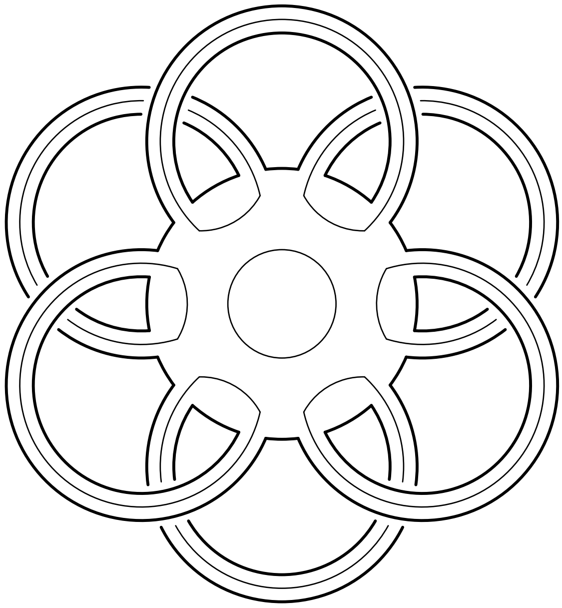
The Moduli Space of Flat Connections on a Surface
Poisson Structures and Quantization



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The Moduli Space of Flat Connections on a Surface
Poisson Structures and Quantization

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Contents

Chapter 1

Introduction

Throughout this dissertation we work in the following set-up: We denote by Σ an oriented, compact and connected surface, possibly with boundary. A basepoint $x_0 \in \Sigma$ is fixed, and we let $\pi_1(\Sigma) = \pi_1(\Sigma, x_0)$ denote the fundamental group. Furthermore, G is a linearly reductive, affine algebraic group over the complex numbers (e.g. $\mathrm{GL}_n(\mathbf{C})$, $\mathrm{SL}_n(\mathbf{C})$, $\mathrm{O}_n(\mathbf{C})$ and $\mathrm{Sp}_{2n}(\mathbf{C})$). By a standard result (cf. [Hu]), G is a closed subgroup of $\mathrm{GL}_n(\mathbf{C})$ so that, in particular, G is a Lie group; we write $\mathfrak{g}$ for its Lie algebra. The moduli space of flat G-connections on Σ is denoted by $\mathcal{M}(\Sigma; G)$. It is well-known that there is a canonical bijection

$$\mathrm{Hol}\colon \mathcal{M}(\Sigma; G) \to {\mathrm{Hom}(\pi_1(\Sigma), G)}/{G} \tag{1.1}$$

where the G-action on $\mathrm{Hom}(\pi_1(\Sigma), G)$ is by conjugation and Hol is given by taking the holonomy with respect to a flat connection along loops on Σ based at x_0. Let $\Gamma_+(\Sigma)$ denote the group of orientation preserving diffeomorphisms of Σ. This group acts on $\mathcal{M}(\Sigma; G)$ via pullback of connections:

$$g \cdot [A] = [(g^{-1})^* A], \quad [A] \in \mathcal{M}(\Sigma; G),\ g \in \Gamma_+(\Sigma)$$

so that the induced action on $\mathrm{Fun}(\mathcal{M}(\Sigma; G)) = \mathrm{Map}(\mathcal{M}(\Sigma; G), \mathbf{C})$ is given by

$$(g \cdot f)([A]) = f(g^{-1} \cdot [A]) = f([g^* A]), \quad f \in \mathrm{Fun}(\mathcal{M}(\Sigma; G)),\ g \in \Gamma_+(\Sigma).$$

For a synopsis of the dissertation, the reader may consult the table of contents and the introductory paragraphs of the individual chapters.

It is presupposed that the reader is familiar with a few basic concepts and results from algebraic geometry and invariant theory (cf. [Fog]). For his convenience we recall the relevant material here. An affine algebraic set $X = V(S) \subseteq \mathbf{C}^N$ is the solution of a set S of polynomial equations in N variables; associated to it is the ideal $I(X) \subseteq \mathbf{C}[x_1, \ldots, x_N]$ of polynomials vanishing on X. The Hilbert Nullstellensatz states that

$$IV(\mathfrak{a}) = \sqrt{\mathfrak{a}}, \quad \mathfrak{a} \text{ an ideal in } \mathbf{C}[x_1, \ldots, x_N].$$

Occasionally the radical ideal $\sqrt{\mathfrak{a}}$ is denoted by $\mathrm{Rad}(\mathfrak{a})$. The ring $\mathcal{O}(X)$ of regular functions on X is isomorphic to $\mathbf{C}[x_1, \ldots, x_N]/I(X)$. Algebraic morphisms between affine sets preserve regular functions; the action on $\mathcal{O}(X)$ induced by an algebraic G-action on X is rational. By the linear reductivity of G, rational actions have well-behaved invariants. To be specific, if G acts rationally on a complex vector space V, then the subspace $V^G \subseteq V$ of fixed points has a unique G-invariant complement V_G. The linear projection $\nabla = \nabla_V \colon V \to V^G$

with kernel V_G is called the Reynolds operator on V. The uniqueness of V_G implies that Reynolds operators are natural with respect to G-equivariant, linear maps $\varphi\colon V \to W$, i.e., the diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \nabla} \\ V^G & \xrightarrow{\varphi_|} & W^G \end{array}$$

is commutative.

Remark 1.1. A simple, but important consequence of this is that $\varphi_|$ is surjective if φ is.

If V is an algebra (and G acts by algebra isomorphisms), then Reynolds' identity

$$\nabla(xy) = \nabla(x)y, \quad x \in V, y \in V^G \tag{1.2}$$

holds.

Chapter 2

The Moduli Space and the Algebra of Chord Diagrams

A Poisson structure on the algebra of functions on $\mathcal{M}(\Sigma; G)$ has been studied by a number of people, e.g., Atiyah and Bott [AB], Goldman [G1, G2], Biswas and Guruprasad [BG], and Fock and Rosly [FR]. The authors approach the subject differently but common to all is that the Poisson bracket $\{\ ,\ \}_B$ is defined in terms of an *orthogonal structure* on G, that is, a non-degenerate, symmetric, bilinear map $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbf{C}$ invariant under the adjoint action.

In this chapter we first construct the algebra $\mathcal{O}(\mathcal{M}(\Sigma; G))$ of regular functions on the moduli space and then adapt the presentation in [FR] to define a Poisson bracket $\{\ ,\ \}_t$ on $\mathcal{O}(\mathcal{M}(\Sigma; G))$ for any symmetric Ad-invariant tensor $t \in \mathfrak{g} \otimes \mathfrak{g}$; this generalizes the aforementioned Poisson structure since $\{\ ,\ \}_B = \{\ ,\ \}_{t_B}$ where $t_B \in \mathfrak{g} \otimes \mathfrak{g}$ is the symmetric Ad-invariant tensor corresponding to $B \in (\mathfrak{g} \otimes \mathfrak{g})^*$ under the isomorphism $\mathfrak{g}^* \cong \mathfrak{g}$ induced by B itself. Afterwards we present the Poisson algebra of chord diagrams $\mathcal{C}(\Sigma; G)$ introduced by Andersen, Mattes and Reshetikhin [AMR1]. One of the main results of this paper is that there exists a Poisson homomorphism $\Psi_B\colon \mathcal{C}(\Sigma; G) \to (\mathcal{O}(\mathcal{M}(\Sigma; G)), \{\ ,\ \}_B)$; we generalize this to all Poisson brackets $\{\ ,\ \}_t$.

2.1 Lattice Gauge Field Theory

A *graph* K is a finite, 1-dimensional CW-complex with an orientation on each 1-cell. Its set of vertices is denoted by $V(K)$ and its set of edges by $E(K)$. We also consider the set $E_\partial(K)$ of all endpoints of edges of K. It is important to notice the distinction between vertices and endpoints; the two concepts are related by the obvious 'incidence' map

$$[\ ]\colon E_\partial(K) \to V(K).$$

In the sequel we identify a vertex with its pre-image under this map. The endpoints of an edge are given by the maps

$$\partial_+, \partial_-\colon E(K) \to E_\partial(K).$$

An edge $\alpha \in E(K)$ may be traversed according to or counter to its orientation, yielding two curves, α and α^{-1}, in K. A *path* in K is a curve in K which is a composition of edge traversals, i.e., they have the form $\alpha_1^{\epsilon_1} \cdots \alpha_n^{\epsilon_n}$ with the compatibility condition that $\partial_+\alpha_i^{\epsilon_i} := \partial_{\epsilon_i}\alpha_i$ and $\partial_-\alpha_{i+1}^{\epsilon_{i+1}} := \partial_{-\epsilon_{i+1}}\alpha_{i+1}$ for $i = 1, \ldots, n-1$ are incident to the same vertex. For *loops* (cyclic paths) we always work with indices $\bmod n$.

A *combinatorial complex* K is a 2-dimensional CW-complex obtained from a graph by attaching a finite number of 2-cells along loops in the graph. The set of *graph connections on* K (more precisely, on its underlying graph) is simply the product $G^{E(K)}$; for a connection $A = (A_\alpha) \in G^{E(K)}$ the element $\mathrm{Hol}^A(\alpha) = A_\alpha \in G$ is called the *holonomy* with respect to A along α. We extend the concept of holonomy to paths in the obvious way:

$$\mathrm{Hol}^A(\alpha_1^{\epsilon_1} \cdots \alpha_n^{\epsilon_n}) = A_{\alpha_1}^{\epsilon_1} \cdots A_{\alpha_n}^{\epsilon_n}, \quad A \in G^{E(K)}.$$

Notice that for a loop without a specified initial point the holonomy is still a well-defined conjugacy class. Therefore the equations

$$\mathrm{Hol}^A(\partial F) = 1, \quad A \in G^{E(K)} \tag{2.1}$$

with F running over all faces in K make sense and define a subset $\mathcal{A}(K) = \mathcal{A}(K; G) \subseteq G^{E(K)}$ called the *G-connections on K*. The *gauge group* of K is by definition $\mathcal{G}(K) = G^{V(K)}$; it acts on the graph connections:

$$(g_v)(A_\alpha) = (g_{[\partial_-\alpha]} A_\alpha g^{-1}_{[\partial_+\alpha]}), \quad (A_\alpha) \in G^{E(K)},\ (g_v) \in \mathcal{G}(K).$$

Since this action conjugates the holonomy along a loop, $\mathcal{A}(K)$ is an invariant subset; the orbit space

$$\mathcal{M}(K; G) = \mathcal{A}(K)/\mathcal{G}(K)$$

of the restricted action is called the *moduli space of G-connections on K*. The natural projection $\pi\colon \mathcal{A}(K) \to \mathcal{M}(K; G)$ sets up a bijection

$$\pi^*\colon \mathrm{Fun}(\mathcal{M}(K; G)) \to (\mathrm{Fun}(\mathcal{A}(K)))^{\mathcal{G}(K)}.$$

As $\mathcal{A}(K)$ is cut out of $G^{E(K)}$ by the algebraic equations (2.1), it is an affine algebraic set. The gauge group action on $\mathcal{A}(K)$ is clearly algebraic whence the induced action on functions preserves the property of being regular. Set

$$\mathcal{O}(\mathcal{M}(K; G)) = (\pi^*)^{-1}\big(\mathcal{O}(\mathcal{A}(K))^{\mathcal{G}(K)}\big) \subseteq \mathrm{Fun}(\mathcal{M}(K; G)). \tag{2.2}$$

The notation $\mathcal{O}(\mathcal{M}(K; G))$ is merely suggestive; we do not claim that $\mathcal{M}(K; G)$ admits the the structure of an algebraic variety.

Now suppose $\iota\colon K \to \Sigma$ is an embedding such that $K \subseteq \Sigma$ is a deformation retract; we call $K = (K, \iota)$ a *model* for Σ. There is a canonical bijection

$$\mathrm{Hol}_\iota\colon \mathcal{M}(\Sigma; G) \to \mathcal{M}(K; G)$$

with the following description (cf. (1.1)): Let A be a flat connection in a principal G-bundle $P \to \Sigma$. Choose for each $v \in V(K)$ a basepoint in the fibre of P over v (a trivialization of $P_{|v}$). This allows the holonomy with respect to A along an edge $\alpha \in E(K)$ to be expressed as an element $A_\alpha \in G$, and we have

$$\mathrm{Hol}_\iota([A]) = [(A_\alpha)] \in \mathcal{M}(K; G), \quad [A] \in \mathcal{M}(\Sigma; G).$$

We define the *algebra of regular functions on the moduli space* by

$$\mathcal{O}(\mathcal{M}(\Sigma; G)) = \mathrm{Hol}_\iota^*(\mathcal{O}(\mathcal{M}(K; G))) \subseteq \mathrm{Fun}(\mathcal{M}(\Sigma; G)).$$

Proposition 2.1. *The subset $\mathcal{O}(\mathcal{M}(\Sigma; G)) \subseteq \mathrm{Fun}(\mathcal{M}(\Sigma; G))$ is independent of the model used to define it.*

Proof. Suppose we have two models $\iota_j \colon K_j \to \Sigma$, $j = 1, 2$. Pick a map $\rho \colon V(K_2) \to V(K_1)$ and for each $v \in V(K_2)$ a curve γ_v on Σ from $\rho(v)$ to v. Consider an edge α of K_2. Since K_1 is a retract of Σ and by cellular approximation, the curve $\gamma_{[\partial_-\alpha]}(\iota_2)_{|\alpha}\gamma^{-1}_{[\partial_+\alpha]}$ on Σ is homotopic rel endpoints to a path P_α in K_1. Define a map $\varphi \colon G^{E(K_1)} \to G^{E(K_2)}$ by

$$\varphi(A)_\alpha = \mathrm{Hol}^A(P_\alpha), \quad A \in G^{E(K_1)}.$$

Assume that $A \in \mathcal{A}(K_1)$ and let $P \to \Sigma$ be a principal G-bundle. Upon trivializing $P_{|V(K_1)}$, A defines a flat connection $\tilde{A}$ on Σ representing $\mathrm{Hol}_{\iota_1}^{-1}([A])$. Now trivialize the fibre over $v \in V(K_2)$ by parallel transporting the basepoint over $\rho(v)$ with respect to $\tilde{A}$ along γ_v; this choice implies that $\mathrm{Hol}_{\iota_2}([\tilde{A}])$ is represented by the graph connection $\varphi(A)$. In consequence, not only does φ map $\mathcal{A}(K_1)$ into $\mathcal{A}(K_2)$, it also fits into the commutative diagram

$$\begin{array}{ccc} G^{E(K_1)} & \xrightarrow{\ \varphi\ } & G^{E(K_2)} \\ \uparrow & & \uparrow \\ \mathcal{A}(K_1) & \xrightarrow{\ \varphi_|\ } & \mathcal{A}(K_2) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ \mathcal{M}(K_1; G) & \xrightarrow{\ T_{21}\ } & \mathcal{M}(K_2; G) \end{array} \tag{2.3}$$

where the bottom map is the bijection $\mathrm{Hol}_{\iota_2} \circ \mathrm{Hol}_{\iota_1}^{-1} \colon \mathcal{M}(K_1; G) \to \mathcal{M}(K_2; G)$.

It is immediate from the construction that φ is an algebraic morphism intertwining the gauge group actions:

$$\varphi((g_v) \cdot A) = \rho^*((g_v)) \cdot \varphi(A), \quad A \in G^{E(K_1)}, \ (g_v) \in \mathcal{G}(K_1).$$

Here $\rho^* \colon \mathcal{G}(K_1) \to \mathcal{G}(K_2)$ is the pullback via ρ. The induced map $\varphi^* \colon \mathrm{Fun}(G^{E(K_2)}) \to \mathrm{Fun}(G^{E(K_1)})$ therefore preserves regular functions and also intertwines the actions:

$$(g_v) \cdot \varphi^* f = \varphi^*(\rho^*((g_v)) \cdot f), \quad f \in \mathrm{Fun}(G^{E(K_2)}), \ (g_v) \in \mathcal{G}(K_1).$$

The same statements hold true for the restriction $\varphi_| \colon \mathcal{A}(K_1) \to \mathcal{A}(K_2)$. In particular, φ^* and $(\varphi_|)^*$ preserve fixed points (invariant functions). It thus follows that

$$T_{21}^* = (\pi_1^*)^{-1} \circ (\varphi_|)^* \circ \pi_2^* \colon \mathrm{Fun}(\mathcal{M}(K_2; G)) \to \mathrm{Fun}(\mathcal{M}(K_1; G))$$

maps $\mathcal{O}(\mathcal{M}(K_2; G))$ to $\mathcal{O}(\mathcal{M}(K_1; G))$; this proves the result since $T_{21}^* = (\mathrm{Hol}_{\iota_1}^*)^{-1} \circ \mathrm{Hol}_{\iota_2}^*$. □

Remark 2.2. During the course of the above proof we established the diagram

$$\begin{array}{ccc} & \mathcal{O}(\mathcal{M}(\Sigma;G)) & \\ {}^{\mathrm{Hol}^*_{\iota_2}}\nearrow & & \nwarrow{}^{\mathrm{Hol}^*_{\iota_1}} \\ \mathcal{O}(\mathcal{M}(K_2;G)) & \xrightarrow{T^*_{21}} & \mathcal{O}(\mathcal{M}(K_1;G)) \\ \downarrow \pi_2^* & & \downarrow \pi_1^* \\ \mathcal{O}(\mathcal{A}(K_2))^{\mathcal{G}(K_2)} & \xrightarrow{(\varphi_|)^*} & \mathcal{O}(\mathcal{A}(K_1))^{\mathcal{G}(K_1)} \\ \uparrow & & \uparrow \\ \mathcal{O}(G^{E(K_2)})^{\mathcal{G}(K_2)} & \xrightarrow{\varphi^*} & \mathcal{O}(G^{E(K_1)})^{\mathcal{G}(K_1)} \end{array} \tag{2.4}$$

We infer, in particular, that although the construction of $\varphi\colon G^{E(K_1)} \to G^{E(K_2)}$ depends on various choices, the induced map $(\varphi_|)^*$ is entirely canonical; namely it is equal to the composition

$$\tau_{12} := \pi_1^* \circ T_{21}^* \circ (\pi_2^*)^{-1}\colon \mathcal{O}(\mathcal{A}(K_2))^{\mathcal{G}(K_2)} \to \mathcal{O}(\mathcal{A}(K_1))^{\mathcal{G}(K_1)}. \tag{2.5}$$

This will be useful later.

Corollary 2.3. *The map* $\mathrm{Hol}_\iota\colon \mathcal{M}(\Sigma;G) \to \mathcal{M}(K;G)$ *depends only on the homotopy class of* $\iota\colon K \to \Sigma$.

Proof. Let $\iota_t\colon K \to \Sigma$ be a homotopy. Consider the construction of the transfer $\varphi\colon G^{E(K)} \to G^{E(K)}$ from the model $\iota_0\colon K \to \Sigma$ to the model $\iota_1\colon K \to \Sigma$. We may take $\rho = \mathrm{Id}_{V(K)}$ and $\gamma_v(t) = \iota_t(v)$. Restricting the homotopy ι_t to an edge α of K, we conclude that

$$\gamma_{[\partial_-\alpha]}(\iota_1)_{|\alpha}\gamma^{-1}_{[\partial_+\alpha]} \simeq (\iota_0)_{|\alpha} \text{ rel endpoints.}$$

Therefore, by definition, $\varphi = \mathrm{Id}_{G^{E(K)}}$, whence the result follows from diagram (2.3). □

We finish this section with an important class of models for Σ. Let $\langle g_\lambda, \lambda \in \Lambda \mid r_\mu, \mu \in M\rangle$ be a finite presentation P of $\pi_1(\Sigma)$. There is an associated complex K_P; its 1-skeleton $K_P^{(1)}$ consists of a single 0-cell v and an edge (loop) for each generator g_λ. The relations r_μ determine glueing maps used to attach the 2-cells of K_P; it follows that $\mathcal{A}(K_P) \subseteq G^{E(K_P)} = G^\Lambda$ is simply defined by the relations of P. Hence the map $\mathrm{Ev}_P\colon \mathrm{Hom}(\pi_1(\Sigma), G) \to \mathcal{A}(K_P)$ given by evaluating a G-representation of $\pi_1(\Sigma)$ on the generators from P is a bijection. As $\mathcal{G}(K_P) = G$ acts by simultaneous conjugation on $\mathcal{A}(K_P)$, there is an induced bijection

$$\mathrm{Ev}_P\colon \mathrm{Hom}(\pi_1(\Sigma), G)/G \to \mathcal{M}(K_P; G).$$

Pre-composing this map with $\mathrm{Hol}\colon \mathcal{M}(\Sigma;G) \to \mathrm{Hom}(\pi_1(\Sigma), G)/G$, we obtain a bijection

$$\mathrm{Ev}_P \circ \mathrm{Hol}\colon \mathcal{M}(\Sigma;G) \to \mathcal{M}(K_P;G).$$

A choice of representatives for the generators $g_\lambda \in \pi_1(\Sigma)$ gives rise to a map $\iota\colon K_P^{(1)} \to \Sigma$ (sending v to x_0). The face boundaries of K_P are mapped to trivial loops on Σ by construction, so ι extends to all of K_P. It is now a triviality that $\mathrm{Hol}_\iota = \mathrm{Ev}_P \circ \mathrm{Hol}$. Thus we have an induced bijection

$$\mathrm{Hol}^* \circ \mathrm{Ev}_P^* = \mathrm{Hol}_\iota^*\colon \mathcal{O}(\mathcal{M}(K_P;G)) \to \mathcal{O}(\mathcal{M}(\Sigma;G)) \tag{2.6}$$

depending solely on P.

2.2 Poisson Structures for Fat Graphs

In this section K denotes a *fat graph*, i.e., a graph equipped with a cyclic order on each of its vertices. In drawings of fat graphs the cyclic order will always agree with the counterclockwise order. Our goal is to define a Poisson bracket $\{\ ,\ \}_t$ on $\mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}$ where $t \in \mathfrak{g} \otimes \mathfrak{g}$ is an Ad-invariant, symmetric element; we achieve this by giving a bivector field on $G^{E(K)}$. Writing down this tensor requires, however, a linearization $\preccurlyeq$ of the cyclic order at the vertices of K; such a choice is termed a *ciliation* since the linear order at a vertex is indicated by a small cilium between the first and the last endpoint.

Let $\Gamma(G^{E(K)})$ denote the set of smooth vector fields on $G^{E(K)}$, and define linear operators

$$X^\kappa \colon \mathfrak{g} \to \Gamma(G^{E(K)}), \quad \kappa \in E_\partial(K)$$

as follows: For $\alpha \in E(K)$ and $b \in \mathfrak{g}$, $X^{\partial_+\alpha}(b)$ is the left-invariant vector field corresponding to b assigned to the factor G^α of $G^{E(K)}$, and $X^{\partial_-\alpha}(b)$ is the right-invariant vector field corresponding to $-b$ assigned to the factor G^α of $G^{E(K)}$. Define bivector fields on $G^{E(K)}$ by

$$B_t(v, \preccurlyeq) = \sum_{\kappa,\lambda \in v} \epsilon(\kappa, \lambda)(X^\kappa \otimes X^\lambda)(t), \quad v \in V(K)$$

where

$$\epsilon(\kappa, \lambda) = \begin{cases} 1 & \text{if } \kappa \prec \lambda \\ 0 & \text{if } \kappa = \lambda \\ -1 & \text{if } \kappa \succ \lambda \end{cases}$$

We set $B_t(\preccurlyeq) = \sum_{v \in V(K)} B_t(v, \preccurlyeq)$, and define

$$\{f, g\}_t = \langle B_t(\preccurlyeq); df \otimes dg \rangle, \quad f, g \in \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}. \tag{2.7}$$

Remark 2.4. Unlike Fock and Rosly, we employ no classical r-matrix in the definition of $B_t(\preccurlyeq)$; this approach is feasible because the corresponding bracket is defined for invariant functions only. In the case where t corresponds to an orthogonal structure on G, the next two results are covered in [FR].

Proposition 2.5. *The formula (2.7) defines a map*

$$\{\ ,\ \}_t \colon \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)} \times \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)} \to \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}$$

which is independent of the ciliation on K.

Theorem 2.6. *The bracket $\{\ ,\ \}_t$ defines a poisson structure on $\mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}$.*

We shall need various basic results concerning t and the maps X^κ for the proofs of these statements. Often we work with a *basis* for t; this is a set $\{e_1, \ldots, e_n\} \subseteq \mathfrak{g}$ such that $t = \sum_i e_i \otimes e_i$. Bases exist since t is symmetric but are by no means unique. In fact, by the Ad-invariance of t the set $\{\mathrm{Ad}_g(e_1), \ldots, \mathrm{Ad}_g(e_n)\}$ is another basis for any $g \in G$; we will use this observation without further mention in the sequel. Applying the shorthand $X_i^\kappa = X^\kappa(e_i) \in \Gamma(G^{E(K)})$, we may write

$$\{f, g\}_t = \sum_{v \in V(K)} \sum_{\kappa,\lambda \in v} \epsilon(\kappa, \lambda) \sum_i X_i^\kappa f X_i^\lambda g, \quad f, g \in \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}. \tag{2.8}$$

Consider the composite map

$$\varphi\colon \mathfrak{g} \xrightarrow{\text{ad}} \text{End}(\mathfrak{g}) \xrightarrow{-\otimes\text{Id}} \text{End}(\mathfrak{g}\otimes\mathfrak{g}) \xrightarrow{\text{Ev}_t} \mathfrak{g}\otimes\mathfrak{g}$$

and define $T = (\varphi\otimes\text{Id})(t) \in \mathfrak{g}^{\otimes 3}$. It is significant that T is invariantly defined; its expression in a basis is

$$T = \sum_{i,j} [e_j, e_i] \otimes e_i \otimes e_j. \tag{2.9}$$

Lemma 2.7. *T is an anti-invariant tensor.*

Proof. Transposing the second and third factors of $\mathfrak{g}^{\otimes 3}$ obviously maps T to $-T$. For any $b \in \mathfrak{g}$ we differentiate the curve

$$s \longmapsto (\text{Ad}_{\exp(sb)} \otimes \text{Ad}_{\exp(sb)})(t) = t, \quad s \in \mathbf{R}$$

at $s = 0$ to obtain

$$(\text{ad}\, b \otimes \text{Id} + \text{Id} \otimes \text{ad}\, b)(t) = 0.$$

Letting the 3-cycle $\sigma = (1,2,3) \in S_3$ act on $\mathfrak{g}^{\otimes 3}$ and applying this fact to $b = e_i$, we get

$$\sigma(T) = \sum_{i,j} e_j \otimes [e_j, e_i] \otimes e_i = \sum_i \Big(\sum_j -e_j \otimes [e_i, e_j]\Big) \otimes e_i = \sum_i \Big(\sum_j [e_i, e_j] \otimes e_j\Big) \otimes e_i = T$$

as desired. □

Remark 2.8. If t comes from an orthogonal structure, then any orthogonal basis of $\mathfrak{g}$ is a basis for t, and T is the structure tensor of $\mathfrak{g}$.

Lemma 2.9. *The linear maps $X^\kappa\colon \mathfrak{g} \to \Gamma(G^{E(K)})$ are independent Lie algebra homomorphisms, i.e.,*

$$[X^\kappa(b_1), X^\lambda(b_2)] = \begin{cases} X^\kappa([b_1, b_2]) & \text{if } \kappa = \lambda \\ 0 & \text{otherwise} \end{cases}$$

for $b_1, b_2 \in \mathfrak{g}$.

Proof. When $\kappa = \lambda$ this is simply by definition of the Lie bracket of a Lie group. If κ and λ are endpoints of distinct edges, the claim is trivial. In case κ and λ are the two endpoints of a single edge, the associativity of G (left and right multiplication commute) implies the result. □

The next two lemmas are easy consequences of the compatibility of the exponential map and the adjoint action:

$$g \exp(sb) = \exp(s\, \text{Ad}_g(b)) g; \quad b \in \mathfrak{g}, g \in G, s \in \mathbf{R}.$$

Lemma 2.10. *Let $\alpha \in E$ and $b \in \mathfrak{g}$. Then*

$$(X^{\partial_+\alpha}(b))_A = -(X^{\partial_-\alpha}(\text{Ad}_{A_\alpha}(b)))_A$$

for $A \in G^{E(K)}$.

Lemma 2.11. *Let $\kappa \in E_\partial(K)$ and $b \in \mathfrak{g}$. Then*

$$(g_v)_* \cdot X^\kappa(b) = X^\kappa(\mathrm{Ad}_{g_{[\kappa]}}(b))$$

where $(g_v) \in \mathcal{G}(K)$ and $(g_v)_*$ is the derivative of $(g_v)\colon G^{E(K)} \to G^{E(K)}$.

Finally, introduce the diagonal operators

$$X^{\Delta(v)} = \sum_{\kappa \in v} X^\kappa \colon \mathfrak{g} \to \Gamma(G^{E(K)}), \quad v \in V$$

whose importance is due to the next lemma.

Lemma 2.12. *Let $v \in V$ and $b \in \mathfrak{g}$. Then*

$$X^{\Delta(v)}(b) f = 0$$

for any $f \in \mathcal{O}(G^E(K))^{\mathcal{G}(K)}$.

Proof. Let γ_b^v be the curve $s \mapsto \exp(-sb)$ assigned to the factor G^v of $\mathcal{G}(K)$. Then, trivially,

$$\frac{d}{ds}_{|s=0}\big(\gamma_b^v(s) \cdot A\big) = X^{\Delta(v)}(b)_A, \quad A \in G^{E(K)}.$$

Hence $X^{\Delta(v)}(b)$ is tangential to the $\mathcal{G}(K)$-orbits of $G^{E(K)}$ along which f is constant. □

Proof (Proposition 2.5). Let $f, g \in \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}$. From Lemma 2.11 it is immediate that $B_t(v, \preccurlyeq)$, $v \in V$ and hence also $B_t(\preccurlyeq)$ are invariant under the gauge group action. Thus $\{f, g\}_t$ is an invariant function since both f and g enjoy this property. As $\mathcal{O}(G^{E(K)})$ is closed under left-invariant and right-invariant derivations (cf. [Hu]), the first statement of the proposition holds true.

For the second claim we must compare any two ciliations $\preccurlyeq$ and $\preccurlyeq'$ of K. It suffices to consider the case in which $\preccurlyeq'$ differs from $\preccurlyeq$ only at a single vertex v where $\kappa_1 \prec \cdots \prec \kappa_n$ and $\kappa_2 \prec' \cdots \prec' \kappa_n \prec' \kappa_1$. Then

$$\begin{aligned} B_t(v, \preccurlyeq) - B_t(v, \preccurlyeq'_v) &= 2 \sum_{\lambda \in v - \{\kappa_1\}} (X^{\kappa_1} \otimes X^\lambda)(t) - (X^\lambda \otimes X^{\kappa_1})(t) \\ &= 2 \sum_{\lambda \in v} (X^{\kappa_1} \otimes X^\lambda)(t) - (X^\lambda \otimes X^{\kappa_1})(t) \\ &= 2[(X^{\kappa_1} \otimes X^{\Delta(v)})(t) - (X^{\Delta(v)} \otimes X^{\kappa_1})(t)]. \end{aligned}$$

An application of Lemma 2.12 then yields

$$\langle B_t(v, \preccurlyeq); df \otimes dg \rangle = \langle B_t(v, \preccurlyeq'); df \otimes dg \rangle$$

as desired. □

Of course, computations with the formula (2.8) still involves a ciliation, but we are free to choose a preferred one.

Proof (Theorem 2.6). The bracket $\{\ ,\ \}_t$ is defined as contraction with a bivector field and is therefore a derivation in each variable with respect to the multiplication of $\mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}$. Its anti-symmetry follows from the same property of $B_t(\preccurlyeq)$. All the difficulty lies in the proof of the Jacobi identity. It is convenient to define the set of *admissible pairs*

$$A = \{(\kappa, \lambda) \in E_\partial(K) \times E_\partial(K) \mid \kappa \neq \lambda,\ [\kappa] = [\lambda]\}$$

to rewrite the bracket as

$$\{f, g\}_t = \sum_{(\kappa,\lambda)\in A} \epsilon(\kappa, \lambda) \sum_i X_i^\kappa f X_i^\lambda g$$

For $f, g, h \in \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}$ we must show that the *Jacobiator*

$$J(f, g, h) = \{\{f, g\}, h\} + \text{cyclic perm. of } f, g, h$$

vanishes. Since

$$\{\{f, g\}, h\} = \sum_{(\kappa_1,\kappa_2)\in A} \sum_{(\lambda_1,\lambda_2)\in A} \epsilon(\kappa_1, \kappa_2)\epsilon(\lambda_1, \lambda_2) \sum_{i,j} X_j^{\lambda_1} (X_i^{\kappa_1} f X_i^{\kappa_2} g) X_j^{\lambda_2} h$$

we are lead to define the set of *parameters*:

$$P = P(f, g, h) = A \times A \times \{(f, g, h); (h, f, g); (g, h, f)\}.$$

To a parameter we associate left and right terms:

$$L((\kappa_1, \kappa_2), (\lambda_1, \lambda_2), (f, g, h)) = \epsilon(\kappa_1, \kappa_2)\epsilon(\lambda_1, \lambda_2) \sum_{i,j} X_j^{\lambda_1} X_i^{\kappa_1} f X_i^{\kappa_2} g X_j^{\lambda_2} h,$$

$$R((\kappa_1, \kappa_2), (\lambda_1, \lambda_2), (f, g, h)) = \epsilon(\kappa_1, \kappa_2)\epsilon(\lambda_1, \lambda_2) \sum_{i,j} X_i^{\kappa_1} f X_j^{\lambda_1} X_i^{\kappa_2} g X_j^{\lambda_2} h$$

By the Leibniz rule we have

$$J(f, g, h) = \sum_{p\in P} L(p) + R(p)$$

We reorganize this sum with the aid of the bijection $\psi\colon P \to P$ given by

$$\psi((\kappa_1, \kappa_2), (\lambda_1, \lambda_2), (f, g, h)) = ((\lambda_2, \lambda_1), (\kappa_1, \kappa_2), (h, f, g)).$$

Since

$$\begin{aligned} R((\lambda_2, \lambda_1), (\kappa_1, \kappa_2), (h, f, g)) &= \epsilon(\lambda_2, \lambda_1)\epsilon(\kappa_1, \kappa_2) \sum_{i,j} X_i^{\lambda_2} h X_j^{\kappa_1} X_i^{\lambda_1} f X_j^{\kappa_2} g \\ &= -\epsilon(\kappa_1, \kappa_2)\epsilon(\lambda_1, \lambda_2) \sum_{i,j} X_i^{\kappa_1} X_j^{\lambda_1} f X_i^{\kappa_2} g X_j^{\lambda_2} h \end{aligned}$$

we associate a third function to a parameter:

$$[(\kappa_1, \kappa_2), (\lambda_1, \lambda_2), (f, g, h)] = \epsilon(\kappa_1, \kappa_2)\epsilon(\lambda_1, \lambda_2) \sum_{i,j} [X_j^{\lambda_1}, X_i^{\kappa_1}] f X_i^{\kappa_2} g X_j^{\lambda_2} h \tag{2.10}$$

and arrive at the more manageable formula

$$J(f,g,h) = \sum_{p \in P} L(p) + R(\psi(p)) = \sum_{p \in P} [p]. \tag{2.11}$$

According to Lemma 2.9, the expression (2.10) is zero unless $\kappa_1 = \lambda_1$. Therefore define

$$P' = \{((\kappa_1, \kappa_2), (\lambda_1, \lambda_2), (f,g,h)) \in P \mid \kappa_1 = \lambda_1\} \cup \text{cyclic perm. of } f,g,h$$

and consider the map $\pi \colon P' \to 2^{E_\partial(K)}$ given by

$$\pi((\kappa,\lambda),(\kappa,\mu),(f,g,h)) = \{\kappa,\lambda,\mu\}.$$

By the admissibility of the two pairs in a parameter, we have

$$\mathrm{Im}(\pi) = \{s \subseteq E_\partial(K) \mid 2 \leqslant |s| \leqslant 3 \wedge \exists v \in V(K) \colon s \subseteq v\}.$$

Consequently, we may rewrite (2.11) as

$$J(f,g,h) = \sum_{p \in P'} [p] = \sum_{v \in V(K)} \sum_{\substack{s \subseteq v, \\ |s| = 2,3}} \sum_{p \in \pi^{-1}(s)} [p]. \tag{2.12}$$

Let us compute the generic term of this sum:

$$\begin{aligned}
[(\kappa,\lambda),(\kappa,\mu),(f,g,h)] &= \epsilon(\kappa,\lambda)\epsilon(\kappa,\mu) \sum_{i,j} [X^\kappa_j, X^\kappa_i] f X^\lambda_i g X^\mu_j h \\
&= \epsilon(\kappa,\lambda)\epsilon(\kappa,\mu) \sum_{i,j} X^\kappa([e_j,e_i]) f X^\lambda(e_i) g X^\mu(e_j) h \\
&= \epsilon(\kappa,\lambda)\epsilon(\kappa,\mu) \langle (X^\kappa \otimes X^\lambda \otimes X^\mu) T; df \otimes dg \otimes dh \rangle.
\end{aligned}$$

Define

$$[(\kappa,\lambda),(\kappa,\mu)] = [(\kappa,\lambda),(\kappa,\mu),(f,g,h)] + \text{cyclic perm. of } f,g,h$$

and apply the cyclic invariance of T (Lemma 2.7) to obtain

$$\begin{aligned}
&[(\kappa,\lambda),(\kappa,\mu)] \\
&\quad = \epsilon(\kappa,\lambda)\epsilon(\kappa,\mu) \big(\langle (X^\kappa \otimes X^\lambda \otimes X^\mu) T; df \otimes dg \otimes dh \rangle + \text{cyclic perm. of } \kappa,\lambda,\mu \big).
\end{aligned}$$

To compute (2.12) let $v \in V(K)$ and consider a subset $s \subseteq v$ of cardinality 2 or 3. In the former case write $s = \{\kappa, \lambda\}$. Then

$$\begin{aligned}
\pi^{-1}(s) &= \pi^{-1}(\{\kappa,\lambda\}) \\
&= \{((\kappa,\lambda),(\kappa,\lambda),(f,g,h)); ((\lambda,\kappa),(\lambda,\kappa),(f,g,h))\} \cup \text{cyclic perm. of } f,g,h
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{p \in \pi^{-1}(s)} [p] &= [(\kappa,\lambda),(\kappa,\lambda)] + [(\lambda,\kappa),(\lambda,\kappa)] \\
&= \sum_{\sigma \colon \{1,2,3\} \twoheadrightarrow s} \langle (X^{\sigma(1)} \otimes X^{\sigma(2)} \otimes X^{\sigma(3)}) T; df \otimes dg \otimes dh \rangle.
\end{aligned}$$

The other case is $|s| = 3$, in which we put $s = \{\kappa, \lambda, \mu\}$. Then

$$\begin{aligned}\pi^{-1}(s) &= \pi^{-1}(\{\kappa,\lambda,\mu\})\\ &= \{((\kappa,\lambda),(\kappa,\mu),(f,g,h));((\kappa,\mu),(\kappa,\lambda),(f,g,h))\}\\ &\quad \bigcup \text{independent cyclic perm. of } f,g,h \text{ and } \kappa,\lambda,\mu.\end{aligned}$$

The computation

$$\begin{aligned}&[(\kappa,\lambda),(\kappa,\mu)] + [(\kappa,\mu),(\kappa,\lambda)]\\ &\quad = \epsilon(\kappa,\lambda)\epsilon(\kappa,\mu)\big(\langle (X^\kappa \otimes X^\lambda \otimes X^\mu)T; df\otimes dg\otimes dh\rangle + \text{perm. of } \kappa,\lambda,\mu\big)\\ &\quad = \epsilon(\kappa,\lambda)\epsilon(\kappa,\mu) \sum_{\sigma\colon \{1,2,3\}\twoheadrightarrow s} \langle (X^{\sigma(1)}\otimes X^{\sigma(2)}\otimes X^{\sigma(3)})T; df\otimes dg\otimes dh\rangle\end{aligned}$$

proves that this contribution is affected only in sign by the cyclic permutation of κ, λ, μ. But

$$\epsilon(\kappa,\lambda)\epsilon(\kappa,\mu) + \epsilon(\lambda,\mu)\epsilon(\lambda,\kappa) + \epsilon(\mu,\kappa)\epsilon(\mu,\lambda) = 1$$

so we end up with the same formula as in the first case:

$$\begin{aligned}\sum_{p\in\pi^{-1}(s)} [p] &= [(\kappa,\lambda),(\kappa,\mu)] + [(\kappa,\mu),(\kappa,\lambda)] + \text{cyclic perm. of } \kappa,\lambda,\mu\\ &= \sum_{\sigma\colon \{1,2,3\}\twoheadrightarrow s} \langle (X^{\sigma(1)}\otimes X^{\sigma(2)}\otimes X^{\sigma(3)})T; df\otimes dg\otimes dh\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}J(f,g,h) &= \sum_{v\in V(K)} \sum_{\substack{s\subseteq v,\\ |s|=2,3}} \sum_{\sigma\colon \{1,2,3\}\twoheadrightarrow s} \big\langle (X^{\sigma(1)}\otimes X^{\sigma(2)}\otimes X^{\sigma(3)})T; df\otimes dg\otimes dh\big\rangle\\ &= \sum_{v\in V(K)} \sum_{\kappa,\lambda,\mu\in v} \langle (X^\kappa\otimes X^\lambda\otimes X^\mu)T; df\otimes dg\otimes dh\rangle\\ &\qquad - \sum_{\kappa\in v} \langle (X^\kappa)^{\otimes 3}(T); df\otimes dg\otimes dh\rangle\\ &= \sum_{v\in V(K)} \langle (X^{\Delta(v)})^{\otimes 3}(T); df\otimes dg\otimes dh\rangle - \sum_{\kappa\in E_\partial(K)} \langle (X^\kappa)^{\otimes 3}(T); df\otimes dg\otimes dh\rangle\end{aligned}$$

The first sum is zero because of Lemma 2.12. So is the second one since for any $\alpha \in E(K)$ and $A \in G^{E(K)}$, Lemma 2.10 implies

$$\begin{aligned}((X^{\partial+\alpha})^{\otimes 3}(T))_A &= ((X^{\partial+\alpha})^{\otimes 3} \sum_{i,j} [e_j,e_i]\otimes e_i\otimes e_j)_A\\ &= \sum_{i,j} X^{\partial+\alpha}([e_j,e_i])_A \otimes X^{\partial+\alpha}(e_i)_A \otimes X^{\partial+\alpha}(e_j)_A\\ &= \sum_{i,j} -X^{\partial-\alpha}(\mathrm{Ad}_{A_\alpha}([e_j,e_i]))_A \otimes -X^{\partial-\alpha}(\mathrm{Ad}_{A_\alpha}(e_i))_A \otimes -X^{\partial-\alpha}(\mathrm{Ad}_{A_\alpha}(e_j))_A\\ &= -\Big((X^{\partial-\alpha})^{\otimes 3} \sum_{i,j} [\mathrm{Ad}_{A_\alpha}(e_j), \mathrm{Ad}_{A_\alpha}(e_i)] \otimes \mathrm{Ad}_{A_\alpha}(e_i)\otimes \mathrm{Ad}_{A_\alpha}(e_j)\Big)_A\\ &= -((X^{\partial-\alpha})^{\otimes 3}(T))_A.\end{aligned}$$

This completes the proof. □

2.3 Poisson Structures on the Moduli Space

In this section K denotes a *regular, fat* combinatorial complex, i.e., the 1-skeleton of K is a fat graph and each face boundary $\alpha_1^{\epsilon_1}\cdots\alpha_n^{\epsilon_n}$ is a simple loop such that $\partial_+\alpha_i^{\epsilon_i}$ is the successor of $\partial_-\alpha_{i+1}^{\epsilon_{i+1}}, i = 1,\dots,n$ in the cyclic order at their common vertex. We shall prove that if K models Σ then the Poisson structure on $\mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}$ induces on $\mathcal{O}(\mathcal{M}(\Sigma;G))$ a Poisson structure which is independent of K; this is accomplished in two steps. Recall that the set of G-connections on K is an invariant, algebraic subset $\mathcal{A}(K) \subseteq G^{E(K)}$; the induced restriction map $\rho\colon \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)} \to \mathcal{O}(\mathcal{A}(K))^{\mathcal{G}(K)}$ is surjective by Remark 1.1.

