iii) of Lemma 2.1.1 by Lemma 2.1.3 on page 10. Thus, $X \hookrightarrow \mathbb{P}(\mathrm{End}_\Bbbk(M)) \times \prod_{i=1}^{\ell} \mathbb{P}(\mathrm{End}_\Bbbk(H^0(-w_o\omega_i)^*))$ by Lemma 2.2.6 on page 18.

Consider the following commutative diagram:

$$\begin{tikzcd} Y \arrow[r, hook] & X \arrow[r, hook] \arrow[dr, "\phi_j"'] & \mathbb{P}(\mathrm{End}_{\Bbbk}(M)) \times \prod_{i=1}^{\ell} \mathbb{P}(\mathrm{H}^0(-w_o\omega_i) \boxtimes \mathrm{H}^0(-w_o\omega_i)^*) \arrow[d, "pr_j"] \\ & & \mathbb{P}(\mathrm{H}^0(-w_o\omega_j) \boxtimes \mathrm{H}^0(-w_o\omega_j)^*) \end{tikzcd}$$

The map $Y \to \mathbb{P}(\mathrm{H}^0(-w_o\omega_j) \boxtimes \mathrm{H}^0(-w_o\omega_j)^*)$ is given by $(gB, g'B) \mapsto (g, g').[x]$ where $x \in \mathrm{H}^0(-w_o\omega_j)\boxtimes\mathrm{H}^0(-w_o\omega_j)^*$ is a highest weight vector wrt. $B \times B$. Consider the twisting sheaf $\mathcal{O}_j(1)$ of $\mathbb{P}(\mathrm{H}^0(-w_o\omega_j) \boxtimes \mathrm{H}^0(-w_o\omega_j)^*)$. Then we have that $\phi_j^*\mathcal{O}_j(1) \simeq \mathcal{L}_X(\omega_j)$ by [B-P1] §2.2 since $\mathcal{L}_X(\lambda)$ is the sheaf whose restriction to Y is $\mathcal{L}_Y(-w_o\lambda, \lambda)$ (cf. 2.3.2).

This induces a map from $\mathcal{O}_j(1)(\mathbb{P}(\mathrm{H}^0(-w_o\omega_j)\boxtimes\mathrm{H}^0(-w_o\omega_j)^*)) \to \mathcal{L}_X(\omega_j)(X)$. Again by [B-P1] §2.2 we identify

$$\mathcal{O}_j(1)(\mathbb{P}(\mathrm{H}^0(-w_o\omega_j) \boxtimes \mathrm{H}^0(-w_o\omega_j)^*)) \simeq \mathrm{H}^0(-w_o\omega_j)^* \boxtimes \mathrm{H}^0(-w_o\omega_j)$$

Therefore we have maps $g_j : \mathrm{H}^0(-w_o\omega_j)^* \boxtimes \mathrm{H}^0(-w_o\omega_j) \to \mathrm{H}^0(X, \mathcal{L}_X(\omega_j))$. Thus since all maps considered are $G \times G$-equivariant we get the $G \times G$-equivariant map

$$g : \bigotimes_{i=1}^{\ell} (\mathrm{H}^0(-w_o\omega_i)^* \boxtimes \mathrm{H}^0(-w_o\omega_i))^{\otimes(p-1)} \to \mathrm{H}^0(X, \mathcal{L}_X((p-1)\rho))$$

Again, we have used that $\sum_{i=1}^{\ell} \omega_i = \rho$.

In order to prove that $res_Y \circ g \neq 0$, we will prove that there exists an element $t \in Z$ such that $g(t)_{|Y} = v$ where v is the $\Delta(G)$-invariant element in $St \otimes St$. Let $t_j \in \mathrm{H}^0(-w_o\omega_j)^* \boxtimes \mathrm{H}^0(-w_o\omega_j)$ denote the $\Delta(G)$-invariant sections. These exist and are unique up to scalar under identification of $\mathrm{H}^0(-w_o\omega_j)^* \boxtimes \mathrm{H}^0(-w_o\omega_j)$ with $\mathrm{End}_{\Bbbk}(\mathrm{H}^0(-w_o\omega_j))$. Let t_j correspond to the identity map in $\mathrm{End}_{\Bbbk}(\mathrm{H}^0(-w_o\omega_j))$. Define the $\Delta(G)$-invariant element $t = \bigotimes_{i=1}^{\ell} t_i^{p-1}$.

Observe that when writing t_j w.r.t. a basis consisting of weight vectors the coefficient of the highest weight vector is non-zero.

Since g is a $G \otimes G$-equivariant map and since $v \in St \boxtimes St$ is the unique (up to scalar) $\Delta(G)$-invariant element we only have to show that $g(t)$ is non-zero when restricted to Y. So consider the map $Z \to St \boxtimes St$. Frobenius reciprocity gives a $B \times B$-equivariant map $Z \to \Bbbk_{(p-1)(\rho,\rho)}$. When writing t in a basis consisting of weight vectors then the coefficient of the highest weight vector of Z is non-zero by the above observation. Thus $g(t)_{|Y} \neq v$ which proves the Lemma. □

The next Lemma shows that $g_i(t_i)$ is an extension of the fundamental characters χ_i in the sense that $g_i(t_i)_{|G_{ad}} \circ \pi = \chi_i$:

Lemma 4.3.3. *We have that $g_i(t_i)_{|G_{ad}} \circ \pi$ equals χ_i up to a scalar.*

Proof. Consider the following maps for some weight $\lambda \in X^*(T)$:

$$\mathrm{H}^0(X, \mathcal{L}_X(\lambda)) \to \mathrm{H}^0(G_{ad}, \mathcal{L}_X(\lambda)_{|G_{ad}}) \to \mathrm{H}^0(G, \pi^*\mathcal{L}_X(\lambda)_{|G_{ad}})$$

Since G is simply connected, the Picard group of G is trivial implying that up to a scalar we have the following isomorphism

$$\mathrm{H}^0(G, \pi^*\mathcal{L}_X(\lambda)_{|G_{ad}}) \simeq \mathrm{H}^0(G, \mathcal{O}_G) = \Bbbk[G]$$

By the realization of X in the above proof of Lemma 4.3.2, we get the following map $G \to \mathbb{P}(\mathrm{H}^0(-w_o\omega_j) \boxtimes \mathrm{H}^0(-w_o\omega_j)^*)$ given by $g \mapsto (g, e).[h_j]$ where h_j is the identity element in $\mathrm{End}_{\Bbbk}(\mathrm{H}^0(-w_o\omega_j)^*)$. Now we have seen that $\mathcal{O}_j(1)(\mathbb{P}(\mathrm{H}^0(-w_o\omega_j)\boxtimes\mathrm{H}^0(-w_o\omega_j)^*)) \simeq \mathrm{H}^0(-w_o\omega_j)^*\boxtimes\mathrm{H}^0(-w_o\omega_j)$. Now the trace map $Tr_j : \mathrm{End}_{\Bbbk}(\mathrm{H}^0(-w_o\omega_j)) \to \Bbbk$ is the image of $[h_j]$ under this isomorphism. Therefore, $g \mapsto Tr_j((g, e).[h_j])$ is the regular function $g_i(t_i)_{|G_{ad}} \circ \pi$ on G. But this is the map

$$g_i(t_i)_{|G_{ad}} \circ \pi : G \to Gl(\mathrm{H}^0(-w_o\omega_j)) \to^{Tr} \Bbbk$$

which is nothing else but the fundamental character χ_i considered in section 3. $\square$

We have now reached our main result, namely that the closure in X of the unipotent variety $\mathcal{U}_{ad}$ of G_{ad} is compatibly F-split in X when $G = Sl_q$. Let again v denote the $\Delta(G)$-invariant element in $St \boxtimes St$.

Proposition 4.3.4. *The element $f(v_- \otimes v_+)g(t)\prod_{i=1}^{\ell} \sigma_i^{p-1} \in H^0(X, \omega_X^{1-p})$ is a F-splitting section of X such that $\cap_{i=1}^{\ell} \mathcal{V}_X(g_i(t_i))$ and $X_1, \dots, X_\ell$ are simultanously compatibly F-split with respect to this element.*
If, furthermore, we assume $\chi_i(e) = 0$ in $\Bbbk$ for $1 \leq i \leq \ell$ then $\bar{\mathcal{U}}$ is also compatibly F-split wrt. the F-splitting section above.

Proof. We have seen that $m^2((v_- \otimes v_+) \otimes v)$ is a F-splitting section of $Y = {}^{G\times G}/_{B\times B}$. Therefore the element $f(v_- \otimes v_+)f(v)\prod_{i=1}^{\ell} \sigma_i^{p-1}$ is a F-splitting section of X by Strickland's result which works for any pullback of $(v_- \otimes v_+) \otimes v$ (cf. Theorem 4.2.1 on page 40). Thus, the element $\phi = f(v_- \otimes v_+)g(t)\prod_{i=1}^{\ell} \sigma_i^{p-1}$ is a F-splitting section of X since $g(t)_{|Y} = v$ by the proof of Lemma 4.3.2 on page 45.

