Extremal Matroid Theory and The Erdös-Pósa Theorem

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Preface

This thesis documents most of my work during the PhD program. To explain the nature of the work, what follows is a short history of the project.

I first heard the word "matroid" about four years ago. I had mainly taken courses in algebra during my undergraduate studies in Århus, when in the fall of 2001 Jørgen Brandt held a one-semester introductory course in combinatorics. The few brief encounters I had had with the field prior to this, I had found intriguing, and Jørgen's course immediately caught my interest. One third of the course material consisted of matroid theory. When at the end of the semester, I was about to apply for a PhD, I made a quick decision and asked Jørgen to supervise me. He agreed and suggested Rota's Conjecture as the subject of my studies.

I quickly learned, that matroid theory is fairly inaccessible to someone inexperienced, since it draws on concepts and methods of several diverse branches of mathematics, such as linear algebra, graph theory and finite geometry. One has to adopt each of these different viewpoints to fully understand many matroid problems.

Since I was quite alone in Århus with my interest in matroids, I spent most of my time reading the matroid literature on my own, without getting any "hands-on experience". In an early stage of the project, in an effort to make things more concrete, I developed a *matroid calculator*, a computer program that performs basic operations on matroids. Though it was a helpful exercise to me, the program has found little use since then.

My main focus, Rota's Conjecture, turned out to be an ambitious one. Indeed, it is perhaps the most important open problem in matroid theory. I was naturally led to reading the recent papers of Geelen, Gerards and Whittle, documenting their progress towards resolving the conjecture. These often quite difficult and technical papers consumed much of my time, and halfway through the course of my studies I wrote my progress report on "Rota's Conjecture: branch-width and grids".

At about the same time I contacted Jim Geelen and Geoff Whittle, both of whom offered to give me supervision. In January 2004 I left for Waterloo, Ontario to visit Jim Geelen for a period of six months.

Working with Jim brought new insights and quickly advanced my understanding of matroid theory. Among many other things, he taught me the geometric perspective on matroids. While I was used to drowning in technicalities and lacking intuition, Jim worked mostly by doing drawings on the blackboard. Most importantly, I gradually became confident, that I could work on matroid problems myself.

I visited Jim in Waterloo a second time in the spring of 2005 for three months. On both occasions we obtained results which I later, back in Denmark, worked out in detail and wrote two papers on. During my first stay in Canada, we worked on an extremal problem related to a result of Jim and his colleagues from their work on Rota's Conjecture. The second time, we mainly worked on different extremal problems related to the Growth Rate Conjecture by Kung. Thus, my project has in the end been concentrated on extremal matroid theory, though it was motivated by extremal aspects of Rota's Conjecture.

During my stays in Canada, with Jim's support, I was also fortunate enough to attend a couple of workshops as well as a major combinatorics conference in Nashville, Tennessee.

Acknowledgements

First of all, I am grateful to my supervisor Jørgen Brandt for introducing me to matroids and for patiently helping with my problems along the way. I also owe thanks to Henning Haahr Andersen for his kind advice on many occasions during both my undergraduate studies and the PhD program. Finally, much gratitude is due to J. Ferdinand Geelen for his willingness to spend time and effort working with an unexperienced PhD student, and for helping in many ways to make my time in Waterloo more enjoyable.

Along with these mentors, I thank all of my family for their patience and support. In particular, I thank my liebling P for keeping me sane throughout the process. Also, respect goes out to Fez, JakeStarr, Jozo, Land, Mo, T-Boz, Teen and V-Sten.

Kasper Kabell, December 2005

The matroid calculator can be found at `www.daimi.au.dk/~kasperk/`

1 Introduction

The field of extremal matroid theory is a broad one, offering many interesting problems and methods. Some of the extremal problems are matroid counterparts of problems in extremal graph theory and many of the methods are derived from there as well. As I know of no precise definition, I shall try here to explain loosely what I consider extremal matroid theory to be.

Extremal matroid theory concerns questions of how different parameters or numerical attributes of matroids behave or relate to each other. Often such questions are studied for certain classes of matroids.

The most common such question and the main focus of Kung's founding paper [**30**] on the subject is "what is the maximum number of points given the rank of a matroid in $\mathcal{M}$", where $\mathcal{M}$ is a given class of matroids. A fundamental result of this type is Kung's Theorem, which bounds the number of points of a rank-n matroid with no $U_{2,q+2}$-minor, by a function of q and n. When q is a prime-power the bound is exact, otherwise it is not.

Another example of an extremal result, which we shall treat, is a matroid version of a theorem of Erdös and Pósa on graphs: A matroid with sufficiently high rank, containing none of a certain set of matroids as a minor, has many disjoint co-circuits.

A third example is the recent theorem of Geelen, Gerards and Whittle, which says that a GF(q)-representable matroid with sufficiently large branch-width contains the cycle matroid of a large grid as a minor.

These examples illustrate, that for some extremal problems an exact relationship can be determined, whereas in other cases one is satisfied simply with the existence of a bound on some parameter. The more difficult extremal problems often fall in the latter category. Therefore the proof techniques tend to be wasteful; one will choose a simpler argument over a better bound.

Nearly all of the contents of this thesis fall under the above description of extremal matroid theory.

1 Outline and main results

The thesis is organized into the following chapters.

Chapters 2 and 3

These chapters present the outcome of my joint work with Jim Geelen. The main results are stated in the introductions to the chapters. Chapter 2 is based on [**19**], *The Erdös-Pósa Property for matroid circuits*, and Chapter 3 is based on [**18**], *Projective geometries in dense matroids.*

The exposition given here differs in a number of respects from that in the papers. It is structured differently and many considerations and remarks have been added, the proofs are somewhat more detailed, the notation differs in some places, and finally, several figures have been added.

Chapter 4

The fourth chapter considers size functions of classes of matroids and Kung's Growth Rate Conjecture. Results of the previous chapters are placed in this context, and two new partial results to the quadratic case of the conjecture are presented. The chapter also contains some minor extensions of Kung's Theorem.

Chapter 5

The fifth chapter briefly surveys the history of Rota's Conjecture and discusses the recent progress, that there has been on the subject. The chapter contains no new material. As described in the preface, I spent much time during the first half of the PhD program studying these matters, and I have included this chapter, in part for my own sense of completeness of the project. The chapter also serves to state some classical matroid results that are referenced elsewhere in the thesis.

2 Prerequisites and notation

The reader is assumed familiar with standard matroid theory, as described in Oxley's excellent book [**37**], and also Aigner [**1**] and Welch [**53**]. Basic results in these sources will be used without reference. We follow the notation and terminology of Oxley, with the exception that we denote the simplification of a matroid M by $\mathrm{si}(M)$. In particular a *graph* means a multigraph, that is, loops and parallel edges are allowed.

The phrase "M contains an N-minor" means that M has a minor *isomorphic* to N. If $N_1, \ldots, N_n$ are matroids, we denote by $\mathcal{EX}(N_1, \ldots, N_n)$ the class of matroids with no N_i-minor, for $i = 1, \ldots, n$. We introduce names for some of the more common classes of matroids as follows.

$\mathcal{G}$	:	The graphic matroids
$\mathcal{G}^*$	:	The co-graphic matroids
$\mathcal{R}(\mathbb{F})$	:	The $\mathbb{F}$-representable matroids, where $\mathbb{F}$ is a field.
$\mathcal{R}(q)$	:	Short for $\mathcal{R}(\mathrm{GF}(q))$, where q is a prime-power.
$\mathcal{U}(q)$	:	Short for $\mathcal{EX}(U_{2,q+2})$, where q is a positive integer.

If q is a prime-power, then $U_{2,q+2}$ is the shortest line, not representable over $\mathrm{GF}(q)$. Hence, $\mathcal{R}(q)$ is contained in $\mathcal{U}(q)$.

We denote by $\mathbb{N}$ the set of positive integers, and by $\mathbb{N}_0$ the set of non-negative integers.

The text contains a number of figures to illustrate various structures. Apart from a few which picture graphs, the figures are all geometric representations of matroids. Strict geometric representations (as in [**37**]) can only be drawn for matroids of rank at most 4. In many cases I have attempted to picture matroids of higher rank by a mixture of Venn diagrams and points, lines and planes. These figures are of a more vague nature; flats of higher rank are drawn as planes and only selected dependencies are shown (see, for instance, page 32). However, the figures are only illustrations, and their intended interpretations should be clear from the context.

2 The Erdös-Pósa Theorem for matroids

This chapter presents a result obtained jointly with Jim Geelen. It is the subject of the article [**19**], which was written by me. The proof of this result takes up most of the chapter. This exposition is different and somewhat more detailed than the one given in [**19**]. The last section on Mader's Theorem for matroids concerns a related result by Geelen, Gerards and Whittle.

1 The Erdös-Pósa property

The number of disjoint co-circuits in a matroid is bounded by its rank. We prove the following theorem, which is a partial converse to this relationship.

Theorem 1.1. *There exists a function $\gamma : \mathbb{N}^3 \to \mathbb{N}$ such that, if M is a matroid with no $U_{a,2a}$-, $M(K_n)$-, or $B(K_n)$-minor and $r(M) \geq \gamma(k, a, n)$, then M has k disjoint co-circuits.*

Here $B(K_n)$ denotes the bicircular matroid of K_n (to be defined below).

A *circuit-cover* of a graph G is a set $X \subseteq E(G)$ such that $G - X$ has no circuits. Thus the maximum number of (edge-) disjoint circuits in a graph is bounded by the minimum size of a circuit cover. This bound is not tight (consider K_4), but Erdös and Pósa in [**10**] proved that the maximum number of disjoint circuits is qualitatively related to the minimum size of a circuit cover.

The Erdös-Pósa Theorem. *There is a function $c : \mathbb{N} \to \mathbb{N}$ such that, if the size of a minimal circuit-cover of G is at least $c(k)$, then G has k disjoint circuits.*

A *circuit-cover* of a matroid is defined analogously. Let M be a matroid. A set $X \subseteq E(M)$ intersects each circuit of M if and only if $E(M) - X$ is independent. Hence, X is a minimal circuit-cover of M if and only if $E(M) - X$ is a basis of M. So, a minimal circuit-cover of M is a basis of M^*. The Erdös-Pósa Theorem can now be rephrased in matroid terminology as follows (note that the statement has been dualized):

Corollary 1.2. *There exists $c : \mathbb{N} \to \mathbb{N}$ such that, if M is a co-graphic matroid with $r(M) \geq c(k)$, then M has k disjoint co-circuits.*

This was generalized by Geelen, Gerards, and Whittle [**15**] who proved the following.

Theorem 1.3. *There exists a function* $c : \mathbb{N}^3 \to \mathbb{N}$ *such that, if* M *is a matroid with no* $U_{2,q+2}$*- or* $M(K_n)$*-minor and* $r(M) \geq c(k, q, n)$*, then* M *has* k *disjoint co-circuits.*

Tutte's excluded minor characterization of the class $\mathcal{G}$ of graphic matroids (see Chapter 5), shows that a co-graphic matroid has no $U_{2,4}$- or $M(K_5)$-minor, so this is indeed a generalization.

The result does not extend to all matroids. A matroid is *round* if it has no two disjoint co-circuits. Equivalently, M is round if each co-circuit in M is a spanning set of M. Or M is round if no two hyperplanes of M cover $E(M)$. There are round matroids of arbitrarily large rank.

Remark 1.4. The matroid $U_{r,n}$, where $n > 2(r-1)$ is round, since it cannot be covered by two hyperplanes (in particular $U_{a,2a}$ is round). Also, for each positive integer n, $M(K_n)$ is a round matroid. Generally, for a graph G, a co-circuit of $M(G)$ is a minimal edge-cut of G. If G is simple it is easily seen, that G has no two disjoint edge-cuts if and only if G is complete (see Figure 1). So $M(G)$ is round if and only if G is complete.

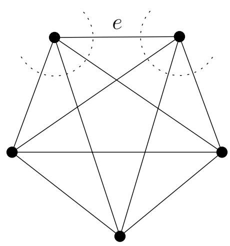


FIGURE 1: The dotted lines indicate two edge-cuts of the graph. If the edge e is removed, the edge cuts are disjoint.

Let $G = (V, E)$ be a graph. Define a matroid $\tilde{B}(G)$ on $V \cup E$ where V is a basis of $\tilde{B}(G)$ and, for each edge $e = uv$ of G, place e freely on the line spanned by $\{u, v\}$ (if e is a loop of G, so $u = v$, place e in parallel with u). Now $B(G) := \tilde{B}(G) \backslash V$ is the *bicircular matroid* of G. A different characterization of $B(G)$ is the following, which gives rise to the term "bicircular". It is easily verified (see [**37**, Prop. 12.1.6]).

Remark 1.5. Let G be a graph. C is a circuit of $B(G)$ if and only if $G[C]$ is a subdivision of one of the following graphs.

The matroid $\tilde{B}(K_n)$ is also round. The bicircular matroid $B(K_n)$ is not round, but it has no three disjoint co-circuits, for $n \neq 3$. These claims will be proved in section 4 of this chapter.

2 Considerations on the main theorem

Before we begin the proof of the main theorem, Theorem 1.1, we make a few observations.

The main theorem is a generalization of Theorem 1.3, as $B(K_n)$ has a $U_{2,q+2}$-minor, for n large (this is shown in Figure 2). It is, in some sense, best possible. Note that each of the families $\{M(K_n) : n \geq 1\}$, $\{B(K_n) : n \geq 1\}$, and $\{U_{a,2a} : a \geq 1\}$ have unbounded rank but they have a bounded number of disjoint co-circuits. Hence, these families must be excluded (from some fixed rank and up), for the Erdös-Pósa property to hold. The main theorem does this by excluding them as minors.

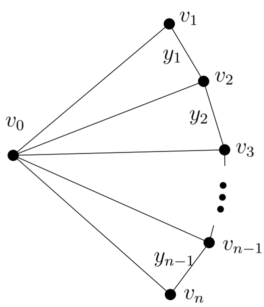


Figure 2: The graph G above is a subgraph of K_{n+1}. It satisfies $B(G)/\{y_1, \ldots, y_{n-1}\} \simeq U_{2,n}$.

Also, we claim that none of the three families are superfluous when we exclude the families as minors. That is, none of the families satisfies, that members of sufficiently high rank contain a high-rank member from one of the other families as a minor.

First consider $\{U_{a,2a} : a \geq 1\}$. For $n \geq 4$, $M(K_n)$ is non-uniform, and for $n \geq 5$, so is $B(K_n)$, which is easily verified. Thus, the uniform family is necessary.

Consider next $\{M(K_n) : n \geq 1\}$. For $a \geq 2$, $U_{a,2a}$ contains a $U_{2,4}$-minor, and for $n \geq 4$, so does $B(K_n)$ (we saw this above for $n \geq 5$, and it is easily checked that $B(K_4) \simeq U_{4,6}$). Hence, none of these can be graphic, and so the graphic family is necessary.

Finally consider $\{B(K_n) : n \geq 1\}$. One can show, that if G is a graph and e an edge of G, then $B(G \backslash e) = B(G) \backslash e$ and if e is not a loop of G, then $B(G/e) = B(G)/e$. It follows, that any loop-less minor of a bicircular matroid is bicircular. It is not hard to check that neither $M(K_4)$ nor $U_{3,7}$ is bicircular (and thus neither is $U_{4,8}$). So the bicircular family is also necessary.

3 Covering number

The proof of Theorem 1.1 takes up most of this chapter. We shall work with dense matroids in the proof. This section develops tools for measuring the size and density of a matroid.

A simple GF(q)-representable matroid M of rank-r has at most $\frac{q^r-1}{q-1}$ elements, since M is isomorphic to a restriction of the projective geometry PG($r-1, q$), which has precisely that many elements. A fundamental result of extremal matroid theory by Kung [**30**] is the extension of this bound to the class $\mathcal{U}(q)$ of matroids with no $U_{2,q+2}$-minor. The bound then holds for all integers $q \geq 2$.

Kung's Theorem. *Let $q \geq 2$ be an integer. If $M \in \mathcal{U}(q)$ is a simple rank-r matroid, then*

$$|E(M)| \leq \frac{q^r - 1}{q - 1}.$$

Proof. The proof is by induction on r, the cases $r = 0$ and $r = 1$ being trivial. Assume $r \geq 2$ and let M be given. Let $e \in E(M)$. For each line L of M containing e, $L - e$ is a rank-1 flat of M/e, and since $M \in \mathcal{U}(q)$, we have $|L - e| \leq q$. Hence, $|E(M)| \leq 1 + q\,|E(\mathrm{si}(M/e))|$. By induction, $|E(M)| \leq 1 + q\frac{q^{r-1}-1}{q-1} = \frac{q^r-1}{q-1}$. ■

For $q = 1$, there is a trivial bound, since the simple matroids in $\mathcal{U}(1)$ are the free matroids $\{U_{n,n} : n \in \mathbb{N}\}$ up to isomorphism. As mentioned, the bound in Kung's Theorem is sharp if q is a prime-power. Kung also proves, that the bound is attained only by projective spaces. By the Fundamental Theorem of Projective Geometry (see Theorem 5.2 of Chapter 3), this implies that for $r \geq 4$, the bound is only sharp if q is a prime-power.

To bound the size of rank-r matroids, it is necessary to restrict the length of lines, or we can have arbitrarily many elements in a rank-2 matroid. As we shall be excluding a uniform matroid of higher rank, we need a new measure of size, for an analogue of Kung's Theorem to hold.

Definition 3.1. Let a be a positive integer. An *a-covering* of a matroid M is a collection $(X_1, \dots, X_m)$ of subsets of $E(M)$, with $E(M) = \cup X_i$ and $r_M(X_i) \leq a$ for all i. The *size* of the covering is m. The *a-covering number* of M, $\tau_a(M)$ is the minimum size of an a-covering of M. If $r(M) = 0$, then we put $\tau_a(M) = 0$.

Note that for a matroid M, $\tau_1(M) = |E(\mathrm{si}(M))|$, so the a-covering number extends the usual notion of size. If M has non-zero rank $r(M) \leq a$, then $\tau_a(M) = 1$. Our first lemma bounds $\tau_a(M)$ in the case $r(M) = a + 1$.

Lemma 3.2. *Let $b > a \geq 1$. If M is a matroid of rank $a + 1$ containing no $U_{a+1,b}$-minor, then*

$$\tau_a(M) \leq \binom{b-1}{a}.$$

Proof. Choose $X \subseteq E(M)$ maximal with $M|X \simeq U_{a+1,l}$. Then $l \leq b - 1$. For an $x \notin X$, by the maximality of X, there exists $Y \subseteq X$ with $|Y| = a$ such that $Y \cup x$ is dependent, and thus $x \in \mathrm{cl}_M(Y)$.

It follows that $(\mathrm{cl}_M(Y) | Y \subseteq X, |Y| = a)$ is an a-covering of M. It has size $\binom{l}{a} \leq \binom{b-1}{a}$. ∎

We obviously always have the inequality $\tau_{a+1}(M) \leq \tau_a(M)$. For matroids with no large rank-$(a+1)$ uniform restriction, using Lemma 3.2, we get a bound in the other direction.

Lemma 3.3. *Let $b > a \geq 1$. If M is a matroid with no $U_{a+1,b}$-restriction, then*

$$\tau_a(M) \leq \binom{b-1}{a} \tau_{a+1}(M).$$

Proof. Let $(X^1, \dots, X^k)$ be a minimal $(a + 1)$-covering of M. By Lemma 3.2 each $M|X^i$ has an a-covering $(X^i_1, \dots, X^i_{m_i})$ of size $m_i \leq \binom{b-1}{a}$. Combining these we get an a-covering $(X^i_j | j = 1, \dots, m_i, i = 1, \dots, k)$ of M. Thus, we have $\tau_a(M) \leq \sum m_i \leq \binom{b-1}{a} k$. ∎

The next result extends Kung's Theorem. The bound we obtain is not sharp, though (in the case where $U_{2,q+2}$ is excluded, it reduces to the expression $(q + 1)^{r-1}$, which exceeds Kung's bound for $r \geq 3$).

Proposition 3.4. *Let $b > a \geq 1$. If M is a matroid of rank $r \geq a$ with no $U_{a+1,b}$-minor, then*

$$\tau_a(M) \leq \binom{b-1}{a}^{r-a}.$$

Proof. The proof is by induction on r. The case $r = a$ is trivial since $(E(M))$ is an a-covering of size 1.

Now let $r > a$ and assume that the result holds for rank $r - 1$. Let x be a non-loop element of M. Then $r(M/x) = r - 1$ and we get by induction $\tau_a(M/x) \leq \binom{b-1}{a}^{r-1-a}$.

Let $(X_1, \ldots, X_k)$ be a minimal a-covering of M/x, so $r_{M/x}(X_i) \leq a$ for all i. This implies $r_M(X_i \cup x) \leq a + 1$, and so $(X_i \cup x | i = 1, \ldots, k)$ is an $(a + 1)$-covering of M. We conclude $\tau_{a+1}(M) \leq \tau_a(M/x)$.

Finally, from Lemma 3.3 we have $\tau_a(M) \leq \binom{b-1}{a}\tau_{a+1}(M)$ and combining inequalities we get the desired result. ∎

Definition 3.5. Let $a \in \mathbb{N}$. The matroid M is called *a-simple*, if M is simple and M has no $U_{k,2k}$-restriction for $k = 2, 3, \ldots, a$.

Equivalently, M is a-simple if it is loop-less and has no $U_{k,2k}$-restriction for $k = 1, 2, 3, \ldots, a$. This concept is just an abbreviation. We shall not define an "a-simplification" operation, since for $a \geq 2$ it would not be well-defined up to isomorphism. For a-simple matroids, the size is proportional to τ_a:

Lemma 3.6. *There exists an integer-valued function $\sigma(a)$ such that, if $a \in \mathbb{N}$ and M is a-simple, then $|E(M)| \leq \sigma(a)\tau_a(M)$.*

Proof. Define σ by

$$\sigma(a) = \prod_{k=2}^{a} \binom{2k-1}{k-1}.$$

Since M has no $U_{k,2k}$-restriction for $k = 2, \ldots, a$, Lemma 3.3 gives

$$\tau_{k-1}(M) \leq \binom{2k-1}{k-1}\tau_k(M), \quad k = 2, \ldots, a.$$

Putting these together, we get $|E(M)| = \tau_1(M) \leq \sigma(a)\tau_a(M)$. ∎

We shall need one more specialized result, which is completely similar to the previous Lemma.

Lemma 3.7. *There exists an integer-valued function $\sigma_2(a,b)$ such that, if $b \geq a \geq 1$ and M is loop-less and has no $U_{k,b}$-restriction for $k = 1,\dots,a$, then $|E(M)| \leq \sigma_2(a,b)\tau_a(M)$.*

Proof. Define σ_2 by

$$\sigma_2(a,b) = (b-1)\prod_{k=2}^{a}\binom{b-1}{k-1}.$$

Now use $|E(M)| \leq (b-1)\tau_1(M)$ and apply Lemma 3.3. ■

4 Round matroids

In this section we investigate the concept of roundness. We first list a few properties.

- If M is round and $Y \subseteq E(M)$, then M/Y is round.
- If N is a spanning minor of M and N is round, then M is round.
- If M is round, then $\mathrm{si}(M)$ is round.

These properties are easily checked directly, but they also follow as special cases from Proposition 4.2 below.

We shall sometimes consider matroids that are only "nearly round". This is made precise using two matroid parameters, that we define next.

Definition 4.1. Let M be a matroid. The *rank-deficiency* of a set of elements $X \subseteq E(M)$ is $r^-_M(X) = r(M) - r_M(X)$. Denote by $\Gamma(M)$ the maximum rank-deficiency among the co-circuits of M. For an integer t we say that M is *t-round* if $\Gamma(M) \leq t$. By $\Theta(M)$ we denote the maximum number of disjoint co-circuits of M.

Notice that a matroid M is round if and only if $\Gamma(M) = 0$, that is, M is 0-round. Also, M is round if and only if $\Theta(M) = 1$. The two parameters are related by

$$1 \leq \Theta(M) \leq \Gamma(M) + 1 \leq r(M).$$

For some integer k, let $C_1,\dots,C_k$ be disjoint co-circuits of M. We then have $r(M\backslash(C_1 \cup \cdots \cup C_k)) \leq r(M) - k$. Notice that, if $\Gamma(M) \leq k-1$, so that $r_M(C) > r(M) - k$ for each co-circuit C of M, then $\Theta(M) \leq k$. From this, the inequality $\Theta(M) \leq \Gamma(M) + 1$ follows. Equality does not hold, as can be seen by considering $M = U_{4,5}$, for which $\Theta(M) = 2$ and $\Gamma(M) = 2$. The following result lists hereditary properties of the two parameters.

Proposition 4.2. *Let M be a matroid and let $X, Y \subseteq E(M)$. Then*
(i) $\Theta(M/Y) \leq \Theta(M)$ *and* $\Gamma(M/Y) \leq \Gamma(M)$.
(ii) $\Theta(M\backslash X) \geq \Theta(M)$ *and* $\Gamma(M\backslash X) \geq \Gamma(M)$, *if X is co-independent.*
(iii) $\Theta(M\backslash X) = \Theta(M)$ *and* $\Gamma(M\backslash X) = \Gamma(M)$, *if for some positive integer a, X is minimal w.r.t. inclusion, such that $M\backslash X$ is a-simple.*

Proof. Every co-circuit C of M/Y is a co-circuit of M. A short calculation shows that $r^{-}_{M/Y}(C) \leq r^{-}_{M}(C)$, so the first assertion of the lemma holds.

To prove the second and third assertions, it is enough to consider $X = \{x\}$, where x is not a co-loop of M. If C is a co-circuit in M, then $C - x$ contains a co-circuit in $M\backslash x$. Thus $\Theta(M\backslash x) \geq \Theta(M)$ and also $\Gamma(M\backslash x) \geq \Gamma(M)$.

We turn to the third assertion. Assume that $x \in W$, where $M|W \simeq U_{k,2k}$, for a $k \in \mathbb{N}$. If C is a co-circuit of $M\backslash x$, then $C = C' - x$ for a co-circuit C' of M, that is, either C or $C \cup x$ is a co-circuit of M. We look at two cases:

- If $C \cap (W - x) = \emptyset$, then C is a co-circuit of M, since the complement of a co-circuit is closed and $x \in \mathrm{cl}_M(W - x)$.
- If $C \cap (W - x) \neq \emptyset$, then we have $|(W - x) - C| < k$, as $M|(W - x) \simeq U_{k,2k-1}$ and the complement of C is closed. Hence, $|C \cap (W - x)| \geq k$.

Note that the second case can happen at most once in a collection of disjoint co-circuits. So, given a collection of disjoint co-circuits of $M\backslash x$, by adding x to at most one of them, we get a collection of disjoint co-circuits of M. Thus $\Theta(M) \geq \Theta(M\backslash x)$. Note also, that for a co-circuit C of $M\backslash x$, if $C \cup x$ is a co-circuit of M, then we are in the second case, and $r_M(C \cup x) = r_{M\backslash x}(C)$. Thus $\Gamma(M) \geq \Gamma(M\backslash x)$. Finally, since no co-circuit can contain a loop, deleting loops also preserves Θ and Γ. ∎

To prove that $\tilde{B}(K_n)$ is round and $B(K_n)$ nearly round, we shall use the following lemma.

Lemma 4.3. *Let M be a matroid, and H_1,H_2 distinct hyperplanes of M, with $H_1 \cup H_2 \neq E(M)$. If $\Theta(M|H_1) \leq m$ and $\Theta(M|H_2) \leq m$, then $\Theta(M) \leq m$.*

Proof. Let $e \in E(M) - (H_1 \cup H_2)$. We may assume, that $E(M) = H_1 \cup H_2 \cup e$, by Proposition 4.2 (ii). Suppose that there exists a collection $\mathcal{C}$ of $m+1$ disjoint co-circuits of M.

If C is a co-circuit of M and $C \cap H_1 \neq \emptyset$, then $C \cap H_1$ is a co-circuit of $M|H_1$. Since $|\mathcal{C}| > \Theta(M|H_1)$, one member of $\mathcal{C}$ must be $C_1 = E(M) - H_1$. Similarly, we argue that $C_2 = E(M) - H_2 \in \mathcal{C}$. But $e \in C_1 \cap C_2$, contradicting that the members of $\mathcal{C}$ are disjoint. ∎

Let M be a matroid and F a flat in M. We say that F is a *round* flat of M if $M|F$ is round. Taking $k = 1$ in the above lemma, we get the following result from [**23**].

Lemma 4.4. *Let M be a matroid, and H_1, H_2 distinct round hyperplanes of M, with $H_1 \cup H_2 \neq E(M)$. Then M is round.*

We now define a family of matroids, that we call Dowling-cliques. They are special cases of a class of combinatorial geometries (simple matroids) introduced by Dowling in [**9**].

Definition 4.5. A matroid M with basis V is a *Dowling-clique*, if $E(M) = V \cup X$, where $V = \{v_1, \ldots, v_n\}$, and $X = \{e_{ij} : 1 \leq i < j \leq n\}$ satisfies that $\{v_i, v_j, e_{ij}\}$ is a triangle, for all $i < j$. We shall sometimes refer to a Dowling-clique by writing (M, V), to emphasize the basis V.

The matroid $\tilde{B}(K_n)$ is a Dowling-clique by construction. Consider $M = M(K_n)$. Let v be a vertex of K_n, and let V be the set of edges incident with v. Then V is a basis of M, and (M, V) is a rank-$(n-1)$ Dowling-clique. It satisfies $M \backslash V \simeq M(K_{n-1})$.

Lemma 4.6. *Dowling-cliques are round. In particular, $M(K_n)$ and $\tilde{B}(K_n)$ are round. For $n > 3$, $\Theta(B(K_n)) = 2$.*

Proof. We first prove that a Dowling-clique (M, V) is round by induction in the rank r. The case $r = 1$ is trivial, so assume $r \geq 2$. Consider a hyperplane H of M spanned by $V - v$ for some $v \in V$. Then $(M|H, V - v)$ is a rank-$(r-1)$ Dowling-clique. So H is round by induction. Two such hyperplanes do not cover $E(M)$, so M is round by Lemma 4.4.

It is easily verified, that $B(K_4) \simeq U_{4,6}$, and so $\Theta(B(K_4)) = 2$. Let (M, V) be the Dowling-clique $\tilde{B}(K_n)$ and let $N = M \backslash V = B(K_n)$. Consider a hyperplane H of M spanned by $V - v$ for some $v \in V$. Using Remark 1.5, it is easy to show, that $H - V$ is a hyperplane of N and $N|(H - V) \simeq B(K_{n-1})$. As above, a simple induction shows, that $\Theta(N) \leq 2$, this time using Lemma 4.3. Since $B(K_n)$ is not round, $\Theta(B(K_n)) = 2$. ∎

The first step in the proof of the main theorem, Theorem 1.1 is to show, that a matroid of large enough rank either has k disjoint co-circuits or a large minor N which is "nearly round", in the sense that $r(N)$ is large compared to $\Gamma(N)$.

Lemma 4.7. *Let $g : \mathbb{N}_0 \to \mathbb{N}_0$ be a non-decreasing function. There exists a function $f_g : \mathbb{N} \to \mathbb{N}$ such that for any $k \in \mathbb{N}$, if M is a matroid with $r(M) \geq f_g(k)$, then either*

(a) *M has k disjoint co-circuits or*

(b) *M has a minor $N = M/Y$ with $r(N) \geq g(\Gamma(N))$.*

Proof. Let g be given and define f_g as follows: $f_g(1) = 1$ and

$$f_g(k) = g(f_g(k-1)), \quad k \geq 2.$$

The proof is by induction on k. If $r(M) \geq 1$, then M has a co-circuit, so the result holds for $k = 1$. Now let $k \geq 2$ and $r(M) \geq f_g(k) = g(f_g(k-1))$.

If $\Gamma(M) \geq f_g(k-1)$, then pick a co-circuit C of M with $r_M^-(C) = \Gamma(M)$. Then $r(M/C) = r_M^-(C) \geq f_g(k-1)$. If M/C has the desired contraction minor, then we are done. If not, then by induction M/C has $k-1$ disjoint co-circuits. These, together with C, give k disjoint co-circuits of M.

If $\Gamma(M) \leq f_g(k-1)$, then as g is non-decreasing, we have $r(M) \geq g(\Gamma(M))$. ∎

5 Building density

The goal of this section is to prove, that a high-rank nearly round matroid with no $U_{a+1,b}$-minor contains a dense minor. We think of a matroid as being dense, if its a-covering number is large compared to the rank (for $a = 1$, this is the usual concept of density). For "nearly round", we use the condition $\Gamma(M) \leq \frac{1}{2}r(M)$. We first prove two technical lemmas.