Theorem 2.13. *The Poisson bracket* $\{\ ,\ \}_t$ *on* $\mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}$ *descends to* $\mathcal{O}(\mathcal{A}(K))^{\mathcal{G}(K)}$ *via the restriction map* $\rho\colon \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)} \to \mathcal{O}(\mathcal{A}(K))^{\mathcal{G}(K)}$.

Proof. It suffices to prove that the kernel of ρ is a Poisson ideal. We begin the proof by considering one face F of K. The algebraic equation

$$\mathrm{Hol}^A(\partial F) = 1, \quad A \in G^{E(K)}$$

defines an affine subset $\mathcal{A}(K,F) \subseteq G^{E(K)}$; associated to it is the ideal $I(\mathcal{A}(K,F)) \subseteq \mathcal{O}(G^{E(K)})$ of regular functions vanishing on $\mathcal{A}(K,F)$.

Claim. $I(\mathcal{A}(K,F)) \cap \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)} \subseteq \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}$ is a Poisson ideal.

Let $f \in I(\mathcal{A}(K,F)) \cap \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}$ and $g \in \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}$. Given $A \in \mathcal{A}(K,F)$, we must show that

$$\{f,g\}_t(A) = 0.$$

Write $\partial F = \alpha_1^{\epsilon_1}\cdots\alpha_n^{\epsilon_n}$, and set

$$\kappa_{j,1} = \partial_-\alpha_j^{\epsilon_j}, \quad \kappa_{j,2} = \partial_+\alpha_{j-1}^{\epsilon_{j-1}}, \quad v_j = [\kappa_{j,1}] = [\kappa_{j,2}]; \quad j = 1,\dots,n. \tag{2.13}$$

By the assumptions on K the vertices $v_1,\dots,v_n$ of F are distinct and $\kappa_{j,2}$ is the successor of $\kappa_{j,1}$ in the cyclic order at v_j. It is convenient to require that the ciliation of K is chosen such that at v_j the minimal elements are $\kappa_{j,1} \prec \kappa_{j,2}$. Since f is constant (in fact, zero) along $\mathcal{A}(K,F)$, we have

$$(X_i^\kappa f)_A = 0, \quad \kappa \in E_\partial(K) - \{\kappa_{1,1},\kappa_{1,2},\dots,\kappa_{n,1},\kappa_{n,2}\}.$$

This implies by Lemma 2.12 that

$$0 = (X^{\Delta(v_j)}(e_i)f)_A = (X_i^{\kappa_{j,1}}f)_A + (X_i^{\kappa_{j,2}}f)_A \tag{2.14}$$

and also simplifies the computation of the bracket to

$$\begin{aligned}\{f,g\}_t(A) &= \sum_j \sum_{\kappa\in v_j-\{\kappa_{j,1}\}} \epsilon(\kappa_{j,1},\kappa)\sum_i (X_i^{\kappa_{j,1}}f)_A(X_i^\kappa g)_A \\ &\quad + \sum_{\kappa\in v_j-\{\kappa_{j,2}\}} \epsilon(\kappa_{j,2},\kappa)\sum_i (X_i^{\kappa_{j,2}}f)_A(X_i^\kappa g)_A \\ &= \sum_j\sum_i (X_i^{\kappa_{j,1}}f)_A(X_i^{\kappa_{j,2}}g)_A + (X_i^{\kappa_{j,1}}f)_A(X_i^{\kappa_{j,1}}g)_A\end{aligned}$$

where the latter equality is due to (2.14). But since $\{\kappa_{j,1}, \kappa_{j+1,2}\} = \{\partial_+\alpha_j, \partial_-\alpha_j\}$ by (2.13), Lemma 2.10 yields

$$\sum_i X_i^{\kappa_{j,1}} f X_i^{\kappa_{j,1}} g = \langle (X^{\kappa_{j,1}})^{\otimes 2}(t); df \otimes dg \rangle = \langle (X^{\kappa_{j+1,2}})^{\otimes 2}(t); df \otimes dg \rangle = \sum_i X_i^{\kappa_{j+1,2}} f X_i^{\kappa_{j+1,2}} g$$

so that another application of (2.14):

$$\begin{aligned}\{f,g\}_t(A) &= \sum_j \sum_i (X_i^{\kappa_{j,1}} f)_A (X_i^{\kappa_{j,2}} g)_A + (X_i^{\kappa_{j+1,2}} f)_A (X_i^{\kappa_{j+1,2}} g)_A \\ &= \sum_i \sum_j (X_i^{\kappa_{j,1}} f)_A (X_i^{\kappa_{j,2}} g)_A - (X_i^{\kappa_{j+1,1}} f)_A (X_i^{\kappa_{j+1,2}} g)_A \\ &= 0\end{aligned}$$

finishes the proof of the claim. To prove the theorem, put $I_K = \sum_F I(\mathcal{A}(K,F))$ and deduce

$$V(I_K) = V\Big(\sum_F I(\mathcal{A}(K,F))\Big) = \bigcap_F V(I(\mathcal{A}(K,F))) = \bigcap_F \mathcal{A}(K,F) = \mathcal{A}(K),$$

the last equality being the definition of $\mathcal{A}(K)$. By Hilbert's Nullstellensatz

$$I(\mathcal{A}(K)) = IV(I_K) = \sqrt{I_K} = \mathrm{Rad}_{\mathcal{O}(G^{E(K)})}(I_K).$$

Therefore

$$\begin{aligned}\operatorname{Ker}\rho &= I(\mathcal{A}(K)) \cap \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)} \\ &= \mathrm{Rad}_{\mathcal{O}(G^{E(K)})}(I_K) \cap \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)} \\ &= \mathrm{Rad}_{\mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}}\big(I_K \cap \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}\big).\end{aligned} \tag{2.15}$$

Claim. We have

$$I_K \cap \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)} = \sum_F \big(I(\mathcal{A}(K,F)) \cap \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}\big). \tag{2.16}$$

This is an identity of ideals in $\mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}$ whence the inclusion $\supseteq$ is automatic. For the other one, let $f \in I_K \cap \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}$ and write $f = \sum_F f_F$, $f_F \in I(\mathcal{A}(K,F))$. The diagram

$$\begin{array}{ccc} \mathcal{O}(G^{E(K)}) & \xrightarrow{\rho_F} & \mathcal{O}(\mathcal{A}(K,F)) \\ \downarrow \nabla & & \downarrow \nabla \\ \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)} & \longrightarrow & \mathcal{O}(\mathcal{A}(K,F))^{\mathcal{G}(K)} \end{array}$$

implies that $\operatorname{Ker}\rho_F = I(\mathcal{A}(K,F))$ is closed under the Reynolds operator. Hence,

$$f = \nabla f = \sum_F \nabla f_F \in \sum_F \big(I(\mathcal{A}(K,F)) \cap \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}\big)$$

proving the claim. The right hand side of (2.16) is by the first claim a sum of Poisson ideals and thus a Poisson ideal. Recalling (2.15), we are done by the next lemma. □

Lemma 2.14. *Let $I \subseteq S$ be a Poisson ideal in a Poisson algebra. Then $\sqrt{I}$ is also a Poisson ideal.*

Proof. Let $x \in \sqrt{I}$ and $y \in S$. Pick $N \in \mathbf{N}$ such that $x^N \in I$. Write $\operatorname{ad} y = \{\ ,y\}\colon S \to S$; this map preserves I, so the Leibniz rule yields

$$\begin{aligned} I \ni (\operatorname{ad} y)^N(x^N) &= \sum_{\sigma\colon \{1,\dots,N\}\circlearrowleft} (\operatorname{ad} y)^{|\sigma^{-1}(1)|}(x) \cdots (\operatorname{ad} y)^{|\sigma^{-1}(N)|}(x) \\ &\equiv \sum_{\sigma\in S_N} \{x,y\}^N \\ &= N!\{x,y\}^N \mod x \end{aligned}$$

implying $N!\{x,y\}^N \in \langle x\rangle + I \subseteq \sqrt{I}$ so that $\{x,y\} \in \sqrt{I}$ as desired. □

Definition 2.15. A model $\iota\colon K \to \Sigma$ for Σ is called a *Poisson model* if ι is a homeomorphism and the cyclic order at each vertex of K agrees with the orientation of Σ. Such a model induces a Poisson structure on $\mathcal{O}(\mathcal{M}(\Sigma;G))$ by insisting that

$$\Psi_\iota\colon \mathcal{O}(\mathcal{A}(K))^{\mathcal{G}(K)} \xrightarrow{(\pi^*)^{-1}} \mathcal{O}(\mathcal{M}(K;G)) \xrightarrow{\operatorname{Hol}_\iota^*} \mathcal{O}(\mathcal{M}(\Sigma;G))$$

is a Poisson isomorphism.

Theorem 2.16. *The Poisson structure $\{\ ,\ \}_t$ on $\mathcal{M}(\Sigma;G)$ is independent of the Poisson model used to define it.*

Remark 2.17. In the case where Σ has non-empty boundary and $t = t_B$ for an orthogonal structure B on G, this result was obtained in [FR].

Proof. Let $\iota_j\colon K_j \to \Sigma$, $j = 1,2$ be two Poisson models. Recalling Remark 2.2, we see that the task is to prove the

Claim. The map $\tau_{12}\colon \mathcal{O}(\mathcal{A}(K_2))^{\mathcal{G}(K_2)} \to \mathcal{O}(\mathcal{A}(K_1))^{\mathcal{G}(K_1)}$ is a Poisson isomorphism.

We verify the claim in three special cases and then reduce the general situation to these cases.

Homotopy. If $K_1 = K_2$ and ι_1 and ι_2 are homotopic homeomorphisms, the claim is trivially true by Corollary 2.3.

For the remaining two cases it is useful to let τ_{12} be induced by a map $\varphi\colon G^{E(K_1)} \to G^{E(K_2)}$ as in the proof of Proposition 2.1. We may assume that φ has been constructed carefully, namely such that if K_1 and K_2 have a common edge α, it holds that

$$\varphi(A)_\alpha = A_\alpha, \quad A \in G^{E(K_1)}. \tag{2.17}$$

As the vector fields $X_i^{\partial_+\alpha}$ and $X_i^{\partial_-\alpha}$ live on the factor G^α common to $G^{E(K_1)}$ and $G^{E(K_2)}$, this implies

$$(X_i^\kappa(f\circ\varphi))_A = (X_i^\kappa f)_{\varphi(A)}, \quad \kappa \in \{\partial_\pm\alpha\},\ A \in G^{E(K_1)} \tag{2.18}$$

for any $f \in \mathcal{O}(G^{E(K_2)})^{\mathcal{G}(K_2)}$. In this formula X_i^κ is interpreted as a vector field on $G^{E(K_1)}$ (respectively, $G^{E(K_2)}$) on the left (respectively, right) hand side. Since we already know that

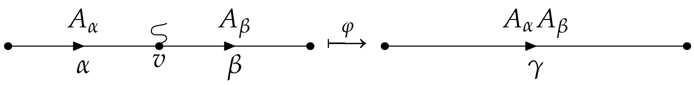
$(\varphi^*)_| = \tau_{12}$ is an isomorphism of algebras, it is enough to prove that it respects the Poisson brackets. But as $\rho_j\colon \mathcal{O}(G^{E(K_j)})^{\mathcal{G}(K_j)} \to \mathcal{O}(\mathcal{A}(K_j))^{\mathcal{G}(K_j)}$, $j = 1,2$ are surjective Poisson homomorphisms by Theorem 2.13, it suffices to verify that $\varphi^*\colon \mathcal{O}(G^{E(K_2)})^{\mathcal{G}(K_2)} \to \mathcal{O}(G^{E(K_1)})^{\mathcal{G}(K_1)}$ is a Poisson homomorphism. In other words, for $f_j \in \mathcal{O}(G^{E(K_2)})^{\mathcal{G}(K_2)}$, $j = 1,2$ and $A \in G^{E(K_1)}$, we must check that

$$\{f_1 \circ \varphi, f_2 \circ \varphi\}_t(A) = \{f_1, f_2\}_t(\varphi(A))$$

which expands to

$$\begin{aligned} \sum_{v \in V(K_1)} \sum_{\kappa,\lambda \in v} \epsilon(\kappa,\lambda) \sum_i (X_i^\kappa(f_1 \circ \varphi))_A (X_i^\lambda(f_2 \circ \varphi))_A \\ = \sum_{v \in V(K_2)} \sum_{\kappa,\lambda \in v} \epsilon(\kappa,\lambda) \sum_i (X_i^\kappa f_1)_{\varphi(A)} (X_i^\lambda f_2)_{\varphi(A)}. \end{aligned} \tag{2.19}$$

Edge division. Suppose that K_1 is obtained from K_2 by dividing an edge γ into two edges α and β with a vertex v. Fix an arbitrary ciliation of K_2. This induces a ciliation on K_1 once we add one of the two possible cilia at v. Almost all edges are common to the two complexes, so we focus attention on the bisected edge; using the trivial fact that $\gamma \simeq \alpha\beta$ rel endpoints, we have the following local picture of φ



Considering the curve $s \longmapsto (A_\alpha \exp(se_i))A_\beta = A_\alpha(\exp(se_i)A_\beta)$ proves

$$\big(X_i^{\partial_+\alpha}(f_j \circ \varphi)\big)_A = -\big(X_i^{\partial_-\beta}(f_j \circ \varphi)\big)_A.$$

Therefore the contribution from v to $\{f_1 \circ \varphi, f_2 \circ \varphi\}_t(A)$ is

$$\sum_i \big(X_i^{\partial_+\alpha}(f_1 \circ \varphi)\big)_A \big(X_i^{\partial_-\beta}(f_2 \circ \varphi)\big)_A - \big(X_i^{\partial_-\beta}(f_1 \circ \varphi)\big)_A \big(X_i^{\partial_+\alpha}(f_2 \circ \varphi)\big)_A = 0.$$

Analogously, the curves

$$s \longmapsto (\exp(se_i)A_\alpha)A_\beta = \exp(se_i)(A_\alpha A_\beta); \quad s \longmapsto A_\alpha(A_\beta \exp(se_i)) = (A_\alpha A_\beta)\exp(se_i)$$

imply the formulas

$$\big(X_i^{\partial_-\alpha}(f_j \circ \varphi)\big)_A = (X_i^{\partial_-\gamma} f_j)_{\varphi(A)}, \quad \big(X_i^{\partial_+\beta}(f_j \circ \varphi)\big)_A = (X_i^{\partial_+\gamma} f_j)_{\varphi(A)}.$$

Together with (2.18) this proves (2.19) and thereby the claim.

Face division. Assume that K_1 is obtained from K_2 by dividing a face F into two faces by a diagonal edge α (oriented arbitrarily). All edges but α are common to the complexes, that is, $E(K_1) = E(K_2) \cup \{\alpha\}$, and in light of (2.17) we see that $\varphi\colon G^{E(K_1)} = G^{E(K_2)} \times G^\alpha \to G^{E(K_2)}$ is the projection on the first factor. It is then obvious that

$$(X_i^\kappa(f_j \circ \varphi))_A = 0, \quad \kappa \in \{\partial_\pm \alpha\}$$

so the two endpoints in K_1 not contained in K_2 contribute nothing to $\{f_1 \circ \varphi, f_2 \circ \varphi\}_t(A)$. Choosing some ciliation of K_1 and employing the induced ciliation on K_2, it follows again from (2.18) that (2.19) and hence the claim holds.

General case. Given a third Poisson model $\iota_3 \colon K_3 \to \Sigma$, it is evident from (2.5) that $\tau_{12} = \tau_{13} \circ \tau_{32}$. Therefore it suffices to prove that K_1 and K_2 are related by a finite sequence of the above three moves homotopy, edge division and face division. First of all, we can assume by edge division that the two complexes contain an equal number of vertices. By homotopy we can ensure that the vertices of K_1 and K_2 are identified and that all intersections in the interiors of their edges are transverse double points. Add a vertex by edge division to both complexes at each of these intersection points. Now edges intersect only at vertices whence the union $K_1 \cup K_2$ is a Poisson model obtainable from either of K_1 and K_2 by successive face divisions. □

2.3.1 Poisson Homomorphisms between Moduli Spaces

Suppose Σ has non-empty boundary and denote by $\partial_0\Sigma$ one of its boundary circles. Let $\overline{\Sigma}$ be the surface obtained from Σ by attaching a disk along $\partial_0\Sigma$. Restricting a flat connection on $\overline{\Sigma}$ to Σ yields a map $r\colon \mathcal{M}(\overline{\Sigma}; G) \to \mathcal{M}(\Sigma; G)$. Let $\iota\colon K \to \Sigma$ be a Poisson model for Σ; attaching a disk along the face boundary of K corresponding to $\partial_0\Sigma$ results in a Poisson model $\bar{\iota}\colon \overline{K} \to \overline{\Sigma}$ for $\overline{\Sigma}$. Obviously, $E(\overline{K}) = E(K)$ and $V(\overline{K}) = V(K)$ so that $\mathcal{G}(\overline{K}) = \mathcal{G}(K)$. By definition, $\mathcal{A}(\overline{K})$ is a subset of $\mathcal{A}(K)$ and the equivariant inclusion $i\colon \mathcal{A}(\overline{K}) \to \mathcal{A}(K)$ models r; more precisely, the diagram

$$\begin{array}{ccccc} \mathcal{A}(\overline{K}) & \xrightarrow{\bar{\pi}} & \mathcal{M}(\overline{K}; G) & \xrightarrow{\mathrm{Hol}_{\bar{\iota}}^{-1}} & \mathcal{M}(\overline{\Sigma}; G) \\ \downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle r} \\ \mathcal{A}(K) & \xrightarrow{\pi} & \mathcal{M}(K; G) & \xrightarrow{\mathrm{Hol}_{\iota}^{-1}} & \mathcal{M}(\Sigma; G) \end{array}$$

commutes. Thus r preserves regular functions, and we have an induced diagram

$$\begin{array}{ccccc} \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)} & \xrightarrow{\rho} & \mathcal{O}(\mathcal{A}(K))^{\mathcal{G}(K)} & \overset{\Psi_\iota}{\text{———}} & \mathcal{O}(\mathcal{M}(\Sigma; G)) \\ \| & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle r^*} \\ \mathcal{O}(G^{E(\overline{K})})^{\mathcal{G}(\overline{K})} & \xrightarrow{\bar{\rho}} & \mathcal{O}(\mathcal{A}(\overline{K}))^{\mathcal{G}(\overline{K})} & \overset{\Psi_{\bar{\iota}}}{\text{———}} & \mathcal{O}(\mathcal{M}(\overline{\Sigma}; G)) \end{array}$$

It now follows from Theorem 2.13 and Definition 2.15 that $r^*\colon \mathcal{O}(\mathcal{M}(\Sigma; G)) \to \mathcal{O}(\mathcal{M}(\overline{\Sigma}; G))$ is a surjective Poisson homomorphism.

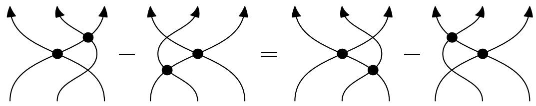
Remark 2.18. In anticipation of what follows we renormalize the Poisson bracket; from now on $\{\ ,\ \}_t$ equals half of the value it had prior to this remark.

2.4 The Poisson Algebra of Chord Diagrams

Following [AMR1], a *chord diagram* D is a finite collection of oriented circles with a finite number of *chords* (undirected line segments connecting two distinct points of the circles) marked upon them. Chords are assumed to be disjoint, in particular no two endpoints of them coincide. The circles in D are called the *core components*, and regarded as a whole their union is termed the *skeleton*. When drawing (parts of) chord diagrams in the plane, chords are depicted as dashed lines.

A *geometric chord diagram on* Σ is a smooth map from a chord diagram to Σ, mapping the chords to points. Images of chords will be drawn as fat dots. A *chord diagram on* Σ is an equivalence class of geometric chord diagrams modulo homotopy. Clearly, every chord diagram on Σ contains a *generic chord diagram*, i.e., a geometric chord diagram such that its skeleton is immersed in Σ and with all intersections being transverse double points.

Let $\mathcal{D}(\Sigma)$ denote the complex vector space with basis the set of chord diagrams on Σ; it is graded by the number of chords. Let $4\mathrm{T}(\Sigma) \subseteq \mathcal{D}(\Sigma)$ be the subspace defined by the local relation



(2.20)

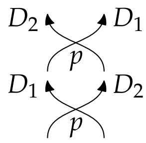
as well as all the relations obtained from this one by reversing orientations of strands (core components) and changing the signs accordingly: For every chord intersecting a component with reversed orientation, there is a factor of -1 for the diagram. These relations are called *4T-relations*. Of course, $4\mathrm{T}(\Sigma)$ is a homogeneous subspace so that the quotient

$$\mathcal{C}(\Sigma) = {}^{\mathcal{D}(\Sigma)}/_{4\mathrm{T}(\Sigma)}$$

is also a graded vector space.

There is an obvious graded algebra structure on $\mathcal{D}(\Sigma)$ given by taking the union of two chord diagrams. This multiplication is evidently commutative, and the empty chord diagram is its unit. Moreover the subspace $4\mathrm{T}(\Sigma)$ is an ideal whence $\mathcal{C}(\Sigma)$ is also a commutative, graded algebra. The Poisson structure on $\mathcal{C}(\Sigma)$ has the following description. Given two chord diagrams on Σ, we can choose geometric chord diagrams $\iota_j \colon D_j \to \Sigma$, $j = 1, 2$ representing them such that their union (product) is a generic chord diagram. For $p \in D_1 \# D_2$ the oriented intersection index is given by

$$\epsilon(p; D_1, D_2) = \begin{cases} 1 & \text{for} \\ -1 & \text{for} \end{cases}$$



Define $D_1 \cup_p D_2$ to be the chord diagram on Σ obtained by joining $\iota_1^{-1}(p)$ and $\iota_2^{-1}(p)$ by a chord mapped to p.

Proposition 2.19 (Andersen, Mattes & Reshetikhin). *The bracket*

$$\{[D_1], [D_2]\} = \sum_{p \in D_1 \# D_2} \epsilon(p; D_1, D_2)[D_1 \cup_p D_2]$$

is well-defined and determines a Poisson structure on $\mathcal{C}(\Sigma)$.

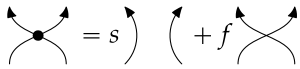
We call $\mathcal{C}(\Sigma)$ the *Poisson algebra of chord diagrams on* Σ.

Remark 2.20. We can colour the core components of chord diagrams with finite dimensional, rational representations of G. The above definitions make sense in this setting, too, and yields a Poisson algebra $\mathcal{C}(\Sigma; G)$.

Remark 2.21. An orientation preserving diffeomorphism of Σ respects the 4T-relation, so there are natural actions of $\Gamma_+(\Sigma)$ on $\mathcal{C}(\Sigma)$ and $\mathcal{C}(\Sigma; G)$. It is immediate from Proposition 2.19 that these actions are by Poisson isomorphisms.

2.4.1 The Poisson Loop Algebras

Various quotients of $\mathcal{C}(\Sigma)$ will occupy us in the sequel. It is therefore important to notice that any subspace defined by local relations is a Poisson ideal and thus yields a quotient Poisson algebra. A recurring example of this is the *loop relation* $L_{s,f}$:



(2.21)

Here s (smooth) and f (forget) are complex parameters. For future reference we calculate the quotient $\mathcal{C}(\Sigma)/L_{s,f}$. The sub-algebra $\mathcal{Z}(\Sigma) = \mathcal{D}^{(0)}(\Sigma) = \mathcal{C}^{(0)}(\Sigma) \subseteq \mathcal{C}(\Sigma)$ is a vector space with basis the set of *diagrams on* Σ (chord diagrams without chords). Let $\hat{\pi}_1(\Sigma)$ denote the set of conjugacy classes in $\pi_1(\Sigma)$, i.e., the set of free homotopy classes of loops on Σ. A diagram on Σ is simply a finite collection of elements from $\hat{\pi}_1(\Sigma)$, implying that $\mathcal{Z}(\Sigma)$ is isomorphic to the polynomial algebra on $\hat{\pi}_1(\Sigma)$; we therefore refer to $\mathcal{Z}(\Sigma)$ as the *loop algebra of* Σ.

Associated to $L_{s,f}$ is the resolving map $R_{s,f}\colon \mathcal{C}(\Sigma) \to \mathcal{Z}(\Sigma)$ given by replacing each chord by the right hand side of (2.21); the loop relation implies the 4T-relation so $R_{s,f}$ is well-defined. It is also obvious that the induced map

$$R_{s,f}\colon \mathcal{C}(\Sigma)/L_{s,f} \to \mathcal{Z}(\Sigma) \tag{2.22}$$

is an algebra isomorphism, its inverse being the composite $\mathcal{Z}(\Sigma) \subseteq \mathcal{C}(\Sigma) \to \mathcal{C}(\Sigma)/L_{s,f}$. To transfer the Poisson bracket to $\mathcal{Z}(\Sigma)$ via this isomorphism is elementary. For generic diagrams D_1 and D_2 on Σ we have

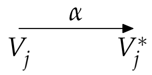
$$\{D_1, D_2\}_{s,f} = \sum_{p \in D_1 \# D_2} \epsilon(p; D_1, D_2)(s(D_1D_2)_p + fD_1D_2) \tag{2.23}$$

where $(D_1D_2)_p$ denotes the diagram obtained from D_1D_2 by the orientation preserving smoothing of the crossing at p. The loop algebra endowed with this Poisson structure is denoted by $\mathcal{Z}_{s,f}(\Sigma)$; for some values of (s, f) these Poisson algebras were studied in [G2], [T], [AMR1] and [AMR2].

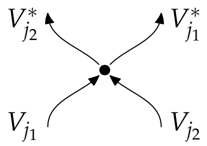
2.5 Chord Diagrams as Functions on the Moduli Space

We provide in this section the generalization of the Poisson homomorphism $\Psi_B\colon \mathcal{C}(\Sigma; G) \to (\mathcal{O}(\mathcal{M}(\Sigma; G)), \{\ ,\ \}_B)$ where B is an orthogonal structure on G (cf. [AMR1]) to the case of an arbitrary symmetric, Ad-invariant tensor $t \in \mathfrak{g} \otimes \mathfrak{g}$.

Let D be a generic chord diagram the ith core component of which is coloured by a finite dimensional, rational representation $\rho_i\colon G \to \mathrm{Aut}(V_i)$. We associate a function on the moduli space to D in the following fashion. Denote by A a flat G-connection in a principal bundle $P \to \Sigma$. A set of *cut points* on D is by definition a finite set C of points on D including all chords and at least one point on each core component of the diagram; such a choice naturally induces a decomposition of D into its chords and a set $E(D, C)$ of arcs (we ignore the non-chord intersections in D). Trivializing the fibre of P over each cut point, the holonomy with respect to A along an arc α becomes an element $T_\alpha^A \in G$. Decorate the arcs and the chords with vector spaces and corresponding tensors as follows (we employ the derived representations of $\mathfrak{g}$):



$$\rightsquigarrow \rho_j(T^A_\alpha) \in \mathrm{Aut}(V_j) \subseteq \mathrm{End}(V_j) \cong V_j \otimes V_j^*, \tag{2.24a}$$



$$\rightsquigarrow (\rho_{j_1} \otimes \rho_{j_2})(t) \in \mathrm{End}(V_{j_1}) \otimes \mathrm{End}(V_{j_2}) \cong V_{j_1} \otimes V_{j_1}^* \otimes V_{j_2} \otimes V_{j_2}^*. \tag{2.24b}$$

It is, of course, understood that the indices on representations are those of the corresponding core components. We have drawn (infinitesimal) parts of the arcs incident to a chord in order to indicate how the vector spaces are associated to these arcs. From the tensors of the individual pieces we get a tensor $\mathcal{T}_t(D, C; A)$ in the tensor product of all the vector spaces involved. Glue the pieces together to get the diagram D back. While doing this, we produce a number $\mathcal{T}_t(D; A) \in \mathbf{C}$ from $\mathcal{T}_t(D, C; A)$ by performing the canonical contraction of a vector space and its dual occurring where two arcs are glued together and where an arc is glued to a chord. The number $\mathcal{T}_t(D; A)$ is independent of the trivialization of $P_{|C}$; this is immediate for the cut points that are not chords, and for the chords it is an easy consequence of the compatibility of the derived representation with the adjoint actions:

$$\rho \circ \mathrm{Ad}_g = \mathrm{Ad}_{\rho(g)} \circ \rho \colon \mathfrak{g} \to \mathrm{End}(V), \quad g \in G$$

and the Ad-invariance of t. It is also clear that $\mathcal{T}_t(D; A)$ is independent of how the non-chord cut points are chosen; omitting C in the notation is hence justified. We set $f^t_D(A) = \mathcal{T}_t(D; A)$.

A core component S_j intersecting no chord evidently contributes the factor $\mathrm{Tr}(\rho_j(T^A_{S_j}))$ to $f^t_D(A)$. Having taken care of such components, we now present a formula for $f^t_D(A)$ in the case where all core components intersect at least one chord. We choose the cut points in the most economical way, namely we choose only the chords. In this way the circle S_j is decomposed into arcs $\alpha_{j,1}, \ldots, \alpha_{j,n_j}$ where the indexing order agrees with the cyclic order of the arcs induced by the orientation of S_j. Pick for each chord c_k a basis $\{e_{i_k}\}_{i_k}$ for t, and define indices $k(j, l)$ by $c_{k(j,l)} = \partial_+ \alpha_{j,l}$; this is possible since every arc of the decomposition ends at a chord. Then

$$f^t_D(A) = \sum_{i_1, \ldots, i_m} \prod_{j=1}^{n} \mathrm{Tr}\Big(\prod_{l=1}^{n_j} \rho_j(T^A_{\alpha_{j,l}}) \rho_j(e_{i_{k(j,l)}}) \Big). \tag{2.25}$$

To verify this formula one simply has to recall the commutative square

$$\begin{array}{ccc} \mathrm{End}(V)^{\otimes N} & \xrightarrow{\ \Pi\ } & \mathrm{End}(V) \\ \big\downarrow{\scriptstyle \cong} & & \big\downarrow{\scriptstyle \mathrm{Tr}} \\ (V \otimes V^*)^{\otimes N} & \longrightarrow & \mathbf{C} \end{array}$$

where Π is multiplication, i.e., composition of endomorphisms. The lower horizontal map is given by performing the canonical contractions of the pairs of spaces indicated by the numbers here:

$$(V \otimes V^*)^{\otimes N} = \overset{1}{V} \otimes \overset{2}{V^*} \otimes \overset{2}{V} \otimes \overset{3}{V^*} \otimes \cdots \otimes \overset{N}{V} \otimes \overset{1}{V^*}.$$

If $t = t_B \in \mathfrak{g} \otimes \mathfrak{g}$ corresponds to an orthogonal structure B, the formula (2.25) agrees with the definition in [AMR1] where it is also proved that $f_D^{t_B}(A)$ depends only on the equivalence class $[A] \in \mathcal{M}(\Sigma; G)$ and the homotopy class $[D] \in \mathcal{D}(\Sigma; G)$. The proof rests on the fact that t_B is Ad-invariant, so it is valid in our setting, too. Therefore the formula

$$(\Psi_t([D]))([A]) = f_D^t(A)$$

determines a linear map $\Psi_t\colon \mathcal{D}(\Sigma; G) \to \mathrm{Fun}(\mathcal{M}(\Sigma; G))$; it is clearly a homomorphism of algebras.

A model (K, ι) for Σ is said to be *compatible* with a chord diagram D with cut points C if $C \subseteq V(K)$ and $E(D, C) \subseteq E(K)$. In this situation we may use the bijection

$$\pi^* \circ (\mathrm{Hol}_\iota^*)^{-1}\colon\ \mathrm{Fun}(\mathcal{M}(\Sigma; G)) \to \mathrm{Fun}(\mathcal{A}(K; G))^{\mathcal{G}(K)}$$

and (2.25) to express $f_D^t \in \mathrm{Fun}(\mathcal{M}(\Sigma; G))$ as

$$f_D^t((A_\alpha)) = \sum_{i_1,\ldots,i_m} \prod_{j=1}^{n} \mathrm{Tr}\Big(\prod_{l=1}^{n_j} \rho_j(A_{\alpha_{j,l}})\rho_j(e_{i_{k(j,l)}})\Big), \quad (A_\alpha) \in \mathcal{A}(K; G). \tag{2.26}$$

It follows immediately from this formula that f_D^t is a regular function on the moduli space.

The last step in the construction is to verify that $\Psi_t\colon \mathcal{D}(\Sigma; G) \to \mathcal{O}(\mathcal{M}(\Sigma; G))$ respects the 4T-relation (2.20). Denote by j_l the index of the core component to which the lth strand (counting from left to right at the bottom) in the local pictures belongs. By an appropriate choice of cut points and trivializations of their fibres, we may assume that the parallel transports $T_\alpha^A = 1$ for all arcs α occuring in the relation. Associating the same basis $\{e_1, \ldots, e_n\}$ for t to all chords, we compute the contribution from the left hand side of (2.20) to be the following endomorphism of $V_{j_1} \otimes V_{j_2} \otimes V_{j_3}$ (recall the formula (2.9)):

$$\begin{aligned}
&\sum_{i_1,i_2} \rho_{j_1}(e_{i_1})\rho_{j_1}(e_{i_2}) \otimes \rho_{j_2}(e_{i_2}) \otimes \rho_{j_3}(e_{i_1}) - \sum_{i_1,i_2} \rho_{j_1}(e_{i_2})\rho_{j_1}(e_{i_1}) \otimes \rho_{j_2}(e_{i_2}) \otimes \rho_{j_3}(e_{i_1})\\
&\qquad = \sum_{i_1,i_2} \rho_{j_1}([e_{i_1}, e_{i_2}]) \otimes \rho_{j_2}(e_{i_2}) \otimes \rho_{j_3}(e_{i_1})\\
&\qquad = (\rho_{j_1} \otimes \rho_{j_2} \otimes \rho_{j_3})(T).
\end{aligned}$$

Analogously, the right hand side of (2.20) contributes the endomorphism

$$\begin{aligned}
&\sum_{i_1,i_2} \rho_{j_1}(e_{i_1}) \otimes \rho_{j_2}(e_{i_2}) \otimes \rho_{j_3}(e_{i_2})\rho_{j_3}(e_{i_1}) - \sum_{i_1,i_2} \rho_{j_1}(e_{i_1}) \otimes \rho_{j_2}(e_{i_2}) \otimes \rho_{j_3}(e_{i_1})\rho_{j_3}(e_{i_2})\\
&\qquad = \sum_{i_1,i_2} \rho_{j_1}(e_{i_1}) \otimes \rho_{j_2}(e_{i_2}) \otimes \rho_{j_3}([e_{i_2}, e_{i_1}])\\
&\qquad = (\rho_{j_1} \otimes \rho_{j_2} \otimes \rho_{j_3})(T)
\end{aligned}$$

by the cylic invariance of T (Lemma 2.7).

Theorem 2.22. *The map* $\Psi_t\colon \mathcal{C}(\Sigma, G) \to (\mathcal{O}(\mathcal{M}(\Sigma; G)), \{\ ,\ \}_t)$ *is a* $\Gamma_+(\Sigma)$*-equivariant Poisson homomorphism.*

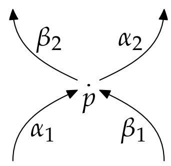
Proof. Let D be a coloured chord diagram on Σ, and let $g \in \Gamma_+(\Sigma)$. From the definition of connection pullback follows

$$(g \cdot f_D^t)([A]) = f_D^t([g^* A]) = \mathcal{T}_t(D; g^* A) = \mathcal{T}_t(g(D); A) = f_{g(D)}^t([A]), \quad [A] \in \mathcal{M}(\Sigma; G)$$

as desired. It only remains to prove that Ψ_t preserves the Poisson brackets. Let D_1 and D_2 be coloured chord diagrams with $D_1 \cup D_2$ in general position. Choose cut points C_j for D_j, $j = 1, 2$ such that every $p \in D_1 \# D_2$ is a cut point for both D_1 and D_2 and such that the arc sets $E(D_j, C_j)$ contain no loops. Let K be a Poisson model for Σ compatible with the decompositions of both diagrams. Notice that formula (2.26) makes sense for any graph connection $(A_\alpha) \in G^{E(K)}$. Therefore $f^t_{D_j} \in \mathcal{O}(\mathcal{A}(K;G))^{\mathcal{G}(K)}$ is naturally the restriction of a function $f_j \in \mathcal{O}(G^{E(K)})^{\mathcal{G}(K)}$, and by Theorem 2.13

$$\{f^t_{D_1}, f^t_{D_2}\}_t((A_\alpha)) = \{f_1, f_2\}_t((A_\alpha)), \quad (A_\alpha) \in \mathcal{A}(K;G).$$

We apply formula (2.8) to compute the right hand side. Since f_j depends only on the factor $G^{E(D_j,C_j)}$ of $G^{E(K)}$, a pair $(\kappa_1, \kappa_2) \in E_\partial(K) \times E_\partial(K)$ cannot contribute to the bracket unless κ_j is an endpoint of an arc in the decomposition of D_j. Hence the vertex $[\kappa_1] = [\kappa_2] \in V(K)$ must be an intersection point $p \in D_1 \# D_2$. Here is a picture of the decomposition of $D_1 \cup D_2$ near p:



Of course, other edges of K may be incident to p, but we have already justified that they can be ignored in the computation. For definiteness we assume that the ciliation at p is such that

$$\partial_+\alpha_1 \prec \partial_+\beta_1 \prec \partial_-\alpha_2 \prec \partial_-\beta_2. \tag{2.27}$$

Since we allow no loops in the decompositions, the arcs $\alpha_1, \alpha_2, \beta_1, \beta_2$ are distinct. The intersection index $\epsilon(p; D_1, D_2)$ distinguishes two cases; suppose p is a positive crossing, i.e., α_1 and α_2 belong to D_1 (and β_1, β_2 to D_2). Denote by j_i, $i = 1, 2$ the index of the relevant core component of D_i. Let $A^0 \in \mathcal{A}(K;G)$ be a connection on K and consider its corresponding inclusions

$$\iota^\alpha_{A^0} \colon G^{\alpha_1} \times G^{\alpha_2} \to G^{E(K)}, \quad \iota^\beta_{A^0} \colon G^{\beta_1} \times G^{\beta_2} \to G^{E(K)}.$$

It is evident from (2.26) that there are naturally defined linear maps $L_i \colon \operatorname{End}(V_{j_i}) \to \mathbf{C}$, $i = 1, 2$ (depending on A^0) such that

$$f_1(\iota^\alpha_{A^0}(A_{\alpha_1}, A_{\alpha_2})) = L_1(\rho_{j_1}(A_{\alpha_1})\rho_{j_1}(A_{\alpha_2})), \quad (A_{\alpha_1}, A_{\alpha_2}) \in G^{\alpha_1} \times G^{\alpha_2},$$
$$f_2(\iota^\beta_{A^0}(A_{\beta_1}, A_{\beta_2})) = L_2(\rho_{j_2}(A_{\beta_1})\rho_{j_2}(A_{\beta_2})), \quad (A_{\beta_1}, A_{\beta_2}) \in G^{\beta_1} \times G^{\beta_2}.$$

Therefore, in a basis $\{e_1, \cdots, e_n\}$ for t,

$$\begin{aligned}(X_i^{\partial_+\alpha_1} f_1)_{A^0} &= \frac{d}{ds}_{|s=0} f_1 \iota^\alpha_{A^0}(A^0_{\alpha_1} \exp(se_i), A^0_{\alpha_2}) \\ &= \frac{d}{ds}_{|s=0} L_1\big(\rho_{j_1}(A^0_{\alpha_1})\rho_{j_1}(\exp(se_i))\rho_{j_1}(A^0_{\alpha_2})\big) \\ &= L_1\big(\rho_{j_1}(A^0_{\alpha_1})\rho_{j_1}(e_i)\rho_{j_1}(A^0_{\alpha_2})\big)\end{aligned} \tag{2.28}$$

since differentiation commutes with linear maps. Analogously,

$$\begin{aligned}(X_i^{\partial_-\alpha_2} f_1)_{A^0} &= \frac{d}{ds}_{|s=0} f_1 \iota^{\alpha}_{A^0}(A^0_{\alpha_1}, \exp(s(-e_i))A^0_{\alpha_2}) \\ &= \frac{d}{ds}_{|s=0} L_1\big(\rho_{j_1}(A^0_{\alpha_1})\rho_{j_1}(\exp(s(-e_i)))\rho_{j_1}(A^0_{\alpha_2})\big) \\ &= -L_1\big(\rho_{j_1}(A^0_{\alpha_1})\rho_{j_1}(e_i)\rho_{j_1}(A^0_{\alpha_2})\big),\end{aligned} \tag{2.29}$$

and in the same way

$$(X_i^{\partial_+\beta_1} f_2)_{A^0} = L_2\big(\rho_{j_2}(A^0_{\beta_1})\rho_{j_2}(e_i)\rho_{j_2}(A^0_{\beta_2})\big), \tag{2.30}$$

$$(X_i^{\partial_-\beta_2} f_2)_{A^0} = -L_2\big(\rho_{j_2}(A^0_{\beta_1})\rho_{j_2}(e_i)\rho_{j_2}(A^0_{\beta_2})\big). \tag{2.31}$$

Thus, up to a sign, the four relevant pairs of endpoints

$$(\partial_+\alpha_1, \partial_+\beta_1), (\partial_+\alpha_1, \partial_-\beta_2), (\partial_-\alpha_2, \partial_+\beta_1) \text{ and } (\partial_-\alpha_2, \partial_-\beta_2)$$

yield the same contribution to $\{f_1, f_2\}_t(A^0)$, namely

$$\frac{1}{2}\sum_i L_1\big(\rho_{j_1}(A^0_{\alpha_1})\rho_{j_1}(e_i)\rho_{j_1}(A^0_{\alpha_2})\big) L_2\big(\rho_{j_2}(A^0_{\beta_1})\rho_{j_2}(e_i)\rho_{j_2}(A^0_{\beta_2})\big).$$

By definition of Ψ_t this expression is exactly $\frac{1}{2} f^t_{D_1 \cup_p D_2}(A^0)$. Regarding the sign, the ciliation (2.27) and the signs in formulas (2.28)–(2.31) imply that the pair $(\partial_+\alpha_1, \partial_-\beta_2)$ yields the total sign -1 whereas the remaining three pairs yield total sign $+1$. Since $(3-1)\frac{1}{2} = 1$, the total contribution from p is $f^t_{D_1 \cup_p D_2}(A^0)$. The case of a negative crossing is, of course, entirely analogous; the contribution to $\{f_1, f_2\}_t(A^0)$ is then $-f^t_{D_1 \cup_p D_2}(A^0)$. Hence,

$$\{f^t_{D_1}, f^t_{D_2}\}_t(A^0) = \{f_1, f_2\}_t(A^0) = \sum_{p \in D_1 \# D_2} \epsilon(p; D_1, D_2) f^t_{D_1 \cup_p D_2}(A^0) = f^t_{\{D_1, D_2\}}(A^0).$$

The proof is complete. □

Theorem 2.23 (Andersen, Mattes & Reshetikhin). *The map* $\Psi_B\colon \mathcal{C}(\Sigma; G) \to \mathcal{O}(\mathcal{M}(\Sigma; G))$ *is surjective if* G *is one of the groups* $\mathrm{GL}_n(\mathbf{C})$, $\mathrm{SL}_n(\mathbf{C})$, $\mathrm{O}_n(\mathbf{C})$ *and* $\mathrm{Sp}_{2n}(\mathbf{C})$ *equipped with a suitable orthogonal structure* B.