Observe that $v_+ = \mu(u_+^{\otimes(p-1)})$ and $v_- = \mu(u_-^{\otimes(p-1)})$ where u_+, u_- are highest, respectively lowest weight vector in $\mathrm{H}^0(\rho)$ and $\mu : \mathrm{H}^0(\rho)^{\otimes(p-1)} \to \mathrm{H}^0((p-1)\rho)$ is the multiplication map. Thus, when writing $f(v_- \otimes v_+) = \nu^{p-1}$ we have that $\phi = (\nu \prod_{i=1}^{\ell} g_i(t_i)\sigma_i)^{p-1}$. Now, by Proposition 4.1.6 on page 39, any irreducible component of $\mathcal{V}_X(\nu \prod_{i=1}^{\ell} g_i(t_i)\sigma_i) = \mathcal{V}_X(\nu) \cup \bigcup_{i=1}^{\ell}(\mathcal{V}_X(g_i(t_i)) \cup \mathcal{V}_X(\sigma_i))$ is compatibly F-split. Hence each of the irreducible components of $\mathcal{V}_X(g_j(t_j))$ is. Furthermore by Proposition 4.1.6 on page 39 any irreducible component of $\cap_{i=1}^{\ell}\mathcal{V}_X(g_i(t_i))$ is compatibly F-split. Thus, by 4.1.4, $\cap_{i=1}^{\ell}\mathcal{V}_X(g_i(t_i))$ is compatibly F-split. Since the zero subset of σ_i is the irreducible divisor X_i by definition, X_i is compatibly F-split by Proposition 4.1.6 on page 39.

Now $\cap_{i=1}^{\ell}\mathcal{V}_X(g_i(t_i)) \cap G_{ad} = \cap_{i=1}^{\ell}\mathcal{V}_{G_{ad}}(g_i(t_i)_{|G_{ad}})$. Recall from Lemma 4.3.3 on the previous page that $g_i(t_i)_{|G_{ad}} \circ \pi = \chi_i$. Since $\mathcal{U} = \cap_{i=1}^{\ell}\mathcal{V}_G(\chi_i - \chi_i(e))$ by Proposition 3.2.3 on page 29, so under our assumption (i.e. $\chi_i(e) = 0$ in $\mathbb{k}$), we get that $\mathcal{U}_{ad} = \cap_{i=1}^{\ell}\mathcal{V}_{G_{ad}}(g_i(t_i)_{|G_{ad}})$. We know that $\mathcal{U}_{ad} \subseteq G_{ad}$ is irreducible and has codimension $\mathrm{rank}(G_{ad}) = \ell$ in G_{ad} (c.f. Theorem 3.1.1 on page 25). Its closure in X satisfies the same and therefore $\bar{\mathcal{U}}$ is an irreducible component of the intersection $\cap_{i=1}^{\ell}\mathcal{V}_X(g_i(t_i))$ (having the highest possible codimension by Krull's Hauptidealsatz). Thus $\bar{\mathcal{U}}$ is compatibly F-split in X. □

Remark 4.3.5. *It is worth observing that the variety $\cap_{i=1}^{\ell}\mathcal{V}_X(g_i(t_i))$ is F-split in all positive characteristics.*

The assumption that $\chi_i(e) = 0$ in $\mathbb{k}$ for all i is rather restrictive in the sense that it is only satisfied in the following case as shown in appendix A.

Type A_n: when $n = p^m$ and $p = char(\mathbb{k}) > 0$ and $m \in \mathbb{N}$.
Type C_n: when $n = 2^m - 1$ and $2 = char(\mathbb{k})$ ($m \in \mathbb{N}$)
Type D_n: when $n = 2^m$ ($m \in \mathbb{N}$) and $2 = char(\mathbb{k})$.
Type E_6: when $char(\mathbb{k}) = 3$
Type E_8: when $char(\mathbb{k}) = 31$
Type F_4: when $char(\mathbb{k}) = 13$
Type G_2: when $char(\mathbb{k}) = 7$

Example 4.3.6. We continue our example (see the previous examples - examples 1.1.1, 1.2.3, 2.1.6, 2.2.8, 2.3.4, 3.1.3, 3.2.11, 4.2.5).

We get that $f(v_- \otimes v_+)f(v)\sigma = Z_{11} \cdot (Z_{11} - Z_{22}) \cdot (Z_{11}Z_{22} - Z_{12}Z_{21})$. Observe that the coefficient of $Z_{11}Z_{12}Z_{21}Z_{22}$ is 1 in the product above. Thus, it is F-splitting section of X.

Now, $\bar{\mathcal{U}} = \mathcal{V}_X((Z_{11} + Z_{22}))$ cf. Example 3.2.11 on page 34. Therefore, we have that $\bar{\mathcal{U}}$ is compatibly split by Proposition 4.1.6 on page 39.

4.4 $\bar{\mathcal{U}}$ is B-Canonical split

In this section we proceed by showing that the F-splitting section of the previous section is actually a B-canonical splitting section of X.

The notion 'B-canonical split' was introduced in [Mat] where Mathieu used it to prove that the global sections of a G-linearized line bundle on a B-canonical split G-scheme allow good filtration cf. the following theorem. We will omit the proof (as we do not use it in this thesis) and refer to the given reference for the details.

Theorem 4.4.1. *([B-K] Theorem 4.2.13)*
Let Z be a G-scheme which is B-canonical split. Then for any G-linearized line bundle $\mathcal{L}$ on Z, the G-module $H^0(X, \mathcal{L})$ admits a good filtration.

This is our motivation for proving that $\bar{\mathcal{U}}$ is B-canonical split. We show that the F-splitting section of X is actually B-canonical when viewing X as a G-variety. As a by-product, we find that it is also $B \times B$-canonical.

Consider the absolute Frobenius morphism $F : Z \to Z$ of the scheme Z. We can identify $F_*\mathcal{O}_Z$ with $\mathcal{O}_Z$ as sheaves of abelian group where the $\mathcal{O}_Z$-structure is $f \cdot g = f^p g$ for $f, g \in \mathcal{O}_Z$. Thus, we can define an $\mathcal{O}_Z$-structure on $\mathrm{End}_F(Z) := \mathrm{Hom}_{\mathcal{O}_Z}(F_*\mathcal{O}_Z, \mathcal{O}_Z)$ by $(f * \phi)(g) = \phi(fg)$ for $f, g \in \mathcal{O}_Z$ and $\phi \in \mathrm{End}_F(Z)$. In particular, we find the $\Bbbk$-linear structure on $\mathrm{End}_F(Z)$ is given by $(z * \phi)(f) = \phi(zf) = z^{1/p}\phi(f)$.

If Z is an H-scheme for an algebraic group H, then H acts $\Bbbk$-linearly on $\mathrm{End}_F(Z)$ by

$$(h \star \phi)(f) = h(\phi(h^{-1}f))$$

where $h \in H$, $\phi \in \mathrm{End}_F(Z)$, and $f \in F_*\mathcal{O}_Z$. The action of H on $F_*\mathcal{O}_Z$ is defined to be the action of H on $\mathcal{O}_Z$ under the identification of $F_*\mathcal{O}_Z$ with $\mathcal{O}_Z$ as sheaves of abelian groups.

Recall from [Jan] §I.7.8 and §I.7.12 that we have a 'Taylor series' for each root $\beta \in \Phi^+$, $z \in \Bbbk$, and m in a G-module:

$$u_\beta(z).m = \sum_{j \geq 0} z^j (e_\beta^{(j)}.m)$$

where $e_\beta \in \mathfrak{g}_\beta$ and $e_\beta^{(j)} = \frac{e_\beta^j}{j!}$.

Definition 4.4.2. ([B-K] Lemma 4.1.6)
Let Z be a B-scheme and let $\phi \in \mathrm{End}_F(Z)$. Then ϕ is called B-*canonical* if ϕ is T-invariant (under the $\star$-action) and for every simple root α_i, $e_{\alpha_i}^{(j)} \star \phi = 0$ for all $1 \leq i \leq \ell$ and $j \geq p$.

If $\phi \in \mathrm{End}_F(Z)$ is a F-splitting of Z and at the same time is B-canonical then ϕ is called a B-canonical splitting of Z. If there is a B-splitting of Z then we call Z B-canonical split .

Lemma 4.4.3. *([B-K] Lemma 4.1.6)*
Let the notation be as in Definition 4.4.2 above. Then if ϕ is assumed furthermore to be T-invariant then ϕ is B-canonical if and only if ϕ is in the image of a B-module map $\theta_\phi : St \otimes \Bbbk_{(p-1)\rho} \to End_F(Z)$.

Proof. Consider the following map $\eta : \mathrm{Dist}(B_u) \to St$ given by $A \mapsto A.v_-$. Here $\mathrm{Dist}(G)$ denotes the algebra of distributions on G (cf. [Jan] §I.7.7). It is surjective by the proof of [Jan] Proposition 2.11 since St is irreducible. By [Polo] Proposition Fondamentale, we find that the kernel

$$\ker(\eta) = \{\sum_{i=1}^{\ell} \sum_{j \geq p} \mathrm{Dist}(G).e_{\alpha_i}^{(j)}\}$$

Assume that $\phi = \theta_\phi(v_- \otimes a)$. Then $e_{\alpha_i}^{(j)} \star \phi = 0$ for all $j \geq p$ and $1 \leq i \leq \ell$ by the description of $\ker(\eta)$ above.