Lemma 5.1. *Let $b > a \geq 1$. Let M be a matroid with no $U_{a+1,b}$-minor and let C be a co-circuit of M of minimal size. If $C_1, \ldots, C_k$ are disjoint co-circuits of $M \backslash C$ with $|C_1| \leq \cdots \leq |C_k|$, then $|C_i| \geq |C| / (a\binom{b-1}{a})$ for $i = a, \ldots, k$.*

Proof. Let C and $C_1 \ldots, C_k$ be given and let $i \in \{a, \ldots, k\}$.

As C_1 is co-dependent in $M \backslash C \backslash C_i$, there exists a co-circuit $C_1' \subseteq C_1$ of $M \backslash (C \cup C_i)$.

Now, C_2 is co-dependent in $M \backslash C \backslash (C_i \cup C_1')$. So there is a co-circuit $C_2' \subseteq C_2$ of $M \backslash (C \cup C_i \cup C_1')$.

Continuing in this fashion, for each $j = 2, \ldots, a-1$ we pick a co-circuit $C_j' \subseteq C_j$ of $M \backslash (C \cup C_i \cup C_1' \cup \cdots \cup C_{j-1}')$.

Denote by F the set $E(M) - (C \cup C_i \cup C_1' \cup \cdots \cup C_{a-1}')$ (see Figure 3). Deleting a co-circuit of a matroid drops its rank by 1, so we get $r_M^-(F) = a+1$.

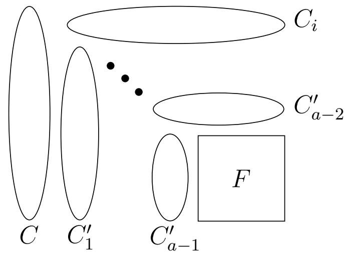


Figure 3

Hence $N = M/F$ has rank $r(N) = a+1$. Since C is a co-circuit of N of minimal size, $E(N) - C$ must be a rank-a set of N of maximal size. We now have

$$\begin{aligned}|C| \le |E(N)| &\le \tau_a(N)\,|E(N) - C|\\ &= \tau_a(N)\,\left|C_i \cup C'_1 \cup \cdots \cup C'_{a-1}\right|\\ &\le \binom{b-1}{a} a\,|C_i|\end{aligned}$$

using Lemma 3.2. This proves the desired result. ■

Lemma 5.2. *There exists an integer-valued function $\kappa(\lambda, a, b)$ such that the following holds: Let $b > a \ge 1$ and $\lambda \in \mathbb{N}$. Let M be an a-simple matroid with no $U_{a+1,b}$-minor, satisfying $\Gamma(M) \le \frac{1}{2}r(M)$. Let C be a minimal sized co-circuit of M. If $M\backslash C$ has $\kappa(\lambda, a, b)$ disjoint co-circuits, then $\tau_a(M) > \lambda r(M)$.*

Proof. Let a, b and λ be given and define

$$\kappa(\lambda, a, b) = \kappa = 2a\binom{b-1}{a}\sigma(a)\lambda + a - 1.$$

Let M and C be given and let $C_1, \ldots, C_\kappa$ be disjoint co-circuits of $M\backslash C$, in non-decreasing order by size. Note that

$$|C| \ge r_M(C) \ge r(M) - \Gamma(M) \ge r(M)/2.$$

By Lemma 3.6 and the above lemma we have

$$\begin{aligned}\sigma(a)\tau_a(M) &\ge |E(M)|\\ &> |C_a| + \cdots + |C_\kappa| \ge (\kappa - a + 1)\frac{r(M)}{2a\binom{b-1}{a}} = \sigma(a)\lambda r(M),\end{aligned}$$

and the result follows. ■

The next lemma is the main result of this section.

Lemma 5.3. *There exists an integer-valued function $\delta(\lambda, a, b)$ such that the following holds: Let $b > a \geq 1$ and $\lambda \in \mathbb{N}$. If M is a matroid with no $U_{a+1,b}$-minor, such that $\Gamma(M) \leq \frac{1}{2}r(M)$ and $r(M) \geq \delta(\lambda, a, b)$, then M has a minor N with $\tau_a(N) > \lambda r(N)$.*

Proof. Let a, b and λ be given and fixed, and let us define $\delta(\lambda, a, b)$. First, we define a sequence of functions $g_n : \mathbb{N}_0 \to \mathbb{N}_0$. Let $g_0(m) = 0$, and for $n \geq 1$ define g_n recursively by

$$g_n(m) = \max(2m, \delta_n),$$
$$\text{where } \delta_n = 2(f_{g_{n-1}}(\kappa(\lambda, a, b)) + 1) \in \mathbb{N}.$$

Finally, let $\delta(\lambda, a, b) = \delta_{n_0}$, where $n_0 = 2\sigma(a)\lambda$. The functions σ, κ and f_{g_n} are defined in previous lemmas. We first prove a partial result.

<u>CLAIM</u>. Let $n \geq 0$. If M is a matroid with no $U_{a+1,b}$-minor, such that $r(M) \geq g_n(\Gamma(M))$, then either

- M has a minor N with $\tau_a(N) > \lambda r(N)$ or
- there exists a sequence of matroids $M = M_0, M_1, \ldots, M_n$, such that for $i = 0, \ldots, n-1$, $M_{i+1} = M_i \backslash C_i / Y_i$, where C_i is a co-circuit of M_i that spans M_i / Y_i.

We prove the claim by induction on n. The case $n = 0$ is trivial, so assume $n \geq 1$ and that the result holds for $n - 1$.

Let $X \subseteq E(M)$ be minimal such that $M \backslash X$ is a-simple. Pick a co-circuit C_0 of M with $|C_0 - X|$ minimal. Note, that $C = C_0 - X$ is a co-circuit of $M \backslash X$ of minimal size.

Choose a basis Z of M/C_0 and let $M' = M/Z$. Then C_0 spans M', and since $r(M) \geq g_n(\Gamma(M))$,

$$r(M') = r_M(C_0) \geq r(M) - \Gamma(M) \geq \tfrac{1}{2}r(M) \geq \tfrac{1}{2}\delta_n.$$

Now $r(M' \backslash C_0) = r(M') - 1 \geq f_{g_{n-1}}(\kappa(\lambda, a, b))$, so by Lemma 4.7 one of the following holds:

(a) $M' \backslash C_0$ has $\kappa(\lambda, a, b)$ disjoint co-circuits.

(b) $M' \backslash C_0$ has a minor $M_1 = M' \backslash C_0 / Y$ with $r(M_1) \geq g_{n-1}(\Gamma(M_1))$.

Assume first that (a) holds. Since $M' \backslash C_0 = M \backslash C_0 / Z$, by Proposition 4.2(i), $M \backslash C_0$ has $\kappa(\lambda, a, b)$ disjoint co-circuits. We claim, that $X - C_0$ is co-independent in $M \backslash C_0$. If not, then there exists a co-circuit $D \subseteq X \cup C_0$ of M with $D \cap (X - C_0) \neq \emptyset$, contradicting our choice of C_0. Now, by Proposition 4.2(ii),

$$\Theta((M \backslash X) \backslash C) = \Theta(M \backslash (C_0 \cup X)) \geq \Theta(M \backslash C_0) \geq \kappa(\lambda, a, b).$$

The proposition also gives $\Gamma(M\backslash X) \leq \frac{1}{2}r(M\backslash X)$. We can now apply Lemma 5.2 to $N = M\backslash X$, and get the desired result.

Assume now that (b) holds. Letting $Y_0 = Z \cup Y$, we have $M_1 = M\backslash C_0/Y_0$ and C_0 spans M/Y_0. Applying the induction hypothesis to M_1 now gives the claim.

Let M be given as in the lemma, and note that $r(M) \geq g_n(\Gamma(M))$, where $n = 2\sigma(a)\lambda$. By the claim, either we are done or there is a sequence of matroids $M = M_0, \ldots, M_n$, such that for $i = 0, \ldots, n-1$, $M_{i+1} = M_i\backslash C_i/Y_i$, where C_i is a co-circuit of M_i that spans M_i/Y_i.

Let $M' = M/(Y_0 \cup \cdots \cup Y_{n-1})$. Notice, that for $i = 0, \ldots, n-1$, C_i is a spanning co-circuit of $M'\backslash(C_0 \cup \cdots \cup C_{i-1})$. Thus $r_{M'}(C_i) = r - i$, where $r = r(M')$. For all i, choose a basis B_i for $M'|C_i$, and define $N = M'|(\cup B_i)$. Then

$$|E(N)| = \sum_{i=0}^{n-1}(r-i) > \frac{nr}{2}.$$

We claim that N is a-simple. Suppose $N|W \simeq U_{k,2k}$ for a $W \subseteq E(N)$ and $k \in \mathbb{N}$. Then $|W \cap B_0| \leq k$, as B_0 is independent. So $|W \cap (E(N) - B_0)| \geq k$, and since $E(N) - B_0$ is closed, $W \cap B_0 = \emptyset$. Repeat this argument in $N\backslash B_0$ to see, that $W \cap B_1 = \emptyset$ etc. We end up with $W \subseteq B_{n-1}$, a contradiction.

Finally, by Lemma 3.6,

$$\sigma(a)\tau_a(N) \geq |E(N)| > \frac{nr}{2} = \sigma(a)\lambda r(N),$$

and the result follows. ∎

6 Arranging circuits

We wish to identify some more concrete structure in a dense matroid. To do this, we need to be able to disentangle some of the many low-rank sets in the matroid.

For a matroid M, we call sets $A_1, \ldots, A_n \subseteq E(M)$ *skew* if $r_M(\cup_i A_i) = \sum_i r_M(A_i)$. This is analogous to subspaces of a vector-space forming a direct sum. The first result of this section is a tool for finding sets in a matroid, that are close to being skew. This is made precise using the following definition.

We define a function μ_M on collections of subsets of $E(M)$ as follows. For sets $A_1, \dots, A_n \subseteq E(M)$, let

$$\begin{aligned}\mu_M(A_1,\dots,A_n) &= r_M(\bigcup_j A_j) - \sum_i \Big(r_M(\bigcup_j A_j) - r_M(\bigcup_{j\neq i} A_j)\Big)\\ &= r_M(\bigcup_j A_j) - \sum_i r_{M/(\cup_{j\neq i}A_j)}(A_i - \cup_{j\neq i}A_j)).\end{aligned}$$

This function can be thought of as a generalized connectivity function. For $n = 2$, μ_M equals the connectivity function $\lambda_M(A_1, A_2) = r_M(A_1) + r_M(A_2) - r_M(A_1 \cup A_2)$. For $n \geq 2$ a recursive formula holds,

$$\mu_M(A_1,\dots,A_n) = \lambda_M(A_1, A_2 \cup \dots \cup A_n) + \mu_{M/A_1}(A_2,\dots,A_n).$$

In showing this, to ease notation, we may assume that $A_1, \dots, A_n$ are disjoint (otherwise we can add elements in parallel wherever the sets intersect, and make them disjoint).

$$\begin{aligned}\mu_M(A_1,\dots,A_n) &= r_M(\bigcup_j A_j) - \sum_i r_{M/(\cup_{j\neq i}A_j)}(A_i)\\ &= r_M(A_1) - r_{M/(A_2\cup\cdots\cup A_n)}(A_1)\\ &\quad + r_{M/A_1}(A_2\cup\dots\cup A_n) - \sum_{i=2}^{n} r_{M/A_1/(\cup_{j\neq 1,i}A_j)}(A_i)\\ &= \lambda_M(A_1, A_2\cup\dots\cup A_n) + \mu_{M/A_1}(A_2,\dots,A_n).\end{aligned}$$

The function μ_M measures in a way the rank of the "overlap" of the sets, though this may not be an actual set in the matroid (see Figure 4). Notice, that $\mu_M(A_1,\dots,A_n) = 0$ if and only if $A_1,\dots,A_n$ are skew. More generally, if there is a set $W \subseteq E(M)$ such that $A_1 - W, \dots, A_n - W$ are skew in M/W, then $\mu_M(A_1,\dots,A_n) \leq r_M(W)$.

Lemma 6.1. *There exists an integer-valued function $\alpha_1(n, r, a, b)$ such that the following holds: Let $b > a \geq 1$, and let r and n be positive integers. If M is a matroid with no $U_{a+1,b}$-minor, and $\mathcal{F}$ is a collection of rank-r subsets of $E(M)$ with $r_M(\cup_{X\in\mathcal{F}}X) \geq \alpha_1(n, r, a, b)$, then there exist $X_1,\dots,X_n \in \mathcal{F}$ satisfying*
(a) *$X_i \not\subseteq \mathrm{cl}_M(\cup_{j\neq i}X_j)$ for $i = 1,\dots,n$ and*
(b) *$\mu_M(X_1,\dots,X_n) \leq (r-1)a$.*

The bound $\mu_M(X_1,\dots,X_n) \leq (r-1)a$ is best possible. To see this, let $m \geq a$, and consider the matroid

$$M = U_{a,m} \oplus \cdots \oplus U_{a,m} \oplus e_1 \oplus \cdots \oplus e_m,$$

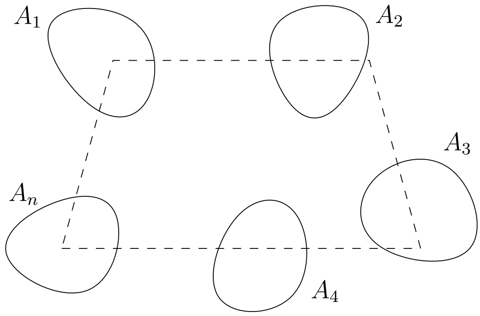


FIGURE 4

consisting of $r-1$ skew copies of $U_{a,m}$ and m co-loops $e_1, \ldots, e_m$. Then M has no $U_{a+1,b}$-minor. For $i = 1, \ldots, m$, let $W_i \subseteq E(M)$ consist of a unique element from each copy of $U_{a,m}$ along with e_i. For any collection $X_1, \ldots, X_n$ of distinct members of $\mathcal{F} = \{W_i\}$ with $n > a$, it is easily seen that $\mu_M(X_1, \ldots, X_n) = (r-1)a$.

Proof. For any positive integers n, c, k, we let $R(n, c, k)$ denote the following Ramsey number: The minimal R, such that if X is a set with $|X| = R$, then for any c-coloring of $[X]^n$, X has a monochromatic subset of size k. Here $[X]^n$ denotes the set of all subsets of X of size n. By a monochromatic subset of X, we mean a subset $Y \subseteq X$ such that the sets in $[Y]^n$ all have the same color. This number exists by Ramsey's Theorem (see [**40**] or [**7**, 9.1.4]).

Let n, r, a, b be given and let us define $\alpha_1(n, r, a, b)$. First we define numbers s_i, l_i for $i = 1, \ldots, r$. Let $s_r = 0, l_r = n$, and for $i = r-1, r-2, \ldots, 1$ define recursively:

$$s_i = s_{i+1} + l_{i+1}, \quad u_i = \binom{b-1}{a}^{rs_i - a}, \quad l_i = n\binom{u_i}{r-i}.$$

Let $m = s_1 + l_1$. So, we have $0 = s_r < s_{r-1} < \cdots < s_1 < m$. Next, define numbers $k_1, \ldots, k_m$ as follows. Let $k_m = m$ and define recursively:

$$k_{i-1} = R(i, r, k_i), \qquad \text{for } \ i = m, m-1, \ldots, 2.$$

Finally, let $\alpha_1(n, r, a, b) = rk_1$.

In the following, for a set of subsets $\mathcal{X} \subseteq 2^{E(M)}$, we use the shorthand notation $r_M(\mathcal{X}) = r_M(\cup_{X \in \mathcal{X}} X)$.

Let M and $\mathcal{F}$ be given, with $r_M(\mathcal{F}) \geq \alpha_1(n,r,a,b) = rk_1$. We can choose sets $Y_1,\ldots,Y_{k_1} \in \mathcal{F}$, such that $Y_i \notin \mathrm{cl}_M(Y_1 \cup \cdots \cup Y_{i-1})$. Let $\mathcal{F}_1 = \{Y_1,\ldots,Y_{k_1}\}$ and put $a_0 = 0, a_1 = r$. We shall iteratively construct sequences:

$$\mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \cdots \supseteq \mathcal{F}_m, \quad a_0 < a_1 < a_2 < \cdots < a_m,$$

such that for $i = 1,\ldots,m$, $|\mathcal{F}_i| = k_i$, and if $\mathcal{F}' \subseteq \mathcal{F}_i$ with $|\mathcal{F}'| = i$, then $r_M(\mathcal{F}') = a_i$. This clearly holds for $\mathcal{F}_1$. Let $i \geq 2$, assume that $\mathcal{F}_{i-1}$ and a_{i-1} satisfy the above, and let us find $\mathcal{F}_i$ and a_i.

Note, that $r_M(\mathcal{F}') \in \{a_{i-1}+1,\ldots,a_{i-1}+r\}$, for $\mathcal{F}' \subseteq \mathcal{F}_{i-1}$ with $|\mathcal{F}'| = i$. This defines an r-coloring of $[\mathcal{F}_{i-1}]^i$. Since $|\mathcal{F}_{i-1}| = k_{i-1} = R(i,r,k_i)$, there exists $\mathcal{F}_i \subseteq \mathcal{F}_{i-1}$ such that, every set in $[\mathcal{F}_i]^i$ has the same rank, and we let a_i be that number.

For $i = 1,\ldots,m$ let $b_i = a_i - a_{i-1}$. Notice that, by submodularity, this gives a decreasing sequence $(a_{i+1} + a_{i-1} \leq a_i + a_i)$,

$$r = b_1 \geq b_2 \geq \cdots \geq b_m \geq 1.$$

Hence, by definition of the pairs (s_i, l_i), there exists an $r' \in \{1,\ldots,r\}$, such that

$$b_{s+1} = \cdots = b_{s+l} = r', \qquad \text{where } s = s_{r'} \text{ and } l = l_{r'}.$$

If $r' = r$, then we get $b_1 = \cdots = b_n = r$. Thus, if we choose any n members $X_1,\ldots,X_n \in \mathcal{F}_m$, then they are skew and we are done.

Assume $r' < r$. Choose s sets $Z_1,\ldots,Z_s \in \mathcal{F}_m$, and let $F = \cup_{i=1}^s Z_i$. Choose another l sets $X_1,\ldots,X_l \in \mathcal{F}_m - \{Z_1,\ldots,Z_s\}$. Since $b_{s+1} = b_{s+l} = r'$, the sets $X_1 - F,\ldots,X_l - F$ are skew of rank r' in M/F. For $i = 1,\ldots,l$, choose an independent set $\overline{B}_i \subseteq X_i$ of size r', skew from F. Expand this set to a basis $B_i \cup \overline{B}_i$ of X_i, so $|B_i| = r_0 = r - r'$.

Let $M' = M/(\cup_i \overline{B}_i)$ and $B = \cup_i B_i$. Then $B_i \subseteq \mathrm{cl}_{M'}(F)$, and thus $r_{M'}(B) \leq r_{M'}(F) \leq sr$. Let $(W_1,\ldots,W_u)$ be a minimal a-covering of $M'|B$. By Proposition 3.4, we have

$$u = \tau_a(M'|B) \leq \binom{b-1}{a}^{sr-a} = u_{r'}.$$

For each B_i, we can find a set of indices $I_i \subseteq \{1,\ldots,u\}$ of size r_0, such that $B_i \subseteq \cup_{j \in I_i} W_j$. There are $\binom{u}{r_0} \leq \binom{u_{r'}}{r-r'}$ possible choices for I_i, and $l = n\binom{u_{r'}}{r-r'}$. By a majority argument, there must exist $I \subseteq \{1,\ldots,u\}$, such that $I_i = I$ for all $i \in J$, where $J \subseteq \{1,\ldots,l\}$ has size n. By possibly re-ordering the $X_i's$ and the $W_j's$ we can assume, that $B_1,\ldots,B_n \subseteq W_1 \cup \cdots \cup W_{r_0}$.

Let $W = W_1 \cup \cdots \cup W_{r_0}$. Then the sets $X_1 - W,\ldots,X_n - W$ are skew in M/W. It follows, that $\mu_M(X_1\ldots,X_n) \leq r_M(W) \leq ar_0 \leq a(r-1)$, and we are done. ∎

The next lemma shows how, by doing suitable contractions, a large collection of nearly (but not completely) skew circuits, can yield a set of nearly skew triangles containing a common element. The idea is to put points in the "overlap" by contracting some of the circuits. The overlap can then be contracted to a point.

Lemma 6.2. *There exists an integer-valued function $\alpha_2(l, r, m)$ such that the following holds: Let $r \geq 2$ and l, m be positive integers. If $n \geq \alpha_2(l, r, m)$ and $C_1 \ldots, C_n$ are rank-r circuits of a matroid M satisfying*

(a) $1 \leq r_M(\cup_j C_j) - r_M(\cup_{j \neq i} C_j) < r$ *for all i, and*

(b) $\mu_M(C_1, \ldots, C_n) \leq m$,

then M has a minor $N = M/Y$ with an element $x \in E(N)$ and triangles $D_1, \ldots, D_l$ of N, such that

- $x \in D_i$ *for all i, and* $r_N(\cup_i D_i) = l + 1$,
- *for all i, $D_i - x \subseteq C_j$ for some $j \in \{1, \ldots, n\}$.*

Proof. Let l and m be fixed. For $r \geq 2$, define $\alpha_2(l, r, m)$ recursively as follows

$$\alpha_2(l, r, m) = 2^m (q_r(r-1) + 1),$$

$$q_2 = l, \qquad q_r = \alpha_2(l, r-1, r-2), \qquad \text{for } r > 2.$$

To facilitate induction, the lemma is proved from the following weaker set of assumptions:

Let $n \geq \alpha_2(l, r, m)$, and $C_1 \ldots, C_n$ be circuits of M with $2 \leq r_M(C_i) \leq r$. Assume there is a set $F \subseteq E(M)$, such that

(a) $1 \leq r_M((\cup_j C_j) \cup F) - r_M((\cup_{j \neq i} C_j) \cup F) < r_M(C_i) - 1$ *for all i.*

(b) $\mu_M(C_1, \ldots, C_n, F) \leq m$.

These assumptions are indeed weaker, since the phrasing in the lemma is the case where $F = \emptyset$ and $r_M(C_i) = r$ for all i. The proof is by induction on r. Let $r = 2$ or let $r > 2$ and assume the result holds for $r - 1$.

Let $c_i = r_M((\cup_j C_j) \cup F) - r_M((\cup_{j \neq i} C_j) \cup F)$ for each i. We first do an easy reduction. If not $c_i = 1$ for all i, then for each i choose a set $Y_i \subseteq C_i$ of size $c_i - 1$, which is skew from $(\cup_{j \neq i} C_j) \cup F$. We may then work with the circuits $C_i - Y_i$ of $M/(Y_1 \cup \cdots \cup Y_n)$ instead. So without loss of generality, we may assume $c_i = 1$ for all i.

Choose $z_i \in C_i - \mathrm{cl}_M(\cup_{j \neq i} C_j)$ for each i, and let $\overline{M} = M/\{z_1, \ldots, z_n\}$. Letting $W = \cup_i (C_i - z_i)$ we have

$$r_{\overline{M}}(W) = r_M(\cup_i C_i) - r_M(\{z_1, \ldots, z_n\}) = \mu_M(C_1, \ldots, C_n, F) \leq m.$$

Let B be a basis of $\overline{M}|W$ and choose a basis B_i of $\overline{M}|(C_i - z_i)$ for each i. Now expand B_i to a basis $B_i \cup X_i$ of $\overline{M}|W$ using elements of B. For all i we have chosen $X_i \subseteq B$ among the $2^{|B|} \leq 2^m$ subsets of B. Hence, there exists an $X_0 \subseteq B$, such that $X_i = X_0$ for $i \in I$, where $|I| = n' \geq n/2^m$. Let $M_1 = M/X_0$ and put $r' = |B| - |X_0| + 1$. Then $r_{M_1}(C_i) = r'$ for all $i \in I$, and $\mu_{M_1}(C_i : i \in I) = r' - 1$. By possibly reordering the circuits, we can assume $I = \{1, \ldots, n'\}$.

Pick an element of one circuit, $z \in C_{n'} - \text{cl}_{M_1}(\cup_{j<n'} C_j)$ and let $M_2 = M_1/z$. Define

$$Z = \text{cl}_{M_2}(C_{n'} - z) \subseteq \text{cl}_{M_2}(C_i), \quad \text{for } i = 1, \ldots, n' - 1,$$

so $r_{M_2}(Z) = r' - 1$. Choose a non-loop element $x \in Z$ and elements $y_i \in C_i - Z$ for $i = 1, \ldots, n' - 1$. Since $x \in \text{cl}_{M_2}(C_i)$, $C_i \cup x$ is connected, so there is a circuit C'_i of M_2 with

$$\{x, y_i\} \subseteq C'_i \subseteq C_i \cup x, \quad \text{and} \quad r_{M_2}(C'_i) \in \{2, \ldots, r'\}.$$

Notice that $C'_i \nsubseteq \text{cl}_{M_2}(\cup_{j \neq i, j<n'} C'_j)$, since $y_i \in C'_i$.

By another majority argument, there exists $s \in \{2, \ldots, r'\}$, such that $r_{M_2}(C'_i) = s$ for $i \in J$, where $|J| \geq (n' - 1)/(r - 1) \geq q_r$. We now have two cases:

$\underline{s = 2}$: Since $q_r \geq q_2 = l$ we can choose $J' \subseteq J$ with $|J'| = l$. We are now done with $\{D_1, \ldots, D_l\} = \{C'_i : i \in J'\}$ and $N = M_2$.

$\underline{2 < s \leq r'}$: Let $M_3 = M_2/x$ and let $\overline{C}_i = C'_i - x$ for $i \in J$. Then $\overline{C}_i$ is a rank-$(s - 1)$ circuit of M_3, with $\overline{C}_i \subseteq C_i$. Letting $F' = Z - x$ we have,

$$\mu_{M_3}(\overline{C}_i : i \in J, F') \leq r_{M_3}(F') = r' - 2 \leq r - 2.$$

As $|J| \geq \alpha_2(l, r - 1, r - 2)$ we get by induction the desired minor. ■

The following result is just a corollary to Lemmas 6.1 and 6.2, that we state for easier reference.

Lemma 6.3. *There exists an integer-valued function $\alpha_3(s, l, a, b)$ such that the following holds: Let $b > a \geq 1$ and let s, l be positive integers. If M is a matroid with no $U_{a+1,b}$-minor, and $\mathcal{C}$ is a set of circuits of M of rank at most $a + 1$, with $r_M(\cup_{C \in \mathcal{C}} C) \geq \alpha_3(s, l, a, b)$, then either:*

(i) *there exist s skew circuits $C_1, \ldots, C_s \in \mathcal{C}$, or*

(ii) *M has a minor $N = M/Y$ with an element $x \in E(N)$ and triangles $D_1, \ldots, D_l$ of N, such that*
  - *$x \in D_i$ for all i, and $r_N(\cup_i D_i) = l + 1$, and*
  - *For all i, $D_i - x \subseteq C$ for some $C \in \mathcal{C}$.*

Proof. Define $\alpha_3(s,l,a,b) = \alpha_3$ by

$$\alpha_3 = \sum_{r=1}^{a+1} \alpha_1(n_r, r, a, b), \quad \text{where } n_r = s + \alpha_2(l, r, (r-1)a),$$

and let M, $\mathcal{C}$ be given. By a majority argument, there exists a number $r \in \{1, \ldots, a+1\}$ and $\mathcal{C}' \subseteq \mathcal{C}$, such that $r_M(C) = r$ for all $C \in \mathcal{C}'$, and $r_M(\cup_{C \in \mathcal{C}'} C) \geq \alpha_1(n_r, r, a, b)$.

Now, by Lemma 6.1, there are $C_1, \ldots, C_n \in \mathcal{C}'$, where $n = n_r = s + \alpha_2(l, r, (r-1)a)$, satisfying

$$c_i = r_M(\cup_j C_j) - r_M(\cup_{j \neq i} C_j) \geq 1,$$

for all i, and $\mu(C_1, \ldots, C_n) \leq (r-1)a$.

Let $I = \{i : c_i = r\}$ and $J = \{i : c_i < r\}$. If $|I| \geq s$, then case (i) holds, since the C_i with $i \in I$ are skew. Otherwise, $|J| \geq \alpha_2(l, r, (r-1)a)$, and the C_i with $i \in J$ still satisfy

$$r_M(\cup_{j \in J} C_j) - r_M(\cup_{j \in J - \{i\}} C_j) < r.$$

Lemma 6.2 now gives case (ii) of the result. ∎

7 Nests

A line in a matroid is *long* if it contains at least 3 points (rank-1 flats). So, a long line in a simple matroid is a line with at least 3 elements. Also, a line is long if and only if it contains a triangle. We need a lot of long lines to construct clique-like structures. We first aim to build an intermediate structure called a nest.

Definition 7.1. A matroid M is a *nest* if M has a basis $B = \{b_1, \ldots, b_n\}$ such that, for each pair of indices $i, j \in \{1, \ldots, n\}$, $i < j$, the set $\{b_i, b_j\}$ spans a long line in $M / \{b_1, \ldots, b_{i-1}\}$. The elements in B are called the *joints* of the nest M.

A Dowling-clique (M, V) is clearly a nest with joints V (the elements in V can be taken in any order, to satisfy Definition 7.1). The main result of this section is the following.

Lemma 7.2. *There exists an integer-valued function $\nu(n,t,a,b)$ such that the following holds: Let $b > a \geq 1$ and let n, t be positive integers. If M is a t-round matroid with no $U_{a+1,b}$-minor and $r(M) \geq \nu(n,t,a,b)$, then M has a rank-n nest as a minor.*

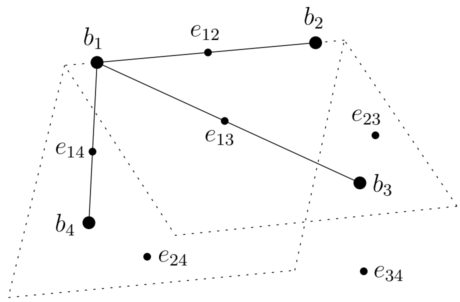


FIGURE 5: A rank-4 nest, where $\{b_i, b_j, e_{ij}\}$ is a triangle of $M/\{b_1, \ldots, b_{i-1}\}$.

We obtain a nest by finding one joint at a time using the next lemma.

Lemma 7.3. *There exists an integer-valued function $\nu_1(m, t, a, b)$ such that the following holds: Let $b > a \geq 1$ and m, t be positive integers. If M is a t-round matroid with no $U_{a+1,b}$-minor, $r(M) \geq \nu_1(m, t, a, b)$ and B is a basis of M, then M has a rank-m minor N, with a basis $B' \subseteq B \cap E(N)$ and an element $b_1 \in B'$, such that $\{b_1, d\}$ spans a long line in N for each $d \in B' - b_1$.*

Let us start by seeing how this result is used to prove Lemma 7.2.

Proof of Lemma 7.2. Let t be fixed. Let $\nu(1, t, a, b) = 1$ and for $n \geq 2$ define ν recursively by

$$\nu(n, t, a, b) = \nu_1(\nu(n-1, t, a, b) + 1, t, a, b).$$

To facilitate induction, we prove the stronger statement:

If M is a t-round matroid with no $U_{a+1,b}$-minor, $r(M) \geq \nu(n, t, a, b)$ and B is a basis of M, then M has a rank-n nest M/Y as a minor, with joints contained in B.

The proof is by induction on n. For $n = 1$ the result is trivial, as any rank-1 matroid is a nest. Let $n \geq 2$ and assume the result holds for $n - 1$. Let M and B be given as above.

By Lemma 7.3, M has a minor N_1 of rank $\nu(n-1, t, a, b) + 1$, with a basis $B_1 \subseteq B$ and $b_1 \in B_1$ such that $\{b_1, d\}$ spans a long line in N_1 for $d \in B_1 - b_1$. We can write $N_1 = M/Y_1 \backslash X_1$, where X_1 is co-independent. Since B_1 is also a basis of M/Y_1, we can assume $N_1 = M/Y_1$.

Let $N_1' = N_1/b_1$. Since t-roundness is preserved under contractions, N_1' is t-round. Now $r(N_1') = \nu(n-1,t,a,b)$ so by induction, N_1' has a rank-$(n-1)$ nest $N_2 = N_1'/Y_2$ as a minor, with joints $B_2 \subseteq B_1 - b_1$.

Now, let $Y = Y_1 \cup Y_2$ and $N = M/Y$, so we have $N_2 = N/b_1$. Then N satisfies the following

- $b_1 \cup B_2 \subseteq B$ is a basis of N,
- For each $d \in B_2$, $\{b_1, d\}$ spans a long line in N,
- $N/b_1 = N_2$ is a nest with joints B_2.

