Contents

1 INTRODUCTION

This thesis contains the results of my work during my study as a Ph.D. student at Aarhus university from 2001-2005. My main interest has been the study of the local cohomology with support in the Schubert variety in G/B. The thesis is divided in 5 different sections with titles

Section (2) : LOCAL COHOMOLOGY
Section (3) : DIFFERENTIAL OPERATORS
Section (4) : G/B IN CHARACTERISTIC ZERO
Section (5) : G/B IN POSITIVE CHARACTERISTIC
Section (6) : THE GRASSMANN VARIETY

Apart from section 2, which simply states known results of local cohomology and section 3.1 and 3.2, there are new results in all the other sections. I will explain them below. For details of the statements one should read the respective sections.

We let X denote an irreducible, smooth variety defined over an algebraic closed field k of characteristic 0. Let $i : Y \hookrightarrow X$ denote a closed immersion with Y irreducible and smooth. It was then proved by Kashiwara, that the category of left modules over the sheaf of differential operators on X, whose support is contained in Y and the category of left modules over the sheaf of differential operators on Y are equivalent. We use this equivalence to establish that for $Z \subset Y$ locally closed, where we let $\mathcal{H}^i_Z$ denote the higher derived sections with support in Z

$$\mathcal{H}^j_Z(\mathcal{O}_X) = 0 \Leftrightarrow \mathcal{H}^{j-\mathrm{codim}(Y)}_Z(\mathcal{O}_Y) = 0$$

and this is the main result of section 3.3.

Let $X = G/B$ denote the flag variety defined over a field of characteristic zero and W the Weyl group. For $v \in W$ let $C(v)$ denote the B-orbit of vB in X and $X(v)$ the Schubert variety, which is the closure of $C(v)$ in X. In 1981 it was proved by Brylinski-Kashiwara in [12] that in the category of holonomic B-equivariant left $\mathcal{D}_X$-modules, which are quasi-coherent $\mathcal{O}_X$-modules (denoted $\mathcal{D}_X - mod$), that the simple modules are parametrized by the Schubert varieties, and if we let $\mathcal{L}(w) \in \mathcal{D}_X - mod$ denote the simple module with support in the Schubert variety $X(w)$ and $\mathrm{codim}(X(w)) = c_w$ that there is an inclusion

$$\mathcal{L}(w) \subset \mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)$$

and $\{\mathcal{L}(w)|\ w \in W\}$ is a basis in the Grothendieck group. These result are proved by Brylinski-Kashiwara by showing that $\mathcal{L}(w)$ corresponds via the Riemann-Hilbert correspondence to the intersection homology $\pi_{X(w)}$ of middle perversity. We hence wish to get information of $\mathrm{Supp}(\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)/\mathcal{L}(w))$. Given $\mathcal{M} \in \mathcal{D}_X - mod$ we denote by $[\mathcal{M}]$ its image in the Grothendieck group and $[\mathcal{M} : \mathcal{L}(w)]$ the coefficient of $[\mathcal{L}(w)]$ in the character formula of $[\mathcal{M}]$. We assume $G = Sl_n$ and therefore $W = S_n$ the group of all permutations on the set $\{1, 2, \ldots, n\}$. Let $\mathrm{Sing}(X(w))$ denote the singular locus of $X(w)$ and $\mathrm{Sing}(X(w)) = \cup_{i=1}^{n} X(v_i)$ be an irreducible decomposition. In 2003 Billey and Warrington gave in [2] a characterization of the v_i. They show $\exists Z_i = \{d_1 < d_2 < \cdots < d_k\} \subset \{1, 2, \ldots, n\}$ such that

$$w(s) = v_i(s)\ \forall s \notin Z_i$$

and there are three different possibilities for $\{v_i(d_1), v_i(d_2), \ldots, v_i(d_k)\}$. The main question we explore in section 4.6 and 4.7 is to find $[\mathcal{H}^{j}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v_i)]$, and we shall give a complete description of the above in one of these three cases. The main result in this case is

$$[\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v_i)] = 1.$$

We also show

$$l(w) - l(v_i) = 3 \Rightarrow\ [\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v_i)] = 1.$$

One of the two other cases will not be examined and the last will be completely treated in the case $Z_i = \{1, 2, \ldots, n\}$ only. We will also prove, if $l(w) - l(v_i) \equiv 0 \mod(2) \Rightarrow \exists j > 0\ j \equiv 1 \mod(2)$ such that $\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) \neq 0$. This is interesting, since to show for $j > 0$ if $\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) \neq 0$ is a very difficult task. The best way in general to do this is to use an algorithm developed by Uli Walther in 2001 in [16]. This algorithm uses Gröbner basis theory on non-commutative rings.

Now we suppose that the ground field is of positive characteristic. It was proved by Bögvad in [5] and [6] that if one replaces the concept of holonomicity with locally finitely generated unit $\mathcal{O}_{F,X}$-modules in $\mathcal{D}_X - mod$ above, then the simple modules are also parameterized by the the Schubert varieties. We prove, that the simple modules are actually $\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)$. This is a major difference from the case of characteristic zero, since we have here shown, that this is far from being the case. Another difference between the two situations is shown by Lauritzen and Kashiwara in [31]. They show, that there is not $\mathcal{D}$-affinity in characteristic greater than zero even though this was proved by Beillinson-Bernstein in [1] to be the case in characteristic zero.

The first ingredient in the proof is to show, that $X(w)$ is globally F-regular. The concept of globally F-regularity was first examined by Hochster and Huneke in 1989 in [26].

The second ingredient is a result by Blickle in 2004 in [4]. He gives some conditions, which ensure for a regular local F-finite ring and $I \subset R$ a prime of height c, that $H_I^c(R)$ is a simple $D(R)$-module. We generalize his result to the case X an irreducible smooth variety and $Y \subset X$ closed and irreducible and gives a condition on Y, which ensures that $\mathcal{H}_Y^{\mathrm{codim}(Y)}(\mathcal{O}_X)$ is simple in $\mathcal{D}_X - mod$. If for example Y was globally F-regular it will satisfy this condition, and we have the result.

Let $\mathrm{Gr}(r,n)$ denote the set of r dimensional subspaces of k^n. The Schubert varieties in $\mathrm{Gr}(r,n)$ are parameterized by $1 \leq a_1 < a_2 < \cdots < a_r \leq n$ $a_i \in \mathbb{N}$, and we denote it as $X(a_1,\ldots,a_r)$. The cohomological dimension of $X(a_1,\ldots,a_r)$ in $\mathrm{Gr}(r,n)$ denoted $\mathrm{cd}_{\mathrm{Gr}(r,n)}(X(a_1,\ldots,a_r))$ is the largest c such that $\mathcal{H}^c_{X(a_1,\ldots,a_r)}(\mathcal{O}_X) \neq 0$. This number is interesting since it locally gives a lower bound on the minimal number of generators of the ideal sheaf of $X(a_1,\ldots,a_r)$. If $\mathrm{char}(k) > 0$ this number is known to be equal to the codimension of the Schubert variety thanks to a result by Peskine and Szpiro in [40]. But if $\mathrm{char}(k) = 0$ this number has only been found in the case $\{a_1,\ldots,a_r\} = \{r-s, r-s+1, \ldots, r, n-r+s+2, n-r+s+3, \ldots, n\}$. This was done by Bruns and Schwänzl in 1990 in [10].

We find $\mathrm{cd}_{\mathrm{Gr}(r,n)}(X(a_s-s+1, a_s-s+2, \ldots, a_s, a_{s+1}, \ldots, a_r))$ whenever $a_s \geq r$. There are two main ingredients in the proof. The first is the result proved in [10] and the second is the Grothendieck-Cousin complex on $\mathrm{Gr}(r,n)$. In the proof of this result we get $\mathcal{H}^{\mathrm{codim}(X(a_1,\ldots,a_r))+j}_{X(a_1,\ldots,a_r)}(\mathcal{O}_{\mathrm{Gr}(r,n)})$ can be decomposable in $\mathcal{D}_{\mathrm{Gr}(r,n)} - mod$ for $j > 0$, which for $j = 0$ is not the case.

Since $\mathrm{Gr}(r,n) = G/P$ for P a maximal parabolic subgroup in G containing B, we let $\pi : G/B \to G/P$ denote the canonical morphism and $X(v) = \pi^{-1}(X(2,3,\ldots,r,n))$. Then the purpose with section 6.3 and 6.4 is to find $[\mathcal{H}^{c_v+j}_{X(v)}(\mathcal{O}_{G/B})]$ $\forall j \geq 0$. Since we also prove $\mathcal{H}^{c_v+j}_{X(v)}(\mathcal{O}_{G/B}) \neq 0 \Leftrightarrow \mathcal{H}^{c_v+j}_{X(2,3,\ldots,r,n)}(\mathcal{O}_{G/P})$, we have also given a new proof of the result proved by Bruns and Schwänzl mentioned above in the case $s = r-2$. But we get a lot more than just the cohomological dimension of $X(v)$ in G/B. We find out for which $j \geq 0$ $\mathcal{H}^{c_v+j}_{X(v)}(\mathcal{O}_{G/B}) \neq 0$ and also get, that $\mathcal{H}^{c_v}_{X(v)}(\mathcal{O}_{G/B})$ is simple in $\mathcal{D}_{G/B} - mod \Leftrightarrow n \neq 2r$. We also show that $X(v)$ is singular and therefore get, that $\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)$ can be simple even if $X(w)$ is singular. If $X(w)$ is non-singular, this is a known result. Another reason for $X(v)$ to be interesting is that, it gives $[\mathcal{H}^{c_v}_{X(v)}(\mathcal{O}_{G/B}) : \mathcal{L}(v_i)]$ for $v_i \in \text{maxSing}(X(v))$ with $Z_i = \{1, 2, \ldots, n\}$ for one of the three possibilities of v_i and could perhaps also give the complete describtion of $[\mathcal{H}^{j}_{X(w)}(\mathcal{O}_{G/B}) : \mathcal{L}(v_i)]$ in this case.

1.1 Notation

Given a ring R or a sheaf of rings $\mathcal{R}$ on a topological space X we denote by respective $R - mod$ and $\mathcal{R} - mod$ as the category of left modules over repective R and $\mathcal{R}$, and $mod - R$ and $mod - \mathcal{R}$ as the category of right modules. Furthermore whenever the notation R is used for a ring, we assume, it is commutative. Whenever we use the notation $:=$, we define what is on the left to be equal to what is on the right. For example $X := G/B$.

1.2 Acknowledgments

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2 LOCAL COHOMOLOGY

2.1 Local cohomology of modules

Throughout this section R will be a commutative Noetherian ring and $I \subset R$ an ideal and $M \in R - mod$. We then define

$$\Gamma_I(M) := \{x \in M | \; \exists n \gg 0 \; I^n x = 0\}.$$

It is then clear, that

$$\Gamma_I(-) : R - mod \to R - mod$$

is a left-exact covariant functor, and we denote its right derived functors as

$$H^i_I(M).$$

These are the local cohomology modules of M with support in I. We will need to calculate the local cohomology of a module M. This is done by using the Cěch complex. For a proof of all the facts below, we refer to section 5.1 of [9]. Since R is Noetherian $\exists f_1, \dots, f_n \in R$ such that

$$I = \langle f_1, \dots, f_n \rangle.$$

Given $a, b \in R$ there is a natural map from M_a to M_{ab} given as

$$\phi : M_a \to M_{ab}, \; \phi(\frac{x}{a^i}) := \frac{b^i x}{(ab)^i}. \tag{2.1}$$

Given $1 \leq k \leq n$ with $k, n \in \mathbb{N}$ we let

$$\mathcal{E}(k, n) := \{(i_1, \dots, i_k) \in \mathbb{N}^k | \; 1 \leq i_1 < i_2 < \dots < i_k \leq n\}.$$

For $k < n$ and $s \in \{1, 2, \dots, k+1\}$ and $\vec{j} = (j_1, \dots, j_{k+1}) \in \mathcal{E}(k+1, n)$ we let $\vec{j}^{\hat{s}} \in \mathcal{E}(k, n)$ be defined as

$$\vec{j}^{\hat{s}} := (j_1, \dots, j_{s-1}, j_{s+1}, \dots, j_{k+1}).$$

We now construct a complex in $R - mod$ $C(M)^\bullet$ defined as

$$C(M)^\bullet := 0 \to C(M)^0 \stackrel{d_0}{\to} C(M)^1 \stackrel{d_1}{\to} \cdots \stackrel{d_{n-2}}{\to} C(M)^{n-1} \stackrel{d_{n-1}}{\to} C(M)^n \to 0$$
$$C(M)^0 := M$$
$$C(M)^k := \bigoplus_{(j_1, \dots, j_k) \in \mathcal{E}(k,n)} M_{\prod_{i=1}^k f_{j_i}}$$

with d_0 defined such that for each $m \in \{1, 2, \ldots, n\}$ the composition of d_0 followed by the canonical projection from $C(M)^1$ to M_{f_m} is just the natural map, and d_k for $k > 0$ is defined for each $\vec{i} = (i_1, \ldots, i_k) \in \mathcal{E}(k, n)$ and each $\vec{j} = (j_1, \ldots, j_{k+1}) \in \mathcal{E}(k+1, n)$ such that the composition

$$0 \to M_{\prod_{s=1}^{k} f_{i_s}} \to C(M)^k \xrightarrow{d_k} C(M)^{k+1} \to M_{\prod_{s=1}^{k+1} f_{j_s}} \to 0$$

with the first map the canonical injection and the last the canonical projection is the natural map defined in (2.1) from $M_{\prod_{s=1}^{k} f_{i_s}} \to M_{\prod_{s=1}^{k+1} f_{j_s}}$ multiplied with $(-1)^{r-1}$ if $\vec{i} = \vec{j}^{\hat{r}}$ for some $r \in \{1, 2, \ldots, k+1\}$ and zero otherwise. We then get an isomorphism in $R - mod$

$$H_I^j(M) \simeq H^j(C(M)^\bullet). \tag{2.2}$$

We define the cohomological dimension of an ideal in the following way

$$\text{cd}(I) := \max\{j |\ H_I^j(M) \neq 0,\ M \in R - mod\}. \tag{2.3}$$

According to [24] page 413

$$\text{cd}(I) := \max\{j |\ H_I^j(R) \neq 0\}. \tag{2.4}$$

So combining this definition with (2.2) we get, that $\text{cd}(I)$ gives a lower bound on the minimal set of generators for I.

Lemma 2.1.1. *Suppose R and S are finitely generated k-algebras with k a field and set $T = R \otimes_k S$ and let $I \subset R$ be an ideal, then*

$$\text{cd}(I) = \text{cd}(IT).$$

Proof. Since R is Noetherian, we get that $I = \langle f_1, \ldots, f_n \rangle$ and also that $IT = \langle f_1 \otimes_k 1_S, \ldots, f_n \otimes_k 1_S \rangle$ with 1_S the identity element in S. So let us consider $C(R)^\bullet$ and $C(T)^\bullet$, where we in the last uses $(f_1 \otimes_k 1_S, \ldots, f_n \otimes_k 1_S)$ and denote the morphisms g_i. Clearly there is an isomorphism in $T - mod$

$$\phi_j : C(R)^j \otimes_k S \simeq C(T)^j$$

$$\phi_j((\bigoplus_{(i_1,\ldots,i_j) \in \mathcal{E}(j,n)} \frac{a_{(i_1,\ldots,i_j)}}{(\prod_{s=1}^{j} f_{i_s})^{b_{(i_1,\ldots,i_j)}}}) \otimes_k s) := \bigoplus_{(i_1,\ldots,i_j) \in \mathcal{E}(j,n)} \frac{a_{(i_1,\ldots,i_j)} \otimes_k s}{(\prod_{s=1}^{j} f_{i_s} \otimes_k 1_S)^{b_{(i_1,\ldots,i_j)}}}$$

and

$$g_j = \phi_{j+1} \circ (d_j \otimes_k id_S) \circ (\phi_j)^{-1}.$$

Since there is an exact sequence in $T - mod$

$$0 \to \operatorname{im}(g_{j-1}) \to \ker(g_j) \to H^j_{IT}(T) \to 0$$

with the convention $\operatorname{im}(f_{-1}) = 0$ and $\operatorname{im}(d_{-1}) = 0$, we get an exact sequence in $T - mod$

$$0 \to \operatorname{im}(d_{j-1} \otimes_k id_S) \to \ker(d_j \otimes_k id_S) \to H^j_{IT}(T) \to 0.$$

Clearly $\operatorname{im}(d_{j-1} \otimes_k id_S) = \operatorname{im}(d_{j-1}) \otimes_k S$ and since S is a free k-module we get $\ker(d_j \otimes_k id_S) = \ker(d_j) \otimes_k S$, and there is thus an exact sequence in $T - mod$

$$0 \to \operatorname{im}(d_{j-1}) \otimes_k S \to \ker(d_j) \otimes_k S \to H^j_{IT}(T) \to 0$$

and since there is an exact sequence in $R - mod$

$$0 \to \operatorname{im}(d_{j-1}) \to \ker(d_j) \to H^j_I(R) \to 0$$

and since S is a free k-algebra, we get an exact sequence in $T - mod$

$$0 \to \operatorname{im}(d_{j-1}) \otimes_k S \to \ker(d_j) \otimes_k S \to H^j_I(R) \otimes_k S \to 0$$

and the result follows. □

By using this Lemma we have therefore shown the following Corollary.

Corollary 2.1.2. *Let $I \subset k[X_i |\ i \in \{1, 2, \dots, r\}]$ be an ideal in the polynomial ring. Let Y_j be indeterminates with $j \in \{1, 2, \dots, s\}$. Consider the ideal $Ik[X_i, Y_j |\ i \in \{1, 2, \dots, r\}, j \in \{1, 2, \dots, s\}] \subset k[X_i, Y_j |\ i \in \{1, 2, \dots, r\}, j \in \{1, 2, \dots, s\}]$. Then*

$$\operatorname{cd}(I) = \operatorname{cd}(Ik[X_i, Y_j]).$$

2.2 Local cohomology of sheaves

Throughout this section X will be a variety and all sheaves considered will be sheaves of abelian groups on X. Let $Z = V \cap U^c \subset X$ be locally closed with $U, V \subset X$ open and $\mathcal{F}$ a sheaf on X, we then define the sheaf of $\mathcal{F}$ with support in Z as for $Y \subset X$ open

$$\underline{\Gamma}_Z(\mathcal{F})(Y) := \{\phi \in \mathcal{F}(Y \cap V) |\ \phi|_{Y \cap V \cap U} = 0\} = \Gamma(Y, \ker(\operatorname{res}(\mathcal{F}|_V \to \mathcal{F}|_{V \cap U}))).$$

That the above definition only depends on Z can be checked, and that it is a sheaf follows by the last equality. Let $\mathcal{R}$ be a sheaf of rings on X and let

$\mathcal{F} \in \mathcal{R} - mod$, it then follows by the last equality in the definition of $\underline{\Gamma}_Z(\mathcal{F})$, that $\underline{\Gamma}_Z(\mathcal{F}) \in \mathcal{R} - mod$ and therefore

$$\underline{\Gamma}_Z(-) : \mathcal{R} - mod \to \mathcal{R} - mod.$$

In the same we define the sections of $\mathcal{F}$ with support in Z as

$$\Gamma_Z(X, \mathcal{F}) := \underline{\Gamma}_Z(\mathcal{F})(X)$$

and once again

$$\Gamma_Z(X, -) : \mathcal{R} - mod \to \mathcal{R}(X) - mod.$$

It follows, that both $\underline{\Gamma}_Z(-)$ and $\Gamma_Z(X,-)$ are left-exact covariant functors, and we denote the right defined functors of respectively $\underline{\Gamma}_Z(-)$ and $\Gamma_Z(X,-)$ as

$$\begin{array}{l}\mathcal{H}^i_Z(\mathcal{F})\\ H^i_Z(X,\mathcal{F})\end{array}$$

and they are called the local cohomology sheaves/groups of X with coefficients in $\mathcal{F}$ and supports in Z. Suppose Z is closed and $Z_1 \subset Z$ is closed, we then get

$$\underline{\Gamma}_{Z_1}(\mathcal{F}) \subset \underline{\Gamma}_Z(\mathcal{F})$$

and define

$$\underline{\Gamma}_{Z/Z_1}(\mathcal{F}) := \underline{\Gamma}_Z(\mathcal{F})/\underline{\Gamma}_{Z_1}(\mathcal{F}).$$

Clearly $\underline{\Gamma}_{Z/Z_1}(-)$ is a covariant functor. If $\mathcal{F}$ is flasque, it follows by Lemma 8.3 (b) in [33], that

$$\underline{\Gamma}_{Z/Z_1}(\mathcal{F}) = \underline{\Gamma}_{Z\cap(Z_1)^c}(\mathcal{F}).$$

So given an injective resolution of $\mathcal{F}$

$$0 \to \mathcal{F} \to \mathcal{M}_0 \to \mathcal{M}_1 \to \dots.$$

We can consider the two new complexes

$$\begin{array}{l}\underline{\Gamma}_{Z/Z_1}(\mathcal{M}_0) \to \underline{\Gamma}_{Z/Z_1}(\mathcal{M}_1) \to \dots\\ \underline{\Gamma}_{Z\cap(Z_1)^c}(\mathcal{M}_0) \to \underline{\Gamma}_{Z\cap(Z_1)^c}(\mathcal{M}_1) \to \dots.\end{array}$$

Due to [23] Lemma 1.5 injective sheaves are flasque, we get that the cohomology of the two complexes above are the same, and the cohomology of the first is in [33] denoted $\mathcal{H}^i_{Z/Z_1}(\mathcal{F})$ and therefore

$$\mathcal{H}^i_{Z/Z_1}(\mathcal{F}) = \mathcal{H}^i_{Z\cap(Z_1)^c}(\mathcal{F}). \tag{2.5}$$

Suppose $\mathcal{F}$ is flasque it then follows by Lemma 8.5 in [33] combined with (2.5), that for Z locally closed

$$\mathcal{H}^i_Z(\mathcal{F}) = 0 \ \forall i > 0$$

and to calculate $\mathcal{H}^i_Z(\mathcal{F})$ we can therefore use a flasque resolution of $\mathcal{F}$. To prove the Lemma below one considers the Godement resolution of $\mathcal{F}$ explained in [18]. A proof may be found in [33] page 361.

Lemma 2.2.1. *Let $\mathcal{R}$ be a sheaf of rings and suppose $\mathcal{F} \in \mathcal{R} - mod$, we then get $\mathcal{H}^i_Z(\mathcal{F}) \in \mathcal{R} - mod$.*

We also need to know what happens, when we compose two left-exact functors. This is simply Grothendieck's spectral sequence one obtains. For a left-exact covariant functor $T : \mathcal{A} \to \mathcal{B}$ with $\mathcal{A}$ and $\mathcal{B}$ categories of modules over some rings, we let $R^pT(-)$ denote its right derived functors, and we say $A \in \mathcal{A}$ is T-acyclic, if $R^pT(A) = 0 \ \forall p > 0$.

Theorem 2.2.2. *Let*

$$T : \mathcal{A} \to \mathcal{B}, \ S : \mathcal{B} \to \mathcal{C}$$

be two left-exact covariant functors. Let $A \in \mathcal{A}$ and suppose A has a resolution, that are carried by T into S-acyclic objects, then there is a spectral sequence

$$E_2^{p,q} := R^pS(R^qT(A)) \Rightarrow R^{p+q}(S \circ T)(A).$$

Proof. One just have to look up chapter XX Theorem 9.6 in [36]. □

We are going to need some exact sequences of local cohomology and they are the Lemmas below.

Lemma 2.2.3. *Let $Z_1 \subset Z$ be closed and set $Z_2 = Z \cap (Z_1)^c$, then there is an exact sequence*

$$0 \to \mathcal{H}^0_{Z_1}(\mathcal{F}) \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{H}^0_{Z_2}(\mathcal{F}) \to \mathcal{H}^1_{Z_1}(\mathcal{F}) \to \mathcal{H}^1_Z(\mathcal{F}) \to \ldots.$$

If furthermore $\mathcal{F} \in \mathcal{R} - mod$ with $\mathcal{R}$ a sheaf of rings the above sequence is exact in $\mathcal{R} - mod$.

Proof. For a proof one should look up Proposition 1.9 in [23]. □

Lemma 2.2.4. *(Mayer-Vietoris) Suppose $Z_1, Z_2 \subset X$ are closed, then there is an exact sequence*

$$0 \to \mathcal{H}^0_{Z_1 \cap Z_2}(\mathcal{F}) \to \mathcal{H}^0_{Z_1}(\mathcal{F}) \bigoplus \mathcal{H}^0_{Z_2}(\mathcal{F}) \to \mathcal{H}^0_{Z_1 \cup Z_2}(\mathcal{F}) \to$$
$$\mathcal{H}^1_{Z_1 \cap Z_2}(\mathcal{F}) \to \mathcal{H}^1_{Z_1}(\mathcal{F}) \bigoplus \mathcal{H}^1_{Z_2}(\mathcal{F}) \to \ldots.$$

If furthermore $\mathcal{F} \in \mathcal{R} - mod$ with $\mathcal{R}$ a sheaf of rings the above sequence is exact in $\mathcal{R} - mod$.

Proof. The Lemma is proved locally in [9] and the proof in the general case is identical. □

Lemma 2.2.5. *Let $U, V, W \subset X$ be open such that $(U^c \cap W) \cap (V^c \cap W) = \emptyset$. Then*

$$\mathcal{H}^j_{U^c \cap W}(\mathcal{F}) \bigoplus \mathcal{H}^j_{V^c \cap W}(\mathcal{F}) \simeq \mathcal{H}^j_{(U^c \cup V^c) \cap W}(\mathcal{F}).$$

If $\mathcal{F} \in \mathcal{R} - mod$ with $\mathcal{R}$ a sheaf of rings the above morphism is a morphism in $\mathcal{R} - mod$

Proof. Let us for $\mathcal{F}$ flasque show

$$\underline{\Gamma}_{V^c \cap W}(\mathcal{F}) \bigoplus \underline{\Gamma}_{U^c \cap W}(\mathcal{F}) \simeq \underline{\Gamma}_{(U^c \cup V^c) \cap W}(\mathcal{F})$$

because we are then done. Let $O \subset X$ be open. Then

$$\begin{aligned}
\underline{\Gamma}_{V^c \cap W}(\mathcal{F})(O) &= \{\pi \in \mathcal{F}(O \cap W) | \; \pi|_{O \cap W \cap V} = 0\} \\
\underline{\Gamma}_{U^c \cap W}(\mathcal{F})(O) &= \{\pi \in \mathcal{F}(O \cap W) | \; \pi|_{O \cap W \cap U} = 0\} \\
\underline{\Gamma}_{(U^c \cup V^c) \cap W}(\mathcal{F})(O) &= \{\pi \in \mathcal{F}(O \cap W) | \; \pi|_{O \cap W \cap V \cap U} = 0\}.
\end{aligned}$$

So let

$$\begin{aligned}
&\beta : \underline{\Gamma}_{V^c \cap W}(\mathcal{F})(O) \bigoplus \underline{\Gamma}_{U^c \cap W}(\mathcal{F})(O) \to \underline{\Gamma}_{(U^c \cup V^c) \cap W}(\mathcal{F})(O) \\
&\beta(\pi, \phi) := \pi - \phi.
\end{aligned}$$

Then we have a morphism. If $\pi = \phi \Rightarrow \pi|_{(O \cap W \cap U) \cup (O \cap W \cap V)} = 0 \Rightarrow \pi|_{O \cap W \cap (U \cup V)} = 0$. Since $\emptyset = (U^c \cap W) \cap (V^c \cap W) = W \cap (U \cup V)^c \Rightarrow O \cap W \cap (U \cup V) = W \cap O \Rightarrow \pi = 0$. So all there is to prove is the surjection. So let

$$\pi \in \underline{\Gamma}_{(U^c \cup V^c) \cap W}(\mathcal{F})(O) \Leftrightarrow \pi \in \mathcal{F}(O \cap W) \wedge \; \pi|_{O \cap W \cap V \cap U} = 0.$$

By construction

$$\pi|_{U \cap O} \in \underline{\Gamma}_{(U^c \cup V^c) \cap W}(\mathcal{F})(O \cap U) = \underline{\Gamma}_{V^c \cap W}(\mathcal{F})(O \cap U).$$

It follows by Lemma 8.3 in [33], that $\underline{\Gamma}_{V^c \cap W}(\mathcal{F})$ is flasque $\Rightarrow \exists \phi \in \underline{\Gamma}_{V^c \cap W}(\mathcal{F})(O)$ such that $\pi|_{U \cap O} = -\phi|_{U \cap O} \Rightarrow (\pi + \phi)|_{U \cap O} = 0 \Rightarrow \pi + \phi \in \underline{\Gamma}_{U^c \cap W}(\mathcal{F})(O) \Rightarrow \pi = \beta(\phi, \pi + \phi)$. □

We also need informations on the local cohomology sheaves with coefficients in $\mathcal{F}$ and support in Z in the case, that X is an affine scheme, and it is in this case, there is a connection with the local cohomolgy modules with support in an ideal.

Propositon 2.2.6. *Let $\mathcal{F} \in \mathcal{O}_X - mod$ be quasi-coherent then $\mathcal{H}^i_Z(\mathcal{F})$ is also quasi-coherent. If X=Spec(A) with A a commutative, Noetherian ring and $Z = V(I)$ is closed*

$$\mathcal{H}^i_Z(\mathcal{F}) = \widetilde{H^i_I(\mathcal{F}(X))}$$
$$(\mathcal{H}^i_Z(\mathcal{F}))_x = H^i_{I_x}(\mathcal{F}(X)_x)\ \forall x \in X.$$

Proof. The first fact is simply Proposition 2.1 in [23]. The second follows by combining Proposition 2.2 and Theorem 2.3 in [23] since Proposition 2.2, shows that

$$\mathcal{H}^i_Z(\mathcal{F}) = H^i_Z(X, \mathcal{F})^{\sim}$$

and then Theorem 2.3 gives the second fact. The last follows due to the fact $H^i_{I_x}(\mathcal{F}(X)_x) = (H^i_I(\mathcal{F}(X)))_x$. □

Given $\mathcal{F}$ we define the support of $\mathcal{F}$ in the following way

$$\text{Supp}(\mathcal{F}) := \{x \in X |\ \mathcal{F}_x \neq 0\}.$$

We are also going to need some conditions on the vanishing of the local cohomology sheaves. To do this we need some commutative algebra. Let R be a ring and I an ideal and $M \in R - mod$. We say $r \in R$ is M-regular if $rx = 0 \Leftrightarrow\ x = 0$ and a sequence of elements $r_1, r_2 \dots r_n \in R$ is M-regular, if r_1 is M-regular and $\forall i \in \{2, 3, \dots, n\}$ r_i is $M/\langle r_1, \dots, r_{i-1}\rangle M$-regular, we then define

$$\text{depth}_I(M) := \max\{n |\ r_1, \dots, r_n \text{ is a regular } M-\text{sequence},\ r_i \in I\}.$$

Suppose $(R, \mathcal{M})$ is a local ring and $M \in R - mod$. Then M is Cohen-Macaulay, if

$$\text{depth}_{\mathcal{M}}(M) = \dim(M)$$

with $\dim(M) = \dim(R/\text{ann}(M))$ and the dim to the right being the Krull dimension. Let $\mathcal{F} \in \mathcal{O}_X - mod$ we say, that $\mathcal{F}$ is Cohen-Macaulay if $\mathcal{F}_x$ is a Cohen-Macaulay module $\forall x \in X$. For $Y \subset X$ closed and $\mathcal{F} \in \mathcal{O}_X - mod$ coherent we define

$$\text{depth}_Y(\mathcal{F}) := \inf_{x \in Y}(\text{depth}_x(\mathcal{F}_x)).$$

Propositon 2.2.7. *Let X be irreducible, $\mathcal{F} \in \mathcal{O}_X-mod$ be Cohen-Macaulay and coherent,* $\mathrm{Supp}(\mathcal{F}) = X$ *and $Y \subset X$ be closed, then*

$$\mathcal{H}_Y^j(\mathcal{F}) = 0 \;\forall j < \mathrm{codim}(Y).$$

Proof. It follows by Theorem 3.8 in [23], that all we have to show, is that

$$\mathrm{depth}_Y(\mathcal{F}) = \mathrm{codim}(Y).$$

Since $\mathrm{Supp}(\mathcal{F}) = X$ we get, that $\dim(\mathcal{F}_x) = \dim(\mathcal{O}_{X,x})$ and thus

$$\mathrm{depth}_Y(\mathcal{F}) := \inf_{x \in Y}(\mathrm{depth}_x(\mathcal{F}_x)) = \inf_{x \in Y}(\dim(\mathcal{F}_x)) = \inf_{x \in Y}(\dim(\mathcal{O}_{X,x})) = \mathrm{codim}(Y)$$

with the last equality stemming from chapter 2 exercise 3.20 (c) in [25]. □

We let $\mathrm{Sing}(Z)$ denote the singular locus of Z.

Lemma 2.2.8. *Let X be smooth and irreducible and Z closed and irreducible. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Then*

$$\mathrm{Supp}(\mathcal{H}_Z^{\mathrm{codim}(Z)+j}(\mathcal{F})) \subset \mathrm{Sing}(Z) \;\forall j > 0.$$

If Z is smooth

$$\mathcal{H}_Z^{\mathrm{codim}(Z)+j}(\mathcal{F}) = 0 \;\forall j > 0.$$

Proof. We just have to prove the first part. In order to prove this it is since $\mathrm{Sing}(Z)$ is closed enough to prove

$$\mathcal{H}_Z^{\mathrm{codim}(Z)+j}(\mathcal{F})|_{(\mathrm{Sing}(Z))^c} = 0 \;\forall j > 0$$

and since

$$\mathcal{H}_{(\mathrm{Sing}(Z))^c \cap Z}^{\mathrm{codim}(Z)+j}(\mathcal{F}|_{(\mathrm{Sing}(Z))^c}) = \mathcal{H}_Z^{\mathrm{codim}(Z)+j}(\mathcal{F})|_{(\mathrm{Sing}(Z))^c}$$

all we have to prove is that

$$(\mathcal{H}_{(\mathrm{Sing}(Z))^c \cap Z}^{\mathrm{codim}(Z)+j}(\mathcal{F}|_{(\mathrm{Sing}(Z))^c}))_x = 0 \;\forall x \in (\mathrm{Sing}(Z))^c \cap Z.$$

Since $(\mathrm{Sing}(Z))^c$ is a dense subset and thus $\mathrm{codim}((\mathrm{Sing}(Z))^c \cap Z) = \mathrm{codim}(Z)$ where the first is the codimension of $(\mathrm{Sing}(Z))^c \cap Z$ considered as a subset of $(\mathrm{Sing}(Z))^c$ and since $(\mathrm{Sing}(Z))^c \cap Z \subset (\mathrm{Sing}(Z))^c$ is smooth and irreducible, it follows by Theorem 8.17 chapter 2 of [25], that the ideal sheaf of $(\mathrm{Sing}(Z))^c \cap Z$ is locally generated by $\mathrm{codim}(Z)$ elements. Let it be $\mathcal{I}$. It then follows by Proposition 2.2.6, that

$$(\mathcal{H}_{(\mathrm{Sing}(Z))^c \cap Z}^{\mathrm{codim}(Z)+j}(\mathcal{F}|_{(\mathrm{Sing}(Z))^c}))_x = H_{\mathcal{I}_x}^{\mathrm{codim}(Z)+j}(\mathcal{F}_x)$$

and then the Lemma follows by (2.2) in section 2.1. □

We shall also need the Grothendieck-Cousin complex.

Theorem 2.2.9. *Let* $X = Z_0 \supset Z_1 \supset \cdots \supset Z_n \supset Z_{n+1} = \emptyset$ *be a decreasing sequence of closed subsets such, that the inclusions* $Z_j \backslash Z_{j+1} \subset X$ *is affine* $\forall j$, $\mathcal{F} \in \mathcal{O}_X - mod$ *is a coherent, Cohen-Macaulay module with* $\mathrm{Supp}(\mathcal{F}) = X$ *and* $\mathrm{codim}(Z_i) \geq i$*, then there is a resolution of* $\mathcal{F}$

$$0 \to \mathcal{H}^0_{Z_0 \cap (Z_1)^c}(\mathcal{F}) \to \mathcal{H}^1_{Z_1 \cap (Z_2)^c}(\mathcal{F}) \to \mathcal{H}^2_{Z_2 \cap (Z_3)^c}(\mathcal{F}) \to \dots.$$

If furthermore $\mathcal{R}$ *is a sheaf of rings and* $\mathcal{F} \in \mathcal{R} - mod$*, then the complex above is a complex in* $\mathcal{R} - mod$*.*

Proof. This is Theorem 10.9 of [33]. □

Let $Y \subset X$ be closed. We define the cohomological dimension of Y in X as

$$\mathrm{cd}_X(Y) := \max\{i \in \mathbb{N} | \ \mathcal{H}^i_Y(\mathcal{M}) \neq 0, \ \mathcal{M} \in \mathcal{O}_X - mod \text{ quasicoherent}\}.$$

Lemma 2.2.10.

$$\mathrm{cd}_X(Y) := \max\{i \in \mathbb{N} | \ \mathcal{H}^i_Y(\mathcal{O}_X) \neq 0\}.$$

Proof. That this Lemma is true follows by considering an affine open covering of $X = \cup_{i=1}^n U_i$ and then using (A.2), (2.4) and Proposition 2.2.6. □

If $Y \subset X$ is locally closed, we define

$$\mathrm{cd}_X(Y) := \max\{i \in \mathbb{N} | \ \mathcal{H}^i_Y(\mathcal{O}_X) \neq 0\}.$$

3 DIFFERENTIAL OPERATORS

3.1 The module of differential operators

In this section we want to define the module of differential operators and to consider some of its properties. For a proof one should look up chapter 16 of [19]. Let k be a field and R a commutative k-algebra and $M, M' \in R{-}mod$. We have $\mathrm{Hom}_k(M, M') \subset R{-}mod$ defined as $(a.\psi)(x) := a\psi(x)$ and also that $\mathrm{Hom}_k(M, M') \subset mod - R$ defined in the following way $(\psi.a)(x) := \psi(ax)$ $\forall \psi \in \mathrm{Hom}_k(M, M'),\ a \in R,\ x \in M$. We will write $a\psi$ instead of $a.\psi$ and ψa instead of $\psi.a$, and when we write $\psi(a)$, we will mean $\psi(a) \in M'$. We define

$$\begin{aligned} &[\ ,\ ] : R \times \mathrm{Hom}_k(M, M') \to \mathrm{Hom}_k(M, M') \\ &[a, \psi] = a\psi - \psi a. \end{aligned}$$

Definition 3.1.1. *We define the module of differential operators of order $\leq m$ of M into M' as*

$$\begin{aligned} &\mathrm{Dif}_{-1}(M, M') = 0 \\ &\mathrm{Dif}_m(M, M') = \{\psi \in \mathrm{Hom}_k(M, M') \mid [f, \psi] \in \mathrm{Dif}_{m-1}(M, M')\ \forall f \in R\}. \end{aligned}$$

Suppose $M'' \in R - mod$ we clearly get

$$(\mathrm{Hom}_k(M', M'') \circ \mathrm{Hom}_k(M, M')) \subset \mathrm{Hom}_k(M, M'').$$

We then get, that

$$\left.\begin{aligned} &\mathrm{Dif}_m(M, M') \subset \mathrm{Dif}_{m+1}(M, M') \\ &(\mathrm{Dif}_n(M', M'') \circ \mathrm{Dif}_m(M, M')) \subset \mathrm{Dif}_{m+n}(M, M'') \\ &\mathrm{Dif}_m(M, M') \in R - mod,\ \mathrm{Dif}_m(M, M') \in mod - R \end{aligned}\right\} \tag{3.1}$$

with the module structure the one stemming from the structure on $\mathrm{Hom}_k(M, M')$.

Definition 3.1.2. $\mathrm{Dif}(M, M') := \bigcup_{m \geq 0} \mathrm{Dif}_m(M, M')$ *is the module of differential operators of M into M'.*

Clearly this is a R-bimodule. If $M = M'$, it follows by (3.1), that $\mathrm{Dif}(M, M)$ is a sub-algebra of $\mathrm{End}_k(M)$. If $M = M' = R$ $\mathrm{Dif}(R, R)$ is a ring containing R, since $\mathrm{Dif}_0(R, R) = \mathrm{Hom}_R(R, R) = R$, and its structure as a respective left and right R-module is equal to the multiplication in the ring structure.

Definition 3.1.3. $D(M) := \mathrm{Dif}(M, M)$ *the module of differential operators on M and* $D_m(M) := \mathrm{Dif}_m(M, M)$*, and if M=R it is the ring of differential operators on R.*

$D(R)$ is a subring of $\mathrm{End}_k(R)$. Given $U \subset R$ a multiplicative subset, we would like a ring isomorphism $D(R)_U \simeq D(R_U)$ which is valid if we consider $D(R) \in R - mod$ or $D(R) \in mod - R$. This result is well known if R is a finitely generated k-algebra, but we have been unable to find an exact reference. We will therefore do it in this case, but all details will be done in Appendix A.3. To do this we briefly sketch the theory of non-commutative localization. For a more thoroughly examination we refer the reader to [35] chapter 4.10. Let A be a non-commutative ring. A subset $S \subset A$ is multiplicative closed, if it satisfies $1 \in S$, $ab \in S \ \forall a, b \in S$ and $0 \notin S$ just as in the commutative case. We also denote $U(A)$ as the units of A.

Definition 3.1.4. *The ring* A_1 *is a left ring of fraction of* A *with respect to* S *if there is a ring homomorphism* $\Theta : A \to A_1$ *such that*

$$\begin{aligned}
&\Theta(S) \subset U(A_1)\\
&\forall a_1 \in A_1 \ \exists a \in A, \ s \in S \text{ with } a_1 = \Theta(s)^{-1}\Theta(a)\\
&\Theta(a) = 0 \Leftrightarrow \ \exists s \in S \text{ such that } sa = 0.
\end{aligned}$$

If it exists, then it is unique, and we denote it $S^{-1}A$. If A is commutative, we see, that A_S satisfies the properties above. We define a right ring of fraction of A with respect to S in a similar way only replace $\Theta(s)^{-1}\Theta(a)$ with $\Theta(a)\Theta(s)^{-1}$and sa with as and denote it as AS^{-1}. The Theorem below shows when a left ring of fraction of A with respect to S exists.

Theorem 3.1.5. $S^{-1}A$ *exists if and only if S satisfies* $\forall s \in S, \ a \in A$

$$\begin{aligned}
&as = 0 \Rightarrow \ \exists s_1 \in S \ s_1 a = 0\\
&Sa \cap As \neq \emptyset.
\end{aligned}$$

In the commutative case these two properties are always satisfied. So we must construct a ring homomorphism

$$\phi : D(R) \to D(R_U)$$

which satisfies the conditions set in definition 3.1.4. All this work is done in Appendix A.3.

Propositon 3.1.6. *Suppose R is a finitely generated k-algebra and $U \subset R$ is a multiplicative closed subset. Then $D(R_U)$ is both a left and right ring of fraction of D(R) with respect to U.*

Proof. This proposition follows by using ϕ defined in appendix A.3 as Θ in definition 3.1.4, and it then follows by combining Proposition A.3.2 and Lemma A.3.4, that ϕ satisfies the required properties. □

If k was of characteristic greater than zero and R still is a finitely generated k-algebra, we could have proved the Proposition above by proving, that if we set $p = \mathrm{char}(k)$ and define

$$R^{(e)} := \{x^{p^e} | \, x \in R\}$$

then it follows by [20] chapter 1, that

$$\begin{aligned} D(R) &\cong \bigcup \mathrm{End}_{R^{(e)}}(R) \\ D(R_U) &\cong \bigcup \mathrm{End}_{(R_U)^{(e)}}(R_U) \end{aligned} \tag{3.2}$$

and we should show, that $\bigcup \mathrm{End}_{(R_U)^{(e)}}(R_U)$ is a respective right and left ring of fraction of $D(R)$ with respect to U. But we have given a characteristic independent argument.

3.2 The sheaf of differential operators

Now we wish to do, what we have done with modules for sheaves also. So let k be an algebraic closed field and X be a smooth variety over k and $\mathcal{F},\ \mathcal{G} \in \mathcal{O}_X - mod$. For a proof of all these standard facts we once again refer to [19] chapter 16. Let $\mathcal{H}om_k(\mathcal{F}, \mathcal{G})$ be the sheaf of k-linear homomorphism. Since $\mathcal{F},\ \mathcal{G} \in \mathcal{O}_X - mod$ $\mathcal{H}om_k(\mathcal{F}, \mathcal{G})$ is an $\mathcal{O}_X$-bimodule, with the left and right module structure defined just as in the case, we are dealing with modules. Let $U \subset X$ be open then

$$\begin{aligned} &[\ ,\ ](U) : \mathcal{O}_X(U) \times \mathcal{H}om_k(\mathcal{F}, \mathcal{G})(U) \to \mathcal{H}om_k(\mathcal{F}, \mathcal{G})(U) \\ &[a, \psi](U) = a\psi - \psi a \ \forall \ a \in \mathcal{O}_X(U), \psi \in \mathcal{H}om_k(\mathcal{F}, \mathcal{G})(U). \end{aligned}$$

In the rest of the text we will write $[\ ,\ ]$ and drop the U. It is clear by definition, that $[\ ,\ ]$ is k-bilinear.

Definition 3.2.1. *We define the sheaf of differential operators of order m in the following way . For $U \subset X$ open*

$$\begin{aligned}
&\mathcal{D}if_{-1}(\mathcal{F},\mathcal{G})(U) = 0\\
&\mathcal{D}if_m(\mathcal{F},\mathcal{G})(U) = \{\psi \in \mathcal{H}om_k(\mathcal{F},\mathcal{G})(U) \mid\\
&\forall V \subset U \text{ open}, f \in \mathcal{O}_X(V)\ [f,\psi|_V] \in \mathcal{D}if_{m-1}(\mathcal{F},\mathcal{G})(V)\}.
\end{aligned}$$

That the above is a sheaf is clear. Just as for rings and modules over the ring, we get for $\mathcal{H} \in \mathcal{O}_X - mod$

$$\begin{aligned}
&\mathcal{D}if_m(\mathcal{F},\mathcal{G}) \in \mathcal{O}_X - mod,\ \mathcal{D}if_m(\mathcal{F},\mathcal{G}) \in mod - \mathcal{O}_X\\
&\mathcal{D}if_m(\mathcal{F},\mathcal{G}) \subset \mathcal{D}if_{m+1}(\mathcal{F},\mathcal{G})\\
&\mathcal{D}if_m(\mathcal{H},\mathcal{G}) \circ \mathcal{D}if_n(\mathcal{F},\mathcal{H}) \subset \mathcal{D}if_{m+n}(\mathcal{F},\mathcal{G}).
\end{aligned} \tag{3.3}$$

We then define

Definition 3.2.2. *The sheaf of differential operators of $\mathcal{F}$ into $\mathcal{G}$ is defined as*

$$\mathcal{D}if(\mathcal{F},\mathcal{G}) = \cup \mathcal{D}if_m(\mathcal{F},\mathcal{G})$$

To simplify notation we set

$$\begin{aligned}
&\mathcal{D}if_m(\mathcal{F},\mathcal{F}) := \mathcal{D}if_m(\mathcal{F})\\
&\mathcal{D}(\mathcal{F}) := \bigcup_{m\geq 0} \mathcal{D}if_m(\mathcal{F})\\
&\mathcal{D}_X := \mathcal{D}(\mathcal{O}_X).
\end{aligned}$$

By (3.3) $\mathcal{D}_X$ is a sheaf of rings.

