Secondary Invariants for Families of Bundles

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Contents

Preface and acknowledgments

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Chapter 1

Introduction

This thesis is concerned with the construction of secondary invariants for families of bundles. By a family of bundles we mean an oriented fibre bundle $Y \to Z$ together with a principal G-bundle $E \to Y$. That is Z is the space parametrising the family of principal G-bundles $E_{|Y_z} \to Y_z$. In [7], which was the main inspiration when we started this work, Dupont-Kamber study families of bundles carrying a fibrewise connection, that is each bundle $E_{|Y_z} \to Y_z$ is equipped with a connection A_z which varies smoothly in z. They extend the fibrewise connection to a full connection in the bundle $E \to Y$ and are in this way getting secondary invariants living in smooth Deligne cohomology $H^*_{\mathcal{D}}(Y, \mathbb{Z})$ of the total space Y. By integrating these over the fibre, they get secondary invariants living in smooth Deligne cohomology $H^{*-n}_{\mathcal{D}}(Z, \mathbb{Z})$ of the parameter space Z – here n is the dimension of the fibre. These classes are in some cases, e.g. if the fibrewise connection A_z is flat, actually independent of the extension to a full connection.

It turns out that it is not that straight forward to construct such an integration map $H^{*+n}_{\mathcal{D}}(Y, \mathbb{Z}) \to H^*_{\mathcal{D}}(Z, \mathbb{Z})$ which satisfies all the properties that one might ask for. In [11], Freed constructs an integration map directly on the cohomology groups. This is, as Freed notes, not entirely satisfactory for many applications, where one would like a map on the cochain level. This is the case in [11] (see also example 7.2.4), where one is not looking for a Deligne 2-class but rather the underlying 2-cocycle, since the former corresponds to an isomorphism class of circle bundles with connection whereas the later corresponds to a specific circle bundle.

The first construction of a cochain model for smooth Deligne cohomology including an integration map appeared in Hopkins-Singer [16]. Their cochain model is quite close to the formulation of smooth Deligne cohomology in terms of Cheeger-Simons differential characters and thus have a global nature. It is probably the most intuitive construction, but it is not possible to see what happens locally and the product in the Hopkins-Singer model depends on a choice of chain homotopy between the wedge product of forms and the cup product of singular cochains. This last problem makes it hard to prove relations between the product and the integration map.

We presents an approach using simplicial forms which is more local in nature and which is compatible with the product in this model.

We show that these two approaches gives the same map in smooth Deligne cohomology, by showing that there is both a unique integration map and a unique product structure which satisfies some natural axioms.

In [7] the invariants for families of bundles are primarily applied to families of foliated bundles, since such a family gives rise to a family of normal bundles with adapted connections. Another direct application is given in the above cited paper by Freed [11], where it is used to give a construction of the determinant line bundle. There is also a less direct application to symplectic fibrations, which we will explain below.

A symplectic fibration is a fibre bundle $Y \to Z$ with fibre (M, ω) a symplectic manifold and with the symplectomorphism group as structure group. Since the structure group preserves the symplectic form, such fibrations carry a fibrewise symplectic form. If the structure group can be reduced to the group of hamiltonian diffeomorphisms then there is a canonical extension of this fibrewise form to a cohomology class $c \in H^2(Y, \mathbb{R})$ on the total space. This class gives rise to characteristic classes

$$\chi_k = \int_{Y/Z} c^{n+k} \in H^{2k}(Z, \mathbb{R}).$$

In [19] Kędra-Mcduff showed that these classes, up to scaling, was equal to a set of characteristic classes which was constructed earlier by Reznikov in [28] using Chern-Weil theory.

We investigate in which cases the χ_k's are actually integral classes and when these classes can be lifted to secondary invariants depending on a choice of connection.

When the symplectic form ω has integral periods, it is possible to find a circle bundle with connection over M with curvature ω. Given such a circle bundle there is a central extension of the hamiltonian group

$$0 \to \mathbb{R}/\mathbb{Z} \to \widetilde{\mathrm{Ham}} \to \mathrm{Ham} \to 1,$$

which was first introduced by Kostant in [20]. We will see that in the universal case this extension splits if and only if we can choose the canonical extension c to have integral periods. If this is the case, the χ_k's are actually integral classes.

If the structure group of our hamiltonian fibration lifts to this central extension, we get an associated family of circle bundles, which gives rise to secondary invariants of the hamiltonian fibration.

In the case where M carries a hamiltonian action of a compact Lie group, it is sometimes possible to show that these classes are non-trivial. We will see that in the case $M = \mathbb{C}P^n$ with its canonical circle bundle. The classes extend the usual Cheeger-Chern-Simons classes on $SU(n+1)$.

There are other ways of constructing characteristic classes for symplectic fibrations. In [21], Kotschick-Morita use the spectral sequence associated with the short exact sequence

$$1 \to \mathrm{Ham}(\Sigma_g) \to \mathrm{Symp}_0(\Sigma_g) \to H^1(\Sigma_g, \mathbb{R}) \to 0$$

to show the existence of characteristic classes for flat symplectic fibrations in the case where $M = \Sigma_g$ is a closed oriented surface of genus $g \geq 2$. We transfer this to the Lie algebra level, and by calculating a differential in the corresponding Lie algebra spectral sequence, we will give an explicit construction of some of these classes. A similar calculation by Vizman, done in another context, suggests that these classes are non-zero also for more general symplectic manifolds than Σ_g.

Summary

The thesis is naturally divided into two parts. The first part consists of chapter 2 to 7 and centres on the construction of an integration map for smooth Deligne cohomology and some direct applications. Much of the material in the first part is taken from the paper Dupont-Ljungmann [8]. The second part, consisting of the last three chapters 8 to 10, is concerned with the construction of characteristic classes for symplectic fibrations, both using the theory developed in the first part and using other more classical methods. Below is a more detailed account of the content of each chapter.

Chapter 2

The second chapter introduces smooth Deligne cohomology and thus contains no new material. The different chain models for smooth Deligne cohomology is described with emphasis on a description in terms of simplicial forms first introduced in Dupont-Kamber [7] which is used in the subsequent chapters.

Chapter 3

In chapter three, we introduce the concept of a prism complex, which is a generalisation of simplicial sets well-suited for handling simplicial constructions involving fibrations.

Chapter 4

Chapter 4 contains the main constructions of the first part of the thesis. We prove

Theorem. *Given a fibre bundle $\pi : Y \to Z$ with compact, oriented n-dimensional fibres and suitable coverings $\mathcal{V}$ and $\mathcal{U}$, then there is an integration map for simplicial forms*

$$\int_{[Y/Z]} : \Omega^{*+n}(|N\mathcal{V}|) \to \Omega^*(|N\mathcal{U}|),$$

compatible with the usual fibre integration map $\int_{Y/Z} : \Omega^{*+n}(Y) \to \Omega^*(Z)$. It satisfies a Stokes' formula and, if $\partial Y = \emptyset$, induces a map

$$\pi_! : H_{\mathcal{D}}^{*+n}(Y, \mathbb{Z}) \to H_{\mathcal{D}}^*(Z, \mathbb{Z})$$

in smooth Deligne cohomology independent of all choices.

In the course of the proof of this theorem, we make a second construction of the integration map defined in a more combinatorial model where the cohomology classes are represented by simplicial forms living in the *triangulated nerve* $|NK|$ associated to a triangulation $|K| \to |L|$ of the bundle. This allows us to prove the following theorem in the case where the fibre has boundary:

Theorem. *Assume that $\partial Y \neq \emptyset$, then for a form $\omega \in \Omega^{*+n}(|N\mathcal{V}|)$ representing an element in smooth Deligne cohomology, the form*

$$\int_{K/L} \omega \in \Omega^*(|NL|)/d\Omega^{*-1}(|NL|)$$

depends only on the triangulation of $\partial Y \to Z$.

Chapter 5

There are other constructions of integration maps in the literature. These constructions are briefly reviewed in chapter 5. We show that all these constructions lead to the same induced integration map in cohomology. This is done by assuming some natural axioms, and then showing that there is a unique integration map in smooth Deligne cohomology satisfying all of these.

Chapter 6

In chapter 6, we review the different ways of giving the smooth Deligne cohomology groups a graded ring structure. We also introduce a new product on simplicial forms, which induces a product on smooth Deligne cohomology. This construction is slightly more complicated than the existing ones, but it fits well together with the integration map constructed in chapter 4. This allows us to prove

Proposition. *Given a fibre bundle $p : Y \to Z$ and classes $a \in H^n_{\mathcal{D}}(Y, \mathbb{Z})$ and $b \in H^k(Z, \mathbb{Z})$, then we have*

$$\int_{[Y/Z]} (a \tilde{\wedge} p^* b) = \left(\int_{[Y/Z]} a \right) \tilde{\wedge} b.$$

We end the chapter by showing that, if we insist that the product satisfies some natural axioms, then there is a unique product in smooth Deligne cohomology.

Chapter 7

In chapter 7, we apply the constructions from the previous chapters in order to construct invariants for families of bundles with connections. The chapter contains no new results, but is included as motivation for the construction of the integration map. The material mainly builds on [7].

Chapter 8

Here we change focus a bit. Chapter 8 contains a short introduction to symplectic topology with emphasis on symplectic fibrations. It contains some preliminary results which is used in chapter 9.

Chapter 9

In chapter 9, we construct secondary classes for certain hamiltonian fibrations using the integration map constructed in chapter 4.

By analysing in which cases the characteristic classes reviewed in chapter 8 can be expected to be integral, we arrive at Kostant's central extension $\widetilde{\mathrm{Ham}}(M)$ of the group of hamiltonian diffeomorphisms $\mathrm{Ham}(M)$. It turns out that we cannot construct our secondary classes for all hamiltonian fibration, but only for those where the structure group lifts to this central extension. In this case, we get a family of circle bundles with a fibrewise connection associated with the hamiltonian fibration. There is a nice interpretation of extensions of this fibrewise connection in terms of connections in the hamiltonian fibrations, which enables us to prove

Theorem. *Given a symplectic manifold* (M,ω) *with a prequantum line bundle* (L,α) *we have well-defined classes*

$$\hat{\chi}_k(\alpha) \in H^{2k-1}(B\widetilde{\mathrm{Ham}}^{\delta}(M), \mathbb{R}/\mathbb{Z}), \quad \textit{for } k \geq 1.$$

If α *and* α' *are gauge equivalent connections in* L *then* $\hat{\chi}_k(\alpha) = \hat{\chi}_k(\alpha')$.

Chapter 10

This last chapter is centred around the work of Kotschick-Morita [21], the chapter is more open ended than the preceding ones, but still contains some new material. Most importantly we are able to give an explicit description of some characteristic classes in $H^2(B\mathrm{Ham}^{\delta}(\Sigma_g), \mathbb{R})^{H^1(\Sigma_g,\mathbb{R})}$. This answers a question raised in [21], where the existence of these classes was proven.

Chapter 2

Smooth Deligne cohomology

In this section, we describe the different ways of looking at smooth Deligne cohomology. We start by explaining the sheaf theoretic approach and what a class looks like in the corresponding Čech complex. Then we reformulate this in terms of simplicial forms, and finally we take a look at the Cheeger-Simons differential characters.

2.1 Sheaf cohomology and the Čech-de Rham model

The main reference for this section is chapter 1 in Brylinski's book [1]. Let Z be a smooth manifold, then we have the following complex of sheaves

$$\mathbb{Z}_{p,\infty} : \underline{\mathbb{R}/\mathbb{Z}} \xrightarrow{\mathrm{d}} \underline{\Omega}^1 \xrightarrow{d} \cdots \xrightarrow{d} \underline{\Omega}^{p-1} \tag{2.1}$$

where $\underline{\mathbb{R}/\mathbb{Z}}$ denotes the sheaf of smooth $\mathbb{R}/\mathbb{Z}$-valued functions on Z, and $\underline{\Omega}^l$ is the sheaf of real-valued differential l-forms on Z.

Ordinarily, to calculate the cohomology of some manifold Z with value in a sheaf $\mathcal{S}$, you pick an injective resolution $\mathcal{I}_*$ of $\mathcal{S}$ and then take the cohomology of the corresponding chain complex of global sections

$$\Gamma(\mathcal{I}_1) \to \Gamma(\mathcal{I}_2) \to \cdots .$$

Given a complex of sheaves

$$\mathcal{S}_* : \mathcal{S}_1 \to \mathcal{S}_2 \to \cdots \to \mathcal{S}_p,$$

as above, one can do something similar. We can find injective resolutions $\mathcal{I}_{i*}$ of each $\mathcal{S}_i$, which are suitably compatible (see [1, ch. 1] for details).

The hyper cohomology of Z with values in the complex of sheaves $\mathcal{S}_*$ is then the cohomology of the total complex of the double complex of global sections

$$\begin{array}{ccccccc} \vdots & & \vdots & & & & \vdots \\ \uparrow & & \uparrow & & & & \uparrow \\ \Gamma(\mathcal{I}_{12}) & \longrightarrow & \Gamma(\mathcal{I}_{22}) & \longrightarrow & \cdots & \longrightarrow & \Gamma(\mathcal{I}_{p2}) \\ \uparrow & & \uparrow & & & & \uparrow \\ \Gamma(\mathcal{I}_{11}) & \longrightarrow & \Gamma(\mathcal{I}_{21}) & \longrightarrow & \cdots & \longrightarrow & \Gamma(\mathcal{I}_{p1}) \end{array}$$

We denote it by $\mathbb{H}^*(Z, \mathcal{S})$. Given a short exact sequence of complexes of sheaves

$$\mathcal{R}_* \to \mathcal{S}_* \to \mathcal{T}_*$$

we get, as in ordinary sheaf cohomology, a long exact sequence of hyper cohomology groups

$$\to \mathbb{H}^{p-1}(Z, \mathcal{T}_*) \to \mathbb{H}^p(Z, \mathcal{R}_*) \to \mathbb{H}^p(Z, \mathcal{S}_*) \to \mathbb{H}^p(Z, \mathcal{T}_*) \to \mathbb{H}^{p+1}(Z, \mathcal{R}_*) \to$$

Definition 2.1.1. The p'th smooth Deligne cohomology group $H^p_{\mathcal{D}}(Z, \mathbb{Z})$ of Z is the $p-1$'st hyper cohomology group

$$\mathbb{H}^{p-1}(Z, \mathbb{Z}_{p.\infty})$$

of the above complex of sheaves.

Remark 2.1.2. The smooth Deligne cohomology groups are usually taken to be the bi-graded groups $\mathbb{H}^q(Z, \mathbb{Z}_{p,\infty})$, but for $q \neq p-1$ we have

$$\mathbb{H}^q(Z, \mathbb{Z}_{p,\infty}) = \begin{cases} H^{q-1}(Z, \mathbb{R}/\mathbb{Z}) & q < p-1 \\ H^q(Z, \mathbb{Z}) & q > p-1 \end{cases}$$

so there is really only new information in the case $q = p-1$. This is most easily seen by observing that we have the following short exact sequence of complexes of sheaves

$$0 \to \mathbb{Z} \to \sigma_{<p}\underline{\Omega}^* \to \mathbb{Z}_{p,\infty} \to 0 \tag{2.2}$$

where the middle complex is the augmented de Rham complex

$$\sigma_{<p}\underline{\Omega}^* : \underline{\Omega}^0 \to \underline{\Omega}^1 \to \cdots \to \underline{\Omega}^{p-1}.$$

Since the hyper cohomology of the augmented de Rham complex is given by

$$\mathbb{H}^q(Z, \sigma_{<p}\underline{\Omega}^*) = \begin{cases} H^q_{dR}(Z) & q < p-1 \\ 0 & q > p-1 \\ \Omega^{p-1}(Z)/d\Omega^{p-2}(Z) & q = p-1 \end{cases} \tag{2.3}$$

the claim follows directly from the long exact cohomology sequence.

The Deligne cohomology groups are in general quite interesting. For $p = 1$ we see that $\mathbb{Z}_{1,\infty}$ is just the sheaf $\underline{\mathbb{R}/\mathbb{Z}}$, so $H^1_{\mathcal{D}}(Z, \mathbb{Z}) = H^0(Z, \underline{\mathbb{R}/\mathbb{Z}})$ is nothing but the group of smooth functions $Z \to \mathbb{R}/\mathbb{Z}$, but already for $p = 2$ the cohomology group is actually isomorphic to the group of isomorphism classes of circle bundles with connection (see proposition 2.1.5 below). The higher cohomology groups correspond to groups of isomorphism classes of *abelian n-gerbes with connection*. There are geometric constructions of 1-gerbes and 2-gerbes, but in general the lack of a proper notion of a weak n-category has so far made it hard to come up with a nice geometric notion of n-gerbes. We will simply identify an n-gerbe with its underlying defining cocycle in $\check{C}^n(\mathcal{U}, \mathbb{R}/\mathbb{Z})$ and by a 'gerbe with connection' simply refer to the underlying defining cocycle that represents a class in smooth Deligne cohomology. To see what such a cocycle looks like we describe the Čech complex corresponding to the complex of sheaves $\mathbb{Z}_{p,\infty}$. It is also this model which is best suited for the reformulation in terms of simplicial forms in section 2.2.

First pick $\mathcal{U} = \{U_i\}_{i \in I}$, a good open cover of Z. Here good means that all intersections are contractible.

Let $\check{\Omega}^{p,q}(\mathcal{U}) = \check{C}^p(\mathcal{U}, \underline{\Omega}^q)$ be the ordinary Čech-de Rham complex and let $\check{\Omega}^*(\mathcal{U})$ denote the corresponding total complex with total differential $D = \delta + (-1)^p d$ on $\Omega^{p,q}(\mathcal{U})$, where δ and d are the Čech and the de Rham differentials respectively.

It is well-known that the chain-map

$$\varepsilon^* : \Omega^q(Z) \to \check{\Omega}^{0,q}(\mathcal{U}),$$

induced by the natural map $\varepsilon : \sqcup U_i \to Z$ gives an isomorphism

$$H^*_{\mathrm{dR}}(Z) \to H^*(\check{\Omega}^*(\mathcal{U}))$$

in cohomology. We also have an inclusion of the ordinary Čech complex with integer coefficients

$$\check{C}^p(\mathcal{U}, \mathbb{Z}) \to \check{\Omega}^{p,0}(\mathcal{U})$$

which gives us the quotient complex

$$\check{\Omega}^*_{\mathbb{R}/\mathbb{Z}}(\mathcal{U}) = \check{\Omega}^*(\mathcal{U})/\check{C}^*(\mathcal{U}, \mathbb{Z}).$$

Finally we have

Proposition 2.1.3. *$H^l_{\mathcal{D}}(Z, \mathbb{Z})$ is the cohomology of the sequence*

$$\check{\Omega}^{l-2}_{\mathbb{R}/\mathbb{Z}}(\mathcal{U}) \xrightarrow{d} \check{\Omega}^{l-1}_{\mathbb{R}/\mathbb{Z}}(\mathcal{U}) \xrightarrow{d} \check{\Omega}^{l}_{\mathbb{R}/\mathbb{Z}}(\mathcal{U})/\varepsilon^*\Omega^l(Z).$$

Proof. Since $\mathcal{U}$ is a good open cover, the hyper cohomology is calculated by its corresponding Čech complex (see e.g. [1, ch. 1]). It is clear that the cohomology of this complex is the same as that of the sequence above. □

Remark 2.1.4. 1. Take a $\omega = (\omega_0, \dots, \omega_l) \in \check{\Omega}^l(\mathcal{U})$, where $\omega_i \in \check{\Omega}^{i,l-i}(\mathcal{U})$. That ω is a cycle in the sequence above is equivalent to the relations

$$\delta\omega_{i-1} + (-1)^i d\omega_i = 0, \quad i = 1, \dots, l$$

and

$$\delta\omega_l \equiv 0 \mod \mathbb{Z}.$$

2. Since $\delta\omega_l \equiv 0 \mod \mathbb{Z}$ we have that $\theta = -\omega_l$ is a cocycle in $\check{C}^l(\mathcal{U}, \mathbb{R}/\mathbb{Z})$ that is a defining cocycle for an l-gerbe. As mentioned above we will refer to the pair (θ, ω) as a gerbe with connection.

The above discussion enables us to show

Proposition 2.1.5. *$H^2_{\mathcal{D}}(Z, \mathbb{Z})$ is isomorphic to the group of isomorphism classes of circle bundles with connection.*

Remark 2.1.6. Here and throughout the thesis, a circle bundle will mean a principal $\mathbb{R}/\mathbb{Z}$-bundle.

Proof. Given a representative (ω_0, ω_1) for a class in $H^2_{\mathcal{D}}(Z, \mathbb{Z})$, then since $\theta = -\omega_1$ is a cocycle in $\check{C}^1(\mathcal{U}, \mathbb{R}/\mathbb{Z})$ it defines a circle bundle in the usual way. Now the first relation in remark 2.1.4 says that ω_0 is a collection of local connection forms with respect to the transition functions given by θ. Another choice of representative (ω_0', ω_1') results in an isomorphic bundle.

Similarly a circle bundle with connection will give cocycles satisfying the relations in 2.1.4. Isomorphic circle bundles give cocycles which differ by a boundary. □

We also have the following useful proposition

Proposition 2.1.7. 1. *We have a commutative diagram*

$$\begin{array}{ccc} H^l_{\mathcal{D}}(Z, \mathbb{Z}) & \xrightarrow{d_*} & \Omega^l_{\text{cl}}(Z) \\ \downarrow{\scriptstyle \delta_*} & & \downarrow{\scriptstyle I} \\ H^l(Z, \mathbb{Z}) & \longrightarrow & H^l(Z, \mathbb{R}) \end{array}$$

where $\Omega^l_{\text{cl}}(Z)$ is the set of closed l-forms with integral periods.

2. *There are short exact sequences*

$$0 \to \Omega^{l-1}(Z)/\Omega^{l-1}_{\text{cl}}(Z) \to H^l_{\mathcal{D}}(Z, \mathbb{Z}) \to H^l(Z, \mathbb{Z}) \to 0$$

and

$$0 \to H^{l-1}(Z, \mathbb{R}/\mathbb{Z}) \to H^l_{\mathcal{D}}(Z, \mathbb{Z}) \xrightarrow{d_*} \Omega^l_{\text{cl}}(Z) \to 0. \tag{2.4}$$

3. *Combining these we get*

$$0 \to H^{l-1}(Z, \mathbb{R})/H^{l-1}(Z, \mathbb{Z}) \to H^l_{\mathcal{D}}(Z, \mathbb{Z}) \to R^l(Z) \to 0,$$

where $R^l(Z) = \{(c, \omega) \in H^l(Z, \mathbb{Z}) \times \Omega^l_{\text{cl}}(Z) \mid c = [\omega]$ in $H^l(Z, \mathbb{R})\}$.

Proof. 1. Let $\omega = (\omega_0, \dots, \omega_{l-1})$ represent a class in $H^l_{\mathcal{D}}(Z,\mathbb{Z})$. Note that since $\delta d\omega_0 = d\delta\omega_0 = d^2\omega_1 = 0$ then $F_\omega = d\omega_0$ is actually a globally defined, closed l-form. F_ω is called the ***curvature*** of ω. d_* is the map sending ω to F_ω. δ_* is just the connecting homomorphism for the short exact sequence

$$0 \to \mathbb{Z} \to \underline{\mathbb{R}} \to \underline{\mathbb{R}/\mathbb{Z}} \to 0$$

and I is the de Rham map. Now, commutativity of the diagram follows from the fact that $d\omega_0 - \delta\theta = d\omega_0 + \delta\omega_{l-1} = D\omega$ in $\check{\Omega}^*_{\mathbb{R}}(\mathcal{U})$, and the cohomology of this complex is $H^*(Z,\mathbb{R})$.

2. The first short exact sequence comes from the long exact sequence associated with the short exact sequence of complexes of sheaves (2.2) together with the result on the cohomology of the augmented de Rham complex (2.3).

The kernel of d_* is the *gerbes with flat connection*. The calculation in 1. shows that ω is a cocycle in the total complex $\check{\Omega}^*_{\mathbb{R}/\mathbb{Z}}(\mathcal{U})$ if and only if the class $[\omega]$ lies in the kernel of d_*. Since the cohomology of $\check{\Omega}^*_{\mathbb{R}/\mathbb{Z}}(\mathcal{U})$ is $H^*(Z,\mathbb{R}/\mathbb{Z})$, the kernel is $H^l(Z,\mathbb{R}/\mathbb{Z})$.

3. Follows from 1. and 2. □

2.2 Simplicial forms

The idea of looking at smooth Deligne cohomology in terms of simplicial forms was introduced in [7]. It is in this setting, the constructions in chapter 4 are carried out. In this section, we will briefly explain the basics on simplicial forms, and then show how they can be used to represent classes in smooth Deligne cohomology.

Definition 2.2.1. A *simplicial set* $X_\bullet = \{X_p\}_{p\geq 0}$ is a collection of sets together with maps

$$d_j : X_p \to X_{p-1}, \quad j = 0, \dots, p$$

called face maps, and maps

$$s_j : X_p \to X_{p+1}, \quad j = 0, \dots p$$

called degeneracy maps, such that the following relations are satisfied

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, && i < j, \\ s_i s_j &= s_{j+1} s_i, && i \leq j, \\ d_i s_j &= \begin{cases} s_{j-1} d_i, & i < j, \\ \text{id}, & i = j, i = j+1, \\ s_j d_{i-1}, & i > j+1. \end{cases} \end{aligned}$$

A *simplicial manifold* is a simplicial set $X_\bullet = \{X_p\}$ where each X_p is a smooth manifold, and the face and degeneracy maps are smooth.

We will mostly be interested in the following example of a simplicial manifold

Example 2.2.2. Given an open cover $\mathcal{U} = \{U_i\}$ of Z we have the *nerve* $N\mathcal{U} = \{N\mathcal{U}(p)\}$ of the covering, where

$$N\mathcal{U}(p) = \coprod_{i_0,\ldots,i_p} U_{i_0} \cap \cdots \cap U_{i_p}.$$

We denote $U_{i_0} \cap \cdots \cap U_{i_p}$ by $U_{i_0\ldots i_p}$ in the following. $N\mathcal{U}$ is a simplicial manifold where the face and degeneracy maps comes from the inclusions

$$d_j : U_{i_0\ldots i_p} \to U_{i_0\ldots \hat{i_j}\ldots i_p}$$

and

$$s_j : U_{i_0\ldots i_p} \to U_{i_0\ldots i_j i_j\ldots i_p}.$$

We also have a corresponding simplicial set $N_d\mathcal{U} = \{N_d\mathcal{U}(p)\}$ called the discrete nerve of the covering. Here $N_d\mathcal{U}(p)$ is simply the set consisting of an element for each non-empty intersection of $p+1$ open sets from $\mathcal{U}$. We have a natural forgetful map $N\mathcal{U} \to N_d\mathcal{U}$.

Definition 2.2.3. A *simplicial n-form* $\omega = \{\omega^{(p)}\}$ on a simplicial manifold $X_\bullet$ consists of a collection of forms $\omega^{(p)} \in \Omega^n(\Delta^p \times X_p)$ which satisfies the relations

$$(\varepsilon_j \times \mathrm{id})^*\omega^{(p)} = (\mathrm{id} \times d_j)^*\omega^{(p-1)},$$

where $\varepsilon_j : \Delta^{p-1} \to \Delta^p$ denotes the ordinary j'th face map. We denote the set of simplicial forms on X by $\Omega^*(||X||)$. If the forms also satisfy the relations

$$(\eta_j \times \mathrm{id})^*\omega^{(p-1)} = (\mathrm{id} \times s_j)^*\omega^{(p)},$$

where $\eta_j : \Delta^p \to \Delta^{p-1}$ is the ordinary j'th degeneracy map, the forms are called *normal*. The set of normal forms is denoted $\Omega^*(|X|)$.