Chapter 3

Quantization of Poisson Algebras

We present in this chapter the concept of deformation quantization of Poisson algebras (à la Turaev [T]). The important special case of a $*$-product is considered, including a vital, general example due to Andersen, Mattes and Reshetikhin [AMR2] and basic properties of quotients and actions. Completion of general quantizations is also addressed. In the course of this presentation we shall need various elementary results about filtered modules and algebras, in particular, about their relation to graded modules, respectively graded (Poisson) algebras; the first section is devoted to these results.

3.1 Filtered and Graded Objects

Throughout this section R is an arbitrary commutative ring; modules and algebras have R as ground ring unless they do not.

3.1.1 Filtered Modules and Algebras

By a *filtered module*, we shall mean a module M with submodules M_n, $n \in \mathbf{N}$ such that

$$M = M_0 \supseteq M_1 \supseteq \cdots$$

As an example let $h \in R$, and set for each $n \in \mathbf{N}$

$$h^n M = \{h^n v \mid v \in M\}, \quad (h^0 = 1).$$

Then $h^n M \subseteq M$ is a submodule, and putting $M_n = h^n M$ defines a filtration termed the *h-filtration* on M.

A map of filtered modules is a module map respecting the filtrations. We define $M_\infty = \bigcap_{n=0}^{\infty} M_n$ and say that M is *Hausdorff* if $M_\infty = 0$. A sequence (v_i) in M is called a *null sequence* if

$$\forall n \in \mathbf{N} \exists N \in \mathbf{N} \colon i \geqslant N \Longrightarrow v_i \in M_n.$$

The sequence is said to converge to $v \in M$ if $(v_i - v)$ is a null sequence. Maps of filtered modules preserve limits. If M is Hausdorff, then limits are unique. Every convergent sequence is *Cauchy*; i.e., it satisfies

$$\forall n \in \mathbf{N} \exists N \in \mathbf{N} \colon i, j \geqslant N \Longrightarrow v_i - v_j \in M_n.$$

In case every Cauchy sequence is convergent, M is said to be *complete*. The *completion of* M is a pair $(\overline{M}, \iota)$ where $\overline{M}$ is a complete, Hausdorff filtered module and $\iota\colon M \to \overline{M}$ is the universal map from M to complete, Hausdorff filtered modules, i.e., if $f\colon M \to N$ is a map of filtered modules of which N is Hausdorff and complete, there is a unique map $\overline{f}\colon \overline{M} \to N$ such that $\overline{f} \circ \iota = f$. It is a consequence of this universal property that $M \mapsto \overline{M}$ becomes a functor on the category of filtered modules and that M is canonically isomorphic to $\overline{M}$ if M is complete and Hausdorff. The completion of M may be constructed as the inverse limit of

$$M/M_1 \leftarrow M/M_2 \leftarrow \cdots$$

Concretely, this means that elements of $\overline{M}$ are sequences $([v_n]_n)$, $[v_n]_n \in M/M_n$ such that $v_{n+1} - v_n \in M_n$; the submodule $\overline{M}_n$ consists of those sequences in which the first n terms vanish, i.e., one may assume that $v_m \in M_n$, $m \in \mathbf{N}$. The universal map is given by $\iota(v) = ([v]_n)$, and the formula for the completed map is $\overline{f}(([v_n]_n)) = \lim_{n\to\infty} f(v_n)$.

Remark 3.1. If M is completed with respect to the h-filtration, then $h^n\overline{M} \subseteq \overline{M}_n$, as is easily verified.

A *filtered algebra* A is an algebra which as a module is filtered in a fashion compatible with the multiplication: $A_n A_{n'} \subseteq A_{n+n'}$. Maps of filtered algebras are maps of algebras which are maps of filtered modules, too. The above discussion of filtered modules carries over to the setting of filtered algebras; in particular, notice that $A_n \subseteq A$ is an ideal so that the construction of the completion $\overline{A}$ makes sense in the category of algebras.

3.1.2 Modules over the Power Series Ring

Let V denote a complex vector space. The set $V[[h]]$ of power series with coefficients in V is naturally a module over $\mathbf{C}[[h]]$ with scalar multiplication

$$\sum_i \lambda_i h^i \sum_j v_j h^j = \sum_{i,j} \lambda_i v_j h^{i+j}, \quad v_j \in V,\ \lambda_i \in \mathbf{C}.$$

Unless explicitly stated otherwise, we employ the h-filtration on $V[[h]]$ making it both complete and Hausdorff. Assume that M is a complete, Hausdorff filtered $\mathbf{C}[[h]]$-module such that $h^n M \subseteq M_n$. Then

$$\mathrm{Hom}_{\mathbf{C}[[h]]}(V[[h]], M) \cong \mathrm{Hom}_{\mathbf{C}}(V, M) \tag{3.1}$$

as complex vector spaces. Namely, if $\varphi\colon V[[h]] \to M$ is $\mathbf{C}[[h]]$-linear, it is also filtered so that

$$\varphi\Big(\sum_i v_i h^i\Big) = \sum_i h^i \varphi(v_i) \tag{3.2}$$

whence φ is determined by its restriction to $V \subseteq V[[h]]$. On the other hand, if $\varphi\colon V \to M$ is complex linear, formula (3.2) provides a well-defined extension of φ to a $\mathbf{C}[[h]]$-linear map $\varphi\colon V[[h]] \to M$ since the right hand side is a Cauchy sequence in M.

Consider the special case $M = W[[h]]$ where W is a complex vector space. Clearly

$$\mathrm{Hom}_{\mathbf{C}}(V, W[[h]]) \cong \mathrm{Hom}_{\mathbf{C}}(V, W)[[h]]$$

as complex vector spaces, and combining this with (3.1) we obtain

$$\mathrm{Hom}_{\mathbf{C}[[h]]}(V[[h]], W[[h]]) \cong \mathrm{Hom}_{\mathbf{C}}(V, W)[[h]]$$

in fact, this is an isomorphism of $\mathbf{C}[[h]]$-modules ($\mathbf{C}[[h]]$-algebras if $V = W$). The formula corresponding to this isomorphism is

$$\sum_j \varphi_j h^j \big(\sum_i v_i h^i\big) = \sum_{i,j} \varphi_j(v_i) h^{i+j}, \quad \varphi_j \in \mathrm{Hom}_{\mathbf{C}}(V, W) \tag{3.3}$$

cf. (3.2). In particular, $\varphi \in \mathrm{Hom}_{\mathbf{C}}(V, W) \subseteq \mathrm{Hom}_{\mathbf{C}}(V, W)[[h]]$ induces a $\mathbf{C}[[h]]$-linear map $\varphi = \varphi_h \colon V[[h]] \to W[[h]]$ given by $\varphi(\sum_i v_i h^i) = \sum_i \varphi(v_i) h^i$.

3.1.3 Relations to Graded Objects

We have a functor Gr from filtered modules to graded modules defined by

$$M_{\mathrm{Gr}} = \bigoplus_{m=0}^{\infty} M^{(m)}, \quad M^{(m)} = M_m / M_{m+1}$$

on objects. A morphism $f \colon M \to N$ is taken to $f_{\mathrm{Gr}} = \bigoplus_{m=0}^{\infty} f_{\mathrm{Gr}}^{(m)}$ where $f_{\mathrm{Gr}}^{(m)}$ is the unique map making the diagram

$$\begin{array}{ccc} M_m & \xrightarrow{\ f\ } & N_m \\ \downarrow & & \downarrow \\ M^{(m)} & \xrightarrow{\ f_{\mathrm{Gr}}^{(m)}\ } & N^{(m)} \end{array}$$

commutative. It is clear that the functors $\mathrm{Gr} \circ \overline{}$ and Gr are naturally isomorphic. Notice that Gr becomes a functor from filtered algebras to graded algebras if we define a multiplication on A_{Gr} by

$$[x][y] = [xy] \in A^{(m+m')}, \quad [x] \in A^{(m)},\ [y] \in A^{(m')}.$$

Observe that A_{Gr} is commutative if and only if

$$xy - yx \in A_{m+m'+1}, \quad x \in A_m,\ y \in A_{m'}.$$

In this case the equivalence class $[xy - yx] \in A^{(m+m'+1)}$ depends only on $[x] \in A^{(m)}$ and $[y] \in A^{(m')}$. In fact we have (cf. [AMR2])

Proposition 3.2. *Let A be a filtered algebra. If A_{Gr} is commutative, then the bracket*

$$\{[x], [y]\} = [xy - yx] \in A^{(m+m'+1)}, \quad [x] \in A^{(m)},\ [y] \in A^{(m')}$$

determines a Poisson structure on A_{Gr}.

Remark 3.3. The natural isomorphism $\mathrm{Gr} \circ \overline{} \cong \mathrm{Gr}$ preserves multiplication and, in case this is commutative, also the Poisson structure.

Also notice that a graded algebra $A = \bigoplus_{m=0}^{\infty} A^{(m)}$ is in a natural way a (Hausdorff) filtered algebra $A^{\mathrm{F}} = A$ with $A^{\mathrm{F}}_n = \bigoplus_{m=n}^{\infty} A^{(m)}$. Any map of graded algebras is clearly a map of filtered algebras, so F is a functor. We have

$$\overline{A} = \overline{A^{\mathrm{F}}} = \prod_{m=0}^{\infty} A^{(m)}$$

with the obvious filtration and multiplication. In the sequel, we sometimes use the notation (a_m) for elements of $\overline{A}$; it is implicit that $a_m \in A^{(m)}$. Of course, the functor $\mathrm{Gr} \circ \mathrm{F}$ is (naturally isomorphic to) the identity functor on graded algebras.

3.1.4 Completions of Quotients

Let A be a filtered algebra, and let $I \subseteq A$ be an ideal with corresponding projection $\pi\colon A \to A/I$. The *induced filtration* on A/I is given by

$$(A/I)_n = \pi(A_n) = (A_n + I)/I. \tag{3.4}$$

Remark 3.4. The h-filtration on A induces the h-filtration on A/I.

Put $A^I = A$ and define a (new) filtration on this algebra by $A^I_n = A_n + I$.

Proposition 3.5. *The projection $\pi\colon A^I \to A/I$ is filtered and induces an isomorphism*

$$\overline{\pi}\colon \overline{A^I} \to \overline{A/I}$$

of filtered algebras.

Proof. By (3.4), π is filtered so it induces the commutative square

$$\begin{array}{ccc} A^I/A^I_{n+1} & \xrightarrow{\overline{\pi}} & A/I\big/(A/I)_{n+1} \\ \downarrow & & \downarrow \\ A^I/A^I_n & \xrightarrow{\overline{\pi}} & A/I\big/(A/I)_n \end{array}$$

But the horizontal maps are simply the canonical isomorphisms

$$A/A_n + I \longrightarrow A/I\big/(A_n + I)/I.$$

This completes the proof. □

3.2 Deformation Quantization and ∗-Products

In this section S denotes a complex Poisson algebra.

Definition 3.6. A *deformation quantization* of S is a $\mathbf{C}[[h]]$-algebra A together with a surjective algebra homomorphism $p\colon A \to S$ such that

$$ab - ba = hp^{-1}(\{p(a), p(b)\}) \mod h \operatorname{Ker} p \tag{3.5}$$

for any $a, b \in A$.

In this definition S is regarded as a $\mathbf{C}[[h]]$-algebra via the augmentation $\epsilon\colon \mathbf{C}[[h]] \to \mathbf{C}$. Notice that (3.5) makes sense since the indeterminacy of the expression $hp^{-1}(\{p(a), p(b)\})$ is exactly $h \operatorname{Ker} p$. Also, the condition need only be verified on a set spanning A. We sometimes omit the word deformation and simply speak of quantizations. A *morphism* from a quantization $p\colon A \to S$ to another one $q\colon B \to T$ is simply an algebra homomorphism $A \to B$ covering a Poisson homomorphism $S \to T$. In this way quantizations form a category with the obvious definitions of identities and morphism composition. We agree that an *equivalence* of quantizations of S is an isomorphism covering Id_S. When looking for quantizations of S, a natural $\mathbf{C}[[h]]$-module to consider is $S[[h]]$. This leads us to a special and very important class of quantizations.

Definition 3.7. A $*$-*product* on S is a deformation quantization of the form $p = \pi_0\colon S[[h]] \to S$. The set of $*$-products on S is denoted by $*(S)$.

Remark 3.8. This definition is equivalent to the traditional one (cf. [BFFLS]) as we shall see shortly.

For a $\mathbf{C}[[h]]$-algebra product $*$ on $S[[h]]$, it is convenient to introduce its *coefficients*, namely the $\mathbf{C}$-bilinear maps $c_r\colon S \times S \to S$ given by

$$x * y = \sum_r c_r(x, y)h^r, \quad x, y \in S.$$

The coefficients determine $*$ completely since

$$\sum_i x_i h^i * \sum_j y_j h^j = \sum_j \big(\big(\sum_i x_i h^i\big) * y_j\big) h^j = \sum_{i,j} x_i * y_j h^{i+j} = \sum_{i,j,r} c_r(x_i, y_j) h^{i+j+r} \tag{3.6}$$

by the $\mathbf{C}[[h]]$-bilinearity, cf. (3.2).

Proposition 3.9. *A $\mathbf{C}[[h]]$-algebra product $*$ on $S[[h]]$ is a $*$-product on S if and only if*

$$x * y = xy \mod h, \tag{3.7a}$$

$$x * y - y * x = \{x, y\}h \mod h^2 \tag{3.7b}$$

for all $x, y \in S \subseteq S[[h]]$.

Proof. If $*$ defines a deformation quantization of S, (3.7) clearly hold. On the other hand (3.7a) means that $c_0\colon S \times S \to S$ is the multiplication on S, so for $x, y \in S[[h]]$ we have

$$x * y = \sum_{i,j,r} c_r(x_i, y_j)h^{i+j+r} \equiv x_0 y_0 + \big(x_0 y_1 + x_1 y_0 + c_1(x_0, y_0)\big)h \mod h^2.$$

This implies that $\pi_0\colon S[[h]] \to S$ is multiplicative. Furthermore, by (3.7b) we may continue the computation to arrive at

$$x * y - y * x \equiv (c_1(x_0, y_0) - c_1(y_0, x_0))h = \{x_0, y_0\}h \mod h^2$$

as required in (3.5). □

Remark 3.10. All the $*$-products considered in the sequel satisfy that the unit $1 \in S$ is also the unit for $*$, as is easily verified in each case.

3.2.1 A Key Example

The next theorem contains an important construction of $*$-products on Poisson algebras that are graded. The statement of the result relies on Proposition 3.2.

Theorem 3.11 (Andersen, Mattes & Reshetikhin). *Assume that S is a graded Poisson algebra, and let F be a complete, Hausdorff filtered complex algebra such that F_{Gr} is commutative. Suppose $V\colon F \to \overline{S}$ is a homomorphism of filtered vector spaces such that $V_{\mathrm{Gr}}\colon F_{\mathrm{Gr}} \to S$ is an isomorphism of Poisson algebras. Then V is an isomorphism of filtered vector spaces, and*

$$x_1 * x_2 = \sum_{r=0}^{\infty} (V(V^{-1}(x_1)V^{-1}(x_2)))^{(m_1+m_2+r)} h^r, \quad x_i \in S^{(m_i)} \tag{3.8}$$

defines a star product on S.

We also need the following complementary result.

Theorem 3.12. *Let S and F be as in Theorem 3.11 and suppose $V_i\colon F \to \overline{S}$, $i = 1,2$ are two maps satisfying the conditions of that theorem. Denote by $*_i$ the $*$-product on S defined by formula (3.8) with $V = V_i$. If $(V_1)_{\mathrm{Gr}} = (V_2)_{\mathrm{Gr}}$, then the $\mathbf{C}[[h]]$-linear map $\tau = \tau_{21}\colon S[[h]] \to S[[h]]$ determined by*

$$\tau(x) = \sum_r (V_2 V_1^{-1}(x))^{(m+r)} h^r, \quad x \in S^{(m)}$$

is an equivalence from $*_1$ to $*_2$.

Proof. By definition of an equivalence we must check that $\tau_0 = \mathrm{Id}_S$; this follows from

$$\tau(x) = (V_2 V_1^{-1}(x))^{(m)} = (V_2)_{\mathrm{Gr}}[((V_1)_{\mathrm{Gr}})^{-1}(x)] = x, \quad x \in S^{(m)}.$$

To prove that τ is multiplicative:

$$\tau(x *_1 y) = \tau(x) *_2 \tau(y), \quad x, y \in S[[h]] \tag{3.9}$$

we take a closer look at the definition of $*_i$. The product on F may be transferred to $\overline{S}$ via the isomorphism V_i:

$$x \circ_i y = V_i(V_i^{-1}(x)V_i^{-1}(y)), \quad x, y \in \overline{S}. \tag{3.10}$$

It is then clear from (3.8) that the map $\eta\colon \overline{S} \to S[[h]]$, $(x_i) \mapsto \sum_i x_i h^i$ satisfies

$$\eta(x \circ_i y) = \eta(x) *_i \eta(y), \quad x, y \in \overline{S}. \tag{3.11}$$

Putting $T = V_2 \circ V_1^{-1}\colon \overline{S} \to \overline{S}$, we see from (3.10) that T takes $\circ_1$ to $\circ_2$. Also, τ is constructed such that

$$\tau \circ \eta = \eta \circ T\colon \overline{S} \longrightarrow S[[h]]. \tag{3.12}$$

By $\mathbf{C}[[h]]$-bilinearity it suffices to verify (3.9) for $x, y \in S$. We may assume that x and y are homogeneous of degree m_1 and m_2, respectively. By applying η to the identity $T(x \circ_1 y) = T(x) \circ_2 T(y)$ and using the properties (3.11) and (3.12), it is straightforward to establish

$$h^{m_1+m_2}\tau(x *_1 y) = h^{m_1+m_2}(\tau(x) *_2 \tau(y))$$

as desired. Reversing the roles of V_1 and V_2 yields the inverse τ_{12} of τ_{21}. The proof is complete. □

Remark 3.13. In the notation of the above theorem, if $V_i\colon F \to \overline{S}$, $i = 1,2,3$ are such that $(V_i)_{\mathrm{Gr}}$ are all equal, then the equivalences obviously satisfy $\tau_{31} = \tau_{32} \circ \tau_{21}$.

3.2.2 Quotients

A situation we frequently encounter is the following. We have a $*$-product on S and a Poisson ideal $I \subseteq S$. Then we want to induce a $*$-product on the quotient S/I and obtain a morphism of quantizations:

$$\begin{array}{ccc} S[[h]] & \xrightarrow{\pi_h} & S/I[[h]] \\ \downarrow & & \downarrow \\ S & \xrightarrow{\pi} & S/I \end{array}$$

The condition for doing so is, of course, that $\operatorname{Ker} \pi_h = I[[h]] \subseteq S[[h]]$ is an ideal with respect to $*$. By (3.6) this requirement translates into

$$c_r(I, S) \subseteq I \supseteq c_r(S, I), \quad r \in \mathbf{N}. \tag{3.13}$$

Abusing terminology we shall often say that I is a $*$-ideal and thereby mean that $I[[h]]$ is a $*$-ideal. Write $*(S, I) \subseteq *(S)$ for the set of $*$-products descending to S/I, and let $\pi\colon *(S, I) \to *(S/I)$ denote the natural map. A morphism

$$\begin{array}{ccc} S[[h]]_{*_1} & \xrightarrow{\varphi} & S[[h]]_{*_2} \\ \downarrow & & \downarrow \\ S & \xrightarrow{\varphi_0} & S \end{array} \tag{3.14}$$

where $*_i \in *(S, I)$, $i = 1, 2$ will induce a morphism

$$\begin{array}{ccc} S/I[[h]]_{\pi(*_1)} & \xrightarrow{\overline{\varphi}} & S/I[[h]]_{\pi(*_2)} \\ \downarrow & & \downarrow \\ S/I & \xrightarrow{\overline{\varphi_0}} & S/I \end{array} \tag{3.15}$$

precisely when $\varphi(I[[h]]) \subseteq I[[h]]$. The formula (3.3) proves that this is equivalent to

$$\varphi_j(I) \subseteq I, \quad j \in \mathbf{N}. \tag{3.16}$$

3.2.3 Actions

Suppose that a group Γ acts on S by Poisson isomorphisms.

Proposition 3.14. *There is an action of* Γ *on* $*(S)$*: For a Poisson isomorphism* $g\colon S \to S$, $g \in \Gamma$ *and a* $*$*-product* $*$ *on S we define* $*' = g \cdot *$ *by*

$$x *' y = g_h(g_h^{-1}(x) * g_h^{-1}(y)) \tag{3.17}$$

for $x, y \in S[[h]]$.

Proof. Trivially, $*'$ is a $\mathbf{C}[[h]]$-algebra product on $S[[h]]$. For $x, y \in S$ we have

$$\begin{aligned} x *' y &= g_h(g_h^{-1}(x) * g_h^{-1}(y)) \\ &= g_h(g^{-1}(x) * g^{-1}(y)) \\ &\equiv g_h\big(g^{-1}(x)g^{-1}(y) + c_1\big(g^{-1}(x), g^{-1}(y)\big)h\big) \\ &\equiv xy + g\big(c_1\big(g^{-1}(x), g^{-1}(y)\big)\big)h \mod h^2 \end{aligned}$$

so that $x *' y = xy \mod h$, and

$$\begin{aligned} x *' y - y *' x &\equiv \big[g\big(c_1\big(g^{-1}(x), g^{-1}(y)\big)\big) - g\big(c_1\big(g^{-1}(y), g^{-1}(x)\big)\big)\big]h \\ &= g\big(\{g^{-1}(x), g^{-1}(y)\}\big)h \\ &= \{x, y\}h \mod h^2 \end{aligned}$$

as desired. It is obvious that $g \mapsto g\cdot$ defines an action. □

It is, of course, contained in this proposition that we have an isomorphism

$$\begin{array}{ccc} S[[h]]_* & \xrightarrow{g_h} & S[[h]]_{g\cdot *} \\ \downarrow & & \downarrow \\ S & \xrightarrow{g} & S \end{array} \tag{3.18}$$

of quantizations. Actions and quotients commute when comparable. More precisely:

Proposition 3.15. *Let $I \subseteq S$ be a Γ-invariant Poisson ideal so that the action of Γ descends to the quotient S/I. Then $*(S, I) \subseteq *(S)$ is a Γ-invariant subset and $\pi\colon *(S, I) \to *(S/I)$ is equivariant.*

Proof. Let $* \in *(S, I)$ and $g \in \Gamma$. Since g and g^{-1} leave I invariant, the induced maps g_h and g_h^{-1} preserve $I[[h]]$. Therefore it follows from (3.17) that $I[[h]]$ is an ideal for $g \cdot *$, that is, $g \cdot * \in *(S, I)$. Moreover, by the criterion (3.16) the diagram (3.18) induces the isomorphism

$$\begin{array}{ccc} S/I[[h]]_{\pi(*)} & \xrightarrow{\overline{g_h}} & S/I[[h]]_{\pi(g\cdot *)} \\ \downarrow & & \downarrow \\ S/I & \xrightarrow{\overline{g}} & S/I \end{array}$$

cf. (3.15). But the action of $\overline{g}$ on $*(S/I)$ yields the isomorphism

$$\begin{array}{ccc} S/I[[h]]_{\pi(*)} & \xrightarrow{\overline{g}_h} & S/I[[h]]_{g\cdot \pi(*)} \\ \downarrow & & \downarrow \\ S/I & \xrightarrow{\overline{g}} & S/I \end{array}$$

As $\overline{g_h} = \overline{g}_h$ we derive $\pi(g \cdot *) = g \cdot \pi(*)$. □

3.2.4 Completion

One advantage of a $*$-product over an ordinary deformation quantization is that $S[[h]]$ is a complete, Hausdorff filtered algebra. Under certain circumstances it is possible to complete a general quantization defined on a filtered algebra:

Theorem 3.16. *Let $p: A \to S$ be a deformation quantization. A filtration $A = A_0 \supseteq A_1 \supseteq \cdots$ is said to be* compatible *with the deformation if*

$$A_1 \subseteq \operatorname{Ker} p, \tag{3.19a}$$

$$ab - ba \in hA_n, \quad a \in A_n, b \in A. \tag{3.19b}$$

In this situation the induced map $\overline{p}: \overline{A} \to S$ given by

$$\overline{p}([a_n]_n) = p(a_1), \quad ([a_n]_n) \in \overline{A} \tag{3.20}$$

is a deformation quantization.

Proof. In the trivial filtration $S = S \supseteq 0 \supseteq \cdots$, S becomes a complete, Hausdorff filtered $\mathbf{C}[[h]]$-algebra receiving the filtered (by (3.19a)) map p. Therefore the induced algebra homomorphism $\overline{p}: \overline{A} \to S$ exists and is given by (3.20); it is obviously surjective. Let $\overline{a} = ([a_n]_n), \overline{b} = ([b_n]_n) \in \overline{A}$. Write

$$x = \{\overline{p}(\overline{a}), \overline{p}(\overline{b})\} = \{p(a_1), p(b_1)\} \tag{3.21}$$

and pick $c \in p^{-1}(x)$. We define inductively a sequence $d_1, d_2, \cdots \in A$ subject to the conditions

$$\begin{aligned} (i)&\ \ d_1 \in \operatorname{Ker} p, \\ (ii)&\ \ a_n b_n - b_n a_n = hc + hd_n, \\ (iii)&\ \ d_{n+1} - d_n \in A_n. \end{aligned}$$

Since $p: A \to S$ is a quantization, we get from (3.21) an element $d_1 \in \operatorname{Ker} P$ such that

$$a_1 b_1 - b_1 a_1 = hc + hd_1.$$

Assume that $d_1, \ldots, d_n$ are defined. We set

$$a_{n+1} = a_n + \alpha, \;\; b_{n+1} = b_n + \beta; \quad \alpha, \beta \in A_n.$$

By hypothesis (ii), we derive

$$\begin{aligned} a_{n+1}b_{n+1} - b_{n+1}a_{n+1} &= a_n b_n - b_n a_n + (a_n\beta - \beta a_n) + (\alpha b_n - b_n\alpha) + (\alpha\beta - \beta\alpha) \\ &= hc + hd_n + hk_n \end{aligned}$$

for a suitable $k_n \in A_n$ the existence of which is guaranteed by (3.19b). Putting $d_{n+1} = d_n + k_n$ completes the induction step. By (iii) the element $\overline{d} = ([d_n]_n) \in \overline{A}$ is well-defined. Setting $\overline{c} = ([c]_n) \in \overline{A}$, we conclude that

$$\overline{a}\overline{b} - \overline{b}\overline{a} = ([a_n b_n - b_n a_n]_n) = ([hc + hd_n]_n) = h\overline{c} + h\overline{d}.$$

Since $\overline{d} \in \operatorname{Ker} \overline{p}$ by (i) and as

$$\overline{p}(\overline{c}) = p(c) = \{\overline{p}(\overline{a}), \overline{p}(\overline{b})\},$$

we have verified the defining equation (3.5) of a deformation quantization. □

One example of this construction is due to

Proposition 3.17. *A quantization $p\colon A \to S$ is compatible with the h-filtration on A.*

Proof. We check the conditions (3.19). Let $a \in A$. Recalling the augmentation $\epsilon\colon \mathbf{C}[[h]] \to \mathbf{C}$,

$$p(ha) = \epsilon(h)p(a) = 0$$

so that $A_1 = hA \subseteq \operatorname{Ker} p$. Let also $b \in A$. Since $ab - ba \in hA$ by the definition of a quantization, we derive

$$(h^n a)b - b(h^n a) = h^n(ab - ba) \in h^n hA = hA_n$$

as desired. □

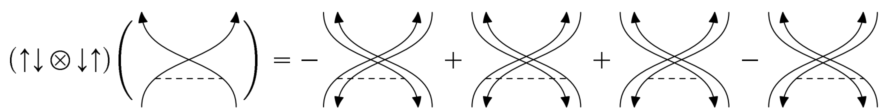
Chapter 4

Quantization of the Algebra of Chord Diagrams

In the case where Σ has non-empty boundary, Andersen, Mattes and Reshetikhin have constructed a $*$-product on the algebra of chord diagrams on Σ by using the machinery of universal Vassiliev invariants of links in the cylinder over Σ [AMR2]. We present their construction in this chapter with emphasis on the fact that the $*$-product obtained depends on the so-called partition of Σ used in the process. This dependence is well-behaved as we shall demonstrate; different partitions yield canonically equivalent $*$-products. When working with the AMR $*$-products and their equivalences, some standard situations arise frequently; we deal with those and a first application of them at the end of the chapter. The first two sections set the scene and are based on Bar-Natan's paper [B] as well as [AMR2].

4.1 Chord Tangles

We generalize the notion of chord diagrams; in a *chord tangle* the core components are allowed to be oriented intervals as well as oriented circles. The boundary of a chord tangle T is a set of oriented points partitioned into two ordered sets $\partial_+ T$ and $\partial_- T$ termed the *top* and *bottom endpoints*, respectively. In drawings of chord tangles their tops and bottoms are consistent with the orientation of the page, and the order of endpoints is from left to right. We may *extend* T by adding vertical, oriented intervals with no chords to the left and right of T. Moreover, T can be *cabled* by substituting bundles of core components (of the same kind, various orientations permitted) for single ones. The result of this operation is the signed sum of all possible liftings of T to the skeleton of core components obtained from the skeleton of T by the prescribed substitution. The sign of a lifting is -1 raised to the number of chord endpoints located on a core component with reversed orientation. Here is an example:



The symbol $\uparrow\downarrow \otimes \downarrow\uparrow$ means: Replace the first (counting at the bottom) strand by the bundle $\uparrow\downarrow$ and the second one by $\downarrow\uparrow$. We remark that cabling preserves the 4T-relation and therefore makes sense for chord diagrams on Σ; restricted to subspaces of chord diagrams with identical skeletons it results in graded linear maps.

Remark 4.1. Given a core component C of a chord tangle T, one possible cabling operation, S_C, is to substitute C with opposite orientation for C; clearly, $S_C(T)$ is the chord tangle obtained from D by reversing the orientation of C and scaling with -1 for each chord endpoint on C. Replacing C by the empty bundle of core components yields another cabling operation ϵ_C; if a chord intersects C then $\epsilon_C(T) = 0$ since it is impossible to lift T to the skeleton obtained from the skeleton of T by erasing C. On the other hand, if no chord intersects C this lifting can be performed in a unique way so that $\epsilon_C(T)$ is the chord tangle obtained from D by simply erasing C.

We now consider an oriented embedded square $S \subseteq \Sigma$ with distinguished top and bottom sides I_+ and I_-. The square is equipped with a *boundary marking*, that is, two finite sets of oriented points $\partial_+ = \partial_+ S \subseteq I_+$ and $\partial_- = \partial_- S \subseteq I_-$. Define *geometric chord tangles in* $(S; \partial_+, \partial_-)$ to be smooth maps $(T; \partial_+ T, \partial_- T) \to (S; \partial_+ S, \partial_- S)$ subject to the condition that $\partial_+ T \to \partial_+ S$ and $\partial_- T \to \partial_- S$ are isomorphisms (bijections preserving order and orientation). *Chord tangles in* $(S; \partial_+, \partial_-)$ are, of course, homotopy classes rel boundary of such maps, and $\mathcal{D}(S; \partial_+, \partial_-)$ is the complex vector space freely generated by them. Since the 4T-relation still makes sense, we obtain in this way a vector space

$$\mathcal{C}(S; \partial_+, \partial_-) = \mathcal{D}(S; \partial_+, \partial_-)/4T$$

graded by the number of chords.

Similarly, we can consider chord tangles in $\Sigma - S$ (more precisely, in $\Sigma - \text{int}(S)$, but for clarity we use the simpler notation); we agree that the top of S is the bottom of $\Sigma - S$ and vice versa, so that the boundary marking on $\Sigma - S$ induced from the one on S becomes

$$(\partial_+(\Sigma - S), \partial_-(\Sigma - S)) = (\partial_- S, \partial_+ S) = (\partial_-, \partial_+).$$

With this convention we obtain a graded vector space

$$\mathcal{C}(\Sigma - S; \partial_-, \partial_+) = \mathcal{D}(\Sigma - S; \partial_-, \partial_+)/4T.$$

Notice that S and $\Sigma - S$ are surfaces in their own right and that $\mathcal{C}(S; \varnothing, \varnothing) = \mathcal{C}(S)$ and $\mathcal{C}(\Sigma - S; \varnothing, \varnothing) = \mathcal{C}(\Sigma - S)$ are the usual Poisson algebras of chord diagrams.

Extension and cabling clearly makes sense for chord tangles in S whereas only cabling is possible for chord tangles in $\Sigma - S$. These operations yield graded linear maps. There is an obvious *composition* of chord tangles

$$\mathcal{C}(\Sigma - S; \partial_-, \partial_+) \times \mathcal{C}(S; \partial_+, \partial_-) \overset{\cdot}{\longrightarrow} \mathcal{C}(\Sigma)$$

defined by glueing the appropriate pairs of boundary points. Note that the map $\mathcal{C}(\Sigma - S) \to \mathcal{C}(\Sigma)$ induced by the inclusion $\Sigma - S \subseteq \Sigma$ can be regarded as composition with $\varnothing \in \mathcal{C}(S)$. Also, if S_1 and S_2 are two embedded squares with boundary markings such that $I_-(S_1) = I_+(S_2)$ and $\partial_- S_1 = \partial_+ S_2$, there is another composition

$$\mathcal{C}(S_1; \partial_+ S_1, \partial_- S_1) \times \mathcal{C}(S_2; \partial_+ S_2, \partial_- S_2) \overset{\cdot}{\longrightarrow} \mathcal{C}(S_1 \cup S_2; \partial_+ S_1, \partial_- S_2)$$

defined analogously. Both compositions are graded bilinear maps.

The union operation which turned $\mathcal{C}(\Sigma)$ into an algebra can be defined for chord tangles in S under certain circumstances. Specifically, if $(\partial_+^i, \partial_-^i)$, $i = 1, 2$ are two disjoint boundary markings on S there is an obvious graded bilinear map

$$\mathcal{C}(S; \partial_+^1, \partial_-^1) \times \mathcal{C}(S; \partial_+^2, \partial_-^2) \overset{\cup}{\longrightarrow} \mathcal{C}(S; \partial_+^1 \cup \partial_+^2, \partial_-^1 \cup \partial_-^2).$$

Figure 4.1: Positive (left) and negative crossings.

As a special case we note that $\mathcal{C}(S;\partial_+,\partial_-)$ becomes a graded module over $\mathcal{C}(S)$. For chord tangles in $\Sigma - S$ analogous considerations of the union operation apply.

Since all the aforementioned operations on chord tangles are graded (bi)linear, they extend to the completions of the chord tangle spaces. Moreover it is clear that the operations commute whenever this makes sense.

4.2 Links and Non-Associative Tangles

Let $\mathcal{L}(\Sigma)$ denote the complex vector space with basis the set of framed, oriented links in (the interior of) the cylinder $\Sigma \times I$. We often think of links as link diagrams on Σ modulo the second and third Reidemeister moves; the usual sign conventions for over- and under-crossings in link diagrams are used throughout, cf. Figure 4.1. (Therefore we sometimes abuse terminology and speak of links on Σ). Introduce an operation ∇ on $\mathcal{L}(\Sigma)$ in the following fashion. For a link diagram L pick a subset $v_1,\dots,v_k$ of its crossings and put

$$\nabla_{v_1,\dots,v_k} L = \sum_{\epsilon_1,\dots,\epsilon_k=\pm 1} \epsilon_1 \cdots \epsilon_k L^{\epsilon_1,\dots,\epsilon_k}$$

where $L^{\epsilon_1,\dots,\epsilon_k}$ is the link obtained from L by adjusting the sign of v_i to be ϵ_i. This allows us to define subspaces

$$\mathcal{L}_m(\Sigma) = \operatorname{span}\{\nabla_{v_1,\dots,v_m} L \mid L \text{ is a link on } \Sigma\} \tag{4.1}$$

easily seen to constitute a filtration on $\mathcal{L}(\Sigma)$; it is called the *Vassiliev filtration*.

Evidently the projection $\Sigma \times I \to \Sigma$ induces a map

$$\pi\colon \mathcal{L}(\Sigma) \to \mathcal{D}^{(0)}(\Sigma) = \mathcal{C}^{(0)}(\Sigma) \subseteq \mathcal{C}(\Sigma).$$

A more interesting coupling of links and chord diagrams on Σ is the graded linear surjection $\lambda\colon \mathcal{C}(\Sigma) \to \mathcal{L}_{\mathrm{Gr}}(\Sigma)$ we now define. Let D be a generic chord diagram on Σ with m chords denoted by $v_1,\dots,v_m$. Pick a link diagram L_D projecting to D, i.e., resolve the crossings (chords and ordinary intersections) of D in some way, and set

$$\lambda(D) = [\nabla_{v_1,\dots,v_m} L_D] \in \mathcal{L}_m(\Sigma)/\mathcal{L}_{m+1}(\Sigma) = \mathcal{L}_{\mathrm{Gr}}^{(m)}(\Sigma).$$

It is inessential how we resolve the crossings in D; this is by definition of ∇ for the chords, and because we divide out $\mathcal{L}_{m+1}(\Sigma)$ for the ordinary intersections. Therefore the map $\lambda\colon \mathcal{D}(\Sigma) \to \mathcal{L}_{\mathrm{Gr}}(\Sigma)$ is well-defined; immediately from (4.1) it is surjective. That λ vanishes on $4\mathrm{T}(\Sigma)$ and thus descends to $\mathcal{C}(\Sigma)$ is an elementary calculation.

For links L_1 and L_2 in $\Sigma \times I$ we define their product by

$$L_1 L_2 = \{(x,t) \in \Sigma \times I \mid t > 1/2 \wedge (x, 2t-1) \in L_1, \text{ or } t < 1/2 \wedge (x, 2t) \in L_2\},$$

that is, $L_1 L_2$ is L_1 stacked on top of L_2. It is evident that $\mathcal{L}(\Sigma)$ endowed with this multiplication becomes a filtered, in general non-commutative, algebra with the empty link as unit.

Proposition 4.2 (Andersen, Mattes & Reshetikhin). *The graded algebra $\mathcal{L}_{\mathrm{Gr}}(\Sigma)$ is commutative, and $\lambda\colon \mathcal{C}(\Sigma) \to \mathcal{L}_{\mathrm{Gr}}(\Sigma)$ is a surjective homomorphism of graded Poisson algebras.*

We now define *non-associative tangles*, a concept corresponding to chord tangles as links correspond to chord diagrams. More precisely, a non-associative tangle is a framed, oriented tangle for which the ordered sets of top and bottom endpoints are (completely) parenthesized. When drawing non-associative tangles, we indicate the parenthesization of the endpoints by the distance between them. *Extension* of non-associative tangles is defined as for chord tangles with the additional requirement that the parenthesization of the extended tangle respects the parenthesization of the original one. Also *cabling* is possible by using the framing to push off a bundle of components from a single component. When this component is an interval a parenthesization on the substituted bundle of intervals must be specified in order that a parenthesization is induced on the boundary of the cabled tangle. As for chord diagrams on Σ cabling makes sense for links on Σ and results in filtered maps.

Again we consider an oriented square S embedded in Σ, the boundary marking now consisting of parenthesized subsets $\partial_+^{()} \subseteq I_+$ and $\partial_-^{()} \subseteq I_-$ of oriented points. We may then define the Vassiliev filtered vector space

$$\mathcal{L}(S; \partial_+^{()}, \partial_-^{()})$$

of non-associative tangles in $(S \times I; \partial_+^{()}, \partial_-^{()})$ by using regular isotopy classes rel boundary of appropriate non-associative tangles. When $\partial_\pm^{()} = \varnothing$ we recover the algebra of links $\mathcal{L}(S)$ defined previously. It is clear how to extend and cable tangles in S. A parallel definition and analogous considerations are valid for non-associative tangles in $(\Sigma - S) \times I$ except that extension of such tangles is not defined.

We have filtered *composition* maps mirroring those for chord tangles

$$\mathcal{L}(\Sigma - S; \partial_-^{()}, \partial_+^{()}) \times \mathcal{L}(S; \partial_+^{()}, \partial_-^{()}) \overset{\cdot}{\longrightarrow} \mathcal{L}(\Sigma),$$
$$\mathcal{L}(S_1; \partial_+^{()} S_1, \partial_-^{()} S_1) \times \mathcal{L}(S_2; \partial_+^{()} S_2, \partial_-^{()} S_2) \overset{\cdot}{\longrightarrow} \mathcal{L}(S_1 \cup S_2; \partial_+^{()} S_1, \partial_-^{()} S_2)$$

subject, of course, to the compatibility conditions $I_-(S_1) = I_+(S_2)$ and $\partial_-^{()} S_1 = \partial_+^{()} S_2$.