Definition 3.2.3. *$\mathcal{D}_X$ is the sheaf of differential operators on X. $\mathcal{M} \in \mathcal{D}_X - mod$ if $\mathcal{M}$ has a structure as a left $\mathcal{D}_X$-module and $\mathcal{M}$ is quasi-coherent with respect to the induced $\mathcal{O}_X = \mathcal{D}if_0(\mathcal{O}_X)$ structure.*

If $\mathcal{F},\mathcal{G}$ are quasi-coherent sheaves in $\mathcal{O}_X - mod$, then $\mathcal{D}if_m(\mathcal{F},\mathcal{G})$ and $\mathcal{D}if(\mathcal{F},\mathcal{G})$ are also quasi-coherent with respect to both the left and right $\mathcal{O}_X$-module structure, and if $U \subset X$ is open and affine

$$\begin{aligned}
&\mathcal{D}if_m(\mathcal{F},\mathcal{G})|_U = (Dif_m(\mathcal{F}(U),\mathcal{G}(U))^{\sim}\\
&\mathcal{D}if(\mathcal{F},\mathcal{G})|_U = (Dif(\mathcal{F}(U),\mathcal{G}(U))^{\sim}.
\end{aligned}$$

Propositon 3.2.4. *Given $x \in X$ and $U \subset X$ open and affine such that $x \in U$ then $(\mathcal{D}_X)_x$ is both a left and right ring of fraction of $D(\mathcal{O}_X(U))$ with respect to x and $D(\mathcal{O}_{X,x}) \simeq (\mathcal{D}_X)_x$ as rings.*

Proof. So let $U = \text{Spec}(A)$ with A a finitely generated k-algebra. Since $\mathcal{D}_X$ is a sheaf of rings $(\mathcal{D}_X)_x$ is a ring. We will only do the left part since the proof of the right part is the same. Remember $x \in \text{Spec}(A)$. So according to Definition 3.1.4 we must construct a ring homomorphism $\Theta : D(A) \to (\mathcal{D}_X)_x$ such that

$$\begin{aligned}
&\Theta((x)^c) \subset U((\mathcal{D}_X)_x),\\
&\forall a_1 \in (\mathcal{D}_X)_x\ \exists a \in D(A),\ s \in (x)^c \text{ with } a_1 = \Theta(s)^{-1}\Theta(a),\\
&\Theta(a) = 0 \Leftrightarrow\ \exists s \in (x)^c \text{ such that } sa = 0.
\end{aligned}$$

So as the ring homomorphism we pick Θ

$$D(A) = \mathcal{D}_X(U) \to \varinjlim_{x \in V \subset X \text{ open}} \mathcal{D}_X(V) = (\mathcal{D}_X)_x.$$

As a map in $A-mod$ Θ is nothing but the localization map $D(A) \to D(A)_x$, and therefore we get, that the last two properties of Θ are satisfied. Since $\mathcal{O}_X = \mathcal{D}if_0(\mathcal{O}_X) \subset \mathcal{D}_X$ as a sheaf of subrings $\mathcal{O}_{X,x} \subset (\mathcal{D}_X)_x$ as a subring and since $\Theta((x)^c) \subset \mathcal{O}_{X,x}$ the first property follows, and we have therefore proved that $(\mathcal{D}_X)_x$ is a left ring of fraction of $D(A)$ with respect to x. But according to Proposition 3.1.6 $D(A_x)$ is also a left ring of fraction of $D(A)$ with respect to x, and since a left ring of fraction is unique, if it exists $D(A_x) \simeq (\mathcal{D}_X)_x$ as rings and since $\mathcal{O}_{X,x} \simeq A_x$ as rings the Proposition follows. □

We define the sheaf of vector fields on X as a subsheaf of $\mathcal{H}om_k(\mathcal{O}_X, \mathcal{O}_X)$ Θ_X such that

$$\Theta_X := \mathcal{D}er_k(\mathcal{O}_X, \mathcal{O}_X). \tag{3.4}$$

Clearly $\Theta_X \subset \mathcal{D}if_1(\mathcal{O}_X)$, since $[a, \phi] = -\phi(a)$ for $a \in \mathcal{O}_X$ and $\phi \in \Theta_X$. We shall also need that there is an equivalence between $\mathcal{D}_X-mod$ and $mod-\mathcal{D}_X$. It is given in the next Proposition for a proof one should look up 1.3.3. in [3] for the case $\text{char}(k) = 0$ and Proposition 6.1 in [21] for $\text{char}(k) > 0$. We let ω_X denote the sheaf of holomorphic forms of maximal degree on X.

Propositon 3.2.5. *The categories $\mathcal{D}_X - mod$ and $mod - \mathcal{D}_X$ are equivalent. For $\mathcal{M} \in \mathcal{D}_X - mod$ and $\mathcal{N} \in mod - \mathcal{D}_X$ the functors are*

$$\begin{aligned}
&\omega_X \bigotimes_{\mathcal{O}_X} \mathcal{M},\\
&\mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{N}).
\end{aligned}$$

We let $f : X \to Y$ denote a morphism between two smooth varieties. For a proof of the Proposition below we refer the reader to respective section VI 4.1 in [7] if $\mathrm{char}(k) = 0$ and section 2 in [21] if $\mathrm{char}(k) > 0$.

Propositon 3.2.6. *Let $\mathcal{M} \in \mathcal{D}_Y - mod$. Then $f^*(\mathcal{M}) \in \mathcal{D}_X - mod$.*

Let B be a linear algebraic group defined over k with an action on X

$$\nu : B \times X \to X,\ \nu((b,x)) := b.x$$

and denote

$$\begin{aligned}
&p : B \times X \to X,\ p((b,x)) := x,\ \forall b \in B,\ x \in X \\
&\text{for } b \in B,\ i_b : X \to B \times X,\ i_b(x) := (b,x),\ \forall x \in X \\
&\text{for } b \in B,\ \phi_b : X \to X,\ \phi_b(x) := \nu(b,x) = b.x \\
&i : X \to B \times X,\ i(x) = (\mathrm{id}_B, x) \\
&p_i : B \times B \times X \to B \times X,\ p_1(b_1,b_2,x) := (b_1, b_2 x) \\
&p_2(b_1,b_2,x) := (b_1 b_2, x),\ p_3(b_1,b_2,x) := (b_2,x)
\end{aligned}$$

Then $\nu \circ i = p \circ i = id_X$, $p \circ p_2 = p \circ p_3$, $p \circ p_1 = \nu \circ p_3$, $\nu \circ p_2 = \nu \circ p_1$ and

$$\nu \circ i_b = \phi_b,\ p \circ i_b = \mathrm{id}. \tag{3.5}$$

Definition 3.2.7. *Let X have a B-action. $\mathcal{M} \in \mathcal{D}_X - mod$ is B-equivariant if there is an isomorphism in $\mathcal{D}_{B\times X} - mod$ $\alpha : \nu^*(\mathcal{M}) \simeq p^*(\mathcal{M})$ such that the the diagrams below commutes*

$$\begin{array}{ccc}
i^*(\nu^*(\mathcal{M})) & \xrightarrow{i^*(\alpha)} & i^*(p^*(\mathcal{M})) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathcal{M} & \xrightarrow{\mathrm{id}} & \mathcal{M}
\end{array}$$

$$\begin{array}{ccc}
p_2^*(\nu^*(\mathcal{M})) & \xrightarrow{p_2^*(\alpha)} & p_2^*(p^*(\mathcal{M})) \\
\downarrow \simeq & & \downarrow \simeq \\
p_1^*(\nu^*(\mathcal{M})) \xrightarrow{p_1^*(\alpha)} p_1^*(p^*(\mathcal{M})) & \simeq & p_3^*(\nu^*(\mathcal{M})) \xrightarrow{p_3^*(\alpha)} p_3^*(p^*(\mathcal{M}))
\end{array}$$

Since $\nu^*(\mathcal{O}_X) \simeq \mathcal{O}_{B\times X}$ and $p^*(\mathcal{O}_X) \simeq \mathcal{O}_{B\times X}$, we get, that $\mathcal{O}_X$ is B-equivariant. It follows by Proposition 4.4 in [5], that if $Z, Y \subset X$ are closed and B-invariant and $\mathcal{M} \in \mathcal{D}_X - mod$ is B-equivariant, then $\mathcal{H}^j_{Z\cap Y^c}(\mathcal{M})$

is also B-equivariant $\forall j$. It follows by (3.5), that if $\mathcal{M} \in \mathcal{D}_X - mod$ is B-equivariant, then

$$\phi_b^*(\mathcal{M}) \simeq \mathcal{M} \ \forall b \in B.$$

Since ϕ_b is an isomorphism, we get

$$\mathcal{M}(b.U) \simeq \mathcal{M}(U), \ \forall U \subset X \text{ open}, \ \forall b \in B$$

and therefore the following Lemma.

Lemma 3.2.8. *Assume X has a B-action and $\mathcal{M} \in \mathcal{D}_X - mod$ is B-equivariant. Then* $\mathrm{Supp}(\mathcal{M})$ *is a union of B-orbits.*

We shall also need to consider the holonomic $\mathcal{D}_X$-modules. This is defined for $\mathrm{char}(k) = 0$, and this will therefore be the case. This is a sub-category of $\mathcal{D}_X - mod$. The main observation in this case is that if one defines

$$\mathrm{gr}(\mathcal{D}_X) := \bigoplus_{m=0}^{\infty} \mathcal{D}if_m(\mathcal{O}_{\mathcal{X}})/\mathcal{D}if_{m-1}(\mathcal{O}_X)$$

then locally this is a commutative, noetherian ring, and one defines the holonomic $\mathcal{D}_X$-modules as those $\mathcal{D}_X$-modules with minimal growth. For $\mathcal{M} \in \mathcal{D}_X - mod$ we say $\mathcal{M}$ is coherent if $\forall x \in X \ \exists U \subset X$ open with $x \in U$, such that there exists an exact sequence in $\mathcal{D}_U - mod$

$$\mathcal{D}_U^q \to \mathcal{D}_U^p \to \mathcal{M}|_U \to 0.$$

According to section VI 1.12 in [7] we get.

Definition 3.2.9. *Let $\mathcal{M} \in \mathcal{D}_X - mod$ be coherent. Then $\mathcal{M}$ is holonomic if $\mathcal{M} = 0$ or $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \neq 0 \Leftrightarrow i = \dim(X)$.*

By the same reference we get, that this subcategory of $\mathcal{D}_X - mod$ is closed with respect to inclusion, quotients and extensions and if $\mathcal{M} \in \mathcal{D}_X - mod$ is holonomic, it has finite length and $\mathcal{O}_X$ is holonomic. It follows by Theorem 1.3 and Theorem 1.4 in [30], that for $\mathcal{M} \in \mathcal{D}_X - mod$ holonomic and $Z \subset X$ locally closed $\mathcal{H}^j_Z(\mathcal{M})$ is also holonomic $\forall j$. If $\mathrm{char}(k) > 0$ one replaces the concept of holonomic modules with F-finite. This is explained in section 5.3.

Given $\mathcal{M} \in \mathcal{D}_X - mod$ we shall denote $\mathcal{D}_X \otimes_{\mathcal{D}_X(X)} \mathcal{M}(X)$ as the sheaf associated to the presheaf

$$U \to \mathcal{D}_X(U) \otimes_{\mathcal{D}_X(X)} \mathcal{M}(X).$$

We say $\mathcal{M}$ is generated by its global sections if the sheaf homomorphism

$$\mathcal{D}_X \otimes_{\mathcal{D}_X(X)} \mathcal{M}(X) \to \mathcal{M}$$

is surjective.

Definition 3.2.10. *X is $\mathcal{D}_X$-affine if $\forall \mathcal{M} \in \mathcal{D}_X - mod$*

$$(1): \ \mathcal{M} \text{ is generated by its global sections.}$$
$$(2): \ H^i(X, \mathcal{M}) = 0 \ \forall i > 0.$$

In [1] the following Theorem is proved. It will be refered to as the Beilinson-Bernstein equivalence or just Beilinson-Bernstein.

Theorem 3.2.11. *Let X be $\mathcal{D}_X$-affine. Then the global section functor*

$$-(X) : \mathcal{D}_X - mod \to \mathcal{D}_X(X) - mod$$

is an equivalence with inverse given as

$$\mathcal{D}_X \otimes_{\mathcal{D}(X)} - : \mathcal{D}_X(X) - mod \to \mathcal{D}_X - mod.$$

If X is affine then X is also $\mathcal{D}_X$-affine according to Theorem 3.7 section 3 in [25]. It is proved in [1], that G/P is $\mathcal{D}_{G/P}$-affine, if G is a semisimple simply connected linear algebraic group defined over a field of characteristic zero and $P \subset G$ is a parabolic subgroup. If the charateristic was greater than zero, then it is proved in [31], that $\mathcal{D}_{G/P}$-affinity fails.

3.3 Kashiwaras equivalence

Let $i : Y \to X$ be a closed immersion of irreducible varieties such that both X and Y are smooth. Before we move on let us examine some relations between the functors i_* and i^{-1}. Let $\mathcal{N}$ be a sheaf on Y. Then there is a natural map

$$\phi : i^{-1} i_*(\mathcal{N}) \to \mathcal{N}.$$

Since $i^{-1} i_*(\mathcal{N})$ is the sheaf associated to the presheaf, which for $U \subset Y$ open is

$$\varinjlim_{i(U) \subset V,\, V \subset X \text{ open}} i_*(\mathcal{N})(V) = \varinjlim_{U \subset V,\, V \subset X \text{ open}} \mathcal{N}(V \cap Y) \qquad (3.6)$$

and since $U \subset V \cap Y$ we can make the restriction equal to ϕ. We also have a natural map for a sheaf $\mathcal{M}$ on X

$$\pi : \mathcal{M} \to i_* i^{-1}(\mathcal{M})$$

defined since $i_*i^{-1}(\mathcal{M})$ is the sheaf associated to the presheaf, which for $U \subset X$ open is

$$\varinjlim_{i(U\cap Y)\subset V,\ V\subset X \text{ open}} \mathcal{M}(V) = \varinjlim_{U\cap Y\subset V,\ V\subset X \text{ open}} \mathcal{M}(V)$$

and by setting $V = U$ above we see, that there is a natural map.

Lemma 3.3.1. *Let $\mathcal{M}$ be a sheaf on X, whose support is contained in Y and $\mathcal{N}$ be a sheaf on Y. Then*

$$\pi(\mathcal{M}) = \mathcal{M},\ \phi(\mathcal{N}) = \mathcal{N}.$$

Proof. We start out with the last fact. Let $U \subset Y$ be open then $\exists W \subset X$ open with $W \cap Y = U$. By setting $V = W$ in (3.6) the last fact follows follows since $U \subset V \Rightarrow U \cap Y \subset V \cap Y$. To prove the first part it follows, all there is to prove, is for given $U \subset X$ open $\forall V \subset X$ open and $U \cap Y \subset V \subset U$ the restriction map

$$\mathcal{M}(U) \to \mathcal{M}(V)$$

is an isomorphism. Since there is an exact sequence

$$0 \to \mathcal{H}^0_{V^c}(\mathcal{M}) \to \mathcal{M} \to \mathcal{H}^0_V(\mathcal{M}) \to \mathcal{H}^1_{V^c}(\mathcal{M}) \to 0$$

we get an exact sequence

$$0 \to \mathcal{H}^0_{V^c\cap U}(\mathcal{M}|_U) \to \mathcal{M}|_U \to \mathcal{H}^0_{V\cap U}(\mathcal{M}|_U) \to \mathcal{H}^1_{V^c\cap U}(\mathcal{M}|_U) \to 0.$$

So we just have to prove, that

$$0 = \mathcal{H}^0_{V^c\cap U}(\mathcal{M}|_U) = \mathcal{H}^1_{V^c\cap U}(\mathcal{M}|_U).$$

Since

$$x \in V^c \cap U \Rightarrow\ x \in V^c \wedge\ x \in U \Rightarrow x \notin U \cap Y \wedge\ x \in U \Rightarrow x \in Y^c \cap U \Rightarrow V^c \cap U \subset Y^c \cap U$$

and $\mathrm{Supp}(\mathcal{M}|_U) \subset U \cap Y$ the result follows. □

Let $\mathcal{D}_X(Y) - mod$ be the subcategory of $\mathcal{D}_X - mod$ and $mod - \mathcal{D}_X(Y)$ the subcategory of $mod - \mathcal{D}_X$ whose elements have support contained in Y.

Theorem 3.3.2. *The following functor called Kashiwaras functor is an equivalence of functors from* $mod - \mathcal{D}_X(Y)$ *to* $mod - \mathcal{D}_Y$

$$i^{-1}(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), -)) : mod - \mathcal{D}_X(Y) \to mod - \mathcal{D}_Y$$

and therefore the functor

$$\mathcal{H}om_{\mathcal{O}_Y}(\omega_Y, i^{-1}(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), - \bigotimes_{\mathcal{O}_X} \omega_X))) : \mathcal{D}_X(Y) - mod \to \mathcal{D}_Y - mod$$

is also an equivalence.

This Theorem is called Kashiwaras equivalence. For a proof look up [3] Theorem 2.6.18 if $\mathrm{char}(k) = 0$ and Corollary 8.11 in [21] if $\mathrm{char}(k) > 0$. We are interested in $\mathrm{char}(k) = 0$, and this will be the case in the rest of this section. It is proved as Proposition 2.6.21 in [3], that $\mathcal{H}_Y^{\mathrm{codim}(Y)}(\mathcal{O}_X)$ is mapped to $\mathcal{O}_Y$. We shall generalize this result and prove that for $Z \subset Y$ locally closed $\mathcal{H}_Z^j(\mathcal{O}_X)$ is mapped to $\mathcal{H}_Z^{j-\mathrm{codim}(Y)}(\mathcal{O}_Y)$. We let $\mathcal{I}_Y$ denote the ideal sheaf of Y in X. Remember the definition of Θ_X in (3.4) in section (3.2). We then define $\Theta_{X|Y}$ as the subsheaf of Θ_X such that $\pi \in \Theta_{X|Y}$ if and only if $\pi(\mathcal{I}_Y) \subset \mathcal{I}_Y$. We set $\mathcal{D}_{X|Y}$ equal to the subring of $\mathcal{D}_X$ generated by $\mathcal{O}_X$ and $\Theta_{X|Y}$ and denote

$$\mathcal{S} := \mathcal{I}_Y \mathcal{D}_X \cap \mathcal{D}_{X|Y}.$$

Clearly $\mathcal{S}$ is a right ideal in $\mathcal{D}_{X|Y}$. That it is also a left ideal follows since for $\delta \in \Theta_{X|Y}$, $g \in \mathcal{I}_Y$ and $d \in \mathcal{D}_X$

$$\delta g d = g \delta d - [g, \delta] d =_{\text{see (3.4)}} g \delta d + \delta(g) d$$

and then it follows since $\delta(g) \in \mathcal{I}_Y$, and therefore $\mathcal{S}$ is a two-sided ideal in $\mathcal{D}_{X|Y}$ and we let

$$\mathcal{R} := \mathcal{D}_{X|Y} / \mathcal{S}$$

denote the sheaf of rings on X. Before we proceed, we need the following Proposition.

Propositon 3.3.3. *There is an isomorphism of rings on* Y

$$i^{-1}(\mathcal{R}) \simeq \mathcal{D}_Y$$

and for $\mathcal{M} \in mod - \mathcal{D}_X$ $\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{M}) \in mod - \mathcal{R}$ *and there is an isomorphism in* $mod - \mathcal{R}$

$$\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}_Y^{\mathrm{codim(Y)}}(\omega_X)) \simeq i_*(\omega_Y).$$

Proof. The first two facts are proved in [3] as Proposition 2.6.5 and Lemma 2.6.10. It follows by Proposition 2.6.21 in [3], that there is an isomorphism in $\mathcal{D}_Y - mod$

$$\mathcal{H}om_{\mathcal{O}_Y}(\omega_Y, i^{-1}(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}_Y^{\text{codim(Y)}}(\omega_X)))) \simeq \mathcal{O}_Y.$$

By tensoring with $\otimes_{\mathcal{O}_Y}\omega_Y$ it follows by Proposition 3.2.5, that there is an isomorphism in $mod - \mathcal{D}_Y$

$$i^{-1}(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}_Y^{\text{codim(Y)}}(\omega_X))) \simeq \omega_Y.$$

By taking the functor i_*, we get an isomorphism in $mod - i_*(\mathcal{D}_Y)$

$$i_*(i^{-1}(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}_Y^{\text{codim(Y)}}(\omega_X)))) \simeq i_*(\omega_Y)$$

and since $\text{Supp}(\mathcal{R}) \subset Y$ and $\text{Supp}(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}_Y^{\text{codim(Y)}}(\omega_X))) \subset Y$ it follows by Lemma 3.3.1 and the already proved in this Proposition, that there is an isomorphism of $mod - \mathcal{R}$

$$\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}_Y^{\text{codim(Y)}}(\omega_X)) \simeq i_*(\omega_Y)$$

and thus the Proposition. □

The Proposition above gives us the reason for given $\mathcal{M} \in mod - \mathcal{D}_X$, that $i^{-1}(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{M})) \in mod - \mathcal{D}_Y$. We need to know the relation between local cohomology on Y and local cohomology on X.

Lemma 3.3.4. *Let $Z \subset Y$ be locally closed and $\mathcal{F} \in \mathcal{O}_X$-mod, we then have the following isomorphism*

$$\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}_Z^0(\mathcal{F})) \simeq \mathcal{H}_Z^0(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{F})).$$

If both sides are left or right $\mathcal{R}$-modules the isomorphism is $\mathcal{R}$-linear.

Proof. Let us start out by assuming Z is an arbitrary closed subset of X, and show that

$$\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}_Z^0(\mathcal{F})) = \mathcal{H}_Z^0(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{F})).$$

Let $U \subset X$ be open and

$$\begin{aligned}
&\pi \in \mathcal{H}_Z^0(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{F}))(U) \Leftrightarrow \\
&\pi : i_*(\mathcal{O}_Y)|_U \to \mathcal{F}|_U \wedge \ \pi|_{Z^c} = 0 \Leftrightarrow \\
&\pi : i_*(\mathcal{O}_Y)|_U \to \mathcal{F}|_U \wedge \ \text{im}(\pi) \subset \mathcal{H}_Z^0(\mathcal{F})|_U \Leftrightarrow \\
&\pi \in \mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}_Z^0(\mathcal{F}))(U).
\end{aligned}$$

Let now $Z \subset Y$ be locally closed. Then $Z = V \cap W^c$ with $V, W \subset X$ open. Let $\mathcal{G} \in \mathcal{O}_X - mod$. Then we have an exact sequence in $\mathcal{O}_X - mod$ for arbitrary $\mathcal{G} \in \mathcal{O}_X - mod$

$$0 \to \mathcal{H}^0_{V^c}(\mathcal{G}) \to \mathcal{G} \xrightarrow{|_V} \mathcal{H}^0_V(\mathcal{G}) \to \mathcal{H}^1_{V^c}(\mathcal{G}) \to 0$$

with $|_V$ the restriction to V. By considering the spectral sequence

$$\begin{aligned} &\mathcal{H}^i_{V^c}(\mathcal{H}^j_Z(\mathcal{G})) \Rightarrow \mathcal{H}^{i+j}_{V^c \cap Z}(\mathcal{G}) =_{V^c \cap Z = \emptyset} 0 \Rightarrow \\ &0 = \mathcal{H}^0_{V^c}(\mathcal{H}^0_Z(\mathcal{G})) = \mathcal{H}^1_{V^c}(\mathcal{H}^0_Z(\mathcal{G})) \end{aligned}$$

and therefore if we let $j : V \to X$ denote the inclusion it follows since $\mathcal{H}^0_V(\mathcal{G}) = j_*(\mathcal{G}|_V)$ that

$$\mathcal{H}^0_Z(\mathcal{G}) \simeq j_*(\mathcal{H}^0_Z(\mathcal{G})|_V) = j_*(\mathcal{H}^0_Z(\mathcal{G}|_V))$$

where $|_V$ the isomorphism and therefore

$$\begin{aligned} &\mathcal{H}^0_Z(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{F})) \simeq j_*(\mathcal{H}^0_Z(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{F})|_V)) = \\ &j_*(\mathcal{H}^0_Z(\mathcal{H}om_{\mathcal{O}_X|_V}(i_*(\mathcal{O}_Y)|_V, \mathcal{F}|_V))) = \\ &j_*(\mathcal{H}^0_Z(\mathcal{H}om_{\mathcal{O}_X|_V}((i|_{V\cap Y})_*(\mathcal{O}_{Y\cap V}), \mathcal{F}|_V))) =_{Z \subset V \text{ closed}} \\ &j_*(\mathcal{H}om_{\mathcal{O}_X|_V}((i|_{V\cap Y})_*(\mathcal{O}_{Y\cap V}), \mathcal{H}^0_Z(\mathcal{F}|_V))) = \\ &j_*(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^0_Z(\mathcal{F}))|_V) = \mathcal{H}^0_V(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^0_Z(\mathcal{F}))) \end{aligned}$$

where the last equality is due to the fact, that $V \subset X$ is open. So we have shown, that

$$\mathcal{H}^0_Z(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{F})) \xrightarrow{|_V} \mathcal{H}^0_V(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^0_Z(\mathcal{F})))$$

is an isomorphism. So if we can show that

$$0 = \mathcal{H}^0_{V^c}(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^0_Z(\mathcal{F}))) = \mathcal{H}^1_{V^c}(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^0_Z(\mathcal{F})))$$

we are done, since in this case

$$\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^0_Z(\mathcal{F})) \xrightarrow{|_V} \mathcal{H}^0_V(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^0_Z(\mathcal{F})))$$

is an isomorphism, and therefore

$$\mathcal{H}^0_Z(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{F})) \simeq \mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^0_Z(\mathcal{F}))$$

and if they are both modules over some sheaf of rings, the isomorphism is linear. Since V^c is closed we get by the already proved that

$$\mathcal{H}^0_{V^c}(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y),\mathcal{H}^0_Z(\mathcal{F}))) = \mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y),\mathcal{H}^0_{V^c}(\mathcal{H}^0_Z(\mathcal{F}))) = 0.$$

According to [23] Lemma 2.9 $\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y),-)$ takes injective $\mathcal{O}_X$-modules to flasque sheaves and due to [25] Lemma 2.4 in chapter 3 injective sheaves in $\mathcal{O}_X - mod$ are flasque and [23] Lemma 1.6 $\mathcal{H}^0_{V^c}(-)$ carries flasque sheaves to flasque sheaves and

$$\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y),\mathcal{H}^0_{V^c}(-)) = \mathcal{H}^0_{V^c}(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y),-))$$

we know that the two spectral sequences converge to the same object $E^{p+q}(-)$

$$\begin{array}{l}\mathcal{H}^q_{V^c}(\mathcal{E}xt^p_{\mathcal{O}_X}(i_*(\mathcal{O}_Y),-))\\ \mathcal{E}xt^p_{\mathcal{O}_X}(i_*(\mathcal{O}_Y),\mathcal{H}^q_{V^c}(-)).\end{array}$$

Since

$$\begin{array}{l}0 = \mathcal{E}xt^0_{\mathcal{O}_X}(i_*(\mathcal{O}_Y),\mathcal{H}^1_{V^c}(\mathcal{H}^0_Z(\mathcal{F}))) = \mathcal{E}xt^1_{\mathcal{O}_X}(i_*(\mathcal{O}_Y),\mathcal{H}^0_{V^c}(\mathcal{H}^0_Z(\mathcal{F}))) \Rightarrow\\ 0 = E^1(\mathcal{H}^0_Z(\mathcal{F})) \Rightarrow 0 = \mathcal{H}^1_{V^c}(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y),\mathcal{H}^0_Z(\mathcal{F}))).\end{array}$$

□

We use this to prove the following Corollary.

Corollary 3.3.5. *There is an isomorphism in* $mod - \mathcal{R}$

$$\mathcal{H}^j_Y(\mathcal{E}xt^r_{\mathcal{O}_X}(i_*(\mathcal{O}_Y),\omega_X)) \simeq \left\{ \begin{array}{cc} 0 & j \neq 0 \vee \ r \neq \text{codim(Y)} \\ i_*(\omega_Y) & j = 0 \wedge r = \text{codim}(Y) \end{array} \right\}$$

Proof. Since $\text{Supp}(i_*(\mathcal{O}_Y)) \subset Y$ we get $\text{Supp}(\mathcal{E}xt^r_{\mathcal{O}_X}(i_*(\mathcal{O}_Y),\omega_X))) \subset Y$ and therefore, we get zero for $j \neq 0$. By using the same spectral sequence argument used in the end of the proof of Lemma 3.3.4 we get, that the two spectral sequences

$$\begin{array}{l}\mathcal{H}^q_Y(\mathcal{E}xt^p_{\mathcal{O}_X}(i_*(\mathcal{O}_Y),\omega_X))\\ \mathcal{E}xt^p_{\mathcal{O}_X}(i_*(\mathcal{O}_Y),\mathcal{H}^q_Y(\omega_X))\end{array}$$

converges to the same object $E^{p+q}(\omega_X)$ and we have

$$E^p(\omega_X) = \mathcal{E}xt^p_{\mathcal{O}_X}(i_*(\mathcal{O}_Y),\omega_X).$$

Since

$$i^{-1}(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y),-)) : mod - \mathcal{D}_X(Y) \to mod - \mathcal{D}_Y$$

is an equivalence and due to 1.2.15 of [3] $\omega_X \in mod - \mathcal{D}_X$ and therefore $\mathcal{H}^j_Y(\omega_X) \in mod - \mathcal{D}_X(Y)$, we get since i^{-1} is exact for $p \neq 0$

$$\mathcal{E}xt^p_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^q_Y(\omega_X)) = 0$$

and therefore there is according to Lemma 3.3.4 an isomorphism in $mod - \mathcal{R}$

$$\mathcal{E}xt^p_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \omega_X) \simeq \mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^p_Y(\omega_X))$$

and now the Corollary follows by combining Proposition 3.3.3, Lemma 2.2.8 and Proposition 2.2.7 since

$$\mathcal{H}^q_Y(\omega_X) \simeq \mathcal{H}^q_Y(\mathcal{O}_X) \bigotimes_{\mathcal{O}_X} \omega_X.$$

□

We are now ready to prove the main Proposition of this section, which tells what happens with local cohomology under Kashiwaras functor.

Propositon 3.3.6. *Let $Z \subset Y$ be locally closed, we then have the following isomorphism in $\mathcal{D}_Y - mod$ $\forall j \geq \operatorname{codim}(Y)$*

$$\mathcal{H}om_{\mathcal{O}_Y}(\omega_Y, i^{-1}(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^j_Z(\mathcal{O}_X) \bigotimes_{\mathcal{O}_X} \omega_X))) \simeq \mathcal{H}^{j-\operatorname{codim}(Y)}_Z(\mathcal{O}_Y).$$

Especially $\mathcal{H}^{j-\operatorname{codim}(Y)}_Z(\mathcal{O}_Y) = 0 \Leftrightarrow \mathcal{H}^j_Z(\mathcal{O}_X) = 0 \ \forall j \geq \operatorname{codim}(Y)$

Proof. After tensoring both sides with $\otimes_{\mathcal{O}_Y}\omega_Y$ it follows by Proposition 3.2.5, that we just have to prove there is an isomorphism in $mod - \mathcal{D}_Y$ $\forall j \geq \operatorname{codim}(Y)$

$$i^{-1}(\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^j_Z(\mathcal{O}_X) \bigotimes_{\mathcal{O}_X} \omega_X)) \simeq \mathcal{H}^{j-\operatorname{codim}(Y)}_Z(\omega_Y).$$

To do this it follows by Proposition 3.3.3 and Lemma 3.3.1, that we just have to prove, there is an isomorphism in $mod - \mathcal{R}$ $\forall j \geq \operatorname{codim}(Y)$

$$\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^j_Z(\mathcal{O}_X) \bigotimes_{\mathcal{O}_X} \omega_X) \simeq i_*(\mathcal{H}^{j-\operatorname{codim}(Y)}_Z(\omega_Y)).$$

Let us start out by proving, that there is an isomorphism in $mod - \mathcal{R}$

$$\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^j_Z(\mathcal{O}_X) \bigotimes_{\mathcal{O}_X} \omega_X) \simeq \mathcal{H}_Z^{j-\mathrm{codim}(Y)}(i_*(\omega_Y)) \ \forall j \geq \mathrm{codim}(Y) \quad (3.7)$$

Since $\mathcal{H}^j_Z(\mathcal{O}_X) \in \mathcal{D}_X(Y) - mod$ we get just as in the proof of Corollary 3.3.5, that

$$\mathcal{E}xt^p_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^j_Z(\mathcal{O}_X) \bigotimes_{\mathcal{O}_X} \omega_X) = \left\{ \begin{array}{cc} 0 & p \neq 0 \\ \mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^j_Z(\mathcal{O}_X) \bigotimes_{\mathcal{O}_X} \omega_X) & p = 0 \end{array} \right\}$$

Thanks to Lemma 3.3.4, we know, that the two spectral sequences converges to the same object $E^{p+q}(\mathcal{O}_X)$

$$\mathcal{H}^q_Z(\mathcal{E}xt^p_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{O}_X \bigotimes_{\mathcal{O}_X} \omega_X))$$
$$\mathcal{E}xt^p_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^q_Z(\mathcal{O}_X \bigotimes_{\mathcal{O}_X} \omega_X)).$$

According to Corollary 3.3.5 we get isomorphisms in $mod - \mathcal{R}$

$$\mathcal{H}^q_Z(\mathcal{E}xt^p_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{O}_X \bigotimes_{\mathcal{O}_X} \omega_X)) \simeq \left\{ \begin{array}{cc} 0 & p \neq \mathrm{codim}(Y) \\ \mathcal{H}^q_Z(i_*(\omega_Y)) & p = \mathrm{codim}(Y) \end{array} \right\}$$

and we have therefore proved (3.7). Since $Y \subset X$ is closed and i is the inclusion, it is affine and therefore for arbitrary sheaf $\mathcal{F}$

$$R^p i_*(\mathcal{F}) \simeq \left\{ \begin{array}{cc} 0 & p \neq 0 \\ i_*(\mathcal{F}) & p = 0 \end{array} \right\}$$

Z is locally closed and therefore $Z = W \cap V^c$ with $V, W \subset X$ open and since $Z \subset Y$ $Z = Z \cap Y = W \cap V^c \cap Y$. Then for $\mathcal{F} \in \mathcal{O}_Y - mod$ and $U \subset X$ open

$$\begin{aligned} &(\mathcal{H}^0_Z \circ i_*)(\mathcal{F})(U) = \{\phi \in i_*(\mathcal{F})(U \cap W) |\ \phi|_{U \cap W \cap V} = 0\} = \\ &\{\phi \in \mathcal{F}(U \cap W \cap Y) |\ \phi|_{U \cap W \cap V \cap Y} = 0\} = \\ &\mathcal{H}^0_Z(\mathcal{F})(U \cap Y) = (i_* \circ \mathcal{H}^0_Z)(\mathcal{F})(U) \end{aligned}$$

with the $\mathcal{H}^0_Z$ in the last line considered in $\mathcal{O}_Y - mod$. This implies that the two spectral sequences

$$\mathcal{H}^p_Z(R^q i_*(\mathcal{F}))$$
$$R^p i_*(\mathcal{H}^p_Z(\mathcal{F}))$$

converges to the same object for arbitrary $\mathcal{F} \in \mathcal{O}_Y - mod$, and there is an isomorphism in $mod - \mathcal{R}$

$$\mathcal{H}^p_Z(i_*(\omega_Y)) \simeq i_*(\mathcal{H}^p_Z(\omega_Y))$$

and also an isomorphism in $mod - \mathcal{R}$

$$\mathcal{H}om_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{H}^j_Z(\mathcal{O}_X) \bigotimes_{\mathcal{O}_X} \omega_X) \simeq i_*(\mathcal{H}^{j-\mathrm{codim}(Y)}_Z(\omega_Y)) \; \forall j \geq \mathrm{codim}(Y)$$

and the first part of the Proposition is true. The last part follows by Theorem 3.3.2. □

For $Z \subset Y$ we use the notation $\mathrm{codim}_Y(Z)$ to denote the codimension of Z in Y. We then get the following Corollaries, which will be crucial in the next section.

Corollary 3.3.7. *Let $Z \subset Y$ be closed and irreducible. If* $\mathrm{codim}_Y(Z) = 1$ $\mathcal{H}^{\mathrm{codim}(Z)+j}_Z(\mathcal{O}_X) = 0 \; \forall j > 0$.

Proof. According to Proposition 3.3.6 all there is to prove is that

$$\mathcal{H}^{\mathrm{codim}_Y(Z)+j}_Z(\mathcal{O}_Y) = 0 \; \forall j > 0.$$

Let $\mathcal{I}$ be the ideal sheaf of $Z \subset Y$. By Proposition 2.2.6 we have, that

$$(\mathcal{H}^{\mathrm{codim}_Y(Z)+j}_Z(\mathcal{O}_Y))_y = H^{\mathrm{codim}_Y(Z)+j}_{\mathcal{I}_y}(\mathcal{O}_{Y,y}) \quad \forall y \in Y.$$

Since $\mathrm{Supp}(\mathcal{H}^{\mathrm{codim}_Y(Z)+j}_Z(\mathcal{O}_Y)) \subset Z$, we may take $y \in Z$. If $\mathrm{codim}_Y(Z) = 1$ we get by Theorem 1 page 91 in [41] that $\mathcal{I}_y$ is generated by one element and it then follows by the Cěch complex. □

Corollary 3.3.8. *Let $Z_1, Z_2 \subset Y$ be closed, irreducible and smooth. Then*

$$\mathcal{H}^{\mathrm{codim}(Y)+\mathrm{codim}_Y(Z_1)+\mathrm{codim}_Y(Z_2)+j}_{Z_1 \cap Z_2}(\mathcal{O}_X) = 0, \; \forall j > 0$$

Proof. According to Proposition 3.3.6 all there is to prove is that

$$\mathcal{H}^{\mathrm{codim}_Y(Z_1)+\mathrm{codim}_Y(Z_2)+j}_{Z_1 \cap Z_2}(\mathcal{O}_Y) = 0 \; \forall j > 0.$$

Let $\mathcal{I}_i$ be the ideal sheaf of $Z_i \subset Y$. By Proposition 2.2.6 we have since the ideal sheaf of $Z_1 \cap Z_2$ is $\mathcal{I}_1 + \mathcal{I}_2$, that

$$(\mathcal{H}^{\mathrm{codim}_Y(Z_1)+\mathrm{codim}_Y(Z_2)+j}_{Z_1 \cap Z_2}(\mathcal{O}_Y))_y = H^{\mathrm{codim}_Y(Z_1)+\mathrm{codim}_Y(Z_2)+j}_{(\mathcal{I}_1)_y+(\mathcal{I}_2)_y}(\mathcal{O}_{Y,y}) \quad \forall y \in Y.$$

So let us pick $y \in Z_1 \cap Z_2$, then the result follows due to the Cěch complex, since Z_i locally is generated by $\mathrm{codim}_Y(Z_i)$ elements due to [25] Theorem 8.17 chapter 2. □

4 G/B IN CHARACTERISTIC ZERO

4.1 The Grothendieck-Cousin complex on G/B

Let G be a semisimple simply connected linear algebraic group defined over a field k of arbitrary characteristic, $B \subset G$ a Borel subgroup, $T \subset G$ a torus contained in B, $W = N_G(T)/T$ the Weyl group and let $X = G/B$. Then there is a left B-action on X. It then follows by [43], that there are finitely many B orbits $C(w)$, and these are parametrized by W. Let $X(w)$ denote the closure of $C(w)$ in X, and these are the Schubert varieties. The Weyl group W is generated by the the simple reflections $s_1, \ldots, s_n$ numbered from left to right in the Dynkin diagram. Let also $l(w)$ denote the length of $w \in W$ and $c_w = \operatorname{codim}(X(w)) = \operatorname{codim}(C(w))$. On W there is the Bruhat order. For $v, w \in W$ we have

$$v \leq w \Leftrightarrow X(v) \subset X(w).$$

We let

$$X_i := \bigcup_{l(w)=\dim(X)-i} X(w).$$

Since X is a smooth variety it is also Cohen-Macaulay, the inclusion morphism $X_i \cap (X_{i+1})^c \to X$ is affine, $\operatorname{codim}(X_i) \geq i$ according to [33] Lemma 12.2 , we get due to Theorem 2.2.9 a resolution of $\mathcal{O}_X$

$$\mathcal{H}^0_{X_0 \cap (X_1)^c}(\mathcal{O}_X) \to \mathcal{H}^1_{X_1 \cap (X_2)^c}(\mathcal{O}_X) \to \ldots . \tag{4.1}$$

Since

$$X_r \cap (X_{r+1})^c = \bigcup_{l(w)=\dim(X)-r} C(w)$$
$$C(w) \cap C(v) \neq \emptyset \Leftrightarrow \; v = w$$

it follows by Lemma 2.2.5, that

$$\mathcal{H}^i_{X_r \cap (X_{r+1})^c}(\mathcal{O}_X) = \bigoplus_{l(w)=\dim(X)-r} \mathcal{H}^i_{C(w)}(\mathcal{O}_X).$$

According to Theorem 10.5 in [33], we therefore get since $c_w + \dim(C(w)) = c_w + l(w) = \dim(X)$

$$\mathcal{H}^i_{X_r \cap (X_{r+1})^c}(\mathcal{O}_X) \neq 0 \Leftrightarrow \; i = r \Rightarrow$$
$$\mathcal{H}^i_{C(w)}(\mathcal{O}_X) \neq 0 \Leftrightarrow \; i = c_w$$

and therefore given $Z \subset X$ locally closed we get, that the spectral sequence

$$\mathcal{H}^i_Z(\mathcal{H}^j_{C(w)}(\mathcal{O}_X)) \Rightarrow \mathcal{H}^{i+j}_{C(w)\cap Z}(\mathcal{O}_X)$$

degenerates and thus

$$\mathcal{H}^i_Z(\mathcal{H}^{c_w}_{C(w)}(\mathcal{O}_X)) = \mathcal{H}^{i+c_w}_{C(w)\cap Z}(\mathcal{O}_X).$$

Let $v_1, \ldots, v_m \in W$ such that

$$Z = \cup_{i=1}^m C(v_i)$$

is locally closed. We get, that since either $C(w) \subset Z$ or $C(w) \cap Z = \emptyset$, either way that

$$\mathcal{H}^i_Z(\mathcal{H}^{c_w}_{C(w)}(\mathcal{O}_X)) = 0 \; i > 0$$

and therefore the resolution (4.1) is acyclic for the functor $\underline{\Gamma}_Z$ and furthermore since $c_w + \dim(C(w)) = c_w + l(v) = \dim(X)$

$$\underline{\Gamma}_Z(\mathcal{H}^i_{X_i \cap (X_{i+1})^c}(\mathcal{O}_X)) = \bigoplus_{c_w=i,\; C(w)\subset Z} \mathcal{H}^i_{C(w)}(\mathcal{O}_X).$$

Since $\mathcal{H}^i_{C(w)}(\mathcal{O}_X) \neq 0 \Leftrightarrow \; i = c_w$ we get the complex

$$0 \to \bigoplus_{i=1}^m \mathcal{H}^0_{C(v_i)}(\mathcal{O}_X) \to \bigoplus_{i=1}^m \mathcal{H}^1_{C(v_i)}(\mathcal{O}_X) \to \ldots \tag{4.2}$$

whose i'th cohomology is $\mathcal{H}^i_Z(\mathcal{O}_X)$ and the complex ends, when the upper index is equal to $\dim(X)$. As a special case we could set

$$Z = X(w) = \bigcup_{v \leq w} C(v).$$

In this case the complex above would look like

$$0 \to \mathcal{H}^{c_w}_{C(w)}(\mathcal{O}_X) \to \bigoplus_{v\leq w,\; l(v)=l(w)-1} \mathcal{H}^{c_w+1}_{C(v)}(\mathcal{O}_X) \to \ldots \tag{4.3}$$

All these results are explained in [31]. Let $K(X)$ be equal to the quotient of the free abelian group generated by all sheaves in $\mathcal{D}_X - mod$ by the subgroup generated by all expressions of the form $\mathcal{F} - \mathcal{F}_1 - \mathcal{F}_2$, whenever there is an exact sequence in $\mathcal{D}_X - mod$

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0.$$

Let $[\mathcal{F}]$ denote the image of $\mathcal{F}$ in $K(X)$. Let $\mathcal{F}_i \in \mathcal{D}_X - mod$ and assume we have a complex in $\mathcal{D}_X - mod$

$$\mathcal{F} := 0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0.$$

Let $\mathcal{H}^i(\mathcal{F}) :=$ i'th cohomology of the complex above.