On the other hand, if ϕ is B-canonical then $A \star \phi = 0$ for all $A \in \ker(\eta)$. Thus, the map $\mathrm{Dist}(G) \to \mathrm{End}_F(Z)$ given by $A \mapsto A \star \phi$ factors through $St \otimes \Bbbk_{(p-1)\rho}$ by the arguments so far. This shows the Lemma. □

Having a B-stable subscheme Z' of a B-scheme Z, it is natural to ask when a B-canonical endomorphism in $\mathrm{End}_F(Z)$ induces a B-canionical endomorphism in $\mathrm{End}_F(Z')$. We find:

Remark 4.4.4. *([B-K] §4.1.16)*
Let Z be B-scheme and Z' a B-stable subscheme. Assume that $\phi \in End_F(Z)$ satisfies $\phi(\mathcal{I}_{Z'}) \subseteq \mathcal{I}_{Z'}$, where $\mathcal{I}_{Z'} \subseteq \mathcal{O}_Z$ is ideal sheaf of Z'. Then the induced endomorphism $\phi' \in End_F(Z')$ is B-canonical. In particular, if ϕ is a B-canonical F-splitting of Z such that Z' is compatibly split wrt. ϕ then Z' is B-canonical split.

Proof. Define $\mathrm{End}_F(Z', Z)$ to be the B-submodule of $\mathrm{End}_F(Z)$ consisting of those ϕ such that $\phi(\mathcal{I}_{Z'}) \subseteq \mathcal{I}_{Z'}$. Then the induced map $\mathrm{End}_F(Z', Z) \to \mathrm{End}_F(Z')$ given by $\phi \to \phi'$ is a B-module map. Thus, if $\phi \in \mathrm{End}_F(Z', Z)$ is B-canonical then the induced endomorphism ϕ' is in the image of a B-module map $St \otimes \Bbbk_{(p-1)\rho} \to \mathrm{End}_F(Z')$ and hence ϕ' is B-canonical by Lemma 4.4.3 above. □

The next Lemma now gives a criterion for a smooth G-variety to be B-canonical split.

Lemma 4.4.5. *([vdK2] Lemma 2.3)*
Let Z be a smooth G-variety. Then Z is B-canonical split if and only if there is a G-module map $\Theta : St \otimes St \to H^0(Z, \omega_Z^{1-p})$ such that $\tau \circ \Theta \neq 0$ where τ is the trace map of Theorem 4.1.5 on page 38.

Proof. By definition $\mathrm{End}_F(Z) = \mathrm{Hom}_{\mathcal{O}_Z}(F_*\mathcal{O}_Z, \mathcal{O}_Z)$ and hence, $\mathrm{End}_F(Z) \simeq \mathrm{H}^0(Z, \omega_Z^{1-p})$. Note also that there is only one (up to scalar) G-equivariant map $\chi : St \otimes St \to \Bbbk$ by Frobenius reciprocity (([Jan] Proposition I.3.4b); see also section 4.2).

If $\tau \circ \Theta \neq 0$ then there is a non-zero B-module map $St \otimes \Bbbk_{(p-1)\rho} \to \Bbbk$ (again by Frobenius reciprocity). Under this map, the T-invariants in $St \otimes \Bbbk_{(p-1)\rho}$ are mapped isomorphically to $\Bbbk$. Hence, identifying $\Bbbk_{(p-1)\rho} \simeq St_{(p-1)\rho}$ (via the map $1 \mapsto v_+$), we have that $\Theta(v_+ \otimes v_+)$ is a B-canonical splitting (under the identification $\mathrm{End}_F(Z) \simeq \mathrm{H}^0(Z, \omega_Z^{1-p})$).

Conversely, assume that Z is B-canonical split. A B-module map $St \otimes \Bbbk_{(p-1)\rho} \to M$ for some G-module M can be extended to a map $St \otimes St \to M$ by [Jan] Lemma II.2.13(a). Let ϕ denote a B-canonical splitting of Z then by Lemma 4.4.3 on the preceding page ϕ is in image of the map $St \otimes \Bbbk_{(p-1)\rho} \to \mathrm{H}^0(Z < \omega_Z^{1-p})$. Thus, there exists a map $\Theta : St \otimes St \to \mathrm{H}^0(Z, \omega_Z^{1-p})$ such that $\phi \in \mathrm{Im}(\Theta)$. Since ϕ is a F-splitting of Z, $\tau(\phi) \neq 0$ and therefore $\tau \circ \Theta \neq 0$. □

Next, we prove the main result of this section:

Proposition 4.4.6. *The F-splitting section $f(v_- \otimes v_+)g(t) \prod_{i=1}^{\ell} \sigma_i^{p-1}$ is B-canonical. In particular, this implies that $\cap_{i=1}^{p-1} \mathcal{V}_X(g_i(t_i))$ is B-canonical split.*
Thus, if $\chi_i(e) = 0$ in $\Bbbk$ for all i, the closure $\bar{\mathcal{U}}$ of the unipotent variety of G_{ad} is B-canonical split.

Proof. Consider the map of G-spaces $St \otimes St \to \mathrm{H}^0(X, \mathcal{L}_X(2(p-1)\rho))$ given by $u \mapsto f(u)g(t)$, $u \in St \otimes St$. This is a G-equivariant map where the action of G is the one of the diagonal of G.

By composing with the map that multiplies with the element $\prod_{i=1}^{\ell} \sigma_i^{p-1}$ we get a map $\Theta : St \otimes St \to \mathrm{H}^0(X, \omega_X^{1-p})$. Now consider the highest and lowest weight vectors $v_+, v_- \in St$. Then $\Theta(v_- \otimes v_+) = f(v_- \otimes v_+)g(t) \prod_{i=1}^{\ell} \sigma_i^{p-1}$. Hence $\tau(\Theta(v_- \otimes v_+)) \neq 0$ by Proposition 4.3.4 on page 47 which then in turn implies that X is B-canonical split by Lemma 4.4.5. Furthermore, $\cap_{i=1}^{\ell} \mathcal{V}_X(g_i(t_i))$ is compatibly F-split with respect to this B-canonical splitting (see again Proposition 4.3.4 on page 47). Thus, $\cap_{i=1}^{\ell} \mathcal{V}_X(g_i(t_i))$ is B-canonical split by Remark 4.4.4 on the preceding page.

Under the additional assumption that $\chi_i(e) = 0$ for all i, $\bar{\mathfrak{U}}$ is compatibly F-split in X wrt. the element $\Theta(v_- \otimes v_+)$ above. Therefore, using Remark 4.4.4 on page 50, we get that $\bar{\mathfrak{U}}$ is B-canonical split. $\square$

The following remark is due to Brion and Polo in [B-P2] but is first written down in [Rit]

Remark 4.4.7. *In the proof of Proposition 4.4.6 above, we showed that X is B-canonical split. More general, X is also $B \times B$-canonical split.*

Proof. Consider the $G \times G$-equivariant map $(St \boxtimes St) \otimes (St \otimes St) \to \mathrm{H}^0(X, \omega_X^{1-p})$ given by $u_1 \otimes u_2 \mapsto f(u_1)f(u_2)\prod_{i=1}^{\ell} \sigma_i^{p-1}$, $u_1, u_2 \in St \boxtimes St$.

The image of the $\Delta(G)$-invariant element $v \in St \boxtimes St$ and the highest and lowest weight vectors $v_+, v_- \in St$ is $f(v_- \otimes v_+)f(v)\prod_{i=1}^{\ell} \sigma_i^{p-1}$ whose image under the natural map $\tau : \mathrm{H}^0(X, \omega_X^{1-p}) \to \Bbbk$ is non-zero as we have already seen in Proposition 4.3.4 on page 47. Hence by Lemma 4.4.5 above X is $B \times B$-canonical split. $\square$

Chapter 5

Further properties of $\bar{\mathcal{U}}$

In this chapter, we look at some applications of Proposition 4.3.4 on page 47. We find that the main result of the previous section implies that $\bar{\mathcal{U}}$ is a locally complete intersection and normal. We also have a partial result on the Picard group of $\bar{\mathcal{U}}$ using that $\bar{\mathcal{U}}$ is a locally complete intersection.

5.1 Geometric properties of $\bar{\mathcal{U}}$

We generalize two known results of $\mathcal{U}$ to its closure in X, namely that $\mathcal{U}$ is a complete intersection (Proposition 3.2.3 on page 29) and normal (Proposition 3.2.9 on page 33). Here, we prove that $\bar{\mathcal{U}}$ is a locally complete intersection and normal when $\chi_i(e) = 0$ in $\Bbbk$ for $1 \leq i \leq \ell$. This implies that $\bar{\mathcal{U}}$ is Cohen-Macauley and Gorenstein in these cases.