Now write $B_2 = \{b_2, \dots, b_n\}$ in accordance with Definition 7.1 for N_2. Then, N is easily seen to be a nest with joints $\{b_1, \dots, b_n\}$. ■

We shall consider coverings of matroids by connected sets. A loop is a trivial connected component of a matroid, that we wish to avoid counting. For a matroid M denote by $\tau_a^c(M)$ the minimum size of an a-covering $(X_1, \dots, X_m)$ of $M\backslash\{loops\}$, where $X_1, \dots, X_m$ are connected sets. Clearly $\tau_a^c(M) \geq \tau_a(M)$. Note also, that a loop-less rank-a matroid N has at most a connected components, so $\tau_a^c(N) \leq a$. Thus, we have in general for a matroid M:

$$\tau_a(M) \leq \tau_a^c(M) \leq a\tau_a(M).$$

We need a technical lemma before we prove Lemma 7.3.

Lemma 7.4. *Let $b > a \geq 1$. Let M be a matroid with no $U_{a+1,b}$-minor, and let $e \in E(M)$. Let $\mathcal{F}$ be the collection of all connected rank-$(a+1)$ sets in M containing e. If $n = r_M(\cup_{X\in\mathcal{F}} X)$, then*

$$\tau_a^c(M) - \tau_a^c(M/e) \leq a^2\binom{b-1}{a}^{n-a} + 1.$$

Proof. If e is a loop, then the result is trivially true, so let e be a non-loop element. We may assume, in fact, that M is loop-less.

Let $(X_1, \dots, X_k)$ be a minimal a-covering of $M/e\backslash\{loops\}$ by connected sets. We shall construct an a-covering of M by connected sets. Consider the following cases:

(1) $e \notin \mathrm{cl}_M(X_i)$. Then X_i is connected already in M, and $r_M(X_i) = r_{M/e}(X_i) \leq a$.
(2) $e \in \mathrm{cl}_M(X_i)$. In this case $X_i \cup e$ is connected in M, and it has rank $r_M(X_i \cup e) = r_{M/e}(X_i) + 1 \leq a+1$. Now either,
  (2a) $r_M(X_i \cup e) \leq a$ or
  (2b) $r_M(X_i \cup e) = a+1$.

We can assume, after possibly reordering the sets, that $X_1, \ldots, X_m$ satisfy (2b), and $X_{m+1}, \ldots, X_k$ satisfy (1) or (2a). For $i = 1, \ldots, m$ we have

$$\tau_a^c(M|(X_i \cup e)) \leq a\tau_a(M|(X_i \cup e)) \leq a\binom{b-1}{a}.$$

The elements of M destroyed when forming $M/e \setminus \{loops\}$ is the connected set $\mathrm{cl}_M(\{e\})$. It is now clear, that we can get an a-covering of size s of M by connected sets, where

$$\begin{aligned}\tau_a^c(M) \leq s &\leq ma\binom{b-1}{a} + (k-m) + 1 \\ &\leq ma\binom{b-1}{a} + \tau_a^c(M/e) + 1.\end{aligned}$$

If $m = 0$, we are done, so assume $m \geq 1$. Define $M' = (M/e)|(\cup_{i=1}^m X_i)$ and note, that $(X_1, \ldots, X_m)$ is a minimal a-covering of M' by connected sets. Hence, by Proposition 3.4,

$$m = \tau_a^c(M') \leq a\tau_a(M') \leq a\binom{b-1}{a}^{r(M')-a}.$$

Also, $r(M') = r_M(\cup_{i=1}^m (X_i \cup e)) - 1 \leq n - 1$, since $X_i \cup e \in \mathcal{F}$ for $i = 1, \ldots, m$. Now, combining the inequalities gives the desired result. ■

Let M be a matroid, $k \in \mathbb{N}$ and let $B \subseteq E(M)$. We say that B *k-dominates* M, if for any element $x \in E(M)$ there is a set $W \subseteq B$ with $r_M(W) \leq k$, such that $x \in \mathrm{cl}_M(W)$. A k-dominating set clearly has to be spanning.

It is easily verified, that k-domination is preserved under contractions in the following sense: If $B, Y \subseteq E(M)$ and B k-dominates M, then $B - Y$ k-dominates M/Y.

Proof of Lemma 7.3. Let m, t, a and b be given, and define the following constants,

$$r_4 = \alpha_3(m+1, m, a, b), \quad l = m + r_4, \quad r_3 = \alpha_3(2, l, a, b),$$

$$\lambda = a^2\binom{b-1}{a}^{r_3-a} + 1, \quad r_1 = \max(2t, \delta(\lambda, a, b)),$$

and let us define $\nu_1(m, t, a, b) = \nu_1 = \sigma(a)\binom{b-1}{a}^{r_1-a}$. Let M and B be given. We first make a quick observation:

(✠) It is enough to find a minor N' of M, with an element $z \in E(N')$ and an m-set $B' \subseteq B \cap E(N')$, such that $B' \cup z$ is independent in N' and $\{z, d\}$ spans a long line in N' for each $d \in B'$.

To see this, we may assume that $B' \cup z$ is a basis of N' (otherwise, we restrict to $\mathrm{cl}_{N'}(B' \cup z)$). Now choose $b_1 \in B'$ and an element y, such that $\{z, b_1, y\}$ is a triangle in N'. Let $N = N'/y$, and note, that z and b_1 are parallel in N. So $\{b_1, d\}$ spans a long line in N for $d \in B' - d$. Since B' is a basis of N we are done.

We start by proving the following.

<u>CLAIM</u>. M has a t-round minor N_1 with $r(N_1) \geq r_1$ and $B \subseteq E(N_1)$, such that B $(a+1)$-dominates N_1.

Let N_1 be a minimal minor of M satisfying, that N_1 is t-round and a-simple and $B \subseteq E(N_1)$. Such a minor exists, since we can choose $X \subseteq E(M)$ minimal, such that $M \backslash X$ is a-simple, and as B is independent we can take X with $X \cap B = \emptyset$. We then have $\Gamma(M \backslash X) = \Gamma(M) \leq t$.

To see that B $(a+1)$-dominates N_1, let $f \in E(N_1) - B$. N_1/f is t-round, as N_1 is t-round. Now $(N_1/f)|B$ cannot be a-simple: If it is, then we may choose $X \subseteq E(N_1/f) - B$ minimal, such that $N_1/f \backslash X$ is a-simple. But then $N_1/f \backslash X$ is t-round by Proposition 4.2, contradicting the minimality of N_1. N_1 is simple, so N_1/f is loop-less. Since $(N_1/f)|B$ is not a-simple, there must be a $W \subseteq B$, with

$$(N_1/f)|W \simeq U_{k,2k},$$

for a $k \in \{1, \dots, a\}$. Then $r_{N_1}(W \cup f) = k+1$, and we must have $r_{N_1}(W) = k+1$. If not, then $N_1|W \simeq U_{k,2k}$, but N_1 is a-simple. Thus, $f \in \mathrm{cl}_{N_1}(W)$, and B $(a+1)$-dominates N_1.

By Lemma 3.6, we have

$$\sigma(a)\tau_a(N_1) \geq |E(N_1)| \geq |B| = r(M) \geq \nu_1,$$

and so, $\tau_a(N_1) \geq \binom{b-1}{a}^{r_1 - a} > 1$. Clearly, $r(N_1) > a$, so we can apply Proposition 3.4, and get $r(N_1) \geq r_1$. This proves the claim.

Now let N_1 be given. By definition of r_1, we have $\Gamma(N_1) \leq t \leq \frac{1}{2} r(N_1)$ and $r(N_1) \geq \delta(\lambda, a, b)$. Lemma 5.3 gives a dense minor N_2 of N_1 that satisfies $\tau_a(N_2) > \lambda r(N_2)$. We may assume, that $N_2 = N_1/Y_1$. Now, N_2 also satisfies $\tau_a^c(N_2) > \lambda r(N_2)$. Let $Y_2 \subseteq E(N_2)$ be maximal, with

$$\tau_a^c(N_2/Y_2) > \lambda r(N_2/Y_2),$$

and let $N_3 = N_2/Y_2$. N_3 must be loop-less, since Y_2 is maximal. Pick an element $e \in E(N_3)$. Then,

$$\tau_a^c(N_3) - \tau_a^c(N_3/e) > \lambda r(N_3) - \lambda r(N_3/e) = \lambda.$$

Let $\mathcal{F}$ denote the collection of all connected rank-$(a+1)$ sets in N_3 containing e, and let $n = r_{N_3}(\cup_{X\in\mathcal{F}})$. By Lemma 7.4, we then have $\lambda < a^2\binom{b-1}{a}^{n-a} + 1$, and by definition of λ, this yields $n \geq r_3$.

Denote by $\mathcal{C}$ the collection of all circuits of N_3 of rank at most $a+1$ containing e. For each $X \in \mathcal{F}$ and non-loop $y \in X - e$, since X is connected, there exists a circuit $C \subseteq X$ containing e and y, so $C \in \mathcal{C}$. Hence, $r_{N_3}(\cup_{C\in\mathcal{C}} C) \geq n$.

Since $n \geq r_3 = \alpha_3(2, l, a, b)$ we can apply Lemma 6.3. As no two circuits in $\mathcal{C}$ are skew, we get case (ii): There is a minor $N_4 = N_3/Y_3$, with $x \in E(N_4)$ and triangles $D_1, \ldots, D_l$ of N_4, such that $x \in D_i$ and $r_{N_4}(\cup_i D_i) = l + 1$. Pick an element $h_i \in D_i - x$ for $i = 1 \ldots, l$.

Let $I = \{i : h_i \in B\}$. If $|I| \geq m$, then we can choose an m-set $B' \subseteq \{h_i : h_i \in B\}$ and we are done by (✠), taking $N' = N_4$ and $z = x$. So, assume $|I| \leq m$. By possibly re-ordering the D_i, we may assume $h_1, \ldots, h_{r_4} \notin B$.

By the remark preceding the proof, $B \cap E(N_4)$ $(a+1)$-dominates N_4. So, for each $i \in \{1, \ldots, r_4\}$, h_i is in the closure of a subset of B of rank at most $a+1$. Choose a circuit C_i of N_4 containing h_i, with $r_{N_4}(C_i) \leq a+1$ and $C_i \subseteq B \cup h_i$ (see Figure 6). Since $\{h_1, \ldots, h_{r_4}\}$ is independent, we have $r_{N_4}(\cup_i C_i) \geq r_4$. As $r_4 = \alpha_3(m+1, m, a, b)$ we can apply Lemma 6.3 again, and we get one of two cases.

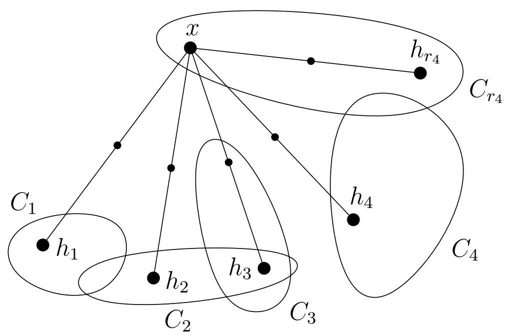


FIGURE 6

Consider case (i): There are $s = m + 1$ skew circuits among $C_1 \ldots, C_{r_4}$ in N_4. After possibly re-ordering we may assume $C_1, \ldots, C_s$ are skew. We would like x to be skew from the C_i's. If not $\{x\}$ and $\cup_{i=1}^s C_i$ are skew, we omit one of the C_i's: Assume $x \in \mathrm{cl}_{N_4}(\cup_{i=1}^{s} C_i)$ and let B_* be a basis for $\cup_{i=1}^{s} C_i$ in N_4. Then there is a unique circuit $C_* \subseteq B_* \cup x$. Again, after re-ordering, we can assume $C_* \cap C_s \neq \emptyset$. Then $x \notin \mathrm{cl}_{N_4}(B_* - C_s) = \mathrm{cl}_{N_4}(\cup_{i=1}^{s-1} C_i)$, and so $C_1, \ldots, C_m, \{x\}$ are skew sets in N_4.

For $i = 1, \ldots, m$, pick an element $b_i \in C_i - h_i$, and let $K_i = C_i - \{h_i, b_i\}$. Define $N_5 = N_4/(\cup_i K_i)$. Then h_i and b_i are parallel in N_5. Letting $B' = \{b_1, \ldots, b_m\}$ we are done by (✠), with $N' = N_5$ and $z = x$.

Consider now case (ii): N_4 has a minor N_5, with $z \in E(N_5)$ and triangles $D'_1, \ldots, D'_m$ in N_5, such that $z \in D'_i$ and $r_{N_5}(\cup_i D'_i) = m + 1$. Also, for each i, $D'_i - z \subseteq C_j$ for some j. Thus, $D'_i - z \subseteq B \cup h_j$ and we can pick an element $b_i \in (D'_i - z) \cap B$. Taking $B' = \{b_1, \ldots, b_m\}$ and $N' = N_5$, we are again done by (✠). ■

8 Dowling-cliques

The goal of this section is to extract from a nest the general kind of clique, that we call a Dowling-clique (defined in Section 4). To do this, we shall first go through yet another intermediate structure.

Definition 8.1. Let $n \in \mathbb{N}$. A matroid M is an *n-storm*, if its ground set is the disjoint union $E(M) = F \cup C_1 \cup \cdots \cup C_m$, where $r_M(F) = n$ and each C_i is a size-$(n+1)$ independent co-circuit of M, with $F \subseteq \mathrm{cl}_M(C_i)$. We call the C_i *clouds* of M.

In an n-storm, the set F must be closed, since it is an intersection of hyperplanes. Note also, that $C_1, \ldots, C_m$ are skew in M/F, and hence that $\mu_M(C_1 \ldots, C_m) = n$.

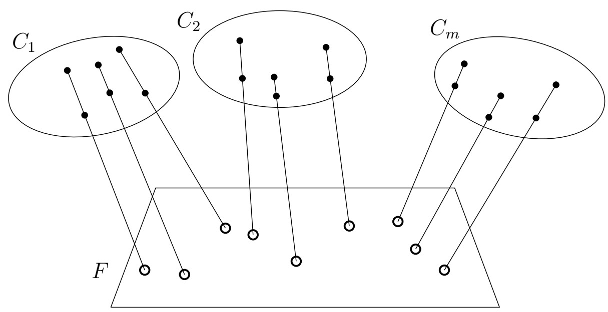


Figure 7: An n-storm with m clouds.

We shall first see that nests contain storms as restrictions.

Lemma 8.2. *Let m and n be positive integers. If M is a nest of rank at least $n+m$, then M has an n-storm N with m clouds as a minor.*

Proof. Let M be a rank-r nest with joints $B = \{b_1, \ldots, b_r\}$, where $r = n + m$. For each pair (i, j), $1 \le i < j \le r$, pick an element e_{ij}, such that $\{b_i, b_j, e_{ij}\}$ is a triangle of $M/\{b_1, \ldots, b_{i-1}\}$. We need two observations:

$$\text{(1)}\ \ e_{1k}, e_{2k}, \ldots, e_{k-1,k} \notin \mathrm{cl}_M(B - b_k), \quad \text{for } k = 2, \ldots, r.$$
$$\text{(2)}\ \ \mathrm{cl}_M(\{b_1, \ldots, b_i, b_k\}) = \mathrm{cl}_M(\{e_{1k}, \ldots, e_{ik}, b_k\}), \quad \text{for } i < k.$$

To see that (1) holds, let i, k be given, with $1 \le i < k$. By definition of e_{ik} we have $e_{ik} \notin \mathrm{cl}_M(\{b_1, \ldots, b_i\})$, but $e_{ik} \in \mathrm{cl}_M(\{b_1, \ldots, b_i, b_k\})$. So the fundamental circuit of e_{ik} in M with respect to the basis B, must contain b_k. Hence, $e_{ik} \notin \mathrm{cl}_M(B - b_k)$.

We prove (2) by induction on i with k fixed. The case $i = 1$ is trivial, since $\{b_1, b_k, e_{1k}\}$ is a circuit in M. Suppose $1 < i < k$ and (2) holds for $i - 1$. Again, by definition of e_{ik} we have $e_{ik} \notin \mathrm{cl}_M(\{b_1, \ldots, b_{i-1}, b_k\})$, but $e_{ik} \in \mathrm{cl}_M(\{b_1, \ldots, b_i, b_k\})$. Thus, by the matroid axioms for closure,

$$\begin{aligned}\mathrm{cl}_M(\{b_1, \ldots, b_i, b_k\}) &= \mathrm{cl}_M(\{b_1, \ldots, b_{i-1}, b_k, e_{ik}\}) \\ &= \mathrm{cl}_M(\{e_{1k}, \ldots, e_{i-1,k}, b_k, e_{ik}\}),\end{aligned}$$

where the induction hypothesis is used in the second step.

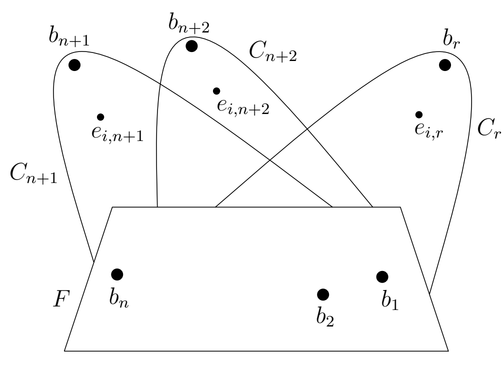


FIGURE 8

Let $S = \{b_1, \ldots, b_n\}$ and $F = \mathrm{cl}_M(S)$, and for each $k = n + 1, \ldots, r$ define $C_k = \{e_{1k}, \ldots, e_{nk}, b_k\}$ (see Figure 8).

Notice that by (1), $C_k \cap F = \emptyset$ for all k, and $C_k \cap C_l = \emptyset$ for $k \neq l$. From (2) we gather, that C_k is independent with $F \subseteq \mathrm{cl}_M(C_k)$ for all k. Define $N = M|(F \cup C_{n+1} \cup \cdots \cup C_r)$. It is easily checked, that the C_k's are co-circuits of N. ∎

In the case $n = 2$, the concept of an n-storm is similar to that of a "book", used by Kung in [**29**], and an analogue of one of his ideas is part of the proof of the following result (the notion of book mentioned here differs slightly from the one we mention in Chapter 4, Section 5).

Lemma 8.3. *There exist integer-valued functions $\phi_1(n,a,b)$, $\phi_2(n,a,b)$ such that the following holds: Let $b > a \geq 1$ and let n be a positive integer. If M is a $\phi_1(n,a,b)$-storm with $\phi_2(n,a,b)$ clouds, and M has no $U_{a+1,b}$-minor, then M contains a rank-n Dowling clique N as a minor.*

Proof. Let n, a, b be given and let us define ϕ_1 and ϕ_2. First, let $l = na$. For $r = 1, \ldots, a$, let $s_r = \alpha_1(n,r,a,b)$ and let $s = \sum_{r=1}^{a} s_r$. Define a sequence of numbers $m_0, \ldots, m_s$ recursively as follows: let $m_s = l$, and

$$m_k = \sigma_2(a, m_{k+1}) \binom{b-1}{a}^{s-a} + 1, \quad \text{for } k = s-1, \ldots, 1, 0.$$

Finally, let $\phi_1(n,a,b) = s$ and $\phi_2(n,a,b) = m_0$.

Let M be an s-storm with $m = m_0$ clouds. Denote the clouds by $C_1, \ldots, C_m$ and their elements $C_i = \{e_0^i, e_1^i, \ldots, e_s^i\}$.

Define $M' = M/\{e_0^1, e_0^2, \ldots, e_0^m\}$. So $r(M') = s$. Let $I_0 = \{1, \ldots, m\}$. We wish to find a subcollection of the clouds, such that elements with the same index lie on a uniform restriction of M'. We shall construct a sequence of subsets,

$$I_0 \supseteq I_1 \supseteq \cdots \supseteq I_s, \quad \text{where } |I_k| = m_k,$$

such that for $k = 1, \ldots, s$, $M'|\{e_k^i : i \in I_k\} \simeq U_{r_k, m_k}$, for some number $r_k \in \{1, \ldots, a\}$. Let $k \geq 1$, suppose I_{k-1} has been defined and let us see how to find I_k. Let $W = \{e_k^i : i \in I_{k-1}\}$ and suppose $M'|W$ has no U_{r,m_k}-restriction for $r = 1, \ldots, a$. Lemma 3.7 and Proposition 3.4 then give

$$m_{k-1} = |W| \leq \sigma_2(a, m_k)\tau_a(M'|W) \leq \sigma_2(a, m_k)\binom{b-1}{a}^{s-a},$$

contradicting our definition of m_{k-1}. So, take $U \subseteq W$ such that $M'|U$ is isomorphic to U_{r_k, m_k}, and let $I_k = \{i : e_k^i \in U\}$.

After possibly re-ordering the clouds in M, we may assume that $I_s = \{1,\dots,l\}$. Let then $L_k = \left\{e_k^1,\dots,e_k^l\right\}$ for $k = 1,\dots,s$, so $M'|L_k \simeq U_{r_k,l}$. Now, by a majority argument, there exist $r \in \{1,\dots,a\}$, and a subset $J \subseteq \{1,\dots,s\}$ with $|J| \geq s_r$, such that $r_k = r$, for all $k \in J$. Define $\mathcal{L} = \{L_k : k \in J\}$. Since $C_1 - e_0^1$ is independent in M', we have

$$r_{M'}(\cup_{L\in\mathcal{L}} L) \geq |\mathcal{L}| = |J| \geq s_r = \alpha_1(n,r,a,b).$$

We now apply Lemma 6.1 and get a subcollection $\mathcal{L}' \subseteq \mathcal{L}$ of size n, such that $L \nsubseteq \mathrm{cl}_{M'}(\cup_{L'\in\mathcal{L}'-L} L')$, for each $L \in \mathcal{L}'$. After possibly permuting the elements of each cloud, we can assume $\mathcal{L}' = \{L_1,\dots,L_n\}$. Let $D_i = \left\{e_0^i, e_1^i,\dots,e_n^i\right\} \subseteq C_i$ (see Figure 9).

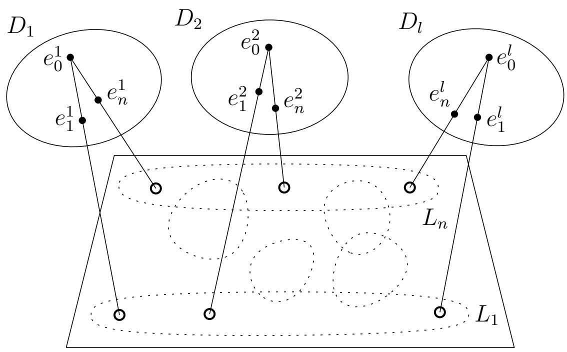


FIGURE 9

By our arrangement of the L_k's, we can find a set $B \subseteq \cup_{k=1}^n L_k$ independent in M', such that $r_{M'}(L_k \cup B) - r_{M'}(B) = 1$, for each $k \in \{1,\dots,n\}$, and the sets $L_1 - B,\dots,L_n - B$ are skew in M'/B.

Now $L_k \nsubseteq \mathrm{cl}_{M'}(B)$, and since $M'|L_k$ is rank-r uniform, we must have $|L_k \cap \mathrm{cl}_{M'}(B)| \leq r-1$. Define $I_B \subseteq \{1,\dots,l\}$ by

$$I_B = \{i : D_i \cap \mathrm{cl}_{M'}(B) \neq \emptyset\}.$$

Then $B \subseteq \cup_{i\in I_B} D_i$ and $|I_B| \leq n(r-1) \leq n(a-1) = l-n$. Notice, that for $i \notin I_B$, D_i and B are skew in M'. We may assume, again after re-ordering the clouds, that $\{1,\dots,n\} \subseteq \{1,\dots,l\} - I_B$. Let

$$M_1 = M/\left\{e_0^i : i \in I_B\right\}/B.$$

Then, by construction, the elements in $\left\{e_k^i : i = 1,\dots,n\right\}$ are in parallel in $M_1/\left\{e_0^i : i = 1,\dots,n\right\}$, for each $k = 1,\dots,n$ (see Figure 10).

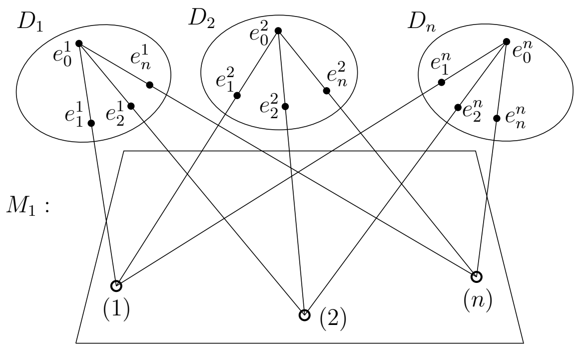


Figure 10

Let $p_k = e_k^n$, for $k = 1, \ldots, n$, and define

$$M_2 = M_1/e_0^n|(D_1 \cup \cdots \cup D_{n-1} \cup \{p_1, \ldots, p_n\}).$$

Now, M_2 is an n-storm with clouds $D_1, \ldots, D_{n-1}$. It satisfies, for each $i = 1, \ldots, n-1$, and each $k = 1, \ldots, n$, that p_k is on the line through e_0^i and e_k^i.

We shall make $\{p_1, \ldots, p_n\}$ the basis of a Dowling clique. Let

$$N = M_2/\left\{e_1^1, e_2^2, \ldots, e_{n-1}^{n-1}\right\}.$$

Then $\{p_1, \ldots, p_n\}$ is a basis for N. Let (i, j) be given with $1 \leq i < j \leq n$. In N, e_0^i and p_i are parallel, so $e_j^i \in \mathrm{cl}_N(\{e_0^i, p_j\}) = \mathrm{cl}_N(\{p_i, p_j\})$, and the set $\{p_i, p_j, e_j^i\}$ is a triangle in N. So, N has a rank-n Dowling clique restriction. ■

9 Cliques

We need Mader's Theorem [**36**] to extract graphic cliques.

Mader's Theorem. *Let H be a graph. There exists $\lambda \in \mathbb{N}$ such that, if G is a simple graph with no H-minor, then $|E(G)| \leq \lambda |V(G)|$.*

An easy corollary is the following matroid version of the theorem. We take H to be a complete graph, and write the contrapositive statement.

Corollary 9.1. *There exists an integer-valued function $\theta(n)$ such that, if M is a graphic and simple matroid with $|E(M)| > \theta(n)r(M)$, then M has an $M(K_n)$-minor.*

Let M be a matroid and V a basis of M, and let $X = E(M) - V$. We call (M, V) a *Dowling matroid* if each $x \in X$ is on a triangle with two elements of V, and any two elements of V span at most one element of X (again, this is only a special case of Dowling's combinatorial geometries [**9**]). By the *associated graph* of (M, V) we mean the graph G on the vertex set V with edge set labeled by X, such that $x \in X$ labels $\{b_1, b_2\}$ if x is on the line through b_1 and b_2 in M. In particular, the associated graph of a rank-n Dowling-clique is the complete graph K_n.

We shall use the following lemma to recognize graphic matroids.

Lemma 9.2. *Let M be a Dowling matroid with basis V, $X = E(M) - V$. If $V \cap \mathrm{cl}_M(X) = \emptyset$, then $M|X$ is graphic.*

Proof. Let G be the associated graph of (M, V). We claim that $M(G) = M|X$. It suffices to prove, that each circuit of $M(G)$ is dependent in $M|X$, and that each independent set of $M(G)$ is independent in $M|X$.

Let C be a cycle of G with vertex set V' and edge set X'. Clearly $X' \subseteq \mathrm{cl}_M(V')$ and by the assumption $V' \cap \mathrm{cl}_M(X') = \emptyset$. Since V' and X' have equal size, X' must be dependent in M.

Let T be a forest in G. We prove by induction on $|E(T)|$ that $E(T)$ is independent in M. Let e be a leaf edge in T and assume $E(T) - e$ is independent in M. Let b be a leaf of T incident on e. Then $E(T) - e \subseteq \mathrm{cl}_M(V - b)$, but $e \notin \mathrm{cl}_M(V - b)$, so $E(T)$ is independent in M. ■

In a similar fashion, using Remark 1.5, the following lemma is easily proved.

Lemma 9.3. *Let M be a Dowling matroid with basis V, $X = E(M) - V$. Let G be the associated graph of (M, V). If for each cycle C in G, $E(C)$ is independent in M, then $M|X = B(G)$ (in fact $M = \tilde{B}(G)$).*

We are ready for the final step in the proof of the main theorem.

Lemma 9.4. *There exists an integer-valued function $\psi(n)$ such that, if M is a Dowling-clique with rank at least $\psi(n)$, then M contains an $M(K_n)$- or $B(K_n)$-minor.*

Proof. Let n be given, and define $\psi(n) = nl$, where $l = 2m\theta(n) + 1$ and $m = 2^n n!$. Let M be a Dowling clique with basis V and assume that $r(M) = nl$. Let $X = E(M) - V$.

Partition V into n sets, $V_1, \ldots, V_n$ of equal size, $|V_i| = l$. We shall contract each V_i to a point. Let $M_i = M|\,\mathrm{cl}_M(V_i)$ (see Figure 11(a)). Choose $Y_i \subseteq E(M_i) \cap X$ such that V_i is a set of parallel elements in M_i/Y_i (e.g. take the edges of a spanning tree in the associated graph of (M_i, V_i)). Define $M' = M/(Y_1 \cup \cdots \cup Y_n)$ and pick a $b_i \in V_i$ for $i = 1, \ldots, n$. For each pair $i < j$, define

$$X_{ij} = \{x \in X : x \in \mathrm{cl}_M(b, d),\ b \in V_i,\ d \in V_j\}.$$

Note, that for each $x \in X_{ij}$, $\{b_i, b_j, x\}$ is a triangle in M' (see Figure 11(b)). We consider two cases.

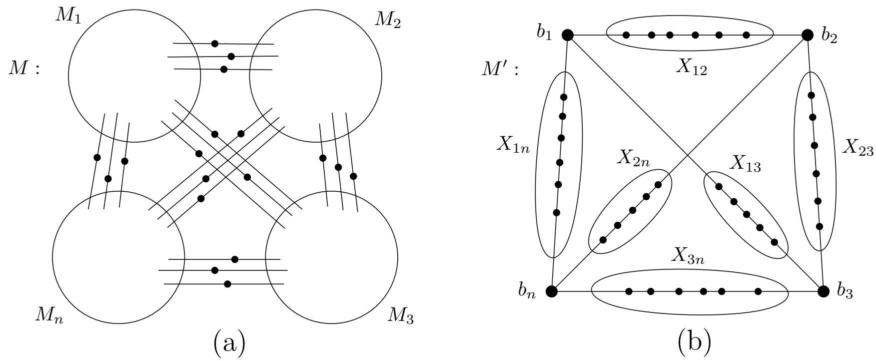


Figure 11

(1). $\tau_1(M'|X_{ij}) > m$, for all pairs $i < j$. Put $V' = \{b_1, \ldots, b_n\}$. We shall choose a set $X' = \{x_{ij} : 1 \le i < j \le n\}$, where $x_{ij} \in X_{ij}$, such that the Dowling-clique $M'|(V' \cup X')$ with basis V' is $\tilde{B}(K_n)$. Let $X' \subseteq \cup X_{ij}$ be maximal, such that $|X' \cap X_{ij}| \le 1$ for all $i < j$, and the cycles in the associated graph of $M'|(V' \cup X')$ all have edge sets independent in M'. We claim, that $X' \cap X_{ij} \ne \emptyset$ for all $i < j$, and thus $M'|(V' \cup X') \simeq \tilde{B}(K_n)$ by Lemma 9.3.

Assume that $X' \cap X_{ij} = \emptyset$ for some $i < j$. Let G be the associated graph of $M'|(V' \cup X')$. If $Z \subseteq X'$ is the edge set of a path from b_i to b_j in G, then $r_{M'}(\mathrm{cl}_{M'}(Z) \cap X_{ij}) \le 1$. There can be at most m such Z, since a simple graph on n vertices has no more than $2^n n! = m$ cycles. Thus, we can pick $x_{ij} \in X_{ij}$ skew from each such Z. So, the cycles created in the associated graph, when adding x_{ij} to X' all have edge sets independent in M', contradicting the maximality of X'.

(2). $\tau_1(M'|X_{ij}) \le m$, for some pair $i < j$. As $|X_{ij}| = l^2$, there is a parallel class $P \subseteq X_{ij}$ of M', with $|P| \ge l^2/m$. Now, since $V \cap \mathrm{cl}_{M'}(P) = \emptyset$, also $V \cap \mathrm{cl}_M(P) = \emptyset$, and Lemma 9.2 gives, that $M|P$ is graphic. And $r(M|P) \le |V_i \cup V_j| = 2l$. We have then

$$|E(M|P)| \ge l^2/m > l2\theta(n) \ge \theta(n) r(M|P),$$

and by Corollary 9.1, we get an $M(K_n)$-minor. ■

Finally, we restate and prove Theorem 1.1.