Propositon 4.1.1. *With the notation above we get*

$$[\mathcal{H}^0(\mathcal{F})] = \sum_{i=0}^{n}(-1)^i[\mathcal{F}_i] + \sum_{i=1}^{n}(-1)^{i-1}[\mathcal{H}^i(\mathcal{F})].$$

Proof. We let

$$\phi_i : \mathcal{F}_i \to \mathcal{F}_{i+1}$$

denote the morphisms. The proof is an induction argument in n. If $n = 1$ we shall consider the complex

$$0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to 0.$$

Since we have two exact sequences in $\mathcal{D}_X - mod$

$$0 \to \mathcal{H}^0(\mathcal{F}) \to \mathcal{F}_0 \to \mathcal{F}_0/\mathcal{H}^0(\mathcal{F}) \to 0$$
$$0 \to \mathrm{im}(\phi_0) \to \ker(\phi_1) \to \mathcal{H}^1(\mathcal{F}) \to 0$$

and since $[\mathcal{F}_0/\mathcal{H}^0(\mathcal{F})] = [\mathrm{im}(\phi_0)]$ and $[\ker(\phi_1)] = [\mathcal{F}_1]$ the Proposition follows. Let now n be arbitrary. Since $\ker(\phi_0) = \mathcal{H}^0(\mathcal{F})$ we have an exact sequence

$$0 \to \mathcal{H}^0(\mathcal{F}) \to \mathcal{F}_0 \to \mathcal{F}_0/\mathcal{H}^0(\mathcal{F}) \to 0.$$

Since we also have an exact sequence

$$0 \to \mathrm{im}(\phi_0) \to \ker(\phi_1) \to \mathcal{H}^1(\mathcal{F}) \to 0$$

we get since $[\mathcal{F}_0/\mathcal{H}^0(\mathcal{F})] = [\mathrm{im}(\phi_0)]$ that $[\mathcal{H}^0(\mathcal{F})] = [\mathcal{F}_0] - [\mathrm{im}(\phi_0)] = [\mathcal{F}_0] + [\mathcal{H}^1(\mathcal{F})] - [\ker(\phi_1)]$. By considering the complex

$$\mathcal{G} := 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n \to 0$$

we get by induction on n, that $[\ker(\phi_1)] = \sum_{i=1}^{n}(-1)^{i+1}[\mathcal{F}_i] + \sum_{i=2}^{n}(-1)^i[\mathcal{H}^i(\mathcal{F})]$ and thus the Proposition. □

That the morphisms in the complex (4.2) are $\mathcal{D}_X$-linear follows by Theorem 2.2.9 and Lemma 2.2.5.

Corollary 4.1.2. *Assume* $Z = X(w) \cap (X(w_1))^c$ *with* $w_1 \leq w$*, then*

$$[\mathcal{H}^{c_w}_{X(w)\cap(X(w_1))^c}(\mathcal{O}_X)] = \sum_{v\leq w,\, v\nleq w_1} (-1)^{l(v)-l(w)}[\mathcal{H}^{c_v}_{C(v)}(\mathcal{O}_X)] +$$

$$\sum_{i=1}^{\dim(X(w))} (-1)^{i-1}[\mathcal{H}^{i+c_w}_{X(w)\cap(X(w_1))^c}(\mathcal{O}_X)].$$

Especially

$$[\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)] = \sum_{v\leq w}(-1)^{l(v)-l(w)}[\mathcal{H}^{c_v}_{C(v)}(\mathcal{O}_X)] + \sum_{i=1}^{\dim(X(w))} (-1)^{i-1}[\mathcal{H}^{i+c_w}_{X(w)}(\mathcal{O}_X)].$$

Proof. Since

$$Z = \bigcup_{v\leq w,\, v\nleq w_1} C(v)$$

we get that the complex (4.2) in this case looks like

$$0 \to \mathcal{H}^{c_w}_{C(w)}(\mathcal{O}_X) \to \bigoplus_{v\leq w,\, v\nleq w_1,\, l(v)=l(w)-1} \mathcal{H}^{c_w+1}_{C(v)}(\mathcal{O}_X) \to \dots$$

and its i'th cohomology is equal to $\mathcal{H}^{c_w+i}_{X(w)\cap(X(w_1))^c}(\mathcal{O}_X)$ and since

$$[\bigoplus_{i=1}^{n} \mathcal{F}_i] = \sum_{i=1}^{n}[\mathcal{F}_i].$$

The first part of the Corollary follows by Proposition 4.1.1. The second part follows by considering the complex (4.3) and using the same arguments. □

4.2 G/B in general

In the rest of this chapter we assume, that the ground field k is algebraic closed of characteristic zero. The category we are going to work in is the category of holonomic B-equivariant $\mathcal{D}_X$-modules. This category is closed with respect to local cohomology with support in Z provided Z is B-invariant, extension, inclusion, quotient and the structure sheaf $\mathcal{O}_X$ is also contained in it. Furthermore any module in this category has a finite decomposition series. All this has been explained in section 3.2. So from now on we denote it as $\mathcal{D}_X - mod$. Corollary 4.1.2 is still true after restriction to this subcategory, since all sheaves in the complex (4.1) are in this subcategory.

We have according to Kazhdan-Lusztig conjecture proved in [1], [12] that the simple modules in $\mathcal{D}_X - mod$ are parametrized by the Schubert varieties and furthermore, if we let $\mathcal{L}(w)$ be the simple module parametrized by $X(w)$ (meaning that $\text{Supp}(\mathcal{L}(w)) = X(w)$), that in the Grothendieck group

$$[\mathcal{L}(w)] = \sum_{v \leq w} (-1)^{l(v)-l(w)} P_{v,w}(1) [\mathcal{H}^{c_v}_{C(v)}(\mathcal{O}_X)] \tag{4.4}$$

with $P_{v,w}$ the associated Kazhdan-Lusztig polynomial defined in appendix A.1. So if $\mathcal{M} \in \mathcal{D}_X - mod$, it follows since $\mathcal{M}$ has a finite decomposition series, that $\exists a_w \in \mathbb{N}$ such that

$$[\mathcal{M}] = \sum_{w \in W} a_w [\mathcal{L}(w)]. \tag{4.5}$$

They are also linearly independent since $\text{Supp}(\mathcal{L}(w)) = X(w)$. They are therefore a basis in the Grothendieck group. Given $\mathcal{M} \in \mathcal{D}_X - mod$, we call (4.5) the character formula for $\mathcal{M}$ and in order to find it, we must know all a_w. In the lines below all morphisms are $\mathcal{D}_X$-linear. The first line below follows due to Theorem 4.1 in [4], where one should be aware, that he gives a sketch of the proof in the end of his proof and the second is due to Proposition 8.5 in [12]

$$0 \to \mathcal{L}(w) \to \mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)$$
$$\mathcal{L}(w)|_{(\text{Sing}(X(w)))^c} \simeq \mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)|_{(\text{Sing}(X(w)))^c}$$

and $\mathcal{L}(w)$ is the only simple module with this property. Therefore it would be interesting to get informations of $\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)/\mathcal{L}(w)$, and we hence wish to get information of the character formula of $\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)$.

Lemma 4.2.1.

$$[\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)] = [\mathcal{L}(w)] +$$
$$\sum_{v<w} (-1)^{l(v)-l(w)} (1 - P_{v,w}(1)) [\mathcal{H}^{c_v}_{C(v)}(\mathcal{O}_X)] + \sum_{i=1}^{\dim(X(w))} (-1)^{i-1} [\mathcal{H}^{i+c_w}_{X(w)}(\mathcal{O}_X)].$$

Proof. By (4.4) we get the following formula in the Grothendieck group since $P_{w,w} = 1$

$$[\mathcal{L}(w)] - \sum_{v<w} (-1)^{l(v)-l(w)} P_{v,w}(1) [\mathcal{H}^{c_v}_{C(v)}(\mathcal{O}_X)] = [\mathcal{H}^{c_w}_{C(w)}(\mathcal{O}_X)].$$

By plugging the above equation into Corollary 4.1.2 we get, that

$$[\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)] = [\mathcal{L}(w)] +$$
$$\sum_{v<w}(-1)^{l(v)-l(w)}(1-P_{v,w}(1))[\mathcal{H}^{c_v}_{C(v)}(\mathcal{O}_X)] + \sum_{i=1}^{\dim(X(w))}(-1)^{i-1}[\mathcal{H}^{i+c_w}_{X(w)}(\mathcal{O}_X)].$$

□

So to find the character formula for $\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)$ in the Grothendieck group, we need to know $P_{v,w}(1)$ and the character formula for $\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X)$ for $j > 0$. But in some cases the last of these two can be done.

Lemma 4.2.2. *Suppose either $X(w)$ is smooth or that $\exists v > w$ satisfying that $X(w) \subset X(v)$, $l(v) - l(w) = 1$ and $X(v)$ is smooth, then*

$$[\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)] = [\mathcal{L}(w)] + \sum_{v<w}(-1)^{l(v)-l(w)}(1-P_{v,w}(1))[\mathcal{H}^{c_v}_{C(v)}(\mathcal{O}_X)].$$

Furthermore we have, that

$$\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) = 0 \ \forall j > 0.$$

Proof. In order to prove this lemma it is according to lemma 4.2.1 enough to prove, that $\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) = 0$ for $j > 0$. If $X(w)$ is smooth we know this is true due to Lemma 2.2.8. If we are in the other situation the result follows by Corollary 3.3.7. □

We need to know some information of the support of $\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X)$. The Lemma below gives this information.

Lemma 4.2.3. $\mathrm{Supp}(\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X)) \subset \mathrm{Sing}(X(w)) \ \forall j > 0.$

Proof. Follows by Lemma 2.2.8. □

4.3 G/B versus G/P

Let $P \subset G$ be a parabolic subgroup containing B. Let

$$\pi : G/B \to G/P$$

be the canonical map. Since $P/B \subset G/B$ is irreducible, closed and the left B-action on G/B leaves P/B invariant, P/B is a Schubert variety. Let

$P/B = X(w_I)$ for some $w_I \in W$. We then have that $\exists s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_r}$ simple reflections in W such that if we set

$$W_J := \langle s_{\alpha_i} | \; i \in \{1, \dots, r\} \rangle$$

then w_I is the longest element in W_J. We also set

$$W^J := \{w \in W | \; l(ww_I) = l(w) + l(w_I)\}.$$

We have for $w \in W^J$, that $X(ww_I) = \pi^{-1}(X(w)_P)$, where $X(w)_P$ is the closure of BwP in G/P and we denote $C(w)_p = BwP \subset G/P$. If $w \in W^J$ then we call $X(w)_P$ a Schubert variety in G/P. Given any $v \in W$ $X(v)_P$ is given in this way. Let us denote $C(v)_P = \pi(C(v))$ and once again given $v \in W$ $\exists w \in W^J$ such that $C(v)_P = C(w)_P$. Whenever we use the notation $C(w)_P$ or $X(w)_P$, we assume, that $w \in W^J$. The whole discussion in section 4.1 is true, when we work in G/P instead of G/B according to [33]. This implies for $v_1, \dots, v_r \in W^J$ if we set

$$Z = \cup_{i=1}^r C(v_i)_P$$

and Z is locally closed the i'th cohomology of the complex

$$0 \to \bigoplus_{i=1}^r \mathcal{H}^0_{C(v_i)_P}(\mathcal{O}_{G/P}) \to \bigoplus_{i=1}^r \mathcal{H}^1_{C(v_i)_P}(\mathcal{O}_{G/P}) \to \dots \tag{4.6}$$

is equal to $\mathcal{H}^i_Z(\mathcal{O}_{G/P})$. We still have

$$\mathcal{H}^r_{C(v_i)_P}(\mathcal{O}_{G/P}) \neq 0 \Leftrightarrow \; r = \operatorname{codim}(C(v_i)_P). \tag{4.7}$$

We need to know the connection between the local cohomologies in G/P and G/B. This is given in the Proposition below.

Propositon 4.3.1. *Let $Z \subset G/P$ be locally closed. We then get an isomorphism in $\mathcal{O}_{G/P} - mod$*

$$\pi_*(\mathcal{H}^j_{\pi^{-1}(Z)}(\mathcal{O}_{G/B})) \simeq \mathcal{H}^j_Z(\mathcal{O}_{G/P}) \; \forall j.$$

Proof. Since Z is locally closed, we know that $Z = V \cap W^c$ with $V, W \subset G/P$ open and $\pi^{-1}(Z) = \pi^{-1}(V) \cap \pi^{-1}(W)^c$. Let $\mathcal{M}$ be an arbitrary sheaf on G/B and let $U \subset G/P$ be open, then

$$\begin{aligned}
&((\pi_* \circ \underline{\Gamma}_{\pi^{-1}(Z)})(\mathcal{M}))(U) = \underline{\Gamma}_{\pi^{-1}(Z)}(\mathcal{M})(\pi^{-1}(U)) = \\
&\{\phi \in \mathcal{M}(\pi^{-1}(U) \cap \pi^{-1}(V)) | \; \phi|_{\pi^{-1}(U) \cap \pi^{-1}(V) \cap \pi^{-1}(W)} = 0\} = \\
&\{\phi \in \mathcal{M}(\pi^{-1}(U \cap V)) | \; \phi|_{\pi^{-1}(U \cap V \cap W)} = 0\} = \\
&\{\phi \in \pi_*(\mathcal{M})(U \cap V) | \; \phi|_{U \cap V \cap W} = 0\} = ((\underline{\Gamma}_Z \circ \pi_*)(\mathcal{M}))(U).
\end{aligned}$$

This shows, that $\pi_* \circ \underline{\Gamma}_{\pi^{-1}(Z)} \simeq \underline{\Gamma}_Z \circ \pi_*$. We then know, the two spectral sequences $R^j\pi_*(\mathcal{H}^i_{\pi^{-1}(Z)}(\mathcal{O}_{G/B}))$ and $\mathcal{H}^j_Z(R^i\pi_*(\mathcal{O}_{G/B}))$ converges to the same object. Since π is a local trivial morphism with fibers P/B and P/B according to [1] is $\mathcal{D}_{P/B}$-affine, it follows by Proposition 3.8.8 in [20], that π_* is an exact functor on $\mathcal{D}_{G/B} - mod$. This implies, that the spectral sequences degenerates, and we have an isomorphism in $\pi_*(\mathcal{O}_{G/B}) - mod$

$$\pi_*(\mathcal{H}^j_{\pi^{-1}(Z)}(\mathcal{O}_{G/B})) \simeq \mathcal{H}^j_Z(\pi_*(\mathcal{O}_{G/B}))$$

and then the Proposition follows since $\pi_*(\mathcal{O}_{G/B}) = \mathcal{O}_{G/P}$. □

We get at once the following Corollary.

Corollary 4.3.2. *Let $Z \subset G/P$ be locally closed. Then*

$$\mathcal{H}^j_{\pi^{-1}(Z)}(\mathcal{O}_{G/B}) = 0 \Leftrightarrow \mathcal{H}^j_Z(\mathcal{O}_{G/P}) = 0.$$

Proof. Since G/B and G/P are $\mathcal{D}$-affine according to [1] and $\pi_*(\mathcal{H}^j_{\pi^{-1}(Z)}(\mathcal{O}_{G/B})) \simeq \mathcal{H}^j_Z(\mathcal{O}_{G/P})$ the result follows by the Beilinson-Bernstein equivalence explained as Theorem 3.2.11. □

We shall also use the following standard result, which relates the singularities of Schubert varieties in G/P to the singularities of Schubert varieties in G/B.

Lemma 4.3.3. *For $w \in W^J$* $\mathrm{Sing}(X(ww_I)) = \pi^{-1}(\mathrm{Sing}(X(w)_P))$.

Proof. Since $\pi|_{X(ww_I)} : X(ww_I) \to X(w)_P$ is locally trivial with fibers P/B all there is to show is for an irreducible variety Y $\mathrm{Sing}(Y \times P/B) = \mathrm{Sing}(Y) \times P/B$. That this is true follows due to [34] exercise 6.3.6. □

4.4 Sl_n/B

We let $G = Sl_n$. Since a Schubert variety $X(w)$ is the closure of a B-orbit, there is a free B-action on $X(w)$, and $x \in \mathrm{Sing}(X(w)) \Leftrightarrow Bx \subset \mathrm{Sing}(X(w))$ and therefore

$$\mathrm{Sing}(X(w)) = \bigcup_{X(v) \subset \mathrm{Sing}(X(w))} X(v).$$

Let us make this into an irreducible union such that

$$\mathrm{Sing}(X(w)) = \bigcup_{i=1}^{n} X(v_i) \tag{4.8}$$

$$\bigcup_{i=1, i\neq j}^{n} X(v_i) \neq \mathrm{Sing}(X(w)) \ \forall j.$$

We denote maxSing$(X(w))$ as

$$\text{maxSing}(X(w)) := \{v_1, \dots, v_n\}.$$

According to [13] $P_{z,w} = 1 \ \forall v \leq z \leq w \Leftrightarrow C(v) \subset (\text{Sing}(X(w)))^c$ and $P_{v_i,w} \neq 1 \ \forall i$. According to [8] page 8 $P_{v_i,w}(0) = 1$ and since the coefficients of the Kazhdan-Lusztig polynomials are natural numbers due to Theorem 4.3 in [32], we get $P_{v_i,w}(1) > 1$.

Lemma 4.4.1. *If $\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) = 0 \ \forall j > 0$, then $X(w)$ is smooth $\Leftrightarrow [\mathcal{L}(w)] = [\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)]$. If $X(w)$ is smooth then $[\mathcal{L}(w)] = [\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)]$.*

Proof. According to Lemma 4.2.1 and the just mentioned properties

$$[\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)] = [\mathcal{L}(w)] + \sum_{i=1}^{n} (-1)^{l(v_i)-l(w)} (1 - P_{v_i,w}(1)) [\mathcal{H}^{c_{v_i}}_{C(v_i)}(\mathcal{O}_X)] + \sum_{v<v_i \text{ for some } i} (-1)^{l(v)-l(w)} (1 - P_{v,w}(1)) [\mathcal{H}^{c_v}_{C(v)}(\mathcal{O}_X)].$$

If $X(w)$ is smooth, the two last conditions are empty, and the Lemma follows. If on the other hand $X(w)$ is not smooth, we get according to (4.4) in section (4.2)

$$[\mathcal{H}^{c_{v_i}}_{C(v_i)}(\mathcal{O}_X)] = [\mathcal{L}(v_i)] - \sum_{x<v_i} (-1)^{l(v_i)-l(x)} P_{x,v_i}(1) [\mathcal{H}^{c_x}_{C(x)}(\mathcal{O}_X)].$$

This implies

$$[\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)] = [\mathcal{L}(w)] + \sum_{i=1}^{n} (-1)^{l(v_i)-l(w)} (1 - P_{v_i,w}(1)) [\mathcal{L}(v_i)] - \sum_{i=1}^{n} (-1)^{l(v_i)-l(w)} (1 - P_{v_i,w}(1)) \sum_{x<v_i} (-1)^{l(v_i)-l(x)} P_{x,v_i}(1) [\mathcal{H}^{c_x}_{C(x)}(\mathcal{O}_X)] + \sum_{v<v_i \text{ for some } i} (-1)^{l(v)-l(w)} (1 - P_{v,w}(1)) [\mathcal{H}^{c_v}_{C(v)}(\mathcal{O}_X)].$$

Since we have an exact sequence

$$0 \to \mathcal{H}^{c_x}_{X(x)}(\mathcal{O}_X) \to \mathcal{H}^{c_x}_{C(x)}(\mathcal{O}_X) \to \mathcal{H}^{c_x+1}_{X(x)\cap C(x)^c}(\mathcal{O}_X) \to \dots$$

$\text{Supp}(\mathcal{H}^{c_x}_{X(x)}(\mathcal{O}_X)) = X(x)$ and $\text{Supp}(\mathcal{H}^{c_x+1}_{X(x)\cap C(x)^c}(\mathcal{O}_X)) \subset X(x) \cap C(x)^c$ $\Rightarrow \text{Supp}(\mathcal{H}^{c_x}_{C(x)}(\mathcal{O}_X)) = X(x)$. So for $x < v_i$ the character formula for

$\mathcal{H}^{c_x}_{C(x)}(\mathcal{O}_X)$, does not contain $\mathcal{L}(v_i)$. We then get since $P_{v_i,w}(1) \neq 1$, that $[\mathcal{L}(v_i)]$ occurs in the character formula for $\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)$, and it is thus not simple. If $X(w)$ is smooth, we get according to Lemma 4.2.2 that $\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) = 0 \ \forall j > 0$, and then we are in the first situation. □

By using the above method we can prove a criterion for non-vanishing of the higher cohomologies.

Lemma 4.4.2. *If* $\exists i$ *with* $l(v_i) - l(w) \equiv 0 \bmod(2) \Rightarrow \exists j > 0 \ j \equiv 1 \bmod(2)$ *such that* $\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) \neq 0$.

Proof. By using the formula above, we get

$$[\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)] =$$
$$[\mathcal{L}(w)] + \sum_{i=1}^{n}(-1)^{l(v_i)-l(w)}(1 - P_{v_i,w}(1))[\mathcal{L}(v_i)] -$$
$$\sum_{i=1}^{n}(-1)^{l(v_i)-l(w)}(1 - P_{v_i,w}(1)) \sum_{x<v_i}(-1)^{l(v_i)-l(x)}P_{x,v_i}(1)[\mathcal{H}^{c_x}_{C(x)}(\mathcal{O}_X)] +$$
$$\sum_{v<v_i \text{ for some } i}(-1)^{l(v)-l(w)}(1 - P_{v,w}(1))[\mathcal{H}^{c_v}_{C(v)}(\mathcal{O}_X)] +$$
$$\sum_{i=1}^{\dim(X(w))}(-1)^{i-1}[\mathcal{H}^{i+c_w}_{X(w)}(\mathcal{O}_X)].$$

Since $[\mathcal{L}(v_i)]$ occurs with negative coefficient in the second line and it does not occur in the third or fourth line, it must occur with positive coefficient in the last line, since it must occur with coefficient greater than or equal to zero in $[\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)]$, and then the Lemma follows. □

4.5 Maximal singular locus of $X(w) \subset Sl_n/B$

The purpose of this subsection is to give a combinatorial description of maxSing$(X(w))$. This is done in [2]. So we shall just explain what is done in this article. We know that the Weyl group of Sl_n/B is S_n the set of bijections on the set $\{1, 2, \ldots, n\}$. Given $v \in S_n$ we sometimes write

$$v = [v(1), \ldots, v(n)]$$

and given $Z = \{a_1, \ldots, a_k\} \subset \{1, 2, \ldots, n\}$ with $a_1 < \cdots < a_k$, we define $\mathrm{fl}_Z(v) \in S_k$ as the bijection, whose elements are in the same relative order as $\{v(a_1), \ldots, v(a_k)\}$. As an example let $v = [4, 6, 1, 3, 5, 2] \in S_6$ and $Z = \{1, 3, 6\}$. Then $a_1 = 1,\ a_2 = 3,\ a_3 = 6$ and $\{v(a_1), \ldots, v(a_3)\} = \{4, 1, 2\}$ and therefore $\mathrm{fl}_Z(v) = [3, 1, 2] \in S_3$.

Lemma 4.5.1. *Let $x, v \in S_n$ and set $A = \{i \in \{1, 2, \ldots, n\} |\ v(i) = x(i)\}$ and let $Z \subset A$. Then*

$$x \leq v \Leftrightarrow \mathrm{fl}_{\{1,\ldots,n\}\setminus Z}(x) \leq \mathrm{fl}_{\{1,\ldots,n\}\setminus Z}(v).$$

Proof. The proof is an induction argument in the number of elements in Z. If Z consists of 1 element it is according to Lemma 17 in [2] true. So let $Z = \{a_1 < \cdots < a_k\}$. We then claim, that

$$\mathrm{fl}_{\{1,\ldots,n\}\setminus Z}(x) = \mathrm{fl}_{\{1,\ldots,n-1\}\setminus\{a_1<\cdots<a_{k-1}\}}(\mathrm{fl}_{\{1,\ldots,n\}\setminus\{a_k\}}(x)).$$

By construction

$$\mathrm{fl}_{\{1,\ldots,n\}\setminus\{a_k\}}(x)(j) = \left\{ \begin{array}{ccc} x(j) & j < a_k & x(j) < x(a_k) \\ x(j) - 1 & j < a_k & x(j) > x(a_k) \\ x(j+1) & j \geq a_k & x(j+1) < x(a_k) \\ x(j+1) - 1 & j \geq a_k & x(j+1) > x(a_k) \end{array} \right\}.$$

So in order to prove the claim we must show that the elements

$$\{x(1), \ldots, x(n)\}\setminus\{x(a_1), \ldots, x(a_k)\}$$
$$\{\mathrm{fl}_{\{1,\ldots,n\}\setminus\{a_k\}}(x)(1), \ldots, \mathrm{fl}_{\{1,\ldots,n\}\setminus\{a_k\}}(x)(n-1)\}\setminus\{\mathrm{fl}_{\{1,\ldots,n\}\setminus\{a_k\}}(x)(a_1), \ldots, \mathrm{fl}_{\{1,\ldots,n\}\setminus\{a_k\}}(x)(a_{k-1})\}$$

occurs in the same relative order. There are some cases to consider. First assume $i, j < a_k$. Then we must show $x(i) < x(j) \Leftrightarrow \mathrm{fl}_{\{1,\ldots,n\}\setminus\{a_k\}}(x)(i) < \mathrm{fl}_{\{1,\ldots,n\}\setminus\{a_k\}}(x)(j)$. This is by construction of $\mathrm{fl}_{\{1,\ldots,n\}\setminus\{a_k\}}(x)$ clear. If $i < a_k$ and $j > a_k$ we must show $x(i) < x(j) \Leftrightarrow \mathrm{fl}_{\{1,\ldots,n\}\setminus\{a_k\}}(x)(i) < \mathrm{fl}_{\{1,\ldots,n\}\setminus\{a_k\}}(x)(j-1)$, which is also satisfied, and finally for $a_k < i, j$ we must show $x(i) < x(j) \Leftrightarrow \mathrm{fl}_{\{1,\ldots,n\}\setminus\{a_k\}}(x)(i-1) < \mathrm{fl}_{\{1,\ldots,n\}\setminus\{a_k\}}(x)(j-1)$ and this is also satisfied. Then the Lemma follows by induction since

$$\mathrm{fl}_{\{1,\ldots,n\}\setminus\{a_k\}}(x)(a_i) = \mathrm{fl}_{\{1,\ldots,n\}\setminus\{a_k\}}(v)(a_i)\ \forall i \in \{1, \ldots, k-1\}.$$

□

For $p, q \in \mathbb{Z}$ and $x, v \in S_n$ define

$$r_v(p, q) := \{i \leq p |\ v(i) \geq q\}$$
$$d_{x,v}(p, q) := \sharp r_v(p, q) - \sharp r_x(p, q).$$

We then have the following Proposition, which is Proposition 7 in chapter 10.5 in [17], and the part with $l(v)$ is in chapter 10.2 in [17].

Propositon 4.5.2. *For* $x, v \in S_n$ *we have*

$$l(v) = \sharp\{i < j \mid v(i) > v(j)\}$$
$$x \leq v \Leftrightarrow d_{x,v}(p,q) \geq 0 \ \forall p,q \in \mathbb{Z}.$$

We set

$$\mathcal{T} := \{t \in S_n \mid \exists a \neq b \in \{1,2,\dots,n\},\ t(j) = j \ \forall j \neq a,b \ t(a) = b,\ t(b) = a\}.$$

So given $t \in \mathcal{T}$ we sometime write $t_{a,b}$. Let $x \leq v \in S_n$, we define

$$\mathcal{R}(x,v) := \{t \in \mathcal{T} \mid x < xt \leq v\}$$
$$\Delta(x,v) := \{i \in \{1,\dots,n\} \mid \exists j \in \{1,\dots,n\},\ t_{i,j} \in \mathcal{R}(x,v)\}.$$

The description of $\Delta(x,v)$ above is difficult to work with. The Lemma below is easier to use.

Lemma 4.5.3. *Let* $x < v$. *Then*

$$\Delta(x,v) = \{i \in \{1,\dots,n\} \mid x(i) \neq v(i) \vee \ d_{x,v}(i, x(i)) \neq 0\}.$$

Proof. That $\supset$ is true follows due to Proposition 14 and Corollary 15 in [2]. So pick $p \in \{1,\dots,n\}$ such that $x(p) = v(p)$ and $d_{x,v}(p, x(p)) = 0$. We get

$$0 = d_{x,v}(p, x(p)) = \sharp\{i \leq p \mid v(i) \geq x(p)\} - \sharp\{i \leq p \mid x(i) \geq x(p)\} \Rightarrow_{x(p)=v(p)}$$
$$0 = \sharp\{i \leq p-1 \mid v(i) \geq x(p)\} - \sharp\{i \leq p-1 \mid x(i) \geq x(p)\} = d_{x,v}(p-1, x(p)) \wedge$$
$$0 = \sharp\{i \leq p \mid v(i) \geq x(p)+1\} - \sharp\{i \leq p \mid x(i) \geq x(p)+1\} = d_{x,v}(p, x(p)+1).$$

Let us assume, that $p \in \Delta(x,v) \Rightarrow \exists b \in \{1,\dots,n\}\ b \neq p$ with $x \leq xt_{p,b} \leq v$. There are two cases to consider.

$(1):\ b < p:$ According to Proposition 4.5.2

$$0 \leq d_{x,xt_{p,b}}(b, x(b)) = \sharp\{i \leq b \mid xt_{p,b}(i) \geq x(b)\} - \sharp\{i \leq b \mid x(i) \geq x(b)\} \Rightarrow x(b) < x(p)$$

with the $\Rightarrow$ true since $xt_{p,b}(i) = x(i)\ \forall i < b$. Thus $1 = d_{x,xt_{p,b}}(p-1, x(p))$. Since $d_{xt_{p,b},v} + d_{x,xt_{p,b}} = d_{x,v} \Rightarrow -1 = d_{xt_{p,b},v}(p-1, x(p))$, and it follows due to Proposition 4.5.2, that we have a contradiction since $xt_{p,b} \leq v$.

$(2):\ p < b:$ Just as in case (1) $x(p) < x(b)$ and $1 = d_{x,xt_{p,b}}(p, x(p)+1) \Rightarrow -1 = d_{xt_{p,b},v}(p, x(p)+1)$, and we have a contradiction. ☐

According to [2] Theorem 11 we have the following Theorem.

Theorem 4.5.4. *$C(v) \subset (\mathrm{Sing}(X(w)))^c \Leftrightarrow \sharp\mathcal{R}(v,w) = l(w) - l(v)$.*

It follows by Lemma 4.5.3, that

$$i \notin \Delta(x,v) \Rightarrow x(i) = v(i). \tag{4.9}$$

Let $\Delta(x,v) = \{d_1 < \cdots < d_k\}$, we then set

$$\begin{aligned}\tilde{x} &:= \mathrm{fl}_{\{d_1<\cdots<d_k\}}(x),\\ \tilde{v} &:= \mathrm{fl}_{\{d_1<\cdots<d_k\}}(v).\end{aligned}$$

By combining Lemma 17 and Proposition 18 in [2] we have the following Proposition.

Propositon 4.5.5. *Let $x \leq v$. Then*

$$\begin{aligned}&\tilde{x} \leq \tilde{v},\\ &l(v) - l(x) = l(\tilde{v}) - l(\tilde{x}),\\ &\mathcal{R}(\tilde{x},\tilde{v}) \simeq \mathcal{R}(x,v),\\ &v_i \in \mathrm{maxSing}(X(w)) \Leftrightarrow\ \tilde{v_i} \in \mathrm{maxSing}(X(\tilde{w})).\end{aligned}$$

For $k, m \geq 2$ define $x_{k,m}, w_{k,m} \in S_{k+m}$

$$\begin{aligned}x_{k,m} &:= [k, k-1, \ldots, 1, k+m, k+m-1, \ldots, k+1], \qquad (4.10)\\ w_{k,m} &:= [k+m, k, k-1, \ldots, 2, k+m-1, k+m-2, \ldots, k+1, 1].\end{aligned}$$

For $k, m \geq 1$, $l \geq 2$ define $x_{k,l,m}, w_{k,l,m} \in S_{k+m+l}$

$$\begin{aligned}x_{k,l,m} &:= [k, \ldots, 1, k+l, \ldots, k+1, k+l+m, \ldots, k+l+1] \qquad (4.11)\\ w_{k,l,m} &:= [k+l, k, \ldots, 2, k+m+l, k+l-1, \ldots, k+2,\\ &1, k+l+m-1, \ldots, k+l+1, k+1].\end{aligned}$$

with the convention if $k = 1$ then the part with $k, \ldots, 2$ is not part of $w_{k,l,m}$, if $m = 1$ the part with $k+l+m-1, \ldots, k+l+1$ is not part of $w_{k,l,m}$ and if $l = 2$ the part with $k+l-1, \ldots, k+2$ is not part of $w_{k,l,m}$. The Theorem below is Theorem 37 in [2].

Theorem 4.5.6. $v \in$ maxSing(X(w)) $\Leftrightarrow$

$$\begin{aligned}&t \in \mathcal{R}(v,w) \Rightarrow\ l(v) + 1 = l(vt) \textit{ and}\\ &\tilde{w} = w_{k,m} \textit{ and } \tilde{v} = x_{k,m}\ k, m \geq 2 \textit{ or}\\ &\tilde{w} = w_{k,l,m} \textit{ and } \tilde{v} = x_{k,l,m}\ k, m \geq 1, l = 2 \textit{ or } k = m = 1,\ l \geq 2.\end{aligned}$$

According to Corollary 40 in [2]

$$P_{\tilde{v},\tilde{w}} = P_{v,w} \tag{4.12}$$

and the Theorem below is Theorem 42 in [2], which therefore gives the Kazhdan-Lusztig polynomial $P_{v,w}$ for $v \in$maxSing$(X(w))$.

Theorem 4.5.7. *For* $k, m \geq 2$

$$P_{x_{k,m},w_{k,m}} = \sum_{j=0}^{\min(k-1,m-1)} q^j.$$

For $k, m \geq 1, l = 2$

$$P_{x_{k,2,m},w_{k,2,m}} = 1 + q.$$

For $k = m = 1, l \geq 2$

$$P_{x_{1,l,1},w_{1,l,1}} = 1 + q^{l-1}.$$

It follows by (9.5) and (9.6) in [2], that

$$\begin{aligned} l(w_{k,m}) &= \binom{k}{2} + \binom{m}{2} + k + m - 1, \\ l(x_{k,m}) &= \binom{k}{2} + \binom{m}{2} \Rightarrow \\ l(w_{k,m}) &- l(x_{k,m}) = k + m - 1. \end{aligned}$$

We therefore get, that

$$\begin{aligned} &3 = l(w_{k,m}) - l(x_{k,m}) \Leftrightarrow\ k = m = 2 \Rightarrow \\ &x_{2,2} = [2, 1, 4, 3],\ w_{2,2} = [4, 2, 3, 1] \end{aligned} \tag{4.13}$$

It follows by Lemma 4.5.3, that

$$\Delta(x_{2,2}, w_{2,2}) = \{1, 2, 3, 4\} \tag{4.14}$$

since $\forall i \in \{1, 2, 3, 4\}$ $x_{2,2}(i) \neq w_{2,2}(i)$. According to (9.9) and (9.10) in [2]

$$\begin{aligned} l(w_{k,l,m}) &= \binom{k}{2} + \binom{l}{2} + \binom{m}{2} + k + m + 2(l-2) + 1, \\ l(x_{k,l,m}) &= \binom{k}{2} + \binom{l}{2} + \binom{m}{2} \Rightarrow \\ l(w_{k,l,m}) &- l(x_{k,l,m}) = k + m + 2(l-2) + 1. \end{aligned} \tag{4.15}$$

Where the $\begin{pmatrix} k \\ 2 \end{pmatrix}$ or $\begin{pmatrix} m \\ 2 \end{pmatrix}$ is not part of the sum above if $k = 1$ respective $m = 1$ and therefore

$$\begin{aligned} &3 = l(w_{k,l,m}) - l(x_{k,l,m}) \Leftrightarrow\ k = m = 1,\ l = 2 \Rightarrow \\ &x_{1,2,1} = [1,3,2,4],\ w_{1,2,1} = [3,4,1,2]. \end{aligned} \tag{4.16}$$

4.6 The case $\tilde{w} = w_{1,l,1}$

Let $\mathcal{M} \in \mathcal{D}_X - mod$. We use the notation $[\mathcal{M} : \mathcal{L}(v)]$ to denote the coefficient of $[\mathcal{L}(v)]$ in the character formula of $\mathcal{M}$. We wish to prove the following Theorem in this section.

Theorem 4.6.1. *Suppose* $v \in \operatorname{maxSing}(X(w))$ *and* $\tilde{w} = w_{1,l,1}$ $\tilde{v} = x_{1,l,1}$ $l \geq 2$. *Then*

$$[\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)] = \left\{ \begin{array}{ll} 0 & j \neq 0 \\ 1 & j = 0 \end{array} \right\}.$$

Especially $X(v) \subset \operatorname{Supp}(\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)/\mathcal{L}(w))$.

This is one of the three possibilities for $\tilde{w}$ according to Theorem 4.5.6, and the Theorem above therefore gives a complete description of how $\mathcal{L}(v)$ behaves in $\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X)$.

Propositon 4.6.2. $\sum_{z \leq w} (-1)^{l(w)-l(z)} [\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X) : \mathcal{L}(v)] = 1.$

Proof. It follows by combining Theorem 4.5.7 and (4.12), that

$$P_{v,w} = 1 + q^{l-1}$$

and according to (4.4) in section 4.2 since $P_{z,w}(1) = 1\ \forall z \leq w\ v < z$ and due to (4.15) combined with Proposition 4.5.5 $l(w) - l(v) \equiv 1 \bmod(2)$ we get

$$\sum_{z \leq w} (-1)^{l(w)-l(z)} [\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X) : \mathcal{L}(v)] = [\mathcal{H}^{c_v}_{C(v)}(\mathcal{O}_X) : \mathcal{L}(v)] = 1.$$

□

The idea in the proof of Theorem 4.6.1 is to construct $w_1, w_2, z \in S_n$ with $X(w_i) \subset X(z)$, $X(w) = X(w_1) \cap X(w_2)$, $c_w = c_z + (c_{w_1} - c_z) + (c_{w_2} - c_z)$ such that, if we set $U := (\operatorname{Sing}(X(z)) \cup_{i=1}^2 \operatorname{Sing}(X(w_i)))^c$ then $C(v) \subset U$. It follows by Corollary 3.3.8 for $j > 0$, that

$$\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X)|_U = \mathcal{H}^{c_w+j}_{(X(w_1)\cap U)\cap(X(w_2)\cap U)}(\mathcal{O}_U) = 0 \Rightarrow [\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)] = 0$$

Since $\tilde{w} = w_{1,l,1}$, $\tilde{v} = x_{1,l,1}$ with $l \geq 2$ we get according to the description given as 4.11

$$\tilde{w} = [l+1, l+2, l, \ldots, 3, 1, 2],\ \tilde{v} = [1, 1+l, \ldots, 2, l+2]$$

with the convention if $l = 2$ the part with $l, \ldots, 2$ is not part of $\tilde{w}$. We therefore get

$$\begin{aligned} &\Delta(v, w) = \{d_1 < d_2 < \cdots < d_{l+1} < d_{l+2}\} \Rightarrow_{(4.9) \text{ in section } 4.5} \\ &v(i) = w(i)\ \forall i \notin \{d_1 < \cdots < d_{l+2}\}. \end{aligned} \tag{4.17}$$

In this case

$$\left.\begin{array}{l} \left.\begin{array}{ccccccc} w(d_{l+1}) & < & w(d_{l+2}) & < & w(d_1) & < & w(d_2) \\ \| & & \| & & \| & & \| \\ v(d_1) & < & v(d_{l+1}) & < & v(d_2) & < & v(d_{l+2}) \end{array}\right\}, \\ v(d_i) = w(d_i),\ w(d_{i+1}) < w(d_i)\ \forall i \in \{3, 4, \ldots, l\}, \\ w(d_3) < w(d_1),\ w(d_l) > w(d_{l+2}) \end{array}\right\} \tag{4.18}$$

Since $d_i \in \Delta(v, w)\ \forall i \in \{3, \ldots, l\}$ it follows due to Lemma 4.5.3, that

$$d_{v,w}((d_i, v(d_i))) \neq 0\ \forall i \in \{3, \ldots, l\}. \tag{4.19}$$

Define two subsets $A, B \subset \{1, 2, \ldots, n\}$

$$\begin{aligned} A &= \{i \in \{d_1, d_1+1, \ldots, d_2\} |\ w(d_1) \leq w(i) \leq w(d_2)\}, \\ B &= \{i \in \{d_{l+1}, d_{l+1}+1, \ldots, d_{l+2}\} |\ w(d_{l+1}) \leq w(i) \leq w(d_{l+2})\}. \end{aligned}$$

Clearly $d_1, d_2 \in A$ and $d_{l+1}, d_{l+2} \in B$. Then we can set

$$\begin{aligned} A &= \{i_1 < i_2 < \cdots < i_r\}, \\ B &= \{k_1 < k_2 < \cdots < k_s\}. \end{aligned}$$

Let us define $w_1, w_2, z \in S_n$ by

$$\begin{aligned} &w(i) = w_1(i)\ \forall i \notin A,\ w_1(i_j) > w_1(i_{j+1})\ \forall j \in \{1, \ldots r-1\}, \\ &w(i) = w_2(i)\ \forall i \notin B,\ w_2(k_j) > w_2(k_{j+1})\ \forall j \in \{1, \ldots s-1\}, \\ &w(i) = z(i)\ \forall i \notin A \cup B,\ z(i_j) > z(i_{j+1})\ \forall j \in \{1, \ldots r-1\}, \\ &z(k_j) > z(k_{j+1})\ \forall j \in \{1, \ldots s-1\},\ z(i_r) > z(k_1). \end{aligned}$$

As sets

$$\{w(i) |\ i \in A \cup B\} = \{z(i) |\ i \in A \cup B\} = \{w_1(i) |\ i \in A \cup B\} = \{w_2(i) |\ i \in A \cup B\}$$

and it then follows due to the construction of A and B and (4.18), that

$$z(i) = w_1(i)\ \forall i \in A,\ z(i) = w_2(i)\ \forall i \in B \tag{4.20}$$

and as sets

$$\left.\begin{array}{c} \{z(i)|\ i \in A\} = \{w_2(i)|\ i \in A\} \\ \{w(i)|\ i \in A\} = \{w_1(i)|\ i \in A\} \\ \{w(i)|\ i \in B\} = \{w_2(i)|\ i \in B\} \\ \{w_1(i)|\ i \in A\} = \{w_2(i)|\ i \in A\} \end{array}\right\}. \tag{4.21}$$

We need a couple of Lemmas.

Lemma 4.6.3. *$w \leq w_i$ and $w_i \leq z$ $i \in \{1, 2\}$.*

Proof. To prove this Lemma we use Lemma 4.5.1. Since $w(i) = w_1(i)\ \forall i \notin A$ we just have to prove, that $\mathrm{fl}_{\{i_1<\cdots<i_r\}}(w) \leq \mathrm{fl}_{\{i_1<\cdots<i_r\}}(w_1)$ and since $\mathrm{fl}_{\{i_1<\cdots<i_r\}}(w_1)$ is the longest element in S_r we get $w \leq w_1$. In exactly the same way we get $w \leq w_2$. Since $w_1(i) = z(i)\ \forall i \notin A \cup B$. We must show

$$\mathrm{fl}_{\{i_1<\cdots<i_r<k_1<\cdots<k_s\}}(w_1) \leq \mathrm{fl}_{\{i_1<\cdots<i_r<k_1<\cdots<k_s\}}(z)$$

and since $\mathrm{fl}_{\{i_1<\cdots<i_r<k_1<\cdots<k_s\}}(z)$ is the longest element in S_{r+s} $w_1 \leq z$. To show $w_2 \leq z$ is done the same way. □

Lemma 4.6.4. *$X(w) = X(w_1) \cap X(w_2)$.*

Proof. That $\subset$ is true is due to Lemma 4.6.3. Let $y \leq w_1$, $y \leq w_2$. We must show $y \leq w$. This is due to Proposition 4.5.2 equivalent to proving $d_{y,w}(p, q) \geq 0\ \forall p, q \in \mathbb{Z}$. Since $d_{y,w_t}(p, q) = \sharp\{i \leq p|\ w_t(i) \geq q\} - \sharp\{i \leq p|\ y(i) \geq q\}$ we get due to Proposition 4.5.2

$$\begin{array}{l} 0 \leq d_{y,w_t}(p, q) \Leftrightarrow \sharp\{i \leq p|\ y(i) \geq q\} \leq \sharp\{i \leq p|\ w_t(i) \geq q\} \Leftrightarrow \\ \min\{\ \sharp\{i \leq p|\ w_t(i) \geq q\}|t \in \{1, 2\}\} \geq \sharp\{i \leq p|\ y(i) \geq q\}. \end{array}$$

So if $\min\{\ \sharp\{i \leq p|\ w_t(i) \geq q\}|t \in \{1, 2\}\} = \sharp\{i \leq p|\ w(i) \geq q\}$ we are done.
If $p < d_{l+1}$ $\min\{\ \sharp\{i \leq p|\ w_t(i) \geq q\}|t \in \{1, 2\}\} = \sharp\{i \leq p|\ w_2(i) \geq q\} = \sharp\{i \leq p|\ w(i) \geq q\}$.
If $p \geq d_{l+1}$ $\min\{\ \sharp\{i \leq p|\ w_t(i) \geq q\}|t \in \{1, 2\}\} =_{(4.21)} \sharp\{i \leq p|\ w_1(i) \geq q\}$ and since $\sharp\{i \leq p|\ w_1(i) \geq q\} - \sharp\{i \leq p|\ w(i) \geq q\} = \sharp\{i \in A|\ w_1(i) \geq q\} - \sharp\{i \in A|\ w(i) \geq q\} =_{(4.21)} 0$ the Lemma follows. □

Lemma 4.6.5. *$l(z) + l(w) = l(w_1) + l(w_2)$.*

Proof. According to Proposition 4.5.2 $l(w) = \sharp\{i < j|\ w(i) > w(j)\}$. For $j \in \{2, \dots, n\}$ set

$$A(j) := \sharp\{i < j|\ w(i) > w(j)\} + \sharp\{i < j|\ z(i) > z(j)\} -$$
$$\sharp\{i < j|\ w_1(i) > w_1(j)\} - \sharp\{i < j|\ w_2(i) > w_2(j)\} \Rightarrow$$
$$l(z) + l(w) - l(w_1) - l(w_2) = \sum_{j=2}^{n} A(j).$$

If $j < d_{l+1}$ we get since $w(i) = w_2(i)\ \forall i \le j$ by construction and due to (4.20) $z(i) = w_1(i)\ \forall i \le j$, $A(j) = 0$. If $j \ge d_{l+1}$ it follows since $w(j) = w_1(j)$ and $w(i) = w_1(i)\ \forall i \notin A$ and due to (4.20) $z(j) = w_2(j)$ and $z(i) = w_2(i)\ \forall i \notin A$

$$A(j) := \sharp\{i \in A|\ w(i) > w(j)\} + \sharp\{i \in A|\ z(i) > z(j)\} -$$
$$\sharp\{i \in A|\ w_1(i) > w(j)\} - \sharp\{i \in A|\ w_2(i) > z(j)\} =_{(4.21)} 0.$$

□

Lemma 4.6.6. $\Delta(w, w_1) = A$, $d_{w,w_1}(i, w(i)) \neq 0\ \forall i \in A \backslash \{d_1, d_2\}$ *and* $\Delta(w, w_2) = B$, $d_{w,w_2}(i, w(i)) \neq 0\ \forall i \in B \backslash \{d_{l+1}, d_{l+2}\}$.