The next two Lemmas are helpful tools. First, we show that if a F-splitting is a product of global sections of some line bundles raised to $(p-1)$'st power then the sections form a regular sequence in the local ring $\mathcal{O}_{X,x}$ in any point where all these sections are zero. We then use this to prove that the zero subsets of some of these sections has codimension equal to the number of sections. More precisely:

Lemma 5.1.1. *Let X denote a projective, smooth variety with canonical sheaf ω_X. Assume $\mathcal{L}_1, \dots, \mathcal{L}_N$ are line bundles such that $\otimes_{i=1}^N \mathcal{L}_i \simeq \omega_X^{-1}$. Assume furthermore that there exist global sections f_i of $\mathcal{L}_i$ such that $\prod_{i=1}^N f_i^{p-1} \in H^0(X, \omega_X^{1-p})$ is a F-split section and such that $\mathcal{V}_X(f_1, \dots, f_N) \neq \emptyset$.*
Then (i) $f_{i_1}, \dots, f_{i_s}$ form a regular sequence in $\mathcal{O}_{X,x}$ for all $x \in \mathcal{V}_X(f_1, \dots, f_N)$ and for all $1 \leq s \leq N$ such that $1 \leq i_j \leq N$ are different for $j = 1, \dots, s$
(ii) Let $C \subseteq \cap_{i \in I} \mathcal{V}_X(f_i)$ be any irreducible component where $I \subseteq \{1, \dots, N\}$. Then codim$(C) = |I|$

Proof. Let $x \in \mathcal{V}_X(f_1, \dots, f_N)$ and let $i_1, \dots, i_s \in \{1, \dots, N\}$. Assume for contradiction that f_{i_j} is a zero divisor in

$$\mathcal{O}_{X,x} / \langle f_{i_1}, \dots, f_{i_{j-1}} \rangle \mathcal{O}_{X,x}$$

Hence there exist $a_r \in \mathcal{O}_{X,x}$ such that $\sum_{r=1}^{j} a_r f_{i_r} = 0$ where some $a_r \neq 0$.

Thus, we have that $\sum_{i=1}^{N} a_i f_i = 0$ (again $a_i \in \mathcal{O}_{X,x}$) and assume without loss of generality that $a_1 \neq 0$ (for example by renumbering). Then $a_1 f_1 = -\sum_{i=2}^{N} a_i f_i$. This implies that

$$(a_1 s)^{p-1} = f_2^{p-1} \dots f_N^{p-1} \sum_{j_2 + \dots + j_N = p-1} \binom{p-1}{j_2 \dots j_N} \prod_{r=2}^{N} (-a_r f_r)^{j_r}$$

Observe that in each summand there exists a $r \geq 2$ such that f_r^p divides that summand.

Consider the trace map $\tau : F_*(\omega_Z^{1-p}) \to \mathcal{O}_Z$ for some non-singular variety Z. Then we know that Z is F-split with respect to $\phi \in \mathrm{H}^0(Z, \omega_Z^{1-p})$ if and only if $\tau(\phi) = 1$ by Theorem 4.1.5 on page 38.

Now, $\tau(f^p g) = f\tau(g)$ which by the above observation shows that $\tau((a_1 s)^{p-1}) \in \langle f_2, \dots, f_N \rangle$ hence also $\tau(a_1^p s^{p-1}) \in \langle f_2, \dots, f_N \rangle$. Since s^{p-1} is a splitting section $\tau(s^{p-1}) = 1$ whence $\tau(a_1^p s^{p-1}) = a_1 \tau(s^{p-1}) = a_1$. This shows that if $a_{i_j} f_{i_j} = 0$ in $\mathcal{O}_{X,x} / \langle f_{i_1}, \dots, f_{i_{j-1}} \rangle \mathcal{O}_{X,x}$ then a_{i_j} is already zero in that ring. This proves the first claim.

We next prove (ii) from (i) by using induction in the number of elements in I. When $|I| = 1$ then Krull's Principal Ideal Theorem ([E] Theorem 10.2) and [Ha1] exercise II.3.20 takes care of it. Now assume $|I| > 1$ and that the claim is true for all subsets $J \subsetneqq I$.

Let $C \subseteq \cap_{i \in I} \mathcal{V}_X(f_i)$ be any irreducible component. Note that Krull's Hauptidealsatz gives that $\operatorname{codim}(C, X) \leq |I|$. Then C is also a closed and irreducible subset of $\cap_{i \in J} \mathcal{V}_X(f_i)$ where $J = I \setminus \{j\}$ for some $j \in I$. Hence C is contained in an irreducible component C' of $\cap_{i \in J} \mathcal{V}_X(f_i)$. Observe that induction gives that $\operatorname{codim}(C', X) = |J| = |I| - 1$. And we claim that $C \subsetneqq C'$ which in turn implies:

$$|I| \geq \operatorname{codim}(C, X) > \operatorname{codim}(C', X) = |I| - 1$$

proving (ii).

To prove the claim that $C \subsetneqq C'$ we assume for contradiction that $C = C'$. Note that if $U \subseteq X$ is open such that $C \cap U \neq \emptyset$ then $\operatorname{codim}(C, X) = \operatorname{codim}(C \cap U, U)$. Let A be the union of the finitely many irreducible components of $\cap_{i \in J} \mathcal{V}_X(f_i)$ except C then the complement $U = X \setminus A$ is open and contains $C \cap U$. Furthermore $C \cap U = \cap_{i \in J} \mathcal{V}_X(f_i) \cap U = \cap_{i \in J} \mathcal{V}_U(f_{i|U}) \neq \emptyset$. Thus $\cap_{i \in J} \mathcal{V}_U(f_{i|U}) \subseteq \mathcal{V}_U(f_{j|U})$. By possibly restricting to an even smaller open subset of X (which we also will denote U) we can assure that $f_{i|U} \in \mathcal{O}_U(U)$ for all $i \in J$. Therefore using Hilbert's Nullstellensatz we get $\sqrt{\langle f_{j|U} \rangle} \subseteq \sqrt{\sum_{i \in J} \langle f_{i|U} \rangle}$ which then implies that $f_{j|U}^M \in \langle f_{i|U} | i \in J \rangle$ for some $M \gg 0$.

Hence $f_{j|U}^{p^d} \in \langle f_{i|U} | i \in J \rangle$ where d satisfies $p^d \geq M$. Using the F-splitting determined by the splitting section s^{p-1} several (actually d) times we get that $f_{j|U} \in \langle f_{i|U} | i \in J \rangle$. Hence f_j is a zero divisor in $\mathcal{O}_{X,x} / \langle f_{i_1}, \ldots, f_{i_{j-1}} \rangle \mathcal{O}_{X,x}$ which contradicts (i). $\square$

The above lemma will become useful when we show that $\bar{\mathcal{U}}$ is a locally complete intersection. But we need another Lemma to prove that it is normal. This Lemma is communicated to us by professor M. Brion:

Lemma 5.1.2. *Let $Y \subseteq X$ be a subvariety of any smooth variety X. And let $D \subseteq X$ be an irreducible divisor of X. Then if the scheme-theoretically intersection $Z := Y \cap D$ is reduced every irreducible component of Z contains a smooth point of Y.*

Proof. If $Y \subseteq D$ then the assertion of Lemma is easily seen to be fullfilled. Assume now that $Y \nsubseteq D$.

Observe that each irreducible component of Z contains a smooth point in Z since the smooth points form a dense subset of a reduced scheme.

The claim is now that every smooth point of Z is also a smooth point of Y. This is a local question so we can assume that X is affine such that there exists a regular function $f \in \Bbbk[X]$ with the property that $\mathcal{V}_X(f) = D$. Hence also D, Y and, Z are affine as they are closed in X.

Let $z \in Z$ be a smooth point (of Z). By Hilbert's Nullstellensatz we get that the ideal of D in $\Bbbk[X]$ is $I_X(D) = \sqrt{\langle f \rangle} = \langle f \rangle$ as D is assumed irreducible. Consider the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_X(Y) & \longrightarrow & \Bbbk[X] & \longrightarrow & \Bbbk[Y] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle id} & & \downarrow & & \\ 0 & \longrightarrow & I_X(Z) & \longrightarrow & \Bbbk[X] & \longrightarrow & \Bbbk[Z] & \longrightarrow & 0 \end{array}$$

Using the snake Lemma we obtain the isomorphism $I_Y(Z) \simeq I_X(Z)/I_X(Y)$. We have that $I_X(Z) = I_X(Y) + I_X(D)$ as Z is the scheme-theoretical intersection $Y \cap D$. Hence $I_Y(D) \simeq I_X(D)/I_X(Y) \cap I_X(D)$ and we therefore conclude that $I_Y(Z) = \langle f_{|Y} \rangle$.

The point z is assumed to be smooth. Hence the local ring $\mathcal{O}_{Z,z} \simeq \Bbbk[Z]_{\mathcal{M}_{Z,z}}$ is regular where $\mathcal{M}_{Z,z}$ denotes the maximal ideal of $\Bbbk[Z]$ that vanishes in z. This implies that $\mathcal{M}_{Z,z}$ is generated by $d = \dim(Z)$ elements. Denote these $f_1, \dots, f_d$.