Theorem 9.5. *There exists an integer-valued function $\gamma(k, a, n)$ such that, if M is a matroid with $r(M) \geq \gamma(k, a, n)$, then either M has k disjoint co-circuits or M has a minor isomorphic to $U_{a,2a}$, $M(K_n)$ or $B(K_n)$.*

Proof. Let k, a, n be positive integers. If $a = 1$, then we let $\gamma(k, a, n) = k$. If $a \geq 2$, then we define the following numbers: Put $a' = a - 1$ and $b = 2a$. Let $k = \psi(n)$ and let $m_1 = \phi_1(k, a', b)$ and $m_2 = \phi_2(k, a', b)$. Let $r = n + m$ and define $g : \mathbb{N} \to \mathbb{N}$ by $g(t) = \nu(r, t, a', b)$. Finally, $\gamma(k, a, n) = f_g(k)$.

Let M be given with $r(M) \geq \gamma(k, a, n)$. If $a = 1$, the result is trivial, since in a matroid with no $U_{1,2}$-minor, every element is a loop or a co-loop.

If $a \geq 2$, then by Lemma 4.7, either M has k disjoint co-circuits or a minor N with $r(N) \geq g(\Gamma(N))$. Assume the second case. Also, if N has a $U_{a'+1,b}$-minor we are done, so assume this is not the case. Applying Lemmas 7.2, 8.2, 8.3 and 9.4 in succession, we obtain an $M(K_n)$- or a $B(K_n)$-minor of N. ∎

10 Mader's Theorem for matroids

In this section we treat a result of Geelen and Whittle [**23**], that extends Mader's Theorem from graphs (or graphic matroids) to the class $\mathcal{U}(q)$. We present their proof of this result with minor changes. To avoid talking about the number of elements in a *simple* matroid, we shall denote by $\varepsilon(M)$ the number of points (rank-1 flats) in M. That is, $\varepsilon(M) = |E(\text{si}(M))|$. With the notation from earlier in this chapter, $\varepsilon(M) = \tau_1(M)$. Geelen and Whittle proved:

Theorem 10.1. *Let n and q be positive integers. There exists an integer λ such that, if $M \in \mathcal{U}(q)$ has $\varepsilon(M) > \lambda r(M)$, then M has an $M(K_n)$-minor.*

Kung had proved this previously in the case $n = 4$ in [**29**], and in the case $n = 5$ for binary matroids ($q = 2$) in [**28**]. By Theorem 1.3, a round matroid in $\mathcal{U}(q)$ of sufficiently high rank contains an $M(K_n)$-minor. Hence, to prove Theorem 10.1 it suffices to prove the following.

Lemma 10.2. *Let m and q be positive integers. There exists an integer λ such that, if $M \in \mathcal{U}(q)$ has $\varepsilon(M) > \lambda r(M)$, then M has a round rank-m minor.*

Recall that a flat F of M is called round if $M|F$ is round. The idea behind the proof of the lemma, is to construct round flats of increasing rank using Lemma 4.4. A rank-1 flat or point is always round. A rank-2 flat or line L is round if and only if it contains at least three points, that is, L is a long line. The following lemma provides the first step in the proof.

Lemma 10.3. *Let $\lambda \in \mathbb{N}$ and let $M \in \mathcal{U}(q)$ be minor-minimal with $\varepsilon(M) > \lambda r(M)$. Then the number of long lines in M is greater than $\frac{\lambda}{q^2}\varepsilon(M)$.*

Proof. Note that by the minor-minimality, M is simple. For $e \in E(M)$ let $\delta(e)$ denote the number of long lines through e in M. Let $e \in E(M)$. When e is contracted each line through e becomes a point. Hence, the number of points destroyed is $\varepsilon(M) - \varepsilon(M/e) \leq 1 + (q-1)\delta(e)$. By the minimality of M, $\varepsilon(M) - \varepsilon(M/e) > \lambda$. So $\delta(e) \geq \lambda/(q-1)$. Since this holds for each $e \in E(M)$, the number of long lines in M is at least

$$\frac{1}{q+1} \sum_{e \in E(M)} \delta(e) > \frac{\lambda}{q^2} |E(M)|,$$

as required. ∎

In the next lemma we shall need an upper bound on the number of hyperplanes of a matroid in $\mathcal{U}(q)$. Let $h_q(r)$ denote the maximum number of hyperplanes in a rank-r matroid in $\mathcal{U}(q)$. Since a hyperplane is spanned by a set of $r-1$ elements, using Kung's Theorem, we get the rough bound:

$$h_q(r) \leq \binom{\frac{q^r-1}{q-1}}{r-1} \leq q^{r(r-1)}.$$

Let M be a matroid and $\mathcal{F}$ a collection of flats of M. We call a rank-k flat F of M *$\mathcal{F}$-constructed*, if there are rank-$(k-1)$ flats $F_1, F_2 \in \mathcal{F}$, such that $F = \mathrm{cl}_M(F_1 \cup F_2)$, but $F \neq F_1 \cup F_2$. If the flats in $\mathcal{F}$ are round, Lemma 4.4 implies that any $\mathcal{F}$-constructed flat is round.

Lemma 10.4. *There exists a function $\lambda : \mathbb{N}^3 \to \mathbb{N}$ such that the following holds: Let $q, n \geq 2$ and $c \geq 0$ be integers. If $M \in \mathcal{U}(q)$ satisfies $\varepsilon(M) > \lambda(n,c,q)r(M)$, then M has a minor N with a collection $\mathcal{F}$ of round rank-$(n-1)$ flats of N, such that the number of $\mathcal{F}$-constructed flats in N is greater than $c\,|\mathcal{F}|$.*

Proof. Let $\lambda(2,c,q) = q^2c$ and for $n \geq 2$ define λ recursively by

$$\lambda(n+1,c,q) = \lambda(n, q^n + q^{(n+1)^2}c, q).$$

The proof is by induction on n. Consider the case $n = 2$. We may assume, that M is minor-minimal with $\varepsilon(M) > q^2cr(M)$. By Lemma 10.3, the number of long lines in M is greater than $c\varepsilon(M)$. Let $\mathcal{F}$ be the set of points in M. Then each long line in M is $\mathcal{F}$-constructed, and we are done.

Now, let $n \geq 2$, assume the lemma holds for n and let us prove it for $n+1$. Let M be given with $\varepsilon(M) > \lambda(n+1, c, q)r(M) = \lambda(n, c', q)r(M)$, where $c' = q^n + q^{(n+1)^2}c$. By induction, M has a minor N with a set $\mathcal{F}$ of round rank-$(n-1)$ flats of N, such that the number of $\mathcal{F}$-constructed flats is greater than $c'|\mathcal{F}|$. Choose N minor-minimal with this property. In particular, N is simple.

Let $\mathcal{F}^1$ denote the set of $\mathcal{F}$-constructed flats, and let $\mathcal{F}^2$ be the set of $\mathcal{F}^1$-constructed flats. For each $e \in E(N)$, let $\mathcal{F}_e$ be the set of rank-$(n-1)$ flats in N/e, that correspond to flats in $\mathcal{F}$, that is

$$\mathcal{F}_e = \left\{ \mathrm{cl}_{N/e}(F) : F \in \mathcal{F}, e \notin F \right\}.$$

The flats in $\mathcal{F}_e$ are round in N/e. Let $\mathcal{F}_e^1$ denote the set of $\mathcal{F}_e$-constructed flats. Note that $\mathcal{F}_e^1 = \left\{ \mathrm{cl}_{N/e}(F) : F \in \mathcal{F}^1, e \notin F \right\}$. Finally, let

$$\Delta = \sum_{e \in E(N)} (|\mathcal{F}| - |\mathcal{F}_e|) \quad \text{and} \quad \Delta_1 = \sum_{e \in E(N)} (\left|\mathcal{F}^1\right| - \left|\mathcal{F}_e^1\right|).$$

We now prove a number of inequalities, that will imply the result.

$$\Delta_1 > c'\Delta. \tag{C1}$$

By the choice of N, $\left|\mathcal{F}^1\right| > c'|\mathcal{F}|$, and as N is minor-minimal with this property, $\left|\mathcal{F}_e^1\right| \leq c'|\mathcal{F}_e|$, for each $e \in E(N)$. Thus, $\left|\mathcal{F}^1\right| - \left|\mathcal{F}_e^1\right| > c'(|\mathcal{F}| - |\mathcal{F}_e|)$. Adding up, we get inequality (C1).

$$\Delta \geq \left|\mathcal{F}^1\right|. \tag{C2}$$

Consider $F \in \mathcal{F}^1$. By definition, there are flats $F_1, F_2 \in \mathcal{F}$ with $F = \mathrm{cl}_N(F_1 \cup F_2)$ and an element $e \in F - (F_1 \cup F_2)$. Clearly $\mathrm{cl}_{N/e}(F_1) = \mathrm{cl}_{N/e}(F_2)$, so F_1 and F_2 give rise to the same flat in $\mathcal{F}_e$. Thus, they contribute to the difference $|\mathcal{F}| - |\mathcal{F}_e|$. This proves (C2).

$$\Delta_1 \leq q^n \left|\mathcal{F}^1\right| + q^{(n+1)^2} \left|\mathcal{F}^2\right|. \tag{C3}$$

For $e \in E(N)$ we wish to measure the difference $\left|\mathcal{F}^1\right| - \left|\mathcal{F}_e^1\right|$, which reflects the fact, that not all flats in $\mathcal{F}^1$ give rise to a flat in $\mathcal{F}_e^1$, and that multiple flats in $\mathcal{F}^1$ may give rise to a single flat in $\mathcal{F}_e^1$.

Consider the first case. A flat $F \in \mathcal{F}^1$ does not give rise to a flat in $\mathcal{F}_e^1$ if $e \in F$. By Kung's Theorem, F has at most $\frac{q^n-1}{q-1} \leq q^n$ elements, so F is counted no more than q^n times in the expression of Δ_1.

Consider the second case. Let $F_1, F_2 \in \mathcal{F}^1$ be flats of N not containing e, such that F_1, F_2 give rise to the same rank-n flat in N/e. Then $F = \mathrm{cl}_N(F_1 \cup F_2)$

has rank $n+1$ and $e \in F$, so $F \in \mathcal{F}^2$. Now F has no more than q^{n+1} elements and by the observation preceding the lemma, F contains no more than $q^{n(n+1)}$ rank-n flats of N. Hence, the rank-n flats in $\mathcal{F}^1$ contained in F are counted no more than $q^{n+1}q^{n(n+1)} = q^{(n+1)^2}$ times in the expression of Δ_1. This proves (C3).

Finally, combining (C1), (C2) and (C3), we get

$$q^n \left|\mathcal{F}^1\right| + q^{(n+1)^2} \left|\mathcal{F}^2\right| \geq \Delta_1 > c'\Delta \geq (q^n + q^{(n+1)^2}c) \left|\mathcal{F}^1\right|.$$

It follows, that $\left|\mathcal{F}^2\right| > c\left|\mathcal{F}^1\right|$, so $\mathcal{F}^1$ is the desired set of flats in N. ■

Lemma 10.2 now follows immediately by taking $\lambda = \lambda(0, m, q)$ in the last result, since a single rank-m constructed flat gives rise to a round minor of rank m.

With the main theorem of this chapter, Theorem 1.1 in mind, we conjecture the following generalization of Theorem 10.1.

Conjecture 10.5. *Let $b > a$ and n be positive integers. There exists an integer λ such that, if M has no $U_{a+1,b}$-minor and $\tau_a(M) > \lambda r(M)$, then M has an $M(K_n)$- or a $B(K_n)$-minor.*

By Theorem 1.1, a round matroid with no $U_{a+1,b}$-minor, of sufficiently high rank contains an $M(K_n)$- or a $B(K_n)$-minor. So, in analogy with the proof of Theorem 10.1, Conjecture 10.5 would follow from the weaker conjecture below.

Conjecture 10.6. *Let $b > a$ and m be positive integers. There exists an integer λ such that, if M has no $U_{a+1,b}$-minor and $\tau_a(M) > \lambda r(M)$, then M has a round rank-m minor.*

In fact, this conjecture can be weakened further to state: *Let $b > a$ and m, t be positive integers. There exists an integer λ such that, if M has no $U_{a+1,b}$-minor and $\tau_a(M) > \lambda r(M)$, then M has a t-round rank-m minor.* This would be enough to imply Conjecture 10.5, as can be seen by a closer inspection of the proof of Theorem 1.1.

3 Projective geometries in dense matroids

This chapter presents joint work with Jim Geelen. The proofs given here are generally expanded versions of the proofs given in the article [**18**], which was written by me. The way the theorems are expressed differs from [**18**], where the results are stated in an equivalent way in terms of size functions of minor-closed classes of matroids. This relationship will be clarified in Chapter 4.

1 The main results

We denote by $\varepsilon(M)$ the number of points of M, i.e. the number of elements in the simplification of M, $\varepsilon(M) = |E(\mathrm{si}(M))|$. With the notation from Chapter 2, $\varepsilon(M) = \tau_1(M)$, the 1-covering number of M.

The main result of the chapter is the following theorem.

Theorem 1.1. *Let q and q_* be integers with $q \geq q_* \geq 2$, and let n be a positive integer. There exists an integer α such that, if $M \in \mathcal{U}(q)$ satisfies $\varepsilon(M) \geq \alpha {q_*}^{r(M)}$, then M contains a $\mathrm{PG}(n-1, q')$-minor, for some prime-power $q' > q_*$.*

While the values of α we obtain are astronomical, the lower bound on the number of elements of M has the correct order of magnitude: Note that we may always take q_* to be a prime-power without weakening the statement of the lemma. In that case, the base q_* in the exponential function $\alpha {q_*}^{r(M)}$ is optimal, since the matroids $\mathrm{PG}(r-1, q_*)$ have in the order of ${q_*}^r$ elements.

Kung's Theorem of Chapter 2 can be rewritten in the following way.

Kung's Theorem. *Let $q \geq 2$ be an integer, and let $M \in \mathcal{U}(q)$ be a rank-r matroid. Then*

$$\varepsilon(M) \leq \frac{q^r - 1}{q - 1}.$$

We already know, that if q is not a prime-power, the bound is not exact. As a consequence of Theorem 1.1, we get an asymptotic improvement on the bound in that case:

Corollary 1.2. *Let $q \geq 2$ be an integer, and let $M \in \mathcal{U}(q)$ be a rank-r matroid. Then*

$$\varepsilon(M) < c{q_*}^r.$$

where q_* is the greatest prime-power with $q_* \leq q$, and c is an integer depending only on q.

Proof. Let q and q_* be given, and let $n = 2$. We take $c = \alpha$, the number given by Theorem 1.1. Since M can have no PG$(1, q')$-minor (that is, $U_{2,q'+1}$-minor) for a prime-power $q' > q_*$, we get the desired bound. ∎

Kung in [**30**] conjectured that the exact bound is $\frac{{q_*}^r - 1}{q_* - 1}$ for sufficiently large r (the bound may not hold for small values of r, indeed it easily fails if $r = 2$ and $q > q_*$). This conjecture would follow from Kung's stronger Growth Rate Conjecture, that we treat in Chapter 4. It has only been verified in the first non-prime-power case $q = 6$. The result is from [**5**].

Theorem 1.3. *Let $M \in \mathcal{U}(6)$ be a simple rank-r matroid, where $r \geq 3$. Then*

$$|E(M)| \leq \frac{5^r - 1}{5 - 1}.$$

Using the same techniques as we do in the proof of Theorem 1.1 we also prove the following theorem. For binary matroids it was proved independently by Sauer [**45**] and Shelah [**48**].

Theorem 1.4. *Let q and n be positive integers. There exist integers γ, m such that, if $M \in \mathcal{U}(q)$ satisfies $\varepsilon(M) > \gamma r(M)^m$, then M contains a* PG$(n-1, q')$-*minor, for some prime-power q'.*

The polynomial bound we prove here is not asymptotically correct. The order of the bound has been conjectured by Kung [**30**] to be quadratic, that is, m should be 2. This is part of his Growth Rate Conjecture, that we consider in Chapter 4. Note that we could omit γ in the statement of the theorem, since the constant can be compensated for by raising the exponent. However, we keep the constant to facilitate the proof.

The following sections contain the proof of Theorem 1.1.

2 Long lines

Let M be a matroid. For a subset $A \subseteq E(M)$, we write $\varepsilon_M(A) = \varepsilon(M|A)$. A line L of M is a rank-2 flat of M. The *length* of L is the number of points on

L, that is $\varepsilon_M(L)$. As in the previous chapter, we call a line L of M *long* if it has length at least 3. For $e \in E(M)$ denote by $\delta_M(e)$ the number of long lines in M containing e. For an integer $q_* \geq 2$, we say that a line L is q_**-long*, if L has length at least $q_* + 2$.

Lemma 2.1. *Let $q \geq q_* \geq 2$. If $M \in \mathcal{U}(q)$ is minor-minimal with $\varepsilon(M) \geq \lambda {q_*}^{r(M)}$, then*

$$\delta_M(e) \geq \frac{\lambda}{2q} {q_*}^{r(M)}, \quad \textit{for each } e \in E(M),$$

and the number of q_*-long lines in M is at least $\frac{\lambda}{q+1} {q_*}^{r(M)}$.

Proof. Note that, by the minor-minimality, M is simple. Consider $e \in E(M)$. Let δ^+ denote the number of q_*-long lines through e, and let $\delta^- = \delta_M(e) - \delta^+$ be the number of long lines through e of length at most $q_* + 1$.

When contracting e, each line L containing e becomes a point, and so $|L| - 2$ points on L other than e are lost. The number of points destroyed is

$$\varepsilon(M) - \varepsilon(M/e) \leq 1 + \delta^-(q_* - 1) + \delta^+(q - 1).$$

By the minimality of M, we have

$$\varepsilon(M) - \varepsilon(M/e) > \lambda {q_*}^{r(M)} - \lambda {q_*}^{r(M)-1} = \lambda(q_* - 1){q_*}^{r(M)-1}.$$

The above inequalities together yield

$$\delta^-(q_* - 1) + \delta^+(q - 1) \geq \lambda(q_* - 1){q_*}^{r(M)-1}. \tag{1}$$

In particular, inequality (1) gives

$$\delta_M(e) = \delta^- + \delta^+ \geq \lambda \frac{q_* - 1}{q - 1} {q_*}^{r(M)-1},$$

which easily implies the first claim of the lemma.

Again, by the minimality of M,

$$\delta^- + \delta^+ \leq \varepsilon(M/e) < \lambda {q_*}^{r(M)-1}. \tag{2}$$

Now notice that if $\delta^+ = 0$, then the inequalities (1) and (2) contradict. So we must have $\delta^+ > 0$. Since this holds for all $e \in E(M)$ and since lines have at most $q + 1$ elements, the number of q_*-long lines in M is at least $\varepsilon(M)/(q + 1)$. This gives the second claim. ∎

Let M be a simple matroid, $e \in E(M)$ and $A \subseteq E(M) - e$ a set of elements of M. We consider the problem of finding a large subset of A, that does not span e in M. Equivalently, we seek a hyperplane H not containing e, such that $|A \cap H|$ is large.

Suppose that M is representable over $\mathrm{GF}(q)$. Then M is isomorphic to a restriction of $\mathrm{PG}(r-1, q)$, where $r = r(M)$. We can assume for convenience, that M is $\mathrm{PG}(r-1, q)$. Let $\mathcal{H}$ be the collection of hyperplanes of M not containing e. The automorphism group of $\mathrm{PG}(r-1, q)$ is doubly transitive, that is, there is an automorphism mapping any pair of elements to any other pair. This implies, that for an element $a \in E(M) - e$, the number $\#\{H \in \mathcal{H} : a \in H\}$ does not depend on a. Therefore,

$$\begin{aligned}\frac{1}{|\mathcal{H}|} \sum_{H \in \mathcal{H}} |A \cap H| &= \frac{1}{|\mathcal{H}|} \sum_{a \in A} \#\{H \in \mathcal{H} : a \in H\} \\ &= \frac{1}{|\mathcal{H}|} \frac{|A|}{|E(M) - e|} |\mathcal{H}| \frac{q^{r-1} - 1}{q - 1} \\ &= \frac{1}{q} |A|\end{aligned}$$

which is easily verified by a direct computation. By a majority argument, there exists $H \in \mathcal{H}$ with $|A \cap H| \geq \frac{1}{q} |A|$.

We do not have representability and need a different argument that works in $\mathcal{U}(q)$. This is provided by the next lemma.

Lemma 2.2. *Let $q \geq q_* \geq 2$. Let $M \in \mathcal{U}(q)$ and let e be a non-loop element of M. If $A \subseteq E(M) - e$ satisfies $\varepsilon_M(A) \geq \lambda {q_*}^{r_M(A)}$, then there exists $X \subseteq A$ such that $e \notin \mathrm{cl}_M(X)$ and $\varepsilon_M(X) \geq \frac{\lambda}{q} {q_*}^{r_M(X)}$.*

Proof. We may assume that A is minimal with $\varepsilon_M(A) \geq \lambda {q_*}^{r_M(A)}$. This implies, that $M|A$ is simple. We can also assume, that $E(M) = A \cup e$. Assume that A spans e, as otherwise we are done with $X = A$.

Choose a flat W not containing e, with $r_M(W) = r(M) - 2$. Let $H_0, H_1, \dots,$ H_m be the hyperplanes of M containing W. It is easily seen, that the sets $H_i - W$ are a disjoint cover of $E(M) - W$. Also, $\mathrm{si}(M/W) \simeq U_{2,m+1}$ and since $M \in \mathcal{U}(q)$, we have $m \leq q$.

Assume that $e \in H_0$. By the minimality of A, $|H_0 \cap A| < \lambda {q_*}^{r(M)-1}$ and so

$$|A - H_0| > \lambda(q_* - 1){q_*}^{r(M)-1}.$$

Since the sets $H_1, \dots, H_m$ cover $E(M) - H_0$, there exists a $k \in \{1, \dots, m\}$ with

$$|H_k \cap A| \geq \frac{1}{m} |A - H_0| > \frac{\lambda}{q}(q_* - 1){q_*}^{r(M)-1}.$$

Taking $X = H_k \cap A$, we have the desired result. ■

3 Pyramids

We now define some intermediate structures that we shall build on our way to constructing a projective geometry.

Definition 3.1. M is a *pyramid* with *joints* $(b_1, \ldots, b_n)$, if $\{b_1, \ldots, b_n\}$ is a basis of M and for each $i = 2, \ldots, n$, b_i is on a long line with every point of $cl_M(\{b_1, \ldots, b_{i-1}\})$. For $q_* \geq 2$, M is a *q_*-strong* pyramid, if each pair of joints spans a q_*-long line.

Pyramids could be defined recursively as follows. The matroid M is a pyramid if M has a hyperplane H and an element $b \notin H$, such that every point of H is on a long line with b and $M|H$ is a pyramid. It follows by an easy induction, that a rank-r pyramid has at least $2^r - 1$ points. In particular, if M is a binary pyramid of rank r, then $\mathrm{si}(M)$ is isomorphic to $\mathrm{PG}(r-1, 2)$, as M has the maximal number of points.

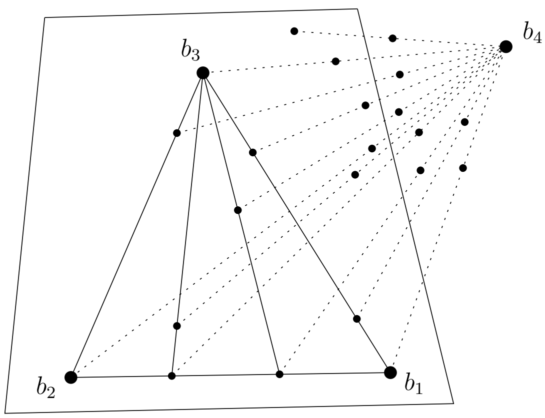


Figure 1: A pyramid of rank 4.

Definition 3.2. M is an *(n, λ, q_*)-prepyramid* if it has a basis $B \cup \{b_1, \ldots, b_n\}$ such that

- $F = \mathrm{cl}_M(B)$ satisfies $\varepsilon_M(F) \geq \lambda {q_*}^{r_M(F)}$ and
- for each $i = 1, \ldots, n$, b_i is on a long line with every point of $\mathrm{cl}_M(B \cup \{b_1, \ldots, b_{i-1}\})$.

So a prepyramid is a pyramid "on top of" a dense flat. The first step in the proof of Theorem 1.1 is to construct a prepyramid.

Lemma 3.3. *Let $q \geq q_* \geq 2$ be given. If $n \geq 0$ and $\lambda \geq 1$ are integers and $M \in \mathcal{U}(q)$ satisfies $\varepsilon(M) \geq \lambda q^{2n} {q_*}^{r(M)}$, then M has an (n, λ, q_*)-prepyramid as a minor.*

Proof. The proof is by induction on n. The case $n = 0$ is trivial, so suppose $n > 0$ and that the result holds for $n-1$. We may assume that M is minor-minimal with $\varepsilon(M) \geq \lambda q^{2n} {q_*}^{r(M)}$. In particular M is simple.

Choose an element $b_n \in E(M)$, and let $A \subseteq E(M) - b_n$ be the set of elements on long lines through b_n. By Lemma 2.1,

$$|A| \geq 2\delta_M(b_n) \geq \frac{\lambda q^{2n}}{q} {q_*}^{r(M)}.$$

By Lemma 2.2, there exists a set $X \subseteq A$ with $b_n \notin \mathrm{cl}_M(X)$ and

$$|X| \geq \frac{\lambda q^{2n}}{q^2} {q_*}^{r_M(X)} = \lambda q^{2(n-1)} {q_*}^{r_M(X)}.$$

By the induction hypothesis $M|X$ has a minor M_1, which is an $(n-1, \lambda, q_*)$-prepyramid. We let $Z = E(M_1)$, and so M_1 can be written $M_1 = (M|X)/Y|Z$. Let $W \subseteq E(M)$ denote the elements that are on a line through b_n and an element $z \in Z$ in M. Then $N = M/Y|W$ is an (n, λ, q_*)-prepyramid, since $b_n \notin \mathrm{cl}_N(Z)$ and every element $z \in Z$ is on a long line with b_n. ∎

4 Getting a strong pyramid

We repeat the definition of skew sets given in the previous chapter. For a matroid M, we call sets $A_1, \ldots, A_n \subseteq E(M)$ *skew* if $r_M(\cup_i A_i) = \sum_i r_M(A_i)$. This is analogous to subspaces of a vector-space forming a direct sum.

The goal of this section is to obtain a strong pyramid from a pre-pyramid. For the first lemma, we shall need a limit on the total number of lines of a matroid in $\mathcal{U}(q)$. Let $m_q(n)$ denote the maximum number of lines of a rank-n matroid in $\mathcal{U}(q)$. From Kung's Theorem, we easily get the following crude upper bound

$$m_q(n) \leq \binom{\frac{q^n-1}{q-1}}{2}.$$

Lemma 4.1. *There exists an integer-valued function $\theta_1(s, \lambda, q)$ such that the following holds: Let $q \geq q_* \geq 2$. If s and λ are positive integers and $M \in \mathcal{U}(q)$ satisfies $\varepsilon(M) \geq \theta_1(s, \lambda, q){q_*}^{r(M)}$, then either*

- *M has a minor N with s skew q_*-long lines or*
- *M has a minor N with a non-loop element $e \in E(N)$ such that the number of q_*-long lines through e in N is at least $\lambda {q_*}^{r(N)}$.*

Proof. Define $\theta_1(1, \lambda, q) = 1$ and for $s \geq 2$,

$$\theta_1(s, \lambda, q) = (q+1)4(s-1)m_q(2s-1)\lambda.$$

We assume that M is minor-minimal with $\varepsilon(M) \geq \theta_1(s, \lambda, q){q_*}^{r(M)}$. Let $\mathcal{L}$ denote the collection of q_*-long lines in M. By Lemma 2.1,

$$|\mathcal{L}| \geq \frac{\theta_1(s, \lambda, q)}{q+1}{q_*}^{r(M)}.$$

In the case $s = 1$ we are now done, since $|\mathcal{L}| > 0$, so assume $s \geq 2$ in the following.

If $\mathcal{L}$ contains s skew lines, then we are done, so assume this is not the case. Pick a maximal set of skew lines from $\mathcal{L}$ and let F be the flat spanned by these lines in M. Let $t = r_M(F) \leq 2(s-1)$. Let $\mathcal{L}' \subseteq \mathcal{L}$ be the lines not contained in F (see Figure 2). Then

$$\left|\mathcal{L}'\right| \geq |\mathcal{L}| - m_q(t) \geq \tfrac{1}{2}\left|\mathcal{L}\right|,$$

provided that $|\mathcal{L}| \geq 2m_q(t)$, which follows easily from the above.

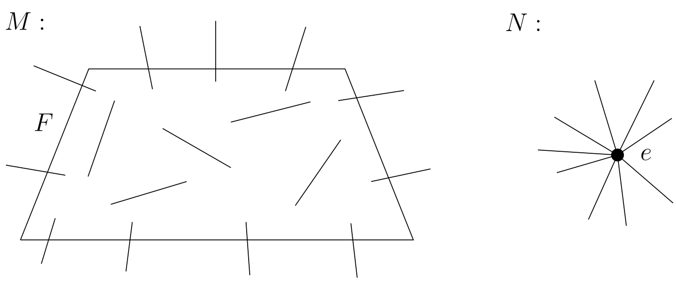


Figure 2

Let B be a basis of F in M. For each $L \in \mathcal{L}'$ pick $B_L \subseteq B$ with $|B_L| = t-1$, such that B_L and L are skew (this can be done by expanding a basis of L to a basis of $L \cup F$ using elements of B). By a majority argument, there is a subcollection $\mathcal{L}'' \subseteq \mathcal{L}'$ with the sets $B_L = B_0$ identical for $L \in \mathcal{L}''$ and such that

$$|\mathcal{L}''| \geq \tfrac{1}{t} |\mathcal{L}'|.$$

Let e be the single element in $B - B_0$ and let $N = M/B_0$. Then each line $L \in \mathcal{L}''$ spans a q_*-long line through e in N. Two lines $L_1, L_2 \in \mathcal{L}''$ give rise to the same line in N if $\mathrm{cl}_M(B_0 \cup L_1) = \mathrm{cl}_M(B_0 \cup L_2)$. Hence, the number of q_*-long lines through e in N is at least

$$\frac{|\mathcal{L}''|}{m_q(t+1)}.$$

By concatenating the inequalities, we get the desired result. ■

We now use the previous lemma to construct a strong pyramid. This is done in exactly the same way as a prepyramid was constructed in Lemma 3.3.

Lemma 4.2. *There exists an integer-valued function $\theta(s, n, q)$ such that the following holds: Let $q \geq q_* \geq 2$. If s and n are positive integers and $M \in \mathcal{U}(q)$ satisfies $\varepsilon(M) \geq \theta(s, n, q){q_*}^{r(M)}$, then either*

- *M has a minor N with s skew q_*-long lines or*
- *M has a rank-n minor N, such that N is a q_*-strong pyramid.*

Proof. Let $\theta(s, 1, q) = 1$, and for $n \geq 2$ define θ recursively by

$$\theta(s, n, q) = \theta_1(s, q\theta(s, n-1, q), q).$$

The proof is by induction on n, the case $n = 1$ being trivial. Suppose $n \geq 2$ and that M does not have a minor with s skew q_*-long lines.

By Lemma 4.1, M has a minor M' with a non-loop element b_n, such that the number of q_*-long lines through b_n is at least

$$q\theta(s, n-1, q){q_*}^{r(M')}.$$

Let $A \subseteq E(M') - b_n$ be the set of elements on q_*-long lines through b_n. Lemma 2.2 gives a set $X \subseteq A$ with $b_n \notin \mathrm{cl}_{M'}(X)$, such that

$$\varepsilon_{M'}(X) \geq \theta(s, n-1, q){q_*}^{r_{M'}(X)}.$$

By induction, $M'|X$ has a minor M_1, which is a q_*-strong rank-$(n-1)$ pyramid. We let $Z = E(M_1)$, and so M_1 can be written $M_1 = (M'|X)/Y|Z$. Let $W \subseteq E(M')$ denote the elements that are on a line in M' through b_n and an element $z \in Z$. Then $N = M'/Y|W$ is a rank-n pyramid, since $b_n \notin \mathrm{cl}_N(Z)$ and every element $z \in Z$ is on a long line with b_n. It is surely q_*-strong, since all the lines through b_n are q_*-long. ■

Lemma 4.3. *Let $q \geq q_* \geq 2$. If n and λ are positive integers with $\lambda \geq \theta(\binom{n}{2}, n, q)$ and $M \in \mathcal{U}(q)$ is an (n, λ, q_*)-prepyramid, then M has a rank-n minor N, such that N is a q_*-strong pyramid.*

Proof. Let M be an (n, λ, q_*)-prepyramid w.r.t. the basis $B \cup \{b_1, \ldots, b_n\}$, and let $F = \mathrm{cl}_M(B)$.

