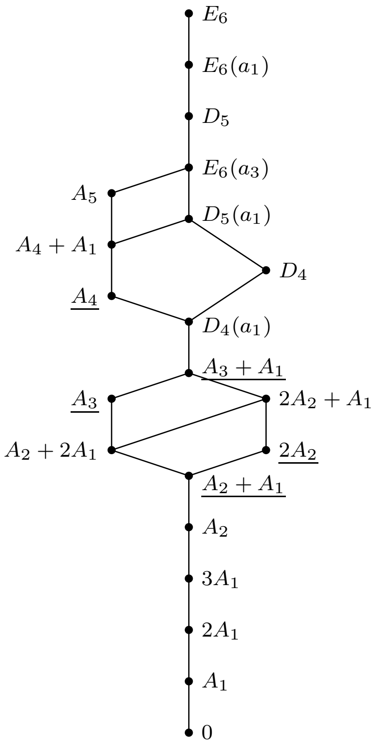
A Classification of the Normal Nilpotent Varieties for Groups of Type E_6



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Contents

Dansk resumé

Lad G være en sammenhængende, semisimpel lineær algebraisk gruppe over et algebraisk lukket legeme k. Så virker G på dens Lie algebra under den adjungerede virkning. I afhandlingen betragter vi de nilpotente baner under denne virkning. Geometrien af de nilpotente baners aflukning er gennem årene blevet undersøgt nøje, specielt om aflukningerne er normale. Hvis k er af karakteristik nul, er dette spørgsmål tidligere blevet besvaret når G er af type A, B, C eller D. I 2003 klassificerede Eric Sommers de nilpotente baner med normal aflukning når G er af type E_6 og karakteristikken af k er nul. I afhandlingen viser vi at denne klassificering også gælder i god karakteristik når G er enkeltsammenhængende.

Introduction

Let G be a connected, semi-simple linear algebraic group over an algebraically closed field k. Then G acts on its Lie algebra $\mathfrak{g}$ under the adjoint action. An element $x \in \mathfrak{g}$ is called nilpotent if there exists a closed, unipotent subgroup H of G such that x belongs to the Lie algebra of H. If $x \in \mathfrak{g}$ is nilpotent and $g \in G$, then also $g.x$ is nilpotent, hence it makes sense to define nilpotent orbits inside $\mathfrak{g}$.

There are only finitely many such nilpotent orbits in $\mathfrak{g}$. In characteristic zero and in characteristic p with $p > 3(h-1)$ where h is the Coxeter number of G, the nilpotent orbits have been classified by Bala and Carter in [BC76a] and [BC76b]. The result was extended by Pommerening to good characteristic in [Pom80] including some case by case studies. Recently Premet has given a conceptual proof of the classification, cf. [Pre03].

The geometry of the closures of the nilpotent orbits has been studied for many years. In particular a great deal of work has been put into deciding whether or not the closures of the nilpotent orbits are normal. For example all adjoint orbits have normal closure when $G = \mathrm{SL}_n$ or $G = \mathrm{GL}_n$. When G is of type B, C or D and the characteristic is zero, the nilpotent orbits with normal closure have been classified, and it turns out that not all orbits have normal closure. For more results on normality one should consult the surveys in Section 8.6. in Jantzen's part of [JN04] and in Section 7.20 in [Hum95].

In the paper [Som03] Eric Sommers characterizes which nilpotent orbits do have normal closure and which do not when G is of type E_6 and the characteristic of k is zero. The aim of this thesis is to prove that the result remains valid when the characteristic of k is a prime number p with $p \geq 5$, i.e. when the characteristic of k is good for G. This is also our main result. In the notation of Bala-Carter the result is:

Theorem 1. Let G be a connected, simply connected, semi-simple linear algebraic group over an algebraically closed field k. Suppose G is of type E_6, and that the characteristic of k is good for G. Then the following nilpotent orbits in $\mathfrak{g}$ have normal closure: E_6, $E_6(a_1)$, D_5, $E_6(a_3)$, $D_5(a_1)$, A_5, $A_4 + A_1$, D_4, $D_4(a_1)$, D_4, $2A_2 + A_1$, $A_2 + 2A_1$, A_2, $3A_1$, $2A_1$, 0.

The last five nilpotent orbits do not have normal closure: A_4, $A_3 + A_1$, A_3, $2A_2$, $A_2 + A_1$.

We will use the same method as Eric Sommers in [Som03] to show normality. The overall idea in Sommers' paper is to start with the orbits of high dimension and to work with orbits of lower and lower dimension. To be more precise we have a partial order $\leq$ on the set of nilpotent orbits: If $\mathcal{O}$ and $\mathcal{O}'$ are two nilpotent orbits, then we define $\mathcal{O} \leq \mathcal{O}'$ if the closure of $\mathcal{O}$ is contained in the closure of $\mathcal{O}'$. Figure 1 shows the partial order on the set of nilpotent orbits in type E_6 in good characteristic. If we know that an orbit $\mathcal{O}$ has normal closure, we can sometimes use this to prove that an orbit $\mathcal{O}'$ with $\mathcal{O}' \leq \mathcal{O}$ (i.e. $\mathcal{O}'$ is below $\mathcal{O}$ in the diagram) has normal closure. Often this smaller orbit $\mathcal{O}'$ will be right below $\mathcal{O}$ in the diagram. In the diagram the underlined orbits do not have normal closure, whereas the rest of the orbits do have normal closure.

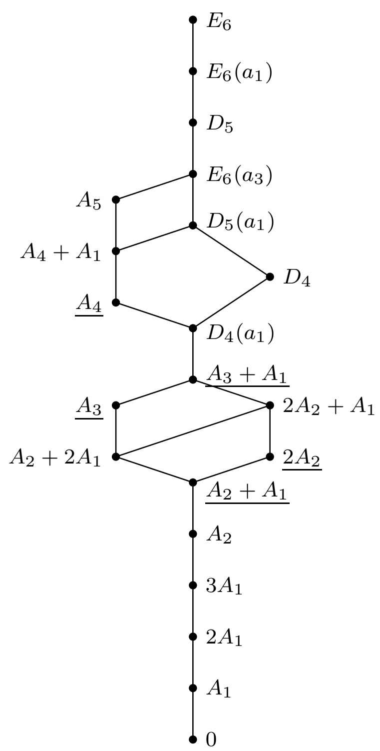

Figure 1: Orbit diagram, cf. Section 13 in [Car85].

The main ideas in [Som03] can be generalized to good characteristic. This generalization will be described thoroughly. The tool is to turn the question about normality into a question concerning cohomology groups where one can apply various vanishing theorems. However Sommers applies a vanishing theorem by Broer, Proposition 4 in [Som03], which is not valid in prime characteristic since it relies on the Grauert-Riemenschneider vanishing theorem. We will avoid this theorem by using a new method, see Example 3.15, and a vanishing theorem by Broer which has been improved by H. H. Andersen.

Summary

The thesis is structured as follows.

Chapter 1: In this chapter we settle the notation and state some well known facts about algebraic groups. Then we introduce the adjoint orbits and discuss when the orbit maps are separable. Furthermore we will define the nilpotent orbits and look at various questions related to these orbits. Finally we introduce Levi factors.

Chapter 2: We will explain the main method used to prove normality of the closures of the nilpotent orbits. This is the same method as Sommers uses in [Som03]. It turns out that a good way to look at the closures of the nilpotent orbits is to consider certain subspaces $V \subseteq \mathfrak{g}$ satisfying that $G.V$ is the closure of a nilpotent orbit. Hence we can work with these subspaces instead of the nilpotent orbits. The next step is to translate the normality question into a question concerning cohomology groups (which depend on these V's) and birationality of certain morphisms. In Section 2.1 we will therefore describe some conditions under which these morphisms are birational.

In Section 2.2 we will explain how the main ideas in [Som03] can be generalized to good characteristic. This generalization will be described in detail. In particular we will explain how to define some of these V's. The V's depend on weighted Dynkin diagrams which are in one to one correspondence with nilpotent orbits. This is explained in Premet's paper [Pre03] in good characteristic. We will also state a new result on birationality which relies on Premet's work, see Lemma 2.8 and Corollary 2.9.

Let P be a parabolic subgroup of G. Richardson's dense orbit theorem states that there exists a unique dense P-orbit in the Lie algebra of the unipotent radical of P. The complement of this orbit is therefore closed. In Section 2.3 we will consider the irreducible components of this variety. We will use the results in a new and easier way to prove that the non special orbit A_5 has normal closure. All the results in this section are new.

Chapter 3: Since the main method of proving normality was reduced to a question concerning cohomology groups, we need some results about vanishing cohomology groups. In [Som] Eric Sommers proves a proposition concerning cohomology groups in characteristic zero, and he also states that the proposition works in a more general setting, see Proposition 6 in [Som03]. Section 3.1 contains a detailed proof of the proposition in this more general setting. This also includes characteristic $p > 0$, but with a lower bound on p, see Proposition 3.3. The proof builds on Lemma 3.4 which Eric Sommers proves in characteristic zero. Again his proof works in characteristic $p > 0$ with a lower bound on p. Using a method based on The Strong Linkage Principle as suggested by H. H. Andersen, the lemma has been improved so it does not require this lower bound on p. This also improves the bound on p in Proposition 3.3.

In Section 3.2 we describe how H. H. Andersen has improved a vanishing theorem by Broer, Theorem 3.9.(iii) in [Bro94]. We explain a new method to obtain vanishing cohomology groups using this vanishing theorem, see Example 3.15. Using the new method we can avoid a vanishing theorem by Broer, Proposition 6 in [Som03], which is only valid in characteristic zero since it relies on the Grauert-Riemenschneider vanishing theorem. When we use Example 3.15 in the actual calculations in Chapter 4, we will do so by using a computer program that we have developed for this purpose.

Chapter 4: In Chapter 4 we prove normality of orbit closures. The calculations are quite similar to the calculations in [Som03], however we have included all details to make the thesis independent of [Som03].

In [Som03] it is proved that the three non special nilpotent orbits A_5, $2A_2+A_1$ and $3A_1$ have normal closure by using some reductions to $\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{SL}_2$ or to $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$. The calculations get very long and tedious, and the method is probably hard to use if one wants to prove that a non special orbit in a group of type E_7 and E_8 has normal closure. In order to avoid these calculations we have used another method to prove the normality of the closure of A_5. The new method uses that A_5 has codimension two in the closure of $E_6(a_3)$. Since $3A_1$ has codimension two in the closure of A_2, the method will probably also work for the orbit $3A_1$. Unfortunately there has not been enough time to finish all the details for $3A_1$, and instead the original proof is included. The idea behind the new method of proving normality of the closure of A_5 is due to Eric Sommers.

Chapter 5: Here we will show that the last five nilpotent orbits do not have normal closure. This result is obtained directly from the result in characteristic zero by introducing group schemes over $\mathbb{Z}$ and making base change.

Appendix A: Here the Java code for the computer program mentioned in the review of Chapter 3 is presented. Furthermore the code for a program that has been used for other calculations is included.

Acknowledgments

This thesis presents the outcome of my work during my graduate studies at University of Aarhus, 2002-2006. Most of the thesis is joint work with my adviser Jesper Funch Thomsen, and I would like to thank him for his guidance. I also thank Henning Haahr Andersen for his suggestions on the work with vanishing cohomology groups and Jens Carsten Jantzen for answering questions from time to time.

As part of my graduate studies I visited University of Massachusetts, Amherst, in the fall 2004 and spring 2005. Here I met great hospitality at the department of mathematics. In particular I would like to thank Eric Sommers for sharing his ideas and time with me.

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Chapter 1

Preliminaries

In this chapter we have gathered well known results about linear algebraic groups and nilpotent orbits. The main references are [Spr98], [Hum75], and Nilpotent Orbits in Representation Theory by Jantzen in [JN04].

1.1 Linear algebraic groups

Let k be an algebraically closed field, and let G be a connected, semi-simple linear algebraic group over k. Let T be a maximal torus in G and B a Borel subgroup containing T. Let $X^*(T)$ respectively $X_*(T)$ denote the character respectively cocharacter group of T, and let $\Phi \subseteq X^*(T)$ be the roots of G relative to T.

We have a pairing of characters and cocharacters

$$\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}.$$

For a root $\alpha \in \Phi$ we let $\alpha^\vee \in X_*(T)$ denote the corresponding coroot. A character $\lambda \in X^*(T)$ is called dominant if

$$\langle \lambda, \alpha^\vee \rangle \geq 0 \quad \text{for all} \quad \alpha \in \Pi.$$

Now G acts on itself by conjugation, and for $g \in G$ we define

$$\operatorname{Int}(g) : G \to G \quad \text{by} \quad \operatorname{Int}(g)(h) = ghg^{-1}.$$

Let $\mathfrak{g}$ denote the Lie algebra of G, and let $[-,-]$ denote the Lie bracket on $\mathfrak{g}$. Then G acts on $\mathfrak{g}$ under the adjoint action

$$\operatorname{Ad} : G \to \operatorname{GL}(\mathfrak{g})$$

where $\operatorname{Ad}(g)$ is the differential of $\operatorname{Int}(g)$. For $g \in G$ and $x \in \mathfrak{g}$, we will write $g.x$ for $\operatorname{Ad}(g)(x)$. Moreover the differential of Ad is

$$\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

where $\operatorname{ad}(x) = [x, -]$ for $x \in \mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ can be written as a direct sum

$$\mathfrak{g} = \mathfrak{t} \oplus \Big(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \Big)$$

where $\mathfrak{t}$ is the Lie algebra of T, and $\mathfrak{g}_\alpha$ is the root space

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | \forall t \in T : \mathrm{Ad}(t)(x) = t.x = \alpha(t)x\}.$$

Let U denote the unipotent radical of B, and let $\mathfrak{u}$ denote its Lie algebra. Now fix the negative roots of Φ to correspond to the T-weights of $\mathfrak{u}$. Let Φ^- and Φ^+ denote the set of negative and positive roots, respectively. Let Π denote the set of simple (positive) roots.

Given a root $\alpha \in \Phi$ there exists a unique connected T-stable subgroup U_α of G having Lie algebra $\mathfrak{g}_\alpha$. We will call U_α a root group. Then there exists an isomorphism $u_\alpha : k \to U_\alpha$ such that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for all $t \in T$ and all $x \in k$. We call such a map an admissible isomorphism.

Let P be a parabolic subgroup of G, and let $\mathfrak{u}_P$ denote the Lie algebra of the unipotent radical of P. If $I \subseteq \Pi$ is a subset of simple roots, we will let P_I denote the corresponding parabolic subgroup containing B. Let Φ_I denote the set of roots which are linear combinations of the roots in I. Then

$$\mathfrak{u}_{P_I} = \bigoplus_{\alpha \in \Phi^- \setminus \Phi_I} \mathfrak{g}_\alpha.$$

Now P_I normalizes its unipotent radical, and hence $\mathfrak{u}_{P_I}$ is a P_I-stable submodule of $\mathfrak{g}$ under the adjoint action.

Let W be the Weyl group of G with respect to T. For $\alpha \in \Phi$ we let $s_\alpha \in W$ denote the corresponding simple reflection in the Weyl group. The Weyl group W acts on $X^*(T)$, and we write $w(\lambda)$ for the action of $w \in W$ on $\lambda \in X^*(T)$. In the notation with coroots we have

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \quad \text{for} \quad \alpha \in \Phi, \lambda \in X^*(T).$$

Now let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha,$$

and define the "dot" action of W on $X^*(T)$ by

$$w \cdot \lambda = w(\lambda + \rho) - \rho \quad \text{for} \quad w \in W, \lambda \in X^*(T). \tag{1.1}$$

The group G is called simple as an algebraic group (or almost simple) if G is non-commutative and has no closed connected normal subgroups other than itself and the group consisting of the identity element in G. Note that this is not the same as being simple as an abstract group. The root system, Φ, of G is irreducible if and only if G is simple as an algebraic group.

Let H be a closed subgroup in G. For $x \in \mathfrak{g}$ we define the centralizer of x in H to be the group

$$Z_H(x) = \{g \in H | \mathrm{Ad}(g)(x)x = g.x = x\}.$$

Let $\mathfrak{h}$ denote the Lie algebra of H, then we also define the centralizer of x in $\mathfrak{h}$ to be

$$\mathfrak{z}_{\mathfrak{h}}(x) = \{z \in \mathfrak{h} | \mathrm{ad}(z)(x) = x\}.$$

By a representation (or module) we will always mean a rational representation. Let V be an H-representation. Then let V^* denote the dual H-representation, let S^nV denote the n'th symmetric power of V and $\wedge^n V$ the n'th exterior power of V. We will use the convention $S^nV = 0$ and $\wedge^n V = 0$ when $n < 0$.

Let $\lambda \in X^*(T) = X^*(B)$. Then λ gives rise to a one dimensional B-representation with weight λ. This representation will sometimes be denoted λ.

1.2 Induced representations and vector bundles

Let H be a closed subgroup in G. We have the induction functor, $\mathrm{Ind}_H^G(-)$, which to an H-representation gives a G-representation. Let $R^i\mathrm{Ind}_H^G(-)$ denote the right derived functor of $\mathrm{Ind}_H^G(-)$.

Let P be a parabolic subgroup in G, and let V be a finite dimensional P-representation. Then P acts on the right on $G \times V$ as $(g, v)p = (gp, p^{-1}.x)$ for $p \in P$, $g \in G$ and $x \in V$. Since $G \to G/P$ has local sections, the quotient of $G \times V/P$ exists. We will write $G \times^P V$ for this quotient. Given (g, x) in $G \times V$ we let $[(g, x)]$ denote the image in $G \times^P V$. Note that G acts on $G \times^P V$ via $g.[(g', x)] = [(gg', x)]$.

The morphism

$$G \times^P V \to G/P \quad \text{given by} \quad [(g, x)] \mapsto gP$$

makes $G \times^P V$ into a vector bundle over G/P of rank equal to $\dim V$. Let $\mathcal{L}(V)$ denote the associated locally free sheaf of sections on G/P, and let $H^i(G/P, \mathcal{L}(V))$ denote the i'th cohomology group of $\mathcal{L}(V)$ on G/P. Then

$$H^i(G/P, \mathcal{L}(V)) = R^i\mathrm{Ind}_P^G(V)$$

and we will write $H^i(G/P, V)$ for these cohomology groups. We will use the convention $H^i(G/P, V) = 0$ if $i < 0$.

Assume furthermore that there exists a finite dimensional G-representation Z such that V is a P-subrepresentation of Z. Let

$$\mathcal{P}_V = \{(gP, x) \in G/P \times Z | g^{-1}.x \in V\}.$$

Then $\mathcal{P}_V$ is closed in $G/P \times Z$. Since G/P is projective, it is complete, and the projection $G/P \times Z \to Z$ is a closed morphism. Since the image of $\mathcal{P}_V$ under this morphism equals $G.V$, we get that $G.V$ is closed in Z. Moreover we have an isomorphism

$$G \times^P V \to \mathcal{P}_V \quad \text{given by} \quad [(g,x)] \mapsto (gP, g.x),$$

and hence the morphism

$$G \times^P V \to G.V \quad \text{given by} \quad [(g,x)] \mapsto g.v \tag{1.2}$$

is projective. Also note that this morphism is G-equivariant.

If X is a variety over k, we will let $k[X]$ denote the set of global regular functions on X. With this notation we have

$$k[G \times^P V] \simeq \bigoplus_{n \geq 0} H^0(G/P, S^n V^*) \tag{1.3}$$

as G-equivariant graded algebras.

1.3 Separability of orbit maps

Throughout this section G is still semi-simple. For $x \in \mathfrak{g}$ we can define the orbit $\mathrm{Ad}(G).x = G.x$ of the adjoint action of G on $\mathfrak{g}$. These adjoint orbits are all even dimensional. Take a look at the orbit map

$$G \to G.x \subseteq \mathfrak{g}$$

sending g to $g.x$. Later on we will need these morphisms to be separable and we will therefore discuss when this is satisfied. By Section 9.1 in [Bor69] the following are equivalent

1. The orbit map $G \to G.x$ is separable.
2. The induced morphism $G/Z_G(x) \to G.x$ is an isomorphism.
3. The Lie algebra of $Z_G(x)$ equals $\mathfrak{z}_{\mathfrak{g}}(x)$.

In characteristic zero these conditions are always satisfied, but in prime characteristic it is not an easy task to decide whether or not the conditions are satisfied.

In order to deal with the separability question and many more issues we will need the notion of good characteristic for G. First we define the set of bad primes:

$p = 2$ is bad if the root system of G has a component not of type A.
$p = 3$ is bad if the root system of G has a component of exceptional type.
$p = 5$ is bad if the root system of G has a component of type E_8.

Then the characteristic $\mathrm{char}(k)$ of G is good if it is either zero or not a bad prime for G.

Now conditions 1-3 above are satisfied under the following "standard hypothesis", see Section 2.9 in Jantzen's part of [JN04].

H1 The derived group of G is simply connected.

H2 The characteristic char(k) is good for G.

H3 The Lie algebra $\mathfrak{g}$ admits a G-invariant non-degenerate bilinear form.

Since G is semi-simple, the derived group equals G, and H1 is equivalent to G being simply connected. Now assume that G satisfies the following conditions

i. G is simply connected.

ii. G is simple as an algebraic group and not of type A.

iii. char(k) is good for G.

Then G satisfies the standard hypothesis, and in particular the orbit maps are separable.

1.4 Nilpotent orbits

In this section we will also assume that G is semi-simple. Now we will define the nilpotent orbits. An element $x \in \mathfrak{g}$ is called nilpotent if there exists a closed, unipotent subgroup H of G such that x belongs to the Lie algebra of H, denoted $\mathfrak{h}$. Let $g \in G$. Since H is a closed, unipotent subgroup of G, also gHg^{-1} is a closed, unipotent subgroup of G with Lie algebra $\mathrm{Ad}(g)\mathfrak{h} = g.\mathfrak{h}$, and therefore $g.x$ is nilpotent. Now it makes sense to define an orbit $G.x$ to be nilpotent if x is nilpotent, i.e. if $G.x$ consists of nilpotent elements. Actually there are only finitely many nilpotent orbits in $\mathfrak{g}$.

Let $\mathcal{N}$ denote the set of nilpotent elements in $\mathfrak{g}$, from above we know that $\mathcal{N}$ is G-stable. We also know that $\mathcal{N}$ is closed in $\mathfrak{g}$. Remember that U is the unipotent radical of the Borel subgroup B. Since U is a closed, unipotent subgroup of G, its Lie algebra $\mathfrak{u}$ consists of nilpotent elements and $\mathfrak{u} \subseteq \mathcal{N}$. Therefore also $G.\mathfrak{u} \subseteq \mathcal{N}$, but actually we have $G.\mathfrak{u} = \mathcal{N}$. This implies that $\mathcal{N}$ is irreducible. Also remember that $\dim \mathcal{N} = 2 \dim \mathfrak{u}$.

Notice that since there are only finitely many nilpotent orbits, and since $\mathcal{N}$ is closed and irreducible, there exists a unique dense (and hence open) nilpotent orbit in $\mathcal{N}$, this orbit is called the regular nilpotent orbit in $\mathfrak{g}$ and denoted $\mathcal{O}_{\mathrm{reg}}$. Now we will define a partial ordering $\leq$ of the nilpotent orbits. Let $\mathcal{O}$ and $\mathcal{O}'$ be nilpotent orbits in $\mathfrak{g}$. Then we define $\mathcal{O}' \leq \mathcal{O}$ if the closure of $\mathcal{O}'$ is contained in the closure of O. For all nilpotent orbits $\mathcal{O}$ we have $\{0\} \leq \mathcal{O} \leq \mathcal{O}_{\mathrm{reg}}$ where $\{0\}$ is the orbit consisting only of the point $0 \in \mathfrak{g}$.

One of the first normality results for closures of nilpotent orbits is the following proposition which was proved by Kostant in characteristic zero, and generalized to characteristic $p > 0$ by Veldkamp and Demazure. See Proposition 8.5. in Jantzen's part of [JN04] for a proof.

Proposition 1.1. Assume that G satisfies the standard hypothesis H1-H3, or that G is simply connected with char(k) good for G. Then the closure of $\mathcal{O}_{\mathrm{reg}}$ is normal. But since the closure of $\mathcal{O}_{\mathrm{reg}}$ equals $\mathcal{N}$, this implies that $\mathcal{N}$ is normal.

Now let $\mathcal{U}$ denote the set of unipotent elements in G. Then $\mathcal{U}$ is a closed subgroup in G.

Theorem 1.2. If char(k) is good for G, and G is simply connected, then there exists a G-equivariant isomorphism between the unipotent variety $\mathcal{U}$ and the nilpotent variety $\mathcal{N}$.

With a small modification this is a theorem by Springer, see Remark 6.1 in Jantzen's part of [JN04]. Now we can use results about the unipotent orbits in $\mathcal{U}$ to get results about the nilpotent orbits in $\mathcal{N}$ and vice versa.

Theorem 1.3 (Richardson's Dense Orbit Theorem)**.** Let P be a parabolic subgroup in G. Then there exists a unique dense P-orbit in $\mathfrak{u}_P$. If $x \in \mathfrak{u}_P$ is an element in this orbit, then

1. The closure of $G.x$ equals $G.\mathfrak{u}_P$ and $G.x \cap \mathfrak{u}_P = P.x$.
2. $\dim(G.x) = 2\dim(\mathfrak{u}_P)$.
3. Let $Z_G(x)^0$ be the identity component of the centralizer of x in G. Then $Z_G(x)^0 \subseteq P$.

An element $x \in \mathfrak{u}_P$ in the dense P-orbit is called a Richardson element.

A proof can be found in Theorem 5.2.3 (including the proof) and Corollary 5.2.4 in [Car85]. In [Car85] it is assumed that the characteristic of k is good for G, but this is only required to make sure that there are finitely many nilpotent orbits in $\mathfrak{g}$. Since there are only finitely many nilpotent orbits in $\mathfrak{g}$ in all characteristics, we do not need this assumption.

Note that if x is a Richardson element for P, then x is nilpotent since $x \in \mathfrak{u}_P$, and $G.x$ is a nilpotent orbit.

As written in the introduction the finitely many nilpotent orbits in $\mathfrak{g}$ have been classified by Bala and Carter, cf. [BC76a] and [BC76b], when char(k) $= 0$ or char(k) $= p > 3(h-1)$ where h is the Coxeter number of G. This classification was extended to good characteristic by Pommerening in [Pom80]. He used some case by case study, but Premet has given a conceptual proof in [Pre03]. The names of the nilpotent orbits given by this classification are called Bala-Carter labels. In the calculations in Chapter 4 we will use these names.

1.5 Levi factors

Let G be a connected linear algebraic group with unipotent radical $R_u(G)$. Then $R_u(G)$ is normal in G. Let L be a reductive, closed subgroup in G. We call L a Levi factor of G if G is the semidirect product as an algebraic group of L and $R_u(G)$. In this case L is isomorphic to $G/R_u(G)$, and hence all Levi factors of G are isomorphic.

If the characteristic of our ground field k is zero, then every connected linear algebraic group G has a Levi factor, see Section 0.8 and 3.14 in [BT65], also for the next results. Then also two Levi factors of G are conjugate by a unique element in $R_u(G)$. Furthermore we can describe the Levi factors of G – they are precisely the centralizers of the maximal tori in the radical of G, i.e. the Levi factors of G

are the subgroups of G of the form $Z_G(T)$ where T is a maximal torus in $R(G)$ where $R(G)$ denotes the radical of G.

Remark: Remember that $R(G)$ is the identity component of the intersection of the Borel groups in G, and that $R_u(G)$ is the identity component of the intersection of the unipotent parts of the Borel subgroups of G.

Lemma 1.4. Let $\pi : G \to G'$ be a surjective morphism of connected linear algebraic groups, and suppose the characteristic of k is zero. Then $\pi(R_u(G)) = R_u(G')$, and if L is a Levi factor of G, then $\pi(L)$ is a Levi factor of G'.

Proof. Since π is surjective, the image of a Borel subgroup in G is a Borel subgroup in G'. And since all Borel subgroups are conjugate we can obtain every Borel subgroup in G' as the image under π of a Borel subgroup in G. Hence – by the remark above the lemma – we have

$$\pi(R(G)) = R(G') \quad \text{and} \quad \pi(R_u(G)) = R_u(G').$$

Let L be a Levi factor of G. Then $L = Z_G(T)$ for a maximal torus $T \subseteq R(G)$. But since $\pi : R(G) \to R(G')$ is surjective, $\pi(T)$ is a maximal torus of $R(G')$, and $Z_{G'}(\pi(T))$ is a Levi factor of G'. Clearly $\pi(Z_G(T)) \subseteq Z_{G'}(\pi(T))$. But since $Z_G(T)$ is a Levi factor of G, we have

$$G' = \pi(G) = \pi(Z_G(T)R_u(G)) = \pi(Z_G(T))R_u(G').$$

Let $g \in Z_{G'}(\pi(T)) \subseteq G'$. Then

$$g = hu \quad \text{where} \quad h \in \pi(Z_G(T)) \subseteq Z_{G'}(\pi(T)), \quad u \in R_u(G'),$$

and hence the element $h^{-1}g = u$ belongs to $Z_{G'}(\pi(T)) \cap R_u(G')$. But since $Z_{G'}(\pi(T))$ is a Levi factor, we get $h^{-1}g = 1 = u$, and hence $g = h \in \pi(Z_G(T))$, and we have proved that $\pi(Z_G(T)) = Z_{G'}(\pi(T))$. □

If the characteristic of k is prime, then Levi factors need not exist, and if they exist, they need not be conjugate.

Chapter 2

Method

In this section we will in more detail describe a method which can be used to prove that a nilpotent orbit has normal closure. Similarly we will explain a method which can be used to prove that the closure of a nilpotent orbit is not normal. Since we are interested in deciding whether or not the the nilpotent orbits have normal closure, we will also describe a way to obtain the orbit closures in a different way.

Let $V \subseteq \mathfrak{u}$ be a closed subspace. If V is P-stable for some parabolic subgroup P containing B, then $G.V$ is closed in $\mathfrak{g}$ and hence affine. But V consists of nilpotent elements, so the elements of $G.V$ are nilpotent. Thus $G.V$ is irreducible, closed and consists of nilpotent elements, and therefore it must equal the closure of a nilpotent orbit. In the following we will therefore formulate the theory using "$G.V$'s".

Let $V_1, V_2 \subseteq \mathfrak{u}$ be closed subspaces stable under some parabolic subgroups P_1, P_2 containing B, respectively. As before $G.V_1$ and $G.V_2$ are affine. Because $V_1 \subseteq V_2$, we have an inclusion $i : G.V_1 \hookrightarrow G.V_2$ and an injective morphism

$$j : G \times^B V_1 \to G \times^B V_2.$$

We also have (surjective) morphisms

$$\pi_i : G \times^B V_i \to G \times^{P_i} V_i, \quad i = 1, 2.$$

We have projective morphisms

$$p_i : G \times^B V_i \to G.V_i, \quad i = 1, 2,$$

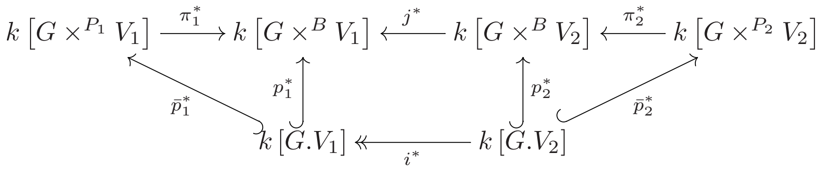
and

$$\bar{p_i} : G \times^{P_i} V_i \to G.V_i, \quad i = 1, 2,$$

which are surjective and make the following diagram commutative

$$\begin{array}{ccccccc} G \times^{P_1} V_1 & \overset{\pi_1}{\twoheadleftarrow} & G \times^B V_1 & \overset{j}{\hookrightarrow} & G \times^B V_2 & \overset{\pi_2}{\twoheadrightarrow} & G \times^{P_2} V_2 \\ & \searrow^{\bar{p_1}} & \downarrow^{p_1} & & \downarrow^{p_2} & \swarrow^{\bar{p_2}} & \\ & & G.V_1 & \underset{i}{\hookrightarrow} & G.V_2 & & \end{array}$$

Taking global regular functions, we get a new commutative diagram



where p_1^*, p_2^*, $\bar{p}_1^*$ and $\bar{p}_2^*$ are injective and i^* is surjective – remember that $G.V_1 \subseteq G.V_2$ is closed and that $G.V_2$ is affine, so i^* is the projection

$$i^* : k[G.V_2] \to k[G.V_1] = k[G.V_2]/I(G.V_1)$$

where $I(G.V_1)$ is the set of functions in $k[G.V_2]$ which vanish on $G.V_1$.

Remember that

$$\begin{aligned} k\left[G \times^B V_i\right] &= \bigoplus_{n \geq 0} H^0(G/B, S^n V_i^*), \quad i = 1, 2, \\ k\left[G \times^{P_i} V_i\right] &= \bigoplus_{n \geq 0} H^0(G/P_i, S^n V_i^*), \quad i = 1, 2, \end{aligned} \tag{2.1}$$

by (1.3). But since V_i is a P_i-representation,

$$H^0(G/B, S^n V_i^*) = H^0(G/P_i, S^n V_i^*) \quad \text{for all} \quad n \in \mathbb{Z},$$

and π_i^* is an isomorphism for i=1,2.

Lemma 2.1. If j^* is injective, then $G.V_1 = G.V_2$.

Proof. Assume that $G.V_1 \neq G.V_2$. Then $I(G.V_1) \neq 0$, so we can choose $f \in I(G.V_1) \setminus \{0\}$. Then $i^*(f) = 0$, and hence

$$0 = p_1^* \circ i^*(f) = j^* \circ p_2^*(f),$$

but this is a contradiction since j^* and p_2^* are injective and $f \neq 0$. □

Lemma 2.2. Assume the following:

1. $G.V_2$ is a normal variety.
2. $\bar{p}_2$ is birational.
3. j^* is surjective.

Then $G.V_1$ is also a normal variety.

Proof. Since $\bar{p}_2$ is birational and $G.V_2$ is normal, we have that $\bar{p}_2^*$ is an isomorphism, cf. Lemma II.14.5 in [Jan87]. Therefore $p_2^* = \pi_2^* \circ \bar{p}_2^*$ is an isomorphism, $p_1^* \circ i^* = j^* \circ p_2^*$ is surjective, and hence p_1^* is surjective and thereby an isomorphism, i.e.

$$k[G.V_1] \simeq k\left[G \times^B V_1\right].$$

But $G \times^B V_1$ is a non-singular variety, and thus $k\left[G \times^B V_1\right]$ is a normal ring. So $G.V_1$ is affine and its coordinate ring is normal, hence $G.V_1$ is a normal variety. □

Lemma 2.3. Assume the following:

1. $\bar{p}_1$ is birational.
2. j^* is not surjective.

Then $G.V_1$ is not normal.

Proof. Assume for contradiction that $G.V_1$ is normal. Then since $\bar{p}_1$ is birational, we know as before that $\bar{p}_1^*$ is an isomorphism, and hence that $p_1^* = \pi_1^* \circ \bar{p}_1^*$ is an isomorphism. But since i^* is surjective, we get that $p_1^* \circ i^* = j^* \circ p_2^*$ is surjective, and therefore j^* is surjective. But this is a contradiction, and $G.V_1$ cannot be normal. □

When we are going to show that a nilpotent orbit has normal closure, we will use Lemma 2.2. On the other hand if we want to show that another nilpotent orbit do not have normal closure, we will use Lemma 2.3.

When we are going to use Lemma 2.2, we will have two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ with closures $\overline{O_1} = G.V_1$ and $\overline{O_2} = G.V_2$ where $V_1 \subseteq V_2$ as above (then $\mathcal{O}_1 \leq O_2$). We will be in the case where $\overline{O_2}$ is normal and by using the lemma we will be able to show that $\overline{\mathcal{O}_1}$ is normal.

Now we will describe how we can prove that j^* is surjective or injective as needed in Lemma 2.1 and Lemma 2.2. If we can show that

$$H^0(G/B, S^n V_2^*) \to H^0(G/B, S^n V_1^*) \tag{2.2}$$

is surjective (or injective) for all n, we have that j^* is surjective (or injective) by (2.1).

We will show that the map in (2.2) is surjective (or injective) by taking the short exact sequence of B-representations

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

where V_3 is the cokernel of the inclusion of V_1 into V_2. We dualize this sequence and get another short exact sequence of B-representations

$$0 \to V_3^* \to V_2^* \to V_1^* \to 0$$

If we take the Koszul resolution of this sequence, we get a new exact sequence of B-representations, cf. [Jan87] Section II.12.12,

$$\cdots S^{n-j} V_2^* \otimes \wedge^j V_3^* \to \cdots \to S^{n-2} V_2^* \otimes \wedge^2 V_3^* \to$$
$$S^{n-1} V_2^* \otimes V_3^* \to S^n V_2^* \to S^n V_1^* \to 0$$

We can split the long exact sequence into short exact sequences. These short exact sequences gives rise to long exact sequences in cohomology. Observing that some of these cohomology groups vanish, we can often show that the map in (2.2) is surjective or an isomorphism (and hence injective). In Chapter 3 we will describe the different vanishing theorems we are going to use.

In order to use Lemma 2.2 and Lemma 2.3 we need to show that $\bar{p}_i : G \times^{P_i} V_i \to G.V_i$ is birational for $i = 1$ or $i = 2$.. In the next section the topic is therefore birationality.

2.1 Birationality

A morphism $f : X \to Y$ of irreducible varieties is called generically one to one if there exists an open subset $U \subseteq Y$ such that for all $y \in U$ the dimension of $f^{-1}(y)$ is zero and $f^{-1}(y)$ consists of exactly one point.

Theorem 2.4. Let $f : X \to Y$ be a dominant morphism of irreducible varieties, and assume that $\dim X = \dim Y$. Then f is birational if and only if f is separable and generically one to one.

Proof. Let $k(X)$ denote the function field of X, and let $k(Y)$ denote the function field of Y. Then $k(Y) \subseteq k(X)$ is a finite algebraic field extension, and from Theorem 5.1.6 in [Spr98] we know that there exists an open subset $U \subseteq Y$ such that

1. For all $y \in U$ the dimension of $f^{-1}(y)$ is zero.
2. For all $y \in U$ the number of points in $f^{-1}(y)$ equals the separable degree, $[k(X) : k(Y)]_s$, of the extension $k(Y) \subseteq k(X)$.

Assume that f is birational. Then $k(X) = k(Y)$ and hence $[k(X) : k(Y)]_s = 1$, and f is generically one to one. Since $k(X) = k(Y)$ the extension $k(Y) \subseteq k(X)$ is clearly separable generated, and f is separable.

If f is separable and generically one to one, we know that $k(Y) \subseteq k(X)$ is separable generated and that $[k(X) : k(Y)]_s = 1$. Since $k(Y) \subseteq k(X)$ is algebraic and separable generated, it is algebraic separable and

$$[k(X) : k(Y)] = [k(X) : k(Y)]_s = 1,$$

and f is birational. □

Lemma 2.5. Suppose G satisfies the standard hypothesis on page 4. Let $V \subseteq \mathfrak{g}$ be a subspace closed under the action of a parabolic subgroup $P \subseteq G$. Assume that there exists an $x \in V$ such that

1. $\dim G \times^P V = \dim G.V$.
2. The closure of $G.x$ equals $G.V$.
3. $Z_G(x) \subseteq P$.
4. $G.x \cap V = P.x$.

Then the morphism

$$\pi : G \times^P V \to G.V$$

defined in (1.2) is birational.

Proof. By Theorem 2.4 we only need to show that π is separable and generically one to one because $\dim G \times^P V = \dim G.V$.

We start by proving separability. The orbit map

$$\varphi : G \to G.x$$

is separable since G satisfies the standard hypothesis on page 4. Let e denote the neutral element in G. Since $G.x$ is open in its closure, $G.V$, the tangent map $\mathrm{d}\varphi_e : \mathfrak{g} = T_e(G) \to T_x(G.V)$ is surjective by Theorem 4.3.7 in [Spr98]. Now define $i : G \to G \times^P V$ by $g \mapsto [(g, x)]$. Then $\varphi = \pi \circ i : G \to G.V$, and on tangent spaces we have $\mathrm{d}\varphi_e = \mathrm{d}\pi_{[(e,x)]} \circ \mathrm{d}i_e$. Since $\mathrm{d}\varphi_e$ is surjective, also $\mathrm{d}\pi_{[(e,x)]}$ is surjective, and since $G \times^P V$ is smooth and $x \in G.V$ is a simple point (since $G.x$ is open in $G.V$), this implies that π is separable by Theorem 4.3.6 in [Spr98].

Now we will show that π is generically one to one. Since π is G-equivariant, and $G.x$ is open in $G.V$, it is enough to show that $\pi^{-1}(x)$ consists of exactly one point. Clearly $\pi([(e, x)]) = e.x = x$, and there is at least one point in $\pi^{-1}(x)$. Now assume that $[(g, y)] \in \pi^{-1}(x)$. Then $x = \pi([(g, y)]) = g.y$, and $y = g^{-1}.x \in G.x \cap V = P.x$. Hence there exists a $p \in P$ such that $y = p.x$, and we have $gp.x = g.y = x$. Therefore $gp \in Z_G(x) \subseteq P$. Hence $g \in P$, and we get

$$[(g, y)] = [(e, gp.x)] = [(e, x)],$$

and $\pi^{-1}(x)$ consists of exactly one point.

□

Corollary 2.6. Suppose G satisfies the standard hypothesis. Let P be a parabolic subgroup in G, and let $x \in \mathfrak{u}_P$ be a Richardson element for P. If $Z_G(x)$ is connected, then $G \times^P \mathfrak{u}_P \to G.\mathfrak{u}_P$ is birational

Proof. Since $x \in \mathfrak{u}_P$ is a Richardson element, we have by Richardson's dense orbit theorem that condition 2, 3 and 4 in Lemma 2.5 are satisfied. Furthermore $\dim G.\mathfrak{u}_P = 2 \dim \mathfrak{u}_P$ and therefore

$$\begin{aligned} \dim G \times^P \mathfrak{u}_P &= \dim G - \dim P + \dim \mathfrak{u}_P \\ &= 2 \dim \mathfrak{u}_P \\ &= \dim G.\mathfrak{u}_P \end{aligned}$$

and condition 1 is satisfied. □

Note that we know that the morphism $G \times^P \mathfrak{u}_P \to G.\mathfrak{u}_P$ is birational if P is a standard parabolic subgroup corresponding to a set of pairwise orthogonal short simple roots, cf. Lemma 11 in [Tho00]. Here one uses the convention that all roots are short if only one root length occur.

2.2 Weighted Dynkin diagrams and Bala-Carter theory

In Section 2 on page 9 we have seen that given a closed subspace $V \subseteq \mathfrak{u}$ which is P-stable for some parabolic subgroup P containing B, then $G.V$ equals the closure of some nilpotent orbit. But if we are given a nilpotent orbit $\mathcal{O}$, we also want to find a closed subspace $V \subseteq \mathfrak{u}$ such that

i. V is P-stable for some parabolic subgroup P containing B.

ii. The closure of $\mathcal{O}$ equals $G.V$.

This is done in Premet's paper [Pre03]. To ensure that the conditions in Section 2 in [Pre03] (and in particular in Theorem 2.3 in [Pre03]) are satisfied, we will throughout Section 2.2 assume that G satisfies the standard hypothesis on page 4. Since G is semi-simple this assumption implies that G does not contain a component of type A_n when $\mathrm{char}(k) = p > 0$ and p divides n; this is exactly the definition of $\mathrm{char}(k)$ being very good for G.

Before going into detail, we will explain what we are going to deal with in the following sections. In [Pre03] Premet gives a new and uniform proof of the Bala-Cater theorem in good characteristic and the Bala-Carter theorem gives a bijection between the set of nilpotent orbits and the set of weighted Dynkin diagrams. Given a weighted Dynkin diagram Δ, we let $\mathcal{O}(\Delta)$ be the nilpotent orbit corresponding to Δ under this bijection. Premet defines a one-parameter subgroup $\lambda_\Delta \in X_*(T)$ only depending on Δ. Then he introduces a corresponding parabolic subgroup $P(\lambda_\Delta)$ with $B \subseteq P(\lambda_\Delta)$ and a closed subspace $V(\lambda_\Delta) \subseteq \mathfrak{u}$ which is $P(\lambda_\Delta)$-stable. Moreover it turns out that $G.V(\lambda_\Delta)$ equals the closure of $\mathcal{O}(\Delta)$, and hence $V(\lambda_\Delta)$ satisfies condition i and ii above.

In the following sections we will explain Premet's results. We will also use his results and Lemma 2.5 to prove that the morphism

$$G \times^{P(\lambda_\Delta)} V(\lambda_\Delta) \to G.V(\lambda_\Delta) \subseteq \mathfrak{g} \quad \text{given by} \quad [(g,x)] \mapsto g.x \tag{2.3}$$

is birational. This result on birationality will be used in the calculations in Chapter 4 and Chapter 5 when we are going to use that the morphism $\bar{p}$ in Lemma 2.2 and Lemma 2.3 is birational.

In Section 2.2.4 we will explain how to generalize characteristic zero results to good characteristic and this involves more than the work of Premet. In that section we will also discuss the bijection mentioned above between the set of nilpotent orbits and the set of weighted Dynkin diagrams.

2.2.1 One-parameter subgroups

Let $\lambda \in X_*(G)$ be a one-parameter subgroup. We assign to λ a grading of the Lie algebra $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}(\lambda, i)$ with

$$\mathfrak{g}(\lambda, i) = \{x \in \mathfrak{g} | \mathrm{Ad}(\lambda(t))(x) = \lambda(t).x = t^i x \text{ for all } t \in k^*\}.$$

For $i_0 \in \mathbb{Z}$ we also define $\mathfrak{g}_{i_0}(\lambda) = \oplus_{i \le i_0}\mathfrak{g}(\lambda, i)$.

Remember that Π denotes the set of simple roots. Let $\lambda \in X_*(T)$ be a one-parameter subgroup satisfying $\langle \alpha, \lambda \rangle \geq 0$ for all $\alpha \in \Pi$. Such a one-parameter subgroup is called Π-dominant. Now define $P(\lambda)$ to be the subgroup of G given by

$$P(\lambda) = \langle T, U_\alpha | \alpha \in \Phi : \langle \alpha, \lambda \rangle \leq 0 \rangle. \tag{2.4}$$

Since $B = \langle T, U_\alpha | \alpha \in \Phi^- \rangle$, we see that $B \subseteq P(\lambda)$. In fact $P(\lambda)$ is the standard parabolic subgroup containing B corresponding to the subset

$$I(\lambda) = \{\alpha \in \Pi | \langle \alpha, \lambda \rangle = 0\}.$$

Let $Z(\lambda)$ denote the Levi subgroup of $P(\lambda)$ containing T, and let $U(\lambda)$ denote the unipotent radical of $P(\lambda)$. Then

$$Z(\lambda) = \langle T, U_\alpha | \alpha \in \Phi : \langle \alpha, \lambda \rangle = 0 \rangle$$
$$U(\lambda) = \langle U_\alpha | \alpha \in \Phi : \langle \alpha, \lambda \rangle < 0 \rangle$$

and $P(\lambda) = Z(\lambda)U(\lambda)$. Moreover $Z(\lambda)$ is the centralizer in G of the image of $\lambda : k^* \to T$. Clearly $\mathfrak{g}(\lambda, i)$ is $Z(\lambda)$-stable, and $\mathfrak{g}_{i_0}(\lambda)$ is $P(\lambda)$-stable. Let $\mathfrak{p}(\lambda)$, $\mathfrak{z}(\lambda)$ and $\mathfrak{u}(\lambda)$ be the Lie algebras of $P(\lambda)$, $Z(\lambda)$ and $U(\lambda)$ respectively. Then

$$\mathfrak{p}(\lambda) = \mathfrak{t} \oplus \left(\bigoplus\nolimits_{\langle \alpha, \lambda \rangle \leq 0} \mathfrak{g}_\alpha \right), \quad \mathfrak{z}(\lambda) = \mathfrak{t} \oplus \left(\bigoplus\nolimits_{\langle \alpha, \lambda \rangle = 0} \mathfrak{g}_\alpha \right), \quad \mathfrak{u}(\lambda) = \bigoplus\nolimits_{\langle \alpha, \lambda \rangle < 0} \mathfrak{g}_\alpha.$$

2.2.2 Chevalley groups

Remember that every connected, semi-simple linear algebraic group is isomorphic to a Chevalley group (considered as an algebraic group), see [Ste68] p. 61. In order to explain the theory in [Pre03] we will first make some well known observations about Chevalley groups. The next paragraphs are mostly taken from Steinberg's book [Ste68] and Chapter 27 in [Hum78].

Each Chevalley group can be obtained in the following way. Let $\mathfrak{g}_\mathbb{C}$ be a complex semi-simple Lie algebra with root system Φ. Let $\mathfrak{t}_\mathbb{C}$ be a Cartan subalgebra in $\mathfrak{g}_\mathbb{C}$, and Π be a set of simple roots in Φ. Choose a Chevalley basis

$$\mathcal{B} = \{x_\alpha | \alpha \in \Phi\} \cup \{h_\alpha | \alpha \in \Pi\}$$

with $x_\alpha \in (\mathfrak{g}_\mathbb{C})_\alpha$ and $h_\alpha \in \mathfrak{t}_\mathbb{C}$. Let V be a finite dimensional faithful $\mathfrak{g}_\mathbb{C}$-representation

$$\pi : \mathfrak{g}_\mathbb{C} \to \mathfrak{gl}(V).$$

Let M be a corresponding admissible lattice in V and let $\mathfrak{g}_\mathbb{Z}$ be its stabilizer in $\mathfrak{g}_\mathbb{C}$. Then $\mathfrak{g}_\mathbb{Z}$ is a lattice in $\mathfrak{g}_\mathbb{C}$ with basis $\{x_\alpha | \alpha \in \Phi\} \cup \{h'_\alpha | \alpha \in \Pi\}$ where $h'_\alpha \in \mathfrak{t}_\mathbb{C}$. Making base change to an algebraically closed field L we get an induced faithful representation

$$\pi_L : \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} L \to \mathfrak{gl}(M \otimes_\mathbb{Z} L).$$

Now the Chevalley group is a certain subgroup $G_L \subseteq \mathrm{GL}(M \otimes_\mathbb{Z} L)$ generated by some elements called $x_\alpha(t)$ where $\alpha \in \Phi$ and $t \in L$, [Ste68] p. 21. The morphisms $u_\alpha : L \to G_L$ sending t to $x_\alpha(t)$ are admissible isomorphisms.