Proof. Due to Lemma 4.6.3 $w \le w_d$ and according to Lemma 4.5.3

$$\Delta(w, w_d) = \{i \in \{1, \dots, n\}|\ w(i) \neq w_d(i) \vee\ d_{w,w_d}(i, w(i)) \neq 0\}.$$

Let us do it for w_1. The part with w_2 is identical. Since $w_1(j) = w(j)\ \forall j \notin A$

$$d_{w,w_1}(i, w(i)) = \sharp\{j \le i|\ w_1(j) \ge w(i)\} - \sharp\{j \le i|\ w(j) \ge w(i)\} =$$
$$\sharp\{j \in A,\ j \le i|\ w_1(j) \ge w(i)\} - \sharp\{j \in A,\ j \le i|\ w(j) \ge w(i)\}.$$

For $i > d_2$ or $i < d_1$ $i \notin \Delta(w, w_1)$ due to (4.21). If $i \in \{d_1, d_1 + 1, \dots, d_2\} \backslash A$ $i \notin \Delta(w, w_1)$ by construction of A. Since $w_1(d_1) = w(d_2)$ and $w_1(d_2) = w(d_1) \Rightarrow d_1, d_2 \in \Delta(w, w_1)$. Let $i = i_t$ $t \in \{2, 3, \dots, r-1\}$. Then since $w(i_1) = w(d_1) < w(i_t)$

$$d_{w,w_1}(i_t, w(i_t)) =$$
$$\sharp\{m \in \{1, \dots, t\}|w_1(i_m) \ge w(i_t)\} - \sharp\{m \in \{1, \dots, t\}|w(i_m) \ge w(i_t)\} =$$
$$\sharp\{m \in \{1, \dots, t\}|\ w_1(i_m) \ge w(i_t)\} - \sharp\{m \in \{2, \dots, t\}|\ w(i_m) \ge w(i_t)\}.$$

Due to (4.21) $\exists u \in \{2, \dots, r-1\}$ with $w_1(i_u) = w(i_t)$. If $t \le u$ $t = \sharp\{m \in \{1, \dots, t\}|\ w_1(i_m) \ge w(i_t)\}$ and $i_t \in \Delta(w, w_1)$. If $u < t$ $u = \sharp\{m \in \{1, \dots, t\}|\ w_1(i_m) \ge w(i_t)\}$ and due to (4.21) there are exactly u elements among $\{w(i_1), \dots, w(i_r)\}$ with $w(i_j) \ge w(i_t)$. One of these is $w(i_r) = w(d_2)$ and thus $\sharp\{m \in \{2, \dots, t\}|\ w(i_m) \ge w(i_t)\} < u \Rightarrow \Delta(w, w_1) = A$. □

Lemma 4.6.7. $(1) : C(v) \subset (\mathrm{Sing}(X(w_1)))^c$, $\Delta(v, w_1) = A \cup \{d_3, \dots, d_{l+2}\}$ *and* $d_{v,w_1}(i, v(i)) \neq 0 \ \forall i \in A \backslash \{d_1\}$.
$(2) : C(v) \subset (\mathrm{Sing}(X(w_2)))^c$, $\Delta(v, w_2) = B \cup \{d_1, \dots, d_l\}$ *and* $d_{v,w_2}(i, v(i)) \neq 0 \ \forall i \in B \backslash d_{l+2}$.

Proof. That $v \leq w_i$ follows since by construction $v \leq w$ and due to Lemma 4.6.3 $w \leq w_i$. According to Lemma 4.5.3

$$\Delta(v, w_1) = \{i \in \{1, \dots, n\} | \ v(i) \neq w_1(i) \vee \ d_{v,w_1}(i, v(i)) \neq 0\}.$$

We want to show $\Delta(v, w_1) = A \cup \{d_3, \dots, d_{l+2}\}$. Pick $i \notin A \cup \{d_3, \dots, d_{l+2}\}$. Then $w_1(i) = w(i) =_{(4.17)} v(i)$. Since $i \notin \Delta(v, w)$ due to (4.17) and $i \notin \Delta(w, w_1)$ due to Lemma 4.6.6 $0 = d_{v,w}(i, v(i)) = d_{w,w_1}(i, w(i))$ and therefore $d_{v,w_1}(i, v(i)) = d_{v,w}(i, v(i)) + d_{w,w_1}(i, v(i)) = 0 \Rightarrow i \notin \Delta(v, w_1)$.

Due to (4.18) $w_1(d_{l+1}) = w(d_{l+1}) = v(d_1) \neq v(d_{l+1})$, $w_1(d_{l+2}) = w(d_{l+2}) = v(d_{l+1}) \neq v(d_{l+2})$ and $w_1(d_1) \neq w_1(d_{l+1}) = v(d_1) \Rightarrow d_1, d_{l+1}, d_{l+2} \in \Delta(v, w_1)$.

$$\begin{aligned}&d_{v,w_1}(d_2, v(d_2)) = \sharp\{j \leq d_2 | \ w_1(j) \geq v(d_2)\} - \sharp\{j \leq d_2 | \ v(j) \geq v(d_2)\} = \\ &\sharp\{j \in A | \ w_1(j) \geq v(d_2)\} - \sharp\{j \in A | \ v(j) \geq v(d_2)\}.\end{aligned}$$

We get due to (4.18) $v(d_2) = w(d_1) = w_1(i_r) = w_1(d_2)$ and hence $r = \sharp\{j \in A | \ w_1(j) \geq v(d_2)\}$ and thanks to (4.18) $v(i_1) = v(d_1) < v(d_2)$ and thus $\sharp\{j \in A | \ v(j) \geq v(d_2)\} < r$ and $d_2 \in \Delta(v, w_1)$ and $d_{v,w_1}((d_2, v(d_2)) \neq 0$.

So pick $i \in A \cup \{d_3, \dots, d_{l+2}\} \backslash \{d_1, d_2, d_{l+1}, d_{l+2}\}$. Then

$$d_{v,w_1}(i, v(i)) = d_{v,w}(i, v(i)) + d_{w,w_1}(i, v(i)).$$

Thanks to Proposition 4.5.2 $d_{v,w}(i, v(i)) \geq 0$ and $d_{w,w_1}(i, v(i)) \geq 0$. If $i \in A \backslash \{d_1, d_2\}$ $v(i) = w(i)$ and it follows by Lemma 4.6.6 $d_{w,w_1}(i, w(i)) \neq 0$. If $i \in \{d_3, \dots, d_l\}$ $d_{v,w}(i, v(i)) \neq 0$ according to (4.19) and $i \in \Delta(v, w_1)$ and hence $\Delta(v, w_1) = A \cup \{d_3, \dots, d_{l+2}\}$. So we have proved (1) but the fact $C(v) \subset (\mathrm{Sing}(X(w_1)))^c$. The proof of (2) but the fact $C(v) \subset (\mathrm{Sing}(X(w_2)))^c$ is identical. The only difference is to show $d_1, d_2, d_{l+1}, d_{l+2} \in \Delta(v, w_2)$. Due to (4.18) $w_2(d_1) = w(d_1) = v(d_2)$, $w_2(d_2) = w(d_2) = v(d_{l+2}) \Rightarrow d_1, d_2, d_{l+2} \in \Delta(v, w_2)$. Since $w_2(i) = w(i) =_{(4.17) \text{ and } (4.18)} v(i) \ \forall i < d_{l+1}$ $i \notin \{d_1, d_2\}$ and $w_2(d_{l+1}) = w(d_{l+2}) =_{(4.18)} v(d_{l+1})$

$$\begin{aligned}&d_{v,w_2}(d_{l+1}, v(d_{l+1})) = \\ &\sharp\{j \leq d_{l+1} | \ w_2(j) \geq v(d_{l+1})\} - \sharp\{j \leq d_{l+1} | \ v(j) \geq v(d_{l+1})\} = \\ &\sharp\{m \in \{1, 2\} | \ w_2(d_m) \geq v(d_{l+1})\} - \sharp\{m \in \{1, 2\} | \ v(d_m) \geq v(d_{l+1})\} =_{(4.18)} \\ &\sharp\{m \in \{1, 2\} | \ w(d_m) \geq v(d_{l+1})\} - 1 = 1.\end{aligned}$$

Let us now prove $C(v) \subset (\mathrm{Sing}(X(w_1)))^c$. Due to Theorem 4.5.4 this is equivalent to proving $\sharp\mathcal{R}(v, w_1) = l(w_1) - l(v)$, which due to Proposition 4.5.5 is equivalent to proving $\sharp\mathcal{R}(\tilde{v}, \tilde{w_1}) = l(\tilde{w_1}) - l(\tilde{v})$, which again due to Theorem 4.5.4 is equivalent to proving $C(\tilde{v}) \subset (\mathrm{Sing}(X(\tilde{w_1})))^c$.

$$\tilde{w_1} = \mathrm{fl}_{\{i_1 < i_2 < \cdots < i_r < d_3 < \cdots < d_{l+2}\}}(w_1).$$

By construction in (4.18) $w_1(d_{l+1}) < w_1(d_{l+2}) < w_1(d_l) < w_1(d_{l-1}) < \cdots < w_1(d_3) < w_1(i_r) < w_1(i_{r-1}) < \cdots < w_1(i_1)$ and therefore $\tilde{w_1} \in S_{r+l}$

$$\tilde{w_1}(j) = \left\{ \begin{array}{cc} r+l-j+1 & j \in \{1, \dots, r+l-2\} \\ 1 & j = r+l-1 \\ 2 & j = r+l \end{array} \right\}.$$

If $X(\tilde{w_1})$ is smooth, we are done. This is equivalent to proving $\mathrm{maxSing}(X(\tilde{w_1})) = \emptyset$. So assume this is not the case, and let $x \in \mathrm{maxSing}(X(\tilde{w_1}))$. It then follows due to Theorem 4.5.6 that $\tilde{\tilde{w_1}} = w_{k,m}$ with $k, m > 1$ or $\tilde{\tilde{w_1}} = w_{k,l,m}$ with $k, m \geq 1,\ l = 2$ or $k = m = 1,\ l \geq 2$. But $\tilde{w_1}$ cannot be equal to those due to the description of $w_{k,m}$ and $w_{k,l,m}$ given as (4.10) and (4.11) in section 4.5. To prove $C(v) \subset (\mathrm{Sing}(X(w_2)))^c$, we do the same

$$\tilde{w_2} = \mathrm{fl}_{\{d_1 < \cdots < d_l < k_1 < k_2 < \cdots < k_s\}}(w_2).$$

By construction in (4.18) $w_2(k_s) < w_2(k_{s-1}) < \cdots < w_2(k_1) < w_2(d_l) < w_2(d_{l-1}) < \cdots < w_2(d_3) < w_2(d_1) < w_2(d_2)$ and therefore $\tilde{w_2} \in S_{s+l}$

$$\tilde{w_2}(j) = \left\{ \begin{array}{cc} s+l-1 & j = 1 \\ s+l & j = 2 \\ s+l-j+1 & j \in \{3, \dots, s+l\} \end{array} \right\}.$$

$X(\tilde{w_2})$ is smooth due to the same arguments as proving $X(\tilde{w_1})$ is smooth. □

Lemma 4.6.8. $C(v) \subset (\mathrm{Sing}(X(z)))^c$.

Proof. That $v \leq z$ is due to Lemma 4.6.3. We first show, that $\Delta(v, z) = A \cup B \cup \{d_3, \dots, d_l\}$. According to Lemma 4.5.3

$$\Delta(v, z) = \{i \in \{1, \dots, n\}|\ v(i) \neq z(i) \vee\ d_{v,z}(i, v(i)) \neq 0\}.$$

Pick $i \in A \backslash \{d_1\}$.

$$d_{v,z}(i, v(i)) = d_{v,w_1}(i, v(i)) + d_{w_1,z}(i, v(i)).$$

Since $w_1 \leq z$ it follows due to Proposition 4.5.2 $d_{w_1,z}(i,v(i)) \geq 0$ and due to Lemma 4.6.7 $d_{v,w_1}(i,v(i)) \neq 0$ and therefore $i \in \Delta(v,z)$. If $i \in B\backslash\{d_{l+2}\}$

$$d_{v,z}(i,v(i)) = d_{v,w_2}(i,v(i)) + d_{w_2,z}(i,v(i))$$

and due to the same arguments $i \in \Delta(v,z)$. If $i = d_1$ $z(i) = w(d_2) =_{(4.18)} v(d_{l+2}) \neq v(d_1) = v(i)$ and if $i = d_{l+2}$ $z(i) = w(d_{l+1}) = v(d_1) \neq v(d_{l+2}) = v(i)$ and thus $d_1, d_{l+2} \in \Delta(v,z) \Rightarrow A \cup B \subset \Delta(v,z)$. If $i \in \{d_3,\dots,d_l\}$ it follows thanks to (4.19) $d_{v,w}(i,v(i)) \neq 0$, and since $w \leq z$ according to Lemma 4.6.3, it follows by Proposition 4.5.2 $d_{w,z}(i,v(i)) \geq 0$ and hence

$$d_{v,z}(i,v(i)) = d_{v,w}(i,v(i)) + d_{w,z}(i,v(i)) \neq 0$$

and $\{d_3,\dots,d_l\} \subset \Delta(v,z)$. So pick $i \notin A \cup B \cup \{d_3,\dots,d_l\}$. Then $v(i) = w(i) = w_d(i) = z(i)$ $d \in \{1,2\}$. It also follows due to Lemma 4.6.7, that $d_{v,w_d}(i,v(i)) = 0$ and therefore

$$d_{v,z}(i,v(i)) = d_{w_d,z}(i,w_d(i)) = \sharp\{j \leq i|\ z(j) \geq w_d(i)\} - \sharp\{j \leq i|\ w_d(j) \geq w_d(i)\}.$$

Let $i > d_2$. It follows thanks to (4.20) that $z(j) = w_2(j)$ $\forall j \notin A$ and hence

$$d_{v,z}(i,v(i)) = \sharp\{j \in A|\ z(j) \geq w_2(i)\} - \sharp\{j \in A|\ w_2(j) \geq w_2(i)\} =_{(4.21)} 0.$$

Let $i < d_2$. It follows due to (4.20) that $z(j) = w_1(j)$ $\forall j \notin B$ and hence

$$d_{v,z}(i,v(i)) = \sharp\{j \leq i|\ z(j) \geq w_1(i)\} - \sharp\{j \leq i|\ w_1(j) \geq w_1(i)\} = 0$$

and we have shown $\Delta(v,z) = A \cup B \cup \{d_3,\dots,d_l\}$.

$$\tilde{z} = \mathrm{fl}_{\{i_1<\cdots<i_r<d_3<d_4<\cdots<d_l<k_1<\cdots<k_s\}}(z).$$

By construction $z(i_1) > z(i_2) > \cdots > z(i_r) > z(k_1) > z(k_2) > \cdots > z(k_s)$ and according to (4.18) $z(i_r) = z(d_2) = w(d_1) > z(d_3) > z(d_4) > \cdots > z(d_l) > w(d_{l+2}) = z(d_{l+1}) = z(k_1)$ and $\tilde{z}$ is the longest element in $S_{r+s+l-2}$, and we see $X(\tilde{z})$ is smooth hence $C(v) \subset (\mathrm{Sing}(X(z)))^c$. □

Propositon 4.6.9. $\mathrm{Supp}(\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X)) \subset \mathrm{Sing}(X(z)) \bigcup_{i=1}^{2} \mathrm{Sing}(X(w_i))$ $\forall j > 0$. *Especially* $[\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)] = 0$ $\forall j > 0$.

Proof. We set $U = (\mathrm{Sing}(X(z)) \bigcup_{i=1}^{2} \mathrm{Sing}(X(w_i)))^c$. Due to Lemma 4.6.7 and 4.6.8, we get

$$C(v) \subset U.$$

Since $[\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)] > 0 \Rightarrow C(v) \subset X(v) \subset \mathrm{Supp}(\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X))$, we have to prove the first part of the Proposition. Let $j > 0$. This is equivalent to proving

$$\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X)|_U = 0 \Leftrightarrow \mathcal{H}^{c_w+j}_{X(w)\cap U}(\mathcal{O}_U) = 0 \Leftrightarrow_{\text{Lemma 4.6.4}}$$
$$\mathcal{H}^{c_w+j}_{(X(w_1)\cap U)\cap(X(w_2)\cap U)}(\mathcal{O}_U) = 0.$$

By construction $\emptyset \neq C(v) \cap U \subset X(w_i) \cap U \subset X(z) \cap U \subset U$ and furthermore $X(w_i) \cap U \subset U$ and $X(z) \cap U \subset U$ is closed, irreducible and smooth. Due to Lemma 4.6.5 $c_w = c_{w_1} + c_{w_2} - c_z = c_z + (c_{w_1} - c_z) + (c_{w_2} - c_z)$. Now we wish to use Corollary 3.3.8 with $X = U$, $Y = X(z) \cap U$ $Z_i = X(w_i) \cap U$ and since $\mathrm{codim}_Y(Z_i) = c_{w_i} - c_z$ and $\mathrm{codim}(Y) = c_z$ the Proposition follows. □

We are now ready to prove Theorem 4.6.1.

Proof. That $[\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)] = 0\ \forall j > 0$ follows due to Proposition 4.6.9. The Theorem now follows due to Corollary 4.1.2 and Proposition 4.6.2 since

$$[\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)] =$$
$$\sum_{z \leq w} (-1)^{l(w)-l(z)} [\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X) : \mathcal{L}(v)] + \sum_{j=1}^{\infty} (-1)^{j-1} [\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)] = 1.$$

□

4.7 $[\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v_i)]$ if $l(w) - l(v_i) = 3$

We wish to prove the following Theorem in this section.

Theorem 4.7.1. *Suppose* $v \in \mathrm{maxSing(X(w))}$ *and* $l(w) - l(v) = 3$. *Then*

$$[\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)] = 0,\ \forall j > 0$$
$$[\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)] = 1,$$
$$[\mathcal{H}^{c_w}_{C(w)}(\mathcal{O}_X) : \mathcal{L}(v)] = 2.$$

Especially $X(v) \subset \mathrm{Supp}(\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)/\mathcal{L}(w))$.

Since $v \in \mathrm{maxSing}(X(w))$ it follows by Theorem 4.5.6, that $\tilde{w} = w_{k,m}$ and $\tilde{v} = x_{k,m}$ or $\tilde{w} = w_{k,l,m}$ and $\tilde{v} = x_{k,l,m}$. According to Proposition 4.5.5

$$3 = l(\tilde{w}) - l(\tilde{v}) \Rightarrow_{\text{see (4.13), (4.16) in section 4.5}}$$
$$\tilde{w} = w_{2,2} \wedge\ \tilde{v} = x_{2,2} \text{ or } \tilde{w} = w_{1,2,1} \wedge\ \tilde{v} = x_{1,2,1}.$$

Since the case $\tilde{w} = w_{1,2,1}$ is treated as Theorem 4.6.1 in section 4.6 apart from the part with $[\mathcal{H}^{c_w}_{C(w)}(\mathcal{O}_X) : \mathcal{L}(v_i)]$, we assume $\tilde{w} = w_{2,2}$.

Propositon 4.7.2. $\sum_{z \le w} (-1)^{l(w)-l(z)} [\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X) : \mathcal{L}(v)] = 1.$

Proof. It follows by combining Theorem 4.5.7 and (4.12), that

$$P_{v,w} = 1 + q.$$

Then the proof is identical to the proof of Proposition 4.6.2. □

The idea in the proof of Theorem 4.7.1 is to construct $z \in S_n$ such that $w \le z$, $l(z) - l(w) = 1$ and $C(v) \subset (\mathrm{Sing}(X(z)))^c = U$. It then follows by Corollary 3.3.7 for $j > 0$

$$\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X)|_U = \mathcal{H}^{c_w+j}_{X(w)\cap U}(\mathcal{O}_U) = 0 \Rightarrow [\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)] = 0$$

It follows by construction of $\tilde{w}$, that

$$\begin{aligned} &\Delta(v,w) = \{d_1 < d_2 < d_3 < d_4\} \Rightarrow_{(4.9) \text{ in section } 4.5} \\ &v(i) = w(i) \ \forall i \notin \{d_1 < d_2 < d_3 < d_4\}. \end{aligned} \tag{4.22}$$

Since $\tilde{w} = w_{2,2}$ and $\tilde{v} = x_{2,2}$ we get

$$\begin{array}{ccccccc} w(d_4) & < & w(d_2) & < & w(d_3) & < & w(d_1) \\ \| & & \| & & \| & & \| \\ v(d_2) & < & v(d_1) & < & v(d_4) & < & v(d_3) \end{array}$$

Pick $p \in \{d_2 + 1, \dots, d_3 - 1\}$. It follows due to Lemma 4.5.3, that

$$\begin{aligned} &0 = d_{v,w}(p, v(p)) = \sharp\{i \le p |\ w(i) \ge v(p)\} - \sharp\{i \le p |\ v(i) \ge v(p)\} = \\ &\sharp\{i \in \{1,2\} |\ w(d_i) \ge v(p)\} - \sharp\{i \in \{1,2\} |\ v(d_i) \ge v(p)\} \Rightarrow \\ &w(p) = v(p) < w(d_4) \vee\ w(p) = v(p) > w(d_1). \end{aligned} \tag{4.23}$$

Let us define $z \in S_n$ as

$$z := wt_{d_2,d_3} \Rightarrow\ z(i) = w(i) \ \forall i \notin \Delta(v,w) \wedge\ z(d_4) < z(d_3) < z(d_2) < z(d_1).$$

Lemma 4.7.3. *$w \le z$, $l(z) - l(w) = 1$ and $C(v) \subset (\mathrm{Sing}(X(z)))^c$.*

Proof. To show $w \le z$ we use Lemma 4.5.1. Since $z(i) = w(i) \ \forall i \notin \{d_2, d_3\}$

$$id = \mathrm{fl}_{\{d_2,d_3\}}(w) \le \mathrm{fl}_{\{d_2,d_3\}}(z)$$

$w \le z$. To prove $l(z) - l(w) = 1$ is thanks to Proposition 4.5.5 the same as proving $l(\tilde{z}) - l(\tilde{w}) = 1$. So if just $\Delta(w,z) = \{d_2, d_3\}$. Since $w(d_2) \ne z(d_2)$

and $w(d_3) \neq z(d_3)$ we have due to Lemma 4.5.3 proved $\supset$. Pick $p \notin \{d_2, d_3\}$. Since $z(p) = w(p)$, we due to Lemma 4.5.3 have to prove $d_{w,z}(p, w(p)) = 0$.

$$d_{w,z}(p, w(p)) = \sharp\{i \leq p|\ wt_{d_2,d_3}(i) \geq w(p)\} - \sharp\{i \leq p|\ w(i) \geq w(p)\}$$

we get $d_{w,z}(p, w(p)) = 0$ if $p \notin \{d_2+1, \ldots, d_3-1\}$. If $p \in \{d_2+1, \ldots, d_3-1\}$ $d_{w,z}(p, w(p)) \neq 0 \Leftrightarrow w(d_3) > w(p) > w(d_2)$. But this is due to (4.23) impossible and $l(z) - l(w) = 1$.
Now we lack to prove the last fact. Let us show $\Delta(v, z) = \{d_1, d_2, d_3, d_4\}$. By construction $z(d_i) \neq v(d_i)$ and $\supset$ follows due to Lemma 4.5.3. So pick $p \notin \{d_1, d_2, d_3, d_4\}$. Since $v(p) = w(p) = z(p)$ we just have to prove $d_{v,z}(p, v(p)) = 0$ due to Lemma 4.5.3. Since $d_{v,z} = d_{v,w} + d_{w,z}$ and also $p \notin \Delta(v, w) \cup \Delta(w, z)$, we get

$$d_{v,z}(p, v(p)) = d_{v,w}(p, v(p)) + d_{w,z}(p, w(p)) = 0.$$

Since $\tilde{z} = \mathrm{fl}_{\{d_1,d_2,d_3,d_4\}}(z)$ is the longest element in S_4 $X(\tilde{z})$ is smooth. Thanks to Theorem 4.5.4 $\sharp\mathcal{R}(\tilde{v}, \tilde{z}) = l(\tilde{z}) - l(\tilde{v})$ and we get due to Proposition 4.5.5 $\sharp\mathcal{R}(z, v) = l(v) - l(z)$ and due to Theorem 4.5.4 the Lemma. □

Propositon 4.7.4. $\mathrm{Supp}(\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X)) \subset \mathrm{Sing}(X(z))\ \forall j > 0$. *Especially* $[\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)] = 0\ \forall j > 0$.

Proof. $[\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)] > 0 \Rightarrow C(v) \subset X(v) \subset \mathrm{Supp}(\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X))$. So if just the first part of the Proposition is true, the other follows due to Lemma 4.7.3. So we just have to prove, that

$$0 = \mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X)|_{(\mathrm{Sing}(X(z)))^c} \simeq \mathcal{H}^{c_w+j}_{X(w)\cap(\mathrm{Sing}(X(z)))^c}(\mathcal{O}_{(\mathrm{Sing}(X(z)))^c})\ \forall j > 0.$$

We use Corollary 3.3.7 with $Z = X(w) \cap (\mathrm{Sing}(X(z)))^c$, $Y = X(z) \cap (\mathrm{Sing}(X(z)))^c$ and $X = (\mathrm{Sing}(X(z)))^c$, and then the Proposition follows since according to Lemma 4.7.3 $l(z) - l(w) = 1 \Rightarrow X(w) \cap (\mathrm{Sing}(X(z)))^c \neq \emptyset \Rightarrow \mathrm{codim}_Y(Z) = 1$. □

We are now ready to prove Theorem 4.7.1.

Proof. That $[\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)] = 0\ \forall j > 0$ follows due to Proposition 4.7.4. Due to Corollary 4.1.2 and Proposition 4.7.2

$$[\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)] =$$

$$\sum_{z \leq w}(-1)^{l(w)-l(z)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X) : \mathcal{L}(v)] + \sum_{j=1}^{\infty}(-1)^{j-1}[\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)] = 1.$$

We now just lack proving $2 = [\mathcal{H}^{c_w}_{C(w)}(\mathcal{O}_X) : \mathcal{L}(v)]$. Now both cases of $\tilde{w}$ can occur. We set

$$[v,w] := \{x \in S_n | v < x < w\}.$$

It follows due to the Corollary in section 7.13 of [28] that

$$\sharp\{x \in S_n | v < x < w,\ l(w) - l(x) = 1\} = \sharp\{x \in S_n | v < x < w,\ l(w) - l(x) = 2\}$$

and furthermore for $x \in [v,w]$ $[\mathcal{H}^{c_x}_{C(x)}(\mathcal{O}_X) : \mathcal{L}(v)] = 1$ since $l(x) - l(v) \leq 2$. It follows by combining Theorem 4.5.7 and (4.12), that

$$P_{v,w}(1) = 2$$

and according to (4.4) in section 4.2

$$[\mathcal{H}^{c_w}_{C(w)}(\mathcal{O}_X) : \mathcal{L}(v)] = -\sum_{x<w} (-1)^{l(w)-l(x)} P_{x,w}(1) [\mathcal{H}^{c_x}_{C(x)}(\mathcal{O}_X) : \mathcal{L}(v)] = 2$$

since $P_{x,w} = 1$ if $l(w) - l(x) \leq 2$, and we are done. □

5 G/B IN POSITIVE CHARACTERISTIC

5.1 F-regularity

Throughout this section we let R denote a finitely generated k-algebra with k a field of positive characteristic $p > 0$ and $M \in R - mod$. We then define $F^e_* M \in R - mod$ as the R-module, which as an abelian group is just R, but $r.m := r^{p^e} m$ where we on the right has used the usual R structure on M. We say that a ring of positive characteristic p is F-finite if $F^1_* R$ is a finitely generated R-module. If k is a perfect field R is F-finite. The concept of global F-regularity was first introduced in [26].

Definition 5.1.1. *If R is F-finite, we say R is strongly F-regular if for every $c \in R$ not in any minimal prime of R, there exists $e \geq 0$ such that the map of R-modules*

$$\begin{aligned} R &\to F^e_* R \\ 1 &\to c \end{aligned}$$

splits.

Let k be algebraic closed and X a projective variety defined over k. The concept of strongly F-regular rings has been extended to projective varieties. This is done in [42].

Definition 5.1.2. *X is globally F-regular, if there exists an ample line bundle $\mathcal{L}$ on X, whose section ring*

$$S(\mathcal{L}) := \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{L}^n)$$

is strongly F-regular.

If X is globally F-regular, then $\mathcal{O}_{X,x}$ is strongly F-regular $\forall x \in X$ according to Proposition 1.2 in [22]. If X is globally F-regular $S(\mathcal{L})$ is strong F-regular for any ample line bundle $\mathcal{L}$ on X according to Theorem 3.10 in [42]. We need to know how global F-regularity behaves under morphisms. To do this we need the concept of stably Frobenius splitting along a divisor D. The absolute Frobenius morphism on X is the morphism $F : X \to X$ of schemes, which is the identity on the set of points, but the associated map of sheaves is

$$\begin{aligned} F^\sharp &: \mathcal{O}_X \to F_* \mathcal{O}_X \\ F^\sharp(x) &= x^p. \end{aligned}$$

For $\mathcal{M} \in \mathcal{O}_X - mod$ we denote $F_*(\mathcal{M}) \in \mathcal{O}_X - mod$ such that as a sheaf of abelian groups $F_*(\mathcal{M})$ is just $\mathcal{M}$, but $x.m := x^p m \ \forall x \in \mathcal{O}_X, m \in \mathcal{M}$. According to [39] X is Frobenius split if the map of $\mathcal{O}_X$-modules above splits. We shall now generalize this definition. For an effective Cartier divisor D we let s denote the associated section of the line bundle $\mathcal{O}_X(D)$. We then define X to be Frobenius split along D if the map in $\mathcal{O}_X - mod$

$$\begin{aligned} &\mathcal{O}_X \to F_*\mathcal{O}_X(D) \\ &1 \to s \end{aligned}$$

splits. The case $D = 0$ corresponds to the case above. We say X is stably Frobenius split along D, if there is a positive integer e, such that the map in $\mathcal{O}_X - mod$

$$\begin{aligned} &\mathcal{O}_X \to F_*^e\mathcal{O}_X(D) \\ &1 \to s \end{aligned}$$

splits. The concept of stably Frobenius split along D is examined in [42]. According to Theorem 3.10, Lemma 3.9 and 3.7 in [42], we get the following Proposition.

Propositon 5.1.3. *Let D, D' be two effective Cartier divisors. If $D' \leq D$ and X is stably Frobenius split along D, then X is stably Frobenius split along pD, and X is also stably Frobenius split along D'. Furthermore the following three conditions are equivalent.*
(a): X is global F-regular.
(b): X is stably Frobenius split along every effective Cartier divisor.
(c): X is stably Frobenius split along some ample effective divisor such that the open set $X \backslash D$ is locally strongly F-regular.

Here the last condition simply means, that $\forall x \in X \backslash D \ \mathcal{O}_{X,x}$ is strongly F-regular. Since regular local rings are strongly F-regular, which is proved in [26], we get that, if X is smooth, X is global F-regular if and only if X is stably Frobenius split along an ample effective divisor. Suppose X is Frobenius split. This means, that

$$\begin{aligned} &\exists \phi \in \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)(X) \\ &\phi \circ F^\sharp = id_{\mathcal{O}_X}. \end{aligned}$$

Let $Y \subset X$ be a closed subset with sheaf of ideals $\mathcal{I}_Y$. We then say, that Y is compatibly Frobenius split if $\phi(F_*\mathcal{I}_Y) \subset \mathcal{I}_Y$. This concept was also introduced in [39]. Given an effective Cartier divisor D we denote $Y(D)$ as the corresponding closed subscheme of X.

Lemma 5.1.4. *Let X be Frobenius split and D be an effective Cartier divisor such that $Y(D)$ is compatibly Frobenius split, then X is stably Frobenius split along (p-1)D.*

Proof. Let $\mathcal{I}_Y$ be the ideal sheaf of $Y(D)$. By construction we get, that the map in $\mathcal{O}_X - mod$

$$\begin{aligned} &\mathcal{I}_Y \rightarrow F_*\mathcal{I}_Y \\ &x \rightarrow x^p \end{aligned}$$

splits, and since $\mathcal{O}_X(-D) \simeq \mathcal{I}_Y$ and if we let s^n be the associated section of $\mathcal{O}_X(nD)$, we get that the map in $\mathcal{O}_X - mod$

$$\begin{aligned} &\mathcal{O}_X(-D) \rightarrow F_*\mathcal{O}_X(-D) \\ &s^{-1} \rightarrow s^{-1} \end{aligned}$$

splits. Since $\mathcal{O}_X(D)$ is a locally free sheaf in $\mathcal{O}_X - mod$, we get that the map in $\mathcal{O}_X - mod$

$$\begin{aligned} &\mathcal{O}_X \rightarrow F_*\mathcal{O}_X(-D) \bigotimes_{\mathcal{O}_X} \mathcal{O}_X(D) \\ &1 \rightarrow s^{-1} \otimes_{\mathcal{O}_X} s \end{aligned}$$

splits. But according to the projection formula

$$\begin{aligned} &F_*\mathcal{O}_X(-D) \bigotimes_{\mathcal{O}_X} \mathcal{O}_X(D) \simeq F_*(\mathcal{O}_X(-D) \bigotimes_{\mathcal{O}_X} F^*(\mathcal{O}_X(D))) \simeq \\ &F_*(\mathcal{O}_X(-D) \bigotimes_{\mathcal{O}_X} \mathcal{O}_X(pD)) \simeq F_*\mathcal{O}_X((p-1)D) \end{aligned}$$

and the isomorphism is given by $s^{-1} \otimes_{\mathcal{O}_X} s \rightarrow s^{p-1}$ and we then get, that the map in $\mathcal{O}_X - mod$

$$\begin{aligned} &\mathcal{O}_X \rightarrow F_*\mathcal{O}_X((p-1)D) \\ &1 \rightarrow s^{p-1} \end{aligned}$$

splits. □

This Lemma enables us to prove what we are looking for.

Propositon 5.1.5. *Let X, Y be two projective varieties over k and $\pi : X \rightarrow Y$ a morphism such that $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$. If X is globally F-regular, then so is Y. So if π is birational, X globally F-regular and Y normal, then Y is also globally F-regular.*

Proof. According to Proposition 5.1.3 we have to show for all D effective Cartier divisors on Y, that Y is stably Frobenius split along D. Let D' be the pull-back of D to X. Let s be the associated section of $\mathcal{O}_Y(D)$ and s' the associated section of $\mathcal{O}_X(D')$. It then follows by Proposition 5.1.3, that there is a positive integer e such that the map in $\mathcal{O}_X - mod$

$$\begin{aligned}&\mathcal{O}_X \to F_*^e \mathcal{O}_X(D')\\&1 \to s'\end{aligned}$$

splits and since $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$, we get $\pi_*(F_*^e\mathcal{O}_X(D')) = F_*^e\mathcal{O}_Y(D)$ and thus also a splitting

$$\begin{aligned}&\mathcal{O}_Y \to F_*^e \mathcal{O}_Y(D)\\&1 \to s.\end{aligned}$$

The last part follows since the conditions ensures, that $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$. □

5.2 Schubert varieties and F-regularity

We now return to the setup of section (4.1), where we considered G/B for G a simply connected semisimple linear algebraic group and $B \subset G$ a Borel subgroup. We let W denote the Weyl group of G, which is generated by the simple reflections $s_1, \dots, s_n$ numbered from left to right in the Dynkin diagram. We know, that

$$G/B \supset X(s_j) = B \cup Bs_jB/B \simeq \mathbb{P}^1$$

and that $B \cup Bs_jB \subset G$ is a minimal parabolic subgroup containing B without being equal to B. We set $P_j := B \cup Bs_jB$ and let $w = (s_{j_1}, \dots, s_{j_r})$ denote a collection of simple reflections. We then set

$$P_w := P_{j_1} \times P_{j_2} \times \dots \times P_{j_r}$$

and see, that there is a right B^r action on P_w defined as

$$(p_1, \dots, p_r).(b_1, \dots, b_r) := (p_1b_1, b_1^{-1}p_2b_2, \dots, b_{r-1}^{-1}p_rb_r).$$

The quotient of this action is denoted $Z(w)$ and is called the Bott-Samelson variety. That it is a smooth, projective variety of dimension r is proved in chapter 13 of [29]. We let $\phi : P_w \to Z(w)$ denote the canonical morphism. Let $Z_i := Z((s_{j_1}, \dots, \widehat{s_{j_i}}, \dots, s_{j_r}))$, then Z_i is a closed irreducible subset of $Z(w)$, and it is given as

$$\phi(\{(p_1, \dots, p_r)|\ p_i = id_G\}) \subset Z(w).$$

Propositon 5.2.1. *Z(w) is globally F-regular.*

Proof. Since $Z(w)$ is smooth we just have to find an ample effective divisor D such that, $Z(w)$ is stably Frobenius split along D according to the discussion following Proposition 5.1.3. According to Theorem 1 in [39] there exists a Frobenius splitting of $Z(w)$ compatibly splitting the effective divisor $\sum_{m=1}^{r} Z_m$. But by combining Lemma 5.1.4 and Proposition 5.1.3, we get, that $Z(w)$ is stably Frobenius split along any divisor of the form $\sum_{m=1}^{r} a_m Z_m$ with $a_m \in \mathbb{N}$, and then the result follows by Lemma 6.1 in [37], since it is here shown, that there exists integers $a_m > 0$ such that $D = \sum_{m=1}^{r} a_m Z_m$ is ample. □

This Proposition enables us to prove the main Theorem of this section.

Theorem 5.2.2. *Let $B \subset P \subset G$ be a parabolic group. All Schubert varieties in G/P are globally F-regular.*

Proof. Let us start out with the case $B = P$ and let $w = s_{j_1} s_{j_2} \dots s_{j_r}$ such that $l(w) = r$, we then get according to [29] chapter 13, that the morphism

$$P_{(s_{j_1},\dots,s_{j_r})} = P_{j_1} \times P_{j_2} \times \cdots \times P_{j_r} \to X(w)$$
$$(p_1, \dots, p_r) \to p_1 p_2 \dots p_r B$$

induces a birational morphism

$$Z((s_{j_1}, s_{j_2} \dots, s_{j_r})) \to X(w)$$

and since according to Proposition 14.15 in [29] $X(w)$ is normal, we get by combining Proposition 5.2.1 and Proposition 5.1.5 that $X(w)$ is globally F-regular, and the Theorem is proved for $B = P$.

Let us consider the canonical map

$$\pi : G/B \to G/P$$

and take a Schubert variety $X(w)_P \subset G/P$. According to page 391 in [29] there exists $v \in W$ such that

$$\pi|_{X(v)} : X(v) \to X(w)_P$$

and $\pi_*(\mathcal{O}_{X(v)}) = \mathcal{O}_{X(w)_P}$, and since $X(v)$ is globally F-regular, the Theorem follows by Proposition 5.1.5. □

5.3 Simplicity of $\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)$

We let $\mathcal{R}$ denote a sheaf of rings on the variety X defined over the algebraic closed field k, which has arbitrary characteristic.

Lemma 5.3.1. *Let $\mathcal{F} \in \mathcal{R} - mod$ be simple and $U \subset X$ be open. Then $\mathcal{F}|_U$ is simple in $\mathcal{R}|_U - mod$.*

Proof. Let $i : U \to X$ be the inclusion. We then get that the restriction map $\mathcal{F} \to i_*(\mathcal{F}|_U)$ is a map in $\mathcal{R} - mod$. Let $\mathcal{M} \subset \mathcal{F}|_U$ be a $\mathcal{R}|_U$-submodule. We then have an exact sequence in $\mathcal{R}|_U - mod$

$$0 \to \mathcal{M} \to \mathcal{F}|_U \to \mathcal{F}|_U/\mathcal{M} \to 0.$$

We then get a map in $i_*(\mathcal{R}|_U) - mod$

$$i_*(\mathcal{F}|_U) \to i_*(\mathcal{F}|_U/\mathcal{M})$$

and since the restriction map $\mathcal{R} \to i_*(\mathcal{R}|_U)$ is a ring homomorphism of sheaves, we get a map in $\mathcal{R} - mod$

$$\phi : \mathcal{F} \to i_*(\mathcal{F}|_U) \to i_*(\mathcal{F}|_U/\mathcal{M}).$$

Then $\ker(\phi)$ is a $\mathcal{R}$-submodule of $\mathcal{F}$ and is thus either 0 or $\mathcal{F}$, which implies, that $\ker(\phi)|_U$ is either 0 or $\mathcal{F}|_U$ and since $\ker(\phi)|_U = \mathcal{M}$, the result follows. □

This Lemma will be used to prove the following Lemma.

Lemma 5.3.2. *Let $\mathcal{F} \in \mathcal{R} - mod$ have finite length. Assume furthermore, that $\mathcal{F} \in \mathcal{O}_X - mod$ is quasi-coherent. Then* $\mathrm{Supp}(\mathcal{F})$ *is closed.*

Proof. Since $\mathcal{F}$ has finite length, there exists $\mathcal{F}_i, \mathcal{L}_i \in \mathcal{R} - mod$ with $\mathcal{L}_i$ simple such that

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{n-1} \subset \mathcal{F}_n = \mathcal{F},$$
$$\mathcal{F}_i/\mathcal{F}_{i-1} = \mathcal{L}_i.$$

Since there is an exact sequence in $\mathcal{R} - mod$

$$0 \to \mathcal{F}_{i-1} \to \mathcal{F}_i \to \mathcal{L}_i \to 0$$

we see by induction, that we just have to prove, that $\mathrm{Supp}(\mathcal{L}_i)$ is closed. We drop the $_i$. So let

$$X = \bigcup_{i=1}^{n} U_i$$

be an affine open covering of X. Then

$$\mathcal{L}|_{U_i} = 0 \Leftrightarrow \mathcal{L}(U_i) = 0.$$

So assume $\mathcal{L}(U_i) \neq 0$ and pick $f_i \in \mathcal{L}(U_i)$. If $\mathcal{L}(U_i) = 0$ we set $f_i = 0$. If $f_i \neq 0$ it follows by Lemma 5.3.1, that $\mathcal{L}|_{U_i}$ is simple in $\mathcal{R}|_{U_i} - mod$ and therefore

$$\mathcal{L}|_{U_i} = \mathcal{R}|_{U_i} f_i.$$

This is also true if $f_i = 0$ and therefore

$$\mathrm{Supp}(\mathcal{L}|_{U_i}) = \mathrm{Supp}(f_i) \tag{5.1}$$

is closed in U_i and therefore $(\mathrm{Supp}(f_i))^c \cap U_i$ is open in X. If just

$$(\mathrm{Supp}(\mathcal{L}))^c = \bigcup_{i=1}^{n}((\mathrm{Supp}(f_i))^c \cap U_i)$$

we are done. This is due to (5.1) clear. □

We now return to the case $\mathrm{char}(k) > 0$ and briefly sketch the theory of unit $R[F]$-modules. The whole point with this is to find a replacement of holonomic $\mathcal{D}_X$-modules, when X is an algebraic smooth variety defined over a field of characteristic zero. The concept $uR[F] - mod$ was first introduced and examined by Lyubeznik in [38]. For a complete treatment of this topic one should look up Lyubezniks article [38] or Blickles article [4]. We now assume, that apart from the already assumed properties of R, R also satisfies R_p is a regular local ring $\forall p \in \mathrm{Spec}(R)$. Let $M \in R - mod$. We then denote M^e as the R bimodule, which in $R - mod$ is just M, but in $mod - R$ is M as an abelian group, and the module structure is defined as $m.r := r^{p^e} m$ $\forall r \in R\ \forall m \in M$.

Definition 5.3.3. *Let $M \in R - mod$. Then $M \in R[F^e] - mod$ if there exists an R-linear map*

$$\vartheta^e : R^e \bigotimes_R M \to M.$$

If ϑ^e is an isomorphism, then (M, ϑ^e) is called a unit $R[F^e]$-module, and it is denoted as $uR[F^e] - mod$.

The module structure on the lefthandside is the one stemming from the left module structure of R^e, and in the tensor product we use the right module structure of R^e, such that $r \otimes_R r_1 m = r_1^{p^e} r \otimes_R m\ \forall r, r_1 \in R,\ m \in M$.

By setting X =spec(R) and $\mathcal{M} = \tilde{M}$, we see that $M \in R[F^1] - mod \Leftrightarrow \exists \phi \in \mathcal{H}om_{\mathcal{O}_X}(F^*(\mathcal{M}), \mathcal{M})(X)$ with F the Frobenius morphism on X. Since $R^e \otimes_R R \simeq R$, $R \in uR[F^e] - mod$. Another example is R_S for $S \subset R$ a multiplicative closed subset. Define

$$\vartheta^e : R^e \otimes_R R_S \to R_S,\ \vartheta^e(r_1 \otimes_R \frac{r}{s}) := \frac{r_1 r^{p^e}}{s^{p^e}},$$
$$(\vartheta^e)^{-1} : R_S \to R^e \otimes_R R_S,\ (\vartheta^e)^{-1}(\frac{r}{s}) := s^{p^e-1} r \otimes_R \frac{1}{s}.$$

With these definitions one sees $R_S \in uR[F^e] - mod$. For $M \in R[F^e] - mod$ set $F^e_M(m) := \vartheta^e(1 \otimes m)$. Then $F^e_M(r.m) = r^{p^e} F^e_M(m)\ \forall r \in R$ and $\forall m \in M$. Let $R[F^e]$ be the non-commutative ring obtained from R by adjoining the noncommutative variable F^e and forcing the relation $r^{p^e} F^e = F^e r\ \forall r \in R$. We then see, that $M \in R[F^e] - mod$ by the above definition implies, that $M \in R[F^e] - mod$ where we consider $R[F^e]$ as a ring. If on the other hand $M \in R[F^e] - mod$ we define ϑ^e as

$$\vartheta^e(r \otimes m) := rF^e(m)$$

and the two identifications of $R[F^e]-mod$ are identical. We have an inclusion of rings $R[F^{re}] \subset R[F^e] \Rightarrow R[F^e] - mod \subset R[F^{re}] - mod$. We denote $R[F] - mod$ as the direct limes of the categories $\{R[F^e] - mod|\ e \in \mathbb{N}\}$. If $M \in uR[F^e] - mod$ then $M \in uR[F^{er}] - mod\ \forall r \in \mathbb{N}$. Since

$$R^{er} \simeq R^e \bigotimes_R R^{e(r-1)}$$
$$r \to\ r \otimes_R 1$$
$$r \otimes_R r_1 \to r_1^{p^e} r$$

we can show the above inductive since

$$R^{er} \bigotimes_R M \simeq R^e \bigotimes_R (R^{e(r-1)} \bigotimes_R M) \simeq R^e \bigotimes_R M \simeq M.$$

This implies that $\{uR[F^e] - mod|\ e \in \mathbb{N}\}$ is a directed system and we denote its direct limes as $uR[F] - mod$. This is contained in $R[F] - mod$. We say, that $M \in R[F] - mod$ is finitely generated if $M \in R[F^e] - mod$ is finitely generated. This is independent of e. We then get the following Theorem, which is proved as Theorem 3.2 in [38].