Remark 2.2.4. When $X_\bullet = N\mathcal{U}$ where

$$N\mathcal{U}(p) = \coprod_{i_0,\ldots,i_p} U_{i_0} \cap \cdots \cap U_{i_p},$$

it is customary to consider only ordered $(p+1)$-tuples, that is for a tuple $(i_0,\ldots,i_p)$ we have $i_0 \leq \cdots \leq i_p$ (our index sets are always assumed to be ordered). Later when we move on to prism complexes, this will in some instances be annoying. Instead we demand that for a permutation $\sigma \in \Sigma(p)$ the normal forms also satisfy the relation

$$\tilde{\sigma}^*\omega = \omega$$

where $\tilde{\sigma} : \Delta^p \times U_{i_0\ldots i_p} \to \Delta^p \times U_{i_{\sigma(0)}\ldots i_{\sigma(p)}}$ on the first factor is the simplicial map that permutes the vertices of Δ^p according to σ, and on the second factor is the identity.

For a simplicial manifold $X_\bullet$ we have a direct sum decomposition

$$\Omega^n(|X|) = \bigoplus_{p+q=n} \Omega^{p,q}(|X|)$$

where $\Omega^{p,q}(|X|)$ is the set of forms that are of degree p in the barycentric coordinates on the simplex in the product $\Delta^k \times X_k$ for $k \geq p$.

For $X_\bullet = N\mathcal{U}$ there is a chain map

$$I_\Delta : \Omega^{p,q}(|N\mathcal{U}|) \to \check{\Omega}^{p,q}(\mathcal{U})$$

given by $I_\Delta(\omega) = \int_{\Delta^p} \omega^{(p)}$. This map gives an isomorphism in cohomology. In fact it has a right inverse given on $\Delta^k \times N\mathcal{U}(k)$ by

$$E(\omega) = p! \sum_{|I|=p} \omega_I \wedge d_I^* \omega,$$

where $I = (i_0, \dots, i_p)$ is a sequence of integers $0 \leq i_0 \leq \cdots \leq i_p \leq k$,

$$\omega_I = \sum_{j=0}^{p} (-1)^j t_{i_j} dt_{i_0} \wedge \hat{dt_{i_j}} \wedge dt_{i_p}$$

are the *elementary forms* on Δ^k and the d_I's are maps $N\mathcal{U}(k) \to N\mathcal{U}(p)$ given by $d_I = d_{j_1} \cdots d_{j_l}$ where $0 \leq j_l \leq \cdots \leq j_1 \leq k$ is the complementary sequence of I (see Dupont [4, ch. 2] for details).

The natural map $\sqcup U_i \to Z$ also induces a map

$$\varepsilon'^* : \Omega^*(Z) \to \Omega^*(|N\mathcal{U}|),$$

so we get the following commutative diagram of homology isomorphisms:

$$\begin{array}{ccc} \Omega^n(Z) & \xrightarrow{\varepsilon'^*} & \Omega^n(|N\mathcal{U}|) \\ & \searrow^{\varepsilon^*} & \downarrow I_\Delta \\ & & \check{\Omega}^n(\mathcal{U}) \end{array}$$

We need a notion of integral simplicial forms in order to imitate the construction in the previous section.

Definition 2.2.5. A form $\omega \in \Omega^*(|N\mathcal{U}|)$ is called *discrete* if it is a pull-back of a form on the discrete nerve $N_d\mathcal{U}$. Furthermore a discrete form is called *integral* if $I_\Delta(\omega) \in \check{C}^*(\mathcal{U}, \mathbb{Z})$. It is easy to see that the integral forms form a subcomplex and we denote this by $\Omega^*_{\mathbb{Z}}(|N\mathcal{U}|)$.

Proposition 2.2.6. 1. *We have*

$$H^n(\Omega^*_{\mathbb{Z}}(|N\mathcal{U}|)) \cong H^n(\check{C}^*(\mathcal{U}, \mathbb{Z})) = H^n(Z, \mathbb{Z}).$$

2. *If we define*

$$\Omega^*_{\mathbb{R}/\mathbb{Z}}(|N\mathcal{U}|) = \Omega^*(|N\mathcal{U}|)/\Omega^*_{\mathbb{Z}}(|N\mathcal{U}|)$$

then

$$H^n(\Omega^*_{\mathbb{R}/\mathbb{Z}}(|N\mathcal{U}|)) \cong H^n(\check{\Omega}^*_{\mathbb{R}/\mathbb{Z}}(\mathcal{U})) \cong H^n(Z, \mathbb{R}/\mathbb{Z}).$$

3. *I_Δ induces an isomorphism from the cohomology of the sequence*

$$\Omega^{l-2}_{\mathbb{R}/\mathbb{Z}}(|N\mathcal{U}|) \xrightarrow{d} \Omega^{l-1}_{\mathbb{R}/\mathbb{Z}}(|N\mathcal{U}|) \xrightarrow{d} \Omega^{l}_{\mathbb{R}/\mathbb{Z}}(|N\mathcal{U}|)/\varepsilon^*\Omega^l(Z) \tag{2.5}$$

to $H^l_{\mathcal{D}}(Z,\mathbb{Z})$.

Proof. 1. The map I_Δ takes integral forms to integral cochains by definition. It induces an isomorphism in cohomology, since the map E takes integral cochains to integral forms, and since the chain homotopies from id to $E \circ I_\Delta$ given in [4, ch. 2] are easily seen to map integral forms to integral forms.

2. This follows from the long exact sequences in cohomology of the short exact sequences

$$0 \to \Omega^*_{\mathbb{Z}}(|N\mathcal{U}|) \to \Omega^*(|N\mathcal{U}|)^* \to \Omega^*_{\mathbb{R}/\mathbb{Z}}(|N\mathcal{U}|) \to 0$$

and

$$0 \to \check{C}^*(\mathcal{U},\mathbb{Z}) \to \check{\Omega}^*(\mathcal{U})^* \to \check{\Omega}^*_{\mathbb{R}/\mathbb{Z}}(\mathcal{U}) \to 0$$

together with 1. and the 5-lemma.

3. Since the cohomology group of (2.5) fits into a short exact sequence analogous to (2.4), the 5-lemma gives us that I_Δ is an isomorphism. ☐

Corollary 2.2.7. *Every class in $H^l_{\mathcal{D}}(Z,\mathbb{Z})$ can be represented by an $(l-1)$-gerbe θ with connection ω, where $\omega = I_\Delta(\Lambda)$ for a $\Lambda \in \Omega^{l-1}(|N\mathcal{U}|)$ with*

$$d\Lambda = \varepsilon^*\alpha - \beta, \quad \alpha \in \Omega^l(Z), \quad \beta \in \Omega^l_{\mathbb{Z}}(|N\mathcal{U}|). \tag{2.6}$$

2.3 Cheeger-Simons differential characters

Cheeger-Simons differential characters were originally introduced in [3] in order to refine the Chern-Weil construction of characteristic classes. We will return to this in section 7.1. For now we will give the basic definitions and see that this is just another way of looking at smooth Deligne cohomology.

Let $C^s_*(Z)$ be the chain complex of smooth singular chains on Z and let $Z^s_*(Z) \subseteq C^s_*(Z)$ denote the smooth cycles. Then we have

Definition 2.3.1. A *differential character* (of degree p) is a pair

$$(h,\omega) \in \mathrm{Hom}(Z_{p-1}(Z), \mathbb{R}/\mathbb{Z}) \times \Omega^p(Z)$$

so that

$$h(\partial\sigma) \equiv \int_\sigma \omega \mod \mathbb{Z}.$$

The group of Cheeger-Simons differential characters of degree p is denoted by $\check{H}^p(Z,\mathbb{Z})$.

It is useful to have a cochain model for the differential characters, that is a chain complex of which the cohomology group is the group of differential characters. Such a cochain model was given in Hopkins-Singer [16] (actually the existence of such a cochain-model is briefly noted already in Esnault [10]). Let

$$C^q(p)(Z) = \begin{cases} \Omega^q(Z) \times C^{q-1}(Z,\mathbb{R}) \times C^q(Z,\mathbb{Z}) & q \geq p \\ C^{q-1}(Z,\mathbb{R}) \times C^q(Z,\mathbb{Z}) & q < p \end{cases} \tag{2.7}$$

where the differential is given by

$$d(\omega, h, c) = (d\omega, \delta h + c - \omega, \delta c)$$

for $q \geq p$ and

$$d(h,c) = \begin{cases} (0, \delta h + c, \delta c) & q = p-1 \\ (\delta h + c, \delta c) & q < p-1 \end{cases}$$

for $q < p$.

It is not hard to see that there is an isomorphism $H^p(C^*(p)(Z)) \cong \check{H}^p(Z,\mathbb{Z})$ given by the map

$$[(\omega, h, c)] \to (\tilde{h}, \omega),$$

where $\tilde{h} : Z_{p-1}(Z) \to \mathbb{R}/\mathbb{Z}$ is the map induced from h.

For $p \neq q$ we have, as in remark 2.1.2 on smooth Deligne cohomology, that the group $H^q(C^*(p)(Z))$ is an ordinary cohomology group.

$\check{H}^p(Z,\mathbb{Z})$ fits into the exact sequence

$$0 \to H^{p-1}(Z, \mathbb{R}/\mathbb{Z}) \to \check{H}^p(Z,\mathbb{Z}) \to \Omega^p_{\text{cl}}(Z) \to 0 \tag{2.8}$$

where the first map is given by $[h] \mapsto [(\delta h, h, 0)]$ and the second by $[(c, h, \omega)] \mapsto \omega$. Given this it is not surprising that we have

Theorem 2.3.2.

$$\check{H}^p(Z,\mathbb{Z}) \cong H^p_{\mathcal{D}}(Z,\mathbb{Z}).$$

Proof. Since the two cohomology groups fit into analogous exact sequences, we only need to come up with a map

$$H^p_{\mathcal{D}}(Z,\mathbb{Z}) \to \check{H}^p(Z,\mathbb{Z})$$

compatible with the short exact sequences, then the 5-lemma will do the work. This is done in [7], but we repeat it here since we need the explicit construction of this map later on.

First let

$$\check{C}_{p,q}(\mathcal{U}) = \bigoplus_{i_0,\dots,i_p} C^s_q(U_{i_0 \dots i_p})$$

be the Čech double complex of smooth singular chains. Here the horizontal differential δ is the Čech differential ,and the vertical differential ∂ is the usual differential on simplices. Then as usual the differential in the total complex $\check{C}_*(\mathcal{U})$ is given by

$$D\tau = \delta\tau + (-1)^p \partial\tau, \quad \tau \in \check{C}_{p,q}(\mathcal{U}).$$

Let $C_*^{\mathcal{U}}(Z) \subseteq C_*^s(Z)$ be the subcomplex generated by simplices with support in some $U \in \mathcal{U}$, i.e. to every simplex $\tau \in C_*^{\mathcal{U}}(Z)$ there is a $U \in \mathcal{U}$ so that $\mathrm{im}\tau \subseteq U$. The map $\varepsilon : \bigsqcup U_i \to Z$ induces a chain map

$$\varepsilon_* : \check{C}_p(\mathcal{U}) \to \check{C}_{0,p}(\mathcal{U}) \to C_p^{\mathcal{U}}(Z),$$

and there is a chain map

$$j : C_*^{\mathcal{U}}(Z) \to \check{C}_*(\mathcal{U})$$

so that $\varepsilon_* \circ j = \mathrm{id}$ and $j \circ \varepsilon_*$ is chain homotopic to id.

Let us first see that in order to determine a differential character $(h, \alpha) \in \check{H}^p(Z, \mathbb{Z})$, it is enough to know $h_{|Z_{p-1}^{\mathcal{U}}(Z)}$, that is to know what values it take on cycles generated by simplices with support in $\mathcal{U}$. From the usual proof of the excision theorem for singular homology, we have chain maps

$$p : C_*(Z) \to C_*^{\mathcal{U}}(Z) \text{ and } i : C_*^{\mathcal{U}}(Z) \to C_*(Z)$$

and a chain homotopy s so that

$$\mathrm{id} - i \circ p = \partial s + s\partial.$$

Now for a cycle σ we have

$$h(\sigma) - h(i \circ p(\sigma)) = h(\partial s \sigma) \equiv \langle I(\alpha), s\sigma \rangle$$

as wanted.

We can now use j to define a map $j_* : H_{\mathcal{D}}^p(Z, \mathbb{Z}) \to \check{H}^p(Z, \mathbb{Z})$ as follows. Take $[\omega] \in H_{\mathcal{D}}^p(Z, \mathbb{Z})$ then let $\alpha = (\varepsilon^*)^{-1} d\omega_0$ and define $h : Z_{p-1}(Z) \to \mathbb{R}/\mathbb{Z}$ by $h(\sigma) = \langle I(\omega), j(\sigma) \rangle$, where $I : \check{\Omega}^{p,q}(\mathcal{U}) \to \check{C}^{p,q}(\mathcal{U})$ is the de Rham map. It is seen that (h, α) is actually a differential character, and we can now define $j_*([\omega]) = (h, \alpha)$. This is independent of the choice of j and ω. □

Remark 2.3.3. 1. One could construct a map on the chain level, but this would include even more choices, since in order to construct a cochain (c, h, ω), the argument above would only determine c and h on cycles with support in $\mathcal{U}$, so there is some ambiguity in lifting them to cochains - this is of course not seen in cohomology.

2. Note that in the case $p = 2$ the above theorem combined with theorem 2.1.5 simply states that the isomorphism class of a circle bundle with connection is determined by its curvature and holonomy. We can think of the theorem as a generalisation of this.

It is not that hard to describe a choice of chain map j explicitly, and we include it here for later use.

Let $S_p^{\mathcal{U}}(Z)$ be the set of p-simplices with support in $\mathcal{U} = \{U_i\}_{i \in I}$. Then we can choose a map $\alpha : S_p^{\mathcal{U}}(Z) \to I$ so that $\mathrm{im}(\tau) \subseteq U_{\alpha(\tau)}$. If we denote a $\tau \in C_q(U_{i_0 \dots i_p})$ by $\tau_{i_0 \dots i_p}$ to emphasise that it maps to $U_{i_0 \dots i_p}$, we have a map

$$s : \check{C}_{p,q}(\mathcal{U}) \to \check{C}_{p+1,q}(\mathcal{U}),$$

given by

$$s(\tau_{i_0\ldots i_p}) = (-1)^{p+1}\tau_{i_0\ldots i_p\alpha(\tau)}.$$

It is not hard to see that

$$\delta s + s\delta = \mathrm{id}.$$

Note that this implies that the rows in the complex $\check{C}_{*,*}(\mathcal{U})$ are exact. We are now ready to construct j. Let $j(\tau)_i$ be the term in $\check{C}_{p-i,i}(\mathcal{U})$. Start by setting $j(\tau)_p = \tau_{\alpha(\tau)}$, then in order to make j into a chain map we have to set $j(\tau)_{p-1} = s\partial j(\tau)_p \in \check{C}_{1,p-1}(\mathcal{U})$ and in general $j(\tau)_{i-1} = (-1)^{p-i+1}s\partial j(\tau)_i$. In order to give a closed formula for $j(\tau)_i$, we need some notation.

The set of flags of length i of a simplex $\tau \in S_p(Z)$ is

$$F(\tau, i) = \{(\tau_{p-i}, \ldots \tau_p) \mid \tau_j \text{ is a face of } \tau_{j+1}, \tau_p = \tau\}.$$

The simplex τ_j in a flag $(\tau_j, \ldots, \tau_p)$ has an induced orientation from the flag. That is τ_{p-1} has an induced orientation from τ_p and so on along the flag. Let $\mathrm{sgn}(\tau_j)$ denote whether or not this orientation coincides with the standard orientation.

We can now write

$$j(\tau)_{p-i} = \sum_{\bar{\tau}\in F(\tau,i)} \mathrm{sgn}(\tau_{p-i})(\tau_{p-i})_{\alpha(\tau_p)\ldots\alpha(\tau_{p-i})}.$$

This formula will be useful in section 5.2 in our analysis of the integration map constructed by Gomi-Terashima in [14].

Chapter 3

Prism complexes

Now we leave smooth Deligne cohomology for a moment in order to introduce the concept of a prism complex. It is a generalisation of a simplicial set (or manifold) well suited for fibre bundles, which is needed in subsequent chapters.

3.1 Definition and first examples

A *prism complex* $P = \{P_p\}$ is a collection of multi-simplicial sets satisfying the following: Each P_p is a $(p+1)$-simplicial set, that is for each set of positive integers $(q_0, \ldots, q_p)$ we have sets $P_{p,q_0\ldots q_p}$ with face and degeneracy maps

$$d^i_j : P_{p,q_0\ldots q_p} \to P_{p,q_0\ldots q_i-1\ldots q_p}$$

and

$$s^i_j : P_{p,q_0,\ldots,q_p} \to P_{p,q_0\ldots q_i+1\ldots q_p}$$

for each $i = 0, \ldots, p$ and $j = 0, \ldots q_i$. These maps satisfy the relations

$$\begin{aligned}
d^i_j \circ d^i_{j'} &= d^i_{j'-1} \circ d^i_j & & j < j' \\
s^i_j \circ s^i_{j'} &= s^i_{j'+1} \circ s^i_j & & j \leq j' \\
d^i_j \circ s^i_{j'} &= \begin{cases} s^i_{j'-1} \circ d^i_j & \\ \text{id} & \\ s^i_{j'} \circ d^i_{j-1} & \end{cases} & & \begin{array}{l} j < j' \\ j = j', j = j'+1 \\ j > j'+1 \end{array}
\end{aligned}$$

and s^i_j and d^i_j commute with $s^{i'}_{j'}$ and $d^{i'}_{j'}$ for $i \neq i'$.

Furthermore, we want another set of simplicial (i.e. commuting with the d^i_j's and s^i_j's) face maps

$$d_i : P_{p,q_0\ldots q_p} \to P_{p-1,q_0\ldots\hat{q_i}\ldots q_p}$$

and degeneracy maps

$$s_i : P_{p,q_0\ldots q_p} \to P_{p+1,q_0,\ldots q_i q_i\ldots q_p}$$

so that (P_p, d_i, s_i) becomes an ordinary simplicial set. Note that in some applications the last set of degeneracy maps does not exist naturally so in these cases

(P_p, d_i) is only a Δ-set. As with ordinary simplicial sets we can for each p form the geometric and fat realisations $|P_p|$ and $||P_p||$, that is, the quotients of

$$\coprod_{q_0\cdots q_p} \Delta^{q_0} \times \cdots \times \Delta^{q_p} \times P_{p,q_0\ldots q_p}$$

where we divide out by the equivalence relations generated by the face and degeneracy maps

$$\varepsilon^i_j : \Delta^{q_0\cdots q_i\cdots q_p} \to \Delta^{q_0\cdots q_i+1\ldots q_p}$$

and (in case of the geometric realisation)

$$\eta^i_j : \Delta^{q_0\cdots q_i\cdots q_p} \to \Delta^{q_0\cdots q_i-1\ldots q_p}$$

(where $\Delta^{q_0\cdots q_p}$ is short hand notation for the prism $\Delta^{q_0} \times \cdots \times \Delta^{q_p}$).

The face and degeneracy maps d_i and s_i now induce a structure of a simplicial set on $|P_p|$ ($||P_p||$) by acting as the projection and the diagonal on $\Delta^{q_0} \times \cdots \times \Delta^{q_p}$ respectively. That is let $\pi_i : \Delta^{q_0\cdots q_p} \to \Delta^{q_0\cdots\hat{q}_i\cdots q_p}$ be the projection that deletes the i'th coordinate and let $\Delta_i : \Delta^{q_0\cdots q_p} \to \Delta^{q_0\cdots q_i q_i\cdots q_p}$ be the diagonal map that repeats the i'th factor. Then we can form the geometric realisation

$$|P.| = \coprod_{p\geq 0} \Delta^p \times |P_p| / \sim$$

where the equivalence relation is generated by

$$(\varepsilon_i t, s, x) \sim (t, \pi_i s, d_i x), \quad t \in \Delta^{p-1},\ s \in \Delta^{q_0\cdots q_p},\ x \in P_{p,q_0\ldots q_p}$$

and

$$(\eta_i t, s, x) \sim (t, \Delta_i s, s_i x), \quad t \in \Delta^{p+1},\ s \in \Delta^{q_0\cdots q_p},\ x \in P_{p,q_0\ldots q_p}.$$

The above definition might seem a bit complicated, so we will give some examples, in which this structure arises naturally. We start with an example that originally motivated the definition.

Example 3.1.1. Given a smooth fibre bundle $\pi : Y \to Z$ with $\dim Y = m + n$, $\dim Z = m$ and compact fibres, possibly with boundary, a theorem of Johnson [18] gives us smooth triangulations K and L of Y and Z respectively and a simplicial map $\pi' : K \to L$ so that the following diagram commutes

$$\begin{array}{ccc} |K| & \xrightarrow{\cong} & Y \\ \downarrow{\scriptstyle |\pi'|} & & \downarrow{\scriptstyle \pi} \\ |L| & \xrightarrow{\cong} & Z \end{array}$$

Here the horizontal maps are homeomorphisms which are smooth on each simplex. Furthermore given such a triangulation of $\partial Y \to Z$, we can also extend it to a triangulation of $Y \to Z$.

Now the geometric idea is that if $z \in Z$ lies in the interior of a p-simplex of L then the fibre over z is in a canonical way decomposed into $p+1$-fold prisms of the form $\Delta^{q_0\cdots q_p}$ as above (see fig. 3.1).

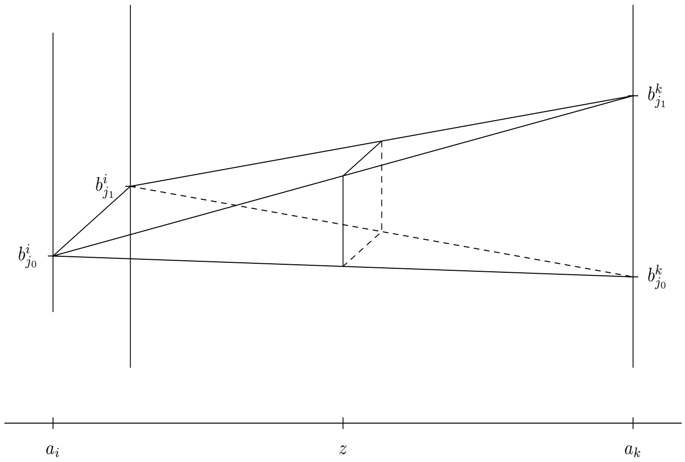


Figure 3.1: A prism in the fibre $\pi^{-1}(z)$.

Formally we define the prismatic complex $PS(K/L)$ by letting

$$PS_p(K/L)_{q_0 \dots q_p} \subseteq S_{p+q_0+\dots+q_p}(K) \times S_p(L)$$

be the subset of pairs of simplices (τ, η) so that $q_i + 1$ of the vertices in τ lies over the i'th vertex in η. Then we have face and degeneracy operators defined in the obvious way. In particular this gives us boundary maps in the fibre direction of the associated chain complex $PC_p(K/L)$ generated by such pairs of simplices,

$$\partial_F^i : PC_p(K/L)_{q_0 \dots q_p} \to PC_p(K/L)_{q_0 \dots q_i - 1 \dots q_p}$$

defined by $\partial_F^i = \sum (-1)^j d_j^i$, ($\partial_F^i = 0$ for $q_i = 0$), and also a total boundary map along the fibre

$$\partial_F = \partial_F^0 + (-1)^{q_0+1} \partial_F^1 + \dots + (-1)^{q_0+\dots+q_{p-1}+p} \partial_F^p.$$

Also there is a horisontal boundary map

$$\partial_H = \partial_0 + (-1)^{q_0+1} \partial_1 + \dots + (-1)^{q_0+\dots+q_{p-1}+p} \partial_p,$$

where

$$\partial_i = \begin{cases} 0 & \text{if } q_i > 0 \\ d_i & \text{if } q_i = 0 \end{cases}$$

so that $\partial = \partial_F + \partial_H$ is a boundary map in the total complex $PC_*(K/L)$. This is actually the cellular chain complex for the geometric realisation and hence calculates the homology of Y.

There is a natural *prismatic triangulation* homeomorphism

$$\ell : |PS(K/L)| \xrightarrow{\cong} |K|$$

induced by

$$\ell(t, s^0, \dots, s^p, (\tau, \eta)) = (t_0 s^0, \dots, t_p s^p, \tau)$$

for $(t,s,\tau) \in \Delta^p \times \Delta^{q_0\cdots q_p} \times PS_p(K/L)_{q_0\ldots q_p}$. Note that if $\overset{\circ}{\sigma}$ is an open p-simplex in L then ℓ provides a natural trivialisation of $|K|$ over $\overset{\circ}{\sigma}$

$$\overset{\circ}{\sigma} \times |PS_p(K/\sigma)| \overset{\cong}{\to} |K|_{|\sigma}$$

Example 3.1.2. Another example in the category of manifolds comes from the nerve of compatible open coverings of the total space and the base space. That is, given a covering $\mathcal{U} = \{U_i\}$ of Z we have a covering $\mathcal{W} = \{W_i = \pi^{-1}(U_i)\}$ of Y, and for each i, $\mathcal{V}^i$ is an open cover of W_i. This gives a covering $\mathcal{V} = \cup\mathcal{V}^i$ of Y (with lexicographically ordered index set). Then we put

$$P_pN(\mathcal{V}/\mathcal{U})_{q_0\ldots q_p} = \bigsqcup V^{i_0}_{j^0_0} \cap \cdots \cap V^{i_0}_{j^0_{q_0}} \cap \cdots \cap V^{i_p}_{j^p_{q_p}}$$

with $V^i_j \in \mathcal{V}^i$, and face and degeneracy maps are inclusions similarly to the simplicial case in section 2.2. In the following, we will denote $V^{i_0}_{j^0_0} \cap \cdots \cap V^{i_p}_{j^p_{q_p}}$ by $V_{j^0_0\ldots j^p_{q_p}}$.

A useful special case of this situation occurs in the context of example 3.1.1 above with the coverings consisting of the (open) stars of the triangulations of K and L. More precisely $\mathcal{U} = \{U_i = \mathrm{st}(a_i)\}$ where $a_i \in L^0$ is a 0-simplex in L and $\mathcal{V}^i = \{V^i_j = \mathrm{st}(b^i_j)\}$ where $b^i_j \in \pi^{-1}(a_i) \cap K^0$. If we define the discrete prismatic nerve in the same way as the discrete simplicial nerve in example 2.2.2, then we note that the discrete prismatic nerve of this covering is nothing but $PS(K/L)$.

3.2 Prismatic forms

As a straightforward generalisation of simplicial forms, we introduce the complex of (normal) prismatic forms on the prism complex in example 3.1.2 above.