The stacking operation defining the product on $\mathcal{L}(S)$ generalizes to filtered maps

$$\mathcal{L}(S) \times \mathcal{L}(S; \partial_+^{()}, \partial_-^{()}) \longrightarrow \mathcal{L}(S; \partial_+^{()}, \partial_-^{()}), \quad \mathcal{L}(S; \partial_+^{()}, \partial_-^{()}) \times \mathcal{L}(S) \longrightarrow \mathcal{L}(S; \partial_+^{()}, \partial_-^{()}).$$

Since S is contractible these maps are equal when flipping the domain factors. Thus $\mathcal{L}(S)$ is a commutative algebra, and $\mathcal{L}(S; \partial_+^{()}, \partial_-^{()})$ is a filtered module over it. This kind of commutativity fails in general for $\Sigma - S$, but it is still true that the stacking maps

$$\mathcal{L}(\Sigma - S) \times \mathcal{L}(\Sigma - S; \partial_-^{()}, \partial_+^{()}) \longrightarrow \mathcal{L}(\Sigma - S; \partial_-^{()}, \partial_+^{()}),$$
$$\mathcal{L}(\Sigma - S; \partial_-^{()}, \partial_+^{()}) \times \mathcal{L}(\Sigma - S) \longrightarrow \mathcal{L}(\Sigma - S; \partial_-^{()}, \partial_+^{()})$$

turn $\mathcal{L}(\Sigma - S; \partial_-^{()}, \partial_+^{()})$ into a filtered bi-module over $\mathcal{L}(\Sigma - S)$.

The operations on non-associative tangles are all filtered so that they extend to the Vassiliev completions, and as for chord tangles they are compatible with each other. The projection $\pi\colon \mathcal{L}(S; \partial_+^{()}, \partial_-^{()}) \to \mathcal{C}(S; \partial_+, \partial_-)$ and the surjective graded linear map $\lambda\colon \mathcal{C}(S; \partial_+, \partial_-) \to \mathcal{L}_{\mathrm{Gr}}(S; \partial_+^{()}, \partial_-^{()})$ defined as in the case of links and chord diagrams on Σ, achieve the coupling between chord tangles and non-associative tangles on S, and similarly for $\Sigma - S$.

4.3 Universal Vassiliev Invariants

A universal Vassiliev invariant of links on Σ is a map $V\colon \mathcal{L}(\Sigma) \to \overline{\mathcal{C}(\Sigma)} = \prod_{m=0}^{\infty} \mathcal{C}^{(m)}(\Sigma)$ of filtered vector spaces such that

$$V_{\mathrm{Gr}} \circ \lambda = \mathrm{Id}_{\mathcal{C}(\Sigma)}\,. \tag{4.2}$$

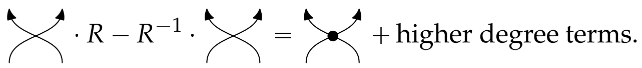
We refer to (4.2) as the defining equation of a universal Vassiliev invariant; it implies the following corollary to Proposition 4.2.

Proposition 4.3. *If V is a universal Vassiliev invariant for Σ, then $\lambda\colon \mathcal{C}(\Sigma) \to \mathcal{L}_{\mathrm{Gr}}(\Sigma)$ is an isomorphism of graded Poisson algebras with inverse $\lambda^{-1} = V_{\mathrm{Gr}}$.*

Assume for the remainder of this chapter that $\partial\Sigma \neq \varnothing$. The construction in [AMR2] of a universal Vassiliev invariant of links on Σ builds on the construction in [B] of a universal Vassiliev invariant of non-associative tangles in the standard square $S = I \times I$ with top $I \times \{1\}$ and bottom $I \times \{0\}$; by this we mean a family of filtered, linear maps parametrized by all boundary markings on S:

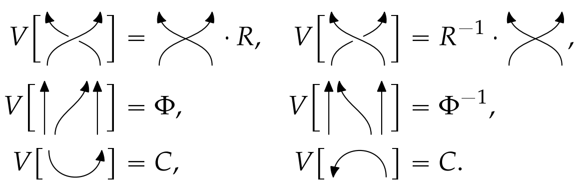
$$V\colon \mathcal{L}(S;\partial_+^{()},\partial_-^{()}) \to \overline{\mathcal{C}(S;\partial_+,\partial_-)}$$

and satisfying the obvious analogue of (4.2). It will be useful to have refined versions of the chord tangle spaces. To be specific, let $T \subseteq S$ be a tangle (chord tangle without chords), and define $\mathcal{C}(S;T) \subseteq \mathcal{C}(S;\partial_+T,\partial_-T)$ to be the (homogeneous) subspace generated by chord tangles with skeleton T. A *perturbation of the skeleton* is any element $(P_m) \in \overline{\mathcal{C}(S;T)}$ such that $P_0 = T$. Also, we allow the second factor of S to shrink and stretch so that the spaces of tangles in this square are closed under composition. In particular, this means that $\mathcal{C}(S;\uparrow\cdots\uparrow)$ is an algebra with composition as multiplication, the unit being the trivial tangle $\uparrow\cdots\uparrow$. To define V we fix, once and for all, two parameters, the *associator* $\Phi \in \overline{\mathcal{C}(S;\uparrow\uparrow\uparrow)}$ and the *R-matrix* $R \in \overline{\mathcal{C}(S;\uparrow\uparrow)}$ subject to various conditions (cf. [B]); we mention a couple of them. Both Φ and R are perturbations of their skeletons; this implies that they are invertible elements in the algebras they belong to. Also, R satisfies the identity



(4.3)

One constructs an element $C \in \overline{\mathcal{C}(S;\uparrow)}$ in terms of Φ; it is a perturbation of its skeleton. Bending C appropriately, it may be regarded as a member of either of the spaces $\overline{\mathcal{C}(S;\circlearrowleft)}$ and $\overline{\mathcal{C}(S;\curvearrowleft)}$; we put



(4.4a) (4.4b) (4.4c)

Any non-associative tangle in S may be obtained from the above six elementary ones by cabling, extension and composition, so requiring that V is compatible with these operations on non-associative tangles and chord tangles, of course, (over)determines V. Only the careful choice of (Φ, R) ensures that this procedure leads to well-defined maps $V\colon \mathcal{L}(S;\partial_+^{()},\partial_-^{()})$

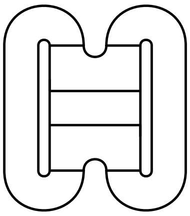


Figure 4.2: A partition of $\Sigma_{0,3}$.

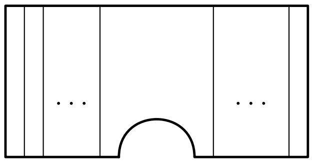


Figure 4.3: The hexagon tangle.

$\to \overline{\mathcal{C}(S;\partial_+,\partial_-)}$. That V is filtered and satisfies the defining equation of a universal Vassiliev invariant follows from (4.3) and the fact that Φ, R and C are perturbations of their skeletons. Note that if $T \subseteq S$ is a non-associative tangle, then

$$V(T) \in \overline{\mathcal{C}(S;\pi(T))}. \tag{4.5}$$

The above construction yields, of course, a universal Vassiliev invariant of non-associative tangles in any embedded square $S \subseteq \Sigma$; we use this below.

Now assume that Σ is not itself a square. The universal Vassiliev invariant of links on Σ depends not only on (Φ, R) but also on a *partition* of Σ. A partition P is determined by a finite collection of embedded intervals $(I_k, \partial I_k) \subseteq (\Sigma, \partial\Sigma)$ chosen such that cutting Σ along these intervals results in a decomposition

$$\Sigma = (\cup_i S_i) \cup (\cup_j H_j)$$

consisting of squares S_i and hexagons H_j. The sides of these polygons are alternately an interval I_k and a piece of $\partial\Sigma$. A possible partition of the three-holed sphere $\Sigma_{0,3}$ is illustrated in Figure 4.2. For technical reasons we assume that no two hexagons are adjacent; in particular, the decomposition contains at least one square. (By the Euler characteristic the number of hexagons is constant, but we will not use that). Also part of the structure is a choice of top and bottom on all polygons; for a square this means that one of the two embedded intervals bordering it is the top of that square and the other one is the bottom, whereas for a hexagon either the top or the bottom consists of two of the I_k bordering it and the opposite side is the remaining one of these intervals. The choice of tops and bottoms must be consistent, i.e., result in an unambiguous direction 'up' on Σ.

Let L be a link on Σ. By isotopy we assume that L is in general position with respect to the embedded intervals and that each intersection $L \cap H_j$ looks like Figure 4.3. (possibly turned upside down). Now choose parenthesizations of all the sets $L \cap I_k$ subject to the condition that the parenthesization of the top (bottom) endpoints of a hexagon is the union of the parenthesizations of the bottom (top) endpoints. In this way all intersections $L \cap S_i$ are non-associative tangles, and we can define

$$V_P(L) = \prod_i V(L \cap S_i) \cdot \prod_j \pi(L \cap H_j) \in \overline{\mathcal{C}(\Sigma)}$$

where the product is composition of chord tangles.

Theorem 4.4 (Andersen, Mattes & Reshetikhin). *The map $V_P \colon \mathcal{L}(\Sigma) \to \overline{\mathcal{C}(\Sigma)}$ is a universal Vassiliev invariant.*

Notice that for any link L on Σ

$$V_P(L) \in \overline{\mathcal{C}(\Sigma;\pi(L))} \tag{4.6}$$

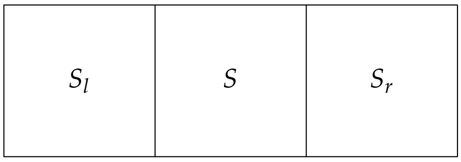


Figure 4.4: A square S_0 containing another square S.

because of (4.5). By definition of the universal Vassiliev invariant of non-associative tangles in a square it is also clear that V_P is compatible with cabling of links and chord diagrams.

It is natural to ask how V_P depends on P. On one hand, we can refine P by bisecting one of the squares S_i with an extra embedded interval running parallel to the top and bottom of S_i. This modification leaves V_P unchanged since the universal Vassiliev invariant for S is compatible with composition. On the other hand, we can consider the action of $\Gamma_+(\Sigma)$ on the set $\mathcal{P}(\Sigma)$ of all partitions of Σ; given a map $g \in \Gamma_+(\Sigma)$, the image $g(P)$ of the embedded intervals constituting P is another partition. Evidently

$$V_{g(P)}(g(L)) = g(V_P(L)), \quad L \in \mathcal{L}(\Sigma). \tag{4.7}$$

We now introduce a useful computational tool, namely universal Vassiliev invariants of non-associative tangles in $\Sigma - S$. They also depend on a partition P of Σ, now required to be *compatible* with S in the sense that there exists a square S_0 in P containing S as depicted in Figure 4.4. For technical reasons we also assume that S_0 is not adjacent to a hexagon; this is no restriction since we can refine P. We begin with the easy case when $S = S_0$ (so that $S_l = S_r = \varnothing$). For a tangle $T \in \mathcal{L}(\Sigma - S_0; \partial_+^{()}(\Sigma - S_0), \partial_-^{()}(\Sigma - S_0))$ we proceed as for links and define

$$V_P(T) = \prod_{i \neq 0} V(T \cap S_i) \cdot \prod_j \pi(T \cap H_j) \in \overline{\mathcal{C}(\Sigma - S_0; \partial_+(\Sigma - S_0), \partial_-(\Sigma - S_0))}$$

remembering, of course, that the parentheses on $T \cap I_\pm(\Sigma - S_0)$ are already fixed to be $\partial_\pm^{()}(\Sigma - S_0)$. Adhering to the rule for parentheses in top and bottom intervals of hexagons is no problem since these polygons do not neighbour S_0.

The general case builds on the first one. Let $T \in \mathcal{L}(\Sigma - S; \partial_+^{()}(\Sigma - S), \partial_-^{()}(\Sigma - S))$. Put $T_l = T \cap S_l$, and choose parentheses on $T_l \cap I_+(S_l)$ and $T_l \cap I_-(S_l)$ to obtain a boundary marking $(\partial_+^{()} S_l, \partial_-^{()} S_l)$ on S_l; in this way $T_l \in \mathcal{L}(S_l; \partial_+^{()} S_l, \partial_-^{()} S_l)$. Similarly for the right hand square. Define boundary markings on S_0 by

$$\partial_+^{()} S_0 = ((\partial_+^{()} S_l \partial_+^{()} S) \partial_+^{()} S_r), \quad \partial_-^{()} S_0 = ((\partial_-^{()} S_l \partial_-^{()} S) \partial_-^{()} S_r)$$

so that $T \cap (\Sigma - S_0)$ can be regarded as an element in $\mathcal{L}(\Sigma - S_0; \partial_+^{()}(\Sigma - S_0), \partial_-^{()}(\Sigma - S_0))$. Put

$$V_P(T) = V_P(T \cap (\Sigma - S_0)) \cdot V(T_l) \cdot V(T_r) \in \overline{\mathcal{C}(\Sigma - S; \partial_+(\Sigma - S), \partial_-(\Sigma - S))}.$$

That these maps are well-defined universal Vassiliev invariants of tangles in $\Sigma - S$ is proved much like Theorem 4.4 (cf. [AMR2]). Compatibility with cabling is immediate from the construction as in the case of links on Σ. For (compatible) non-associative tangles $T_S \in \mathcal{L}(S; \partial_+^{()}, \partial_-^{()})$ and $T_{\Sigma - S} \in \mathcal{L}(\Sigma - S; \partial_-^{()}, \partial_+^{()})$, the composition $T_{\Sigma-S} \cdot T_S$ is a link on Σ, and the three kinds of universal Vassiliev invariants fit together in

$$V_P(T_{\Sigma - S} \cdot T_S) = V_P(T_{\Sigma - S}) \cdot V(T_S)$$

as one readily deduces from the definitions. This formula is ubiquitous in calculations in the sequel.

4.4 $*$-Products and Standard Situations

For partitions P of Σ we consider the completed map

$$V_P = \overline{V_P} \colon \overline{\mathcal{L}(\Sigma)} \to \overline{\mathcal{C}(\Sigma)}.$$

By Proposition 4.3 and Remark 3.3, Theorem 3.11 applies to $S = \mathcal{C}(\Sigma), F = \overline{\mathcal{L}(\Sigma)}$ and $V = V_P$:

Theorem 4.5 (Andersen, Mattes & Reshetikhin). *For any partition P of Σ there is a $*$-product $*_P$ with coefficients*

$$c_r(D, E) = (V_P(V_P^{-1}(D)V_P^{-1}(E)))^{(m_1+m_2+r)}$$

for chord diagrams D and E with m_1 and m_2 chords, respectively.

Different partitions may yield different $*$-products as we shall see, but at least we have

Theorem 4.6. *If P_1 and P_2 are two partitions of Σ, then the endomorphism τ of $\mathcal{C}(\Sigma)[[h]]$ determined by*

$$\tau(D) = \sum_r (V_{P_2} V_{P_1}^{-1}(D))^{(m+r)} h^r, \quad D \in \mathcal{C}^{(m)}(\Sigma)$$

is an equivalence from $*_{P_1}$ to $*_{P_2}$.

Proof. Theorem 3.12 applies since $(V_{P_1})_{\mathrm{Gr}} = \lambda = (V_{P_2})_{\mathrm{Gr}}$. □

Remark 4.7. From formula (4.6) follows immediately that the AMR $*$-products and the equivalences between them preserve the skeletons of the chord diagrams. Therefore the above two theorems also hold for $\mathcal{C}(\Sigma; G)$ if we simply carry along the representations associated to core components in the definitions.

Remark 4.8. With a little more effort one can show that $*_P$ preserves more than skeletons; if D and E are chord diagrams then $c_r(D, E)$ is a linear combination of chord diagrams each of which is obtained from DE by adding r chords appropriately. Formally, this is proved by generalizing the results about non-associative tangles in the complement of an embedded square $S \subseteq \Sigma$ to the case of two disjoint embedded squares assumed (by isotopy) to contain the 'non-trivial' parts of D, respectively E.

Proposition 4.9. *The map $\mathcal{P}(\Sigma) \to *(\mathcal{C}(\Sigma))$ given by $P \mapsto *_P$ is $\Gamma_+(\Sigma)$-equivariant.*

Proof. Let P be a partition of Σ, and let $g \in \Gamma_+(\Sigma)$. For chord diagrams D and E we have by (4.7) and the analogous identity for V_P^{-1}

$$\begin{aligned} gV_P(V_P^{-1}(D)V_P^{-1}(E)) &= V_{g(P)}(g(V_P^{-1}(D)V_P^{-1}(E))) \\ &= V_{g(P)}(gV_P^{-1}(D)gV_P^{-1}(E)) \\ &= V_{g(P)}(V_{g(P)}^{-1}(g(D))V_{g(P)}^{-1}(g(E))). \end{aligned}$$

It follows from Theorem 4.5 that $g(D *_P E) = g(D) *_{g(P)} g(E)$; this is exactly the statement $g \cdot *_P = *_{g(P)}$, cf. Proposition 3.14. □

4.4.1 Standard Situations

We shall encounter a couple of standard situations in computations involving $*_P$, $P \in \mathcal{P}(\Sigma)$ and the equivalences between these $*$-products. The common set-up of the standard situations is as follows: There is an embedded square $(S; I_+, I_-) \subseteq \Sigma$ with boundary marking $(\partial_+^{()}, \partial_-^{()})$, and we have compatible elements $L \in \mathcal{C}^{(m)}(S; \partial_+, \partial_-)$ and $T \in \mathcal{C}^{(m_1)}(\Sigma - S; \partial_-, \partial_+)$. We write $D = T \cdot L \in \mathcal{C}^{(m_1+m)}(\Sigma)$.

In the first standard situation E is a chord diagram on Σ with m_2 chords, and we want to calculate $D *_P E$ where P is some partition of Σ. By a homotopy we may assume firstly that S is contained in the interior of a square S_0 from P and secondly that E is represented by an element $E \in \mathcal{C}(\Sigma - S)$ so that $E = E \cdot \varnothing \in \mathcal{C}(\Sigma)$. We refine P with two embedded intervals as illustrated in Figure 4.5 in order to make P compatible with S. Having settled these technical issues, we derive

$$\begin{aligned} V_P\big(V_P^{-1}(D)V_P^{-1}(E)\big) &= V_P\big(V_P^{-1}(T \cdot L)V_P^{-1}(E \cdot \varnothing)\big) \\ &= V_P([V_P^{-1}(T) \cdot V^{-1}(L)][V_P^{-1}(E) \cdot \varnothing]) \\ &= V_P[(V_P^{-1}(T)V_P^{-1}(E)) \cdot V^{-1}(L)] \\ &= V_P\big(V_P^{-1}(T)V_P^{-1}(E)\big) \cdot L \end{aligned}$$

so that

$$\begin{aligned} c_r(D, E) &= \big(V_P\big(V_P^{-1}(D)V_P^{-1}(E)\big)\big)^{(m_1+m+m_2+r)} \\ &= \big(V_P\big(V_P^{-1}(T)V_P^{-1}(E)\big) \cdot L\big)^{(m_1+m+m_2+r)} \\ &= \big(V_P\big(V_P^{-1}(T)V_P^{-1}(E)\big)\big)^{(m_1+m_2+r)} \cdot L. \end{aligned}$$

Of course, we can reverse the roles of D and E and get a parallel result. A first application of this standard situation yields

Theorem 4.10 (Andersen, Mattes & Reshetikhin). *A subspace $\mathcal{I} \subseteq \mathcal{C}(\Sigma)$ spanned by local relations is a $*$-ideal with respect to $*_P$ for any partition P of Σ.*

Proof. Previously (cf. 2.4.1) we noted that $\mathcal{I}$ is a Poisson ideal so the statement of the theorem makes sense. To prove it we must verify condition (3.13). Consider a generator $D \in \mathcal{I}$. There exists an embedded square $(S; I_+, I_-) \subseteq \Sigma$ with boundary marking (∂_+, ∂_-) such that $D = T \cdot \sum_i \lambda_i L_i$; here $L_i \in \mathcal{C}(S, \partial_+, \partial_-)$ and $\lambda_i \in \mathbf{C}$ are the chord tangles and the scalars defining the relevant local relation, and $T \in \mathcal{C}^{(m_1)}(\Sigma - S; \partial_-, \partial_+)$ is an arbitrary

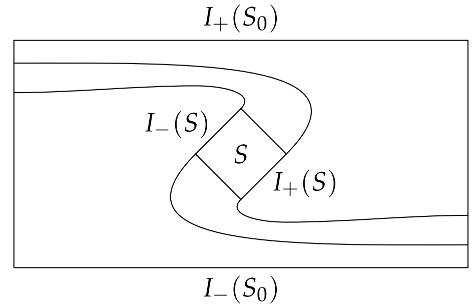


Figure 4.5: Two intervals in S_0 refining P.

chord tangle. Letting E be a chord diagram with m_2 chords and choosing an arbitrary parenthesization on (∂_+, ∂_-), the standard situation yields

$$\begin{aligned} c_r(D, E) &= \sum_i \lambda_i c_r(T \cdot L_i, E) \\ &= \sum_i \lambda_i \big(V_P\big(V_P^{-1}(T) V_P^{-1}(E)\big)\big)^{(m_1+m_2+r)} \cdot L_i \\ &= \big(V_P\big(V_P^{-1}(T) V_P^{-1}(E)\big)\big)^{(m_1+m_2+r)} \cdot \sum_i \lambda_i L_i \in \mathcal{I} \end{aligned}$$

as desired. In the same way, $c_r(E, D) \in \mathcal{I}$. □

In the second standard situation we aim to compute $\tau(D) \in \mathcal{C}(\Sigma)[[h]]$ where τ is the canonical equivalence from $*_{P_1}$ to $*_{P_2}$ for partitions P_1 and P_2 of Σ, cf. Theorem 4.6. By homotopy and the refinement procedure illustrated in Figure 4.5, we may assume that S is compatible with both P_1 and P_2. We get

$$V_{P_2} V_{P_1}^{-1}(D) = V_{P_2} V_{P_1}^{-1}(T \cdot L) = V_{P_2}(V_{P_1}^{-1}(T) \cdot V^{-1}(L)) = (V_{P_2} V_{P_1}^{-1}(T)) \cdot L$$

so that

$$\tau_r(D) = (V_{P_2} V_{P_1}^{-1}(D))^{(m_1+m+r)} = (V_{P_2} V_{P_1}^{-1}(T) \cdot L)^{(m_1+m+r)} = (V_{P_2} V_{P_1}^{-1}(T))^{(m_1+r)} \cdot L.$$

Not surprisingly the first application of the second standard situation is the following result.

Theorem 4.11. *Let $\mathcal{I} \subseteq \mathcal{C}(\Sigma)$ be a subspace spanned by local relations, and let P_1 and P_2 be two partitions of Σ. The canonical equivalence $\tau\colon \mathcal{C}(\Sigma)[[h]] \to \mathcal{C}(\Sigma)[[h]]$ from $*_{P_1}$ to $*_{P_2}$ descends to $\mathcal{C}(\Sigma)/\mathcal{I}$ to yield an equivalence between the induced $*$-products.*

Proof. This is analogous to the proof of Theorem 4.10; in the notation of that proof we deduce

$$\tau_r(D) = \sum_i \lambda_i \tau_r(T \cdot L_i) = \sum_i \lambda_i (V_{P_2} V_{P_1}^{-1}(T))^{(m_1+r)} \cdot L_i = (V_{P_2} V_{P_1}^{-1}(T))^{(m_1+r)} \cdot \sum_i \lambda_i L_i \in \mathcal{I}$$

as required, cf. (3.16). □

Applying the preceding two theorems to the loop relation (2.21), we obtain

Theorem 4.12. *The AMR $*$-products on $\mathcal{C}(\Sigma)$ and the canonical equivalences between them descend to the Poisson loop algebra $\mathcal{Z}_{s,f}(\Sigma)$ via the resolving map $R_{s,f}\colon \mathcal{C}(\Sigma) \to \mathcal{Z}_{s,f}(\Sigma)$.*

Chapter 5

Quantization of the Moduli Space

In this chapter we prove, under the assumption $\partial\Sigma \neq \emptyset$, that the $*$-products $*_P$, $P \in \mathcal{P}(\Sigma)$ on $\mathcal{C}(\Sigma)$ and the canonical equivalences between them descend to $\mathcal{O}(\mathcal{M}(\Sigma; G))$ if G is one of the groups $\mathrm{GL}_n(\mathbf{C})$ and $\mathrm{SL}_n(\mathbf{C})$. These results are achieved by presenting $\mathcal{O}(\mathcal{M}(\Sigma; G))$ as an explicit quotient of $\mathcal{C}(\Sigma)$; the description we give is also valid in the case where Σ is closed. Obtaining it relies on Sikora's work [S]; in the general linear case we adapt the methods of his paper to derive a parallel version of its main result, and in the special linear case we simply translate the main result into our context.

We round off the chapter with Andersen's explicit formula for $*_P$ in the abelian case $G = \mathrm{GL}_1(\mathbf{C})$ [A]; it is a corollary that $*_P$ is independent of P and $\Gamma_+(\Sigma)$-invariant. We also provide counterexamples illustrating that this corollary fails in general.

5.1 The General Linear Case

We consider the group $G = \mathrm{GL}_n(\mathbf{C})$ equipped with the orthogonal structure

$$B(X, Y) = \mathrm{Tr}(XY), \quad X, Y \in \mathfrak{gl}_n(\mathbf{C}).$$

That is, we fix $t \in \mathfrak{gl}_n(\mathbf{C}) \otimes \mathfrak{gl}_n(\mathbf{C})$ to be the Ad-invariant symmetric tensor corresponding to the pairing B. Colouring all core components of chord diagrams with the defining representation $\iota = \mathrm{Id}_{\mathrm{GL}_n(\mathbf{C})}$ of $\mathrm{GL}_n(\mathbf{C})$ yields a Poisson homomorphism $\mathcal{C}(\Sigma) \to \mathcal{C}(\Sigma; \mathrm{GL}_n(\mathbf{C}))$. We write

$$\Psi\colon \mathcal{C}(\Sigma) \to \mathcal{C}(\Sigma; \mathrm{GL}_n(\mathbf{C})) \xrightarrow{\Psi_t} \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{GL}_n(\mathbf{C})))$$

for the composite Poisson homomorphism, cf. Theorem 2.22.

Theorem 5.1. *Assume that $\partial\Sigma \neq \emptyset$. For any partition P of Σ the $*$-product $*_P$ on $\mathcal{C}(\Sigma)$ descends via Ψ to a $*$-product on $\mathcal{O}(\mathcal{M}(\Sigma; \mathrm{GL}_n(\mathbf{C})))$.*

Theorem 5.2. *Assume that $\partial\Sigma \neq \emptyset$, and let P_1 and P_2 be two partitions of Σ. The canonical equivalence from $*_{P_1}$ to $*_{P_2}$ on $\mathcal{C}(\Sigma)$ descends via Ψ to $\mathcal{O}(\mathcal{M}(\Sigma; \mathrm{GL}_n(\mathbf{C})))$ to yield an equivalence between the induced $*$-products.*

Remark 5.3. Theorem 5.1 was also stated in [AMR2]. The proof appearing below roughly follows the outline of the justification supplied in that paper. The primary deviation is that we shall not claim that the kernel of Ψ is generated by local relations.

5.1.1 The Relevant Loop Relation

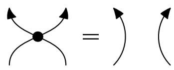
Let $E_{i,j} \in \mathfrak{gl}_n(\mathbf{C})$ be the matrix whose sole non-zero entry is a 1 in the (i, j)th entry, and define

$$B^+_{i,j} = E_{i,j} + E_{j,i}, \quad B^-_{i,j} = E_{i,j} - E_{j,i}; \quad 1 \leqslant i < j \leqslant n.$$

These matrices along with $E_{i,i}$, $i = 1, \dots, n$ are readily seen to constitute an orthogonal basis for $\mathfrak{gl}_n(\mathbf{C})$. Recalling Remark 2.8, we normalize suitably and perform a simple calculation to obtain

$$(\iota \otimes \iota)(t) = t = \sum_{i,j} E_{i,j} \otimes E_{j,i} \in \mathrm{End}(\mathbf{C}^n) \otimes \mathrm{End}(\mathbf{C}^n)$$

which under the isomorphism $\mathrm{End}(\mathbf{C}^n) \otimes \mathrm{End}(\mathbf{C}^n) \cong \mathrm{End}(\mathbf{C}^n \otimes \mathbf{C}^n)$ corresponds to the transposition of the factors. Hence Ψ satisfies the relation (cf. (2.24b))


(5.1)

that is, the loop relation (2.21) with parameters $(s, f) = (1, 0)$. Thus we derive a triangle of Poisson homomorphisms

$$\begin{array}{ccc} \mathcal{C}(\Sigma) & & \\ \downarrow{\scriptstyle R_{1,0}} & \searrow^{\Psi} & \\ \mathcal{Z}_{1,0}(\Sigma) & \xrightarrow{\Psi} & \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{GL}_n(\mathbf{C}))) \end{array}$$

The Poisson structure on the loop algebra will not occupy us in the study of $\mathrm{Ker}\,\Psi$, so we agree to write $\mathcal{Z}(\Sigma) = \mathcal{Z}_{1,0}(\Sigma)$.

5.1.2 The Universal GL_n-Representation

Our strategy is to introduce a commutative complex algebra $\mathcal{R}_n(\Sigma) = \mathcal{R}(\Sigma; \mathrm{GL}_n(C))$ endowed with a $\mathrm{GL}_n(\mathbf{C})$-action such that Ψ factors through the algebra of fixed points:

$$\begin{array}{ccc} & & \mathcal{R}_n(\Sigma)^{\mathrm{GL}_n(\mathbf{C})} \\ & \nearrow & \downarrow \\ \mathcal{Z}(\Sigma) & \xrightarrow{\Psi} & \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{GL}_n(\mathbf{C}))) \end{array}$$

A universal property defines $\mathcal{R}_n(\Sigma)$; it admits a representation $\rho_\Sigma = \rho_{\Sigma, \mathrm{GL}_n(\mathbf{C})} \colon \pi_1(\Sigma) \to \mathrm{GL}_n(\mathcal{R}_n(\Sigma))$ (the *universal* GL_n*-representation of* $\pi_1(\Sigma)$) such that for any representation $\rho \colon \pi_1(\Sigma) \to \mathrm{GL}_n(A)$, A being a commutative complex algebra, there exists a unique homomorphism $h_\rho \colon \mathcal{R}_n(\Sigma) \to A$ fitting into the diagram

$$\begin{array}{ccc} & & \mathrm{GL}_n(\mathcal{R}_n(\Sigma)) \\ & \nearrow^{\rho_\Sigma} & \downarrow{\scriptstyle \mathrm{GL}_n(h_\rho)} \\ \pi_1(\Sigma) & \xrightarrow{\rho} & \mathrm{GL}_n(A) \end{array} \tag{5.2}$$

Here is an explicit construction of $\mathcal{R}_n(\Sigma)$. Let $\langle g_\lambda, \lambda \in \Lambda \mid r_\mu, \mu \in M\rangle$ be a presentation P of $\pi_1(\Sigma)$ satisfying that all relations r_μ are written as products of generators g_λ. Let $Q_n(\Lambda)$ be the polynomial algebra $\mathbf{C}[x^\lambda_{i,j}, d_\lambda]$ where $\lambda \in \Lambda$ and $i, j = 1, \dots, n$. Define matrices $A_\lambda = (x^\lambda_{i,j}) \in M_n(Q_n(\Lambda))$, $\lambda \in \Lambda$. Here M_n denotes the functor assigning the complex algebra $M_n(R)$ of $n \times n$ matrices to a commutative complex algebra R; note that R is included in $M_n(R)$ as the central subalgebra of scalar matrices. Let $I(P) \subseteq Q_n(\Lambda)$ be the ideal generated by $d_\lambda \det A_\lambda - 1$ and all entries in $A_{\lambda_1} \cdots A_{\lambda_k} - 1$ for each relation $r_\mu = g_{\lambda_1} \cdots g_{\lambda_k}$. Set

$$\mathcal{R}_n(\Sigma; P) = \mathcal{R}(\Sigma; \mathrm{GL}_n(\mathbf{C}), P) = {Q_n(\Lambda)}/{I(P)}; \quad q\colon Q_n(\Lambda) \to \mathcal{R}_n(\Sigma; P).$$

We now prove that $\mathcal{R}_n(\Sigma; P)$ satisfies the universal property, implying in particular that different presentations of $\pi_1(\Sigma)$ yield canonically isomorphic algebras (all denoted by $\mathcal{R}_n(\Sigma)$). The formulas

$$\det(M_n(q)(A_\lambda)) = q(\det A_\lambda), \quad q(d_\lambda)q(\det A_\lambda) = 1$$

prove that $M_n(q)(A_\lambda) \in \mathrm{GL}_n(\mathcal{R}_n(\Sigma; P))$, and it is clear that we have a representation

$$\rho_\Sigma = \rho_{\Sigma,\mathrm{GL}_n(\mathbf{C}),P}\colon \pi_1(\Sigma) \to \mathrm{GL}_n(\mathcal{R}_n(\Sigma)); \quad \rho_\Sigma(g_\lambda) = M_n(q)(A_\lambda). \tag{5.3}$$

If A is a commutative complex algebra admitting a representation $\rho\colon \pi_1(\Sigma) \to \mathrm{GL}_n(A)$, we define $h_\rho\colon Q_n(\Lambda) \to A$ by

$$h_\rho(x^\lambda_{i,j}) = \rho(g_\lambda)_{i,j}, \quad h_\rho(d_\lambda) = \det \rho(g_\lambda)^{-1}. \tag{5.4}$$

Since $M_n(h_\rho)(A_\lambda) = \rho(g_\lambda)$, it follows that $I(P) \subseteq \mathrm{Ker}\, h_\rho$; the induced map $h_\rho\colon \mathcal{R}_n(\Sigma) \to A$ is obviously the unique homomorphism making the triangle (5.2) commutative.

The action of $\mathrm{GL}_n(\mathbf{C})$ on $\mathcal{R}_n(\Sigma)$ is the prime application of the universal property. Elements $A \in \mathrm{GL}_n(\mathbf{C})$ give rise to representations

$$A^{-1}\rho_\Sigma A\colon \pi_1(\Sigma) \to \mathrm{GL}_n(\mathcal{R}_n(\Sigma))$$

and the corresponding endomorphisms $A*\colon \mathcal{R}_n(\Sigma) \to \mathcal{R}_n(\Sigma)$ define a $\mathrm{GL}_n(\mathbf{C})$-action; this is a direct consequence of the uniqueness of (5.2). By (5.4) and (5.3) we have

$$A * q(x^\lambda_{i,j}) = (A^{-1}M_n(q)(A_\lambda)A)_{i,j}, \quad A * q(d_\lambda) = \det(M_n(q)(A_\lambda))^{-1} = q(d_\lambda). \tag{5.5}$$

We extend the action of $\mathrm{GL}_n(\mathbf{C})$ to $M_n(\mathcal{R}_n(\Sigma))$ by

$$A * M = A(A * M_{i,j})A^{-1}, \quad M \in M_n(\mathcal{R}_n(\Sigma)),\ A \in \mathrm{GL}_n(\mathbf{C}). \tag{5.6}$$

Consequently,

Lemma 5.4. *The inclusion $\mathcal{R}_n(\Sigma) \subseteq M_n(\mathcal{R}_n(\Sigma))$ is equivariant.*

We lift the $\mathrm{GL}_n(\mathbf{C})$-actions to $Q_n(\Lambda)$ and $M_n(Q_n(\Lambda))$. By definition we can regard $Q_n(\Lambda)$ as the algebra of polynomial functions from $(M_n(\mathbf{C}) \times \mathbf{C})^\Lambda$ to $\mathbf{C}$, and hence $M_n(Q_n(\Lambda))$ as the algebra of polynomial functions from $(M_n(\mathbf{C}) \times \mathbf{C})^\Lambda$ to $M_n(\mathbf{C})$. Since $\mathrm{GL}_n(\mathbf{C})$ acts on $M_n(\mathbf{C})$ by conjugation and trivially on $\mathbf{C}$, it acts on the product $(M_n(\mathbf{C}) \times \mathbf{C})^\Lambda$. The induced actions on the function sets $\mathrm{Map}((M_n(\mathbf{C}) \times \mathbf{C})^\Lambda, \mathbf{C})$ and $\mathrm{Map}((M_n(\mathbf{C}) \times \mathbf{C})^\Lambda, M_n(\mathbf{C}))$ preserve the property of being polynomial, thereby defining the desired actions denoted also by $*$.

Lemma 5.5. *We have* $d_\lambda, \det A_\lambda \in Q_n(\Lambda)^{\mathrm{GL}_n(\mathbf{C})}$ *and* $A_\lambda \in M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})}$.

Proof. This is immediate when regarding $d_\lambda, \det A_\lambda$ and A_λ as functions on $(M_n(\mathbf{C}) \times \mathbf{C})^\Lambda$. □

The analogues of (5.6) and Lemma 5.4 hold:

Lemma 5.6. *For any* $F \in M_n(Q_n(\Lambda))$ *and* $A \in \mathrm{GL}_n(\mathbf{C})$ *we have*

$$A * F = A(A * F_{i,j})A^{-1}$$

Proof. Thinking of F and $F_{i,j}$ as functions we derive

$$\begin{aligned}(A * F)((M_\lambda, s_\lambda)_\lambda) &= AF(A^{-1} * (M_\lambda, s_\lambda)_\lambda)A^{-1} \\ &= A(F_{i,j}(A^{-1} * (M_\lambda, s_\lambda)_\lambda))A^{-1} \\ &= A((A * F_{i,j})((M_\lambda, s_\lambda)_\lambda))A^{-1}\end{aligned}$$

where $(M_\lambda, s_\lambda)_\lambda \in (M_n(\mathbf{C}) \times \mathbf{C})^\Lambda$. □

Corollary 5.7. *The inclusion* $Q_n(\Lambda) \subseteq M_n(Q_n(\Lambda))$ *is equivariant.*

All four $\mathrm{GL}_n(\mathbf{C})$-actions are related by

Proposition 5.8. *The maps in the commutative square*

$$\begin{array}{ccc} M_n(Q_n(\Lambda)) & \xrightarrow{M_n(q)} & M_n(\mathcal{R}_n(\Sigma)) \\ \big\downarrow \scriptstyle{\mathrm{Tr}} & & \big\downarrow \scriptstyle{\mathrm{Tr}} \\ Q_n(\Lambda) & \xrightarrow{\quad q \quad} & \mathcal{R}_n(\Sigma) \end{array}$$

are equivariant.

Proof. The traces are invariant under conjugation with complex matrices and hence equivariant by (5.6) and Lemma 5.6, respectively. The equivariance of q need only be verified on the generators $x^\lambda_{i,j}, d_\lambda$. Fix $A \in \mathrm{GL}_n(\mathbf{C})$. For any $(M_\lambda, s_\lambda)_\lambda \in (M_n(\mathbf{C}) \times \mathbf{C})^\Lambda$ we have

$$\begin{aligned}(A * x^{\lambda_0}_{i,j})((M_\lambda, s_\lambda)_\lambda) &= x^{\lambda_0}_{i,j}(A^{-1} * (M_\lambda, s_\lambda)_\lambda) \\ &= x^{\lambda_0}_{i,j}((A^{-1}M_\lambda A, s_\lambda)_\lambda) \\ &= (A^{-1}M_{\lambda_0}A)_{i,j} \\ &= (A^{-1}A_{\lambda_0}((M_\lambda, s_\lambda)_\lambda)A)_{i,j} \\ &= ((A^{-1}A_{\lambda_0}A)((M_\lambda, s_\lambda)_\lambda))_{i,j}\end{aligned}$$

so that

$$q(A * x^{\lambda_0}_{i,j}) = q((A^{-1}A_{\lambda_0}A)_{i,j}) = (A^{-1}M_n(q)(A_{\lambda_0})A)_{i,j} = A * q(x^{\lambda_0}_{i,j})$$

by (5.5). The elements d_λ are invariant by Lemma 5.5, so (5.5) also takes care of those. Regarding $M_n(q)$, we derive for $M \in M_n(Q_n(\Lambda))$

$$\begin{aligned} M_n(q)(A * M) &= M_n(q)(A(A * M_{i,j})A^{-1}) \\ &= A(q(A * M_{i,j}))A^{-1} \\ &= A(A * (q(M_{i,j})))A^{-1} \\ &= A * M_n(q)(M) \end{aligned}$$

by Lemma 5.6, the equivariance of q and formula (5.6). □

Proposition 5.9. *The image of the universal* GL_n*-representation* $\rho_\Sigma\colon \pi_1(\Sigma) \to M_n(\mathcal{R}_n(\Sigma))$ *is invariant under the action of* $\mathrm{GL}_n(\mathbf{C})$*.*

Proof. It suffices to consider a generator g_λ. The matrix $A_\lambda \in M_n(Q_n(\Lambda))$ is invariant by Lemma 5.5; the result now follows from the previous proposition since $M_n(q)(A_\lambda) = \rho_\Sigma(g_\lambda)$ is then invariant, too. □

Remark 5.10. Before continuing the investigation of $\mathcal{R}_n(\Sigma)$, we explore its relationship with $\mathcal{M}(\Sigma; \mathrm{GL}_n(\mathbf{C}))$. We think of $\mathrm{GL}_n(\mathbf{C})$ as an affine subset of $M_n(\mathbf{C}) \times \mathbf{C} \cong \mathbf{C}^{n^2+1}$, namely

$$\mathrm{GL}_n(\mathbf{C}) = \{(A, d) \in M_n(\mathbf{C}) \times \mathbf{C} \mid d \det A = 1\}.$$

Recalling the construction of the combinatorial complex K_P (cf. 2.1), we infer (at least in the case $|\Lambda| < \infty$) that the vanishing set of the ideal $I(P) \subseteq Q_n(\Lambda) \cong \mathcal{O}((M_n(\mathbf{C}) \times \mathbf{C})^\Lambda)$ is exactly $\mathcal{A}(K_P; \mathrm{GL}_n(\mathbf{C})) \subseteq \mathrm{GL}_n(\mathbf{C})^{E(K_P)} = \mathrm{GL}_n(\mathbf{C})^\Lambda \subseteq (M_n(\mathbf{C}) \times \mathbf{C})^\Lambda$. By Hilbert's Nullstellensatz, restriction of functions provides an isomorphism

$$Q_n(\Lambda)/\sqrt{I(P)} \to \mathcal{O}(\mathcal{A}(K_P; \mathrm{GL}_n(\mathbf{C}))).$$

The former space is, of course, nothing but $\mathcal{R}_n(\Sigma; P)/\sqrt{0}$, so we have a commutative triangle

$$\begin{array}{ccc} Q_n(\Lambda) & \xrightarrow{\;q\;} & \mathcal{R}_n(\Sigma; P) \\ & \searrow & \downarrow p \\ & & \mathcal{O}(\mathcal{A}(K_P; \mathrm{GL}_n(\mathbf{C}))) \end{array}$$

where $\operatorname{Ker} p = \sqrt{0}$. It also follows that p is a $\mathrm{GL}_n(\mathbf{C})$-equivariant surjection since the other two maps in the diagram enjoy this property.

5.1.3 Diagrams and Relative Diagrams

Recall that a diagram on Σ is simply the homotopy class of a map from a finite collection of oriented circles to Σ, or, in other words, a set of conjugacy classes in $\pi_1(\Sigma)$. We need a relative version of this concept; a *relative diagram* D is the unit interval I union a finite collection of oriented circles, and a *relative diagram on* Σ is a map $f\colon D \to \Sigma \times I$ such that $f(i) = (x_0, i)$, $i = 0, 1$, regarded up to homotopy rel ∂I. Post-composing with the projection $p\colon \Sigma \times I \to \Sigma$ (a homotopy equivalence), one sees that such an object is nothing but an

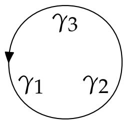


Figure 5.1: A decorated diagram.