We apply Lemma 4.2 to $M|F$ and get one of two outcomes. If $M|F$ has a rank-n q_*-strong pyramid minor, then we are done, so assume this is not the case. Then $M|F$ has a minor $M|F/Y$ (we can assume it is a contraction) containing $\binom{n}{2}$ skew q_*-long lines. Let B' be a basis for $M|F/Y$. Then $M' = M/Y$ is an (n, λ', q_*)-prepyramid with respect to the basis $B' \cup \{b_1 \ldots, b_n\}$, for some λ'. If we let $F' = \mathrm{cl}_{M'}(B') = F - Y$, then $M'|F'$ has $\binom{n}{2}$ skew q_*-long lines. Since we make no reference to λ in the following, we may assume that $Y = \emptyset$ and $M' = M$. We shall also assume that M is simple.

Let L_{ij}, for $1 \leq i < j \leq n$ denote skew q_*-long lines in $M|F$. We shall contract these lines in place between $b_1, \ldots, b_n$. For each pair of indices $i < j$ choose two different elements $x_{ij}, y_{ij} \in L_{ij}$. Also, choose points on the lines between b_i, b_j and L_{ij} as follows (see Figure 3),

$$e_{ij} \in \mathrm{cl}_M(\{b_i, x_{ij}\}) - \{b_i, x_{ij}\},$$
$$f_{ij} \in \mathrm{cl}_M(\{b_j, y_{ij}\}) - \{b_j, y_{ij}\}.$$

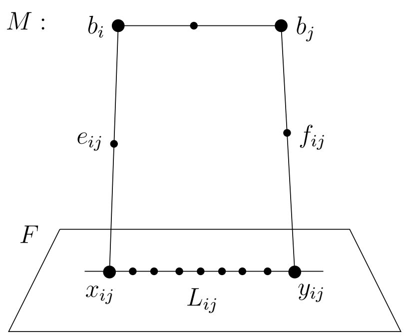


Figure 3

Notice, that $\{b_1, \ldots, b_n\} \cup \{e_{ij}, f_{ij} : i < j\}$ is an independent set. We define the minor N in the following way:

$$N = M/\{e_{ij}, f_{ij}\}|W,$$
$$\text{where } W = \mathrm{cl}_M(\{b_1, \ldots, b_n\}) \cup (\cup_{i<j} W_{ij}),$$
$$W_{ij} = \mathrm{cl}_M(L_{ij} \cup \{b_{j+1}, \ldots, b_n\}), \quad 1 \leq i < j \leq n.$$

We leave it to the reader to verify in details, that N is a pyramid with joints $(b_1, \ldots, b_n)$. To do this, note that

$$W_{ij} \cap \mathrm{cl}_N(\{b_1, \ldots, b_k\}) = \begin{cases} \emptyset & \text{if } k < i, \\ \{x_{ij}\} & \text{if } i \leq k < j, \\ \mathrm{cl}_M(L_{ij} \cup \{b_{j+1}, \ldots, b_k\}) & \text{if } j \leq k. \end{cases}$$

The pyramid N is q_*-strong, since the line spanned by b_i and b_j contains L_{ij}. ■

5 Pyramids contain projective geometries

Let M be a pyramid with joints $(b_1, \ldots, b_n)$, and for each i, let $H_i = \mathrm{cl}_M(\{b_1, \ldots, b_i\})$. We call M *modular* if for each i, if $x, y \in H_i - H_{i-1}$ are non-parallel elements, then the line through x and y intersects H_{i-1} in a point.

The first step towards getting a projective geometry minor of a pyramid, will be to find a modular pyramid.

Lemma 5.1. *Let $q \geq 2$ and let m be a positive integer. If $M \in \mathcal{U}(q)$ is a pyramid with $r(M) \geq mq^{\binom{m}{2}}$, then M has a rank-m modular pyramid minor N. If for some number $q_* \geq 2$, M is q_*-strong, then N is q_*-strong.*

Proof. Let m be a fixed positive integer. To each pyramid $N \in \mathcal{U}(q)$ of rank $r(N) = n \geq m$ with joints $(a_1, \ldots, a_n)$, we assign a vector

$$Q(N) = (\varepsilon_N(H_2), \varepsilon_N(H_3), \ldots, \varepsilon_N(H_{m-1})) \in \mathbb{Z}^{m-2},$$

where $H_k = \mathrm{cl}_N(\{a_1, \ldots, a_k\})$. By Kung's Theorem, the number of values that $Q(N)$ can attain is bounded by

$$\prod_{k=2}^{m-1} \frac{q^k - 1}{q - 1} \leq \prod_{k=2}^{m-1} q^k \leq q^{\binom{m}{2}}.$$

We shall also consider the lexicographic ordering on $\mathbb{Z}^{m-2}$ defined by:

$$(a_1, \ldots, a_{m-2}) <_{LEX} (b_1, \ldots, b_{m-2})$$

if there is a $k \in \{1, \ldots, m-2\}$, such that $a_i = b_i$ for $i = 1, \ldots, k-1$ and $a_k < b_k$. This is a total order.

Let $N \in \mathcal{U}(q)$ be a pyramid with joints $(a_1, \ldots, a_n)$, $n \geq 2m$, and let $H_k = \mathrm{cl}_N(\{a_1, \ldots, a_k\})$. Assume that the pyramid $N|H_m$ with joints $(a_1, \ldots, a_m)$ is not modular. We now describe an operation, that gives a minor of N, with an increased value of $Q(\cdot)$ in the above order. We can assume, that N is simple. There exists an $i \leq m$ and an element $y \in H_i - H_{i-1}$, such that $\left|\mathrm{cl}_{\mathrm{si}(N/y)}(\{a_1, \ldots, a_{i-1}\})\right| > |H_{i-1}|$. Choose $k \in \{2, \ldots, i-1\}$ minimal, with

$$\left|\mathrm{cl}_{\mathrm{si}(N/y)}(\{a_1, \ldots, a_k\})\right| > |H_k|.$$

See Figure 4. Now let $B' = (a_1, \ldots, a_k, a_{i+1}, \ldots, a_n)$ and define

$$N' = N/y|\,\mathrm{cl}_{N/y}(B').$$

By construction, N' is a pyramid with joints B', which is easily verified. It has a higher value in the order $Q(N) <_{LEX} Q(N')$, and rank $r(N') \geq r(N) - m$. Also, if N is q_*-strong, N' is q_*-strong.

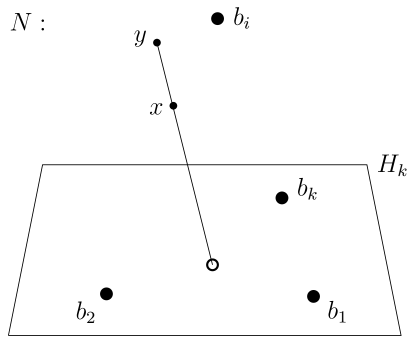


FIGURE 4

Now, let $M \in \mathcal{U}(q)$ be a pyramid, with $r(M) \geq mq^{\binom{m}{2}}$. By the bound on the number of possible values of $Q(\cdot)$, the process of repeating the above operation must terminate with a rank-m modular pyramid minor. This pyramid is q_*-strong if M is q_*-strong. ∎

The projective geometries $\mathrm{PG}(n-1, q)$ are examples of *projective spaces*. We shall not define this concept in general, only state that a matroid is a projective space if every line has at least three points, and every pair of coplanar lines intersect.

The following theorem is the finite case of what is sometimes referred to as the Fundamental Theorem of Projective Geometry (see [**6**, pp. 27–28] for a detailed account of the theorem and [**2**, cpt. VII] for a proof. Note that there is at least one other theorem that goes by the same name). The result does not hold in rank 3.

Theorem 5.2. *Every finite projective space of rank $n \geq 4$ is isomorphic to $\mathrm{PG}(n-1, q')$ for some prime-power q'.*

In the next lemma we use the theorem to identify a projective geometry in a modular pyramid.

Lemma 5.3. *There exists an integer-valued function $\psi(n, q)$ such that the following holds: Let $q \geq 2$. If $M \in \mathcal{U}(q)$ is a modular pyramid with $r(M) \geq \psi(n, q)$, then M has a $\mathrm{PG}(n-1, q')$-restriction for some prime-power q'. If for some number $q_* \geq 2$, M is q_*-strong, then $q' > q_*$.*

Proof. For $n \geq 4$, define

$$\psi(n, q) = (q-1)(n-1) + 2,$$

and let $\psi(n, q) = \psi(4, q)$, for $n = 1, 2, 3$. Hence, the cases $n = 1, 2, 3$ follow from the case $n = 4$, so let $n \geq 4$ be given in the following.

Let M be a modular pyramid with joints $(b_1, \dots, b_r)$, where $r = r(M) \geq \psi(n, q)$. Assume that M is simple. Let $H_i = \mathrm{cl}_M(\{b_1, \dots, b_i\})$, for $i = 1, \dots, r$.

Notice first, that every line $L \subseteq H_{r-1}$ has length at least 3. Otherwise, looking at the plane spanned by L and b_r, we find a contradiction to the modularity of M.

Define a sequence of numbers $l_2, \dots, l_{r-1}$, by $l_i = \min\{|L| : L \subseteq H_i\}$, where the minimum is over all lines of M contained in H_i. The sequence is clearly descending,

$$q + 1 \geq l_2 \geq l_3 \geq \cdots \geq l_{r-1} \geq 3.$$

Since $r - 2 \geq (q-1)(n-1)$, by a majority argument there must be a constant subsequence of length $n - 1$,

$$l_k = l_{k+1} = \cdots = l_{k+n-2} = l$$

with value l. Choose a line $L_* \subseteq H_k$ with $|L_*| = l$, and let $p_1, p_2 \in L_*$ be different elements. Let $p_3 = b_{k+1}, \dots, p_n = b_{k+n-2}$. We define the minor $N = M|\mathrm{cl}_M(\{p_1, \dots, p_n\})$. By construction, N is a modular pyramid. Let $F_i = \mathrm{cl}_N(\{p_1, \dots, p_i\})$ for each i.

We claim that every line in N has length l. Clearly, there are no shorter lines. Suppose the claim fails and let i be minimal, such that there is a line $L \subseteq F_i$ with $|L| > l$. We must have $i > 2$, since $F_2 = L_*$ has length l. Choose an element $x \in F_i - F_{i-1}$, not on L (note, that this is possible). Now, by modularity each element in L is on a line through x that intersects F_{i-1} in a point. This gives $|L|$ colinear points in F_{i-1}, contradicting the minimality of i (see Figure 5a).

Observe, that if M is a q_*-strong pyramid, $l \geq q_* + 2$, since N contains the line spanned by b_{k+1} and b_{k+2} which is a q_*-long line of M.

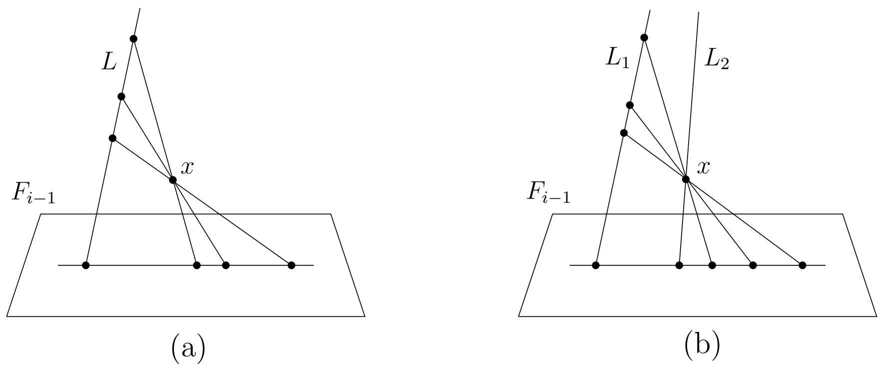


Figure 5

To prove that N is a projective space, we show that coplanar lines intersect. Let L_1 and L_2 be coplanar lines of N and let $P = \mathrm{cl}_N(L_1 \cup L_2)$. Let i be minimal with $P \subseteq F_i$. If L_1 is contained in F_{i-1}, then L_2 must intersect L_1 by the modularity of N. Similarly if L_2 is contained in F_{i-1}. Suppose $L_1, L_2 \not\subseteq F_{i-1}$, and assume that L_1 and L_2 do not intersect. Let $x \in L_2 - F_{i-1}$. Each point on L_1 is on a line through x than intersects F_{i-1} in a point. These, together with the point of intersection of L_2 and F_{i-1} account for $l+1$ points of the line $P \cap F_{i-1}$, a contradiction (see Figure 5b).

Finally by Theorem 5.2, N is isomorphic to $\mathrm{PG}(n-1, q')$, and we must have $l = q' + 1$. ∎

Theorem 1.1 is now proved by applying Lemmas 3.3, 4.3, 5.1 and 5.3 in succession. The bound α in the theorem, depending on n and q becomes:

$$\alpha = \lambda q^{2n'},$$
$$\text{where } \lambda = \theta(\binom{n'}{2}, n', q),$$
$$n' = mq^{\binom{m}{2}} \text{ and } m = \psi(n, q).$$

6 Proof of the polynomial result

We now turn to Theorem 1.4. To prove the theorem, by the previous results, it is enough to get a large pyramid. This is done in the same way that we obtained a prepyramid in Lemma 3.3, the proof of which rested on Lemmas 2.1 and 2.2. The arguments are the same, only the calculations differ. The following result parallels Lemma 2.2.

Lemma 6.1. *Let $q \geq 2$ and let λ and n be positive integers. Let $M \in \mathcal{U}(q)$ and let e be a non-loop element of M. If $A \subseteq E(M) - e$ satisfies $\varepsilon_M(A) > \lambda r_M(A)^n$, then there exists $X \subseteq A$ such that $e \notin \mathrm{cl}_M(X)$ and $\varepsilon_M(X) > \frac{\lambda n}{q} r_M(X)^{n-1}$.*

Proof. The proof mimics the proof of Lemma 2.2. We may assume that A is minimal with $\varepsilon_M(A) > \lambda r_M(A)^n$, implying that $M|A$ is simple. We also assume, that $E(M) = A \cup e$. Assume that A spans e, as otherwise we are done.

Choose a flat W not containing e, with $r_M(W) = r-2$, and let $H_0, H_1, \dots, H_m$ be the hyperplanes of M containing W. Then $\mathrm{si}(M/e) \simeq U_{2,m+1}$ and so $m \leq q$, since $M \in \mathcal{U}(q)$.

We may assume $e \in H_0$. By the minimality of A, $|H_0 \cap A| \leq \lambda(r-1)^n$ and thus

$$|A - H_0| > \lambda(r^n - (r-1)^n) \geq \lambda n(r-1)^{n-1},$$

where we have used the inequality $(x+1)^n - x^n \geq nx^{n-1}$ for a non-negative number x. Since the sets $H_1, \dots, H_m$ cover $E(M) - H_0$, by a majority argument we have

$$|H_i \cap A| \geq \frac{1}{m}|A - H_0| > \frac{\lambda n}{q}(r-1)^{n-1},$$

for some i, and we are done with $X = H_i \cap A$. ∎

In the following lemma a pyramid is constructed.

Lemma 6.2. *There exists an integer-valued function $\phi(n,q)$ such that the following holds: Let $q \geq 2$ and n be positive integers. If $M \in \mathcal{U}(q)$ has $\varepsilon(M) > \phi(n,q)r(M)^{2(n-1)}$, then M has a rank-n pyramid minor.*

Proof. Let $\phi(1,q) = 1$, and for $n \geq 2$ define

$$\phi(n,q) = \frac{q^2\phi(n-1,q)}{4n-6}.$$

The proof is by induction on n. The case $n = 1$ is trivial, so assume $n \geq 2$, and that the result holds for $n-1$. We write $\phi = \phi(n,q)$ for brevity.

Let $r = r(M)$, and let $k = 2(n-1)$. We may assume that M is minimal with $\varepsilon(M) > \phi r^k$. Choose an element e of M. Then $\varepsilon(M/e) \leq \phi(r-1)^k$ and

$$\varepsilon(M) - \varepsilon(M/e) > \phi(r^k - (r-1)^k) \geq \phi r^{k-1}.$$

When contracting e, $|L| - 2$ points other that e are lost from each line L containing e. Hence $\varepsilon(M) - \varepsilon(M/e) \leq 1 + (q-1)\delta_M(e)$ and

$$(q-1)\delta_M(e) \geq \phi r^{k-1}.$$

Let $A \subseteq E(M) - e$ be the set of points on long lines through e. Then $|A| \geq 2\delta_M(e) > \frac{2\phi}{q} r^{k-1}$. The previous lemma now gives a set $X \subseteq A$, with $e \notin \mathrm{cl}_M(X)$ and

$$|X| > \frac{2\phi(k-1)}{q^2} r_M(X)^{k-2} = \phi(n-1, q) r_M(X)^{2(n-2)}.$$

Applying the induction hypothesis to $M|X$ we get a minor of $M|X$ that is a rank-$(n-1)$ pyramid. We can now argue, as in the proofs of Lemmas 3.3 and 4.2, that M has a rank-n pyramid minor. ■

When $q \geq 2$, Theorem 1.4 now follows from Lemmas 6.2, 5.1 and 5.3. For the case $q = 1$, note that a simple matroid M in $\mathcal{U}(1)$ has no circuits, and thus $|E(M)| = r(M)$. So, taking $\gamma = m = 1$, the condition $|E(M)| > \gamma r(M)^m$ of the theorem is never satisfied.

4 The Growth Rate Conjecture

This chapter considers a number of problems in the field of extremal matroid theory. It is largely motivated by (and contains numerous references to) Kung's founding paper [**30**] on the subject. The main focus of the chapter is on Kung's Growth Rate Conjecture, that we state in Section 2. We begin with a few observations about graphs.

For a graph G, let $v(G)$ denote the number of vertices in G and let $e(G)$ denote the number of edges in the simplification of G. For any graph G, $e(G)$ is bounded by a function of $v(G)$, namely $e(G) \leq \binom{v(G)}{2}$. Given a class of graphs $\mathcal{C}$, one can ask for a better bound on $e(G)$ as a function of $v(G)$, for $G \in \mathcal{C}$. Exact bounds of this sort are known for many common classes of graphs. Turán's Theorem [**50**] (or see [**7**]) gives the exact bound for the class of graphs with no K_m-subgraph. This bound is quadratic in $v(G)$ for $m \geq 3$. The theorem illustrates that, except for some degenerate cases, for a class of graphs defined by excluding certain graphs as subgraphs, the bound is quadratic.

Mader's Theorem states that, given a graph H, if $\mathcal{C}$ is the class of graphs with no H-minor, then $e(G) \leq cv(G)$ for each $G \in \mathcal{C}$, where c is a constant depending only on H. This shows that any minor-closed class $\mathcal{C}$ of graphs satisfies a linear bound, unless $\mathcal{C}$ is the class of all graphs. So there is a *gap* in the possible bounds for minor-closed classes of graphs between linear and quadratic functions. We shall see that this phenomenon arises for minor-closed classes of matroids as well.

1 Size functions of classes of matroids

For classes of matroids we are interested in the maximum number of points as a function of the rank. Since, for a connected graph G, the rank of $M(G)$ is $v(G)-1$, this is the natural counterpart of the above question for graphs. When we talk of a *class* of matroids in the following, we shall mean a set of matroids closed under isomorphism, and containing members of any rank; this is to avoid degenerate cases in definitions and theorems. We shall be particularly interested in minor-closed classes of matroids. Any minor-closed class $\mathcal{M}$ contains the free matroids $\{U_{n,n} : n \in \mathbb{N}_0\}$. It follows that arbitrary intersections and unions of minor-closed classes are again minor-closed classes.

For a class of matroids $\mathcal{M}$, Kung (see [**30**],[**31**]) defines the *size function* $h(\mathcal{M};\cdot) : \mathbb{N}_0 \to \mathbb{N}_0 \cup \{\infty\}$ by

$$h(\mathcal{M};r) = \max\{\varepsilon(M) : M \in \mathcal{M}, r(M) = r\},$$

if the maximum exists, and $h(\mathcal{M};r) = \infty$ otherwise. Since $\mathcal{M}$ contains matroids of any rank, $h(\mathcal{M};r) \geq r$ for all r. Clearly, $h(\mathcal{M};0) = 0$ and $h(\mathcal{M};1) = 1$.

Let $\mathcal{M}$ be a minor-closed class of matroids. To bound the number of points in rank 2 is to bound the length of lines. If $h(\mathcal{M};2)$ is finite, then $\mathcal{M}$ does not contain arbitrarily long lines, that is, $\mathcal{M} \subseteq \mathcal{U}(q)$ for some integer q. On the other hand, by Kung's theorem, if $\mathcal{M} \subseteq \mathcal{U}(q)$ for some q, then $h(\mathcal{M};r)$ is finite for all r. One can imagine minor-closed classes for which $h(\mathcal{M};2) = \infty$, but $h(\mathcal{M};r)$ is finite for higher ranks (e.g. $\mathcal{M} = \mathcal{G} \cup \{U_{2,q+2} : q \in \mathbb{N}\}$). However, every common class of matroids has non-decreasing size function; this is the case, in particular, if the class is closed under direct sums. So, apart from these exceptions, the classes $\mathcal{U}(q)$ are the largest minor-closed classes for which the study of size functions makes sense.

Exact size functions are known for some classes of matroids. Denote by $\mathcal{R}$ the class of regular matroids.

$$h(\mathcal{G}^*;r) = 3r-3, \text{ for } r \geq 2 \qquad h(\mathcal{G};r) = \binom{r+1}{2}$$

$$h(\mathcal{R};r) = \binom{r+1}{2} \qquad h(\mathcal{R}(q);r) = \frac{q^r-1}{q-1}$$

The linear bound for the class of co-graphic matroids was proved by Jaeger (see [**30**] for a reference). Since graphic matroids are regular, $h(\mathcal{R};r) \geq \binom{r+1}{2}$. That equality holds, can be deduced from Seymour's famous decomposition theorem for regular matroids [**47**]. Alternatively, it follows from a result we prove in Section 4. Let F_7 denote the Fano plane PG(2, 2). We prove, that $h(\mathcal{R}(2) \cap \mathcal{EX}(F_7);r) \leq \binom{r+1}{2}$. Tutte's excluded minor characterization of regular matroids (see Chapter 5) says $\mathcal{R} = \mathcal{EX}(U_{2,4}, F_7, F_7^*)$. Hence, $\mathcal{R}$ is contained in $\mathcal{R}(2) \cap \mathcal{EX}(F_7)$ and both classes must have size function $\binom{r+1}{2}$. For exact size functions of several other classes of matroids, see [**30**].

Kung's Theorem states that for $q \geq 2$,

$$h(\mathcal{U}(q);r) \leq \frac{q^r-1}{q-1},$$

with equality for all r if q is a prime-power. If q is not a prime-power, we obtained an asymptotic improvement on the size function in Corollary 1.2 of

Chapter 3,

$$h(\mathcal{U}(q); r) \le c{q_*}^r,$$

where q_* denotes the greatest prime-power with $q_* \le q$, and c is a constant depending on q. Since $\mathcal{R}(q_*) \subseteq \mathcal{U}(q)$, we have the lower bound $h(\mathcal{U}(q); r) \ge \frac{{q_*}^r - 1}{q_* - 1}$. Hence, $h(\mathcal{U}(q); r)$ has order of magnitude ${q_*}^r$. Bonin in [**4**] proves a number of more specific bounds on $h(\mathcal{U}(q); r)$, where q is not a prime-power. However, each of these bounds have order of magnitude q^r.

2 The Growth Rate Conjecture

For each of the classes above, the size function is either linear, quadratic or an exponential function with a prime-power base. When we call a function *linear*, *quadratic* or *exponential*, we mean up to order of magnitude. For instance, h is quadratic if there are positive constants c_1, c_2, such that $c_1 r^2 \le h(r) \le c_2 r^2$ for all $r \ge 1$. There is currently no minor-closed class for which the size function is known to be cubic or any order of magnitude other than linear, quadratic or exponential. This observation leads us to the following beautiful conjecture of Kung.

The Growth Rate Conjecture. *Let $\mathcal{M}$ be a minor-closed class of matroids, that does not contain arbitrarily long lines.*

(1) *If $\mathcal{G} \nsubseteq \mathcal{M}$, then $h(\mathcal{M}; r)$ is linear.*
(2) *If $\mathcal{G} \subseteq \mathcal{M}$ and $\mathcal{R}(q) \nsubseteq \mathcal{M}$ for every prime-power q, then $h(\mathcal{M}; r)$ is quadratic.*
(3) *If for some prime-power q, $\mathcal{R}(q) \subseteq \mathcal{M}$ and q is maximal with this property, then $h(\mathcal{M}; r) = \frac{q^r - 1}{q - 1}$ for all r sufficiently large.*

The conjecture says, that for any minor-closed class in $\mathcal{U}(q)$, the size function is either linear, quadratic or exponential. Kung first made this conjecture for binary matroids in [**28**] and later in a version equivalent to the above in [**30**] (though it is phrased quite differently). By the term *growth rate* of a minor-closed class $\mathcal{M}$, Kung refers to the difference function $h(\mathcal{M}; r) - h(\mathcal{M}; r-1)$. If this function is bounded, then $h(\mathcal{M}; r)$ is linear. We shall not be using the concept of growth rate here.

Notice that the conjectured bounds on $h(\mathcal{M}; r)$ in case (1) and (2) are up to order of magnitude, while in case (3) it is exact (for r sufficiently large). It is necessary to allow arbitrarily large constants in the linear and quadratic bounds. In the linear case, this is true already for classes of graphic matroids, since there are lower bounds on the constant in Mader's Theorem, that tend to infinity with the size of the clique being excluded. For the quadratic case, Kung in [**30**] constructs minor-closed classes in $\mathcal{R}(q)$ with quadratic size functions with arbitrarily large constants ("framed gain-graphic matroids").

Case (1) in the conjecture follows from the extension of Mader's Theorem to $\mathcal{U}(q)$ by Geelen and Whittle. The two main results of Chapter 3 imply weaker versions of (2) and (3). We accumulate these results in the following weakened modification of the conjecture.

Theorem 2.1. *Let $\mathcal{M}$ be a minor-closed class of matroids, that does not contain arbitrarily long lines.*

(1) *If $\mathcal{G} \not\subseteq \mathcal{M}$, then $h(\mathcal{M}; r)$ is linear.*
(2) *If $\mathcal{G} \subseteq \mathcal{M}$ and $\mathcal{R}(q) \not\subseteq \mathcal{M}$ for every prime-power q, then there is an integer m such that $\binom{r+1}{2} \leq h(\mathcal{M}; r) \leq r^m$ for all r.*
(3) *If for some prime-power q, $\mathcal{R}(q) \subseteq \mathcal{M}$ and q is maximal with this property, then $h(\mathcal{M}; r)$ has order of magnitude q^r.*

Proof. Let $\mathcal{M}$ be given. Since $\mathcal{M}$ does not contain arbitrarily long lines, we have $\mathcal{M} \subseteq \mathcal{U}(q)$ for some q.

To prove (1) assume that $\mathcal{M}$ does not contain all graphic matroids. Since any graph is a minor of a complete graph, there exists an integer n, such that $M(K_n) \notin \mathcal{M}$. By Theorem 10.1 of Chapter 2, there exists an integer λ, such that $h(\mathcal{M}; r) \leq \lambda r$ for all r. Since $h(\mathcal{M}; r) \geq r$, the size function is linear.

Consider (2). For each prime-power $q' \leq q$, $\mathcal{M}$ does not contain $\mathcal{R}(q')$, so there is an integer $n_{q'}$, such that $\mathrm{PG}(n_{q'} - 1, q') \notin \mathcal{M}$. Let n be the maximum of these integers $n_{q'}$ over prime-powers $q' \leq q$. So $\mathrm{PG}(n-1, q') \notin \mathcal{M}$ for every prime-power q'. By Theorem 1.4 of Chapter 3, there exists an integer m, such that $h(\mathcal{M}; r) \leq r^m$. Since $\mathcal{G} \subseteq \mathcal{M}$, $h(\mathcal{M}; r) \geq \binom{r+1}{2}$.

Case (3) follows in much the same way as (2). Let q_* be the maximal prime-power with $\mathcal{R}(q_*) \subseteq \mathcal{M}$. For each prime-power q' with $q_* < q' \leq q$, choose $n_{q'}$, such that $\mathrm{PG}(n_{q'} - 1, q') \notin \mathcal{M}$, and let n be the maximum of these integers. So $\mathrm{PG}(n-1, q') \notin \mathcal{M}$ for every prime-power $q' > q_*$. By Theorem 1.1 of Chapter 3, there exists an integer α, such that $h(\mathcal{M}; r) \leq \alpha {q_*}^r$. Since $\mathcal{R}(q_*) \subseteq \mathcal{M}$, $h(\mathcal{M}; r) \geq \frac{{q_*}^r - 1}{q-1} \geq \frac{1}{q_*} {q_*}^r$. We conclude that the size function has order of magnitude ${q_*}^r$. ■

The conjectured exact size function in case (3) of the Growth Rate Conjecture seems very difficult to prove. If true, case (3) of the conjecture would readily imply the weaker conjecture mentioned in Chapter 3:

Let $q \geq 2$ and let q_* be the largest prime-power with $q_* \leq q$. Then

$$h(\mathcal{U}(q); r) = \frac{{q_*}^r - 1}{q_* - 1},$$

for all sufficiently large r.

We shall see next, that if we assume representability over some finite field, the gaps between possible orders of magnitude of the size function become much larger. Let q be a prime-power and $\mathbb{F}$ a field. Then

$$\mathcal{R}(q) \subseteq \mathcal{R}(\mathbb{F}) \quad \textit{if and only if} \quad \mathrm{GF}(q) \textit{ is a subfield of } \mathbb{F}.$$

The "if" assertion is trivial. The "only if" assertion follows from the fact, that the projective plane $\mathrm{PG}(2,q)$ is representable only over extension fields of $\mathrm{GF}(q)$ (this holds for the projective plane over an infinite field as well. See [**1**], and [**2**, Cpt. 7] for a proof). Furthermore, it is well known that $\mathrm{GF}(q)$ is a subfield of $\mathrm{GF}(q')$ if and only if there is a prime p, such that $q = p^d$, $q' = p^s$ and d divides s (see for instance [**33**]).

Corollary 2.2. *Let p be a prime, s a positive integer and let $\mathcal{M}$ be a minor-closed class of matroids with $\mathcal{R}(p) \subseteq \mathcal{M} \subseteq \mathcal{R}(p^s)$. Then $h(\mathcal{M};r)$ has order of magnitude q^r, where $q = p^d$ and d is the largest divisor in s, such that $\mathcal{R}(p^d) \subseteq \mathcal{M}$.*

Proof. Let q be the largest prime-power with $\mathcal{R}(q) \subseteq \mathcal{M}$. By the third case in Theorem 2.1, $h(\mathcal{M};r)$ has order of magnitude q^r. Since $\mathcal{R}(q) \subseteq \mathcal{R}(p^s)$, by the observation above, q can be written $q = p^d$, where d divides s. ■

The fewer divisors there are in s, the fewer possible orders of magnitude there are for $h(\mathcal{M};r)$. As an example, taking $p = 2$ and $s = 5$ in the corollary, we can deduce the following. Given a positive integer n, there exists an integer λ such that, if $M \in \mathcal{R}(32)$ satisfies $\varepsilon(M) > \lambda 2^{r(M)}$, then M has a $\mathrm{PG}(n-1,32)$-minor.

3 Extensions of Kung's Theorem

Kung's Theorem provides a bound on the number of points as a function of the rank. As we have seen, it does this, in some sense, in the widest setting possible, namely for matroids in $\mathcal{U}(q)$.

The results on covering numbers in Chapter 2 show one direction in which Kung's Theorem can be extended. Proposition 3.4 gives an exponential bound on the a-covering number as a function of the rank, for matroids in the minor-closed class $\mathcal{EX}(U_{a+1,b})$. The matroid $M = U_{a+1,b}$ has a-covering number $\tau_a(M) = \lceil \frac{b}{a} \rceil$. This shows that it is necessary to exclude $U_{a+1,b}$ for some b, for a bound on τ_a to hold. So $\mathcal{EX}(U_{a+1,b})$ is the largest minor-closed class for which the a-covering number is bounded by a function of the rank, generalizing our considerations on Kung's Theorem.

The covering number results also show that we can bound the number of points without excluding a line *as a minor*. The following is another extension of Kung's Theorem, which is similar to Lemma 3.7 of Chapter 2.