Theorem 5.3.4. *Let $M \in uR[F] - mod$ be finitely generated. Then M has finite length in $R[F] - mod$.*

It follows by section 3.1 (3.2), that

$$D(R) = \cup \mathrm{End}_{R^{(e)}}(R) = \cup \mathrm{End}_R(R^e)$$

where we use the right R module structure in the last equality. We let $M \in uR[F^e] - mod$ and $\delta \in \mathrm{End}_R(R^r)$. Since $M \in uR[F^{er}] - mod$, we get by composition an action of δ on M

$$M \overset{(\vartheta^{er})^{-1}}{\longrightarrow} R^{er} \bigotimes_R M \overset{\delta \bigotimes_R \mathrm{id}}{\longrightarrow} R^{er} \bigotimes_R M \overset{\vartheta^{er}}{\longrightarrow} M.$$

This is independent of r and the inclusion $uR[F^e] - mod \subset uR[F^{ne}] - mod$. We have therefore a well defined functor from $uR[F] - mod$ to $D(R) - mod$. According to Theorem 5.7 (b) in [38] the Theorem below is true.

Theorem 5.3.5. *Let $I \subset R$ be an ideal. Then $H^j_I(R)$ has finite length in $D(R) - mod$.*

We also need the concept of a F-rational ring and F-regular. Given $I \subset R$ an ideal, we denote $I^{[p^e]}$ the ideal in R generated by all p^e 'th powers of the elements in I. For a complete treatment of these topics one should look up [26]. We denote the tight closure of I as I^*. It is defined as those $r \in R$ such that $\exists c \in R$ not contained in any minimal prime of R such that

$$cr^{p^e} \in I^{[p^e]} \; \forall e \gg 0.$$

Then I^* is an ideal containing I. R is weakly F-regular if $I^* = I$ for all ideals in R. An ideal is generated by parameters if it has height t and there exists t elements in R generating the ideal or it is the unit ideal. Then R is F-rational if $I^* = I$ for all ideals generated by parameters. This means

$$R \text{ weakly } F\text{−regular} \Rightarrow R \; F\text{−rational}.$$

Now it follows by Theorem 5.5 (d) in [27] along with definition 3.2 in [27]

$$R \text{ strongly } F\text{−regular} \Rightarrow R \text{ weakly } F\text{−regular}.$$

Propositon 5.3.6. *$\mathcal{O}_{X(w),x}$ is F-rational $\forall x \in X(w)$.*

Proof. Due to Theorem 5.2.2 $X(w)$ is globally F-regular, and thus $\mathcal{O}_{X(w),x}$ is strongly F-regular $\forall x \in X(w)$ and then the Proposition follows. □

For a proof of the Theorem below one should look up corollary 4.10. in [4].

Theorem 5.3.7. *Let R be regular, local and F-finite and $I \subset R$ a prime ideal of height c. If R/I is F-rational, then $H_I^c(R)$ is simple in $D(R)$-mod.*

We start out by proving a sheaf version of the Theorem above.

Propositon 5.3.8. *Let $Y \subset X$ be a closed irreducible subset of codimension c and X smooth and irreducible. If $\mathcal{O}_{Y,y}$ is F-rational $\forall y \in Y$, then $\mathcal{H}_Y^c(\mathcal{O}_X)$ is simple in $\mathcal{D}_X - mod$.*

Proof. Let us start out by assuming X affine. $X = \text{Spec}(A)$ $Y = V(I)$ with $I \in \text{Spec}(A)$ and $\text{height}(I) = c$. Let us have an exact sequence in $\mathcal{D}_X - mod$

$$0 \to \mathcal{L} \to \mathcal{H}_Y^c(\mathcal{O}_X) \to \mathcal{M} \to 0.$$

Due to Proposition 3.2.4 we get an exact sequence in $D(A_y) - mod$ $\forall y \in Y$

$$0 \to (\mathcal{L})_y \to H_{IA_y}^c(A_y) \to (\mathcal{M})_y \to 0.$$

Since $\text{Supp}(\mathcal{H}_Y^c(\mathcal{O}_X)) = Y$ and the middle term above is simple in $D(A_y) - mod$ according to Theorem 5.3.7, we get that

$$Y = \text{Supp}(\mathcal{L}) \cup \text{Supp}(\mathcal{M}) \ , \ \emptyset = \text{Supp}(\mathcal{L}) \cap \text{Supp}(\mathcal{M}).$$

So if only $\text{Supp}(\mathcal{L})$ and $\text{Supp}(\mathcal{M})$ are closed we are done, and therefore we just have to prove according to Lemma 5.3.2, that $\mathcal{M}$ and $\mathcal{L}$ have finite length in $\mathcal{D}_X - mod$. That this is true follows, since we have an exact sequence in $D(A) - mod$

$$0 \to \mathcal{L}(X) \to H_I^c(A) \to \mathcal{M}(X) \to 0$$

and the middle term according to Theorem 5.3.5 has finite length in $D(A) - mod$. If X is not affine let $X = \cup_{i=1}^n U_i$ be an open affine covering of X. Let $\mathcal{L} \subset \mathcal{H}_Y^c(\mathcal{O}_X)$ be a $\mathcal{D}_X$-submodule. Then since $\mathcal{H}_Y^c(\mathcal{O}_X)|_{U_i}$ is simple $\mathcal{L}|_{U_i} = \mathcal{H}_Y^c(\mathcal{O}_X)|_{U_i}$ or $\mathcal{L}|_{U_i} = 0$. Then the Proposition follows since X is irreducible and therefore $U_i \cap U_j \neq \emptyset$. □

This Proposition enables us to prove the main Theorem of this section.

Theorem 5.3.9. *$\mathcal{H}_{X(w)}^{c_w}(\mathcal{O}_X)$ is simple in $\mathcal{D}_X - mod$.*

Proof. The Theorem follows by combining Proposition 5.3.8 and 5.3.6. □

We denote by $\mathcal{O}_{F,X}$ the quasi-coherent sheaf of rings, which on an open affine subset spec(R) is $R[F^1]$. We define $u\mathcal{O}_{F,X} - mod$ as $\mathcal{M} \in u\mathcal{O}_{F,X} - mod$ if $\mathcal{M}$ is quasi-coherent in $\mathcal{O}_X - mod$ and $F^*(\mathcal{M}) \simeq \mathcal{M}$ with F the Frobenius morphism on X. So by construction if $U \subset X$ is open and affine $\mathcal{M}(U) \in u\mathcal{O}_X(U)[F^1]-mod$. For $\mathcal{M} \in u\mathcal{O}_{F,X}-mod$ we shall say $\mathcal{M}$ is locally finitely generated if $X = \cup_{i=1}^n U_i$ $U_i \subset X$ open and affine such that $\mathcal{M}(U)$ is finitely generated in $\mathcal{O}_X(U)[F^1]-mod$. For $X = G/B$ we let $\mathcal{D}_X - mod$ denote the category of modules $\mathcal{M}$, which has a $\mathcal{D}_X$-module structure such that it is quasi-coherent with respect to the induced $\mathcal{O}_X \subset \mathcal{D}_X$ module structure, is B-equivariant and $\mathcal{M}$ is locally a finitely generated unit $\mathcal{O}_{F,X}$-module. We then have the following Theorem.

Theorem 5.3.10. *The simple modules in $\mathcal{D}_X - mod$ are $\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)$.*

Proof. If the concept of locally finitely generated unit $\mathcal{O}_{F,X}-mod$ is replaced with the concept of filtration holonomic introduced in [5], we get due to Theorem 4.6 in [5], that in this category the simple objects are parameterized by the Schubert varieties $X(w)$. According to Proposition 9.1 in [6] any locally finitely generated unit $\mathcal{O}_{F,X}$ module is also filtration holonomic. So if just $\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X) \in \mathcal{D}_X - mod$ we are due to Theorem 5.3.9 done. So all there is to show, is that it is locally a finitely generated unit $\mathcal{O}_{F,X}$ module. That this is true follows due to Proposition 2.10 in [38]. □

As has been carefully inspected in section 3, this is far from being the case if $\mathrm{char}(k) = 0$. We set $\mathcal{L}(w) := \mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X)$. Let us derive another difference between the two situations.

Propositon 5.3.11.

$$[\mathcal{H}^{c_w}_{C(w)}(\mathcal{O}_X)] = \sum_{v \leq w} [\mathcal{L}(v)].$$

Proof. Due to Proposition 4.1 in [40] $\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) \neq 0 \Leftrightarrow \; j = 0$. We then get due to Corollary 4.1.2, that

$$[\mathcal{L}(v)] = [\mathcal{H}^{c_v}_{X(v)}(\mathcal{O}_X)] = \sum_{z \leq v} (-1)^{l(z)-l(v)} [\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] \Rightarrow$$

$$\sum_{v \leq w} [\mathcal{L}(v)] = \sum_{v \leq w} \sum_{z \leq v} (-1)^{l(z)-l(v)} [\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)].$$

According to [28] Corollary 7.13

$$\sum_{\{v | z \leq v \leq w\}} (-1)^{l(z)-l(v)} = \delta_{z,w}.$$

and therefore we have the Proposition. □

If $\mathrm{char}(k) = 0$ then the coefficients in the Proposition above would be equal to the inverse Kazhdan-Lusztig polynomials evaluated in 1.

6 THE GRASSMANN VARIETY

6.1 Gr(r, n)

We let Gr(r, n) denote the set of r dimensional subspaces in the k^n. We still assume $G = Sl_n$. We then get by [29] chapter 13.10 part II, that

$$\mathrm{Gr}(r, n) = G/P$$

with $B \subset P \subset G$ a maximal parabolic subgroup. Therefore Gr(r, n) is a smooth variety. We know, that $W = S_n$ the group of all permutations on $\{1, 2, \ldots, n\}$. We let $s_i \in W$ denote the simple reflection such that

$$s_i(i) = i + 1,\ s_i(i + 1) = i,\ s_i(j) = j\ \forall j \notin \{i, i + 1\}.$$

Then $W = \langle s_i | i \in \{1, 2, \ldots, n - 1\}\rangle$. It also follows by [29] chapter 13.10 part II, that $\exists j \in \{1, 2, \ldots, n-1\}$ where we use the notation of Appendix A such that

$$\begin{aligned} W^J &= \{\sigma \in W |\ \sigma(1) < \cdots < \sigma(n - r),\ \sigma(n - r + 1) < \cdots < \sigma(n)\}, \\ W_J &= \langle s_i | i \in \{1, 2, \ldots, n - 1\}\ i \neq j\rangle \end{aligned}$$

and w_I is the longest element in W_J. To find j we do the following. Pick $\sigma \in W^J$ such that

$$\sigma(j) = \left\{ \begin{array}{cc} r + j & j \in \{1, \ldots, n - r\} \\ j - n + r & j \in \{n - r + 1, \ldots, n\} \end{array} \right\}.$$

If $\sigma s_{n-r} \leq \sigma$ it follows by the description of W^J given in Lemma A.1.1 in Appendix A, that

$$W_J = \langle s_i | i \in \{1, 2, \ldots, n - 1\}\ i \neq n - r\rangle. \tag{6.1}$$

To prove $\sigma s_{n-r} \leq \sigma$ is done by using Lemma 4.5.1 since $\sigma s_{n-r}(j) = \sigma(j)$ $\forall j \neq n - r, n - r + 1$ and since $\mathrm{fl}_{n-r,n-r+1}(\sigma s_{n-r}) = id$ the result follows. By the description above, we see for $w \in W^J$ combined with Lemma 4.5.2

$$l(w) = \sum_{j=1}^{r} n - (r - j) - \sigma(n - r + j). \tag{6.2}$$

We get a bijection from W^J to all integers $1 \leq a_1 < a_2 < \cdots < a_r \leq n$ given in the following way $a_s := n + 1 - \sigma(n - s + 1)$, and therefore given $\sigma \in W^J$, we denote the Schubert variety $X(w)_P$ as $X(a_1, \ldots, a_r)$ and $C(w)_P$

as $C(a_1,\dots,a_r)$. Let us denote this bijection as Δ. Thus given $X(a_1,\dots,a_r)$ with $1 \le a_1 < a_2 < \dots < a_r \le n$, we get $(\Delta)^{-1}(a_1,\dots,a_r) \in W^J$ such that $X((\Delta)^{-1}(a_1,\dots,a_r))_P = X(a_1,\dots,a_r)$ by

$$((\Delta)^{-1}(a_1,\dots,a_r))(n-r+i) = n+1-a_{r-i+1} \ \forall i \in \{1,2,\dots,r\}.$$

It also follows by [29] chapter 13.10 part II, that

$$\forall v,w \in W^J \ v \le w \Leftrightarrow \Delta(v) = (a_1,\dots,a_r), \Delta(w) = (b_1,\dots,b_r) \wedge a_i \le b_i \ \forall i.$$

Now we need another description of Gr(r,n) and its Schubert varieties. For a proof of all the facts below one should look up [11] chapter 1.D. Gr(r,n) can be described in the following way. Let us denote

$$\mathbf{Mat}^r_{r,n} = \{A \in \mathbf{Mat}_{r,n}(k) | \operatorname{rang}(A) = r\}.$$

Then

$$\mathrm{Gr}(r,n) = \mathbf{Mat}^r_{r,n}/\sim, \ A \sim B \Leftrightarrow \ \exists C \in \mathbf{Gl}_r(k) \text{ with } A = CB.$$

So let $A \in \mathbf{Mat}^r_{r,n}$ and let $[A]$ denote it as an element in Gr(r,n). To make Gr(r,n) into a variety we use the Plücker embedding. Let $i_1 < i_2 < \dots < i_r$ with $i_j \in \{1,2,\dots,n\}$ $\forall j$ and let $P_{i_1,\dots,i_r}(A)$ denote the determinant of the matrix obtained from A by taking the respective columns. Then define

$$\begin{aligned}&\phi : \mathrm{Gr}(r,n) \to \mathbb{P}^{\frac{n!}{r!(n-r!)}-1} = \operatorname{proj}(k[X_{i_1,\dots,i_r}|1 \le i_1 < i_2 < \dots < i_r \le n]),\\ &\phi([A]) = (P_{i_1,\dots,i_r}(A)).\end{aligned}$$

That the above choice is independent of the choice of A and is a closed embedding can be shown. Therefore $\exists I_{r,n} \subset k[X_{i_1,\dots,i_r}]$ a homogeneous ideal such that

$$\begin{aligned}&\phi(\mathrm{Gr}(r,n)) = V(I_{r,n}),\\ &\mathrm{Gr}(r,n) = \operatorname{proj}(k[X_{i_1,\dots,i_r}]/I_{r,n}).\end{aligned}$$

Another way to consider Gr(r,n) is to do the following. Let

$$R = k[X_{ij} | \ i \in \{1,2,\dots,r\}, \ j \in \{1,2,\dots,n\}]$$

be the polynomial ring and denote

$$\begin{aligned}&[i_1,\dots,i_r] := \det \begin{pmatrix} X_{1i_1} & X_{1i_2} & \dots & X_{1i_r} \\ X_{2i_1} & X_{2i_2} & \dots & X_{2i_r} \\ \dots & \dots & \dots & \dots \\ X_{ri_1} & X_{ri_2} & \dots & X_{ri_r} \end{pmatrix},\\ &S := k[\ [i_1,\dots,i_r] \mid 1 \le i_1 < i_2 < \dots < i_r \le n].\end{aligned}$$

So S is the k-algebra generated by $[i_1, \ldots, i_r]$. Then

$$\mathrm{Gr}(r,n) = \mathrm{proj}(S)$$

which is seen by setting $X_{i_1,\ldots,i_r} = [i_1, \ldots, i_r]$. Now we wish to get a describtion of $X(a_1, \ldots, a_r)$ in proj(S). Let $e_1, \ldots, e_n$ be the standard basis of k^n and set

$$V_j := \sum_{i=n+1-j}^{n} ke_i \;\; j > 0, \; V_0 := 0 \Rightarrow \; V_0 \subset V_1 \subset \cdots \subset V_n = k^n.$$

By identifying $\mathrm{Gr}(r,n)$ as the set of r-dimensional subspaces of k^n

$$X(a_1, \ldots, a_r) = \{W \in \mathrm{Gr}(r,n) |\, \dim(W \cap V_{a_i}) \geq i \; \forall i \in \{1, 2, \ldots, r\} \;\}.$$

Due to Theorem 1.4 in [11] we get the following describtion of $X(a_1, \ldots, a_r)$ in proj(S)

$$X(a_1, \ldots, a_r) = V(\langle [i_1, \ldots, i_r] |\; \exists j \in \{1, \ldots, r\} \; i_j < n + 1 - a_{r+1-j} \rangle).$$

Given a Schubert variety $X(a_1, \ldots, a_r) \subset \mathrm{Gr}(r,n)$ let

$$S(a_1, \ldots, a_r) := \{j \in \{1, 2, \ldots r - 1\} |\; a_{j+1} - a_j > 1\} \cup \{r\}.$$

We then get the following couple of Lemmas.

Lemma 6.1.1.

$$X(a_1, \ldots, a_r) \cap X(b_1, \ldots, b_r) = X(\min(a_1, b_1), \ldots, \min(a_r, b_r)).$$

Proof. Let $A = \{1, 2, \ldots, r\}$. Then

$$\begin{aligned}
&X(a_1, \ldots, a_r) \cap X(b_1, \ldots, b_r) = \\
&V(\langle [i_1, \ldots, i_r] |\; \exists j \in A \; i_j < n + 1 - a_{r+1-j} \rangle) \cap \\
&V(\langle [i_1, \ldots, i_r] |\; \exists j \in A \; i_j < n + 1 - b_{r+1-j} \rangle) = \\
&V(\langle [i_1, \ldots, i_r] |\; \exists j \in A \; i_j < n + 1 - b_{r+1-j} \vee \; i_j < n + 1 - a_{r+1-j} \rangle) = \\
&V(\langle [i_1, \ldots, i_r] |\; \exists j \in A \; i_j < \max(n + 1 - b_{r+1-j}, n + 1 - a_{r+1-j}) \rangle) = \\
&V(\langle [i_1, \ldots, i_r] |\; \exists j \in A \; i_j < n + 1 - \min(b_{r+1-j}, a_{r+1-j}) \rangle) = \\
&X(\min(a_1, b_1), \ldots, \min(a_r, b_r)).
\end{aligned}$$

□

This Lemma will be used often without reference.

Lemma 6.1.2.

$$X(a_1,\dots,a_r) = \cap_{r+1-j\in S(a_1,\dots,a_r)} V(\langle [i_1,\dots,i_r]|\ i_j < n+1-a_{r+1-j}\rangle).$$

Proof. We have

$$X(a_1,\dots,a_r) = V(\langle [i_1,\dots,i_r]|\ \exists j\in\{1,\dots,r\}\ i_j < n+1-a_{r+1-j}\rangle) = \\ \cap_{j=1}^{r} V(\langle [i_1,\dots,i_r]|\ i_j < n+1-a_{r+1-j}\rangle).$$

So if we can show, that for $j\in\{1,\dots,r\}$ and $r+1-j\notin S(a_1,\dots,a_r)$

$$V(\langle [i_1,\dots,i_r]|\ i_j < n+1-a_{r+1-j}\rangle) \supset V(\langle [i_1,\dots,i_r]|\ i_{j-1} < n+1-a_{r+1-(j-1)}\rangle) \Leftarrow \\ \langle [i_1,\dots,i_r]|\ i_j < n+1-a_{r+1-j}\rangle \subset \langle [i_1,\dots,i_r]|\ i_{j-1} < n+1-a_{r+1-(j-1)}\rangle$$

we are done. Let us take $[i_1,\dots,i_r]$ satisfying that $i_j < n+1-a_{r+1-j}$. Since $r+1-j\notin S(a_1,\dots,a_r) \Rightarrow a_{r+1-j+1}-a_{r+1-j} = 1$. By construction $i_{j-1}\le i_j-1$ and we get $i_j < n+1-a_{r+1-j} \Rightarrow i_{j-1}\le i_j-1 < n-a_{r+1-j} = n+1-a_{r+1-j+1} = n+1-a_{r+1-(j-1)}$. □

Lemma 6.1.3. *There are closed immersions* $i:\mathrm{Gr}(r-1,n-1)\to\mathrm{Gr}(r,n)$ *and* $j:\mathrm{Gr}(r,n-1)\to\mathrm{Gr}(r,n)$ *such that*

$$i(X(a_1,\dots,a_{r-1})) = X(1,a_1+1,a_2+1,\dots,a_{r-1}+1), \\ j(X(a_1,\dots,a_r)) = X(a_1,\dots,a_r).$$

Proof. Consider the k-algebra homomorphisms defined in the following ways

$$R = k\begin{bmatrix} X_{1,1} & \dots & X_{1,n-1} & X_{1,n} \\ X_{2,1} & \dots & X_{2,n-1} & X_{2,n} \\ \dots & \dots & \dots & \dots \\ X_{r-1,1} & \dots & X_{r-1,n-1} & X_{r-1,n} \\ X_{r,1} & \dots & X_{r,n-1} & X_{r,n}\end{bmatrix} \to k\begin{bmatrix} X_{1,1} & \dots & X_{1,n-1} & 0 \\ X_{2,1} & \dots & X_{2,n-1} & 0 \\ \dots & \dots & \dots & \dots \\ X_{r-1,1} & \dots & X_{r-1,n-1} & 0 \\ 0 & \dots & 0 & 1\end{bmatrix}$$

$$R = k\begin{bmatrix} X_{1,1} & X_{1,2} & \dots & X_{1,n-1} & X_{1,n} \\ X_{2,1} & X_{2,2} & \dots & X_{2,n-1} & X_{2,n} \\ \dots & \dots & \dots & \dots & \dots \\ X_{r,1} & X_{r,2} & \dots & X_{r,n-1} & X_{r,n}\end{bmatrix} \to k\begin{bmatrix} 0 & X_{1,2} & \dots & X_{1,n-1} & X_{1,n} \\ 0 & X_{2,2} & \dots & X_{2,n-1} & X_{2,n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & X_{r,2} & \dots & X_{r,n-1} & X_{r,n}\end{bmatrix} \xrightarrow{\phi} \\ k\begin{bmatrix} X_{1,1} & X_{1,2} & \dots & X_{1,n-1} \\ X_{2,1} & X_{2,2} & \dots & X_{2,n-1} \\ \dots & \dots & \dots & \dots \\ X_{r,1} & X_{r,2} & \dots & X_{r,n-1}\end{bmatrix}$$

with $\phi(X_{i,j}) := X_{i,j-1}$. These induces k-algebra homomorphisms such that

$$\begin{aligned}
&i_* : k[\,[i_1,\ldots,i_r] \mid 1 \le i_1 < i_2 < \cdots < i_r \le n] \to \\
&k[\,[j_1,\ldots,j_{r-1}] \mid 1 \le j_1 < j_2 < \cdots < j_{r-1} \le n-1], \\
&i_*([i_1,\ldots,i_r]) = \left\{ \begin{array}{cc} 0 & i_r \ne n \\ [i_1,\ldots,i_{r-1}] & i_r = n \end{array} \right\}, \\
&j_* : k[\,[i_1,\ldots,i_r] \mid 1 \le i_1 < i_2 < \cdots < i_r \le n] \to \\
&k[\,[j_1,\ldots,j_r] \mid 1 \le j_1 < j_2 < \cdots < j_r \le n-1], \\
&j_*([i_1,\ldots,i_r]) = \left\{ \begin{array}{cc} 0 & i_1 = 1 \\ [i_1-1,\ldots,i_r-1] & i_1 > 1 \end{array} \right\}.
\end{aligned}$$

Then

$$\begin{aligned}
\ker(i_*) &= \langle [i_1,\ldots,i_r] | \; i_r < n \rangle, \\
\ker(j_*) &= \langle [i_1,\ldots,i_r] | \; i_1 < 2 \rangle.
\end{aligned}$$

and since $S(1, n-r+2, \ldots, n) = \{1, r\}$ and $S(n-r, n-r+1, \ldots, n-1) = \{r\}$ we get according to Lemma 6.1.2, that

$$\begin{aligned}
X(1, n-r+2, \ldots, n) &= V(\langle [i_1,\ldots,i_r] | \; i_r < n \rangle), \\
X(n-r, n-r+1, \ldots, n-1) &= V(\langle [i_1,\ldots,i_r] | \; i_1 < 2 \rangle)
\end{aligned}$$

and therefore the first part follows. Since

$$\begin{aligned}
&X(1, a_1+1, \ldots, a_{r-1}+1) = \\
&V(\langle [i_1,\ldots,i_r] | \; \exists j \in \{1,\ldots,r-1\} \; i_j < n+1-(a_{r-1+1-j}+1) \; \vee \; i_r < n \rangle), \\
&X(a_1,\ldots,a_{r-1}) = V(\langle [i_1,\ldots,i_{r-1}] | \; \exists j \in \{1,\ldots,r-1\} \; i_j < n - a_{r-j} \rangle)
\end{aligned}$$

it follows

$$\begin{aligned}
&i_*(\langle [i_1,\ldots,i_r] | \; \exists j \in \{1,\ldots,r-1\} \; i_j < n+1-(a_{r-1+1-j}+1) \; \vee \; i_r < n \rangle) = \\
&\langle [i_1,\ldots,i_{r-1}] | \; \exists j \in \{1,\ldots,r-1\} \; i_j < n - a_{r-j} \rangle
\end{aligned}$$

and then the first half of the second part is proved. To prove the remaining part pick $a_1 < a_2 < \cdots < a_r \le n-1$. Then

$$\begin{aligned}
&\mathrm{Gr}(r,n) \supset X(a_1,\ldots,a_r) = V(\langle [i_1,\ldots,i_r] | \; \exists j \in \{1,\ldots,r\} \; i_j < n+1-a_{r+1-j} \rangle), \\
&\mathrm{Gr}(r,n-1) \supset X(a_1,\ldots,a_r) = V(\langle [i_1,\ldots,i_r] | \; \exists j \in \{1,\ldots,r\} \; i_j < n - a_{r+1-j} \rangle)
\end{aligned}$$

and since

$$\begin{aligned}
&j_*(\langle [i_1,\ldots,i_r] | \; \exists j \in \{1,\ldots,r\} \; i_j < n+1-a_{r+1-j} \rangle) = \\
&\langle [i_1,\ldots,i_r] | \; \exists j \in \{1,\ldots,r\} \; i_j < n - a_{r+1-j} \rangle
\end{aligned}$$

the Lemma is proved. □

Lemma 6.1.4. *Let* $X(a_1, \ldots, a_r) \subset \mathrm{Gr}(r, n)$*. Then*

$$\mathrm{codim}(X(a_1, \ldots, a_r)) = rn - \sum_{j=1}^{r} a_j - \frac{r(r-1)}{2}.$$

Especially

$$\begin{aligned}
\mathrm{codim}(j(X(a_1, \ldots, a_r))) &= \mathrm{codim}(X(a_1, \ldots, a_r)) + r, \\
\mathrm{codim}(i(X(a_1, \ldots, a_{r-1}))) &= \mathrm{codim}(X(a_1, \ldots, a_{r-1})) + n - r.
\end{aligned}$$

Proof. According to page 391 of [29]

$$\dim(X(a_1, \ldots, a_r)) = l((\Delta)^{-1}(a_1, \ldots a_r)) =_{\text{see (6.2)}}$$
$$\sum_{j=1}^{r} (n - r + j) - ((\Delta)^{-1}(a_1, \ldots a_r))(n - r + j)).$$

Since $X(a_1, \ldots, a_r)$ is irreducible and $\mathrm{Gr}(r, n)$ is smooth, we get, that

$$\begin{aligned}
&\mathrm{codim}(X(a_1, \ldots, a_r)) = \dim(\mathrm{Gr}(r, n)) - \dim(X(a_1, \ldots, a_r)) = \\
&r(n-r) - \sum_{j=1}^{r} (n - r + j - ((\Delta)^{-1}(a_1, \ldots a_r))(n - r + j)) = \\
&r(n-r) - \sum_{j=1}^{r} (n - r + j - (n + 1 - a_{r+1-j})) = \\
&rn - r^2 - \sum_{s=1}^{r} (a_s - s) = rn - r^2 + \frac{r(r+1)}{2} - \sum_{s=1}^{r} a_s = \\
&rn - \sum_{s=1}^{r} a_s - \frac{r(r-1)}{2}.
\end{aligned}$$

To prove the second part we use Lemma 6.1.3. ☐

We now follow the description given in section 4.D. of [11]. Let

$$\epsilon = (-1)^{\frac{n(n-1)}{2}}.$$

We use the notation of [25] chapter 2.2 and see

$$D_+(\epsilon[n - r + 1, \ldots, n]) \subset \mathrm{Gr}(r, n)$$

is open and affine and for given $X(a_1,\dots,a_r)\subset \mathrm{Gr}(r,n)$ we have

$$X(a_1,\dots,a_r)\cap D_+(\epsilon[n-r+1,n-r+2,\dots,n])\neq\emptyset \tag{6.3}$$

since $X(1,2,\dots,r)=V([i_1,\dots,i_r]|\ \{i_1,\dots,i_r\}\neq\{n-r+1,n-r+2,\dots,n\})$ due to Lemma 6.1.2 hence $X(1,\dots,r)\cap D_+(\epsilon[n-r+1,n-r+2,\dots,n])\neq\emptyset$, and then it follows since $X(1,2,\dots,r)\subset X(a_1,\dots,a_r)$. Let $R_1\subset R$ be the subring defined as

$$R_1=k[X_{ij}|\ i\in\{1,2,\dots r\},\ j\in\{1,2,\dots,n-r\}]$$

and let us for $1\leq a_1<a_2<\dots<a_u\leq r$ and $1\leq b_1<b_2<\dots<b_u\leq n-r$ with $a_i,b_i\in\mathbb{N}$ denote

$$[a_1,a_2,\dots,a_u|b_1,b_2,\dots,b_u]:=\det\begin{pmatrix} X_{a_1b_1} & X_{a_1b_2} & \dots & X_{a_1b_u}\\ X_{a_2b_1} & X_{a_2b_2} & \dots & X_{a_2b_u}\\ \dots & \dots & \dots & \dots\\ X_{a_ub_1} & X_{a_ub_2} & \dots & X_{a_ub_u}\end{pmatrix}.$$

Let us define a k-algebra homomorphism $\phi: S\to R_1$ in the following way. We start out with a k-algebra homomorphism $\tau: R\to R_1$

$$\tau(X_{ij}):=\left\{\begin{array}{cc} X_{ij} & j\notin\{n-r+1,n-r+2,\dots,n\}\\ 1 & j=n-r+s\wedge\ s\in\{1,2,\dots,r\}\wedge\ i=r-s+1\\ 0 & j=n-r+s\wedge\ s\in\{1,2,\dots,r\}\wedge\ i\neq r-s+1\end{array}\right\}.$$

With this definition of τ we define

$$\phi([s_1,\dots,s_r]):=\tau([s_1,\dots,s_r])$$

and furthermore, we have ring isomorphisms

$$S_{(\epsilon[n-r+1,n-r+2,\dots,n])}\simeq S/\langle[n-r+1,\dots,n]-\epsilon\rangle\simeq R_1$$

where $S_{(\epsilon[n-r+1,\dots,n])}$ is the subring of elements of degree 0 in the localized ring $S_{\epsilon[n-r+1,\dots,n]}$, and therefore we have an isomorphism of varieties

$$D_+(\epsilon[n-r+1,\dots,n])\simeq k^{r(n-r)}$$

with the isomorphism given above. We need to get a description of the following $X(a_1,\dots,a_r)\cap D_+(\epsilon[n-r+1,\dots,n])$. This is done since

$$X(a_1,\dots,a_r)=V(\langle[i_1,\dots,i_r]|\ \exists j\in\{1,\dots,r\}\ i_j<n+1-a_{r+1-j}\rangle)\Rightarrow$$
$$D_+(\epsilon[n-r+1,\dots,r])\supset X(a_1,\dots,a_r)\cap D_+(\epsilon[n-r+1,\dots,r])=$$
$$V(\phi(\langle[i_1,\dots,i_r]|\ \exists j\in\{1,\dots,r\}\ i_j<n+1-a_{r+1-j}\rangle)).$$

So in order to do this we will need to get a description of $\phi([i_1, \dots, i_r])$. Let $[i_1, \dots, i_r] \neq [n-r+1, \dots, n]$. Since $i_1 < \cdots < i_r \Rightarrow \exists j \in \{1, \dots, r\}$ such that $i_j \leq n-r$, $i_{j+1} \geq n-r+1$ with the convention $i_{r+1} > n$ then

$$\{i_1, \dots, i_r\} \backslash \{n-r+1, \dots, n\} = \{i_1, \dots, i_j\},$$
$$\{i_1, \dots, i_r\} \cap \{n-r+1, \dots, n\} = \{i_{j+1}, \dots, i_r\}.$$

Then choose $\{c_1, \dots, c_j\}$ such that $c_1 < \cdots < c_j$ and

$$\{c_1, \dots, c_j, i_{j+1} - (n-r), \dots, i_r - (n-r)\} = \{1, \dots, r\}.$$

Then

$$\phi([i_1, \dots, i_r]) = \pm[c_1, c_2, \dots, c_j | i_1, i_2, \dots, i_j].$$

Let us as an example pick $X(2,5) \subset \mathrm{Gr}(2,5)$. In this case

$$R = k[X_{ij} |\ i \in \{1,2\},\ j \in \{1,2,3,4,5\}],$$
$$R_1 = k[X_{ij} |\ i \in \{1,2\},\ j \in \{1,2,3\}]$$

and therefore $X(2,5) = V([i_1, i_2] |\ i_2 < 4) \Rightarrow X(2,5) \cap D_+(\epsilon[4,5]) = V($ideal generated by all $2{\times}2$ minors in $R_1) = V(I)$. In [31] it is proved that $\mathrm{Gr}(2,5)$ is not $\mathcal{D}_{\mathrm{Gr}(2,5)}$-affine when $\mathrm{char}(k) > 0$. One of the main ingredients in the proof, is that $H^3_I(R_1) \neq 0$ when $\mathrm{char}(k) = 0$.

Lemma 6.1.5. *Let $X(a_1, \dots, a_r) \subset \mathrm{Gr}(r,n)$ be a Schubert variety. Then for $a_{r+1-j} \geq r$*

$$V(\phi(\langle [i_1, \dots, i_r] |\ i_j < n+1-a_{r+1-j} \rangle)) =$$
$$V(\langle [b_1, \dots, b_j | c_1, \dots, c_j] |\ c_j < n+1-a_{r+1-j} \wedge\ 1 \leq b_1 < b_2 < \cdots < b_j \leq r \rangle).$$

Proof. Let us start out by proving $\subset$ this is the same as proving

$$\langle \phi[i_1, \dots, i_r] |\ i_j < n+1-a_{r+1-j} \rangle \supset$$
$$\langle [b_1, \dots, b_j | c_1, \dots, c_j] |\ c_j < n+1-a_{r+1-j} \wedge\ 1 \leq b_1 < b_2 < \cdots < b_j \leq r \rangle.$$

To do this all there is to prove, is that

$$\langle \phi[i_1, \dots, i_r] |\ i_j < n+1-a_{r+1-j}\ \wedge i_{j+1} \geq n-r+1 \rangle =$$
$$\langle [b_1, \dots, b_j | c_1, \dots, c_j] |\ c_j < n+1-a_{r+1-j} \wedge\ 1 \leq b_1 < b_2 < \cdots < b_j \leq r \rangle.$$

So assume $\{i_1, \dots, i_r\}$ satisfies $i_j < n+1-a_{r+1-j} \wedge i_{j+1} \geq n-r+1$. Then since $a_{r+1-j} \geq r$

$$\{i_1, \dots, i_r\} \backslash \{n-r+1, \dots, n\} = \{i_1, \dots, i_j\},$$
$$\{i_1, \dots, i_r\} \cap \{n-r+1, \dots, n\} = \{i_{j+1}, \dots, i_r\}.$$

By the description above

$$\phi([i_1,\dots,i_r]) = \pm[e_1,e_2,\dots,e_j|i_1,i_2,\dots,i_j]$$

with $e_1 < e_2 < \cdots < e_j$ chosen such that

$$\{e_1,\dots,e_j,i_{j+1}-(n-r),\dots,i_r-(n-r)\} = \{1,\dots,r\}$$

and since $n-r+1 \leq i_{j+1} < i_{j+2} < \cdots < i_r \leq n$ can be chosen arbitrary so can $e_1,e_2,\dots,e_j$, and we have shown one inclusion. If we just can show

$$\langle\phi[i_1,\dots,i_r]|\ i_j < n+1-a_{r+1-j}\ \wedge i_{j+1} \geq n-r+1\rangle =$$
$$\langle\phi([i_1,\dots,i_r])|\ i_j < n+1-a_{r+1-j}\rangle$$

we are done. So let us take $i_1 < i_2 < \cdots < i_r$ with $i_j < n+1-a_{r+1-j}$. Let us choose s such that

$$\{i_1,\dots,i_r\}\backslash\{n-r+1,\dots,n\} = \{i_1,\dots,i_s\}.$$

Then $s \geq j$ and furthermore

$$\phi[i_1,\dots,i_r] = \pm[e_1,\dots,e_s|\ i_1,\dots,i_s]$$

where $e_1 < e_2 < \cdots < e_s$ is chosen by the procedure above. Since $s \geq j$

$$[e_1,\dots,e_s|\ i_1,\dots,i_s] \in \langle[b_1,\dots,b_j|i_1,\dots,i_j]|\ 1 \leq b_1 < b_2 < \cdots < b_j \leq r\rangle$$

and we are therefore done since

$$\langle\phi[i_1,\dots,i_r]|\ i_j < n+1-a_{r+1-j}\ \wedge i_{j+1} \geq n-r+1\rangle =$$
$$\langle[b_1,\dots,b_j|c_1,\dots,c_j]|\ c_j < n+1-a_{r+1-j} \wedge\ 1 \leq b_1 < b_2 < \cdots < b_j \leq r\rangle.$$

□

6.2 Cohomological dimension of $X(a_1,\dots,a_r)$ in $\mathrm{Gr}(r,n)$

In this section we are going to find the cohomological dimension of all Schubert varieties in $\mathrm{Gr}(r,n)$ of the form $X(a_s-s+1, a_s-s+2,\dots,a_s,a_{s+1},\dots,a_r)$ with $r \leq a_s$ and $s \geq 1$. The cohomological dimension is defined in section 2.2. We assume $\mathrm{char}(k)=0$. If $\mathrm{char}(k)>0$ Proposition 4.1 in [40] gives, that

$$\mathrm{cd}_{\mathrm{Gr}(r,n)}(X(a_1,\dots,a_r)) = \mathrm{codim}(X(a_1,\dots,a_r)).$$

If $\{a_{s+1},a_{s+2}\dots,a_r\}=\{n-r+s+1,n-r+s+2,\dots,n\}$ the cohomological dimension is described in [10], but for all other cases the result is new. The two main ingredients in the proof are the Grothendieck-Cousin complex on $\mathrm{Gr}(r,n)$ and the result given in [10]. Let $s\in\{0,1,\dots,\min(r-2,n-2r+1)\}$

$$Y_{r,n}=\mathrm{Gr}(r,n),$$
$$X_s=X(r-s,r-s+1,\dots,r,n-r+s+2,n-r+s+3,\dots,n).$$

We get, that

Lemma 6.2.1.

$$\mathcal{H}^j_{X(a_1,\dots,a_r)}(\mathcal{O}_{Y_{r,n}})\neq 0 \Leftrightarrow \mathcal{H}^j_{X(a_1,\dots,a_r)}(\mathcal{O}_{Y_{r,n}})|_{D_+(\epsilon[n-r+1,\dots,n])}\neq 0 \Leftrightarrow$$
$$\mathcal{H}^j_{X(a_1,\dots,a_r)\cap D_+(\epsilon[n-r+1,\dots,n]}(\mathcal{O}_{Y_{r,n}}|_{D_+(\epsilon[n-r+1,\dots,n])})\neq 0.$$

Proof. The last $\Leftrightarrow$ is true is a general fact. All there is to prove is therefore $\Rightarrow$ in the first $\Leftrightarrow$. That this is true follows since $\mathcal{H}^j_{X(a_1,\dots,a_r)}(\mathcal{O}_{Y_{r,n}})$ according to Theorem 1.4 in [30] is holonomic and has then finite length in $\mathcal{D}_{Y_{r,n}}-mod$, and hereby $\mathrm{Supp}(\mathcal{H}^j_{X(a_1,\dots,a_r)}(\mathcal{O}_{Y_{r,n}}))$ is according to Lemma 5.3.2 closed. Since $X(a_1,\dots,a_r)$ is B-invariant and $\mathcal{O}_{Y_{r,n}}$ is B-equivariant $\mathcal{H}^j_{X(a_1,\dots,a_r)}(\mathcal{O}_{Y_{r,n}})$ is also B-equivariant. Due to Lemma 3.2.8 its support is a union of B-orbits and therefore a union of Schubert varieties and then $\Rightarrow$ is a consequence of (6.3) in section 6.1. □

In the rest of this section we shall write $\mathrm{cd}(Y)$ instead of $\mathrm{cd}_{\mathrm{Gr}(r,n)}(Y)$, and we shall also use Lemma 2.2.10 frequently without giving a reference. In [10] Bruns and Schwänzl found the cohomological dimension of the ideal generated by all $r\times r$ minors in a $n\times m$ polynomial ring defined over a field of characteristic zero. This is done by using étale cohomology. The Proposition below simply uses this result to find $\mathrm{cd}(X_s)$, which is therefore known.

Propositon 6.2.2. *Assume* $n-r+s+2>r+1$ *then*

$$\mathrm{cd}(X_s)=(n-r)r-(r-s)^2+1.$$

Proof. Let $R=k[X_{ij}]$ be the polynomial ring in the indeterminate X_{ij} with $i\in\{1,2,\dots,r\}$, $j\in\{1,2,\dots,n-r\}$. Using the notation of section 6.1

$S(r-s, r-s+1, \ldots, r, n-r+s+2, n-r+s+3, \ldots, n) = \{s+1, r\}$.

$X_s \cap D_+(\epsilon[n-r+1, \ldots, n]) =_{\text{Lemma 6.1.2}}$
$V([i_1, \ldots, i_r] |\ i_{r-s} < n+1-a_{s+1}\rangle \vee\ i_1 < n+1-a_r) \cap D_+(\epsilon[n-r+1, \ldots, n]) =_{a_r=n}$
$V([i_1, \ldots, i_r] |\ i_{r-s} < n+1-r\rangle) \cap D_+(\epsilon[n-r+1, \ldots, n]) =_{\text{Lemma 6.1.5}}$
$V(\langle [b_1, \ldots, b_{r-s} | c_1, \ldots, c_{r-s}] |\ 1 \leq b_1 < b_2 < \cdots < b_{r-s} \leq r\rangle) =$
V(ideal generated by all (r − s) × (r − s) minors of R).

Let this ideal be I. Due to Proposition 2.2.6

$$\mathcal{H}^j_{X_s}(\mathcal{O}_{Y_{r,n}})|_{D_+(\epsilon[n-r+1,\ldots,n])} = \widetilde{H^j_I(R)}$$

and now the Proposition follows by combining Lemma 6.2.1 with [10], since it is here shown, that $\mathrm{cd}(I) = r(n-r) - (r-s)^2 + 1$. The Proposition follows in the case $s = r-2$ by Theorem 6.4.1 in section 6.4 combined with Corollary 4.3.2. □

Lemma 6.2.3. *Let $a_s > r$ in $X(a_s - s+1, a_s - s+2, \ldots, a_s, a_{s+1}, \ldots, a_r) \subset Y_{r,n}$ with $s \in \{1, \ldots, r-1\}$ and consider $X(r-s+1, r-s+2, \ldots, r, a_{s+1} - (a_s - r), \ldots, a_r - (a_s - r)) \subset Y_{r,n-(a_s-r)}$. Then*

$\mathrm{cd}_{Y_{r,n}}(X(a_s - s+1, a_s - s+2, \ldots, a_s, a_{s+1}, \ldots, a_r)) =$
$\mathrm{cd}_{Y_{r,n-(a_s-r)}}(X(r-s+1, r-s+2, \ldots, r, a_{s+1} - (a_s - r), \ldots, a_r - (a_s - r)))$.

Proof. Since $S(a_s - s+1, a_s - s+2, \ldots, a_s, a_{s+1}, \ldots, a_r) \subset \{s, s+1, \ldots, r\}$ and $S(r-s+1, r-s+2, \ldots, r, a_{s+1} - (a_s - r), \ldots, a_r - (a_s - r))) \subset \{s, s+1, \ldots, r\}$ we get due to Lemma 6.1.2 that

$X(a_s - s+1, a_s - s+2, \ldots, a_s, a_{s+1}, \ldots, a_r) =$
$\cap_{r+1-j \in \{s, s+1, \ldots, r\}} V(\langle [i_1, \ldots, i_r] |\ i_j < n+1-a_{r+1-j}\rangle) =$
$\cap_{j=1}^{r+1-s} V(\langle [i_1, \ldots, i_r] |\ i_j < n+1-a_{r+1-j}\rangle) =$
$V(\langle [i_1, \ldots, i_r] |\ \exists j \in \{1, 2, \ldots, r+1-s\}\ i_j < n+1-a_{r+1-j}\rangle)$
$X(r-s+1, r-s+2, \ldots, r, a_{s+1} - (a_s - r), \ldots, a_r - (a_s - r)) =$
$V(\langle [i_1, \ldots, i_r] |\ \exists j \in \{1, 2, \ldots, r+1-s\}\ i_j < n - (a_s - r) + 1 - (a_{r+1-j} - (a_s - r))\rangle) =$
$V(\langle [i_1, \ldots, i_r] |\ \exists j \in \{1, 2, \ldots, r+1-s\}\ i_j < n+1-a_{r+1-j}\rangle)$.