Definition 3.2.1. A *prismatic n-form* is a collection $\omega = \{\omega_{q_0\ldots q_p}\}$ of forms $\omega_{q_0\ldots q_p} \in \Omega^n(\Delta^p \times \Delta^{q_0\cdots q_p} \times P_pN\mathcal{V}/\mathcal{U}_{q_0\ldots q_p})$ satisfying the relations

$$(\mathrm{id} \times \varepsilon^i_j \times \mathrm{id})^*\omega_{q_0\ldots q_p} = (\mathrm{id} \times \mathrm{id} \times d^i_j)^*\omega_{q_0\ldots q_i-1\ldots q_p}$$

and

$$(\varepsilon_i \times \mathrm{id} \times \mathrm{id})^*\omega_{q_0\ldots q_p} = (\mathrm{id} \times \pi_i \times d_i)^*\omega_{q_0\ldots\hat{q_i}\ldots q_p}.$$

A form is called *normal* if it also satisfies the relations

$$(\mathrm{id} \times \eta^i_j \times \mathrm{id})^*\omega_{q_0\ldots q_i-1\ldots q_p} = (\mathrm{id} \times \mathrm{id} \times s^i_j)^*\omega_{q_0\ldots q_p}$$

and

$$(\eta_i \times \mathrm{id} \times \mathrm{id})^*\omega_{q_0\ldots q_p} = (\mathrm{id} \times \Delta_i \times s_i)^*\omega_{q_0\ldots q_i q_i\ldots q_p}.$$

The complex of normal prismatic forms is denoted by

$$\Omega^*(|PN\mathcal{V}/\mathcal{U}|).$$

As in the simplicial case we have a direct sum decomposition of this complex

$$\begin{aligned}\Omega^n(|PN\mathcal{V}/\mathcal{U}|) &= \bigoplus_{p+q+r=n} \Omega^{p,q,r}(|PN\mathcal{V}/\mathcal{U}|)\\ &= \bigoplus_{p+q_0+\cdots+q_p+r=n} \Omega^{p,q_0,\dots,q_p,r}(|PN\mathcal{V}/\mathcal{U}|),\end{aligned}$$

where $\Omega^{p,q_0,\dots,q_p,r}(|PN\mathcal{V}/\mathcal{U}|)$ is the set of forms of degree p in the barycentric coordinates of the first simplex, of degree q_0 in the second and so on and finally of degree r in some local coordinates on the nerve of the covering. This makes $\Omega^*(|PN\mathcal{V}/\mathcal{U}|)$ into a triple complex. There is also a corresponding Čech-de Rham triple complex

$$\check{\Omega}^{p,q,r}(\mathcal{V}/\mathcal{U}) = \bigoplus_{q_0+\cdots+q_p=q} \Omega^r(P_pN\mathcal{V}/\mathcal{U}_{q_0,\dots,q_p})$$

with differentials

$$\begin{aligned}\partial' &: \check{\Omega}^{p,q,r}(\mathcal{V}/\mathcal{U}) \to \check{\Omega}^{p+1,q,r}(\mathcal{V}/\mathcal{U})\\ \partial'' &: \check{\Omega}^{p,q,r}(\mathcal{V}/\mathcal{U}) \to \check{\Omega}^{p,q+1,r}(\mathcal{V}/\mathcal{U})\\ \partial''' &: \check{\Omega}^{p,q,r}(\mathcal{V}/\mathcal{U}) \to \check{\Omega}^{p,q,r+1}(\mathcal{V}/\mathcal{U})\end{aligned}$$

Here $\partial' = \sum(-1)^i\partial_i'$ where

$$\partial_i'\alpha_{|j_0^0\cdots j_{q_{p+1}}^{p+1}} = \begin{cases} 0 & \text{if } q_i > 0\\ \alpha_{|j_0^0\cdots\hat{j}_0^i\cdots j_{q_{p+1}}^{p+1}} & \text{if } q_i = 0\end{cases}$$

∂'' and ∂''' are usual Čech and de Rham differentials.

As in the simplicial case, we have

Proposition 3.2.2. *The map*

$$I_\Delta : \Omega^{p,q,r}(|PN\mathcal{V}/\mathcal{U}|) \to \check{\Omega}^{p,q,r}(\mathcal{V}/\mathcal{U})$$

given by

$$I_\Delta(\omega) = \int_{\Delta^p\times\Delta^{q_0\cdots q_p}} \omega_{q_0\dots q_p}, \quad \textit{for } \omega \in \Omega^{p,q_0,\dots,q_p,r}(|PN\mathcal{V}/\mathcal{U}|)$$

induces an isomorphism in cohomology. The right inverse is given on $\Delta^{k_0\cdots k_p} \times P_pN\mathcal{V}/\mathcal{U}_{k_0\dots k_p}$ by

$$E(\omega) = p!q_0!\cdots q_p! \sum_{|J|=p}\sum_{|J_0|=q_0}\cdots\sum_{|J_p|=q_p} \omega_J\wedge\omega_{J_0}\wedge\cdots\wedge\omega_{J_p}\wedge d^*_{J_0\cdots J_p}\omega.$$

The ω_{J_j}'s are the elementary forms on Δ^{q_j}, and $d_{J_0\cdots J_p}$ are face maps as in the simplicial case.

Proof. The proof is the same as in the simplicial case (see e.g. [4, ch. 2]). □

Proposition 3.2.3. *The natural map $\sqcup V_j^i \to W_i$ induces the maps ε_1^* and ε_2^* in the following commutative diagram*

$$\begin{array}{ccc} \Omega^n(|N\mathcal{W}|) & \xrightarrow{\varepsilon_1^*} & \Omega^n(|PN\mathcal{V}/\mathcal{U}|) \\ \downarrow{\scriptstyle I_\Delta} & & {\scriptstyle I_\Delta}\downarrow \\ \check{\Omega}^n(\mathcal{W}) & \xrightarrow{\varepsilon_2^*} & \check{\Omega}^n(\mathcal{V}/\mathcal{U}) \end{array}$$

Both of these maps induce isomorphisms in cohomology.

Proof. Since both vertical maps induce isomorphisms in cohomology, it is enough to see that also ε_2^* induces an isomorphism. This map factors as

$$\check{\Omega}^n(\mathcal{W}) \to \check{\Omega}^n(\mathcal{V}) \to \check{\Omega}^n(\mathcal{V}/\mathcal{U}).$$

The first of these maps is induced by a refinement map, so it induces an isomorphism in cohomology. The only difference between the middle and the rightmost complex is that on the right side we have split the Čech differential in two different maps, and this is not seen in the total complex, so the cohomology of these complexes is isomorphic. □

Remark 3.2.4. The above result could also have been obtained by showing directly that ε_1^* induces an isomorphism. In section 4.1, we construct a right inverse ϕ to ε'. Given the construction of ϕ it will be easy to see that we have a linear homotopy $\phi \circ \varepsilon' \sim \mathrm{id}$ which in turn gives a chain homotopy directly on $\Omega^*(|PN\mathcal{V}/\mathcal{U}|)$.

Chapter 4

Integration of simplicial forms

In this chapter, we will construct an integration map for simplicial forms and see that it induces a well-defined map in smooth Deligne cohomology. The first construction given in section 4.1 is quite simple, and the only thing that turns out to be difficult is to show that the map takes integral forms to integral forms. For the proof of this fact we will need another more combinatorial integration map which we construct in section 4.3.

4.1 The integration map

Again, let $\pi : Y \to Z$ be a fibre bundle with compact oriented n-dimensional fibre and m-dimensional base. Then with notation as in example 3.1.2, we want to define an integration map

$$\int : \Omega^{*+n}(|N\mathcal{V}|) \to \Omega^*(|N\mathcal{U}|),$$

for coverings $\mathcal{U}$ and $\mathcal{V}$ coming from compatible triangulations. To do so, we define a map $|N\mathcal{W}| \to |N\mathcal{V}|$ (recall that $\mathcal{W} = \pi^{-1}(\mathcal{U})$), and then our integration is given by pulling back forms by this map and then integrating along the fibre in $|N\mathcal{W}| \to |N\mathcal{U}|$. We define the map in two steps. First we have, similar to the 'prismatic triangulation' map in example 3.1.1, a map $\ell : |PN\mathcal{V}/\mathcal{U}| \to |N\mathcal{V}|$, where

$$\ell : \Delta^p \times \Delta^{q_0 \cdots q_p} \times V_{j_0^0 \cdots j_{q_p}^p} \to \Delta^{p+q_0+\cdots+q_p} \times V_{j_0^0 \cdots j_{q_p}^p}$$

is given by

$$\ell(t, s^0, \ldots, s^p, x) = (t_0 s^0, \ldots, t_p s^p, x).$$

Now recall that each W_i is covered by $\mathcal{V}^i = \{\mathrm{st}(b_j^i)\}_{j \in J_i}$. Choose partitions of unity $\{\phi_j^i\}$ on W_i subordinate $\mathcal{V}^i$ for each i. We are now ready to define

$$\tilde{\phi} : |N\mathcal{W}| \to |PN\mathcal{V}/\mathcal{U}|$$

on $\Delta^p \times W_{i_0 \dots i_p}$. Take $x \in W_{i_0 \dots i_p}$ then for each $i = i_0, \dots, i_p$ there is a minimal set $\{j_0^i, \dots j_{q_i}^i\} \in J_i$ so that

$$\sum_{r=0}^{q_i} \phi^i_{j^i_r}(x) = 1.$$

We then map

$$(t, x) \in \Delta^p \times W_{i_0 \dots i_p}$$

to

$$(t, \phi^{i_0}_{j^0_0}(x), \dots, \phi^{i_0}_{j^0_{q_0}}(x), \dots, \phi^{i_p}_{j^p_{q_p}}(x), x) \in \Delta^p \times \Delta^{q_0 \dots q_p} \times V_{j^0_0 \dots j^p_{q_p}}.$$

Remark 4.1.1. Note that since the covering $\mathcal{V}$ comes from a triangulation it has covering dimension $n + m$. This implies that we have $q = \sum q_i \leq n$ for non-degenerate simplices. This observation will be important in some of the proofs in the end of this chapter.

Now for $\omega \in \Omega^{n+k}(|N\mathcal{V}|)$ define $\int_{[Y/Z]} \omega \in \Omega^k(|N\mathcal{U}|)$ by

$$\left(\int_{[Y/Z]} \omega \right)_{|\Delta^p \times U_{i_0 \dots i_p}} = \int_{\Delta^p \times W_{i_0 \dots i_p} / \Delta^p \times U_{i_0 \dots i_p}} \tilde{\phi}^* \ell^* \omega,$$

where the right hand side denotes usual integration along the fibres.

Theorem 4.1.2. *Given triangulations and partitions of unity as above, the following holds.*
1. *Let $\omega \in \Omega^{*+n}(|N\mathcal{V}|)$ be a normal simplicial form, then $\int_{[Y/Z]} \omega$ is a well-defined normal simplicial form.*
2. *For $\omega \in \Omega^{*+n-1}(|N\mathcal{V}|)$ we have*

$$\int_{[Y/Z]} d\omega = \int_{[\partial Y/Z]} \omega + (-1)^n d \int_{[Y/Z]} \omega.$$

Proof. 1. It is clear that $\int_{[Y/Z]} \omega$ is a well-defined simplicial form i.e. is compatible with respect to the degeneracy operators. Let us see that it is normal, that is

$$(\eta_j \times \mathrm{id})^* \left(\int_{[Y/Z]} \omega \right)^{(p)} = (\mathrm{id} \times s_j)^* \left(\int_{[Y/Z]} \omega \right)^{(p+1)}.$$

We first notice that

$$\begin{aligned}
(\eta_j \times \mathrm{id})^* \left(\int_{[Y/Z]} \omega \right)_{|\Delta^p \times U_{i_0 \dots i_p}} &= (\eta_j \times \mathrm{id})^* \int_{\Delta^p \times W_{i_0 \dots i_p} / \Delta^p \times U_{i_0 \dots i_p}} \tilde{\phi}^* \ell^* \omega \\
&= \int_{\Delta^{p+1} \times W_{i_0 \dots i_p} / \Delta^{p+1} \times U_{i_0 \dots i_p}} (\eta_j \times \mathrm{id})^* \tilde{\phi}^* \ell^* \omega \\
&= \int_{\Delta^{p+1} \times W_{i_0 \dots i_p} / \Delta^{p+1} \times U_{i_0 \dots i_p}} \tilde{\phi}^* (\eta_j \times \mathrm{id})^* \ell^* \omega \\
&= \int_{\Delta^{p+1} \times W_{i_0 \dots i_p} / \Delta^{p+1} \times U_{i_0 \dots i_p}} \tilde{\phi}^* (\ell \circ (\eta_j \times \mathrm{id}))^* \omega
\end{aligned}$$

and at the same time

$$
\begin{aligned}
(\mathrm{id}\times s_j)^*&\left(\int_{[Y/Z]}\omega\right)_{|\Delta^{p+1}\times U_{i_0\dots i_j i_j\dots i_p}}\\
&= (\mathrm{id}\times s_j)^*\int_{\Delta^{p+1}\times W_{i_0\dots i_j i_j\dots i_p}/\Delta^{p+1}\times U_{i_0\dots i_j i_j\dots i_p}}\tilde\phi^*\ell^*\omega\\
&= \int_{\Delta^{p+1}\times W_{i_0\dots i_j\dots i_p}/\Delta^{p+1}\times U_{i_0\dots i_j\dots i_p}}(\mathrm{id}\times s_j)^*\tilde\phi^*\ell^*\omega\\
&= \int_{\Delta^{p+1}\times W_{i_0\dots i_p}/\Delta^{p+1}\times U_{i_0\dots i_p}}\tilde\phi^*(\mathrm{id}\times s_j)^*\ell^*\omega\\
&= \int_{\Delta^{p+1}\times W_{i_0\dots i_p}/\Delta^{p+1}\times U_{i_0\dots i_p}}\tilde\phi^*(\ell\circ(\mathrm{id}\times s_j))^*\omega.
\end{aligned}
$$

Hence we only need to show that $(\ell\circ(\mathrm{id}\times s_j))^*\omega = (\ell\circ(\eta_j\times\mathrm{id}))^*\omega$. This can be seen from the following commutative diagram

$$
\begin{array}{ccc}
\Delta^p\times\Delta^{q_0\cdots q_p}\times V_{j_0^0\cdots j_{q_p}^p} & \xrightarrow{\ \ell\ } & \Delta^{p+q}\times V_{j_0^0\cdots j_{q_p}^p}\\
\Big\uparrow{\scriptstyle \eta_j\times\mathrm{id}} & & \Big\uparrow{\scriptstyle \tilde\eta\times\mathrm{id}}\\
\Delta^{p+1}\times\Delta^{q_0\cdots q_p}\times V_{j_0^0\cdots j_{q_p}^p} & \xrightarrow{\ \tilde\ell\ } & \Delta^{p+q+q_j+1}\times V_{j_0^0\cdots j_{q_p}^p}\\
\Big\downarrow{\scriptstyle \mathrm{id}\times s_j} & & \Big\downarrow{\scriptstyle \tilde\sigma\circ(\mathrm{id}\times\tilde s)}\\
\Delta^{p+1}\times\Delta^{q_0\cdots q_j q_j\cdots q_p}\times V_{j_0^0\cdots j_0^j\cdots j_{q_j}^j j_0^j\cdots j_{q_p}^p} & \xrightarrow{\ \ell\ } & \Delta^{p+q+q_j+1}\times V_{j_0^0\cdots j_0^j\cdots j_{q_j}^j j_0^j\cdots j_{q_p}^p}
\end{array}
$$

where $q=\sum q_i$, $\tilde\ell$ is given by

$$
\begin{aligned}
\tilde\ell(t,s^0,\ \dots\ ,s^p,x)&\\
&= (t_0 s^0,\dots,t_j s_0^j, t_{j+1}s_0^j, t_j s_1^j,\dots,t_{j+1}s_{q_j}^j,\dots,t_{p+1}s^p,x),
\end{aligned}
$$

$\tilde\eta$ (and similarly for $\tilde s$) is given by

$$\tilde\eta = \eta_{q_0+\dots+q_{j-1}+j}\circ\eta_{q_0+\dots+q_{j-1}+j+2}\circ\dots\circ\eta_{q_0+\dots+q_{j-1}+j+2q_j}$$

and finally $\tilde\sigma$ is the map that permutes the vertices in the simplex as in remark 2.2.4, so that by assumption $\tilde\sigma^*\omega=\omega$.

2. Follows from the analogous formula for usual fibre integration. □

There is a map $\varepsilon' : |N\mathcal{V}|\to|N\mathcal{W}|$ induced by the natural map $\sqcup V_j^i\to W_i$ given on $\Delta^{p+q_0+\dots+q_p}\times V_{j_0^0\dots j_{q_p}^p}$ by

$$\varepsilon'(t_0^0,\dots,t_{q_0}^0,\dots,t_{q_p}^p,x) = (\sum_j t_j^0,\dots,\sum_j t_j^p,x)\in\Delta^p\times W_{i_0\dots i_p}.$$

Since ε' is left inverse to $\ell\circ\tilde\phi$ the following lemma follows easily from the construction of the integral.

Lemma 4.1.3. *The following diagrams commute*

$$\begin{array}{ccc} \Omega^{*+n}(Y) & \xrightarrow{\varepsilon^*} & \Omega^{*+n}(|N\mathcal{V}|) \\ {\scriptstyle \int_{Y/Z}}\downarrow & & \downarrow{\scriptstyle \int_{[Y/Z]}} \\ \Omega^*(Z) & \xrightarrow{\varepsilon^*} & \Omega^*(|N\mathcal{U}|) \end{array} \qquad \begin{array}{ccc} \Omega^{*+n}(|N\mathcal{W}|) & \xrightarrow{\varepsilon'^*} & \Omega^{*+n}(|N\mathcal{V}|) \\ {\scriptstyle \int_{Y/Z}}\downarrow & \swarrow{\scriptstyle \int_{[Y/Z]}} & \\ \Omega^*(|N\mathcal{U}|) & & \end{array}$$

that is, the integration of simplicial forms is compatible with the usual fibre integration.

The integration map is natural in the following sense:

Proposition 4.1.4. *Let $Y \to Z$ be a fibre bundle and $f : Z' \to Z$ be an embedding, then in the pull-back diagram*

$$\begin{array}{ccc} f^*Y & \xrightarrow{\hat{f}} & Y \\ \downarrow & & \downarrow \\ Z' & \xrightarrow{f} & Z \end{array}$$

we have

$$f^* \int_{[Y/Z]} \omega = \int_{[f^*Y/Z']} f^*\omega \in \Omega^*(|Nf^*\mathcal{U}|),$$

where the integration on the right is with respect to the pull-back cover and the induced partitions of unity on f^*Y.

Proof. The diagram

$$\begin{array}{ccccc} |Nf^*\mathcal{W}| & \xrightarrow{\tilde{\phi}_f} & |PNf^*\mathcal{V}/f^*\mathcal{U}| & \xrightarrow{\ell} & |Nf^*\mathcal{V}| \\ \downarrow{\scriptstyle \hat{f}} & & \downarrow{\scriptstyle \hat{f}} & & \downarrow{\scriptstyle \hat{f}} \\ |N\mathcal{W}| & \xrightarrow{\tilde{\phi}} & |PN\mathcal{V}/\mathcal{U}| & \xrightarrow{\ell} & |N\mathcal{V}| \end{array}$$

obviously commutes, so we get

$$\begin{aligned} \left(f^* \int_{[Y/Z]} \omega \right)_{|\Delta^p \times U_{i_0 \cdots i_p}} &= f^* \int_{\Delta^p \times W_{i_0 \cdots i_p}/\Delta^p \times U_{i_0 \cdots i_p}} \tilde{\phi}^* \ell^* \omega \\ &= \int_{\Delta^p \times f^* W_{i_0 \cdots i_p}/\Delta^p \times f^* U_{i_0 \cdots i_p}} \hat{f}^* \tilde{\phi}^* \ell^* \omega \\ &= \int_{\Delta^p \times f^* W_{i_0 \cdots i_p}/\Delta^p \times f^* U_{i_0 \cdots i_p}} \tilde{\phi}_f^* \ell^* \hat{f}^* \omega \\ &= \left(\int_{[f^*Y/Z']} \hat{f}^* \omega \right)_{|\Delta^p \times f^* U_{i_0 \cdots i_p}} . \end{aligned}$$

□

So far we have constructed an integration map for simplicial forms which is compatible with the usual fibre integration map for differential forms. In order to see that this map actually induces a map in smooth Deligne cohomology, we need to see that it takes integral forms to integral forms. This is the goal of the next sections.

4.2 The triangulated nerve

In chapter 5, we will see that Gomi-Terashima have constructed a combinatorial integration formula for smooth Deligne cohomology in the case of a product bundle. This inspired us to search for a similar combinatorial formula for simplicial forms in the hope that it would be easier to see what happened on integral forms. Such a construction is carried out in section 4.3, however, the resulting forms are only piece-wise smooth, so in this section we introduce some tools to handle this situation. It is indeed true that it is easier to see that this combinatorial map takes integral forms to integral forms. This will, in turn, solve the problem of showing that the map in section 4.1 does the same.

We start by introducing a new complex consisting of these piece-wise smooth forms.

Given a triangulation L of a smooth manifold Z we have as mentioned earlier an open cover $\mathcal{U}$ given by the stars $\mathrm{st}(a)$ where $a \in L^0$ is a 0-simplex.

For every simplex $\sigma \in L$, the closed star

$$\overline{\mathrm{st}(\sigma)} = \bigcup_{\tau \in L^m, \sigma \subseteq \tau} |\tau|$$

is the union of all top-dimensional simplices containing σ. It inherits a natural triangulation L_σ from L. This gives a realisation $|L_\sigma| \cong \overline{\mathrm{st}(\sigma)}$.

Definition 4.2.1. The *triangulated nerve* NL is the simplicial complex

$$N_p L = \bigsqcup_{\sigma \in L^p} |L_\sigma|,$$

and for $\sigma = [a_0, \dots, a_p]$ the face and degeneracy operators

$$d_j : |L_{a_0 \dots a_p}| \to |L_{a_0 \dots \hat{a_j} \dots a_p}| \text{ and } s_j : |L_{a_0 \dots a_p}| \to |L_{a_0 \dots a_j a_j \dots a_p}|$$

are given by inclusions.

Our construction will give simplicial forms on $|NL|$.

Recall that a form ω on a simplicial complex is a collection of forms $\omega = \{\omega^{(p)}\}$ with $\omega^{(p)} \in \Omega^*(\Delta^p \times N_p L)$ satisfying the relation $(\varepsilon^j \times \mathrm{id})^* \omega^{(p)} = (\mathrm{id} \times d_j)^* \omega^{(p-1)}$. But the L_σ's, $\sigma \in L^p$ are simplicial sets too, so our forms $\omega^{(p)}$ actually live on

$$\sqcup_{\sigma \in L^p} \sqcup_i \Delta^p \times \Delta^i \times L_\sigma^{(i)},$$

where $L_\sigma^{(i)}$ is the discrete set of i-simplices in L_σ.

Now much of what has been done in the previous sections carry over. We can define integral forms $\Omega^*_{\mathbb{Z}}(|NL|) \subseteq \Omega^*(|NL|)$ exactly as before and given triangulations of a fibre bundle as in example 3.1.1 we also get triangulated nerves both of the base and the total space. We can also associate a prism complex to this situation in exactly the same way as in example 3.1.2. There is obviously also a map $\ell : |PNK/L| \to |NK|$ as before.

Now let us show that with regard to cohomology it does not matter whether we use ordinary simplicial forms or simplicial forms on the triangulated nerves.

We introduce the simplicial manifold (with corners) $N\overline{\mathcal{U}}$ with

$$N_p\overline{\mathcal{U}} = \bigsqcup_{i_0,\dots,i_p} \overline{U_{i_0\dots i_p}}.$$

Since the cohomology of $\Omega^*(|N\mathcal{U}|)$ does not depend on the open cover and since forms on a closed subset are restrictions of forms on a larger open subset, we get that the restriction $\Omega^*(|N\overline{\mathcal{U}}|) \to \Omega^*(|N\mathcal{U}|)$ induces an isomorphism in cohomology.

Proposition 4.2.2. *The map*

$$\iota : \Omega^*(|N\bar{\mathcal{U}}|) \to \Omega^*(|NL|)$$

induced by the homeomorphisms

$$|L_{a_0\dots a_p}| \cong \overline{\mathrm{st}([a_0,\dots,a_p])}$$

is an isomorphism in cohomology.

Proof. The result follows readily from the following commutative diagram

$$\begin{array}{ccc} \Omega^{p,q}(|N\bar{\mathcal{U}}|) & \xrightarrow{\iota} & \Omega^{p,q}(|NL|) \\ \downarrow I_\Delta & & \downarrow I_\Delta \\ \Omega^q(N_p\overline{\mathcal{U}}) & \xrightarrow{\iota'} & \Omega^q(N_pL) \end{array}$$

since both vertical maps are isomorphisms in cohomology by the simplicial de Rham theorem. In fact, the de Rham theorem also implies that the lower horizontal map induces an isomorphism in cohomology for fixed p, since the map $\Omega^q(\overline{\mathrm{st}(\sigma)}) \to \Omega^q(|L_\sigma|)$ is a cohomology isomorphism for all $\sigma \in L$. That implies that there is an isomorphism between the two spectral sequences already at the E^1-term, so ι' induces an isomorphism in cohomology of the total complexes. □

Now let us show that we can also represent a class in Deligne cohomology by a simplicial form on a triangulated nerve.

First, let $\mathcal{U} = \{U_i\}_{i\in I}$ be a covering of Z and let L be a triangulation, so that every closed star of L lies inside an open set of $\mathcal{U}$. That is we have a map $\alpha : L^0 \to I$ so that $\overline{\mathrm{st}(a)} \subseteq U_{\alpha(a)}$. This gives a chain map

$$T : \Omega^*(|N\mathcal{U}|) \to \Omega^*(|NL|).$$

Proposition 4.2.3. *The map T induces an isomorphism both in ordinary cohomology and between the cohomology of (2.5) and*

$$\Omega^{l-1}_{\mathbb{R}/\mathbb{Z}}(|NL|) \xrightarrow{d} \Omega^{l}_{\mathbb{R}/\mathbb{Z}}(|NL|) \xrightarrow{d} \Omega^{l+1}(|NL|)/\varepsilon^*\Omega^{l+1}(Z) \tag{4.1}$$

Proof. Follows from the last proposition since T is the composition of a refinement map and ι. □

Hence the smooth Deligne cohomology group is isomorphic to the cohomology of the sequence (4.1).

4.3 The combinatorial integration map

In this section we finally define a combinatorial integration map

$$\int_{K/L} : \Omega^{*+n}(|NK|) \to \Omega^*(|NL|).$$

For this we will be needing the chain complex introduced in example 3.1.1. Here we simplify the notation a little. Let $PC_{k,l}(K/L)$ be the chain complex generated by pairs of simplices $(\tau, \eta) \in K^{(k+l)} \times L^l$ such that τ maps to η under the projection map. First, for a simplex $\sigma = [a_0, \dots, a_p] \in L$ we define a map

$$\mathrm{AW} : PC_{k,m}(K/L_\sigma) \to \bigoplus_{k_1+k_2=k} PC_{k_1,p}(K/\sigma) \otimes PC_{k_2,m}(K/L_\sigma).$$

Let $(\tau, \eta) \in PC_{k,m}(K/L_\sigma)$. Since η is a top-dimensional simplex in L_σ (recall that $\dim Z = m$) we have $\sigma \subseteq \eta$. Let $i_0, \dots, i_p \in \{0, \dots, n\}$ denote the indices of the corresponding vertices of σ in η. Let us write τ as $\tau = [b^0_0, \dots b^0_{q_0} | \dots | b^m_0, \dots, b^m_{q_m}]$, where the i'th block, $|b^i_0, \dots, b^i_{q_i}|$ consists of the $q_i + 1$ vertices in τ which lies over the i'th vertex in η. For $0 \le s_j \le q_{i_j}$ we define

$$\tau^{s_0 \cdots s_p} = [b^{i_0}_0, \dots, b^{i_0}_{s_0} | \dots | b^{i_p}_0, \dots, b^{i_p}_{s_p}] \in K^{s_0 + \cdots + s_p + p}$$

and

$$\tau_{s_0 \cdots s_p} = [b^0_0, \dots, b^0_{q_0} | \dots | b^{i_j}_{s_j}, \dots, b^{i_j}_{q_{i_j}} | \dots | b^m_0, \dots, b^m_{q_m}] \in K^{k - s_0 - \cdots - s_p}.$$

Now our map is given by

$$\mathrm{AW}(\tau, \eta) = \sum_{0 \le s_j \le q_{i_j}} (\tau^{s_0 \cdots s_p}, \sigma) \otimes (\tau_{s_0 \cdots s_p}, \eta),$$

that is, our map is an Alexander-Whitney type map with respect to each block of vertices in τ lying over a vertex in σ.