Let $\mathrm{Lie}(G_L)$ denote the Lie algebra of G_L. Since $G_L \subseteq \mathrm{GL}(M \otimes_{\mathbb{Z}} L)$, we have $\mathrm{Lie}(G_L) \subseteq \mathfrak{gl}(M \otimes_{\mathbb{Z}} L)$. It turns out that $\mathrm{Lie}(G_L)$ is the image of π_L. Hence

$$\pi_L : \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} L \to \pi_L(\mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} L) = \mathrm{Lie}(G_L)$$

is an isomorphism of Lie algebras, and we can identify $\mathrm{Lie}(G_L)$ with $\mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} L$. Under this identification the set

$$\{x_\alpha \otimes 1 | \alpha \in \Phi\} \cup \{h'_\alpha \otimes 1 | \alpha \in \Pi\} \subseteq \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} L = \mathrm{Lie}(G_L)$$

is a basis for $\mathrm{Lie}(G_L)$. Now we can choose a maximal torus $T_L \subseteq G_L$ such that the Lie algebra of T_L is

$$\mathrm{Lie}(T_L) = \mathrm{span}_L(h'_\alpha \otimes_{\mathbb{Z}} 1 | \alpha \in \Pi).$$

Now one can identify the roots of G_L with respect to T_L with Φ.

Note that if G_L is simply connected, then the lattice generated by $\mathcal{B}$ is actually $\mathfrak{g}_{\mathbb{Z}}$ and we may assume that $h'_\alpha = h_\alpha$ for $\alpha \in \Pi$, see Section A.2.5 in Borel's part of [MR070].

Also notice that if $L = \mathbb{C}$ above, then under our identifications we have $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{Lie}(G_{\mathbb{C}})$ and $\mathfrak{t}_{\mathbb{C}} = \mathrm{Lie}(T_{\mathbb{C}})$.

2.2.3 Weighted Dynkin diagrams

Now we are finally ready to follow [Pre03]. We keep our notation from the last section, so $\mathfrak{g}_{\mathbb{C}}$ is the complex semi-simple Lie algebra with root system Φ. Now we choose a faithful finite dimensional representation

$$\pi : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(V)$$

such that the corresponding Chevalley groups, G_L, are simply connected. If we let $L = k$, we observe that our group G can be identified with the Chevalley group G_k. Then the Lie algebra, $\mathfrak{g}$, of G is identified with $\mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$. In the following we will also consider the case $L = \mathbb{C}$, i.e. the Chevalley group $G_{\mathbb{C}}$.

Let $x \in \mathfrak{g}_{\mathbb{C}}$ be a nilpotent element. Then by classical theory x is $\mathrm{Ad}(G_{\mathbb{C}})$-conjugate to an element $x' \in \mathfrak{g}_{\mathbb{C}}$ such that x' is part of a standard triple $\{x', h, y\} \subseteq \mathfrak{g}_{\mathbb{C}}$ with $h \in \mathfrak{t}_{\mathbb{C}} = \mathrm{Lie}(T_{\mathbb{C}})$ and with $r_\alpha := \alpha(h)$ a nonnegative integer for all simple roots $\alpha \in \Pi$. It turns out that $r_\alpha \in \{0, 1, 2\}$, see Proposition 5.6.6 in [Car85]. Furthermore we see that $h \in \mathfrak{g}_{\mathbb{Z}}$: Restricting π to the $\mathfrak{sl}_2(\mathbb{C})$-copy generated by $\{x', h, y\}$, we see that the eigenvalues of h on V are integers since V is a direct sum of irreducible $\mathfrak{sl}_2(\mathbb{C})$-representations, and hence $h \in \mathfrak{g}_{\mathbb{Z}}$. Therefore we can write $h = \sum_{\alpha \in \Pi} q_\alpha h_\alpha$ with $q_\alpha \in \mathbb{Z}$.

Now we are ready to define the weighted Dynkin diagram $\Delta(x)$ of x. It is defined to be the Dynkin diagram of Π with the number $r_\alpha = \alpha(h)$ attached to the node corresponding to the simple root α. The weighted Dynkin diagram only depends on x, see Proposition 5.6.7 in [Car85]. Also $\Delta(x) = \Delta(x')$ if and only if x and x' are $\mathrm{Ad}(G_{\mathbb{C}})$-conjugate. Let $\mathcal{D}(\Pi)$ denote the set of weighted Dynkin diagrams.

Let Δ be a weighted Dynkin diagram. Then we will define a one-parameter subgroup $\lambda_\Delta \in X_*(T)$ by $\lambda_\Delta = \sum_{\alpha\in\Pi} q_\alpha \alpha^\vee$ where $\alpha^\vee$ is the coroot corresponding to the simple root $\alpha \in \Pi$, and the q_α's are defined as above. Note that the q_α's are uniquely determined by Δ, hence λ_Δ only depends on Δ. We have $\langle \alpha, \lambda_\Delta \rangle = \alpha(h) = r_\alpha$ for all simple roots $\alpha \in \Pi$. Hence $\langle \alpha, \lambda_\Delta \rangle \geq 0$ for all simple roots $\alpha \in \Pi$, so λ_Δ is a Π-dominant one-parameter subgroup. Hence we can define the parabolic subgroup $P(\lambda_\Delta)$ as in equation (2.4).

Remember that $Z(\lambda_\Delta)$ is the centralizer of the image of $\lambda_\Delta : k \to T$ in G. Since there are only finitely many nilpotent orbits in $\mathfrak{g}$ it follows that there are only finitely many $Z(\lambda_\Delta)$-orbits in $\mathfrak{g}(\lambda_\Delta, -2)$, see Theorem E in [Ric85]. Hence $Z(\lambda_\Delta)$ has a unique dense open orbit in $\mathfrak{g}(\lambda_\Delta, -2)$. We will call this dense orbit $\mathfrak{g}(\lambda_\Delta, -2)_{\text{reg}}$. Now let $\mathcal{O}(\Delta)$ be the G-orbit $G.\mathfrak{g}(\lambda_\Delta, -2)_{\text{reg}}$ in $\mathfrak{g}$. We know that $\mathfrak{g}(\lambda_\Delta, -2) \subseteq \mathfrak{u}$ where $\mathfrak{u}$ is the Lie algebra of the unipotent radical of the Borel group B. Hence $\mathfrak{g}(\lambda_\Delta, -2)$ consists of nilpotent elements, and $\mathcal{O}(\Delta)$ is a nilpotent orbit.

Since G satisfies the standard hypothesis, Theorem 2.3 in [Pre03] holds. We state it here:

Theorem 2.7. Let $x \in \mathfrak{g}(\lambda_\Delta, -2)_{\text{reg}}$. Then the following hold:

(i) The centralizer $Z_G(x)$ is contained in $P(\lambda_\Delta)$.

(ii) Let $C_G(\lambda_\Delta, x) = Z_G(x) \cap Z(\lambda_\Delta)$. Then $C_G(\lambda_\Delta, x)$ is a reductive group. Moreover the centralizer $Z_G(x)$ is a semidirect product of $C_G(\lambda_\Delta, x)$ and $Z_{U(\lambda_\Delta)}(x)$ as algebraic groups, and $Z_{U(\lambda_\Delta)}(x)$ is the unipotent radical of $Z_G(x)$. Hence $C_G(\lambda_\Delta, x)$ is a Levi factor of $Z_G(x)$.

(iii) $\mathfrak{z}_{\mathfrak{g}}(x) \subseteq \mathfrak{p}(\lambda_\Delta)$ and $[\mathfrak{p}(\lambda_\Delta), x] = \mathfrak{g}_{-2}(\lambda_\Delta)$.

Actually Premet's theorem states a bit more, but since we do not need that part, and since it would require a lot of explanation, we have omitted that statement. Note that McNinch has proved the theorem without the condition that G satisfies the standard hypothesis, see Proposition 16 in [McN04], but in the following lemma we still need G to satisfy the standard hypothesis since we need the orbit maps to be separable.

We have a few remarks to the above notation. First note that $P(\lambda_\Delta)$ is the parabolic subgroup containing B corresponding to the subset

$$\begin{aligned} I(\lambda_\Delta) &= \{\alpha \in \Pi | \langle \alpha, \lambda_\Delta \rangle = 0\} \\ &= \{\alpha \in \Pi | r_\alpha = 0\}. \end{aligned}$$

So given the weighted Dynkin diagram we can directly determine $P(\lambda_\Delta)$. Next remember that $\mathfrak{g}_{-2}(\lambda_\Delta)$ is $P(\lambda_\Delta)$-stable, and that r_α is the number in the weighted Dynkin diagram Δ attached to the node corresponding to the simple root α. Finally we notice that $\mathfrak{u}_{P(\lambda_\Delta)}$ denotes the Lie algebra of the unipotent radical of $P(\lambda_\Delta)$. If $r_\alpha \in \{0, 2\}$ for all simple roots $\alpha \in \Pi$, then

$$\mathfrak{g}(\lambda_\Delta, -2) = \bigoplus_{\alpha \in \Phi^- \setminus \Phi_{I(\lambda_\Delta)}} \mathfrak{g}_\alpha = \mathfrak{u}_{P(\lambda_\Delta)}.$$

Using Premet's theorem above we can prove the following lemma

Lemma 2.8. Let $x \in \mathfrak{g}(\lambda_\Delta, -2)_{\text{reg}}$. Then the following hold:

(i) The orbit $P(\lambda_\Delta).x$ is open and dense in $\mathfrak{g}_{-2}(\lambda_\Delta)$, and the closure of $\mathcal{O}(\Delta) = G.x$ equals $G.\mathfrak{g}_{-2}(\lambda_\Delta)$.

(ii) The dimension of $G \times^{P(\lambda_\Delta)} \mathfrak{g}_{-2}(\lambda_\Delta)$ equals the dimension of $G.\mathfrak{g}_{-2}(\lambda_\Delta)$.

(iii) $G.x \cap \mathfrak{g}_{-2}(\lambda_\Delta) = P(\lambda_\Delta).x$

Proof. (i): First note that $G.\mathfrak{g}_{-2}(\lambda_\Delta)$ is closed in $\mathfrak{g}$ since $\mathfrak{g}_{-2}(\lambda_\Delta)$ is $P(\lambda_\Delta)$-stable. Therefore the closure of $G.x$ is contained in $G.\mathfrak{g}_{-2}(\lambda_\Delta)$.

We need Theorem 2.7 to prove the other direction. By this theorem we know that $[\mathfrak{p}(\lambda_\Delta), x] = \mathfrak{g}_{-2}(\lambda_\Delta)$. Hence the morphism $-\text{ad}(x) : \mathfrak{p}(\lambda_\Delta) \to \mathfrak{g}_{-2}(\lambda_\Delta)$ is surjective. The rest of the proof of (i) follows from the proof of Proposition 5.7.3 in [Car85], but we include it here for completeness.

Since $\mathfrak{g}_{-2}(\lambda_\Delta)$ is $P(\lambda_\Delta)$-stable, we have a morphism $\varphi_x : P(\lambda_\Delta) \to \mathfrak{g}_{-2}(\lambda_\Delta)$ given by $\varphi_x(p) = \text{Ad}(p)(x) = p.x$. The differential of this morphism is the surjective map $-\text{ad}(x) : \mathfrak{p}(\lambda_\Delta) \to \mathfrak{g}_{-2}(\lambda_\Delta)$ from above, and by Theorem 4.3.6.(i) in [Spr98] we know that φ_x is dominant and separable. In particular the orbit $P(\lambda_\Delta).x$ is a dense open subset of $\mathfrak{g}_{-2}(\lambda_\Delta)$. Consequently $\mathfrak{g}_{-2}(\lambda_\Delta)$ is contained in the closure of $G.x$, and hence $G.\mathfrak{g}_{-2}(\lambda_\Delta)$ is also contained in the closure of $G.x$.

(ii): Since $\mathcal{O}(\Delta) = G.x$ is dense in $G.\mathfrak{g}_{-2}(\lambda_\Delta)$, we have

$$\dim G.\mathfrak{g}_{-2}(\lambda_\Delta) = \dim \mathcal{O}(\Delta),$$

but from the proof of Theorem 2.6 in [Pre03] we have

$$\dim \mathcal{O}(\Delta) = \dim \mathfrak{g} - \dim(\mathfrak{g}(\lambda_\Delta, 0) \oplus \mathfrak{g}(\lambda_\Delta, -1)).$$

Therefore

$$\begin{aligned} \dim\bigl(G \times^{P(\lambda_\Delta)} \mathfrak{g}_{-2}(\lambda_\Delta)\bigr) &= \dim G - \dim P(\lambda_\Delta) + \dim \mathfrak{g}_{-2}(\lambda_\Delta) \\ &= \dim \mathfrak{g} - \dim \mathfrak{p}(\lambda_\Delta) + \dim \mathfrak{g}_{-2}(\lambda_\Delta) \\ &= \dim \mathfrak{g} - \dim(\oplus_{i \le 0} \mathfrak{g}(\lambda_\Delta, i)) + \dim(\oplus_{i \le -2} \mathfrak{g}(\lambda_\Delta, i)) \\ &= \dim \mathcal{O}(\Delta), \end{aligned} \tag{2.5}$$

and (ii) is satisfied.

(iii): It is clear that $P(\lambda_\Delta).x \subseteq G.x \cap \mathfrak{g}_{-2}(\lambda_\Delta)$. Let $y \in G.x \cap \mathfrak{g}_{-2}(\lambda_\Delta)$. We want to show that $y \in P(\lambda_\Delta).x$. Now $y = g.x$ for some $g \in G$, and $G.x = G.y$. We have

$$\begin{aligned} \dim(P(\lambda_\Delta).y) &= \dim P(\lambda_\Delta) - \dim Z_{P(\lambda_\Delta)}(y) \\ &\ge \dim P(\lambda_\Delta) - \dim Z_G(y) \\ &= \dim P(\lambda_\Delta) - \dim G + \dim G.y \\ &= \dim P(\lambda_\Delta) - \dim G + \dim G.x \\ &= \dim P(\lambda_\Delta) - \dim G + \dim \mathcal{O}(\Delta) \\ &= \dim \mathfrak{g}_{-2}(\lambda_\Delta) \end{aligned}$$

where the last equality follows by (2.5). But $P(\lambda_\Delta).y \subseteq \mathfrak{g}_{-2}(\lambda_\Delta)$, and therefore

$$\dim(P(\lambda_\Delta).y) = \dim \mathfrak{g}_{-2}(\lambda_\Delta),$$

and the closure of the orbit $P(\lambda_\Delta).y$ equals $\mathfrak{g}_{-2}(\lambda_\Delta)$. Hence $P(\lambda_\Delta).y$ is open in $\mathfrak{g}_{-2}(\lambda_\Delta)$. But similarly $P(\lambda_\Delta).x$ is open in $\mathfrak{g}_{-2}(\lambda_\Delta)$, and hence the orbits $P(\lambda_\Delta).x$ and $P(\lambda_\Delta).y$ intersects, and $y \in P(\lambda_\Delta).x$. □

A consequence of the lemma is the following corollary.

Corollary 2.9. The morphism

$$G \times^{P(\lambda_\Delta)} \mathfrak{g}_{-2}(\lambda_\Delta) \to G.\mathfrak{g}_{-2}(\lambda_\Delta) \quad \text{given by} \quad [(g, y)] \mapsto g.y$$

is birational.

Proof. This follows from Lemma 2.5, Lemma 2.8 and Theorem 2.7 (i). □

Now we are ready to define

$$V(\lambda_\Delta) := \mathfrak{g}_{-2}(\lambda_\Delta) = \bigoplus_{\substack{\alpha\in\Phi \\ \langle\alpha,\lambda_\Delta\rangle\leq -2}} \mathfrak{g}_\alpha.$$

Then $P(\lambda_\Delta)$, $V(\lambda_\Delta)$ and $\mathcal{O}(\Delta)$ satisfy condition i and ii on page 14 by Lemma 2.8. Moreover the morphism in (2.3) is birational by this corollary.

Now return to the case where the weighted Dynkin diagram Δ only consists of the numbers 0 and 2. Then we have already seen that $\mathfrak{g}_{-2}(\lambda_\Delta) = \mathfrak{u}_{P(\lambda_\Delta)}$. By Lemma 2.8(i) we observe that elements in $\mathfrak{g}(\lambda_\Delta, -2)_{\text{reg}}$ are Richardson elements in $\mathfrak{u}_{P(\lambda_\Delta)}$ for $P(\lambda_\Delta)$.

On the other hand if $y \in \mathfrak{g}(\lambda_\Delta, -2)$ is a Richardson element for $P(\lambda_\Delta)$, then y is $P(\lambda_\Delta)$-conjugate to an element $x \in \mathfrak{g}(\lambda_\Delta, -2)_{\text{reg}}$. By definition $\mathfrak{g}(\lambda_\Delta, -2)_{\text{reg}}$ is a $Z(\lambda_\Delta)$-orbit, and we have $\mathfrak{g}(\lambda_\Delta, -2)_{\text{reg}} = Z(\lambda_\Delta).x$. Since $P(\lambda_\Delta) = Z(\lambda_\Delta)U(\lambda_\Delta)$ we can write

$$y = (zu).x \quad \text{with} \quad z \in Z(\lambda_\Delta), \quad u \in U(\lambda_\Delta).$$

Now $u.x = x + x'$ with $x' \in \mathfrak{g}_{-3}(\lambda_\Delta)$. Since $\mathfrak{g}(\lambda_\Delta, i)$ is $Z(\lambda_\Delta)$-stable, and since $\mathfrak{g}(\lambda_\Delta, -2)_{\text{reg}} = Z(\lambda_\Delta).x$, this implies that

$$y = (zu).x = z.x + z.x' \quad \text{with} \quad z.x \in \mathfrak{g}(\lambda_\Delta, -2)_{\text{reg}}, \quad z.x' \in \mathfrak{g}_{-3}(\lambda_\Delta).$$

But $y \in \mathfrak{g}(\lambda_\Delta, -2)$, and hence $z.x' = 0$. Consequently $y = z.x \in \mathfrak{g}(\lambda_\Delta, -2)_{\text{reg}}$.

To summarize: We have seen that $\mathfrak{g}(\lambda_\Delta, -2)_{\text{reg}}$ is exactly the set of Richardson elements in $\mathfrak{u}_{P(\lambda_\Delta)}$ for $P(\lambda_\Delta)$ which are contained in $\mathfrak{g}(\lambda_\Delta, -2)$.

Now if $k = \mathbb{C}$, we see that $\mathfrak{p}(\lambda_\Delta)$ is a nice parabolic subalgbra in the sense of [BW05].

2.2.4 Why we can use characteristic zero results

Remember that $G = G_k$ is the connected, semi-simple, simply connected linear algebraic group over k with root system Φ and Lie algebra $\mathfrak{g} = \mathrm{Lie}(G_k)$.

Again we take a look at Premet's paper. He shows that the map from the set of weighted Dynkin diagrams to the set of all nilpotent G-orbits in $\mathfrak{g}$ sending $\Delta \in \mathcal{D}(\Pi)$ to the orbit $\mathcal{O}(\Delta)$ is a bijection, see Proposition 2.4, Theorem 2.6 and Theorem 2.7 in [Pre03]. But the Bala-Carter theorem in good characteristic (which Premet proves in a new, uniform way, see Theorem 2.6 and Theorem 2.7 in [Pre03]), assigns to each nilpotent orbit a Bala-Carter label, and hence we get a bijection between the set of weighted Dynkin diagrams and the set of Bala-Carter labels. Premet proves that this bijection between the set of weighted Dynkin diagrams and the set of Bala-Carter labels is independent of our field k, and hence we denote it by ψ (not indexed by k).

From classical theory over $\mathbb{C}$ we already have a bijection between weighted Dynkin diagrams and nilpotent $G_{\mathbb{C}}$-orbits in $\mathfrak{g}_{\mathbb{C}}$, and also a bijection between the set of Bala-Carter labels and nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ from the Bala-Carter theorem. Composing these two maps we get a new bijection between the set of weighted Dynkin diagrams and the set of Bala-Carter labels. Again following Premet it turns out that this bijection is equal to the bijection ψ from the last paragraph. In Chapter 13.1 in [Car85] this bijection is explicitly calculated for each root system, and consequently we can use these results.

Let $\Delta \in \mathcal{D}(\Pi)$ be a weighted Dynkin diagram. Let $\mathcal{O}(\Delta)$ be the corresponding nilpotent G-orbit in $\mathfrak{g}$, and let $\mathcal{O}_{\mathbb{C}}(\Delta)$ be the corresponding $G_{\mathbb{C}}$-orbit in $\mathfrak{g}_{\mathbb{C}}$. Then

$$\dim_k \mathcal{O}(\Delta) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}}(\Delta)$$

by Theorem 2.6.iv in [Pre03], and hence we can use the dimension results over $\mathbb{C}$ which e.g. can be found in Chapter 8.4 and Corollary 6.1.4 in [CM93].

Let $x \in \mathcal{O}(\Delta)$, and let $x' \in \mathcal{O}_{\mathbb{C}}(\Delta)$. Then remember that $Z_G(x)$ denotes the centralizer of x in G, and that $Z_{G_{\mathbb{C}}}(x')$ denotes the centralizer of x' in $G_{\mathbb{C}}$. By Theorem 2.7(ii) we know that there exists Levi factors $L = C_G(\lambda_\Delta, x)$ and $L_{\mathbb{C}} = C_{G_C}(\lambda_\Delta, x')$ of $Z_G(x)$ and $Z_{G_{\mathbb{C}}}(x')$ respectively.

In the beginning of Section 2.2 we observed that the characteristic of k is very good for G since G is semi-simple and satisfies the standard hypothesis. Hence we can apply the following results from [McN]. Theorem A in [McN] states that the root datum of L and $L_{\mathbb{C}}$ can be identified, in particular L is semi-simple if and only if $L_{\mathbb{C}}$ is semi-simple. Theorem B in [McN] which is an extension of Theorem 36 in [MS03] shows that the component groups of the centralizers are isomorphic finite groups, i.e. it tells us that $Z_G(x)/Z_G(x)^0$ and $Z_{G_{\mathbb{C}}}(x')/Z_{G_{\mathbb{C}}}(x')^0$ are isomorphic. Therefore we can use the results in Chapter 13.1 in [Car85] about the root datum of $L_{\mathbb{C}}$ and the component group $Z_{G_{\mathbb{C}}}(x')/Z_{G_{\mathbb{C}}}(x')^0$ to find the root datum of L and to find the component group $Z_G(x)/Z_G(x)^0$.

2.3 The complement of the set of Richardson elements

Let G be a connected linear algebraic group. Let $I \subseteq \Pi$ be a subset of simple roots, and let P be the standard parabolic subgroup containing B corresponding to I. Remember that P acts linearly on the Lie algebra of its unipotent radical $\mathfrak{u}_P$. According to Richardson's Dense Orbit Theorem there exists a unique dense and open P-orbit in $\mathfrak{u}_P$, see Theorem 1.3. Let $\mathcal{O}_P$ denote this P-orbit. Then the elements in $\mathcal{O}_P$ are called Richardson elements. Since $\mathcal{O}_P$ is open in $\mathfrak{u}_P$, the complement $\mathfrak{u}_P \setminus \mathcal{O}_P$ is closed in $\mathfrak{u}_P$, and hence an affine variety. Moreover the dimension of $\mathfrak{u}_P \setminus \mathcal{O}_P$ is strictly less than the dimension of $\mathfrak{u}_P$ since $\mathfrak{u}_P$ is irreducible.

Since

$$\mathfrak{u}_P = \bigoplus_{\alpha \in \Phi^- \setminus \Phi_I} \mathfrak{g}_\alpha,$$

we can identify the coordinate ring of $\mathfrak{u}_P$, denoted $k[\mathfrak{u}_P]$, with the polynomial ring

$$k[x_\alpha | \alpha \in \Phi^- \setminus \Phi_I]$$

with the usual grading $\deg(x_\alpha) = 1$.

Lemma 2.10. Let V be an irreducible component in $\mathfrak{u}_P \setminus \mathcal{O}_P$, and let $I(V)$ be the defining ideal of V in $\mathfrak{u}_P$. Then $I(V) \subseteq k[\mathfrak{u}_P]$ is a homogeneous ideal.

Proof. In addition to the P-action on $\mathfrak{u}_P$, we also have an action of k^* on $\mathfrak{u}_P$ because $\mathfrak{u}_P$ is a k-vector space. Since P acts linearly on $\mathfrak{u}_P$, the P-action commutes with the k^*-action.

Let $x \in \mathcal{O}_P$ be a Richardson element, and let $t \in k^*$. Then $\mathcal{O}_P = P.x$ and

$$t.\mathcal{O}_P = t.(P.x) = P.(t.x).$$

But then $t.\mathcal{O}_P$ is a P-orbit in $\mathfrak{u}_P$ of the same dimension as $\mathcal{O}_P$, but there exists only one such orbit, and hence $t.\mathcal{O}_P = \mathcal{O}_P$, and $\mathcal{O}_P$ is k^*-stable.

Now also the complement $\mathfrak{u}_P \setminus \mathcal{O}_P$ is k^*-stable, and since k^* is irreducible, the irreducible components of $\mathfrak{u}_P \setminus \mathcal{O}_P$ are k^*-stable too. In particular V is k^*-stable, and $I(V)$ is homogeneous with respect to the chosen grading of $k[\mathfrak{u}_P]$. □

We want to study the components in $\mathfrak{u}_P \setminus \mathcal{O}_P$ of maximal dimension, i.e. the components of dimension equal to $\dim \mathfrak{u}_P - 1$. Since $\mathcal{O}_P$ is P-stable, also $\mathfrak{u}_P \setminus \mathcal{O}_P$ is P-stable. But P is irreducible, and hence all components in $\mathfrak{u}_P \setminus \mathcal{O}_P$ are P-stable. Now let $V \subseteq \mathfrak{u}_P \setminus \mathcal{O}_P$ be a component of maximal dimension. Then

$$I(V) = \langle f \rangle \subseteq k[\mathfrak{u}_P] \quad \text{for some irreducible element} \quad f \in k[\mathfrak{u}_P]$$

since $\mathfrak{u}_P$ is just affine space. Since P-acts on $\mathfrak{u}_P$, it also acts on the coordinate ring $k[\mathfrak{u}_P]$. Now V is P-stable, and hence also $I(V) = \langle f \rangle$ is P-stable. Therefore there exists a P-character $\lambda \in X^*(P)$ such that

$$p.f = \lambda(p) f \quad \text{for all} \quad p \in P.$$

Let $k(\mathfrak{u}_P)$ denote the function field of $\mathfrak{u}_P$. Then $k(\mathfrak{u}_P)$ is just the fraction field of $k[\mathfrak{u}_P]$, so P also acts on $k(\mathfrak{u}_P)$. Now define

$$k(\mathfrak{u}_P)^{(P)} = \{h \in k(\mathfrak{u}_P) | \exists \lambda \in X^*(P) : p.h = \lambda(p)h \text{ for all } p \in P\},$$

and let $k[\mathfrak{u}_P]^{(P)} = k[\mathfrak{u}_P] \cap k(\mathfrak{u}_P)^{(P)}$. The elements in $k(\mathfrak{u}_P)^{(P)}$ are called P-semistable.

We have seen that given a component in $\mathfrak{u}_P \setminus \mathcal{O}_P$, we can construct an irreducible element in $k[\mathfrak{u}_P]^{(P)}$ which is unique up to multiplication by scalars in k^*. On the other hand, given an irreducible element $f \in k[\mathfrak{u}_P]^{(P)}$, then $V(f)$ is irreducible of dimension equal to $\dim \mathfrak{u}_P - 1$. But since f is P-semistable, $V(f)$ is P-stable. Therefore if $V(f)$ intersects $\mathcal{O}_P$, then $\mathcal{O}_P \subseteq V(f)$. But since $\mathcal{O}_P$ is dense in $\mathfrak{u}_P$, we get $V(f) = \mathfrak{u}_P$ which is a contradiction. Therefore $V(f) \subseteq \mathfrak{u}_P \setminus \mathcal{O}_P$, and $V(f)$ is a component in $\mathfrak{u}_P \setminus \mathcal{O}_P$ of dimension equal to $\dim \mathfrak{u}_P - 1$. It is clear that if we multiply f by a scalar in k^*, we get the same component.

To summarize: We have shown that there is a bijection between the set of components in $\mathfrak{u}_P$ of dimension $\dim \mathfrak{u}_P - 1$ and the set of irreducible elements in $k[\mathfrak{u}_P]^{(P)}$ modulo scalars in k^*.

Let

$$\left(k[\mathfrak{u}_P]^{(P)}\right)_0$$

denote the fraction field of $k[\mathfrak{u}_P]^{(P)}$.

Lemma 2.11. Now the following are satisfied.

i. If $f \in k[\mathfrak{u}_P]^{(P)}$, then all irreducible components of f belongs to $k[\mathfrak{u}_P]^{(P)}$.

ii. We have the identity

$$\left(k[\mathfrak{u}_P]^{(P)}\right)_0 = \left(k[\mathfrak{u}_P]_0\right)^{(P)}.$$

Proof. i: Let $f \in k[\mathfrak{u}_P]^{(P)} \setminus \{0\}$ and assume that f is not a unit. Write $f = \prod_i f_i$ as a product of irreducibles. Then since f is P-semistable, we know that $V(f) \subseteq \mathfrak{u}_P$ is P-stable. As above we therefore know that $V(f) \subseteq \mathfrak{u}_P \setminus P.x$. But $V(f) = \cup_i V(f_i)$, and $V(f_i)$ is irreducible of dimension equal to $\dim(\mathfrak{u}_P) - 1$. Hence $V(f_i)$ is a component in $\mathfrak{u}_P \setminus P.x$, and since P is connected, $V(f_i)$ is P-stable. But then $I(V(f_i)) = \langle f_i \rangle$ is P-stable, and f_i is P-semistable.

ii: Clearly $\left(k[\mathfrak{u}_P]^{(P)}\right)_0 \subseteq k(\mathfrak{u}_P)^{(P)}$. Let $h \in k(\mathfrak{u}_P)^{(P)}$. Then we can write $h = \frac{f}{g}$ with $f, g \in k[\mathfrak{u}_P]$, $g \neq 0$, such that f and g have no common irreducible factors. Since h is P-semistable, there exists a character $\lambda \in X^*(P)$ such that $p.h = \lambda(p)h$ for all $p \in P$. Hence

$$\lambda(p)\tfrac{f}{g} = p.\tfrac{f}{g} = \tfrac{p.f}{p.g} \in k(\mathfrak{u}_P),$$

and we have

$$\lambda(p)(p.g)f = (p.f)g \in k[\mathfrak{u}_P].$$

Now $k[\mathfrak{u}_P]$ is a UFD, and f and g have no common factors, hence g divides $p.g$, and f divides $p.f$ for all $p \in P$. Let $p \in P$. Then g divides $p^{-1}.g$, and $p^{-1}.g = g'g$ for some $g' \in k[\mathfrak{u}_P]$. Now we have

$$g = p.(p^{-1}.g) = p.(g'g) = (p.g')(p.g),$$

and $p.g$ divides g. Now since g divides $p.g$, and $p.g$ divides g for all $p \in P$, we must have $p.g = \lambda(p)g$ for some character $\lambda \in X^*(P)$, and $g \in k[\mathfrak{u}_P]^{(P)}$. Similarly we can show that $f \in k[\mathfrak{u}_P]^{(P)}$. ☐

Remember that P is the standard parabolic subgroup corresponding to the subset $I \subseteq \Pi$ of simple roots. Assume that G is simply connected. Then $X^*(P)$ is a finitely generated free abelian group, and the set of fundamental weights corresponding to the simple roots in $\Pi \setminus I$ is a basis for $X^*(P)$. Let $x \in \mathcal{O}_P$ be a Richardson element. Let $Z_P(x)$ denote the centralizer of x in P, and let $Z_P(x)^0$ be its identity component. Then $Z_P(x)^0$ has finite index in $Z_P(x)$.

The inclusions $Z_P(x)^0 \subseteq Z_P(x) \subseteq P$ induces restriction maps between their character groups

$$\begin{array}{ccc} X^*(P) & \xrightarrow{\quad\psi_x\quad} & X^*(Z_P(x)^0) \\ & {\scriptstyle\varphi_x}\searrow \qquad \nearrow & \\ & X^*(Z_P(x)) & \end{array}$$

Since $X^*(P)$ is a finitely generated free abelian group, also Ker φ_x and Ker ψ_x are finitely generated free groups. Clearly we have Ker $(\varphi_x) \subseteq$ Ker (ψ_x). Now let N denote the finite index of $Z_P(x)^0$ in $Z_P(x)$. Then we claim that NKer $(\psi_x) \subseteq$ Ker (φ_x):

Let $\lambda \in$ Ker (ψ_x), then $\lambda(p) = 1$ for all $p \in Z_P(x)^0$. Let $q \in Z_P(x)$. Then $N\lambda(q) = \lambda(q^N) = 1$ since $q^N \in Z_P(x)^0$, and $N\lambda \in$ Ker (φ_x).

Hence the rank of Ker (φ_x) equals the rank of Ker (ψ_x).

Also notice that if we choose another Richardson element $x' \in \mathcal{O}_P$, then there exists a $p \in P$, such that $x' = p.x$. Hence $Z_P(x') = pZ_P(x)p^{-1}$, and

$$\text{Ker } \varphi_x = \text{Ker } \varphi_{x'} \quad \text{and} \quad \text{Ker } \psi_x = \text{Ker } \psi_{x'}.$$

Therefore we define $K_\varphi =$ Ker φ_x and $K_\psi =$ Ker ψ_x.

Lemma 2.12. The number of components in $\mathfrak{u}_P \setminus \mathcal{O}_P$ with dimension equal to $\dim(\mathfrak{u}_P) - 1$ is less than or equal to the rank of K_φ. In particular it is less than the rank of $X^*(P)$.

Proof. Let $V_1, \ldots, V_n$ be the components of dimension $\dim(\mathfrak{u}_P) - 1$ in $\mathfrak{u}_P \setminus \mathcal{O}_P$. Then for $i = 1, \ldots, n$ we have $I(V_i) = \langle f_i \rangle \subseteq k[\mathfrak{u}_P]$ for some irreducible element $f_i \in k[\mathfrak{u}_P]^{(P)}$. So there exist $\lambda_1, \ldots, \lambda_n \in X^*(P)$ such that for $i = 1, \ldots, n$ we have $p.f_i = \lambda_i(p)f_i$ for all $p \in P$. Notice that since V_i does not intersect $\mathcal{O}_P$, we know that $f_i(x) \neq 0$ for all $x \in \mathcal{O}_P$. Now choose an element $x \in \mathcal{O}_P$. Since $f_i(x) \neq 0$, we may assume that $f_i(x) = 1$ by multiplying with a scalar. Now for all $p \in Z_P(x)$ we have

$$\lambda_i(p) = \lambda_i(p)f_i(x) = (p.f_i)(x) = f_i(p^{-1}.x) = f_i(x) = 1$$

hence $\lambda_i \in \text{Ker } \varphi_x$.

Assume that the rank of $K_\varphi = \text{Ker } \varphi_x$ is strictly less than n. Then there exist $m_1, \ldots, m_n \in \mathbb{Z}$ such that $\sum_{i=1}^n m_i\lambda_i = 0$ and not all m_i's are zero. Now look at the element $\prod_{i=1}^n f_i^{m_i} \in k(\mathfrak{u}_P)$. Since all f_i's are nonzero on $\mathcal{O}_P$, the element $\prod_{i=1}^n f_i^{m_i}$ is a regular function on $\mathcal{O}_P$. For all $p \in P$ we have

$$\begin{aligned}\Big(\prod_{i=1}^n f_i^{m_i}\Big)(p^{-1}.x) &= \prod_{i=1}^n (p.f_i)^{m_i}(x) = \prod_{i=1}^n \left(\lambda_i(p)^{m_i} f_i^{m_i}\right)(x) \\ &= \prod_{i=1}^n \left(\lambda_i(p)^{m_i}(f_i(x))^{m_i}\right) = \prod_{i=1}^n \left(\lambda_i(p)^{m_i}\cdot 1\right) \\ &= \Big(\sum_{i=1}^n m_i\lambda_i\Big)(p) = 1,\end{aligned}$$

and

$$\Big(\prod_{i=1}^n f_i^{m_i}\Big)(z) = 1 \quad \text{for all} \quad z \in P.x = \mathcal{O}_P. \tag{2.6}$$

Now define

$$f^+ = \prod_{i:m_i>0} f_i^{m_i} \in k[\mathfrak{u}_P], \qquad f^- = \prod_{i:m_i<0} f_i^{-m_i} \in k[\mathfrak{u}_P].$$

From (2.6) it follows that $f^+(z) = f^-(z)$ for all $z \in \mathcal{O}_P$. But $\mathcal{O}_P$ is dense in $\mathfrak{u}_P$, and hence $f^+(z) = f^-(z)$ for all $z \in \mathfrak{u}_P$, i.e.

$$\prod_{i:m_i>0} f_i^{m_i} = \prod_{i:m_i<0} f_i^{-m_i} \in k[\mathfrak{u}_P] \tag{2.7}$$

Since the f_i's corresponds to different components, we know that for all constants $a \in k$ we have $f_i \neq af_j$ when $i \neq j$. But $k[\mathfrak{u}_P]$ is a UFD, and the f_i's are irreducible, and hence (2.7) implies that all m_i's must be zero. But this is a contradiction, so n is less than or equal to the rank of K_φ. □

Lemma 2.13. Let $x \in \mathcal{O}_P$ be a Richardson element. If the orbit map $P \to P.x = \mathcal{O}_P$ sending p to $p.x$ is separable, then the number of components in $\mathfrak{u}_P \setminus \mathcal{O}_P$ with dimension equal to $\dim(\mathfrak{u}_P) - 1$ equals the rank of K_φ.

Proof. Let s denote the rank of K_φ, and let n denote the number of components in $\mathfrak{u}_P \setminus \mathcal{O}_P$ with dimension equal to $\dim(\mathfrak{u}_P) - 1$. From the preceding lemma we know that $n \leq s$, so we want to show that $s \leq n$.

We us the notation from the preceding proof. Again $V_1, \ldots, V_n$ are the components in $\mathfrak{u}_P \setminus \mathcal{O}_P$ of dimension $\dim(\mathfrak{u}_P) - 1$, and $I(V_i) = \langle f_i \rangle \subseteq k[\mathfrak{u}_P]$ for some irreducible element $f_i \in k[\mathfrak{u}_P]^{(P)}$ with $f_i(x) = 1$ for $i = 1, \dots, n$. Now there exist $\lambda_1 \dots, \lambda_n \in X^*(P)$ such that $p.f_i = \lambda_i(p) f_i$ for all $p \in P$. Now remember that up to multiplication with scalars in k^*, the f_i's are the only irreducible elements in $k[\mathfrak{u}_P]^{(P)}$.

Since the orbit map $P \to P.x = \mathcal{O}_P$ is separable, the induced morphism $P/Z_P(x) \to \mathcal{O}_P$ is an isomorphism. Therefore all $\mu \in \operatorname{Ker} \varphi_x = K_\varphi$ induce morphisms $\bar{\mu} : \mathcal{O}_P \to k^*$ given by $\bar{\mu}(p.x) = \mu(p^{-1})$. Now $\bar{\mu}$ is a non-vanishing regular function on $\mathcal{O}_P$. Since $\mathcal{O}_P$ is open in $\mathfrak{u}_P$, we can consider $\bar{\mu}$ as an element in the function field of $\mathfrak{u}_P$, i.e. $\bar{\mu} \in k(\mathfrak{u}_P)$. Note that

$$(p.\bar{\mu})(q.x) = \bar{\mu}(p^{-1}.q.x) = \mu(q^{-1}p) = \mu(p)\mu(q^{-1}) = \mu(p)\bar{\mu}(q.x)$$

for all $p, q \in P$. In particular $p.\bar{\mu} = \mu(p)\bar{\mu}$, and $\bar{\mu} \in k(\mathfrak{u}_P)^{(P)}$.

Now we can choose a basis $\mu_1, \dots, \mu_s$ for K_φ. Using the above method we get induced elements $\bar{\mu}_1, \dots, \bar{\mu}_s \in k(\mathfrak{u}_P)^{(P)}$. Since the f_i's are the only irreducible elements in $k[\mathfrak{u}_P]^{(P)}$ up to scalars, Lemma 2.11 tells us that we can write

$$\bar{\mu}_j = c_j \prod_{i=1}^{n} f_i^{m_{i,j}}$$

for some $c_j \in k$ and some $m_{j,i} \in \mathbb{Z}$. Since $p.f_i = \lambda_i(p) f_i$ for all $p \in P$, we have

$$p.\bar{\mu}_j = \Big(\sum_{i=1}^{n} m_{j,i}\lambda_i\Big)(p)\bar{\mu}_j.$$

But we also have $p.\bar{\mu}_j = (\mu_j)(p)\bar{\mu}_j$, and hence $\mu_j = \sum_{i=1}^{s} m_{j,i}\lambda_i$.

As in the proof of the preceding lemma we know that $\lambda_i \in \operatorname{Ker} \varphi_x = K_\varphi$. But the set of μ_j's is a basis for K_φ, and we can write $\lambda_i = \sum_{j=1}^{s} a_{i,j}\mu_j$ for some $a_{i,j} \in \mathbb{Z}$.

Let A denote the $n \times s$-matrix with entries $a_{i,j} \in \mathbb{Z}$, and let M denote the $s \times n$-matrix with entries $m_{j,i} \in \mathbb{Z}$. Let $\underline{\mu}$ be the vector with entries μ_j, and let $\underline{\lambda}$ denote the vector with entries λ_i.

We have seen that $\underline{\lambda} = A\underline{\mu}$ and that $\underline{\mu} = M\underline{\lambda}$. Hence $\underline{\mu} = MA\underline{\mu}$, but since the μ_j's are linearly independent in K_φ, we get that $MA = I_s$ where I_s is the $s \times s$ identity matrix, and we must have $n \geq s$. $\square$

Let $x \in \mathcal{O}_P$. Notice that in characteristic zero the orbit map $P \to P.x$ sending p to $p.x$ is always separable. In the following situation the orbit map $P \to P.x$ is also separable. Suppose G satisfies the standard hypothesis on page 4. We use the notation of Section 2.2.3. Assume that $P = P(\lambda_\Delta)$ for some weighted Dynkin diagram Δ. Since $\mathfrak{g}(\lambda_\Delta, -2)_{\text{reg}}$ consists of Richardson elements, we know that x is P-conjugate to an element $x' \in \mathfrak{g}(\lambda_\Delta, -2)_{\text{reg}}$. Then $Z_G(x') \subseteq P$ by Theorem 2.7. i, and since $Z_G(x)$ and $Z_G(x')$ are P-conjugate, we have $Z_G(x) \subseteq P$. Therefore $Z_G(x) = Z_P(x)$. Since G satisfies the standard hypothesis, the orbit map $G \to G.x$ is separable, and the induced map $G/Z_P(x) = G/Z_G(x) \to G.x$ is an isomorphism. Restricting this isomorphism to the closed set $P/Z_P(x)$, we get an isomorphism $P/Z_P(x) \to P.x$ which is equivalent to $P \to P.x$ being separable.

From now on we will assume that G satisfies the standard hypothesis, and that $P = P(\lambda_\Delta)$ for some weighted Dynkin diagram Δ. Then we know from Theorem 2.7.ii that $Z_P(x') = Z_G(x')$ is a semidirect product of $C_G(\lambda_\Delta, x')$ and $Z_{U(\lambda_\Delta)}(x')$ as algebraic groups where $C_G(\lambda_\Delta, x')$ is reductive and $Z_{U(\lambda_\Delta)}(x')$ equals the unipotent radical of $Z_P(x')$. Let $R_u(Z_P(x))$ denote the unipotent radical of $Z_P(x)$. Then since x is P-conjugate to x', we get that $Z_P(x) = Z_G(x)$ is a semidirect product of a reductive group L and $R_u(Z_P(x))$ where L is P-conjugate to $C_G(\lambda_\Delta, x')$. Now also $Z_P(x)^0 = Z_G(x)^0$ is a semidirect product of the identity component L^0 and $R_u(Z_G(x))$.

Since L^0 is reductive and connected, we can write $L^0 = R(L^0)(L^0, L^0)$ where $R(L^0)$ is the radical of L^0 and (L^0, L^0) is the commutator subgroup of L^0. The radical $R(L^0)$ is a central torus, hence $X^*(R(L^0))$ is a finitely generated free abelian group.

Lemma 2.14. Let r denote the rank of $X^*(R(L^0))$. Then also $X^*(Z_P(x)^0)$ is free abelian of rank r.

Proof. The considerations above imply that

$$Z_P(x)^0 = Z_G(x)^0 = R(L^0)(L^0, L^0)R_u(Z_G(x)).$$

Since $R(L^0) \subseteq Z_P(x)^0$ we get an induced map

$$\Gamma : X^*(Z_P(x)^0) \to X^*(R(L^0)).$$

Since (L^0, L^0) is commutative, the character group of (L^0, L^0) is trivial. Since $R_u(Z_G(x))$ consists of unipotent elements, the character group of $R_u(Z_G(x))$ is also trivial. Hence Γ is injective, and $X^*(Z_P(x)^0)$ is free abelian of rank less than or equal to r.

Now we want to define a group homomorphism

$$\Psi : X^*(R(L^0)) \to X^*(Z_P(x)^0).$$

Let K be the number of elements in the finite set $(L^0, L^0) \cap R(L^0)$. Let $f \in X^*(R(L^0))$. Let $x \in Z_P(x)^0$ and write $x = x_1x_2x_3$ with $x_1 \in R(L^0)$, $x_2 \in (L^0, L^0)$ and $x_3 \in R_u(Z_G(x))$. Then define $\Psi(f)(x) = x_1^K$. It turns out that this is well defined and that Ψ becomes a group homomorphism. It is clear that

$$\Gamma \circ \Psi : X^*(R(L^0)) \to X^*(Z_G(x)^0) \to X^*(R(L^0)),$$

is just multiplication by K, and hence Ψ is injective. It follows that the rank of $X^*(Z_G(x)^0)$ equals r. □

Corollary 2.15. If L^0 is semisimple, then $X^*(Z_P(x)^0) = 0$.

Proof. If L^0 is semisimple, then $R(L^0)$ is trivial, and we have $X^*(R(L^0)) = 0$. Hence $X^*(Z_P(x)^0) = 0$ by Lemma 2.14. □

Corollary 2.16. If L^0 is semisimple, then the number of components in $\mathfrak{u}_P \setminus P.x$ with dimension $\dim(\mathfrak{u}_P) - 1$ is equal to the rank of $X^*(P)$.

Proof. Corollary 2.15 and Lemma 2.13 □

As explained in Section 2.2.4 we can use the results in Chapter 13.1 in [Car85] to find the root datum of L^0 given the root system Φ and the weighted Dynkin diagram Δ. In particular we can check whether or not L^0 is semi-simple.

Chapter 3

Vanishing theorems

We need some vanishing results for cohomology groups, cf. Section 2. The main theorem used by Eric Sommers is the following theorem by Demazure, found in [Dem76] in characteristic 0, and e.g. in [Tho00] in characteristic $p > 0$.

Theorem 3.1. *Let α denote a simple root. Let V be a P_α-representation. Let $\lambda \in X^*(T)$ and $m = \langle \lambda, \alpha^\vee \rangle$. Assume that*

$$\begin{aligned} m \leq -1 &\quad \text{if } \operatorname{char}(k) = 0 \\ -p-1 \leq m \leq -1 &\quad \text{if } \operatorname{char}(k) = p. \end{aligned}$$

Then there exists an isomorphism of G-representations

$$H^i(G/B, V \otimes \lambda) = H^{i-1}(G/B, V \otimes (s_\alpha(\lambda) - \alpha)) \quad \text{for all} \quad i \in \mathbb{Z}.$$

In particular, if $m = -1$ all cohomology groups vanish.

Remember that

$$\lambda - (m+1)\alpha = s_\alpha(\lambda) - \alpha.$$

Eric Sommers has shown the following proposition, Proposition 3.2, which relies on Theorem 3.1. We need some notation to be able to explain it.

Let G be of type A_l, and label the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_l$ such that

$$\langle \alpha_i, \alpha_{i+1}^\vee \rangle = -1 \quad \text{for all} \quad i = 1, 2, \ldots, l-1.$$

Let $\varpi_1, \varpi_2, \ldots \varpi_l$ denote the corresponding fundamental weights. Let $\mathfrak{u}_i$ be the Lie algebra of the unipotent radical of the *maximal* standard parabolic subgroup containing B corresponding to all the simple roots except α_i.

Proposition 3.2. *Let $\operatorname{char}(k) = 0$. Choose m with $1 \leq m \leq l$, and let $m' = \min\{m, l+1-m\}$. If r is an integer satisfying*

$$2m' - 2 - l \leq r \leq 0,$$

then there exists an isomorphism of G-representations

$$H^i\left(G/B, S^n \mathfrak{u}_m^* \otimes r\varpi_m\right) = H^i\left(G/B, S^{n+rm'} \mathfrak{u}_{l+1-m}^* \otimes -r\varpi_{l+1-m}\right)$$

for all $i, n \in \mathbb{Z}$.

Eric Sommers gives a proof of the proposition in [Som]. He also states that the proposition works in a more general setting, Proposition 6 in [Som03]. The following section contains a detailed proof of the proposition in this more general setting, also when $\operatorname{char}(k) = p > 0$. However there will be a lower bound on p in characteristic p.