Using the surjectivity of the map $\mathcal{O}_{Y,z} \to \mathcal{O}_{Z,z}$ we let $f_i' \in \mathcal{O}_{Y,z}$ denote a lift of f_i and consider the ideal I generated by $f_1', \dots, f_d'$ and $f_{|Y}$. Notice that $I \subseteq \mathcal{M}_{Y,z}$ where $\mathcal{M}_{Y,z} \subseteq \mathcal{O}_{Y,z}$ is the maximal ideal. This is because $f_i'(z) = f_i(z) = 0$ since $f'_{i|Z} = f_i \in \mathcal{M}_{Z,z}$. And $f_{|Y}$ vanishes on Z and hence especially in z so also $f_{|Y}$ belongs to $\mathcal{M}_{Z,z}$. We claim that $I = \mathcal{M}_{Y,z}$.

Let $g \in \mathcal{M}_{Y,z}$ then $g_{|Z} \in \mathcal{M}_{Z,z}$ hence there exist functions $\lambda_1, \dots \lambda_d$ such that $g_{|Z} = \sum_{i=1}^d \lambda_i f_i$ Therefore $g_{|Z} - \sum_{i=1}^d \lambda_i f_i$ is identically zero on Z. Thus letting λ_i' denote a lift of λ_i we have that

$$g - \sum_{i=1}^{d} \lambda_i' f_i' \in I_Y(Z) = \langle f_{|Y} \rangle$$

Thus $\mathcal{M}_{Y,z} = I$ which implies that $\dim_{\Bbbk}(\mathcal{M}_{Y,z}/\mathcal{M}_{Y,z}^2) = d + 1 = \dim(Y)$. Therefore $\mathcal{O}_{Y,z}$ is regular, which is equivalent to z being a smooth point of Y. $\square$

We are now ready to prove the main result of this section:

Proposition 5.1.3. *Assume that $\chi_i(e) = 0$ for all i. Then*
i) $\bar{\mathcal{U}}$ is a locally complete intersection
ii) $\bar{\mathcal{U}}$ is normal

Proof. We claim that $\bar{\mathcal{U}} = \mathcal{V}_X(g_1(t_1), \dots, g_\ell(t_\ell)) =: V$ which proves i).

To prove the claim, notice that $\bar{\mathcal{U}}$ is actually an irreducible component of V since it is irreducible, closed, and of codimension (in X) ℓ. Let $C \subseteq V$ be an irreducible component. Note that $\operatorname{codim}(C, X) = \ell$ by Lemma 5.1.1 on page 53. Then consider the intersection $C \cap G_{ad}$.

If $G_{ad} \cap C = \emptyset$ then $C \subseteq X \setminus G_{ad} = \cup_{i=1}^{\ell} X_i$. Since C is irreducible $C \subseteq X_j$ for some j. But then $C \subseteq V \cap X_j$ is an irreducible component since it is maximal closed and irreducible subset and hence has codimension $\ell + 1$ by Lemma 5.1.1 on page 53 contradicting $\operatorname{codim}(C, X) = \ell$.

Hence $C \cap G_{ad} \neq \emptyset$. Now, since $C \cap G_{ad} \subseteq V \cap G_{ad} = \mathcal{U}_{ad}$ we get that $C \subseteq \bar{\mathcal{U}}$ implying that $C = \bar{\mathcal{U}}$. Hence $\bar{\mathcal{U}} = \cap_{i=1}^{\ell} \mathcal{V}_X(g_i(t_i))$ and therefore a locally complete intersection.

To prove (ii) it suffices to prove that $\bar{\mathcal{U}}$ is regular in codimension one by (i) and [Ha1] II.8.23(b).

Now by Theorem 2.2.1 on page 13 we have that $X = G_{ad} \cup \bigcup_{i=1}^{\ell} X_i$. The intersections $\bar{\mathcal{U}} \cap X_i$ are reduced since X_i and $\bar{\mathcal{U}}$ are simultaneously F-split in X by Proposition 4.3.4 on page 47. Now the preceding Lemma 5.1.2 gives that each irreducible component of the intersections $\bar{\mathcal{U}} \cap X_i$ contains smooth points of $\bar{\mathcal{U}}$.

Let $\bar{\mathcal{U}}^{sing}$ be the singular locus of $\bar{\mathcal{U}}$ and let C denote an irreducible component of $\bar{\mathcal{U}}^{sing}$. If $C \cap G_{ad} \neq \emptyset$ then $\operatorname{codim}(C, \bar{\mathcal{U}}) = \operatorname{codim}(C \cap G_{ad}, \mathcal{U}_{ad})$ since G_{ad} is an open subset of X. Observe that $C \cap G_{ad}$ is irreducible (since open in the irreducible component C) and closed in G_{ad}. Therefore, C is contained in an irreducible component C' of G_{ad}. Thus $\operatorname{codim}(C \cap G_{ad}, G_{ad}) \geq \operatorname{codim}(C', G_{ad})$. Note also that $\bar{\mathcal{U}}^{sing} \cap G_{ad} = \mathcal{U}_{ad}^{sing}$ is the singular locus of $\mathcal{U}_{ad}$. Thus, $C' \subseteq \mathcal{U}_{ad}^{sing}$ and hence it suffices to prove that $\dim(\mathcal{U}_{ad}^{sing}) \leq \dim(\mathcal{U}) - 2$.

Recall that the regular elements $\mathcal{U}^{reg}$ in $\mathcal{U}$ form an open G-orbit (in $\mathcal{U}$) by Proposition 3.2.7 on page 31. Hence we get that $\pi(\mathcal{U}^{reg})$ is an open subset of $\mathcal{U}_{ad}$ which then implies that $\mathcal{U}_{ad}^{sing} \subseteq \mathcal{U}_{ad} \setminus \pi(\mathcal{U}^{reg})$. Thus the pull-back $\pi^{-1}(\mathcal{U}_{ad}^{sing}) \subseteq \mathcal{U} \setminus \mathcal{U}^{reg}$. Restricting π to $\pi^{-1}(\mathcal{U}_{ad}^{sing})$ we get a surjective map $\pi^{-1}(\mathcal{U}_{ad}^{sing}) \to \mathcal{U}_{ad}^{sing}$ which gives that $\dim(\mathcal{U}_{ad}^{sing}) \leq \dim(\pi^{-1}(\mathcal{U}_{ad}^{sing})) \leq \dim(\mathcal{U} \setminus \mathcal{U}^{reg}) \leq \dim(\mathcal{U}) - 2$ since $\mathcal{U}$ is regular in codimension 1 by Proposition 3.2.7 on page 31. Thus, the codimension of C in $\bar{\mathcal{U}}$ is at least 2 when $C \cap G_{ad} \neq \emptyset$.

If $C \cap G_{ad} = \emptyset$ then C is an closed and irreducible subset of $X_i \cap \bar{\mathcal{U}}$ and hence contained in a irreducible component C' of $X_i \cap \bar{\mathcal{U}}$. Because every irreducible component of $X_i \cap \bar{\mathcal{U}}$ contains smooth points (cf. Lemma 5.1.2 on page 55), $C \subsetneq C'$. This implies that $\operatorname{codim}(C, \bar{\mathcal{U}}) > \operatorname{codim}(C', \bar{\mathcal{U}}) = 1$ by Krull's Hauptidealsatz.

Therefore, in either case the codimension of C is greater than (or equal to) 2 showing that $\bar{\mathcal{U}}$ is regular in codimension 1 and hence normal by (i) and [Ha1] proposition II.8.23. □

Remark 5.1.4. *Assume that $\chi_i(e) = 0$ for all i.*
(i) Since $\mathcal{U}_{ad}$ is an open subvariety of $\bar{\mathcal{U}}$ the proposition tells us that $\mathcal{U}_{ad}$ is also normal and a locally complete intersection[1].
(ii) The Proposition 5.1.3 implies that $\bar{\mathcal{U}}$ is Cohen-Macauley by [Ha1] Proposition II.8.23(a). It is also Gorenstein by [E] Corollary 21.19 since $\bar{\mathcal{U}}$ is a locally complete intersection.

Example 5.1.5. Since $\bar{\mathcal{U}} = \mathcal{V}_X(Z_{11} + Z_{22})$, we easily get that $\bar{\mathcal{U}}$ is a locally complete intersection. Furthermore, it shows that $\bar{\mathcal{U}} \simeq \mathbb{P}^2$. Hence, it is normal.

[1]This is propably well-known although I have not been able to find a specific reference

5.2 The Picard group of $\bar{\mathfrak{U}}$

We now exploit the Picard group of $\bar{\mathfrak{U}}$. Unfortunately, we only have a partial result, namely that the natural map $Pic(X) \to \mathrm{Pic}(\bar{\mathfrak{U}})$ induced by the inclusion map is injective.

First, we consider the following commutative diagram:

$$\begin{tikzcd} \bar{\mathfrak{U}} \cap Y \arrow[r] \arrow[d] & Y \arrow[d] \\ \bar{\mathfrak{U}} \arrow[r] & X \end{tikzcd} \tag{5.1}$$

There is a G-equivariant morphism

$$\xi : G \times^B {}^{G}\!/_{B} \to {}^{G}\!/_{B} \times {}^{G}\!/_{B}\,, \qquad (x, yB) \mapsto (xB, xyB)$$

Let S_w denote the Schubert variety , i.e. the closure in ${}^{G}\!/_{B}$ of ${}^{B\dot{w}B}\!/_{B}$. Then $G \times^B S_w$ is called the G-Schubert variety . All closed, irreducible G-stable subsets of ${}^{G}\!/_{B} \times {}^{G}\!/_{B}$ are the image of G-Schubert varieties under the isomorphism ξ (see [B-K] Definition 2.2.6). Observe that $\xi(G \times^B \{eB\}) = \Delta({}^{G}\!/_{B})$.