The following lemma is a straightforward computation similar to the proof of the usual AW map being a chain map.

Lemma 4.3.1. *The map*

$$\mathrm{AW}: PC_{k,m}(K/L_\sigma) \to \bigoplus_{k_1+k_2=k} PC_{k_1,p}(K/\sigma) \otimes PC_{k_2,m}(K/L_\sigma)$$

is a chain map with respect to the boundary map ∂_F from example 3.1.1, that is

$$\mathrm{AW}\partial_F = \partial_F \mathrm{AW}.$$

□

We have to specify $\int_{K/L} \omega \in \Omega^*(|NL|)$ as a form on $\Delta^p \times \eta$ for $\eta \in L_\sigma$. If, for the moment, we let η be an m-simplex the formula is quite simple.

First pick an orientation of η. Since the fibres of π are oriented, this gives us an orientation of $Y_{|\eta}$ and hence a fundamental class $[Y_{|\eta}] \in PC_{n+m}(K/\sigma)$.

Now consider $NK_{|\pi^{-1}(|L_\sigma|)}$ as a subset of $|K|_{|\sigma} \times |K|_{||L_\sigma|}$. We will define $\int_{K/L} \omega_{|\Delta^p \times \eta}$ by restricting ω to $\mathrm{AW}([Y_{|\eta}])$ and integrate along the fibre over $\Delta^p \times \eta$.

Set $s = \sum_{i=0}^{p} s_i$, then the formula is given by

$$\left(\int_{K/L} \omega \right)_{|\Delta^p \times \eta} = \sum_{\tau \in PS_{n,m}(K/\eta)} \sum_{0 \le s_j \le q_{i_j}} \varepsilon(\tau) \int_{\Delta^{p+s} \times \tau_{s_0 \cdots s_p} / \Delta^p \times \eta} \omega^{(p+s)}_{\tau^{s_0 \cdots s_p}}, \quad (4.2)$$

where $\omega^{(p+s)}_{\tau^{s_0 \cdots s_p}} \in \Omega^{*+n}(\Delta^p \times K_{\tau^{s_0 \cdots s_p}})$, and $\varepsilon(\tau)$ is the sign of τ in $[Y_{|\eta}]$. The integration shall be understood as follows: We restrict ω to $\Delta^{p+s} \times \tau_{s_0 \dots s_p}$ and then integrate it along the fibres over $\Delta^p \times \eta$ with respect to the map $\Delta^{p+s} \to \Delta^p$ given by

$$(t_0, \dots t_{p+s}) \mapsto (\sum_{i=0}^{s_0} t_i, \sum_{i=s_0+1}^{s_0+s_1+1} t_i, \dots, \sum_{i=s_0+\dots+s_{p-1}+p}^{s_0+\dots+s_p+p} t_i)$$

and the map $\tau_{s_0 \dots s_p} \to \eta$ which is just the restriction of π.

Remark 4.3.2. In the above, we could also have chosen to use ℓ to pull ω back to $|PNK/L_\sigma|$ and then integrate with respect to the map

$$\Delta^p \times \Delta^{s_0 \dots s_p} \times \tau_{s_0 \dots s_p} \to \Delta^p \times \eta.$$

This gives the same result, but will be more convenient when we shall see that the two approaches to integration give the same result.

We still need to define the integral on $\Delta^p \times \eta'$ for $\eta' \in S_k(L_\sigma)$ a lower-dimensional simplex. This will actually just be the restriction of the integral on $\Delta^p \times \eta$ for η a top-simplex such that $\eta' \subseteq \eta$. We shall see that this is independent of which top-simplex we choose (this also shows that the resulting form is really simplicial on $|L_\sigma|$).

Let us first take a look at what happens to the formula (4.2) when the integral is restricted to $\Delta^p \times \eta' \subseteq \Delta^p \times \eta$. We define the dimension in the direction of the fibre, for a simplex in K, to be $\dim_F \tau = \dim \tau - \dim \pi(\tau)$.

For a $\tau \in PS_{n,m}(K/\eta)$ we see that

$$\int_{\Delta^{p+s}\times\tau_{s_0\dots s_p}/\Delta^p\times\eta} \omega^{(p+s)}_{\tau^{s_0\dots s_p}} \tag{4.3}$$

restricted to $\Delta^p\times\eta'$ is non-zero exactly when $\tau_{s_0\dots s_p}\cap\pi^{-1}(\eta')$ and $\tau_{s_0\dots s_p}$ have the same dimension $r=n-s$ in the direction of the fibre. That is $\tau_{s_0\dots s_p}\cap\pi^{-1}(\eta')\in PS_{r,k}(K/\eta')$ and $\tau_{s_0\dots s_p}\in PS_{r,m}(K/\eta)$.

Now in this case, let α be the simplex in L_σ 'spanned' by η' and σ, then $\tau\cap\pi^{-1}(\alpha)$ is n-dimensional in the fibre direction, so over each $\tilde\eta\in S_m(L_\sigma)$, with $\eta'\subseteq\tilde\eta$, there is exactly one $\tilde\tau\in PS_{n,m}(K/\tilde\eta)$ with $\tau\cap\pi^{-1}(\alpha)\subseteq\tilde\tau$ and in the expression (4.3) it would make no difference if we used $\tilde\tau$ instead of τ.

We can also give an explicit formula in this case, but first we need some notation. For a top-simplex $\mu\in PS_{n,m}(K/L_\sigma)$ set $\tilde\mu=\mu\cap\pi^{-1}(\sigma)$. For a simplex $\rho\in PS_{r,k}(K/\eta')$ let

$$\begin{aligned} F\rho=\{\mu\in PS_{n,m}(K/L_\sigma)\ \mid\ &\rho=\mu\cap\pi^{-1}(\eta'),\\ &\dim_F\tilde\mu+\dim_F\rho-\dim_F(\tilde\mu\cap\rho)=n\}.\end{aligned}$$

Now write

$$\rho=[c^0_0,\dots,c^0_{r_0}|\dots|c^k_0,\dots,c^k_{r_k}]\text{ and }\tilde\mu=[b^0_0,\dots,b^0_{q_0}|\dots|b^p_0,\dots,b^p_{q_p}]$$

with $\mu\in F\rho$ and let $i_0,\dots,i_l\in\{0,\dots,p\}$ and $j_0,\dots,j_l\in\{0,\dots,k\}$ denote the coinciding blocks in $\tilde\mu$ and ρ, that is

$$\tilde\mu\cap\rho=[b^{i_0}_0,\dots,b^{i_0}_{q_{i_0}}|\dots|b^{i_l}_0,\dots,b^{i_l}_{q_{i_l}}]=[c^{j_0}_0,\dots,c^{j_0}_{q_{j_0}}|\dots|c^{j_l}_0,\dots,c^{j_l}_{q_{j_l}}].$$

As before we set

$$\rho_{s_0\dots s_l}=[c^0_0,\dots,c^0_{r_0}|\dots|c^{j_\nu}_{s_\nu},\dots,c^{j_\nu}_{r_{j_\nu}}|\dots|c^k_0,\dots,c^k_{r_k}]$$

and

$$\tilde\mu^{s_0\dots s_l}=[b^0_0,\dots,b^0_{q_0}|\dots|b^{i_\nu}_0,\dots,b^{i_\nu}_{s_\nu}|\dots|b^p_0,\dots,b^p_{q_p}],$$

and then finally the integration formula is given on $\Delta^p\times\eta'$ by

$$\sum_{\rho\in PS_{*,k}(K/\eta')}\ \sum_{\{\tilde\mu|\mu\in F\rho\}}\ \sum_{0\le s_\nu\le q_{i_\nu}}\varepsilon(\mu)\int_{\Delta^{p+s}\times\rho_{s_0\dots s_l}/\Delta^p\times\eta}\omega^{(p+s)}_{\tilde\mu^{s_0\dots s_l}}. \tag{4.4}$$

Theorem 4.3.3. 1. *Let $\omega\in\Omega^{*+n}(|NK|)$ be a piece-wise smooth normal simplicial form; then $\int_{K/L}\omega$ is a well-defined piece-wise smooth normal simplicial form.*

2. *Let $\omega\in\Omega^{k+n-1}(|NK|)$, then we have a Stokes' theorem*

$$\int_{K/L}d\omega=\int_{\partial_F K/L}\omega+(-1)^n d\int_{K/L}\omega$$

3. *The map $\int_{K/L}:\Omega^{*+n}(|NK|)\to\Omega^*(|NL|)$ takes integral forms to integral forms and, if $\partial Y=\emptyset$, it induces a map $\pi_!:H^{*+n}_{\mathcal D}(Y,\mathbb Z)\to H^*_{\mathcal D}(Z,\mathbb Z)$ in smooth Deligne cohomology.*

Proof. 1. This follows at once from the construction.

2. First we observe that for $\omega \in \Omega^{k+n-1}(|NK|)$ we have on $\Delta^p \times \eta$ ($\eta \in L_\sigma^{(n)}$)

$$\begin{aligned}
\int_{K/L} d\omega &= \sum_{\tau \in PS_{n,m}(K/\eta)} \sum_{0 \le s_j \le q_{i_j}} \int_{\Delta^{p+s} \times \tau_{s_0 \cdots s_p}/\Delta^p \times \eta} (d\omega)^{\tau^{s_0 \cdots s_p}} \\
&= \sum_{\tau \in PS_{n,m}(K/\eta)} \sum_{0 \le s_j \le q_{i_j}} \int_{\partial_F (\Delta^{p+s} \times \tau_{s_0 \cdots s_p})/\Delta^p \times \eta} \omega^{(p+s)}_{\tau^{s_0 \cdots s_p}} + \\
&\quad + (-1)^n \sum_{\tau \in PS_{n,m}(K/\eta)} \sum_{0 \le s_j \le q_{i_j}} d \int_{\Delta^{p+s} \times \tau_{s_0 \cdots s_p}/\Delta^p \times \eta} \omega^{(p+s)}_{\tau^{s_0 \cdots s_p}} \\
&= \sum_{\tau \in PS_{n,m}(K/\eta)} \sum_{0 \le s_j \le q_{i_j}} \int_{\partial_F (\Delta^{p+s} \times \tau_{s_0 \cdots s_p})/\Delta^p \times \eta} \omega^{(p+s)}_{\tau^{s_0 \cdots s_p}} + \\
&\quad + (-1)^n d \int_{K/L} \omega.
\end{aligned}$$

In this formula, we recognise the first terms as $\int_{\partial_F K/L} \omega$ since lemma 4.3.1 gives us that $\partial_F AW([Y_{|\eta}]) = AW(\partial_F [Y_{|\eta}])$. Hence we have verified the formula for η a top-dimensional simplex, and since the value of the integral on the other simplices is given by restrictions, the formula holds in general.

3. If $\omega \in \Omega^{*+n}(|NK|)$ is integral, then we observe that the only non-zero terms in (4.2) are those for $s = n$, that is, the integration is only with respect to the map $\Delta^{p+n} \to \Delta^p$, and the resulting forms are then clearly integral. We also see that there is a result similar to lemma 4.1.3, so it is now clear that we have an induced map in Deligne cohomology. □

Now we are ready to compare the two integration maps. This comparison will also quite easily show that the first, smooth version of the integration map also takes integral forms to integral forms.

First, choose a triangulation of the fibre bundle $Y \to Z$ and let $\mathcal{V} = \{V_j\}_{j \in J}$ and $\mathcal{U} = \{U_i\}_{i \in I}$ be the associated coverings by the stars. Now let K and L be subdivisions of these triangulations so that every closed star of K and L lies inside an open set of $\mathcal{V}$ and $\mathcal{U}$ respectively. Then we get maps

$$T : \Omega^*(|PN\mathcal{V}/\mathcal{U}|) \to \Omega^*(|PNK/L|), \quad T' : \Omega^*(|N\mathcal{U}|) \to \Omega^*(|NL|)$$

inducing isomorphisms in cohomology.

Lemma 4.3.4. *If $\partial Y = \emptyset$ then the map $\int_{[Y/Z]} : \Omega^{*+n}(|N\mathcal{V}|) \to \Omega^*(|N\mathcal{U}|)$ takes integral forms to integral forms and hence induces a map in smooth Deligne cohomology.*

Proof. Let $\beta \in \Omega^{k+n}(|N\mathcal{V}|)$ be an integral form. Now remark 4.1.1 ensures us that the pull back $\ell^*\beta \in \Omega^{k+n}(|PN\mathcal{V}/\mathcal{U}|)$ lies completely in the subcomplex $\bigoplus_{q \le n} \Omega^{k+n-q,q,0}(|PN\mathcal{V}/\mathcal{U}|)$. Note also that under the integration map everything, besides the term in $\Omega^{k,n,0}(|PN\mathcal{V}/\mathcal{U}|)$, is mapped to zero.

In the following, we will make use of remark 4.3.2, that is, we will look at the integration map in terms of the prism complex. We will thus repress the map ℓ^* from the notation and simply write β for $\ell^*\beta \in \Omega^{k+n}(|PN\mathcal{V}/\mathcal{U}|)$.

Now in the diagram

$$\begin{array}{ccccc} \Omega^{*+n}(|N\mathcal{W}|) & \underset{\tilde{\phi}^*}{\overset{\varepsilon_1^*}{\rightleftarrows}} & \Omega^{*+n}(|PN\mathcal{V}/\mathcal{U}|) & \xrightarrow{T} & \Omega^{*+n}(|PNK/L|) \\ & \searrow^{\int_{Y/Z}} & \downarrow{\scriptstyle \int_{[Y/Z]}} & & \downarrow{\scriptstyle \int_{K/L}} \\ & & \Omega^*(|N\mathcal{U}|) & \xrightarrow{T'} & \Omega^*(|NL|) \end{array} \tag{4.5}$$

we have commutativity of the triangle and the outer square. If we put $\beta' = \varepsilon_1^*\tilde{\phi}^*\beta$ then by remark 3.2.4 we have

$$\beta' - \beta = hd\beta + dh\beta,$$

where h is the homotopy operator inducing the chain homotopy $\varepsilon_1^*\tilde{\phi}^* \sim \mathrm{id}$. Note that since $\tilde{\phi} \circ \varepsilon_1$ is the identity in the variable of the first simplex and in those on the nerve, h maps $\Omega^{p,q,r}(|PN\mathcal{V}/\mathcal{U}|)$ into $\Omega^{p,q-1,r}(|PN\mathcal{V}/\mathcal{U}|)$, so the image of the integral forms under h is mapped to zero by the integration map. Now by definition we have

$$\int_{[Y/Z]} \beta = \int_{Y/Z} \tilde{\phi}^*\beta$$

and hence commutativity of the outer square in (4.5) gives

$$\begin{aligned} T' \int_{[Y/Z]} \beta &= T' \int_{Y/Z} \tilde{\phi}^*\beta \\ &= \int_{K/L} T\varepsilon_1^*\tilde{\phi}^*\beta = \int_{K/L} T\beta' \\ &= \int_{K/L} T\beta + \int_{K/L} T(hd\beta + dh\beta). \end{aligned}$$

Because of the above remark and since $\int_{K/L} Tdh\beta = (-1)^{n-1} d \int_{K/L} Th\beta$ the last integral is zero. We therefore finally get

$$T' \int_{[Y/Z]} \beta = \int_{K/L} T\beta,$$

and since the right side is clearly integral, as noted above, we conclude that $\int_{[Y/Z]}$ maps integral forms to integral forms. □

Theorem 4.3.5. *If $\partial Y = \emptyset$ then the maps*

$$\int_{[Y/Z]} : \Omega^{*+n}(|N\mathcal{V}|) \to \Omega^*(|N\mathcal{U}|)$$

and

$$\int_{K/L} : \Omega^{*+n}(|NK|) \to \Omega^*(|NL|)$$

induce the same map (under the obvious identifications by T and T')

$$\pi_! : H_{\mathcal{D}}^{*+n}(Y, \mathbb{Z}) \to H_{\mathcal{D}}^*(Z, \mathbb{Z})$$

in smooth Deligne cohomology.

Proof. This is similar to the proof above. Taking $\omega \in \Omega^{*+n}(|PN\mathcal{V}/\mathcal{U}|)$ with $d\omega = \varepsilon^*\alpha - \beta$, we set $\omega' = \varepsilon_1^*\tilde{\phi}^*\omega$ and get $\omega' - \omega = dh\omega + hd\omega$. As above we have

$$\begin{aligned} T' \int_{[Y/Z]} \omega &= T' \int_{Y/Z} \tilde{\phi}^*\omega = \int_{K/L} T\varepsilon_1^*\tilde{\phi}^*\omega \\ &= \int_{K/L} T\omega' = \int_{K/L} T\omega + d\int_{K/L} Th\omega + \int_{K/L} Thd\omega. \end{aligned}$$

Because $\varepsilon'^*\tilde{\phi}^*$ obviously acts as the identity on $\varepsilon^*\alpha$, we see that $hd\omega = h\varepsilon^*\alpha - h\beta = -h\beta$, and then as in the proof of lemma 4.3.4 the last term vanishes, since β is integral. Hence we get

$$T' \int_{[Y/Z]} \omega = \int_{K/L} T\omega + d\tau,$$

where $\tau = \int_{K/L} Th\omega$. □

Corollary 4.3.6. *The induced integration map*

$$\pi_! : H_{\mathcal{D}}^{*+n}(Y, \mathbb{Z}) \to H_{\mathcal{D}}^*(Z, \mathbb{Z})$$

is independent of choice of coverings, partitions of unity and triangulations.

Proof. The result follows directly from the above theorem, since the combinatorial map does not depend on the choice of coverings and partitions of unity so in cohomology none of the integration maps do. A similar argument applies to the independence of triangulations. □

Recall from example 3.1.1 that in the case of a fibre bundle $Y \to Z$ with compact oriented fibres, where $\partial Y \neq \emptyset$ and a given triangulation of the bundle $\partial Y \to Z$, it is possible to extend this triangulation to a triangulation of the bundle $Y \to Z$. The integration is actually independent of this extension in the following sense.

Theorem 4.3.7. *Given a form $\omega \in \Omega^{*+n}(|N\mathcal{V}|)$ representing a class in Deligne cohomology and a triangulation of $\partial Y \to Z$ compatible with the covering $\mathcal{V}$ and two extensions $|K_1| \to |L|$ and $|K_2| \to |L|$ of this to $Y \to Z$ then*

$$\int_{K_1/L} T_1\omega \sim \int_{K_2/L} T_2\omega \quad \text{in } \Omega^*(|NL|),$$

where $T_i : \Omega^*(|N\mathcal{V}|) \to \Omega^*(|NK_i|)$, $i = 1, 2$ are given as above.

Proof. As in the proof of theorem 4.3.5 we set $\omega' = \varepsilon_1^* \tilde{\phi}^* \omega$, so that $\omega' = \omega + dh\omega + hd\omega$, and we get, for $i = 1, 2$,

$$\begin{aligned} T' \int_{[Y/Z]} \omega &= T' \int_{Y/Z} \tilde{\phi}^* \omega = \int_{K/L} T \varepsilon_1 \tilde{\phi} \omega \\ &= T' \int_{[Y/Z]} \omega' = \int_{K_i/L} T_i \omega' \\ &= \int_{K_i/L} T_i \omega + \int_{K_i/L} T_i (dh\omega + hd\omega) \\ &= \int_{K_i/L} T_i \omega + \int_{K_i/L} T_i dh\omega. \end{aligned}$$

Now the theorem follows from the fact that

$$\int_{K_i/L} T_i dh\omega = d \int_{K_i/L} T_i h\omega \pm \int_{\partial_F K_i/L} T_i h\omega,$$

where the last term is easily seen to be independent of i. □

Finally let us see that the integration map is compatible with the usual integration maps.

Proposition 4.3.8. *The integration map constructed above fits into the following commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{k+n-1}(Y,\mathbb{R})/H^{k+n-1}(Y,\mathbb{Z}) & \longrightarrow & H^{*+n}_{\mathcal{D}}(Y,\mathbb{Z}) & \xrightarrow{r} & R^{k+n}(Y) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_!} & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle \pi_!} & & \\ 0 & \longrightarrow & H^{k-1}(Z,\mathbb{R})/H^{k-1}(Z,\mathbb{Z}) & \longrightarrow & H^{*}_{\mathcal{D}}(Z,\mathbb{Z}) & \xrightarrow{r} & R^{k}(Z) & \longrightarrow & 0 \end{array}$$

Proof. From lemma 4.1.3 we know that the integration map is compatible with the usual fibre integration map for differential forms. Let $\Lambda \in \Omega^{k+n}(|NK|)$ be a discrete form, i.e. a form which is the pull-back of a form on the discrete nerve. For such a form, the combinatorial integration map presented above simplifies a great deal. We see that the value of $\int_{K/L} \Lambda \in \Omega^k(|NL|)$ integrated over a simplex $\tau \in L$ is simply the integral of Λ over all simplices lying over τ. This is consistent with the usual definition of the push forward map. Since integral forms are discrete, and since the classes in $H^{k+n-1}(Y,\mathbb{R})/H^{k+n-1}(Y,\mathbb{Z})$ can be represented by discrete forms, the result follows. □

Chapter 5

Integration in the other models

In this chapter, we will take a brief look at the constructions that exist in the other models for Deligne cohomology. We will end this chapter by showing that all these maps induce the same map in smooth Deligne cohomology. This is done by showing that there is a unique integration map which is both natural and compatible with the usual fibre integration maps.

5.1 The Hopkins-Singer construction

In this and the following sections, $Y \to Z$ will (unless otherwise mentioned) be a fibre bundle with compact oriented n-dimensional fibres.

In [16], Hopkins-Singer have constructed an integration map in the cochain model we described in section 2.3. We will only sketch their construction in the following, and refer to [16, ch. 3] for the details.

They start with the case of a product bundle $\mathbb{R}^N \times Z \to Z$, where they only consider cochains with compact support. In this case, the map

$$C_c^{p+N}(p+N)(\mathbb{R}^N \times Z) \to C^p(p)(Z)$$

is simply a slant product

$$(\omega, h, c) \mapsto \left(\int_{\mathbb{R}^N} \omega, h/Z_N, c/Z_N \right)$$

where Z_N is a representative for the fundamental class of $\mathbb{R}^N$ (in general this does not make sense since $\mathbb{R}^N$ is not compact, but since the cochains on which we take the slant product are compactly supported we can in each case choose an ordinary finite chain that does the job).

For a general bundle $Y \to Z$ the concept of a $\check{H}$-orientation of the bundle is needed. That is

1. An embedding $Y \to \mathbb{R}^N \times Z$ over Z for some N.
2. A tubular neighbourhood W.
3. A choice of a (compactly supported) Thom cocycle

$$U \in C_c^{N-n}(N-n)(\mathbb{R}^N \times Z)$$

on the normal bundle.

Then for a bundle $Y \to Z$ with a choice of $\check{H}$-orientation, the integration map

$$\int_{Y/Z} : C^{p+n}(p+n)(Y) \to C^p(p)(Z)$$

is the composite

$$C^{p+n}(p+n)(Y) \stackrel{\cup U}{\to} C^{p+N}(p+N)(\mathbb{R}^N \times Z) \stackrel{/[\mathbb{R}^N]}{\to} C^p(p)(Z).$$

In the case where $\partial X = \emptyset$, this construction induces a well-defined map in cohomology only depending on the usual orientation of the fibre.

The above construction gives an integration map for a general fibre bundle, but has some small drawbacks. It does not give an explicit local formula, so it can be hard to keep track of what happens with a cocycle when one takes the product with the Thom cocycle. Another problem is the fact that the product in this model is not as nicely behaved as in the other two models (see chapter 6), this makes it hard to use this model to prove a projection formula as the one in proposition 6.4.2.

5.2 The Gomi-Terashima construction

In the Čech-de Rham model, Gomi-Terashima [14] have created a more explicit integration map in the case of a product bundle. For a bundle $X \times Z \to Z$ (again with X compact and oriented), they construct a map

$$\int_X : \check{\Omega}^{n+p}(\mathcal{U}' \times \mathcal{U}) \to \check{\Omega}^p(\mathcal{U}),$$

where $\mathcal{U}$ is an open covering of Z, and $\mathcal{U}'$ is an open covering of X.

To describe this map we need some notation. A product of unions of open sets $U'_{i_0} \cap \cdots \cap U'_{i_p} \times U_{j_0} \cap \cdots \cap U_{j_q}$ can be rewritten as a union of products $U'_{i_0} \times U_{j_0} \cap \cdots \cap U'_{j_p} \times U_{j_p}$, with the (now lexicographic) order preserved, in several different ways, each corresponding to a shuffle $\nu \in S(p,q)$. We denote the union of products corresponding to a certain $\nu \in S(p,q)$ by $U' \times U_{\nu(i_0 \ldots i_p; j_0 \ldots j_q)}$.

Recall also the notion of a flag of a simplex introduced in the end of section 2.3. Now let $\omega = (\omega_0, \ldots, \omega_{p+n}) \in \check{\Omega}^{n+p}(\mathcal{U}' \times \mathcal{U})$ then we want to describe $\tau = (\tau_0, \ldots, \tau_p) = \int_X \omega$. We have

$$(\tau_k)_{i_0 \ldots i_k} = \sum_{j=0}^{n} \sum_{\sigma} \sum_{\bar{\sigma} \in F(\sigma,j)} \sum_{\nu \in S(n-j,k)} \operatorname{sgn}(\nu) \int_{\sigma_j} (\omega_{n-j+k})_{\nu(\alpha(\sigma_n) \ldots \alpha(\sigma_j); i_0 \ldots i_k)},$$

where the orientation of σ_j is induced from the flag $\bar{\sigma}$, and the sum over σ is a sum over all simplices in a representative for the fundamental class of X (in [14] this is done by picking an explicit triangulation of X and summing over all top-dimensional simplices in this).

This formula might look a bit strange to come up with and [14] does not give much of an idea as to what is going on. A little analysis will show that it is simply a formula for taking the slant product in this model. This implies that the induced map on cohomology is independent of all choices, and furthermore this map coincides with the Hopkins-Singer map in cohomology when $\partial X = \emptyset$.

To do this we need the map $j : C_*^{\mathcal{U}}(Z) \to \check{C}_*(\mathcal{U})$ from section 2.3. It fits into the following commutative diagram

$$\begin{array}{ccc} C_*^{\mathcal{U}'}(X) \times C_*^{\mathcal{U}}(Z) & \xrightarrow{j \times j} & \check{C}_*(\mathcal{U}') \times \check{C}_*(\mathcal{U}) \\ \downarrow EZ & & \downarrow EZ' \\ C_*^{\mathcal{U}' \times \mathcal{U}}(X \times Z) & \xrightarrow{j} & \check{C}_*(\mathcal{U}' \times \mathcal{U}) \end{array}$$

where the two vertical maps are the usual Eilenberg-Zilber maps. To describe them explicitly let $\nu \in S(p,q)$ be a (p,q)-shuffle, and let it also denote the corresponding map $\Delta^{p+q} \to \Delta^p \times \Delta^q$, then the left Eilenberg-Zilber map on a p-simplex τ and a q-simplex η is given by

$$EZ'(\tau, \eta) = \sum_{\nu \in S(p,q)} \mathrm{sgn}(\nu)(\tau \times \eta) \circ \nu.$$

Similarly, given a p-chain $\tau = (\tau_0, \ldots, \tau_p)$ and a q-chain $\eta = (\eta_0, \ldots, \eta_q)$ where $\tau_i \in \check{C}_{p-i,i}(\mathcal{U}')$ and $\eta_i \in \check{C}_{q-i,i}(\mathcal{U})$ are simplices, the right map is given by

$$EZ(\tau, \eta)_l = \sum_{r+s=l} \sum_{\nu_1 \in S(r,s)} \sum_{\nu_2 \in S(p-r,q-s)} \mathrm{sgn}(\nu_1)\mathrm{sgn}(\nu_2)\iota_{\nu_2} \circ (\tau_r \times \eta_s) \circ \nu_1$$

where, for $\nu \in S(p,q)$, $\iota_\nu : U'_{i_0 \ldots i_p} \times U_{j_0 \ldots j_q} \to U' \times U_{\nu(i_0 \ldots i_p; j_0 \ldots j_q)}$ is simply the identity.