Let us consider and denote

$X := X(a_s - s+1, a_s - s+2, \ldots, a_s, a_{s+1}, \ldots, a_r) \cap D_+(\epsilon[n-r+1, \ldots, n]) \subset D_+(\epsilon[n-r+1, \ldots, n])$
$Y := X(r-s+1, \ldots, r, a_{s+1} - (a_s - r), \ldots, a_r - (a_s - r)) \cap D_+(\epsilon[n-a_s+1, \ldots, n-(a_s - r)]) \subset D_+(\epsilon[n-a_s+1, \ldots, n-(a_s - r)])$

It then follows by Lemma 6.2.1, that

$$\begin{aligned}
&\mathrm{cd}(X) = \mathrm{cd}_{Y_{r,n}}(X(a_s - s + 1, a_s - s + 2, \dots, a_s, a_{s+1}, \dots, a_r))\\
&\mathrm{cd}(Y) = \mathrm{cd}_{Y_{r,n-(a_s-r)}}(X(r - s + 1, r - s + 2, \dots, r, a_{s+1} - (a_s - r), \dots, a_r - (a_s - r))).
\end{aligned}$$

It now follows by Lemma 6.1.5, that

$$\begin{aligned}
&X = V(\langle [b_1,\dots,b_j|c_1,\dots,c_j]|\\
&j \in \{1,\dots,r+1-s\}\ c_j < n+1-a_{r+1-j} \wedge\ 1 \le b_1 < b_2 < \dots < b_j \le r\rangle)\\
&Y = V(\langle [b_1,\dots,b_j|c_1,\dots,c_j]|\ 1 \le b_1 < b_2 < \dots < b_j \le r \wedge\\
&j \in \{1,\dots,r+1-s\}\ c_j < n-(a_s-r)+1-(a_{r+1-j}-(a_s-r))\rangle) =\\
&V(\langle [b_1,\dots,b_j|c_1,\dots,c_j]|\\
&j \in \{1,\dots,r+1-s\}\ c_j < n+1-a_{r+1-j} \wedge\ 1 \le b_1 < b_2 < \dots < b_j \le r\rangle).
\end{aligned}$$

Let $X = V(I)$ and $Y = V(J)$. Then $J \subset k[X_{ij}|\ i \in \{1,2,\dots,r\},\ j \in \{1,2,\dots,n-(a_s-r)-r\}] = S$ is an ideal and $I \subset k[X_{ij}|\ i \in \{1,2,\dots,r\},\ j \in \{1,2,\dots,n-r\}] = R$ is an ideal and since $JR = I$ the Lemma follows by using Corollary 2.1.2 since

$$\begin{aligned}
&\mathcal{H}^j_X(\mathcal{O}_{\mathrm{Spec}(R)}) = \widetilde{H^j_I(R)},\\
&\mathcal{H}^j_Y(\mathcal{O}_{\mathrm{Spec}(S)}) = \widetilde{H^j_J(S)}.
\end{aligned}$$

□

Let us as an example find $\mathrm{cd}(X(4679))$ with $X(4679) \subset Y_{4,9}$. All the reults will later be generalized.

$$X(w) := X(4679) =_{\text{Lemma 6.1.1}} X(4789) \cap X(5679).$$

We then have the Mayer-Vietoris exact sequence

$$\dots \to \mathcal{H}^j_{X(w)}(\mathcal{O}_{Y_{4,9}}) \to \mathcal{H}^j_{X(4789)}(\mathcal{O}_{Y_{4,9}}) \bigoplus \mathcal{H}^j_{X(5679)}(\mathcal{O}_{Y_{4,9}}) \to \mathcal{H}^j_{X(4789)\cup X(5679)}(\mathcal{O}_{Y_{4,9}}) \to \dots$$

Since

$$(X(4789) \cup X(5679))^c = C(6789) \cup C(5789) \cup C(5689)$$

we get according to the Grothendieck-Cousin complex on G/P explained as (4.6) in section 4.3, that

$$\mathrm{cd}((X(4789) \cup X(5679))^c) \le 2.$$

This implies, since we according to Lemma 2.2.3 get

$$\mathcal{H}^j_{X(4789)\cup X(5679)}(\mathcal{O}_{Y_{4,9}}) \simeq \mathcal{H}^{j-1}_{(X(4789)\cup X(5679))^c}(\mathcal{O}_{Y_{4,9}})\ \forall j > 1$$

that $\mathrm{cd}(X(4789)\cup X(5679)) \leq 3$ and due to Proposition 6.2.2 $\mathrm{cd}(X(4789)) = 5*4-4^2+1=5$ and to Lemma 6.2.3

$$\begin{aligned}&\mathrm{cd}(X(5679)) = \mathrm{cd}(2346) =_{\text{Proposition 6.2.2}} 4*2-2^2+1=5 \Rightarrow\\ &\mathrm{cd}(X(w)) = 5\ \wedge\\ &\mathcal{H}^5_{X(w)}(\mathcal{O}_{Y_{4,9}}) \simeq \mathcal{H}^5_{X(4789)}(\mathcal{O}_{Y_{4,9}}) \bigoplus \mathcal{H}^5_{X(5679)}(\mathcal{O}_{Y_{4,9}})\end{aligned}$$

and we get the following Corollary.

Corollary 6.2.4. *$\mathcal{H}^5_{X(4679)}(\mathcal{O}_{\mathrm{Gr}(4,9)})$ is decomposable in $\mathcal{D}_{\mathrm{Gr}(4,9)}-mod$.*

If the degree was the codimension, this could never be the case thanks to Theorem 4.1 in [4]. One of the main steps in finding $\mathrm{cd}(X(4679))$ was to prove $\mathrm{cd}(X(4789)\cup X(5679)) < \mathrm{cd}(X(4789))$. This result will be generalized at once. Let $r+1 < a_{s+1} < \cdots < a_{r-1} < n$ with $1 \leq s < r-1$ and $r \geq 3$ and set

$$X_{s,(a_{s+1},\ldots,a_{r-1})} := X(r-s+1, r-s+2, \ldots, r, a_{s+1}, \ldots, a_{r-1}, n)$$

we then get the following Lemma, which is going be the key in the general proof of the cohomological dimension as the example showed.

Lemma 6.2.5. *Assume $X_{s,(a_{s+1},\ldots,a_{r-1})} \neq X_{s-1}$, then*

$$\mathrm{cd}(X_{s,(a_{s+1},\ldots,a_{r-1})}) = \max\{\mathrm{cd}(X_{s-1}), \mathrm{cd}(X(a_{s+1}-s, a_{s+1}-s+1, \ldots, a_{s+1}, \ldots, a_{r-1}, n))\}.$$

Proof. We let

$$Y = X(a_{s+1}-s, a_{s+1}-s+1, \ldots, a_{s+1}, \ldots, a_{r-1}, n).$$

Since

$$\begin{aligned}&X_{s-1} \cap Y = X(r-s+1, \ldots, r, n-r+s+1, n-r+s+2, \ldots, n) \cap\\ &X(a_{s+1}-s, a_{s+1}-s+1, \ldots, a_{s+1}, \ldots, a_{r-1}, n) =_{\text{Lemma 6.1.1}}\\ &X(r-s+1, r-s+2, \ldots, r, a_{s+1}, \ldots, a_{r-1}, n) = X_{s,(a_{s+1},\ldots,a_{r-1})}\end{aligned}$$

we get according to the Mayer-Vietoris an exact sequence

$$\cdots \to \mathcal{H}^j_{X_{s,(a_{s+1},\ldots,a_{r-1})}}(\mathcal{O}_{Y_{r,n}}) \to \mathcal{H}^j_{X_{s-1}}(\mathcal{O}_{Y_{r,n}}) \bigoplus \mathcal{H}^j_Y(\mathcal{O}_{Y_{r,n}}) \to \mathcal{H}^j_{Y\cup X_{s-1}}(\mathcal{O}_{Y_{r,n}}) \to \ldots$$

So if we can just show, that

$$\mathrm{cd}(Y \cup X_{s-1}) < \mathrm{cd}(X_{s-1})$$

we are done. Since $r+1 < a_{s+1} \leq n-r+s+1$ we according to Proposition 6.2.2 get that

$$\mathrm{cd}(X_{s-1}) = (n-r)r - (r-s+1)^2 + 1.$$

and we must therefore show, that

$$\mathrm{cd}(Y \cup X_{s-1}) < (n-r)r - (r-s+1)^2 + 1$$

Since $r+1 < a_{s+1} < \cdots < a_{r-1} < n \Rightarrow n > 2r-s \Rightarrow (n-r)r-(r-s+1)^2+1 > 1$ we are done if $\mathrm{cd}(Y \cup X_{s-1}) \in \{0,1\}$. So assume this is not the case. We have the following isomorphism in $\mathcal{D}_{Y_{r,n}} - mod$

$$\begin{aligned}&\mathcal{H}^j_{Y \cup X_{s-1}}(\mathcal{O}_{Y_{r,n}}) \simeq \mathcal{H}^{j-1}_{(Y\cup X_{s-1})^c}(\mathcal{O}_{Y_{r,n}})\ \forall j>1 \Rightarrow_{\mathrm{cd}(Y\cup X_{s-1})>1}\\ &\mathrm{cd}(Y\cup X_{s-1}) - 1 = \mathrm{cd}((Y \cup X_{s-1})^c).\end{aligned} \tag{6.4}$$

So all we have to prove is that

$$\mathrm{cd}((Y \cup X_{s-1})^c) < (n-r)r - (r-s-1)^2. \tag{6.5}$$

In the rest of this proof we assume $b_1 < \cdots < b_r$. Then

$$(Y \cup X_{s-1})^c = \bigcup_{C(b_1,\dots,b_r)\cap(Y\cup X_{s-1})=\emptyset} C(b_1,\dots,b_r).$$

Let us examine the condition $C(b_1,\dots,b_r) \cap (Y \cup X_{s-1}) = \emptyset$

$$\begin{aligned}&C(b_1,\dots,b_r) \cap (Y \cup X_{s-1}) = \emptyset \Leftrightarrow\\ &C(b_1,\dots,b_r) \cap Y = \emptyset \ \wedge\ C(b_1,\dots,b_r) \cap X_{s-1} = \emptyset \Leftrightarrow\\ &C(b_1,\dots,b_r) \cap X(a_{s+1}-s, a_{s+1}-s+1,\dots,a_{s+1},\dots,a_{r-1},n) = \emptyset \wedge\\ &C(b_1,\dots,b_r) \cap X(r-s+1,\dots,r,n-r+s+1,n-r+s+2,\dots,n) = \emptyset \Leftrightarrow\\ &\exists j \in \{s+1,s+2,\dots,r-1\}\ b_j > a_j \ \wedge\ b_s > r.\end{aligned}$$

According to Lemma 6.1.4

$$\mathrm{codim}(C(b_1,\dots,b_r)) = rn - \sum_{m=1}^{r} b_m - \frac{r(r-1)}{2}.$$

According to the Grothendieck-Cousin complex explained as (4.6) in section 4.3, we get, that

$$\begin{aligned}
&\mathrm{cd}((Y \cup X_{s-1})^c) \leq \\
&\max\{\mathrm{codim}(C(b_1, \ldots, b_r))|\ \exists j \in \{s+1, s+2, \ldots, r-1\}\ b_j > a_j\ \wedge\ b_s > r\} = \\
&\max\{rn - \sum_{m=1}^{r} b_m - \frac{r(r-1)}{2}|\ \exists j \in \{s+1, s+2, \ldots, r-1\}\ b_j > a_j\ \wedge\ b_s > r\} = \\
&rn - \frac{r(r-1)}{2} - \min\{\sum_{m=1}^{r} b_m|\ \exists j \in \{s+1, s+2, \ldots, r-1\}\ b_j > a_j\ \wedge\ b_s > r\}.
\end{aligned}$$

Since

$$\begin{aligned}
&A := \min\{\sum_{m=1}^{r} b_m|\ \exists j \in \{s+1, s+2, \ldots, r-1\}\ b_j > a_j\ \wedge\ b_s > r\} = \\
&\min\{(\sum_{m=1}^{s-1} m) + (\sum_{m=1}^{j-s} r+m) + (\sum_{m=1}^{r-(j-1)} a_j + m)|\ j \in \{s+1, s+2, \ldots, r-1\}\} = \\
&\min\{\frac{s(s-1)}{2} + (j-s)r + \frac{(j-s)(j-s+1)}{2} + \frac{(r-j+1)(r-j+2)}{2} + (r-j+1)a_j| \\
&j \in \{s+1, s+2, \ldots, r-1\}\}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{s(s-1)}{2} + (j-s)r + \frac{(j-s)(j-s+1)}{2} + \frac{(r-j+1)(r-j+2)}{2} = \\
&\frac{s^2-s}{2} + jr - sr + \frac{j^2+s^2-2js+j-s}{2} + \frac{r^2+j^2-2rj+3r-3j+2}{2} = \\
&s^2 - s - sr + j^2 - j - js + \frac{r^2+3r}{2} + 1
\end{aligned}$$

we get

$$\begin{aligned}
&A = \min\{s^2 - s + j^2 - j - rs - js + \frac{3r}{2} + \frac{r^2}{2} + 1 + (r-j+1)a_j| \\
&j \in \{s+1, s+2, \ldots, r-1\}\}.
\end{aligned}$$

Let $j \in \{s+1, s+2, \ldots, r-1\}$. Since $a_{s+1} > r+1 \Rightarrow a_j > r+1+j-(s+1) = r+j-s$ and $r-j+1 > 0 \Rightarrow (r-j+1)a_j > (r-j+1)(r+j-s) =$

$r^2 - j^2 - rs + sj + r + j - s \Rightarrow$

$$A > \min\{s^2 - s + j^2 - j + \frac{r^2}{2} + \frac{3r}{2} + 1 - sr - js + (r-j+1)(r+j-s)|$$
$$j \in \{s+1, s+2, \dots, r-1\}\} = s^2 - 2s + \frac{3r^2}{2} + \frac{5r}{2} - 2rs + 1 \Rightarrow$$
$$\mathrm{cd}((Y \cup X_{s-1})^c) < rn - \frac{r(r-1)}{2} - (s^2 - 2s + \frac{3r^2}{2} + \frac{5r}{2} - 2rs + 1) =$$
$$rn - s^2 + 2s + 2rs - 2r^2 - 2r - 1 =$$
$$rn - r^2 - (r^2 + s^2 - 2rs + 2r - 2s + 1) = r(n-r) - (r-s+1)^2$$

which is (6.5) and the Lemma is true. □

Lemma 6.2.6. *Given* $n - 1 > a_{r-1} > r + 1$ *and* $r \geq 3$

$$\mathrm{cd}(X_{r-2,(a_{r-1})}) = \max\{(n-r)r-8, r(n-a_{r-1})-3\} = \mathrm{cd}(X_{r-3}) = (n-r)r-8.$$

Proof. The last equality is simply Proposition 6.2.2. According to Lemma 6.2.5

$$\mathrm{cd}(X_{r-2,(a_{r-1})}) = \max\{\mathrm{cd}(X_{r-3}), \mathrm{cd}(X(a_{r-1}-r+2, a_{r-1}-r+3, \dots, a_{r-1}, n))$$

So to prove the Lemma we simply have to show, that

$$r(n - a_{r-1}) - 3 = \mathrm{cd}(X(a_{r-1} - r + 2, a_{r-1} - r + 3, \dots, a_{r-1}, n)) < \mathrm{cd}(X_{r-3}).$$

Let

$$Z = X(2, 3, \dots, r, n - (a_{r-1} - r)) \subset Y_{r,n-(a_{r-1}-r)}.$$

It then follows by Lemma 6.2.3 and Proposition 6.2.2, that

$$\mathrm{cd}(X(a_{r-1} - r + 2, a_{r-1} - r + 3, \dots, a_{r-1}, n)) =$$
$$\mathrm{cd}(Z) = (n - (a_{r-1} - r) - r)r - (r - (r-2))^2 + 1 = nr - ra_{r-1} - 3$$

and since $a_{r-1} > r + 1$ and $r \geq 3$

$$nr - ra_{r-1} - 3 < r(n-r) - 8.$$

□

We are now ready to prove the main Theorem of this section. We use the convention, that

$$\{a_1, \dots, a_r\} = \{r - s + 1, r - s + 2, \dots, r, a_{s+1}, \dots, a_{r-1}, n\},\ a_i < a_{i+1}.$$

Theorem 6.2.7. *Assume* $a_{s+1} > r+1$ *and* $r \geq 3$ *then*

$$\begin{aligned}&\operatorname{cd}(X_{s,(a_{s+1},\ldots,a_{r-1})}) = \\ &\max\{r(n-a_j)-(r-(j-1))^2+1 |\ a_{j+1}-a_j > 1\}.\end{aligned}$$

Proof. If

$$X_{s,(a_{s+1},\ldots,a_{r-1})} = X_{s-1}$$

we are due to Proposition 6.2.2 done. So assume this is not the case. Then let $m \geq 1$ be defined such that

$$\begin{aligned}&m = 1 \text{ if } a_{s+2}-a_{s+1} > 1 \\ &a_{s+1+t}-a_{s+1+t-1} = 1\ \forall t \in \{1,\ldots,m-1\} \wedge\ a_{s+1+m}-a_{s+1+m-1} > 1 \text{ otherwise.}\end{aligned}$$

The proof is an induction proof in s, and according to Lemma 6.2.6, Theorem 6.2.7 is true for $s = r-2$. So assume it is true $\forall s \in \{j, j+1, \ldots, r-2\}$ and let us prove it for $s = j-1$. Due to Proposition 6.2.2

$$\operatorname{cd}(X_{s-1}) = r(n-r)-(r-(s-1))^2+1 = r(n-a_s)-(r-(s-1))^2+1.$$

According to Lemma 6.2.5 it is therefore enough to prove, that

$$\begin{aligned}&\operatorname{cd}(X(a_{s+1}-s, a_{s+1}-s+1, \ldots, a_{s+1}, \ldots, a_{r-1}, n)) = \\ &\max\{r(n-a_j)-(r-(j-1))^2+1 |\ a_{j+1}-a_j > 1,\ j \neq s\}.\end{aligned}$$

Let

$$\begin{aligned}&Y_{r,n-(a_{s+m}-r)} \supset Y = \\ &X(r-s-m+1, \ldots, r, a_{s+m+1}-(a_{s+m}-r), \ldots, a_{r-1}-(a_{s+m}-r), n-(a_{s+m}-r)).\end{aligned}$$

It then follows by Lemma 6.2.3, that

$$\begin{aligned}&\operatorname{cd}(X(a_{s+1}-s, a_{s+1}-s+1, \ldots, a_{s+1}, \ldots, a_{r-1}, n)) = \operatorname{cd}(Y) =_{\text{induction}} \\ &\max\{r(n-(a_{s+m}-r)-(a_j-(a_{s+m}-r))-(r-(j-1))^2+1 | \\ &a_{j+1}-(a_{s+m}-r)-(a_j-(a_{s+m}-r)) > 1,\ j \neq s\} = \\ &\max\{r(n-a_j)-(r-(j-1))^2+1 |\ a_{j+1}-a_j > 1,\ j \neq s\}\end{aligned}$$

and the Theorem is proved. □

Let $r+1 < a_{s+1} < s_{s+2} < \cdots < a_{r-1} < a_r \leq n$ with $r \geq 3$ and $1 \leq s < r-1$ and set

$$X_{s,(a_{s+1},\ldots,a_r)} := X(r-s+1, r-s+2, \ldots, r, a_{s+1}, \ldots, a_r).$$

We will now find $\operatorname{cd}(X_{s,(a_{s+1},\ldots,a_r)})$. If $a_r = n$ we have done it.

Corollary 6.2.8.

$$\mathrm{cd}(X_{s,(a_{s+1},\dots,a_r)}) = \max\{r(n-a_j)-(r-(j-1))^2+1 |\ a_{j+1}-a_j>1\}.$$

Proof. Let

$$Z = X_{s,(a_{s+1},\dots,a_{r-1})} \subset Y_{r,a_r}.$$

It then follows by Theorem 6.2.7, that

$$\mathrm{cd}(Z) = \max\{r(a_r-a_j)-(r-(j-1))^2+1 |\ a_{j+1}-a_j>1\}.$$

By combining Lemma 6.1.3 and Proposition 3.3.6 we get

$$\mathrm{cd}(X_{s,(a_{s+1},\dots,a_r)}) = \mathrm{cd}(Z)+\mathrm{codim}(X(a_r-r+1,a_r-r+2,\dots,a_r)) =_{\text{Lemma 6.1.4}}$$
$$\mathrm{cd}(Z)+rn-(\sum_{j=1}^{r}(a_r-r+j))-\frac{r(r-1)}{2} = \mathrm{cd}(Z)+rn-ra_r =$$
$$\max\{r(n-a_j)-(r-(j-1))^2+1 |\ a_{j+1}-a_j>1\}.$$

□

By combining this Corollary with Lemma 6.2.3, we get the main Theorem of this section.

Theorem 6.2.9. *Let* $a_s \geq r$ *and* $a_{s+1} > a_s+1$ *then*

$$\mathrm{cd}(X(a_s-s+1,a_s-s+2,\dots,a_s,a_{s+1},\dots,a_r)) =$$
$$\max\{r(n-a_j)-(r-(j-1))^2+1 |\ a_{j+1}-a_j>1\}.$$

Let us end this chapter by showing why our methods above sometimes fail, when we are in another situation. Let us consider $X(2,a,n)$ with $a>3$ and hereby show, that the methods used above must be improved. In this case

$$D_+(\epsilon[n-2,n-1,n]) = \mathrm{Spec}(k\begin{bmatrix} X_{1,1} & \dots & X_{1,n-4} & X_{1,n-3} \\ X_{2,1} & \dots & X_{2,n-4} & X_{2,n-3} \\ X_{3,1} & \dots & X_{3,n-4} & X_{3,n-3} \end{bmatrix}).$$

By using the same arguments as in the proof of Lemma 6.1.5

$$X(2,n-1,n)\cap D_+(\epsilon[n-2,n-1,n]) = V(\langle \begin{vmatrix} X_{1,i} & X_{1,j} \\ X_{2,i} & X_{2,j} \end{vmatrix} \mid i<j\rangle).$$

We then get

$$\begin{aligned}
&\mathrm{cd}_{\mathrm{Gr}(3,n)}(X(2,n-1,n)) =_{\text{Lemma 6.2.1}}\\
&\mathrm{cd}_{\mathrm{Gr}(3,n)\cap D_+(\epsilon[n-2,n-1,n])}(X(2,n-1,n)\cap D_+(\epsilon[n-2,n-1,n])) =_{\text{Corollary 2.1.2}}\\
&\mathrm{cd}_{\mathrm{Gr}(2,n-1)\cap D_+(\epsilon[n-2,n-1])}(X(2,n-1)\cap D_+(\epsilon[n-2,n-1])) =_{\text{Theorem 6.2.9}}\\
&2(n-1-2)-2^2+1 = 2n-9,\\
&\mathrm{cd}_{\mathrm{Gr}(3,n)}(X(a-1,a,n)) =_{\text{Theorem 6.2.9}} 3(n-a)-(3-1)^2+1 = 3(n-a-1),\\
&X(2,a,n) = X(2,n-1,n)\cap X(a-1,a,n).
\end{aligned}$$

We drop the $_{\mathrm{Gr}(3,n)}$ in $\mathrm{cd}_{\mathrm{Gr}(3,n)}$. If the proof of Theorem 6.2.9 should work, it would due to the Mayer-Vietoris require $\mathrm{cd}(X(2,n-1,n)\cup X(a-1,a,n)) < \max\{\mathrm{cd}(X(a-1,a,n)), \mathrm{cd}(X(a-1,a,n))\}$. Thanks to Lemma 2.2.3 $\mathrm{cd}(X(2,n-1,n)\cup X(a-1,a,n)) = \mathrm{cd}((X(2,n-1,n)\cup X(a-1,a,n))^c)+1$.

$$\begin{aligned}
&(X(2,n-1,n)\cup X(a-1,a,n))^c = \bigcup_{j=a+1}^{n-1}\bigcup_{k=j+1}^{n}\bigcup_{i=3}^{j-1} C(i,j,k) \Rightarrow\\
&\mathrm{cd}((X(2,n-1,n)\cup X(a-1,a,n))^c) \leq \mathrm{codim}(X(3,a+1,a+2)) =\\
&3n-(3+a+1+a+2)-3 = 3n-2a-9 \Rightarrow\\
&\mathrm{cd}(X(2,n-1,n)\cup X(a-1,a,n)) \leq 3n-2a-8
\end{aligned}$$

With the first $\Rightarrow$ stemming from the Grothensieck-Cousin complex on $\mathrm{Gr}(3,n)$. So pick $a=5$ and $n=10$. Then

$$\begin{aligned}
&\mathrm{cd}(X(2,9,10)\cup X(4,5,10)) \leq 12,\\
&\mathrm{cd}(X(2,9,10)) = 11,\ \mathrm{cd}(X(4,5,10)) = 12
\end{aligned}$$

and we are now unable to conclude if $\mathcal{H}^{13}_{X(2,5,10)}(\mathcal{O}_{\mathrm{Gr}(3,10)}) \neq 0$, and our method does therefore not generalize.

6.3 Kazhdan-Lusztig polynomials of $X(2,3,\ldots,r,n)$

The purpose with the two next subsections is to find the character formula of $\mathcal{H}^j_{\pi^{-1}(X(2,3,\ldots,r,n))}(\mathcal{O}_{G/B})\ \forall j$. Besides being interesting in itself, it is interesting since with the notation of section 4.5 $\pi^{-1}(X(2,3,\ldots,r,n)) = X(w_{n-r,r})$ and therefore our results might be generalized to the general setting when $x \in \mathrm{maxSing}(X(w))$ and $\tilde{w} = w_{k,m}$ to describe $[\mathcal{H}^j_{X(w)}(\mathcal{O}_{G/B}) : \mathcal{L}(x)]$, but we will return to these questions in the end. In this section we let $r>1$ and consider $X(2,3,\ldots,r,n) \subset \mathrm{Gr}(r,n) = Sl_n/P$ with P a maximal parabolic subgroup of Sl_n. We let once again $G = Sl_n$ and

$$\pi : G/B \rightarrow G/P$$

denote the canonical morphism and $\pi^{-1}(P) := X(w_I)$. We then have that $\exists w_{r,n} \in W^J$ such that $X(w_{r,n}w_I) = \pi^{-1}(X(2,3,\ldots,r,n))$. We then get by section 6.1 that

$$w_{r,n}(j) = \left\{ \begin{array}{cc} j+1 & j \in \{1,2,\ldots,n-r-1\} \\ n & j = n-r \\ 1 & j = n-r+1 \\ j-1 & j \in \{n-r+2,\ldots,n\} \end{array} \right\}.$$

Let us take $\tau \in W^J$ such that

$$\tau(j) = \left\{ \begin{array}{cc} j & j \in \{1,2,\ldots,n-r-1\} \\ n & j = n-r \\ j-1 & j \in \{n-r+1,\ldots,n\} \end{array} \right\}.$$

Then $X(\tau w_I) = \pi^{-1}(X(2,3,\ldots,r+1))$. An important observation is that

$$\tau = s_{n-r-1}s_{n-r-2}\ldots s_2 s_1 w_{r,n}.$$

This is true since

$$s_{n-r-1}s_{n-r-2}\ldots s_2 s_1(j) = \left\{ \begin{array}{cc} n-r & j = 1 \\ j-1 & j \in \{2,\ldots,n-r\} \\ j & j \in \{n-r+1,\ldots,n\} \end{array} \right\}$$

and we then see it. Let us for $k \in \{r+1,\ldots,n\}$ set

$$w_{r,k,n} := s_{n-k}\ldots s_2 s_1 w_{r,n} \text{ if } k \neq n,\ w_{r,n,n} = w_{r,n}$$

we then get

$$s_{n-k+1}w_{r,k,n} = w_{r,k-1,n}\ k > r+1,\ \ \tau = w_{r,r+1,n}$$

$$w_{r,k,n}(j) = \left\{ \begin{array}{cc} j & j \in \{1,2,\ldots,n-k\} \\ j+1 & j \in \{n-k+1,\ldots,n-r-1\} \\ n & j = n-r \\ n-k+1 & j = n-r+1 \\ j-1 & j \in \{n-r+2,\ldots,n\} \end{array} \right\} \text{ for } k \neq n, r+1$$

and thus that

$$\begin{aligned} &X(w_{r,k,n}w_I) = \pi^{-1}(X(2,3,\ldots,r,k)) \Rightarrow \\ &w_{r,k-1.n} = s_{n-k+1}w_{r,k,n} \leq w_{r,k,n}\ ,\ k \neq r+1. \end{aligned}$$

We wish to find the Kazhdan-Lusztig polynomials. To do this we first find the parabolic Kazhdan-Lusztig polynomials, which has been described in appendix A.1. The Lemma below is Proposition A.1.3. We drop the subscript -1 in $P^{J,-1}_{\sigma,\omega}$.

Lemma 6.3.1. $\forall \sigma, \omega \in W^J$

$$P^J_{\sigma,\omega} = P_{\sigma w_I, \omega w_I}.$$

We wish to find $P^J_{\sigma, w_{r,k,n}}$. Since this is not difficult but demands long calculations this is done in appendix A.2. The Proposition below is proven as Proposition A.2.3 there.

Propositon 6.3.2. *Let $\sigma \in W^J$. Then*

$$P^J_{\sigma, w_{r.k.n}} = \left\{ \begin{array}{ll} 1 & \sigma \leq w_{r,k,n},\ \sigma \neq id \\ \sum_{j=0}^{\min(r-1,k-r-1)} q^j & \sigma = id \end{array} \right\}.$$

We shall at once give the equivalent description on G/B.

Lemma 6.3.3. *Let $\sigma \leq w_{r,k,n} w_I$ then*

$$P_{\sigma, w_{r,k,n} w_I} = \left\{ \begin{array}{ll} 1 & \sigma \not\leq w_I \\ \sum_{j=0}^{min(r-1,k-r-1)} q^j & \sigma \leq w_I \end{array} \right\}.$$

Proof. That it is true for $\sigma = w_I$ follows by combining Proposition 6.3.2 and Lemma 6.3.1. Assume $\sigma \leq w_I$. According to the discussion in the beginning of section 4.3 w_I is the longest element in the subgroup of W generated by $s_{\alpha_1}, \ldots, s_{\alpha_r}$ with s_{α_i} some simple reflections, which is W_J. We have that $\sigma \leq w_I \Rightarrow \exists \omega \in W_J$ such that $\sigma\omega = w_I$. Let s be a simple reflection such that $w_{r,k,n} w_I s < w_{r,k,n} w_I$, it then follows by Corollary 7.14 in [28], that $P_{\sigma s, w_{r,k,n} w_I} = P_{\sigma, w_{r,k,n} w_I}$. Since $w_{r,k,n} w_I s_{\alpha_i} \leq w_{r,k,n} w_I$ we get by repeated use of this property that

$$P_{\sigma, w_{r,k,n} w_I} = P_{w_I, w_{r,k,n} w_I}\ \forall \sigma \leq w_I.$$

Suppose $\sigma \not\leq w_I$. It follows by Lemma A.1.1 in Appendix A, that $\exists \sigma_1 \in W_J$ and $\sigma_2 \in W^J$ such that $\sigma = \sigma_2 \sigma_1$ and $\sigma_2 \neq id$ since otherwise $\sigma \leq w_I$. By using the same arguments as above we get, that

$$P_{\sigma, w_{r,k,n} w_I} = P_{\sigma_2 w_I, w_{r,k,n} w_I} =_{\text{Lemma 6.3.1}} P^J_{\sigma_2, w_{r,k,n}} =_{\text{Proposition 6.3.2}} 1$$

and thus the Lemma. □

Let us introduce more notation. Let $s \in \{1, \ldots, r-2\}$, $k \in \{r+1, \ldots, n\}$ and then define $w_{s,r,k,n} \in W^J$ such that

$$\pi^{-1}(X(1, \ldots, s, s+2, \ldots, r, k)) = X(w_{s,r,k,n} w_I).$$

It follows by the description given in the beginning of section 6.1 that

$$w_{s,r,k,n}(j) = \left\{ \begin{array}{cc} j & j \in \{1, 2, \dots, n-k\} \\ j+1 & j \in \{n-k+1, \dots, n-r-1\} \\ n-s & j = n-r \\ n-k+1 & j = n-r+1 \\ j-1 & j \in \{n-r+2, \dots, n-s\} \\ j & j \in \{n-s+1, \dots, n\} \end{array} \right\} \text{ for } k \neq n, r+1$$

$$w_{s,r,n,n}(j) = \left\{ \begin{array}{cc} j+1 & j \in \{1, \dots, n-r-1\} \\ n-s & j = n-r \\ 1 & j = n-r+1 \\ j-1 & j \in \{n-r+2, \dots, n-s\} \\ j & j \in \{n-s+1, \dots, n\} \end{array} \right\}$$

$$w_{s,r,r+1,n}(j) = \left\{ \begin{array}{cc} j & j \in \{1, 2, \dots, n-r-1\} \\ n-s & j = n-r \\ j-1 & j \in \{n-r+1, \dots, n-s\} \\ j & j \in \{n-s+1, \dots, n\} \end{array} \right\}$$

and thus, that

$$s_{n-k+1} w_{s,r,k,n} = w_{s,r,k-1,n}, \; k > r+1.$$

We get the following Lemma, which is proven as Lemma A.2.4 in appendix A.2.

Lemma 6.3.4.

$$P^J_{\sigma, w_{s,r,k,n}} = \left\{ \begin{array}{ll} 1 & \sigma \leq w_{s,r,k,n}, \; \sigma \neq id \\ \sum_{j=0}^{\min(r-s-1, k-r-1)} q^j & \sigma = id \end{array} \right\}.$$

We get the following Corollary, whose proof is exactly identical to the proof of Lemma 6.3.3.

Corollary 6.3.5. *Let* $\sigma \leq w_{s,r,k,n} w_I$ *then*

$$P_{\sigma, w_{s,r,k,n} w_I} = \left\{ \begin{array}{ll} 1 & \sigma \nleq w_I \\ \sum_{j=0}^{\min(r-s-1, k-r-1)} q^j & \sigma \leq w_I \end{array} \right\}.$$

So now we know all the Kazhdan-Lusztig polynomials and by using these informations, we get the following Lemma, which is essential in finding the character formula of $\mathcal{H}^{c_{w_{r,k,n} w_I}}_{X(w_{r,k,n} w_I)}(\mathcal{O}_{G/B})$. We denote $X = G/B$.

Lemma 6.3.6.

$$\sum_{z\le w_{r,k,n}w_I}(-1)^{l(z)-l(w_{r,k,n}w_I)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] =$$
$$[\mathcal{L}(w_{r,k,n}w_I)] + \min(r-1,k-r-1)(-1)^k[\mathcal{L}(w_I)].$$
$$\sum_{z\le w_{s,r,k,n}w_I}(-1)^{l(z)-l(w_{s,r,k,n}w_I)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] =$$
$$[\mathcal{L}(w_{s,r,k,n}w_I)] + \min(r-s-1,k-r-1)(-1)^{k-s}[\mathcal{L}(w_I)].$$

Proof. According to (4.4) in section 4.2

$$[\mathcal{L}(w_{r,k,n}w_I)] = \sum_{z\le w_{r,k,n}w_I}(-1)^{l(z)-l(w_{r,k,n}w_I)}P_{z,w_{r,k,n}w_I}(1)[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] =_{\text{Lemma 6.3.3}}$$
$$\sum_{z\le w_{r,k,n}w_I}(-1)^{l(z)-l(w_{r,k,n}w_I)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] +$$
$$(\min(r-1,k-r-1))\sum_{z\le w_I}(-1)^{l(z)-l(w_{r,k,n}w_I)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] =_{l(w_{r,k,n}w_I)=l(w_{r,k,n})+l(w_I)}$$
$$\sum_{z\le w_{r,k,n}w_I}(-1)^{l(z)-l(w_{r,k,n}w_I)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] +$$
$$(-1)^{l(w_{r,k,n})}(\min(r-1,k-r-1))\sum_{z\le w_I}(-1)^{l(z)-l(w_I)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)].$$

Since $l(w_{r,k,n})$ =dim$(X(2,3,\dots,r,k)) = r-1+k-r = k-1$ and $X(w_I) = P/B$ is smooth, which implies, that the last sum is $[\mathcal{L}(w_I)]$ thanks to (4.4) in section 4.2 along with [13], and therefore the first part of the Lemma is true. Since $l(w_{s,r,k,n})$ =dim$(X(1,\dots,s,s+2,\dots r,k)) = k-s-1$ the other part is proved exactly the same way. □

6.4 Character formula of $X(2,3,\dots,r,n)$

Throughout this section we use the notation introduced in section 6.3. We assume $n > r+1$ and $r > 1$. For $k \in \{r+1,\dots,n\}$ we set

$$X := G/B,\ Z_{r,k,n} := X(w_{r,k,n}w_I)\cap(X_{r,k-1,n}w_I)^c\ k > r+1.$$

The goal of this section is to prove the following Theorem.

Theorem 6.4.1. *Let $k > r+1$ and $r > 1$ then*

$$[\mathcal{H}^{c_{w_{r,k,n}w_I}}_{X(w_{r,k,n}w_I)}(\mathcal{O}_X)] = \left\{\begin{array}{ll}[\mathcal{L}(w_{r,k,n}w_I)] & k\ne 2r\\ [\mathcal{L}(w_{r,k,n}w_I)]+[\mathcal{L}(w_I)] & k=2r\end{array}\right\},$$
$$[\mathcal{H}^{c_{w_{r,k,n}w_I}+j}_{X(w_{r,k,n}w_I)}(\mathcal{O}_X)] = \left\{\begin{array}{ll}[\mathcal{L}(w_I)] & j\in\{k-2t \mid t\in\{2,\dots,\min(r,k-r)\}\}\ j>0\\ 0 & \text{otherwise } j\ne 0\end{array}\right\}.$$

As a Corollary we get

Corollary 6.4.2. *$\mathcal{H}^{cw}_{X(w)}(\mathcal{O}_X)$ can be simple in $\mathcal{D}_X - mod$ even if $X(w)$ is not smooth.*

If $X(w)$ was smooth it follows by Kashiwaras equivalence. If $X(w)$ is not smooth and $v \in \mathrm{maxSing}(X(w))$ the cases, we so far have examined as Theorem 4.6.1 and 4.7.1 in section 4, have shown $X(v) \subset \mathrm{Supp}(\mathcal{H}^{cw}_{X(w)}(\mathcal{O}_X)/\mathcal{L}(w))$, but this is due to the Corollary not true in general, and one can therefore not conclude $\mathrm{Supp}(\mathcal{H}^{cw}_{X(w)}(\mathcal{O}_X)/\mathcal{L}(w)) = \mathrm{Sing}(X(w))$. We have the following Lemma.

Lemma 6.4.3. *There is an exact sequence in $\mathcal{D}_X - mod$ for $k > r+1$*

$$0 \to \mathcal{H}^{cw_{r,k,n}w_I}_{X(w_{r,k,n}w_I)}(\mathcal{O}_X) \to \mathcal{H}^{cw_{r,k,n}w_I}_{Z_{r,k,n}}(\mathcal{O}_X) \to \mathcal{H}^{cw_{r,k,n}w_I+1}_{X(w_{r,k-1,n}w_I)}(\mathcal{O}_X) \to \mathcal{H}^{cw_{r,k,n}w_I+1}_{X(w_{r,k,n}w_I)}(\mathcal{O}_X) \to \dots$$

Proof. This follows from Lemma 2.2.3 since $X(w_{r,k-1,n}w_I) \subset X(w_{r,k,n}w_I)$ is closed. □

To prove Theorem 6.4.1 the following Proposition along with the Lemma above and induction will turn out to be enough. The proof of it is rather technical and can be found in the next section.

Propositon 6.4.4. *Let $k > r+1$. Then*

$$[\mathcal{H}^{cw_{r,k,n}w_I+j}_{Z_{r,k,n}}(\mathcal{O}_X)] = \left\{ \begin{array}{ll} 0 & j \neq 0 \wedge\ k \geq 2r \\ 0 & j \notin \{0, 2r-k\} \wedge k < 2r \\ {[\mathcal{L}(w_I)]} & k < 2r \wedge\ j = 2r-k \\ {[\mathcal{L}(w_{r,k,n}w_I)] + [\mathcal{L}(w_{r,k-1,n}w_I)]} & j = 0 \wedge\ k \neq 2r \\ {[\mathcal{L}(w_{r,k,n}w_I)] + [\mathcal{L}(w_{r,k-1,n}w_I)] + [\mathcal{L}(w_I)]} & j = 0 \wedge\ k = 2r \end{array} \right\}.$$

Now we are ready to prove Theorem 6.4.1.

Proof. Given r the proof is an induction proof in k. We must therefore prove it in the case $k = r+2$ first. What we must prove is, that

$$[\mathcal{H}^{cw_{r,r+2,n}w_I+j}_{X(w_{r,r+2,n}w_I)}(\mathcal{O}_X)] = \left\{ \begin{array}{ll} {[\mathcal{L}(w_{r,r+2,n}w_I)]} & r \neq 2,\ j = 0 \\ {[\mathcal{L}(w_{r,r+2,n}w_I)] + [\mathcal{L}(w_I)]} & r = 2,\ j = 0 \\ {[\mathcal{L}(w_I)]} & j = r-2 \wedge\ r \geq 3 \\ 0 & r = 2 \wedge\ j \neq 0 \vee\ j \neq 0, r-2 \wedge\ r \geq 3 \end{array} \right\}.$$

We have an exact sequence due to Lemma 6.4.3

$$0 \to \mathcal{H}^{c_{w_{r,r+2,n}w_I}}_{X(w_{r,r+2,n}w_I)}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{r,r+2,n}w_I}}_{Z_{r,r+2,n}}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{r,r+2,n}w_I}+1}_{X(w_{r,r+1,n}w_I)}(\mathcal{O}_X) \to \dots$$

Since $X(w_{r,r+1,n}w_I) = \pi^{-1}(X(2,3,\dots,r+1))$ and due to Lemma A.2.1 $X(2,3,\dots,r+1)$ is smooth, and therefore according to Lemma 4.3.3 $X(w_{r,r+1,n}w_I)$ is also smooth. By combining Lemma 4.2.2 and Lemma 4.4.1 we see, that

$$[\mathcal{H}^{c_{w_{r,r+2,n}w_I}+1}_{X(w_{r,r+1,n}w_I)}(\mathcal{O}_X)] = [\mathcal{L}(w_{r,r+1,n}w_I)] \wedge \mathcal{H}^{c_{w_{r,r+2,n}w_I}+1+j}_{X(w_{r,r+1,n}w_I)}(\mathcal{O}_X) \neq 0 \Leftrightarrow \ j = 0.$$

Due to Lemma 4.2.3 Supp$(\mathcal{H}^{c_{w_{r,r+2,n}w_I}+1}_{X(w_{r,r+2,n}w_I)}(\mathcal{O}_X)) \subset$Sing$(X(w_{r,r+2,n}w_I) = X(w_I)$ with the last equality stemming from Lemma 6.3.3 and [13], we get an exact sequence in $\mathcal{D}_X - mod$

$$0 \to \mathcal{H}^{c_{w_{r,r+2,n}w_I}}_{X(w_{r,r+2,n}w_I)}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{r,r+2,n}w_I}}_{Z_{r,r+2,n}}(\mathcal{O}_X) \to \mathcal{L}(w_{r,r+1,n}w_I) \to 0$$
$$[\mathcal{H}^{c_{w_{r,r+2,n}w_I}+j}_{X(w_{r,r+2,n}w_I)}(\mathcal{O}_X)] = [\mathcal{H}^{c_{w_{r,r+2,n}w_I}+j}_{Z_{r,r+2,n}}(\mathcal{O}_X)] \ \forall j \geq 1$$

and then it is true due to Proposition 6.4.4. So assume $k \geq r+3$ and the Theorem is true $\forall j \leq k-1$. We once again use the exact sequence

$$0 \to \mathcal{H}^{c_{w_{r,k,n}w_I}}_{X(w_{r,k,n}w_I)}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{r,k,n}w_I}}_{Z_{r,k,n}}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{X(w_{r,k-1,n}w_I)}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{X(w_{r,k,n}w_I)}(\mathcal{O}_X) \to \dots$$

There are again some cases to consider.

(1) : $k < 2r$: We then know due to Proposition 6.4.4, that

$$[\mathcal{H}^{c_{w_{r,k,n}w_I}+j}_{Z_{r,k,n}}(\mathcal{O}_X)] = \left\{ \begin{array}{ll} 0 & j \notin \{0, 2r-k\} \\ \ [\mathcal{L}(w_I)] & j = 2r-k \\ \ [\mathcal{L}(w_{r,k,n}w_I)] + [\mathcal{L}(w_{r,k-1,n}w_I)] & j = 0 \end{array} \right\}$$

and since $r+1 < k-1 < 2r$ by induction since $c_{w_{r,k-1,n}w_I} = c_{w_{r,k,n}w_I} + 1$

$$[\mathcal{H}^{c_{w_{r,k,n}w_I}+1+j}_{X(w_{r,k-1,n}w_I)}(\mathcal{O}_X)] = \left\{ \begin{array}{ll} [\mathcal{L}(w_{r,k-1,n}w_I)] & j = 0 \\ \ [\mathcal{L}(w_I)] & j \in \{k-1-2t | t \in \{2,\dots,k-1-r\} \} \ j > 0 \\ 0 & \text{otherwise } j \neq 0 \end{array} \right\}.$$

By combining these informations we get since $\mathrm{Supp}(\mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{X(w_{r,k,n}w_I)}(\mathcal{O}_X)) \subset X(w_I)$ an exact sequence

$$0 \to \mathcal{H}^{c_{w_{r,k,n}w_I}}_{X(w_{r,k,n}w_I)}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{r,k,n}w_I}}_{Z_{r,k,n}}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{X(w_{r,k-1,n}w_I)}(\mathcal{O}_X) \to 0 \Rightarrow$$

$$[\mathcal{H}^{c_{w_{r,k,n}w_I}+j}_{X(w_{r,k,n}w_I)}(\mathcal{O}_X)] = \left\{ \begin{array}{ll} [\mathcal{L}(w_{r,k,n}w_I)] & j=0 \\ [\mathcal{L}(w_I)] & j \in \{k-2t | t \in \{2,\ldots,k-r\}\ \}\ j>0 \\ 0 & \text{otherwise } j \neq 0 \end{array} \right\}$$

which is the Theorem in the case $k < 2r$.