3.1 Proof and explanation of Proposition 4.4 in [Som03]

Let $I \subseteq \Delta$, and let P_I be the corresponding subgroup containing B. Let L be the Levi subgroup of P containing T. We assume that L is of type A_l, i.e. the root system of L relative to T, Φ_I, is a root system of type A_l. Let w_0 be the longest element in the Weyl group of L. The Weyl group of L is a subgroup of the Weyl group, W, of G so we can consider w_0 as an element in W.

We now enumerate the simple roots as follows:

- The roots in I are denoted $\alpha_1, \alpha_2, \ldots, \alpha_l$ such that $\langle \alpha_i, \alpha_{i+1} \rangle = -1$.
- The roots $\beta \in \Delta \setminus I$ for which there exists an α_i satisfying $\langle \beta, \alpha_i^\vee \rangle < 0$ (we will call β and α_i for neighbors), are denoted $\beta_1, \beta_2, \ldots, \beta_s$.
- The rest of the simple roots are denoted $\gamma_1, \gamma_2, \ldots, \gamma_t$.

We start by choosing a set of simple roots $\{\gamma_{k_1}, \gamma_{k_2}, \ldots, \gamma_{k_u}\}$, and let P_i be the parabolic subgroup corresponding to the set

$$I_i = I \cup \{\gamma_{k_1}, \gamma_{k_2}, \ldots, \gamma_{k_u}\} \setminus \{\alpha_i\}, \quad 1 \leq i \leq l.$$

Let $\mathfrak{u}_i$ denote the unipotent radical of P_i. Then $\mathfrak{u}_i$ is a P_i-module, and

$$\mathfrak{u}_i = \bigoplus_{\alpha \in \Phi^- \setminus \Phi_i} \mathfrak{g}_\alpha,$$

where Φ_i is the set of roots which are linear combinations of the roots in I_i.

Note that if $\lambda \in X^*(T)$ satisfy $\langle \lambda, \alpha^\vee \rangle = 0$ for some simple root α, we know that $\lambda \in X^*(P_\alpha)$ – this will be used in the proof of the following proposition. Also remember that the one dimensional B-representation with weight $\lambda \in X^*(T) = X^*(B)$ is just denoted λ.

Proposition 3.3. Let $1 \leq m \leq l$ and $m' = \min\{m, l + 1 - m\}$. Let $\lambda \in X^*(T)$, and set $r = \langle \lambda, \alpha_m^\vee \rangle$. Suppose that

$$\langle \lambda, \alpha_i^\vee \rangle = 0 \quad \text{for } i = 1, \ldots, m-1, m+1, \ldots l$$

and that $2m' - 2 - l \leq r \leq 0$.

If $\operatorname{char}(k) = 0$, or $\operatorname{char}(k) = p$ with $m' - 1 \leq p$, then there exists an isomorphism of G-modules

$$H^i(G/B, S^n \mathfrak{u}_m \otimes \lambda) = H^i(G/B, S^{n+rm'} \mathfrak{u}_{l+1-m} \otimes w_0(\lambda)) \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

We start by proving the following lemma.

Lemma 3.4. Let $1 \leq a \leq b \leq l$, let $J = \{\alpha_a, \alpha_{a+1}, \dots, \alpha_b\}$, and let Q be a P_J-module. Let $\lambda \in X^*(T)$, $s = \langle \lambda, \alpha_a^\vee \rangle$, and suppose that

$$\langle \lambda, \alpha_i^\vee \rangle = 0 \quad \text{for } a < i \leq b$$

and that $a - b - 1 \leq s \leq -1$. Then

$$H^i(G/B, Q \otimes \lambda) = 0 \quad \text{for all} \quad i \in \mathbb{Z}. \tag{3.1}$$

This lemma is a generalization of a lemma by Eric Sommers which is valid when $\mathrm{char}(k) = 0$. If we use his method to prove the lemma when $\mathrm{char}(k) = p > 0$, then we get a lower bound on p. The following proof, due to H. H. Andersen, has no bound on p.

Proof. The Grothendieck Spectral Sequence

$$E_2^{i,j} = H^i(G/P_J, H^j(P_J/B, Q \otimes \lambda))$$

abuts to

$$H^{i+j}(G/B, Q \otimes \lambda)$$

But since Q is a P_J-module, the generalized tensor identity gives

$$H^j(P_J/B, Q \otimes \lambda) = Q \otimes H^j(P_J/B, \lambda) \quad \text{for all} \quad j \in \mathbb{Z}.$$

If we can show that

$$H^j(P_J/B, \lambda) = 0 \quad \text{for all} \quad j \in \mathbb{Z}.$$

then $E_2^{i,j} = 0$ for all i and j, the spectral sequence already collapses at the E_2-level, and (3.1) is satisfied.

Let L_J denote the Levi subgroup of P_J containing T, and let L' be the commutator subgroup (L_J, L_J). Then L' is semi-simple and connected with Borel subgroup $B' = B \cap L'$ and maximal torus $T' = (T \cap L')^0$. By Remark I.6.13 in [Jan87] we have

$$H^j(P_J/B, \lambda)|_{L'} = H^j(L'/B', \lambda|_{B'}) \quad \text{for all} \quad j \in \mathbb{Z}. \tag{3.2}$$

Since $H^j(P_J/B, \lambda)$ equals $H^j(P_J/B, \lambda)|_{L'}$ as vector spaces, it is enough to show that the right hand side of (3.2) vanishes for all $j \in \mathbb{Z}$.

Now J is a simple system of roots in the root system Φ_J of L'. We denote the corresponding fundamental weights

$$\varpi_a, \varpi_{a+1} \dots, \varpi_b \in X^*(T') = X^*(B').$$

But

$$\langle \lambda, \alpha_i^\vee \rangle = 0 \quad \text{for } a < i \leq b$$
$$\langle \lambda, \alpha_a^\vee \rangle = s,$$

and hence $\lambda|_{B'} = s\varpi_a$ in $X^*(B')$. So we need to show

$$H^j(L'/B', s\varpi_a) = 0 \quad \text{for all} \quad j \in \mathbb{Z}.$$

We will show this by using the Strong Linkage Principle. Let W_J be the Weyl group of L' with respect to T', and let $s_i \in W_J$ be the simple reflections corresponding to the simple root α_i for $i = a, \dots, b$. We will use the "dot" action defined in (1.1) in Section 1.1. We want to show that

$$(s_{a-s-1}s_{a-s-2}\cdots s_a)\cdot(s\varpi_a) = -\varpi_{a-s-1}. \tag{3.3}$$

Using that $\alpha_a = 2\varpi_a - \varpi_{a+1}$, we get

$$\begin{aligned} s_a \cdot (s\varpi_a) &= s_a(s\varpi_a) - \alpha_a \\ &= s\varpi_a - (\langle s\varpi_a, \alpha_a^\vee\rangle + 1)\alpha_a \\ &= s\varpi_a - (s+1)\alpha_a \\ &= -(s+2)\varpi_a + (s+1)\varpi_{a+1}. \end{aligned}$$

Using that $\alpha_{a+i-1} = -\varpi_{a+i-2} + 2\varpi_{a+i-1} - \varpi_{a+i}$ for $2 \le i \le b-a$, we get by induction

$$(s_{a+i-1}s_{a+i-2}\cdots s_a)\cdot(s\varpi_a) = -(s+i+1)\varpi_{a+i-1} + (s+i)\varpi_{a+i} \tag{3.4}$$

for $1 \le i \le b-a$. If $s \ne a-b-1$, we can set $i = -s$ in this equation, obtaining

$$(s_{a-s-1}s_{a-s-2}\cdots s_a)\cdot(s\varpi_a) = -\varpi_{a-s-1}.$$

and (3.3) is satisfied if $a - b \le s \le -1$. For $i = b-a$ and $s = a-b-1$ equation (3.4) gives

$$(s_{b-1}s_{b-2}\cdots s_a)\cdot(s\varpi_a) = -\varpi_b = -\varpi_{a-s-1}.$$

But since

$$s_b \cdot (-\varpi_b) = -\varpi_b - (\langle -\varpi_b, \alpha_b^\vee\rangle + 1)\alpha_b = -\varpi_b,$$

(3.3) is valid also if $s = a-b-1$.

By (3.3) we get

$$\begin{aligned} H^j(L'/B', \lambda|_{B'}) &= H^j(L'/B', s\varpi_a) \\ &= H^j(L'/B', (s_{a-s-1}s_{a-s-2}\cdots s_a)^{-1}\cdot(-\varpi_{a-s-1})) \end{aligned}$$

for all $j \in \mathbb{Z}$. But since $\langle \varpi_{a-s-1} + \rho, \alpha_i^\vee\rangle \ge 0$ for $i = a, \dots, b$, and since there exists no dominant weight $\mu \in X^*(T')$ with $\mu \le -\varpi_{a-s-1}$ the latter vanishes for all $j \in \mathbb{Z}$ by the Strong Linkage Principle, Proposition II.6.13 in [Jan87]. □

Identically we can prove a symmetric version of Lemma 3.4:

Let $1 \le a \le b \le l$, let $J = \{\alpha_a, \alpha_{a+1}, \dots, \alpha_b\}$, and let Q be a P_J-module. Let $\lambda \in X^*(T)$. In this symmetric version we let $s = \langle \lambda, \alpha_b^\vee\rangle$! Suppose that

$$\langle \lambda, \alpha_i^\vee\rangle = 0 \quad \text{for } a \le i < b$$

and that $a - b - 1 \le s \le -1$. Then

$$H^i(G/B, Q \otimes \lambda) = 0 \quad \text{for all} \quad i \in \mathbb{Z}.$$

Proof of Proposition 3.3. The proof works in characteristic zero and in characteristic p if p is "big enough". Unfortunately we cannot avoid the bound on p using the Strong Linkage Principle instead of Theorem 3.1 in Step 3 in the following proof. But the bound on p is still better than without using the Strong Linkage Principle in the proof of Lemma 3.4.

In this proof we will write $H^i(-)$ instead of $H^i(G/B,-)$.

Step 1: Let $V = \mathfrak{u}_m \cap \mathfrak{u}_{l+1-m}$, and suppose that $m \leq l+1-m$. Now assume that $\mu \in X^*(T)$ satisfies $\langle \mu, \alpha_t^\vee \rangle = 0$ for $m < t \leq l$.

We wish to show that

$$H^i(S^n\mathfrak{u}_m^* \otimes \mu) = H^i(S^nV^* \otimes \mu) \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

This is obvious if $m = l+1-m$, since in this case $V = \mathfrak{u}_\mathfrak{m}$. We will therefore assume that $m < l+1-m$.

There exists a short exact sequence (U is the kernel)

$$0 \to U \to \mathfrak{u}_m^* \to V^* \to 0. \tag{3.5}$$

Taking the Koszul resolution of the short exact sequence, and tensoring with μ we get the exact sequence

$$0 \to \ldots \to S^{n-j}\mathfrak{u}_m^* \otimes \wedge^j U \otimes \mu \to \ldots \to S^n\mathfrak{u}_m^* \otimes \mu \to S^nV^* \otimes \mu \to 0. \tag{3.6}$$

The weights of U (T-weights) are the the weights of $\mathfrak{u}_m^*$, which are not weights of V^*. The set of weights of $\mathfrak{u}_m^*$ is $\Phi^+ \setminus \Phi_m$, and the set of weights of V^* is $\Phi^+ \setminus (\Phi_m \cup \Phi_{l+1-m})$. Thus the set of weights of U is

$$\begin{aligned}
&\Phi^+ \cap (\Phi_{l+1-m} \setminus \Phi_m) \\
&= \{\alpha \in \Phi^+ \cap \Phi_{l+1-m} | \alpha \text{ has a nonzero coefficient to } \alpha_m\} \\
&= \{\alpha \in \Phi^+ \cap \Phi_I | \alpha \text{ has a nonzero coefficient to } \alpha_m, \\
&\qquad\qquad \text{and the coefficient to } \alpha_{l+1-m} \text{ is zero}\} \\
&= \{\alpha_i + \alpha_{i+1} + \ldots + \alpha_k | 1 \leq i \leq m, m \leq k < l+1-m\}.
\end{aligned}$$

Here we have used the fact that if $\alpha \in \Phi_s$, it may be written as

$$\alpha = \sum_{i=1}^{l} c_i\alpha_i + \sum_{j=1}^{u} d_j\gamma_{k_j}, \quad \text{where } c_s = 0.$$

Since $\langle \gamma_{k_j}, \alpha_i^\vee \rangle = 0$ for all i, j, it follows that either all c_i's are zero or all d_i's are zero. Hence, if $\alpha \in \Phi_{l+1-m}$ has a nonzero coefficient to α_m, then all d_i's are zero and $\alpha \in \Phi_I$.

Thus, if η is a weight of $\wedge^j U$, there exists t_0 with $m < t_0 \leq l+1-m$, $-m \leq \langle \eta, \alpha_{t_0}^\vee \rangle \leq -1$ and $\langle \eta, \alpha_t^\vee \rangle = 0$ for $t_0 < t \leq l$.

We can use Lemma 3.4 with $Q = S^{n-j}\mathfrak{u}_m^* \otimes \mu$, $a = t_0$, $b = l$ and $s = \langle \eta, \alpha_{t_0}^\vee \rangle$ because

$$a - b - 1 = t_0 - l - 1 \leq -m \leq \langle \eta, \alpha_{t_0}^\vee \rangle \leq -1,$$

and because $\mathfrak{u}_m^*$ is a P_{α_t}-module for $t \neq m$, and μ a P_{α_t}-module for $m < t \leq l$. Hence

$$H^i(S^{n-j}\mathfrak{u}_m^* \otimes \mu \otimes \eta) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

We can filter $\wedge^j U$ by B-submodules such that the consecutive quotients are one dimensional with weights equal to the weights of $\wedge^j U$, and we get for all $1 \leq j \leq \dim(U)$

$$H^i(S^{n-j}\mathfrak{u}_m^* \otimes \mu \otimes \wedge^j U) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z},$$

Splitting the Koszul resolution in (3.6) into short exact sequences and taking long exact sequences in cohomology, we see that

$$H^i(S^n\mathfrak{u}_m^* \otimes \mu) = H^i(S^n V^* \otimes \mu) \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

Identically the proof may be carried out if $m \geq l+1-m$ and $\mu \in X^*(T)$ satisfy $\langle \mu, \alpha_t^\vee \rangle = 0$ for $1 \leq t < m$.

Thus, if $\mu \in X^*(T)$ satisfy $\langle \mu, \alpha_t^\vee \rangle = 0$ for $t \neq m$, we have

$$H^i(S^n\mathfrak{u}_m^* \otimes \mu) = H^i(S^n V^* \otimes \mu) \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

Step 2: From now on we will assume that $m \leq l+1-m$. Let $V_1 = V \cap \mathfrak{u}_{m-1}$, and $V_2 = V_1 \cap \mathfrak{u}_{l+2-m}$. If $m = 1$ we consider $\mathfrak{u}_{m-1}$ and $\mathfrak{u}_{l+2-m}$ to be the zero vector space. Let $\mu \in X^*(T)$, and assume that μ satisfy

$$\begin{array}{ll} \langle \mu, \alpha_i^\vee \rangle = 0 & \text{for } i = m+1, \dots, l-m, l+2-m, \dots, l, \\ r' = \langle \mu, \alpha_m^\vee \rangle & \text{with } 2m-2-l \leq r' \leq -1, \\ \langle \mu, \alpha_{l+1-m}^\vee \rangle = 0 & \text{if } r' = 2m-2-l. \end{array}$$

We wish to show that

$$H^i(S^n V_1^* \otimes \mu) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

Since $V_2 \subseteq V_1$, we have a short exact sequence (U_2 is the kernel)

$$0 \to U_2 \to V_1^* \to V_2^* \to 0$$

Taking the Koszul resolution of this short exact sequence and tensoring with μ, we get an exact sequence

$$0 \to \ldots \to S^{n-j}V_1^* \otimes \wedge^j U_2 \otimes \mu \to \ldots \to S^n V_1^* \otimes \mu \to S^n V_2^* \otimes \mu \to 0.$$

Thus it is enough to show that

$$H^i(S^{n-j}V_1^* \otimes \wedge^j U_2 \otimes \mu) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}$$
$$H^i(S^n V_2^* \otimes \mu) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

Actually it holds that

$$V_2 = \mathfrak{u}_{m-1} \cap \mathfrak{u}_m \cap \mathfrak{u}_{l+1-m} \cap \mathfrak{u}_{l+2-m} = \mathfrak{u}_{m-1} \cap \mathfrak{u}_{l+2-m}.$$

This can be seen by comparing the weights:

$$(\Phi^- \setminus \Phi_I) \cup \{\alpha \in \Phi^- \cap \Phi_I | \alpha \text{ has nonzero coefficients to } \alpha_{m-1}, \alpha_m, \alpha_{l+1-m} \text{ and } \alpha_{l+2-m}\}$$

is the set of weights of V_2. But this set equals the set

$$(\Phi^- \setminus \Phi_I) \cup \{\alpha \in \Phi^- \cap \Phi_I | \alpha \text{ has nonzero coefficients to } \alpha_{m-1} \text{ and } \alpha_{l+2-m}\},$$

which is exactly the set of weights of $\mathfrak{u}_{m-1} \cap \mathfrak{u}_{l+2-m}$. Thus $V_2 = \mathfrak{u}_{m-1} \cap \mathfrak{u}_{l+2-m}$, and V_2^* is a P_{α_t}-module for $t \neq m-1, l+2-m$.

If $r' \neq 2m-l-2$ we can use Lemma 3.4 with $Q = S^n V_2^*$, $a = m$, $b = l-m$ and $s = \langle \mu, \alpha_m^\vee \rangle = r'$ because $a-b-1 = 2m-1-l \leq r' \leq -1$, and because V_2^* is a P_{α_t}-module for $m \leq t \leq l-m$. Thus

$$H^i(S^n V_2^* \otimes \mu) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

If $r' = 2m-l-2$ we can also use Lemma 3.4, but this time with $Q = S^n V_2^*$, $a = m$, $b = l+1-m$ and $s = \langle \mu, \alpha_m^\vee \rangle = r'$ because $a-b-1 = 2m-2-l \leq r' \leq -1$, and because V_2^* is a P_{α_t}-module for $m \leq t \leq l+1-m$. Again we have

$$H^i(S^n V_2^* \otimes \mu) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

The set of weights of V_1^* is

$$\Phi^+ \setminus (\Phi_{m-1} \cup \Phi_m \cup \Phi_{l+1-m}),$$

and

$$\Phi^+ \setminus (\Phi_{m-1} \cup \Phi_m \cup \Phi_{l+1-m} \cup \Phi_{l+2-m})$$

is the set of weights of V_2^*. Thus the set of weights of U_2 is

$$\begin{aligned}
&\Phi^+ \cap (\Phi_{l+2-m} \setminus (\Phi_{m-1} \cup \Phi_m \cup \Phi_{l+1-m}))\\
&= \{\alpha \in \Phi^+ \cap \Phi_{l+2-m} | \alpha \text{ has a nonzero coefficient to } \alpha_{m-1}, \alpha_m \text{ and } \alpha_{l+1-m}\}\\
&= \{\alpha \in \Phi^+ \cap \Phi_I | \alpha \text{ has a nonzero coefficient to } \alpha_{m-1}, \alpha_m \text{ and } \alpha_{l+1-m},\\
&\qquad \text{and the coefficient to } \alpha_{l+2-m} \text{ is zero}\}\\
&= \{\alpha_i + \alpha_{i+1} + \ldots + \alpha_{l+1-m} | 1 \leq i \leq m-1\}.
\end{aligned}$$

Hence, if η is a weight of $\wedge^j U_2$, we have

$$\begin{aligned}
&\langle \eta, \alpha_{l+2-m}^\vee \rangle = -j,\\
&\langle \eta, \alpha_t^\vee \rangle = 0 \text{ for } l+2-m \leq t \leq l.
\end{aligned}$$

Remark that $\dim(U_2) = m-1$. Now we use Lemma 3.4 again. This time with $Q = S^{n-j} V_1^* \otimes \mu$, $a = l+2-m$, $b = l$ and $s = \langle \eta, \alpha_{l+2-m}^\vee \rangle = -j$. V_1^* is a P_{α_t}-module for $t \neq m-1, m, l+1-m$ [1], and μ is a P_{α_t}-module for $t =$

[1] Actually V_1^* is a P_{α_t}-module for $t \neq m-1, l+1-m$ since $V_1^* = \mathfrak{u}_{m-1} \cap \mathfrak{u}_{l+1-m}$, compare with the proof of $V_2^* = \mathfrak{u}_{m-1} \cap \mathfrak{u}_{l+2-m}$.

$m+1,\dots,l-m,l+2-m,\dots,l$. Thus Q is a P_{α_t}-module for $l+2-m \le t \le l$. We also have

$$a-b-1 = 1-m \le -j \le -1$$

because $1 \le j \le \dim(U_2) = m-1$. Therefore

$$H^i(S^{n-j}V_1^* \otimes \mu \otimes \eta) = 0 \quad \text{for all} \quad i,n \in \mathbb{Z}$$

for all weights η of $\wedge^j U_2$. But filtering $\wedge^j U_2$ (as in Step 1) gives us

$$H^i(S^{n-j}V_1^* \otimes \wedge^j U_2 \otimes \mu) = 0 \quad \text{for all} \quad i,n \in \mathbb{Z}.$$

Step 3: Suppose $\mu \in X^*(T)$ satisfy

$$\begin{array}{ll} \langle \mu, \alpha_i^\vee \rangle = 0 & \text{for } i \neq m, l+1-m, \\ r' = \langle \mu, \alpha_m^\vee \rangle & \text{with } 2m-2-l \le r' \le -1, \\ \langle \mu, \alpha_{l+1-m}^\vee \rangle = 0 & \text{if } r' = 2m-2-l. \end{array} \tag{3.7}$$

We want to show that

$$H^i(S^nV^* \otimes \mu) = H^i(S^{n-m}V^* \otimes \mu + \nu_0) \quad \text{for all} \quad i,n \in \mathbb{Z},$$

where

$$\begin{aligned} \nu_0 = \alpha_1 + 2\alpha_2 + \ldots + (m-1)\alpha_{m-1} + m(\alpha_m + \alpha_{m+1} + \ldots + \alpha_{l+1-m}) \\ + (m-1)\alpha_{l+2-m} + \ldots + 2\alpha_{l-1} + \alpha_l. \end{aligned} \tag{3.8}$$

Since $V_1 \subseteq V$, we get a short exact sequence (U_1 is the kernel)

$$0 \to U_1 \to V^* \to V_1^* \to 0$$

Taking the corresponding Koszul resolution and tensoring with μ, we get an exact sequence

$$0 \to \ldots \to S^{n-j}V^* \otimes \wedge^j U_1 \otimes \mu \to \ldots \to S^nV^* \otimes \mu \to S^nV_1^* \otimes \mu \to 0 \tag{3.9}$$

The set of weights of V^* is

$$\Phi^+ \setminus (\Phi_m \cup \Phi_{l+1-m}),$$

and the set of weights of V_1^* is

$$\Phi^+ \setminus (\Phi_{m-1} \cup \Phi_m \cup \Phi_{l+1-m}).$$

Thus the set of weights of U_1 is

$$\begin{aligned} &\Phi^+ \cap (\Phi_{m-1} \setminus (\Phi_m \cup \Phi_{l+1-m})) \\ &= \{\alpha \in \Phi^+ \cap \Phi_{m-1} | \alpha \text{ has a nonzero coefficient to } \alpha_m \text{ and } \alpha_{l+1-m}\} \\ &= \{\alpha \in \Phi^+ \cap \Phi_I | \alpha \text{ has a nonzero coefficient to } \alpha_m \text{ and } \alpha_{l+1-m}, \\ &\qquad\qquad \text{and the coefficient to } \alpha_{m-1} \text{ is zero}\} \\ &= \{\alpha_m + \alpha_{m+1} + \ldots + \alpha_i | l+1-m \le i \le l\}. \end{aligned}$$

Thus $\dim(U_1) = m$, and if η is a weight of $\wedge^j U_1$, it satisfy

$$\langle \eta, \alpha_{m-1}^\vee \rangle = -j,$$
$$\langle \eta, \alpha_t^\vee \rangle = 0 \text{ for } 1 \leq t \leq m-1.$$

If $j \leq m-1$ we can use the symmetric version of Lemma 3.4 with $Q = S^{n-j}V^* \otimes \mu$, $a = 1$, $b = m-1$ and $s = \langle \eta, \alpha_{m-1}^\vee \rangle = -j$. Now Q is a P_{α_t}-module for $1 \leq t \leq m-1$ because V^* and μ are P_{α_t}-modules for $t \neq m, l+1-m$, and

$$a - b - 1 = 1 - m \leq -j \leq 1.$$

Thus we get

$$H^i(S^{n-j}V^* \otimes \mu \otimes \eta) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

But filtering U_1 (as in Step 1), we get

$$H^i(S^{n-j}V^* \otimes \wedge^j U_1 \otimes \mu) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}, \tag{3.10}$$

when $j \leq m-1$.

We now concentrate on the case $j = m$. Define

$$\eta = m(\alpha_m + \alpha_{m+1} \ldots + \alpha_{l+1-m}) + (m-1)\alpha_{l+2-m} + \ldots + 2\alpha_{l-1} + \alpha_l.$$

Then $\wedge^m U_1 = \eta$. We wish to use Theorem 3.1 $m-1$ times. Still, V^* and μ are P_{α_t}-modules for $t \neq m, l+1-m$, especially for $1 \leq t \leq m-1$. Now $\langle \eta, \alpha_{m-1}^\vee \rangle = -m$ and the theorem gives

$$H^i(S^{n-m}V^* \otimes \mu \otimes \wedge^m U_1) = H^{i-1}(S^{n-m}V^* \otimes \mu \otimes (m-1)\alpha_{m-1} + \eta)$$

for all $i, n \in \mathbb{Z}$. In characteristic p we will need $m - 1 \leq p$. Again $\langle (m-1)\alpha_{m-1} + \eta, \alpha_{m-2}^\vee \rangle = -m + 1$, so the latter cohomology group equals

$$H^{i-1}(S^{n-m}V^* \otimes \mu \otimes (m-2)\alpha_{m-2} + (m-1)\alpha_{m-1} + \eta)$$

by Theorem 3.1 (here $m - 2 \leq p$). After $m - 1$ times we see that

$$\begin{aligned} &H^i(S^{n-m}V^* \otimes \mu \otimes \wedge^m U_1) \\ &\quad = H^{i-(m-1)}(S^{n-m}V^* \otimes \mu \otimes \alpha_1 + \alpha_2 + \ldots (m-1)\alpha_{m-1} + \eta) \\ &\quad = H^{i-m+1}(S^{n-m}V^* \otimes \mu \otimes \nu_0) \end{aligned} \tag{3.11}$$

for all $i, n \in \mathbb{Z}$ where we must have $m-3, m-4, \ldots, 1 \leq p$. So all in all we need

$$m - 1 \leq p \tag{3.12}$$

to use Theorem 3.1. We have

$$H^i(S^n V_1^* \otimes \mu) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z} \tag{3.13}$$

using Step 2.

Splitting the exact sequence in (3.9) into short exact sequences and using (3.10) and (3.13) we see that

$$H^i(S^nV^*\otimes\mu) = H^{i+m-1}(S^{n-m}V^*\otimes\wedge^m U_1\otimes\mu) \quad \text{for all} \quad i,n\in\mathbb{Z},$$

which by (3.11) equals

$$H^i(S^{n-m}V^*\otimes\mu\otimes\nu_0) \quad \text{for all} \quad i,n\in\mathbb{Z}.$$

Step 4: We want to show that

$$H^i(S^n\mathfrak{u}_m^*\otimes\lambda) = H^i(S^{n+rm}\mathfrak{u}_{l+1-m}^*\otimes\lambda - r\nu_0) \quad \text{for all} \quad i,n\in\mathbb{Z}.$$

We will use Step 3 inductively $-r$ times. Since $\lambda + s\nu_0$ satisfies (3.7) for $1\leq s\leq -r-1$, we have

$$\begin{aligned}
H^i(S^nV^*\otimes\lambda) &= H^i(S^{n-m}V^*\otimes\lambda+\nu_0)\\
&= H^i(S^{n-2m}V^*\otimes\lambda+2\nu_0)\\
&= \dots\\
&= H^i(S^{n-(-r)m}V^*\otimes\lambda+(-r-1+1)\nu_0)
\end{aligned}$$

for all $i,n\in\mathbb{Z}$. But by Step 1 we have

$$\begin{aligned}
H^i(S^n\mathfrak{u}_m\otimes\lambda) &= H^i(S^nV^*\otimes\lambda)\\
&= H^i(S^{n+rm}V^*\otimes\lambda-r\nu_0)\\
&= H^i(S^{n+rm}\mathfrak{u}_{l+1-m}\otimes\lambda-r\nu_0)
\end{aligned}$$

for all $i,n\in\mathbb{Z}$.

Step 5: All that is left to show is that $w_0(\lambda)=\lambda-r\nu_0$.

We know that w_0 is a product of simple reflections corresponding to the simple roots $\alpha_1,\alpha_2,\dots,\alpha_l$. Therefore $\lambda-w_0(\lambda)$ is a linear combination of $\alpha_1,\alpha_2,\dots,\alpha_l$. But $r\nu_0$ is also a linear combination of $\alpha_1,\alpha_2,\dots,\alpha_l$. Hence it is enough to show that

$$\langle\lambda-w_0(\lambda),\alpha_i^\vee\rangle = \langle r\nu_0,\alpha_i^\vee\rangle, \quad i=1,2,\dots,l.$$

From Planche I, p. 250 [Bou81] we know that $w_0(\alpha_i)=-\alpha_{l+1-i}$. Remembering

$$\langle\lambda,\alpha_i^\vee\rangle = \begin{cases} r & \text{if } i=m\\ 0 & \text{otherwise}\end{cases}$$

we therefore get

$$\langle\lambda-w_0(\lambda),\alpha_i^\vee\rangle = \langle\lambda,\alpha_i^\vee\rangle+\langle\lambda,\alpha_{l+1-i}^\vee\rangle = \begin{cases} r & \text{if } i=m,l+1-m\\ 0 & \text{otherwise}\end{cases}$$

But

$$\langle r\nu_0,\alpha_i^\vee\rangle = \begin{cases} r & \text{if } i=m,l+1-m\\ 0 & \text{otherwise}\end{cases}$$

according to (3.8). Hence $w_0(\lambda) = \lambda - r\nu_0$.

Step 6: In characteristic p, the limit for p is found in equation (3.12), so we need $m - 1 \leq p$.

All what we have done in Step 2 - 5 may also be done with the assumption $m \geq l + 1 - m$, so Proposition 3.3 is correct for all m. In general we need $m' - 1 \leq p$. □

3.2 A new vanishing method

Whenever Eric Sommers proves that a nilpotent orbit has normal closure, he applies the following proposition in the calculations. The proposition is a small extension of a vanishing theorem by Broer in [Bro94].

Proposition 3.5 (Proposition 4 in [Som03])**.** Suppose $\operatorname{char}(k) = 0$. Let P be a parabolic subgroup containing B, and let $\lambda \in X^*(P)$ be a dominant P-character. Then

1. For all $i > 0, n \in \mathbb{Z}$ we have

$$H^i(G/P, S^n\mathfrak{u}_P^* \otimes \lambda) = H^i(G/B, S^n\mathfrak{u}_P^* \otimes \lambda) = 0$$

2. Assume P is the parabolic subgroup with Lie algebra $\oplus_{i\geq 0}\mathfrak{g}_i$ where $\mathfrak{g}_i$ is the i-eigenspace for the semi-simple element of an $\mathfrak{sl}_2$-triple normalized so that P contains B. Let $V = \oplus_{i\geq 2}\mathfrak{g}_i$, and let $w = \wedge^{\dim(\mathfrak{g}_1)}\mathfrak{g}_1$. Then

$$H^i(G/P, S^nV^* \otimes w \otimes \lambda) = H^i(G/B, S^nV^* \otimes w \otimes \lambda) = 0$$

   for all $i > 0, n \in \mathbb{Z}$.

Broer's result relies on the Grauert-Riemenschneider vanishing theorem which is only valid in characteristic zero, hence this proposition is unfortunately only valid in characteristic zero. However we have found another method to obtain vanishing cohomology groups. There are two important ingredients in this method. The first is a vanishing theorem by Broer in characteristic zero, Theorem 3.9.(iii) in [Bro94], which was generalized to characteristic $p > 0$ by Thomsen in Theorem 1 in [Tho00] with a lower bound on p. Now H. H. Andersen has given another version of this theorem without any bound on p, this is the following Theorem 3.11. The second ingredient in the new method to obtain vanishing cohomology groups is Example 3.15 where we combine this theorem with Koszul resolutions.

Note that Broer's/Thomsen's vanishing theorem would work in the actual calculations in Chapter 4. However, in prime characteristic $p > 0$ we would need to require that $p \geq 7$ in the proof of normality of the closures of some of the smallest nilpotent orbits. But 5 is a good prime when G is of type E_6, and hence we need to use Andersen's result.

In Theorem 3.11 we need a function $X^*(T) \to \mathbb{N}$ satisfying some conditions to make the proof work. This is the motivation for the following definition.

Definition 3.6. Let $d : X^*(T) \to \mathbb{N}$ be a function satisfying

1. $d(\lambda) = 0$ if $\lambda \in X^*(T)$ is dominant.

If $\lambda \in X^*(T)$ is not dominant, there exists a simple root α such that either 2 or 3 is satisfied

2. $\langle \lambda, \alpha^\vee \rangle = -1$ and $d(s_\alpha(\lambda)) \leq d(\lambda)$.
3. $\langle \lambda, \alpha^\vee \rangle < -1$ with $d(\lambda + i\alpha) < d(\lambda)$ for all $i = 1, 2, \ldots, -\langle \lambda, \alpha^\vee \rangle - 1$ and $d(s_\alpha(\lambda)) \leq d(\lambda)$.

Then we call d a *vanishing function* since it can be used in the vanishing theorem, Theorem 3.11.

First we give an example of a vanishing function. In order to do this we need some notation. Let $\leq$ denote the partial order on $X^*(T)$ defined by $\mu \leq \lambda$ if $\lambda - \mu$ is a sum of positive roots. For all $\lambda \in X^*(T)$ there exists a unique dominant weight $\lambda^+ \in X^*(T)$ in the W-orbit of λ. We know that $\lambda \leq \lambda^+$.

For each weight $\lambda \in X^*(T)$ there exist a dominant weight $\lambda^\star \in X^*(T)$ with $\lambda \leq \lambda^\star$ which is minimal in the sense that if $\mu \in X^*(T)$ is dominant with $\lambda \leq \mu$, then $\lambda^\star \leq \mu$, see e.g. Proposition 2 in [Tho00].

Now we can define the Chevalley height of $\lambda \in X^*(T)$ to be the largest integer r such that there exist dominant weights $\mu_0, \mu_1, \ldots, \mu_r \in X^*(T)$ with

$$\lambda^\star = \mu_0 < \mu_1 < \ldots < \mu_{r-1} < \mu_r = \lambda^+.$$

We will let $\mathrm{Cht}(\lambda)$ denote the Chevalley height of λ, and clearly $\mathrm{Cht}(\lambda) \in \mathbb{N}$.

The following proposition can be found in [Tho00].

Proposition 3.7. Let $\lambda \in X^*(T)$ be a character, and let $\alpha \in \Pi$ be a simple root.

1. If $\langle \lambda, \alpha^\vee \rangle = -1$ then $\mathrm{Cht}(\lambda) = \mathrm{Cht}(\lambda + \alpha)$.
2. If $\langle \lambda, \alpha^\vee \rangle \leq -2$ then $\mathrm{Cht}(\lambda) > \mathrm{Cht}(\lambda + \alpha)$.
3. If $\langle \lambda, \alpha^\vee \rangle \leq 0$ then $\mathrm{Cht}(\lambda) \geq \mathrm{Cht}(s_\alpha(\lambda))$.
4. If $\langle \lambda, \alpha^\vee \rangle \leq -2$ then $\mathrm{Cht}(\lambda) > \mathrm{Cht}(s_\alpha(\lambda) - \alpha)$.

Corollary 3.8. The function $\mathrm{Cht} : X^*(T) \to \mathbb{N}$ is a vanishing function.

Proof. If $\lambda \in X^*(T)$ is dominant, then clearly $\mathrm{Cht}(\lambda) = 0$, and condition 1 in Definition 3.6 is satisfied. If λ is not dominant, there exists a simple root α with $\langle \lambda, \alpha^\vee \rangle \leq -1$. If $\langle \lambda, \alpha^\vee \rangle = -1$, then $\mathrm{Cht}(\lambda) = \mathrm{Cht}(\lambda + \alpha)$ and condition 2 in Definition 3.6 is satisfied since in this case $s_\alpha(\lambda) = \lambda + \alpha$. If $\langle \lambda, \alpha^\vee \rangle < -1$, then

$$\langle \lambda + i\alpha, \alpha^\vee \rangle \leq -2 \quad \text{for} \quad 0 \leq i \leq \tfrac{1}{2}(-\langle \lambda, \alpha^\vee \rangle - 2)$$

and the proposition above gives that

$$\begin{aligned} &\mathrm{Cht}(\lambda + i\alpha) > \mathrm{Cht}(\lambda + (i+1)\alpha) \\ &\mathrm{Cht}(\lambda + i\alpha) > \mathrm{Cht}(s_\alpha(\lambda + i\alpha) - \alpha) = \mathrm{Cht}(\lambda - (\langle \lambda, \alpha^\vee \rangle + i + 1)\alpha) \end{aligned}$$

for $0 \leq i \leq \frac{1}{2}(-\langle \lambda, \alpha^\vee \rangle - 2)$. This implies that

$$\mathrm{Cht}(\lambda) > \mathrm{Cht}(\lambda + j\alpha) \quad \text{for} \quad j = 1, 2, \ldots, -\langle \lambda, \alpha^\vee \rangle - 1.$$

But also $\mathrm{Cht}(\lambda) \geq \mathrm{Cht}(s_\alpha(\lambda))$ by the above proposition, and hence condition 3 in Definition 3.6 is satisfied. □

Later we will actually define a minimal vanishing function. For $\lambda \in X^*(T)$ we define $l(\lambda)$ to be the number of elements in

$$\{\beta \in \Phi^+ | \langle \lambda, \beta^\vee \rangle < 0\}$$

We call $l(\lambda)$ the length of λ.

Corollary 3.9. Let $\lambda \in X^*(T)$. If $\langle \lambda, \alpha \rangle < 0$ for some simple root α, then

$$l(s_\alpha(\lambda)) = l(\lambda) - 1.$$

Proof. Let $\beta \in \Phi^+$ be a positive root. Then $\langle s_\alpha(\lambda), \beta^\vee \rangle = \langle \lambda, (s_\alpha(\beta))^\vee \rangle$. But since s_α permutes all the positive roots except α, and since $s_\alpha(\alpha) = -\alpha$, the result follows. □

Remember that ρ denotes half the sum of the positive roots, i.e. $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. Also remember that $\langle \rho, \alpha^\vee \rangle = 1$ for all simple roots $\alpha \in \Pi$.

Corollary 3.10. Let $d : X^*(T) \to \mathbb{N}$ be a vanishing function. If $d(\lambda) = 0$, then $\lambda + \rho$ is dominant.

Proof. Assume that $d(\lambda) = 0$. We will show the corollary by induction in $l(\lambda)$. If $l(\lambda) = 0$, then λ is dominant and so is $\lambda + \rho$. If $l(\lambda) > 0$, then since $d(\lambda) = 0$, either λ is dominant (and the result follows), or there exists a simple root α such that condition 2 in Definition 3.6 is satisfied, i.e.

$$\langle \lambda, \alpha^\vee \rangle = -1 \quad \text{and} \quad d(s_\alpha(\lambda)) \leq d(\lambda) = 0.$$

But $l(s_\alpha(\lambda)) = l(\lambda) - 1$ so by induction $s_\alpha(\lambda) + \rho = \lambda + \alpha + \rho$ is dominant, i.e. $\langle \lambda + \rho + \alpha, \beta^\vee \rangle \geq 0$ for all simple roots β. But for $\beta \neq \alpha$ we have $\langle \alpha, \beta^\vee \rangle \leq 0$ and hence

$$\langle \lambda + \rho, \beta^\vee \rangle \geq \langle \lambda + \rho + \alpha, \beta^\vee \rangle \geq 0.$$

But $\langle \lambda, \alpha^\vee \rangle = -1$, and hence $\langle \lambda + \rho, \alpha^\vee \rangle = 0$, and we see that $\lambda + \rho$ is dominant. □

Now we are ready to state the vanishing theorem by H. H. Andersen.

Theorem 3.11. Let $d : X^*(T) \to \mathbb{N}$ be a vanishing function. Let $\lambda \in X^*(T)$. Then if the characteristic of k is good for G we have

$$H^j(G/B, S^n \mathfrak{u}^* \otimes \lambda) = 0 \quad \text{for all} \quad j > d(\lambda), n \in \mathbb{Z}.$$

Proof. We will proof the theorem by induction first on n and then on $d(\lambda)$. Suppose $n \leq 0$.

Assume that $d(\lambda) = 0$. Then we are in condition 1 or 2 of Definition 3.6, and either λ is dominant or there exists a simple root α with $\langle \lambda, \alpha^\vee \rangle = -1$ (and $d(s_\alpha(\lambda)) \leq d(\lambda)$). If λ is dominant, Kempf's vanishing theorem (Proposition II.4.5 in [Jan87]) gives us

$$H^j(G/B, \lambda) = 0 \quad \text{for all} \quad j > 0,$$

and if $\langle \lambda, \alpha^\vee \rangle = -1$, Theorem 3.1 gives us

$$H^j(G/B, \lambda) = 0 \quad \text{for all} \quad j \in \mathbb{Z},$$

and hence the theorem holds in this case.

If $d(\lambda) > 0$ there exists a simple root α satisfying condition 2 or 3 in Definition 3.6. If α satisfies condition 2, we have $\langle \lambda, \alpha^\vee \rangle = -1$ and again the theorem holds by Theorem 3.1.

If α is satisfying condition 3 we do the following. Let P_α denote the minimal parabolic subgroup containing B corresponding to $\{\alpha\}$. Then since $\langle \lambda, \alpha^\vee \rangle < -1$ we have by Proposition II.5.4 in [Jan87]

$$\begin{aligned} H^j(G/B, \lambda) &= H^{j-1}(G/P_\alpha, H^1(P_\alpha/B, \lambda)) \\ &= H^{j-1}(G/B, H^1(P_\alpha/B, \lambda)) \end{aligned}$$

for all $j \in \mathbb{N}$. Now by Proposition II.5.2 in [Jan87] the weights of $H^1(P_\alpha/B, \lambda)$ are

$$\lambda + \alpha, \lambda + 2\alpha, \ldots, \lambda + (-\langle \lambda, \alpha^\vee \rangle - 1)\alpha. \tag{3.14}$$

We can filter $H^1(P_\alpha/B, \lambda)$ with B-subrepresentations such that the quotients are one dimensional with the same weights. For any such weight μ we have $d(\mu) < d(\lambda)$ since α satisfies condition 3 in Definition 3.6. So by our second induction we have

$$H^{j-1}(G/B, \mu) = 0 \quad \text{for all} \quad j - 1 > d(\mu),$$

and hence for all $j > d(\lambda)$. So for all quotients Q in the filtration of $H^1(P_\alpha/B, \lambda)$ we have

$$H^{j-1}(G/B, Q) = 0 \quad \text{for all} \quad j > d(\lambda).$$

and hence

$$H^j(G/B, \lambda) = H^{j-1}(G/B, H^1(P_\alpha/B, \lambda)) = 0 \quad \text{for all} \quad j > d(\lambda).$$

Now suppose that $n > 0$, and assume that the theorem holds for $n - 1$. If $\lambda \in X^*(T)$ is dominant, the theorem holds by Theorem 2 in [KLT99] since the characteristic is good for G. If λ is not dominant, there exists a simple root α satisfying condition 2 or 3 in Definition 3.6. Now let $\mathfrak{u}_{P_\alpha}$ denote the Lie algebra of the unipotent radical of P_α. This is a P_α-module, and as a B-module $\mathfrak{u}/\mathfrak{u}_{P_\alpha}$ is just the one dimensional B-module with weight $-\alpha$. Remember that we denote this B-module just by $-\alpha$. With this notation we have a short exact sequence of B-modules

$$0 \to \mathfrak{u}_{P_\alpha} \to \mathfrak{u} \to -\alpha \to 0$$

The Koszul resolution corresponding to the dual sequence is the short exact sequence

$$0 \to S^{n-1}\mathfrak{u}^* \otimes \alpha \to S^n\mathfrak{u}^* \to S^n\mathfrak{u}^*_{P_\alpha} \to 0$$

Tensoring with the one dimensional B-module with weight λ we get the short exact sequence of B-modules

$$0 \to S^{n-1}\mathfrak{u}^* \otimes (\lambda + \alpha) \to S^n\mathfrak{u}^* \otimes \lambda \to S^n\mathfrak{u}^*_{P_\alpha} \otimes \lambda \to 0$$

which induces this long exact sequence in cohomology

$$\begin{aligned}\dots,\to H^j(G/B,&S^{n-1}\mathfrak{u}^*\otimes(\lambda+\alpha))\\&\to H^j(G/B,S^n\mathfrak{u}^*\otimes\lambda)\to H^j(G/B,S^n\mathfrak{u}^*_{P_\alpha}\otimes\lambda)\to\dots\end{aligned}\tag{3.15}$$

If α satisfies condition 2 in Definition 3.6 we have $\langle\lambda,\alpha^\vee\rangle=-1$ and

$$H^j(G/B,S^n\mathfrak{u}^*_{P_\alpha}\otimes\lambda)=0\quad\text{for all}\quad j\in\mathbb{Z}$$

by Theorem 3.1, and by the long exact sequence above we get

$$H^j(G/B,S^n\mathfrak{u}^*\otimes\lambda)=H^j(G/B,S^{n-1}\mathfrak{u}^*\otimes(\lambda+\alpha))\quad\text{for all}\quad j\in\mathbb{Z}.$$

By induction on n we know that the latter vanishes for $j>d(\lambda+\alpha)$. But since α satisfies condition 2 in Definition 3.6, we know that

$$d(\lambda+\alpha)=d(s_\alpha(\lambda))\le d(\lambda).$$

In particular

$$H^j(G/B,S^n\mathfrak{u}^*\otimes\lambda)=H^j(G/B,S^{n-1}\mathfrak{u}^*\otimes(\lambda+\alpha))=0\quad\text{for all}\quad j>d(\lambda),$$

and the theorem holds in this case.

Now assume that α instead satisfies condition 3 in Definition 3.6. Then we have $d(\lambda+\alpha)<d(\lambda)$, and by induction

$$H^j(G/B,S^{n-1}\mathfrak{u}^*\otimes(\lambda+\alpha))=0\quad\text{for all}\quad j>d(\lambda+\alpha),$$

in particular it holds for $j>d(\lambda)$. If we can show that

$$H^j(G/B,S^n\mathfrak{u}^*_{P_\alpha}\otimes\lambda)=0\quad\text{for all}\quad j>d(\lambda)\tag{3.16}$$

we are done by (3.15). Now since $\langle\lambda,\alpha^\vee\rangle<-1$ we obtain by Proposition II.5.4 in [Jan87]

$$\begin{aligned}H^j(G/B,S^n\mathfrak{u}^*_{P_\alpha}\otimes\lambda)&=H^{j-1}(G/P_\alpha,S^n\mathfrak{u}^*_{P_\alpha}\otimes H^1(P_\alpha/B,\lambda))\\&=H^{j-1}(G/B,S^n\mathfrak{u}^*_{P_\alpha}\otimes H^1(P_\alpha/B,\lambda))\end{aligned}\tag{3.17}$$

for all $j\in\mathbb{Z}$. Let μ be a weight of $H^1(P_\alpha/B,\lambda)$, i.e. μ is one of the weights in (3.14). Then look at the long exact sequence in (3.15) with μ instead of λ. In order to show (3.16), it is by (3.17) enough to show that

$$H^j(G/B,S^{n-1}\mathfrak{u}\otimes(\mu+\alpha))=0\qquad\text{for all}\quad j>d(\lambda)\tag{3.18}$$

$$H^{j-1}(G/B,S^n\mathfrak{u}^*\otimes\mu)=0\qquad\text{for all}\quad j>d(\lambda).\tag{3.19}$$

By induction on n the first cohomology group vanishes for $j>d(\mu+\alpha)$. If $\mu\neq\lambda+(-\langle\lambda,\alpha^\vee\rangle-1)\alpha$, then $d(\mu+\alpha)<d(\lambda)$ by condition 3 in Definition 3.6, and if $\mu=\lambda+(-\langle\lambda,\alpha^\vee\rangle-1)\alpha$, then $\mu+\alpha=s_\alpha(\lambda)$ and we have $d(\mu+\alpha)\le d(\lambda)$ again by condition 3. Hence (3.18) is satisfied.

By condition 3 we know that $d(\mu)<d(\lambda)$, and by induction (3.19) is satisfied for $j>d(\mu)$. But then it also holds for $j>d(\lambda)$, and we are done. □

We know that the Chevalley height, Cht, is a vanishing function, and we can use it in the above theorem. But Cht is not small enough for our calculations in Chapter 4, so we will define a minimal vanishing function recursively.

Definition 3.12. We will define a function $m : X^*(T) \to \mathbb{N}$ as follows. If $\lambda \in X^*(T)$ is dominant, we define $m(\lambda) = 0$. If λ is not dominant, we may inductively assume that $m(\mu)$ is defined for $\mu \in X^*(T)$ satisfying one of the following two conditions

i. $\mathrm{Cht}(\mu) < \mathrm{Cht}(\lambda)$

ii. $\mathrm{Cht}(\mu) = \mathrm{Cht}(\lambda)$ and $l(\mu) < l(\lambda)$.