Assume again that $\partial X = \emptyset$, take a class in $H_{\mathcal{D}}^{n+p}(X \times Z, \mathbb{Z})$ represented by a $\omega \in \check{\Omega}^{n+p}(\mathcal{U}' \times \mathcal{U})$ and consider $j_*([\omega]) = (h, \alpha)$ then from the construction in theorem 2.3.2 we get

$$\begin{aligned} h([X] \times \sigma) &= \langle I(\omega), j(EZ([X] \times \sigma)) \rangle \\ &= \langle I(\omega), EZ(j[X], j\sigma) \rangle = \sum_{l=0}^{n+p} \langle I(\omega_{n+p-l}), EZ(j[X], j\sigma)_l \rangle, \end{aligned}$$

where

$$\langle I(\omega_{n+p-l}), EZ(j[X], j\sigma)_l \rangle$$

is given by the expression

$$\sum_{r+s=l} \sum_{\tau} \sum_{\bar{\tau} \in F(\tau,r)} \sum_{\bar{\sigma} \in F(\sigma,s)} \sum_{\nu \in S(n-r,p-s)} \mathrm{sgn}(\nu) \int_{\tau_r \times \sigma_s} (\omega_{n+p-l})_{\nu(\alpha(\tau_n) \ldots \alpha(\tau_r); \beta(\sigma_p) \ldots \beta(\sigma_s))}.$$

Given this, it is not that hard to see that

$$h([X] \times \sigma) = \langle I(\int_X \omega)), j\sigma \rangle.$$

5.3 Equivalence of the constructions

In this section, we will show that all the integration maps presented above actually coincide in Deligne cohomology. We will do this by showing that there is a unique integration map which is natural with respect to pull-backs and which is compatible with the usual integration maps.

Theorem 5.3.1. *Given a smooth fibre bundle $Y \to Z$ with closed, oriented n-dimensional fibre, there is precisely one map $I : H^{*+n}_{\mathcal{D}}(Y,\mathbb{Z}) \to H^*_{\mathcal{D}}(Z,\mathbb{Z})$ satisfying*

1. Compatibility
   The following diagram is commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{*+n-1}(Y,\mathbb{R})/H^{*+n-1}(Y,\mathbb{Z}) & \longrightarrow & H^{*+n}_{\mathcal{D}}(Y,\mathbb{Z}) & \xrightarrow{r} & R^{*+n}(Y) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_!} & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle \pi_!} & & \\
0 & \longrightarrow & H^{*-1}(Z,\mathbb{R})/H^{*-1}(Z,\mathbb{Z}) & \longrightarrow & H^*_{\mathcal{D}}(Z,\mathbb{Z}) & \xrightarrow{r} & R^*(Z) & \longrightarrow & 0
\end{array}$$

2. Naturality
   Let $f : Z' \to Z$ be an embedding. In the pull-back diagram

$$\begin{array}{ccc}
f^*Y & \xrightarrow{\hat{f}} & Y \\
\downarrow & & \downarrow \\
Z' & \xrightarrow{f} & Z
\end{array}$$

we have $f^*I(x) = I(f^*x)$ for $x \in H^{*+n}_{\mathcal{D}}(Y,\mathbb{Z})$, i.e. the following diagram commutes

$$\begin{array}{ccc}
H^{*+n}_{\mathcal{D}}(Y,\mathbb{Z}) & \xrightarrow{\hat{f}^*} & H^{*+n}_{\mathcal{D}}(f^*Y,\mathbb{Z}) \\
\downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} \\
H^*_{\mathcal{D}}(Z,\mathbb{Z}) & \xrightarrow{f^*} & H^*_{\mathcal{D}}(Z',\mathbb{Z})
\end{array}$$

Proof. The existence of an integration map is already clear from chapter 4, where naturality is shown in proposition 4.1.4, and the compatibility is shown in proposition 4.3.8, so we only need to prove uniqueness. Let I_1, I_2 be two maps satisfying 1. and 2. above and let $h = I_1 - I_2$. We see that since

$$r \circ h = r \circ I_1 - r \circ I_2 = \pi_! \circ r - \pi_1 \circ r = 0$$

we have

$$h : H^{*+n}_{\mathcal{D}}(Y,\mathbb{Z}) \to H^{*-1}(Z,\mathbb{R})/H^{*-1}(Z,\mathbb{Z}).$$

We shall see that $h = 0$ by showing that $h(x) = 0$ in $H^{k-1}(Z,\mathbb{R})/H^{k-1}(Z,\mathbb{Z})$ for all $x \in H^{k+n}_{\mathcal{D}}(Y,\mathbb{Z})$. Given such an x let $a_x \in H^{k-1}(Z,\mathbb{R}) = \hom_{\mathbb{R}}(H_{k-1}(Z,\mathbb{R}),\mathbb{R})$ be a lift of $h(x)$. We shall see that $a_x \in \hom(H_{k-1}(Z),\mathbb{Z})$.

Recall that one can always choose a basis for $H_{k-1}(Z, \mathbb{R})$, which can be represented by embeddings of closed $(k-1)$-dimensional manifold $Z' \to Z$. We can furthermore choose such a basis so that it generates the lattice $\mathrm{im} H_{k-1}(Z, \mathbb{Z}) \subseteq H_{k-1}(Z, \mathbb{R})$. It is enough to show that a_x takes values in $\mathbb{Z}$ on this basis. So let σ be such a class represented by the map $f : Z' \to Z$.

Now since both I_1 and I_2 are natural, we get that

$$f^*h(x) = f^*(I_1(x) - I_2(x)) = I_1(f^*x) - I_2(f^*x) = \pi_!(x) - \pi_!(x) = 0,$$

where the second to last equality follows from dimensional reasons. Since $\dim Z' = k-1$ we have $\dim f^*Y = n + k - 1$ which implies that

$$H^{k+n}_{\mathcal{D}}(f^*Y, \mathbb{Z}) \cong H^{k+n-1}(f^*Y, \mathbb{R})/H^{k+n-1}(f^*Y, \mathbb{Z}),$$

and here the maps I_1 and I_2 coincide with the usual push-forward map $\pi_!$. Now we get

$$a_x(\sigma) = a_x(f_*[Z']) = f^*a_x([Z']),$$

and since

$$[f^*a_x] = f^*[a_x] = f^*h(x) = 0 \text{ in } H^{k-1}(Z', \mathbb{R})/H^{k-1}(Z', \mathbb{Z}),$$

we conclude that $f^*a_x \in \hom(H_{k-1}(Z'), \mathbb{Z})$, so we have $a_x(\sigma) \in \mathbb{Z}$. It follows that $h = 0$ and the two integration maps coincide. □

In [16, ch. 3], it is shown that the Hopkins-Singer map satisfies property 1. and 2. above, so we see that the two maps do induce the same map in cohomology.

Chapter 6

Products

All the models for smooth Deligne cohomology in chapter 2 carry product structures. We will briefly describe these well-known constructions in the Čech-de Rham and the Hopkins-Singer models, and then we will present a construction of a product in the simplicial case. We will end the chapter by showing that these constructions induce the same map in Deligne cohomology. This is done by showing that there is a unique product satisfying some natural axioms.

6.1 The Čech-de Rham model

This is the easiest case. First we fix a good open covering $\mathcal{U}$ of our manifold Z, and then at the cochain-level we have

Definition 6.1.1. With the notation from remark 2.1.4 i), let $\omega = (\omega_0, \dots, \omega_p) \in \check{\Omega}^p(\mathcal{U})$ and $\tau = (\tau_0, \dots, \tau_q) \in \check{\Omega}^q(\mathcal{U})$ then $\omega \cup \tau \in \check{\Omega}^{p+q+1}(\mathcal{U})$ is given by

$$\omega \cup \tau = (\omega_0 \wedge d\tau_0, \dots, \omega_p \wedge d\tau_0, (-1)^{p+1}\delta\omega_p \wedge \tau_0, \dots, (-1)^{p+1}\delta\omega_p \wedge \tau_q).$$

Here δ and d is the usual Čech and exterior differentials and $\wedge$ should be read as taking the ordinary cup product on the Čech cochains and the wedge product on the coefficients.

Remark 6.1.2. The above definition is just a reformulation of the usual definition (see e.g. [1, ch. 1]).

Proposition 6.1.3. *The map*

$$\cup : \check{\Omega}^p(\mathcal{U}) \times \check{\Omega}^q(\mathcal{U}) \to \check{\Omega}^{p+q+1}(\mathcal{U})$$

induces a well-defined graded-commutative product in smooth Deligne cohomology.

Proof. Simple checking. Note for instance that $\omega \cup \tau$ does represent a class in Deligne cohomology since

$$D(\omega \cup \tau) = d\omega_0 \wedge d\tau_0 - \delta\omega_p \cup \delta\tau_q,$$

where D is the differential in the total complex. □

6.2 The Hopkins-Singer model

There is a small problem when defining the product in the Hopkins-Singer model. This is due to the fact that the usual cup product of two forms is not a form, thus we have to choose a chain homotopy between the cup and wedge product i.e. a map

$$B : (\Omega^*(Z) \otimes \Omega^*(Z))^n \to C^{n-1}(Z, \mathbb{R})$$

so that

$$\omega \wedge \tau - \omega \cup \tau = \delta B(\omega \otimes \tau) + B(\delta(\omega \otimes \tau)).$$

Given this we can make the following definition:

Definition 6.2.1. Let $(\omega, h, c) \in C^p(p)(Z)$ and $(\omega', h', c') \in C^q(q)(Z)$ then

$$(\omega, h, c) \cup (\omega', h', c') = (\omega \wedge \omega', h \cup c' + (-1)^p \omega \cup h' + B(\omega \otimes \omega'), c \cup c') \in C^{p+q}(p+q)(Z).$$

Again we have

Proposition 6.2.2. *The map*

$$\cup : C^p(p)(Z) \times C^q(q)(Z) \to C^{p+q}(p+q)(Z)$$

induces a well-defined graded-commutative product in smooth Deligne cohomology.

Proof. Again this is simple checking. For instance the product becomes independent of the chain homotopy B since two such maps are chain homotopic. □

Remark 6.2.3. One way of constructing the chain homotopy B is by going through the proof of the de Rham isomorphism using simplicial forms as in [4, ch. 2].

6.3 The simplicial model

Initially, we had a problem similar to the one in the Hopkins-Singer model, since the subcomplex of integral forms is not closed under the ordinary simplicial wedge product. This problem is overcome by introducing a new product on the complex of simplicial forms which also turns out to be compatible with the integration map introduced in section 4. This is done in the first part of this section, and then this is used to construct a product in Deligne cohomology. Again let $\mathcal{U}$ be fixed good open covering of Z.

First consider the maps

$$\pi_i : |P_1 N\mathcal{U}| \to |N\mathcal{U}|, \quad i = 1, 2$$

where

$$\pi_1 : \Delta^{q_0 q_1} \times U_{i_0 \dots i_{q_0+q_1+1}} \to \Delta^{q_0} \times U_{i_0 \dots i_{q_0}}$$

is given by

$$(r^0, r^1, x) \mapsto (r^0, x)$$

and similarly

$$\pi_2 : \Delta^{q_0 q_1} \times U_{i_0 \ldots i_{q_0+q_1+1}} \to \Delta^{q_1} \times U_{i_{q_0+1} \ldots i_{q_0+q_1+1}}$$

is given by

$$(r^0, r^1, x) \mapsto (r^1, x).$$

For $t \in \Delta^1$ we have the map

$$\ell_t : |P_1 N\mathcal{U}| \to |N\mathcal{U}|$$

given by

$$\ell_t(r^0, r^1, x) = (tr^0, (1-t)r^1, x).$$

It is clearly a homeomorphism for $t \in \overset{\circ}{\Delta^1}$.

The inverse is given as follows. Take an $(r_0, \ldots, r_n, x) \in \Delta^n \times U_{i_0 \ldots i_n}$ and choose p so that $\sum_{i=0}^{p-1} r_i \leq t < \sum_{i=0}^{p} r_i$. Then

$$\ell_t^{-1}(r_0, \ldots, r_n, x) = \\ ((\frac{r_0}{t}, \ldots, \frac{r_{p-1}}{t}, 1 - \frac{\sum_{i=0}^{p-1} r_i}{t}), (1 - \frac{\sum_{i=p+1}^{n} r_i}{1-t}, \frac{r_{p+1}}{1-t}, \ldots, \frac{r_n}{1-t}), s_p x).$$

Choose a smooth bump function $\phi : \mathbb{R} \to \mathbb{R}$ with $\mathrm{supp}(\phi) \subseteq [0,1]$, so that the following holds

1. $\int_0^1 \phi(t) dt = 1$.
2. $\lim_{t \to 0} \phi(t)/t^p = 0$ and $\lim_{t \to 1} \phi(t)/(1-t)^p = 0$ for all $p \in \mathbb{N}$. We now have

Definition 6.3.1. The product

$$\wedge_1 : \Omega^*(|N\mathcal{U}|) \times \Omega^*(|N\mathcal{U}|) \to \Omega^*(|N\mathcal{U}|).$$

is given by

$$\omega_1 \wedge_1 \omega_2 := \int_{\Delta^1} \phi(t) dt \wedge (\ell_t^{-1})^* (\pi_1^* \omega_1 \wedge \pi_2^* \omega_2)$$

The choice of bump function ensures that there are no convergence problems, so the construction is well-defined and gives a normal simplicial form.

Most of the following proposition is trivial.

Proposition 6.3.2. 1. *Two different choices of bump function give chain homotopic products.*

2. *For $\omega_1 \in \Omega^p(|N\mathcal{U}|)$ and $\omega_2 \in \Omega^q(|N\mathcal{U}|)$ we have*

$$d(\omega_1 \wedge_1 \omega_2) = d\omega_1 \wedge_1 \omega_2 + (-1)^p \omega_1 \wedge_1 d\omega_2.$$

3. *$I_\Delta : \Omega^*(|N\mathcal{U}|) \to \check{\Omega}^*(\mathcal{U})$ is multiplicative.*
4. *If $\omega_1, \omega_2 \in \Omega^*_{\mathbb{Z}}(|N\mathcal{U}|)$ then $\omega_1 \wedge_1 \omega_2 \in \Omega^*_{\mathbb{Z}}(|N\mathcal{U}|)$.*

Proof. 1. This is trivial since two choices of bump functions that satisfy the required conditions are certainly homotopic by a linear homotopy through such bump functions.

2. Follows from the corresponding formula for the wedge product.

3. Suppose $\omega_1 \wedge_1 \omega_2 \in \Omega^{n,m}(|N\mathcal{U}|)$ then we have

$$\begin{aligned}
\int_{\Delta^n} (\omega_1 \wedge_1 \omega_2)_{i_0 \dots i_n} &= \int_{\Delta^n} \int_{\Delta^1} \phi(t) dt \wedge (\ell_t^{-1*}(\pi_1^* \omega_1 \wedge \pi_2^* \omega_2))_{i_0 \dots i_n} \\
&= \int_{\Delta^1 \times \Delta^n} \phi(t) dt \wedge (\ell_t^{-1*}(\pi_1^* \omega_1 \wedge \pi_2^* \omega_2))_{i_0 \dots i_n} \\
&= \sum_{p+q=n} \int_{\Delta^1 \times \Delta^p \times \Delta^q} \phi(t) dt \wedge \ell_t^* \eta_p^* (\ell_t^{-1*}(\pi_1^* \omega_1 \wedge \pi_2^* \omega_2))_{i_0 \dots i_n} \\
&= \sum_{p+q=n} \int_{\Delta^1 \times \Delta^p \times \Delta^q} \phi(t) dt \wedge (\omega_1)_{i_0 \dots i_p} \wedge (\omega_2)_{i_p \dots i_n} \\
&= \sum_{p+q=n} \int_{\Delta^p} (\omega_1)_{i_0 \dots i_p} \wedge \int_{\Delta^q} (\omega_2)_{i_p \dots i_n}.
\end{aligned}$$

So $I(\omega_1 \wedge_1 \omega_2) = I(\omega_1) \wedge I(\omega_2)$ as claimed.

4. This follows directly from the proof of 3. □

Remark 6.3.3. Unfortunately the product is neither commutative nor associative on the chain level but 3. above insures us that it is up to chain homotopy.

Let us move on to Deligne cohomology where the product structure is a little different.

Definition 6.3.4. Let $\omega_1 \in \Omega^p(|N\mathcal{U}|)$ and $\omega_2 \in \Omega^q(|N\mathcal{U}|)$ be two forms representing classes in Deligne cohomology. That is $d\omega_i = \varepsilon^* \alpha_i - \beta_i$, where α_i is a global form and β_i is integral. Then we define

$$\omega_1 \tilde{\wedge} \omega_2 := \omega_1 \wedge_1 \varepsilon^* \alpha_2 + (-1)^{p+1} \beta_1 \wedge_1 \omega_2.$$

Some calculations show that this induces a well-defined product in Deligne cohomology. E.g. take another representative $\omega_1 + \beta$ for the class $[\omega_1]$ then we have

$$\begin{aligned}
(\omega_1 + \beta) \tilde{\wedge} \omega_2 &= (\omega_1 + \beta) \wedge_1 \varepsilon^* \alpha_2 + (-1)^{p+1} (\beta_1 + d\beta) \wedge_1 \omega_2 \\
&= \omega_1 \tilde{\wedge} \omega_2 + \beta \wedge_1 \varepsilon^* \alpha_2 + (-1)^{p+1} d\beta \wedge_1 \omega_2 \\
&= \omega_1 \tilde{\wedge} \omega_2 + d(\beta \wedge_1 \omega_2) + \beta \wedge_1 \beta_2 \\
&\sim \omega_1 \tilde{\wedge} \omega_2.
\end{aligned}$$

Notably $d(\omega_1 \tilde{\wedge} \omega_2) = \varepsilon^*(\alpha_1 \wedge_1 \alpha_2) - \beta_1 \wedge_1 \beta_2$.

With this product on $\Omega^*(|N\mathcal{U}|)$, the map I_Δ of section 2.2 between the simplicial and the Čech-de Rham model for Deligne cohomology becomes an isomorphism of graded rings.

6.4 A projection formula

It turns out that the product constructed in section 6.3 is well-behaved with respect to the integration map above.

So let $Y \to Z$ be a fibre bundle with compact oriented fibres and let, as before, $\mathcal{V}$ and $\mathcal{U}$ be good open coverings of the total space and base space respectively.

Proposition 6.4.1. *For* $\omega_1 \in \Omega^{p+n}(|N\mathcal{V}|)$ *and* $\omega_2 \in \Omega^q(|N\mathcal{U}|)$ *we have*

$$\left(\int_{[Y/Z]} \omega_1\right) \wedge_1 \omega_2 = \int_{[Y/Z]} \omega_1 \wedge_1 \pi^*\omega_2. \tag{6.1}$$

Proof. Note that

$$(\pi_1 \times \pi_2) \circ \ell_t^{-1} \circ \ell \circ \tilde{\phi} = (\ell \circ \tilde{\phi} \circ \pi_1 \times \ell \circ \tilde{\phi} \circ \pi_2) \circ \ell_t^{-1}$$

as maps

$$|N\mathcal{W}| \to |N\mathcal{V}| \times |N\mathcal{V}|,$$

so for a pair of forms $\omega, \tau \in \Omega^*(|N\mathcal{U}|)$ we get the relation

$$(\ell \circ \tilde{\phi})^*(\omega \wedge_1 \tau) = (\ell \circ \tilde{\phi})^*\omega \wedge_1 (\ell \circ \tilde{\phi})^*\tau.$$

This implies that

$$\begin{aligned}
\int_{[Y/Z]} \omega_1 \wedge_1 \pi^*\omega_2 &= \int_{Y/Z} (\ell \circ \tilde{\phi})^*\omega_1 \wedge_1 (\ell \circ \tilde{\phi})^*\pi^*\omega_2 \\
&= \int_{Y/Z} (\ell \circ \tilde{\phi})^*\omega_1 \wedge_1 \pi^*\omega_2 \\
&= \left(\int_{Y/Z} (\ell \circ \tilde{\phi})^*\omega_1\right) \wedge_1 \omega_2 \\
&= \left(\int_{[Y/Z]} \omega_1\right) \wedge_1 \omega_2
\end{aligned}$$

as stated above. □

The corresponding result in Deligne cohomology now follows immediately from the above and a look at definition 6.3.4.

Proposition 6.4.2. *For* $\omega_1 \in \Omega^{p+n}(|N\mathcal{V}|)$ *and* $\omega_2 \in \Omega^q(|N\mathcal{U}|)$ *representing classes in Deligne cohomology, we have*

$$\left(\int_{[Y/Z]} \omega_1\right) \tilde{\wedge} \omega_2 = \int_{[Y/Z]} \omega_1 \tilde{\wedge} \pi^*\omega_2. \tag{6.2}$$

Remark 6.4.3. If we look at the proposition above in the light of theorem 5.3.1, we see that the *unique* integration map in smooth Deligne cohomology satisfies such a projection formula with respect to the product introduced in section 6.3.

6.5 Uniqueness

In this section, we will show that there is a unique product which is natural and compatible with the usual cup and wedge products. The proof follows the same line as the proof of uniqueness of the integration map.

Theorem 6.5.1. *Given a smooth manifold X, there is a unique graded ring structure on $H^*_{\mathcal{D}}(X,\mathbb{Z})$ such that*

1. Compatibility *The product is compatible with the usual cup and wedge products in the sense that*

a) the following diagram commutes

$$\begin{array}{ccc} H^n_{\mathcal{D}}(X,\mathbb{Z}) \times H^m_{\mathcal{D}}(X,\mathbb{Z}) & \xrightarrow{\wedge} & H^{m+n}_{\mathcal{D}}(X,\mathbb{Z}) \ , \\ \downarrow & & \downarrow \\ R^n(X) \times R^m(X) & \xrightarrow{\cup \times \wedge} & R^{m+n}(X) \end{array}$$

whereas before $R^n(X) = \{(c,\omega) \in H^n(X,\mathbb{Z}) \times \Omega^n(X) \mid c = [\omega] \in H^n(X,\mathbb{R})\}$.

b) If we restrict the product to $H^n_{\mathcal{D}}(X,\mathbb{Z}) \times H^{m-1}(X,\mathbb{R})/H^{m-1}(X,\mathbb{Z})$, then the product factors as

$$\begin{aligned} & H^n_{\mathcal{D}}(X,\mathbb{Z}) \times H^{m-1}(X,\mathbb{R})/H^{m-1}(X,\mathbb{Z}) \to \\ & \to H^n(X,\mathbb{Z}) \times H^{m-1}(X,\mathbb{R})/H^{m-1}(X,\mathbb{Z}) \to \\ & \xrightarrow{\cup} H^{n+m-1}(X,\mathbb{R})/H^{n+m-1}(X,\mathbb{Z}) \to \\ & \to H^{n+m-1}_{\mathcal{D}}(X,\mathbb{Z}), \end{aligned}$$

and similarly if we restrict to $H^{n-1}(X,\mathbb{R})/H^{n-1}(X,\mathbb{Z}) \times H^m_{\mathcal{D}}(X,\mathbb{Z})$.

2. Naturality *Let $f : X' \to X$ be a differential map then the following diagram commutes*

$$\begin{array}{ccc} H^n_{\mathcal{D}}(X,\mathbb{Z}) \times H^m_{\mathcal{D}}(X,\mathbb{Z}) & \xrightarrow{\wedge} & H^{m+n}_{\mathcal{D}}(X,\mathbb{Z}) \\ \Big\downarrow{\scriptstyle f^* \times f^*} & & \Big\downarrow{\scriptstyle f^*} \\ H^n_{\mathcal{D}}(X',\mathbb{Z}) \times H^m_{\mathcal{D}}(X',\mathbb{Z}) & \xrightarrow{\wedge} & H^{m+n}_{\mathcal{D}}(X',\mathbb{Z}) \end{array}$$

Proof. Instead of showing directly that the product is unique, we will show that the corresponding exterior product is unique.

Assume that there are two products $\wedge_1$ and $\wedge_2$ satisfying 1) and 2) above. Using $1a$) we define (for the corresponding exterior products)

$$h = \wedge_1 - \wedge_2 : H^n_{\mathcal{D}}(X,\mathbb{Z}) \times H^m_{\mathcal{D}}(Y,\mathbb{Z}) \to H^{n+m-1}(X \times Y,\mathbb{R})/H^{n+m-1}(X \times Y,\mathbb{Z}).$$

We will show that $h(x,y) = 0$ for all $x \in H^n_{\mathcal{D}}(X,\mathbb{Z})$ and $y \in H^n_{\mathcal{D}}(Y,\mathbb{Z})$. So take such x and y and let $a_{x,y}$ be a lift of $h(x,y)$ to $H^{n+m-1}(X\times Y,\mathbb{R}) = \hom_{\mathbb{R}}(H_{n+m-1}(X\times Y,\mathbb{R}),\mathbb{R})$. Now the Künneth formula says that

$$H_{n+m-1}(X \times Y,\mathbb{R}) = \bigoplus_{1=0}^{n+m-1} H_i(X,\mathbb{R}) \otimes H_{n+m-1-i}(Y,\mathbb{R}).$$

As in the proof of theorem 5.3.1, it is enough to evaluate $a_{x,y}$ on cycles represented by embedded manifolds of the form $f \times f' : Z \times Z' \to X \times Y$ where $\dim Z = i$ and $\dim Z' = n+m-1-i$. It is clear for dimensional reasons that the only interesting cases are $i = n-1$ and $i = n$. So assume that a cycle $\sigma \in H_{n+m-1}(X \times Y, \mathbb{R})$ has a representative of the form $f \times f' : Z \times Z' \to X \times Y$ where $\dim Z = n$ and $\dim Z' = m-1$. Now

$$a_{x,y}(\sigma) = a_{x,y}((f \times f')_*[Z \times Z']) = (f \times f')^* a_{x,y}([Z \times Z'])$$

and since $(f \times f')^* a_{x,y}$ represents $(f \times f')^* h(x,y) = h(f^*x, f'^*y)$, it is integral. This follows from property 1*b*) since $R^m(Z') = 0$, so $h(f^*x, f'^*y) = 0$ in

$$H^{n+m-1}(X \times Y, \mathbb{R}) / H^{n+m-1}(X \times Y, \mathbb{Z}).$$

$\square$

Clearly all the products introduced in this chapter satisfy the above axioms, so they induce the same product in smooth Deligne cohomology.

Chapter 7

Applications of the integration map

In this chapter, we will apply the integration map to the construction of invariants for families of bundles with connection as in [7]. This application was the original motivation for the construction of an integration for smooth Deligne cohomology. We will describe another interesting application in chapter 9.

7.1 Secondary invariants

In this section, we will briefly describe the original Cheeger-Simons (see [3]) construction of secondary invariants in the context of simplicial forms as in [7].