(2) : $k = 2r$: We get again by induction since $k-1 = 2r-1 < 2r$ and since $c_{w_{r,2r-1,n}w_I} = c_{w_{r,2r,n}w_I} + 1$

$$[\mathcal{H}^{c_{w_{r,2r,n}w_I}+1+j}_{X(w_{r,2r-1,n}w_I)}(\mathcal{O}_X))] = \left\{ \begin{array}{ll} [\mathcal{L}(w_I)] & j \in \{2r-2t-1 |\ t \in \{2,\ldots,r-1\}\} \\ [\mathcal{L}(w_{r,2r-1,n}w_I)] & j=0 \\ 0 & \text{otherwise} \end{array} \right\}$$

and this time due to Proposition 6.4.4

$$[\mathcal{H}^{c_{w_{r,2r,n}w_I}+j}_{Z_{r,2r,n}}(\mathcal{O}_X)] = \left\{ \begin{array}{ll} 0 & j \neq 0 \\ [\mathcal{L}(w_{r,2r,n}w_I)] + [\mathcal{L}(w_{r,2r-1,n}w_I)] + [\mathcal{L}(w_I)] & j=0 \end{array} \right\} \Rightarrow$$

$$[\mathcal{H}^{c_{w_{r,2r,n}w_I}+j}_{X(w_{r,2r,n}w_I)}(\mathcal{O}_X))] = \left\{ \begin{array}{ll} [\mathcal{L}(w_I)] & j \in \{2r-2t |\ t \in \{2,\ldots,r-1\}\} \\ [\mathcal{L}(w_{r,2r,n}w_I] + [\mathcal{L}(w_I)] & j=0 \\ 0 & \text{otherwise} \end{array} \right\}$$

and once again the Theorem.
(3) : $k > 2r$ We get due to Proposition 6.4.4, that

$$[\mathcal{H}^{c_{w_{r,k,n}w_I}+j}_{Z_{r,k,n}}(\mathcal{O}_X)] = \left\{ \begin{array}{ll} 0 & j \neq 0 \\ [\mathcal{L}(w_{r,k,n}w_I)] + [\mathcal{L}(w_{r,k-1,n}w_I)] & j=0 \end{array} \right\}.$$

This implies that there is an exact sequence in $\mathcal{D}_X - mod$

$$0 \to \mathcal{H}^{c_{w_{r,k,n}w_I}}_{X(w_{r,k,n}w_I)}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{r,k,n}w_I}}_{Z_{r,k,n}}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{X(w_{r,k-1,n}w_I)}(\mathcal{O}_X) \to$$
$$\mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{X(w_{r,k,n}w_I)}(\mathcal{O}_X) \to 0,$$
$$[\mathcal{H}^{c_{w_{r,k,n}w_I}+1+j}_{X(w_{r,k-1,n}w_I)}(\mathcal{O}_X)] = [\mathcal{H}^{c_{w_{r,k,n}w_I}+1+j}_{X(w_{r,k,n}w_I)}(\mathcal{O}_X)]\ \forall j > 0.$$

If $k > 2r+1 \Rightarrow k-1 > 2r$ it follows by induction

$$[\mathcal{H}^{c_{w_{r,k,n}w_I}+j}_{X(w_{r,k-1,n}w_I)}(\mathcal{O}_X))] = \left\{ \begin{array}{ll} [\mathcal{L}(w_I)] & j \in \{k-2t |\ t \in \{2,\ldots,r\}\} \\ [\mathcal{L}(w_{r,n-1,n}w_I)] & j=1 \\ 0 & \text{otherwise} \end{array} \right\}$$

and the Theorem. If $k = 2r+1 \Rightarrow k-1 = 2r$ and the Theorem since

$$[\mathcal{H}^{c_{w_{r,2r+1,n}w_I}+j}_{X(w_{r,2r,n}w_I)}(\mathcal{O}_X))] =_{\text{induction}}$$
$$\left\{\begin{array}{ll} [\mathcal{L}(w_I)] & j \in \{2r+1-2t |\ t \in \{2,\dots,r-1\}\} \\ [\mathcal{L}(w_{r,2r,n}w_I)] + [\mathcal{L}(w_I)] & j = 1 \\ 0 & \text{otherwise} \end{array}\right\} \Rightarrow$$
$$[\mathcal{H}^{c_{w_{r,2r+1,n}w_I}+j}_{X(w_{r,2r+1,n}w_I)}(\mathcal{O}_X))] = \left\{\begin{array}{ll} [\mathcal{L}(w_I)] & j \in \{2r+1-2t |\ t \in \{2,\dots,r\}\} \\ [\mathcal{L}(w_{r,2r+1,n}w_I)] & j = 0 \\ 0 & \text{otherwise} \end{array}\right\}.$$

□

Let us try to combine these results with section 4.5. We have shown $\mathrm{Sing}(X(w_{r,n,n}w_I)) = X(w_I)$. According to section 6.1 (6.1)

$$W_J = \langle s_i | i \in \{1, 2, \dots, n-1\}\ i \neq n-r\rangle$$

and since w_I is the longest element in W_J

$$w_I(j) = \left\{\begin{array}{cc} n-r+1-j & j \in \{1,\dots,n-r\} \\ 2n-r+1-j & j \in \{n-r+1,\dots,n\} \end{array}\right\} \Rightarrow$$
$$w_I = [n-r, n-r-1, \dots, 1, n, n-1, \dots, n-r+1] = x_{n-r,r}$$

$w_{r,n,n}w_I = \pi^{-1}(X(2,\dots,r,n))$ and due to the beginning of section 6.1

$$w_{r,n,n}(j) = \left\{\begin{array}{cc} j+1 & j \in \{1,\dots,n-r-1\} \\ n & j = n-r \\ 1 & j = n-r+1 \\ j-1 & j \in \{n-r+2,\dots,n\} \end{array}\right\} \Rightarrow$$
$$w_{r,n,n}w_I(j) = \left\{\begin{array}{cc} n & j = 1 \\ n-r+2-j & j \in \{2,\dots,n-r\} \\ 2n-r-j & j \in \{n-r+1,\dots,n-1\} \\ 1 & j = n \end{array}\right\} \Rightarrow$$
$$w_{r,n,n}w_I = w_{n-r,r}.$$

So we have proved the following Corollary due to Theorem 6.4.1.

Corollary 6.4.5. *Let $r > 1$ and $n > r+1$*

$$X(x_{n-r,r}) \subset \mathrm{Supp}(\mathcal{H}^{c_{w_{n-r,r}}}_{X(w_{n-r,r})}(\mathcal{O}_X)/\mathcal{L}(w_{n-r,r})) \Leftrightarrow n = 2r,$$
$$[\mathcal{H}^{c_{w_{n-r,r}}+j}_{X(w_{n-r,r})}(\mathcal{O}_X) : \mathcal{L}(x_{n-r,r})] = \left\{\begin{array}{ll} 1 & j \in \{n-2t | t \in \{2,\dots,\min(r, n-r)\}\},\ j > 0 \\ 1 & j = 0 \wedge\ n = 2r \\ 0 & \text{otherwise} \end{array}\right\}$$

6.5 $[\mathcal{H}^j_{Z_{r,k,n}}(\mathcal{O}_X)]$

The whole point with this section is to prove Proposition 6.4.4. We use the notation introduced in section 6.3. We assume $n > r+1$ and $r > 1$. Furthermore we set for $s \in \{1, \dots, r-2\}$, $k \in \{r+1, \dots, n\}$

$$\begin{aligned}
&Y_{r,n} := \mathrm{Gr}(r,n),\\
&X := G/B,\\
&Z_{r,k,n} := X(w_{r,k,n}w_I) \cap (X_{r,k-1,n}w_I)^c \ k > r+1,\\
&Z_{s,r,k,n} := X(w_{s,r,k,n}w_I) \cap (X(w_{s,r,k-1,n}w_I))^c \ k > r+1,\\
&X(v_{r,k,n}w_I) := \pi^{-1}(X(1,\dots,r-1,k)),\\
&Z_{r-1,r,k,n} := X(v_{r,k,n}w_I) \cap (X(v_{r,k-1,n}w_I))^c.
\end{aligned}$$

So $Z_{s,r,k,n}$ is also defined for $s = r-1$. The key fact in the proof of Proposition 6.4.4 is, that by combining Corollary 4.3.2 and section 4.3 (4.7)

$$\mathcal{H}^{\mathrm{codim}(\pi^{-1}(X(a_1,\dots,a_r)))+j}_{\pi^{-1}(C(a_1,\dots,a_r))}(\mathcal{O}_X) \neq 0 \Leftrightarrow j = 0.$$

Let us start out with the following couple of Lemmas.

Lemma 6.5.1. *Let $k > r+1$ and $s \leq r-2$. Then there is an exact sequence in $\mathcal{D}_X - mod$*

$$\begin{aligned}
&0 \to \mathcal{H}^{c_{w_{s,r,k,n}w_I}}_{Z_{s,r,k,n}}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{s,r,k,n}w_I}}_{Z_{s,r,k,n}\cap(Z_{s+1,r,k,n})^c}(\mathcal{O}_X) \to\\
&\mathcal{H}^{c_{w_{s,r,k,n}w_I}+1}_{Z_{s+1,r,k,n}}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{s,r,k,n}w_I}+1}_{Z_{s,r,k,n}}(\mathcal{O}_X) \to 0,\\
&\mathcal{H}^{c_{w_{s,r,k,n}w_I}+1+j}_{Z_{s,r,k,n}}(\mathcal{O}_X) \simeq \mathcal{H}^{c_{w_{s,r,k,n}w_I}+1+j}_{Z_{s+1,r,k,n}}(\mathcal{O}_X) \ \forall j > 0,\\
&0 \to \mathcal{H}^{c_{w_{r,k,n}w_I}}_{Z_{r,k,n}}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{r,k,n}w_I}}_{Z_{r,k,n}\cap(Z_{1,r,k,n})^c}(\mathcal{O}_X) \to\\
&\mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{Z_{1,r,k,n}}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{Z_{r,k,n}}(\mathcal{O}_X) \to 0,\\
&\mathcal{H}^{c_{w_{r,k,n}w_I}+1+j}_{Z_{r,k,n}}(\mathcal{O}_X) \simeq \mathcal{H}^{c_{w_{r,k,n}w_I}+1+j}_{Z_{1,r,k,n}}(\mathcal{O}_X) \ \forall j > 0.
\end{aligned}$$

Especially we get for $k < 2r-1$ and $s \in \{1, \dots, 2r-k-1\}$ $\forall j \geq s$

$$[\mathcal{H}^{c_{w_{1,r,k,n}w_I}+j}_{Z_{1,r,k,n}}(\mathcal{O}_X)] = [\mathcal{H}^{c_{w_{s,r,k,n}w_I}+j-s+1}_{Z_{s,r,k,n}}(\mathcal{O}_X)].$$

Proof. If we can just prove, that $Z_{s+1,r,k,n} \subset Z_{s,r,k,n}$ is closed, and that $0 = \mathcal{H}^{c_{w_{s,r,k,n}w_I}+j}_{Z_{s,r,k,n}\cap(Z_{s+1,r,k,n})^c}(\mathcal{O}_X)$ $\forall j \neq c_{w_{s,r,k,n}w_I}$ we are due to Lemma 2.2.3 done with the first part. The proof of the other is apart from the remark identical. We will do it in the case $s < r-2$, since the case $s = r-2$ is identical. We let

$$\begin{aligned}&Z = X(1,\dots,s,s+2,\dots,r,k)\cap(X(1,\dots,s,s+2,\dots,r,k-1))^c \Rightarrow\\ &Z_{s,r,k,n} = \pi^{-1}(Z),\\ &Z_{s+1,r,k,n} = \pi^{-1}(X(1,\dots,s+1,s+3,\dots,r,k)\cap(X(1,\dots,s+1,s+3,\dots,r,k-1))^c),\\ &X(1,\dots,s+1,s+3,\dots,r,k)\cap(X(1,\dots,s+1,s+3,\dots,r,k-1))^c =\\ &X(1,\dots,s+1,s+3,\dots,r,k)\cap Z\end{aligned}$$

and the closed part follows. Since

$$(X(1,\dots,s+1,s+3,\dots,r,k))^c\cap Z = C(1,\dots,s,s+2,\dots,r,k)$$

we get according to section 4.3 (4.7), that

$$\mathcal{H}^{\operatorname{codim}(C(1,\dots,s,s+2,\dots,r,k))+j}_{(X(1,\dots,s+1,s+3,\dots,r,k))^c\cap Z}(\mathcal{O}_{Y_{r,n}})\neq 0 \Leftrightarrow\ j=0$$

and since $\operatorname{codim}(C(1,\dots,s,s+2,\dots,r,k)) = c_{w_{s,r,k,n}w_I}$ and

$$Z_{s,r,k,n}\cap(Z_{s+1,r,k,n})^c = \pi^{-1}((X(1,\dots,s+1,s+3,\dots,r,k))^c\cap Z)$$

the last part is a consequence of Corollary 4.3.2. Now we lack to prove the remark. This is an induction proof in s. Since by the just proved

$$[\mathcal{H}^{c_{w_{s,r,k,n}w_I}+1+j}_{Z_{s,r,k,n}}(\mathcal{O}_X)] = [\mathcal{H}^{c_{w_{s,r,k,n}w_I}+1+j}_{Z_{s+1,r,k,n}}(\mathcal{O}_X)]\ \forall j>0$$

and $c_{w_{s,r,k,n}w_I}+1 = c_{w_{s+1,r,k,n}w_I}$, we get

$$[\mathcal{H}^{c_{w_{s,r,k,n}w_I}+j}_{Z_{s,r,k,n}}(\mathcal{O}_X)] = [\mathcal{H}^{c_{w_{s+1,r,k,n}w_I}+j-1}_{Z_{s+1,r,k,n}}(\mathcal{O}_X)]\ \forall j>1$$

and then it follows. □

We use the notation of Lemma 6.1.4.

Lemma 6.5.2.

$$\mathcal{H}^{j+n-r}_{Z_{s,r,k,n}}(\mathcal{O}_X)\neq 0 \Leftrightarrow \mathcal{H}^{j}_{i^{-1}(\pi(Z_{s,r,k,n}))}(\mathcal{O}_{Y_{r-1,n-1}})\neq 0.$$

Proof. It follows by Corollary 4.3.2, that

$$\mathcal{H}^{j+n-r}_{Z_{s,r,k,n}}(\mathcal{O}_X)\neq 0 \Leftrightarrow \mathcal{H}^{j+n-r}_{(\pi(Z_{s,r,k,n}))}(\mathcal{O}_{Y_{r,n}})\neq 0$$

and since

$$\pi(Z_{s,r,k,n}) = X(1,\dots,s,s{+}2,\dots,r,k)\cap(X(1,\dots,s,s{+}2,\dots,r,k{-}1))^c \subset i(Y_{r-1,n-1})$$

it follows by Proposition 3.3.6 combined with Lemma 6.1.4, that

$$\mathcal{H}^{j+n-r}_{(\pi(Z_{s,r,k,n}))}(\mathcal{O}_{Y_{r,n}}) \neq 0 \Leftrightarrow \mathcal{H}^{j}_{i^{-1}(\pi(Z_{s,r,k,n}))}(\mathcal{O}_{Y_{r-1,n-1}}) \neq 0$$

and then the Lemma. □

We now have all the tools to prove Proposition 6.4.4. It is written below.

Propositon 6.5.3. *Let $k > r+1$. Then*

$$[\mathcal{H}^{c_{w_{r,k,n}w_I}+j}_{Z_{r,k,n}}(\mathcal{O}_X)] = \left\{\begin{array}{ll} 0 & j \neq 0 \wedge\ k \geq 2r \\ 0 & j \notin \{0, 2r-k\} \wedge k < 2r \\ \ [\mathcal{L}(w_I)] & k < 2r \wedge\ j = 2r-k \\ \ [\mathcal{L}(w_{r,k,n}w_I)] + [\mathcal{L}(w_{r,k-1,n}w_I)] & j = 0 \wedge\ k \neq 2r \\ \ [\mathcal{L}(w_{r,k,n}w_I)] + [\mathcal{L}(w_{r,k-1,n}w_I)] + [\mathcal{L}(w_I)] & j = 0 \wedge\ k = 2r \end{array}\right\}.$$

Proof. According to Lemma 6.4.3 there is an exact sequence in $\mathcal{D}_X - mod$

$$\dots \to \mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{X(w_{r,k,n}w_I)}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{Z_{r,k,n}}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{r,k,n}w_I}+2}_{X(w_{r,k-1,n}w_I)}(\mathcal{O}_X) \to \dots$$

Due to Lemma 4.2.3, Lemma 6.3.3 and [13] we get, that $\forall j \geq 0$

$$\mathrm{Supp}(\mathcal{H}^{c_{w_{r,k,n}w_I}+1+j}_{X(w_{r,k,n}w_I)}(\mathcal{O}_X)) \subset X(w_I) \wedge\ \mathrm{Supp}(\mathcal{H}^{c_{w_{r,k,n}w_I}+2+j}_{X(w_{r,k-1,n}w_I)}(\mathcal{O}_X)) \subset X(w_I)$$

and therefore

$$\mathrm{Supp}(\mathcal{H}^{c_{w_{r,k,n}w_I}+1+j}_{Z_{r,k,n}}(\mathcal{O}_X)) \subset X(w_I)\ \forall j \geq 0. \tag{6.6}$$

The proof is an induction proof in r, and we therefore must prove it for $r = 2$ first. We must then show, that

$$[\mathcal{H}^{c_{w_{2,k,n}w_I}+j}_{Z_{2,k,n}}(\mathcal{O}_X)] = \left\{\begin{array}{ll} 0 & j \neq 0 \\ \ [\mathcal{L}(w_{2,4,n}w_I)] + [\mathcal{L}(w_{2,3,n}w_I)] + [\mathcal{L}(w_I)] & j = 0 \wedge\ k = 4 \\ \ [\mathcal{L}(w_{2,k,n}w_I)] + [\mathcal{L}(w_{2,k-1,n}w_I)] & j = 0 \wedge\ k > 4 \end{array}\right\}.$$

We know, that $Y_{1,n} = \mathbb{P}^{n-1}$ and that $\mathbb{P}^{k-2} = X(k-1) \subset Y_{1,n}$. Therefore $X(k-1)$ is smooth, and so is $X(1,k) = i(X(k-1))$ and due to Lemma 4.3.3

also $\pi^{-1}(X(1,k)) = X(v_{2,k,n}w_I)$. We then get according to Lemma 4.2.2 and Lemma 4.4.1

$$\mathcal{H}^{\mathrm{codim}(X(1,k))+j}_{\pi^{-1}(X(1,k))}(\mathcal{O}_X) \neq 0 \Leftrightarrow j = 0,\ [\mathcal{H}^{\mathrm{codim}(X(1,k))}_{\pi^{-1}(X(1,k))}(\mathcal{O}_X)] = [\mathcal{L}(v_{2,k,n}w_I)].$$

This implies, since $\pi^{-1}(X(1,k-1)) \subset \pi^{-1}(X(1,k))$ is closed, and there is an exact sequence

$$\cdots \to \mathcal{H}^m_{\pi^{-1}(X(1,k-1))}(\mathcal{O}_X) \to \mathcal{H}^m_{\pi^{-1}(X(1,k))}(\mathcal{O}_X) \to \mathcal{H}^m_{\pi^{-1}(X(1,k)\cap(X(1,k-1))^c)}(\mathcal{O}_X) \to \ldots$$

since $Z_{1,2,k,n} = \pi^{-1}(X(1,k) \cap (X(1,k-1))^c)$ that

$$\begin{aligned}&\mathcal{H}^{\mathrm{codim}(X(1,k))+j}_{Z_{1,2,k,n}}(\mathcal{O}_X) \neq 0 \Leftrightarrow j = 0,\\ &[\mathcal{H}^{\mathrm{codim}(X(1,k))}_{Z_{1,2,k,n}}(\mathcal{O}_X)] = [\mathcal{L}(v_{2,k-1,n}w_I] + [\mathcal{L}(v_{2,k,n}w_I].\end{aligned} \tag{6.7}$$

By construction $k > r + 1 = 3$ and therefore

$$X(1,k-1) \neq X(1,2) \Rightarrow X(w_I) \subsetneq \pi^{-1}(X(1,k-1))) = X(v_{2,k-1,n}w_I) \tag{6.8}$$

It then follows by Lemma 6.5.1, that we have the following exact sequences in $\mathcal{D}_X - mod$ since $\mathrm{codim}(X(1,k)) = c_{w_{2,k,n}w_I} + 1$

$$\begin{aligned}&0 \to \mathcal{H}^{c_{w_{2,k,n}w_I}}_{Z_{2,k,n}}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{2,k,n}w_I}}_{(Z_{1,2,k,n})^c\cap Z_{2,k,n}}(\mathcal{O}_X) \to\\ &\mathcal{H}^{c_{w_{2,k,n}w_I}+1}_{Z_{1,2,k,n}}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{2,k,n}w_I}+1}_{Z_{2,k,n}}(\mathcal{O}_X) \to 0,\\ &\mathcal{H}^{c_{w_{2,k,n}w_I}+j}_{Z_{2,k,n}}(\mathcal{O}_X) = 0\ \forall j > 1.\end{aligned}$$

It now follows by combining (6.8), (6.6) and (6.7), that

$$\mathcal{H}^{c_{w_{2,k,n}w_I}+1}_{Z_{2,k,n}}(\mathcal{O}_X) = 0.$$

Since $1 = l(w_{2,k,n}w_I) - l(w_{2,k-1,n}w_I)$ it follows by Corollary 4.1.2 and Lemma 6.3.6, that

$$\begin{aligned}&[\mathcal{H}^{c_{w_{2,k,n}w_I}}_{Z_{2,k,n}}(\mathcal{O}_X)] =\\ &\sum_{z\leq w_{2,k,n}w_I,\ z\not\leq w_{2,k-1,n}w_I} (-1)^{l(w_{2,k,n}w_I)-l(z)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] =\\ &\sum_{z\leq w_{2,k,n}w_I} (-1)^{l(w_{2,k,n}w_I)-l(z)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] +\\ &\sum_{z\leq w_{2,k-1,n}w_I} (-1)^{l(w_{2,k-1,n}w_I)-l(z)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] =_{\text{Lemma 6.3.6}}\\ &\left\{\begin{array}{ll}[\mathcal{L}(w_{2,4,n}w_I)] + [\mathcal{L}(w_{2,3,n}w_I)] + [\mathcal{L}(w_I)] & k = 4\\ [\mathcal{L}(w_{2,k,n}w_I)] + [\mathcal{L}(w_{2,k-1,n}w_I)] & k > 4\end{array}\right\}.\end{aligned}$$

and we have the Proposition for $r = 2$. Let us assume, it is true $\forall j < r$ $r > 2$. We let

$$\begin{aligned} &Y := Sl_{n-1}/B_{n-1}, \\ &\pi_1 : Y \to Y_{r-1,n-1} = Sl_{n-1}/P_{n-1}, \\ &X(w_J) := \pi_1^{-1}(P_{n-1}) \end{aligned}$$

with $B_{n-1} \subset Sl_{n-1}$ a Borel subgroup, $P_{n-1} \subset Sl_{n-1}$ a maximal parabolic subgroup containing B_{n-1} such that the above is satisfied and π_1 the canonical morphism. Now we must divide the proof in two cases.

(1) : $k \geq 2r - 1$. Since $k \geq 2r - 1 \Rightarrow k - 1 \geq 2(r - 1)$ we know by induction, that

$$\begin{aligned} &\mathcal{H}^{c_{w_{r-1,k-1,n-1}w_J}+t}_{Z_{r-1,k-1,n-1}}(\mathcal{O}_Y) \neq 0 \Leftrightarrow t = 0 \Rightarrow_{\text{Corollary 4.3.2}} \\ &\mathcal{H}^{c_{w_{r-1,k-1,n-1}w_J}+t}_{X(2,\ldots,r-1,k-1)\cap(X(2,\ldots,r-1,k-2))^c}(\mathcal{O}_{Y_{r-1,n-1}}) \neq 0 \Leftrightarrow t = 0. \end{aligned}$$

We first prove the part of the vanishing of the higher cohomologies. Since

$$\pi(Z_{1,r,k,n}) = i(X(2,\ldots,r-1,k-1) \cap (X(2,\ldots,r-1,k-2))^c)$$

we get according to Proposition 3.3.6 and Lemma 6.1.4 since $i : Y_{r-1,n-1} \to Y_{r,n}$ and both of these varieties are smooth and irreducible, and it is a closed immersion, that

$$\begin{aligned} &\mathcal{H}^{c_{w_{r-1,k-1,n-1}w_J}+n-r+t}_{\pi(Z_{1,r,k,n})}(\mathcal{O}_{Y_{r,n}}) \neq 0 \Leftrightarrow t = 0 \Rightarrow_{\text{Corollary 4.3.2}} \\ &\mathcal{H}^{c_{w_{r-1,k-1,n-1}w_J}+n-r+t}_{Z_{1,r,k,n}}(\mathcal{O}_X) \neq 0 \Leftrightarrow t = 0 \end{aligned}$$

and due to Lemma 6.1.4

$$\begin{aligned} &c_{w_{r-1,k-1,n-1}w_J} + n - r = \operatorname{codim}(X(2,\ldots,r-1,k-1)) + n - r = \\ &\operatorname{codim}(i(X(2,\ldots,r-1,k-1))) = \operatorname{codim}(X(1,3,\ldots,r,k)) = c_{w_{r,k,n}w_I} + 1. \end{aligned}$$

By using these informations along with Lemma 6.5.1, we get a surjection

$$\mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{Z_{1,r,k,n}}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{Z_{r,k,n}}(\mathcal{O}_X) \to 0 \tag{6.9}$$

and

$$\mathcal{H}^{c_{w_{r,k,n}w_I}+1+j}_{Z_{r,k,n}}(\mathcal{O}_X) = 0 \; \forall j > 0. \tag{6.10}$$

Furthermore it follows by combining Corollary 4.1.2 and Corollary 6.3.6, that

$$[\mathcal{H}_{Z_{1,r,k,n}}^{c_{w_{r,k,n}w_I}+1}(\mathcal{O}_X)] = \sum_{z\leq w_{1,r,k,n}w_I,\ z\nleq w_{1,r,k-1,n}w_I} (-1)^{l(w_{1,r,k,n}w_I)-l(z)}[\mathcal{H}_{C(z)}^{c_z}(\mathcal{O}_X)] = \sum_{z\leq w_{1,r,k,n}w_I} (-1)^{l(w_{1,r,k,n}w_I)-l(z)}[\mathcal{H}_{C(z)}^{c_z}(\mathcal{O}_X)] + \sum_{z\leq w_{1,r,k-1,n}w_I} (-1)^{l(w_{1,r,k-1,n}w_I)-l(z)}[\mathcal{H}_{C(z)}^{c_z}(\mathcal{O}_X)] =_{\text{Lemma 6.3.6}} \begin{cases} [\mathcal{L}(w_{1,r,k,n}w_I)] + [\mathcal{L}(w_{1,r,k-1,n}w_I)] + [\mathcal{L}(w_I)] & k = 2r-1 \\ [\mathcal{L}(w_{1,r,k,n}w_I)] + [\mathcal{L}(w_{1,r,k-1,n}w_I)] & k > 2r-1 \end{cases} \tag{6.11}$$

Since $r \geq 3$ $X(w_I) \subsetneq X(w_{1,r,k-1,n}w_I) \subsetneq X(w_{1,r,k,n}w_I)$ and we then get by combining (6.9), (6.6) and (6.11) for $k > 2r-1$

$$\mathcal{H}_{Z_{r,k,n}}^{c_{w_{r,k,n}w_I}+1}(\mathcal{O}_X) = 0.$$

It follows by combining Corollary 4.1.2, Corollary 6.3.6 and (6.10), that

$$[\mathcal{H}_{Z_{r,k,n}}^{c_{w_{r,k,n}w_I}}(\mathcal{O}_X)] = \sum_{z\leq w_{r,k,n}w_I,\ z\nleq w_{r,k-1,n}w_I} (-1)^{l(w_{r,k,n}w_I)-l(z)}[\mathcal{H}_{C(z)}^{c_z}(\mathcal{O}_X)] + [\mathcal{H}_{Z_{r,k,n}}^{c_{w_{r,k,n}w_I}+1}(\mathcal{O}_X)] = \sum_{z\leq w_{r,k,n}w_I} (-1)^{l(w_{r,k,n}w_I)-l(z)}[\mathcal{H}_{C(z)}^{c_z}(\mathcal{O}_X)] + \sum_{z\leq w_{r,k-1,n}w_I} (-1)^{l(w_{r,k-1,n}w_I)-l(z)}[\mathcal{H}_{C(z)}^{c_z}(\mathcal{O}_X)] + [\mathcal{H}_{Z_{r,k,n}}^{c_{w_{r,k,n}w_I}+1}(\mathcal{O}_X)] =_{\text{Lemma 6.3.6}} \left\{ \begin{array}{ll} [\mathcal{L}(w_{r,k,n}w_I)] + [\mathcal{L}(w_{r,k-1,n}w_I)] + [\mathcal{L}(w_I)] & k = 2r \\ [\mathcal{L}(w_{r,k,n}w_I)] + [\mathcal{L}(w_{r,k-1,n}w_I)] & k > 2r \\ [\mathcal{L}(w_{r,k,n}w_I)] + [\mathcal{L}(w_{r,k-1,n}w_I)] - [\mathcal{L}(w_I)] + [\mathcal{H}_{Z_{r,k,n}}^{c_{w_{r,k,n}w_I}+1}(\mathcal{O}_X)] & k = 2r-1 \end{array} \right\}$$

and the Proposition is true for $k > 2r-1$. By combining this with (6.11), (6.9) and (6.6), we see that for $k = 2r-1$

$$[\mathcal{H}_{Z_{r,2r-1,n}}^{c_{w_{r,2r-1,n}w_I}+1}(\mathcal{O}_X)] = [\mathcal{L}(w_I)]$$

and the Proposition is true for $k = 2r-1$.

$(2):\ k < 2r-1$: Since $k-1 < 2(r-1)$ we know by induction, that

$$\mathcal{H}^{c w_{r-1,k-1,n-1} w_J + t}_{Z_{r-1,k-1,n-1}}(\mathcal{O}_Y) \neq 0 \Leftrightarrow\ t \in \{0, 2(r-1)-(k-1)\}.$$

By using exactly the same arguments as in the case $k \geq 2r-1$, we get

$$\begin{gathered}\mathcal{H}^{c w_{r,k,n} w_I + t}_{Z_{1,r,k,n}}(\mathcal{O}_X) \neq 0 \Leftrightarrow\ t \in \{1, 2r-k\} \Leftrightarrow \\ \mathcal{H}^{c w_{1,r,k,n} w_I + t}_{Z_{1,r,k,n}}(\mathcal{O}_X) \neq 0 \Leftrightarrow\ t \in \{0, 2r-k-1\}\end{gathered}$$

and by combining this with Lemma 6.5.1, we get, since $2r-k > 1$

$$\left.\begin{gathered}\mathcal{H}^{c w_{r,k,n} w_I + t}_{Z_{r,k,n}}(\mathcal{O}_X) \neq 0 \Rightarrow\ t \in \{0, 1, 2r-k\} \\ t \in \{0, 2r-k\} \Rightarrow \mathcal{H}^{c w_{r,k,n} w_I + t}_{Z_{r,k,n}}(\mathcal{O}_X) \neq 0 \\ [\mathcal{H}^{c w_{r,k,n} w_I + 2r-k}_{Z_{r,k,n}}(\mathcal{O}_X)] = [\mathcal{H}^{c w_{r,k,n} w_I + 2r-k}_{Z_{1,r,k,n}}(\mathcal{O}_X)]\end{gathered}\right\}. \tag{6.12}$$

By use of Lemma 6.5.1 we get

$$\begin{aligned}&\mathcal{H}^{c w_{s,r,k,n} w_I + t}_{Z_{s,r,k,n}}(\mathcal{O}_X) \neq 0 \Leftrightarrow\ t \in \{0, 2r-k-s\}, \forall s \in \{1, \dots, 2r-k\}, \\ &[\mathcal{H}^{c w_{1,r,k,n} w_I + 2r-k-1}_{Z_{1,r,k,n}}(\mathcal{O}_X)] = [\mathcal{H}^{c w_{2r-k-1,r,k,n} w_I + 1}_{Z_{2r-k-1,r,k,n}}(\mathcal{O}_X)] \Rightarrow \\ &[\mathcal{H}^{c w_{r,k,n} w_I + 2r-k}_{Z_{r,k,n}}(\mathcal{O}_X)] = [\mathcal{H}^{c w_{2r-k-1,r,k,n} w_I + 1}_{Z_{2r-k-1,r,k,n}}(\mathcal{O}_X)]\end{aligned} \tag{6.13}$$

and we get according to (6.6) and (6.12)

$$\emptyset \subsetneq \mathrm{Supp}(\mathcal{H}^{c w_{2r-k-1,r,k,n} w_I + 1}_{Z_{2r-k-1,r,k,n}}(\mathcal{O}_X)) \subset X(w_I) \tag{6.14}$$

along with a surjection in $\mathcal{D}_X - mod$ due to Lemma 6.5.1

$$\mathcal{H}^{c w_{2r-k,r,k,n} w_I}_{Z_{2r-k,r,k,n}}(\mathcal{O}_X) \to \mathcal{H}^{c w_{2r-k-1,r,k,n} w_I + 1}_{Z_{2r-k-1,r,k,n}}(\mathcal{O}_X) \to 0. \tag{6.15}$$

According to Corollary 4.1.2, we see that

$$
\begin{aligned}
&[\mathcal{H}^{c_{w_{2r-k,r,k,n}w_I}}_{Z_{2r-k,r,k,n}}(\mathcal{O}_X)] = \\
&\sum_{z\le w_{2r-k,r,k,n}w_I,\; z\not\le w_{2r-k,r,k-1,n}w_I} (-1)^{l(w_{2r-k,r,k,n}w_I)-l(z)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] = \\
&\sum_{z\le w_{2r-k,r,k,n}w_I} (-1)^{l(w_{2r-k,r,k,n}w_I)-l(z)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] + \\
&\sum_{z\le w_{2r-k,r,k-1,n}w_I} (-1)^{l(w_{2r-k,r,k-1,n}w_I)-l(z)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] =_{\text{Lemma 6.3.6}} \\
&[\mathcal{L}(w_{2r-k,r,k,n}w_I)] + [\mathcal{L}(w_{2r-k,r,k-1,n}w_I)] + \\
&((-1)^{k-(2r-k)}\min(r-(2r-k)-1, k-r-1) + \\
&(-1)^{k-1-(2r-k)}\min(r-(2r-k)-1, k-1-r-1))[\mathcal{L}(w_I)] = \\
&[\mathcal{L}(w_{2r-k,r,k,n}w_I)] + [\mathcal{L}(w_{2r-k,r,k-1,n}w_I)] + [\mathcal{L}(w_I)]
\end{aligned}
$$

and since $X(w_I) \subsetneq X(w_{2r-k,r,k-1,n}w_I) \subsetneq X(w_{2r-k,r,k,n}w_I)$ since $k > r+1$, it follows by combining (6.14), (6.15) and (6.13), that

$$
[\mathcal{H}^{c_{w_{r,k,n}w_I}+2r-k}_{Z_{r,k,n}}(\mathcal{O}_X)] = [\mathcal{H}^{c_{w_{r,k,n}w_I}+2r-k}_{Z_{1,r,k,n}}(\mathcal{O}_X)] = [\mathcal{L}(w_I)]. \tag{6.16}
$$

By using Corollary 4.1.2 we get

$$
\begin{aligned}
&[\mathcal{H}^{c_{w_{r,k,n}w_I}}_{Z_{r,k,n}}(\mathcal{O}_X)] = [\mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{Z_{r,k,n}}(\mathcal{O}_X)] + (-1)^{2r-k-1}[\mathcal{H}^{c_{w_{r,k,n}w_J}+2r-k}_{Z_{r,k,n}}(\mathcal{O}_X)] + \\
&\sum_{z\le w_{r,k,n}w_I,\; z\not\le w_{r,k-1,n}w_I} (-1)^{l(w_{r,k,n}w_I)-l(z)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] = \\
&\sum_{z\le w_{r,k,n}w_I} (-1)^{l(w_{r,k,n}w_I)-l(z)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] - (-1)^k[\mathcal{L}(w_I)] + \\
&\sum_{z\le w_{r,k-1,n}w_I} (-1)^{l(w_{r,k-1,n}w_I)-l(z)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] + [\mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{Z_{r,k,n}}(\mathcal{O}_X)] =_{\text{Lemma 6.3.6}} \\
&[\mathcal{L}(w_{r,k,n}w_I)] + [\mathcal{L}(w_{r,k-1,n}w_I)] + (-1)^k[\mathcal{L}(w_I)] + \\
&[\mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{Z_{r,k,n}}(\mathcal{O}_X)] - (-1)^k[\mathcal{L}(w_I)] = \\
&[\mathcal{L}(w_{r,k,n}w_I)] + [\mathcal{L}(w_{r,k-1,n}w_I)] + [\mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{Z_{r,k,n}}(\mathcal{O}_X)],
\end{aligned} \tag{6.17}
$$

$$\begin{aligned}
&[\mathcal{H}^{c_{w_{1,r,k,n}w_I}}_{Z_{1,r,k,n}}(\mathcal{O}_X)] = (-1)^{2r-k}[\mathcal{H}^{c_{w_{1,r,k,n}w_I}+2r-k-1}_{Z_{1,r,k,n}}(\mathcal{O}_X)] + \\
&\sum_{z\leq w_{1,r,k,n}w_I,\, z\not\leq w_{1,r,k-1,n}w_I} (-1)^{l(w_{1,r,k,n}w_I)-l(z)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] = \\
&\sum_{z\leq w_{1,r,k,n}w_I} (-1)^{l(w_{1,r,k,n}w_I)-l(z)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] + (-1)^k[\mathcal{L}(w_I)] + \\
&\sum_{z\leq w_{1,r,k-1,n}w_I} (-1)^{l(w_{1,r,k-1,n}w_I)-l(z)}[\mathcal{H}^{c_z}_{C(z)}(\mathcal{O}_X)] =_{\text{Lemma 6.3.6}} \\
&[\mathcal{L}(w_{1,r,k,n}w_I)] + [\mathcal{L}(w_{1,r,k-1,n}w_I)] + (-1)^{k-1}[\mathcal{L}(w_I)] + (-1)^k[\mathcal{L}(w_I)] = \\
&[\mathcal{L}(w_{1,r,k,n}w_I)] + [\mathcal{L}(w_{1,r,k-1,n}w_I)].
\end{aligned} \tag{6.18}$$

According to Lemma 6.5.1 there is a surjection in $\mathcal{D}_X - mod$

$$H^{c_{w_{r,k,n}w_I}+1}_{Z_{1,r,k,n}}(\mathcal{O}_X) \to \mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{Z_{r,k,n}}(\mathcal{O}_X) \to 0.$$

Since $r \geq 3$ $X(w_I) \subsetneq X(w_{1,r,k-1,n}w_I) \subsetneq X(w_{1,r,k,n}w_I)$ we get by combining (6.18) and (6.6), that

$$0 = [\mathcal{H}^{c_{w_{r,k,n}w_I}+1}_{Z_{r,k,n}}(\mathcal{O}_X)] \;\Rightarrow_{(6.17)}\; [\mathcal{H}^{c_{w_{r,k,n}w_I}}_{Z_{r,k,n}}(\mathcal{O}_X)] = [\mathcal{L}(w_{r,k,n}w_I)] + [\mathcal{L}(w_{r,k-1,n}w_I)]$$

and we have the Proposition by combining these results with (6.16) and (6.12). □

7 PROBLEMS

Independently of the characteristic of the ground field $\exists \mathcal{L}(w) \in \mathcal{D}_{G/B} - mod$ such that $\mathrm{Supp}(\mathcal{L}(w)) = X(w)$ and $\mathcal{L}(w) \subset \mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_{G/B})$. If the characteristic is greater than zero, we have shown $\mathrm{Supp}(\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_{G/B})/\mathcal{L}(w)) = \emptyset$. If the characteristic is zero and $X = Sl_n/B$, we have proven that for $v \in \mathrm{maxSing}(X(w))$ with either $l(w) - l(v) = 3$ or $\tilde{w} = w_{1,l,1}$ and $l \geq 2$, that $1 = [\mathcal{H}^{c_w}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)]$ and

$$[\mathcal{H}^{c_{w_{n-r,r}}+j}_{X(w_{n-r,r})}(\mathcal{O}_X) : \mathcal{L}(x_{n-r,r})] = \left\{ \begin{array}{lc} 1 & j \in \{n - 2t | t \in \{2, \ldots, \min(r, n-r)\}\},\ j > 0 \\ 1 & j = 0 \wedge\ n = 2r \\ 0 & \text{otherwise} \end{array} \right\}.$$

This raises the natural question. Let $w \in S_n$ and $v \in \mathrm{maxSing}(X(w))$ and suppose $\tilde{w} = w_{k,m}$ and $\tilde{v} = x_{k,m}$ is it true that

$$[\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)] = \left\{ \begin{array}{lc} 1 & j \in \{k + m - 2t | t \in \{2, \ldots, \min(k, m)\}\} \wedge\ j > 0 \\ 1 & j = 0 \wedge\ m = k \\ 0 & \text{otherwise} \end{array} \right\}.$$

The problem in proving the above, is that the methods in proving it in the special case $w = w_{k,m}$ and $v = x_{k,m}$ builds heavily on the fact, that $X(w_{k,m}) = \pi^{-1}(X(2, 3, \ldots, m, k + m))$ with $\pi : G/B \to \mathrm{Gr}(m, k + m)$ the canonical map. The main use of this result, is $\mathcal{H}^j_{\pi^{-1}(C(a_1,\ldots,a_m))}(\mathcal{O}_X) \neq 0 \Leftrightarrow j = \mathrm{codim}(\pi^{-1}(X(a_1, \ldots, a_m)))$. If only one could get a similar result in the general setting, the above question could perhaps be answered, but we have been unable to accomplish this. Suppose $v \in \mathrm{maxSing}(X(w))$ and $\tilde{w} = w_{k,2,m}$ and $\tilde{v} = x_{k,2,m}$ with $k, m \geq 1$. We have not dealt with this last possibility, but we never the less have a suspicion. It states

$$[\mathcal{H}^{c_w+j}_{X(w)}(\mathcal{O}_X) : \mathcal{L}(v)] = \left\{ \begin{array}{lc} 1 & j = l(w) - l(v) - 3 \\ 0 & \text{otherwise} \end{array} \right\}.$$

If $l(w) - l(v) = 3$ we have proven the above. We have not included the arguments, but if $w = w_{1,2,2}, w_{2,2,1}, w_{3,2,1}, w_{1,2,3}$ then the above can be accomplished. But since we have been unable to generalize these arguments further, they are not included.

We have also found $\mathrm{cd}_{\mathrm{Gr}(r,n)}(X(a_s - s + 1, \ldots a_s, a_{s+1}, \ldots, a_r))$ for $a_s \geq r$. We would like to generalize our arguments to find $\mathrm{cd}_{\mathrm{Gr}(r,n)}(X(a_1, \ldots, a_r))$, but as indicated by the example in the end of section 6.2 our methods must be improved, and we have not been able to accomplish this. Even further we would like to find $\mathrm{cd}_X(X(w))$. But if this can be achieved just by using the Grothendieck-Cousin complex is doubtful, and one therefore must be more innovative.

Although the author has been to stupid or unlucky in answering these questions, they ought not to be impossible.

A APPENDIX

A.1 Parabolic Kazhdan-Lusztig polynomials

The purpose with this appendix is to sketch the theory of parabolic Kazhdan-Lusztig polynomials. For a full treatment of this subject one should look up [14]. We let (W, S) denote a Coxeter group and let W_J denote the subgroup generated by any subset $J \subset S$. We define W^J in the following way

$$w \in W^J \Leftrightarrow \; vW_J = wW_J \Rightarrow \; l(w) \leq l(v)$$

where l is the length function on (W, S). So W^J is those $w \in W$, whose length is minimal among wW_J. It then follows by Lemma 2.1 in [14].

Lemma A.1.1. *(i) : $W^J = \{\sigma \in W |\, l(\sigma s) \geq l(\sigma) \; \forall s \in J\}$.*
(ii) : $\forall w \in W \; \exists \sigma \in W^J, \; \tau \in W_J$ such that $w = \sigma\tau$ and $l(w) = l(\sigma) + l(\tau)$ σ, τ are unique.
(iii) If $\sigma \in W^J$ and $s \in S$ satisfies $l(s\sigma) \leq l(\sigma)$ then $s\sigma \in W^J$.

Let M^J be the free $A = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-module with basis $\{m^J_\sigma |\; \sigma \in W^J\}$. We let $H = H(W)$ denote the Hecke algebra over A. This is an A-algebra, which as an A-module is free and generated by $\{T_w\}_{w \in W}$ and the multiplication is defined as

$$T_w T_v = T_{wv} \text{ if } l(wv) = l(w) + l(v),$$
$$(T_s)^2 = (q-1)T_s + qT_{\mathrm{id}}, \; \forall s \in S.$$

By setting $(T_s)^{-1} = q^{-1}T_s - (1 - q^{-1})T_{\mathrm{id}}$ it follows, that T_s is invertible $\forall s \in S$ and therefore also $T_w \; \forall w \in W$. H comes equipped with an involution $\bar{\ }$ defined as

$$\overline{\sum_{w \in W} r_w T_w} = \sum_{w \in W} \overline{r_w} (T_{w^{-1}})^{-1} \;, \; r_w \in A$$

with $\bar{\ }$ defined on A the ringhomomorphism sending $q^{\frac{1}{2}}$ to $q^{-\frac{1}{2}}$ and being the identity on $\mathbb{Z}$. We let $u \in \{-1, q\}$ and $\varphi_J \in \mathrm{Hom}_A(H, M^J)$ be defined as $\varphi_J(T_w) := u^{l(\tau)} m^J_\sigma$ with $w = \sigma\tau \; \sigma \in W^J \; \tau \in W_J$. By defining

$$\overline{\sum_{\sigma \in W^J} r_\sigma m^J_\sigma} := \sum_{\sigma \in W^J} \overline{r_\sigma} \varphi_J(\overline{T_\sigma}) \;, \; r_\sigma \in A$$

we get an involution $\bar{\ }$ on M^J according to (2.6) in [14]. We can now define the parabolic Kazhdan-Lusztig polynomials. A proof of the Proposition below may be found as Proposition 3.2 in [14].