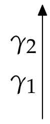


Figure 5.2: A decorated relative diagram.

element of $\pi_1(\Sigma)$ together with a finite set of conjugacy classes in this group. Let $\mathcal{Z}(\Sigma, x_0)$ denote the complex vector space freely generated by relative diagrams on Σ; it is equipped with a natural algebra structure: For relative diagrams $f_i \colon D_i \to \Sigma \times I$, $i = 1, 2$ we define

$$D = D_1 \cup D_2 / \{\partial_+ I_1 = \partial_- I_2\} \tag{5.7}$$

and $f = f_1 f_2 \colon D \to \Sigma \times I$ by

$$f(d) = \begin{cases} (x, 1/2t), & d \in D_1 \wedge f_1(d) = (x,t) \\ (x, 1/2t + 1/2), & d \in D_2 \wedge f_2(d) = (x,t) \end{cases}$$

The unit for the multiplication is

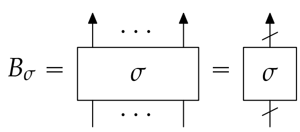
$$e \colon I \to \Sigma \times I, \quad e(t) = (x_0, t).$$

It is convenient to represent (relative) diagrams on Σ by *decorated (relative) diagrams*. By this we mean (relative) diagrams along each component of which one or more elements of $\pi_1(\Sigma)$ are written. We give a couple of examples of how decorated (relative) diagrams represent (relative) diagrams on Σ. The decorated diagram in Figure 5.1 determines the diagram on Σ given by a map $S^1 \to \Sigma$ representing the conjugacy class of $\gamma_1\gamma_2\gamma_3 \in \pi_1(\Sigma)$. Similarly, the relative diagram on Σ represented by the decorated relative diagram in Figure 5.2 is defined by a map $f \colon I \to \Sigma \times I$ such that the loop $p \circ f$ is in the homotopy class $\gamma_1\gamma_2 \in \pi_1(\Sigma)$. How to interpret general decorated (relative) diagrams is obvious from these examples. Decoration of a component with $1 \in \pi_1(\Sigma)$ is sometimes suppressed in the notation. If the component in question is the interval of a relative diagram, it may be omitted entirely; the potential confusion with a (non-relative) decorated diagram is non-serious as we shall later.

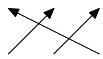
Remark 5.11. It is obvious that two decorated (relative) diagrams represent the same (relative) diagram on Σ if and only if there exists a bijection between the circles of the two diagrams such that the products along corresponding circles are conjugate elements of $\pi_1(\Sigma)$, and, in the relative case, the products along the two intervals are equal.

Multiplication of (relative) diagrams on Σ lifts to the setting of decorated (relative) diagrams in the obvious way; take the union of all components carrying along the decoration, and glue the intervals in the relative case (cf. (5.7)).

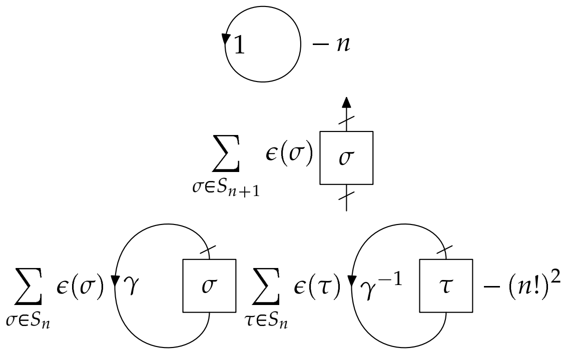
Parts (local and non-local on Σ) of decorated (relative) diagrams play an important role in the sequel. The ubiquitous example is the braid (over- and undercrossings being ignored) B_σ corresponding to a permutation $\sigma \in S_m$; we depict it as (notice the notation for bundles of strands):



For example, if $\sigma = (1,2,3) \in S_3$ we have

$B_\sigma =$ 

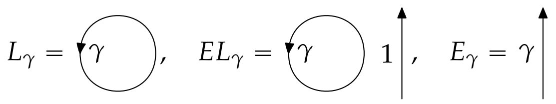
We define ideals $\mathcal{I}_n(\Sigma) = \mathcal{I}(\Sigma; \mathrm{GL}_n(\mathbf{C})) \subseteq \mathcal{Z}(\Sigma)$ and $\mathcal{I}_n(\Sigma; x_0) \subseteq \mathcal{Z}(\Sigma, x_0)$ to be generated by the following three kinds of expressions:



The first two relations are local on Σ, whereas the third one is not. We write

$$\mathcal{D}_n(\Sigma) = \mathcal{Z}(\Sigma)/\mathcal{I}_n(\Sigma), \quad \mathcal{D}_n(\Sigma, x_0) = \mathcal{Z}(\Sigma, x_0)/\mathcal{I}_n(\Sigma; x_0)$$

for the quotient algebras. Certain elementary decorated (relative) diagrams deserve special attention:



For easy reference we record the following simple fact about them.

Proposition 5.12. *$\mathcal{D}_n(\Sigma)$ is generated by L_γ, $\gamma \in \pi_1(\Sigma)$, and $\mathcal{D}_n(\Sigma, x_0)$ is generated by $E_{g_\lambda^{\pm 1}}$, $\lambda \in \Lambda$ and EL_γ, $\gamma \in \pi_1(\Sigma)$.*

The algebras $\mathcal{D}_n(\Sigma)$ and $\mathcal{D}_n(\Sigma, x_0)$ are related by a pair of maps. In one direction $\iota\colon \mathcal{Z}(\Sigma) \to \mathcal{Z}(\Sigma, x_0)$ is given on decorated diagrams by simply adding an interval decorated by $1 \in \pi_1(\Sigma)$. This is well-defined on the level of diagrams on Σ and clearly induces an algebra homomorphism $\iota\colon \mathcal{D}_n(\Sigma) \to \mathcal{D}_n(\Sigma, x_0)$; its image is central, so we may view $\mathcal{D}_n(\Sigma, x_0)$ as an algebra over $\mathcal{D}_n(\Sigma)$. On the other hand, closing up the interval of a decorated relative diagram to a circle and thereby obtaining a decorated diagram results in a map $\mathcal{Z}(\Sigma, x_0) \to \mathcal{Z}(\Sigma)$; it descends to a linear map $\mathrm{Tr}\colon \mathcal{D}_n(\Sigma, x_0) \to \mathcal{D}_n(\Sigma)$. By relation (5.8a) we have

$$\mathrm{Tr} \circ \iota = n\,\mathrm{Id}\colon \mathcal{D}_n(\Sigma) \to \mathcal{D}_n(\Sigma).$$

In particular, ι embeds $\mathcal{D}_n(\Sigma)$ as a subalgebra of $\mathcal{D}_n(\Sigma, x_0)$; this justifies the aforementioned convention of occasionally omitting a trivially decorated interval of a decorated relative diagram, and we often suppress ι in the notation.

5.1.4 Diagrams and Invariant Functions

Theorem 5.13. *There exist homomorphisms of complex algebras* $\theta\colon \mathcal{D}_n(\Sigma) \to \mathcal{R}_n(\Sigma)^{\mathrm{GL}_n(\mathbf{C})}$ *and* $\Theta\colon \mathcal{D}_n(\Sigma, x_0) \to M_n(\mathcal{R}_n(\Sigma))^{\mathrm{GL}_n(\mathbf{C})}$ *uniquely determined by*

$$\theta(L_\gamma) = \mathrm{Tr}(\rho_\Sigma(\gamma)), \quad \gamma \in \pi_1(\Sigma); \tag{5.9a}$$

$$\Theta(E_\gamma) = \rho_\Sigma(\gamma), \quad \Theta(EL_\gamma) = \mathrm{Tr}(\rho_\Sigma(\gamma)), \quad \gamma \in \pi_1(\Sigma). \tag{5.9b}$$

Furthermore the diagrams

$$\begin{array}{ccc} \mathcal{D}_n(\Sigma, x_0) & \xrightarrow{\Theta} & M_n(\mathcal{R}_n(\Sigma))^{\mathrm{GL}_n(\mathbf{C})} \\ \iota \uparrow & & \uparrow \iota \\ \mathcal{D}_n(\Sigma) & \xrightarrow{\theta} & \mathcal{R}_n(\Sigma)^{\mathrm{GL}_n(\mathbf{C})} \end{array} \qquad \begin{array}{ccc} \mathcal{D}_n(\Sigma, x_0) & \xrightarrow{\Theta} & M_n(\mathcal{R}_n(\Sigma))^{\mathrm{GL}_n(\mathbf{C})} \\ \downarrow \mathrm{Tr} & & \downarrow \mathrm{Tr} \\ \mathcal{D}_n(\Sigma) & \xrightarrow{\theta} & \mathcal{R}_n(\Sigma)^{\mathrm{GL}_n(\mathbf{C})} \end{array} \tag{5.10}$$

commute.

Proof. The uniqueness is immediate from Proposition 5.12. We construct algebra homomorphisms $\theta\colon \mathcal{Z}(\Sigma) \to \mathcal{R}_n(\Sigma)$ and $\Theta\colon \mathcal{Z}(\Sigma, x_0) \to M_n(\mathcal{R}_n(\Sigma))$ by paralleling the construction of $\Psi_t\colon \mathcal{C}(\Sigma; G) \to \mathcal{O}(\mathcal{M}(\Sigma; G))$, cf. (2.24). Let $V = \mathcal{R}_n(\Sigma)^n$ be the free $\mathcal{R}_n(\Sigma)$-module of rank n with its standard basis $\{e_1, \ldots, e_n\}$; the standard dual basis of V^* is denoted by $\{e^1, \ldots, e^n\}$. Represent a (relative) diagram D on Σ by a decorated (relative) diagram. Cut D into arcs, one for each element $\gamma \in \pi_1(\Sigma)$ of the decoration, and assign tensors to the arcs as follows

$$\mathcal{T}\Big(\xrightarrow{\ \gamma\ }\Big) = \underset{V}{\ }\xrightarrow{\ \rho_\Sigma(\gamma)\ }\underset{V^*}{\ }$$

By definition, $\theta(D)$ (respectively, $\Theta(D)$) is the contracted tensor $\mathcal{T}(D)$; this element belongs to the right codomain (we use, of course, the canonical identification $M_n(\mathcal{R}_n(\Sigma)) \cong V \otimes V^*$ throughout). It follows from Remark 5.11 that θ and Θ are well-defined, and it is evident that the conditions (5.9) hold. These identities prove that θ factors through $\mathcal{R}_n(\Sigma)^{\mathrm{GL}_n(\mathbf{C})}$ and Θ through $M_n(\mathcal{R}_n(\Sigma))^{\mathrm{GL}_n(\mathbf{C})}$ by Propositions 5.12, 5.9, 5.8 and Lemma 5.4.

Thus it only remains to prove that θ and Θ descend to the quotients, since the commutativity of the diagrams (5.10) is then obvious. This verification can be performed simultaneously for θ and Θ as the relations (5.8) to be checked are the same in the two cases. The first relation is easy:

$$\mathcal{T}(L_1) = \mathrm{Tr}(\rho_\Sigma(1)) = n.$$

For relation (5.8b) number the sources and sinks of B_σ from left to right, and index the copies of V and V^* assigned to them accordingly. The strand connecting the kth source to the $\sigma(k)$th sink is implicitly decorated with 1 and hence contributes

$$\rho_\Sigma(1) = \mathrm{Id}_V = \sum_{1 \leqslant i_k \leqslant n} e_{i_k} \otimes e^{i_k} \in V_k \otimes V^*_{\sigma(k)}.$$

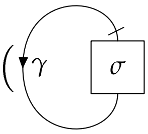
Therefore

$$\mathcal{T}(B_\sigma) = \sum_{1 \leqslant i_1, \ldots, i_{n+1} \leqslant n} e_{i_1} \otimes \cdots \otimes e_{i_{n+1}} \otimes e^{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes e^{i_{\sigma^{-1}(n+1)}}$$

so that

$$\mathcal{T}\Big(\sum_{\sigma\in S_{n+1}} \epsilon(\sigma)B_\sigma\Big) = \sum_{1\leqslant i_1,\ldots,i_{n+1}\leqslant n}\ \sum_{\sigma\in S_{n+1}} \epsilon(\sigma) e_{i_1}\otimes\cdots\otimes e_{i_{n+1}}\otimes e^{i_{\sigma^{-1}(1)}}\otimes\cdots\otimes e^{i_{\sigma^{-1}(n+1)}}$$
$$= 0.$$

This is because among any $1 \leqslant i_1,\ldots,i_{n+1} \leqslant n$ exist $i_j = i_k$ with $j \neq k$. Pre-composition with the transposition (j,k) yields a fixed point free involution on S_{n+1}, and the permutations matched by this map contribute terms differing only in sign. Regarding relation (5.8c), recall that any permutation $\sigma \in S_n$ can be decomposed essentially uniquely into a product of disjoint cycles (including those of length 1). Denoting the lengths of all cycles in σ by $c_1,\ldots,c_k$, we obtain

$\mathcal{T}\Big($  $\Big) = \mathrm{Tr}(\rho_\Sigma(\gamma)^{c_1})\cdots\mathrm{Tr}(\rho_\Sigma(\gamma)^{c_k}).$

The result thus follows by applying the lemma below to $\rho_\Sigma(\gamma)$, $\rho_\Sigma(\gamma^{-1}) \in M_n(\mathcal{R}_n(\Sigma))$. □

Lemma 5.14 (Formanek [For]). *Let R be a commutative ring. For any matrix $M \in M_n(R)$,*

$$\det M = \frac{1}{n!}\sum_{\sigma\in S_n} \epsilon(\sigma)\,\mathrm{Tr}(M^{c_1})\cdots\mathrm{Tr}(M^{c_k})$$

where $c_1,\ldots,c_k$ are the lengths of all cycles in σ.

Theorem 5.15. *The maps $\theta\colon \mathcal{D}_n(\Sigma) \to \mathcal{R}_n(\Sigma)^{\mathrm{GL}_n(\mathbf{C})}$ and $\Theta\colon \mathcal{D}_n(\Sigma, x_0) \to M_n(\mathcal{R}_n(\Sigma))^{\mathrm{GL}_n(\mathbf{C})}$ are algebra isomorphisms.*

This theorem will be proved in the course of the next two subsections. To this end we fix a presentation $\langle g_\lambda, \lambda \in \Lambda \mid r_\mu, \mu \in M\rangle$ of $\pi_1(\Sigma)$ such that

- Λ is an infinite set.
- The inverse of every generator is a generator.

Note that the latter condition implies that all relations are products of generators so that the presentation yields a model for $\mathcal{R}_n(\Sigma)$.

5.1.5 Fundamental Theorems of Invariant Theory

Let T_y be the complex polynomial algebra in variables $\mathrm{Tr}(X_{\lambda_1}\cdots X_{\lambda_k})$, $\lambda_1,\ldots,\lambda_k \in \Lambda$ and y_λ, $\lambda \in \Lambda$. Here it is understood that $\mathrm{Tr}(M) = \mathrm{Tr}(N)$ if and only if the monomials M and N are related by a cyclic permutation. Denote by $T_y\{X_\lambda\}$ the free T_y-algebra generated by X_λ, $\lambda \in \Lambda$. As usual T_y is naturally a central subalgebra of $T_y\{X_\lambda\}$. Moreover, the assignment $X_{\lambda_1}\cdots X_{\lambda_k} \mapsto \mathrm{Tr}(X_{\lambda_1}\cdots X_{\lambda_k})$ (for $k = 0$ this should be interpreted as $1 \mapsto n$) determines a T_y-linear map $\mathrm{Tr}\colon T_y\{X_\lambda\} \to T_y$. Define a $\mathbf{C}$-algebra homomorphism $\pi\colon T_y\{X_\lambda\} \to M_n(Q_n(\Lambda))$ by

$$\pi(X_\lambda) = A_\lambda, \tag{5.11a}$$
$$\pi(\mathrm{Tr}(X_{\lambda_1}\cdots X_{\lambda_k})) = \mathrm{Tr}(A_{\lambda_1}\cdots A_{\lambda_k}), \tag{5.11b}$$
$$\pi(y_\lambda) = d_\lambda. \tag{5.11c}$$

We remark that the images of $\mathrm{Tr}(X_{\lambda_1}\cdots X_{\lambda_k})$, $\lambda_1,\ldots,\lambda_k \in \Lambda$ and d_λ, $\lambda \in \Lambda$ belong to the central subalgebra $Q_n(\Lambda) \subseteq M_n(Q_n(\Lambda))$ so that π is well-defined. It follows from Lemma 5.5, Proposition 5.8 and Corollary 5.7 that π factors through $M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})}$ and that $\pi_{|T_y}$ factors through $Q_n(\Lambda)^{\mathrm{GL}_n(\mathbf{C})}$. Moreover these maps commute with the trace, that is, we have the square

$$\begin{array}{ccc} T_y\{X_\lambda\} & \xrightarrow{\pi} & M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})} \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} \\ T_y & \xrightarrow{\pi} & Q_n(\Lambda)^{\mathrm{GL}_n(\mathbf{C})} \end{array} \tag{5.12}$$

The following version of the First Fundamental Theorem of invariant theory of $n \times n$ matrices is due to Procesi [P1].

Theorem 5.16 (Procesi). *The map* $\pi\colon T_y\{X_\lambda\} \to M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})}$ *is surjective.*

Corollary 5.17 (Procesi). *The map* $\pi\colon T_y \to Q_n(\Lambda)^{\mathrm{GL}_n(\mathbf{C})}$ *is surjective.*

Proof. Diagram (5.12) and Remark 1.1. □

Procesi also accomplished a description of the kernel of $\pi\colon T_y\{X_\lambda\} \to M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})}$ in [P1]. Formulating the result requires a little preparation. To simplify notation we assume that $\mathbf{N} \subseteq \Lambda$. Decomposing a permutation $\sigma \in S_m$ into disjoint cycles

$$\sigma = (i_1,\ldots,i_s)(j_1,\ldots,j_t)\cdots(k_1,\ldots,k_v),$$

we may define

$$\Phi_\sigma(X_1,\ldots,X_m) = \mathrm{Tr}(X_{i_1}\cdots X_{i_s})\,\mathrm{Tr}(X_{j_1}\cdots X_{j_t})\cdots\mathrm{Tr}(X_{k_1}\cdots X_{k_v}) \in T_y$$

unambiguously. Put

$$F(X_1,\ldots,X_m) = \sum_{\sigma\in S_m} \epsilon(\sigma)\Phi_\sigma(X_1,\ldots,X_m).$$

If $\sigma \in S_{m+1}$, we may arrange the cycle decomposition in such a way that $m+1$ is the last element of the first cycle:

$$\sigma = (i_1,\ldots,i_s,m+1)(j_1,\ldots,j_t)\cdots(k_1,\ldots,k_v)$$

allowing us to set

$$\Psi_\sigma(X_1,\ldots,X_m) = X_{i_1}\cdots X_{i_s}\,\mathrm{Tr}(X_{j_1}\cdots X_{j_t})\cdots\mathrm{Tr}(X_{k_1}\cdots X_{k_v}) \in T_y\{X_\lambda\}. \tag{5.13}$$

According to Procesi there is a unique element $G(X_1,\ldots,X_n) \in T_y\{X_\lambda\}$ involving only the variables $X_1,\ldots,X_n$ and traces of monomials in these variables such that

$$F(X_1,\ldots,X_{n+1}) = \mathrm{Tr}[G(X_1,\ldots,X_n)X_{n+1}]. \tag{5.14}$$

We shall need an explicit formula for G.

Lemma 5.18. $G(X_1,\ldots,X_n) = \sum_{\sigma\in S_{n+1}} \epsilon(\sigma)\Psi_\sigma(X_1,\ldots,X_n)$.

Proof. Since

$$\begin{aligned}\operatorname{Tr}\Big(\sum_{\sigma\in S_{n+1}} \epsilon(\sigma)\Psi_\sigma(X_1,\dots,X_n)X_{n+1}\Big) &= \sum_{\sigma\in S_{n+1}} \epsilon(\sigma)\operatorname{Tr}(\Psi_\sigma(X_1,\dots,X_n)X_{n+1})\\ &= \sum_{\sigma\in S_{n+1}} \epsilon(\sigma)\Phi_\sigma(X_1,\dots,X_{n+1})\\ &= F(X_1,\dots,X_{n+1}),\end{aligned}$$

the result follows from the uniqueness. □

It is clear that the formulas for $\Phi_\sigma, \Psi_\sigma, F$ and G make sense for monomials in the variables X_λ, $\lambda \in \Lambda$. The Second Fundamental Theorem of invariant theory of $n \times n$ matrices now takes the form of

Theorem 5.19 (Procesi). *The kernel of* $\pi\colon T_y\{X_\lambda\} \to M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})}$ *is generated by the expressions* $F(M_1,\dots,M_{n+1})$ *and* $G(N_1,\dots,N_n)$ *where* M_i *and* N_j *are monomials in* X_λ, $\lambda \in \Lambda$.

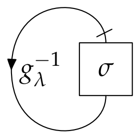
5.1.6 Proof of Theorem 5.15

We initiate the construction of the inverse to $\Theta\colon \mathcal{D}_n(\Sigma, x_0) \to M_n(\mathcal{R}_n(\Sigma))^{\mathrm{GL}_n(\mathbf{C})}$; define a $\mathbf{C}$-algebra homomorphism $\psi\colon T_y\{X_\lambda\} \to \mathcal{D}_n(\Sigma, x_0)$ by

$$\psi(X_\lambda) = E_{g_\lambda}, \tag{5.15a}$$

$$\psi(\operatorname{Tr}(X_{\lambda_1}\cdots X_{\lambda_k})) = EL_{g_{\lambda_1}\cdots g_{\lambda_k}}, \tag{5.15b}$$

$$\psi(y_\lambda) = \frac{1}{n!}\sum_{\sigma\in S_n} \epsilon(\sigma) \tag{5.15c}$$



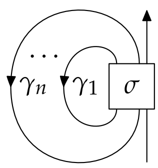
We note that the images of $\operatorname{Tr}(X_{\lambda_1}\cdots X_{\lambda_k})$ and y_λ belong to $\mathcal{D}_n(\Sigma) \subseteq \mathcal{D}_n(\Sigma, x_0)$ and are thus central. This implies that ψ is well-defined and that $\psi_{|T_y}$ factors through $\mathcal{D}_n(\Sigma)$. Another property of ψ is its compatibility with the trace:

$$\begin{array}{ccc} T_y\{X_\lambda\} & \xrightarrow{\ \psi\ } & \mathcal{D}_n(\Sigma, x_0)\\ \Big\downarrow{\scriptstyle\mathrm{Tr}} & & \Big\downarrow{\scriptstyle\mathrm{Tr}}\\ T_y & \xrightarrow{\ \psi\ } & \mathcal{D}_n(\Sigma)\end{array} \tag{5.16}$$

Proposition 5.20. *The kernel of* $\psi\colon T_y\{X_\lambda\} \to \mathcal{D}_n(\Sigma, x_0)$ *contains the kernel of* $\pi\colon T_y\{X_\lambda\} \to M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})}$.

Proof. Owing to Theorem 5.19, this is not difficult. Let $N_1,\dots,N_n$ be monomials, and pick $\gamma_1,\dots,\gamma_n \in \pi_1(\Sigma)$ such that $\psi(N_i) = E_{\gamma_i}$. For $\sigma \in S_{n+1}$ we apply formula (5.13) to obtain

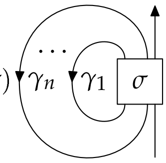
$$\psi(\Psi_\sigma(N_1,\dots,N_n)) =$$



Therefore Lemma 5.18 and relation (5.8b) yield

$$\psi(G(N_1,\dots,N_n)) = \sum_{\sigma\in S_{n+1}} \epsilon(\sigma)\psi(\Psi_\sigma(N_1,\dots,N_n))$$

$$= \sum_{\sigma\in S_{n+1}} \epsilon(\sigma)$$



$$= 0$$

as desired. Let $M_1,\dots,M_{n+1}$ be monomials. Using the defining property (5.14) of G and the diagram (5.16), we derive

$$\begin{aligned}\psi(F(M_1,\dots,M_{n+1})) &= \psi(\mathrm{Tr}[G(M_1,\dots,M_n)M_{n+1}]) \\ &= \mathrm{Tr}(\psi(G(M_1,\dots,M_n))\psi(M_{n+1})) \\ &= 0\end{aligned}$$

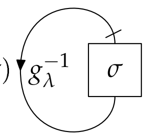
according to the first part of the proof. □

Corollary 5.21. *There exists a* $\mathbf{C}$*-algebra homomorphism* $\psi'\colon M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})} \to \mathcal{D}_n(\Sigma,x_0)$ *satisfying*

$$\psi'(A_\lambda) = E_{g_\lambda}, \tag{5.17a}$$

$$\psi'(\mathrm{Tr}(A_{\lambda_1}\cdots A_{\lambda_k})) = EL_{g_{\lambda_1}\cdots g_{\lambda_k}}, \tag{5.17b}$$

$$\psi'(d_\lambda) = \frac{1}{n!}\sum_{\sigma\in S_n}\epsilon(\sigma) \tag{5.17c}$$



Proof. This follows from Theorems 5.16 and 5.19 along with the definitions of π (5.11) and ψ (5.15). □

Due to Corollary 5.17, the subalgebra $Q_n(\Lambda)^{\mathrm{GL}_n(\mathbf{C})} \subseteq M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})}$ is generated by $\mathrm{Tr}(A_{\lambda_1}\cdots A_{\lambda_k})$ and d_λ. We deduce that $\psi'_{|Q_n(\Lambda)^{\mathrm{GL}_n(\mathbf{C})}}$ factors through $\mathcal{D}_n(\Sigma)$; the diagram (5.16) induces

$$\begin{array}{ccc} M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})} & \xrightarrow{\psi'} & \mathcal{D}_n(\Sigma,x_0) \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} \\ Q_n(\Lambda)^{\mathrm{GL}_n(\mathbf{C})} & \xrightarrow{\psi'} & \mathcal{D}_n(\Sigma) \end{array} \tag{5.18}$$

It requires more effort to factor ψ' through the surjection $M_n(q)_{|}\colon M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})} \to M_n(\mathcal{R}_n(\Sigma))^{\mathrm{GL}_n(\mathbf{C})}$ (cf. Proposition 5.8 and Remark 1.1). For each relation $r_\mu = g_{\lambda_1}\cdots g_{\lambda_k}$ in the presentation P of $\pi_1(\Sigma)$, we let $M_\mu = A_{\lambda_1}\cdots A_{\lambda_k} \in M_n(Q_n(\Lambda))$ denote the corresponding matrix. Consider the ideal

$$I'(P) = \langle d_\lambda \det A_\lambda - 1, M_\mu - 1 \mid \lambda\in\Lambda, \mu\in M\rangle \subseteq M_n(Q_n(\Lambda)).$$

Recall that the ideal $I(P) \subseteq Q_n(\Lambda)$ determining $\mathcal{R}_n(\Sigma)$ is, by definition, generated by the entries of the generators of $I'(P)$. It is then an elementary result about matrix rings that

$$I'(P) = M_n(I(P)). \tag{5.19}$$

Proposition 5.22. $d_\lambda \det A_\lambda - 1, M_\mu - 1 \in \operatorname{Ker} \psi'$, $\lambda \in \Lambda, \mu \in M$

Proof. By Lemma 5.5, $M_\mu = A_{\lambda_1} \cdots A_{\lambda_k} \in M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})}$, and by (5.17a),

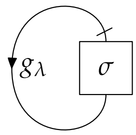
$$\psi'(M_\mu) = E_{g_{\lambda_1}} \cdots E_{g_{\lambda_k}} = E_{g_{\lambda_1} \cdots g_{\lambda_k}} = E_1 = 1$$

as desired. Also, $\det A_\lambda \in Q_n(\Lambda)^{\mathrm{GL}_n(\mathbf{C})} \subseteq M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})}$ by Lemma 5.5, and applying Lemma 5.14 to A_λ, we obtain

$$\det A_\lambda = \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma) \operatorname{Tr}(A_\lambda^{c_1}) \cdots \operatorname{Tr}(A_\lambda^{c_k}).$$

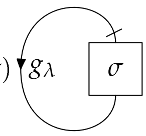
Since (5.17b) implies

$$\psi'(\operatorname{Tr}(A_\lambda^{c_1}) \cdots \operatorname{Tr}(A_\lambda^{c_k})) =$$



we have

$$\psi'(\det A_\lambda) = \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma)$$



It now follows from (5.17c) and relation (5.8c) that $\psi'(d_\lambda \det A_\lambda) = 1$. □

The following result can be extracted from the proof of Theorem 2.6 of [P2].

Proposition 5.23 (Procesi). *Let* $J \subseteq M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})}$ *be an ideal closed under* Tr. *Then*

$$J = M_n(Q_n(\Lambda)) J M_n(Q_n(\Lambda)) \cap M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})}.$$

Proof. The inclusion $\subseteq$ is trivial. To prove the other one we shall need gradings, depending on an index $\lambda_0 \in \Lambda$, on $Q_n(\Lambda)$ and $M_n(Q_n(\Lambda))$. We define $Q_n^{(d)}(\Lambda) \subseteq Q_n(\Lambda)$ to be the subspace of polynomials homogeneous of degree d in the variables $x_{i,j}^{\lambda_0}$, $i, j = 1, \ldots, n$. Parallelling this, we denote by $M_n^{(d)}(Q_n(\Lambda)) \subseteq M_n(Q_n(\Lambda))$ the subspace of matrices all entries of which have degree d. In this way

$$Q_n(\Lambda) = \bigoplus_{d=0}^{\infty} Q_n^{(d)}(\Lambda); \quad M_n(Q_n(\Lambda)) = \bigoplus_{d=0}^{\infty} M_n^{(d)}(Q_n(\Lambda))$$

are graded algebreas, and $\operatorname{Tr} \colon M_n(Q_n(\Lambda)) \to Q_n(\Lambda)$ is graded linear. The $\mathrm{GL}_n(\mathbf{C})$-actions on $Q_n(\Lambda)$ and $M_n(Q_n(\Lambda))$ respect the gradings: For $Q_n(\Lambda)$ we need only verify this for the generators $x^\lambda_{i,j}, d_\lambda$. Letting $A \in \mathrm{GL}_n(\mathbf{C})$ and recalling the notation $A_\lambda = (x^\lambda_{i,j})$, Lemmas 5.6 and 5.5 imply

$$(A * x^\lambda_{i,j}) = A^{-1}(x^\lambda_{i,j})A, \quad A * d_\lambda = d_\lambda$$

thereby establishing the claim for $Q_n(\Lambda)$. Using this and Lemma 5.6 again, we deduce

$$M \in M_n^{(d)}(Q_n(\Lambda)) \Longrightarrow A * M = A(A * M_{i,j})A^{-1} \in M_n^{(d)}(Q_n(\Lambda))$$

as desired. The vital consequence is: The inclusions $Q_n^{(d)}(\Lambda) \subseteq Q_n(\Lambda)$ and $M_n^{(d)}(Q_n(\Lambda)) \subseteq M_n(Q_n(\Lambda))$ are $\mathrm{GL}_n(\mathbf{C})$-equivariant and therefore commute with the Reynolds operators, that is, the Reynolds operators on $Q_n(\Lambda)$ and $M_n(Q_n(\Lambda))$ preserve degrees. To prove the proposition let

$$c = \sum_k a_k c_k b_k \in M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})}$$

where $a_k, b_k \in M_n(Q_n(\Lambda))$ and $c_k \in J$. Since a_k, b_k, c_k involve only finitely many variables and $|\Lambda| = \infty$, there exists $\lambda_0 \in \Lambda$ such that $x^{\lambda_0}_{i,j}$, $i, j = 1, \ldots, n$ do not occur. We employ the gradings with respect to λ_0; obviously

$$\deg a_k = \deg b_k = \deg c_k = 0 = \deg c.$$

As $\mathrm{Tr}(cA_{\lambda_0}) \in Q_n(\Lambda)^{\mathrm{GL}_n(\mathbf{C})}$ by Lemma 5.5 and Proposition 5.8, we have

$$\begin{aligned}\mathrm{Tr}(cA_{\lambda_0}) &= \nabla\,\mathrm{Tr}(cA_{\lambda_0}) = \nabla\,\mathrm{Tr}\Big(\sum_k a_k c_k b_k A_{\lambda_0}\Big)\\ &= \nabla\,\mathrm{Tr}\Big(\sum_k b_k A_{\lambda_0} a_k c_k\Big) = \mathrm{Tr}\sum_k \nabla(b_k A_{\lambda_0} a_k)c_k\end{aligned}$$

by Reynolds' identity (1.2). Notice that

$$\deg \nabla(b_k A_{\lambda_0} a_k) = \deg(b_k A_{\lambda_0} a_k) = 0 + 1 + 0 = 1.$$

For the generators of $M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})}$ (cf. Theorem 5.16), we have

$$\begin{aligned}\deg A_\lambda &= \delta_{\lambda,\lambda_0},\\ \deg \mathrm{Tr}(A_{\lambda_1}\cdots A_{\lambda_k}) &= \delta_{\lambda_1,\lambda_0} + \cdots + \delta_{\lambda_k,\lambda_0},\\ \deg d_\lambda &= 0.\end{aligned}$$

Thus $\nabla(b_k A_{\lambda_0} a_k) \in M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})}$ may be written

$$\nabla(b_k A_{\lambda_0} a_k) = \sum_l p_{k,l} A_{\lambda_0} q_{k,l} + \sum_m \mathrm{Tr}(s_{k,m} A_{\lambda_0}) t_{k,m}$$

for suitable $p_{k,l}, q_{k,l}, s_{k,l}, t_{k,l} \in M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})}$ all of degree 0. This leads to

$$\begin{aligned}\mathrm{Tr}(cA_{\lambda_0}) &= \mathrm{Tr}\Big(\sum_{k,l} p_{k,l} A_{\lambda_0} q_{k,l} c_k + \sum_{k,m} \mathrm{Tr}(s_{k,m} A_{\lambda_0}) t_{k,m} c_k\Big)\\ &= \mathrm{Tr}\Big(\Big[\sum_{k,l} q_{k,l} c_k p_{k,l} + \sum_{k,m} \mathrm{Tr}(t_{k,m} c_k) s_{k,m}\Big] A_{\lambda_0}\Big).\end{aligned}$$

Hence $x = c - \sum_{k,l} q_{k,l}c_k p_{k,l} + \sum_{k,m} \mathrm{Tr}(t_{k,m}c_k)s_{k,m}$ is a matrix of degree 0 satisfying $\mathrm{Tr}(xA_{\lambda_0}) = 0$. This can only be true if $x = 0$. Finally,

$$c = \sum_{k,l} q_{k,l}c_k p_{k,l} + \sum_{k,m} \mathrm{Tr}(t_{k,m}c_k)s_{k,m} \in J$$

since J is closed under Tr. □

Proposition 5.24. *There exists a* $\mathbf{C}$*-algebra homomorphism* $\psi'' \colon M_n(\mathcal{R}_n(\Sigma))^{\mathrm{GL}_n(\mathbf{C})} \to \mathcal{D}_n(\Sigma, x_0)$ *satisfying*

$$\begin{aligned} \psi''(\rho_\Sigma(g_\lambda)) &= E_{g_\lambda}, \\ \psi''(\mathrm{Tr}(\rho_\Sigma(g_{\lambda_1}\cdots g_{\lambda_k}))) &= EL_{g_{\lambda_1}\cdots g_{\lambda_k}}. \end{aligned}$$

Proof. Using (5.19) and Proposition 5.22 and applying the preceding proposition to $J = \mathrm{Ker}\,\psi'$ which is closed under Tr by diagram (5.18), we obtain

$$\begin{aligned} \mathrm{Ker}\, M_n(q)_| &= \mathrm{Ker}\, M_n(q) \cap M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})} \\ &= M_n(\mathrm{Ker}\, q) \cap M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})} \\ &= M_n(I(P)) \cap M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})} \\ &= \langle d_\lambda \det A_\lambda - 1, M_\mu - 1 \mid \lambda \in \Lambda, \mu \in M \rangle \cap M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})} \\ &\subseteq M_n(Q_n(\Lambda))\,\mathrm{Ker}\,\psi' M_n(Q_n(\Lambda)) \cap M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})} \\ &= \mathrm{Ker}\,\psi'. \end{aligned}$$

Consequently, $\psi' \colon M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})} \to \mathcal{D}_n(\Sigma, x_0)$ descends to $M_n(\mathcal{R}_n(\Sigma))^{\mathrm{GL}_n(\mathbf{C})}$ via the map $M_n(q)_|$. The result now follows from Corollary 5.21 and the definition (5.3) of ρ_Σ. □

At last,

Proof (Theorem 5.15). According to Theorem 5.16, $M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})}$ is generated by the elements

$$A_\lambda,\ \mathrm{Tr}(A_{\lambda_1}\cdots A_{\lambda_k}),\ d_\lambda, \quad \lambda, \lambda_1, \ldots, \lambda_k \in \Lambda.$$

Using the surjection $M_n(q)_| \colon M_n(Q_n(\Lambda))^{\mathrm{GL}_n(\mathbf{C})} \to M_n(\mathcal{R}_n(\Sigma))^{\mathrm{GL}_n(\mathbf{C})}$, we conclude that $M_n(\mathcal{R}_n(\Sigma))^{\mathrm{GL}_n(\mathbf{C})}$ is generated by

$$\rho_\Sigma(g_\lambda),\ \mathrm{Tr}(\rho_\Sigma(g_{\lambda_1}\cdots g_{\lambda_k})),\ q(d_\lambda), \quad \lambda, \lambda_1, \ldots, \lambda_k \in \Lambda.$$

By assumption the inverse of any generator g_λ in the presentation of $\pi_1(\Sigma)$ is also a generator. On one hand, this fact allows us to pick $\bar{\lambda} \in \Lambda$ such that $g_{\bar{\lambda}} = g_\lambda^{-1}$ and deduce

$$\begin{aligned} q(d_\lambda) &= \det(M_n(q)(A_\lambda))^{-1} = \det \rho_\Sigma(g_\lambda)^{-1} = \det \rho_\Sigma(g_{\bar{\lambda}}) \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma)\,\mathrm{Tr}(\rho_\Sigma(g_{\bar{\lambda}})^{c_1}) \cdots \mathrm{Tr}(\rho_\Sigma(g_{\bar{\lambda}})^{c_k}) \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma)\,\mathrm{Tr}(\rho_\Sigma(g_{\bar{\lambda}}^{c_1})) \cdots \mathrm{Tr}(\rho_\Sigma(g_{\bar{\lambda}}^{c_k})) \end{aligned}$$

so that $q(d_\lambda)$ may be omitted from the list of generators of $M_n(\mathcal{R}_n(\Sigma))^{\mathrm{GL}_n(\mathbf{C})}$. On the other hand, it implies that Proposition 5.12 can be strengthened to say that $\mathcal{D}_n(\Sigma, x_0)$ is generated by E_{g_λ}, $\lambda \in \Lambda$ and $EL_{g_{\lambda_1} \cdots g_{\lambda_k}}$, $\lambda_1, \ldots, \lambda_k \in \Lambda$. Therefore Theorem 5.13 and Proposition 5.24 imply that Θ and ψ'' are inverse maps since this need only be verified on generators. As Θ is an isomorphism, the left (right) hand diagram of (5.10) implies that θ is a monomorphism (epimorphism) thereby completing the proof. □

5.1.7 Returning to the Moduli Space

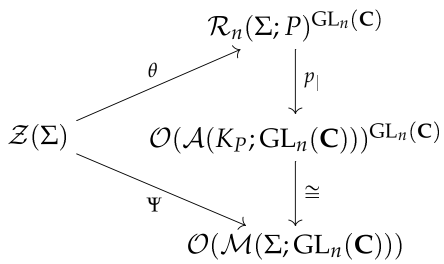
Theorem 5.25. *The map* $\Psi \colon \mathcal{Z}(\Sigma) \to \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{GL}_n(\mathbf{C})))$ *induces a Poisson isomorphism*

$$\Psi \colon \mathcal{Z}_{1,0}(\Sigma) \Big/ \sqrt{\mathcal{I}_n(\Sigma)} \to \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{GL}_n(\mathbf{C}))).$$

Proof. It is a simple matter to verify that the composite bijection (cf. (2.2) and (2.6))

$$\mathcal{O}(\mathcal{A}(K_P; \mathrm{GL}_n(\mathbf{C})))^{\mathrm{GL}_n(\mathbf{C})} \xrightarrow{(\pi^*)^{-1}} \mathcal{O}(\mathcal{M}(K_P; \mathrm{GL}_n(\mathbf{C}))) \xrightarrow{\mathrm{Hol}^* \circ \mathrm{Ev}_P^*} \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{GL}_n(\mathbf{C})))$$

fits into a commutative diagram



Since $\operatorname{Ker} p_| = \sqrt{0}$ by Remark 5.10, it follows from Theorem 5.15 that Ψ is a surjective map with kernel $\sqrt{\mathcal{I}_n(\Sigma)}$. □

Corollary 5.26. *If* $\partial\Sigma \neq \emptyset$, *then*

$$\Psi \colon \mathcal{Z}_{1,0}(\Sigma) / \mathcal{I}_n(\Sigma) \to \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{GL}_n(\mathbf{C})))$$

is a Poisson isomorphism.