Let G be a Lie group G with finitely many connected components. We denote the set of invariant homogeneous polynomials of degree k by $I^k(G)$. Let $I^k_{\mathbb{Z}}(G) \subseteq I^k(G)$ be the subset of polynomials that map to the image of $H^{2k}(BG,\mathbb{Z})$ inside $H^{2k}(BG,\mathbb{R})$ under the Chern-Weil homomorphism.

Fix a $Q \in I^k_{\mathbb{Z}}(G)$ and let $u \in H^{2k}(BG,\mathbb{Z})$ be the corresponding integral class. Then to a principal G-bundle $P \to M$ with connection A, we will associate a class $[\Lambda(Q,u,A)] \in H^{2k}_{\mathcal{D}}(M,\mathbb{Z})$.

The idea is to pick an integral cocycle representing u and then compare this to the form coming from Chern-Weil theory. They represent the same class in real cohomology, but are not identical since differential forms do not give integral cocycles. Now since they represent the same class, their difference is a boundary. This bounding cochain is well-defined up to a boundary since the odd dimensional cohomology of BG vanishes.

To do this properly one would have to replace $EG \to BG$ with a smooth approximation (which we also denote $EG \to BG$) and find a connection $\bar{A}$ in this bundle, so that the classifying map $f : M \to BG$ for the bundle $P \to M$ satisfy $f^*\bar{A} = A$. This is done in detail in e.g. [7].

Now pick compatible coverings $\mathcal{U}$ of M and $\mathcal{U}'$ of BG and a representative $\gamma \in \Omega^{2k}_{\mathbb{Z}}(|N\mathcal{U}'|)$ for u. Since γ and $Q(F^k_{\bar{A}})$ represent the same class in $H^{2k}(BG,\mathbb{R})$, there is a simplicial form $\bar{\Lambda} \in \Omega^{2k-1}(|N\mathcal{U}'|)$ so that $d\bar{\Lambda} = \varepsilon^* Q(F^k_{\bar{A}}) - \gamma$. Since the odd dimensional cohomology of BG vanishes, $\bar{\Lambda}$ is well-defined modulo exact forms. Finally we set $\Lambda(Q,u,A) = f^*\bar{\Lambda} \in \Omega^{2k-1}(|N\mathcal{U}|)$, and we have the following result (which proof is in [7, sec. 5]):

Theorem 7.1.1. $\Lambda(Q,u,A)$ *represents a well-defined class*

$$[\Lambda(Q,u,A)] \in H^{2k}_{\mathcal{D}}(M,\mathbb{Z})$$

independent of all choices.

The product introduced in section 6.3 enables us to simplify proposition 5.17 ii) in [7] a little.

Proposition 7.1.2. *Given* $Q_1 \in I^{k_1}_{\mathbb{Z}}(G)$ *and* $Q_2 \in I^{k_2}_{\mathbb{Z}}(G)$ *with corresponding integral classes* u_1 *and* u_2*, then*

$$[\Lambda(Q_1Q_2, u_1 \cup u_2, A)] = [\Lambda(Q_1,u_1,A)\tilde{\wedge}\Lambda(Q_2,u_2,A)]$$

in $H^{2(k_1+k_2)}_{\mathcal{D}}(M,\mathbb{Z})$*.*

Proof. Follows directly from the observation that

$$d(\Lambda(Q_1,u_1,A)\tilde{\wedge}\Lambda(Q_2,u_2,A)) = Q_1(F_A^{k_1}) \wedge Q_2(F_A^{k_2}) - \gamma_1 \wedge_1 \gamma_2$$

and that $\gamma_1 \wedge_1 \gamma_2$ represents $u_1 \cup u_2$. □

7.2 Invariants for families of connections

Now we are ready to construct invariants for families of bundles with connection. We start with the definition.

Definition 7.2.1. A family of principal G-bundles with connections is

1. A smooth fibre bundle $\pi : Y \to Z$ with oriented fibre X.
2. A principal G-bundle $p : E \to Y$.
3. A smooth family $A = \{A_z \mid z \in Z\}$ of connections in the G-bundles $P_z \to X_z$ where $X_z = \pi^{-1}(z)$ and $P_z = E_{|X_z}$.

Remark 7.2.2. The above definition is taken from [7]. In the corresponding definition in [11], the family in 3. is a connection for the whole bundle $E \to Y$. In some situations there are however a point in not letting the 'horizontal' information be part of the structure, since the invariants for families of bundles with connection might be independent of this extension to a connection in the whole bundle. Note that it is always possible to construct such an extension using a partition of unity.

Now given such a family of bundles with connection, with X a closed manifold, we can create invariants living in Deligne cohomology of the parameter space. This is done as follows: First we extend the family of connections A to a connection B in $E \to Y$, then we proceed as in the section above where we to a polynomial $Q \in I^k_{\mathbb{Z}}(G)$ obtained a simplicial form $\Lambda(Q,u,B) \in \Omega^{2k-1}(|N\mathcal{V}|)$, where $\mathcal{V}$ is a good open cover of Y. Now we apply integration along the fibres in $Y \to Z$ and get a form

$$\Lambda_{Y/Z}(Q,u,B) = \int_{[Y/Z]} \Lambda(Q,u,B) \in \Omega^{2k-n-1}(|N\mathcal{U}|),$$

where $\mathcal{U}$ is a good open cover of Z, and n is the dimension of the fibre X. We now have the following result from [7, sec. 6].

Theorem 7.2.3. 1. $\Lambda_{Y/Z}(Q,u,B)$ *represents a well-defined class*

$$[\Lambda_{Y/Z}(Q,u,B)] \in H^{2k-n}_{\mathcal{D}}(Z,\mathbb{Z})$$

with

$$d\Lambda_{Y/Z}(Q,u,B) = \varepsilon^* \int_{Y/Z} Q(F_B^k) - \int_{[Y/Z]} f^*\gamma.$$

2. *If* $F_{A_z}^{n-k+1} = 0$ *for all* $z \in Z$ *then the class in Deligne cohomology is independent of the extension* B.
3. *If* $F_{A_z}^{n-k} = 0$ *then* $\Lambda_{Y/Z}(Q,u,B)$ *is flat, i.e.* $\int_{[Y/Z]} Q(F_B^k) = 0$.

In the paper [7], this is applied to families of foliated bundles, and we refer to this paper for examples of that type. Instead we will apply the above in the following example which is mostly a reformulation in terms of simplicial forms of the idea in [11, 13]. See also [6] for a generalisation of this example which originally motivated the work in [7].

Example 7.2.4. Let $k = 2$ and take a $Q \in I^2_{\mathbb{Z}}(G)$. Now let M be a smooth compact oriented 3-manifold and assume at first that $\partial M = \emptyset$. Let $Y \to Z$ be a fibre bundle with oriented fibre M and let $E \to Y$ be a family of bundles with connections.

At first we get a form $\Lambda(Q,u,B) \in \Omega^3(|N\mathcal{V}|)$ representing a class

$$[\Lambda(Q,u,B)] \in H^4_{\mathcal{D}}(Y,\mathbb{Z}).$$

The procedure above then gives us a 0-form

$$\Lambda_M = \int_{[Y/Z]} \Lambda(Q,u,B) \in \Omega^0_{\mathbb{R}/\mathbb{Z}}(|N\mathcal{U}|),$$

representing a class in $H^1_{\mathcal{D}}(Z,\mathbb{Z})$, that is, simply a smooth function $\Lambda_M : Z \to \mathbb{R}/\mathbb{Z}$. Since $n = 3$, we have $n - k + 1 = 2$, and since $F^2_{A_z} = 0$ on a 3-manifold, Λ_M is independent of the extension B.

If now $\partial M = \Sigma \neq \emptyset$, then Λ_M is not a globally defined function anymore, instead the Stokes formula gives us

$$d\Lambda_M = \int_{[\partial Y/Z]} \Lambda(Q,u,B) - \int_{[Y/Z]} d\Lambda(Q,u,B). \tag{7.1}$$

If we restrict $\Lambda(Q,u,B)$ to $\partial Y \to Z$, we get

$$\Lambda_\Sigma = \int_{[\partial Y/Z]} \Lambda(Q,u,B),$$

which represents a class in $H^2_{\mathcal{D}}(Z,\mathbb{Z})$. As we saw in the proof of proposition 2.1.5, Λ_Σ then defines a circle bundle with connection over Z. If we write $\Lambda_\Sigma = \Lambda^0_\Sigma + \Lambda^1_\Sigma \in \Omega^{0,1}(|N\mathcal{U}|) \oplus \Omega^{1,0}(|N\mathcal{U}|)$, then $c = \int_{\Delta^1} \Lambda^1_\Sigma$ is the defining cocycle, and $\omega = \int_{\Delta^0} \Lambda^0_\Sigma$ is the connection.

The relation that comes from integrating - over Δ^1 - the part of (7.1) that lies in $\Omega^{1,0}(|N\mathcal{U}|)$ implies that Λ_M is actually a section in the circle bundle given by the cocycle c. The corresponding relation in $\Omega^{0,1}(|N\mathcal{U}|)$ gives that the covariant derivative of Λ_M is $\int_M Q(F_B^2)$.

Note that the circle bundle and the section do depend on the choices made e.g. of the integral form representing u, the partition of unity used in the integration and so on, but the isomorphism class does not. Although it still does depend on the extension B.

However, if we restrict ourselves to a family of flat connections i.e. $F_{A_z} = 0$ for all $z \in Z$ then the isomorphism class of Λ_Σ becomes independent of the extension B. Furthermore since $\int_M Q(F_B^2)$ vanishes, Λ_M actually becomes a covariant constant section in Λ_Σ.

In this example, one usually takes Z to be the smooth part of the representation variety $\hom(\pi_1(M), G)/G$ and let $Y = M \times Z$ be the trivial fibre bundle. In this case, E is the canonical G-bundle over Y with fibrewise flat connection. If we at the same time let Z' be the smooth part of $\hom(\pi_1(\Sigma), G)/G$ and similarly let $Y' = \Sigma \times Z'$ and let E' be the canonical G-bundle over Y', then we get from the above a circle bundle with connection over Z', and if we pull this circle bundle back to Z with the map $\iota : Z \to Z'$ induced from the inclusion $\Sigma \to M$ we have a covariantly constant section Λ_M in $\iota^*\Lambda_\Sigma$.

Chapter 8

Symplectic topology

In this chapter, we will go through some of the basics of symplectic topology that will be needed in the next chapters, mainly focusing on the theory of symplectic fibrations. A more comprehensive introduction to symplectic topology can be found in e.g. McDuff and Salamon's book [25].

8.1 Symplectomorphism groups

A *symplectic manifold* (M, ω) is a manifold M equipped with a closed, non-degenerate 2-form ω called the *symplectic form*. That ω is non-degenerate means that the map $T_pM \to T_p^*M$ given by sending a tangent vector $X \in T_pM$ to $\omega_p(X, -) \in T_p^*M$ is an isomorphism.

This isomorphism gives a one-to-one correspondence between vector fields and 1-forms on M. The vector fields that correspond to closed forms are called *symplectic vector fields*, and the vector fields that correspond to exact forms are called *hamiltonian vector fields*.

That ω is non-degenerate also implies that M is even dimensional, of say dimension $2n$, and that the top power ω^n is an orientation form so M comes with a preferred orientation.

The Lie bracket of two symplectic vector fields has a very nice description in terms of the symplectic form ω. Given two symplectic vector fields X and Y then $[X, Y]$ is the hamiltonian vector field associated to the exact 1-form $d(\omega(X, Y))$, so the symplectic vector fields form a Lie algebra which we denote by $\chi_\omega(M)$, and the hamiltonian vector fields form an ideal in $\chi_\omega(M)$ which we denote by $\chi_h(M)$. We have that the Lie algebra of hamiltonian vector fields fit into the exact sequence

$$0 \to \mathbb{R} \to C^\infty(M) \to \chi_h(M) \to 0, \tag{8.1}$$

where the rightmost map is given by sending a $h \in C^\infty(M)$ to X_h the unique hamiltonian vector field that satisfies $\iota_{X_h}\omega = dh$. We call h the hamiltonian function associated to $X_h \in \chi_h(M)$. We give $C^\infty(M)$ a Poisson structure by setting $\{f, g\} = \omega(X_f, X_g)$.

In general (8.1) is not split, but if M is closed, we obtain a splitting, by identifying $\chi_h(M)$ with $C_0^\infty(M) = \{f \in C^\infty(M) \mid \int_M f\omega^n = 0\}$ the functions that integrate to zero. Here we see that since we generally have

$$\omega(X,Y)\omega^n = n\omega(X,-)\omega(Y,-)\omega^{n-1},$$

we get

$$\int_M \{f,g\}\omega^n = n\int_M df\wedge dg\wedge\omega^{n-1} = n\int_M d(f\wedge dg\wedge\omega^{n-1}) = 0,$$

so $C_0^\infty(M)$ is an ideal in $C^\infty(M)$.

A *symplectomorphism* $\phi : M \to M$ is a diffeomorphism such that $\phi^*\omega = \omega$. We denote the group of symplectomorphisms by $\mathrm{Symp}(M) \subseteq \mathrm{Diff}(M)$.

Both the diffeomorphism group and the symplectomorphism group are *ILH-groups*. That is they are smooth infinite dimensional Lie groups modeled on a vector space $E = \varprojlim H_i$ which is an inverse limit of Hilbert spaces H_i. *ILH*-groups are, although infinite dimensional, quite well-behaved, so we will not need to be too concerned with these underlying analytical details. We refer to the survey [30] and references therein for details.

Let us see that the Lie algebra Lie(Symp) of Symp = Symp(M) is the Lie algebra of symplectic vector fields. Let $\gamma : (-\varepsilon,\varepsilon) \to \mathrm{Symp}$ be a curve with $\gamma(0) = \mathrm{id}_M$, then we get a vector field X on M where $X(p) = \frac{d}{dt}\phi_t(p)_{|t=0}$. Since ω is fixed by ϕ_t, the homotopy formula gives

$$0 = \frac{d}{dt}\phi_t^*\omega = \mathcal{L}_X\omega = \iota_X d\omega + d\iota_X\omega = d\iota_X\omega,$$

and we see that X is a symplectic vector field. Similarly we see that the flow of a symplectic vector field fixes ω.

A symplectomorphism is called a hamiltonian symplectomorphism or a *hamiltonian diffeomorphism* if it is generated by a time-dependent hamiltonian vector field, i.e. if there is a curve of symplectomorphisms $\phi_t \in \mathrm{Symp}$ such that $\frac{d}{dt}\phi_t(p) = X_t(\phi_t(p))$ with $\phi_0 = \mathrm{id}$ and $\phi_1 = \phi$, where $\iota_{X_t}\omega = dh_t$. It is not hard to see that the hamiltonian diffeomorphisms form a connected normal subgroup $\mathrm{Ham}(M) \subseteq \mathrm{Symp}(M)$ of the symplectomorphism group.

It turns out that the Lie algebra of Ham = Ham(M) is actually the Lie algebra of hamiltonian vector fields, so now we have Lie groups corresponding to two of the Lie algebras in the short exact sequence

$$0 \to \chi_h(M) \to \chi_\omega(M) \to H^1(M,R) \to 0. \tag{8.2}$$

We will also need the *flux homomorphism*, which is a lift of the Lie algebra homomorphism $\chi_\omega(M) \to H^1(M,\mathbb{R})$ to the Lie group level.

The flux homomorphism is most easily defined on the universal cover $\widetilde{\mathrm{Symp}}_0$ of the identity component of the symplectomorphism group, where it is a homomorphism

$$\widetilde{\mathrm{Flux}} : \widetilde{\mathrm{Symp}}_0 \to H^1(M,\mathbb{R}) = \hom(\pi_1(M),\mathbb{R}).$$

It is given by

$$\widetilde{\text{Flux}}([\phi_t])([\sigma]) = \int_{T_{\phi_t,\sigma}} \omega, \quad [\phi_t] \in \widetilde{\text{Symp}}_0, [\sigma] \in \pi_1(M),$$

where $T_{\phi_t,\sigma} : S^1 \times I \to M$ is given by $(s,t) \mapsto \phi_t(\sigma(s))$. That is, the flux homomorphism measures the area of the cylinder given by moving σ around with the curve ϕ_t. It is not hard to see that this is indeed a well-defined homomorphism. If we let $\Gamma = \widetilde{\text{Flux}}(\pi_1(\text{Symp}_0))$ then we get a well-defined homomorphism $\text{Symp} \to H^1(M,\mathbb{R})/\Gamma$, which we also call the flux homomorphism. It turns out that the kernel of this homomorphism is given by the hamiltonian diffeomorphisms, so we have a short exact sequence

$$1 \to \text{Ham} \to \text{Symp}_0 \to H^1(M,\mathbb{R})/\Gamma \to 0.$$

The group Γ is called the flux group. It has recently been proved by Ono that $\Gamma \subseteq H^1(M;\mathbb{R})$ is a discrete subgroup, so we see that Ham is a closed subgroup of Symp of codimension $b_1 = \dim H^1(M,\mathbb{R})$.

8.2 Symplectic fibrations

A *symplectic fibration* is a fibre bundle $Y \to Z$ with fibre (M,ω), a symplectic manifold, and with structure group $\text{Symp}(M)$. Similarly a *hamiltonian fibration* is a fibre bundle with fibre (M,ω) and structure group $\text{Ham}(M)$.

Since the structure group preserves the symplectic form ω, such a fibration carries a fibrewise symplectic form $\{\omega_z\}_{z\in Z}$. There are several reasons as to why one wants to extend this to a globally defined 2-form on the total space Y.

One reason is that any extension $\Omega \in \Omega^2(Y)$ defines a horizontal distribution $H_\Omega \subseteq TY$ by setting

$$H_{\Omega x} = \{v \in T_xY \mid \Omega(v,w) = 0, \forall w \in T_x^vY\},$$

where $T^vY = \ker \pi_*$ is the vertical tangent bundle.

It is not too hard to see that $H_\Omega \oplus T^vY \cong TY$. That $H_{\omega x} \cap T_x^vY = \{0\}$ follows from $\omega_{\pi(x)}$ being non-degenerate. That $H_{\omega x} \oplus T_x^vY \to T_xY$ is surjective is seen by using Ω to project onto $H_{\Omega x}$ and T_x^vY respectively.

Such a horizontal distribution defines a connection in $Y \to Z$, and it turns out that we have

Proposition 8.2.1. *The connection H_Ω has symplectic holonomy around all loops in Z if and only if Ω is fibrewise closed, that is if and only if we have*

$$d\Omega_x(X,Y,-) = 0 \text{ for all } X,Y \in T_x^vY.$$

We refer to [25, ch. 6] for a proof.

Assume now that Ω_1 and Ω_2 are two closed extensions defining the same connection $H_{\Omega_1} = H_{\Omega_2}$. We have that their difference $\tau = \Omega_1 - \Omega_2$ is a basic 2-form, this is clear, since if $X \in T^vY$ is a vertical vector, then $\iota_X\tau = \iota_X\Omega_1 - \iota_X\Omega_2$

vanishes both on a vertical and a horizontal vector, and τ is closed. On the other hand, if we have closed extensions Ω_1 and Ω_2 differing by a basic 2-form τ then they define the same connection.

This means that if we have a connection $H = H_\Omega$ coming from a closed extension form Ω, then there is a unique extension Ω' with $H_{\Omega'} = H_\Omega$ and satisfying the normalisation condition

$$\int_M \Omega'^{n+1} = 0.$$

Here Ω' is given by

$$\Omega' = \Omega - \frac{1}{(n+1)\mathrm{vol}} p^* \int_M \Omega^{n+1}. \tag{8.3}$$

Such a normalised extension is called a *coupling form*. In [15], Guillemin-Lerman-Sternberg showed that in a symplectic fibration with simply connected fibre and connection H with hamiltonian holonomy around every contractible loop there is a closed extension Ω such that $H = H_\Omega$. Their construction of Ω is quite analytic. Below we sketch a more topological proof by McDuff.

Remark 8.2.2. It is easy to construct a fibrewise closed extension by using a partition of unity. Let $\mathcal{U} = \{U_i\}$ be an open cover of the base Z such that we have local trivialisations $\phi_i : Y_{|U_i} \to U_i \times M$ of Y. Let $\omega_i = \phi_i^* \omega$ and let $\{\rho_i\}$ be a partition of unity subordinate $\mathcal{U}$. Then we define

$$\Omega = \sum \rho_i \circ \pi \cdot \omega_i$$

and $d\Omega = \sum d(\rho_i \circ \pi) \wedge \omega_i$, so if $X, Y \in T_x^v Y$ and $Z \in T_x Y$ then we get

$$\begin{aligned} d\Omega_x(X,Y,Z) &= \sum d(\rho_i \circ \pi)(Z)\omega_i(X,Y) \\ &= \sum d(\rho_i \circ \pi)(Z)\omega(X,Y) \\ &= (\sum d(\rho_i \circ \pi)(Z))\omega(X,Y) = 0, \end{aligned}$$

where the second equality comes from the fact that X and Y are vertical, and the last equality is true because $\sum \rho_i = 1$.

A priori it is less clear whether or not one can construct a closed extension. A computation of Thurston (see e.g. [25] for a proof) shows that it is enough to ask for a cohomology class $c \in H^2(Y, \mathbb{R})$ extending the fibrewise class $[\omega_z] \in H^2(Y_z, \mathbb{R}) = H^2(M, \mathbb{R})$, because if such a class exists then one can always find an extension form Ω with $[\Omega] = c$. This fact was used by Lalonde-McDuff [23] to show that ω extends for hamiltonian fibrations.

The class $[\omega] \in H^2(M, \mathbb{R})$ extends to a class in $H^2(Y, \mathbb{R})$ exactly when $[\omega] \in E_2^{0,2}$ survives to the E_∞-term in the Serre spectral sequence for the fibration $Y \to Z$. We have the following lemma from [23].

Lemma 8.2.3. *If $Y \to Z$ is a hamiltonian fibration, the class $[\omega] \in E_2^{0,2} = H^2(M, \mathbb{R})$ survives to $E_\infty^{0,2}$, and there exists a closed extension Ω of ω.*

Proof. There are two differentials to look at. The first is $d_2 : E_2^{0,2} \to E_2^{2,1}$, and the second is $d_3 : E_3^{0,2} \to E_3^{3,0}$. It is enough to consider the universal case $Z = B\mathrm{Ham}$, so we can assume that $\pi_1(Z) = 0$. This implies that $E_3^{3,0} = E_2^{3,0} = H^3(Z, \mathbb{R})$ and of course $E_2^{2,1} = H^2(Z, H^1(M, \mathbb{R})) = H^2(Z, \mathbb{R}) \otimes H^1(M, \mathbb{R})$.

To see that $d_2([\omega]) = 0$, we note that since $\pi_1(Z) = 0$ we have $H^2(Z, \mathbb{R}) = \hom(\pi_2(Z), \mathbb{R})$ so it is enough to check the case $Z = S^2$, and here it is possible to give an explicit construction of a coupling form so in this case $d_2([\omega]) = 0$. We will come back to the construction during our analysis in section 9.1.

Now let us see that also $d_3([\omega]) = 0$. Now since $\omega^{n+1} = 0$ for dimensional reasons, we get that

$$0 = d\omega^{n+1} = (n+1) d\omega \otimes \omega^n,$$

and since the map $- \wedge [\omega^n] : H^0(M, \mathbb{R}) \to H^{2n}(M, \mathbb{R})$ is an isomorphism, we get that $d\omega = 0$. □

Remark 8.2.4. 1. If $\tau \in H^2(M, \mathbb{Z})$ is an integral lift of the class $[\omega] \in H^2(M, \mathbb{R})$, we see that we still have $d_2\tau = 0$ in $H^2(B, H^1(M, \mathbb{Z}))$. This follows from the fact that $H^2(Z, H^1(M, \mathbb{Z}))$ is torsion free in the universal case $Z = B\mathrm{Ham}$, so here

$$H^2(Z, H^1(M, \mathbb{Z})) \to H^2(Z, H^1(M, \mathbb{R})$$

is injective. Unfortunately we can only conclude from the above argument that $d_3\tau$ is a torsion class and that $(n+1)\mathrm{vol} \cdot d_3\tau = 0$. In [24], there is an example where $d_3\tau$ does not vanish.

2. Note that in the universal case $Z = B\mathrm{Ham}$, we get the following exact sequence from the Serre spectral sequence

$$H^2(Z, \mathbb{R}) \to H^2(Y, \mathbb{R}) \to H^2(M, \mathbb{R})$$

so we see that in this case there is a unique normalised extension class $c \in H^2(Y, \mathbb{R})$ with $\pi_! c^{n+1} = 0$, so in general there is a canonical choice of extension class given by the pull-back of the universal extension class.

We end this section by showing how to construct an extension form when the structure group is finite dimensional.

Example 8.2.5. [25, 32] Let (M, ω) be a symplectic manifold and let G be a Lie group with a hamiltonian action on M with moment map $\mu : M \to \mathfrak{g}^*$. Given a principal G-bundle $P \to Z$ with connection A, we shall see that we can construct a connection 2-form in the associated bundle $P \times_G M \to Z$ from these data. The construction goes as follows. The connection in P gives a projection $TP \to T^V P = P \times_G TG$ and dually a map $\iota_A : P \times_G T^*G \to T^*P$. We can use ι_A to pull back the canonical symplectic form ω_0 on T^*P to a 2-form on $P \times_G T^*G$. Now we have the 2-form $\omega' = \iota_A^*\omega_0 + \omega$ on $P \times_G T^*G \times M$. We also have a fibrewise action of G and a corresponding fibrewise moment map $\mu' = \mu_0 \circ \iota_A + \mu$, where μ_0 is the moment map on T^*P. Now we can perform symplectic reduction, and we see that $P \times_G T^*G \times M//G = \mu'^{-1}(0)/G = P \times_G M$. The induced 2-form on the quotient restricts to ω on each fibre, so it is a connection 2-form. We will elaborate further on this example in section 9.4.

8.3 Reznikov's classes

In [28], Reznikov noticed that there is a number of invariant polynomials on Lie(Ham). This enabled him to construct characteristic classes for hamiltonian fibrations using the usual Chern-Weil construction. In this section, we will look closer at this construction and see how far Reznikov was able to go using the original finite dimensional approach and where this approach runs into problems.

As mentioned in section 8.1, we have for a symplectic manifold (M, ω) that the Lie algebra of $\mathrm{Ham}(M)$ can be identified with

$$\mathrm{Lie}(\mathrm{Ham}(M)) = C_0^\infty(M) = \{f : M \to \mathbb{R} \mid \int_M f\omega^n = 0\},$$

equipped with the Poisson bracket $\{f, g\} = \omega(X_f, X_g)$. The adjoint action of $\mathrm{Symp}(M)$ on this Lie algebra is given by

$$\mathrm{Ad}(\phi)(f) = f \circ \phi^{-1}, \quad \phi \in \mathrm{Symp}, f \in C_0^\infty(M).$$

There is a non-degenerate inner product on the Lie algebra given by

$$\langle f, g\rangle = \int_M fg\omega^n,$$

which is easily seen to be invariant under the adjoint action of Symp, since ω is invariant under Symp. For a finite dimensional Lie algebra this would make Lie(Ham) semi-simple. What is known about Lie(Ham) suggests that we can think of it as being semi-simple, i.e. we have that $\mathrm{Lie(Ham)} = \{\mathrm{Lie(Ham)}, \mathrm{Lie(Ham)}\}$ [2]. This is the general idea of Reznikov [28], which is pursued further by e.g. McDuff [23], that Symp_0 behaves much like a finite dimensional Lie group, and that we should think of Ham as a maximal compact subgroup of Symp_0.