We can give a very explicit description of $\bar{\mathfrak{U}} \cap Y$:

Lemma 5.2.1. *Assume that $\chi_i(e) = 0$ for all i.*
Then $\bar{\mathfrak{U}} \cap Y = \bigcap_{i=1}^{\ell} G \times^B S_{w_o s_{\alpha_i}}$

Proof. Consider the $\Delta(G)$-invariant element $t_i \in \mathrm{H}^0(-w_o\omega_i)^* \boxtimes \mathrm{H}^0(-w_o\omega_i)$ of Lemma 4.3.2 on page 45. We know that $d := \dim(\mathrm{H}^0(-w_o\omega_i))$ is finite. Let $I = \{1, \ldots, d\}$. Take a basis $\{t_{ij}\}_{j \in I}$ for $\mathrm{H}^0(-w_o\omega_i)$ consisting of weight vectors and let $\{t_{ij}^*\}_{j \in I}$ denote the dual basis for $\mathrm{H}^0(-w_o\omega_i)^*$. As in the proof of Lemma 4.3.2 $t_i = \sum_{j \in I} t_{ij}^* \otimes t_{ij}$. Choose $t_{i1} \in \mathrm{H}^0(-w_o\omega_i)$ to be the highest weight vector, i.e. the evaluation map $Ev : \mathrm{H}^0(-w_o\omega_i) \to \Bbbk_{-w_o\omega_i}$ given by $f \mapsto f(1)$ maps t_{i1} to 1 and $Ev(t_{ij}) = 0$ for $j > 1$.

Using Frobenius reciprocity ([Jan] Proposition I.3.4b) we find that $\mathrm{H}^0(\omega_i) \boxtimes \mathrm{H}^0(-w_o\omega_i)$ has a $\Delta(G)$-invariant element. Furthermore, there is a G-module map $\theta : \mathrm{H}^0(-w_o\omega_i)^* \to \mathrm{H}^0(\omega_i)$ (also by Frobenius reciprocity) which maps highest weight vectors to highest weight vectors. Let $x_i \in \mathrm{H}^0(-w_o\omega_i) \boxtimes \mathrm{H}^0(\omega_i)$ be the image of t_i under the map $\theta \times id$. Thus, x_i is $\Delta(G)$-invariant and we can write $x_i = \sum_{j \in I} x_{ij} \otimes t_{ij}$ where $x_{ij} = \theta(t_{ij}^*)$.

Now, consider $x_i(\dot{w}, 1)$ for $w \in W$. First note that $x_i(\dot{w}, 1) = \sum_{j \in I} x_{ij}(\dot{w}) \otimes t_{ij}(1) = x_{i1}(\dot{w}) \otimes t_{i1}(1)$ by the above observation. Let $t \in T$ then

$$t.t_{i1}(1) = t_{i1}(t^{-1}.1) = \omega_i(t^{-1})t_{i1}(1)$$

hence $t_{i1}(1)$ has weight $-\omega_i$. The T-action on the first factor reveals

$$t.x_{i1}(\dot{w}) = x_{i1}(t^{-1}\dot{w}) = x_{i1}(\dot{w}(\dot{w}^{-1}t^{-1}\dot{w})) = (-w_o.\omega_i)(\dot{w}^{-1}t^{-1}\dot{w})x_{i1}(\dot{w}) = (ww_o.\omega_i)(t)x_{i1}(\dot{w})$$

Hence $x_{i1}(\dot{w})$ has weight $w^{-1}w_o.\omega_i$. Since x_i is $\Delta(G)$-invariant, we get

$$x_i(\dot{w},1) = (t,t).x_i(\dot{w},1) = (-\omega_i + ww_o.\omega_i)(t)x_i(\dot{w},1)$$

Thus, if $ww_o.\omega_i \neq \omega_i$ then $x_i(\dot{w},1) = 0$. Observe that if $w = s_j w_o$ then $ww_o.\omega_i = \omega_i$ for $j \neq i$ while for $j = i$ we have $ww_o\omega_i = \omega_i - \alpha_i \neq \omega_i$. Hence $\mathcal{V}_Y(x_i) \supseteq \xi(G \times^B S_{s_i w_o})$.

Now, by Lemma 5.1.1 on page 53, $V_X(g_i(t_i)) \cap Y = V_Y(x_i)$ has pure codimension 1 in Y. Therefore $V_Y(x_i)$ is a union of codimension 1 G-Schubert varieties. From the same Lemma (Lemma 5.1.1) we get that $V_Y(x_i) \cap V_Y(x_j)$ has pure dimension 2 for $i \neq j$. Hence, we conclude that a codimension 1 G-Schubert variety only lies in $V_Y(x_i)$ for one $1 \leq i \leq \ell$. Therefore, $\mathcal{V}_Y(x_i) = \xi(G \times^B S_{s_i w_o})$. Thus, we have proved the Lemma since $\bar{\mathfrak{U}} \cap Y = \cap_{i=1}^{\ell} V_Y(x_i)$. □

It follows from this description that $G \times^B \{eB\} \subseteq \bar{\mathfrak{U}} \cap Y$. Using [Ha1] Exercise II.6.8(a) we get from the diagram in (5.1) the following commutative diagram of homomorphisms

$$\begin{tikzcd} \mathrm{Pic}(X) \arrow[r] \arrow[d] & \mathrm{Pic}(Y) \arrow[d] & \\ \mathrm{Pic}(\bar{\mathfrak{U}}) \arrow[r] & \mathrm{Pic}(\bar{\mathfrak{U}} \cap Y) \arrow[r] & \mathrm{Pic}(G \times^B \{eB\}) \end{tikzcd} \qquad (5.2)$$

By Proposition 2.3.2 on page 21, we have that the homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is injective and the image is line bundles on the form $\mathcal{L}_Y(-w_o\lambda, \lambda)$ for $\lambda \in X^*(T)$. Hence, $\mathrm{Pic}(X) \simeq X^*(T)$. Note also that $G \times^B \{eB\} \simeq \Delta({}^G\!/_B) \subseteq Y = {}^G\!/_B \times {}^G\!/_B$. Therefore, $\mathrm{Pic}(Y) \to \mathrm{Pic}(G \times^B \{eB\}) \simeq \mathrm{Pic}({}^G\!/_B) \simeq X^*(T)$ is given by $(\lambda, \mu) \mapsto \lambda + \mu$ where we have identified $\mathrm{Pic}(Y)$ with $X^*(T) \times X^*(T)$.

For our main result of this section:

Proposition 5.2.2. *Assume that $\chi_i(e) = 0$ for all i.*
Let $i : \bar{\mathfrak{U}} \to X$ denote the inclusion map. Then the induced map $i^{\#}$: Pic$(X) \to$ Pic$(\bar{\mathfrak{U}})$ is injective.

Proof. Let $\mathcal{L} \in \mathrm{Pic}(X)$ and assume that $\mathcal{L}_{|\bar{\mathfrak{U}}}$ is the trivial line bundle on $\bar{\mathfrak{U}}$. Then $\mathcal{L}$ has trivial image in $\mathrm{Pic}(\bar{\mathfrak{U}} \cap Y)$ and hence also in $\mathrm{Pic}(G \times^B \{eB\})$.

We know from Proposition 2.3.2 on page 21 that $\mathcal{L} = \mathcal{L}_X(\lambda)$ for some $\lambda \in X^*(T)$. Further, the image of $\mathcal{L}_X(\lambda)$ in $\mathrm{Pic}(Y)$ is $\mathcal{L}_Y(-w_o\lambda, \lambda) \simeq$

$\mathcal{L}_{G/B}(-w_o\lambda) \boxtimes \mathcal{L}_{G/B}(\lambda)$. The image in $\mathrm{Pic}(G \times^B \{eB\})$ of the latter line bundle is $\mathcal{L}_{G/B}(-w_o\lambda + \lambda)$ as observed above. Since the diagram in (5.2) commutes, $\mathcal{L}_{G/B}(-w_o\lambda + \lambda)$ is trivial. Thus, $w_o\lambda = \lambda$ which only $\lambda = 0$ satisfies. Therefore, $\mathcal{L}_X(\lambda)$ is the trivial line bundle on X proving that the homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(\bar{\mathfrak{U}})$ is injective. □

Example 5.2.3. We have seen that $\bar{\mathfrak{U}} \simeq \mathbb{P}^2$ and hence $\mathrm{Pic}(\bar{\mathfrak{U}}) \simeq \mathbb{Z}$. By [Ha1] Exercises II.6.2(d) and II.6.8(c), we get that $\mathrm{Pic}(X) \to \mathrm{Pic}(\bar{\mathfrak{U}})$ is multiplication with 2 as $\bar{\mathfrak{U}}$ is hypersurface in $X = \mathbb{P}^3$ of degree 2.