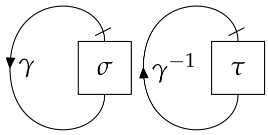
Proof. The group $\pi_1(\Sigma)$ is free so we may take its presentation P to be $\langle g_1, \ldots, g_N \rangle$. Then

$$I(P) = \langle d_\lambda \det A_\lambda - 1 \mid \lambda = 1, \ldots, N \rangle \subseteq Q_n(\Lambda).$$

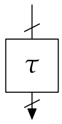
It is a standard fact that $I(P)$ is its own radical ideal so that $\mathcal{R}_n(\Sigma; P) = Q_n(\Lambda)/I(P)$ contains no nilpotents. □

This corollary enables us to prove the main theorems of this section:

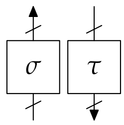
Proof (Theorem 5.1). By Corollary 5.26 the Poisson algebra $\mathcal{O}(\mathcal{M}(\Sigma; \mathrm{GL}_n(\mathbf{C})))$ can be regarded as the quotient of $\mathcal{C}(\Sigma)$ by the loop relation (5.1) and the relations (5.8). Thus we must verify that $*_P$ preserves these relations. Due to Theorem 4.10 we can concentrate on the only non-local relation, namely (5.8c). It will be convenient to write this relation as

$$\sum_{\sigma,\tau\in S_n} \epsilon(\sigma)\epsilon(\tau)$$

$$= (n!)^2$$

where we use the convention


$$= (B_\tau \text{ rotated } 180^\circ).$$

Now consider an instance of the relation; we have an embedded square $S \subseteq \Sigma$, a curve $\gamma \in \mathcal{C}^{(0)}(\Sigma - S; \uparrow, \uparrow)$ (The $\uparrow$'s should be thought of as having infinitesimal length; they denote oriented points on the boundary of $\Sigma - S$) and the tangle

$$L = \sum_{\sigma,\tau\in S_n} \epsilon(\sigma)\epsilon(\tau)$$

$$\in \mathcal{C}^{(0)}(S; \uparrow \cdots \uparrow\downarrow \cdots \downarrow, \uparrow \cdots \uparrow\downarrow \cdots \downarrow).$$

The instance then has the form

$$D(\Delta\gamma \cdot L) - (n!)^2 D$$

where D is some chord diagram on Σ with m_1 chords and Δ is the cabling operation $\Delta^{\uparrow\cdots\uparrow\downarrow\cdots\downarrow}$ acting on both chord tangles and non-associative tangles in $\Sigma - S$ (we use an arbitrary, but fixed parenthesization on $\uparrow \cdots \uparrow\downarrow \cdots \downarrow$). Now let E be a chord diagram on Σ with m_2 chords. We may push D away from S so that $D \in \mathcal{C}^{(m_1)}(\Sigma - S)$ and then apply the standard situation with

$$T = D\Delta\gamma \in \mathcal{C}^{(m_1)}(\Sigma - S; \uparrow \cdots \uparrow\downarrow \cdots \downarrow, \uparrow \cdots \uparrow\downarrow \cdots \downarrow)$$

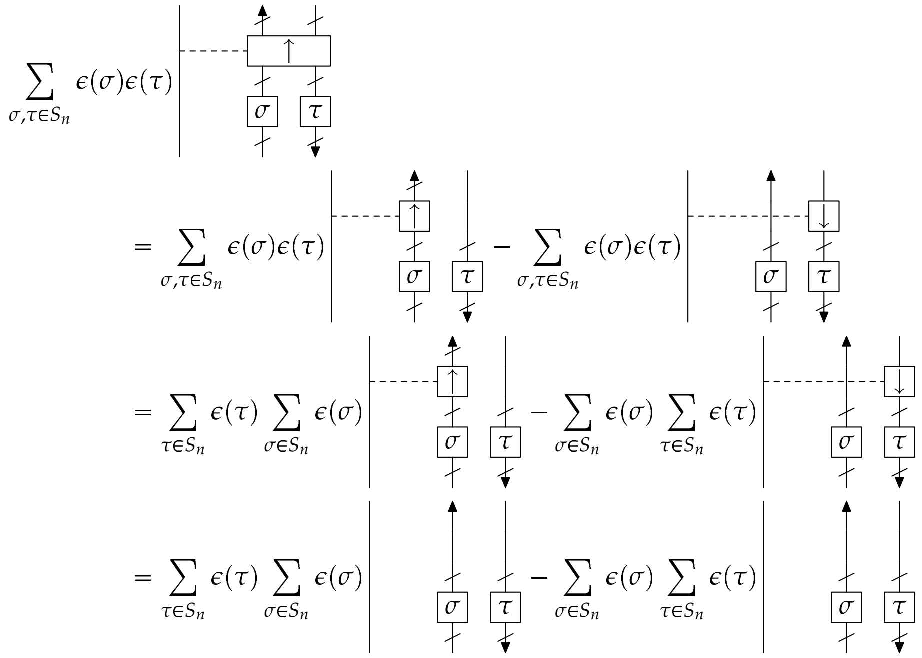
to derive

$$\begin{aligned} c_r(D(\Delta\gamma \cdot L), E) &= c_r((D\Delta\gamma) \cdot L, E) \\ &= \big(V_P\big(V_P^{-1}(D\Delta\gamma)V_P^{-1}(E)\big)\big)^{(m_1+m_2+r)} \cdot L \\ &= \Delta\big(\big(V_P\big(V_P^{-1}(D\gamma)V_P^{-1}(E)\big)\big)^{(m_1+m_2+r)}\big) \cdot L. \end{aligned}$$

Recall that (cf. Remark 4.8)

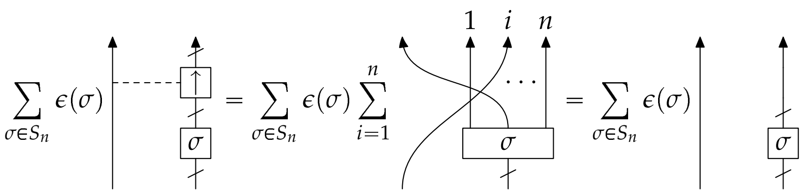
$$X = \big(V_P\big(V_P^{-1}(D\gamma)V_P^{-1}(E)\big)\big)^{(m_1+m_2+r)} \in \mathcal{C}(\Sigma - S; \uparrow, \uparrow)$$

is a linear combination of chord tangles obtained from $(D\gamma)E$ by suitably adding r chords. We distinguish two kinds of terms in X. A term for which at least one chord interescts γ contributes to $\Delta(X) \cdot L$ an element of the following form (the notation ---$\boxed{\uparrow}$ is a shorthand for the cabling summation; the orientation of the original strand (needed to determine the signs) is indicated by ↑)



$= 0.$

The penultimate equality is justified by relations (5.1) and (5.8b):



The cases with other orientations are analogous. In conclusion, terms of the first kind may be ignored. For a term t in X with no chord intersecting γ we may apply the relation (5.8c) to $\Delta(t) \cdot L$ to remove L and the copies of γ and γ^{-1} at the cost of a factor $(n!)^2$. Recalling Remark 4.1 about the cabling operation ϵ_γ, we can continue our computation:

$$\begin{aligned}
c_r(D(\Delta\gamma \cdot L), E) &= \Delta\big(\big(V_P\big(V_P^{-1}(D\gamma)V_P^{-1}(E)\big)\big)^{(m_1+m_2+r)}\big) \cdot L \\
&= (n!)^2 \epsilon_\gamma\big(\big(V_P\big(V_P^{-1}(D\gamma)V_P^{-1}(E)\big)\big)^{(m_1+m_2+r)}\big) \\
&= (n!)^2 \big(V_P\big(V_P^{-1}\epsilon_\gamma(D\gamma)V_P^{-1}(E)\big)\big)^{(m_1+m_2+r)} \\
&= (n!)^2 \big(V_P\big(V_P^{-1}(D)V_P^{-1}(E)\big)\big)^{(m_1+m_2+r)} \\
&= c_r((n!)^2 D, E)
\end{aligned}$$

as desired. Analogously $c_r(E, D(\Delta\gamma \cdot L)) = c_r(E, (n!)^2 D)$ completing the proof by the criterion (3.13). □

Proof (Theorem 5.2). This is analogous to the proof of Theorem 5.1. By Theorem 4.11 we need only consider the relation (5.8c). With set-up and notation as above we may apply the standard situation for equivalences to get

$$\begin{aligned}\tau_r((D\Delta\gamma)\cdot L) &= (V_{P_2}V_{P_1}^{-1}(D\Delta\gamma))^{(m_1+r)}\cdot L\\ &= \Delta((V_{P_2}V_{P_1}^{-1}(D\gamma))^{(m_1+r)})\cdot L\\ &= (n!)^2\epsilon_\gamma((V_{P_2}V_{P_1}^{-1}(D\gamma))^{(m_1+r)})\\ &= (n!)^2(V_{P_2}V_{P_1}^{-1}(D))^{(m_1+r)}\\ &= \tau_r((n!)^2D)\end{aligned}$$

as required in (3.16). □

5.2 The Special Linear Case

We consider the group $G = \mathrm{SL}_n(\mathbf{C})$ with Lie algebra $\mathfrak{g} = \mathfrak{sl}_n(\mathbf{C}) \subseteq M_n(\mathbf{C})$ consisting of the traceless matrices. As in the general linear case, we let $t \in \mathfrak{g}\otimes\mathfrak{g}$ be given by the orthogonal structure

$$B(X,Y) = \mathrm{Tr}(XY), \quad X, Y \in \mathfrak{sl}_n(\mathbf{C}).$$

Colouring all core components of chord diagrams on Σ with the standard representation $\iota\colon \mathrm{SL}_n(\mathbf{C}) \to \mathrm{GL}_n(\mathbf{C})$, we have a Poisson homomorphism

$$\Psi = \Psi_t\colon \mathcal{C}(\Sigma) \to \mathcal{O}(\mathcal{M}(\Sigma;\mathrm{SL}_n(\mathbf{C}))).$$

As announced the main results are:

Theorem 5.27. *Assume that $\partial\Sigma \neq \varnothing$. For any partition P of Σ the $*$-product $*_P$ on $\mathcal{C}(\Sigma)$ descends via Ψ to a $*$-product on $\mathcal{O}(\mathcal{M}(\Sigma;\mathrm{SL}_n(\mathbf{C})))$.*

Theorem 5.28. *Assume that $\partial\Sigma \neq \varnothing$. Let P_1 and P_2 be two partitions of Σ. The canonical equivalence from $*_{P_1}$ to $*_{P_2}$ on $\mathcal{C}(\Sigma)$ descends via Ψ to $\mathcal{O}(\mathcal{M}(\Sigma;\mathrm{SL}_n(\mathbf{C})))$ to yield an equivalence between the induced $*$-products.*

Remark 5.29. Theorem 5.27 was stated in [AMR2] with justification analogous to the one supplied in the general linear case, cf. Theorem 5.1 and Remark 5.3.

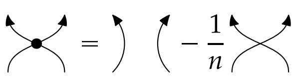
From the general linear case we recall that the matrices $B^+_{i,j}, B^-_{i,j} \in \mathfrak{sl}_n(\mathbf{C})$, $1 \leqslant i < j \leqslant n$ are orthogonal to each other. Adding the diagonal matrices

$$E_{i,i} - \frac{1}{n-i}\sum_{j=i+1}^{n} E_{j,j}, \quad i = 1,\dots,n-1,$$

we obtain an orthogonal basis for $\mathfrak{sl}_n(\mathbf{C})$. An elementary computation yields

$$(\iota\otimes\iota)(t) = \sum_{i,j} E_{i,j}\otimes E_{j,i} - \frac{1}{n}\sum_{i,j} E_{i,i}\otimes E_{j,j} \in \mathrm{End}(\mathbf{C}^n)\otimes\mathrm{End}(\mathbf{C}^n).$$

Of course, $\sum_{i,j} E_{i,i}\otimes E_{j,j} = \mathrm{Id}\otimes\mathrm{Id}$ so that Ψ satisfies the relation

 (5.20)

that is, the loop relation (2.21) with parameters $(s, f) = (1, -\frac{1}{n})$. Thus we derive a commutative triangle of Poisson homomorphisms

$$\begin{array}{ccc} \mathcal{C}(\Sigma) & & \\ \downarrow R_{1,-\frac{1}{n}} & \searrow^{\Psi} & \\ \mathcal{Z}_{1,-\frac{1}{n}}(\Sigma) & \xrightarrow{\Psi} & \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_n(\mathbf{C}))) \end{array}$$

5.2.1 Translating Sikora's Result

Following [S], there is a commutative complex algebra $\mathcal{R}(\Sigma; \mathrm{SL}_n(\mathbf{C}))$ admitting the universal SL_n-representation of $\pi_1(\Sigma)$:

$$\rho_{\Sigma,\mathrm{SL}_n} \colon \pi_1(\Sigma) \to \mathrm{SL}_n(\mathcal{R}(\Sigma; \mathrm{SL}_n(\mathbf{C}))).$$

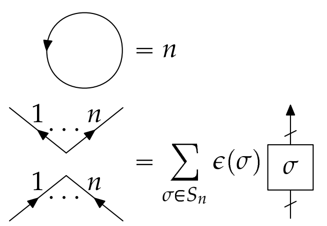
The $\mathrm{GL}_n(\mathbf{C})$-action on $\mathcal{R}(\Sigma; \mathrm{SL}_n(\mathbf{C}))$ (analogous to the general linear case) is defined essentially by conjugation and therefore restricts to an $\mathrm{SL}_n(\mathbf{C})$-action with the same orbits. In particular, the algebras of fixed points are equal:

$$\mathcal{R}(\Sigma; \mathrm{SL}_n(\mathbf{C}))^{\mathrm{SL}_n(\mathbf{C})} = \mathcal{R}(\Sigma; \mathrm{SL}_n(\mathbf{C}))^{\mathrm{GL}_n(\mathbf{C})}.$$

Recall that in our terminology a graph is a finite, 1-dimensional CW-complex with an orientation on each 1-cell. An *n-graph D* is a finite collection of oriented circles together with a graph each vertex of which is either an n-valent source or an n-valent sink, endowed with a numbering from 1 to n of its incident edges. An *n-graph on* Σ is a homotopy class of continuous maps $D \to \Sigma$. Let $\mathcal{G}_n(\Sigma)$ be the complex vector space freely generated by the set of n-graphs on Σ. Two n-graphs (on Σ) can be multiplied by taking their union; in this way $\mathcal{G}_n(\Sigma)$ becomes a commutative $\mathbf{C}$-algebra with the empty n-graph as unit. A diagram on Σ can be considered as an n-graph on Σ; this induces an injective homomorphism of algebras:

$$\iota \colon \mathcal{Z}(\Sigma) \to \mathcal{G}_n(\Sigma).$$

Let $\mathcal{J}_n(\Sigma) \subseteq \mathcal{G}_n(\Sigma)$ be the subspace generated by the local relations



(5.21a)

(5.21b)

Clearly, $\mathcal{J}_n(\Sigma)$ is an ideal.

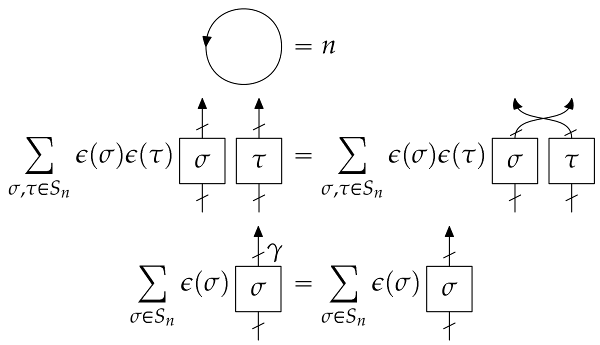
Theorem 5.30 (Sikora). *There exists an isomorphism of complex algebras*

$$\theta \colon \mathcal{G}_n(\Sigma)/\mathcal{J}_n(\Sigma) \to \mathcal{R}(\Sigma; \mathrm{SL}_n(\mathbf{C}))^{\mathrm{SL}_n(\mathbf{C})}$$

uniquely determined by

$$\theta(L_\gamma) = \mathrm{Tr}(\rho_{\Sigma,\mathrm{SL}_n}(\gamma)), \quad \gamma \in \pi_1(\Sigma).$$

Let $\mathcal{I}_n(\Sigma) = \mathcal{I}(\Sigma; \mathrm{SL}_n(\mathbf{C})) \subseteq \mathcal{Z}(\Sigma)$ be the ideal generated by the relations

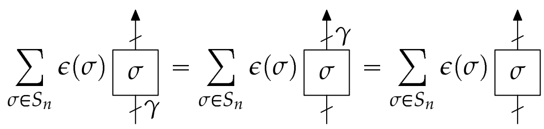


(5.22a)

(5.22b)

(5.22c)

Notice that the third relation is non-local. Using a homotopy, it implies



(5.23)

Proposition 5.31. *The inclusion* $\iota\colon \mathcal{Z}(\Sigma) \to \mathcal{G}_n(\Sigma)$ *induces an isomorphism*

$$\iota\colon \mathcal{Z}(\Sigma)/\mathcal{I}_n(\Sigma) \to \mathcal{G}_n(\Sigma)/\mathcal{J}_n(\Sigma)$$

of complex algebras.

Proof. Since an n-graph must contain an equal number of sources and sinks, it follows from relation (5.21b) that the map

$$\mathcal{Z}(\Sigma) \xrightarrow{\iota} \mathcal{G}_n(\Sigma) \to \mathcal{G}_n(\Sigma)/\mathcal{J}_n(\Sigma)$$

is surjective; it is immediate that its kernel contains $\mathcal{I}_n(\Sigma)$. We now describe the inverse, φ, to the induced map $\iota\colon \mathcal{Z}(\Sigma)/\mathcal{I}_n(\Sigma) \to \mathcal{G}_n(\Sigma)/\mathcal{J}_n(\Sigma)$. Given an n-graph D we choose

- a curve from each vertex of D to the basepoint x_0;
- a bijection σ between the sources and sinks of D.

Then $\varphi(D)$ is constructed by homotoping the vertices of D to x_0 along the chosen curves and using relation (5.21b) on each pair $(i, \sigma(i))$ of a source and a sink to obtain a linear combination of diagrams on Σ. We check that $\varphi(D)$ is well-defined. Independence of σ follows from relation (5.22b). Two different curves α_1 and α_2 from a vertex to x_0 combine to give a loop $\gamma = \alpha_2^{-1}\alpha_1$ at x_0; working with a fixed bijection it is immaterial whether we choose α_1 or α_2 by relation (5.22c) or (5.23).

Next we verify that $\varphi(\mathcal{J}_n(\Sigma)) = 0$. Relation (5.21a) is also a generator of $\mathcal{I}_n(\Sigma)$ and poses therefore no problem. The other relation (5.21b) is also easy since in the computation of φ we can move the source and the sink involved along the same curve to x_0 and then choose σ to match these two vertices. Therefore φ descends to

$$\varphi\colon \mathcal{G}_n(\Sigma)/\mathcal{J}_n(\Sigma) \to \mathcal{Z}(\Sigma)/\mathcal{I}_n(\Sigma)$$

Trivially, $\varphi \circ \iota$ is the identity on $\mathcal{Z}(\Sigma)/\mathcal{I}_n(\Sigma)$ so that ι is also injective. □

Corollary 5.32. *There exists an isomorphism of complex algebras*

$$\theta\colon \mathcal{Z}(\Sigma)/\mathcal{I}_n(\Sigma) \to \mathcal{R}(\Sigma; \mathrm{SL}_n(\mathbf{C}))^{\mathrm{SL}_n(\mathbf{C})}$$

uniquely determined by

$$\theta(L_\gamma) = \mathrm{Tr}(\rho_{\Sigma,\mathrm{SL}_n}(\gamma)), \quad \gamma \in \pi_1(\Sigma). \tag{5.24}$$

Remark 5.33. As is clear from (5.24), the homomorphism $\theta\colon \mathcal{Z}(\Sigma) \to \mathcal{R}(\Sigma; \mathrm{SL}_n(\mathbf{C}))^{\mathrm{SL}_n(\mathbf{C})}$ can be constructed by the procedure, mutatis mutandis, used to define the map $\mathcal{Z}(\Sigma) \to \mathcal{R}(\Sigma; \mathrm{GL}_n(\mathbf{C}))^{\mathrm{GL}_n(\mathbf{C})}$, cf. the proof of Theorem 5.13. It therefore also follows from this proof that elements of the form (5.8b) are contained in $\operatorname{Ker}\theta$ which by the above corollary equals $\mathcal{I}_n(\Sigma)$.

5.2.2 Proofs of the Main Results

Letting $\langle g_\lambda, \lambda \in \Lambda \mid r_\mu, \mu \in M\rangle$ be a presentation P of $\pi_1(\Sigma)$ in which each relation is written as a product of generators results in an explicit model for $\mathcal{R}(\Sigma; \mathrm{SL}_n(\mathbf{C}))$. Namely, consider the polynomial algebra $\mathbf{C}[x^\lambda_{i,j}] = \mathbf{C}[x^\lambda_{i,j} \mid \lambda \in \Lambda,\ i,j = 1,\dots,n]$ and its ideal $I(P; \mathrm{SL}_n(\mathbf{C}))$ generated by $\det A_\lambda - 1$ and all entries in $A_{\lambda_1}\cdots A_{\lambda_k} - 1$ for each relation $r_\mu = g_{\lambda_1}\cdots g_{\lambda_k}$. Then

$$\mathcal{R}(\Sigma; \mathrm{SL}_n(\mathbf{C})) = \mathcal{R}(\Sigma; \mathrm{SL}_n(\mathbf{C}), P) = \mathbf{C}[x^\lambda_{i,j}]/I(P; \mathrm{SL}_n(\mathbf{C})).$$

It is also readily observed that $\mathcal{A}(K_P; \mathrm{SL}_n(\mathbf{C})) \subseteq \mathrm{SL}_n(\mathbf{C})^{|\Lambda|} \subseteq M_n(\mathbf{C})^{|\Lambda|}$ is the vanishing set of $I(P; \mathrm{SL}_n(\mathbf{C})) \subseteq \mathbf{C}[x^\lambda_{i,j}] \cong \mathcal{O}(M_n(\mathbf{C})^{|\Lambda|})$ so that the natural map

$$p\colon \mathcal{R}(\Sigma; \mathrm{SL}_n(\mathbf{C}), P) \to \mathcal{O}(\mathcal{A}(K_P; \mathrm{SL}_n(\mathbf{C})))$$

is an $\mathrm{SL}_n(\mathbf{C})$-equivariant surjection with kernel $\sqrt{0}$. As in the general linear case we obtain a commutative diagram

$$\begin{array}{ccc}
 & & \mathcal{R}(\Sigma; \mathrm{SL}_n(\mathbf{C}), P)^{\mathrm{SL}_n(\mathbf{C})} \\
 & \overset{\theta}{\nearrow} & \downarrow p_| \\
\mathcal{Z}(\Sigma) & & \mathcal{O}(\mathcal{A}(K_P; \mathrm{SL}_n(\mathbf{C})))^{\mathrm{SL}_n(\mathbf{C})} \\
 & \underset{\Psi}{\searrow} & \downarrow \cong \\
 & & \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_n(\mathbf{C})))
\end{array}$$

proving

Theorem 5.34. *The map* $\Psi\colon \mathcal{Z}(\Sigma) \to \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_n(\mathbf{C})))$ *induces a Poisson isomorphism*

$$\Psi\colon \mathcal{Z}_{1,-\frac{1}{n}}(\Sigma)\Big/\sqrt{\mathcal{I}(\Sigma; \mathrm{SL}_n(\mathbf{C}))} \to \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_n(\mathbf{C})))$$

The vital corollary is also analogous.

Corollary 5.35. *If $\partial\Sigma \neq \varnothing$, then*

$$\Psi\colon \mathcal{Z}_{1,-\frac{1}{n}}(\Sigma)\Big/\mathcal{I}(\Sigma;\mathrm{SL}_n(\mathbf{C})) \to \mathcal{O}(\mathcal{M}(\Sigma;\mathrm{SL}_n(\mathbf{C})))$$

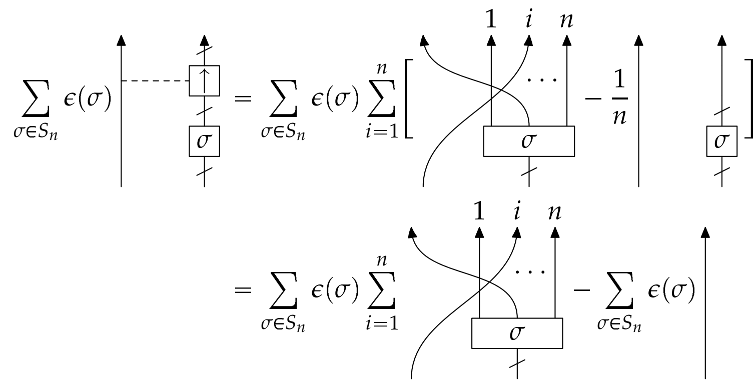
is a Poisson isomorphism

Proof. The group $\pi_1(\Sigma)$ is free so we may take its presentation P to be $\langle g_1, \ldots, g_N\rangle$. Then

$$I(P;\mathrm{SL}_n(\mathbf{C})) = \langle \det A_\lambda - 1 \mid \lambda = 1, \ldots, N\rangle \subseteq \mathbf{C}[x^\lambda_{i,j}].$$

It is a standard fact that $I(P;\mathrm{SL}_n(\mathbf{C}))$ is its own radical ideal so that $p\colon \mathcal{R}(\Sigma;\mathrm{SL}_n(\mathbf{C}), P) \to \mathcal{O}(\mathcal{A}(K_P;\mathrm{SL}_n(\mathbf{C})))$ is an isomorphism. □

Proof (Theorems 5.27 and 5.28). By Corollary 5.35 the Poisson algebra $\mathcal{O}(\mathcal{M}(\Sigma;\mathrm{SL}_n(\mathbf{C})))$ can be regarded as the quotient of $\mathcal{C}(\Sigma)$ by relation (5.20) and relations (5.22). We may focus attention on (5.22c) entirely by Theorems 4.10 and 4.11. This is analogous to the general linear case; one applies the standard situations in the natural way, the crucial observation being that chords intersecting the curve γ on the left hand side of (5.22c) may be ignored:

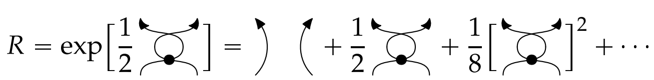


$= 0$

because of relations (5.20) and (5.8b), cf. Remark 5.33. □

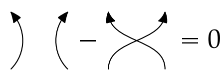
5.3 Miscellaneous Results

In this section Σ is required to have non-empty boundary; we shall employ the description of $\mathcal{O}(\mathcal{M}(\Sigma;\mathrm{GL}_n(\mathbf{C})))$ given in Corollary 5.26. To be able to do some calculations with the AMR $*$-products, we assume, now and for the remainder of this thesis, that the R-matrix used to define universal Vassiliev invariants of links on Σ is given by



(5.25)

Consider the case $G = \mathrm{GL}_1(\mathbf{C})$. The relation (5.8b) reduces to



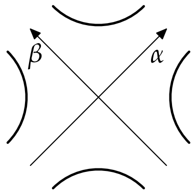


Figure 5.3: Two loops on the punctured torus.

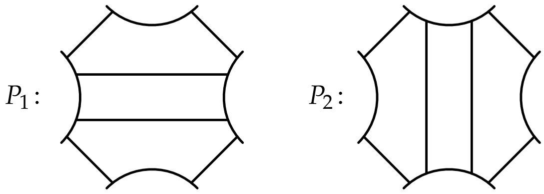
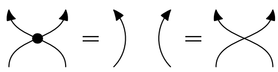
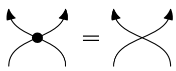


Figure 5.4: Two partitions of the punctured torus.

and combining this with the loop relation (5.1) we obtain



Call an associator Φ *regular* if

$\Phi \equiv \uparrow\uparrow\uparrow \quad \text{mod}$ 

For example, the associator given in [LM] is easily seen to be regular. The following theorem can be found in [A].

Theorem 5.36 (Andersen). *Let P be a partition of Σ. If the $*$-product $*_P$ on $\mathcal{O}(\mathcal{M}(\Sigma; \mathrm{GL}_1(\mathbf{C})))$ is defined in terms of a regular associator, then*

$$f_D *_P f_E = \exp\Big(\frac{1}{2}\sum_{p\in D\# E} \epsilon(p; D, E)h\Big) f_D f_E$$

for diagrams D and E on Σ.

Corollary 5.37. *The $*$-product $*_P$ on $\mathcal{O}(\mathcal{M}(\Sigma; \mathrm{GL}_1(\mathbf{C})))$ is independent of P, and it is invariant under orientation preserving diffeomorphisms of Σ.*

Proof. For the second assertion, recall the equivariance statement of Theorem 2.22. □

This corollary does not generalize to the case $n > 1$ as we now demonstrate. Fix Σ to be the punctured torus; think of Σ as a disk D with two handles appropriately attached. Let α and β be loops running along these handles. Figure 5.3 is an illustration of the D part of Σ and the loops. Consider the two partitions of Σ depicted in Figure 5.4; for the present purpose it is insignificant how bottoms and tops of the polygons are chosen. Denote by c_r^i the coefficients of $*_{P_i}$ on $\mathcal{O}(\mathcal{M}(\Sigma; \mathrm{GL}_n(\mathbf{C})))$.

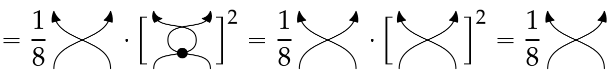
Claim. If $n > 1$, then $c_2^1(f_\alpha, f_\beta) \neq c_2^2(f_\alpha, f_\beta)$.

Proof. We first notice that α and β are nicely compatible with the partitions in the sense that

$$V_{P_i}^{-1}(\alpha) = \alpha, \quad V_{P_i}^{-1}(\beta) = \beta, \quad i = 1, 2$$

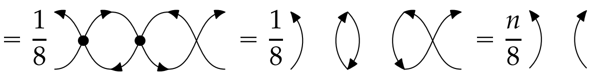
where we regard the loops as link diagrams on the right hand sides of these equations. Thus, to compute $f_\alpha *_{P_i} f_\beta = f_{\alpha *_{P_i} \beta}$, we should apply the universal Vassiliev invariant V_{P_i} to the link L obtained by stacking α on top of β, cf. Theorem 4.5. This is simple since we only need to apply (4.4a) to the crossing in L, remembering, of course, that this must be done in accordance with the partition P_i. Depicting only the disk D, we get

$$c_2^1(\alpha, \beta) =$$



whereas

$$c_2^2(\alpha, \beta) =$$



We used relations (5.1) and (5.8a). Since in general $\operatorname{Tr} A \operatorname{Tr} B \neq n \operatorname{Tr}(AB)$, $A, B \in \mathrm{GL}_n(\mathbf{C})$, the claim is proved. □

It follows, of course, that $*_{P_1} \neq *_{P_2}$ for $n > 1$. There is an obvious orientation preserving diffeomorphism g of Σ such that $g(P_1) = P_2$. By Propositions 4.9 and, tacitly, 3.15, we have

$$g \cdot *_{P_1} = *_{g(P_1)} = *_{P_2} \neq *_{P_1}$$

so that $*_{P_1}$ is not diffeomorphism invariant.

Remark 5.38. The same example leads to the same conclusions in the special linear case.

Chapter 6

Quantization of the Loop Algebras

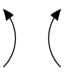
By the results in the previous chapter, the $*$-products $*_P$, $P \in \mathcal{P}(\Sigma)$ ($\partial\Sigma \neq \emptyset$) on the loop algebras $\mathcal{Z}_{1,0}(\Sigma)$ and $\mathcal{Z}_{1,-\frac{1}{n}}(\Sigma)$ are not independent of P (they induce different $*$-products on $\mathcal{O}(\mathcal{M}(\Sigma;\mathrm{GL}_n(\mathbf{C})))$, respectively $\mathcal{O}(\mathcal{M}(\Sigma;\mathrm{SL}_n(\mathbf{C})))$). Motivated by this fact we define in this chapter a canonical deformation quantization of the loop algebra $\mathcal{Z}_{s,f}(\Sigma)$ using an approach due to Turaev [T]; we emphasize that the construction works also when Σ is a closed surface. In the case $\partial\Sigma \neq \emptyset$ we demonstrate, under mild restrictions on the parameters (s, f), that the quantization obtained is equivalent to each of the AMR $*$-products on $\mathcal{Z}_{s,f}(\Sigma)$.

6.1 The Turaev-Vassiliev Quantization

Let $\mathcal{L}_h(\Sigma)$ be the free $\mathbf{C}[[h]]$-module with basis the set of framed, oriented links in $\Sigma \times I$, that is $\mathcal{L}_h(\Sigma) = \mathcal{L}(\Sigma) \otimes \mathbf{C}[[h]]$. Of course, the Vassiliev filtration

$$\mathcal{L}_h(\Sigma) = \mathcal{L}_h^V(\Sigma)_0 \supseteq \mathcal{L}_h^V(\Sigma)_1 \supseteq \cdots$$

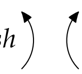
and the compatible stack multiplication turn $\mathcal{L}_h(\Sigma)$ into a filtered $\mathbf{C}[[h]]$-algebra. The projection $\Sigma \times I \to \Sigma$ induces a homomorphism of $\mathbf{C}[[h]]$-modules $p\colon \mathcal{L}_h(\Sigma) \to \mathcal{Z}_{s,f}(\Sigma)$; recall that $\mathcal{Z}_{s,f}(\Sigma)$ is a $\mathbf{C}[[h]]$-algebra via the augmentation $\epsilon\colon \mathbf{C}[[h]] \to \mathbf{C}$. Clearly p preserves multiplication and is therefore a $\mathbf{C}[[h]]$-algebra homomorphism. Let $\mathcal{I}_{s,f}(\Sigma) \subseteq \mathcal{L}_h(\Sigma)$ be the subspace generated by the skein relation

$\exp(-\frac{f}{2}h)$  $-\exp(\frac{f}{2}h)$  $= 2\sinh(\frac{s}{2}h)$  (6.1)

As usual $I_{s,f}(\Sigma)$ is an ideal, and the quotient

$$\mathcal{A}_{s,f}(\Sigma) = {}^{\mathcal{L}_h(\Sigma)}/_{\mathcal{I}_{s,f}(\Sigma)}$$

is called the *Turaev-Vassiliev skein algebra of* $\mathcal{Z}_{s,f}(\Sigma)$. Modulo h^2 the relation (6.1) takes the form

$(1-\frac{f}{2}h)$  $-(1+\frac{f}{2}h)$  $= sh$  (6.2)

Recalling the definition of the Vassiliev filtration (4.1), this implies the important inequality

$$\mathcal{L}_h^V(\Sigma)_n \subseteq \mathcal{I}_{s,f}(\Sigma) + h^n\mathcal{L}_h(\Sigma). \tag{6.3}$$

It also follows from (6.2) that $\mathcal{I}_{s,f}(\Sigma) \subseteq \operatorname{Ker} p$, and we write

$$p_{s,f}\colon \mathcal{A}_{s,f}(\Sigma) \to \mathcal{Z}_{s,f}(\Sigma)$$

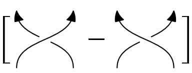
for the induced map. The next result is essentially due to Turaev [T].

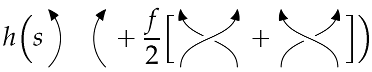
Theorem 6.1. *The pair $(\mathcal{A}_{s,f}(\Sigma), p_{s,f})$ is a deformation quantization of $\mathcal{Z}_{s,f}(\Sigma)$.*

Proof. Obviously, $p_{s,f}$ is surjective. To prove the defining equation (3.5) of a quantization, let L and L' be two links on Σ in general position. Put $D = p(L)$ and $D' = p(L')$. Since $L'L$ can be obtained from LL' by moving L 'down through' L', we derive the telescoping sum

$$LL' - L'L = \sum_{q\in D\#D'} L_1^q - L_2^q$$

where L_1^q and L_2^q are links differing only by a crossing change at q and satisfying $p(L_i^q) = DD'$, $i = 1, 2$. Focusing attention on the point q, we get

$$L_1^q - L_2^q = \epsilon(q;D,D')$$ 

$$\equiv \epsilon(q;D,D')h$$ 

$$= \epsilon(q;D,D')h\big(sL^q + \frac{f}{2}(L_1^q + L_2^q)\big) \quad \operatorname{mod} h^2\mathcal{A}_{s,f}(\Sigma).$$

Here L^q is the link obtained from either of L_i^q by smoothing the crossing at q so that $p(L^q) = (DD')_q$. Collecting the terms,

$$LL' - L'L = h \sum_{q\in D\#D'} \epsilon(q;D,D')\big(sL^q + \frac{f}{2}(L_1^q + L_2^q)\big) \quad \operatorname{mod} h \operatorname{Ker} p_{s,f}$$

since $h\mathcal{A}_{s,f}(\Sigma) \subseteq \operatorname{Ker} p_{s,f}$. But by (2.23),

$$\begin{aligned} p\Big(\sum_{q\in D\#D'} \epsilon(q;D,D')\big(sL^q + \frac{f}{2}(L_1^q + L_2^q)\big)\Big) &= \sum_{q\in D\#D'} \epsilon(q;D,D')(s(DD')_q + fDD') \\ &= \{D, D'\}_{s,f} \\ &= \{p(L), p(L')\}_{s,f}. \end{aligned}$$

This completes the proof. □

Remark 6.2. We have used only the property (6.2) of the skein relation (6.1) in the construction of the quantization. Thus, varying the skein relation subject to (6.2) and/or working with unframed links leads to other quantizations of $\mathcal{Z}_{s,f}(\Sigma)$, cf. [T].

By Theorem 3.16 and Proposition 3.17 we may complete $\mathcal{A}_{s,f}(\Sigma)$ with respect to the h-filtration to obtain another deformation quantization $(\overline{\mathcal{A}_{s,f}(\Sigma)}, \overline{p_{s,f}})$ of $\mathcal{Z}_{s,f}(\Sigma)$. Applying Proposition 3.5 to the case $A = \mathcal{L}_h(\Sigma)$ with the h-filtration and the ideal $I = \mathcal{I}_{s,f}(\Sigma)$ and recalling Remark 3.4, we infer that $\overline{\mathcal{A}_{s,f}(\Sigma)}$ can be viewed as the completion of $\mathcal{L}_h(\Sigma)$ with respect to the filtration

$$\mathcal{L}_h(\Sigma)_n = \mathcal{I}_{s,f}(\Sigma) + h^n\mathcal{L}_h(\Sigma).$$

For future reference we notice that the Vassiliev completion and the skein algebra completion of $\mathcal{L}_h(\Sigma)$ are related by the map

$$\overline{\mathrm{Id}}\colon \overline{\mathcal{L}_h(\Sigma)} \to \overline{\mathcal{A}_{s,f}(\Sigma)} \tag{6.4}$$

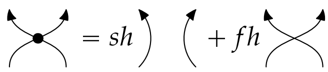
induced by the identity on $\mathcal{L}_h(\Sigma)$, cf. (6.3).

6.2 The AMR $*$-Products and the Turaev-Vassiliev Quantization

In this section we assume that $\partial\Sigma \neq \varnothing$ and that $s \neq \pm f$. We shall prove that the deformation quantization $(\overline{\mathcal{A}_{s,f}(\Sigma)}, \overline{p_{s,f}})$ is equivalent to the $*$-product $*_P$ on $\mathcal{Z}_{s,f}(\Sigma)$ for any partition P of Σ. Let $\mathcal{C}_h(\Sigma)$ denote the algebra of chord diagrams on Σ with ground ring $\mathbf{C}[[h]]$ in place of $\mathbf{C}$. Of course, $\mathcal{C}_h(\Sigma)$ is graded by the number of chords so that the completion $\overline{\mathcal{C}_h(\Sigma)} = \prod_{m=0}^{\infty} \mathcal{C}_h^{(m)}(\Sigma)$ is at our disposal. The construction of the universal Vassiliev invariant of links on Σ carries over verbatim to yield a filtered, $\mathbf{C}[[h]]$-linear map

$$V_P\colon \mathcal{L}_h(\Sigma) \to \overline{\mathcal{C}_h(\Sigma)}.$$

The h-version of the loop relation (2.21) is the $\mathbf{C}[[h]]$-submodule $L^h_{s,f} \subseteq \mathcal{C}_h(\Sigma)$ generated by the local relation



Define a homomorphism of $\mathbf{C}[[h]]$-modules $\eta\colon \mathcal{C}_h(\Sigma) \to \mathcal{C}(\Sigma)[[h]]$ by

$$\eta(D) = Dh^m, \quad D \text{ a chord diagram with } m \text{ chords.}$$

By construction η is filtered with respect to the chord filtration on $\mathcal{C}_h(\Sigma)$ so it extends to a $\mathbf{C}[[h]]$-linear map $\eta\colon \overline{\mathcal{C}_h(\Sigma)} \to \mathcal{C}(\Sigma)[[h]]$. Let

$$q_{s,f} = R_{s,f} \circ \eta\colon \overline{\mathcal{C}_h(\Sigma)} \to \mathcal{C}(\Sigma)[[h]] \to \mathcal{Z}_{s,f}(\Sigma)[[h]] \tag{6.5}$$

be the composite $\mathbf{C}[[h]]$-module homomorphism, cf. (2.22).

Remark 6.3. By construction, $L^h_{s,f} \subseteq \operatorname{Ker} q_{s,f}$.

Put

$$T_P = q_{s,f} \circ V_P\colon \mathcal{L}_h(\Sigma) \to \overline{\mathcal{C}_h(\Sigma)} \to \mathcal{Z}_{s,f}(\Sigma)[[h]]. \tag{6.6}$$

Theorem 6.4. *The map $T_P\colon \mathcal{L}_h(\Sigma) \to \mathcal{Z}_{s,f}(\Sigma)[[h]]$ induces an equivalence of deformation quantizations*

$$\begin{array}{ccc} \overline{\mathcal{A}_{s,f}(\Sigma)} & \xrightarrow{T_P} & \mathcal{Z}_{s,f}(\Sigma)[[h]]_{*_P} \\ & {\scriptstyle \overline{p_{s,f}}} \searrow \quad \swarrow {\scriptstyle \pi_0} & \\ & \mathcal{Z}_{s,f}(\Sigma) & \end{array}$$

for any partition P of Σ.

Theorem 6.5. *For two partitions P_1 and P_2 of Σ, the composite equivalence*

$$T_{P_2} \circ T_{P_1}^{-1} \colon \mathcal{Z}_{s,f}(\Sigma)[[h]]_{*_{P_1}} \to \mathcal{Z}_{s,f}(\Sigma)[[h]]_{*_{P_2}}$$

is equal to the canonical equivalence from $*_{P_1}$ to $*_{P_2}$, cf. Theorem 4.12.

We prove part of Theorem 6.4 in

Lemma 6.6. *The map $T_P \colon \mathcal{L}_h(\Sigma) \to \mathcal{Z}_{s,f}(\Sigma)[[h]]_{*_P}$ is a $\mathbf{C}[[h]]$-algebra homomorphism and induces a morphism of deformation quantizations*

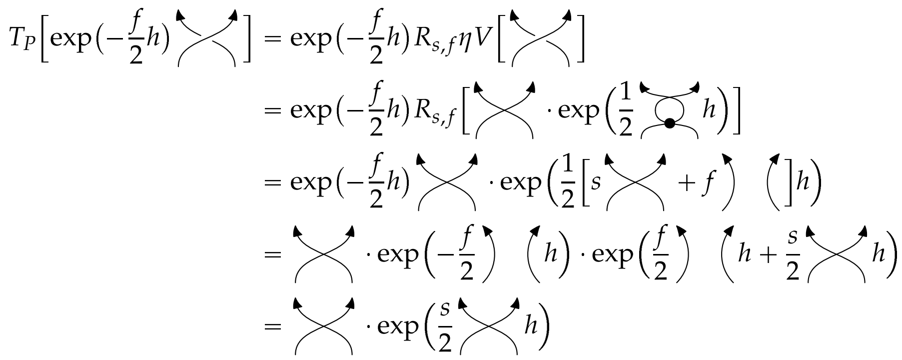
$$\begin{array}{ccc} \overline{\mathcal{A}_{s,f}(\Sigma)} & \xrightarrow{\;T_P\;} & \mathcal{Z}_{s,f}(\Sigma)[[h]]_{*_P} \\ & {\scriptstyle \overline{p_{s,f}}} \searrow \quad \swarrow {\scriptstyle \pi_0} & \\ & \mathcal{Z}_{s,f}(\Sigma) & \end{array} \tag{6.7}$$

for any partition P of Σ.