Besides the inner product, we have in general the invariant polynomials

$$Q_k(f_1, \dots, f_k) = \int_M f_1 \cdots f_k\omega^n, \quad \text{for } k \geq 2. \tag{8.4}$$

on Lie(Ham). These invariant polynomials give rise to characteristic classes $q_k \in H^{2k}(B\mathrm{Ham}(M), \mathbb{R})$ for $k \geq 2$ by the exact same construction as for a finite dimensional Lie group G and principal G-bundles.

In [28], Reznikov notes that if there is a hamiltonian action of a compact Lie group G on M, then one can try to show that the classes $q_k \in H^{2k}(B\mathrm{Ham}, \mathbb{R})$ are non-trivial by showing that the pull-back of the invariant polynomials Q_k to $\mathfrak{g}$ are non-trivial, then the usual Chern-Weil theory will imply that the pull-back and hence the q_k's themselves are non-trivial. Reznikov does this explicitly in the case where $M = \mathbb{C}P^n$ and $G = SU(n+1)$ and concludes that the Q_k's pull back to multiplicative generators of the ring of invariant polynomials, $I^*(SU(n+1))$. We go through this computation, since a little extra work will show that they up to scaling actually pull back to the symmetric polynomials - this will be useful in the next section.

Proposition 8.3.1. *Let* $p : SU(n+1) \to \mathrm{Symp}(\mathbb{C}P^n)$ *then we have* $q_k \circ p_* = \mathrm{const} \cdot \sigma_k$ *where* σ_k *is the k'th elementary symmetric polynomial.*

Proof. Recall that

$$\mathfrak{su}(n+1) = \{A \in M(n+1,\mathbb{C}) \mid A + A^* = 0, \mathrm{tr}A = 0\}.$$

The map $p_* : \mathfrak{su}(n+1) \to \mathrm{Lie}(\mathrm{Symp})(\mathbb{C}P^n) = C_0^\infty(\mathbb{C}P^n)$ is given by

$$A \mapsto ([z] \mapsto \langle Az, z\rangle),$$

where $z \in S^{2n+1}$ is a representative of $[z] \in \mathbb{C}P^n$.

Using the Fubini theorem we then get

$$q_k \circ p_*(A) = \int_{\mathbb{C}P^n} (p_*A)^k = \mathrm{const} \cdot \int_{S^{2n+1}} \langle Az, z\rangle^k dz.$$

Furthermore if $B \in M(n+1,\mathbb{C})$ is a positive definit hermitian matrix, we have that

$$\int_{\mathbb{C}^{n+1}} e^{-\langle Bz,z\rangle} = \mathrm{const} \cdot (\det B)^{-1}.$$

This is trivial if B is diagonal with positive eigenvalues, and since the expression is invariant under unitary conjugation it is true for all positive hermitian matrices.

We also have

$$\int_{\mathbb{C}^{n+1}} e^{-\langle Bz,z\rangle} dz = \int_{S^{2n+1}} \int_0^\infty r^{2n+1} e^{-r^2\langle Bv,v\rangle} dr dv = \mathrm{const} \cdot \int_{S^{2n+1}} \langle Bv, v\rangle^{-n-1},$$

where we first changes to polar coordinates and then use the equality

$$\int_0^\infty r^{2n+1} e^{-ar^2} dr = \mathrm{const} \cdot a^{-n-1}$$

which comes from differentiating $\int_0^\infty re^{-ar^2}dr = \frac{1}{2a}$ n times with respect to a.

Now take $A \in \mathfrak{su}(n+1)$. For $t > 0$ big enough $B = tI + \frac{1}{2\pi i}A$ is a positive definit hermitian matrix, so we get

$$\int_{S^{2n+1}} t^{-n-1}(1 + t^{-1}\langle Av, v\rangle)^{-n-1} dv = \mathrm{const} \cdot \det\left(tI + \frac{1}{2\pi i}A\right)^{-1},$$

and using the binomial series we get, again for $t > 0$ big enough,

$$\int_{S^{2n+1}} t^{-n-1}(1 + t^{-1}\langle Av, v\rangle)^{-n-1} = \sum_{k=0}^{\infty} \binom{-n-1}{k} t^{-k-n-1} \int_{S^{2n+1}} \langle Av, v\rangle^k.$$

From this, Reznikov concludes that ring of invariant polynomials on $\mathfrak{su}(n+1)$ is generated by the $p_* \circ q_k$'s, but it is possible to be a little more precise. If we denote the eigenvalues of A by $\lambda_1, \ldots, \lambda_{n+1}$ then we have

$$\det\left(tI + \frac{1}{2\pi i}A\right)^{-1} = \prod_{j=1}^{n+1}\left(t + \frac{1}{2\pi i}\lambda_j\right)^{-1} = t^{-n-1}\prod_{j=1}^{n+1}\left(1 + \frac{1}{2\pi i}\frac{\lambda_j}{t}\right)^{-1},$$

so we get

$$\sum_{k=0}^{\infty} \binom{-n-1}{k} t^{-k} \int_{S^{2n+1}} \langle Av, v\rangle^k = \text{const} \cdot \prod_{j=1}^{n+1} \left(\sum_{k=0}^{\infty} \left(\frac{-1}{2\pi i} \frac{\lambda_j}{t} \right)^k \right).$$

Comparing coefficients, we see that

$$q_k \circ p_*(A) = \text{const} \int_{S^{2n+1}} \langle Av, v\rangle^k = \text{const} \cdot \sigma_k(A)$$

as claimed. □

In the case of $M = \mathbb{C}P^2$ Gromov has shown that Ham $=$ Symp_0 $=$ Symp homotopy retracts onto $PSU(2)$, so here Reznikov is, in the flat case, able to lift q_2 to a class in $H^3(B\text{Ham}(\mathbb{C}P^2)^\delta, \mathbb{R}/\mathbb{Z})$. This is not possible for $n \geq 3$ because of the lack of knowledge about the topology of the symplectomorphism group, and whether or not the q_k's lie in the integral lattice in $H^{2k}(B\text{Ham}(\mathbb{C}P^n), \mathbb{R})$. In chapter 9, we will see that if we lift the problem to a central extension of Ham, we can ensure that the classes are actually integral, and we can lift them to secondary classes depending only on a connection in the fibration and a choice of prequantum line bundle for (M, ω).

8.4 Characteristic classes from the coupling class

In the paper [17], Januszkiewicz-Kędra used the existence of a coupling class to define another set of characteristic classes. For a hamiltonian fibration $Y \to Z$, we can pick a normalised extension Ω of the fibrewise symplectic form $\{\omega_z\}$, this is possible by lemma 8.2.3. Furthermore, the cohomology class of this extension is unique in the universal case. Given such a class they consider the classes

$$\chi_k = \int_M [\Omega]^{n+k} \in H^{2k}(Z, \mathbb{R}).$$

We have the following result from [19]

Proposition 8.4.1. *We have* $\chi_k = \text{const} \cdot q_k \in H^{2k}(B\text{Ham}, \mathbb{R})$.

Proof. Fix a normalised closed extension form Ω and let $H = H_\Omega$ be the corresponding connection.

The proposition essentially follows from the fact that the curvature of H is given by $F_{Hz}(v, w) = \Omega(v^\#, w^\#) \in C^\infty(Y_z)$, where $v^\#$ and $w^\#$ are the horizontal lifts of v and w respectively (see [25, ch. 6]). Recall that the classes are represented by the forms $\int_M \Omega^{n+k}$ and $\int_M (F_H^k)\omega^n$ respectively. For $v_1, \dots, v_{2k}$ vector fields on BHam and $w_1, \dots, w_{2n}$ vectors tangent to the fibre at some $x \in M_{\text{Ham}}$ we have

$$\Omega^{n+k}(v_1^\#, \dots, v_{2k}^\#, w_1, \dots, w_{2n}) = \text{const} \cdot \Omega^k(v_1^\#, \dots, v_{2k}^\#)\omega^n(w_1, \dots, w_{2n}),$$

and the result follows. □

Chapter 9

Secondary invariants for hamiltonian fibrations

In the last chapter, we saw how Reznikov constructed characteristic classes $q_k \in H^{2k}(B\mathrm{Ham}(\mathbb{C}P^n), \mathbb{R})$ extending the usual Chern classes. In this chapter, we will try to construct secondary classes lifting the q_k's to Deligne cohomology. We cannot do this on Ham, but in the case where we can lift the structure group to a central extension we will see that we get a family of bundles with connections, and we can then use the machinery from chapter 4. The classes constructed here differ from the examples in chapter 7 in that the fibrewise connection is far from being flat.

9.1 Preliminary analysis

In the following, we will assume that the symplectic form ω has integral periods. This implies that we can pick a prequantum circle bundle, i.e. a circle bundle with connection (L, α), such that α has curvature ω (in order to keep the same notation as in the first chapters of this thesis we look at a prequantum circle bundle instead of the usual hermitian complex prequantum line bundle). In this case, one could hope to construct secondary invariants for a symplectic fibration $Y \to Z$ with connection in the following way. If we could extend ω to an integral coupling form Ω on Y compatible with the connection in $Y \to Z$, we could hope to construct a circle bundle with connection on Y such that its Deligne class $\Lambda \in H^2_{\mathcal{D}}(Y, \mathbb{Z})$ only depended on the prequantum circle bundle and the connection in the hamiltonian fibration. Now applying the machinery of chapter 4, we could then define classes $\hat{\chi}_k \in H^{2k}_{\mathcal{D}}(Z, \mathbb{Z})$ as follows

$$\hat{\chi}_k = \int_{[Y/Z]} \Lambda^{\tilde{\wedge}(n+k)}.$$

There are several problems in the procedure suggested above. First of all, it is not clear that we can pick an integral extension of ω which is normalised. Secondly, even though the group Ham fixes the class $[(L, \alpha)] \in H^2_{\mathcal{D}}(M, \mathbb{Z})$ there is no action of Ham on L so even if we have a circle bundle extending L we do not a priori have a fibrewise connection in the same way that we have a fibrewise symplectic form. We will see that these two problems are very much related.

First we take a deeper look at the second problem. We start with the following lemma which implies that Ham fixes the Deligne class of (L, α) as claimed above.

Lemma 9.1.1. *Let* $\phi \in \mathrm{Symp}_0(M)$ *then* $\mathrm{Flux}(\phi)$ *is (*$\mathrm{mod}\ \mathbb{Z}$*) the holonomy of the flat bundle* $(\phi^* L)^{-1} \otimes L$.

Proof. To see this let $h : Z^1(M) \to \mathbb{R}/\mathbb{Z}$ be the holonomy homomorphism of L. Then the holonomy of $(\phi^* L)^{-1} \otimes L$ is given by $h - h \circ \phi$. Since ϕ is isotopic to the identity we have a chain homotopy H_ϕ such that $\mathrm{id} - \phi = \partial H_\phi + H_\phi \partial$. So for $a \in Z^1(M)$ we get

$$(h - h \circ \phi)(a) = h(\partial H_\phi(a)) \equiv \int_{H_\phi(a)} \omega = \mathrm{Flux}(\phi)(a)$$

□

So for $\phi \in \mathrm{Ham}$ the lemma gives us that $(\phi^* L, \phi^* \alpha)$ and (L, α) are isomorphic as circle bundles with connection. This implies that we can lift any $\phi \in \mathrm{Ham}$ to a bundle map $\hat{\phi} : L \to L$ that preserves the connection. Let $\mathcal{G}_\alpha$ be the group of such maps. This group is a central extension

$$0 \to \mathbb{R}/\mathbb{Z} \to \mathcal{G}_\alpha \to \mathrm{Ham} \to 0. \tag{9.1}$$

The extension was introduced by Kostant in [20] and is called the quantomorphism group. We will see in section 9.3 that the isomorphism class of this central extension does not depend on the choice of connection α, but for now it is convenient with this description of the central extension.

We see that we can restate the second problem mentioned above to whether or not we can lift the structure group of a given hamiltonian fibration to $\mathcal{G}_\alpha$.

We will need the following lemma form [20] later on

Lemma 9.1.2. *We can identify the Lie algebra* $\mathfrak{g}_\alpha$ *of* $\mathcal{G}_\alpha$ *with* $C^\infty(M)$ *equipped with the usual Poisson bracket.*

Proof. Take a curve $\phi_t \in \mathcal{G}_\alpha$ with $\phi_0 = \mathrm{id}$ and let X be the vector field on L given by $X(p) = \frac{d}{dt}\phi_t(p)_{|t=0}$. We see that since the ϕ_t's are equivariant with respect to the $\mathbb{R}/\mathbb{Z}$-action, X also becomes equivariant. This implies that it descents to a vector field on M which we denote by X_M, and furthermore that the function $p \mapsto \alpha_p(X(p))$ is invariant, so it descents to a function $h_X = -\alpha(X) \in C^\infty(M)$ on M. We have

$$0 = \frac{d}{dt}\phi_t^* \alpha = \mathcal{L}_X \alpha = d\iota_X \alpha + \iota_{X_M} d\alpha = -dh_X + \iota_X \omega,$$

so X_M is a hamiltonian vector field on M, and h_X is a, not necessarily, normalised hamiltonian function for X_M.

The above says that we have a map $\mathfrak{g}_\alpha \to C^\infty(M)$ given by $X \mapsto h_X$, and since both Lie algebras are extensions of $C_0^\infty(M)$ by $\mathbb{R}$, the 5-lemma gives that it is an isomorphism, so we only need to show that it is a Lie algebra homomorphism.

Take $X, Y \in \mathfrak{g}_\alpha$ then we have

$$\begin{aligned} h_{[X,Y]} &= -\alpha([X,Y]) = -\omega(X,Y) - Y\alpha(X) + X\alpha(Y) \\ &= -dh_X(Y) + dh_X(Y) - dh_Y(X) = -dh_Y(X) = -\omega(Y,X) = \{h_X, h_Y\}. \end{aligned}$$

□

Let us now see where the first problem takes us. We see from remark 8.2.4 that the obstruction to extend ω to an integral form is a torsion class in $H^3(Z,\mathbb{Z})$, so this is really just a question of rescaling the symplectic form on M. The real problem lies in the normalising condition $\int_M \Omega^{n+1} = 0$. As we saw in section 8.2 we have to divide by $(n+1)$vol when we normalise the coupling class. That is, we can normalise our integral class if and only if $[\int_M \Omega^{n+1}]$ lies in the image of the map

$$H^2(Z,\mathbb{Z}) \overset{\times(n+1)\mathrm{vol}}{\to} H^2(Z,\mathbb{Z}).$$

In the universal case $Z = B\mathrm{Ham}$, we have $\pi_1(Z) = 0$ so the Serre spectral sequence gives us the exact sequence

$$H^2(Z) \to H^2(Y) \to H^2(M),$$

so we see that the only way of changing the cohomology class of the extension is by adding a class from $H^2(Z)$. If we change our extension form Ω by adding a 2-form $\tau \in \Omega^2(Z)$ from the base, we get that

$$\int_M (\Omega + p^*\tau)^{n+1} = \int_M \Omega^{n+1} + (n+1)\mathrm{vol}\tau,$$

and we see that there is a single well-defined obstruction $o \in H^2(Z, \mathbb{Z}/(n+1)\mathrm{vol})$ to getting an integral coupling class. We have the following

Proposition 9.1.3. *The obstruction class $o \in H^2(B\mathrm{Ham}, \mathbb{Z}/(n+1)\mathrm{vol})$ vanishes if and only if the central extension (9.1) splits.*

The proof is a bit technical, and the *only if* part will occupy the rest of this section. We will postpone the *if* part of the proof until the end of the next section, after we have developed more of the theory.

Before we go into the details of the proof, we will need some preliminary results on hamiltonian fibrations over S^2. This is because of the following calculation: We have that $B\mathrm{Ham}$ is simply connected so the Hurewitz and the universal coefficient theorems give us that

$$\begin{aligned} H^2(B\mathrm{Ham}, \mathbb{Z}/(n+1)\mathrm{vol}) &= \hom(\pi_2(B\mathrm{Ham}), \mathbb{Z}/(n+1)\mathrm{vol})) \\ &= \hom(\pi_1(\mathrm{Ham}), \mathbb{Z}/(n+1)\mathrm{vol})). \end{aligned}$$

This implies that we only have to look at the case of hamiltonian fibrations over S^2.

Such a fibration is given by a clutching function, i.e. a map $\phi : S^1 \to \mathrm{Ham}$, and it is possible to make quite explicit calculations in this case see e.g. [27] which we follow in the construction of a coupling form below.

Let ϕ_t be a loop in Ham. Let D^2_+ and D^2_- be two copies of D^2 with the usual and the reversed orientation respectively. Let $\Phi : M \times \partial D^2_- \to M \times \partial D^2_+$ be given by $\Phi(x,t) = (\phi_t(x), t)$ and set

$$Y_\phi = M \times D^2_+ \cup_\Phi M \times D^2_-.$$

This defines a fibration $Y_\phi \to S^2$. It is not hard to see that homotopic loops define isomorphic bundles.

A coupling form in the fibration $Y_\phi \to S^2$ is constructed as follows. Let h_t be the normalised (that is $h_t \in C^\infty_0(M)$) time-dependent hamiltonian function corresponding to the loop of hamiltonian diffeomorphisms ϕ_t. Let $c : [0,1] \to [0,1]$ be a monotonely increasing function which is equal to 0 near 0 and equal to 1 near 1. The coupling form is then given by

$$\Omega = \begin{cases} \omega & \text{on } M \times D^2_+ \\ \omega + d(c(r)H_t(x)) \wedge dt & \text{on } M \times D^2_-, \end{cases}$$

where $(r,t) \in D^2_-$ is the radius and the normalised angle respectively and $H_t(x) = h_t(\phi_t(x))$. It is clear that Ω is closed and extends ω so we only need to see that

$$\int_{Y_\phi} \Omega^{n+1} = 0$$

in order to conclude that Ω is a coupling form. We have

$$\Omega^{n+1} = \begin{cases} 0 & \text{on } M \times D^2_+ \\ (n+1)\omega^n \wedge d(c(s)H_t(x)) \wedge dt & \text{on } M \times D^2_-, \end{cases}$$

so

$$\begin{aligned} \int_{Y_\phi} \Omega^{n+1} &= (n+1) \int_{M \times D^2_-} \omega^n \wedge d(c(r)H_t(x)) \wedge dt \\ &= -(n+1) \int_{M \times S^1} H_t(x)\omega^n \wedge dt \\ &= -(n+1) \int_{S^1} \left(\int_M H_t(x)\omega^n \right) \wedge dt \\ &= -(n+1) \int_{S^1} \left(\int_M h_t(x)\omega^n \right) \wedge dt = 0, \end{aligned} \tag{9.2}$$

since $h_t \in C^\infty_0(M)$.

The coupling form Ω constructed above does not necessarily have integral periods. By changing the above construction a little we can construct an extension that has, but then we can no longer guarantee that h_t has zero mean, and the last calculation above will fail in general, i.e. we will not get a coupling form.

Since $\mathbb{R}/\mathbb{Z}$ is connected, the long exact sequence of homotopy groups we get from (9.1) implies that $\pi_1(\mathcal{G}_\alpha) \to \pi_1(\mathrm{Ham})$ is surjective, so we can pick a loop $\tilde{\phi}_t \in \mathcal{G}_\alpha$ over $\phi_t \in \mathrm{Ham}$. In the same way as before, we have a function $\tilde{\Phi} : L \times \partial D^2_- \to L \times \partial D^2_+$ given by $\tilde{\Phi}(x,t) = (\tilde{\phi}_t(x), t)$, and we set

$$Y_{L\tilde{\phi}} = L \times D^2_+ \cup_{\tilde{\Phi}} L \times D^2_-$$

An explicit connection in the circle bundle $Y_{L\tilde{\phi}} \to Y_\phi$ is given as follows

$$A = \begin{cases} \alpha & \text{on } L \times D^2_+ \\ \alpha + (c(r)\tilde{H}_t(x)) \wedge dt & \text{on } L \times D^2_-, \end{cases}$$

where $\tilde{H}_t(x) = \tilde{h}_t(\tilde{\phi}_t(x))$ and h_t is the (time-dependent) function associated to the unique vector field that satisfies

$$\frac{d}{dt}\tilde{\phi}_t = \tilde{X}_t \circ \tilde{\phi}_t.$$

We have as usual that the curvature $\Omega_{\mathbb{Z}} = dA$ descents to Y_ϕ, and we see that

$$\Omega_{\mathbb{Z}} = \begin{cases} \omega & \text{on } M \times D^2_+ \\ \omega + d(c(r)\tilde{H}_t(x)) \wedge dt & \text{on } M \times D^2_-. \end{cases}$$

Since $\Omega_{\mathbb{Z}}$ is a curvature form, it is clearly an integral extension, but we see that a calculation similar to (9.2) does not give us that $\Omega_{\mathbb{Z}}$ is a coupling form, but we can at least conclude that

$$(n+1)\int_{S^1}\left(\int_M \tilde{h}_t(x)\omega^n\right) \wedge dt = \int_{Y_\phi} \Omega_{\mathbb{Z}}^{n+1} \in \mathbb{Z}. \tag{9.3}$$

Lemma 9.1.4. *If the expression*

$$\int_{S^1}\left(\int_M \tilde{h}_t(x)\omega^n\right) \wedge dt \tag{9.4}$$

takes values in $\mathrm{vol} \cdot \mathbb{Z}$ *for all loops* $\tilde{\phi}_t$ *in* $\mathcal{G}_\alpha$*, then the short exact sequence*

$$\mathbb{R}/\mathbb{Z} \to \mathcal{G}_\alpha \to \mathrm{Ham}$$

splits.

Proof. Let $\tilde{\mathcal{G}}_\alpha$ be the universal cover of $\mathcal{G}_\alpha$, then we will define a map

$$H : \tilde{\mathcal{G}}_\alpha \to \mathbb{R}.$$

First we define a continuous map $H' : P\mathcal{G}_\alpha \to \mathbb{R}$ from the path space of $\mathcal{G}_\alpha$. Let, for a path ϕ_t,

$$H'(\phi_t) = (n+1)\int_M \int_I h_t^\phi dt \wedge \omega^n,$$

where h_t^ϕ is the function associated to the unique vector field that satisfies

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t.$$

We see that H' only depends on the homotopy class of the curve, since we know from the discussion above, that H' restricted to the loop space $\Omega\mathcal{G}_\alpha$ takes integral values, so in particular we have that H' vanishes on null-homotopic curves. This means that H' descents to a map $H : \tilde{\mathcal{G}}_\alpha \to \mathbb{R}$.

Note that H restricted to $\mathbb{R} = \widetilde{\mathbb{R}/\mathbb{Z}} \subseteq \tilde{\mathcal{G}}_\alpha$ is multiplication by $(n+1)\text{vol}$. Take $s \in \mathbb{R}$, then the natural representing curve γ for s is the curve that at constant speed s goes around $\mathbb{R}/\mathbb{Z}$, i.e. at time 1 it has travelled $\lfloor s \rfloor$ full times around $\mathbb{R}/\mathbb{Z}$ and ends in $[s] \in \mathbb{R}/\mathbb{Z}$. Such a curve in $\mathcal{G}_\alpha$ has the constant function $h_t(x) = s$ associated to the time-dependent vector field given by the curve, so $H([\gamma]) = (n+1)\text{vol} \cdot s$. This together with our assumption that $H(\pi_1(\mathcal{G}_\alpha)) \subseteq \text{vol} \cdot \mathbb{Z}$ implies that $H : \tilde{\mathcal{G}}_\alpha \to \mathbb{R}$ induces a map $\mathcal{G}_\alpha \to \mathbb{R}/\mathbb{Z}$ which is a left inverse to the inclusion $\mathbb{R}/\mathbb{Z} \to \mathcal{G}_\alpha$, and the sequence splits. □

Now we are finally ready to prove the first half of proposition 9.1.3.

Proof. **(of the *only if* part of proposition 9.1.3)** Assume that the obstruction vanishes. As we saw before

$$H^2(B\text{Ham}, \mathbb{Z}/(n+1)\text{vol}) = \text{hom}(\pi_1(\text{Ham}), \mathbb{Z}/(n+1)\text{vol})),$$

so we can view the obstruction as a homomorphism $o : \pi_1(\text{Ham}) \to \mathbb{Z}/(n+1)\text{vol}$ the value of o on a representing loop $\phi_t \in \text{Ham}$ is given as follows. Form the corresponding hamiltonian fibration over S^2 and pick any integral extension $\Omega_\mathbb{Z}$, then $o(\phi_t) = \int_{Y_\phi} \Omega_\mathbb{Z}^{n+1} \mod (n+1)\text{vol}$. That the obstruction vanishes implies that the expression (9.4) takes values in $\text{vol} \cdot \mathbb{Z}$, so lemma 9.1.4 gives us that the exact sequence (9.1) splits. □

The following commutative diagram with exact columns illustrates the situation above

$$\begin{array}{ccc}
\pi_1(\mathbb{R}/\mathbb{Z}) & \xrightarrow{\times(n+1)\text{vol}} & (n+1)\text{vol} \cdot \mathbb{Z} \ , \\
\downarrow & & \downarrow \\
\pi_1(\mathcal{G}_\alpha) & \xrightarrow{H} & \mathbb{Z} \\
\downarrow & & \downarrow \\
\pi_1(\text{Ham}) & \xrightarrow{o} & \mathbb{Z}/(n+1)\text{vol}
\end{array}$$

where one sees that o vanishes if $H(\pi(\mathcal{G}_\alpha)) \subseteq (n+1)\text{vol} \cdot \mathbb{Z}$.

9.2 $\mathcal{G}_\alpha$-fibrations

The above results show that it is most natural to consider $\mathcal{G}_\alpha$-fibrations when trying to carry out the construction suggested in the beginning of the last section.

In this section, we show that such fibrations have properties similar to those of symplectic fibrations mentioned in chapter 8.

A *$\mathcal{G}_\alpha$-fibration* is a fibre bundle $Y_L \to Z$ with fibre L and structure group $\mathcal{G}_\alpha$. Since $\mathcal{G}_\alpha$ is a group of bundle maps, we have an action of $\mathbb{R}/\mathbb{Z}$ on Y_L and if we set $Y = Y_L/(\mathbb{R}/\mathbb{Z})$ we get a hamiltonian fibration $Y \to Z$ with a circle bundle $Y_L \to Y$ which carries a fibrewise connection $\{\alpha_z\}_{z\in Z}$ in $Y_{Lz} \to Y_z$.

A *map of $\mathcal{G}_\alpha$- fibrations* is a bundle map

$$\begin{array}{ccc} Y'_L & \xrightarrow{\hat{f}} & Y_L \\ \downarrow & & \downarrow \\ Z' & \xrightarrow{f} & Z \end{array} ,$$

where $\hat{f}$ is equivariant with respect to the fibrewise action and preserves the fibrewise connection i.e. $\hat{f}^*\alpha_{f(z)} = \alpha_z$. This implies that we get an induced map $\bar{f} : Y' \to Y$ which gives us both a map of circle bundles and a map of hamiltonian fibrations.

We noted already in remark 7.2.2 that using a partition of unity we can extend a family of fibrewise connections to a full connection in $Y_L \to Y$.

We have the following result:

Proposition 9.2.1. *There is a one-to-one correspondence between extensions of the family $\{\alpha_z\}$ to full connections in the circle bundle $Y_L \to Y$ and connections in $Y_L \to Z$, that is, horizontal distribution $H \subseteq TY_L$, that satisfy*

$$R_{g*}H_z = H_{zg} \text{ for } g \in \mathbb{R}/\mathbb{Z}. \tag{9.5}$$

Proof. Given an extension A of $\{\alpha_z\}$ we get an extension $\Omega = dA$ of the family of symplectic forms $\{\omega_z\}$ in $Y \to Z$, which in turn gives us a horizontal distribution $H_\Omega \subseteq TY$ as in section 8.2. We define a horizontal distribution $H \subseteq TY_L$ by

$$H = \ker A \cap p_*^{-1}(H_\Omega) \subseteq TY_L.$$

Let us see that this is indeed a horizontal distribution, that is for each $z \in Y_L$ we have to show that $T_zY_L = T_z^vY_L \oplus H_z$.