Note that $S_{w_o s_\alpha} = \{eB\}$ and therefore $Y \cap \bar{\mathfrak{U}} \simeq \Delta({}^G\!/\!_B)$. In our example, we then know the maps in the diagram (5.2). The map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is given by $m \mapsto (m, m)$. And the map $\mathrm{Pic}(X) \to \mathrm{Pic}(\Delta({}^G\!/\!_B))$ is the map $(m, n) \mapsto m + n$.

We conjecture that $\mathrm{Pic}(\bar{\mathfrak{U}}) \simeq \mathbb{Z}^\ell$ and that the map $\mathrm{Pic}(\bar{\mathfrak{U}}) \to \mathrm{Pic}(\Delta({}^G\!/\!_B))$ is bijective.

Appendices

Appendix A

Dimensions of the fundamental representations

Note that the fundamental character evaluated on the identity element e of G is nothing but the dimension of the representation corresponding to the fundamental weight, ω_i, i.e. $\chi_i(e) = \dim(\mathrm{H}^0(\omega_i))$. These dimensions can be found in various tables - the following has been taken from [M-P-R].Our goal in this section is to determine when $\chi_i(e) = 0$ in $\Bbbk$ for all $1 \leq i \leq \ell$.

We use the following fact about binomial coefficients: If p is a prime then writing $n = n_0 p^0 + n_1 p^1 + \cdots + n_m p^m$ and similarly for i we get

$$\binom{n}{i} \equiv \binom{n_0}{i_0}\binom{n_1}{i_1}\cdots\binom{n_m}{i_m} \text{ modulo } p \tag{A.1}$$

Notice that $p|\binom{n}{i}$ if and only if $\exists j : n_j < i_j$.

<u>*Type* A_n:</u>

$$\chi_i(e) = \binom{n+1}{i} \qquad \text{for } i = 1, \ldots, n$$

These binomial coefficients are zero modulo p if and only if $n+1 = p^m$ for some integer m by the equation (A.1)

<u>*Type* B_n:</u>

$$\chi_n(e) = 2^n\,, \quad \chi_i(e) = \binom{2n+1}{i} \qquad \text{for } i = 1, \ldots, n-1$$

Since $\binom{2n+1}{1} = 2n+1$ is odd and $\chi_n(e) = 2^n$ these numbers do not all vanish in any charateristic p.

Type C_n:

$$\chi_i(e) = \binom{2n}{i} - \binom{2n}{i-2} \qquad \text{for } i = 1, \ldots n$$

(Note that we set $\binom{m}{l} = 0$ when $l < 0$). First observe that $\binom{2n}{1} = 2n$ and $\binom{2n}{2} - 1 = n(2n-1) - 1$. From this we conclude that only the prime 2 can divide both these expressions and the latter only when n is odd. Using equation (A.1) above we note that if i is odd then $2|\binom{2n}{i}$ and furthermore we get for $i = 2j$:

$$\binom{2n}{2j} - \binom{2n}{2j-2} \equiv \binom{n}{j} - \binom{n}{j-1} \text{ modulo } 2 \equiv \binom{n}{j} + \binom{n}{j-1} \text{ modulo } 2$$

Now $\binom{n}{j} + \binom{n}{j-1} = \binom{n+1}{j}$ and by using equation (A.1) we get that $2|\binom{n+1}{j}$ for all $1 \leq j \leq \frac{n-1}{2}$ if and only if $n = 2^m - 1$ for some integer m.

Type D_n:

$$\chi_{n-1}(e) = 2^{n-1},\ \chi_n(e) = 2^{n-1},\ \chi_i(e) = \binom{2n}{i} \qquad \text{for } i = 1, \ldots, n-2$$

The only case where all these dimensions can be zero modulo p is when $p = 2$. Again, by equation (A.1) we get that $2|\binom{2n}{i}$ for all $i = 1, \ldots, n-2$ if and only if $n = 2^m$ for some m.

Type E_6:

$$\begin{aligned}
\chi_1(e) &= 27 && = \mathbf{3}^3 \\
\chi_2(e) &= 351 && = \mathbf{3}^3 \cdot 13 \\
\chi_3(e) &= 2925 && = \mathbf{3}^2 \cdot 5^2 \cdot 13 \\
\chi_4(e) &= 351 && = \mathbf{3}^3 \cdot 13 \\
\chi_5(e) &= 27 && = \mathbf{3}^3 \\
\chi_6(e) &= 78 && = 2 \cdot \mathbf{3} \cdot 13
\end{aligned}$$

Type E_7:

$$\begin{aligned}
\chi_1(e) &= 133 && = 7 \cdot 19 \\
\chi_2(e) &= 8645 && = 5 \cdot 7 \cdot 13 \cdot 19 \\
\chi_3(e) &= 365750 && = 2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 19 \\
\chi_4(e) &= 27664 && = 2^4 \cdot 7 \cdot 13 \cdot 19 \\
\chi_5(e) &= 1539 && = 3^4 \cdot 19 \\
\chi_6(e) &= 56 && = 2^3 \cdot 7 \\
\chi_7(e) &= 912 && = 2^4 \cdot 3 \cdot 19
\end{aligned}$$

Type E_8:

$$\begin{aligned}
\chi_1(e) &= 248 &&= 2^3 \cdot \mathbf{31}\\
\chi_2(e) &= 30380 &&= 2^2 \cdot 5 \cdot 7^2 \cdot \mathbf{31}\\
\chi_3(e) &= 2450240 &&= 2^6 \cdot 5 \cdot 13 \cdot 19 \cdot \mathbf{31}\\
\chi_4(e) &= 146325270 &&= 2 \cdot 3 \cdot 5 \cdot 7^2 \cdot 13^2 \cdot 19 \cdot \mathbf{31}\\
\chi_5(e) &= 6899079264 &&= 2^5 \cdot 3 \cdot 7^2 \cdot 11^2 \cdot 17 \cdot 23 \cdot \mathbf{31}\\
\chi_6(e) &= 6696000 &&= 2^6 \cdot 3^3 \cdot 5^3 \cdot \mathbf{31}\\
\chi_7(e) &= 3875 &&= 5^3 \cdot \mathbf{31}\\
\chi_8(e) &= 147250 &&= 2 \cdot 5^3 \cdot 19 \cdot \mathbf{31}
\end{aligned}$$

Type F_4:

$$\begin{aligned}
\chi_1(e) &= 52 &&= 2^2 \cdot \mathbf{13}\\
\chi_2(e) &= 1274 &&= 2 \cdot 7^2 \cdot \mathbf{13}\\
\chi_3(e) &= 273 &&= 3 \cdot 7 \cdot \mathbf{13}\\
\chi_4(e) &= 26 &&= 2 \cdot \mathbf{13}
\end{aligned}$$

Type G_2: $\chi_1(e) = 14 = 2 \cdot \mathbf{7} \qquad \chi_2(e) = \mathbf{7}$

To summarize, our arguments show that $\chi_i(e) = 0$ for all i in the following cases:

Type A_n: when $n = p^m$ and $p = \mathrm{char}(\mathbb{k}) > 0$ and $m \in \mathbb{N}$.
Type C_n: when $n = 2^m - 1$ and $2 = \mathrm{char}(\mathbb{k})$ $(m \in \mathbb{N})$
Type D_n: when $n = 2^m$ $(m \in \mathbb{N})$ and $2 = \mathrm{char}(\mathbb{k})$.
Type E_6: when $\mathrm{char}(\mathbb{k}) = 3$
Type E_8: when $\mathrm{char}(\mathbb{k}) = 31$
Type F_4: when $\mathrm{char}(\mathbb{k}) = 13$
Type G_2: when $\mathrm{char}(\mathbb{k}) = 7$

Appendix B

Comments

Here, I have gathered some questions that I have not been able to answer due to lack of time as well as my mathematical limitations.

Question 1: Is $\bar{\mathcal{U}}$ F-split always?

The hard thing is to find global sections of line bundles on X that restrict to constants on G_{ad}. That is why we assume that $\chi_i(e) = 0$ for all i.

Note that in the cases summarized in the previous appendix, $\pi : G \to G_{ad}$ is bijective. This suggests that it may be more natural to switch the point of view to the simply connected group G in stead of the adjoint group G_{ad}. But for a simply connected group G there is not a canonical compactification like the wonderful compactification of G_{ad}. Normality and smoothness of a compactification can not be taken for granted.

Note that the subset $V := \cap_{i=1}^{\ell} \mathcal{V}_X(g_i(t_i))$ of Proposition 4.3.4 on page 47 satisfies that $V \cap G = \kappa^{-1}(0, \ldots, 0)$ where $\kappa : G \to \mathbb{A}^\ell$ is the Steinberg map considered in chapter 3. More generally, question 1 can be restated as "Is the Steinberg fibres F-split in any reductive embedding[1] and any positive charateristic?"

Some recent but so far not published results of Jesper Funch Thomsen indicate that the closure of any Steinberg fibre in any reductive embedding is compatibly split.

Question 2: Find the line bundles on $\bar{\mathcal{U}}$[2].

If one can give an explicit description of the Picard Group of $\bar{\mathcal{U}}$ then one can probably prove some vanishing results. Also, one could look more closely at the good filtration that the global sections of an ample line bundle on $\bar{\mathcal{U}}$ admit.