Now define

$$m(\lambda) = \min\{m_\alpha(\lambda) | \alpha \text{ is a simple root with } \langle \lambda, \alpha \rangle \leq -1\}$$

where $m_\alpha(\lambda)$ is defined for all simple roots α with $\langle \lambda, \alpha^\vee \rangle \leq -1$ as follows:

Suppose $\langle \lambda, \alpha^\vee \rangle = -1$: Then $\lambda + \alpha = s_\alpha(\lambda)$, and hence

$$\mathrm{Cht}(\lambda) = \mathrm{Cht}(\lambda + \alpha) = \mathrm{Cht}(s_\alpha(\lambda))$$

by Proposition 3.7 and $l(s_\alpha(\lambda)) < l(\lambda)$ by Corollary 3.9. Hence $s_\alpha(\lambda)$ satisfies condition ii above, and we may assume that $m(s_\alpha(\lambda))$ is defined. Now define

$$m_\alpha(\lambda) = m(s_\alpha(\lambda)).$$

Suppose $\langle \lambda, \alpha^\vee \rangle < -1$: Then

$$\mathrm{Cht}(\lambda) > \mathrm{Cht}(\lambda + j\alpha) \quad \text{for} \quad j = 1, 2, \ldots, -\langle \lambda, \alpha^\vee \rangle - 1$$

as in the proof of Corollary 3.8, so for these j's the weights $\lambda + j\alpha$ satisfies condition i above, and we may assume that $m(\lambda + j\alpha)$ is defined. Now $\mathrm{Cht}(\lambda) \geq \mathrm{Cht}(s_\alpha(\lambda))$ by Proposition 3.7, but again $l(s_\alpha(\lambda)) < l(\lambda)$ by Corollary 3.9, so one of the two conditions i and ii is satisfied, and $m(s_\alpha(\lambda))$ is defined. Now we can define

$$m_\alpha(\lambda) = \max\{m(s_\alpha(\lambda)), m(\lambda + j\alpha) + 1 | j = 1, 2, \ldots, -\langle \lambda, \alpha^\vee \rangle - 1\}$$

Since a weight $\lambda \in X^*(T)$ is dominant, if and only if $\mathrm{Cht}(\lambda) = 0$ and $l(\lambda) = 0$, and since $l(\lambda) \leq |\Phi^+|$ for all $\lambda \in X^*(T)$, our function m is well defined.

It is clear from the definition that m is a vanishing function, and it is constructed as a minimal vanishing function in the following sense.

Lemma 3.13. Let $d : X^*(T) \to \mathbb{N}$ be a vanishing function. Then for all $\lambda \in X^*(T)$ we have

$$m(\lambda) \leq d(\lambda).$$

Proof. We will prove it by induction first on $d(\lambda)$ and then on $l(\lambda)$.

Assume $d(\lambda) = 0$. Then Corollary 3.10 gives us that $\lambda + \rho$ is dominant. If λ is dominant, then $m(\lambda) = 0$. If not, there exists a simple root α with

$$\langle \lambda, \alpha^\vee \rangle \leq -1 \quad \text{and} \quad m(\lambda) = m_\alpha(\lambda)$$

by construction of m. But since $\lambda + \rho$ is dominant this α must satisfy $\langle \lambda, \alpha^\vee \rangle = -1$ and hence $m(\lambda) = m_\alpha(\lambda) = m(s_\alpha(\lambda))$. Since $l(s_\alpha(\lambda)) < l(\lambda)$ by Corollary 3.9, we have by induction

$$m(\lambda) = m(s_\alpha(\lambda)) \leq d(s_\alpha(\lambda)).$$

But by condition 2 in Definition 3.6 we know that $d(s_\alpha(\lambda)) \leq d(\lambda)$ and therefore

$$m(\lambda) = m(s_\alpha(\lambda)) \leq d(s_\alpha(\lambda)) \leq d(\lambda).$$

Now assume that $d(\lambda) > 0$. Then λ is not dominant, and hence there exists a simple root α such that condition 2 or 3 in Definition 3.6 is satisfied.

If condition 2 is satisfied we have $\langle \lambda, \alpha^\vee \rangle = -1$ and $d(s_\alpha(\lambda)) \leq d(\lambda)$. Then $l(s_\alpha(\lambda)) < l(\lambda)$ so by induction we have

$$m(s_\alpha(\lambda)) \leq d(s_\alpha(\lambda)) \leq d(\lambda).$$

But remember that $m(\lambda) \leq m_\alpha(\lambda) = m(s_\alpha(\lambda))$ by construction of m, and we are done.

If condition 3 is satisfied we know that $\langle \lambda, \alpha^\vee \rangle < -1$, and

$$\begin{aligned} &d(\lambda + j\alpha) < d(\lambda) \quad \text{for} \quad j = 1, 2, \ldots, -\langle \lambda, \alpha^\vee \rangle - 1, \\ &d(s_\alpha(\lambda)) \leq d(\lambda). \end{aligned}$$

So by induction

$$m(\lambda + i\alpha) \leq d(\lambda + i\alpha) < d(\lambda) \tag{3.20}$$

for $i = 1, 2, \ldots, -\langle \lambda, \alpha^\vee \rangle - 1$, and since $l(s_\alpha(\lambda)) < l(\lambda)$ we also have

$$m(s_\alpha(\lambda)) \leq d(s_\alpha(\lambda)) \leq d(\lambda) \tag{3.21}$$

by induction. But

$$m(\lambda) \leq m_\alpha(\lambda) = \max(\{m(s_\alpha(\lambda))\} \cup \{m(\lambda + j\alpha) + 1 | j = 1, 2, \ldots, -\langle \lambda, \alpha^\vee \rangle - 1\})$$

and hence $m(\lambda) \leq d(\lambda)$ by (3.20) and (3.21). □

Another nice property of this minimal vanishing function is that we know exactly where it vanishes. We will show that it vanishes on the following set

$$C = \{\lambda \in X^*(T) | \langle \lambda, \beta^\vee \rangle \geq -1 \text{ for all } \beta \in \Phi^+\}.$$

Now assume that $\lambda \in C$ and $\langle \lambda, \alpha^\vee \rangle \leq 1$ for some simple root α. Then $s_\alpha(\lambda) \in C$: First observe that

$$\langle s_\alpha(\lambda), \beta^\vee \rangle = \langle \lambda, (s_\alpha(\beta))^\vee \rangle \quad \text{for all} \quad \beta \in \Phi^+.$$

Since s_α permutes all the positive roots except α, and since

$$\langle s_\alpha(\lambda), \alpha^\vee \rangle = \langle \lambda, (s_\alpha(\alpha))^\vee \rangle = -\langle \lambda, \alpha^\vee \rangle \geq -1,$$

this implies that $s_\alpha(\lambda) \in C$. We will use this observation in the proof of the following corollary.

Corollary 3.14. Let $\lambda \in X^*(T)$. Then $m(\lambda) = 0$ if and only if $\lambda \in C$.

Proof. We will prove the corollary by induction on $l(\lambda)$.

Assume that $\lambda \in C$. If $l(\lambda) = 0$, then λ is dominant and $m(\lambda) = 0$. If on the other hand $l(\lambda) > 0$, then since $\lambda \in C$, there exists a simple root α with

$$\langle \lambda, \alpha^\vee \rangle = -1 \quad \text{and} \quad m(\lambda) = m_\alpha(\lambda) = m(s_\alpha(\lambda)).$$

Hence $s_\alpha(\lambda) \in C$. Since $l(s_\alpha(\lambda)) < l(\lambda)$ we therefore have by induction $m(\lambda) = m(s_\alpha(\lambda)) = 0$.

Now assume that $m(\lambda) = 0$. Then either λ is dominant, or there exists a simple root α with

$$\langle \lambda, \alpha^\vee \rangle = -1 \quad \text{and} \quad 0 = m(\lambda) = m_\alpha(\lambda) = m(s_\alpha(\lambda)).$$

Now $l(s_\alpha(\lambda)) < l(\lambda)$, and by induction we have $s_\alpha(\lambda) \in C$. Hence we have $\lambda = s_\alpha(s_\alpha(\lambda)) \in C$. □

Example 3.15. We will use Theorem 3.11 in the following setup. Assume that $V \subseteq \mathfrak{u}$ is a B-subrepresentation. Let $\lambda \in X^*(T)$, and let $i_0 \in \mathbb{Z}$. Suppose we want to show that

$$H^i(G/B, S^n V^* \otimes \lambda) = 0 \quad \text{for all} \quad i > i_0, n \in \mathbb{Z}. \tag{3.22}$$

We take the exact sequence of B-modules (W is the cokernel)

$$0 \to V \to \mathfrak{u} \to W \to 0$$

If we take the Koszul resolution of the dual of this short exact sequence and tensor it with λ, we get the exact sequence

$$\begin{aligned} 0 \to S^{n-\dim(W)}\mathfrak{u}^* \otimes \wedge^{\dim(W)} W^* \otimes \lambda \to \cdots \to S^{n-j}\mathfrak{u}^* \otimes \wedge^j W^* \otimes \lambda \to \\ \cdots \to S^{n-1}\mathfrak{u}^* \otimes W^* \otimes \lambda \to S^n\mathfrak{u}^* \otimes \lambda \to S^n V^* \otimes \lambda \to 0 \end{aligned} \tag{3.23}$$

We may filter $\wedge^j W^* \otimes \lambda$ with B-subrepresentations such that the quotients are one dimensional with T-weights equal to the T-weights of $\wedge^j W^* \otimes \lambda$ counted with multiplicities. Hence, if we for all these weights μ can show that $m(\mu) \leq j + i_0$, then Theorem 3.11 gives us

$$H^i(G/B, S^{n-j}\mathfrak{u}^* \otimes \mu) = 0 \quad \text{for all} \quad i > j + i_0, n \in \mathbb{Z}$$

for all these weights μ. Then all quotients Q in the filtration of $\wedge^j W^* \otimes \lambda$ satisfy

$$H^i(G/B, S^{n-j}\mathfrak{u}^* \otimes Q) = 0 \quad \text{for all} \quad i > j + i_0, n \in \mathbb{Z}.$$

Hence we get

$$H^i(G/B, S^{n-j}\mathfrak{u}^* \otimes \wedge^j W^* \otimes \lambda) = 0 \quad \text{for all} \quad i > j + i_0, n \in \mathbb{Z}.$$

If this is satisfied for all $j = 0, 1, 2, \ldots, \dim(W)$, we get, by splitting the exact sequence in (3.23) into short exact sequences and taking long exact sequences in cohomology, that

$$H^i(G/B, S^n V^* \otimes \lambda) = 0 \quad \text{for all} \quad i > i_0, n \in \mathbb{Z}.$$

Remark 3.16. In type E_6 we have created a computer program which does the following. Given λ, i_0 and the weights of V, the program checks if $m(\mu) \leq j + i_0$ for all weights μ of $\wedge^j W^* \otimes \lambda$ where $j = 0, \ldots, \dim(W)$. If $m(\mu) \leq j + i_0$ for all appropriate μ we know that (3.22) holds.

Because m is inductively defined it is easy to make a computer program which calculates it. The code for the computer program can be seen in appendix A.

When we are going to use the method of Example 3.15 to show the vanishing of a cohomology group in the actual calculations in Chapter 4, we will refer to this example, meaning that we have used the computer program to get the result.

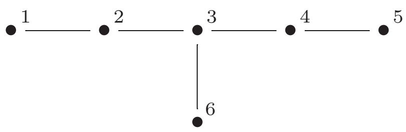
Chapter 4

Calculations

Throughout this chapter we will assume that G is a connected, simply connected, semi-simple linear algebraic group of type E_6 over an algebraically closed field k. Moreover we will assume that the characteristic of k is good for G, i.e. either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p \geq 5$. Then G satisfies the standard hypothesis on page 4, and all the results in the preceding chapters apply.

In this chapter we will prove that the orbits E_6, $E_6(a_1)$, D_5, $E_6(a_3)$, $D_5(a_1)$, A_5, $A_4 + A_1$, D_4, $D_4(a_1)$, D_4, $2A_2 + A_1$, $A_2 + 2A_1$, A_2, $3A_1$, $2A_1$, 0 all have normal closure. Before we begin our calculations, we will establish some notation specifically for type E_6.

Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_6\}$ be a numbering of the simple roots corresponding to the Dynkin diagram with vertices numbered as



Let $\lambda = \sum_{i=1}^{6} n_i\alpha_i \in X^*(T)$ be a character. Then we will sometimes use the notation

$$\left\{ \begin{smallmatrix} n_1 & n_2 & n_3 & n_4 & n_5 \\ & & n_6 & & \end{smallmatrix} \right\}$$

for the character λ and for the one dimensional B-representation with weight λ. For $b_1, \ldots, b_6 \in \mathbb{Z}$ we define

$$\left[\begin{smallmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \\ & & b_6 & & \end{smallmatrix} \right] = \bigoplus_{\substack{\alpha = \sum n_i\alpha_i \in \Phi, \\ \sum b_i n_i \leq -2}} \mathfrak{g}_\alpha \subseteq \mathfrak{g}.$$

Let $I \subseteq \Pi$ be a subset of simple roots. Let P_I be the standard parabolic subgroup containing B corresponding to I, and let $\mathfrak{u}_{P_I}$ be the Lie algebra of the unipotent radical of P_I. Now define

$$b_i = \begin{cases} 0 & \text{if } \alpha_i \in I \\ 2 & \text{if } \alpha_i \notin I \end{cases}$$

Then

$$\left[\begin{smallmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \\ & & b_6 & & \end{smallmatrix}\right] = \bigoplus_{\alpha\in\Phi^-\setminus\Phi_I} \mathfrak{g}_\alpha = \mathfrak{u}_{P_I}$$

because

$$\begin{aligned} \{\textstyle\sum n_i\alpha_i \in \Phi | \sum b_i n_i \leq -2\} &= \{\textstyle\sum n_i\alpha_i \in \Phi^- | \sum b_i n_i \leq -2\} \\ &= \{\textstyle\sum n_i\alpha_i \in \Phi^- | \exists i : b_i = 2, n_i \neq 0\} \\ &= \{\textstyle\sum n_i\alpha_i \in \Phi^- | \exists i : \alpha_i \notin I, n_i \neq 0\} \\ &= \Phi^- \setminus \Phi_I. \end{aligned}$$

Let $\mathfrak{u}$ denote the Lie algebra of the unipotent radical of the Borel group B. Then $\mathfrak{u}$ consists of nilpotent elements. If all the b_i's are non-negative, then

$$\left[\begin{smallmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \\ & & b_6 & & \end{smallmatrix}\right] \subseteq \mathfrak{u}$$

is a B-stable subspace of $\mathfrak{u}$ consisting of nilpotent elements.

In this chapter we will often write $H^i(-)$ instead of $H^i(G/B, -)$.

4.1 The orbits E_6, $E_6(a_1)$, D_5, and $E_6(a_3)$

These orbits have normal closure by work of Kostant, Kumar, Lauritzen and Thomsen.

4.2 The orbit $D_5(a_1)$

We will use that the closure of $E_6(a_3)$ is normal to show that also the closure of $D_5(a_1)$ is normal.

Step 1: Let $\overline{D_5(a_1)}$ denote the closure of $D_5(a_1)$. This notation will also be used for other orbits. We want to show that

$$\overline{D_5(a_1)} = G.\left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right].$$

Let Δ be the weighted Dynkin diagram $\left\{\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right\}$. Then with the notation of Section 2.2.3 we have $V(\lambda_\Delta) = \mathfrak{g}_2(\lambda_\Delta) = \left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right]$, and because the weighted Dynkin diagram of D_4 is $\left\{\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right\}$, it follows from Lemma 2.8 that

$$\overline{D_4} = G.\left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right].$$

Let $P = P_I$ be the standard parabolic subgroup corresponding to $I = \{\alpha_1, \alpha_2, \alpha_5\}$. Then $\mathfrak{u}_P = \left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right]$, and $\dim(G.\mathfrak{u}_P) = 2\cdot(36-4) = 64$, cf. Richardson's dense orbit theorem, Theorem 1.3. Since $G.\mathfrak{u}_P$ is the closure of a nilpotent orbit, and since the only two nilpotent orbits of dimension 64 are A_5 and $D_5(a_1)$ according to the table p. 129 in [CM93], $G.\mathfrak{u}_P$ must equal the closure of either A_5 or $D_5(a_1)$. But

$$\left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right] \subseteq \left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right] = \mathfrak{u}_P,$$

hence $\overline{D_4} = G.\left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right] \subseteq G.\mathfrak{u}_P$. But $\overline{D_4}$ is not a subset of $\overline{A_5}$, cf. Figure 1. Thus $G.\mathfrak{u}_P = \overline{D_5(a_1)}$.

Step 2: We want to show that

$$\overline{D_5(a_1)} = G.\left[\begin{smallmatrix}0&1&1&2&0\\&&2&&\end{smallmatrix}\right].$$

Note that $\left[\begin{smallmatrix}0&1&1&2&0\\&&2&&\end{smallmatrix}\right] \subseteq \left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right]$. If we let V denote the cokernel of this inclusion, we get a short exact sequence of B-modules

$$0 \to \left[\begin{smallmatrix}0&1&1&2&0\\&&2&&\end{smallmatrix}\right] \to \left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right] \to V \to 0.$$

Then V is one dimensional with T-weight $-\alpha_3 = \left\{\begin{smallmatrix}0&0&-1&0&0\\&&0&&\end{smallmatrix}\right\}$, and the Koszul resolution for the dual sequence is

$$0 \to S^{n-1}\left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}\right\} \to S^n\left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right]^* \to S^n\left[\begin{smallmatrix}0&1&1&2&0\\&&2&&\end{smallmatrix}\right]^* \to 0. \tag{4.2.1}$$

Since $\langle \alpha_3, \alpha_2^\vee\rangle = -1$, and since $\mathfrak{u}_P = \left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right]$ is a P-module (and in particular a P_{α_2}-module), we can use Theorem 3.1 to see that

$$H^i(S^{n-1}\left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}\right\}) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

The short exact sequence in (4.2.1) gives rise to a long exact sequence in cohomology. Using this sequence we therefore have

$$H^i(S^n \left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right]^*) = H^i(S^n \left[\begin{smallmatrix}0&1&1&2&0\\&&2&&\end{smallmatrix}\right]^*) \quad \text{for all} \quad i, n \in \mathbb{Z},$$

and Lemma 2.1 gives

$$\overline{D_5(a_1)} = G.\left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right] = G.\left[\begin{smallmatrix}0&1&1&2&0\\&&2&&\end{smallmatrix}\right].$$

Step 3: Now remember that we are going to use that $E_6(a_3)$ has normal closure. The closure of $E_6(a_3)$ equals $G.\left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]$ because the dimension of $G.\left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]$ is $2 \cdot (36 - 3) = 66$ by Richardson's dense orbit theorem and because the only orbit of dimension 66 is $E_6(a_3)$ according to the table p. 129 in [CM93].

Look at the short exact sequence of B-modules (V is the cokernel)

$$0 \to \left[\begin{smallmatrix}0&1&1&2&0\\&&2&&\end{smallmatrix}\right] \to \left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right] \to V \to 0.$$

Then V^* is 2 dimensional with T-weights

$$\left\{\begin{smallmatrix}0&1&0&0&0\\&&0&&\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}1&1&0&0&0\\&&0&&\end{smallmatrix}\right\}.$$

Thus the Koszul resolution of the dual of the short exact sequence is

$$\begin{aligned} 0 \to S^{n-2} \left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes \wedge^2 V^* \to S^{n-1} \left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes V^* \\ \to S^n \left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]^* \to S^n \left[\begin{smallmatrix}0&1&1&2&0\\&&2&&\end{smallmatrix}\right]^* \to 0. \end{aligned} \tag{4.2.2}$$

We will show that

$$H^i(S^{n-1} \left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes V^*) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

There exists a short exact sequence of B-modules

$$0 \to \left\{\begin{smallmatrix}0&1&0&0&0\\&&0&&\end{smallmatrix}\right\} \to V^* \to \left\{\begin{smallmatrix}1&1&0&0&0\\&&0&&\end{smallmatrix}\right\} \to 0.$$

Since $\left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right] = \mathfrak{u}_P$ is a P_{α_3}-module, and since

$$\langle \left\{\begin{smallmatrix}0&1&0&0&0\\&&0&&\end{smallmatrix}\right\}, \alpha_3^\vee \rangle = -1, \quad \langle \left\{\begin{smallmatrix}1&1&0&0&0\\&&0&&\end{smallmatrix}\right\}, \alpha_3^\vee \rangle = -1,$$

Theorem 3.1 gives

$$\begin{aligned} H^i(S^{n-1} \left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&1&0&0&0\\&&0&&\end{smallmatrix}\right\}) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z} \\ H^i(S^{n-1} \left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}1&1&0&0&0\\&&0&&\end{smallmatrix}\right\}) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}. \end{aligned}$$

Hence

$$H^i(S^{n-1} \left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes V^*) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

Thus, by splitting the exact sequence in (4.2.2) into short exact sequences and taking long exact sequences in cohomology, we get

$$\begin{aligned} H^i(\text{Ker } (S^n \left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]^* \to S^n \left[\begin{smallmatrix}0&1&1&2&0\\&&2&&\end{smallmatrix}\right]^*)) \\ = H^{i+1}(S^{n-2} \left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes \wedge^2 V^*) \quad \text{for all} \quad i, n \in \mathbb{Z}. \end{aligned}$$

Step 4: If we can show that

$$H^{i+1}(S^{n-2}\left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes \wedge^2 V^*) = 0 \quad \text{for all} \quad i \geq 1, n \in \mathbb{Z} \tag{4.2.3}$$

we have the exact sequence

$$\begin{aligned} 0 \to H^1(S^{n-2}\left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes \wedge^2 V^*) \\ \to H^0(S^n\left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]^*) \to H^0(S^n\left[\begin{smallmatrix}0&1&1&2&0\\&&2&&\end{smallmatrix}\right]^*) \to 0 \end{aligned}$$

for all $n \in \mathbb{Z}$, and we are able to prove that $D_5(A_1)$ has normal closure: Let $J = \{\alpha_1, \alpha_3, \alpha_5\} \subseteq \Pi$, and let P_J be the corresponding parabolic subgroup. Then $\mathfrak{u}_{P_J} = \left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]$, and by Lemma 11 in [Tho00] the map

$$G \times^{P_J} \left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right] \to G.\left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]$$

is birational. Since $E_6(a_3) = G.\left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]$, and since $E_6(a_3)$ has normal closure, Lemma 2.2 gives that

$$G.\left[\begin{smallmatrix}0&1&1&2&0\\&&2&&\end{smallmatrix}\right] = G.\left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right] = \overline{D_5(a_1)}$$

is normal.

Now we are going to show (4.2.3). Remark that $\wedge^2 V^* = \left\{\begin{smallmatrix}1&2&0&0&0\\&&0&&\end{smallmatrix}\right\}$ and that $\left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]$ is a P_{α_3}-module, so using Theorem 3.1 with $\langle \left\{\begin{smallmatrix}1&2&0&0&0\\&&0&&\end{smallmatrix}\right\}, \alpha_3^\vee \rangle = -2$, we get

$$H^{i+1}(S^{n-2}\left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes \wedge^2 V^*) = H^i(S^{n-2}\left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}1&2&1&0&0\\&&0&&\end{smallmatrix}\right\})$$

for all $i, n \in \mathbb{Z}$.

We now start using Proposition 3.3. We will use the notation from Section 3.1 with the small exception that α_i in Section 3.1 here will be denoted α_i', so the notation α_i is reserved for simple roots in type E_6. Let $l = 3$, $m = 2$, $\alpha_1' = \alpha_3$, $\alpha_2' = \alpha_4$, $\alpha_3' = \alpha_5$, $\Gamma = \{\alpha_1\}$. Then

$$l + 1 - m = 2, \quad m' = 2, \quad I_2 = \{\alpha_1, \alpha_3, \alpha_5\}$$

and

$$\langle \left\{\begin{smallmatrix}1&2&1&0&0\\&&0&&\end{smallmatrix}\right\}, \alpha_3^\vee \rangle = 0, \quad \langle \left\{\begin{smallmatrix}1&2&1&0&0\\&&0&&\end{smallmatrix}\right\}, \alpha_5^\vee \rangle = 0, \quad r = \langle \left\{\begin{smallmatrix}1&2&1&0&0\\&&0&&\end{smallmatrix}\right\}, \alpha_4^\vee \rangle = -1,$$

so by the proposition we get

$$\begin{aligned} H^i(S^{n-2}\left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}1&2&1&0&0\\&&0&&\end{smallmatrix}\right\}) \\ = H^i(S^{n-4}\left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}1&2&2&2&1\\&&0&&\end{smallmatrix}\right\}) \quad \text{for all} \quad i, n \in \mathbb{Z}. \end{aligned} \tag{4.2.4}$$

We use Proposition 3.3 again with $l = 2$, $m = 2$, $\alpha_1' = \alpha_3$, $\alpha_2' = \alpha_6$, $\Gamma = \{\alpha_1, \alpha_5\}$. Then

$$l + 1 - m = 1, \quad m' = 1, \quad I_2 = \{\alpha_1, \alpha_3, \alpha_5\}, \quad I_1 = \{\alpha_1, \alpha_5, \alpha_6\}$$

and

$$\langle \left\{\begin{smallmatrix}1&2&2&2&1\\&&0&&\end{smallmatrix}\right\}, \alpha_3^\vee \rangle = 0, \quad r = \langle \left\{\begin{smallmatrix}1&2&2&2&1\\&&0&&\end{smallmatrix}\right\}, \alpha_6^\vee \rangle = -2,$$

and the proposition gives

$$H^i(S^{n-4} \left[\begin{smallmatrix}0&2&0&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}1&2&2&2&1\\&&0&&\end{smallmatrix}\right\}) = H^i(S^{n-6} \left[\begin{smallmatrix}0&2&2&2&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}1&2&4&2&1\\&&2&&\end{smallmatrix}\right\}) \tag{4.2.5}$$

for all $i, n \in \mathbb{Z}$. At this point Eric Sommers uses Proposition 3.3 two times, and then Proposition 3.5 to show that the latter cohomology group vanishes for all $i > 0$ in characteristic 0. Instead we obtain by Example 3.15 that

$$H^i(S^{n-6} \left[\begin{smallmatrix}0&2&2&2&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}1&2&4&2&1\\&&2&&\end{smallmatrix}\right\}) = 0 \quad \text{for all} \quad i > 0, n \in \mathbb{Z}.$$

Remembering all the isomorphisms of the cohomology groups, we see that (4.2.3) is satisfied and hence that $D_5(a_1)$ has normal closure.

4.3 The orbit A_5

We want to show that also the closure of A_5 is normal using that the closure of $E_6(a_3)$ is normal. As explained in the summary in the introduction, we will use a method which is different from the one in [Som03]. This new method does not require as many calculations as the old one, even though the new method is not easy. The main ingredient is the observation that A_5 has codimension two in $E_6(a_3)$, see the table on page 129 in [CM93], and remember that we are allowed to use characteristic zero results as explained in Section 2.2.4. The idea behind the method is due to Eric Sommers.

The weighted Dynkin diagram of A_5 is $\Delta = \left\{\begin{smallmatrix}2&1&0&1&2\\&&1&&\end{smallmatrix}\right\}$, and since $V(\lambda_\Delta) = \left[\begin{smallmatrix}2&1&0&1&2\\&&1&&\end{smallmatrix}\right]$, we have by Lemma 2.8 that

$$\overline{A_5} = G.\left[\begin{smallmatrix}2&1&0&1&2\\&&1&&\end{smallmatrix}\right].$$

Similarly since $\Delta' = \left\{\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right\}$ is the weighted Dynkin diagram of $E_6(a_3)$, we know that $V(\lambda_{\Delta'}) = \left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right]$, and that

$$\overline{E_6(a_3)} = G.\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right].$$

Let $P = P(\lambda_{\Delta'})$. Then the morphism

$$\bar{p} : G \times^P \left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right] \to G.\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right]$$

is birational by Corollary 2.9. Note that P is the standard parabolic subgroup containing B corresponding to $J = \{\alpha_2, \alpha_4, \alpha_6\}$.

We know that $\left[\begin{smallmatrix}2&1&0&1&2\\&&1&&\end{smallmatrix}\right] \subseteq \left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right]$. We want to show that the closure of $\overline{A_5}$ is normal. By Lemma 2.2 it is enough to show that the inclusion $\left[\begin{smallmatrix}2&1&0&1&2\\&&1&&\end{smallmatrix}\right] \subseteq \left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right]$ induces a surjection on cohomology groups

$$H^0(G/B, S^n \left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right]^*) \to H^0(G/B, S^n \left[\begin{smallmatrix}2&1&0&1&2\\&&1&&\end{smallmatrix}\right]^*) \to 0 \qquad (4.3.1)$$

for all $n \in \mathbb{N}$ because $\bar{p}$ is birational, and because $\overline{E_6(a_3)} = G.\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right]$ is normal.

The idea is as follows: We will find a normal, irreducible P-stable subvariety

$$W \subseteq \left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right] \qquad (4.3.2)$$

of codimension one. Let $k\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right]$ be the coordinate ring of $\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right]$. Then we can identify $k\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right]$ with the polynomial ring $k[x_\alpha | \alpha \in \Phi^- \setminus \Phi_J]$ graded with $\deg(x_\alpha) = 1$ for all α. Let $I(W)$ be the defining ideal of W in $\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right]$. Then it turns out that

$$I(W) = \langle f \rangle \subseteq k\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right]$$

for some irreducible, homogeneous element $f \in k\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right]^{(P)}$.

Since f is P-semistable, there exists a P-character $\lambda \in X^*(P)$ such that $p.f = \lambda(p)f$ for all $p \in P$. We also claim that $\lambda = \left\{\begin{smallmatrix}0&2&4&2&0\\&&2&&\end{smallmatrix}\right\}$. Since f is homogeneous, the coordinate ring of W is graded. Let $k^n[W]$ denote the graded piece of degree n in the coordinate ring of W.

We will show that the inclusion $W \subseteq \left[\begin{smallmatrix} 2 & 0 & 2 & 0 & 2 \\ & & 0 & & \end{smallmatrix}\right]$ induces a short exact sequence of cohomology groups

$$\begin{aligned} 0 \to H^0(G/B, S^{n-4}\left[\begin{smallmatrix} 2 & 0 & 2 & 0 & 2 \\ & & 0 & & \end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix} 0 & 2 & 4 & 2 & 0 \\ & & 2 & & \end{smallmatrix}\right\}) \\ \to H^0(G/B, S^n \left[\begin{smallmatrix} 2 & 0 & 2 & 0 & 2 \\ & & 0 & & \end{smallmatrix}\right]^*) \\ \to H^0(G/B, k^n[W]) \to 0 \end{aligned} \tag{4.3.3}$$

for all $n \in \mathbb{N}$.

Late we will show that $P.\left[\begin{smallmatrix} 2 & 1 & 0 & 1 & 2 \\ & & 1 & & \end{smallmatrix}\right] = W$, and that the inclusion $\left[\begin{smallmatrix} 2 & 1 & 0 & 1 & 2 \\ & & 1 & & \end{smallmatrix}\right] \subseteq W$ induces an isomorphism

$$H^0(G/B, k^n[W]) \simeq H^0(G/B, S^n \left[\begin{smallmatrix} 2 & 1 & 0 & 1 & 2 \\ & & 1 & & \end{smallmatrix}\right]^*) \tag{4.3.4}$$

for all $n \in \mathbb{N}$. To prove this, we will need that W is normal. Combining (4.3.3) and (4.3.4) we get the desired surjection in (4.3.1).

4.3.1 Restricting to a subgroup of type D_4

We will define W in (4.3.2) by restricting to a subgroup of type D_4.

Let P_I be the standard parabolic subgroup containing B corresponding to the set $I = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6\}$ of simple roots. Let L_I be the Levi subgroup of P_I containing T. The commutator group $G' := (L_I, L_I)$ is semi-simple and simply connected (cf. Exercise 8.4.6,6 in [Spr98]) with root system $\Phi' := \Phi_I$ of type D_4. Then $T' = (T \cap G')^0$ is a maximal torus in G', and $B' = B \cap G'$ is a Borel subgroup in G' containing T'. Let P' be the standard parabolic subgroup in G' containing B' corresponding to the subset $J = \{\alpha_2, \alpha_4, \alpha_6\} \subseteq \Phi'$.

Let $\alpha = \sum_{i=2,3,4,6} a_i \alpha_i \in \Phi'$ be a root. We will also use the notation

$$\left\{\begin{smallmatrix} a_2 & a_3 & a_4 \\ & a_6 & \end{smallmatrix}\right\} \tag{4.3.5}$$

for the root α.

Let $\mathfrak{g}'$ be the Lie algebra of G'. Then $\mathfrak{g}'$ is the subset of $\mathfrak{g}$ given by

$$\mathfrak{g}' = \mathfrak{t}' \oplus \Big(\bigoplus_{\alpha \in \Phi'} \mathfrak{g}_\alpha \Big) \subseteq \mathfrak{g}$$

where t' is the Lie algebra of T'. In $\mathfrak{g}'$ we define

$$\left[\begin{smallmatrix} b_2 & b_3 & b_4 \\ & b_6 & \end{smallmatrix}\right] = \bigoplus_{\substack{\alpha = \sum_{i=2,3,4,6} n_i \alpha_i \in \Phi', \\ \sum b_i n_i \leq -2}} \mathfrak{g}_\alpha \subseteq \mathfrak{g}',$$

and we will consider this as a subspace of $\mathfrak{g}$. Let $\mathfrak{u}_{P'}$ denote the Lie algebra of the unipotent radical of P', and note that $\mathfrak{u}_{P'} = \left[\begin{smallmatrix} 0 & 2 & 0 \\ & 0 & \end{smallmatrix}\right]$. Let $\mathcal{O}_{P'} \in \mathfrak{u}_{P'}$ be the dense, open P'-orbit of Richardson elements, and let $k[\mathfrak{u}_{P'}]$ denote the coordinate ring of $\mathfrak{u}_{P'}$. Let $\varpi_3 \in X^*(T')$ denote the fundamental weight corresponding to α_3. Now since G' is simply connected, we have $X^*(P') = \mathbb{Z}\varpi_3$. We will use the notation and results from Section 2.3 in the following lemma.

Lemma 4.1. The closed set $\mathfrak{u}_{P'} \setminus \mathcal{O}_{P'}$ has exactly one component V of dimension equal to $\dim(\mathfrak{u}_{P'}) - 1$. Furthermore $I(V) = \langle f \rangle$ where $f \in k[\mathfrak{u}_{P'}]^{(P)}$ and $p.f = 2\varpi_3(p)f$ for all $p \in P'$.

Proof. Let $x \in \mathcal{O}_{P'}$ be a Richardson element, and let Δ' be the weighted Dynkin diagram $\{ {}^{0\ 2\ 0}_{\ \ 0} \}$. Then $P' = P(\lambda_{\Delta'})$, and by Lemma 2.13 we know that the number of components in $\mathfrak{u}_{P'} \setminus \mathcal{O}_{P'}$ of dimension equal to $\dim(\mathfrak{u}_{P'}) - 1$ equals the rank of the kernel of the restriction map

$$\varphi_x : X^*(P') \to X^*(Z_{P'}(x)).$$

Remember that $X^*(P') = \mathbb{Z}\varpi_3$, and that Ker φ_x is independent of the Richardson element x, cf. Section 2.3. We will show that Ker $\varphi_x = \mathbb{Z}2\varpi_3$.

First we show that $\varpi_3 \notin$ Ker (φ). To do this we will choose a specific Richardson element $x \in \mathcal{O}_{P'}$ and an element $p_0 \in Z_{P'}(x)$ such that $\varpi_3(p_0) \neq 1$.

In order to find a Richardson element $x \in \mathfrak{u}_{P'}$ and an element $p_0 \in Z_{P'}(x)$ such that $\varpi_3(p_0) \neq 1$, we are going to use Lemma 4.6 which is stated and proved in Section 4.3.2.

Let l' be the rank of G'. Lemma 4.6 gives the existence of a basis

$$\{x_\alpha | \alpha \in \Phi'\} \cup \{h'_i | i = 1, 2, \dots, l'\} \tag{4.3.6}$$

for $\mathfrak{g}'$ with $x_\alpha \in \mathfrak{g}_\alpha$ and $h'_i \in \mathfrak{t}'$ and the existence of admissible isomorphisms $u_\alpha : k \to U_\alpha$ for $\alpha \in \Phi'$ such that

$$\mathrm{Ad}(u_\alpha(t))(x_\beta) = x_\beta + \sum_{\substack{i \geq 1 \\ \beta + i\alpha \in \Phi}} c_i^{\alpha,\beta} t^i x_{\beta + i\alpha} \quad \text{for all } t \in k \tag{4.3.7}$$

where $c_i^{\alpha,\beta}$ are constants with $c_1^{\alpha,\beta} = \pm(r+1)$ where $r \geq 0$ is the greatest integer satisfying that $\beta - r\alpha$ is a root.

By Remark 4.7 we can choose the signs of $c_1^{\alpha,\beta}$ using the process in [Sam69] page 54. We choose the signs of $c_1^{\alpha,\beta}$ to positive whenever we use equation (4.3.7) in the following calculations.

With this notation and the notational convention in (4.3.5) we define

$$x = x_{\left\{ {}^{-1\ -1\ -1}_{\quad -1} \right\}} + x_{\left\{ {}^{0\ -1\ 0}_{\quad 0} \right\}} \in \mathfrak{u}_{P'}.$$

Then one can directly check that x is a Richardson element in $\mathcal{O}_{P'}$. Define

$$n_i = u_{\alpha_i}(1) u_{-\alpha_i}(-1) u_{\alpha_i}(1) \in N_{G'}(T'),$$

and let $n = n_{\alpha_6} n_{\alpha_4} n_{\alpha_2}$. Let L' be the Levi subgroup of P' containing T'. Since P' is the standard parabolic subgroup containing B' corresponding to $J = \{\alpha_2, \alpha_4, \alpha_6\}$, we see that the image of n in the Weyl group $N_{G'}(T')/T'$ of G' is the longest element in the Weyl group of L'. Let $t = \alpha_3^\vee(-1) \in T'$, and define $p_0 = tn \in P'$.

Now we are ready to prove that $p_0 \in Z_{P'}(x)$:

$$\begin{aligned} n_2.x_{\left\{\begin{smallmatrix}-1&-1&-1\\&-1&\end{smallmatrix}\right\}} &= \mathrm{Ad}(u_{\alpha_2}(1)u_{-\alpha_2}(-1)u_{\alpha_2}(1))x_{\left\{\begin{smallmatrix}-1&-1&-1\\&-1&\end{smallmatrix}\right\}} \\ &= \mathrm{Ad}(u_{\alpha_2}(1)u_{-\alpha_2}(-1))\Big(x_{\left\{\begin{smallmatrix}-1&-1&-1\\&-1&\end{smallmatrix}\right\}} + x_{\left\{\begin{smallmatrix}0&-1&-1\\&-1&\end{smallmatrix}\right\}}\Big) \\ &= \mathrm{Ad}(u_{\alpha_2}(1))\Big((1-1)x_{\left\{\begin{smallmatrix}-1&-1&-1\\&-1&\end{smallmatrix}\right\}} + x_{\left\{\begin{smallmatrix}0&-1&-1\\&-1&\end{smallmatrix}\right\}}\Big) \\ &= \mathrm{Ad}(u_{\alpha_2}(1))x_{\left\{\begin{smallmatrix}0&-1&-1\\&-1&\end{smallmatrix}\right\}} \\ &= x_{\left\{\begin{smallmatrix}0&-1&-1\\&-1&\end{smallmatrix}\right\}} \end{aligned}$$

Continuing this way we get

$$n.x_{\left\{\begin{smallmatrix}-1&-1&-1\\&-1&\end{smallmatrix}\right\}} = x_{\left\{\begin{smallmatrix}0&-1&0\\&0&\end{smallmatrix}\right\}}$$

and therefore

$$p_0.x_{\left\{\begin{smallmatrix}-1&-1&-1\\&-1&\end{smallmatrix}\right\}} = \alpha_3^\vee(-1).x_{\left\{\begin{smallmatrix}0&-1&0\\&0&\end{smallmatrix}\right\}} = x_{\left\{\begin{smallmatrix}0&-1&0\\&0&\end{smallmatrix}\right\}} \tag{4.3.8}$$

Now

$$\begin{aligned} n_2.x_{\left\{\begin{smallmatrix}0&-1&0\\&0&\end{smallmatrix}\right\}} &= \mathrm{Ad}(u_{\alpha_2}(1)u_{-\alpha_2}(-1)u_{\alpha_2}(1))x_{\left\{\begin{smallmatrix}0&-1&0\\&0&\end{smallmatrix}\right\}} \\ &= \mathrm{Ad}(u_{\alpha_2}(1)u_{-\alpha_2}(-1))x_{\left\{\begin{smallmatrix}0&-1&0\\&0&\end{smallmatrix}\right\}} \\ &= \mathrm{Ad}(u_{\alpha_2}(1))\Big(x_{\left\{\begin{smallmatrix}0&-1&0\\&0&\end{smallmatrix}\right\}} - x_{\left\{\begin{smallmatrix}-1&-1&0\\&0&\end{smallmatrix}\right\}}\Big) \\ &= (1-1)x_{\left\{\begin{smallmatrix}0&-1&0\\&0&\end{smallmatrix}\right\}} - x_{\left\{\begin{smallmatrix}-1&-1&0\\&0&\end{smallmatrix}\right\}} \\ &= -x_{\left\{\begin{smallmatrix}-1&-1&0\\&0&\end{smallmatrix}\right\}} \end{aligned}$$

and we get

$$n.x_{\left\{\begin{smallmatrix}0&-1&0\\&0&\end{smallmatrix}\right\}} = (-1)^3 x_{\left\{\begin{smallmatrix}-1&-1&-1\\&-1&\end{smallmatrix}\right\}}$$

Hence

$$p_0.x_{\left\{\begin{smallmatrix}0&-1&0\\&0&\end{smallmatrix}\right\}} = \alpha_3^\vee(-1)\Big(-x_{\left\{\begin{smallmatrix}-1&-1&-1\\&-1&\end{smallmatrix}\right\}}\Big) = -\Big(-x_{\left\{\begin{smallmatrix}-1&-1&-1\\&-1&\end{smallmatrix}\right\}}\Big) = x_{\left\{\begin{smallmatrix}-1&-1&-1\\&-1&\end{smallmatrix}\right\}} \tag{4.3.9}$$

Putting (4.3.8) and (4.3.9) together we get $p_0.x = x$, and so $p_0 \in Z_{P'}(x)$. Since n is a product of unipotent elements, we know that $\varpi_3 : P \to k^*$ satisfies $\varpi_3(n) = 1$, and therefore

$$\varpi_3(p_0) = \varpi_3(\alpha_3^\vee(-1))\varpi_3(n) = -1 \cdot 1 = -1$$

and $p_0 \notin \mathrm{Ker}\ \varphi_x$.

Now we need to show that $2\varpi_3 \in \text{Ker } \varphi_x$. Since $P' = P(\lambda_{\Delta'})$, Theorem 2.7. i gives that $Z_{G'}(x) = Z_{P'}(x)$ as mentioned on page 25. Hence we have

$$Z_{P'}(x)/Z_{P'}(x)^0 = Z_{G'}(x)/Z_{G'}(x)^0.$$

The latter is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ by Chapter 13.1 in [Car85] – as described in Section 2.2.4 we are allowed to use the characteristic zero results. Hence $Z_{P'}(x)^0$ has index two in $Z_{P'}(x)$. Let

$$\psi_x : X^*(P) \to X^*(Z_{P'}(x)^0).$$

be the restriction map. As observed in Section 2.3 we have $2\text{Ker } \psi_x \subseteq \text{Ker } \varphi_x$, and we only need to show that $\varpi_3 \in \text{Ker } \psi_x$.

Let $S = \text{Ker } (\alpha_3) \cap \text{Ker } (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_6) \subseteq T'$. Then S is a two-dimensional torus, and $S \subseteq Z_{P'}(x)$. Hence $S \subseteq Z_{P'}(x)^0$. Also note that ϖ_3 restricted to S, denoted $\varpi_3|_S$, is constantly 1 since $\varpi_3 = (\alpha_3) + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_6)$.

Let $R_u(Z_{G'})$ be the unipotent radical of $Z_{G'}(x)$. If we can show that $Z_{P'}(x)^0 = Z_{G'}(x)^0$ is the semidirect product of S and $R_u(Z_{G'})$ (i.e. if S is a Levi factor for $Z_{G'}(x)^0$), then $\varpi_3 \in \text{Ker } \psi_x$ since $\varpi_3|_S = 1$ and since $R_u(Z_{G'}(x))$ consists of unipotent elements. So in this case $\varpi_3 \in \text{Ker } \psi_x$, and we are done.

In Chapter 13.1 in [Car85] we see that if $G^{\text{ad}}_{\mathbb{C}}$ is an adjoint, semi-simple, connected, linear algebraic group of type D_4 over the complex numbers, and $x_{\mathbb{C}}$ is a Richardson element for $P(\lambda_{\Delta'})$ where Δ' is the weighted Dynkin diagram $\Delta' = \{ {}^{0\ 2\ 0}_{\ \ 0} \}$, then $Z_{G^{\text{ad}}_{\mathbb{C}}}(x_{\mathbb{C}})^0$ has a Levi factor isomorphic to $(\mathbb{C}^*)^2$.

Let $G^{\text{sc}}_{\mathbb{C}}$ be the simply connected, connected, semi-simple linear algebraic group of type D_4 over $\mathbb{C}$. Then we have the surjective morphism $\pi : G^{\text{sc}}_{\mathbb{C}} \to G^{\text{ad}}_{\mathbb{C}}$ with finite fibers and with a differential which is an isomorphism of the Lie algebras of the two groups. Under this isomorphism $x_{\mathbb{C}}$ is also a Richardson element for $G^{\text{sc}}_{\mathbb{C}}$. Now π restricts to a surjective map

$$\pi : Z_{G^{\text{sc}}_{\mathbb{C}}}(x_{\mathbb{C}})^0 \to Z_{G^{\text{ad}}_{\mathbb{C}}}(x_{\mathbb{C}})^0.$$

Let $L^{\text{sc}}_{\mathbb{C}}$ be a Levi factor of $Z_{G^{\text{sc}}_{\mathbb{C}}}(x_{\mathbb{C}})^0$. Then $L^{\text{sc}}_{\mathbb{C}}$ is connected. Moreover the image $\pi(L^{\text{sc}}_{\mathbb{C}})$ is a Levi factor of $Z_{G^{\text{ad}}_{\mathbb{C}}}(x_{\mathbb{C}})^0$ by Lemma 1.4, and hence $\pi(L^{\text{sc}}_{\mathbb{C}}) \simeq (\mathbb{C}^*)^2$. Now

$$\pi : L^{\text{sc}}_{\mathbb{C}} \to \pi(L^{\text{sc}}_{\mathbb{C}}) \tag{4.3.10}$$

is surjective with finite fibers, and hence $\dim_{\mathbb{C}} L^{\text{sc}}_{\mathbb{C}} = 2$. Let $T^{\text{sc}}_{\mathbb{C}}$ be a maximal torus in $L^{\text{sc}}_{\mathbb{C}}$. Since the morphism in (4.3.10) is surjective, $\pi(T^{\text{sc}}_{\mathbb{C}})$ is a maximal torus in $\pi(L^{\text{sc}}_{\mathbb{C}}) \simeq (\mathbb{C}^*)^2$. But then $\dim_{\mathbb{C}} T^{\text{sc}}_{\mathbb{C}} \geq 2$. Since $L^{\text{sc}}_{\mathbb{C}}$ is connected of dimension two, we can conclude that $L^{\text{sc}}_{\mathbb{C}} = T^{\text{sc}}_{\mathbb{C}} \simeq (\mathbb{C}^*)^2$.

Now Theorem 2.7 and Section 2.2.4 tells us that $Z_{G'}(x)^0 = Z_{P'}(x)^0$ has a Levi factor L with the same root datum as $L^{\text{sc}}_{\mathbb{C}}$. Hence $L \simeq (k^*)^2$. Then L is a unique maximal torus in $Z_{G'}(x)^0$, and since the dimension of S is two, we have $S = L^{\text{sc}}_{\mathbb{C}}$. Therefore S is a Levi factor of $Z_{G'}(x)^0$, and we are done.

□

We can identify the coordinate ring $k[\mathfrak{u}_{P'}]$ with the polynomial ring

$$k[x_\alpha | \alpha \in (\Phi')^- \setminus \Phi_J]$$

where $J = \{\alpha_2, \alpha_4, \alpha_6\}$, and we grade it with $\deg(x_\alpha) = 1$ for all α.

Corollary 4.2. The element

$$f \in k[\mathfrak{u}_{P'}] = k[x_\alpha | \alpha \in (\Phi')^- \setminus \Phi_J]$$

from Lemma 4.1 is homogeneous of degree 4.

Proof. By Lemma 2.10 f is homogeneous. Since f is homogeneous, irreducible and P-semistable with $p.f = 2\varpi_3(p)f$ for all $p \in P'$ and $2\varpi_3 = \left\{\begin{smallmatrix}2&4&2\\&2&\end{smallmatrix}\right\}$, we see that the terms of f must be products of two of these four monomials in $k[\mathfrak{u}_{P'}]$:

$$x_{\left\{\begin{smallmatrix}1&1&1\\&1&\end{smallmatrix}\right\}} x_{\left\{\begin{smallmatrix}0&1&0\\&0&\end{smallmatrix}\right\}}, \quad x_{\left\{\begin{smallmatrix}1&1&1\\&0&\end{smallmatrix}\right\}} x_{\left\{\begin{smallmatrix}0&1&0\\&1&\end{smallmatrix}\right\}}, \quad x_{\left\{\begin{smallmatrix}1&1&0\\&1&\end{smallmatrix}\right\}} x_{\left\{\begin{smallmatrix}0&1&1\\&0&\end{smallmatrix}\right\}}, \quad x_{\left\{\begin{smallmatrix}0&1&1\\&1&\end{smallmatrix}\right\}} x_{\left\{\begin{smallmatrix}1&1&0\\&0&\end{smallmatrix}\right\}}.$$

□

Since f is homogeneous, the coordinate ring $k[V] = k[\mathfrak{u}_{P'}]/\langle f \rangle$ is graded. Let $k^n[V]$ denote the degree n graded piece.