Proposition 3.1. *Let a, b be positive integers with $b \geq 2a$. Let M be a matroid with no $U_{k,b}$-restriction for $k = 2, \ldots, a$ and no $U_{a+1,b}$-minor. Then*

$$\varepsilon(M) \leq \binom{b-1}{a}^{r(M)-1}.$$

Proof. Assume first that $r(M) \geq a$. Recall that $\varepsilon(M) = \tau_1(M)$. By Lemma 3.3 and Proposition 3.4 of Chapter 2, we have

$$\tau_1(M) \leq \prod_{k=2}^{a} \tbinom{b-1}{k-1} \tau_a(M) \leq \prod_{k=2}^{a} \tbinom{b-1}{k-1} \tbinom{b-1}{a}^{r(M)-a} \leq \tbinom{b-1}{a}^{r(M)-1}.$$

In case $r(M) < a$, the result follows just by applying Lemma 3.3. ■

For $b > a \geq 1$, let $\mathcal{M}_{a,b}$ be the class of matroids in $\mathcal{EX}(U_{a+1,b})$ that have no $U_{k,b}$-restriction for $k = 2, \ldots, a$. Then by the proposition, if $b \geq 2a$,

$$h(\mathcal{M}_{a,b}; r) \leq \tbinom{b-1}{a}^{r-1}.$$

If $b < 2a$, then $\mathcal{M}_{a,b} \subseteq \mathcal{M}_{a,2a}$ and we get another bound, so in any case $\mathcal{M}_{a,b}$ has finite size function. Yet, if a and b are large enough, $\mathcal{M}_{a,b}$ is not contained in $\mathcal{U}(q)$ for any q, as we show below. This does not contradict our earlier conclusions, since $\mathcal{M}_{a,b}$ is not a minor-closed class.

Consider $\mathcal{M}_{2,7}$, the class of matroids having no $U_{2,7}$-restriction and no $U_{3,7}$-minor. We observed in Chapter 2, that no bicircular matroid $B(G)$ has a $U_{3,7}$-minor. It is immediate from the definition of bicircular matroids, that every three element set of $B(K_n)$ is independent. In particular $B(K_n)$ has no 7-point line-restriction. Hence, $\mathcal{M}_{2,7}$ contains all bicircular cliques. We also showed in Chapter 2, that $B(K_{n+1})$ has an n-point line-minor. So $\mathcal{M}_{2,7}$ is not contained in $\mathcal{U}(q)$ for any q.

We now mention one last direction in which Kung's Theorem can be extended. Let k be a positive integer. We consider the problem of bounding the number of rank-k flats of a matroid as a function of the rank. In the class of all matroids there is clearly no such bound. The rank-$(k+1)$ matroid $M = U_{k-1,k-1} \oplus U_{2,b}$ has at least b rank-k flats ($b+k-1$ to be exact). This shows that if we consider minor-closed classes, then it is necessary to exclude a line, for a bound to hold (at least if the class is closed under direct sums; exceptions can be constructed, e.g. $\mathcal{R}(2) \cup \{U_{2,b} : b \in \mathbb{N}\}$).

On the other hand, excluding a line as a minor is sufficient for a bound to hold. Let M be a rank-r matroid in $\mathcal{U}(q)$. Then $\varepsilon(M) \leq \frac{q^r-1}{q-1} \leq q^r$, by Kung's Theorem. Since a rank-k flat is spanned by k points, we get the following rough bound on the number of rank-k flats in M,

$$\binom{\varepsilon(M)}{k} \leq \varepsilon(M)^k \leq q^{kr(M)}.$$

The number of rank-k flats in the projective geometry $\mathrm{PG}(n-1, q)$ is given by the Gaussian coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^{n-i}-1}{q^{k-i}-1}.$$

Hence, any rank-r matroid in $\mathcal{R}(q)$ has at most $\begin{bmatrix} r \\ k \end{bmatrix}_q$ rank-k flats. Since for q a prime-power, the maximal number of points among rank-r matroids of $\mathcal{U}(q)$ is attained by $\mathrm{PG}(r-1, q)$, the following conjecture of Bonin [**4**] is natural.

Conjecture 3.2. *Let $q \geq 2$. Any rank-r matroid in $\mathcal{U}(q)$ has at most $\begin{bmatrix} r \\ k \end{bmatrix}_q$ rank-k flats.*

The Gaussian coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by a rational function of q. In fact, it is a polynomial in q with integer coefficients. When evaluated in $q = 1$, this polynomial yields the binomial coefficient $\binom{n}{k}$ (see [**35**]). Therefore, it is custom to define $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$. The conjecture can then be extended to $q \geq 1$. A simple rank-r matroid in $\mathcal{U}(1)$ is isomorphic to the free matroid $U_{r,r}$, which has precisely $\binom{r}{k}$ rank-k flats.

For $q = 2$ the conjecture is true, since $\mathcal{U}(2) = \mathcal{R}(2)$, the binary matroids. The case $k = 1$ in the conjecture is Kung's Theorem. To my knowledge, the conjecture is unsettled for $k \geq 2$. I have proved the first non-trivial case (besides Kung's Theorem), the case $k = 2$ and $q = 3$. That is, counting lines of matroids in $\mathcal{U}(3)$.

Theorem 3.3. *Any rank-r matroid $M \in \mathcal{U}(3)$ has at most $\begin{bmatrix} r \\ 2 \end{bmatrix}_3$ lines.*

We start by showing how the full conjecture would be implied by the following conjecture.

Conjecture 3.4. *For any $q, k \geq 1$, if $M \in \mathcal{U}(q)$ has rank $k+1$ and $e \in E(M)$, then the number of rank-k flats in M avoiding e is at most*

$$\begin{bmatrix} k+1 \\ k \end{bmatrix}_q - \begin{bmatrix} k \\ k-1 \end{bmatrix}_q = q^k.$$

Note that this conjecture holds for $\mathrm{PG}(k, q)$ and thus for all rank-$(k+1)$ matroids in $\mathcal{R}(q)$.

Proof that 3.4 implies 3.2. Conjecture 3.2 is trivial in case $k = 0$, so assume $k > 0$. The proof is by induction on r. The result is trivial if $r \leq k$, so assume $r > k$ and that the result holds for $r - 1$. Let $e \in E(M)$ be a non-loop element.

Each rank-k flat in M containing e gives rise to a unique rank-$(k-1)$ flat in M/e. So, by induction, there are at most $\begin{bmatrix} r-1 \\ k-1 \end{bmatrix}_q$ such flats.

A rank-k flat in M avoiding e gives rise to a rank-k flat in M/e. Two of these give rise to the same flat in M/e if and only if they are contained in the same rank-$(k+1)$ flat in M. So by induction on M/e and Conjecture 3.4, there are at most $q^k \begin{bmatrix} r-1 \\ k \end{bmatrix}_q$ rank-k flats in M avoiding e.

Adding up, we get the desired bound

$$\begin{bmatrix} r \\ k \end{bmatrix}_q = \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} r-1 \\ k \end{bmatrix}_q.$$

This formula, which is a q-analogue of the well-known recursive formula for the binomial coefficient, can be found in [**35**] (or verified directly). ∎

The above proof shows, that to prove Theorem 3.3, it is sufficient to show that a rank-3 matroid in $\mathcal{U}(3)$ has no more than $3^2 = 9$ lines avoiding a given element. I can do this by a case analysis which, unfortunately, I haven't been able to complete for higher values of q or k.

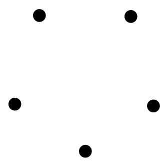
A quicker way to realize it (which has no chance of generalizing to a proof of the full conjecture), is to use the excluded minor characterization of ternary matroids (see Chapter 5):

$M \in \mathcal{R}(3)$ if and only if M has no $U_{2,5}$-, $U_{3,5}$-, F_7- or F_7^*-minor.

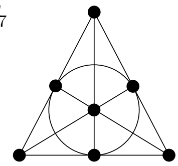
The matroids $U_{3,5}$ and F_7 have rank 3, and F_7^* has rank 4. Hence, if M is a rank-3 matroid in $\mathcal{U}(3)$, then either $M \in \mathcal{R}(3)$ or M is an extension of $U_{3,5}$ or F_7.

It is easy to check, that neither of $U_{3,5}$ and F_7 can be extended by a point without creating a matroid with a 5-point line-minor. Thus, if $M \in \mathcal{U}(3)$ is a simple rank-3 matroid, then either $M \in \mathcal{R}(3)$ or M is isomorphic to $U_{3,5}$ or F_7. In any case, M has no more than 9 lines avoiding a given element. This concludes the proof of Theorem 3.3.

$U_{3,5}$



F_7



4 The quadratic conjecture

We shall refer to the second case in the Growth Rate Conjecture as "the quadratic conjecture". The remainder of the chapter is concerned with this conjecture, and some partial results will be presented. The conjecture can be stated in the following equivalent way.

The quadratic conjecture. *Let $q \geq 2$ and n be positive integers. There exists an integer λ such that, if $M \in \mathcal{U}(q)$ with $\varepsilon(M) > \lambda r(M)^2$, then M has a* $\mathrm{PG}(n-1, q')$*-minor for some prime-power q'.*

This conjecture implies case (2) in the Growth Rate Conjecture with the exact same argument we used in the proof of Theorem 2.1. The reverse implication is realized by defining a class $\mathcal{M}$ given q and n as follows

$$\mathcal{M} = \mathcal{U}(q) \cap \mathcal{EX}(\mathrm{PG}(n-1, q') : q' \text{ a prime-power}),$$

and then applying case (2) in the Growth Rate Conjecture to $\mathcal{M}$.

The quadratic conjecture is trivial in the case $n = 2$, since if $\varepsilon(M) > r(M)$, then M contains a circuit with at least three elements, and hence a $U_{2,3}$-minor, which is a $\mathrm{PG}(1,2)$-minor. Kung has proved the conjecture in the case where $n = 4$ and $q = 2$ (M is binary), and in the restricted case $n = 3$ and $M \in \mathcal{R}(q)$ (M is representable). These results are mentioned without proof in [**30**]. The only easy non-trivial case is the case $n = 3$ and $q = 2$, that we prove below.

Lemma 4.1. *If M is a binary matroid with $\varepsilon(M) > \binom{r(M)+1}{2}$, then M has an F_7-minor.*

Proof. Let M be minor-minimal with $\varepsilon(M) > \binom{r(M)+1}{2}$. In particular, M is simple. Let $r = r(M)$ and consider $e \in E(M)$. By the minimality of M, we have

$$\varepsilon(M) - \varepsilon(M/e) > \binom{r+1}{2} - \binom{r}{2} = r.$$

As M is binary, $\varepsilon(M) - \varepsilon(M/e) = 1 + \delta_M(e)$, so $\delta_M(e) \geq r$. Let $x_1, \ldots, x_r \in E(M) - e$ be elements on different long lines through e. Then $\{x_1, \ldots, x_r\}$ is dependent in M/e, so after contracting some of the x_i from M, we obtain a minor N with three long coplanar lines through e. So N has a plane $P \subseteq E(N)$ with at least 7 points. Since N is binary, $\mathrm{si}(N|P) \simeq F_7$. ■

This implies $h(\mathcal{R}(2) \cap \mathcal{EX}(F_7); r) \leq \binom{r+1}{2}$, as mentioned in Section 1.

The proof of the above lemma uses a technique, which Kung calls “the method of cones”. We have seen similar arguments in the constructions of pyramids in Chapter 3 and also in the proof of Mader's Theorem for $\mathcal{U}(q)$ in Chapter 2. The key observation is, that a matroid $M \in \mathcal{U}(q)$ with a certain minimum density contains a minor N with an element e, such that $\delta_N(e)$ is large. It gives a method for finding more concrete structures in a dense matroid. Kung attributes this idea to Mader among others.

A matroid C is a *cone* if it has a non-loop element a called the *apex*, such that every other element of C is on a long line with a. For instance, a projective geometry is a cone with any point as the apex. We review the basic argument in obtaining a cone.

Let M be a matroid in $\mathcal{U}(q)$ and $e \in E(M)$ a non-loop element. The number of long lines through e estimates the number of points lost when contracting e,

$$1 + \delta_M(e) \leq \varepsilon(M) - \varepsilon(M/e) \leq 1 + (q-1)\delta_M(e).$$

Suppose that M satisfies a lower bound on the density given by an increasing function f, that is $\varepsilon(M) > f(r(M))$. If M is minor-minimal with this property, then

$$\varepsilon(M) - \varepsilon(M/e) > f(r(M)) - f(r(M)-1) \approx f'(r(M)-1),$$

provided f is smooth (if f is convex, ‘$\approx$’ can be replaced by ‘$\geq$’). Thus, M has a cone-restriction C with apex e, and roughly $f'(r(M)-1)$ lines through e (up to a constant).

We can now contract e in C, and C/e has at least $f'(r(C))$ points (up to a constant). With quadratic density this argument can be done twice, to yield two “nested” cones. As we saw in the above lemma, this is enough to construct a rank-3 projective geometry in the binary case. However, to construct projective geometries of higher rank, something more is needed.

The following sections present two partial results to the quadratic conjecture, that I have obtained in my failed attempts to prove it. Interestingly, they utilize ideas and results used in the proofs of the linear and exponential cases in Theorem 2.1.

Both results provide a “second half” of a proof of the conjecture. That is, they reach the conclusion in the quadratic conjecture from a different or stronger set of assumptions than quadratic density. This reflects, that I have not been able to exchange quadratic density for a more concrete and sufficiently strong structure, as in the exponential case, where the method of cones basically yields pyramids. In the linear case, this was difficult as well, and the concept of roundness was the key.

5 Books

The following special case of the quadratic conjecture has been solved by Kung (it is mentioned in [**30**]).

Theorem 5.1. *There is an integer λ such that, if M is a simple binary matroid with $|E(M)| > \lambda r(M)^2$, then M contains a* PG(3, 2)*-minor.*

The proof of this is in [**32**], and uses the concept of a book defined as follows. A simple matroid $B \in \mathcal{R}(2)$ is a *book* if it has a collection of flats called the *pages*, whose union is $E(B)$, such that for each page X, $B|X$ is isomorphic to $M(K_4)$ or F_7, and such that all the pages contain the same 3-point line L, called the *spine*. The proof is in two steps:

(a) Given λ_1, λ_2 there exists λ, such that if M is simple and binary with $\delta_M(e) > \lambda r(M)$ for all elements $e \in E(M)$, then one of the following holds.
  - M has a cone minor C, with at least $\lambda_1 r(C)^2$ points.
  - M has a book minor B, with at least $\lambda_2 r(B)$ pages.

(b) There exists λ_2 such that, if B is a book with at least $\lambda_2 r(B)$ pages, then B has a PG(3, 2)-minor.

Step (a) is the tricky part, while (b) is an easy argument. Note, that a cone C with apex a and at least $\lambda_1 r(C)^2$ points (λ_1 large) yields a PG(3, 2)-minor, by applying Lemma 4.1 to C/a.

Inspired by Kung's definition, we define a more general notion of book.

Definition 5.2. Let k be a positive integer. A matroid B is a *k-book* with *spine* S, if S is a rank-k flat of B. The rank-$(k+1)$ flats of B containing S are called its *pages*.

The condition of being a k-book is rather weak. Indeed, each rank-$r \geq k$ matroid is a k-book, with any rank-k flat as the spine. We shall be interested in k-books where all the pages are round flats. The roundness serves to guarantee, that the flats are somewhat dense.

Let M be a simple binary rank-3 matroid. If M is round, then it is isomorphic to $M(K_4)$ or F_7 (If M has less than 6 points, then it is isomorphic to a proper restriction of $M(K_4)$, which is not round by Remark 1.4 of Chapter 2). Hence, a matroid B is a book in Kung's sense, if B is a simple binary 2-book with round pages. A 1-book with round pages is a cone, whose apex is the spine.

For a k-book B we denote by $\pi(B)$ the number of pages of B. This number depends implicitly on the choice of spine S, and $\pi(B) = \varepsilon(B/S)$. The following result generalizes part (b) of Kung's proof. The result was first conjectured together with Jim Geelen.

Theorem 5.3. *Let $q \geq 2$ and let n be a positive integer. There exist integers k and λ such that, if $B \in \mathcal{U}(q)$ is a k-book with round pages and $\pi(B) \geq \lambda r(B)$, then B has a* $\mathrm{PG}(n-1, q')$*-minor for some q'.*

The proof of this theorem occupies the next section. With this in place, we needed a suitable extension of part (a). The idea in Kung's argument is roughly as follows. Consider a cone C, with the number of lines through the apex linear in the rank. Let a be the apex of C and pick another point a' on a line L of C through a. Let y be a third point on L, and let C' be the cone with apex a' (see Figure 1).

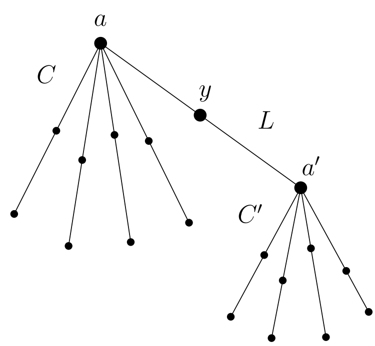


FIGURE 1

The idea is to add lines to C by contracting y. Two things can happen: Either a linear (in the rank) number of lines of C' are co-planar with lines of C, giving rise to pages of a book with spine L. Or a linear number of lines of C' become new lines of C by contracting y (at the same time the number of lines is reduced by a constant factor, as lines get identified). Iterating this operation a linear number of times (e.g. $\frac{1}{2}r(M)$ times) gives the result.

We proposed the following generalization of (a).

(a*) Given $k, \lambda_1, \lambda_2, \lambda_3$, there exist λ, λ', such that: If $M \in \mathcal{U}(q)$ has a k-book restriction B with round pages and $\pi(B) \geq \lambda r(B)$, and if $\delta_M(e) > \lambda' r(M)$ for all $e \in E(M)$, then one of the following holds.
- M has a cone minor C, with at least $\lambda_1 r(C)^2$ points.
- M has a minor N with a $(k+1)$-book restriction B_* with round pages and $\pi(B_*) \geq \lambda_2 r(B_*)$, and where $\delta_N(e) > \lambda_3 r(N)$ for all $e \in E(N)$.

This would be sufficient to prove the quadratic conjecture. Unfortunately, we were unable to prove this or a similar generalization using the technique described above. Still, I feel Theorem 5.3 is interesting, since k-books with dense pages are quite natural structures to consider.

6 Proof of the book result

This section contains the proof of Theorem 5.3. The following lemma is the key to the proof.

Lemma 6.1. *Let B be a k-book with spine S. If $r(B) = k+2$ and B has exactly three pages X_1, X_2 and X_3, then one of the following holds.*
(i) *There exists an element $y \notin S$, such that $\varepsilon(B/y) > \varepsilon(B|X_i)$, for some $i \in \{1,2,3\}$.*
(ii) *There is a subset $S' \subseteq S$, such that in $B \backslash S'$, every line is long.*

Proof. We may assume that B is simple. Suppose (i) fails and let us prove that (ii) holds. We define

$$S_* = \{e \in S : e \text{ is on a long line } L, L \not\subseteq S\},$$

and let $S' = S - S_*$. Let x and y be different elements of $B \backslash S'$ and let L be the line through x and y. We consider a number of cases.

If $x \in X_1 - S$ and $y \in X_2 - S$, then L intersects X_3, as otherwise we have $\varepsilon(B/y) > |X_3|$. Thus, L is long. This argument holds for any permutation of X_1, X_2 and X_3.

If $x, y \in X_i - S$, then L intersects S, as otherwise $\varepsilon(B/y) > |X_j|$ for $j \neq i$. Thus, L is long.

Assume $x \in X_1 - S$ and $y \in S_*$. Suppose y is on a long line with $z \in X_2 - S$. Let $z' \in X_2 - S$ be another point on this line. The line through x and z meets X_3 in a point w, and the line through w and z' meets X_1 in a point, which must be on L, proving it long (see Figure 2a). Again, by repeating this argument with X_1, X_2 and X_3 permuted, we have covered the cases where $x \in X_i - S$ and $y \in S_*$.

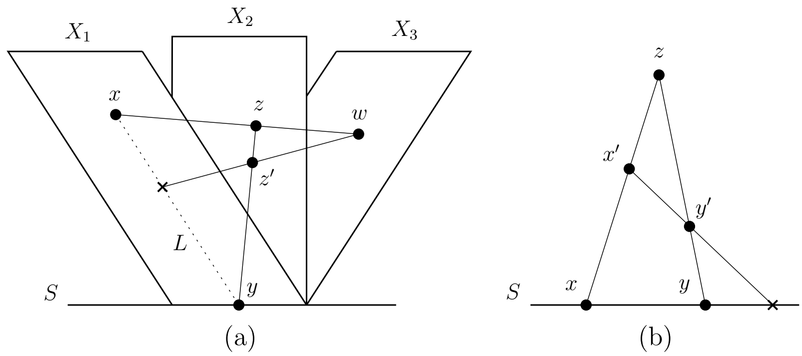


FIGURE 2

Finally, assume $x, y \in S_*$. Choose a point $z \notin S$. Let x' and y' be other points on the lines through $\{z, x\}$ and $\{z, y\}$ respectively (see Figure 2b). The line through x' and y' meets S_* in a point on L, so it is long. ∎

Remark 6.2. We make a few easy observations concerning the above lemma:

- If a matroid M satisfies, that every pair of points is on a long line, then M is a pyramid with respect to any basis as the set of joints.
- If B is a k-book with spine S and X is a round page, then since $X - S$ is a co-circuit in $B|X$, we have $r_B(X - S) = k + 1$.

Hence, if all three pages are round, case (ii) in the lemma gives a pyramid-minor of rank $k + 2$.

Theorem 5.3 is an immediate consequence of the next result and the result of Chapter 3, that a pyramid in $\mathcal{U}(q)$ of sufficiently large rank has a $\mathrm{PG}(n-1, q')$-minor for some prime-power q' (Lemmas 5.1 and 5.3 of Chapter 3).

Proposition 6.3. *Let $q \geq 2$ and let k be a positive integer. There exists an integer λ, such that if $B \in \mathcal{U}(q)$ is a k-book with round pages and $\pi(B) \geq \lambda r(B)$, then B has a rank-$(k + 2)$ pyramid-minor.*

Notice that, for $q = 2$ and $k = 2$ the proposition gives part (b) of Kung's proof, since a simple binary rank-4 pyramid is isomorphic to $\mathrm{PG}(3, 2)$. Another corollary is the following easy fact (taking $q = 2$ and $k = 1$): There is a λ, such that a binary cone C with $\varepsilon(C) > \lambda r(C)$ has an F_7-minor.

Proof of Proposition 6.3. Let q and k be given, and define

$$m = \frac{q^{k+1} - 1}{q - 1} + 1, \quad n = \beta(k, m), \quad \lambda = \lambda(n, q),$$

where β is given by the lemma following this proof, and λ is given by Theorem 10.1 of Chapter 2 (Mader's Theorem for $\mathcal{U}(q)$). Let B be given satisfying the assumptions in the lemma, and let S be the spine of B. Then

$$\varepsilon(B/S) = \pi(B) \geq \lambda r(B) > \lambda(n,q) r(B/S).$$

By Theorem 10.1 of Chapter 2, B/S contains an $M(K_n)$-minor. Choose Y independent, such that $B/S/Y$ has an $M(K_n)$-restriction, and let $B' = B/Y$. Then B' is a k-book with spine $S' = \mathrm{cl}_{B'}(S)$.

We now apply the technical lemma below to B'. This gives either a rank-$(k+2)$ pyramid-minor, or a rank-$(k+1)$ minor with at least m points. By the choice of m and Kung's Theorem, the latter case is impossible. ∎

Lemma 6.4. *There exists a function $\beta(k,m)$ defined on positive integers k and m, such that the following holds: Let B be a k-book with spine S and round pages. If B/S has an $M(K_n)$-restriction, where $n = \beta(k,m)$, then either*
(i) *There is a page X of B, and a set $Y \subseteq E(B)$ skew from X, such that $\varepsilon(B/Y|\,\mathrm{cl}_{B/Y}(X)) \geq m$, or*
(ii) *B has a rank-$(k+2)$ pyramid-minor.*

Proof. We define $\beta(k,m) = 2$ for $m \leq k+1$, and recursively $\beta(k,m) = 3\beta(k,m-1)$ for $m > k+1$. The proof is by induction on m. For the basis case ($m \leq k+1$), note that (i) holds trivially, since there is at least one page, and any rank-$(k+1)$ flat has $k+1$ points.

Let $m > k+1$ and let B and S be given, where B/S has an $M(K_n)$-restriction and $n = 3n' = 3\beta(k,m-1)$. We can assume, after possibly renaming elements, that $B/S|W = M(G)$, where G is a K_n graph and $W = E(G)$. Let G_1, G_2 and G_3 be vertex-disjoint complete subgraphs of G on n' vertices (see Figure 3a).

For each $w \in W$, let X^w denote the page in B corresponding to w, that is $X^w = \mathrm{cl}_B(S \cup w)$. Let $W_i = E(G_i)$ for $i = 1,2,3$. Then W_1, W_2, W_3 are skew in B/S. We define three sub-books B_1, B_2 and B_3 with spine S by $B_i = \cup_{w \in W_i} X^w$. We can now apply induction to each B_i. If any of the B_i satisfies (ii), we are done, so assume otherwise. Then B_i has a page X_i and and a set $Y_i \subseteq E(B_i)$, skew from X_i, such that

$$\varepsilon(B_i/Y_i|\,\mathrm{cl}_{B_i/Y_i}(X_i)) \geq m-1.$$

We can take Y_i to be independent. The sets X_1, X_2 and X_3 are pages of B. Let $e_1, e_2, e_3 \in W$ be the elements with $X_i = X^{e_i}$.

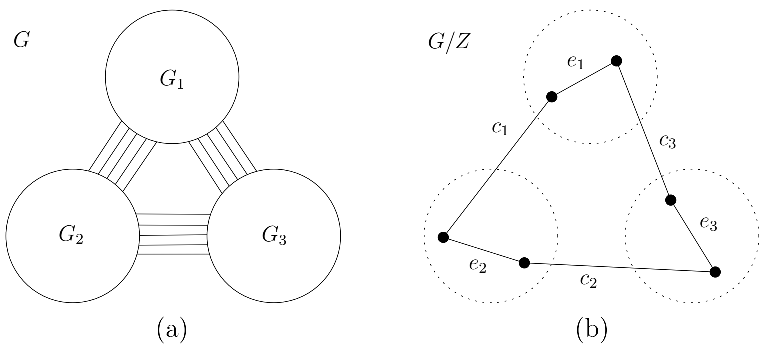


FIGURE 3

Let $Y_* = Y_1 \cup Y_2 \cup Y_3$ and let $B_* = B/Y_*$. Then B_* is a k-book and $\varepsilon(B_*|\operatorname{cl}_{B_*}(X_i)) \geq m-1$ for $i = 1,2,3$. Note that Y_* is independent in B/S, and every element $y \in Y_*$ is in parallel with a $w \in W$ in B/S. So, there is a $Z \subseteq W$ with

$$B/S/Z = B/S/Y_* = B_*/S.$$

Hence B_*/S has as a restriction $M(G/Z)$. The edges e_1, e_2 and e_3 are vertex-disjoint in the complete graph G/Z, so they lie together on a 6-edge cycle. Let the last three edges of the cycle be $c_1, c_2, c_3 \in W$ (see Figure 3b).

Let $B' = B_*/c_1, c_2, c_3$ and let $X'_i = \operatorname{cl}_{B'}(X_i)$ be the page in B' corresponding to X_i for $i = 1,2,3$. Since $\{e_1, e_2, e_3\}$ is the edge-set of a triangle in the graph $G/Z/c_1, c_2, c_3$, it is a circuit of B'/S. Hence $r_{B'}(X'_1 \cup X'_2 \cup X'_3) = k+2$.

We now apply Lemma 6.1 to the 3-page k-book $B'|(X'_1 \cup X'_2 \cup X'_3)$ and get one of two outcomes. In the first case, there is an element y whose contraction throws at least one more point into one of the pages. Since $\varepsilon(B'|X'_i) \geq m-1$ for $i = 1,2,3$, we have proved (i) taking $Y = Y_* \cup \{c_1, c_2, c_3, y\}$. In the second case, we obtain a rank-$(k+2)$ pyramid-minor as noted in Remark 6.2. ■

We could still achieve Theorem 5.3 with a weaker assumption in place of roundness of the pages. What is needed in Remark 6.2, is a lower bound on the rank of $X - S$ for each page X, that grows with k. It is sufficient, for instance, that the pages be near-round.

7 A spanning clique-minor

The following theorem is another partial result to the quadratic conjecture, obtained by me. Its proof is based on an idea for binary matroids by Jim Geelen, Bert Gerards and Geoff Whittle, explained to me by Jim Geelen. It gives strong evidence to the conjecture, and seems more promising than the book result, to be part of a complete proof of the conjecture.

Theorem 7.1. *Let q and n be positive integers. There exists an integer λ such that, if $M \in \mathcal{U}(q)$ has a spanning clique-minor and satisfies $\varepsilon(M) > \lambda r(M)^2$, then M has a* $\mathrm{PG}(n-1, q')$*-minor for some prime-power q'.*

By a spanning clique-minor, we mean a spanning Dowling-clique-restriction. That is, we shall need M to have a basis V, such that every pair of elements of V is on a long line in M. This is satisfied, for instance, if M has a spanning $M(K_m)$-minor.

With this result, the quadratic conjecture would follow from the weaker conjecture:

Conjecture 7.2. *Let $q \geq 2$ and λ' be given. There exists an integer λ such that, if $M \in \mathcal{U}(q)$ satisfies $\varepsilon(M) > \lambda r(M)^2$, then M has a minor N with a spanning clique-minor, and $\varepsilon(N) > \lambda' r(N)^2$.*

We know from Mader's Theorem for $\mathcal{U}(q)$, that high linear density (with a sufficiently large constant) is enough to guarantee a large graphic clique-minor. Unfortunately, I have been unable to obtain such a minor, while preserving the quadratic density.

The next result is similar to the above theorem. It has a different set of assumptions, that I have tried (in vain) to reach from quadratic density. It will follow by a short argument during the proof of Theorem 7.1.

Theorem 7.3. *Let q and n be positive integers. There exists an integer λ such that, if $M \in \mathcal{U}(q)$ contains a set X with $\varepsilon_M(X) > \lambda r(M)$ and every pair of elements of X is on a long line, then M has a* $\mathrm{PG}(n-1, q')$*-minor for some prime-power q'.*

The proof of Theorem 7.1 occupies the next two sections.

8 Support-sets

Let M be a matroid with basis V. For $e \in E(M) - V$, define $V_e = C_e(V) - e$, where $C_e(V)$ denotes the fundamental circuit of e with respect to V. The choice of (M, V) is not reflected by this notation, but will be clear from the context. The set V_e can be thought of as the "support" for e. Given (M, V), we call a set $X \subseteq V$ a *support-set*, if $X = V_e$ for some $e \in E(M) - V$.

Note, that there is a Dowling-clique on V in M if and only if every subset of V of size 2 is a support-set. Our first lemma shows how support-sets can be used to build a dense minor.

Lemma 8.1. *There exists a function $\psi : \mathbb{N} \to \mathbb{N}$ such that, if M is a matroid with basis V, there is a Dowling-clique on V in M and M has at least $\psi(k)$ disjoint support-sets $X \subseteq V$, with $|X| \geq 3$, then M has a rank-k minor N, with $\varepsilon(N) = 2^k - 1$.*

Proof. Define

$$\psi(k) = \sum_{i=3}^{k} \binom{k}{i}(i-2).$$

Let (M, V) be given and let $\mathcal{X} \subseteq 2^V$ be a collection of disjoint support-sets of size at least 3, with $|\mathcal{X}| = \psi(k)$. We may assume that $|X| = 3$ for all $X \in \mathcal{X}$ (otherwise, contract points in X).

We first describe an operation to "merge" two support-sets into one of larger size. Let $e, e' \in E(M) - V$ be given, with V_e and $V_{e'}$ disjoint and $|V_e| = i$, $|V_{e'}| = 3$. Choose points $v \in V_e$ and $v' \in V_{e'}$, and let y be a third point on the line through v and v'. Then v and v' are parallel in $M' = M/y$. Note that $C = V_e \cup e$ and $C' = (V_{e'} - v') \cup \{v, e'\}$ are circuits in M' and $C \cap C' = \{v\}$ (see Figure 4).

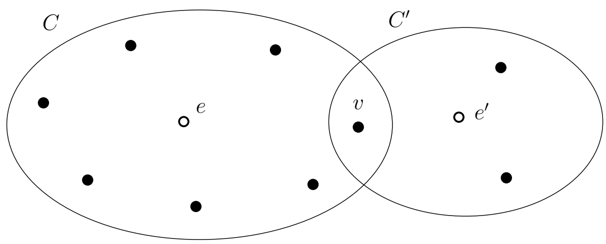


FIGURE 4

By the (weak) circuit elimination axiom, $C \cap C' - v$ contains a circuit of M'. We can write $C \cap C' - v = V_* \cup \{e, e'\}$, where $V_* = (V_e - v) \cup (V_{e'} - v')$. Observe, that every proper subset of $V_* \cup \{e, e'\}$ is independent in M', and hence $V_* \cup \{e, e'\}$ must be a circuit of M'.