[1] see [Rit] or [B-K] Section 6.2

[2] This still denotes the wonderful compactification of $\mathcal{U}_{ad}$

Question 3: Is $\bar{\mathcal{U}}$ globally F-regular?

The paper [B-P2] on large Schubert varieties have inspired us, and therefore it would be natural to seek ideas to prove globally F-regularity of $\bar{\mathcal{U}}$ by 'similar' methods as in [B-T].

Question 4: Find a desingularization of $\bar{\mathcal{U}}$.

It can be proved that the set Z' considered in the proof of Theorem 3.1.1 on page 25 is actually a desingularization of $\mathcal{U}$. This problem is related to find a equivariant desingularization of the large Schubert variety $\bar{B}_{ad}$; see [B-P2]. To my knowledge, this has not been constructed so far.

I would like to end by drawing the attension to a recent article [He] of Xuhua He where he proves that the boundary of $\bar{\mathcal{U}}$ ($\bar{\mathcal{U}} \setminus \mathcal{U}$) is a union of certain G-stable sets defined by Lutztig. This can prove helpful in further studies of $\bar{\mathcal{U}}$.

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List of Notations

Unless explicitly stated otherwise, the following notation is used throughout this thesis

Chapter 1

Section 1.1

$\Bbbk$	algebraically closed field of characteristic $p > 0$
G_{ad}	connected, semi-simple, adjoint linear algebraic group over $\Bbbk$
T_{ad}	maximal torus of G_{ad}
B_{ad}	Borel subgroup of G_{ad}
$\pi : G \to G_{ad}$	simply connected covering of G_{ad}, i.e. G is a connected, semi-simple, simply connected linear algebraic group over $\Bbbk$ with a surjective morphism of algebraic groups $\pi : G \to G_{ad}$ and the kernel of π is central.
T	maximal torus of G such that $\pi(T) = T_{ad}$
ℓ	$= \dim(T)$, the rank of G.
B	Borel subgroup of G such that $\pi(B) = B_{ad}$
B^-	Borel subgroup of G such that $\pi(B^-) = B_{ad}^-$ and $B \cap B^- = T$
Φ	Root system of G wrt. T (is the root system of G_{ad} wrt. T_{ad})
Φ^+	The positive roots in Φ wrt. B
Δ	The simple roots in Φ^+. We index the simple roots $\{\alpha_1, \ldots, \alpha_\ell\}$.
Λ	$= X^*(T)$; the weight lattice is equal to the characters of T.
$X^*(T_{ad})$	The characters of T_{ad} is the root lattice, the sublattice of Λ spanned by the roots over $\mathbb{Z}$.
ω_i	the fundmental weight corresponding to the simple root α_i for $1 \leq i \leq \ell$. They form a basis of $X^*(T)$.
ρ	$= \sum_{i=1}^{\ell} \omega_i = 1/2 \sum_{\alpha \in \Phi^+} \alpha$
B_u	The unipotent radical of B
W	$= N_G(T)/T$; the Weyl group of G wrt. T

s_i	$= s_{\alpha_i}$; the reflection corresponding to the the simple root α_i
w_o	the unique element in W such that $w_o.\Phi^+ = -\Phi^+$.
$u_\alpha : \mathbb{G}_a \simeq U_\alpha$	For each root $\alpha \in \Phi$ there is an isomorphism u_α of $\mathbb{G}$ onto a closed subgroup U_α such that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for all $t \in T, x \in \Bbbk$. U_α is called the root subgroup.
$\Phi(w)$	$= \{\alpha \in \Phi^+ \| w(\alpha) \in -\Phi^+\}$
U_w	$= \prod_{\alpha\in\Phi(w)} U_\alpha$
$l(w)$	the length of the element w of W

Section 1.2

$\Bbbk_\lambda$	The one dimensional vector space with B-action $b.x = \lambda(b)^{-1}x$
$\mathcal{L}_{G/B}(\lambda)$	The locally free sheaf corresponding to the line bundle $G \times^B \Bbbk_\lambda$
Λ^+	the dominant weights of Λ
$\Bbbk[G]$	the coordinate ring of G
$\mathrm{H}^0(\lambda)$	$= \mathrm{H}^0({}^G\!/\!_B, \mathcal{L}_{G/B}(\lambda))$. Note that this G-module has highest weight $-w_o\lambda$ and lowest weight $-\lambda$ for a dominant weight λ. Else, it is zero.
$V(\lambda)$	$= \mathrm{H}^0(\lambda)^*$. Note that this G-module has highest weight λ and lowest weight $w_o\lambda$ for a dominant weight λ. Else it is zero
$M \boxtimes N$	The external product of 2 G-modules is regarded as a $G \times G$-module via the action $(g,h).m \otimes n = g.m \otimes h.n$.
$\mathrm{H}^0(\lambda,\mu)$	$= \mathrm{H}^0({}^G\!/\!_B \times {}^G\!/\!_B, \mathcal{L}_{G/B\times G/B}(\lambda,\mu)) = \mathrm{H}^0(\lambda) \boxtimes \mathrm{H}^0(\mu)$.
St	$\mathrm{H}^0((p-1)\rho)$; the Steinberg module. It is irreducible and selfdual.

Section 1.3

$T(\lambda)$	The indecomposable tilting module of highest weight λ

Chapter 2

h	the identity element of $\mathrm{End}_{\Bbbk}(M)$ where M satisfies conditions (i)–(iii) of Lemma 2.1.1.
$[h]$	the image of h in $\mathbb{P}(\mathrm{End}_{\Bbbk}(M))$
X	the closure in $\mathbb{P}(\mathrm{End}_{\Bbbk}(M))$ of $(G \times G).[h]$. It is the wonderful compactification of G_{ad}.
∂X	$= X \setminus G_{ad}$; the boundary of G_{ad} which is a union of ℓ smooth $G_{ad} \times G_{ad}$-stable divisor with normal crossings.
X_i	a smooth $G \times G$-stable boundary divisor.
Y	$= \bigcap_{i=1}^{\ell} X_i \simeq {}^{G}/_{B} \times {}^{G}/_{B}$; the unique closed orbit in X.
$\mathbb{P}_o$	The affine open subset of $\mathbb{P}(\mathrm{End}_{\Bbbk}(M))$ such that when writing the element in the basis defined in Remark 2.1.4 then the coefficient of $h_\lambda = m_\lambda^* \otimes m_\lambda$ equals 1. $\mathbb{P}_o$ is $B \times B^-$-stable.
$\overline{T_{ad}}$	The closure in X of $(T \times T).[h] \simeq T_{ad}$ by Lemma 2.1.5.
$\overline{T_{ad,o}}$	$= \overline{T_{ad}} \cap \mathbb{P}_o$
X_o	$= X \cap \mathbb{P}_o$. For the very important properties see Proposition 2.2.3, Lemma 2.2.5, Lemma 2.2.4.
$\mathrm{Cl}(X)$	The divisor class group of X.
$\mathrm{Pic}(X)$	The Picard group of X.
σ	The unique (up to a scalar) $G \times G$-invariant global section of the line bundle $\mathcal{L}_X(X_i)$ such that the zero subset of σ_i is X_i
D_i	$:= \overline{Bs_i w_o B}$; a prime divisor on X in the boundary of X_o.
τ	The unique (up to a scalar) $B \times B$-stable global section of $\mathcal{L}_X(D_i)$ such that the zero subset is D_i.
$\mathcal{L}_X(\lambda)$	The line bundle on X such that the restriction to Y is th eline bundle $\mathcal{L}_Y(-w_o\lambda, \lambda)$.

Chapter 3

$\mathcal{U}$	The unipotent variety in G, i.e. the subset of all unipotent elements in G
χ_i	The fundamental character $G \to Gl(\mathrm{H}^0(-w_o\omega_i)) \to \Bbbk$. The first map is the representation and the last one is the trace map.
$\kappa : G \to \mathbb{A}^\ell$	The Steinberg map defined by $g \mapsto (\chi_1(g), \dots, \chi_\ell(g))$.
$\mathcal{U}_{ad}$	The unipotent variety of G_{ad}
$\bar{\mathcal{U}}$	The closure in X of $\mathcal{U}_{ad}$.

Chapter 4

$\boldsymbol{t^c}$	$= \prod_{i=1}^N t_i^{c_i}$ where $\boldsymbol{t} = (t_1, \dots, t_N)$
$\boldsymbol{p-1}$	$= (p-1, \dots, p-1)$
Tr	The map is defined on page 38.
τ	The trace map is defined in Theorem 4.1.5 on page 38.
v_-, v_+	The lowest, respectively highest weight vector in St.
v	The $\delta(G)$-invariant element in $St \boxtimes St$.
$\mathrm{End}_F(X)$	$:= \mathcal{H}om_{\mathcal{O}_Z}(F_*\mathcal{O}_Z, \mathcal{O}_Z)$.

Chapter 5

S_w	The Schubert variety associated to $w \in W$, i.e. the closure in ${}^{G}/_{B}$ of ${}^{B\dot{w}B}/_{B}$.
$\mathrm{Pic}(\bar{\mathcal{U}})$	The Picard group of $\bar{\mathcal{U}}$

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