We want to prove that the component V from Lemma 4.1 is normal, and we also want to find a good description of $k^n[V]$. In order to do this we will prove that $V = P'.\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$ and that the morphism $P' \times^{B'} \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right] \to P'.\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$ is birational.

First notice that $\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right] \subseteq \left[\begin{smallmatrix}0&2&0\\&0&\end{smallmatrix}\right] = \mathfrak{u}_{P'}$ and therefore $P'.\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right] \subseteq \left[\begin{smallmatrix}0&2&0\\&0&\end{smallmatrix}\right]$. Next observe that $P'.\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$ is closed in $\left[\begin{smallmatrix}0&2&0\\&0&\end{smallmatrix}\right]$ since P'/B' is projective, cf. Section 1.2 page 4. Clearly $P'.\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$ is irreducible.

Lemma 4.3. The morphism

$$\Psi : P' \times^{B'} \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right] \to P'.\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$$

is birational.

Proof. Let

$$\pi : P' \times \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right] \to P' \times^{B'} \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$$

be the projection, and let $U' = U_{\alpha_2}U_{\alpha_4}U_{\alpha_6}$. Then U' is a subgroup of P'. Now $\pi(U' \times \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right])$ is open in $P' \times^{B'} \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$ since

$$\pi^{-1}(\pi(U' \times \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right])) = U'B' \times \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$$

and since $U'B'$ is open in P' by the Bruhat decomposition. Therefore it is enough to show that

$$\Psi : \pi(U' \times \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]) \to P'.\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$$

is birational. Since Ψ is dominant, it is by Theorem 2.4 enough to show that Ψ is generically one to one and separable.

Consider the composition

$$\Phi = \Psi \circ \pi|_{U' \times \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]} : U' \times \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right] \to P' . \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right].$$

We will use the same notation as in the proof of Lemma 1.4. First notice that $\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$ is the span of the x_α's where α is one of the following roots

$$\left\{\begin{smallmatrix}-1&-2&-1\\&-1&\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}-1&-1&-1\\&-1&\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}0&-1&-1\\&-1&\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}-1&-1&-1\\&0&\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}-1&-1&0\\&-1&\end{smallmatrix}\right\}. \tag{4.3.11}$$

Let $\bar{x} \in \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$. Then we can write

$$\begin{aligned}\bar{x} =& a x_{\left\{\begin{smallmatrix}-1&-2&-1\\&-1&\end{smallmatrix}\right\}} + b x_{\left\{\begin{smallmatrix}-1&-1&-1\\&-1&\end{smallmatrix}\right\}} + c x_{\left\{\begin{smallmatrix}0&-1&-1\\&-1&\end{smallmatrix}\right\}} \\ &+ d x_{\left\{\begin{smallmatrix}-1&-1&-1\\&0&\end{smallmatrix}\right\}} + e x_{\left\{\begin{smallmatrix}-1&-1&0\\&-1&\end{smallmatrix}\right\}}\end{aligned}$$

for some constants $a, b, c, d, e \in k$. Then

$$\begin{aligned}&\Psi \circ \pi \left(u_{\alpha_2}(v_2) u_{\alpha_4}(v_4) u_{\alpha_6}(v_6), \bar{x}\right) \\ =&\mathrm{Ad}(u_{\alpha_2}(v_2) u_{\alpha_4}(v_4) u_{\alpha_6}(v_6)) \, (\bar{x}) \\ =& a x_{\left\{\begin{smallmatrix}-1&-2&-1\\&-1&\end{smallmatrix}\right\}} + b x_{\left\{\begin{smallmatrix}-1&-1&-1\\&-1&\end{smallmatrix}\right\}} + (c + v_2 b) x_{\left\{\begin{smallmatrix}0&-1&-1\\&-1&\end{smallmatrix}\right\}} + (d + v_6 b) x_{\left\{\begin{smallmatrix}-1&-1&-1\\&0&\end{smallmatrix}\right\}} \\ &+ (e + v_4 b) x_{\left\{\begin{smallmatrix}-1&-1&0\\&-1&\end{smallmatrix}\right\}} + (v_2 d + v_6 c + v_2 v_6 b) x_{\left\{\begin{smallmatrix}0&-1&-1\\&0&\end{smallmatrix}\right\}} \\ &+ (v_2 e + v_4 c + v_2 v_4 b) x_{\left\{\begin{smallmatrix}0&-1&0\\&-1&\end{smallmatrix}\right\}} + (v_4 d + v_6 e + v_4 v_6 b) x_{\left\{\begin{smallmatrix}-1&-1&0\\&0&\end{smallmatrix}\right\}} \\ &+ (v_2 v_4 d + v_2 v_6 e + v_4 v_6 c + v_2 v_4 v_6 b) x_{\left\{\begin{smallmatrix}0&-1&0\\&0&\end{smallmatrix}\right\}}\end{aligned} \tag{4.3.12}$$

Now we will see that $\Psi : \pi(U' \times \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]) \to P' . \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$ is generically one to one. Let V'' be the complement in $\left[\begin{smallmatrix}0&2&0\\&0&\end{smallmatrix}\right] = \mathfrak{u}_{P'}$ of the zero set of the three polynomials

$$\begin{aligned}&x_{\left\{\begin{smallmatrix}-1&-1&-1\\&-1&\end{smallmatrix}\right\}} x_{\left\{\begin{smallmatrix}0&-1&-1\\&0&\end{smallmatrix}\right\}} - x_{\left\{\begin{smallmatrix}0&-1&-1\\&-1&\end{smallmatrix}\right\}} x_{\left\{\begin{smallmatrix}-1&-1&-1\\&0&\end{smallmatrix}\right\}} \in k[\mathfrak{u}_{P'}] \\ &x_{\left\{\begin{smallmatrix}-1&-1&-1\\&-1&\end{smallmatrix}\right\}} x_{\left\{\begin{smallmatrix}0&-1&0\\&-1&\end{smallmatrix}\right\}} - x_{\left\{\begin{smallmatrix}0&-1&-1\\&-1&\end{smallmatrix}\right\}} x_{\left\{\begin{smallmatrix}-1&-1&0\\&-1&\end{smallmatrix}\right\}} \in k[\mathfrak{u}_{P'}] \\ &x_{\left\{\begin{smallmatrix}-1&-1&-1\\&-1&\end{smallmatrix}\right\}} x_{\left\{\begin{smallmatrix}-1&-1&0\\&0&\end{smallmatrix}\right\}} - x_{\left\{\begin{smallmatrix}-1&-1&0\\&-1&\end{smallmatrix}\right\}} x_{\left\{\begin{smallmatrix}-1&-1&-1\\&0&\end{smallmatrix}\right\}} \in k[\mathfrak{u}_{P'}].\end{aligned}$$

Let $V' = V'' \cap P' . \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$. Then by direct calculations using (4.3.12) one can check that

$$\Phi : (\Phi)^{-1}(V') \to V'$$

is injective. Since $\pi : U' \times \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right] \to \pi(U' \times \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right])$ is bijective, this implies that $\Psi : \pi(U' \times \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]) \to P' . \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$ is generically one to one.

Now we want to show that Ψ is separable. Let $\bar{u} \in U'$ and $\bar{x} \in \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$. We look at the differential of Φ as a map of tangent spaces

$$d\Phi_{(\bar{u},\bar{x})} = d(\Psi \circ \pi)_{(\bar{u},\bar{x})} : T_{(\bar{u},\bar{x})}(U' \times \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]) \to T_{\Psi \circ \pi(\bar{u},\bar{x})}(P' . \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]).$$

Using (4.3.12) one can again directly calculate that this differential is surjective for $(\bar{u}, \bar{x}) \in U' \times W$ where W is the complement in $\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$ of the zero set of

$$x_{\left\{\begin{smallmatrix}0&-1&-1\\&-1&\end{smallmatrix}\right\}} x_{\left\{\begin{smallmatrix}-1&-1&-1\\&0&\end{smallmatrix}\right\}} x_{\left\{\begin{smallmatrix}-1&-1&-1\\&0&\end{smallmatrix}\right\}} \in k\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right].$$

Hence $\mathrm{d}\Psi_z$ is surjective for all z in the open set $\pi(U' \times W)$. Since $\Psi : \pi(U' \times \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]) \to P'.\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$ is dominant, this implies by Theorem 4.3.6 in [Spr98] that Ψ is separable.

□

Corollary 4.4. Let V be the component given by Lemma 4.1. Then

$$V = P'.\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right].$$

Proof. From the above lemma we see that

$$\begin{aligned}\dim(P'.\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]) &= \dim(P' \times^{B'} \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]) \\ &= (\dim P' - \dim B') + \dim\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right] \\ &= 3 + 5 = 8,\end{aligned}$$

and $P'.\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$ has codimension one in $\left[\begin{smallmatrix}0&2&0\\&0&\end{smallmatrix}\right] = \mathfrak{u}_{P'}$. Furthermore $P'.\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right] \subseteq \mathfrak{u}_{P'}$ is closed, irreducible and P'-stable, and hence $P'.\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$ is a component in $\mathfrak{u}_{P'} \setminus \mathcal{O}_{P'}$ of dimension equal to $\dim \mathfrak{u}_{P'} - 1$. But by Lemma 4.1 V is the only component in $\mathfrak{u}_{P'} \setminus \mathcal{O}_{P'}$ of this dimension, and we have $V = P'.\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$. □

Note that in this proof we see that $P'.\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$ is a component in $\mathfrak{u}_{P'} \setminus \mathcal{O}_{P'}$ of dimension $\dim \mathfrak{u}_{P'} - 1$. Since $X^*(P') = \mathbb{Z}\varpi_3$, there is at most one such component by Lemma 2.12. We also know that the defining ideal of $P'.\left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$ in $\mathfrak{u}_{P'}$ is equal to $\langle f \rangle \subseteq k[\mathfrak{u}_{P'}]$, and that there exists a character $\lambda \in X^*(P)$ such that $p.f = \lambda(p)f$ for all $p \in P'$. Now one can ask why we made such an effort to prove Lemma 4.1? The answer is that we got some extra information from Lemma 4.1, namely the information that $\lambda = 2\varpi_3$. This will be important later.

We will prove that the component V from Lemma 4.1 is normal. In the proof we need to know that there are only finitely many P'-orbits in $\mathfrak{u}_{P'} = \left[\begin{smallmatrix}0&2&0\\&0&\end{smallmatrix}\right]$. To see this we will use the theorem on page iii in the introduction of [Röh] which is a generalization of Theorem 1.1 in [HR99]. Actually we could use Theorem 1.1 in [HR99], but if we wanted to prove normality of $\overline{3A_1}$ by the same method we are now using to prove normality of $\overline{A_5}$, we would need the generalized version. The setup is the following:

Let G be a reductive linear algebraic group, and P a parabolic subgroup with Levi factor L_P and unipotent radical U_P. Let $\mathfrak{u}_P$ denote the Lie algebra of U_P. Now U_P is a nilpotent group, and we define the descending central series of U_P: Let $\mathcal{C}^0(U_P) = U_P$, and let $\mathcal{C}^{i+1}(U_P)$ be the commutator $(\mathcal{C}^i(U_P), U_P)$ for $i \geq 0$. Since U_P is nilpotent, we can define $l(U_P)$ to be the smallest integer m such that $\mathcal{C}^m(U_P)$ is trivial.

For G of type D_r we let τ denote the graph automorphism of G of order two.

Suppose G is simple as an algebraic group, and the characteristic is good for G. Then the generalized theorem in [Röh] states that P acts on $\mathfrak{u}_P$ with a finite number of orbits if and only if one of the following conditions hold.

i. $l(U_P) \leq 4$.

ii. G is of type D_r, $l(U_P) = 5$, $\tau P \neq P$, and the semi-simple part of L_P consists of two components which are simple as algebraic groups.

iii. G is of type E_6, $l(U_P) = 5$, and P is of type $2A_1 + A_2$ or A_3.

iv. G is of type E_7, $l(U_P) = 5$, and P is of type $A_1 + A_5$.

A method to compute $l(U_P)$ is given by the formula on page 4 in [Röh]: Let T be a maximal torus in G. Let Φ be the roots of G with respect to T, and let $\Pi \subseteq \Phi$ be a set of simple roots. Suppose P is a standard parabolic subgroup corresponding to the subset $I \subseteq \Pi$ of simple roots. Let ϱ denote the highest root in Φ, and write $\varrho = \sum_{\alpha\in\Pi} n_\alpha \alpha$ as a linear combination of the simple roots. If the characteristic of G is very good, then the formula is

$$l(U_P) = \sum_{\alpha\in\Pi\setminus I} n_\alpha. \tag{4.3.13}$$

Now we are ready to prove that V is normal.

Lemma 4.5. The component V from Lemma 4.1 is normal.

Proof. Since V is a hypersurface in $\mathfrak{u}_{P'} = [\begin{smallmatrix}0 & 2 & 0\\ & 0 & \end{smallmatrix}]$, it is by Proposition III.8.2 in [Mum99] enough to show that the set of singular points in V has codimension at least two, i.e. it is enough show that the set of singular points in V has dimension six or less.

Now we want to use the above theorem to conclude that there are only finitely many P'-orbits in $[\begin{smallmatrix}0 & 2 & 0\\ & 0 & \end{smallmatrix}] = \mathfrak{u}_{P'}$. Now G' is simple as an algebraic group and of type D_4. Since $\mathrm{char}(k) \geq 5$ is good for G of type E_6, it is also good for G' of type D_4. Then it is also very good for G', since G' is not of type A. The highest root in Φ' is

$$\varrho = \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6 = \{\begin{smallmatrix}1 & 2 & 1\\ & 1 & \end{smallmatrix}\}.$$

Since P' corresponds to the subset $\{\alpha_2, \alpha_4, \alpha_6\}$ of simple roots, formula (4.3.13) gives us that $l(U_{P'}) = 2$. Hence Röhrle and Hille's theorem above tells us, that there are only finitely many P'-orbits in $\mathfrak{u}_{P'}$.

Let $\mathcal{O}$ denote the nilpotent G'-orbit in $\mathfrak{g}'$ with weighted Dynkin diagram $\{\begin{smallmatrix}1 & 0 & 1\\ & 1 & \end{smallmatrix}\}$. According to Corollary 6.1.4 in [CM93] the dimension of $\mathcal{O}$ is 16 – as explained in Section 2.2.4 we are allowed to use the dimension results from characteristic zero. Moreover Corollary 6.1.4 in [CM93] tells us that there are no nilpotent G'-orbits in $\mathfrak{g}'$ of dimension 14 (remember that the dimension of an orbit is always even). We want to conclude that V contains no P'-orbit of dimension 7:

The main ingredient to prove this is the following result from [Kaw87]. Let H be a connected, semi-simple, linear algebraic group over an algebraically closed field of good characteristic. Let $D \subseteq H$ be a one dimensional torus, and let $\lambda \in X^*(D)$ be a generator for the character group of D. Then λ induces a grading on the Lie algebra of H, denoted $\mathfrak{h}$, defined by

$$\mathfrak{h}(i, \lambda) = \{x \in \mathfrak{h} | \mathrm{Ad}(t)x = t.x = \lambda(t)^i x \quad \forall t \in D\}.$$

Now let $Q \subseteq G$ be the parabolic subgroup with Lie algebra

$$\bigoplus_{i \leq 0} \mathfrak{h}(i, \lambda).$$

Formula 3.1.8. in [Kaw87] states that

$$\dim(Q.x) = \tfrac{1}{2}\dim(H.x) \quad \text{for all} \quad x \in \mathfrak{h}(-1, \lambda). \tag{4.3.14}$$

We will for contradiction assume that V contains a P'-orbit of dimension 7, and then use Kawanaka's result to show that there must exist a G'-orbit of dimension 14. Since this is not the case, we have obtained a contradiction.

Now since the fundamental weight ϖ_3 is actually a root, we can define $\varpi_3^\vee : k^* \to T'$ to be its coroot. Let D be the image of $\varpi_3^\vee$. Then D is a one dimensional torus with character group $X^*(D) \simeq \mathbb{Z}$ and $\alpha_3 \in X^*(D)$ is a generator for this group. As described before α_3 induces a grading of the Lie algebra $\mathfrak{g}'$ given by

$$\begin{aligned} \mathfrak{g}'(i, \alpha_3) &= \{x \in \mathfrak{g}' | \mathrm{Ad}(d)x = d.x = (\alpha_3(d))^i x \quad \forall d \in D\} \\ &= \{x \in \mathfrak{g}' | \varpi_3^\vee(t).x = (\alpha_3(\varpi_3^\vee(t)))^i x = t^i x \quad \forall t \in k^*\}. \end{aligned}$$

Hence

$$\mathfrak{g}'(i, \alpha_3) = \bigoplus_{\substack{\alpha \in \Phi' \cup \{0\} \\ \langle \alpha, \varpi_3^\vee \rangle = i}} \mathfrak{g}_\alpha = \bigoplus_{\substack{\alpha \in \Phi' \cup \{0\} \\ \alpha = \sum_{j=2,3,4,6} n_j \alpha_j, n_3 = i}} \mathfrak{g}_\alpha,$$

and

$$\mathrm{Lie}(P') = \mathfrak{t}' \oplus \Big(\bigoplus_{\substack{\alpha \in \Phi' \\ \alpha = \sum_j n_j \alpha_j, n_3 \leq 0}} \mathfrak{g}_\alpha \Big) = \bigoplus_{i \leq 0} \mathfrak{g}'(i, \alpha_3). \tag{4.3.15}$$

We have assumed for contradiction that we have a 7-dimensional P'-orbit contained in V. By Corollary 4.4 we have $V = P'.\left[{}^{1\,0\,1}_{\;\;1} \right]$, and we may assume that the P'-orbit is of the form $P'.x$ for some $x \in \left[{}^{1\,0\,1}_{\;\;1} \right]$. In the following we will find an element $b \in B'$ such that $b.x \in \mathfrak{g}'(-1, \alpha_3)$. Then we have $P'.x = P'.(b.x)$, and since $b.x \in \mathfrak{g}'(-1, \alpha_3)$ we can use Kawanaka's result in (4.3.14) to conclude that

$$\dim(G'.x) = 2\dim(P'.(b.x)) = 2\dim(P'.x) = 14$$

which is a contradiction.

Now we are going to find the above element $b \in B'$. We use the same notation as we used in the proof of Lemma 4.1. In particular we use the basis for $\mathfrak{g}'$ given in (4.3.6) and the formula for the action of the root groups given in (4.3.7). This time we do not care about the signs of the constants $c_1^{\alpha,\beta}$.

Remember that $\left[{}^{1\,0\,1}_{\;\;1} \right]$ is the span of the x_α's where α is one of the roots in (4.3.11), and notice that

$$\left[{}^{1\,0\,1}_{\;\;1} \right] = \mathfrak{g}'(-1, \alpha_3) \oplus \mathfrak{g}'(-2, \alpha_3).$$

The element $x \in \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$ can be written

$$x = \sum_{\alpha:\mathfrak{g}_\alpha \subseteq \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]} a_\alpha x_\alpha, \quad \text{with} \quad a_\alpha \in k$$

where the sum is taken over all $\alpha \in \Phi'$ such that $\mathfrak{g}_\alpha \subseteq \left[\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}\right]$ (i.e. where α is one of the roots in (4.3.11)). If $a_{-\left\{\begin{smallmatrix}1&2&1\\&1&\end{smallmatrix}\right\}} = 0$, then $x \in \mathfrak{g}'(-1, \alpha_3)$ and we are done, so we may assume that

$$a_{-\left\{\begin{smallmatrix}1&2&1\\&1&\end{smallmatrix}\right\}} \neq 0.$$

Since $\mathfrak{g}'(2, \alpha_3) = kx_{-\left\{\begin{smallmatrix}1&2&1\\&1&\end{smallmatrix}\right\}}$ is P'-stable, and $P'.x$ is 7-dimensional, we have $x \notin \mathfrak{g}'(2, \alpha_3)$. Hence at least one of the coefficients

$$a_{-\left\{\begin{smallmatrix}1&1&1\\&1&\end{smallmatrix}\right\}}, \quad a_{-\left\{\begin{smallmatrix}1&1&0\\&1&\end{smallmatrix}\right\}}, \quad a_{-\left\{\begin{smallmatrix}1&1&1\\&0&\end{smallmatrix}\right\}}, \quad a_{-\left\{\begin{smallmatrix}0&1&1\\&1&\end{smallmatrix}\right\}}$$

is nonzero. Using (4.3.7) we get

$$u_{-\alpha_3}(t).x = \sum_{\alpha:\mathfrak{g}_\alpha \subseteq \mathfrak{g}'(-1,\alpha_3)} a_\alpha x_\alpha + \big(\pm t a_{-\left\{\begin{smallmatrix}1&1&1\\&1&\end{smallmatrix}\right\}} + a_{-\left\{\begin{smallmatrix}1&2&1\\&1&\end{smallmatrix}\right\}}\big) x_{-\left\{\begin{smallmatrix}1&2&1\\&1&\end{smallmatrix}\right\}}. \tag{4.3.16}$$

Now assume that $a_{-\left\{\begin{smallmatrix}1&1&1\\&1&\end{smallmatrix}\right\}} \neq 0$. Then letting

$$t = \pm \frac{a_{-\left\{\begin{smallmatrix}1&2&1\\&1&\end{smallmatrix}\right\}}}{a_{-\left\{\begin{smallmatrix}1&1&1\\&1&\end{smallmatrix}\right\}}}$$

(with an appropriate sign) the last term in (4.3.16) vanishes and we get

$$u_{-\alpha_3}\Big(\pm \frac{a_{-\left\{\begin{smallmatrix}1&2&1\\&1&\end{smallmatrix}\right\}}}{a_{-\left\{\begin{smallmatrix}1&1&1\\&1&\end{smallmatrix}\right\}}}\Big).x = \sum_{\alpha:\mathfrak{g}_\alpha \subseteq \mathfrak{g}'(-1,\alpha_3)} a_\alpha x_\alpha \in \mathfrak{g}'(-1, \alpha_3).$$

And since $U_{-\alpha_3} \subseteq B'$ we are done.

If on the contrary we have $a_{-\left\{\begin{smallmatrix}1&1&1\\&1&\end{smallmatrix}\right\}} = 0$, then one of the three coefficients

$$a_{-\left\{\begin{smallmatrix}1&1&0\\&1&\end{smallmatrix}\right\}}, \quad a_{-\left\{\begin{smallmatrix}1&1&1\\&0&\end{smallmatrix}\right\}}, \quad a_{-\left\{\begin{smallmatrix}0&1&1\\&1&\end{smallmatrix}\right\}}$$

is nonzero. By symmetry we may assume that $a_{-\left\{\begin{smallmatrix}1&1&0\\&1&\end{smallmatrix}\right\}} \neq 0$. Then choosing appropriate signs and using (4.3.7) again, we see that the coefficient to $x_{-\left\{\begin{smallmatrix}1&2&1\\&1&\end{smallmatrix}\right\}}$ in

$$\Big(u_{-\alpha_2}\Big(\pm \frac{a_{-\left\{\begin{smallmatrix}1&2&1\\&1&\end{smallmatrix}\right\}}}{a_{-\left\{\begin{smallmatrix}1&1&0\\&1&\end{smallmatrix}\right\}}}\Big) u_{-\alpha_3}(\pm 1)\Big).x$$

is zero, and hence the above element belongs to $\mathfrak{g}'(-1, \alpha_3)$. Also $U_{-\alpha_3} U_{-\alpha_2} \subseteq B'$, so we are done. $\square$

4.3.2 The adjoint action of a root group on the root spaces

In this section we will prove Lemma 4.6 which we have used many times in Section 4.3.3.

Let G be a connected, semi-simple linear algebraic group of rank l with root system Φ, and let $\mathrm{Lie}(G)$ be its Lie algebra. Let T be a maximal torus with Lie algebra $\mathrm{Lie}(T)$, and let Φ be the root system of G with respect to T. For $\alpha \in \Phi$ let $\mathrm{Lie}(G)_\alpha$ be the root space with weight α, and let U_α be the corresponding root group, i.e. U_α is the unique T-stable subgroup of G with Lie algebra $\mathrm{Lie}(G)_\alpha$. We know that there exist admissible isomorphisms $u_\alpha : k \to U_\alpha$. An admissible isomorphism $u_\alpha : k \to U_\alpha$ is unique up to a scalar factor, i.e. up to choosing a basis for $\mathrm{Lie}(G)_\alpha$.

Lemma 4.6. There exists a basis

$$\{x_\alpha | \alpha \in \Phi\} \cup \{h'_i | i = 1, 2, \ldots, l\}$$

for $\mathrm{Lie}(G)$, and there exist admissible isomorphisms $u_\alpha : k \to U_\alpha$ such that

$$\mathrm{Ad}(u_\alpha(t))(x_\beta) = x_\beta + \sum_{\substack{i \geq 1 \\ \beta + i\alpha \in \Phi}} c_i^{\alpha,\beta} t^i x_{\beta + i\alpha} \quad \text{for all } t \in k \tag{4.3.17}$$

where $c_i^{\alpha,\beta}$ are constants with $c_1^{\alpha,\beta} = \pm(r+1)$ where $r \geq 0$ is the greatest integer satisfying that $\beta - r\alpha$ is a root. Furthermore $x_\alpha \in \mathrm{Lie}(G)_\alpha$ and $h'_i \in \mathrm{Lie}(T)$.

Sketch of proof. Since every semi-simple linear algebraic group is isomorphic to a Chevalley group (considered as an algebraic group), see [Ste68] p. 61, it is enough to show the lemma for Chevalley groups.

We will use the notation and results about Chevalley groups in Section 2.2.2, so $G = G_k$ with this notation. Remember that $\pi_k : \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} k \to \mathrm{Lie}(G)$ is an isomorphism. By abuse of notation we will let x_α denote $\pi_k(x_\alpha \otimes 1) \in \mathrm{Lie}(G)$ for $\alpha \in \Phi$ and h'_i denote $\pi_k(h'_i \otimes 1) \in \mathrm{Lie}(G)$ for $i = 1, 2, \ldots, l$. Then

$$\{x_\alpha | \alpha \in \Phi\} \cup \{h'_i | i = 1, 2, \ldots, l\}$$

is a basis for $\mathrm{Lie}(G)$. Let $u_\alpha = \bar{x}_\alpha : k \to U_\alpha$ be the admissible isomorphism. Then $\mathrm{d}u_\alpha : k \to \mathrm{Lie}(G)_\alpha$, and it turns out that

$$x_\alpha = \pi_k(x_\alpha \otimes 1) = \mathrm{d}u_\alpha(1). \tag{4.3.18}$$

We will try to calculate $\mathrm{Ad}(u_\alpha(t))(x_\beta)$ for all $t \in k$. Therefore consider the composition

$$\theta = \mathrm{Int}(u_\alpha(t)) \circ u_\beta : k \to G.$$

This is given by $\theta(u) = u_\alpha(t)u_\beta(u)(u_\alpha(t))^{-1}$. According to Chevalley's commutator formula we have

$$u_\alpha(t)u_\beta(u)(u_\alpha(t))^{-1}(u_\beta(u))^{-1} = \prod_{\substack{i,j \geq 1 \\ i\alpha + j\beta \in \Phi}} u_{i\alpha + j\beta}(c_{i,j}^{\alpha,\beta} t^i u^j)$$

where the product is taken in some fixed order over the roots $i\alpha + j\beta \in \Phi$ and where $c_{i,j}^{\alpha,\beta} \in k$ are constants depending on α, β and the chosen ordering, but not on t and u. Now consider the x_β's as elements in $\mathfrak{g}_{\mathbb{C}}$. Then the x_β's are part of the Chevalley basis, so we know that when $\alpha+\beta$ is a root then $c_{1,1}^{\alpha,\beta}$ equals the constant $N_{\alpha,\beta}$ that satisfies $[x_\alpha, x_\beta] = N_{\alpha,\beta}x_{\alpha+\beta} \in \mathfrak{g}_{\mathbb{C}}$. Hence $c_{1,1}^{\alpha,\beta} = \pm(r+1)$ where $r \geq 0$ is the largest integer such that $\beta - r\alpha$ is a root. Now we have a formula for $\theta(u)$

$$\theta(u) = u_\alpha(t)u_\beta(u)(u_\alpha(t))^{-1} = \prod_{\substack{i,j\geq 1\\ i\alpha+j\beta\in\Phi}} u_{i\alpha+j\beta}(c_{i,j}^{\alpha,\beta}t^i u^j)u_\beta(u).$$

The differential $\mathrm{d}\theta = \mathrm{Ad}(u_\alpha(t)) \circ \mathrm{d}u_\beta : k \to \mathrm{Lie}(G)$ can be calculated. For all $a, t \in k$ we get

$$\begin{aligned}\mathrm{Ad}(u_\alpha(t))(u_\beta(a)) = \mathrm{d}\theta(a) &= \sum_{\substack{i\geq 1\\ i\alpha+\beta\in\Phi}} c_{i,1}^{\alpha,\beta}t^i \mathrm{d}u_{\alpha+\beta}(a) + \mathrm{d}u_\beta(a)\\ &= \mathrm{d}u_\beta(a) + \sum_{\substack{i\geq 1\\ \beta+i\alpha\in\Phi}} c_{i,1}^{\alpha,\beta}t^i \mathrm{d}u_{\alpha+\beta}(a).\end{aligned}$$

Letting $a = 1$ in the this equation and using (4.3.18), we get

$$\mathrm{Ad}(u_\alpha(t))(x_\beta) = x_\beta + \sum_{\substack{i\geq 1\\ \beta+i\alpha\in\Phi}} c_{i,1}^{\alpha,\beta}t^i x_{\alpha+\beta}$$

for all $t \in k$, and we are done. □

Remark 4.7. Remember that the x_α's considered as elements in $\mathfrak{g}_{\mathbb{C}}$ are part of a Chevalley basis for $\mathfrak{g}_{\mathbb{C}}$. Hence it is possible to choose the x_α's such that the signs of the $N_{\alpha,\beta}$'s can be chosen by the process in [Sam69] p. 54.

Therefore we can choose the basis for $\mathrm{Lie}(G)$ and the admissible isomorphisms as described in the lemma such that the signs of $c_1^{\alpha,\beta} = N_{\alpha,\beta}$ can be chosen by the process in [Sam69].

4.3.3 Using the D_4-case to show the general case

We will use the results from Section 4.3.1 in the D_4-case to obtain similar results in our original setup. First notice that

$$\mathfrak{u}_P = \left[\begin{smallmatrix} 0 & 2 & 0 \\ & 0 & \end{smallmatrix}\right] \oplus \Big(\bigoplus_{\alpha\in\Phi^-\setminus\Phi_I} \mathfrak{g}_\alpha\Big) = \mathfrak{u}_{P'} \oplus \Big(\bigoplus_{\alpha\in\Phi^-\setminus\Phi_I} \mathfrak{g}_\alpha\Big).$$

Next remember the identifications

$$k\left[\begin{smallmatrix} 0 & 2 & 0 \\ & 0 & \end{smallmatrix}\right] = k[x_\alpha | \alpha \in (\Phi')^- \setminus \Phi_J], \quad k\left[\begin{smallmatrix} 2 & 0 & 2 & 0 & 2 \\ & & 0 & & \end{smallmatrix}\right] = k[x_\alpha | \alpha \in \Phi^- \setminus \Phi_J].$$

This gives us an inclusion

$$i : k\left[\begin{smallmatrix} 0 & 2 & 0 \\ & 0 & \end{smallmatrix}\right] \to k\left[\begin{smallmatrix} 2 & 0 & 2 & 0 & 2 \\ & & 0 & & \end{smallmatrix}\right]$$

with $i(x_\alpha) = x_\alpha$ for all $\alpha \in (\Phi')^- \setminus \Phi_J$, or written differently

$$i\left(x_{\left\{\begin{smallmatrix}a_1&a_2&a_3\\&a_4&\end{smallmatrix}\right\}}\right) = x_{\left\{\begin{smallmatrix}0&a_1&a_2&a_3&0\\&&a_4&&\end{smallmatrix}\right\}}.$$

Now let $V \subseteq \mathfrak{u}_{P'} = \left[\begin{smallmatrix}0&2&0\\&0&\end{smallmatrix}\right]$ be the component given by Lemma 4.1, and define

$$W = V \times \bigoplus_{\alpha \in \Phi^- \setminus \Phi_I} \mathfrak{g}_\alpha \subseteq \left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right] = \mathfrak{u}_P.$$

By Lemma 4.1 and Lemma 4.5 we know that V is a normal, irreducible, affine subvariety of codimension one in $\left[\begin{smallmatrix}0&2&0\\&0&\end{smallmatrix}\right]$. Hence W is a normal, irreducible, affine subvariety of codimension one in $\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right] = \mathfrak{u}_P$. Since $I(V) = \langle f \rangle \subseteq k\left[\begin{smallmatrix}0&2&0\\&0&\end{smallmatrix}\right]$ where $f \in k\left[\begin{smallmatrix}0&2&0\\&0&\end{smallmatrix}\right]$ is the element from Lemma 4.1, we see that the defining ideal of W is given by

$$I(W) = \langle i(f) \rangle \subseteq k\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right].$$

If $h \in k\left[\begin{smallmatrix}0&2&0\\&0&\end{smallmatrix}\right]$ is T'-semistable with T'-weight $\left\{\begin{smallmatrix}a_1&a_2&a_3\\&a_4&\end{smallmatrix}\right\}$, then $i(h)$ is T-semistable with T-weight $\left\{\begin{smallmatrix}0&a_1&a_2&a_3&0\\&&a_4&&\end{smallmatrix}\right\}$. Since f is P'-semistable with T'-weight $2\varpi_3 = \left\{\begin{smallmatrix}2&4&2\\&2&\end{smallmatrix}\right\}$ by Lemma 4.1, the element $i(f)$ is T-semistable with weight $\left\{\begin{smallmatrix}0&2&4&2&0\\&&2&&\end{smallmatrix}\right\}$. As in Lemma 4.3 and Corollary 4.4 it is possible to show that

$$P \times^B \left[\begin{smallmatrix}2&1&0&1&2\\&&1&&\end{smallmatrix}\right] \to P.\left[\begin{smallmatrix}2&1&0&1&2\\&&1&&\end{smallmatrix}\right]$$

is birational and that $W = P.\left[\begin{smallmatrix}2&1&0&1&2\\&&1&&\end{smallmatrix}\right]$. In particular W is P-stable, and $i(f)$ is P-semistable with $p.f = \lambda(p)f$ where $\lambda = \left\{\begin{smallmatrix}0&2&4&2&0\\&&2&&\end{smallmatrix}\right\}$ (since f was T-semistable with this weight).

Since W is P-stable, it is certainly B-stable, hence we get a short exact sequence of B-modules

$$0 \longrightarrow k\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right] \xrightarrow{\phi} k\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right] \xrightarrow{\psi} k\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right]/\langle i(f)\rangle \longrightarrow 0 \quad (4.3.19)$$

where $\phi(h) = hi(f)$ for $h \in k\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right]$ and ψ is the projection.

Also note that

$$k[W] = k\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right]/\langle i(f)\rangle$$

and that

$$k\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right] = \bigoplus_{n\geq 0} S^n \left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right]^*.$$

Since $f \in k\left[\begin{smallmatrix}0&2&0\\&0&\end{smallmatrix}\right]$ is homogeneous of degree four by Corollary 4.2, also $i(f) \in k\left[\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}\right]$ is homogeneous of degree four. In particular $k[W]$ is graded. Let $k^n[W]$ denote the degree n graded piece.

Remember that $\left\{\begin{smallmatrix}0&2&4&2&0\\&&2&&\end{smallmatrix}\right\}$ also denotes the one dimensional B-module with weight $\left\{\begin{smallmatrix}0&2&4&2&0\\&&2&&\end{smallmatrix}\right\}$. Moreover remember that $i(f)$ is homogeneous of degree four, and that the weight of $i(f)$ is $\lambda = \left\{\begin{smallmatrix}0&2&4&2&0\\&&2\end{smallmatrix}\right\}$. Therefore, investigating each degree of the sequence in (4.3.19) we get the following exact sequence

$$0 \longrightarrow S^{n-4}\left[\begin{smallmatrix}2&0&2&0&2\\&&0\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&2&4&2&0\\&&2\end{smallmatrix}\right\} \xrightarrow{\widetilde{\phi}} S^{n}\left[\begin{smallmatrix}2&0&2&0&2\\&&0\end{smallmatrix}\right]^* \xrightarrow{\widetilde{\psi}} k^n[W] \longrightarrow 0 \tag{4.3.20}$$

for all $n \in \mathbb{Z}$. Here $\widetilde{\phi}(h \otimes a) = a \cdot h \cdot i(f)$ for all $h \in S^{n-4}\left[\begin{smallmatrix}2&0&2&0&2\\&&0\end{smallmatrix}\right]^*$ and all $a \in \left\{\begin{smallmatrix}0&2&4&2&0\\&&2\end{smallmatrix}\right\}$.

Using Theorem 3.1 three times gives

$$\begin{aligned} &H^i(G/B, S^{n-4}\left[\begin{smallmatrix}2&0&2&0&2\\&&0\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&2&4&2&0\\&&2\end{smallmatrix}\right\}) \\ &\qquad = H^{i+3}(G/B, S^{n-4}\left[\begin{smallmatrix}2&0&2&0&2\\&&0\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&1&4&1&0\\&&1\end{smallmatrix}\right\}) \end{aligned}$$

for all $i \in \mathbb{Z}$ and all $n \in \mathbb{Z}$. By Example 3.15 we know that

$$H^j(G/B, S^{n-4}\left[\begin{smallmatrix}2&0&2&0&2\\&&0\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&1&4&1&0\\&&1\end{smallmatrix}\right\}) = 0 \quad \text{for all} \quad j > 3.$$

Hence we have

$$H^i(G/B, S^{n-4}\left[\begin{smallmatrix}2&0&2&0&2\\&&0\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&2&4&2&0\\&&2\end{smallmatrix}\right\}) = 0 \quad \text{for all} \quad i > 0. \tag{4.3.21}$$

Now the long exact sequence in cohomology arising from (4.3.20) gives us the following short exact sequence for all $n \in \mathbb{Z}$

$$\begin{aligned} 0 \to H^0(G/B, S^{n-4}\left[\begin{smallmatrix}2&0&2&0&2\\&&0\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&2&4&2&0\\&&2\end{smallmatrix}\right\}) \\ \to H^0(G/B, S^{n}\left[\begin{smallmatrix}2&0&2&0&2\\&&0\end{smallmatrix}\right]^*) \\ \to H^0(G/B, k^n[W]) \to 0 \end{aligned}$$

which is exactly (4.3.3).

In order to prove that A_5 has normal closure, it now remains to show that the inclusion

$$\left[\begin{smallmatrix}2&1&0&1&2\\&&1\end{smallmatrix}\right] \subseteq P.\left[\begin{smallmatrix}2&1&0&1&2\\&&1\end{smallmatrix}\right] = W$$

of B-modules induces an isomorphism

$$H^0(G/B, S^n\left[\begin{smallmatrix}2&1&0&1&2\\&&1\end{smallmatrix}\right]^*) \simeq H^0(G/B, k^n[W])$$

for all $n \in \mathbb{Z}$, cf. the discussion on page 55. Remember the birational morphism

$$\Psi : P \times^B \left[\begin{smallmatrix}2&1&0&1&2\\&&1\end{smallmatrix}\right] \to P.\left[\begin{smallmatrix}2&1&0&1&2\\&&1\end{smallmatrix}\right] = W$$

Let $O_{P\times^B\left[\begin{smallmatrix}2&1&0&1&2\\&&1\end{smallmatrix}\right]}$ and O_W denote the structure sheaves of $P \times^B \left[\begin{smallmatrix}2&1&0&1&2\\&&1\end{smallmatrix}\right]$ and W respectively. Now Ψ is clearly surjective and projective, and since W is normal, Lemma II.14.5 in [Jan87] gives us that Ψ induces an isomorphism of sheaves

$$\varphi^* O_{P\times^B\left[\begin{smallmatrix}2&1&0&1&2\\&&1\end{smallmatrix}\right]} \simeq O_W.$$

Hence the two sets of global regular functions are isomorphic, i.e.

$$k[W] \simeq k[P \times^B \left[\begin{smallmatrix} 2 & 1 & 0 & 1 & 2 \\ & & 1 & & \end{smallmatrix}\right]].$$

But by (2.1) the latter equals

$$H^0(P/B, \oplus_{n\geq 0} S^n \left[\begin{smallmatrix} 2 & 1 & 0 & 1 & 2 \\ & & 1 & & \end{smallmatrix}\right]^*),$$

and we get

$$\begin{aligned} H^i(G/B, k[W]) &= H^i(G/B, H^0(P/B, \oplus_{n\geq 0} S^n \left[\begin{smallmatrix} 2 & 1 & 0 & 1 & 2 \\ & & 1 & & \end{smallmatrix}\right]^*)) \\ &= H^i(G/P, H^0(P/B, \oplus_{n\geq 0} S^n \left[\begin{smallmatrix} 2 & 1 & 0 & 1 & 2 \\ & & 1 & & \end{smallmatrix}\right]^*)). \end{aligned} \tag{4.3.22}$$

Take a look at the Grothendieck spectral sequence

$$E_2^{i,j} = H^i(G/P, H^j(P/B, \oplus_{n\geq 0} S^n \left[\begin{smallmatrix} 2 & 1 & 0 & 1 & 2 \\ & & 1 & & \end{smallmatrix}\right]^*)).$$

We know that it abuts to

$$H^{i+j}(G/B, \oplus_{n\geq 0} S^n \left[\begin{smallmatrix} 2 & 1 & 0 & 1 & 2 \\ & & 1 & & \end{smallmatrix}\right]^*).$$

But by Example 3.15 we see that

$$H^j(P/B, \oplus_{n\geq 0} S^n \left[\begin{smallmatrix} 2 & 1 & 0 & 1 & 2 \\ & & 1 & & \end{smallmatrix}\right]^*) = 0 \quad \text{for all} \quad j > 0.$$

Hence $E_2^{i,j} = 0$ for $j > 0$, and the spectral sequence already collapses at the E_2-term. Therefore we get

$$H^i(G/P, H^0(P/B, \oplus_{n\geq 0} S^n \left[\begin{smallmatrix} 2 & 1 & 0 & 1 & 2 \\ & & 1 & & \end{smallmatrix}\right]^*)) = E_2^{i,0} = H^{i+0}(G/B, \oplus_{n\geq 0} S^n \left[\begin{smallmatrix} 2 & 1 & 0 & 1 & 2 \\ & & 1 & & \end{smallmatrix}\right]^*)$$

for all $i \in \mathbb{Z}$. Combining this with (4.3.22) we get the desired isomorphism (in (4.3.4))

$$H^i(G/B, k[W]) = H^i(G/B, \oplus_{n\geq 0} S^n \left[\begin{smallmatrix} 2 & 1 & 0 & 1 & 2 \\ & & 1 & & \end{smallmatrix}\right]^*) \quad \text{for all} \quad i \in \mathbb{Z}.$$

Now we have proved (4.3.3) and (4.3.4), and hence we have finally proved that the closure of A_5 is normal.

4.4 The orbit $A_4 + A_1$

$A_4 + A_1$ is the only nilpotent orbit of dimension 62 according to the table on p. 129 in [CM93], and the dimension of $G.\left[\begin{smallmatrix}0&0&2&2&0\\&&0&&\end{smallmatrix}\right]$ is also 62, cf. Richardson's dense orbit theorem, Theorem 1.3, hence

$$\overline{A_4 + A_1} = G.\left[\begin{smallmatrix}0&0&2&2&0\\&&0&&\end{smallmatrix}\right].$$

We have proved that the closure of $D_5(a_1)$ is normal. We will use this to show that also the closure of $A_4 + A_1$ is normal. In Step 1 in Section 4.2 we observed that $\overline{D_5(a_1)} = G.\left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right]$. Consider the short exact sequence of B-modules (V is the cokernel)

$$0 \to \left[\begin{smallmatrix}0&0&2&2&0\\&&0&&\end{smallmatrix}\right] \to \left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right] \to V \to 0.$$

Then V^* is one dimensional with T-weight $\left\{\begin{smallmatrix}0&0&0&0&0\\&&1&&\end{smallmatrix}\right\}$, and the Koszul resolution of the dual sequence is

$$0 \to S^n \left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&0&0&0&0\\&&1&&\end{smallmatrix}\right\} \to S^n \left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right]^* \to S^n \left[\begin{smallmatrix}0&0&2&2&0\\&&0&&\end{smallmatrix}\right]^* \to 0.$$

Taking the long exact sequence in cohomology and observing that

$$H^i(S^n \left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&0&0&0&0\\&&1&&\end{smallmatrix}\right\}) = 0 \quad \text{for all} \quad i > 0, n \in \mathbb{Z} \tag{4.4.1}$$

by Example 3.15, we get a short exact sequence

$$\begin{aligned} 0 \to H^0(S^n \left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&0&0&0&0\\&&1&&\end{smallmatrix}\right\}) \\ \to H^0(S^n \left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right]^*) \to H^0(S^n \left[\begin{smallmatrix}0&0&2&2&0\\&&0&&\end{smallmatrix}\right]^*) \to 0 \end{aligned} \tag{4.4.2}$$

for all $n \in \mathbb{Z}$. Now let $x \in \left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right]$ be a Richardson element. By Richardson's dense orbit theorem, Theorem 1.3, we get

$$\overline{G.x} = G.\left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right] = \overline{D_5(a_1)}$$

and hence $G.x = D_5(a_1)$, and x belongs to the orbit $D_5(a_1)$. But for all elements y in the orbit $D_5(a_1)$ we know that $Z_G(y) = Z_G(y)^0$, cf. the table on pp. 428-429 in [Car85], since we are allowed to us the characteristic zero result as described in Section 2.2.4. Now Lemma 2.6 gives that

$$G \times^{P_{\{\alpha_1,\alpha_2,\alpha_5\}}} \left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right] \to G.\left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right]$$

is birational. Since $\overline{D_5(a_1)} = G.\left[\begin{smallmatrix}0&0&2&2&0\\&&2&&\end{smallmatrix}\right]$ is normal this implies by Lemma 2.2 (and by (4.4.2)) that $A_4 + A_1 = G.\left[\begin{smallmatrix}0&0&2&2&0\\&&0&&\end{smallmatrix}\right]$ is normal.

Note that Eric Sommers uses Proposition 3.3 three times and then Proposition 3.5 to get the vanishing in (4.4.1).

4.5 The orbit D_4

Step 1: The weighted Dynkin diagram of D_4 is $\Delta = \left\{\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right\}$ and since $V(\lambda_\Delta) = \left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right]$ we have by Lemma 2.8

$$\overline{D_4} = G.\left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right].$$

Again we will use that $\overline{D_5(a_1)}$ is normal. By Richardson's dense orbit theorem, Theorem 1.3, we know that $G.\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]$ has dimension $2(36-4) = 64$ so it must equal $\overline{D_5(a_1)}$ or $\overline{A_5}$, cf. the table p. 129 in [CM93]. But

$$\overline{D_4} = G.\left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right] \subseteq G.\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right],$$

and $\overline{D_4}$ is not contained in $\overline{A_5}$. Hence $\overline{D_5(a_1)} = G.\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]$.

Step 2: Look at the short exact sequence (V is the cokernel)

$$0 \to \left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right] \to \left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right] \to V \to 0.$$

V^* has dimension two with T-weights

$$\left\{\begin{smallmatrix}0&0&0&0&1\\&&0&&\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}0&0&0&1&1\\&&0&&\end{smallmatrix}\right\}.$$

We take the Koszul resolution of the dual sequence and get the exact sequence

$$\begin{aligned} 0 \to S^{n-2}\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]^* \otimes \wedge^2 V^* \to S^{n-1}\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]^* \otimes V^* \\ \to S^n\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]^* \to S^n\left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right]^* \to 0 \end{aligned}$$

for all $n \in \mathbb{Z}$. In the following two steps we will show that

$$\begin{aligned} H^i(S^{n-2}\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]^* \otimes \wedge^2 V^*) = 0 \quad \text{for all} \quad i > 1, n \in \mathbb{Z} \\ H^i(S^{n-1}\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]^* \otimes V^*) = 0 \quad \text{for all} \quad i > 0, n \in \mathbb{Z}. \end{aligned}$$

For a moment we assume this. Then splitting the Koszul resolution into short exact sequences, and taking long exact sequences in cohomology, we get the short exact sequence

$$\begin{aligned} 0 \to H^0(\mathrm{Ker}\,(S^n\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]^* \to S^n\left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right]^*)) \\ \to H^0(S^n\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]^*) \to H^0(S^n\left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right]^*) \to 0. \end{aligned}$$

Let $x \in \left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]$ be a Richardson element. By Richardson's dense orbit theorem, Theorem 1.3, we have

$$\overline{G.x} = G.\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right] = \overline{D_5(a_1)}$$

and hence $G.x = D_5(a_1)$, and x belongs to the orbit $D_5(a_1)$. But we just observed that for all elements y in the orbit $D_5(a_1)$ we have $Z_G(y) = Z_G(y)^0$, and Lemma 2.6 gives that

$$G \times^{P_{\{\alpha_1,\alpha_2,\alpha_4\}}} \left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right] \to G.\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]$$

is birational. Since $\overline{D_5(a_1)} = G.\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]$ is normal, this implies that $\overline{D_4} = G.\left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right]$ is normal by Lemma 2.2.