Let $M_1 = M'/e'$ and note that $V_1 = V - \{v, v'\}$ is a basis of M_1. The set V_* is a support-set of (M_1, V_1) with respect to e, and $|V_*| = i + 1$.

Using the above operation we can build a support-set of size $i \geq 3$ (in a minor of M with a basis contained in V) out of $i-2$ sets from $\mathcal{X}$. Thus we may assume, that $M, V, \mathcal{X}$ are given, such that $\mathcal{X}$ is a collection of disjoint support-sets of (M, V), and $\mathcal{X}$ contains $\binom{k}{i}$ sets of size i, for $i = 3, \ldots, k$ (and no sets of other sizes).

Let $W_0 \in \mathcal{X}$ be the unique member with $|W_0| = k$. To each set $W \in \mathcal{X} - W_0$, we assign a subset $X_W \subseteq W_0$, with $|X_W| = |W|$, such that this assignment is one-to-one. Also, choose a set $Y_W \subseteq E(M) - V$ with $|Y_W| = |W|$ as follows: pair up the elements of W and X_W, and for each pair let Y_W contain a third point on the line through the two points (see Figure 5).

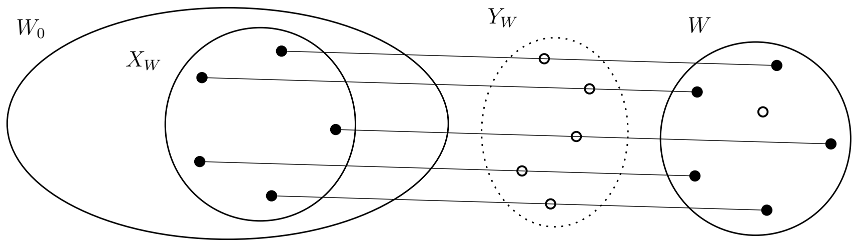


FIGURE 5

Let $M_0 = M/\cup_{W \in \mathcal{X} - W_0} Y_W$, and define $N = M_0|\operatorname{cl}_{M_0}(W_0)$. Then W_0 is a basis for N and every $X \subseteq W_0$, with $|X| \geq 2$ is a support-set for (N, W_0), and thus determines a unique point of N. Hence,

$$\varepsilon(N) \geq k + \#\{X \subseteq W_0 : |X| \geq 2\} = 2^k - 1,$$

which concludes the proof. ■

The next lemma is a corollary to the previous one.

Lemma 8.2. *There exists a function $\phi : \mathbb{N}^2 \to \mathbb{N}$ such that, if $M \in \mathcal{U}(q)$ has basis V, there is a Dowling-clique on V in M and M has at least $\phi(n, q)$ disjoint support-sets $X \subseteq V$, with $|X| \geq 3$, then M has a* PG$(n-1, q')$*-minor for some q'.*

Proof. Let n and q be given. By Theorem 1.4 of Chapter 3, there are integers a, m such that any matroid $N \in \mathcal{U}(q)$ with $\varepsilon(N) > ar(N)^m$ has a PG$(n-1, q')$-

minor. Since

$$\lim_{k\to\infty} \frac{2^k}{k^m} = \infty,$$

we can pick a $k \in \mathbb{N}$ with $2^k - 1 > ak^m$. We then define $\phi(n,q) = \psi(k)$.

The result now follows by applying the previous lemma and Theorem 1.4 of Chapter 3. ■

We can now prove Theorem 7.3.

Proof of Theorem 7.3. Let q and n be given. Define $m = 3\phi(n,q) + 1$, and let $\lambda = \lambda(m,q)$, the function given by Mader's Theorem for $\mathcal{U}(q)$ (Theorem 10.1 of Chapter 2). Let M and X be given, where $\varepsilon(M|X) > \lambda r(M)$ and every pair of elements of X is on a long line.

By Mader's Theorem for $\mathcal{U}(q)$, $M|X$ has an $M(K_m)$-minor. Hence, M has a contraction minor N, such that $N|X_N \simeq M(K_m)$, where $X_N \subseteq X \cap E(N)$ and X_N spans N.

Choose a basis $V \subseteq X_N$ of N corresponding to the edges incident with a fixed vertex in K_m. So, there is a Dowling-clique on V in N in which every pair of elements is on a long line of N. It follows easily, that every 3-element subset of V is a support-set of (N,V). The result now follows from Lemma 8.2. ■

We return focus to the main theorem, Theorem 7.1. The next result is where the quadratic density is brought to use. We first make an observation. Let λ be a positive integer, and let $M \in \mathcal{U}(q)$ satisfy $\varepsilon(M) > \lambda r(M)^2$. Then, if e is a non-loop element of M,

$$\varepsilon(M/e) \geq \frac{\varepsilon(M) - 1}{q} \geq \frac{\lambda r(M)^2}{q} > \frac{\lambda}{q} r(M/e)^2.$$

So M/e has quadratic density with the constant reduced by a factor q.

Lemma 8.3. *There exists a function $\theta_1 : \mathbb{N}^3 \to \mathbb{N}$ such that, if $M \in \mathcal{U}(q)$ has basis V, there is a Dowling-clique on V in M, $|V_e| \leq d$ for all $e \in E(M) - V$, and $\varepsilon(M) > \lambda r(M)^2$ where $\lambda = \theta_1(n,d,q)$, then M has a $\mathrm{PG}(n-1,q')$-minor for some q'.*

Proof. For $d \leq 2$ let $\theta_1(n,d,q) = q$, and for $d \geq 3$ define θ_1 recursively by

$$\theta_1(n,d,q) = q^{\phi(n,q)d}\theta_1(n,d-1,q).$$

Let (M, V) be given. We may assume, that M is simple. The proof is by induction on d. Consider first the case $d \leq 2$. Then every point of $E(M) - V$ is on a line with two points of V, and since $M \in \mathcal{U}(q)$,

$$\varepsilon(M) \leq r(M) + \binom{r(M)}{2}(q-1) \leq qr(M)^2.$$

By definition of θ_1, this contradicts our assumptions.

Consider next the case $d \geq 3$ and assume the result holds for $d-1$. Let $e_1, \dots, e_m \in E(M) - V$ be a maximal collection, satisfying that $V_{e_1}, \dots, V_{e_m}$ are disjoint and that $|V_{e_i}| \geq 3$ for all i. Let $m_0 = \phi(n, q)$. We look at two cases.

If $m \geq m_0$, we are done by Lemma 8.2. Assume now that $m \leq m_0$. Let $Y = V_{e_1} \cup \dots \cup V_{e_m} \subseteq V$. Then, by assumption $|Y| \leq m_0 d$. Now, in $M' = M/Y$ with basis $V - Y$, all support-sets have size at most $d-1$. And by the remark preceding the lemma,

$$\varepsilon(M') > \frac{\lambda}{q^{m_0 d}} r(M')^2 = \theta_1(n, d-1, q) r(M')^2.$$

So, we can apply induction to M', and we are done. ■

Let M be a matroid with basis V. For $e, f \in E(M) - V$, we define a notion of distance:

$$d_V(e, f) = \min \left\{ |Y| : Y \subseteq V \text{ and } r_{M/Y}(e, f) < 2 \right\}.$$

Observe, that M is simple if and only if $d_V(e, f) > 0$ for all $e \neq f$. Also, trivially $d_V(e, f) \leq |V_e|, |V_f|$.

As an example, suppose M is represented by the matrix A over some field, and V labels the columns of an identity sub-matrix. Then V can be thought of as labeling the rows, and V_e is then the support of the column labeled by e. Note, that $d_V(e, f)$ is the minimum number of rows of A, whose deletion makes the columns $\{e, f\}$ dependent.

The next lemma generalizes Lemma 8.3.

Lemma 8.4. *There exists a function $\theta : \mathbb{N}^4 \to \mathbb{N}$ such that, if $M \in \mathcal{U}(q)$ has basis V, there is a Dowling-clique on V in M, and $\varepsilon(M) > \lambda r(M)^2$ where $\lambda = \theta(n, m, d, q)$, then either*

(1) *M has a $\mathrm{PG}(n-1, q')$-minor for some q', or*

(2) *there are $e_1 \dots, e_m \in E(M) - V$, with $d_V(e_i, e_j) > d$ for $i \neq j$.*

Proof. Define θ by

$$\theta(n, m, d, q) = m(d + 1)q\theta_1(n, d, q).$$

Let M and V be given and assume that M is simple. Let $e_1, \ldots, e_s \in E(M) - V$ be a maximal collection of elements satisfying $d_V(e_i, e_j) > d$ for all $i \neq j$, and $|V_{e_i}| > d$ for each i (note that, by the remarks preceding the lemma, the second condition is superfluous in case $s \geq 2$).

If $s \geq m$, then we are done immediately, since (2) is satisfied. If $s = 0$, then $|V_e| \leq d$ for all $e \in E(M) - V$. Since $\theta(n, m, d, q) \geq \theta_1(n, d, q)$ we are done by Lemma 8.3 in this case.

Suppose now, that $1 \leq s \leq m$. Let $X = E(M) - V - \{e_1, \ldots, e_s\}$. Since $\lambda > m$ by definition, we easily get $|X| > \lambda(r(M) - 1)^2$. Now, by the maximality of s, for every element $f \in X$, $d_V(e_i, f) \leq d$ for some $i = 1, \ldots, s$. By a majority argument, there is a set $X' \subseteq X$ and an $i \in \{1, \ldots, m\}$, such that $d_V(e, f) \leq d$ for all $f \in X'$, where $e = e_i$, and $|X'| \geq \frac{1}{m}|X|$.

For each element $f \in X'$, choose $Y_f \subseteq V$ minimal such that $Y_f \cup \{e, f\}$ is dependent in M. So $|Y_f| = d_V(e, f) \leq d$. Since $|V_e| \geq d + 1$, by a majority argument, there is a set $X'' \subseteq X'$ and an element $v \in V_e$, such that $v \in V_e - Y_f$ for all $f \in X''$, and $|X''| \geq \frac{1}{d+1}|X'|$.

Since $v \in V_e$, $V' = V - v$ is a basis of $M' = M/e$. For any $f \in X''$, $Y_f \subseteq V'$, and $Y_f \cup f$ is dependent in M', which shows that the support-set of f in (M', V') is contained in Y_f, and so has size at most d. By the preceding inequalities, we get

$$\varepsilon(M'|X'') \geq \frac{|X''|}{q} > \frac{\lambda}{m(d + 1)q}(r(M) - 1)^2 = \theta_1(n, d, q)r(M')^2.$$

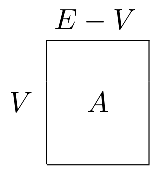
Hence, we can apply Lemma 8.3 to $M'|(V' \cup X'')$, and we are done. ∎

9 Pseudo-matrices

We now define a generalized type of matrix representation of a matroid, that need not be representable over a field. This construction is equivalent to the notion of "abstract matrix" in Truemper [**49**], though Truemper defines abstract matrices from a set of axioms.

The proofs in this section make extensive use of figures to describe special structures of such matrix representations. The proofs are dependent on the figures and cannot be read without them. I have found this circumstance hard to avoid, as the information described by some of these figures would be exceedingly cumbersome to present in the text, which would clutter the exposition.

Definition 9.1. Let M be a matroid on the ground set E, and let V be a basis of M. By the *pseudo-matrix* of (M, V), we shall mean the symbolic matrix A, with rows indexed by V and columns indexed by $E - V$, characterized by its rank function $r_A : 2^V \times 2^{E-V} \to \mathbb{N}$, that associates a number to each sub-matrix of A, its *rank*.



The rank-function is given by $r_A(Y, X) = r_{M/(V-Y)}(X)$.

Notice, that $r_A(Y, X) \leq |Y|, |X|$. The entries of A have no values, but we shall view them as being zero or non-zero depending on whether the rank is 0 or 1. An entry $A[v, e]$ is non-zero if and only if $v \in V_e$, the support-set of e. In our figures, when an entry is marked by a "0" or a "1", it means the entry has rank 0 or 1, respectively.

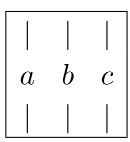
Suppose M is representable over some field, and it is represented by $[I|B]$, where V labels the columns of I (the identity-matrix). Then we can view V as labeling the rows of B. Note, that if $Y \subseteq V$ and $X \subseteq E - V$, then the sub-matrix $B[Y, X]$ has rank exactly $r_A(Y, X)$. So, the pseudo-matrix can be thought of as a generalized matrix representation.

Remark 9.2. Let M, V and A be given as above. If $Y \subseteq V$ is held fixed, then $r_M(Y, \cdot)$ is the rank-function of a matroid on $E - V$, namely $M/(V - Y)\backslash Y$. On the other hand, let $X \subseteq E - V$ be fixed, and let $Z = V \cup X$. We calculate the rank-function of $(M|Z)^*$ on $Y \subseteq V$:

$$\begin{aligned} r_{(M|Z)^*}(Y) &= |Y| - r(M|Z) + r_{M|Z}(Z - Y) \\ &= |Y| - |V| + r_M(V \cup X - Y) \\ &= r_M(V - Y \cup X) - r_M(V - Y) \\ &= r_A(Y, X). \end{aligned}$$

Thus $r_A(\cdot, X)$ is also the rank-function of a matroid.

A consequence of these observations, that we shall use extensively is the following. If a pseudo-matrix has three non-zero columns a, b, c,



and the sub-matrices $[ab]$ and $[bc]$ have rank 1, then $[ac]$ has rank 1. The same thing holds if we look at rows instead of columns.

Let A be the pseudo-matrix of (M, V). Then the sub-matrix of A, obtained by deleting rows $Y_0 \subseteq V$ and columns $X_0 \subseteq E(M) - V$ is the pseudo-matrix of a minor of M, namely $M/Y_0 \backslash X_0$ with basis $V - Y_0$. When we picture sub-matrices of a pseudo-matrix, we allow permutations of the rows and of the columns.

If the matroid M with basis V has as pseudo-matrix, a $2 \times m$ matrix, in which every 2×2 sub-matrix has rank 2, then $M \backslash V \simeq U_{2,m}$. Thus, M can have no sub-matrix of this form with $m = q + 2$, if $M \in \mathcal{U}(q)$. We use this fact in the following Ramsey-like lemma.

Lemma 9.3. *There exists a function $\kappa_1 : \mathbb{N}^2 \to \mathbb{N}$ such that, if A is the pseudo-matrix of a simple matroid in $\mathcal{U}(q)$, and A has a row (denoted v) with at least s non-zero entries, where $s = \kappa_1(t, q)$, then A has a $(t+1) \times t$ sub-matrix, of one of the following forms:*

$$(I): \quad \begin{array}{c|cccc|} \cline{2-5} v & 1 & 1 & \cdots & 1 \\ \cline{2-5} & \beta & \alpha & \cdots & \alpha \\ & \alpha & \beta & \ddots & \vdots \\ & \vdots & \ddots & \ddots & \alpha \\ & \alpha & \cdots & \alpha & \beta \\ \cline{2-5} \end{array} \qquad \textit{or} \quad (II): \quad \begin{array}{c|cccc|} \cline{2-5} v & 1 & 1 & \cdots & 1 \\ \cline{2-5} & \beta & \alpha & \cdots & \alpha \\ & \beta & \beta & \ddots & \vdots \\ & \vdots & & \ddots & \alpha \\ & \beta & \beta & \cdots & \beta \\ \cline{2-5} \end{array}$$

where 2×2 sub-matrices of the following forms have the indicated rank:

$$\begin{array}{c|cc|} \cline{2-3} v & 1 & 1 \\ & \alpha & \alpha \\ \cline{2-3} \end{array} \ \textit{rank } 1 \qquad \begin{array}{c|cc|} \cline{2-3} v & 1 & 1 \\ & \beta & \beta \\ \cline{2-3} \end{array} \ \textit{rank } 1 \qquad \begin{array}{c|cc|} \cline{2-3} v & 1 & 1 \\ & \beta & \alpha \\ \cline{2-3} \end{array} \ \textit{rank } 2 \qquad (\star)$$

Note, that entries marked "α" or "β" may be zero or non-zero, and both can occur at once in different entries. Also, we say nothing about the rank of 2×2 sub-matrices, that are not of one of the forms in $(\star)$.

Proof. Define $\kappa_1(t, q) = (q + 1)^{t_1}$, where $t_1 = (q + 1)^{t_2}$ and $t_2 = 3t$.

Let A be the pseudo-matrix of a simple matroid $M \in \mathcal{U}(q)$. We may assume, that v is the first row of A, v has no zero entries, and A has width $s = \kappa_1(t, q)$. That M is simple implies that no two columns of A are parallel (i.e. the sub-matrix consisting of any two columns has rank 2). We first prove the following.

Claim. Let $d \in \mathbb{N}$. If $s \geq (q+1)^d$, then A has a $(d+1) \times d$ sub-matrix of the form:

$$\begin{array}{c|cccc|} \cline{2-5} v & 1 & 1 & \cdots & 1 \\ \cline{2-5} & \beta & \alpha & \cdots & \alpha \\ & ? & \beta & \ddots & \vdots \\ & \vdots & \ddots & \ddots & \alpha \\ & ? & \cdots & ? & \beta \\ \cline{2-5} \end{array} \tag{1}$$

where 2×2 sub-matrices satisfy $(\star)$.

The proof of the claim is by induction on d, the case $d = 1$ being trivial. Let $d > 1$ and assume the claim holds for $d - 1$. Choose a row v' in A, such that the $2 \times s$ matrix with rows v, v' has rank 2 (if this is not possible, A has rank 1, contradicting that M is simple).

We define a relation on the columns of A as follows. Two columns are related, $e_1 \sim e_2$ if the 2×2 sub-matrix,

$$\begin{array}{c|cc|} & e_1 & e_2 \\ \cline{2-3} v & 1 & 1 \\ v' & ? & ? \\ \cline{2-3} \end{array} \tag{2}$$

has rank 1. By Remark 9.2, this is an equivalence relation. By the choice of v', there are at least two equivalence classes. Also, since $M \in \mathcal{U}(q)$, the number of classes is at most $q + 1$. By a majority argument, we can find a class X of size at least $s/(q+1)$. Let e be a column not in X. Deleting all columns not in $X \cup e$, we obtain the sub-matrix:

$$A' : \quad \begin{array}{c|c|cccc|} & e & & X & & \\ \cline{2-6} v & 1 & 1 & 1 & \cdots & 1 \\ \cline{2-6} v' & \beta & \alpha & \alpha & \cdots & \alpha \\ & ? & & & & \\ & \vdots & & ? & & \\ & ? & & & & \\ \cline{2-6} \end{array}$$

Consider the matrix A'' obtained by deleting column e and row v' from A'. Note, that A'' has no parallel columns, as otherwise A would. Since, $|X| = (q+1)^{d-1}$, by induction A'' has a $d \times (d-1)$ sub-matrix of the form (1). Hence, A' has the desired sub-matrix. This concludes the proof of the claim.

By definition of κ_1, A has a sub-matrix B of the form (1), such that B has width t_1. We prove another partial result.

CLAIM. Let $d \in \mathbb{N}$. If $t_1 \geq (q+1)^d$, then B has a $(d+1) \times d$ sub-matrix of the form:

$$\begin{array}{c|cccc|} \cline{2-5} v & 1 & 1 & \cdots & 1 \\ \cline{2-5} & \beta & \alpha & \cdots & \alpha \\ & \gamma & \beta & \ddots & \vdots \\ & \vdots & \ddots & \ddots & \alpha \\ & \gamma & \cdots & \gamma & \beta \\ \cline{2-5} \end{array} \qquad \begin{array}{c|cc|} \cline{2-3} v & 1 & 1 \\ & \gamma & \gamma \\ \cline{2-3} \end{array} \text{ rank } 1 \tag{3}$$

where the entries marked β or α are subsets of the corresponding entries in B, and where in addition to $(\star)$, 2×2 sub-matrices of the form shown on the right have rank 1.

The proof is by induction on d. The case $d = 1$ is trivial, so assume $d > 1$ and that the claim holds for $d - 1$. Let v' denote the bottom row in B, and e the right-most column.

We define an equivalence relation on the set of columns excluding e, as before, by $e_1 \sim e_2$ if the sub-matrix shown in (2) has rank 1. By a majority argument, there exists a class X of size at least $(q+1)^{d-1}$, since

$$t_1 - 1 > ((q+1)^{d-1} - 1)(q+1).$$

By deleting the columns not in $X \cup e$, and the corresponding rows (meeting at the diagonal marked β), we get the sub-matrix:

$$B' : \quad \begin{array}{c|cccc|c|} \multicolumn{1}{c}{} & \multicolumn{4}{c}{X} & \multicolumn{1}{c}{e} \\ \cline{2-6} v & 1 & 1 & \cdots & 1 & 1 \\ \cline{2-6} & \beta & \alpha & \cdots & \alpha & \alpha \\ & ? & \beta & \ddots & \vdots & \vdots \\ & \vdots & \ddots & \ddots & \alpha & \vdots \\ & ? & \cdots & ? & \beta & \alpha \\ \cline{2-6} v' & \gamma & \gamma & \cdots & \gamma & \beta \\ \cline{2-6} \end{array}$$

Let B'' be given by deleting column e and row v' from B'. Then, by induction B'' has a sub-matrix of the form (3), and of width $d - 1$, which shows that B' has the desired sub-matrix.

By definition of t_1, B has a sub-matrix C of the form (3), such that C has width t_2. Let Y denote the set of rows of C excluding v. Observe that, for a fixed row $w \in Y$, 2×2 sub-matrices of the form:

$$\begin{array}{c|cc|} \cline{2-3} v & 1 & 1 \\ w & \gamma & \alpha \\ \cline{2-3} \end{array}$$

either all have rank 1 or all have rank 2. Let us say, that w has type 1/type 2 respectively (the top and bottom row of Y have both).

If there is a set $Y' \subseteq Y$ of t type 1 rows, then by deleting the rows in $Y - Y'$ and the corresponding columns, we arrive in case (I), and we are done. Otherwise, there is a set $Y' \subseteq Y$ of $2t$ type 2 rows. Delete first the rows in $Y - Y'$ and the corresponding columns, to obtain a sub-matrix C' of the form (3), of width $2t$, and where every row has type 2. We then delete the entries marked β, by deleting every other row and column in the following fashion (rows and columns to be deleted are marked by a "$*$"):

$$\begin{array}{c|ccccc} & & * & & * & \\ \hline & 1 & \cdot & 1 & \cdot & 1 \\ \hline * & \beta & \cdot & \cdot & \cdot & \cdot \\ & \gamma & \beta & \alpha & \cdot & \alpha \\ * & \cdot & \cdot & \beta & \cdot & \cdot \\ & \gamma & \cdot & \gamma & \beta & \alpha \end{array} \quad \longrightarrow \quad \begin{array}{|ccc} \hline 1 & 1 & 1 \\ \hline \gamma & \alpha & \alpha \\ \gamma & \gamma & \alpha \end{array}$$

This gives a $(t+1) \times t$ sub-matrix as in case (II), and the lemma is proved. ∎

In the next lemma we assume instead an upper bound on the number of non-zero entries in a row, and get a result about 0/1-matrices.

Lemma 9.4. *There exists a function* $\kappa_2 : \mathbb{N}^2 \to \mathbb{N}$ *such that, if* A *is a* 0/1-*matrix, with no* 0-*columns, where no row has more than* s *non-zero entries, and* A *has width at least* m, *where* $m = \kappa_2(t, s)$, *then* A *has a* $t \times t$ *identity sub-matrix.*

Proof. Define $\kappa_2(t, s) = s(t-1) + 1$. The proof is by induction on t, the case $t = 1$ being trivially true. Let $t > 1$ be given and assume the result holds for $t - 1$.

Let A be given, with width $m = \kappa_2(t, s)$. We may assume, that A has a column with only one non-zero entry (if not, we can delete rows without creating 0-columns, until it holds). After permuting the rows and the columns, moving this entry to the top left corner, we obtain the sub-matrix:

$$\begin{array}{|c|cccc|cccc|} \hline 1 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & & & & & \\ \vdots & & ? & & & & A' & & \\ 0 & & & & & & & & \\ \hline \end{array}$$

Since the first row has at most s non-zero entries, A' has width at least $m - s = \kappa_2(t-1, s)$. By induction, A' has a $(t-1) \times (t-1)$ identity sub-matrix, and we are done. ∎

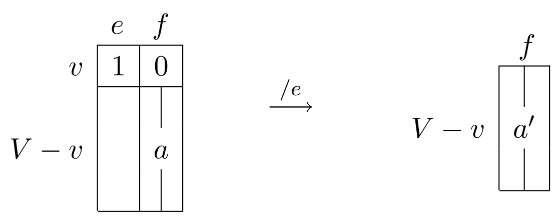
Remark 9.5. The above lemma marks one of the rare occasions in this text, where the bound in an extremal result is exact. Let I_t denote the $t \times t$ identity matrix, and consider the matrix consisting of s copies of I_{t-1}, as follows:

I_{t-1}	I_{t-1}	$\cdots$	I_{t-1}

This matrix has no I_t sub-matrix, so we must have $\kappa_2(t, s) > s(t-1)$.

We are now ready for the last part of the proof of Theorem 7.1. We first make a few observations. Let M be a matroid with basis V, and let A be the pseudo-matrix of (M, V). A sub-matrix of A corresponds to a minor of M, formed by contracting a subset of V and deleting a subset of $E(M) - V$. In the next result we shall need to contract elements of $E(M) - V$ as well. The effect on the pseudo-matrix is analogous to pivoting in a real matrix:

Let $v \in V$ and $e, f \in E(M) - V$, and suppose that $v \in V_e$, that is, the entry $A[v, e]$ is non-zero. Then $V - v$ is a basis of M/e. Let B be the pseudo-matrix of $M/e \backslash v$ with respect to $V - v$. Suppose the entry $A[v, f]$ is zero. Then f is spanned by $V - v$, so the support-set V_f of f is unchanged by the contraction (in the figure below, a and a' have the same 0/1-pattern).



If we do not assume, that $A[v, f]$ is zero, then we need to consider 2×2 sub-matrices. Let $w \in V - v$. If the sub-matrix of A,

$$\begin{array}{c|cc|} & e & f \\ \hline v & 1 & ? \\ w & ? & ? \\ \hline \end{array}$$

has rank 2, then the entry $B[w, f]$ is non-zero, and if the sub-matrix has rank 1, then the entry is zero.

Finally, notice that 2×2 pseudo-matrices of the following forms always have the indicated rank:

$$\begin{array}{|cc|}\hline 1 & ? \\ 0 & 0 \\ \hline\end{array}\ \text{rank } 1 \qquad \begin{array}{|cc|}\hline 1 & ? \\ 0 & 1 \\ \hline\end{array}\ \text{rank } 2$$

Lemma 9.6. *There exists a function $\kappa : \mathbb{N}^2 \to \mathbb{N}$ such that, if $M \in \mathcal{U}(q)$ has basis V, there is a Dowling-clique on V in M, and there are elements $e_1, \ldots, e_m \in E(M) - V$, with $d_V(e_i, e_j) > 4$ for $i \neq j$, where $m = \kappa(n, q)$, then M has a $\mathrm{PG}(n-1, q')$-minor for some q'.*

Proof. Let n and q be given. We first define a function $w : \mathbb{N} \to \mathbb{N}$ by

$$w(1) = 3, \qquad w(d) = (w(d-1)+1)(q+1)^3 + 3, \text{ for } d > 1.$$

Next, we define numbers b_0, b_1, b_2, b_3 by

$$b_0 = w(\phi(n, q)) + 1, \qquad b_i = \kappa_1(b_{i-1}, q), \text{ for } i = 1, 2, 3.$$

Let $s = b_3$, and define numbers t_0, t_1, t_2, t_3 by

$$t_0 = \phi(n, q), \qquad t_i = \kappa_2(t_{i-1}, s), \text{ for } i = 1, 2, 3.$$

Finally, let $\kappa(n, q) = t_3$.

Let (M, V) be given, and let $X = \{e_1, \ldots, e_m\}$, where $m = \kappa(n, q)$. Let A be the pseudo-matrix of $M|(V \cup X)$ with respect to V.

Note that, since $|V_{e_i}| \geq 5 > 2$, for all i, the Dowling-clique on V contains no elements from X. Any construction of minors that we do in the proof is a combination of the following operations: contracting elements of V, and deleting or contracting elements of X. Note that every such minor, N has a basis contained in V, and thus has a spanning clique-minor. When we consider the pseudo-matrix of such a minor below, we always restrict the column set to elements in X (as in the definition of A above). We consider three cases in the proof.

Case 1: No row in A has more than s non-zero entries. Mark the entries in A "0" or "1", depending on whether they are zero or non-zero. Every column in A has at least $5 \geq 3$ non-zero entries. Since $m = t_3 = \kappa_2(t_2, s)$, by Lemma 9.4 A has a $t_2 \times t_2$ identity sub-matrix. Thus, A has the sub-matrix,

$$\begin{array}{|c|}\hline I \\ \hline A' \\ \hline\end{array}$$

where A' has width t_2, and every column in A' has at least 2 non-zero entries. We now apply Lemma 9.4 to A' and repeat the argument. After applying the lemma three times, we obtain the sub-matrix of width t_0:

$$\begin{array}{|c|}\hline I \\ \hline I \\ \hline I \\ \hline \end{array}$$

This shows, than M has a minor N with basis $V' \subseteq V$ such that, there is a Dowling-clique on V' in N, and $t_0 = \phi(n, q)$ elements with disjoint support-sets of size 3. Lemma 8.2 now gives the desired PG-minor.

Case 2: There is a row v in A with at least s non-zero entries, and A has no sub-matrix of width b_0 of the form (II), shown in Lemma 9.3. Since $s = b_3 = \kappa_1(b_2, q)$, by Lemma 9.3, A has a sub-matrix of width b_2 of one of the forms (I) and (II) shown in the lemma. Because $b_2 \geq b_0$, the first must be the case. Hence, A has a sub-matrix (formed only by deleting columns) of width b_2, of the form:

$$\begin{array}{c|c|}\cline{2-2} v & 1 \;\cdots\; 1 \\ \cline{2-2} Y & B \\ \cline{2-2} & A' \\ \cline{2-2} \end{array} \qquad \text{where} \qquad \begin{array}{c|c|}\cline{2-2} v & 1 \;\cdots\; 1 \\ \cline{2-2} Y & B \\ \cline{2-2} \end{array} \qquad \text{has type (I).}$$

Let A'' be the sub-matrix obtained by deleting the set of rows Y in the above. Notice, that in a pseudo-matrix of type (I), any pair of columns can be made parallel by deleting two rows. It follows that no pair of columns $\{e, f\}$ of A'' can have rank < 2, since in that case, the elements e, f of M satisfy $d_V(e, f) \leq 2$, contradicting our assumptions.

We can now apply Lemma 9.3 to A'', and get a sub-matrix of A'' of width b_1 of the form (I). We repeat the above argument (this time using the fact that $d_V(e, f) > 4$, for distinct elements $e, f \in X$), and apply Lemma 9.3 a third time. This shows, that A has a sub-matrix C of width b_0, of the form shown below.

$$
C:\quad
\begin{array}{c|cccc|}
 & e & & & \\
\hline
v & 1 & 1 & \cdots & 1 \\
\hline
 & \beta & \alpha & \cdots & \alpha \\
 & \alpha & \beta & \ddots & \vdots \\
 & \vdots & \ddots & \ddots & \alpha \\
 & \alpha & \cdots & \alpha & \beta \\
\hline
 & \beta & \alpha & \cdots & \alpha \\
 & \alpha & \beta & \ddots & \vdots \\
 & \vdots & \ddots & \ddots & \alpha \\
 & \alpha & \cdots & \alpha & \beta \\
\hline
 & \beta & \alpha & \cdots & \alpha \\
 & \alpha & \beta & \ddots & \vdots \\
 & \vdots & \ddots & \ddots & \alpha \\
 & \alpha & \cdots & \alpha & \beta \\
\hline
\end{array}
\qquad\qquad
D:\quad
\begin{array}{|ccc|}
\hline
1 & \cdots & 1 \\
\hline
 & I & \\
\hline
1 & \cdots & 1 \\
\hline
 & I & \\
\hline
1 & \cdots & 1 \\
\hline
 & I & \\
\hline
\end{array}
$$

Let V' be the row set and X' the column set of C. Now, M has a minor M' with basis V', satisfying the restrictions described first in the proof, such that C is the pseudo-matrix of M' with respect to V' (actually of $M'|(V' \cup X')$). By the observations preceding the lemma, the minor formed by contracting e has pseudo-matrix D of the form shown above, with respect to $V' - v$.

Now, delete the three 1-rows in D. Since the resulting sub-matrix has width $b_0 - 1 \geq \phi(n, q)$, Lemma 8.2 now yields the desired PG-minor, as in Case 1.