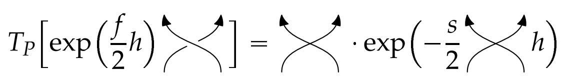
Proof. By construction, T_P is $\mathbf{C}[[h]]$-linear. To verify that the stack multiplication is taken to $*_P$, let L_1, L_2 be links on Σ. Recalling Theorem 4.12, the definition of the product $\circ$ on $\overline{\mathcal{C}(\Sigma)}$ (3.10), and the identity (3.11), we obtain

$$\begin{aligned} T_P(L_1) *_P T_P(L_2) &= R_{s,f}\eta V_P(L_1) *_P R_{s,f}\eta V_P(L_2) \\ &= R_{s,f}(\eta V_P(L_1) *_P \eta V_P(L_2)) \\ &= R_{s,f}\eta(V_P(L_1) \circ V_P(L_2)) \\ &= R_{s,f}\eta V_P(L_1 L_2) \\ &= T_P(L_1 L_2). \end{aligned}$$

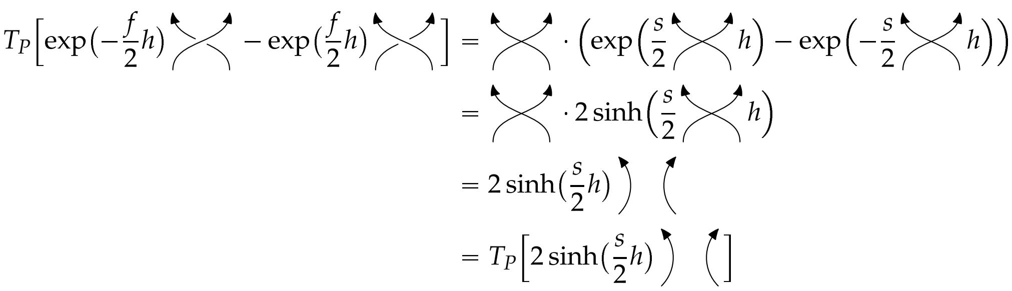
The next step is to prove that T_P descends to the skein algebra, i.e., that $\mathcal{I}_{s,f}(\Sigma) \subseteq \operatorname{Ker} T_P$. By the first part of the proof, $\operatorname{Ker} T_P$ is an ideal so it suffices to consider a generator of $\mathcal{I}_{s,f}(\Sigma)$. As T_P can be computed locally, we simply consider the skein relation (6.1); we have (cf. (5.25))



and analogously



so that



as desired. The induced map $T_P\colon \mathcal{A}_{s,f}(\Sigma) \to \mathcal{Z}_{s,f}(\Sigma)[[h]]$ is $\mathbf{C}[[h]]$-linear and thereby h-filtered whence it can be completed to a homomorphism of (filtered) $\mathbf{C}[[h]]$-algebras $T_P\colon \overline{\mathcal{A}_{s,f}(\Sigma)} \to \mathcal{Z}_{s,f}(\Sigma)[[h]]$. It is immediate from the definitions that the triangle (6.7) commutes. □

The proof of Theorem 6.4 is complete once we establish that $T_P\colon \overline{\mathcal{A}_{s,f}(\Sigma)} \to \mathcal{Z}_{s,f}(\Sigma)[[h]]$ is an isomorphism of $\mathbf{C}[[h]]$-algebras. The strategy is to show that the $\mathbf{C}[[h]]$-linear homomorphism (recall (6.4))

$$F_P = \overline{\mathrm{Id}} \circ V_P^{-1}\colon \overline{\mathcal{C}_h(\Sigma)} \to \overline{\mathcal{L}_h(\Sigma)} \to \overline{\mathcal{A}_{s,f}(\Sigma)}$$

descends via $q_{s,f}\colon \overline{\mathcal{C}_h(\Sigma)} \to \mathcal{Z}_{s,f}(\Sigma)[[h]]$ to yield the inverse of T_P. To this end it will be useful to introduce the *large* filtration (larger than the chord filtration) on $\overline{\mathcal{C}_h(\Sigma)}$; it is given by

$$\overline{\mathcal{C}_h(\Sigma)}_n = \eta^{-1}\big(h^n\mathcal{C}(\Sigma)[[h]]\big) \subseteq \overline{\mathcal{C}_h(\Sigma)}, \quad n \in \mathbf{N}.$$

Remark 6.7. By definition, η and hence $q_{s,f} = R_{s,f} \circ \eta$ are filtered with respect to the large filtration.

Intuitively, the large filtration measures a 'degree' defined in terms of both chords and powers of h; to make this idea precise we introduce $\mathbf{C}$-linear maps $p_n\colon \mathcal{C}_h(\Sigma) \to \mathcal{C}_h(\Sigma) \subseteq \overline{\mathcal{C}_h(\Sigma)}$, $n \in \mathbf{N}$ determined by the formula (D is a chord diagram with m chords)

$$p_n\Big(\big(\sum_i \lambda_i h^i\big)D\Big) = \begin{cases} 0, & m > n \\ \lambda_{n-m}h^{n-m}D, & m \leqslant n \end{cases}$$

Clearly, these maps are independent projections in the sense that

$$p_n \circ p_{n'} = \delta_{n,n'}p_n, \quad n, n' \in \mathbf{N}. \tag{6.8}$$

Elements in the subset $\operatorname{Im} p_n \subseteq \mathcal{C}_h(\Sigma)$ are said to have *chord-h degree n*. Of course, p_n extends to $p_n\colon \overline{\mathcal{C}_h(\Sigma)} \to \mathcal{C}_h(\Sigma) \subseteq \overline{\mathcal{C}_h(\Sigma)}$, and it is a consequence of the definitions that

$$\bigcap_{i=0}^{n-1} \operatorname{Ker} p_i = \overline{\mathcal{C}_h(\Sigma)}_n, \quad n \in \mathbf{N}. \tag{6.9}$$

Remark 6.8. Evidently, the concepts of the large filtration and the chord-h degree make sense locally, that is, they can be defined for chord tangles in an embedded square $S \subseteq \Sigma$. Composition of chord tangles is chord-h graded.

Lemma 6.9. *The map* $q_{s,f}\colon \overline{\mathcal{C}_h(\Sigma)} \to \mathcal{Z}_{s,f}(\Sigma)[[h]]$ *is surjective, and the induced map*

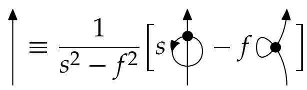
$$q_{s,f}\colon \overline{\mathcal{C}_h(\Sigma)}/\mathrm{Ker}\, q_{s,f} \to \mathcal{Z}_{s,f}(\Sigma)[[h]]$$

is an isomorphism of filtered $\mathbf{C}[[h]]$*-modules when the domain is equipped with the filtration induced from the large filtration.*

Proof. Let $\tilde{z} = \sum_i \tilde{z}_i h^i \in \mathcal{Z}_{s,f}(\Sigma)[[h]]$. Write $\tilde{z}_i = \lambda_i \varnothing + z_i$ where z_i is a complex linear combination of non-empty diagrams on Σ, and put $z = \sum_i z_i h^i$. As

$$q_{s,f}\big(\big(\sum_i \lambda_i h^i\big)\varnothing\big) = \sum_i (\lambda_i \varnothing) h^i = \tilde{z} - z,$$

it suffices to show $z \in \mathrm{Im}\, q_{s,f}$. By repeated application of the relation


mod $L_{s,f}$,

we may find $D_i \in \mathcal{C}^{(i)}(\Sigma)$ such that $R_{s,f}(D_i) = z_i$, i.e.,

$$R_{s,f}\big(\sum_i D_i h^i\big) = \sum_i z_i h^i = z.$$

But $D = (D_i) \in \overline{\mathcal{C}(\Sigma)} \subseteq \overline{\mathcal{C}_h(\Sigma)}$ satisfies $\eta(D) = \sum_i D_i h^i$ so that $q_{s,f}(D) = z$ as desired. The induced isomorphism of $\mathbf{C}[[h]]$-modules

$$q_{s,f}\colon \overline{\mathcal{C}_h(\Sigma)}/\mathrm{Ker}\, q_{s,f} \to \mathcal{Z}_{s,f}(\Sigma)[[h]]$$

is a filtered map. The argument for surjectivity reveals that if $z \in h^n \mathcal{Z}_{s,f}(\Sigma)[[h]]$, then the inverse image $q_{s,f}^{-1}(z)$ contains an element in $\overline{\mathcal{C}_h(\Sigma)}_n$ so that the inverse map $q_{s,f}^{-1}$ is also filtered. □

Lemma 6.10. $\mathrm{Ker}\, q_{s,f} = \{c \in \overline{\mathcal{C}_h(\Sigma)} \mid c \in L^h_{s,f} + \overline{\mathcal{C}_h(\Sigma)}_n,\ n \in \mathbf{N}\}$.

Proof. Let $c \in \overline{\mathcal{C}_h(\Sigma)}$. For the inclusion $\subseteq$ suppose that $c \in \mathrm{Ker}\, q_{s,f}$. Letting $\pi_i\colon \mathcal{Z}_{s,f}(\Sigma)[[h]] \to \mathcal{Z}_{s,f}(\Sigma)$ denote the projection on the ith coefficient, we derive

$$0 = \pi_i q_{s,f}(c) = \pi_i R_{s,f}\eta(c) = R_{s,f}\pi_i\eta(c), \quad i \in \mathbf{N}$$

implying that $\pi_i\eta(c) \in L_{s,f}$; this means that $p_i(c) \in L^h_{s,f}$. Given $n \in \mathbf{N}$, (6.8) and (6.9) yield

$$c - \sum_{i=0}^{n-1} p_i(c) \in \overline{\mathcal{C}_h(\Sigma)}_n$$

so that $c \in L^h_{s,f} + \overline{\mathcal{C}_h(\Sigma)}_n$. For the other inclusion we assume that c is in the right hand set. Since

$$q_{s,f}\big(L^h_{s,f} + \overline{\mathcal{C}_h(\Sigma)}_n\big) \subseteq h^n \mathcal{C}(\Sigma)[[h]], \quad n \in \mathbf{N}$$

by Remarks 6.3 and 6.7, it follows that $q_{s,f}(c) = 0$ as desired. □

Lemma 6.11. *The map $F_P\colon \overline{\mathcal{C}_h(\Sigma)} \to \overline{\mathcal{A}_{s,f}(\Sigma)}$ is filtered with respect to the large filtration.*

Proof. Suppose that $c = (c_i) \in \overline{\mathcal{C}_h(\Sigma)}$ belongs to $\overline{\mathcal{C}_h(\Sigma)}_n$. Since F_P is filtered with respect to the chord filtration on $\overline{\mathcal{C}_h(\Sigma)}$, it is enough to show that

$$F_P\Big(\sum_{i=0}^{n-1} c_i\Big) \in \overline{\mathcal{A}_{s,f}(\Sigma)}_n. \tag{6.10}$$

By the characterization of the large filtration (6.9), each c_i may be written as a $\mathbf{C}[[h]]$-linear combination of elements of the form $h^{n-i}D$ where D is a chord diagram with i chords. Choose $L_j \in \mathcal{L}_h^V(\Sigma)_i$ such that $([L_j]_j) = V_P^{-1}(D) \in \overline{\mathcal{C}_h(\Sigma)}_i$. Since

$$F_P(h^{n-i}D) = h^{n-i}([L_j]_j) = ([h^{n-i}L_j]_j) \in \overline{\mathcal{A}_{s,f}(\Sigma)}$$

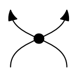
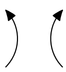
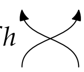
with (cf. (6.3))

$$h^{n-i}L_j \in h^{n-i}\mathcal{L}_h^V(\Sigma)_i \subseteq h^{n-i}(\mathcal{I}_{s,f}(\Sigma) + h^i\mathcal{L}_h(\Sigma)) \subseteq \mathcal{I}_{s,f}(\Sigma) + h^n\mathcal{L}_h(\Sigma) = \mathcal{L}_h(\Sigma)_n,$$

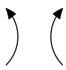
we infer that $F_P(h^{n-i}D) \in \overline{\mathcal{A}_{s,f}(\Sigma)}_n$. This implies (6.10) and thereby the lemma. □

Lemma 6.12. $L^h_{s,f} \subseteq \operatorname{Ker} F_P$.

Proof. Any generator of $L^h_{s,f}$ is the composition of the element

$g =$  $- sh$  $- fh$  $\in \mathcal{C}_h(S;\uparrow\uparrow,\uparrow\uparrow)$

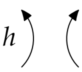
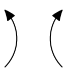
located in some square $S \subseteq \Sigma$, with a suitable chord tangle in $\Sigma - S$. By the compatibility of the universal Vassiliev invariant with this decomposition, we need only consider the square S and show that $F_P(g) = 0$. Define

$X = V\Big[\exp(-\frac{f}{2}h)$  $- \exp(\frac{f}{2}h)$  $- 2\sinh(\frac{s}{2}h)$  $\Big] \in \overline{\mathcal{C}_h(S;\uparrow\uparrow,\uparrow\uparrow)}.$

By definition of $\mathcal{I}_{s,f}(\Sigma)$,

$$F_P(X) = 0. \tag{6.11}$$

Refining the computation in the proof of Lemma 6.6 a little, one establishes

$X = 2\sinh\Big[\frac{1}{2}g + \frac{s}{2}h$  $\Big] - 2\sinh(\frac{s}{2}h)$ 

Since g is homogeneous of chord-h degree 1, it follows that

$$p_n(X) = \begin{cases} 0, & n = 0 \\ g, & n = 1 \\ g \cdot x_n,\ x_n \in \mathcal{C}_h(S;\uparrow\uparrow,\uparrow\uparrow) \text{ has chord-}h\text{ degree } n-1; & n \geqslant 2 \end{cases} \tag{6.12}$$

Applying induction we construct a sequence $Y_i \in \overline{\mathcal{C}_h(S;\uparrow\uparrow,\uparrow\uparrow)}$, $i = 1, 2, \ldots$ of the form

$$Y_i = X \cdot y_i, \quad y_i \in \mathcal{C}_h(S;\uparrow\uparrow,\uparrow\uparrow) \text{ has chord-}h\text{ degree } i-1$$

and satisfying

$$\sum_{i=1}^{n} Y_i - g \in \overline{\mathcal{C}_h(S;\uparrow\uparrow,\uparrow\uparrow)}_{n+1}. \tag{6.13}$$

To initiate the process we set $y_1 = \uparrow\uparrow$ so that $Y_1 = X$; by (6.12) and (6.9) this is sound. Assume that $Y_1, \ldots, Y_n$ have already been defined. Then

$$p_{n+1}\Big(\sum_{i=1}^{n} Y_i\Big) = \sum_{i=1}^{n} p_{n+1}(X \cdot y_i) = \sum_{i=1}^{n} p_{n+2-i}(X) \cdot y_i = \sum_{i=1}^{n} g \cdot x_{n+2-i} \cdot y_i.$$

Thus we are lead to put $y_{n+1} = -\sum_{i=1}^{n} x_{n+2-i} \cdot y_i$, which by induction has chord-h degree n, and thereby obtain

$$p_{n+1}\Big(\sum_{i=1}^{n+1} Y_i\Big) = p_{n+1}\Big(\sum_{i=1}^{n} Y_i\Big) + p_1(X) \cdot y_{n+1} = 0.$$

Taken together with the hypothesis (6.13), this implies

$$p_j\Big(\sum_{i=1}^{n+1} Y_i - g\Big) = 0, \quad j = 0, \ldots, n+1$$

completing the induction step by (6.9). Now, (6.13) means by definition that $\sum_{i=1}^{n} Y_i \to g$, $n \to \infty$ in the large filtration so by (6.11) and Lemma 6.11

$$0 = \sum_{i=1}^{n} F_P(X) \cdot y_i = F_P\Big(\sum_{i=1}^{n} Y_i\Big) \to F_P(g), \quad n \to \infty.$$

Since $\overline{\mathcal{A}_{s,f}(S;\uparrow\uparrow,\uparrow\uparrow)}$ is Hausdorff, this implies $F_P(g) = 0$ as desired. □

Lemma 6.13. *The map $T_P \colon \overline{\mathcal{A}_{s,f}(\Sigma)} \to \mathcal{Z}_{s,f}(\Sigma)[[h]]$ is a $\mathbf{C}[[h]]$-algebra isomorphism.*

Proof. From Lemmas 6.10, 6.11 and 6.12 follow that $\operatorname{Ker} q_{s,f} \subseteq \operatorname{Ker} F_P$. Consequently, Lemma 6.9 yields an induced homomorphism

$$F_P \colon \mathcal{Z}_{s,f}(\Sigma)[[h]] \xrightarrow{\cong} \overline{\mathcal{C}_h(\Sigma)}/\operatorname{Ker} q_{s,f} \longrightarrow \overline{\mathcal{A}_{s,f}(\Sigma)}$$

of filtered $\mathbf{C}[[h]]$-modules. We verify that F_P is the inverse of T_P. From the diagram

$$\begin{array}{ccccccc}
\mathcal{L}_h(\Sigma) & \xrightarrow{V_P} & \overline{\mathcal{C}_h(\Sigma)} & = & \overline{\mathcal{C}_h(\Sigma)} & \xrightarrow{V_P^{-1}} & \overline{\mathcal{L}_h(\Sigma)} \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle q_{s,f}} & & \downarrow & & \downarrow{\scriptstyle \overline{\mathrm{Id}}} \\
\overline{\mathcal{A}_{s,f}(\Sigma)} & \xrightarrow{T_P} & \mathcal{Z}_{s,f}(\Sigma)[[h]] & \xrightarrow{\cong} & \overline{\mathcal{C}_h(\Sigma)}/\operatorname{Ker} q_{s,f} & \xrightarrow{F_P} & \overline{\mathcal{A}_{s,f}(\Sigma)}
\end{array}$$

follows that $F_P \circ T_P \circ \iota = \iota \colon \mathcal{L}_h(\Sigma) \to \overline{\mathcal{A}_{s,f}(\Sigma)}$. Since $F_P \circ T_P$ is a filtered endomorphism of $\overline{\mathcal{A}_{s,f}(\Sigma)}$, this means that $F_P \circ T_P = \mathrm{Id}$. That $T_P \circ F_P$ is the identity on $\mathcal{Z}_{s,f}(\Sigma)[[h]]$ need only be verified on $\mathcal{Z}_{s,f}(\Sigma) \subseteq \mathcal{Z}_{s,f}(\Sigma)[[h]]$ by the $\mathbf{C}[[h]]$-linearity, cf. (3.1). Let D be a diagram on Σ. We may consider D as an element of $\overline{\mathcal{C}_h(\Sigma)}$, and clearly $q_{s,f}(D) = D$. Therefore

$$F_P(D) = \overline{\mathrm{Id}}(V_P^{-1}(D)) \tag{6.14}$$

so that

$$T_P(F_P(D)) = T_P\overline{\mathrm{Id}}V_P^{-1}(D) = q_{s,f}V_P V_P^{-1}(D) = D$$

as desired. □

Proof (Theorem 6.4). Lemmas 6.6 and 6.13. □

Proof (Theorem 6.5). For a diagram D on Σ the canonical equivalence from $*_{P_1}$ to $*_{P_2}$ on $\mathcal{Z}_{s,f}(\Sigma)$ is given by

$$D \mapsto R_{s,f}\tau(D) = R_{s,f}\sum_r (V_{P_2}V_{P_1}^{-1}(D))^{(r)}h^r = R_{s,f}\eta V_{P_2}V_{P_1}^{-1}(D),$$

cf. Theorems 4.6 and 4.12. But (cf. (6.14))

$$T_{P_2}T_{P_1}^{-1}(D) = T_{P_2}\overline{\mathrm{Id}}V_{P_1}^{-1}(D) = q_{s,f}V_{P_2}V_{P_1}^{-1}(D) = R_{s,f}\eta V_{P_2}V_{P_1}^{-1}(D).$$

This completes the proof since by (3.1) it suffices to consider the restrictions of the two maps to $\mathcal{Z}_{s,f}(\Sigma)$. □

Chapter 7

The Case $\mathrm{SL}_2(\mathbf{C})$ Revisited

We present in this chapter a canonical $\Gamma_+(\Sigma)$-invariant $*$-product on $\mathcal{O}(\mathcal{M}(\Sigma;\mathrm{SL}_2(\mathbf{C})))$ due to Bullock, Frohman and Kania-Bartoszyńska [BFK]. This $*$-product is defined on a model for $\mathcal{O}(\mathcal{M}(\Sigma;\mathrm{SL}_2(\mathbf{C})))$ especially suited for the purpose; we develop the model in the first section. Subsequently we prove, in the case $\partial\Sigma \neq \varnothing$, that the BFK $*$-product is canonically equivalent to each of the AMR $*$-products on $\mathcal{O}(\mathcal{M}(\Sigma;\mathrm{SL}_2(\mathbf{C})))$. We end the dissertation with an investigation of the differentiability of the BFK $*$-product.

7.1 A Good Model for the Moduli Space

Using the abbreviations $\mathcal{Z}(\Sigma) = \mathcal{Z}_{1,-\frac{1}{2}}(\Sigma)$ and $\mathcal{I}_2(\Sigma) = \mathcal{I}(\Sigma;\mathrm{SL}_2(\mathbf{C}))$, Theorem 5.34 implies that we have an algebra homomorphism

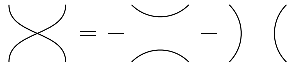
$$\Psi\colon \mathcal{Z}(\Sigma)/\mathcal{I}_2(\Sigma) \to \mathcal{O}(\mathcal{M}(\Sigma;\mathrm{SL}_2(\mathbf{C}))).$$

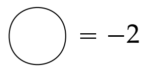
Remark 7.1. We also learn from Theorem 5.34 that Ψ is a surjection with $\operatorname{Ker}\Psi = \sqrt{0}$. Shortly we shall see that $\sqrt{0} = 0$ so that, by the same theorem, Ψ is actually a Poisson isomorphism (we know this already in the case $\partial\Sigma \neq \varnothing$ by Corollary 5.35).

Ignoring the orientation of the loops in diagrams on Σ induces an equivalence relation; the equivalence classes are called *unoriented diagrams on* Σ and we denote by $\mathcal{Z}^\circ(\Sigma)$ the free complex vector space generated by them. Of course, $\mathcal{Z}^\circ(\Sigma)$ is a commutative algebra under union of unoriented diagrams, and the orientation-forgetting map

$$u\colon \mathcal{Z}(\Sigma) \to \mathcal{Z}^\circ(\Sigma), \quad u(D) = D^\circ$$

is an algebra homomorphism. Define $\mathcal{K}_0(\Sigma) \subseteq \mathcal{Z}^\circ(\Sigma)$ to be the subspace generated by the local relations


(7.1a)


(7.1b)

As usual, $\mathcal{K}_0(\Sigma)$ is an ideal. Consider the linear map $\widetilde{u}\colon \mathcal{Z}(\Sigma) \to \mathcal{Z}^\circ(\Sigma)$ given by

$$\widetilde{u}(D) = (-1)^n D^\circ = (-1)^n u(D), \quad D \text{ a diagram with n loops.} \tag{7.2}$$

It is a homomorphism of complex algebras.

Remark 7.2. Unlike many other maps defined on (chord) diagrams, $\widetilde{u}$ cannot be computed locally because of the sign. In expressions below where $\widetilde{u}$ is seemingly evaluated on a tangle, it is understood that this tangle is part of a particular diagram on Σ.

Proposition 7.3. *The map* $\widetilde{u}\colon \mathcal{Z}(\Sigma) \to \mathcal{Z}^\circ(\Sigma)$ *induces an isomorphism*

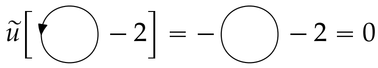
$$\widetilde{u}\colon \mathcal{Z}(\Sigma)/\mathcal{I}_2(\Sigma) \to \mathcal{Z}^\circ(\Sigma)/\mathcal{K}_0(\Sigma) \tag{7.3}$$

of complex algebras.

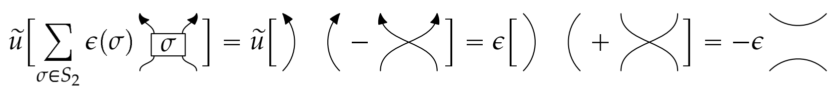
Proof. The first step of the proof is to show that $\mathcal{I}_2(\Sigma)$ maps to 0 under the composition

$$\mathcal{Z}(\Sigma) \xrightarrow{\widetilde{u}} \mathcal{Z}^\circ(\Sigma) \longrightarrow \mathcal{Z}^\circ(\Sigma)/\mathcal{K}_0(\Sigma).$$

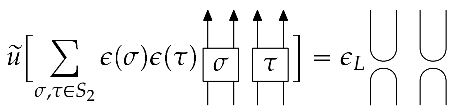
It is sufficient to consider the generators (cf. (5.22)) of the ideal $\mathcal{I}_2(\Sigma)$. We have



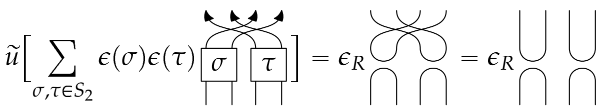
taking care of (5.22a). For the other two relations we need the following intermediate result


(7.4)

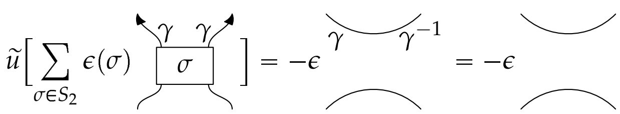
where ϵ is equal to -1 raised to the number of loops in the diagram corresponding to $\uparrow\uparrow$. This formula implies (5.22b) since



and



with appropriate signs ϵ_L and ϵ_R easily seen to be equal. Applying (7.4) to the left hand side of (5.22c) yields

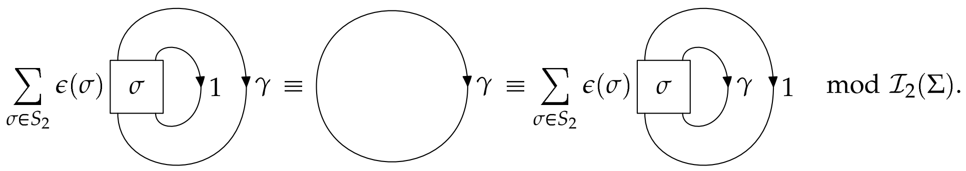


Here the decoration of the middle diagram means that traversing the upper strand from right to left amounts to first traversing γ^{-1} and then γ. Comparing this formula with (7.4) verifies relation (5.22c). Consequently $\widetilde{u}$ descends to an algebra homomorphism

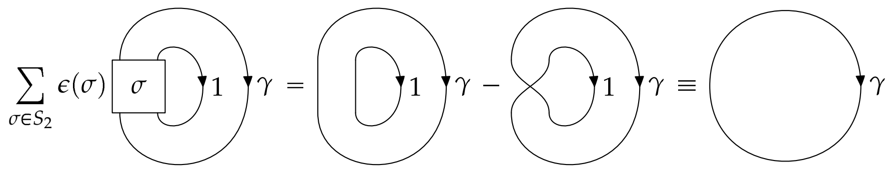
$$\widetilde{u}\colon \mathcal{Z}(\Sigma)/\mathcal{I}_2(\Sigma) \to \mathcal{Z}^\circ(\Sigma)/\mathcal{K}_0(\Sigma)$$

In order to invert $\widetilde{u}$, we record a simple observation.

Claim. For any $\gamma \in \pi_1(\Sigma)$ we have



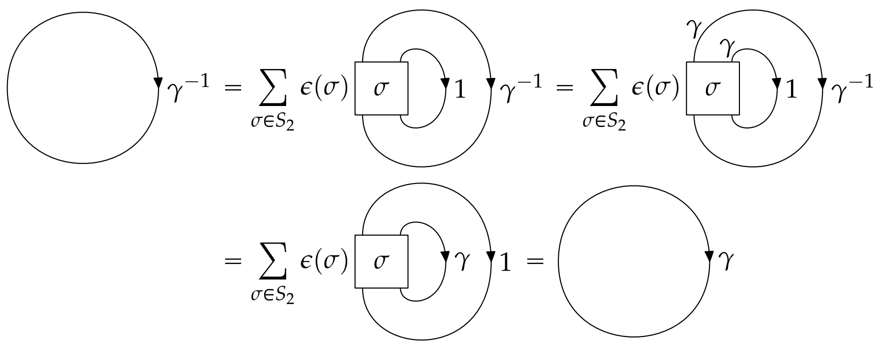
The two identities are analogous; we prove the former one:



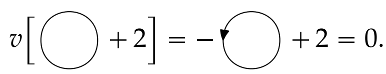
Now let $v\colon \mathcal{Z}^\circ(\Sigma) \to \mathcal{Z}(\Sigma)/\mathcal{I}_2(\Sigma)$ be the $\mathbf{C}$-algebra homomorphism determined by

$$v(\gamma) = -\vec{\gamma} \tag{7.5}$$

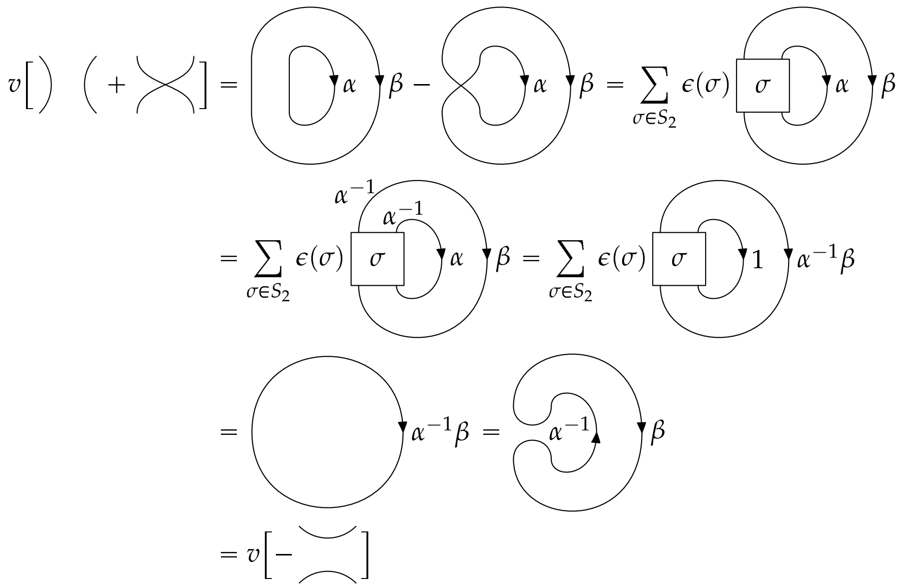
where γ denotes some unoriented loop on Σ and $\vec{\gamma}$ is one of the two possible oriented versions of it. That v is well-defined follows from the claim and an application of relation (5.22c):



The next step is to verify that $\mathcal{K}_0(\Sigma) \subseteq \operatorname{Ker} v$. Relation (7.1b) is trivial:



Given an instance of relation (7.1a) we assume without loss of generality that the two vertical strands belong to separate loops. Letting α and β denote oriented versions of them we deduce

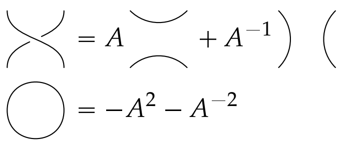


as desired. It is obvious from (7.2) and (7.5) that the induced map

$$v\colon \mathcal{Z}^\circ(\Sigma)/\mathcal{K}_0(\Sigma) \to \mathcal{Z}(\Sigma)/\mathcal{I}_2(\Sigma)$$

is the inverse of u. □

The usefulness of this proposition stems from the fact that the relations (7.1) are similar to the skein relations defining the Kauffman bracket; recall that this is a polynomial invariant $\langle L\rangle \in \mathbf{Z}[A^{\pm 1}]$ of the framed, unoriented link $L \subseteq \mathbf{R}^3$, satisfying (and determined by) the conditions



(7.6a)

(7.6b)

Substituting $A = -1$, over- and undercrossings cannot be distinguished, and the skein relations reduce to (7.1).

Remark 7.4. In light of what we have just said, it is easy to see that $\mathcal{Z}^\circ(\Sigma)/\mathcal{K}_0(\Sigma)$ is isomorphic to the complex algebra $\mathcal{S}_{2,\infty}(\Sigma \times I; \mathbf{C}, -1)$ studied in [PS]. The latter algebra has no zero-divisors, in particular no nilpotent elements, by Theorem 4.7 of that paper. It thus follows from Proposition 7.3 (cf. Remark 7.1) that $\Psi\colon \mathcal{Z}(\Sigma)/\mathcal{I}_2(\Sigma) \to \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C})))$ is a Poisson isomorphism. We transfer the Poisson structure on $\mathcal{Z}(\Sigma)/\mathcal{I}_2(\Sigma)$ to $\mathcal{Z}^\circ(\Sigma)/\mathcal{K}_0(\Sigma)$ by requiring that the algebra isomorphism $\widetilde{u}$ is a Poisson isomorphism.

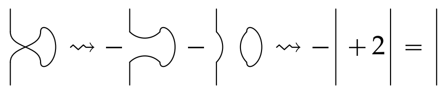
Pursuing the similarity of (7.1) and (7.6), we define a *BFK-diagram on* Σ to be the isotopy class of a finite collection of unoriented circles embedded into Σ such that no loop bounds a disk in Σ; informally speaking, a BFK-diagram is an unoriented diagram on Σ with no crossings and no homotopically trivial components. Let $\mathcal{B}(\Sigma)$ be the complex vector space freely generated by all BFK-diagrams on Σ. Define a linear map $\kappa\colon \mathcal{Z}^\circ(\Sigma) \to \mathcal{B}(\Sigma)$ by the Kauffman bracket procedure, i.e., given a generic unoriented diagram D replace all crossings by the right hand side of (7.1a) and remove all arising trivial loops at the cost of a factor -2 to obtain a linear combination $\kappa(D) \in \mathcal{B}(\Sigma)$.

Proposition 7.5. *The map* $\kappa\colon \mathcal{Z}^\circ(\Sigma) \to \mathcal{B}(\Sigma)$ *is well-defined and descends to an isomorphism*

$$\kappa\colon \mathcal{Z}^\circ(\Sigma)/\mathcal{K}_0(\Sigma) \to \mathcal{B}(\Sigma) \tag{7.7}$$

of complex vector spaces.

Proof. Two generic unoriented diagrams on Σ are homotopic if and only if they are related by isotopy and the three Reidemeister moves. Of course, $\kappa(D)$ is invariant under isotopies of D. Regarding the first Reidemeister move, we follow the steps in the computation of κ to obtain:



as required. For the second and third Reidemeister moves one can simply substitute $A = -1$ (and ignore over-/undercrossing information) in Kauffman's proof of the invariance of his bracket under these moves, cf. [K]. Hence κ is well-defined, and by construction it descends to a quotient map as in (7.7).

Any BFK-diagram on Σ can be considered as an unoriented diagram on Σ; this defines a linear map $\iota\colon \mathcal{B}(\Sigma) \to \mathcal{Z}^\circ(\Sigma)$. Evidently, the composition

$$\mathcal{B}(\Sigma) \xrightarrow{\iota} \mathcal{Z}^\circ(\Sigma) \longrightarrow \mathcal{Z}^\circ(\Sigma)/\mathcal{K}_0(\Sigma)$$

is the inverse of κ. □

We equip $\mathcal{B}(\Sigma)$ with the Poisson algebra structure induced by κ and have thus constructed the diagram

$$\begin{array}{ccccccccc} \mathcal{C}(\Sigma) & \xrightarrow{R_{1,-\frac{1}{2}}} & \mathcal{Z}(\Sigma) & \longrightarrow & \mathcal{Z}(\Sigma)/\mathcal{I}_2(\Sigma) & \xrightarrow{\tilde{u}} & \mathcal{Z}^\circ(\Sigma)/\mathcal{K}_0(\Sigma) & \xrightarrow{\kappa} & \mathcal{B}(\Sigma) \\ & \searrow^{\Psi} & & & \downarrow{\scriptstyle\Psi} & & & \swarrow_{\nu} & \\ & & & & \mathcal{O}(\mathcal{M}(\Sigma;\mathrm{SL}_2(\mathbf{C}))) & & & & \end{array} \tag{7.8}$$

of Poisson homomorphisms. Here $\nu\colon \mathcal{B}(\Sigma) \to \mathcal{O}(\mathcal{M}(\Sigma;\mathrm{SL}_2(\mathbf{C})))$ is the unique map (isomorphism) making the diagram commutative. From (7.5) follows that

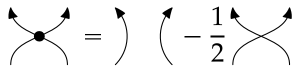
$$\nu(\gamma) = -\Psi(\vec{\gamma})$$

where γ is a non-trivial loop on Σ. Therefore ν is equivariant with respect to the natural action of $\Gamma_+(\Sigma)$ on $\mathcal{B}(\Sigma)$, cf. Theorem 2.22.

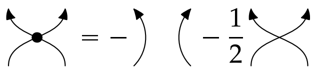
It will facilitate computations later on to adapt diagram (7.8) slightly. Consider the map $\widetilde{\Psi}\colon \mathcal{C}(\Sigma) \to \mathcal{O}(\mathcal{M}(\Sigma;\mathrm{SL}_2(\mathbf{C})))$ given by

$$\widetilde{\Psi}(D) = (-1)^n \Psi(D), \quad D \text{ a chord diagram with } n \text{ core components}$$

Since $\Psi\colon \mathcal{C}(\Sigma) \to \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C})))$ is a Poisson homomorphism and the Poisson structure on $\mathcal{C}(\Sigma)$ preserves the skeletons of chord diagrams, $\widetilde{\Psi}$ is also a Poisson homomorphism. We know that Ψ maps the loop relation



to 0. Since smoothing a chord in-/decreases the number of core components by 1, it follows that $\widetilde{\Psi}$ satisfies a different loop relation:



Thus we obtain a triangle of Poisson homomorphisms

$$\begin{array}{ccc} \mathcal{C}(\Sigma) & \xrightarrow{R_{-1,-\frac{1}{2}}} & \mathcal{Z}_{-1,-\frac{1}{2}}(\Sigma) \\ & \searrow^{\widetilde{\Psi}} & \downarrow \widetilde{\Psi} \\ & & \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C}))) \end{array}$$

For a diagram D with n loops we have by (7.2)

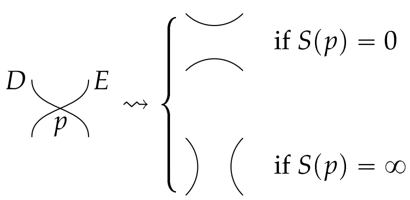
$$\nu\kappa u(D) = (-1)^n \nu\kappa\tilde{u}(D) = (-1)^n \Psi(D) = \widetilde{\Psi}(D)$$

so that (7.8) transforms into the commutative diagram

$$\begin{array}{ccccccc} \mathcal{C}(\Sigma) & \xrightarrow{R_{-1,-\frac{1}{2}}} & \mathcal{Z}_{-1,-\frac{1}{2}}(\Sigma) & \xrightarrow{u} & \mathcal{Z}^{\circ}(\Sigma)/\mathcal{K}_0(\Sigma) & \xrightarrow{\kappa} & \mathcal{B}(\Sigma) \\ & \searrow^{\widetilde{\Psi}} & \downarrow \widetilde{\Psi} & & & \swarrow_{\nu} & \\ & & \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C}))) & & & & \end{array} \tag{7.9}$$

As a composition of Poisson homomorphisms $u = \kappa^{-1}\nu^{-1}\widetilde{\Psi}$ is a Poisson homomorphism.

We are now set to derive formulas for the product and the Poisson bracket on $\mathcal{B}(\Sigma)$. Let D and E be BFK-diagrams in general position. Regarding D and E as unoriented diagrams on Σ, their product is simply the union $D \cup E$. Hence relation (7.1a) leads us to define a *state for* (D, E) to be any map $S\colon D\#E \to \{0, \infty\}$ and its corresponding diagram $D(S)$ to be the one obtained from $D \cup E$ by resolving all crossings $D\#E$ as follows



(7.10)

Notice that $D(S)$ is not necessarily a BFK-diagram since it may contain trivial loops. Obviously we have

$$DE = (-1)^{|D\#E|} \sum_S D(S) \in \mathcal{Z}^{\circ}(\Sigma)/\mathcal{K}_0(\Sigma). \tag{7.11}$$

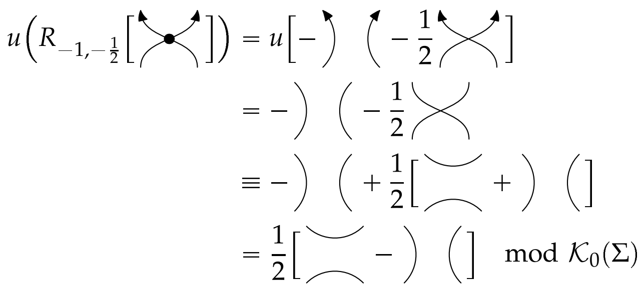
To calculate $\{D, E\}$, lift D and E to diagrams $\vec{D}$ and $\vec{E}$ on Σ so that $u(\vec{D}) = D$ and $u(\vec{E}) = E$. Since

$$\{\vec{D}, \vec{E}\} = \sum_{p \in \vec{D}\#\vec{E}} \epsilon(p; \vec{D}, \vec{E}) \vec{D} \cup_p \vec{E} \in \mathcal{C}(\Sigma),$$

diagram (7.9) implies

$$\begin{aligned} \{D, E\} &= \{uR_{-1,-\frac{1}{2}}(\vec{D}), uR_{-1,-\frac{1}{2}}(\vec{E})\} \\ &= u\big(R_{-1,-\frac{1}{2}}(\{\vec{D}, \vec{E}\})\big) \\ &= \sum_{p \in D\#E} \epsilon(p; \vec{D}, \vec{E}) u\big(R_{-1,-\frac{1}{2}}(\vec{D} \cup_p \vec{E})\big) \in \mathcal{Z}^\circ(\Sigma)/\mathcal{K}_0(\Sigma). \end{aligned}$$

Focusing on the chord in $\vec{D} \cup_p \vec{E}$, we get



(7.12)

so that

$$\epsilon(p; \vec{D}, \vec{E}) u\big(R_{-1,-\frac{1}{2}}(\vec{D} \cup_p \vec{E})\big) = \frac{1}{2}[D \cup_{p,\infty} E - D \cup_{p,0} E]$$

where $D \cup_{p,s} E$, $s = 0, \infty$ is obtained from $D \cup E$ by resolving only the crossing at p according to the rule (7.10). Hence we have

$$\{D, E\} = \frac{1}{2} \sum_{p \in D\#E} [D \cup_{p,\infty} E - D \cup_{p,0} E] \in \mathcal{Z}^\circ(\Sigma)/\mathcal{K}_0(\Sigma).$$

The diagram $D \cup_{p,s} E$ contains the crossings $D\#E - \{p\}$ which may also be resolved via (7.1a); doing so leads to the various state diagrams for (D, E). Putting $0(S) = |S^{-1}(0)|$ and $\infty(S) = |S^{-1}(\infty)|$ for a state S, we therefore derive

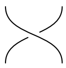
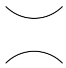
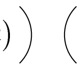
$$\{D, E\} = (-1)^{|D\#E|} \frac{1}{2} \sum_S (0(S) - \infty(S)) D(S) \in \mathcal{Z}^\circ(\Sigma)/\mathcal{K}_0(\Sigma). \tag{7.13}$$

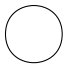
This formula (up to a sign) was obtained in [BFK].