Propositon A.1.2. *$\forall \mu \leq \sigma\ \exists ! P^{J,u}_{\mu,\sigma} \in \mathbb{Z}[q]$ satisfying:*
$(i) : P^{J,u}_{\sigma,\sigma} = 1$ and if $\mu \neq \sigma$ $\deg(P^{J,u}_{\mu,\sigma}) \leq \frac{l(\sigma)-l(\mu)-1}{2}$.
$(ii) : \forall \sigma \in W^J\ \sum_{\mu \leq \sigma} (-1)^{l(\sigma)+l(\mu)} q^{\frac{l(\sigma)}{2}-l(\mu)} \overline{P^{J,u}_{\mu,\sigma}} m^J_\mu$ is invariant under $^-$.

If $J = \emptyset$ it follows since $\varphi_\emptyset(T_w) = m^\emptyset_w$, that $M^\emptyset = H$ and the parabolic Kazhdan-Lusztig polynomials in this case are the ordinary Kazhdan-Lusztig polynomials. We shall be interested in the case $u = -1$ and the following two Propositions, which are Proposition 3.4 and 3.9 in [14].

Propositon A.1.3. *If W_J is finite, then $P^{J,-1}_{\mu,\sigma} = P_{\mu w_J, \sigma w_J}$ with w_J the longest element in W_J.*

We denote $\mu(\sigma, \omega)$ as the coefficient of $q^{\frac{l(\omega)-l(\sigma)-1}{2}}$ in $P^{J,-1}_{\sigma,\omega}$.

Propositon A.1.4. *Let $s \in S$ satisfy $l(s\omega) \leq l(\omega)$. Then for $\sigma \leq \omega$*

$$P^{J,-1}_{\sigma,\omega} = \overline{P} - \sum_{\{\sigma \leq \phi \leq s\omega,\ \phi \in W^J \mid\ s\phi \leq \phi \text{ or } s\phi \notin W^J\}} \mu(\phi, s\omega) q^{\frac{l(\omega)-l(\phi)}{2}} P^{J,-1}_{\sigma,\phi}$$

with

$$\overline{P} = \left\{ \begin{array}{lllll} P^{J,-1}_{s\sigma,s\omega} + qP^{J,-1}_{\sigma,s\omega} & if & l(s\sigma) \leq l(\sigma) & & \\ P^{J,-1}_{\sigma,s\omega} + qP^{J,-1}_{s\sigma,s\omega} & if & l(s\sigma) > l(\sigma) & and & s\sigma \in W^J \\ (1+q)P^{J,-1}_{\sigma,s\omega} & if & l(s\sigma) > l(\sigma) & and & s\sigma \notin W^J \end{array} \right\}.$$

A.2 Some parabolic Kazhdan-Lusztig polynomials

We use the notation of section 6.3.

Lemma A.2.1. *$X(2, 3, \ldots, r+1) \subset \mathrm{Gr}(r, n)$ is smooth.*

Proof. This is simply Lemma 6.1.3. □

We drop the subscript -1 in $P^{J,-1}_{\sigma,\omega}$.

Corollary A.2.2.

$$P^J_{\sigma,\tau} = P^J_{\sigma, w_{r,r+1,n}} = 1\ \forall \sigma \leq \tau.$$

Proof. Since $X(2, 3, \ldots, r+1)$ is smooth, we get according to Lemma 4.3.3 that $X(\tau w_I) = \pi^{-1}(X(2, 3, \ldots, r+1))$ is smooth, and it then follows by [13], that $P_{\theta, \tau w_I} = 1\ \forall \theta \leq \tau w_I$ and then the Corollary follows by Lemma 6.3.1. □

Propositon A.2.3. *Let $\sigma \in W^J$. Then*

$$P^J_{\sigma,w_{r.k.n}} = \left\{ \begin{array}{ll} 1 & \sigma \leq w_{r,k,n},\ \sigma \neq id \\ \sum_{j=0}^{\min(r-1,k-r-1)} q^j & \sigma = id \end{array} \right\}.$$

Proof. The proof is an induction proof in k. It is due to Corollary A.2.2 true for $k = r+1$, and this case is thus done. So let us assume it is true $\forall j \leq k-1$ and prove it for k, with $k \geq r+2$. Since $s_{n-k+1}w_{r,k,n} = w_{r,k-1,n} \leq w_{r,k,n}$ we use Proposition A.1.4 to prove this Proposition. Let us pick $\phi \in W^J$ such that $\sigma \leq \phi \leq s_{n-k+1}w_{r,k,n} = w_{r,k-1,n}$. It then follows by induction that $P^J_{\phi,w_{r,k-1,n}} = 1$ if $\phi \neq id$, and thus that

$$\begin{array}{l} \mu(\phi, s_{n-k+1}w_{r,k,n}) \neq 0 \Rightarrow \\ \phi = id \vee\ \phi \leq w_{r,k-1,n}\ \wedge\ l(w_{r,k-1,n}) - l(\phi) = 1 \Rightarrow_{\text{last case}}\ 1 = \mu(\phi, s_{n-k+1}w_{r,k,n}). \end{array}$$

So in the sum in Proposition A.1.4 it is among these, we have to do a search. So let $\phi \in W^J$ $\phi \neq id$ satisfy the above. Since $\pi(X(w_{r,k-1,n}w_I)) = X(2,3,\ldots,r,k-1)$, we get that either $\pi(X(\phi w_I)) = X(1,3,\ldots,r,k-1)$ or $\pi(X(\phi w_I)) = X(2,3,\ldots,r,k-2)$ where the last possibility is provided $k \neq r+2$. If we choose the first situation, we see that

$$\phi(j) = \left\{ \begin{array}{cc} j & j \in \{1,2,\ldots,n-k+1\} \\ j+1 & j \in \{n-k+2,\ldots,n-r-1\} \\ n-1 & j = n-r \\ n-k+2 & j = n-r+1 \\ j-1 & j \in \{n-r+2,\ldots,n-1\} \\ n & j = n \end{array} \right\} \text{if } k > r+2 \wedge$$

$$\phi(j) = \left\{ \begin{array}{cc} j & j \in \{1,2,\ldots,n-k+1\} \\ n-1 & j = n-r \\ n-k+2 & j = n-r+1 \\ j-1 & j \in \{n-r+2,\ldots,n-1\} \\ n & j = n \end{array} \right\} \text{if } k = r+2 \Rightarrow$$

$$s_{n-k+1}\phi(j) = \left\{ \begin{array}{cc} j & j \in \{1,2,\ldots,n-k\} \\ j+1 & j \in \{n-k+1,\ldots,n-r-1\} \\ n-1 & j = n-r \\ n-k+1 & j = n-r+1 \\ j-1 & j \in \{n-r+2,\ldots,n-1\} \\ n & j = n \end{array} \right\} \text{if } k > r+2$$

and

$$s_{n-k+1}\phi(j) = \left\{ \begin{array}{cc} j & j \in \{1,2,\dots,n-k\} \\ n-k+2 & j = n-k+1 \\ n-1 & j = n-r \\ n-k+1 & j = n-r+1 \\ j-1 & j \in \{n-r+2,\dots,n-1\} \\ n & j = n \end{array} \right\} \text{ if } k = r+2.$$

We see $\pi^{-1}(X(1,3,\dots,r,k)) = X((s_{n-k+1}\phi)w_I)$ and thus $s_{n-k+1}\phi \in W^J$ and $\phi \leq s_{n-k+1}\phi$, and this choice of ϕ does not appear in the sum in Proposition A.1.4. Let us choose the last situation. In this case $\phi = w_{r,k-2,n}$ and therefore

$$s_{n-k+1}w_{r,k-2,n}(n-k+1) = n-k+2,\ s_{n-k+1}w_{r,k-2,n}(n-k+2) = n-k+1$$

we see, that $s_{n-k+1}\phi \notin W^J$ since $n-k+2 < n-r+1$ and, this is the only choice of ϕ occurring in the sum along with $\phi = id$ since $s_{n-k+1} = s_{n-k+1}id \notin W^J$ due to the same reason.

Let us now assume $\sigma \neq id$. Then

$$P^J_{\sigma,w_{r,k,n}} = \left\{ \begin{array}{ll} \overline{P} - qP^J_{\sigma,w_{r,k-2,n}} & k \neq r+2 \\ \overline{P} & k = r+2 \end{array} \right\}.$$

Since $\sigma \leq w_{r,k,n}$ and $\sigma \in W^J$, we know

$$\begin{aligned} &X(\sigma w_I) = \pi^{-1}(X(a_1,\dots,a_r)), \\ &\sigma(n-r+j) = n+1-a_{r-j+1}\ \forall j \in \{1,2,\dots,r\}. \end{aligned}$$

Due to the fact that $X(w_{r,k,n}w_I) = \pi^{-1}(X(2,3,\dots,r,k))$, we know $a_i \leq i+1$ if $i \leq r-1$ and $a_r \leq k$. So assume $k \neq r+2$. It then follows since $X(w_{r,k-2,n}w_I) = \pi^{-1}(X(2,3,\dots,r,k-2))$ that $\sigma \leq w_{r,k-2,n} \Leftrightarrow a_r \leq k-2$ and therefore

$$P^J_{\sigma,w_{r,k,n}} = \left\{ \begin{array}{ll} \overline{P} - qP^J_{\sigma,w_{r,k-2,n}} & k \neq r+2,\ a_r \leq k-2 \\ \overline{P} & k \neq r+2,\ a_r \in \{k-1,k\} \\ \overline{P} & k = r+2 \end{array} \right\}.$$

Since $s_{n-k+1}w_{r,k,n} = w_{r,k-1,n}$ and $X(w_{r,k-1,n}w_I) = \pi^{-1}(X(2,3,\dots,r,k-1))$ and therefore $a_r \leq k-1 \Leftrightarrow \sigma \leq w_{r,k-1,n} = s_{n-k+1}w_{r,k,n}$. There are now some cases to consider.

$(1): k \neq r+2,\ a_r \leq k-2$: In this case since $\sigma(n-r+1) = n+1-a_r \geq n-k+3$ we get, that $\sigma(j) = j\ \forall j \in \{1,\dots,n-k+2\}$ and therefore, that $s_{n-k+1}\sigma(n-k+1) = n-k+2$ and $s_{n-k+1}\sigma(n-k+2) = n-k+1$, which implies, that $s_{n-k+1}\sigma \notin W^J$. It follows by Lemma A.1.1 in Appendix A, that $l(s_{n-k+1}\sigma) \leq l(\sigma) \Rightarrow s_{n-k+1}\sigma \in W^J$. This implies, that we are in the last of the three situations in Proposition A.1.4 and thus $\overline{P} = (1+q)P^J_{\sigma,w_{r,k-1,n}} = 1+q$ thanks to induction and the fact that $\sigma \neq id$ from which, we also get $1 = P^J_{\sigma,w_{r,k-2,n}}$ and therefore $1 = P^J_{\sigma,w_{r,k,n}}$.

$(2): k = r+2,\ a_r \leq k-2$: This implies $a_r \leq r$, and therefore $\sigma = id$, and this case will be considered afterward.

$(3): a_r = k-1$: In this case

$$\sigma(j) = j\ \forall j \in \{1,\dots,n-k+1\},\ \sigma(n-k+2) = n-k+3,\ \sigma(n-r+1) = n+1-a_r = n-k+2.$$

Therefore

$$s_{n-k+1}\sigma(j) = j\ \forall j \in \{1,\dots,n-k\},$$
$$s_{n-k+1}\sigma(n-k+1) = n-k+2,\ s_{n-k+1}\sigma(n-r+1) = n-k+1.$$

Thus $X(s_{n-k+1}\sigma) = \pi^{-1}(X(a_1,\dots,a_{r-1},k)) \Rightarrow l(s_{n-k+1}\sigma) \geq l(\sigma)$ and $s_{n-k+1}\sigma \in W^J$. Hence we are in the middle part of Proposition A.1.4. Since $X(s_{n-k+1}w_{r,k,n}) = X(w_{r,k-1,n}) = \pi^{-1}(X(2,3,\dots,r,k-1))$ and also $s_{n-k+1}\sigma \not\leq s_{n-k+1}w_{r,k,n}$ we get $\overline{P} = P^J_{\sigma,w_{r,k-1,n}}$. It then follows by induction and since $\sigma \neq id$ that $P^J_{\sigma,w_{r,k,n}} = \overline{P} = P^J_{\sigma,w_{r,k-1,n}} = 1$.

$(4): a_r = k$: In this case

$$\sigma(j) = j\ \forall j \in \{1,\dots,n-k\},\ \sigma(n-k+1) = n-k+2,\ \sigma(n-r+1) = n-k+1.$$

Therefore

$$s_{n-k+1}\sigma(j) = j\ \forall j \in \{1,\dots,n-k,n-k+1\},$$
$$s_{n-k+1}\sigma(n-k+2) = n-k+3,\ s_{n-k+1}\sigma(n-r+1) = n-k+2.$$

Thus $X(s_{n-k+1}\sigma) = \pi^{-1}(X(a_1,\dots,a_{r-1},k-1)) \Rightarrow l(s_{n+k+1}\sigma) \leq l(\sigma)$, and we are thus in the first part of Proposition A.1.4. Since $X(s_{n-k+1}w_{r,k,n}) = X(w_{r,k-1,n}) = \pi^{-1}(X(2,3,\dots,r,k-1))\ \sigma \not\leq s_{n-k+1}w_{r,k,n}$. Hence $P^J_{\sigma,w_{r,k,n}} = \overline{P} = P^J_{s_{n-k+1}\sigma,w_{r,k-1,n}}$. Since $k \geq r+2 \Rightarrow k-1 \geq r+1$ and therefore

$s_{n-k+1}\sigma \neq id$ it follows by induction, that $P^J_{\sigma,w_{r,k,n}} = 1$.

Now we just lack to prove it in the case $\sigma = id$. Now as noticed earlier $\phi = id$ also occurs in the sum in Proposition A.1.4. Since $\mu(id, s_{n-k+1}w_{r,k,n}) = \mu(id, w_{r,k-1,n})$ is equal to the coefficient of $q^{\frac{l(w_{r,k-1,n})-l(id)-1}{2}}$ in $P^J_{id,w_{r,k-1,n}}$ and $l(id) = 0$ and $l(w_{r,k-1,n})$ =dim$(X(2,3,\dots,r,k-1)) = r-1+k-1-r = k-2 \Rightarrow$

$$\frac{l(w_{r,k-1,n}) - l(id) - 1}{2} = \frac{k-3}{2}$$

since by induction deg$(P^J_{id,w_{r,k-1,n}})$ =min$(r-1, k-1-r-1)$, we get, that also by induction

$$\begin{aligned}
&\mu(id, w_{r,k-1,n}) \neq 0 \Leftrightarrow\ \mu(id, w_{r,k-1,n}) = 1 \Leftrightarrow \\
&2|(k-3) \wedge\ \frac{k-3}{2} \leq \min(r-1, k-r-2) \Leftrightarrow \\
&2|(k-3) \wedge\ \frac{k-3}{2} \leq r-1 \wedge \frac{k-3}{2} \leq k-r-2 \Leftrightarrow \\
&2|(k-3) \wedge\ k \leq 2r+1 \wedge\ k \geq 2r+1 \Leftrightarrow k = 2r+1.
\end{aligned}$$

So there occurs an extra part in the sum in Proposition A.1.4 if and only if $k = 2r+1$. Furthermore

$$\min(r-1, k-r-1) = r-1 \Leftrightarrow\ k \geq 2r.$$

Since $s_{n-k+1} \in W^J \Leftrightarrow n-k+1 = n-r \Leftrightarrow k = r+1$, which is not the case, we get that, we are in the last part of Proposition A.1.4 and therefore $\overline{P} = (1+q)P^J_{id,w_{r,k-1,n}}$. Once again there are some cases to consider.

(1) : $k = r+2$: Since $2r+1 = r+2 \Leftrightarrow r = 1$ and $r \geq 2$ this does not happen and therefore

$$P_{id,w_{r,k,n}} = (1+q)P^J_{id,w_{r,k-1,n}} = 1+q$$

since in this case $w_{r,k-1,n} = w_{r,r+1,n} = \tau$ and it then follows by induction.

(2) : $k = 2r+1 \wedge\ k > r+2$: Since $k = 2r+1 > r+2$ we have

$$P^J_{id,w_{r,k,n}} = (1+q)P^J_{id,w_{r,k-1,n}} - qP^J_{id,w_{r,k-2,n}} - q^{\frac{l(w_{r,k,n})-l(id)}{2}} P^J_{id,id}.$$

In this setup by induction $P^J_{id,w_{r,k-1,n}} = \sum_{j=0}^{r-1} q^j$ and $P^J_{id,w_{r,k-2,n}} = \sum_{j=0}^{r-2} q^j$ and $l(w_{r,k,n})$ =dim$(X(2,\dots,r,k)) = r-1+k-r = 2r$. According to

Lemma 6.3.1 $P^J_{id,id} = P_{w_I,w_I} = 1$ therefore

$$P^J_{id,w_{r,k,n}} = (1+q)\sum_{j=0}^{r-1} q^j - q\sum_{j=0}^{r-2} q^j - q^r = \sum_{j=0}^{r-1} q^j$$

and since $\min(r-1, k-r-1) = r-1$ the Proposition is true in this case.

$(3) : k > 2r+1 \wedge\ k > r+2$: Now we get

$$P^J_{id,w_{r,k,n}} = (1+q)P^J_{id,w_{r,k-1,n}} - qP^J_{id,w_{r,k-2,n}}$$

since $\min(r-1, k-1-r-1) = \min(r-1, k-2-r-1) = r-1$ we get by induction, that $P^J_{id,w_{r,k-1,n}} = P^J_{id,w_{r,k-2,n}} = \sum_{j=0}^{r-1} q^j$ and therefore

$$P^J_{id,w_{r,k,n}} = \sum_{j=0}^{r-1} q^j$$

which proves the Proposition since $\min(r-1, k-r-1) = r-1$.

$(4) : k < 2r+1 \wedge\ k > r+2$: In this case $\min(r-1, k-2-r-1) = k-r-3$ and $\min(r-1, k-1-r-1) = k-r-2$ and therefore

$$P^J_{id,w_{r,k,n}} = (1+q)P^J_{id,w_{r,k-1,n}} - qP^J_{id,w_{r,k-2,n}} =$$
$$(1+q)\sum_{j=0}^{k-r-2} q^j - q\sum_{j=0}^{k-r-3} q^j = \sum_{j=0}^{k-r-1} q^j$$

and this proves the Proposition in the last case, since $\min(r-1, k-r-1) = k-r-1$. □

Lemma A.2.4.

$$P^J_{\sigma,w_{s,r,k,n}} = \left\{ \begin{array}{ll} 1 & \sigma \leq w_{s,r,k,n},\ \sigma \neq id \\ \sum_{j=0}^{\min(r-s-1,k-r-1)} q^j & \sigma = id \end{array} \right\}.$$

Proof. Let us consider $X(2,\ldots,r-s,k-s) \subset Y_{r-s,n-s}$. It follows by combining Lemma 6.3.3, Lemma 4.3.3 and [13], that $\mathrm{Sing}(X(2,\ldots,r-s,k-s)) = X(1,2,\ldots,r-s)$. It then follows by repeated use of Lemma 6.1.3, that $\mathrm{Sing}(X(1,\ldots,s,s+2,\ldots,r,k)) = X(1,2,\ldots,r)$ and therefore again due to Lemma 4.3.3 and [13] it follows for $W^J \ni \sigma \neq id$ $P_{\sigma w_I, w_{s,r,k,n} w_I} = 1$ and by Lemma 6.3.1, that $1 = P^J_{\sigma,w_{s,r,k,n}}$.

If $k = r+1$ it follows since $X(2,\dots,r-s,r+1-s) \subset Y_{r-s,n-s}$ is smooth due to Lemma A.2.1 by the same arguments, that $X(1,\dots,s,s+2,\dots,r,r+1)$ is smooth and thus $P^J_{id,w_{s,r,r+1,n}} = 1$. Assume now the Lemma is true $\forall j \leq k-1$. By using the same arguments as in the proof of Proposition A.2.3, we get for $k > r+2$

$$P^J_{id,w_{s,r,k,n}} = (1+q)P^J_{id,w_{s,r,k-1,n}} - \mu(id,w_{s,r,k-1,n})q^{\frac{l(w_{s,r,k,n})-l(id)}{2}} P^J_{id,id} - qP^J_{id,w_{s,r,k-2,n}}$$

and for $k = r+2$

$$\begin{aligned}&P^J_{id,w_{s,r,r+2,n}} = (1+q)P^J_{id,w_{s,r,r+1,n}} - \mu(id,w_{s,r,r+1,n})q^{\frac{l(w_{s,r,r+2,n})-l(id)}{2}} P^J_{id,id} = \\ &1+q-\mu(id,w_{s,r,r+1,n})q^{\frac{l(w_{s,r,r+2,n})-l(id)}{2}}.\end{aligned}$$

$l(w_{s,r,k,n})$ =dim$(X(1,\dots,s,s+2,\dots,r,k)) = k-r+r-s-1 = k-s-1$ and

$$\begin{aligned}&\mu(id,w_{s,r,k-1,n}) \neq 0 \Leftrightarrow \mu(id,w_{s,r,k-1,n}) = 1 \Leftrightarrow \\ &2|k-s-3 \wedge \frac{k-s-3}{2} \leq \min(r-s-1,k-1-r-1) \Leftrightarrow \\ &k = 2r-s+1.\end{aligned}$$

If $k = r+2 \wedge\ k = 2r-s+1 \Rightarrow 1 = r-s$ and since $s \in \{1,\dots,r-2\}$ the Lemma is true for $k = r+2$. If $k \geq r+3$ the Lemma follows as the proof of Proposition A.2.3 since $\min(r-s-1,k-2-r-1)$ =$\min(r-s-1,k-1-r-1) = r-s-1 \Leftrightarrow\ k \geq 2r-s+2 \Rightarrow P^J_{id,w_{s,r,k,n}} = \sum_{j=0}^{r-s-1} q^j = \sum_{j=0}^{\min(k-r-1,r-s-1)} q^j$ and the other cases are proved in the same way. □

A.3 Localization of differential operators

We let R denote a k algebra with k a field and $M \in R-mod$. We wish to construct a morphism

$$\phi : D(M) \to D(M_U).$$

Let us find some simple relations with [,]. Let $u,t \in R$, $\psi \in \mathrm{Hom}_k(M,M')$ and $b \in k$, then

$$\begin{aligned}&[u,[t,\psi]] = ut\psi - u\psi t - t\psi u + \psi tu = t[u,\psi] - [u,\psi]t = [t,[u,\psi]], \\ &[ut,\psi] = ut\psi - \psi ut = u(t\psi - \psi t) + (u\psi - \psi u)t = u[t,\psi] + [u,\psi]t, \\ &[u+t,\psi] = (u+t).\psi - \psi.(u+t) = [u,\psi] + [t,\psi], \\ &[bu,\psi] = bu\psi - \psi.(bu) = b[u,\psi], \\ &[u,t\psi] = ut\psi - t\psi u = t[u,\psi].\end{aligned} \tag{A.1}$$

To create ϕ we shall need the following Lemma.

Lemma A.3.1. *Let* $d \in D(M_U)$ *and suppose* $d(\frac{m}{1}) = 0 \ \forall m \in M \Rightarrow$ *d=0.*

Proof. We must show $d(\frac{m}{u}) = 0 \ \forall m \in M, \ u \in U$. Pick $\frac{m}{u}$.

$$0 = d(\frac{m}{1}) = d(u\frac{m}{u}) = -[u,d](\frac{m}{u}) + ud(\frac{m}{u})$$

we are done, if we can show $[u,d] = 0$. This is done by induction. If $d \in D_0(M_U) = \mathrm{Hom}_{R_U}(M_U, M_U)$, we are done, since $d(\frac{m}{u}) = \frac{1}{u}d(\frac{m}{1}) = 0$. So assume $d \in D_r(M_U)$, then $[u,d] \in D_{r-1}(M_U)$ and since

$$[u,d](\frac{m}{1}) = ud(\frac{m}{1}) - d(\frac{um}{1}) = 0$$

we get by induction $[u,d] = 0$ and thus the Lemma. □

This Lemma enables us to construct the morphism above.

Propositon A.3.2. *Let* $d, d' \in D_r(M)$*. By setting*

$$\phi(d)(\frac{m}{u}) := \frac{\phi([u,d])(\frac{m}{u}) + d(m)}{u} \ \forall m \in M, \ u \in U$$

we get $\phi(d) \in D_r(M_U)$ *satisfying*

$$\phi(d)(\frac{m}{1}) = \frac{d(m)}{1}, \ \phi(d + d') = \phi(d) + \phi(d'),$$
$$\phi(ad)(\frac{m}{u}) = \frac{a}{1}(\phi(d)(\frac{m}{u})), \ \phi(da)(\frac{m}{u}) = \phi(d)(\frac{am}{u}) \ \forall a \in R,$$
$$\phi(d \circ d') = \phi(d) \circ \phi(d') \ (d' \in D(M) \text{ arbitrary}).$$

Proof. The proof is an induction proof. Assume $r = 0$. By construction $D_0(M) = \mathrm{End}_R(M)$ and $[r,d] = 0 \ \forall r \in R$. We must prove, $\phi(d)$ is independent of $\frac{m}{u}$. Assume $\frac{m}{u} = \frac{n}{t} \Leftrightarrow \exists u' \in U \ u'(tm - un) = 0$. By construction

$$\phi(d)(\frac{m}{u}) := \frac{d(m)}{u} = \frac{u't(d(m))}{u'tu} = \frac{d(u'tm)}{u'tu} = \frac{d(u'un)}{u'tu} = \frac{u'u(d(n))}{u'tu} = \frac{d(n)}{t} = \phi(d)(\frac{n}{t}).$$

So $\phi(d) \in \mathrm{End}_k(M_U)$. So given $d' \in D_0(M)$, $m \in M$, $u, t \in U$ and $a \in R$

$$\phi(d)(\frac{m}{1}) = \frac{d(m)}{1},$$
$$\phi(d)(\frac{a}{t}\frac{m}{u}) = \frac{d(am)}{ut} = \frac{a(d(m))}{ut} = \frac{a}{t}\frac{d(m)}{u} = \frac{a}{t}\phi(\frac{m}{u}) \Rightarrow \ \phi(d) \in D_0(M_U),$$
$$\phi(ad)(\frac{m}{u}) = \frac{ad(m)}{u} = \frac{a}{1}\frac{d(m)}{u} = \frac{a}{1}(\phi(d)(\frac{m}{u})),$$
$$\phi(da)(\frac{m}{u}) = \frac{d(am)}{u} = \phi(d)(\frac{am}{u}),$$
$$\phi(d + d')(\frac{m}{u}) = \frac{(d + d')(m)}{u} = \frac{d(m) + d'(m)}{u} = \phi(d)(\frac{m}{u}) + \phi(d')(\frac{m}{u})$$

and the Lemma is proved for $r = 0$ apart from the last property, but this will be proved in the end. So assume it is proved $\forall j < r$ with $r > 0$. We let $\frac{m}{u} = \frac{n}{t} \Leftrightarrow \ \exists u' \in U \ u'(tm - un) = 0$. Then we get, since we know the Lemma is true for $[a, d]$ $\forall a \in R$

$$\phi(d)(\frac{m}{u}) := \frac{\phi([u,d])(\frac{m}{u}) + d(m)}{u} = \frac{t\phi([u,d])(\frac{m}{u}) + td(m)}{tu} =$$
$$\frac{\phi(t[u,d])(\frac{m}{u}) + td(m)}{tu} = \frac{\phi([t,[u,d]] + [u,d]t)(\frac{m}{u}) + td(m)}{tu} =_{\text{see } (A.1)}$$
$$\frac{\phi([u,[t,d]])(\frac{m}{u}) + \phi([u,d])(\frac{tm}{u}) + td(m)}{tu} =$$
$$\frac{\phi(u[t,d] - [t,d]u)(\frac{m}{u}) + \phi([u,d])(\frac{tn}{t}) + td(m)}{tu} =$$
$$\frac{\phi(u[t,d])(\frac{m}{u}) - \phi([t,d]u)(\frac{m}{u}) + [u,d](n) + td(m)}{tu} =$$
$$\frac{\phi([t,d])(\frac{m}{u})}{t} + \frac{[u,d](n) + td(m) - \phi([t,d])(\frac{m}{1})}{tu} =$$
$$\frac{\phi([t,d])(\frac{n}{t})}{t} + \frac{[u,d](n) + td(m) - [t,d](m)}{tu} =$$
$$\frac{\phi([t,d])(\frac{n}{t})}{t} + \frac{ud(n) + d(tm - un)}{tu} = \phi(d)(\frac{n}{t}) + \frac{d(tm - un)}{tu}.$$

To show, $\phi(d)(\frac{m}{u})$ is independent of $\frac{m}{u}$, we just have to show, $0 = \frac{d(tm-un)}{1}$

$$\frac{d(tm - un)}{1} = \frac{u'd(tm - un)}{u'} = \frac{[u',d](tm - un) + d(u'(tm - un))}{u'} = \frac{[u',d](tm - un)}{u'}.$$

Therefore we just have to show, $\frac{[u',d](tm-un)}{1} = 0$ and since $[u', d]$ is a differential operator of order $< r$ the result follows by induction. We have therefore shown, that $\phi(d) \in \mathrm{End}_k(M_U)$. We need to prove, that $\phi(d) \in D_r(M_U) \Leftrightarrow [\frac{a}{u}, \phi(d)] \in D_{r-1}(M_U)\ \forall a \in R,\ u \in U$. According to (A.1)

$$[\frac{a}{u}, \phi(d)] = a[\frac{1}{u}, \phi(d)] + [\frac{a}{1}, \phi(d)]\frac{1}{u}$$

and since $D_{r-1}(M_U)$ is a R_U-bimodule, we have to show, $[\frac{a}{1}, \phi(d)], [\frac{1}{u}, \phi(d)] \in D_{r-1}(M_U)$. Since

$$0 = [1, \phi(d)] = [\frac{u}{u}, \phi(d)] = u[\frac{1}{u}, \phi(d)] + [\frac{u}{1}, \phi(d)]\frac{1}{u} \Rightarrow$$
$$[\frac{1}{u}, \phi(d)] = \frac{-1}{u}[\frac{u}{1}, \phi(d)]\frac{1}{u}$$

we see, we just have to prove $\forall a \in R$ that $[\frac{a}{1}, \phi(d)] \in D_{r-1}(M_U)$. So pick $a \in R$, $\frac{n}{v} \in M_U$, then

$$[\frac{a}{1}, \phi(d)](\frac{n}{v}) = \frac{a}{1}(\frac{\phi([v,d])(\frac{n}{v}) + d(n)}{v}) - \frac{\phi([v,d])(\frac{an}{v}) + d(an)}{v} =$$
$$\frac{a\phi([v,d])(\frac{n}{v}) - \phi([v,d])(\frac{an}{v}) + [a,d](n)}{v} = \frac{\phi(a[v,d])(\frac{n}{v}) - \phi([v,d]a)(\frac{n}{v}) + [a,d](n)}{v} =$$
$$\frac{\phi(a[v,d] - [v,d]a)(\frac{n}{v}) + [a,d](n)}{v} = \frac{\phi([a,[v,d]])(\frac{n}{v}) + [a,d](n)}{v} =_{\text{see } (A.1)}$$
$$\frac{\phi([v,[a,d]])(\frac{n}{v}) + [a,d](n)}{v} = \phi([a,d])(\frac{n}{v}) \Rightarrow [\frac{a}{1}, \phi(d)] = \phi([a,d])$$

and the result follows by induction. Since $[1,d] = 0$, $[u, ad] = a[u,d]$ and $[u, da] = [u,d]a\ \forall a \in R$ we get

$$\phi(d)(\frac{m}{1}) = \frac{d(m)}{1},\ \phi(ad)(\frac{m}{u}) = \frac{a}{1}(\phi(d)(\frac{m}{u})),\ \phi(da)(\frac{m}{u}) = \phi(d)(\frac{am}{u}).$$

It follows by Lemma A.3.1, that

$$\phi(d+d') = \phi(d) + \phi(d') \Leftrightarrow\ \phi(d+d')(\frac{m}{1}) = \phi(d)(\frac{m}{1}) + \phi(d')(\frac{m}{1})\ \forall m \in M \Leftrightarrow$$
$$\frac{(d+d')(m)}{1} = \frac{d(m)}{1} + \frac{d'(m)}{1}\ \forall m \in M$$

which by construction is true. We just lack to prove the last property. Let $d, d' \in D(M)$ and let us show, $\phi(d \circ d') = \phi(d) \circ \phi(d')$. Since all elements are in $D(M_U)$ we just have to show according to Lemma A.3.1, that $\forall m \in M$

$$(\phi(d \circ d'))(\frac{m}{1}) = (\phi(d) \circ \phi(d'))(\frac{m}{1}) \Leftrightarrow$$
$$\frac{d(d'(m))}{1} = (\phi(d))(\frac{d'(m)}{1}) = \frac{d(d'(m))}{1}$$

and we get the Proposition. □

So we have now constructed a map

$$\phi : D(M) \to D(M_U)$$

which is R-linear with respect to both the right and left R-module structure and in the case $R = M$ is a ring homomorphism. We want to know the kernel of this ring homomorphism, and its image, and in the case that R is a finitely generated k-algebra, this can be achieved. We shall need the following Lemma.

Lemma A.3.3. *Let $R = k[x_1, \ldots, x_n]$ be a finitely generated k-algebra and let $d \in \mathrm{End}_k(R)$, then $d \in D_m(R) \Leftrightarrow [x_i, d] \in D_{m-1}(R)\ \forall i \in \{1, 2, \ldots, n\}$.*

Proof. That $\Rightarrow$ is true follows by definition of $D_m(R)$. So let us assume $[x_i, d] \in D_{m-1}(R)\ \forall i \in \{1, 2, \ldots, n\}$. It follows by (A.1), that we just have to show, that $[\prod_{j=1}^n x_j^{a_j}, d] \in D_{m-1}(R)$ since $D_{m-1}(R)$ is a k-vectorspace since it is a left R module and $k \subset R$. So pick $a_1, \ldots, a_n \in \mathbb{N}$ and let $J = \{j_1, \ldots, j_t\} \subset \{1, 2, \ldots, n\}$ be defined such that $a_j \neq 0 \Leftrightarrow j \in J$. It then follows by (A.1)

$$[\prod_{j=1}^n x_j^{a_j}, d] = [\prod_{r=1}^t x_{j_r}^{a_{j_r}}, d] = [x_{j_1}, d] x_{j_1}^{a_{j_1}-1} \prod_{r=2}^t x_{j_r}^{a_{j_r}} + x_{j_1} [x_{j_1}^{a_{j_1}-1} \prod_{r=2}^t x_{j_r}^{a_{j_r}}, d]$$

and the Lemma follows by induction in $s = \sum_{j=1}^n a_j$ since $D_{m-1}(R) \in R - mod$ and $D_{m-1}(R) \in mod - R$. □

Lemma A.3.4. *Suppose R is a finitely generated k-algebra and $M = R$ then*

$$\phi(d) = 0 \Leftrightarrow\ \exists u \in U\ ud = 0 \Leftrightarrow \exists v \in U\ dv = 0,$$
$$\forall \pi \in D_r(R_U)\ \exists d, d_1 \in D_r(R),\ u, v \in U\ \pi = \frac{1}{u}\phi(d) = \phi(d_1)\frac{1}{v}.$$

Proof. Since R is a finitely generated k-algebra $R = k[x_1, \ldots, x_n]$. We shall only prove the first implication in the first line above. The other is proved in exactly the same manner. So assume

$$\exists u \in U\ ud = 0 \Rightarrow 0 = \phi(ud) = \phi(u)\phi(d) = \frac{u}{1}\phi(d) \Rightarrow \phi(d) = 0.$$

Assume $\phi(d) = 0$. The proof of this implication is an induction proof in the order of the differential operator. So assume $d \in D_0(R) = R$. Then $\phi(d) = \frac{d}{1} = 0 \Rightarrow \exists u \in U\ ud = 0$, and we are done. So assume $d \in D_r(R)$

$r > 0$, and it is proved for all differential operators of order less than r. By construction $[x_i, d] \in D_{r-1}(R)$ and

$$\phi([x_i, d]) = \phi(x_i d - dx_i) = \phi(x_i)\phi(d) - \phi(d)\phi(x_i) = 0 \Rightarrow$$
$$\exists u_i \in U \ u_i[x_i, d] = 0,$$
$$0 = \phi(d)(\frac{1}{1}) = \frac{d(1)}{1} \Rightarrow \ \exists u_{n+1} \in U \ u_{n+1}d(1) = 0.$$

Now we will show, that by setting $u = \prod_{i=1}^{n+1} u_i$ we get $ud = 0$. Since $ud \in \mathrm{End}_k(R)$ we just have to show $ud(\prod_{j=1}^{n} x_j^{a_j}) = 0$. This is an induction proof in $m = \sum_{j=1}^{n} a_j$. If $m = 0 \Rightarrow a_j = 0$ and since $ud(1) = \prod_{j=1}^{n} u_j u_{n+1} d(1) = 0$ it is true for $m = 0$. So assume $m > 0$ and let $J = \{j_1, \dots, j_s\} \subset \{1, 2, \dots, n\}$ such that $a_j = 0 \Leftrightarrow j \notin J$. Then

$$ud(\prod_{j=1}^{n} x_j^{a_j}) = ud(\prod_{k=1}^{s} x_{j_k}^{a_{j_k}}) =$$
$$-(u[x_{j_1}, d](x_{j_1}^{a_{j_1}-1} \prod_{k=2}^{s} x_{j_k}^{a_{j_k}}) - ux_{j_1} d(x_{j_1}^{a_{j_1}-1} \prod_{k=2}^{s} x_{j_k}^{a_{j_k}})) = 0$$

and we have proved the first line of the Lemma. We shall only prove the first equality in the second line, since the proof of the second is the same. Once again the proof is an induction proof in r. So assume $r = 0$. Then $\pi = \frac{a}{u}$ with $a \in R = D_0(R)$ and $u \in U$ and then it is clearly true. So let $\pi \in D_m(R_U)$. Assume $\pi(\frac{1}{1}) = 0$. By induction $\exists d_i \in D_{m-1}(R)$ and $u_i \in U$ such that

$$[\frac{x_i}{1}, \pi] = \frac{1}{u_i}\phi(d_i).$$

So let $s \in U$ be defined as

$$s := \prod_{j=1}^{n} u_j$$

we then get by combining (A.1) and Proposition A.3.2, that

$$[\frac{x_i}{1}, \frac{s}{1}\pi] = \frac{s}{1}[\frac{x_i}{1}, \pi] = \frac{\prod_{j=1, j\neq i}^{n} u_j}{1}\phi(d_i) = \phi(\prod_{j=1, j\neq i}^{n} u_j d_i). \tag{A.2}$$

We now wish to construct a morphism, which satisfies

$$\omega : R \to R, \ \omega \in \mathrm{End}_k(R),$$
$$\frac{s}{1}\pi(\frac{z}{1}) = \frac{\omega(z)}{1}.$$

Due to Proposition 1.1 in [44] $\exists J \subset \mathbb{N}^n$ with $\{\prod_{j=1}^n x_i^{a_i} |\ (a_1,\ldots,a_n) \in J\}$ a k-vector space basis for $R = k[x_1,\ldots,x_n]$ and if $(a_1,\ldots,a_n) \in J \Rightarrow (b_1,\ldots,b_n) \in J$ provided $b_i \leq a_i\ \forall i$. Since $\frac{s}{1}\pi \in \mathrm{End}_k(R_U)$ it is enough to show $\forall (a_1,\ldots,a_n) \in J\ \exists z_{a_1,\ldots,a_n} \in R$ such that

$$\frac{s}{1}\pi(\frac{\prod_{j=1}^n x_j^{a_j}}{1}) = \frac{z_{a_1,\ldots,a_n}}{1}$$

because we then define $\omega(\prod_{j=1}^n x_j^{a_j}) = z_{a_1,\ldots,a_n}$. Since $k \subset R$ $(0,\ldots,0) \in J$ and

$$0 = \frac{s}{1}\pi(\frac{1}{1})$$

we are done if $J = \{(0,\ldots,0)\}$. Otherwise it follows by the last property of J explained above $\exists (a_1,\ldots,a_n) \in J$ $1 = \sum_{j=1}^n a_j \Leftrightarrow \exists r \in \{1,\ldots,n\}$ $a_j = 0\ \forall j \neq r$, $a_r = 1$ According to (A.2) and Proposition A.3.2

$$\frac{s}{1}\pi(\frac{x_r}{1}) = -[\frac{x_r}{1}, \frac{s}{1}\pi](\frac{1}{1}) = \frac{-\prod_{j=1,j\neq r}^n u_j d_r(1)}{1}$$

and we have the property in this case. To prove the rest of this property is an induction argument in $m = \sum_{j=1}^n a_j$ with $(a_1,\ldots,a_n) \in J$. We assume $m > 1$. Let $S = \{j_1,\ldots,j_t\} \subset \{1,2,\ldots,n\}$ be defined such that $a_j \neq 0 \Leftrightarrow j \in S$. Then

$$\begin{aligned}
&\frac{s}{1}\pi(\frac{\prod_{j=1}^n x_j^{a_j}}{1}) = \frac{s}{1}\pi(\frac{\prod_{k=1}^t x_{j_k}^{a_{j_k}}}{1}) = \\
&-[\frac{x_{j_1}}{1}, \frac{s}{1}\pi](\frac{x_{j_1}^{a_{j_1}-1}}{1}\frac{\prod_{k=2}^t x_{j_k}^{a_{j_k}}}{1}) + \frac{x_{j_1}}{1}\frac{s}{1}\pi(\frac{x_{j_1}^{a_{j_1}-1}}{1}\frac{\prod_{k=2}^t x_{j_k}^{a_{j_k}}}{1}) =_{(A.2)} \\
&-\phi(\prod_{j=1,j\neq j_1}^n u_j d_{j_1})(\frac{x_{j_1}^{a_{j_1}-1}}{1}\frac{\prod_{k=2}^t x_{j_k}^{a_{j_k}}}{1}) + \frac{x_{j_1}}{1}\frac{s}{1}\pi(\frac{x_{j_1}^{a_{j_1}-1}}{1}\frac{\prod_{k=2}^t x_{j_k}^{a_{j_k}}}{1}) =_{\text{Proposition } A.3.2} \\
&\frac{-\prod_{j=1,j\neq j_1}^n u_j d_{j_1}(x_{j_1}^{a_{j_1}-1}\prod_{k=2}^t x_{j_k}^{a_{j_k}})}{1} + \frac{x_{j_1}}{1}\frac{s}{1}\pi(\frac{x_{j_1}^{a_{j_1}-1}}{1}\frac{\prod_{k=2}^t x_{j_k}^{a_{j_k}}}{1})
\end{aligned}$$

and we have the property since $(a_{j_1}-1, a_{j_2}\ldots,a_{j_k}) \in J$. We have thus constructed $\omega \in \mathrm{End}_k(R)$ satisfying

$$\frac{s}{1}\pi(\frac{z}{1}) = \frac{\omega(z)}{1}.$$

Let $J \subset R$ be the ideal defined as

$$J = \langle [x_i,\omega](z) - \prod_{j=1,j\neq i}^n u_j d_i(z) |\ i \in \{1,2,\ldots,n\},\ z \in R\rangle$$

then for $z \in R$ and $i \in \{1, 2, \ldots, n\}$

$$\frac{[x_i, \omega](z) - \prod_{j=1,j\neq i}^{n} u_j d_i(z)}{1} = [\frac{x_i}{1}, \frac{s}{1}\pi](\frac{z}{1}) - \frac{\prod_{j=1,j\neq i}^{n} u_j d_i(z)}{1} =_{\text{Proposition } A.3.2}$$

$$[\frac{x_i}{1}, \frac{s}{1}\pi](\frac{z}{1}) - \phi(\prod_{j=1,j\neq i}^{n} u_j d_i)(\frac{z}{1}) =_{(A.2)} [\frac{x_i}{1}, \frac{s}{1}\pi](\frac{z}{1}) - [\frac{x_i}{1}, \frac{s}{1}\pi](\frac{z}{1}) = 0.$$

Since R is a finitely generated k-algebra, it is Noetherian and $J \subset R$ is an ideal and thus finitely generated and $J_U = 0$, it follows by Proposition 2.1 in [15] $\exists u_{n+1} \in U$ such that $u_{n+1}J = 0$. Let us set

$$\begin{aligned}
&u_{n+1}s = u \Rightarrow \\
&\frac{u}{1}\pi(\frac{z}{1}) = \frac{u_{n+1}\omega(z)}{1} \ \forall z \in R \wedge \\
&[x_i, u_{n+1}\omega](z) - \prod_{j=1,j\neq i}^{n+1} u_j d_i(z) = 0 \ \forall z \in R, \ \forall i \in \{1, 2, \ldots, n\} \Rightarrow \\
&[x_i, u_{n+1}\omega] = \prod_{j=1,j\neq i}^{n+1} u_j d_i \in D_{m-1}(R) \Rightarrow_{\text{Lemma } A.3.3} u_{n+1}\omega \in D_m(R)
\end{aligned}$$

and according to Proposition A.3.2

$$\begin{aligned}
&\frac{u}{1}\pi(\frac{z}{1}) = \frac{u_{n+1}\omega(z)}{1} = \phi(u_{n+1}\omega)(\frac{z}{1}) \ \forall z \in R \Rightarrow \\
&(\frac{u}{1}\pi - \phi(u_{n+1}\omega))(\frac{z}{1}) = 0 \ \forall z \in R \Rightarrow_{\text{Lemma } A.3.1} \\
&\frac{u}{1}\pi = \phi(u_{n+1}\omega)
\end{aligned}$$

and we have proved the first equality in the last line in the case $\pi(\frac{1}{1}) = 0$. If this is not the case set $\pi_1 = \pi - \pi(\frac{1}{1})$ and let $\pi(\frac{1}{1}) = \frac{a}{v}$. Then $\pi_1 \in D_m(R_U)$ and $\pi_1(\frac{1}{1}) = 0 \Rightarrow \exists u \in U, d \in D_m(R)$ such that

$$\pi - \pi(\frac{1}{1}) = \frac{1}{u}\phi(d) \Rightarrow \pi = \frac{1}{uv}\phi(vd + ua)$$

and since $vd + ua \in D_m(R)$ the Lemma is proved. □

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