<u>Case 3:</u> There is a row v in A with at least s non-zero entries, and A has a sub-matrix B of width b_0 of the form (II), shown in Lemma 9.3. Let e denote the first column of B as below:

$$
B:\quad
\begin{array}{c|cccc|}
 & e & & & \\
\hline
v & 1 & 1 & \cdots & 1 \\
\hline
 & \beta & \alpha & \cdots & \alpha \\
 & \beta & \beta & \ddots & \vdots \\
 & \vdots & & \ddots & \alpha \\
 & \beta & \beta & \cdots & \beta \\
\hline
\end{array}
\qquad\qquad
C_1:\quad
\begin{array}{|ccc|}
\hline
1 & \cdots & 1 \\
0 & \ddots & \vdots \\
\vdots & \ddots & 1 \\
0 & \cdots & 0 \\
\hline
\end{array}
$$

Now, B is the pseudo-matrix of a minor M_1 of M, with basis V_1. Let C_1 be the pseudo-matrix of M_1/e with the basis $V_1 - v$ (as always, restricted to the column set in X). Then C_1 has the form shown above. Let C be the sub-matrix obtained by deleting the last row of C_1. We call C *upper-triangular*, meaning that C is square and an entry of C is non-zero if and only if it lies on or above the diagonal. We prove an intermediate result:

CLAIM. Let $d \in \mathbb{N}$. Let C be the pseudo-matrix of a minor N_1 of M (satisfying the restrictions made first in the lemma), with basis contained in V. If C is upper-triangular, with width at least $w(d)$, then N_1 has a minor N with basis contained in V, and pseudo-matrix of width d, of the form:

$$\begin{array}{|c|} \hline I \\ \hline I \\ \hline I \\ \hline \end{array} \tag{1}$$

The proof of the claim is by induction on d, the case $d = 1$ being trivial, since $w(1) = 3$. Let $d > 1$, and assume the claim holds for $d - 1$. Let N_1 and C be given. Let v_0, v_1, v_2 and v_3 be the top four rows of C, in that order. Let X_1 denote the set of columns of C excluding the leftmost three columns. For each $i \in \{1, 2, 3\}$ we define an equivalence relation $\sim_i$ on X_1, by $e_1 \sim_i e_2$ if the 2×2 sub-matrix,

$$\begin{array}{c} \\ v_0 \\ v_i \end{array} \begin{array}{c} \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{array}{|cc|} \hline 1 & 1 \\ 1 & 1 \\ \hline \end{array} \end{array}$$

has rank 1. Since $M \in \mathcal{U}(q)$, $\sim_i$ defines at most $q + 1$ equivalence classes. Define $e_1 \sim e_2$, if $e_1 \sim_i e_2$, for $i = 1, 2, 3$. Then $\sim$ is an equivalence relation on X_1 with at most $(q + 1)^3$ classes. Hence, there exists a class $X_2 \subseteq X_1$, with $|X_2| \geq |X_1| / (q + 1)^3$. So, C has a sub-matrix C':

$$C' : \begin{array}{c|cccccc|} & & e & & & \\ \hline v_0 & 1 & \alpha & \alpha & \cdots & \alpha \\ v_1 & 1 & \alpha & \alpha & \cdots & \alpha \\ v_2 & 1 & \alpha & \alpha & \cdots & \alpha \\ v_3 & 0 & \alpha & \alpha & \cdots & \alpha \\ & 0 & 0 & 1 & \cdots & 1 \\ & \vdots & & \ddots & \ddots & \vdots \\ & 0 & 0 & \cdots & 0 & 1 \\ \hline \end{array} \qquad D : \begin{array}{c|c|cccc|} \hline v_0 & 1 & & & & \\ v_1 & 1 & & 0 & & \\ v_2 & 1 & & & & \\ \hline & 0 & 1 & 1 & \cdots & 1 \\ & 0 & 0 & 1 & & 1 \\ & \vdots & \vdots & \ddots & \ddots & \vdots \\ & 0 & 0 & \cdots & 0 & 1 \\ \hline \end{array}$$

where the set of columns of C' excluding the first is X_2, and such that entries marked "α" are non-zero, and 2×2 sub-matrices with all entries marked "α"

have rank 1. Suppose C' is the pseudo-matrix of a minor M_2 of M with basis V_2. Then $M_D = M_2/e$ with basis $V_2 - v_3$ has pseudo-matrix D of the form shown above. Let f denote the first column of D.

Deleting the top three rows and the first column of D, we get an upper-triangular sub-matrix of width $|X_2| - 1 \geq w(d-1)$. Hence, by induction we get a minor with pseudo-matrix of width $d - 1$ of the form (1). By the structure of D, we can construct this minor from M_D, keeping the elements v_0, v_1, v_2 and f, without affecting the support-set of f. Hence, N_1 has a minor with pseudo-matrix of width d, of the form (1). This concludes the proof of the claim.

Now, C has width $b_0 - 1 = w(\phi(n, q))$, so by the above, M has a minor with pseudo-matrix of the form (1), and of width $\phi(n, q)$. Again, by Lemma 8.2, we are done. ■

Theorem 7.1 now follows immediately from Lemmas 8.4 and 9.6.

5 Rota's Conjecture, branch-width and grids

Already in his 1935 seminal paper on matroid theory [**54**], *On the abstract properties of linear dependence*, Whitney considered the problem of characterizing the matroids representable over a given field. This problem has been at the heart of matroid theory ever since. The central task in this area has become to resolve a 1971 conjecture of Rota. There has been recent progress toward settling Rota's Conjecture, which we shall describe in this chapter, without going into technicalities.

1 Rota's Conjecture

The problem of determining whether a given matroid is representable over a given field is far from trivial. Even for the simple class of uniform matroids $U_{r,n}$, the answer is not known in general.

Problem 1.1. *Let $n \geq r$ and let $\mathbb{F}$ be a field. Is $U_{r,n}$ $\mathbb{F}$-representable?*

This problem has received much attention in projective geometry, where the term *k-arc* is used for a set X of points in $\mathrm{PG}(r-1,q)$, which satisfies $\mathrm{PG}(r-1,q)|X \simeq U_{r,k}$. Hirschfeld has written a survey paper [**26**] on the cases of Problem 1.1 that have been settled.

In [**54**], Whitney gave a characterization of binary matroids in terms of their circuits. Tutte [**51**] later used Whitney's result to prove, that a matroid is binary if and only if it has no $U_{2,4}$-minor. This characterization is possible, since the class of binary matroids is minor-closed. Any minor-closed class $\mathcal{M}$ of matroids may be described by listing the minor-minimal matroids *not* in $\mathcal{M}$ (up to isomorphism). These are called *excluded minors* for $\mathcal{M}$. Such excluded minor characterizations have proved quite successful.

We first mention the situation for graphs. In their Graph Minors project, Robertson and Seymour [**43**] proved the following amazing theorem.

Graph Minor Theorem. *In any infinite set of graphs there is one that is isomorphic to a minor of another.*

A set of graphs or a set of matroids is called an *antichain* if no member of the set is isomorphic to a minor of another member of the set. (The relation $H \preccurlyeq G$

if H is isomorphic to a minor of G defines a *quasi-order* on graphs or matroids, that is, a partial order that may lack anti-symmetry. We consider antichains in this order). So the Graph Minor Theorem says, that there is no infinite antichain of graphs. It follows, that any minor-closed class of graphs has a finite number of excluded minors, as these form an antichain. This was known as Wagner's Conjecture. Tutte [**52**] obtained the excluded minor characterization of the graphic matroids $\mathcal{G}$.

Theorem 1.2. *A matroid is graphic if and only if it has no minor isomorphic to any of the matroids $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$ and $M^*(K_{3,3})$.*

By dualizing, we get the excluded minors for the co-graphic matroids $\mathcal{G}^*$. The last two of the excluded minors are related to Kuratowski's (see [**7**]) characterization of planar graphs: *A graph G is planar if and only if G has no K_5- or $K_{3,3}$-minor.* By the above two theorems, we can conclude that any minor-closed class of graphic matroids has a finite number of excluded minors. In particular, using Kuratowski's Theorem, one can determine the excluded minors for the planar matroids (i.e. cycle matroids of planar graphs). Tutte in [**51**] also characterized the regular matroids.

Theorem 1.3. *A matroid is regular if and only if it has no minor isomorphic to any of the matroids $U_{2,4}$, F_7 and F_7^*.*

We return to the question of characterizing $\mathcal{R}(\mathbb{F})$. In 1979 Bixby [**3**] and Seymour [**46**] independently published the excluded minor characterization for the ternary matroids. The quaternary matroids were characterized in 2000 by Geelen, Gerards and Kapoor [**11**]. We summarize the results in the following theorem.

Theorem 1.4. *The binary, ternary and quaternary matroids have the excluded minor characterizations,*

(a) $\mathcal{R}(2) = \mathcal{EX}(U_{2,4})$
(b) $\mathcal{R}(3) = \mathcal{EX}(U_{2,5}, U_{3,5}, F_7, F_7^*)$
(c) $\mathcal{R}(4) = \mathcal{EX}(U_{2,6}, U_{4,6}, P_6, F_7^-, (F_7^-)^*, P_8, P_8^=)$

See [**11**] or [**38**] for a description of the matroids in (c). Proving (b) and especially (c) was far more difficult than proving (a). The major obstruction lies in the fact, that matroids in $\mathcal{R}(q)$ lack unique representability, for $q \geq 4$ (see [**37**] for the definition and a short treatment of unique representability). Kahn [**27**] showed that 3-connected matroids in $\mathcal{R}(4)$ are uniquely representable, which is crucial in the proof of (c). Kahn also conjectured, that for each prime-power q, there is a bound $n(q)$ on the number of inequivalent representations of any 3-connected matroid in $\mathcal{R}(q)$. This conjecture was proved by Oxley, Vertigan and Whittle [**39**] in the case $q = 5$ with $n(q) = 6$, and disproved for all $q > 5$ by constructing counterexamples. Attempts have been made to overcome this obstacle by introducing a notion of connectivity stronger than 3-connectivity, yet weaker than the very restrictive property of 4-connectivity. Among others, the paper [**21**] considers *sequentially* 4*-connected* matroids, and in [**25**] a variant called *fork-connectivity* is suggested.

Continuing along the lines of Theorem 1.4 seems exceedingly difficult, since the current approach for each of the cases in the theorem relies heavily on unique representability. Instead, attention has turned to the following conjecture of Rota [**44**], which he boldly made after (a) and (b) had been announced.

Rota's Conjecture. *For a prime-power q, the set of excluded minors for $\mathcal{R}(q)$ is finite.*

In other words, given q there are matroids $N_1, \ldots, N_k$, such that $\mathcal{R}(q) = \mathcal{E}\mathcal{X}(N_1, \ldots, N_k)$. The conjecture remains open for $q \geq 5$. Theorem 1.4 and Rota's Conjecture contrast with the result of Lazarson [**34**], that there are an infinite number of excluded minors for $\mathcal{R}(\mathbb{Q})$ (or $\mathcal{R}(\mathbb{F})$, for any field $\mathbb{F}$ of characteristic zero).

Lazarson's result also shows, that a generalization of the Graph Minor Theorem to the class of all matroids is impossible. Another infinite antichain of matroids is the set $\{\mathrm{PG}(2,p) : p \text{ prime}\}$ (it is an antichain by an observation in Chapter 4, Section 2). A more modest possible generalization is the following.

Problem 1.5. *Let q be a prime power. Is there no infinite antichain in $\mathcal{R}(q)$?*

This is of course a difficult question, as it extends the Graph Minor Theorem, whose proof is long and difficult. The above problem asks about the existence of an infinite antichain within the class $\mathcal{R}(q)$, while Rota's Conjecture concerns an antichain of matroids not in $\mathcal{R}(q)$. The results that we shall discuss in this chapter add weight to the plausibility of both Rota's Conjecture and the non-existence of an infinite antichain in $\mathcal{R}(q)$.

2 A new approach

As mentioned, progress has recently been bade on Rota's Conjecture. Much of this work utilizes ideas and techniques developed by Robertson and Seymour during their Graph Minors project. The following generalization of Kuratowski's Theorem on planar graphs is an immediate consequence of the Graph Minor Theorem; it was proved earlier in the Graph Minors project [**43**].

General Kuratowski Theorem. *For any surface Σ, the class of graphs that embed in Σ has finitely many excluded minors.*

Here, a *surface* is a compact connected 2-manifold without boundary. By Kuratowski's Theorem, if Σ is the plane or equivalently the sphere, then there are two excluded minors, K_5 and $K_{3,3}$. If Σ is the projective plane, it is known that there are 103 excluded minors! (see [**43**] for a reference).

The above theorem is similar in nature to Rota's Conjecture. There is now a relatively short proof based on Robertson and Seymour's ideas, that we describe next. Let $\#_m$ denote the m by m *grid*, a graph defined as follows. The vertex set is $V(\#_m) = \{v_{ij} : i, j = 0, \dots, m\}$ and the edge set $E(\#_m)$ is

$$\{e_{ij} : i = 1, \dots, m, \ j = 0, \dots, m\} \cup \{f_{ij} : i = 0, \dots, m, \ j = 1, \dots, m\},$$

where e_{ij} labels $\{v_{i-1,j}, v_{ij}\}$, and f_{ij} labels $\{v_{i,j-1}, v_{ij}\}$. Grids are universal among planar graphs in the sense, that grids themselves are planar and any planar graph is isomorphic to a minor of some grid (see [**7**]). The notion of *tree-width* introduced by Robertson and Seymour is an integer parameter for a graph. Roughly speaking, it measures how much the graph looks like a tree (see [**7**] for the definition). The new proof of the General Kuratowski Theorem is a 3-part argument, that can be summarized as follows.

(1) Let k be an integer. The class of graphs with tree-width at most k contains no infinite antichain.
(2) For any integer m, there exists an integer k such that, if G is a graph with tree-width at least k, then G contains a $\#_m$-minor.
(3) For any surface Σ there exists an integer $m \in \mathbb{N}$ such that, if G is an excluded minor for the graphs embeddable in Σ, then G has no $\#_m$-minor.

Geelen, Gerards and Whittle proved Part (1) in [**14**]. A surprisingly simple proof of part (2) was given by Diestel, Jensen, Gorbunov and Thomassen [**8**]. Finally, a short proof of part (3) was given first by Thomassen and later by Geelen, Richter and Salazar [**20**].

The concept of tree-width has an analogue called *branch-width*. Robertson and Seymour [**42**] showed, that they are equivalent in the sense, that a set of graphs has bounded tree-width if and only if it has bounded branch width. However, while tree-width has perhaps the more intuitive definition of the two, branch-width extends naturally to matroids (we avoid the definition here, see for instance [**14**]).

The 3-part proof outlined above has suggested an analogous strategy for proving Rota's Conjecture. A matroid M is *almost representable* over the field $\mathbb{F}$ if M has an element e, such that $M\backslash e$ and M/e are $\mathbb{F}$-representable. We denote by $\mathcal{AR}(q)$ the class of almost GF(q)-representable matroids. Note that $\mathcal{R}(q) \subseteq \mathcal{AR}(q)$ and that also excluded minors for $\mathcal{R}(q)$ belong to $\mathcal{AR}(q)$.

(i) Let q be a prime-power and k an integer. The matroids in $\mathcal{AR}(q)$ with branch-width at most k contain no infinite antichain.
(ii) Let q be a prime-power and m an integer. There exists an integer k such that, if $M \in \mathcal{R}(q)$ has branch-width at least k, then M contains an $M(\#_m)$-minor.
(iii) Let q be a prime-power. There exists an integer $m \in \mathbb{N}$, such that no excluded minor for $\mathcal{R}(q)$ contains an $M(\#_m)$-minor.

We first argue that these three steps would prove Rota's Conjecture. Part (i) implies, that among the excluded minors for $\mathcal{R}(q)$ there are finitely many with branch-width at most k. A basic property of branch-width is, that deleting an element of a matroid drops it's branch-width by at most one. Thus, if k is sufficiently large, by (ii), an excluded minor for $\mathcal{R}(q)$ with branch-width exceeding k contains an $M(\#_m)$-minor. If m is sufficiently large, by (iii), this is impossible.

The first two parts have both been proved recently by Geelen, Gerards and Whittle. This promising progress leaves only the third part, which clearly has to hold if Rota's Conjecture is true. Opinion has previously been divided on the plausibility of Rota's Conjecture, but now it seems unlikely that (iii) should fail.

Let $\mathcal{B}_k$ denote the class of matroids with branch-width at most k. In [**14**] it was proved, that $\mathcal{R}(q) \cap \mathcal{B}_k$ contains no infinite antichain. As representability is crucial in this proof, it was not clear how to cover excluded minors for $\mathcal{R}(q)$. However, using basically the same techniques, in [**22**] the result was extended to $\mathcal{AR}(q) \cap \mathcal{B}_k$, by associating GF$(q)$-representable 2-polymatroids to matroids in $\mathcal{AR}(q)$. This establishes part (i). The fact that $\mathcal{R}(q) \cap \mathcal{B}_k$ contains no infinite antichain is also an important step towards resolving Problem 1.5. The finiteness of the field is necessary in this result. Indeed, in [**14**], an infinite antichain (of "spikes") is constructed within the class $\mathcal{R}(\mathbb{F}) \cap \mathcal{B}_3$ for any infinite field $\mathbb{F}$.

We discuss Part (ii) in the next section. Geelen, Gerards and Whittle [**17**] have also proved a partial result to part (iii).

Theorem 2.1. *Let q be a prime-power. There exists an integer $n \in \mathbb{N}$, such that no excluded minor for $\mathcal{R}(q)$ contains a* PG$(n-1, q)$*-minor.*

While there is still quite a gap between grids and projective geometries, it now remains to show, that (given q and n, there exists an m such that) no excluded minor for $\mathcal{R}(q)$ has an $M(\#_m)$-minor and no PG$(n-1, q)$-minor.

3 The Grid Theorem

The notion of branch-width for matroids has some desirable properties. It is invariant under duality and non-increasing under taking minors. Thus $\mathcal{B}_k$ is a minor-closed class, closed under duality. Furthermore, $\mathcal{B}_k$ is closed under direct sums and 2-sums (see for instance [**24**] for proofs). It is shown in [**42**], that $\mathcal{B}_2$ is the class of direct sums of series-parallel networks. The class $\mathcal{B}_3$ is much larger and, as noted earlier, contains infinite antichains. Nonetheless, Hall, Oxley, Semple and Whittle [**24**] proved that $\mathcal{B}_3$ has finitely many excluded minors, by showing that they have size at most 14. This was extended to $\mathcal{B}_k$ for k arbitrary by Geelen, Gerards, Robertson and Whittle [**12**].

Define $\mathcal{U}^*(q) = \mathcal{EX}(U_{q,q+2})$, the dual class of $\mathcal{U}(q)$. It can be shown, that the branch-width of $M(\#_m)$ is m. Thus a large grid-minor is a certificate for large branch-width. On the other hand, the following result states that within the class $\mathcal{U}(q) \cap \mathcal{U}^*(q)$, sufficiently large branch-width implies the existence of a large grid-minor. Thus, within this class grid-minors provide a qualitative characterization of large branch-width.

The Grid Theorem. *Let q and m be positive integers. There exists an integer k such that, if $M \in \mathcal{U}(q) \cap \mathcal{U}^*(q)$ has branch-width at least k, then M contains an $M(\#_m)$-minor.*

The theorem was proved by Geelen, Gerards and Whittle in [**16**] as the culmination of a series of papers ([**15**], [**23**], [**12**], [**13**]). The proof is long and technical. Part (ii) in the three-part proof sketch of Rota's Conjecture is implied by the theorem, since $\mathcal{R}(q) \subseteq \mathcal{U}(q) \cap \mathcal{U}^*(q)$ for a prime-power q.

The Grid Theorem also has consequences for Problem 1.5. Let q be a prime-power and let H be any planar graph. Since H is a minor of some grid, by the Grid Theorem, there is an integer k, such that $\mathcal{R}(q) \cap \mathcal{EX}(M(H)) \subseteq \mathcal{R}(q) \cap \mathcal{B}_k$. Since the latter contains no infinite antichain, $\mathcal{R}(q) \cap \mathcal{EX}(M(H))$ contains no infinite antichain.

Though considering the class $\mathcal{U}(q) \cap \mathcal{U}^*(q)$ is sufficient for the purpose of Rota's Conjecture or Problem 1.5, a generalization of the Grid Theorem to the class of all matroids would be of interest. In this case, a grid-minor can not be the only certificate for large branch-width. It is easily seen, that $U_{n,2n}$ has branch-width at least $\frac{2}{3}n$, but no $M(\#_2)$-minor for all n. So we need to consider uniform minors as well. The dual of a grid, $M^*(\#_n)$ also has branch-width n. However, the plane dual of the graph $\#_n$ clearly has a $\#_{n-1}$-subgraph. So $M(\#_n^*)$ has an $M(\#_{n-1})$-minor, and we can ignore duals of grids. Another type of matroid for which the branch-width grows with the size is the bicircular matroid of a grid $B(\#_n)$ or its dual. Johnson, Robertson and Seymour have made the following conjecture (unpublished, see [**38**]).

Conjecture 3.1. *Let n be a positive integer. There exists an integer k such that, if M has branch-width at least k, then M contains a minor isomorphic to $U_{n,2n}$, $M(\#_n)$, $B(\#_n)$ or $B(\#_n)^*$.*

This would indeed be a generalization of the Grid Theorem, since both $U_{n,2n}$ and $B(\#_n)$ contains a $U_{2,n+2}$-minor for $n \geq 2$ (see Chapter 2). Moreover, it would in a sense be best possible, as it gives a qualitative characterization of large branch-width for all matroids.

Two extremal results that we have treated play a part in the proof of the Grid Theorem. Namely the Erdös-Pósa Theorem and Mader's Theorem for matroids in $\mathcal{U}(q)$ (Theorems 1.3 and 10.1 of Chapter 2). Note that $\#_m$ is a subgraph of K_n, where $n = (m+1)^2$. Hence, by Mader's Theorem for $\mathcal{U}(q)$, (given q and m there exists a λ, such that)

if $M \in \mathcal{U}(q)$ satisfies $\varepsilon(M) > \lambda r(M)$, then M has an $M(\#_m)$-minor.

The proof of the Grid Theorem considers a number of different cases, many of which are handled using the above implication.

It is my hope, that our generalization of the Erdös-Pósa Theorem and conjectured generalization of Mader's Theorem (Theorem 1.1 and Conjecture 10.5 of Chapter 2) may serve a similar purpose in a proof of Conjecture 3.1. When $U_{n,2n}$ is excluded, we have developed tools for measuring the size and density of matroids, using the a-covering number (with $a = n - 1$). Hopefully this will contribute to the derivation of a general grid theorem.

Summary

Dansk resumé

Afhandlingen omhandler aspekter af den matematiske disciplin kaldet *matroideteori*, som blev påbegyndt i 1935 af Hassler Whitney. Matroider abstraherer kombinatoriske egenskaber ved en række forskellige typer matematiske objekter, som for eksempel grafer, matricer og endelige geometrier.

Ekstremal matroide teori beskæftiger sig med sammenhænge mellem forskellige numeriske parametre knyttet til matroider, såsom størrelse og rang. Afhandlingen præsenterer resultater indenfor dette emne, herunder en generalisation af et klassisk resultat af Erdös og Pósa i graf teori samt en svagere version af Kungs "vækstrate formodning".

Summary in English

This thesis is associated to the area of mathematics known as *matroid theory*. It was founded in 1935 by Hassler Whitney, and has since then grown to be one of the major branches of combinatorics.

Matroids abstract combinatorial properties of a number of different types of mathematical objects, such as graphs, matrices, and finite geometries. Many key concepts carry over to matroids, including circuits, rank, and points and lines.

Extremal matroid theory deals with questions of how different parameters of matroids relate to each other. Many such questions are derived from extremal graph theory.

The Growth Rate Conjecture posed by Kung concerns bounds on the number of points as a function of the rank within different classes of matroids. The thesis presents several partial results to this conjecture, as well as some minor related results.

It also contains a matroid generalization of a classical graph-theoretic result known as the Erdös-Pósa Theorem. This result relates the rank of a matroid to the number of disjoint circuits in the dual matroid. In the process, tools are developed for measuring the size of matroids without a given uniform matroid as a minor.

Bibliography

[1] M. Aigner, *Combinatorial Theory*, Springer-Verlag, New York, 1979.
[2] R. Baer, *Linear algebra and projective geometry*, Academic Press, New York, 1952.
[3] R. E. Bixby, *On Reid's characterization of the ternary matroids*, J. Combin. Theory, Ser. B **26** (1979), 174–204.
[4] J. Bonin, *An introduction to extremal matroid theory with an emphasis on the geometric perspective* (course notes), Universitat Politecnica de Catalunya, Barcelona, 2003.
[5] J. Bonin and J. P. S. Kung, The number of points in a combinatorial geometry with no 8-point-line minors, in *Mathematical essays in honor of Gian-Carlo Rota*, B. Sagan and R. Stanley, eds., Birkhäuser Boston, Boston, MA, 1998, 271–284.
[6] P. Dembowski, *Finite geometries*, Springer-Verlag, New York, 1968.
[7] R. Diestel, *Graph Theory*, Springer-Verlag, New York, 1997.
[8] R. Diestel, T. R. Jensen, K. Y. Gorbunov, and C. Thomassen, *Highly connected sets and the excluded grid theorem*, J. Combin. Theory, Ser. B **75** (1999), 61–73.
[9] T. A. Dowling, *A class of geometric lattices based on finite groups*, J. Combin. Theory, Ser. B **14** (1973), 61–86.
[10] P. Erdös and L. Pósa, *On independent circuits contained in graphs*, Canad. J. Math. **17** (1965), 347–352.
[11] J. F. Geelen, A. M. H. Gerards, and A. Kapoor, *The excluded minors for* GF(4)*-representable matroids*, J. Combin. Theory, Ser. B **79** (2000), 247–299.
[12] J. F. Geelen, A. M. H. Gerards, N. Robertson, and G. P. Whittle, *On the excluded minors for the matroids of branch-width k*, J. Combin. Theory, Ser. B **88** (2003), 261–265.
[13] J. F. Geelen, A. M. H. Gerards, N. Robertson, and G. P. Whittle, *Obstructions to branch-decomposition of matroids*, Research Report 03-2, School of Mathematical and Computing Sciences, Victoria University of Wellington (2003).
[14] J. F. Geelen, A. M. H. Gerards, and G. P. Whittle, *Branch-width and well-quasi-ordering in matroids and graphs*, J. Combin. Theory, Ser. B **84** (2002), 270–290.
[15] J. F. Geelen, A. M. H. Gerards, and G. P. Whittle, *Disjoint cocircuits in matroids with large rank*, J. Combin. Theory, Ser. B **87** (2003), 270–279.
[16] J. F. Geelen, A. M. H. Gerards, and G. P. Whittle, *Excluding a planar graph from GF(q)-representable matroids*, Research Report 03-4, School of Mathematical and Computing Sciences, Victoria University of Wellington (2003).
[17] J. F. Geelen, A. M. H. Gerards, and G. P. Whittle, *On Rota's Conjecture and excluded minors containing large projective geometries*, J. Combin. Theory, Ser. B (2005).

[18] J. F. Geelen, and K. Kabell, *Projective geometries in dense matroids*, preprint at `www.imf.au.dk/publs?id=582&dlang=en` (2005), submitted to J. Combin. Theory, Ser. B.

[19] J. F. Geelen, and K. Kabell, *The Erdös-Pósa Property for matroid circuits*, preprint at `www.imf.au.dk/publs?id=587&dlang=en` (2005), to be submitted to J. Combin. Theory, Ser. B.

[20] J. F. Geelen, R. B. Richter, and G. Salazar, *Embedding grids in surfaces*, European J. Combin. **25** (2004), 785–792.

[21] J. F. Geelen, and G. P. Whittle, *Matroid* 4*-connectivity: a deletion-contraction theorem*, J. Combin. Theory, Ser. B **83** (2001), 15–37.

[22] J. F. Geelen, and G. P. Whittle, *Branch-width and Rota's Conjecture*, J. Combin. Theory, Ser. B **86** (2002), 315–330.

[23] J. F. Geelen, and G. P. Whittle, *Cliques in dense GF(q)-representable matroids*, J. Combin. Theory, Ser. B **87** (2003), 264–269.

[24] R. Hall, J. G. Oxley, C. A. Semple, and G. P. Whittle, *On matroids of branch-width three*, J. Combin. Theory, Ser. B **86** (2002), 148–171.

[25] R. Hall, J. G. Oxley, C. A. Semple, and G. P. Whittle, *Fork-decomposition of matroids*, Adv. In Appl. Math. **32** (2004), 523–575.

[26] J. W. P. Hirschfeld, Maximum sets in finite projective spaces, in *Surveys in combinatorics*, London Math. Soc. Lecture Notes **82**, 55–76.

[27] J. Kahn, *On the uniqueness of matroid representations over* GF(4), Bull. London Math. Soc. **20** (1988), 5–10.

[28] J. P. S. Kung, *Excluding the cycle geometries of the Kuratowski graphs from binary geometries*, Proc. London Math. Soc. **55** (1987), 209–242.

[29] J. P. S. Kung, *The long-line graph of a combinatorial geometry. I. Excluding* $M(K_4)$ *and the* $(q+2)$*-point line as minors*, Quart. J. Math. Oxford **39** (1988), 223–234.

[30] J. P. S. Kung, Extremal matroid theory, in *Graph structure theory*, N. Robertson and P.D. Seymour, eds., Amer. Math. Soc., Providence, Rhode Island, 1993, 21–61.

[31] J. P. S. Kung, Critical problems, in *Matroid Theory* (Seattle, WA, 1995), Contemp. Math., **197**, Amer. Math. Soc., Providence, Rhode Island, 1996, 1–127.

[32] J. P. S. Kung, *The size of a binary matroid with no* PG(3, 2)*-minor is quadratic in the rank*, Private communications.

[33] S. Lang, *Algebra*, 3-rd. ed., Springer-Verlag, New York, 2002.

[34] T. Lazarson, *The representation problem for independence functions*, J. London Math. Soc. **33** (1958), 21–25.

[35] J. H. van Lint, and R. M. Wilson, *A Course in Combinatorics*, Cambridge University Press, 1992.

[36] W. Mader, *Homomorphieeigenschaften und mittlere Kantendichte von Graphen*, Math. Ann. **174** (1967), 265–268.

[37] J. G. Oxley, *Matroid Theory*, Oxford University Press, New York, 1992.

[38] J. G. Oxley, *On the interplay between graphs and matroids*, Survey in Combinatorics, 2001, ed. J. W. P. Hirschfeld, Cambridge University Press, Cambridge, (2001), 199–239.

[39] J. G. Oxley, D. Vertigan, and G. P. Whittle, *On inequivalent representations of matroids over finite fields*, J. Combin. Theory, Ser. B **67** (1996), 325–343.
[40] F. P. Ramsey, *On a problem for formal logic*, Proc. London Math. Soc. (2) **30** (1930), 264–286.
[41] N. Robertson, and P. D. Seymour, *Graph minors VIII. A Kuratowski theorem for general surfaces*, J. Combin. Theory, Ser. B **48** (1990), 255–288.
[42] N. Robertson, and P. D. Seymour, *Graph minors X. Obstructions to tree-decomposition*, J. Combin. Theory, Ser. B **52** (1991), 153–190.
[43] N. Robertson, and P. D. Seymour, *Graph minors XX. Wagner's Conjecture*, J. Combin. Theory, Ser. B **92** (2004), 325–357.
[44] G.-C. Rota, *Combinatorial theory, old and new*, in Proc. Internat. Cong. Math. (Nice, 1970), Vol. 3, Gauthier-Villars, Paris (1971), 229–233.
[45] N. Sauer, *On the density of families of sets*, J. Combin. Theory, Ser. A **13** (1972), 145–147.
[46] P. D. Seymour, *Matroid representation over* GF(3), J. Combin. Theory, Ser. B **26** (1979), 159–173.
[47] P. D. Seymour, *Decomposition of regular matroids*, J. Combin. Theory, Ser. B **28** (1980), 305–359.
[48] S. Shelah, *A combinatorial problem: stability and order for models and theories in infinitary languages*, Pacific J. Math. **41** (1972), 247–261.
[49] K. Truemper, *Matroid Decomposition*, Academic Press, San Diego, 1992.
[50] P. Túran, *On an extremal problem in graph theory*, in Hungarian, Mat. Fiz. Lapok **48** (1941), 436–452.
[51] W. T. Tutte, *A homotopy theorem for matroids, I, II*, Trans. Amer. Math. Soc. **88** (1958), 144–174.
[52] W. T. Tutte, *Matroids and graphs*, Trans. Amer. Math. Soc. **90** (1958), 527–552.
[53] D. J. A. Welch, *Matroid Theory*, Academic Press, London, 1976.
[54] H. Whitney, *On the abstract properties of linear dependence*, Amer. J. Math **57** (1935), 509–533.