Remark 7.6. In the case $\partial\Sigma \neq \emptyset$ we have the $*$-product $*_P$, $P \in \mathcal{P}(\Sigma)$ on $\mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C})))$ induced via the $*$-equivalence $\Psi\colon \mathcal{Z}(\Sigma)/\mathcal{I}_2(\Sigma) \to \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C})))$, cf. Theorem 5.27 and Corollary 5.35. We may transfer $*_P$ to $\mathcal{Z}^\circ(\Sigma)/\mathcal{K}_0(\Sigma)$ and $\mathcal{B}(\Sigma)$ by requiring $\tilde{u}$, κ and, thus, ν to be $*$-equivalences. The adapted map $\widetilde{\Psi}\colon \mathcal{C}(\Sigma) \to \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C})))$ is a morphism of $*_P$ since this $*$-product preserves skeletons of chord diagrams. From diagram (7.9) follows that $u\colon \mathcal{Z}_{-1,-\frac{1}{2}}(\Sigma) \to \mathcal{Z}^\circ(\Sigma)/\mathcal{K}_0(\Sigma)$ is also a morphism of $*_P$.

7.2 The BFK $*$-Product

Let $\mathcal{L}_h^\circ(\Sigma)$ denote the free $\mathbf{C}[[h]]$-module generated by the set of framed, unoriented links in $\Sigma \times I$; endowed with stack multiplication it is a $\mathbf{C}[[h]]$-algebra. Let $\mathcal{K}(\Sigma) \subseteq \mathcal{L}_h^\circ(\Sigma)$ be the submodule generated by the skein relations

 $= -\exp\bigl(\frac{1}{4}h\bigr)$  $- \exp\bigl(-\frac{1}{4}h\bigr)$  (7.14a)

 $= -2\cosh\bigl(\frac{1}{2}h\bigr)$ (7.14b)

As usual $\mathcal{K}(\Sigma)$ is an ideal; the quotient

$$\mathcal{A}^\circ(\Sigma) = \mathcal{L}_h^\circ(\Sigma)/\mathcal{K}(\Sigma)$$

is called the *Kauffman bracket skein algebra of* Σ. In the sequel we shall mainly be interested in the completion $\overline{\mathcal{A}^\circ(\Sigma)}$ with respect to the h-filtration on $\mathcal{A}^\circ(\Sigma)$. Notice that (7.14) are the Kauffman bracket skein relations (7.6) with parameter $A = -\exp(\frac{1}{4}h) \in \mathbf{C}[[h]]$. Hence we can define a $\mathbf{C}[[h]]$-linear map

$$K\colon \mathcal{L}_h^\circ(\Sigma) \to \mathcal{B}(\Sigma)[[h]]$$

by the Kauffman bracket procedure (cf. the construction of $\kappa\colon \mathcal{Z}^\circ(\Sigma) \to \mathcal{B}(\Sigma)$).

Theorem 7.7. *The map* $K\colon \mathcal{L}_h^\circ(\Sigma) \to \mathcal{B}(\Sigma)[[h]]$ *induces an isomorphism*

$$K\colon \overline{\mathcal{A}^\circ(\Sigma)} \to \mathcal{B}(\Sigma)[[h]] \tag{7.15}$$

of $\mathbf{C}[[h]]$*-modules.*

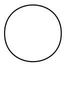
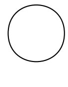
Proof. It is immediate from the definition that $\mathcal{K}(\Sigma) \subseteq \operatorname{Ker} K$. The induced map $K\colon \mathcal{A}^\circ(\Sigma) \to \mathcal{B}(\Sigma)[[h]]$ is filtered with respect to the h-filtrations since it is $\mathbf{C}[[h]]$-linear. Therefore we obtain a homomorphism of $\mathbf{C}[[h]]$-modules as in (7.15).

Inverting K is simple. A BFK-diagram may be considered as a diagram of a framed, unoriented link; the complex linear map

$$\mathcal{B}(\Sigma) \to \mathcal{L}_h^\circ(\Sigma) \to \mathcal{A}^\circ(\Sigma) \to \overline{\mathcal{A}^\circ(\Sigma)}$$

induced hereby determines (cf. (3.1) and Remark 3.1) a $\mathbf{C}[[h]]$-linear map $\mathcal{B}(\Sigma)[[h]] \to \overline{\mathcal{A}^\circ(\Sigma)}$ easily seen to be the inverse of K. □

Remark 7.8. Thinking of $\mathcal{B}(\Sigma)$ as the quotient $\mathcal{Z}^\circ(\Sigma)/\mathcal{K}_0(\Sigma)$, we may treat trivial loops a little differently in the computation of K. Namely

$K\bigl[$  $\bigr] = \cosh\bigl(\frac{1}{2}h\bigr)$ 

by (7.14b) and (7.1b).

Theorem 7.9 (Bullock, Frohman & Kania-Bartoszyńska). *The multiplication on* $\overline{\mathcal{A}^\circ(\Sigma)}$ *induces a* $*$*-product* $*_\Sigma$ *on* $\mathcal{B}(\Sigma)[[h]]$ *via the isomorphism* $K\colon \overline{\mathcal{A}^\circ(\Sigma)} \to \mathcal{B}(\Sigma)[[h]]$.

Proof. Let D and E be BFK-diagrams in general position. To compute $D *_\Sigma E$ one must stack D on top of E, resolve all crossings using (7.14a) and collect the factor $\cosh(\frac{1}{2}h)$ for each trivial loop (keeping this loop). Given a state S for (D, E) we denote by $\tau(S)$ the number of trivial loops in its diagram $D(S)$. Putting $a = \exp(\frac{1}{4}h)$, we get

$$D *_\Sigma E = \sum_S (-a)^{0(S)-\infty(S)} \cosh\big(\frac{1}{2}h\big)^{\tau(S)} D(S). \tag{7.16}$$

Since $\cosh(\frac{1}{2}h) \equiv 1 \mod h^2$, this simplifies to

$$\begin{aligned} D *_\Sigma E &\equiv \sum_S (-a)^{0(S)-\infty(S)} D(S) \\ &= (-1)^{|D\# E|} \sum_S a^{0(S)-\infty(S)} D(S) \\ &\equiv (-1)^{|D\# E|} \sum_S \big(1 + (0(S) - \infty(S))\frac{1}{4}h\big) D(S) \\ &= (-1)^{|D\# E|} \sum_S D(S) + \frac{1}{2}h(-1)^{|D\# E|}\frac{1}{2}\sum_S (0(S) - \infty(S)) D(S) \\ &= DE + \frac{1}{2}\{D, E\}h \mod h^2 \end{aligned}$$

by formulas (7.11) and (7.13). This proves the theorem, cf. Proposition 3.9. □

The BFK $*$-product is $\Gamma_+(\Sigma)$-invariant by (7.16); so is the induced $*$-product on the moduli space by the equivariance of $\nu\colon \mathcal{B}(\Sigma) \to \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C})))$.

7.2.1 Relating BFK $*$-Products for Different Surfaces

Suppose Σ has non-empty boundary and denote by $\partial_0\Sigma$ one of its boundary circles. Let $\overline{\Sigma}$ be the surface obtained from Σ by attaching a disk along $\partial_0\Sigma$. In this subsection we define a morphism from $*_\Sigma$ to $*_{\overline{\Sigma}}$ covering the Poisson homomorphism $r^*\colon \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C}))) \to \mathcal{O}(\mathcal{M}(\overline{\Sigma}; \mathrm{SL}_2(\mathbf{C})))$ given by restricting regular functions, cf. 2.3.1.

Let D be a BFK-diagram on Σ. Some, n say, of the loops of D are isotopic to $\partial_0\Sigma$, so we may write $D = D' \sqcup (\partial_0\Sigma)^n$ where D' is the remaining part of D. Notice that D' can be regarded as a BFK-diagram on $\overline{\Sigma}$. Letting $x = -2\cosh(\frac{1}{2}h)$, we set

$$\varphi(D) = x^n D' \in \mathcal{B}(\overline{\Sigma})[[h]].$$

This formula determines a $\mathbf{C}[[h]]$-linear map $\varphi\colon \mathcal{B}(\Sigma)[[h]] \to \mathcal{B}(\overline{\Sigma})[[h]]$, cf. (3.1). In the notation $\varphi = \sum_j \varphi_j h^j$ the map $\varphi_0\colon \mathcal{B}(\Sigma) \to \mathcal{B}(\overline{\Sigma})$ is the BFK-version of r^* in the sense that the diagram

$$\begin{array}{ccc} \mathcal{B}(\Sigma) & \xrightarrow{\quad\varphi_0\quad} & \mathcal{B}(\overline{\Sigma}) \\ \Big\downarrow \nu & & \Big\downarrow \nu \\ \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C}))) & \xrightarrow{\;r^*\;} & \mathcal{O}(\mathcal{M}(\overline{\Sigma}; \mathrm{SL}_2(\mathbf{C}))) \end{array}$$

is commutative. Let E be another BFK-diagram on Σ with decomposition $E = E' \sqcup (\partial_0\Sigma)^m$. When stacking D on top of E to compute $D *_\Sigma E$, we may assume that all crossings are between D' and E'. Then

$$\varphi(D *_\Sigma E) = x^{n+m} D' *_{\overline{\Sigma}} E' = x^n D' *_{\overline{\Sigma}} x^m E' = \varphi(D) *_{\overline{\Sigma}} \varphi(E)$$

so that φ is the desired $*$-morphism. Note that we had to twist the restriction map to obtain a $*$-homomorphism. In other words, if $*_\Sigma$ induces a $*$-product on $\mathcal{O}(\mathcal{M}(\overline{\Sigma}))$ via r^*, this $*$-product is not equal to $*_{\overline{\Sigma}}$.

7.3 The AMR $*$-Products and the BFK $*$-Product

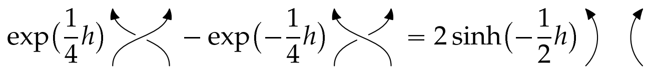
In this section we assume that $\partial\Sigma \neq \varnothing$; we prove that the AMR $*$-product $*_P$, $P \in \mathcal{P}(\Sigma)$ on $\mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C})))$ is equivalent to the BFK $*$-product. The strategy is to show that the equivalence

$$T_P \colon \overline{\mathcal{A}_{-1,-\frac{1}{2}}(\Sigma)} \to \mathcal{Z}_{-1,-\frac{1}{2}}(\Sigma)[[h]]_{*_P} \tag{7.17}$$

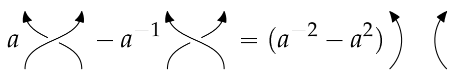
of quantizations (cf. Theorem 6.4) descends to an isomorphism

$$K_P \colon \overline{\mathcal{A}^\circ(\Sigma)} \to \mathcal{B}(\Sigma)[[h]]_{*_P} \tag{7.18}$$

of $\mathbf{C}[[h]]$-algebras. The co-domains of (7.17) and (7.18) are related by the $*$-homomorphism $\kappa \circ u \colon \mathcal{Z}_{-1,-\frac{1}{2}}(\Sigma)[[h]]_{*_P} \to \mathcal{B}(\Sigma)[[h]]_{*_P}$ (cf. Remark 7.6), so let us establish the connection between the domain spaces. Recall (6.1) that the Turaev-Vassiliev skein algebra $\mathcal{A}_{-1,-\frac{1}{2}}(\Sigma)$ is the quotient of $\mathcal{L}_h(\Sigma)$ by the ideal $\mathcal{I}_{-1,-\frac{1}{2}}(\Sigma)$ generated by

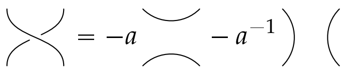


Putting $a = \exp(\frac{h}{4})$, this may be written

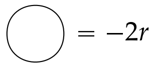


(7.19)

On the other hand, the Kauffman bracket skein algebra $\mathcal{A}^\circ(\Sigma)$ is the quotient of $\mathcal{L}_h^\circ(\Sigma)$ by the ideal $\mathcal{K}(\Sigma)$ generated by



(7.20a)

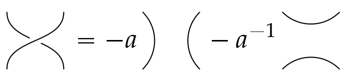


(7.20b)

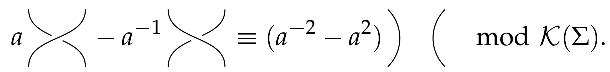
where $r = \cosh(\frac{h}{2})$, cf. (7.14). Forgetting the orientation of a link induces a homomorphism of $\mathbf{C}[[h]]$-algebras

$$U \colon \mathcal{L}_h(\Sigma) \to \mathcal{L}_h^\circ(\Sigma).$$

It is easy to see that U maps $\mathcal{I}_{-1,-\frac{1}{2}}(\Sigma)$ into $\mathcal{K}(\Sigma)$. Namely, rotate (7.20a) 90 degrees to see that the skein relation



holds in $\mathcal{A}^\circ(\Sigma)$. A suitable linear combination of this identity and (7.20a) proves



Comparing this formula with (7.19) yields the claim. The induced map $U\colon \mathcal{A}_{-1,-\frac{1}{2}}(\Sigma) \to \mathcal{A}^\circ(\Sigma)$ on the quotients is h-filtered (it is $\mathbf{C}[[h]]$-linear) and therefore extends to the h-completions

$$U\colon \overline{\mathcal{A}_{-1,-\frac{1}{2}}(\Sigma)} \to \overline{\mathcal{A}^\circ(\Sigma)}.$$

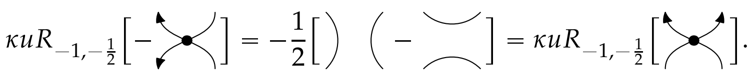
Consider the homomorphism of $\mathbf{C}[[h]]$-algebras

$$K_P\colon \mathcal{L}_h(\Sigma) \xrightarrow{T_P} \mathcal{Z}_{-1,-\frac{1}{2}}(\Sigma)[[h]]_{*_P} \xrightarrow{u} \mathcal{Z}^\circ(\Sigma)/\mathcal{K}_0(\Sigma)[[h]]_{*_P} \xrightarrow{\kappa} \mathcal{B}(\Sigma)[[h]]_{*_P}.$$

First we prove that K_P factors through $U\colon \mathcal{L}_h(\Sigma) \to \mathcal{L}_h^\circ(\Sigma)$, i.e., that K_P cannot detect the orientation of its input link. Let L be a link on Σ, and let C be some component of it. Denote by S_C the cabling operation reversing the orientation of C. By the definition of T_P (cf. (6.6) and (6.5)) and the fact that the universal Vassiliev invariant is compatible with cabling, we derive

$$K_P(S_C(L)) = \kappa u R_{-1,-\frac{1}{2}} \eta V_P S_C(L) = \kappa u R_{-1,-\frac{1}{2}} S_C \eta V_P(L) = \kappa u R_{-1,-\frac{1}{2}} \eta V_P(L) = K_P(L)$$

as desired. To justify the third equality, we recall Remark 4.1 about S_C and apply (7.12) to get

 (7.21)

Remark 7.10. Strictly speaking (7.21) should be performed as a calculation in $\mathcal{Z}^\circ(\Sigma)/\mathcal{K}_0(\Sigma)$ since κ is not locally computable.

Theorem 7.11. *The map* $K_P\colon \mathcal{L}_h^\circ(\Sigma) \to \mathcal{B}(\Sigma)[[h]]$ *extends to an isomorphism*

$$K_P\colon \overline{\mathcal{A}^\circ(\Sigma)} \to \mathcal{B}(\Sigma)[[h]]_{*_P}$$

of $\mathbf{C}[[h]]$*-algebras fitting into the diagram*

$$\begin{array}{ccccc}
\overline{\mathcal{A}_{-1,-\frac{1}{2}}(\Sigma)} & & \xrightarrow{\quad T_P \quad} & & \mathcal{Z}_{-1,-\frac{1}{2}}(\Sigma)[[h]] \\
 & \nwarrow & & \nearrow{\scriptstyle T_P} & \\
\Big\downarrow{\scriptstyle U} & & \mathcal{L}_h(\Sigma) & & \Big\downarrow{\scriptstyle \kappa\circ u} \\
 & & \Big\downarrow{\scriptstyle U} & & \\
 & & \mathcal{L}_h^\circ(\Sigma) & & \\
 & \swarrow & & \searrow{\scriptstyle K_P} & \\
\overline{\mathcal{A}^\circ(\Sigma)} & & \xrightarrow{\quad K_P \quad} & & \mathcal{B}(\Sigma)[[h]]
\end{array} \tag{7.22}$$

Corollary 7.12. *The composition*

$$K_P \circ K^{-1} \colon \mathcal{B}(\Sigma)[[h]] \to \mathcal{B}(\Sigma)[[h]]$$

is an equivalence from the BFK $*$-product $*_\Sigma$ to $*_P$.

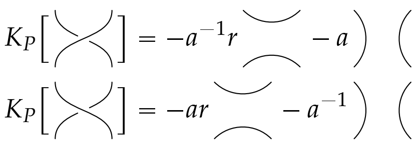
Proof. By Theorem 7.9, one need only verify that the triangle

$$\begin{array}{ccc} \mathcal{B}(\Sigma)[[h]] & \xrightarrow{K_P \circ K^{-1}} & \mathcal{B}(\Sigma)[[h]] \\ & \searrow \quad \swarrow & \\ & \mathcal{B}(\Sigma) & \end{array}$$

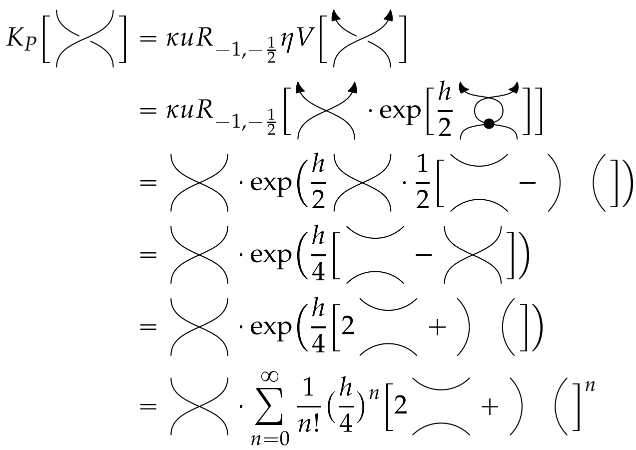
commutes; this is immediate from the definitions. □

We shall need a couple of lemmas for the proof of Theorem 7.11.

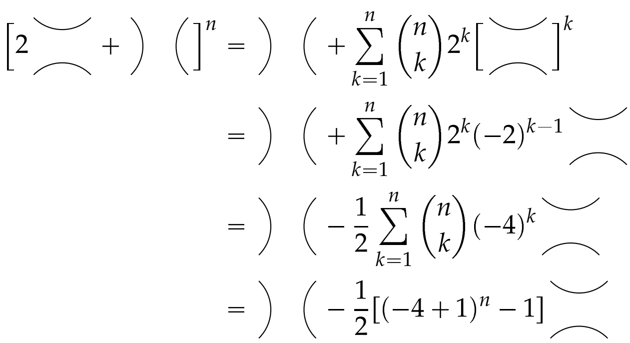
Lemma 7.13.



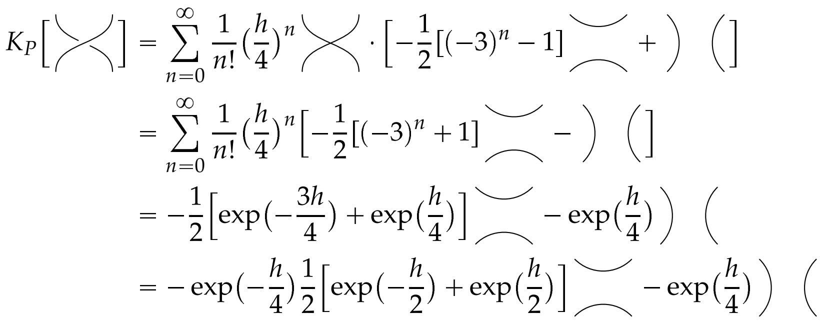
Proof. We prove the former identity; the proof of the latter one is analogous. Applications of (7.12) and (7.1a) yield



But by (7.1b),

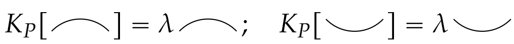


so we may continue and obtain

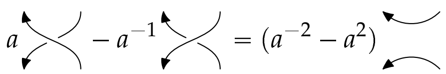


as desired. □

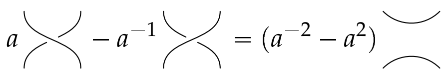
Lemma 7.14. *There exists $\lambda \in \mathbf{C}[[h]]$ such that $\lambda^2 = r$ and*


(7.23)

Proof. Recall the definition (4.4c) of the universal Vassiliev invariants of cups and caps. No matter what particular value C has (this depends on the associator Φ), it is clear from (7.12) and (7.1a) that there is a $\lambda \in \mathbf{C}[[h]]$ such that (7.23) holds. Determining λ by a direct computation is not possible since we allow different associators, and even for a particular associator such as the one given in [LM], the computation is not feasible. We circumvent these difficulties by exploiting the isotopy invariance of V_P to calculate λ. The skein relation (7.19) defining $\mathcal{I}_{-1,-\frac{1}{2}}(\Sigma)$ may be depicted as

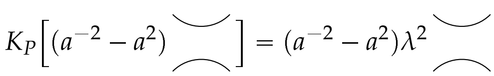

(7.24)

Since T_P respects this identity, K_P satisfies the unoriented version of it:



in the cases where the orientations in (7.24) are consistent. But according to the preceding lemma

and by (7.23),

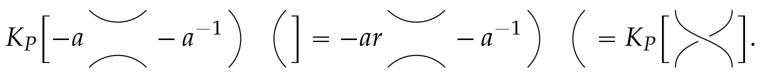


Equating the above two formulas yields the result since the scalar multiplication

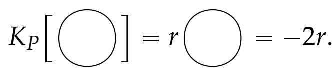
$$\mathbf{C}[[h]] \times \mathcal{B}(\Sigma)[[h]] \longrightarrow \mathcal{B}(\Sigma)[[h]]$$

is faithful. □

Proof (Theorem 7.11). By Lemmas 7.14 and 7.13, we have



Another application of Lemma 7.14 provides



Recalling (7.20) this means that $K_P(\mathcal{K}(\Sigma)) = 0$; the induced map $K_P\colon \mathcal{A}^\circ(\Sigma) \to \mathcal{B}(\Sigma)[[h]]$ is h-filtered and therefore extends to a homomorphism $K_P\colon \overline{\mathcal{A}^\circ(\Sigma)} \to \mathcal{B}(\Sigma)[[h]]_{*_P}$ of $\mathbf{C}[[h]]$-algebras. The diagram (7.22) is obviously commutative.

Define a $\mathbf{C}[[h]]$-algebra homomorphism by

$$F_P^\circ = U \circ T_P^{-1}\colon \mathcal{Z}_{-1,-\frac{1}{2}}(\Sigma)[[h]]_{*_P} \longrightarrow \overline{\mathcal{A}_{-1,-\frac{1}{2}}(\Sigma)} \longrightarrow \overline{\mathcal{A}^\circ(\Sigma)}.$$

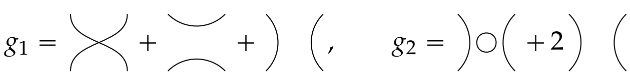
We shall prove that F_P° factors through the surjection $\kappa \circ u\colon \mathcal{Z}(\Sigma)[[h]] \to \mathcal{B}(\Sigma)[[h]]$ to a map $F_P^\circ\colon \mathcal{B}(\Sigma)[[h]] \to \overline{\mathcal{A}^\circ(\Sigma)}$ being the inverse of K_P, cf. diagram (7.22). Since

$$\mathrm{Ker}\big(\mathcal{Z}(\Sigma)[[h]] \xrightarrow{\kappa\circ u} \mathcal{B}(\Sigma)[[h]]\big) = \mathrm{Ker}[\mathcal{Z}(\Sigma) \xrightarrow{\kappa\circ u} \mathcal{B}(\Sigma)][[h]]$$

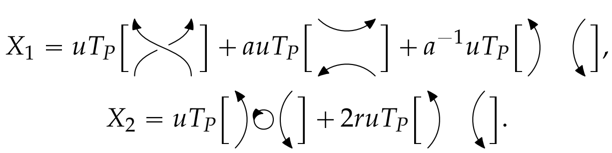
and because the formula (3.2) applies to F_P°, it suffices to prove that the restriction of F_P° to $\mathcal{Z}(\Sigma)$ descends to $\mathcal{B}(\Sigma)$ via $\kappa \circ u\colon \mathcal{Z}(\Sigma) \to \mathcal{B}(\Sigma)$. For a diagram D on Σ we have by (6.14)

$$F_P^\circ(D) = U\overline{\mathrm{Id}}V_P^{-1}(D). \tag{7.25}$$

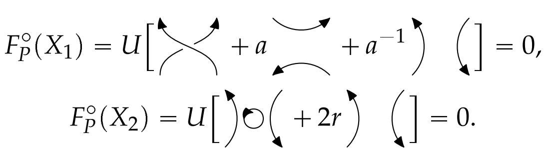
As the universal Vassiliev invariant is compatible with orientation changes in D, it follows from (7.25) that F_P° factors through $u\colon \mathcal{Z}(\Sigma) \to \mathcal{Z}^\circ(\Sigma)$. By a method similar to the one used in the proof of Lemma 6.12, we now demonstrate that $\mathcal{K}_0(\Sigma) = \mathrm{Ker}\,\kappa$ is contained in the kernel of $F_P^\circ\colon \mathcal{Z}^\circ(\Sigma) \to \overline{\mathcal{A}^\circ(\Sigma)}$. Consider an embedded square $S \subseteq \Sigma$, and let $\mathcal{Z}^\circ(S;\partial)$ denote the complex vector space of unoriented tangles in S with two top and two bottom boundary points. Notice that $\mathcal{Z}^\circ(S;\partial)$ is a commutative algebra under composition of tangles. Recalling (7.1) it is enough to prove that the two elements



of $\mathcal{Z}^\circ(S;\partial)$ are mapped to 0 by F_P°. Define power series $X_1, X_2 \in \mathcal{Z}^\circ(S;\partial)[[h]]$ by



By definition of F_P° we have (cf. (7.20))



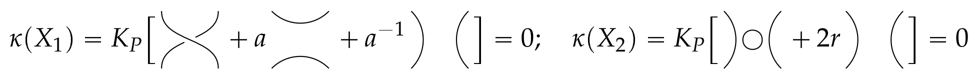
(7.26a)

(7.26b)

Let $\pi_i\colon \mathcal{Z}^\circ(S;\partial)[[h]] \to \mathcal{Z}^\circ(S;\partial)$ denote the projection on the ith coefficient. We notice that

$$\pi_0(X_1) = g_1, \quad \pi_0(X_2) = g_2. \tag{7.27}$$

More generally, the identities



imply that there exist elements $x^j_{k,i} \in \mathcal{Z}^\circ(S;\partial)$, $j,k = 1,2,\ i = 1,2,\dots$ such that

$$\pi_i(X_1) = g_1 \cdot x^1_{1,i} + g_2 \cdot x^1_{2,i}, \quad \pi_i(X_2) = g_1 \cdot x^2_{1,i} + g_2 \cdot x^2_{2,i}.$$

We use this observation to inductively construct a sequence $Y_i \in \mathcal{Z}^\circ(S;\partial)[[h]]$, $i = 0,1,\dots$ of the form

$$Y_i = X_1 \cdot y_{1,i} + X_2 \cdot y_{2,i}, \quad y_{1,i}, y_{2,i} \in \mathcal{Z}^\circ(S;\partial) \tag{7.28}$$

and satisfying

$$\sum_{i=0}^{n} h^i Y_i - g_1 \in h^{n+1}\mathcal{Z}^\circ(S;\partial)[[h]]. \tag{7.29}$$

The process is initiated by setting $y_{1,0} = |\,|$ and $y_{2,0} = 0$ so that $Y_0 = X_1$; by (7.27) this is sound. Assume that $Y_0, \dots, Y_n$ have already been defined. The computation

$$\begin{aligned}
\pi_{n+1}\Big(\sum_{i=0}^{n} h^i Y_i\Big) &= \sum_{i=0}^{n} \pi_{n+1-i}(Y_i) \\
&= \sum_{i=0}^{n} \pi_{n+1-i}(X_1)\cdot y_{1,i} + \pi_{n+1-i}(X_2)\cdot y_{2,i} \\
&= \sum_{i=0}^{n} [g_1 \cdot x^1_{1,n+1-i} + g_2 \cdot x^1_{2,n+1-i}]\cdot y_{1,i} + [g_1 \cdot x^2_{1,n+1-i} + g_2 \cdot x^2_{2,n+1-i}]\cdot y_{2,i} \\
&= g_1 \cdot \Big[\sum_{i=0}^{n} x^1_{1,n+1-i}\cdot y_{1,i} + x^2_{1,n+1-i}\cdot y_{2,i}\Big] \\
&\quad + g_2 \cdot \Big[\sum_{i=0}^{n} x^1_{2,n+1-i}\cdot y_{1,i} + x^2_{2,n+1-i}\cdot y_{2,i}\Big]
\end{aligned}$$

leads us to set

$$y_{1,n+1} = -\sum_{i=0}^{n} x^1_{1,n+1-i}\cdot y_{1,i} + x^2_{1,n+1-i}\cdot y_{2,i}, \quad y_{2,n+1} = -\sum_{i=0}^{n} x^1_{2,n+1-i}\cdot y_{1,i} + x^2_{2,n+1-i}\cdot y_{2,i}$$

and thereby obtain (cf. (7.28) and (7.27))

$$\pi_{n+1}\Big(\sum_{i=0}^{n+1} h^i Y_i - g_1\Big) = \pi_{n+1}\Big(\sum_{i=0}^{n} h^i Y_i\Big) + \pi_0(Y_{n+1}) = 0.$$

This formula and the induction hypothesis

$$\pi_j\big(\sum_{i=0}^{n+1} h^i Y_i - g_1\big) = \pi_j\big(\sum_{i=0}^{n} h^i Y_i - g_1\big) = 0, \quad j = 0, \dots, n$$

complete the induction step. By definition, (7.29) means that

$$\sum_{i=0}^{n} h^i Y_i \to g_1, \quad n \to \infty$$

in the h-filtration. Therefore, using (7.26), we derive

$$0 = \sum_{i=0}^{n} h^i [F_P^\circ(X_1) \cdot F_P^\circ(y_{1,i}) + F_P^\circ(X_2) \cdot F_P^\circ(y_{2,i})] = F_P^\circ\big(\sum_{i=0}^{n} h^i Y_i\big) \to F_P^\circ(g_1), \quad n \to \infty.$$

By the Hausdorff property, this means that $F_P^\circ(g_1) = 0$. Analogously, one constructs another sequence $Y_i' \in \mathcal{Z}^\circ(S;\partial)[[h]]$ to prove that $F_P^\circ(g_2) = 0$. It follows from diagram (7.22) that the induced map $F_P^\circ\colon \mathcal{B}(\Sigma)[[h]] \to \overline{\mathcal{A}^\circ(\Sigma)}$ is the inverse of K_P. □

Remark 7.15. Assume we have two partitions P_1 and P_2 of Σ. By Corollary 7.12,

$$K_{P_2} \circ K_{P_1}^{-1} = (K_{P_2} \circ K^{-1}) \circ (K_{P_1} \circ K^{-1})^{-1} \colon \mathcal{B}(\Sigma)[[h]] \to \mathcal{B}(\Sigma)[[h]]$$

is an equivalence from $*_{P_1}$ to $*_{P_2}$. But from Theorem 6.5 and the commutative diagrams (7.9) and (7.22) follows that this map corresponds under the isomorphism $\nu\colon \mathcal{B}(\Sigma) \to \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C})))$ to the canonical equivalence between these $*$-products given in Theorem 5.28.

Remark 7.16. Recall the set-up in 7.2.1 where we constructed a $*$-morphism

$$\varphi\colon \mathcal{B}(\Sigma)[[h]] \to \mathcal{B}(\overline{\Sigma})[[h]]$$

from $*_\Sigma$ to $*_{\overline{\Sigma}}$. Suppose $\overline{D}$ and $\overline{E}$ are BFK-diagrams on $\overline{\Sigma}$. We may represent them by BFK-diagrams on Σ denoted by D, respectively E so that $\varphi(D) = \overline{D}$ and $\varphi(E) = \overline{E}$. Then

$$\varphi\big((K_P K^{-1})^{-1}\big(K_P K^{-1}(D) *_P K_P K^{-1}(E)\big)\big) = \varphi(D *_\Sigma E) = \varphi(D) *_{\overline{\Sigma}} \varphi(E) = \overline{D} *_{\overline{\Sigma}} \overline{E}.$$

Thus, the leftmost expression can be interpreted as a recipe for constructing a $*$-product on $\mathcal{O}(\mathcal{M}(\overline{\Sigma}; \mathrm{SL}_2(\mathbf{C})))$ out of $*_P$ on $\mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C})))$, i.e., we have one possible answer to the question of how one defines a $*$-product on the moduli space of a closed surface in terms of the AMR $*$-products for punctured surfaces.

7.4 Differentiability of the BFK $*$-Product

A $*$-product on the algebra of smooth functions on a Poisson manifold M is often required to be *differential*, that is, in the notation

$$f * g = \sum_{n=0}^{\infty} c_n(f,g) h^n, \quad f, g \in C^\infty(M)$$

the maps $c_n\colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ must be given by bi-differential operators. In order to investigate the question of differentiability for the BFK $*$-product, we shall derive expressions for its coefficients $c_n = c_n(\Sigma)$. Recalling formula (7.16), we define complex numbers a_i^0, a_i^∞, b_i, $i \in \mathbf{N}$ by

$$\sum_{i=0}^{\infty} a_i^0 h^i = -\exp\big(\frac{1}{4}h\big), \quad \sum_{i=0}^{\infty} a_i^\infty h^i = -\exp\big(-\frac{1}{4}h\big), \quad \sum_{i=0}^{\infty} b_i h^i = \cosh\big(\frac{1}{2}h\big)$$

and notice that

$$a_0^0 = a_0^\infty = -1, \quad b_0 = 1. \tag{7.30}$$

Let D and E be BFK-diagrams. Given a state S for (D, E) we denote by $T(S)$ the set of trivial loops in the corresponding diagram $D(S)$. We define a *graded state for* (D, E) to be a pair (S, d) of a state S for (D, E) and a map $d\colon D\#E \sqcup T(S) \to \mathbf{N}$. Its *coefficient* is

$$C(S, d) = \prod_{p \in D\#E} a_{d(p)}^{S(p)} \times \prod_{t \in T(S)} b_{d(t)}, \tag{7.31}$$

and its *total degree* is $\sum_{x \in D\#E \sqcup T(S)} d(x)$. A graded state of total degree n is called an *n-state*. Writing $f_D = \nu(D)$ for the isomorphism $\nu\colon \mathcal{B}(\Sigma) \to \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C})))$, we infer from (7.16) that

$$c_n(f_D, f_E) = \sum_{n\text{-states } (S,d)} C(S, d) f_{D(S)}. \tag{7.32}$$

A graded state is said to be *degenerate* if $d_{|T(S)} = 0$. For such states we identify d with $d_{|D\#E}$, and (7.31) simplifies to

$$C(S, d) = \prod_{p \in D\#E} a_{d(p)}^{S(p)} \tag{7.33}$$

because of (7.30). Consider the special case in which $D = \bigsqcup_i D_i$ is a (disjoint) union of $m > n$ BFK-diagrams so that $f_D = \prod_i f_{D_i}$. Define BFK-diagrams

$$D_{\hat{k}} = \bigsqcup_{i \neq k} D_i, \quad k = 1, \ldots, m.$$

As $D_{\hat{k}}\#E \subseteq D\#E$, a state S for (D, E) induces a state $S_{\hat{k}}$ for $(D_{\hat{k}}, E)$ by restriction. If (S, d) is a degenerate state for (D, E), then $(S, d)_{\hat{k}} = (S_{\hat{k}}, d_{\hat{k}})$, where $d_{\hat{k}} = d_{|D_{\hat{k}}\#E}$, determines a degenerate state for $(D_{\hat{k}}, E)$. The pigeon hole principle allows us to define the *index* of a (degenerate) n-state for (D, E) to be the minimal $k \in \{1, \ldots, m\}$ such that $d_{|D_k\#E} = 0$. Two degenerate n-states of the same index k are said to be *equivalent* if their restrictions to degenerate states for $(D_{\hat{k}}, E)$ are equal. For an equivalence class $\mathcal{S} = [S, d]$ we may set

$$D(\mathcal{S}) = D([S, d]) = D(S_{\hat{k}}), \quad C(\mathcal{S}) = C([S, d]) = C((S, d)_{\hat{k}}). \tag{7.34}$$

We note moreover that $\mathcal{S}$ determines all of (S, d) except $S_{|D_k\#E}$ which in turn is unrestricted. Hence $\mathcal{S}$ consists of $2^{|D_k\#E|}$ elements, and $|D_k\#E|$ applications of relation (7.1a) restores the crossings $D_k\#E$ and thereby proves

$$\sum_{(S,d) \in \mathcal{S}} f_{D(S)} = (-1)^{|D_k\#E|} f_{D(\mathcal{S})} f_{D_k}.$$

By the definition of index and formulas (7.30), (7.33) and (7.34), we derive

$$C(S,d) = (-1)^{|D_k \# E|} C(\mathcal{S}), \quad (S,d) \in \mathcal{S}.$$

Combining the above two formulas:

$$\sum_{(S,d)\in\mathcal{S}} C(S,d) f_{D(S)} = C(\mathcal{S}) f_{D(\mathcal{S})} f_{D_k}.$$

Consequently,

$$\sum_{\text{deg. } n\text{-states } (S,d)} C(S,d) f_{D(S)} \in \langle f_{D_1}, \dots, f_{D_m} \rangle \subseteq \mathcal{O}(\mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C}))). \tag{7.35}$$

Reversing the roles of D and E yields an analogous result. Comparing formulas (7.32) and (7.35), we see that only the non-degenerate n-states can prevent c_n from being a bi-differential operator of order at most $m-1$. We now give an example demonstrating that this obstruction is non-trivial in general. Let $\Sigma = \Sigma_{2,3}$ be the genus 2 surface with 3 boundary components. In Figure 7.1 is an illustration of a BFK-diagram $D = D_1 \sqcup D_2 \sqcup D_3$ on Σ stacked on top of another BFK-diagram $E(= E_1 \sqcup E_2 \sqcup E_3)$ on Σ. It is immediate that all but one of the states for (D,E) permit no trivial loops in their diagrams; the single exception is called S_0 (see Figure 7.2). Of course, any non-degenerate graded state for (D,E) must have S_0 as its underlying state. Since $b_1 = 0$, the only non-degenerate 2-state for (D,E) with non-zero coefficient is (S_0, d_0) where $d_0 \colon D\#E \sqcup T(S_0) \to \mathbf{N}$ is the function vanishing on $D\#E$ and taking the value 2 on the singleton $T(S_0)$. By (7.31) we have $C(S_0,d_0) = (-1)^6 \frac{1}{8}$ so that (7.32) yields

$$c_2(f_D, f_E) = \frac{1}{8} f_{D(S_0)} + \sum_{\text{deg. 2-states } (S,d)} C(S,d) f_{D(S)}.$$

But we may find a connection $[A] \in \mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C}))$ such that

$$f_{D_i}([A]) = 0, \quad i = 1,2,3; \quad f_{D(S_0)}([A]) \neq 0. \tag{7.36}$$

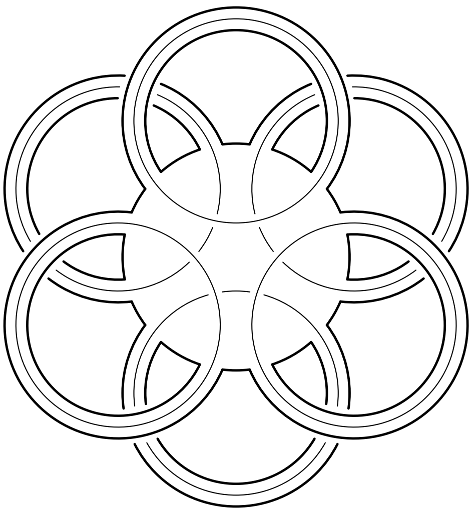


Figure 7.1: Two BFK-diagrams on $\Sigma_{2,3}$.

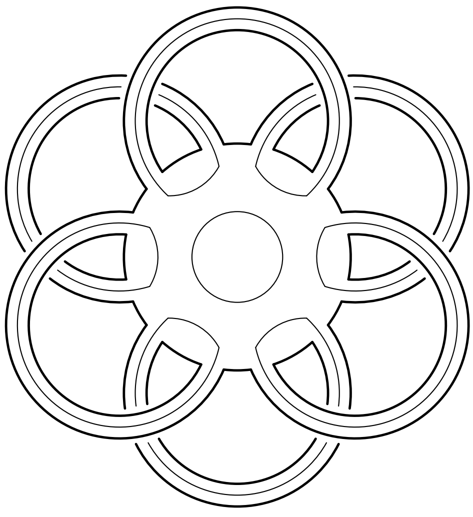


Figure 7.2: The diagram $D(S_0)$.

It thus follows from (7.35) that

$$c_2(f_D, f_E)([A]) = \frac{1}{8} f_{D(S_0)}([A]) \neq 0$$

despite the fact that $f_D = \prod f_{D_i}$ has a third order zero at $[A]$. Hence $c_2(\Sigma_{2,3})$ is not a bi-differential operator of order at most 2 (at $[A]$).

This example can readily be generalized to prove for any $m \in \mathbf{N}$ that $c_2(\Sigma)$ is not a bi-differential operator of order at most m if Σ has sufficiently large genus. In the same vein one can obtain analogous results for coefficients $c_n(\Sigma)$, $n > 2$. Refining the examples a little, it is also possible to derive a more generic condition than (7.36) for a connection to provide a counterexample. In fact, endowing $\mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C}))$ with the topology induced from the Zariski topology on the space $\mathcal{A}(K; \mathrm{SL}_2(\mathbf{C}))$ of $\mathrm{SL}_2(\mathbf{C})$-connections on a complex K modelling Σ (this is independent of K, cf. 2.1), we may summarize our considerations in

Theorem 7.17. *Let $n \geqslant 2$ be an even integer, and let $m \geqslant n$ be another integer. If Σ has sufficiently large genus (the lower bound is linear in m), then there exists a non-empty, open subset of $\mathcal{M}(\Sigma; \mathrm{SL}_2(\mathbf{C}))$ at which $c_n(\Sigma)$ fails to be a bi-differential operator of bi-degree at most m.*

Acknowledgement

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