Step 3: We want to show that

$$H^i(S^{n-2}\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]^* \otimes \wedge^2 V^*) = 0 \quad \text{for all} \quad i > 1, n \in \mathbb{Z}.$$

We know that $\wedge^2 V^*$ is one dimensional with T-weight $\left\{\begin{smallmatrix}0&0&0&1&2\\&&0&&\end{smallmatrix}\right\}$. We will use Proposition 3.3 with $l = 4$, $m = 3$, $\alpha_i' = \alpha_i$ for $i = 1, 2, 3, 4$ and $\Gamma = \emptyset$. Then

$$l + 1 - m = 2, \quad m' = 2, \quad I_2 = \{\alpha_1, \alpha_3, \alpha_4\}, \quad I_3 = \{\alpha_1, \alpha_2, \alpha_4\},$$

and

$$\langle \left\{\begin{smallmatrix}0&0&0&1&2\\&&0&&\end{smallmatrix}\right\}, \alpha_i^\vee \rangle = 0 \quad i = 1, 2, 4, \quad r = \langle \left\{\begin{smallmatrix}0&0&0&1&2\\&&0&&\end{smallmatrix}\right\}, \alpha_3^\vee \rangle = -1,$$

and the proposition gives

$$\begin{aligned} &H^i(S^{n-2}\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&0&0&1&2\\&&0&&\end{smallmatrix}\right\}) \\ &\qquad = H^i(S^{n-4}\left[\begin{smallmatrix}0&2&0&0&2\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}1&2&2&2&2\\&&0&&\end{smallmatrix}\right\}) \quad \text{for all} \quad i, n \in \mathbb{Z}. \end{aligned}$$

But Example 3.15 gives that the latter vanishes for $i > 1$ and for all $n \in \mathbb{Z}$, and we are done. Here Eric Sommers has to use Proposition 3.3 before he gets vanishing by Proposition 3.5.

Step 4: We want to show that

$$H^i(S^{n-1}\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]^* \otimes V^*) = 0 \quad \text{for all} \quad i > 0, n \in \mathbb{Z}.$$

Since $\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]$ is a P_{α_4}-representation and $\langle \left\{\begin{smallmatrix}0&0&0&0&1\\&&0&&\end{smallmatrix}\right\}, \alpha_4^\vee \rangle = -1$, we have by Theorem 3.1

$$H^i(S^{n-1}\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]^* \otimes V^*) = H^i(S^{n-1}\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&0&0&1&1\\&&0&&\end{smallmatrix}\right\}) \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

Consider the short exact sequence of B-modules (W is the cokernel)

$$0 \to \left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right] \to \left[\begin{smallmatrix}0&0&2&2&2\\&&2&&\end{smallmatrix}\right] \to W \to 0.$$

Then W^* is one dimensional with T-weight $\left\{\begin{smallmatrix}0&0&0&1&0\\&&0&&\end{smallmatrix}\right\}$. Taking the Koszul resolution of the dual sequence and tensoring with $\left\{\begin{smallmatrix}0&0&0&1&1\\&&0&&\end{smallmatrix}\right\}$, we get the short exact sequence

$$\begin{aligned} 0 \to S^{n-1}\left[\begin{smallmatrix}0&0&2&2&2\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&0&0&2&1\\&&0&&\end{smallmatrix}\right\} &\to S^n\left[\begin{smallmatrix}0&0&2&2&2\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&0&0&1&1\\&&0&&\end{smallmatrix}\right\} \\ &\to S^n\left[\begin{smallmatrix}0&0&2&0&2\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&0&0&1&1\\&&0&&\end{smallmatrix}\right\} \to 0. \end{aligned}$$

Using Example 3.15 we have

$$H^i(S^n\left[\begin{smallmatrix}0&0&2&2&2\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&0&0&1&1\\&&0&&\end{smallmatrix}\right\}) = 0 \quad \text{for all} \quad i > 0, n \in \mathbb{Z}. \tag{4.5.1}$$

Proposition 3.3 with $l = 3$, $m = 3$, $\alpha'_i = \alpha_i$ for $i = 1, 2, 3$ and $\Gamma = \emptyset$ gives

$$\begin{aligned} &H^i(S^{n-1} \left[\begin{smallmatrix} 0 & 0 & 2 & 2 & 2 \\ & & 2 & & \end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix} 0 & 0 & 0 & 2 & 1 \\ & & 0 & & \end{smallmatrix}\right\}) \\ &\qquad = H^i(S^{n-2} \left[\begin{smallmatrix} 2 & 0 & 0 & 2 & 2 \\ & & 2 & & \end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix} 2 & 2 & 2 & 2 & 1 \\ & & 0 & & \end{smallmatrix}\right\}) \quad \text{for all} \quad i, n \in \mathbb{Z}. \end{aligned} \tag{4.5.2}$$

Using Example 3.15 we see that the latter vanish for all $i > 1$ and all $n \in \mathbb{Z}$. Hence

$$H^i(S^n \left[\begin{smallmatrix} 0 & 0 & 2 & 0 & 2 \\ & & 2 & & \end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix} 0 & 0 & 0 & 1 & 1 \\ & & 0 & & \end{smallmatrix}\right\}) = 0 \quad \text{for all} \quad i > 0, n \in \mathbb{Z},$$

and we are done.

Eric Sommers uses Proposition 3.3 three times more and then Proposition 3.5 before he obtains the vanishing in (4.5.1). Similarly he gets the desired vanishing of (4.5.2).

4.6 The orbit $D_4(a_1)$

By Richardson's Dense Orbit Theorem, Theorem 1.3, we know that $G.\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]$ has dimension 58, and the only orbit of dimension 58 is $D_4(a_1)$, cf. the table p. 129 in [CM93], hence

$$\overline{D_4(a_1)} = G.\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]. \tag{4.6.1}$$

In the last section we observed that $\overline{D_4} = G.\left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right]$ is normal. We study the following short exact sequence (V is the cokernel)

$$0 \to \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right] \to \left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right] \to V \to 0$$

where V^* is one dimensional with T-weight $\left\{\begin{smallmatrix}0&0&0&0&0\\&&1&&\end{smallmatrix}\right\}$. The Koszul resolution of the dual sequence is

$$0 \to S^{n-1}\left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&0&0&0&0\\&&1&&\end{smallmatrix}\right\} \to S^n\left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right]^* \to S^n\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \to 0.$$

But

$$H^i(S^{n-1}\left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&0&0&0&0\\&&1&&\end{smallmatrix}\right\}) = 0 \quad \text{for all} \quad i > 0, n \in \mathbb{Z}$$

by Example 3.15, so we have the short exact sequence

$$\begin{aligned}0 \to H^0(S^{n-1}\left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&0&0&0&0\\&&1&&\end{smallmatrix}\right\})&\\ \to H^0(S^n\left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right]^*) \to H^0(S^n\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^*) &\to 0\end{aligned}$$

for all $n \in \mathbb{Z}$. Remember that the weighted Dynkin diagram of D_4 is $\Delta = \left\{\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right\}$. Then $V(\lambda_\Delta) = \left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right]$, and by Corollary 2.9 the morphism

$$G \times^{P(\lambda_\Delta)} \left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right] \to G.\left[\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}\right] = \overline{D_4}$$

is birational. Since $\overline{D_4}$ is normal, this implies by Lemma 2.2 that $\overline{D_4(a_1)} = G.\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]$ is normal.

4.7 The orbit $2A_2 + A_1$

Now we will prove that $2A_2 + A_1$ has normal closure by using the normality of the closure of $D_4(a_1)$.

Step 1: The weighted Dynkind diagram of $D_4(a_1)$ is $\Delta' = \{\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\}$, and by Lemma 2.8

$$\overline{D_4(a_1)} = G.V(\lambda_\Delta) = G.\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right].$$

Let $P = P(\lambda_{\Delta'})$. Then P is the standard parabolic subgroup containing B corresponding to the subset $\{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6\}$ of simple roots. The morphism

$$\bar{p} : G \times^P \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right] \to G.\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]$$

is birational by Corollary 2.9.

Assume we can find a closed B-stable subspace $U \subseteq \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]$ such that $\overline{2A_2 + A_1} = G.U$, and such that the inclusion $U \subseteq \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]$ induces a surjection

$$H^0(G/B, S^n \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^*) \to H^0(G/B, S^n U^*) \to 0$$

Then since $\overline{D_4(a_1)} = G.\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]$ is normal, and since $\bar{p}$ is birational, Lemma 2.2 gives us that $G.U = \overline{2A_2 + A_1}$ is normal.

Step 2: In this step we define U from above, and see that $G.U = \overline{2A_2 + A_1}$.

Since the weighted Dynkin diagram of $2A_2 + A_1$ is $\Delta = \{\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\}$, we know by Lemma 2.8 that

$$\overline{2A_2 + A_1} = G.V(\lambda_\Delta) = G.\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right].$$

Define U' to be the direct sum of the root spaces in $\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]$ except the two root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ where

$$\alpha = \left\{\begin{smallmatrix}-1&-1&-1&0&0\\&&0&&\end{smallmatrix}\right\} \quad \text{and} \quad \beta = \left\{\begin{smallmatrix}0&0&-1&-1&-1\\&&0&&\end{smallmatrix}\right\}.$$

Then U' is B-stable and we have the short exact sequence of B-modules (W is the cokernel)

$$0 \to U' \to \left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right] \to W \to 0$$

where W^* is two dimensional with T-weights

$$\left\{\begin{smallmatrix}1&1&1&0&0\\&&0&&\end{smallmatrix}\right\} \quad \text{and} \quad \left\{\begin{smallmatrix}0&0&1&1&1\\&&0&&\end{smallmatrix}\right\}. \tag{4.7.1}$$

Taking the Koszul resolution of the dual sequence, we get for all $n \in \mathbb{Z}$ the exact seqeuence

$$0 \to S^{n-2}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^2 W^* \to S^{n-1}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^* \otimes W^*$$
$$\to S^n\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^* \to S^n(U')^* \to 0$$

Note that $\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]$ is $P_{\{\alpha_2,\alpha_4,\alpha_6\}}$-stable. Since the weight of $\wedge^2 W^*$ is $\left\{\begin{smallmatrix}1&1&2&1&1\\&&0&&\end{smallmatrix}\right\}$, and since $\langle\left\{\begin{smallmatrix}1&1&2&1&1\\&&0&&\end{smallmatrix}\right\}, \alpha_2^\vee\rangle = -1$, we have by Theorem 3.1 that

$$H^i(G/B, S^{n-2}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^2 W^*) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

We can filter W^* by B-submodules such that the quotients are one dimensional with weights equal to the weights of W^*, i.e. the weights in (4.7.1). Since these weights satisfies $\langle\left\{\begin{smallmatrix}1&1&1&0&0\\&&0&&\end{smallmatrix}\right\}, \alpha_4^\vee\rangle = -1$ and $\langle\left\{\begin{smallmatrix}0&0&1&1&1\\&&0&&\end{smallmatrix}\right\}, \alpha_2^\vee\rangle = -1$, we therefore have by Theorem 3.1

$$H^i(G/B, S^{n-1}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^* \otimes W^*) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

Splitting the Koszul resolution into short exact sequences and taking long exact sequences in cohomology we therefore have

$$H^i(G/B, S^n\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^*) = H^i(G/B, S^n(U')^*) \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

By Lemma 2.1 we see that

$$\overline{2A_2 + A_1} = G.\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right] = G.U'.$$

Now let

$$U = U' \oplus \mathfrak{g}_\alpha \subseteq \mathfrak{u}, \quad \text{where} \quad \alpha = \left\{\begin{smallmatrix}0&-1&-1&-1&0\\&&-1&&\end{smallmatrix}\right\},$$

and notice that U is B-stable. Taking the Koszul resolution of the dual of the short exact sequence of the inclusion $U' \subseteq U$ of B-modules we get the exact sequence

$$0 \to S^{n-1}U^* \otimes \left\{\begin{smallmatrix}0&1&1&1&0\\&&1&&\end{smallmatrix}\right\} \to S^n U^* \to S^n(U')^* \to 0$$

Actually U is $P_{\{\alpha_3\}}$-stable, and since $\langle\left\{\begin{smallmatrix}0&1&1&1&0\\&&1&&\end{smallmatrix}\right\}, \alpha_3^\vee\rangle = -1$ we have by Theorem 3.1 that

$$H^i(G/B, S^n U^*) = H^i(G/B, S^n(U')^*) \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

Hence

$$\overline{2A_2 + A_1} = G.U' = G.U$$

again by Lemma 2.1

Step 3: Now look at the short exact sequence of B-modules

$$0 \to U \to \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right] \to V \to 0 \tag{4.7.2}$$

where V is the cokernel. Then V^* is of dimension nine with the following T-weights

$$\begin{gathered}\left\{\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}0&1&1&0&0\\&&0&&\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}0&0&1&0&0\\&&1&&\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}0&0&1&1&0\\&&0&&\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}1&1&1&0&0\\&&0&&\end{smallmatrix}\right\}, \\ \left\{\begin{smallmatrix}0&1&1&0&0\\&&1&&\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}0&1&1&1&0\\&&0&&\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}0&0&1&1&0\\&&1&&\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}0&0&1&1&1\\&&0&&\end{smallmatrix}\right\}.\end{gathered} \tag{4.7.3}$$

The Koszul resolution of the dual sequence in (4.7.2) is

$$\begin{aligned}0 \to S^{n-11}\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^{11}V^* \to \cdots \to S^{n-1}\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes V^*\\ \to S^{n}\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \to S^nU^* \to 0\end{aligned}$$

Our goal is to show that if $j = 1, 2, \ldots, 8$, $j \neq 6$, then

$$H^i(G/B, S^{n-j}\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^jV^*) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}, \tag{4.7.4}$$

and to show that

$$H^i(G/B, S^{n-6}\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^6V^*) = H^i(G/B, S^{n-6}\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}1&2&6&2&1\\&&1&&\end{smallmatrix}\right\}) \tag{4.7.5}$$

for all $i, n \in \mathbb{Z}$. Before we are going to prove this, we will show why it is enough to prove the normality of the closure of $2A_2 + A_1$.

Let π_n denote the map

$$S^n\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \xrightarrow{\pi_n} S^nU^* \tag{4.7.6}$$

from the Koszul resolution. Let $K_n = \text{Ker } \pi_n$. Now we split the Koszul resolution into short exact sequences, take long exact sequences in cohomology and use (4.7.4) and (4.7.5). This gives rise to a long exact sequence

$$\begin{aligned}\cdots \to H^{i+7}(G/B, S^{n-9}\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^9V^*)\\ \to H^{i+5}(G/B, S^{n-6}\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}1&2&6&2&1\\&&1&&\end{smallmatrix}\right\})\\ \to H^i(G/B, K_n) \to H^{i+8}(G/B, S^{n-9}\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^9V^*) \to \cdots\end{aligned} \tag{4.7.7}$$

But

$$\begin{aligned}H^{i+5}(G/B, S^{n-6}\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}1&2&6&2&1\\&&1&&\end{smallmatrix}\right\})\\ = H^{i}(G/B, S^{n-6}\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}2&4&6&4&2\\&&4&&\end{smallmatrix}\right\})\end{aligned}$$

by Theorem 3.1 used five times, and by Example 3.15 the latter vanishes for all $i > 0$ and all $n \in \mathbb{Z}$. Also

$$\begin{aligned}H^{i+7}(G/B, S^{n-9}\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^9V^*)\\ = H^{i+7}(G/B, S^{n-9}\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}1&4&9&4&1\\&&3&&\end{smallmatrix}\right\})\\ = H^{i}(G/B, S^{n-9}\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}3&6&9&6&3\\&&5&&\end{smallmatrix}\right\})\end{aligned}$$

by Theorem 3.1 used seven times, and again the latter vanishes for all $i > 0$ and all $n \in \mathbb{Z}$ by Example 3.15. But then we get from (4.7.7) that

$$H^i(G/B, K_n) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z},$$

and thus we have the desired surjection

$$0 \to H^0(G/B, \text{Ker } \pi_n) \to H^0(G/B, S^n\left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^*) \to H^0(G/B, S^nU^*) \to 0$$

induced from (4.7.6).

4.7.1 The easy terms in the Koszul resolution

For $j \neq 1, 2, 4, 5, 7, 8$ it is easy to prove (4.7.4). We simply filter $\wedge^j V^*$ by B-submodules such that the quotients are one dimensional with the same weights as in $\wedge^j V^*$. Then it is enough to show that for all the weights, λ, in $\wedge^j V^*$ we have

$$H^i(G/B, S^{n-j} \left[\begin{smallmatrix} 0 & 0 & 2 & 0 & 0 \\ & & 0 & & \end{smallmatrix}\right]^* \otimes \lambda) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}. \tag{4.7.8}$$

Remembering that $\left[\begin{smallmatrix} 0 & 0 & 2 & 0 & 0 \\ & & 0 & & \end{smallmatrix}\right] = \mathfrak{u}_P$ is the unipotent radical of the parabolic subgroup P (from page 76), we can prove (4.7.8) by using Theorem 3.1 – sometimes several times for each weight λ. This is very easy calculations, but since we have to check this for so many weights, we have constructed a computer program that can do the calculations for us, see Appendix A, Section A.3.

4.7.2 The seventh term in the Koszul resolution

In this section we will prove (4.7.4) for $j = 3$, i.e. we will prove that

$$H^i(G/B, S^{n-3} \left[\begin{smallmatrix} 0 & 0 & 2 & 0 & 0 \\ & & 0 & & \end{smallmatrix}\right]^* \otimes \wedge^3 V^*) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}. \tag{4.7.9}$$

Look at the Grothendieck spectral sequence

$$E_2^{i,j} = H^i(G/P, H^j(P/B, S^{n-3} \left[\begin{smallmatrix} 0 & 0 & 2 & 0 & 0 \\ & & 0 & & \end{smallmatrix}\right]^* \otimes \wedge^3 V^*)).$$

We know it abuts to

$$H^{i+j}(G/B, S^{n-3} \left[\begin{smallmatrix} 0 & 0 & 2 & 0 & 0 \\ & & 0 & & \end{smallmatrix}\right]^* \otimes \wedge^3 V^*).$$

If we can show that

$$H^j(P/B, S^{n-3} \left[\begin{smallmatrix} 0 & 0 & 2 & 0 & 0 \\ & & 0 & & \end{smallmatrix}\right]^* \otimes \wedge^3 V^*) = 0 \quad \text{for all} \quad j \in \mathbb{Z},$$

then $E_2^{i,j} = 0$ for all i and j and hence it already collapses at the E_2-term and (4.7.9) is satisfied.

Since $\left[\begin{smallmatrix} 0 & 0 & 2 & 0 & 0 \\ & & 0 & & \end{smallmatrix}\right] = \mathfrak{u}_P$, it is a P-module, and we have by the generalized tensor identity

$$H^j(P/B, S^{n-3} \left[\begin{smallmatrix} 0 & 0 & 2 & 0 & 0 \\ & & 0 & & \end{smallmatrix}\right]^* \otimes \wedge^3 V^*) = S^{n-3} \left[\begin{smallmatrix} 0 & 0 & 2 & 0 & 0 \\ & & 0 & & \end{smallmatrix}\right]^* \otimes H^j(P/B, \wedge^3 V^*).$$

So it is enough to show that

$$H^j(P/B, \wedge^3 V^*) = 0 \quad \text{for all} \quad j \in \mathbb{Z}. \tag{4.7.10}$$

We will do this by restricting even more.

Let L denote the Levi subgroup of P containing T. Let $L' = (L, L)$. Then L' is semi-simple with Borel group $B' = B \cap L'$ and maximal torus $T' = (T \cap L')^0$. Since $H^j(P/B, \wedge^3 V^*)$ equals $H^j(P/B, \wedge^3 V^*)|_{L'}$ as vectorspaces, it is enough to show that the latter vanishes for all $j \in \mathbb{Z}$. But according to Remark I.6.13 in [Jan87] we have

$$H^j(P/B, \wedge^3 V^*)|_{L'} = H^j(L'/B', (\wedge^3 V^*)|_{B'}). \tag{4.7.11}$$

Remember that $P = P_I$ where $I = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6\}$. Now L' is connected and semi-simple with rootsystem $A_2 \times A_2 \times A_1$. But since G is simply connected, also L' is simply connected, cf. Exercise 6, Section 8.4.6 in [Spr98]. Hence L' is isomorphic to $\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{SL}_2$ by the isomorphism theorem of algebraic groups, see e.g. Theorem 9.6.2 in [Spr98].

For $i = 2, 3$ let B_i denote a Borel subgroup in SL_i, and let $T_i \subseteq B_i$ be a maximal torus in SL_i. Now we identify L' with $\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{SL}_2$ in such a way that B' is identified with $B_3 \times B_3 \times B_2$, and T' is identified with $T_3 \times T_3 \times T_2$. Moreover we may asume that the fundamental weights ϖ_1, ϖ_2 are identified with the fundamental weights of the first SL_3-factor, the fundamental weights ϖ_4, ϖ_5 are identified with the fundamental weights of the second SL_3-factor, and ϖ_6 is identified with the fundamental weight of the SL_2-factor.

Let $V^{\mathrm{std}}_{\mathrm{SL}_i}$ denote the standard SL_i-module for $i = 2, 3$, and let V' be the $\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{SL}_2$-module $V^{\mathrm{std}}_{\mathrm{SL}_3} \otimes V^{\mathrm{std}}_{\mathrm{SL}_3} \otimes V^{\mathrm{std}}_{\mathrm{SL}_2}$ where the first SL_3-factor acts on the first $V^{textstd}_{\mathrm{SL}_3}$-factor etc. We will show that $(V|_{B'})^*$ is the B'-submodule of V' given by the direct sum of the weight spaces corresponding to the T'-weights

$$\begin{gathered} -\varpi_2 - \varpi_4 - \varpi_6, \quad -\varpi_1 + \varpi_2 - \varpi_4 - \varpi_6, \quad -\varpi_2 - \varpi_4 + \varpi_6, \\ -\varpi_2 + \varpi_4 - \varpi_5 - \varpi_6, \quad \varpi_1 - \varpi_4 - \varpi_6, \quad -\varpi_1 + \varpi_2 - \varpi_4 + \varpi_6, \\ -\varpi_1 + \varpi_2 + \varpi_4 - \varpi_5 - \varpi_6, \quad -\varpi_2 + \varpi_4 - \varpi_5 + \varpi_6, \quad -\varpi_2 + \varpi_5 - \varpi_6. \end{gathered} \tag{4.7.12}$$

Once we have proved this, Lemma 4.8 (which we will state and prove later) gives us that $H^i(L'/B', (\wedge^3 V^*)|_{B'}) = 0$ for all $i \in \mathbb{Z}$.

Let Z be the direct sum of the rootspaces corresponding to the roots where the coefficient to α_3 is -1, i.e.

$$Z = \bigoplus_{\substack{\alpha \in \Phi^- \\ \alpha = \sum_{j=1}^6 n_j \alpha_j, n_3 = -1}} \mathfrak{g}_\alpha$$

Then Z is clearly L'-stable, and $Z \subseteq \left[\begin{smallmatrix} 0 & 0 & 2 & 0 & 0 \\ & & 0 & & \end{smallmatrix}\right]$. Remember our short exact sequence of B-modules, cf. (4.7.2),

$$0 \to U \to \left[\begin{smallmatrix} 0 & 0 & 2 & 0 & 0 \\ & & 0 & & \end{smallmatrix}\right] \to V \to 0$$

Considering this sequence as a sequence of B'-modules, we have the commutative diagram of B'-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z \cap U & \longrightarrow & Z & \longrightarrow & \tilde{V} & \longrightarrow & 0 \\ & & \cap\!| & & \cap\!| & & & & \\ 0 & \longrightarrow & U & \longrightarrow & \left[\begin{smallmatrix} 0 & 0 & 2 & 0 & 0 \\ & & 0 & & \end{smallmatrix}\right] & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

where $\tilde{V}$ is the cokernel of the first row. Notice that it is also a diagram of T-modules. Since the two rows are exact, we get an injective map of B'-modules and T-modules from $\tilde{V}$ to V. But since

$$\dim V = 9 \quad \text{and} \quad \dim(\tilde{V}) = \dim Z - \dim(Z \cap U) = 9,$$

it must be an isomorphism, and therefore $\tilde{V}$ and V are isomorphic as B'- and T-modules. Hence we have $V^*|_{B'} \simeq (\tilde{V})^* \subseteq Z^*$.

Now we are going to show that Z^* is actually isomorphic to $V^{\text{std}}_{\text{SL}_3} \otimes V^{\text{std}}_{\text{SL}_3} \otimes V^{\text{std}}_{\text{SL}_2}$ and that $(\tilde{V})^*$ is the B'-subrepresentation with weights as in (4.7.12).

The T-weights of Z^* are (written as usual in the basis of the simple roots)

$$
\begin{gathered}
\left\{\begin{smallmatrix} 0&0&1&0&0\\ &&0&& \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} 0&1&1&0&0\\ &&0&& \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} 0&0&1&0&0\\ &&1&& \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} 0&0&1&1&0\\ &&0&& \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} 0&1&1&0&0\\ &&1&& \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} 1&1&1&0&0\\ &&0&& \end{smallmatrix}\right\}, \\
\left\{\begin{smallmatrix} 0&1&1&1&0\\ &&0&& \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} 0&0&1&1&1\\ &&0&& \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} 0&0&1&1&0\\ &&1&& \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} 1&1&1&1&0\\ &&0&& \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} 0&1&1&1&1\\ &&0&& \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} 0&1&1&1&0\\ &&1&& \end{smallmatrix}\right\}, \\
\left\{\begin{smallmatrix} 1&1&1&0&0\\ &&1&& \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} 0&0&1&1&1\\ &&1&& \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} 1&1&1&1&1\\ &&0&& \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} 1&1&1&1&0\\ &&1&& \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} 0&1&1&1&1\\ &&1&& \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} 1&1&1&1&1\\ &&1&& \end{smallmatrix}\right\}.
\end{gathered}
$$

and hence the T'-weights of Z^* are the weights in (4.7.12) together with the following weights

$$
\begin{gathered}
\varpi_1 + \varpi_4 - \varpi_5 - \varpi_6, \quad -\varpi_1 + \varpi_2 + \varpi_5 - \varpi_6, \quad -\varpi_1 + \varpi_2 + \varpi_4 - \varpi_5 + \varpi_6, \\
\varpi_1 - \varpi_4 + \varpi_6, \quad -\varpi_2 + \varpi_5 + \varpi_6, \quad \varpi_1 + \varpi_5 - \varpi_6, \quad \varpi_1 + \varpi_4 - \varpi_5 + \varpi_6, \\
-\varpi_1 + \varpi_2 + \varpi_5 + \varpi_6, \quad \varpi_1 + \varpi_5 + \varpi_6.
\end{gathered}
$$

Now let $k_{\varpi_1+\varpi_5+\varpi_6}$ denote the one dimensional B'-module with T'-weight $\varpi_1 + \varpi_5 + \varpi_6$. Then the projection map

$$Z^* \to k_{\varpi_1+\varpi_5+\varpi_6}$$

is a map of B'-modules. By Fronbenius reciprocity we have

$$\text{Hom}_{B'}(Z^*, k_{\varpi_1+\varpi_5+\varpi_6}) \simeq \text{Hom}_{L'}(Z^*, H^0(L'/B', k_{\varpi_1+\varpi_5+\varpi_6}))$$

and hence we get a nonzero map of L'-modules

$$\phi : Z^* \to H^0(L'/B', k_{\varpi_1+\varpi_5+\varpi_6}) \tag{4.7.13}$$

Now

$$
\begin{aligned}
H^0(L'/B', k_{\varpi_1+\varpi_5+\varpi_6}) &= H^0(\text{SL}_3 \times \text{SL}_3 \times \text{SL}_2/B_3 \times B_3 \times B_2, k_{\varpi_1+\varpi_5+\varpi_6}) \\
&= H^0(\text{SL}_3/B_3 \times \text{SL}_3/B_3 \times \text{SL}_2/B_2, k_{\varpi_1} \otimes k_{\varpi_5} \otimes k_{\varpi_6})
\end{aligned}
$$

Since $\varpi_1 \in X^*(T_3)$, $\varpi_5 \in X^*(T_3)$ and $\varpi_6 \in X^*(T_2)$ are dominant we know that

$$
\begin{aligned}
H^i(\text{SL}_3/B_3, k_{\varpi_1}) &= 0 \quad \text{for} \quad i > 0, \\
H^i(\text{SL}_3/B_3, k_{\varpi_5}) &= 0 \quad \text{for} \quad i > 0, \\
H^i(\text{SL}_2/B_2, k_{\varpi_6}) &= 0 \quad \text{for} \quad i > 0,
\end{aligned}
$$

according to Kempf's vanishing theorem (Proposition II4.5 in [Jan87]). Hence we have by the Künneth fomula that

$$
\begin{aligned}
&H^0(\text{SL}_3/B_3 \times \text{SL}_3/B_3 \times \text{SL}_2/B_2, k_{\varpi_1} \otimes k_{\varpi_5} \otimes k_{\varpi_6}) \\
&= H^0(\text{SL}_3/B_3, k_{\varpi_1}) \otimes H^0(\text{SL}_3/B_3, k_{\varpi_5}) \otimes H^0(\text{SL}_2/B_2, k_{\varpi_6})
\end{aligned}
$$

We want to show that this module is irreducible and isomorphic to the module $V' = V_{\mathrm{SL}_3}^{\mathrm{std}} \otimes V_{\mathrm{SL}_3}^{\mathrm{std}} \otimes V_{\mathrm{SL}_2}^{\mathrm{std}}$. To prove this we need to use a corollary to the Borel-Bott-Weil theorem, but in order to use this corollary we need a bit of notation.

For $i = 2, 3$ let Φ_i^+ denote the set of positive roots in $X^*(T_i)$, and let ρ_i denote half the sum of the positive roots, i.e. $\rho_i = \frac{1}{2}\sum_{\alpha\in\Phi_i^+}\alpha$. For $i = 2, 3$ define

$$\overline{C}_{\mathbb{Z},i} = \{\lambda \in X^*(T_i) | 0 \leq \langle \lambda + \rho_i, \beta^\vee\rangle \text{ for all } \beta \in \Phi_i^+\}$$

if $\mathrm{char}(k) = 0$, and

$$\overline{C}_{\mathbb{Z},i} = \{\lambda \in X^*(T_i) | 0 \leq \langle \lambda + \rho_i, \beta^\vee\rangle \leq p \text{ for all } \beta \in \Phi_i^+\}$$

if $\mathrm{char}(k) = p > 0$.

Now $\varpi_1 \in \overline{C}_{\mathbb{Z},3}$, and ϖ_1 is dominant. Therefore the module $H^0(\mathrm{SL}_3/B_3, k_{\varpi_1})$ is an irreducible SL_2-module with highest weight ϖ_1 by Corollary II.5.6 in [Jan87], and $H^0(\mathrm{SL}_3/B_3, k_{\varpi_1})$ is the standard SL_3-representation $V_{\mathrm{SL}_3}^{\mathrm{std}}$. Similarly

$$H^0(\mathrm{SL}_3/B_3, k_{\varpi_5}) = V_{\mathrm{SL}_3}^{\mathrm{std}} \quad \text{and} \quad H^0(\mathrm{SL}_2/B_2, k_{\varpi_6}) = V_{\mathrm{SL}_2}^{\mathrm{std}}, \tag{4.7.14}$$

and these modules are irreducible. Therefore

$$H^0(\mathrm{SL}_3/B_3 \times \mathrm{SL}_3/B_3 \times \mathrm{SL}_2/B_2, k_{\varpi_1} \otimes k_{\varpi_5} \otimes k_{\varpi_6}) \simeq V_{\mathrm{SL}_3}^{\mathrm{std}} \otimes V_{\mathrm{SL}_3}^{\mathrm{std}} \otimes V_{\mathrm{SL}_2}^{\mathrm{std}} = V',$$

and this module is irreducible. Hence the map ϕ from (4.7.13) must be surjective since it is nonzero. But $\dim Z^* = 18$ and

$$\dim(V_{\mathrm{SL}_3}^{\mathrm{std}} \otimes V_{\mathrm{SL}_3}^{\mathrm{std}} \otimes V_{\mathrm{SL}_2}^{\mathrm{std}}) = 3 \cdot 3 \cdot 2 = 18$$

and hence ϕ is an isomorphism and Z^* is isomorphic to $V' = V_{\mathrm{SL}_3}^{\mathrm{std}} \otimes V_{\mathrm{SL}_3}^{\mathrm{std}} \otimes V_{\mathrm{SL}_2}^{\mathrm{std}}$.

We have seen that $V^*|_{B'} \simeq (\tilde{V})^*$ is a B'-submodule of Z^*, and it is easy to see from (4.7.3) that the T'-weights of $(\tilde{V})^*$ are the ones listed in (4.7.12).

Now we are ready to apply Lemma 4.8. We get

$$H^j(L'/B', (\wedge^3 V^*)|_{B'}) = H^j(\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{SL}_2/B_3 \times B_3 \times B_2, \wedge^3(\tilde{V})^*) = 0$$

for all $j \in \mathbb{Z}$, and by (4.7.11) we have proved (4.7.10) and hence (4.7.9) which was the goal of this section. Now it only remains to prove the following lemma.

Lemma 4.8. Let $G = \mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{SL}_2$, let $B = B_3 \times B_3 \times B_2$ be a Borel subgroup in G, and let $T = T_3 \times T_3 \times T_2$ be a maximal torus contained in B. Let ϖ_i, $i = 1, 2, 4, 5, 6$ be the fundamental weights as described on page 80.

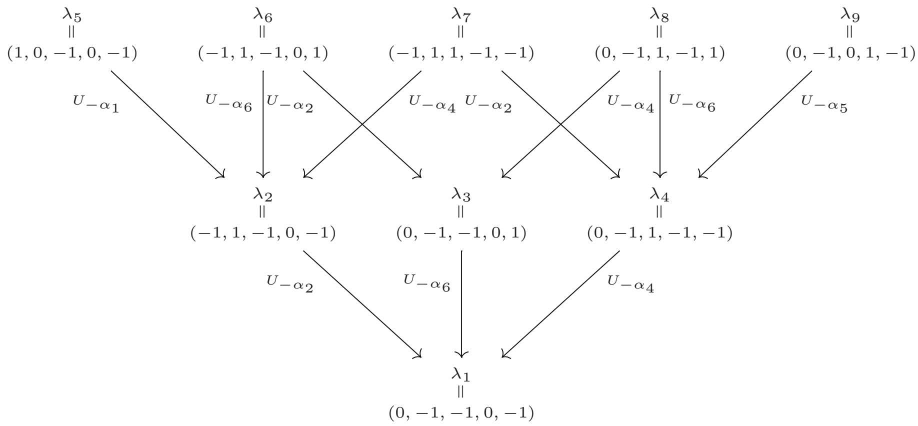
Let U be the G-module $V_{\mathrm{SL}_2}^{\mathrm{std}} \otimes V_{\mathrm{SL}_2}^{\mathrm{std}} \otimes V_{\mathrm{SL}_2}^{\mathrm{std}}$, and let U' be the B-submodule of U given by the T-weights in (4.7.12). Then

$$H^j(G/B, \wedge^3 U') = 0 \quad \text{for all} \quad j \in \mathbb{Z}$$

Proof. We will compute $H^j(G/B, \wedge^3 U')$ by making a filtration of $\wedge^3 U'$ with B-modules and computing the cohomology groups of the quotients in the filtration.

Remember that the T-weights of U' are the ones written down in (4.7.12). In the next diagram we will write the coefficients of these weights in the ordered basis

$(\varpi_1, \varpi_2, \varpi_4, \varpi_5, \varpi_6)$. Here the arrows indicate how the root groups $U_{-\alpha_i} \subseteq B$ acts on U'.



(4.7.15)

Let $v_i \in U'$ be a nonzero weight vector in U' of weight λ_i, and let

$$v_{s,t,u} = v_s \wedge v_t \wedge v_u \in \wedge^3 U'.$$

Then

$$\{v_{s,t,u} | 1 \leq s < t < u \leq 9\}$$

is a basis for $\wedge^3 U'$. Also define

$$\lambda_{s,t,u} = \lambda_s + \lambda_t + \lambda_u \quad \text{for} \quad 1 \leq s < t < u \leq 9.$$

Then the weight of $v_{s,t,u}$ is exactly $\lambda_{s,t,u}$. We say that $v_{s,t,u}$ is of the form $[a, b, c]$ where $a, b, c \in \{1, 2, 3\}$, if λ_s can be found in row number a (counted from below) in diagram (4.7.15), and if λ_t can be found in row b, and λ_u in row c. For example $v_{1,8,9}$ is of the form $[1, 3, 3]$.

Now we are ready to describe the filtration $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq \wedge^3 U'$ of B-submodules of $\wedge^3 U'$. Define

$$V_0 = 0,\ V_1 = kv_{1,2,3},\ V_2 = V_1 \oplus kv_{1,2,4},\ V_3 = V_2 \oplus kv_{1,3,4},\ V_4 = V_3 \oplus kv_{2,3,4}.$$

Then the quotients $Q_l = V_l/V_{l-1}$, $l = 1, 2, 3, 4$ are one dimensional with T-weights μ_i that satisfies $\langle \mu_l, \alpha_m^\vee \rangle = -1$ for $m = 1, 2, 4, 5$ or 6. Hence $H^j(G/B, Q_i) = 0$ for all $j \in \mathbb{Z}$ by Theorem 3.1. Remark that there are only 14 $v_{s,t,u}$'s with weight $\lambda_{s,t,u}$ which do not satisfy $\langle \lambda_{s,t,u}, \alpha_m^\vee \rangle = -1$ for $m = 1, 2, 4, 5$ or 6. They are

$$\begin{array}{ccccccc} v_{1,2,5}, & v_{1,4,9}, & v_{1,5,9}, & v_{3,4,8}, & v_{2,3,6}, & v_{2,4,7}, & v_{2,5,9}, \\ v_{4,5,7}, & v_{4,5,9}, & v_{2,7,9}, & v_{3,5,6}, & v_{3,8,9}, & v_{6,7,8}, & v_{5,7,9}. \end{array} \qquad (4.7.16)$$

Let

$$V_5 = V_4 \oplus kv_{1,2,5}, \quad V_6 = V_5 \oplus kv_{1,4,9}.$$

Now we construct the next parts of the filtration such that the quotients are one dimensional with weights of the form $[1,2,3]$. All weights of this form occur however we omit the two weights $\lambda_{1,2,5}$ and $\lambda_{1,4,9}$ (since we allready have quotients with these weigths). All these quotients have vanishing cohomology by Theorem 3.1 since they are not in the list (4.7.16).

The next B-modules in the filtrations are chosen such that the quotients are one dimensional with weights of the form $[2,2,3]$. Again all weights of this form occur except the three weights $\lambda_{2,4,7}$, $\lambda_{2,3,6}$ and $\lambda_{3,4,8}$. Again these quotients have vanishing cohomology by Theorem 3.1.

Now we take B-modules with quotients of dimension one with all the weights of the form [1,3,3] except $\lambda_{1,5,9}$. These quotients have vanishing cohomology by the same argument.

Let V_N be the last B-module we constructed, and define the B-modules

$$V_{N+1} = V_N \oplus kv_{3,5,6} \oplus kv_{2,3,6}, \qquad V_{N+2} = V_{N+1} \oplus kv_{2,7,9} \oplus kv_{2,4,7},$$
$$V_{N+3} = V_{N+2} \oplus kv_{3,8,9} \oplus kv_{3,4,8}, \qquad V_{N+4} = V_{N+3} \oplus kv_{1,5,9} \oplus kv_{2,5,9}.$$

The next B-modules in the filtration are again constructed such that the quotients are one dimensional with all the weights of the form $[2,3,3]$ except $\lambda_{3,5,6}$, $\lambda_{2,7,9}$, $\lambda_{3,8,9}$ and except $\lambda_{4,5,7}$ and $\lambda_{4,5,9}$. Again all these quotients have vanishing cohomology by Theorem 3.1.

Let again V_M be the last module constructed, and define

$$V_{M+1} = V_M \oplus kv_{4,5,7}, \quad V_{M+2} = V_{M+1} \oplus kv_{4,5,9} \oplus kv_{5,7,9}$$
$$V_{M+3} = V_{M+2} \oplus v_{6,7,8}$$

Now the next modules in the filtration are again one dimensional with all the weights of the form $[3,3,3]$ omitting the weights $\lambda_{5,7,9}$ and $\lambda_{6,7,8}$. This finishes the filtration of $\wedge^3 U'$.

Let $Q_l = V_l/V_{l-1}$ denote the quotients of the filtration. The only quotients with non vanishing cohomology are Q_5, Q_6, Q_{N+1}, Q_{N+2}, Q_{N+3}, Q_{N+4}, Q_{M+1}, Q_{M+2} and Q_{M+3}. We will study these quotients a bit.

Note that Q_{N+1}, Q_{N+2}, Q_{N+3}, Q_{N+4} and Q_{M+2} are two dimensional. The two weights of Q_{N+1} are $\lambda_{3,5,6} = \left\{\begin{smallmatrix}1&2&3&0&0\\&&2&&\end{smallmatrix}\right\}$ and $\lambda_{2,3,6} = \left\{\begin{smallmatrix}0&2&3&0&0\\&&2&&\end{smallmatrix}\right\}$. Note that $\lambda_{3,5,6} = \alpha_1 + \lambda_{2,3,6}$, and that

$$\langle \lambda_{2,3,6}, \alpha_1^\vee \rangle = \langle \left\{\begin{smallmatrix}0&2&3&0&0\\&&2&&\end{smallmatrix}\right\}, \alpha_1^\vee \rangle = -2,$$

Hence Lemme 1 in [Dem77] gives that

$$H^j(G/B, Q_{N+1}) = 0 \quad \text{for all} \quad j \in \mathbb{Z}.$$

Similarly Q_{N+2}, Q_{N+3}, Q_{N+4} and Q_{M+2} have vanishing cohomology.

We will use the Borel-Bott-Weil Theorem to show that Q_5, Q_6, Q_{M+1} and Q_{M+3} has only one non vanishing cohomology group which is either in degree two or three. Remember that Q_5 is one dimensional with weight

$$\lambda_{1,2,5} = -3\varpi_4 - 3\varpi_6$$

Hence

$$\begin{aligned} H^j(G/B, Q_5) &= H^j(G/B, k_{-3\varpi_4 - 3\varpi_6}) \\ &= H^j(\mathrm{SL}_3/B_3 \times \mathrm{SL}_3/B_3 \times \mathrm{SL}_2/B_2, k_0 \otimes k_{-3\varpi_4} \otimes k_{-3\varpi_6}) \end{aligned} \tag{4.7.17}$$

To calculate this cohomology group we will use the Künneth formula, so we need to compute

$$H^j(\mathrm{SL}_3/B_3, k_0), \quad H^j(\mathrm{SL}_3/B_3, k_{-\varpi_4}), \quad \text{and} \quad H^j(\mathrm{SL}_2/B_2, k_{-3\varpi_6}).$$

First notice that $0 \in X^*(T_3)$ is dominant, so according to Kempf's vanishing theorem (Proposition II.4.5 in [Jan87]) we have

$$H^j(\mathrm{SL}_3/B_3, k_0) = 0 \quad \text{for} \quad j > 0.$$

Also notice that k_0 is the one dimensional B_3-module with T_3-weight 0. Hence k_0 is the one dimensional trivial B_3-module, and therefore $H^0(\mathrm{SL}_3/B_3, k_0)$ is the one dimensional trivial SL_3-module which we will denote k. With this notation

$$H^j(\mathrm{SL}_3/B_3, k_0) = \begin{cases} k & \text{for} \quad j = 0 \\ 0 & \text{for} \quad j \neq 0 \end{cases}$$

Remember the "dot" action defined in (1.1), and notice that $s_{\alpha_5} s_{\alpha_4} \cdot (-3\varpi_4) = 0$ and that $0 \in \overline{C}_{\mathbb{Z},3}$ is dominant ($\overline{C}_{\mathbb{Z},3}$ is defined on page 82). By the Borel-Bott-Weil theorem (Corollary II.5.5 in [Jan87]) we therefore know that

$$H^j(\mathrm{SL}_3/B_3, k_{-3\varpi_4}) = \begin{cases} H^0(\mathrm{SL}_3/B_3, k_0) = k & \text{for} \quad j = 2 \\ 0 & \text{for} \quad j \neq 2 \end{cases}$$

Since $s_{\alpha_6} \cdot (-3\varpi_6) = \varpi_6$, and since $\varpi_6 \in \overline{C}_{\mathbb{Z},2}$ is dominant, we get by the Borel-Bott-Weil theorem

$$H^j(\mathrm{SL}_2/B_2, k_{-3\varpi_6}) = \begin{cases} H^0(\mathrm{SL}_2/B_2, k_{\varpi_6}) = V^{\mathrm{std}}_{\mathrm{SL}_2} & \text{for} \quad j = 1 \\ 0 & \text{for} \quad j \neq 1 \end{cases}$$

where we remember that $H^0(\mathrm{SL}_2/B_2, k_{\varpi_6}) = V^{\mathrm{std}}_{\mathrm{SL}_2}$, cf. (4.7.14). Let W denote G-representation $k \otimes k \otimes V^{\mathrm{std}}_{\mathrm{SL}_2}$ where the first SL_3-factor acts on the first k trivially etc. Also notice that W is an irreducible G-representation – we will need it later. Using all these cohomology results we get by the Künneth formula applied to (4.7.17)

$$H^j(G/B, Q_5) = \begin{cases} W & \text{for} \quad j = 3 \\ 0 & \text{for} \quad j \neq 3 \end{cases} \tag{4.7.18}$$

Similarly we can calculate that

$$\begin{aligned} H^j(G/B, Q_6) &= \begin{cases} W & \text{for} \quad j = 3 \\ 0 & \text{for} \quad j \neq 3 \end{cases} \\ H^j(G/B, Q_{M+1}) = H^j(G/B, Q_{M+3}) &= \begin{cases} W & \text{for} \quad j = 2 \\ 0 & \text{for} \quad j \neq 2 \end{cases} \end{aligned} \tag{4.7.19}$$

Since all the quotients in the filtration of $\wedge^3 U'$ only has non vanishing cohomology groups in degree two and three the same must be true for $\wedge^3 U'$, i.e.

$$H^j(G/B, \wedge^3 U') = 0 \quad \text{when} \quad j \neq 2, 3. \tag{4.7.20}$$

Now we need to prove that $H^2(G/B, \wedge^3 U') = 0$ and $H^3(G/B, \wedge^3 U') = 0$. Consider the short exact sequence of B-modules

$$0 \to U' \to U \to U/U' \to 0$$

and the corresponding Koszul resolution with five terms

$$0 \to \wedge^3 U' \to \wedge^2 U' \otimes U \to U' \otimes S^2 U \to S^3 U \to S^3(U/U') \to 0$$

Again we notice that

$$H^i(G/B, \wedge^2 U') = 0 \quad \text{for all} \quad i \in \mathbb{N}$$
$$H^i(G/B, U') = 0 \quad \text{for all} \quad i \in \mathbb{N}$$

by Theorem 3.1 just as we proved (4.7.4) for $j = 1, 2$, cf. Section 4.7.1. Since U is a G-representation this implies by the generalized tensor identity that

$$H^i(G/B, \wedge^2 U' \otimes U) = 0 \quad \text{for all} \quad i \in \mathbb{N}$$
$$H^i(G/B, U' \otimes S^2 U) = 0 \quad \text{for all} \quad i \in \mathbb{N}$$

and therefore

$$H^2(G/B, \wedge^3 U') = \text{Ker}\ (H^0(G/B, S^3 U) \to H^0(G/B, S^3(U/U')))$$

But again since U is a G-representation we have $H^0(G/B, S^3 U) = S^3 U$ and we see that $H^2(G/B, \wedge^3 U')$ is a G-submodule in $S^3 U$. We will use this to show that $H^j(G/B, \wedge^3 U') = 0$ for $j = 2, 3$.

Remember our filtration $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq \wedge^3 U'$. This gives us the following short exact sequences

$$0 \to V_{l-1} \to V_l \to Q_l \to 0$$

Remember that for $l \neq 5, 6, M+1, M+3$ we have $H^j(G/B, Q_l) = 0$ for all $j \in \mathbb{N}$. Hence taking long exact sequences in cohomology corresponding to these short exact sequences and using the results about the cohomology groups $H^j(G/B, Q_l)$ for $l = 5, 6, M+1, M+3$ in (4.7.18) and (4.7.19), we get three exact sequences

$$0 \to W \to H^3(G/B, V_M) \to W \to 0$$

$$0 \to H^2(G/B, V_{M+2}) \to W \to H^3(G/B, V_M) \to H^3(G/B, V_{M+2}) \to 0$$

and

$$\begin{aligned} 0 \to H^2(G/B, V_{M+2}) \to & H^2(G/B, \wedge^3 U') \\ & \to W \to H^3(G/B, V_{M+2}) \to H^3(G/B, \wedge^3 U') \to 0 \end{aligned}$$

Remember that $H^2(G/B, \wedge^3 U') \subseteq S^3 U$ is a G-submodule. We will show that W is not a submodule of S^3U. For a moment assume this. Then we are ready to show that $H^i(G/B, \wedge^3 U') = 0$ for all $i \in \mathbb{N}$.

Since W is irreducible, we get that

$$H^2(G/B, V_{M+2}) = W \quad \text{or} \quad H^2(G/B, V_{M+2}) = 0$$

by the second short exact sequence. If $H^2(G/B, V_{M+2}) = W$, then by the third exact sequence W is a submodule of $H^2(G/B, \wedge^3 U')$ which is a submodule of S^3U and we have reached a contradiction. Therefore $H^2(G/B, V_{M+2}) = 0$.

But then by the third exact sequence $H^2(G/B, \wedge^3 U')$ is a submodule of W. Again W is irreducible so either

$$H^2(G/B, \wedge^3 U') = W \quad \text{or} \quad H^2(G/B, \wedge^3 U') = 0$$

But since W is not a submodule of S^2U by assumption, and since $H^2(G/B, \wedge^3 U')$ is a submodule of S^2U, we are in the case where $H^2(G/B, \wedge^3 U') = 0$.

Now the dimension of W is two, and hence $H^3(G/B, V_M)$ has dimension four according to the first short exact sequence. But then the second short exact sequence gives that the dimension of $H^3(G/B, V_{M+2})$ is two, and the third exact sequence shows that $H^3(G/B, \wedge^3 U') = 0$. Now we have proved the lemma under the assumption that W is not a submodule of S^3U.

In order to show that W is not a G-submodule of S^3U we will prove that S^3U has a good filtration, i.e. a filtration with quotients of the form $H^0(G/B, k_\lambda)$ with $\lambda \in X^*(T)$ dominant. We will also find the quotients in a good filtration (remember that the quotients are independent of the actual filtration).

Remember that $U = V^{\text{std}}_{\text{SL}_3} \otimes V^{\text{std}}_{\text{SL}_3} \otimes V^{\text{std}}_{\text{SL}_2}$. Using the earlier methods to compute cohomology – remember for example how we computed $H^j(G/B, Q_5)$ – we see that

$$\begin{aligned} U &= H^0(\text{SL}_3/B_3, k_{\varpi_1}) \otimes H^0(\text{SL}_3/B_3, k_{\varpi_5}) \otimes H^0(\text{SL}_2/B_2, k_{\varpi_6}) \\ &= H^0(G/B, k_{\varpi_1+\varpi_5+\varpi_6}). \end{aligned}$$

But $\varpi_1 + \varpi_5 + \varpi_6$ is dominant, and therefore U is itself a good filtration of U. But then $U \otimes U$ also admits a good filtration, cf. Proposition II.4.19 in [Jan87]. Now S^3U is a quotient of $U \otimes U$, and we have a spitting

$$S^3U \to U \otimes U$$

given by

$$f_1 \otimes f_2 \otimes f_3 \mapsto \tfrac{1}{6} \sum_{\sigma \in S_3} f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes f_{\sigma(3)}$$

where S_3 is the symmetric group on three letters. Thus S^3U is a direct summand of $U \otimes U$, and it has a good filtration, cf. Proposition II.4.16 in [Jan87].

We can compute that

$$W = k \otimes k \otimes V^{\text{std}}_{\text{SL}_2} = H^0(G/B, k_{\varpi_6}).$$

Now notice that ϖ_6 is dominant. Hence, in order to show that W is not a submodule of S^3U, it is enough to observe that $H^0(G/B, k_{\varpi_6})$ does not occur as a quotient in a good filtration of S^3U. Now we will find these quotients.

Let $\lambda_1, \dots, \lambda_m \in X^*(T)$ be dominant weights such that the quotients in a good filtration of S^3U are of the form $H^0(G/B, k_{\lambda_i})$. Then

$$S^3U \simeq \bigoplus_{i=1}^{m} H^0(G/B, k_{\lambda_i})$$

as T-representations. We want to find the λ_i's.

We know all the weights of S^3U counted with multiplicities. The weight $3\varpi_1 + 3\varpi_5 + 3\varpi_6$ is a highest weight of S^3U, and hence one of the λ_i's must be equal to $3\varpi_1 + 3\varpi_5 + 3\varpi_6$. By reordering the λ_i's we may assume that $\lambda_m = 3\varpi_1 + 3\varpi_5 + 3\varpi_6$. Now

$$S^3U/H^0(G/B, k_{3\varpi_1+3\varpi_5+3\varpi_6}) \simeq \bigoplus_{i=1}^{m-1} H^0(G/B, k_{\lambda_i}) \tag{4.7.21}$$

as T-representations.

There are many ways to find the weights of $H^0(G/B, k_{3\varpi_1+3\varpi_5+3\varpi_6})$ counted with multiplicities. One can for example use Kostant's multiplicity formula (see e.g. Theorem 24.2 in [Hum78]), and one can reduce to the SL_3- or SL_2-case by Kempf's vanishing theorem and the Künneth formula as described in the calculation of $H^j(G/B, Q_5)$. Hence we can find the weights of

$$S^3U/H^0(G/B, k_{3\varpi_1+3\varpi_5+3\varpi_6}).$$

We see that $\varpi_1 + \varpi_2 + \varpi_4 + \varpi_5 + 3\varpi_6$ is a highest weight of this module, and by (4.7.21) we may assume that $\lambda_{m-1} = \varpi_1 + \varpi_2 + \varpi_4 + \varpi_5 + 3\varpi_6$.

Continuing this way we can find all the λ_i's and hence the quotients in a good filtration of S^3U. They are

$$\begin{array}{ll} H^0(G/B, k_{3\varpi_1+3\varpi_5+3\varpi_6}), & H^0(G/B, k_{\varpi_1+\varpi_2+\varpi_4+\varpi_5+3\varpi_6}), \\ H^0(G/B, k_{3\varpi_6}), & H^0(G/B, k_{3\varpi_1+\varpi_4+\varpi_5+\varpi_6}), \\ H^0(G/B, k_{\varpi_1+\varpi_2+3\varpi_5+\varpi_6}), & H^0(G/B, k_{\varpi_1+\varpi_2+\varpi_4+\varpi_5+\varpi_6}). \end{array}$$

But $W = H^0(G/B, k_{\varpi_6})$ is not one of these modules, and hence W is not a submodule of S^3U. □

4.7.3 The fourth term in the Koszul resolution

In this section we will prove (4.7.5), i.e. we will prove that

$$H^i(G/B, S^{n-6}\left[\begin{smallmatrix} 0\ 0\ 2\ 0\ 0 \\ 0 \end{smallmatrix}\right]^* \otimes \wedge^6 V^*) = H^i(G/B, S^{n-6}\left[\begin{smallmatrix} 0\ 0\ 2\ 0\ 0 \\ 0 \end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix} 1\ 2\ 6\ 2\ 1 \\ 1 \end{smallmatrix}\right\}) \tag{4.7.22}$$

for all $i \in \mathbb{Z}$ and all $n \in \mathbb{Z}$. We will do this using the same method as in the last section, but with a little twist.

Let Q be the B-submodule of $\wedge^6 V^*$ corresponding to the T-weights λ of $\wedge^6 V^*$ that satisfies $\langle \lambda, \alpha_6^\vee \rangle \in \{-4, -6\}$. Then we have a short exact sequence of B-modules

$$\begin{aligned} 0 \to S^{n-6} \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes Q \to S^{n-6} \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^6 V^* \\ \to S^{n-6} \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes (\wedge^6 V^*/Q) \to 0 \end{aligned}$$

The idea is to show that

$$H^i(G/B, S^{n-6} \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes Q) = H^i(G/B, S^{n-6} \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}1&2&6&2&1\\&&1&&\end{smallmatrix}\right\}) \tag{4.7.23}$$

for all $i \in \mathbb{Z}$, and to show that

$$H^i(G/B, S^{n-6} \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes (\wedge^6 V^*/Q)) = 0 \tag{4.7.24}$$

for all $i \in \mathbb{Z}$.

It is easiest to show (4.7.23). As in the last section we can filter Q with B-modules $0 \subseteq V_0 \subseteq V_1 \subseteq \ldots \subseteq V_N = Q$ with quotients $Q_s = V_s/V_{s-1}$ such that all the quotients except one (call it Q_{s_0}) satisfies

$$H^j(P/B, Q_s) = 0$$

for all $j \in \mathbb{Z}$. This vanishing is obtained by observing that either Q_s can be chosen to be one dimensional with vanishing cohomology by Theorem 3.1, or to be two dimensional with vanishing cohomology by Lemme 1 in [Dem77]. We can make the filtration such that Q_{s_0} is one dimensional with T-weight $\left\{\begin{smallmatrix}1&2&6&2&1\\&&1&&\end{smallmatrix}\right\}$. Take a look at the Grothendick spectral sequence

$$E_2^{i,j}(Q_s) = H^i(G/P, H^j(P/B, S^{n-6} \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes Q_s)).$$

We know it abuts to

$$H^{i+j}(G/B, S^{n-6} \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes Q_s).$$

The generalized tensor identity gives that

$$H^j(P/B, S^{n-6} \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes Q_s)) = S^{n-6} \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes H^j(P/B, Q_s).$$

Since $H^j(P/B, Q_s) = 0$ for all $j \in \mathbb{Z}$ when $s \neq s_0$, we know that $E_2^{i,j}(Q_s) = 0$ for all i and j. Hence

$$H^i(G/B, S^{n-6} \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes Q_s) = 0 \tag{4.7.25}$$

for all $i \in \mathbb{Z}$ and all $s \neq s_0$. This implies the desired equation (4.7.23).

Section 4.7.2 was devoted to show that

$$H^i(G/B, S^{n-3} \left[\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^3 V^*) = 0$$

for all $i \in \mathbb{Z}$ and all $n \in \mathbb{Z}$. We will use the same method to show (4.7.24), but since Section 4.7.2 is quite long we will not include all details this time. We will use the same notation as in Section 4.7.2. Again we restrict to $L' = (L, L) \simeq \mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{SL}_2$ and observe that it is enough to show that

$$H^j(L'/B', (\wedge^6 V^*/Q)|_{B'}) = 0 \tag{4.7.26}$$

for all $j \in N$ in order to prove (4.7.24). Let $Q' = Q|_{B'}$. Again we show the vanishing result by showing the following Lemma.

Lemma 4.9. Let $G = \mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{SL}_2$, let $B = B_3 \times B_3 \times B_2$ be a Borel subgroup in G, and let $T = T_3 \times T_3 \times T_2$ be a maximal torus of G contained in B. Let U be the G-module $V^{\mathrm{std}}_{\mathrm{SL}_3} \otimes V^{\mathrm{std}}_{\mathrm{SL}_3} \otimes^{\mathrm{std}}_{\mathrm{SL}_3}$, and let U' be the B-module described in Lemma 4.8. Furthermore let Q' denote the B-submodule of $\wedge^6 U'$ arising as above. Then

$$H^j(G/B, (\wedge^6 U')/Q') = 0 \quad \text{for all} \quad j \in \mathbb{Z}.$$

Proof. Again we proceed by making a filtration of $\wedge^6 U'/Q'$ with B-submodules. This time we find a filtration

$$0 = U_0 \subseteq U_1 \subseteq U_2 \subseteq \ldots \subseteq U_M = \wedge^6 U'/Q'$$

with quotients $P_i = U_i/U_{i-1}$. Now we can find $i_1 < i_2 < i_3 < i_4$ such that if $i \neq i_1, \ldots, i_4$, then

$$H^j(G/B, P_i) = 0$$

for all $j \in \mathbb{Z}$. Again this result is obtained by using Theorem 3.1 and Lemme 1 in [Dem77]. Now the P_{i_j}'s are one dimensional with T-weight λ_{i_j} where

$$\lambda_{i_1} = -3\varpi_1 - 3\varpi_5 - 2\varpi_6, \qquad \lambda_{i_2} = -3\varpi_1 - 2\varpi_4 - 2\varpi_5,$$
$$\lambda_{i_3} = -2\varpi_1 + \varpi_2 - 2\varpi_4 - 2\varpi_5, \qquad \lambda_{i_4} = -3\varpi_1 + \varpi_4 - 2\varpi_5 - 2\varpi_6.$$

Let k denote the trivial G-representation. Again one can calculate that

$$H^j(G/B, P_{i_1}) = H^j(G/B, P_{i_2}) = \begin{cases} k & \text{for} \quad j = 5 \\ 0 & \text{for} \quad j \neq 5 \end{cases}$$
$$H^j(G/B, P_{i_3}) = H^j(G/B, P_{i_4}) = \begin{cases} k & \text{for} \quad j = 4 \\ 0 & \text{for} \quad j \neq 4 \end{cases}$$

and hence $H^j(G/B, \wedge^6 U'/Q') = 0$ for all $j \neq 4, 5$. Furthermore we obtain the following exact sequences in cohomology

$$0 \to k \to H^5(G/B, U_2) \to k \to 0$$

$$0 \to H^4(G/B, U_3) \to k \to H^5(G/B, U_2) \to H^5(G/B, U_3) \to 0$$

$$0 \to H^4(G/B, U_3) \to H^4(G/B, \wedge^6 U'/Q') \to k \to H^5(G/B, U_3) \to H^5(G/B, \wedge^6 U'/Q') \to 0$$

This implies that the trivial G-module k is a submodule of $H^4(G/B, \wedge^6 U'/Q')$ or that

$$H^4(G/B, \wedge^6 U'/Q') = H^5(G/B, \wedge^6 U'/Q') = 0. \tag{4.7.27}$$

Therefore we just need to show that k is a not submodule of $H^4(G/B, \wedge^6 U'/Q')$.

To prove this we will use the Koszul resolution with eight terms induced by the inclusion $U' \subseteq U$:

$$0 \to \wedge^6 U' \to U \otimes \wedge^5 U' \to \cdots \to S^5 U \otimes U' \to S^6 U \to S^6(U/U') \to 0$$

Notice that $H^j(G/B, \wedge^t U') = 0$ for all $j \in \mathbb{Z}$ when $t = 1, 2, \ldots, 5$. For $t \neq 3$ this is shown by using Theorem 3.1 just as we proved (4.7.4) and for $t = 3$ it follows from Lemma 4.8. Since U is a G-module this implies by the generalized tensor identity that

$$H^j(G/B, S^{6-t} U \otimes \wedge^t U') = 0$$

for all $j \in \mathbb{Z}$ and for $t = 1, \ldots, 5$. Therefore

$$H^j(G/B, \wedge^6 U') = 0$$

for $j = 0, 1, \ldots, 4$, and hence the short exact sequence

$$0 \to Q' \to \wedge^6 U' \to \wedge^6 U'/Q' \to 0$$

induces this exact sequence in cohomology

$$\begin{aligned} 0 \to H^4(G/B, \wedge^6 U'/Q') &\to H^5(G/B, Q') \\ &\to H^5(G/B, \wedge^6 U') \to H^5(G/B, \wedge^6 U'/Q') \to \cdots \end{aligned}$$

But using the filtration of Q' on page 89 we can show that $H^5(G/B, Q') = H^5(G/B, k_{-3\varpi_2 - 3\varpi_4 - 4\varpi_6})$ which is irreducible of dimension three. Therefore the trivial G-module k cannot be a submodule of $H^4(G/B, \wedge^6 U'/Q')$ and (4.7.27) is satisfied. □

Now we have proved (4.7.22) and hence (4.7.5) which was the missing result in order to prove that the closure of $2A_2 + A_1$ is normal.

4.8 The orbit $A_2 + 2A_1$

Step 1: We are going to use that $2A_2+A_1$ has normal closure in order to show that A_2+2A_1 has normal closure. By Richardson's dense orbit theorem, Theorem 1.3, the dimension of $G.\left[\begin{smallmatrix}0&0&0&2&0\\&&0&&\end{smallmatrix}\right]$ is $= 50$, and the only orbit of dimension 50 is A_2+2A_1 by the table p. 129 in [CM93]. Therefore

$$\overline{A_2+2A_1} = G.\left[\begin{smallmatrix}0&0&0&2&0\\&&0&&\end{smallmatrix}\right].$$

The weighted Dynkin diagram of $2A_2 + A_1$ is $\Delta = \left\{\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right\}$, and $V(\lambda_\Delta) = \left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]$. Therefore Lemma 2.8 gives

$$\overline{2A_2+A_1} = G.\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right].$$

Let U be the B-subrepresentation of $\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]$ obtained by omitting the two root spaces

$$\mathfrak{g}_{\left\{\begin{smallmatrix}-1&-1&-1&0&0\\&&0&&\end{smallmatrix}\right\}}, \quad \mathfrak{g}_{\left\{\begin{smallmatrix}-1&-1&-1&0&0\\&&-1&&\end{smallmatrix}\right\}}.$$

We will show that $G.\left[\begin{smallmatrix}0&0&0&2&0\\&&0&&\end{smallmatrix}\right] = G.U$. Look at the short exact sequence (U_1 is the cokernel)

$$0 \to U \to \left[\begin{smallmatrix}0&0&0&2&0\\&&0&&\end{smallmatrix}\right] \to U_1 \to 0. \tag{4.8.1}$$

Then U_1^* is six dimensional with T-weights

$$\left\{\begin{smallmatrix}0&0&0&1&0\\&&0&&\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}0&0&1&1&0\\&&0&&\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}0&0&0&1&1\\&&0&&\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}0&0&1&1&0\\&&1&&\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}0&1&1&1&0\\&&0&&\end{smallmatrix}\right\}, \left\{\begin{smallmatrix}0&1&1&1&0\\&&1&&\end{smallmatrix}\right\}.$$

We look at the first six terms of the Koszul resolution of the dual sequence. Applying Theorem 3.1 several times we get

$$H^i(S^{n-j}\left[\begin{smallmatrix}0&0&0&2&0\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^j U_1^*) = 0 \quad \text{for all} \quad i,n \in \mathbb{Z}, j = 1,2,\ldots,6.$$

In practice the calculations were done using the computer program in Appendix A, Section A.3. Hence

$$H^i(S^n\left[\begin{smallmatrix}0&0&0&2&0\\&&0&&\end{smallmatrix}\right]^*) = H^i(S^n U^*) \quad \text{for all} \quad i,n \in \mathbb{Z},$$

and $\overline{A_2+2A_1} = G.\left[\begin{smallmatrix}0&0&0&2&0\\&&0&&\end{smallmatrix}\right] = G.U$ by Lemma 2.1.

Step 2: Look at the short exact sequence of B-modules (defining V)

$$0 \to U \to \left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right] \to V \to 0.$$

Then V^* is two dimensional with weights

$$\left\{\begin{smallmatrix}1&1&1&0&0\\&&0&&\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}1&1&1&0&0\\&&1&&\end{smallmatrix}\right\}.$$

We take the Koszul resolution of the dual sequence

$$0 \to S^{n-2}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^2 V^* \to S^{n-1}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^* \otimes V^* \to S^n\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^* \to S^n U^* \to 0. \tag{4.8.2}$$

Theorem 3.1 gives that

$$H^i(S^{n-1}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^*\otimes\left\{\begin{smallmatrix}1&1&1&0&0\\&&0&&\end{smallmatrix}\right\})=0\quad\text{for all}\quad i,n\in\mathbb{Z}$$
$$H^i(S^{n-1}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^*\otimes\left\{\begin{smallmatrix}1&1&1&0&0\\&&1&&\end{smallmatrix}\right\})=0\quad\text{for all}\quad i,n\in\mathbb{Z}$$

because $\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]$ is a P_{α_4}-representation. Hence

$$H^i(\text{Ker }(S^n\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^*\to S^nU^*))=H^{i+1}(S^{n-2}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^*\otimes\wedge^2V^*)$$

for all $i,n\in\mathbb{Z}$. If we can show that the latter vanishes for all $i>0$, we have the short exact sequence

$$0\to H^1(S^{n-2}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^*\otimes\wedge^2V^*)\to H^0(S^n\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^*)\to H^0(S^nU^*)\to 0$$

for all $n\in\mathbb{Z}$. We assume this for a moment. Remember that $\Delta=\left\{\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right\}$ is the weighted Dynkin diagram for $2A_2+A_1$ and $V(\lambda_\Delta)=\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]$. Then

$$G\times^{P(\lambda_\Delta)}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]\to G.\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]=\overline{2A_2+A_1}$$

is birational by Corollary 2.9. Since $\overline{2A_2+A_1}$ is normal, Lemma 2.2 gives that $\overline{A_2+2A_1}=G.U$ is normal.

Step 3: We need to show that

$$H^{i+1}(S^{n-2}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^*\otimes\wedge^2V^*)=0\quad\text{for all}\quad i>0,n\in\mathbb{Z}.$$

We know that $\wedge^2V^*$ is one dimensional with T-weight $\left\{\begin{smallmatrix}2&2&2&0&0\\&&1&&\end{smallmatrix}\right\}$, and by Theorem 3.1 we get

$$\begin{aligned}&H^{i+1}(S^{n-2}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^*\otimes\left\{\begin{smallmatrix}2&2&2&0&0\\&&1&&\end{smallmatrix}\right\})\\&\qquad=H^i(S^{n-2}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^*\otimes\left\{\begin{smallmatrix}2&2&2&1&0\\&&1&&\end{smallmatrix}\right\})\quad\text{for all}\quad i,n\in\mathbb{Z}.\end{aligned}$$

By symmetry

$$\begin{aligned}&H^{i}(S^{n-2}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^*\otimes\left\{\begin{smallmatrix}2&2&2&1&0\\&&1&&\end{smallmatrix}\right\})\\&\qquad=H^i(S^{n-2}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^*\otimes\left\{\begin{smallmatrix}0&1&2&2&2\\&&1&&\end{smallmatrix}\right\})\quad\text{for all}\quad i,n\in\mathbb{Z}.\end{aligned}$$

We tensor the exact sequence of (4.8.2) (writing $n-2$ instead of n) with $\left\{\begin{smallmatrix}0&1&2&2&2\\&&1&&\end{smallmatrix}\right\}$ and obtain the following exact sequence

$$\begin{aligned}0\to\ &S^{n-2}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^*\otimes\wedge^2V^*\otimes\left\{\begin{smallmatrix}0&1&2&2&2\\&&1&&\end{smallmatrix}\right\}\\&\to S^{n-1}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^*\otimes V^*\otimes\left\{\begin{smallmatrix}0&1&2&2&2\\&&1&&\end{smallmatrix}\right\}\\&\qquad\to S^{n}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^*\otimes\left\{\begin{smallmatrix}0&1&2&2&2\\&&1&&\end{smallmatrix}\right\}\to S^nU^*\otimes\left\{\begin{smallmatrix}0&1&2&2&2\\&&1&&\end{smallmatrix}\right\}\to 0\end{aligned}$$

Using Theorem 3.1 two times we get

$$H^i(S^{n-3}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^*\otimes V^*\otimes\left\{\begin{smallmatrix}0&1&2&2&2\\&&1&&\end{smallmatrix}\right\})=0\quad\text{for all}\quad i,n\in\mathbb{Z}.$$

In Step 4 we will show that

$$H^i(S^{n-2}U^* \otimes \left\{\begin{smallmatrix}0&1&2&2&2\\&&1&&\end{smallmatrix}\right\}) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

Assuming this, we have

$$\begin{aligned}
&H^i(S^{n-2}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&1&2&2&2\\&&1&&\end{smallmatrix}\right\})\\
&\qquad = H^{i+1}(S^{n-4}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^2 V^* \otimes \left\{\begin{smallmatrix}0&1&2&2&2\\&&1&&\end{smallmatrix}\right\})\\
&\qquad = H^{i+1}(S^{n-4}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}2&3&4&2&2\\&&2&&\end{smallmatrix}\right\})\\
&\qquad = H^{i}(S^{n-4}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}2&3&4&3&2\\&&2&&\end{smallmatrix}\right\})
\end{aligned}$$

for all $i, n \in \mathbb{Z}$ where we use Theorem 3.1 to get the last equality. But the latter vanishes for all $i > 0$ by Example 3.15, and since

$$H^{i+1}(S^{n-2}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^2 V^*) = H^i(S^{n-2}\left[\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&1&2&2&2\\&&1&&\end{smallmatrix}\right\})$$

for all $i, n \in \mathbb{Z}$, we are done.

Step 4: We will now show that

$$H^i(S^{n-2}U^* \otimes \left\{\begin{smallmatrix}0&1&2&2&2\\&&1&&\end{smallmatrix}\right\}) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

Take a look at the Koszul resolution of the dual sequence of (4.8.1) (writing $n-2$ instead of n). Then tensor this sequence with $\left\{\begin{smallmatrix}0&1&2&2&2\\&&1&&\end{smallmatrix}\right\}$. Using Theorem 3.1 several times – i.e. using the computer program in Appendix A, Section A.3 – we observe that

$$H^i(S^{n-2-j}\left[\begin{smallmatrix}0&0&0&2&0\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^j U_1^* \otimes \left\{\begin{smallmatrix}0&1&2&2&2\\&&1&&\end{smallmatrix}\right\}) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}, j = 1, 2, \ldots, 6$$

where $6 = \dim(U_1^*)$. Hence

$$H^i(S^{n-2}U^* \otimes \left\{\begin{smallmatrix}0&1&2&2&2\\&&1&&\end{smallmatrix}\right\}) = H^i(S^{n-2}\left[\begin{smallmatrix}0&0&0&2&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&1&2&2&2\\&&1&&\end{smallmatrix}\right\}) \quad \text{for all} \quad i, n \in \mathbb{Z}.$$

But the latter vanishes for all $i, n \in \mathbb{Z}$ by Theorem 3.1, and we are done.

4.9 The orbit A_2

Step 1: We are going to use that the closure of $A_2 + 2A_1$ is normal to prove that A_2 has normal closure. The weighted Dynkin diagram of A_2 is $\Delta = \{\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}\}$ and $V(\lambda_\Delta) = [\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}]$, and by Lemma 2.8 we have

$$\overline{A_2} = G.\,[\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}]\,.$$

The weighted Dynkin diagram of $A_2 + 2A_1$ is $\Delta' = \{\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\}$. Similarly we have $V(\lambda_{\Delta'}) = [\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}]$ and

$$\overline{A_2 + 2A_1} = [\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}]\,.$$

Let

$$U = [\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}] \cap [\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}]\,. \tag{4.9.1}$$

We will show that $\overline{A_2} = G.U$. We study the short exact sequence of B-modules (V is the cokernel)

$$0 \to U \to [\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}] \to V \to 0.$$

The dimension of V^* is 6, and its T-weights are

$$\{\begin{smallmatrix}0&0&0&0&0\\&&1&&\end{smallmatrix}\},\ \{\begin{smallmatrix}0&0&1&0&0\\&&1&&\end{smallmatrix}\},\ \{\begin{smallmatrix}0&1&1&0&0\\&&1&&\end{smallmatrix}\},\ \{\begin{smallmatrix}0&0&1&1&0\\&&1&&\end{smallmatrix}\},\ \{\begin{smallmatrix}1&1&1&0&0\\&&1&&\end{smallmatrix}\},\ \{\begin{smallmatrix}0&0&1&1&1\\&&1&&\end{smallmatrix}\}.$$

We take the Koszul resolution of the dual sequence, and observe that

$$H^i(S^{n-j}\,[\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}]^* \otimes \wedge^j V^*) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}, j = 1, 2, \dots, 6 = \dim(V^*)$$

by Theorem 3.1 used several times, again we have used the computer program in Appendix A, Section A.3 for the calculations. Therefore we get

$$H^i(S^n\,[\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}]^*) = H^i(S^n U^*) \quad \text{for all} \quad i, n \in \mathbb{Z}, \tag{4.9.2}$$

and $\overline{A_2} = G.\,[\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}] = G.U$ by Lemma 2.1.

Step 2: Look at the short exact sequence (W is the cokernel)

$$0 \to U \to [\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}] \to W \to 0 \tag{4.9.3}$$

Then W^* is four dimensional with weights

$$\{\begin{smallmatrix}0&1&1&1&0\\&&0&&\end{smallmatrix}\},\ \{\begin{smallmatrix}1&1&1&1&0\\&&0&&\end{smallmatrix}\},\ \{\begin{smallmatrix}0&1&1&1&1\\&&0&&\end{smallmatrix}\},\ \{\begin{smallmatrix}1&1&1&1&1\\&&0&&\end{smallmatrix}\}.$$

We take the Koszul resolution of the dual sequence and observe that

$$H^i(S^{n-j}\,[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}]^* \otimes \wedge^j W^*) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}, j = 1, 2 \tag{4.9.4}$$

by using Theorem 3.1 repeatedly. We also observe that

$$H^i(S^{n-3}\,[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}]^* \otimes \lambda) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}$$

for all weights λ in $\wedge^3 W^*$ except for $\lambda = \left\{\begin{smallmatrix}2&3&3&3&2\\&&0&&\end{smallmatrix}\right\}$. Filtering $\wedge^3 W^*$ by one dimensional B-modules we therefore have

$$H^i(S^{n-3}\left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^3 W^*) = H^i(S^{n-3}\left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}2&3&3&3&2\\&&0&&\end{smallmatrix}\right\}) \qquad (4.9.5)$$

for all $i, n \in \mathbb{Z}$. The last part of the Koszul resolution is here

$$\cdots \to S^{n-1}\left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right]^* \otimes W^* \to S^n\left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right]^* \xrightarrow{\pi_n} S^n U^* \to 0$$

Let $K_n = \operatorname{Ker} \pi_n$. Splitting the Koszul resolution into short exact sequences, taking long exact sequences in cohomology and using (4.9.4) and (4.9.5), we get the long exact sequence

$$\begin{aligned}\ldots \to H^{i+2}(S^{n-4}\left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^4 W^*) \to H^{i+2}(S^{n-3}\left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}2&3&3&3&2\\&&0&&\end{smallmatrix}\right\})\\ \to H^i(K_n) \to H^{i+3}(S^{n-4}\left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^4 W^*) \to \ldots\end{aligned}$$

Thus, if we can show

$$\begin{aligned}&H^{i+3}(S^{n-4}\left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right]^* \otimes \wedge^4 W^*) = 0 \quad \text{for all} \quad i > 0, n \in \mathbb{Z}\\ &H^{i+2}(S^{n-3}\left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}2&3&3&3&2\\&&0&&\end{smallmatrix}\right\}) = 0 \quad \text{for all} \quad i > 0, n \in \mathbb{Z}\end{aligned}$$

we get $H^i(K_n) = 0$ for all $i > 0$ and all $n \in \mathbb{Z}$. Then we have the short exact sequence

$$0 \to H^0(K_n) \to H^0(S^n\left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right]^*) \to H^0(S^n U^*) \to 0$$

Now remeber that $\Delta' = \left\{\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right\}$ is the weighted Dynkin diagram of the orbit $A_2 + 2A_1$. Hence the morphism

$$G \times^{P(\lambda_{\Delta'})} \left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right] \to G.\left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right] = \overline{A_2 + 2A_1}$$

is birational by Corollary 2.9. Since $\overline{A_2 + 2A_1}$ is normal, all the conditions of Lemma 2.2 are satisfied, and $\overline{A_2} = G.U$ is normal.

We know that $\wedge^4 W^* = \left\{\begin{smallmatrix}2&4&4&4&2\\&&0&&\end{smallmatrix}\right\}$. Using Theorem 3.1 we get

$$\begin{aligned}&H^{i+3}(S^{n-4}\left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}2&4&4&4&2\\&&0&&\end{smallmatrix}\right\})\\ &\qquad = H^{i+2}(S^{n-4}\left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}2&4&4&4&2\\&&3&&\end{smallmatrix}\right\})\\ &\qquad = H^{i+1}(S^{n-4}\left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}2&4&6&4&2\\&&3&&\end{smallmatrix}\right\}) \quad \text{for all} \quad i, n \in \mathbb{Z}.\end{aligned}$$

The latter vanishes for all $i > 0$ and all $n \in \mathbb{Z}$ by Example 3.15.

Theorem 3.1 also gives

$$\begin{aligned}&H^{i+2}(S^{n-3}\left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}2&3&3&3&2\\&&0&&\end{smallmatrix}\right\})\\ &\qquad = H^{i+1}(S^{n-3}\left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}2&3&3&3&2\\&&2&&\end{smallmatrix}\right\}) \quad \text{for all} \quad i, n \in \mathbb{Z}.\end{aligned}$$

Again the latter vanishes for all $i > 0$ and all $n \in \mathbb{Z}$ by Example 3.15. The last two vanishing results given by Example 3.15 are difficult to obtain for Eric Sommers, so Example 3.15 sometimes gives strong vanishing results.

4.10 The orbit $3A_1$

We have just shown that the closure of A_2 is normal. We can use this result to prove that also $3A_1$ has normal closure.

Step 1: The weighted Dynkin diagram of A_2 is $\Delta = \left\{\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}\right\}$ and since $V(\lambda_\Delta) = \left[\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}\right]$ we have by Lemma 2.8 that

$$\overline{A_2} = G.\left[\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}\right].$$

Now we let $P = P(\lambda_\Delta)$, i.e. P is the standard parabolic subgroup containing B corresponding to the subset $I = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ of simple roots. By Corollary 2.9 the morphism

$$\bar{p} : G \times^P \left[\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}\right] \to G.\left[\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}\right]$$

is birational. Now we know that $\overline{A_2} = G.\left[\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}\right]$ is normal, and $\bar{p}$ is birational. Therefore, if we can find a closed B-stable subspace $W \subseteq \left[\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}\right]$ such that $G.W = \overline{3A_1}$ and such that the inclusion $W \subseteq \left[\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}\right]$ induces a surjection

$$H^0(G/B, S^n \left[\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}\right]^*) \to H^0(G/B, S^n W^*) \to 0 \tag{4.10.1}$$

for all $n \in \mathbb{N}$, then $\overline{3A_1}$ has normal closure by Lemma 2.2.

Step 2: In this step we find the desired W and show that $G.W = \overline{3A_1}$. Since the weighted Dynkin diagram of $3A_1$ is $\left\{\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}\right\}$ we have by Lemma 2.8 that

$$\overline{3A_1} = G.\left[\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}\right].$$

Let W_1 be the B-submodule of $\left[\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}\right]$ obtained by omitting the root space $\mathfrak{g}_\alpha$ where

$$\alpha = \left\{\begin{smallmatrix}0&-1&-2&-1&0\\&&-1&&\end{smallmatrix}\right\},$$

and let W be the $P_{\alpha_2,\alpha_3,\alpha_4}$-module obtained from W_1 by adding the root space $\mathfrak{g}_\beta$ where

$$\beta = \left\{\begin{smallmatrix}-1&-1&-1&-1&-1\\&&-1&&\end{smallmatrix}\right\}.$$

We will show that $G.\left[\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}\right] = G.W$. Consider the short exact sequence (W_2 is the cokernel)

$$0 \to W_1 \to \left[\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}\right] \to W_2 \to 0$$

Then W_2 is one dimensional with T-weight

$$\left\{\begin{smallmatrix}0&-1&-2&-1&0\\&&-1&&\end{smallmatrix}\right\}.$$

Looking at the Koszul resolution of the dual sequence and observing that by Theorem 3.1 we have

$$H^i(G/B, S^{n-1}\left[\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0&1&2&1&0\\&&1&&\end{smallmatrix}\right\}) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z},$$

we get that

$$H^i(G/B, S^n \left[\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}\right]^*) = H^i(G/B, S^n W_1^*) \quad \text{for all} \quad i,n \in \mathbb{Z}.$$

From Lemma 2.1 we see that $\overline{3A_1} = G.\left[\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}\right] = G.W_1$.

Similarly the short exact sequence defined by the inclusion $W_1 \subseteq W$ gives rise to a Koszul resolution of its dual. And again since W is a $P_{\alpha_2,\alpha_3,\alpha_4}$-module we get by Theorem 3.1 that

$$H^i(G/B, S^{n-1}W^* \otimes \left\{\begin{smallmatrix}1&1&1&1&1\\&&1&&\end{smallmatrix}\right\}) = 0 \quad \text{for all} \quad i,n \in \mathbb{Z}.$$

Hence

$$H^i(G/B, S^n W^*) = H^i(G/B, S^n W_1^*) \quad \text{for all} \quad i,n \in \mathbb{N},$$

and again Lemma 2.1 tells us that $G.W = G.W_1 = \overline{3A_1}$.

Step 3: In this step we will begin proving that $W \subseteq \left[\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}\right]$ induces a surjection

$$H^0(G/B, S^n \left[\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}\right]^*) \to H^0(G/B, S^n W^*) \to 0$$

for all $n \in \mathbb{Z}$ as in (4.10.1). First let

$$U = \left[\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}\right] \cap \left[\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}\right]$$

as in (4.9.1). In Section 4.9, cf. (4.9.2), we proved that the inclusion $U \subseteq \left[\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}\right]$ induced an isomorphism

$$H^i(G/B, S^n \left[\begin{smallmatrix}0&0&0&2&0\\&&0&&\end{smallmatrix}\right]^*) = H^i(G/B, S^n U^*) \quad \text{for all} \quad i,n \in \mathbb{Z}.$$

Hence it is enough to prove that the inclusion $W \subseteq U$ induces a surjection

$$H^0(G/B, S^n U^*) \to H^0(G/B, S^n W^*) \to 0 \tag{4.10.2}$$

for all $n \in \mathbb{Z}$. Consider the short exact sequence of B-modules (V is the cokernel)

$$0 \to W \to U \to V \to 0. \tag{4.10.3}$$

Then V^* is four dimensional with T-weights

$$\left\{\begin{smallmatrix}0&1&1&1&0\\&&1&&\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}1&1&1&1&0\\&&1&&\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}0&1&1&1&1\\&&1&&\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}0&1&2&1&0\\&&1&&\end{smallmatrix}\right\}. \tag{4.10.4}$$

Take a look at the Koszul resolution of the dual sequence

$$0 \to S^{n-4}U^* \otimes \wedge^4 V^* \to \cdots \to S^{n-1}U^* \otimes V^* \to S^n U^* \to S^n W^* \to 0 \tag{4.10.5}$$

and notice that U is a $P_{\alpha_1,\alpha_3,\alpha_5}$-module. By Theorem 3.1 used several times we know that

$$H^i(G/B, S^{n-j}U^* \otimes \lambda) = 0 \quad \text{for all} \quad i,n \in \mathbb{Z}$$

for all weights λ in $\wedge^j V^*$ when $j = 1, 3$. By filtering V^* and $\wedge^3 V^*$ by appropriate B-submodules we therefore get

$$H^i(G/B, S^{n-j}U^* \otimes \wedge^j V^*) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}$$

when $j = 1, 3$.

In the next section we will prove that

$$H^i(G/B, S^{n-2}U^* \otimes \wedge^2 V^*) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z} \tag{4.10.6}$$

using the same method as in Section 4.7.2 and Section 4.7.3. Assume this for a moment. By splitting the Koszul resolution in (4.10.5) into short exact sequences and taking long exact sequences in cohomology we obtain the following long exact sequence using all the above vanishing results

$$\begin{aligned} \cdots \to H^{i+3}(G/B, S^{n-4}U^* \otimes \wedge^4 V^*) \to H^i(G/B, S^n U^*) \\ \to H^i(G/B, S^n W^*) \to H^{i+3}(G/B, S^{n-4}U^* \otimes \wedge^4 V^*) \to \cdots \end{aligned}$$

Note that $\wedge^4 V^*$ is the one dimensional B-module with T-weight $\{\begin{smallmatrix} 1 & 4 & 5 & 4 & 1 \\ & & 4 & & \end{smallmatrix}\}$. Using Theorem 3.1 three times we get

$$\begin{aligned} H^{i+3}(G/B, S^{n-4}U^* \otimes \wedge^4 V^*) &= H^{i+3}(G/B, S^{n-4}U^* \otimes \wedge^4 \{\begin{smallmatrix} 1 & 4 & 5 & 4 & 1 \\ & & 4 & & \end{smallmatrix}\}) \\ &= H^i(G/B, S^{n-4}U^* \otimes \wedge^4 \{\begin{smallmatrix} 2 & 4 & 6 & 4 & 2 \\ & & 4 & & \end{smallmatrix}\}) \end{aligned}$$

for all $i, n \in \mathbb{Z}$. But the latter vanishes for $j > 0$ by Example 3.15, and we get a short exact sequence

$$\begin{aligned} 0 \to H^0(G/B, S^{n-4}U^* \otimes \wedge^4 \{\begin{smallmatrix} 2 & 4 & 6 & 4 & 2 \\ & & 4 & & \end{smallmatrix}\}) \\ \to H^0(G/B, S^n U^*) \to H^0(G/B, S^n W^*) \to 0 \end{aligned}$$

and (4.10.2) is satisfied.

4.10.1 The third term in the Koszul resolution

In order to know that the closure of $3A_1$ is normal, it only remains to prove (4.10.6) which states that

$$H^i(G/B, S^{n-2}U^* \otimes \wedge^2 V^*) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}. \tag{4.10.7}$$

We will use the same method as in Section 4.7.2 and Section 4.7.3 to prove this vanishing result.

Let $P = P_{\alpha_1, \alpha_3, \alpha_5}$. The Grothendieck spectral sequence

$$E_2^{i,j} = H^i(G/P, H^j(P/B, S^{n-2}U^* \otimes \wedge^2 V^*))$$

abuts to

$$H^{i+j}(G/B, S^{n-2}U^* \otimes \wedge^2 V^*).$$

If we can show that

$$H^j(P/B, S^{n-2}U^* \otimes \wedge^2 V^*) = 0 \quad \text{for all} \quad j \in \mathbb{Z},$$

then $E_2^{i,j} = 0$ for all i, j and the spectral sequence collapses allready at the E_2-term. Hence (4.10.7) is satisfied.

Since U is a P-module, the generalized tensor identity gives us that

$$H^j(P/B, S^{n-2}U^* \otimes \wedge^2 V^*) = S^{n-2}U^* \otimes H^j(P/B, \wedge^2 V^*),$$

so it is enough to show that

$$H^j(P/B, \wedge^2 V^*) = 0 \quad \text{for all} \quad j \in \mathbb{Z}. \tag{4.10.8}$$

Let L denote the Levi subgroup of P containing T, and let L' be the commutator subgroup of L. Then L' is semi-simple and connected with Borel subgroup $B' = B \cap L'$ and maximal torus $T' = (T \cap L')^0$. According to Remark I.6.13 in [Jan87] we know that

$$H^j(P/B, \wedge^2 V^*)|_{L'} = H^j(L'/B', (\wedge^2 V^*)|_{B'}),$$

so in order to prove (4.10.8), it is enough to prove

$$H^j(L'/B', (\wedge^2 V^*)|_{B'}) = 0 \quad \text{for all} \quad j \in \mathbb{Z}. \tag{4.10.9}$$

Since $P = P_{\alpha_1,\alpha_3,\alpha_5}$, we know that the root system of L' is of type $A_1 \times A_1 \times A_1$. But G is simply connected, and hence also L' is simply connected by Exercise 6 in Section 8.4.6 in [Spr98]. Therefore L' is isomorphic to $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ by the isomorphism theorem of algebraic groups, see e.g. Theorem 9.6.2 in [Spr98].

Let B_2 denote a Borel subgroup in SL_2, and let $T_2 \subseteq B_2$ be a maximal torus in SL_2. Now we identify L' with $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ in such a way that B' is identified with $B_2 \times B_2 \times B_2$, and T' is identified with $T_2 \times T_2 \times T_2$. Moreover we may assume that the fundamental weight ϖ_1 is identified with the fundamental weight of the first SL_2-factor, the fundamental weight ϖ_3 is identified with the fundamental weight of the second SL_2-factor, and ϖ_5 is identified with the fundamental weight of the third SL_2-factor.

Let $V_{\mathrm{SL}_2}^{\mathrm{std}}$ denote the standard SL_2-module, and let V' be the $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$-module $V_{\mathrm{SL}_2}^{\mathrm{std}} \otimes V_{\mathrm{SL}_2}^{\mathrm{std}} \otimes V_{\mathrm{SL}_2}^{\mathrm{std}}$ where the first SL_2-factor acts on the first $V_{\mathrm{SL}_2}^{\mathrm{std}}$-factor etc. We will prove that $V^*|_{B'}$ is the B'-submodule of V' given by the T'-weights

$$-\varpi_1 - \varpi_3 - \varpi_5, \quad -\varpi_1 - \varpi_3 + \varpi_5, \quad -\varpi_1 + \varpi_3 - \varpi_5, \quad \varpi_1 - \varpi_3 - \varpi_5. \tag{4.10.10}$$

When we have proved this result, equation (4.10.9) follows from Lemma 4.10 which we will state and prove later.

Now define Z to be the direct sum of the root spaces $\mathfrak{g}_\alpha \subseteq W$ which satisfies $\alpha = \sum_{i=1}^{6} n_i \alpha_i$ and $n_i \geq -1$ for $i = 2, 4, 6$, i.e. where α equals one of the following roots (written in the basis of simple roots)

$$\begin{gathered}
\left\{\begin{smallmatrix} 0 & -1 & -1 & -1 & 0 \\ & & -1 & & \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} -1 & -1 & -1 & -1 & 0 \\ & & -1 & & \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} 0 & -1 & -1 & -1 & -1 \\ & & -1 & & \end{smallmatrix}\right\}, \\
\left\{\begin{smallmatrix} 0 & -1 & -2 & -1 & 0 \\ & & -1 & & \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} -1 & -1 & -1 & -1 & -1 \\ & & -1 & & \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} -1 & -1 & -2 & -1 & 0 \\ & & -1 & & \end{smallmatrix}\right\}, \\
\left\{\begin{smallmatrix} 0 & -1 & -2 & -1 & -1 \\ & & -1 & & \end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix} -1 & -1 & -2 & -1 & -1 \\ & & -1 & & \end{smallmatrix}\right\}.
\end{gathered} \tag{4.10.11}$$

Note that Z is L'-stable and in particular B'-stable.

Remember the short exact sequence of B-modules, cf. (4.10.3),

$$0 \to W \to U \to V \to 0$$

Considering this as a sequence of B'-modules, we have the following commutative diagram of B'-modules ($\tilde{V}$ is the cokernel)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z \cap W & \longrightarrow & Z & \longrightarrow & \tilde{V} & \longrightarrow & 0 \\ & & \cap\!\mid & & \cap\!\mid & & & & \\ 0 & \longrightarrow & W & \longrightarrow & U & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

Since the two rows are exact, we have an induced injective map $\tilde{V} \to V$ of B'-modules. But since the dimension of V equals the dimension of $\tilde{V}$ (they are both four), the B'-mdoules $\tilde{V}$ and V are isomorphic. Hence

$$V^*|_{B'} \simeq \tilde{V}^* \subseteq Z^*.$$

Now we will show that Z^* is actually isomorphic to $V' = V^{\mathrm{std}}_{\mathrm{SL}_2} \otimes V^{\mathrm{std}}_{\mathrm{SL}_2} \otimes V^{\mathrm{std}}_{\mathrm{SL}_2}$ and that $\tilde{V}^*$ is the B'-submodule of Z^* given by the T'-weights in (4.10.10). The T-weights of Z are given in (4.10.11), and therefore the T'-weights of Z^* are the four weights in (4.10.10) together with the four T'-weights

$$-\varpi_1 + \varpi_3 + \varpi_5, \quad \varpi_1 - \varpi_3 + \varpi_5, \quad \varpi_1 + \varpi_3 - \varpi_5, \quad \varpi_1 + \varpi_3 + \varpi_5.$$

Let $k_{\varpi_1+\varpi_3+\varpi_5}$ denote the one dimensional B'-module with T'-weight $\varpi_1 + \varpi_3 + \varpi_5$. Then the projection map

$$Z^* \to k_{\varpi_1+\varpi_3+\varpi_5}$$

is a map of B'-modules. By Frobenius reciprocity we have

$$\mathrm{Hom}_{B'}(Z^*, k_{\varpi_1+\varpi_3+\varpi_5}) \simeq \mathrm{Hom}_{L'}(Z^*, H^0(L'/B', k_{\varpi_1+\varpi_3+\varpi_5}))$$

and hence we get a nonzero map of L'-modules

$$\phi : Z^* \to H^0(L'/B', k_{\varpi_1+\varpi_3+\varpi_5}).$$

Just as in Section 4.7.2, page 81, we get that

$$H^0(L'/B', k_{\varpi_1+\varpi_3+\varpi_5})$$

is irreducible and isomorphic to $V' = V^{\mathrm{std}}_{\mathrm{SL}_2} \otimes V^{\mathrm{std}}_{\mathrm{SL}_2} \otimes V^{\mathrm{std}}_{\mathrm{SL}_2}$. Hence $Z^* \simeq V'$. Since the T-weights of V^* are the weights listed in (4.10.4), and since $V^*|_{B'} \simeq \tilde{V}^*$, the T'-weights of $\tilde{V}^*$ are the ones listed in (4.10.10). Hence $V^*|_{B'}$ is the desired B'-submodule of $Z^* \simeq V'$.

Now

$$H^j(L'/B', (\wedge^2 V^*)|_{B'}) = H^j(\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 / B_2 \times B_2 \times B_2, \wedge^2 \tilde{V}^*),$$

and by the following lemma the latter vanishes for all $j \in \mathbb{Z}$, and (4.10.9) is satisfied. Then also (4.10.7) is satisfied, and we can conclude that it only remains to prove Lemma 4.10 in order to prove the normality of the closure of $3A_1$.

Lemma 4.10. Let $G = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$, let $B = B_2 \times B_2 \times B_2$ be a Borel subgroup in G, and let $T = T_2 \times T_2 \times T_2$ be a maximal torus contained in B. Let ϖ_i, $i = 1, 3, 5$ be the fundamental weights as described on page 100.

Let U be the G-module $V^{\mathrm{std}}_{\mathrm{SL}_2} \otimes V^{\mathrm{std}}_{\mathrm{SL}_2} \otimes V^{\mathrm{std}}_{\mathrm{SL}_2}$, and let U' be the B-submodule of U given by the T-weights

$$-\varpi_1 - \varpi_3 - \varpi_5, \quad -\varpi_1 - \varpi_3 + \varpi_5, \quad -\varpi_1 + \varpi_3 - \varpi_5, \quad \varpi_1 - \varpi_3 - \varpi_5.$$

Then

$$H^j(G/B, \wedge^2 U') = 0 \quad \text{for all} \quad j \in \mathbb{Z}.$$

Proof. The idea of the proof is to filter $\wedge^2 U'$ with B-submodules and then calculate the cohomology groups of the corresponding quotients.

Let

$$\lambda_1 = -\varpi_1 - \varpi_3 - \varpi_5 \qquad \lambda_2 = -\varpi_1 - \varpi_3 + \varpi_5$$
$$\lambda_3 = -\varpi_1 + \varpi_3 - \varpi_5 \qquad \lambda_4 = \varpi_1 - \varpi_3 - \varpi_5.$$

For $s = 1, 2, 3, 4$ let $v_s \in U'$ be a nonzero weight vector of weight λ_s. Then

$$\{v_s \wedge v_t \in \wedge^2 U' | 1 \le s < t \le 4\}$$

is a basis for $\wedge^2 U'$. Define the filtration

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_4 = \wedge^2 U'$$

of B-submodules by

$$V_1 = k(v_1 \wedge v_2)$$
$$V_2 = V_1 \oplus k(v_1 \wedge v_3) \oplus k(v_2 \wedge v_3)$$
$$V_3 = V_2 \oplus k(v_1 \wedge v_4) \oplus k(v_2 \wedge v_4).$$

Define the quotients $Q_l = V_l / V_{l-1}$ for $l = 1, 2, 3, 4$. Then Q_2 is two dimensional with weights $\lambda_1 + \lambda_3 = -2\varpi_1 - 2\varpi_3$ and $\lambda_2 + \lambda_3 = -2\varpi_1$. Since $\lambda_2 + \lambda_3 = (\alpha_3) + (\lambda_1 + \lambda_3)$, and

$$\langle -2\varpi_1 - 2\varpi_3, \alpha_3^\vee \rangle = -2,$$

Lemme 1 in [Dem77] gives that

$$H^j(G/B, Q_2) = 0 \quad \text{for all} \quad j \in \mathbb{Z}. \tag{4.10.12}$$

Similarly

$$H^j(G/B, Q_3) = 0 \quad \text{for all} \quad j \in \mathbb{Z}. \tag{4.10.13}$$

Now consider Q_1 and Q_4. They are both of dimension one with weights $\lambda_1 + \lambda_2 = -2\varpi_1 - 2\varpi_3$ and $\lambda_3 + \lambda_4 = -2\varpi_5$ respectively, i.e.

$$Q_1 = k_{-2\varpi_1 - 2\varpi_3}, \quad \text{and} \quad Q_4 = k_{-2\varpi_5}.$$

Now

$$\begin{aligned} H^j(G/B, Q_1) &= H^j(G/B, k_{-2\varpi_1 - 2\varpi_3}) \\ &= H^j(\mathrm{SL}_2/B_2 \times \mathrm{SL}_2/B_2 \times \mathrm{SL}_2/B_2, k_{-2\varpi_1} \otimes k_{-\varpi_3} \otimes k_0). \end{aligned} \tag{4.10.14}$$

To calculate this cohomology group we will use the Künneth formula, so we need to compute

$$H^j(\mathrm{SL}_2/B_2, k_{-2\varpi_1}) = H^j(\mathrm{SL}_2/B_2, k_{-2\varpi_3}) \quad \text{and} \quad H^j(\mathrm{SL}_2/B_2, k_0).$$

By Kempf's vanishing theorem, see e.g. Proposition I.4.5 in [Jan87], we have

$$H^j(\mathrm{SL}_2/B_2, k_0) = 0 \quad \text{for all} \quad j > 0$$

since $0 \in X^*(T_2)$ is a dominant weight. But k_0 is the one dimensional B_2-module with weight 0, so it is just the trivial one dimensional B_2-module. Hence $H^0(\mathrm{SL}_2/B_2, k_0)$ is the one dimensional trivial SL_2-module which we will denote k. Hence

$$H^j(\mathrm{SL}_2/B_2, k_0) = \begin{cases} k & \text{for} \quad j = 0 \\ 0 & \text{for} \quad j \neq 0 \end{cases}$$

To compute $H^j(\mathrm{SL}_2/B_2, k_{-2\varpi})$ we will use the Borel-Bott-Weil theorem, see e.g. Corollary II.5.5 in [Jan87]. Now remember the "dot" action defined in (1.1). Since $s_{\alpha_1} \cdot (-2\varpi_1) = 0$, and since $0 \in X^*(T_2)$ is dominant with $0 \in \overline{C}_{\mathbb{Z},2}$ (remember the definition of $\overline{C}_{\mathbb{Z},2}$ on page 82), we have

$$H^j(\mathrm{SL}_2/B_2, k_{-2\varpi_1}) = \begin{cases} H^0(\mathrm{SL}_2/B_2, k_0) = k & \text{for} \quad j = 1 \\ 0 & \text{for} \quad j \neq 1 \end{cases}$$

Now we can us the Künneth formula on the cohomology group in (4.10.14) using the above cohomology results. We get

$$H^j(G/B, Q_1) = \begin{cases} k & \text{for} \quad j = 2 \\ 0 & \text{for} \quad j \neq 2 \end{cases} \tag{4.10.15}$$

where k is the trivial one dimensional G-module obtained as the tensor product of three trivial one dimensional SL_2-modules. Similarly we see that

$$H^j(G/B, Q_4) = \begin{cases} k & \text{for} \quad j = 1 \\ 0 & \text{for} \quad j \neq 2 \end{cases} \tag{4.10.16}$$

Now look at the short exact sequences

$$0 \to V_{l-1} \to V_l \to Q_l \to 0$$

Taking long exact sequences in cohomology and using the results in (4.10.12), (4.10.13), (4.10.15) and (4.10.16) we see that

$$H^j(G/B, \wedge^2 U') = 0 \quad \text{for} \quad j \neq 1, 2,$$

and that we have an exact sequence

$$0 \to H^1(G/B, \wedge^2 U') \to k \to k \to H^2(G/B, \wedge^2 U') \to 0$$

Now we just have to show that

$$H^1(G/B, \wedge^2 U') = 0 \quad \text{and} \quad H^2(G/B, \wedge^2 U') = 0.$$

But either

$$H^1(G/B, \wedge^2 U') = 0 \quad \text{or} \quad H^1(G/B, \wedge^2 U') = k.$$

If $H^1(G/B, \wedge^2 U') = 0$, then also $H^2(G/B, \wedge^2 U') = 0$, and hence it is enough to show that $H^1(G/B, \wedge^2 U') \neq k$.

Now consider the short exact sequence of B-modules

$$0 \to U' \to U \to U/U' \to 0$$

and take the the corresponding Koszul resolution with four terms

$$0 \to \wedge^2 U' \to U \otimes U' \to S^2 U \to S^2(U/U') \to 0$$

Since U is a G-module, we have by the generalized tensor identity

$$H^j(G/B, U \otimes U') = U \otimes H^j(G/B, U'). \tag{4.10.17}$$

Now U' can be filtered by B-submodules such that the quotients in the filtration are one dimensional with vanishing cohomology by Theorem 3.1, and hence

$$H^j(G/B, U') = 0 \quad \text{for all} \quad j \in \mathbb{Z}.$$

But then also the cohomology group in (4.10.17) vanishes for all $j \in \mathbb{Z}$. Splitting the Koszul resolution into short exact sequences and taking long exact sequences in cohomology, we therefore get

$$H^1(G/B, \wedge^2 U') = \text{Ker}\,(H^0(G/B, S^2 U) \to H^0(G/B, S^2(U/U'))).$$

Since U is a G-module, we have $H^0(G/B, S^2 U) = S^2 U$, and we see that the G-module $H^1(G/B, \wedge^2 U')$ is a submodule of $S^2 U$. Now it is enough to show that the trivial one dimensional G-module, k, is not a submodule of $S^2 U$. We will do this by proving that $S^2 U$ admits a good filtration without $k = H^0(G/B, k_0)$ as a quotient.

Remember that U is the standard representation of SL_2 tensored with itself three times. But $H^0(\text{SL}_2/B_2, k_{\varpi_i})$, $i = 1, 3, 5$, equals the standard representation of SL_2 as seen in (4.7.14) in Section 4.7.2. Since $\varpi_i \in X^*(T_2)$ is dominant, Kempf's vanishing theorem tells us that

$$H^j(\mathrm{SL}_2/B_2, k_{\varpi_i}) = 0 \quad \text{for} \quad j > 0,$$

for $i = 1, 3, 5$, so by the Künneth formula we have

$$\begin{aligned} U &= H^0(\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2/B_2 \times B_2 \times B_2, k_{\varpi_1} \otimes k_{\varpi_3} \otimes k_{\varpi_5}) \\ &= H^0(G/B, k_{\varpi_1+\varpi_3+\varpi_5}). \end{aligned}$$

But $\varpi_1 + \varpi_3 + \varpi_5 \in X^*(T)$ is dominant, and therefore U is itself a good filtration of U. But then $U \otimes U$ also admits a good filtration, cf. Proposition II.4.19 in [Jan87]. Now S^2U is a quotient of $U \otimes U$, and we have a spitting

$$S^2U \to U \otimes U$$

given by

$$f \otimes g \mapsto \tfrac{1}{2}(f \otimes g + g \otimes f).$$

Thus S^2U is a direct summand of $U \otimes U$, and it has a good filtration, cf. Proposition II.4.16 in [Jan87].

Now we will find the quotients in a good filtration of S^2U. Let $\lambda_1, \dots, \lambda_m \in X^*(T)$ be dominant weights such that the quotients in a good filtration of S^2U are of the form $H^0(G/B, k_{\lambda_i})$. Then

$$S^2U \simeq \bigoplus_{i=1}^{m} H^0(G/B, k_{\lambda_i})$$

as T-representations. We want to find the λ_i's.

The dimension of S^2U is 36, and we know all the weights of S^2U counted with multiplicities. The weight $2\varpi_1 + 2\varpi_3 + 2\varpi_5$ is a highest weight of S^2U, and hence one of the λ_i's must be equal to $2\varpi_1 + 2\varpi_3 + 2\varpi_5$. By reordering the λ_i's we may assume that $\lambda_m = 2\varpi_1 + 2\varpi_3 + 2\varpi_5$. Now

$$S^2U/H^0(G/B, k_{2\varpi_1+2\varpi_3+2\varpi_5}) \simeq \bigoplus_{i=1}^{m-1} H^0(G/B, k_{\lambda_i}) \tag{4.10.18}$$

as T-representations.

But $H^0(G/B, k_{2\varpi_1+2\varpi_3+2\varpi_5})$ is 27 dimensional, and we can find its weights counted with multiplicities for example by Kostant's multiplicity formula (see e.g. Theorem 24.2 in [Hum78]). Then we see that the weights of

$$S^2U/H^0(G/B, k_{2\varpi_1+2\varpi_3+2\varpi_5})$$

becomes

$$2\varpi_1, \quad 2\varpi_3, \quad 2\varpi_5, \quad 0, \quad 0, \quad 0, \quad -2\varpi_1, \quad -2\varpi_3, \quad -2\varpi_5$$

counted with multiplicities, and that $2\varpi_1$, $2\varpi_3$ and $2\varpi_5$ are highest weights. By (4.10.18) we may assume that $\lambda_{m-1} = 2\varpi_1$, $\lambda_{m-2} = 2\varpi_3$ and $\lambda_{m-3} = 2\varpi_5$. Now the module in (4.10.18) is of dimension $36 - 27 = 9$, and $H^0(G/B, k_{2\varpi_j})$ is of dimension 3 for $j = 1, 3, 5$. Hence $m = 4$, and the quotients in a good filtration of S^2U are

$$H^0(G/B, k_{2\varpi_1+2\varpi_3+2\varpi_5}), \quad H^0(G/B, k_{2\varpi_1}), \quad H^0(G/B, k_{2\varpi_3}), \quad H^0(G/B, k_{2\varpi_5}).$$

Since $k = H^0(G/B, k_0)$ is not one of these modules, it cannot be a subrepresentation of S^2U. Hence $H^i(G/B, \wedge^2 U') = 0$ for all $i \in \mathbb{Z}$.

□

4.11 The orbit $2A_1$

We will use the normality of $\overline{3A_1}$ to show that $2A_1$ has normal closure. Since the weighted Dynkin diagram of $2A_1$ is $\Delta = \{\begin{smallmatrix}1&0&0&0&1\\&&0&&\end{smallmatrix}\}$, and since $V(\lambda_\Delta) = [\begin{smallmatrix}1&0&0&0&1\\&&0&&\end{smallmatrix}]$, we have

$$\overline{2A_1} = G.[\begin{smallmatrix}1&0&0&0&1\\&&0&&\end{smallmatrix}]$$

by Lemma 2.8. Similarly the weighted Dynkin diagram of $3A_1$ is $\Delta' = \{\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}\}$, and we have

$$\overline{3A_1} = G.[\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}].$$

Define now

$$U = [\begin{smallmatrix}1&0&0&0&1\\&&0&&\end{smallmatrix}] \cap [\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}].$$

We want to show that $\overline{2A_1} = G.U$. We have a short exact sequence (V is the cokernel)

$$0 \to U \to [\begin{smallmatrix}1&0&0&0&1\\&&0&&\end{smallmatrix}] \to V \to 0$$

where V^* is two dimensional with T-weights

$$\{\begin{smallmatrix}1&1&1&1&1\\&&0&&\end{smallmatrix}\}, \quad \{\begin{smallmatrix}1&1&1&1&1\\&&1&&\end{smallmatrix}\}.$$

Hence the Koszul resolution of the dual sequence is

$$\begin{aligned}0 \to S^{n-2}[\begin{smallmatrix}1&0&0&0&1\\&&0&&\end{smallmatrix}]^* \otimes \wedge^2 V^* \to S^{n-1}[\begin{smallmatrix}1&0&0&0&1\\&&0&&\end{smallmatrix}]^* \otimes V^* \\ \to S^n[\begin{smallmatrix}1&0&0&0&1\\&&0&&\end{smallmatrix}]^* \to S^n U^* \to 0\end{aligned}$$

Theorem 3.1 gives that

$$H^i(G/B, S^{n-2}[\begin{smallmatrix}1&0&0&0&1\\&&0&&\end{smallmatrix}]^* \otimes \wedge^2 V^*) = 0$$
$$H^i(G/B, S^{n-1}[\begin{smallmatrix}1&0&0&0&1\\&&0&&\end{smallmatrix}]^* \otimes V^*) = 0$$

for all $i \in \mathbb{N}$ and all $n \in \mathbb{Z}$, so by Lemma 2.1 we have

$$\overline{2A_1} = G.[\begin{smallmatrix}1&0&0&0&1\\&&0&&\end{smallmatrix}] = G.U$$

Now we are ready to show that $\overline{2A_1} = G.U$ is normal using that $\overline{3A_1}$ is normal. Consider the short exact sequence of B-modules (W is the cokernel)

$$0 \to U \to [\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}] \to W \to 0$$

Then W^* is five dimensional with T-weights

$$\{\begin{smallmatrix}0&1&2&1&0\\&&1&&\end{smallmatrix}\}, \quad \{\begin{smallmatrix}1&1&2&1&0\\&&1&&\end{smallmatrix}\}, \quad \{\begin{smallmatrix}0&1&2&1&1\\&&1&&\end{smallmatrix}\}, \quad \{\begin{smallmatrix}1&2&2&1&0\\&&1&&\end{smallmatrix}\}, \quad \{\begin{smallmatrix}0&1&2&2&1\\&&1&&\end{smallmatrix}\}.$$

Again we take a look at the Koszul resolution for the dual sequence

$$\begin{aligned}0 \to & S^{n-5}\left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right]^* \otimes \wedge^5 W^* \to \\ & \ldots \to S^{n-j}\left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right]^* \otimes \wedge^j W^* \to \ldots \to S^{n}\left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right]^* \to S^n U^* \to 0\end{aligned}$$

Let K_n denote the kernel of

$$S^{n}\left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right]^* \to S^n U^* \to 0 \tag{4.11.1}$$

Remeber that $\Delta' = \left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right]$ is the weighted Dynkin diagram of the orbit $3A_1$. Then the morphism

$$G \times^{P(\lambda_{\Delta'})} \left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right] \to G.\left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right]$$

is birational by Corollary 2.9. In order to show that $\overline{2A_1} = G.U$ is normal, it follows from Lemma 2.2 that we just have to show that the morphism in (4.11.1) induces a surjection in cohomology since $\overline{3A_1} = G.\left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right]$ is normal.

Filtering $\wedge^j W^*$ by one dimensional B-modules and using Theorem 3.1 several times we get

$$H^i(G/B, S^{n-j}\left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right]^* \otimes \wedge^j W^*) = 0$$

for alle $i \in \mathbb{N}$ and all $n \in \mathbb{Z}$ when $j \neq 3$. Hence

$$H^i(G/B, K_n) = H^{i+2}(G/B, S^{n-3}\left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right]^* \otimes \wedge^3 W^*)$$

for all $i \in \mathbb{N}$ and all $n \in \mathbb{Z}$, and we just need to show that

$$H^{i+2}(G/B, S^{n-3}\left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right]^* \otimes \wedge^3 W^*) = 0 \quad \text{for all} \quad i > 0, n \in \mathbb{Z}. \tag{4.11.2}$$

But now we filter $\wedge^3 W^*$ by B-submodules such that the quotients are one dimensional with the same weights as the weights of $\wedge^3 W^*$. Using Theorem 3.1 we can show that the quotients have vanishing cohomology except the quotients with weights of this form

$$\left\{\begin{smallmatrix}2\,4\,6\,3\,0\\3\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}0\,3\,6\,4\,2\\3\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}1\,4\,6\,4\,1\\3\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}2\,4\,6\,4\,1\\3\end{smallmatrix}\right\}, \quad \left\{\begin{smallmatrix}1\,4\,6\,4\,2\\3\end{smallmatrix}\right\}$$

But by Theorem 3.1 we see that

$$\begin{aligned} & H^i(G/B, S^{n-3}\left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}2\,4\,6\,4\,2\\3\end{smallmatrix}\right\}) \\ &\quad = H^{i+2}(G/B, S^{n-3}\left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}2\,4\,6\,3\,0\\3\end{smallmatrix}\right\}) \\ &\quad = H^{i+2}(G/B, S^{n-3}\left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}0\,3\,6\,4\,2\\3\end{smallmatrix}\right\}) \\ &\quad = H^{i+2}(G/B, S^{n-3}\left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}1\,4\,6\,4\,1\\3\end{smallmatrix}\right\}) \\ &\quad = H^{i+1}(G/B, S^{n-3}\left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}2\,4\,6\,4\,1\\3\end{smallmatrix}\right\}) \\ &\quad = H^{i+1}(G/B, S^{n-3}\left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}1\,4\,6\,4\,2\\3\end{smallmatrix}\right\}) \end{aligned}$$

so in order to show (4.11.2), it is enough to show that

$$H^i(G/B, S^{n-3}\left[\begin{smallmatrix}0\,0\,1\,0\,0\\0\end{smallmatrix}\right]^* \otimes \left\{\begin{smallmatrix}2\,4\,6\,4\,2\\3\end{smallmatrix}\right\}) = 0 \quad \text{for all} \quad i > 0, n \in \mathbb{Z}.$$

But this is satisfied by Example 3.15, and hence we have proved that $\overline{2A_1}$ is normal.

4.12 The orbits A_1 and 0

The orbit A_1 is the minimal orbit with closure equal to the union of A_1 and 0. Since G is semisimple, simply connected and simple as an algebraic group, the orbit A_1 has normal closure, see e.g. Remark 1 in Section 8.13 in Jantzen's part of [JN04].

The orbit 0 is clearly closed and normal since it consists of the single point $0 \in \mathfrak{g}$.

Chapter 5

The orbits without normal closure

As in Chapter 4 we let G denote a connected, simply connected, semi-simple linear algebraic group over k of type E_6 where k is an algebraically closed field of good characteristic for G. In this chapter we will show that the nilpotent G-orbits in the Lie algebra $\mathfrak{g}$ with Bala-Carter labels A_4, $A_3 + A_1$, A_3, $2A_2$ and $A_2 + A_1$ do not have normal closure. We will prove it by using that this is the case over $\mathbb{C}$ as shown by Eric Sommers, cf. Theorem 1 in [Som03].

The idea of the proof is to connect the characteristic zero case with the characteristic $p > 0$ case by defining everything over $\mathbb{Z}$ and making base change. Before we start discussing the question of normality, we need quite a lot of notation. The notation with definitions etc. is taken from [Jan87] mainly from Section II.1.

For each algebraically closed field L let G_L be a semi-simple, simply connected, connected linear algebraic group over L of type E_6. By the theory of Chevalley groups [1] there exists a split, connected, reductive algebraic $\mathbb{Z}$-group, $G_\mathbb{Z}$, which is flat over $\mathbb{Z}$, such that for each algebraically closed field L we get G_L as the fibered product

$$G_L = G_\mathbb{Z} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(L).$$

Furthermore $G_\mathbb{Z}$ can be chosen with a split maximal torus $T_\mathbb{Z}$, i.e.

$$T_\mathbb{Z} \simeq \mathrm{Spec}(\mathbb{Z}[T_1, T_1^{-1}, \dots, T_r, T_r^{-1}]).$$

Now we can define

$$T_L = T_\mathbb{Z} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(L),$$

and T_L is a maximal torus in G_L.

In general let R be an integral domain, and define

$$G_R = G_\mathbb{Z} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(R), \quad T_R = T_\mathbb{Z} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(R).$$

Let $X^*(T_R)$ be the character group of T_R. Then $X^*(T_R)$ is a free abelian group of rank r, and $X^*(T_R)$ identifies with $X^*(T_\mathbb{Z})$. Let $X_*(T_R)$ be the cocharacter group

[1] See e.g. Theorem 5.6 in [Ste68] and Borel's Section A.3.4-A.3.5, A.4 in [MR070].

of T_R, then also $X_*(T_R)$ can be identified with $X_*(T_\mathbb{Z})$. Let

$$\langle\cdot,\cdot\rangle : X^*(T_R) \times X_*(T_R) \to \mathbb{Z}$$

denote the pairing of characters and cocharacters.

Now let $\mathfrak{g}_R$ denote the Lie algebra of G_R and remember that G_R acts on $\mathfrak{g}_R$ by the adjoint action. Now

$$\mathfrak{g}_R = \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} R$$

and we have a direct decomposition into root spaces

$$\mathfrak{g}_R = \mathfrak{t}_R \oplus \bigoplus_{\alpha\in\Phi} (\mathfrak{g}_R)_\alpha \tag{5.1}$$

where $\mathfrak{t}_R$ denotes the Lie algebra of T_R, and Φ are the roots of G_R with respect to T_R. Since the roots of G_R with respect to T_R can be identified with the roots of $G_\mathbb{Z}$ with respect to $T_\mathbb{Z}$, the set of roots Φ in (5.1) does not depend on R. For all roots $\alpha \in \Phi$ we have

$$(\mathfrak{g}_R)_\alpha = (\mathfrak{g}_\mathbb{Z})_\alpha \otimes_\mathbb{Z} R.$$

Let $x_\alpha \in (\mathfrak{g}_\mathbb{Z})_\alpha \setminus \{0\}$, and define

$$x_{\alpha,R} = x_\alpha \otimes 1 \in (\mathfrak{g}_\mathbb{Z})_\alpha \otimes_\mathbb{Z} R = (\mathfrak{g}_R)_\alpha.$$

Let Φ^- be a negative system of roots in Φ.

Let $U_{\alpha,R}$ be the root subgroup of G_R corresponding to $\alpha \in \Phi$. Also notice that

$$U_{\alpha,R} = U_{\alpha,\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(R).$$

Let U_R be the closed subgroup of G_R generated by all $U_{\alpha,R}$ with $\alpha \in \Phi^-$. Now we can define a Borel subgroup B_R as the semidirect product of T_R and U_R. We identify B_R with the image of this product in G_R and write $B_R = T_R U_R$.

If M_R is a T_R-module, then there is a direct decomposition into weight spaces

$$M_R = \bigoplus_{\lambda\in X^*(T_R)} (M_R)_\lambda$$

If furthermore M_R is a $T_R U_{\alpha,R}$-module, then we have

$$U_{\alpha,R}.(M_R)_\lambda \subseteq \bigoplus_{n\geq 0} (M_R)_{\lambda+n\alpha}. \tag{5.2}$$

Now we turn to the more specific setting. Let $\mathfrak{L}$ be one of the Bala-Carter labels A_4, $A_3 + A_1$, A_3, $2A_2$ or $A_2 + A_1$, and let $\Delta_\mathfrak{L}$ be the weighted Dynkin diagram corresponding to this Bala-Carter label. Let $\lambda_{\Delta_\mathfrak{L}} \in X_*(T_\mathbb{Z}) = X_*(T_R)$ denote the cocharacter defined in Section 2.2.3.

Now define

$$\mathfrak{u}_R = \bigoplus_{\alpha\in\Phi^-} (\mathfrak{g}_R)_\alpha \qquad \text{and} \qquad V(\lambda_{\Delta_\mathfrak{L}})_R = \bigoplus_{\substack{\alpha\in\Phi^- \\ \langle\alpha,\lambda_{\Delta_\mathfrak{L}}\rangle\leq -2}} (\mathfrak{g}_R)_\alpha,$$

and note that the sets

$$\{x_{\alpha,R}|\alpha \in \Phi^-\} \qquad \text{and} \qquad \{x_{\alpha,R}|\alpha \in \Phi^-, \langle \alpha, \lambda_{\Delta_{\mathfrak{L}}}\rangle \leq -2\} \tag{5.3}$$

are bases of $\mathfrak{u}_R$ and $V(\lambda_{\Delta_{\mathfrak{L}}})_R$ respectively. Also note that

$$\mathfrak{u}_R = \mathfrak{u}_{\mathbb{Z}} \otimes_{\mathbb{Z}} R, \qquad V(\lambda_{\Delta_{\mathfrak{L}}})_R = V(\lambda_{\Delta_{\mathfrak{L}}})_{\mathbb{Z}} \otimes_{\mathbb{Z}} R.$$

Since $\mathfrak{g}_R$ is a G_R-module under the adjoint action, it is also a B_R-module, and by (5.2) we see that $\mathfrak{u}_R$ and $V(\lambda_{\Delta_{\mathfrak{L}}})_R$ are B_R-submodules of $\mathfrak{g}_R$.

Now look at the inclusion of B_R-modules $V(\lambda_{\Delta_{\mathfrak{L}}})_R \subseteq \mathfrak{u}_R$ and at the induced surjection $S^n(\mathfrak{u}_R)^* \to S^n(V(\lambda_{\Delta_{\mathfrak{L}}})_R)^*$ of B_R-modules. Let $K_{R,n}$ denote the kernel of this surjection. Then we have a short exact sequence of B_R-modules

$$0 \to K_{R,n} \to S^n(\mathfrak{u}_R)^* \to S^n(V(\lambda_{\Delta_{\mathfrak{L}}})_R)^* \to 0$$

Using the bases of $\mathfrak{u}_R$ and $V(\lambda_{\Delta_{\mathfrak{L}}})_R$ in (5.3) we see that the above short exact sequence arises from the one over $\mathbb{Z}$ by tensoring with R, and that $K_{R,n}$ is a finitely generated free R-module. The short exact sequence above gives rise to a long exact sequence in cohomology

$$\begin{aligned} 0 \to H^0(G_R/B_R, K_{R,n}) &\to H^0(G_R/B_R, S^n(\mathfrak{u}_R)^*) \\ &\to H^0(G_R/B_R, S^n(V(\lambda_{\Delta_{\mathfrak{L}}})_R)^*) \to H^1(G_R/B_R, K_{R,n}) \to \cdots \end{aligned}$$

Notice that since k is the ground field of G, we have $G = G_k$ and $\mathfrak{g} = \mathfrak{g}_k$. Let $\mathcal{O}_k^{\mathfrak{L}}$ denote the nilpotent G_k-orbit in $\mathfrak{g}_k$ with Bala-Carter label $\mathfrak{L}$, and let $\mathcal{O}_{\mathbb{C}}^{\mathfrak{L}}$ denote the nilpotent $G_{\mathbb{C}}$-orbit in $\mathfrak{g}_{\mathbb{C}}$ with Bala-Carter label $\mathfrak{L}$. We know that $\mathcal{O}_{\mathbb{C}}^{\mathfrak{L}}$ does not have normal closure, and we want to show that also $\mathcal{O}_k^{\mathfrak{L}}$ does not have normal closure. Note that $G_{\mathbb{C}}.\mathfrak{u}_{\mathbb{C}}$ is the closure of the regular orbit E_6 ($G_{\mathbb{C}}.\mathfrak{u}_{\mathbb{C}}$ is the full nilpotent variety), and hence it is normal. By Corollary 2.9 the morphism

$$G_{\mathbb{C}} \times^{B_{\mathbb{C}}} \mathfrak{u}_{\mathbb{C}} \to G_{\mathbb{C}}.\mathfrak{u}_{\mathbb{C}}$$

is birational. Since $\mathcal{O}_{\mathbb{C}}^{\mathfrak{L}}$ does not have normal closure, Lemma 2.2 therefore gives that there exists a number $n \in \mathbb{N}$ such that

$$H^0(G_{\mathbb{C}}/B_{\mathbb{C}}, S^n(\mathfrak{u}_{\mathbb{C}})^*) \to H^0(G_{\mathbb{C}}/B_{\mathbb{C}}, S^n(V(\lambda_{\Delta_{\mathfrak{L}}})_{\mathbb{C}})^*)$$

is not surjective, and hence

$$H^1(G_{\mathbb{C}}/B_{\mathbb{C}}, K_{\mathbb{C},n}) \neq 0.$$

But by Proposition I.4.13 in [Jan87] we have (since $\mathbb{C}$ is a flat $\mathbb{Z}$-module)

$$H^1(G_{\mathbb{Z}}/B_{\mathbb{Z}}, K_{\mathbb{Z},n}) \otimes_{\mathbb{Z}} \mathbb{C} = H^1(G_{\mathbb{C}}/B_{\mathbb{C}}, K_{\mathbb{C},n}). \tag{5.4}$$

Now $G_{\mathbb{Z}}/B_{\mathbb{Z}}$ is projective by Section II.1.8 in [Jan87], and since $K_{\mathbb{Z},n}$ is a finitely generated $\mathbb{Z}$-module, the module $H^1(G_{\mathbb{Z}}/B_{\mathbb{Z}}, K_{\mathbb{Z},n})$ is a finitely generated $\mathbb{Z}$-module by Proposition I.5.12 in [Jan87]. But then (5.4) shows that $H^1(G_{\mathbb{Z}}/B_{\mathbb{Z}}, K_{\mathbb{Z},n})$ is not a torsion module over $\mathbb{Z}$.

Since $\mathbb{Z}$ is a Dedekind domain, and $K_{\mathbb{Z},n}$ is a finitely generated free $\mathbb{Z}$-module (and hence a flat $\mathbb{Z}$-module), Proposition I.4.18 in [Jan87] gives an injective map

$$H^1(G_{\mathbb{Z}}/B_{\mathbb{Z}}, K_{\mathbb{Z},n}) \otimes_{\mathbb{Z}} k \hookrightarrow H^1(G_k/B_k, K_{k,n})$$

and since $H^1(G_{\mathbb{Z}}/B_{\mathbb{Z}}, K_{\mathbb{Z},n})$ is not a torsion $\mathbb{Z}$-module, we know that

$$H^1(G_{\mathbb{Z}}/B_{\mathbb{Z}}, K_{\mathbb{Z},n}) \otimes_{\mathbb{Z}} k \neq 0,$$

and hence $H^1(G_k/B_k, K_{k,n}) \neq 0$. But since $H^1(G_k/B_k, S^n(\mathfrak{u}_k)^*) = 0$ by Theorem 2 in [KLT99] the morphism

$$H^0(G_k/B_k, S^n(\mathfrak{u}_k)^*) \to H^0(G_k/B_k, S^n(V(\lambda_{\Delta_{\mathfrak{L}}})_k)^*) \tag{5.5}$$

is not surjective. Let $P(\lambda_{\Delta_{\mathfrak{L}}})_k$ be the parabolic subgroup in G_k defined in Section 2.2.3. Then the morphism

$$G_k \times^{P(\lambda_{\Delta_{\mathfrak{L}}})_k} V(\lambda_{\Delta_{\mathfrak{L}}})_k \to G_k.V(\lambda_{\Delta_{\mathfrak{L}}})_k$$

is birational by Corollary 2.9, and since the map in (5.5) is not surjective, we get by Lemma 2.3 that $\overline{\mathcal{O}_k^{\mathfrak{L}}} = G_k.V(\lambda_{\Delta_{\mathfrak{L}}})_k$ is not normal.

This finishes the proof of Theorem 1.

Appendix A

Computer programs

This appendix contains the Java code for the computer programs mentioned in the thesis. Note that the programs only work for groups of type E_6. The two basic classes are

- Vaegt.java
- Wedge.java

The main programs are

- MindsteAfunktionTensorListe.java
- PTensorListe.java

A.1 MindsteAfunktionTensorListe.java – the program from Remark 3.16

This section contains the Java code for the main program mentioned in Example 3.15 and Remark 3.16. The idea behind the program is explained in Example 3.15, so we will only mention the setup.

Let $V \subseteq \mathfrak{u}$ be a B-subrepresentation. Let $\lambda \in X^*(T)$, and let $i_0 \in \mathbb{N}$. Using Theorem 3.11 we want to show that

$$H^i(G/B, S^n V^* \otimes \lambda) = 0 \quad \text{for all} \quad i > i_0, n \in \mathbb{Z}. \tag{A.1}$$

Let $\lambda_1, \ldots, \lambda_l$ be the T-weights of $\mathfrak{u}$ which are not weights of V.

Input: The λ_i's, then λ and at last i_0.

Output: If the program prints

```
Vi kan desvaerre ikke sige noget, oev!
```

we know that we cannot make the conclusion in (A.1) by the method described in Example 3.15.

If the program prints

```
H^i(S^n[bla] tensor lambda) = 0 for i > i_0
```

then (A.1) is satisfied.

```java
import java.io.*;
import java.util.*;

class MindsteAfunktionTensorListe {

   /* We use the above notation */
   public static void main(String[] args) {
      int antal = args.length-2; // The number of weights in V^*
      int cohgraense = Integer.valueOf(args[args.length-1]).intValue();
      // cohgraense is the number i_0
      Vaegt tensor = new Vaegt(args[args.length-2]); // The weight lambda
      Vaegt[] vaegte = new Vaegt[antal]; // The weights in V^*
      for (int i=0;i<antal;++i) {
         vaegte[i]= new Vaegt(args[i]);
      }
      Wedge wedge = new Wedge(tensor, vaegte ,cohgraense);
      // See Wedge.java for comments
      boolean cohErNul = wedge.erCohNulMindsteA(0,0);
      if (cohErNul) { // The cohomology is zero
         System.out.println("H^i(S^n[bla]␣tensor␣" +tensor.n(0) +
              tensor.n(1) + tensor.n(2) + tensor.n(3) + tensor.n(4) +
              tensor.n(5) + ")␣=␣0␣for␣i␣>" + cohgraense);
      } else { // The cohomology is not necessarily zero
         System.out.println("Vi␣kan␣desvaerre␣ikke␣sige␣noget,␣oev!");
      }
   }
}
```

A.2 Wedge.java – the wedge class

```java
class Wedge{

   /* This class is only used from MindsteAfunktionTensorListe. Hence
    * we will use the notation from Section A.1. Moreover let u be
    * the Lie algebra of the unipotent radical of the Borel. Let
    * W=u/V */

   Vaegt v; // This v will be a weight in \wedge^j W^* \otimes \lambda.
        To
          // begin with we have v = \lambda.
   Vaegt[] liste; // The weights of W^*
```

```
    int cohgraense; // i_0
    int antalvaegte; // The number of weights in W^*

    // Constructor
    public Wedge(Vaegt tensorVaegt, Vaegt[] vaegtListe, int cohGraense){
       cohgraense = cohGraense;
       v = tensorVaegt;
       antalvaegte = vaegtListe.length;
       liste = vaegtListe;
    }

    /* Let m be the minimal vanishing function from Section
      3.2. Inductively we check if H^l(G/B, S^{n-i} u* \otimes \mu) =
      0 for l > i+i_0 where \mu is a T-weight in \wedge^i W^* \otimes
      \lambda. By Theorem 3.11 this is satisfied if m(\mu) =<
      i+i_0. See example 3.15 for more details. The i below equals
      the i above. */
    public boolean erCohNulMindsteA(int i, int j) {
       boolean tmp1 = true;
       boolean tmp2 = true;
       int mindsteA;
       if (j<antalvaegte) {
          tmp1 = erCohNulMindsteA(i,j+1);
          v.plus(liste[j]);
          tmp2 = erCohNulMindsteA(i+1,j+1);
          v.minus(liste[j]);
          return (tmp1 && tmp2);
       } else {
          mindsteA = v.mindsteA();
          if (mindsteA > cohgraense +i) { // m(\mu) > i+i_0
             v.print();
          }
          return (mindsteA < cohgraense + i + 1); // m(\mu) =< i+i_0
       }
    }

}
```

A.3 PTensorListe.java – the program that checks for vanishing cohomology using Theorem 3.1

This section contains the Java code for the program which is used for example in Section 4.7.1.

Let $I \subseteq \Pi$ be a subset of the simple roots. Let $U \subseteq \mathfrak{u}$ be a P_I-module, and let $V \subseteq U$ be a B-submodule. The inclusion $V \subseteq U$ induces a short exact sequence of B-modules

$$0 \to V \to U \to U/V \to 0$$

Take the Koszul resolution of the dual short exact sequence, and tensor it with the one dimensional B-module with weight $\lambda \in X^*(T)$. Then we obtain the following exact sequence

$$0 \to \cdots \to S^{n-j}U^* \otimes \wedge^j V^* \otimes \lambda \to \cdots \to S^{n-1}U^* \otimes V^* \otimes \lambda \\ \to S^n U^* \otimes \lambda \to S^n(U/V)^* \otimes \lambda \to 0$$

Using Theorem 3.1 we want to show that

$$H^i(G/B, S^{n-j}U^* \otimes \wedge^j V^* \otimes \lambda) = 0 \quad \text{for all} \quad i, n \in \mathbb{Z}. \tag{A.2}$$

for some j's.

In the program we use Theorem 3.1 on cohomology groups of the form

$$H^i(G/B, S^{n-j}U^* \otimes \mu)$$

where μ is a T-weight of $\wedge^j V^* \otimes \lambda$. We can use Theorem 3.1 again on the resulting cohomology group if this group is not zero. In the program we have to specify how many times we will at most repeat this process, we call this number N.

Input: The weights in V^*; the subset I; the number N; the weight λ.

Output: Now the program prints something like

```
Dimension 0
Dimension 1
01410
  1
Dimension 2
```

and so forth. Finally it prints

```
Vi må kræve at p >= n
```

If no weights are printed between "Dimension j" and "Dimension $j+1$", then (A.2) holds for this j in characteristic zero and in characteristic $p > 0$ when $p \geq n$.

On the contrary if a weight is printed, then we cannot conclude that (A.2) holds for this j.

```
import java.io.*;
import java.util.*;

class PTensorListe {

```

```
/* We will use the notation above */

public static void main(String[] args) {
   int antal = args.length-3; // The number of weights in V^*
   int lgd = Integer.valueOf(args[args.length-2]).intValue();
   Vaegt tensor = new Vaegt(args[args.length-1]); // lambda
   Vaegt[] vores = new Vaegt[antal]; // The weights in V^*
   for (int i=0;i<antal;++i) {
      vores[i]= new Vaegt(args[i]);
   }
   int antalgodkendte = args[antal].length(); // The number of
   // simple roots in I
   int[] godkendte = new int[antalgodkendte]; // The simple roots
                                   // in I
   for (int i=0;i<antalgodkendte;++i) {
      godkendte[i] =
          Integer.valueOf(args[antal].substring(i,i+1)).intValue();
   }
   int[] hvilke = new int[antal];
   int[] hvilketmp = new int[antal];
   int sidste1 = 0;
   Vaegt vaegt;
   Vector liste = new Vector();
   Vector listetmp = new Vector();
   int[] svar = new int[2];
   int pgraense = 0; // The limit we should put on p in
                 // characteristic p
   System.out.println("Dimension␣0"); // We check if
   // H^i(G/B, S^nU^* \otimes \lambda)=0 for all i
   svar = tensor.pErCohomologi0(godkendte, lgd, 0);
   if(svar[0] == 1) { // The cohomology is not 0 by Theorem 3.1
                 // of Demazure
      tensor.print();
   } else { // The cohomology is 0 by Theorem 3.1, and we have a
         // new limit for p in characteristic p.
      pgraense = Math.max(pgraense, svar[1]);
   }
   System.out.println("Dimension␣1"); // We check if
   // H^i(G/B, S^{n-1}U^* \otimes V^* \otimes \lambda)=0 for all i
   for (int i=0;i<antal;++i) {
      hvilke = new int[antal];
      hvilke[i]=1;
      liste.add(hvilke);
      vores[i].plus(tensor);
      svar = vores[i].pErCohomologi0(godkendte, lgd, 0);
      if (svar[0] == 1) { // The cohomology is not 0 by Theorem 3.1
         vores[i].print();
```

```
            } else { // The cohomology is 0 by Theorem 3.1, and we have a
                // new limit for p in characteristic p.
               pgraense = Math.max(pgraense, svar[1]);
            }
            vores[i].minus(tensor);
         }
         hvilke = new int[antal];
         for (int i=1;i<antal;++i) {
            System.out.println("Dimension␣"+(i+1)); // We check if
         // H^l(G/B, S^{n-{i+1}}U^* \otimes \wedge^{i+1}V^* \otimes \lambda)=0 for all l
            for (int j=0;j<liste.size();++j) {
               hvilke = (int[]) liste.get(j);
               for (int m=0;m<antal;++m) {
                  if (hvilke[m] == 1) {
                     sidste1 = m+1;
                  }
               }
               for (int m=sidste1;m<antal;++m) {
                  hvilketmp = new int[antal];
                  for (int k=0; k<antal;++k) {
                     hvilketmp[k] = hvilke[k];
                  }
                  hvilketmp[m]=1;
                  listetmp.add(hvilketmp);
                  vaegt = Vaegt.nyvaegt(vores,hvilketmp);
                  vaegt.plus(tensor);
                  svar = vaegt.pErCohomologi0(godkendte, lgd, 0);
                  if (svar[0] == 1) { // The cohomology is not 0 by
                               // Theorem 3.1
                     vaegt.print();
                  } else { // The cohomology is 0 by Theorem 3.1,
                        // and we have a new limit for p in
                        // characteristic p.
                     pgraense = Math.max(pgraense, svar[1]);
                  }
                  vaegt.minus(tensor);
               }
               hvilke = new int[antal];
            }
            // We do not need liste anymore, only the new listetmp.
            liste = listetmp;
            listetmp = new Vector();
         }
         System.out.println("Vi␣må␣kræve␣at␣p␣>=␣" + pgraense);
      }
}
```

A.4 Vaegt.java – the class of weights

```java
import java.io.*;
import java.util.*;
import java.math.*;

public class Vaegt
{
   int[] vaegt; // Write a T-weight as a linear combination of simple
            // roots. Then the integers here are the coefficients
            // to the simple roots. We will call "vaegt" for "this
            // weight" in the following.

   // Constructor
   public Vaegt(String a){
      vaegt = new int[6];
      for (int i=0;i<6;i++){
         vaegt[i] = Integer.valueOf(a.substring(i,i+1)).intValue();
      }
   }

   // Constructor
   public Vaegt(int[] a){
      vaegt = new int[6];
      for (int i=0;i<6;i++){
         vaegt[i] = a[i];
      }
   }

   // Makes a copy of the weight
   public Vaegt kopi(){
      Vaegt nyvaegt = new Vaegt(vaegt);
      return nyvaegt;
   }

   // This method prints the weight
   public void print() {
      System.out.println(n(0)+"␣"+n(1)+"␣"+n(2)+"␣"+n(3)+"␣"+n(4));
      System.out.println("␣␣␣␣"+n(5));
   }

   // If the weight is written as a linear combination of simple
   // roots, then this method returns the coefficient to the i'th
   // simple root
   public int n(int i) {
      return vaegt[i];
```

```
    }


    // Adds a weight v to this weight
    public void plus(Vaegt v) {
       for (int i=0;i<6;++i) {
          vaegt[i] = n(i) +v.n(i);
       }
    }


    // Adds the simple root alpha_i to this weight
    public void plusAlfa(int i) {
       vaegt[i] = vaegt[i]+1;
    }

    // Subtracts the simple root alpha_i from this weight
    public void minusAlfa(int i) {
       vaegt[i] = vaegt[i]-1;
    }


    // Subtract the weight v from this weight
    public void minus(Vaegt v) {
       for (int i=0;i<6;++i) {
          vaegt[i] = n(i) - v.n(i);
       }
    }


    /* Input: An array of weights, liste. An array of 0's and 1's,
      hvilke. The lists should be of the same size. */
    /* Output: The sum over i of the weights liste[i] where i
      satisfies that hvilke[i]=1. I.e. we sum over some of the
      weights in the list, liste. */
    public static Vaegt nyvaegt(Vaegt[] liste, int[] hvilke) {
       Vaegt ny = new Vaegt("000000");
       for (int i=0;i<hvilke.length;i++){
          if (hvilke[i] == 1){
             ny.plus(liste[i]);
          }
       }
       return ny;
    }


    // Output: True, if this weight is dominant. Else false.
```

```java
    public boolean erDominant() {
        boolean retur = true;
        if (firkant(1) < 0) {
            retur = false;
        }
        if (firkant(2) < 0) {
            retur = false;
        }
        if (firkant(3) < 0) {
            retur = false;
        }
        if (firkant(4) < 0) {
            retur = false;
        }
        if (firkant(5) < 0) {
            retur = false;
        }
        if (firkant(6) < 0) {
            retur = false;
        }
        return retur;
    }

    /* Let alpha_i be the i'th simple root. Then we let alpha_i^v
     * denote the corresponding coroot */

    /* Returns the pairing of this weight and alpha_i^v, i.e.
       <this weight, alpha_i^v >, i = 1,2,3,4,5,6 */
    public int firkant(int i) {
        if (i==1) {
            return 2 * n(0) - n(1);
        }
        else if (i==2) {
            return 2 * n(1) - n(0) - n(2);
        }
        else if (i==3) {
            return 2 * n(2) - n(1) - n(3) - n(5);
        }
        else if (i==4) {
            return 2 * n(3) - n(2) - n(4);
        }
        else if (i==5) {
            return 2 * n(4) - n(3);
        }
        else {
            return 2 * n(5) - n(2);
        }
```

```
    }

    /* Returns the pairing of this weight and alpha_i^v, i.e.
       <this weight, alpha_i^v >, i = 0,1,2,3,4,5. */
    public int firkant0(int i) {
       if (i==0) {
          return 2 * n(0) - n(1);
       }
       else if (i==1) {
          return 2 * n(1) - n(0) - n(2);
       }
       else if (i==2) {
          return 2 * n(2) - n(1) - n(3) - n(5);
       }
       else if (i==3) {
          return 2 * n(3) - n(2) - n(4);
       }
       else if (i==4) {
          return 2 * n(4) - n(3);
       }
       else {
          return 2 * n(5) - n(2);
       }
    }


    /* Let m be the minimal vanishing function from Chapter 3.2. This
     * method returns the value of m on this weight.*/
    public int mindsteA() {
       if (this.erDominant() ) { // If this weight is dominant, m is
                             // zero on this weight.
          return 0;
       }
       int mindsteA = -1; // This will eventually be the value of m
                       // on this weight
       int mindsteAtmp = -1;
       int firkant = 0;
       Vaegt v;
       for (int i=0;i<6;i++) {
          firkant = firkant0(i);
          if ( firkant == -1) { // 1: <this weight, alpha_i^v>=-1
             // See the definition of m.
             this.plusAlfa(i);
             mindsteAtmp = this.mindsteA();
             this.minusAlfa(i);
          }
          if ( firkant < -1) { // 2: <this weight, alpha_i^v><-1,
```

```
            // Se the definition of m.
            v = this.kopi();
            mindsteAtmp = -1;
            for (int j=0; j<-firkant-1; ++j) {
               v.plusAlfa(i); // Now v = this weight + r*alpha_i
               // where 0 <= r <= -<this weight, alpha_i^v> -1.
               mindsteAtmp = Math.max(mindsteAtmp,v.mindsteA()+1);
            }
            v.plusAlfa(i); // Now v = s_alpha_i(this weight), where
                       // s_alpha_i is the reflection in the
                       // Weyl group corresponding to alpha_i
            mindsteAtmp = Math.max(mindsteAtmp,v.mindsteA());
         }
         if (mindsteA==-1) {
            mindsteA = mindsteAtmp;
         } else {
            mindsteA = Math.min(mindsteA,mindsteAtmp); // m is the
            // minimum over all i with <this weight, alpha_i^v> of
            // mindsteAtmp in either 1 or 2. See the definition of m.
         }
      }
      return mindsteA;
   }

   /* In the following, godkendte, is the number such that
   alpha_{godkendte-1} is the simple root use in Theorem 3.1. The method
   changes this weight to be s_alpha_{godkendt-1}(this
   weight)-alpha_{godkendt-1}, see Theorem 3.1. The method returns
   how big p should be in characteristic p in order to use Theorem 3.1
   with this weight.*/
   public int pCohVaegt(int godkendt) {
      int firkant = firkant(godkendt);
      vaegt[godkendt-1] += - (firkant+1); // this weight =
      // s_alpha_{godkendt-1}(this weight)-alpha_{godkendt-1},
      // see Theorem 3.1.
      if (firkant < 0) {
         return (-firkant-1);
      } else {
         return (firkant-1);
      }
   }

   /* We want to see if H^i(G/B, V \otimes (this weight))=0 by using
     Theorem 3.1 by Demazure. Let P_alpha_i be the minimal standard
     parabolic subgroup corresponding to alpha_i. Let p the
     characteristic, if it is not 0. */
   /* Input: godkendte: An array of numbers j_0, ... , j_k satisfying
```

```
     that V is a P_alpha_{j_k}-module.
     lgd: The number of times we should apply Theorem 3.1 in a
     row.
     n: The number that counts how big p should be in order to
     use Theorem 3.1. */
   /* Output: An array with two numbers. The first number is 0 if we
     can conclude H^i(-)=0 by Theorem 3.1. Else it is 1. If the
     first number is 0, the second number, s, tells us that Hi(-)=0
     if p>=s */
   public int[] pErCohomologi0(int[] godkendte, int lgd, int n) {
      int[] retur = new int[2];
      for (int i=0;i<godkendte.length;i++) {
         if (firkant(godkendte[i]) == -1) { // Since
            // <this weight, alpha_{godkendte[i]}^v>=-1, Theorem 3.1.
            // tells us that H^i(-)=0.
            retur[0] = 0;
            retur[1] = n;
            return retur;
         }
      }
      Vaegt tmpv;
      if (lgd > 0) {
         for (int i=0;i<godkendte.length;i++) {
            tmpv = kopi();
            n = Math.max(n,tmpv.pCohVaegt(godkendte[i]));
            retur = tmpv.pErCohomologi0(godkendte, lgd -1,n);
            if (retur[0]==0) { // H^i(-)=0 for all i
               return retur;
            }
         }
      }
      retur[0]=1; // H^i(-) is not necessarily zero
      retur[1]=n;
      return retur;
   }


}
```

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