First, we see that $H_z \cap T_z^vY_L = \{0\}$. Take $v \in H_z \cap T_z^vY_L$ then, if $p : Y_L \to Y$ is the projection map, we have $p_*v \in H_{\Omega p(z)} \cap T^v_{p(z)}Y = \{0\}$ so $v \in \ker p_*$, but at the same time we have $v \in \ker A$ so we conclude that $v = 0$. Now take any $v \in T_zY_L$ then we can write $p_*v = v^h + v^v$, where v^h and v^v are the horizontal and vertical parts of p_*v respectively with regard to the horizontal distribution H_Ω. If we denote by $v^{h\#} \in T_zY_L$ the horizontal lift of $v^h \in H_{\Omega p(z)} \subseteq T_{p(z)}Y$ with respect to A we see that $v^{h\#} \in H_z$ and $v - v^{h\#} \in T_z^vY_L$ since $p_*(v - v^{h\#}) = v^v \in T_z^vY$.

We also see that $R_{g*}H_z = H_{zg}$ for all $g \in \mathbb{R}/\mathbb{Z}$, since we have $R_{g*}\ker A_z = \ker A_{zg}$ and $p : Y_L \to Y$ is of course invariant under the $\mathbb{R}/\mathbb{Z}$-action.

On the other hand, let $H \subseteq T^vY_L$ be a horizontal distribution satisfying (9.5) then we can define a connection A in the circle bundle $Y_L \to Y$ as follows.

The family of connections defines splittings $T_z^vY_L \cong \mathbb{R} \oplus H_z^\alpha$ for each $z \in Y_L$, so combining this with our distribution $H \subseteq TY_L$ we get that

$$T_zY_L \cong T_z^vY_L \oplus H_z \cong \mathbb{R} \oplus H_z^\alpha \oplus H_z \cong \mathbb{R} \oplus (H_z^\alpha \oplus H_z)\,.$$

If we set $H_{Az} = H_z^\alpha \oplus H_z$, we see that we have defined a horizontal distribution in the circle bundle $Y_L \to Y$ and since $R_{g*}H_{Az} = H_{Azg}$ for all $g \in \mathbb{R}/\mathbb{Z}$, this defines a connection A with $\ker A = H_A$, and this connection clearly extends the family $\{\alpha\}$. □

Given a connection $H \subseteq TY_L$ satisfying (9.5), the curvature is

$$F_H(v,w) = [v^\#, w^\#] - [v,w]^\# = [v^\#, w^\#]^v,$$

where $v, w : Z \to TZ$ are vector fields on the base space, and $v^\#, w^\# : Y_L \to TY_L$ are their horizontal lifts. Note that since H satisfies (9.5), the lifted vector fields are equivariant under the $\mathbb{R}/\mathbb{Z}$-action. This implies that the vector field $F_H(v,w)$ is also equivariant, so plugging it into the connection form A gives a function on Y, which can be seen as a section in the adjoint bundle. We have the following curvature formula:

Lemma 9.2.2. *Let $\Omega = dA$ be the curvature of the connection A in the circle bundle $Y_L \to Y$ and let F_H be the curvature of the associated connection H in $Y_L \to Z$ then we have*

$$\Omega(v^\#, w^\#) = \alpha \circ F_H(v,w)$$

for v, w vector fields on Z.

Proof. This is a straightforward calculation

$$\begin{aligned}\Omega(v^\#, w^\#) &= dA(v^\#, w^\#) = v^\# A(w^\#) - w^\# A(v^\#) + A([v^\#, w^\#]) \\ &= A([v^\#, w^\#]) = A([v^\#, w^\#]^v) \\ &= \alpha([v^\#, w^\#]^v)\end{aligned}$$

□

Given a symplectic fibration one can define a corresponding frame bundle. This is done in e.g. [25, ch. 6]. We generalise this to $\mathcal{G}_\alpha$-fibrations.

Definition 9.2.3. Given a $\mathcal{G}_\alpha$-fibration $Y_L \to Z$ we define the associated frame bundle $\mathcal{P} \to Z$ to be the bundle over Z with fibre

$$\mathcal{P}_z = \{f : L \to Y_L \mid f^*\alpha_z = \alpha \text{ and } f \text{ is equivariant}\}.$$

This defines a principal $\mathcal{G}_\alpha$-bundle $\pi : \mathcal{P} \to Z$. If we have a $\mathcal{G}_\alpha$-connection in $Y_L \to Z$, we can define a connection 1-form in the frame bundle as follows. First note that the tangent space at a point $f \in \mathcal{P}$ is given by

$$\begin{aligned}T_f\mathcal{P} = \{&s \text{ is a section in } f^*TY_L \to L \mid \pi_* \circ \bar{f}_* \circ s = \text{const}, \\ &\Omega(s(z), -) = -dA(s(z)), \text{ and s is equivariant, i.e. } s(zg) = R_{g*}(s(z))\},\end{aligned}$$

where A is the connection form in $Y_L \to Y$ induced by the connection in $Y_L \to Z$. The vertical tangent bundle is given by

$$\mathcal{V}_f = \{s \in T_f\mathcal{P} \mid s = \bar{f}_* \circ s' \text{ for } s' \text{ a section in } TL \to L\}.$$

We have a connection $\mathcal{A}$ in $\mathcal{P}$ given as follows. For $s \in T_f\mathcal{P}$ we have

$$\mathcal{A}_f(s) = A \circ s \in C^\infty(M),$$

this defines a function on M since s is equivariant and A is invariant under the action of R_{g*}.

As noted in section 9.1, the Lie algebra of $\mathcal{G}_\alpha$ can be identified with $C^\infty(M)$, so we still have the invariant polynomials Q_k, that where defined in section 8.3. Recall that they where given by

$$Q_k(f_1, \dots, f_k) = \int_M f_1 \cdots f_k \omega^n, \quad f_1, \dots, f_k \in C_0^\infty(M).$$

These are of course defined on all of $C^\infty(M)$, where furthermore we have the invariant polynomial Q_1 given by

$$Q_1(f) = \int_M f\omega^n,$$

which clearly vanishes on the subalgebra $C_0^\infty(M)$.

As usual Chern-Weil theory gives us corresponding classes $q_k \in H^{2k}(B\mathcal{G}_\alpha, \mathbb{R})$. We see that if the short exact sequence 9.1 splits, then the pull-back of these classes by the map $\mathrm{Ham} \to \mathcal{G}_\alpha$ (which is unique, since Ham is perfect) coincides with Reznikov's classes in $H^{2k}(B\mathrm{Ham}, \mathbb{R})$. On $B\mathcal{G}_\alpha$, we furthermore have

Proposition 9.2.4. *Let $Y_L \to Z$ be a $\mathcal{G}_\alpha$-fibration then we have classes*

$$\chi_k \in H^{2k}(B\mathcal{G}_\alpha, \mathbb{Z}), \quad \textit{for } k \geq 1,$$

and $\chi_k = \mathrm{const} \cdot q_k$ *in* $H^{2k}(B\mathcal{G}_\alpha, \mathbb{R})$.

Under the map $B\mathbb{R}/\mathbb{Z} \to B\mathcal{G}_\alpha$ χ_1 *pulls back to* $\mathrm{vol} \cdot c_1 \in H^2(B\mathbb{R}/\mathbb{Z}, \mathbb{Z})$.

Proof. The first Chern class $c_1(Y_L) \in H^2(Y, \mathbb{Z})$ of the circle bundle $Y_L \to Y$ gives rise to the classes

$$\chi_k = \int_M c_1(Y_L)^{n+k} \in H^{2k}(B\mathcal{G}_\alpha, \mathbb{Z}).$$

If we pick a connection in $Y_L \to Z$ we get an extension A of $\{\alpha_z\}$. Since the χ_k's maps to the classes $\int_M [\Omega]^{n+k} \in H^{2k}(Z, \mathbb{R})$ and since the curvature of the connection is given by $F(v, w) = \Omega(v^\#, w^\#)$ the proof of proposition 8.4.1 applies in this setting as well, so up to a scalar, the classes χ_k and q_k agree in $H^{2k}(Z, \mathbb{R})$.

To see that χ_1 pulls back to $\mathrm{vol} \cdot c_1$, note that $Q_1(t) = \mathrm{vol} \cdot t$ for $t \in \mathbb{R} \subseteq C^\infty(M)$, and the invariant polynomial that maps to c_1 under the Chern-Weil homomorphism is just the identity map $\mathrm{id} : \mathbb{R} \to \mathbb{R}$. □

In [12], Gal-Kędra presents a somewhat different approach to the construction of integral classes.

Now let us return to the proof of proposition 9.1.3, where we still have to prove that we can construct a normalised integral coupling class if the central extension 9.1 splits.

Proof. **(of the *if* part of proposition 9.1.3)** From the discussion above, we see why we cannot expect an integral extension to be normalisable in general. The class $\int_M [\Omega]^{n+1} \in H^2(Z,\mathbb{Z})$ is the characteristic class corresponding to the invariant polynomial Q_1 defined above. If the central extension (9.1) splits, we can reduce the connection in the $\mathcal{G}_\alpha$-bundle to a connection that takes values in $C_0^\infty(M)$. Now Q_1 vanishes when restricted to $C_0^\infty(M)$, so in this case $\int_M [\Omega]^{n+1} = 0$. □

9.3 Secondary invariants

In this section, we use the theory developed above to carry out the idea described in the beginning of the chapter.

Theorem 9.3.1. *Given a $\mathcal{G}_\alpha$-fibration $Y_L \to Z$ with connection we have well-defined secondary classes*

$$\hat{\chi}_k \in H^{2k}_{\mathcal{D}}(Z,\mathbb{Z}),$$

which are natural with respect to maps of $\mathcal{G}_\alpha$-fibrations and induced connections.

Proof. As we saw in the last section, a connection in a $\mathcal{G}_\alpha$-fibration $Y_L \to Z$ gives an extension A of the fibrewise connection $\{\alpha_z\}$ to a connection in the circle bundle $Y_L \to Y$. This gives us a well-defined class $\Lambda = [(Y_L, A)] \in H^2_{\mathcal{D}}(Y,\mathbb{Z})$, and we now have $\hat{\chi}_k = \int_{[Y/Z]} \Lambda^{\tilde{\wedge} n+k}$. □

Since the group $\mathcal{G}_\alpha$ itself depends on the choice of connection α in the prequantum circle bundle L, it is not so easy to see directly from the construction above what happens if we change the connection. In order to say something about this issue, we first need a construction of the structure group which is independent of α.

First, recall that the action of Ham on M fixes the Deligne class of the circle bundle L with connection, so in particular it fixes the isomorphism class of L, so we can lift any $\phi \in \mathrm{Ham}$ to a map of circle bundles $\tilde{\phi} : L \to L$ over ϕ - now we do not care whether or not it preserves a connection in L. The group $\mathcal{G}_{\mathrm{Ham}}$ of such maps is an extension of Ham

$$\mathcal{G} \to \mathcal{G}_{\mathrm{Ham}} \to \mathrm{Ham}$$

by the gauge group $\mathcal{G} = \mathrm{Map}(M, \mathbb{R}/\mathbb{Z})$. Let $(\mathcal{G}_{\mathrm{Ham}})_0 \subseteq \mathcal{G}_{\mathrm{Ham}}$ denote the connected component that contains the identity map, then we have the extension

$$\mathcal{G}_0 \to (\mathcal{G}_{\mathrm{Ham}})_0 \to \mathrm{Ham},$$

where $\mathcal{G}_0 = \mathrm{Map}(M, \mathbb{R}/\mathbb{Z})_0 = \mathrm{Map}(M, \mathbb{R})/\mathbb{Z}$ is the identity component of $\mathcal{G}$. Note that inside this group we have $C_0^\infty(M)$, and if we set $\widetilde{\mathrm{Ham}} = (\mathcal{G}_{\mathrm{Ham}})_0 / C_0^\infty(M)$ we get the extension

$$\mathbb{R}/\mathbb{Z} \to \widetilde{\mathrm{Ham}} \to \mathrm{Ham}.$$

This extension is clearly equivalent to the extension (9.1) for any connection α, since we have a map $\iota_\alpha \mathcal{G}_\alpha \subseteq \mathcal{G}_{\mathrm{Ham}} \to \widetilde{\mathrm{Ham}}$ which is then an isomorphism by the (short) 5-lemma.

Since there is no canonical action of $\widetilde{\mathrm{Ham}}$ on L, we do not have a natural notion of $\widetilde{\mathrm{Ham}}$-fibration, but we still have the following:

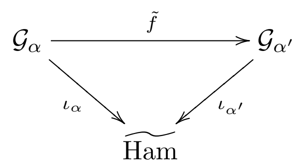
Theorem 9.3.2. *Given a symplectic manifold (M, ω) with a prequantum circle bundle (L, α), then for a principal $\widetilde{\mathrm{Ham}}$-bundle $\mathcal{P} \to Z$ with connection, there are well-defined classes*

$$\hat{\chi}_k(\alpha) \in H^{2k}_{\mathcal{D}}(Z, \mathbb{Z}),$$

which are natural with respect to bundle maps and induced connections.

If α and α' are gauge equivalent connections in L then $\hat{\chi}_k(\alpha) = \hat{\chi}_k(\alpha')$.

Proof. By identifying $\widetilde{\mathrm{Ham}}$ with $\mathcal{G}_\alpha$, we get an associated $\mathcal{G}_\alpha$-fibration $Y_L \to Z$ with connection from the principal bundle $\mathcal{P} \to Z$. The first claim follows then trivially from theorem 9.3.1. So assume that we have two gauge equivalent connections α and α'. That is, we have $f \in \mathcal{G}$ such that $f^*\alpha' = \alpha$. This gives a map $\tilde{f} : \mathcal{G}_\alpha \to \mathcal{G}_{\alpha'}$ where $\tilde{f}(\phi) = f \circ \phi \circ f^{-1}$. We see that $\tilde{f}$ fits into the following commutative diagram



This follows from the fact that

$$f \circ \phi \circ f^{-1} = \phi \circ (\phi^{-1} \circ f \circ \phi \circ f^{-1})$$

and $\phi^{-1} \circ f \circ \phi \circ f^{-1} \in \mathcal{G}_0$ because $\phi^{-1} \circ f \circ \phi$ is isotopic to f.

We have an action of $\mathcal{G}_\alpha$ on $\mathcal{P} \times L$ given by $g.(x, z) = (x.\iota_\alpha(g^{-1}), g(z))$ and similarly $\mathcal{G}_{\alpha'}$ acts on $\mathcal{P} \times L$ through $\iota_{\alpha'}$.

The map

$$F : \mathcal{P} \times L \to \mathcal{P} \times L$$

given by

$$F(x, z) = (x, f(z))$$

is equivariant in the sense that

$$\tilde{f}(g).F(x, z) = F(g.(x, z)).$$

The induced map on the quotients fits into the following commutative diagram

$$\begin{array}{ccc} \mathcal{P} \times_{\mathcal{G}_\alpha} L & \xrightarrow{F} & \mathcal{P} \times_{\mathcal{G}_{\alpha'}} L \\ & \searrow \quad \swarrow & \\ & \mathcal{P} \times_{\mathcal{G}_\alpha} M = \mathcal{P} \times_{\mathcal{G}_{\alpha'}} M & \end{array}$$

and the claim follows. □

Corollary 9.3.3. *Given a symplectic manifold (M, ω) with a prequantum circle bundle (L, α) we have well-defined classes*

$$\hat{\chi}_k(\alpha) \in H^{2k-1}(B\widetilde{\mathrm{Ham}}^\delta, \mathbb{R}/\mathbb{Z}), \quad \textit{for } k \geq 1.$$

Under the map $B\mathbb{R}/\mathbb{Z} \to B\mathcal{G}_\alpha$ $\hat{\chi}_1$ pulls back to $\mathrm{vol}{\cdot}\hat{c}_1 = \mathrm{vol}{\cdot}\mathrm{id} \in H^1(B\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = \mathrm{hom}(\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z})$.

If α and α' are gauge equivalent connections in L then $\hat{\chi}_k(\alpha) = \hat{\chi}_k(\alpha')$.

Proof. It follows directly from the interpretation of the extension form Ω as the curvature form, or equivalently from proposition 9.2.4, that the image of the class in $\Omega^{2k}_{\mathrm{cl}}(B\widetilde{\mathrm{Ham}}^\delta)$ vanishes when the connection is flat. This implies that the class lives in $H^{2k-1}(B\widetilde{\mathrm{Ham}}^\delta, \mathbb{R}/\mathbb{Z})$.

To see that $\hat{\chi}_1$ pulls back to $\hat{c}_1$, pick a connection in the universal $\widetilde{\mathrm{Ham}}$ bundle over $B\widetilde{\mathrm{Ham}}$ that pulls back to the flat connection in the bundle over $B\widetilde{\mathrm{Ham}}^\delta$. The diagram

$$\begin{array}{ccc} B\mathbb{R}/\mathbb{Z}^\delta & \longrightarrow & B\mathbb{R}/\mathbb{Z} \\ \downarrow & & \downarrow \\ B\widetilde{\mathrm{Ham}}^\delta & \longrightarrow & B\widetilde{\mathrm{Ham}} \end{array}$$

combined with the result from proposition 9.2.4 then gives that on $B\mathbb{R}/\mathbb{Z}^\delta$ $\hat{\chi}_1$ and $\mathrm{vol} \cdot \hat{c}_1$ coincide, since both classes are pull-backs of the class $\hat{\chi}_1 \in H^2_{\mathcal{D}}(B\widetilde{\mathrm{Ham}}, \mathbb{Z})$. □

9.4 Examples

Example 9.4.1. Recall from example 8.2.5 that if we have a symplectic manifold (M, ω) with a hamiltonian action of a Lie group G with moment map $\mu : M \to \mathfrak{g}^*$, then, for a principal G-bundle $P \to Z$ with connection A, we can construct a closed extension of ω in the hamiltonian bundle $P \times_G M \to Z$.

Now let us furthermore assume that we have a circle bundle with connection (L, α) on M such that the G-action on M lifts to an action on L that leaves α invariant. Then we can proceed in much the same way as in example 8.2.5 and construct a global connection form in $P \times_G L \to P \times_G M$ as follows. If λ_0 is the canonical 1-form on T^*P and ι_A as before is the dual of the projection map induced from the connection in $P \to Z$, then $\alpha' = \iota_A^* \lambda_0 + \alpha$ is a connection in $P \times_G T^*G \times L \to P \times_G T^*G \times M$. This connection descents to a connection in the circle bundle $P \times_G L \to P \times_G M$ if the moment map μ is constructed from the invariant connection, i.e. if $\mu_x(\chi) = \alpha_{\tilde{x}}(X_\chi(\tilde{x}))$

This gives, as in the general case above, a class

$$\Lambda = [(P \times_G L, \alpha')] \in H^2_{\mathcal{D}}(P \times_G M, \mathbb{Z})$$

and thus classes $\hat{\chi}_k \in H^{2k}_{\mathcal{D}}(Z, \mathbb{Z})$.

Example 9.4.2. In this example, we look more closely at the case $M = \mathbb{C}P^n$ with $\omega = \omega_{FS}$ - the usual Fubini-Study symplectic form. Since $\pi_1(\mathbb{C}P^n) = 0$, we have $\mathrm{Ham}(\mathbb{C}P^n) = \mathrm{Symp}_0(\mathbb{C}P^n)$, and there is, up to equivalence, only one prequantum circle bundle - the canonical circle bundle $H \to \mathbb{C}P^n$. Recall from section 8.3 that there is a hamiltonian action of $PSU(n+1)$ on $\mathbb{C}P^n$, and over this action we have an action of $SU(n+1)$ on the canonical bundle H. Let α be an $SU(n+1)$-invariant connection on H, such a connection can be constructed by averaging over $SU(n+1)$, i.e. pick any connection α' and let

$$\alpha = \int_{g \in SU(n+1)} g^* \alpha'$$

where the integration is done with respect to the biinvariant Haar measure on $SU(n+1)$.

All in all, we have a commutative diagram

$$\begin{array}{ccc} SU(n+1) & \longrightarrow & \mathcal{G}_\alpha \\ \downarrow & & \downarrow \\ PSU(n+1) & \longrightarrow & \mathrm{Ham} \end{array}$$

Now following example 9.4.1 for any $SU(n+1)$-bundle $P \to Z$ with connection, we get a fibration $Y_H \to Z$ with connection, and from theorem 9.3.1 we get classes

$$\tilde{\chi}_k \in H^{2k}_{\mathcal{D}}(Z, \mathbb{Z}),$$

which are natural with respect to bundle maps and induced connections. Since proposition 8.3.1 shows that the q_k's are a multiple of the usual Chern classes, we see that the $\tilde{\chi}_k$'s are a multiple of the usual Cheeger-Simons lifts of these classes, since both classes are lifts of (a multiple of) the usual Chern classes and are natural.

In the case of flat bundles, we see that classes $\hat{\chi}_k \in H^{2k-1}(B\widetilde{\mathrm{Ham}}^{\delta}, \mathbb{R}/\mathbb{Z})$ extend a multiple of the usual Chern-Simons classes $\hat{c}_k \in H^{2k-1}(BSU(n+1)^\delta, \mathbb{R}/\mathbb{Z})$.

Chapter 10

Symplectic surface bundles

This last chapter is more open-ended than the preceding ones. It builds on the interesting work of Kotschick-Morita [21, 22] on the cohomology of the discrete hamiltonian and symplectic groups of a closed oriented surface Σ_g of genus g. Our idea was to see how much of their work we could generalise to general symplectic manifolds, using classical constructions of secondary classes and by possibly carrying out these constructions on extensions of $\mathrm{Ham}(\Sigma_g)$ and $\mathrm{Symp}(\Sigma_g)$. The proofs in [21] rely heavily on techniques from the theory of surfaces, and it turned out to be harder than expected to generalise their ideas. However along the way, we where able to give an explicit description of some characteristic classes in $H^2(B\mathrm{Ham}(\Sigma_g)^\delta, \mathbb{R})^{H^1(M,\mathbb{R})}$ and thus answer a question posed by Kotschick-Morita in the end of [21]. In the first section, we will quickly review the work of Kotschick-Morita [21] in order to put the construction in section 10.2 into a proper context. We will end the chapter with a section, in which we will try to explain some of the ideas we had about generalising the work of Kotschick-Morita.

Below we will abbreviate $H^1(\Sigma_g, \mathbb{R})$ by $H^1_{\mathbb{R}}$ in order to make the notation a bit easier.

10.1 Overview of the work of Kotschick-Morita

In the papers [21, 22], Kotschick-Morita focus on the closed oriented surfaces Σ_g of genus $g \geq 2$, so let us first state some preliminary facts about this special case.

First of all, Moser's stability theorem (see e.g. [25, ch. 3]) implies that the group of orientation preserving diffeomorphisms $\mathrm{Diff}(\Sigma_g)$ homotopy retracts onto $\mathrm{Symp}(\Sigma_g)$, and that we have the short exact sequence

$$1 \to \mathrm{Symp}_0(\Sigma_g) \to \mathrm{Symp}(\Sigma_g) \to \mathcal{M}_g \to 1, \tag{10.1}$$

where $\mathcal{M}_g$ is the usual mapping class group.

Since $\mathrm{Diff}_0(\Sigma_g)$ is contractible by a theorem of Earle-Eells [9], we get that $\mathrm{Symp}_0(\Sigma_g)$ too is contractible, so $\pi_1(\mathrm{Symp}_0) = 0$ and the flux group $\Gamma \subseteq H^1_{\mathbb{R}}$ vanish. This gives us the short exact sequence

$$1 \to \mathrm{Ham}(\Sigma_g) \to \mathrm{Symp}_0(\Sigma_g) \to H^1_{\mathbb{R}} \to 0. \tag{10.2}$$

In [22], Kotschick-Morita noticed that for Σ_g as above, it is possible to extend the flux homomorphism Flux : $\mathrm{Symp}_0 \to H^1_{\mathbb{R}}$ to a crossed homomorphism $\widehat{\mathrm{Flux}}$: $\mathrm{Symp} \to H^1_{\mathbb{R}}$, i.e. a map satisfying

$$\widehat{\mathrm{Flux}}(\phi\psi) = \widehat{\mathrm{Flux}}(\psi) + \psi^* \widehat{\mathrm{Flux}}(\phi).$$

This is possible because, for the universal flat symplectic fibration $\Sigma_{\mathrm{Symp}} = E\mathrm{Symp}^\delta \times_{\mathrm{Symp}^\delta} \Sigma_g \to B\mathrm{Symp}^\delta$, there are two essentially different classes that extend the fibrewise symplectic form. There is the connection 2-form Ω which defines the flat connection. This is just the form induced from the invariant form ω on $E\mathrm{Symp}^\delta \times \Sigma_g$. Furthermore, we have on Σ_{Symp} the Euler class of the fibrewise tangent bundle $e = e(T^v\Sigma_{\mathrm{Symp}}) \in H^2(\Sigma_{\mathrm{Symp}}, \mathbb{Z})$. If we scale the symplectic form such that $\int_{\Sigma_g} \omega = 2g-2$ then $-e$ will extend $[\omega]$.

Since both $[\Omega]$ and $-e$ extend $[\omega]$, we get that $F = [\Omega] + e$ is a class in $H^1(\mathrm{Symp}^\delta, H^1_{\mathbb{R}})$. Kotschick-Morita show that any representative for F will be a crossed homomorphism extending Flux. Furthermore, such an extension is essentially unique in the sense that the cohomology class $F \in H^1(\mathrm{Symp}, H^1_{\mathbb{R}})$ is unique. This follows from the exact sequence

$$H^1(\mathcal{M}_g, H^1_{\mathbb{R}}) \to H^1(\mathrm{Symp}^\delta, H^1_{\mathbb{R}}) \to H^1(\mathrm{Symp}^\delta_0, H^1_{\mathbb{R}})^{\mathcal{M}_g},$$

since $H^1(\mathcal{M}_g, H^1_{\mathbb{R}}) = 0$ by a calculation of Morita [26]. The exact sequence is obtained from the spectral sequence associated with the short exact sequence (10.1) and with coefficients in $H^1_{\mathbb{R}}$.

Note that it is too much to expect the extension to be a homomorphism, since the usual flux homomorphism satisfies the relation

$$\mathrm{Flux}(\psi^{-1}\phi\psi) = \psi^*\mathrm{Flux}(\phi), \qquad \text{for } \phi \in \mathrm{Symp}_0 \text{ and } \psi \in \mathrm{Symp}.$$

The above can be generalised to any symplectic manifold M where there is a universal extension of the fibrewise symplectic form in $M_{\mathrm{Symp}} = E\mathrm{Symp} \times_{\mathrm{Symp}} M \to B\mathrm{Symp}$, e.g. when ω is proportional to the first Chern class $c_1 = c_1(TM)$. McDuff elaborates further on this in [24].

The existence of $\widehat{\mathrm{Flux}} : \mathrm{Symp} \to H^1_{\mathbb{R}}$ was used in [21] to construct characteristic invariants as follows. First one notes that since Ham is perfect $H_1(\mathrm{Ham}) = 0$. This together with the spectral sequence associated with the short exact sequence (10.2) implies that the flux homomorphism induces a surjective homomorphism

$$H_2(\mathrm{Symp}^\delta_0) \to H_2(H^{1\,\delta}_{\mathbb{R}}) = \Lambda^2_{\mathbb{Z}} H^1_{\mathbb{R}}.$$

Kotschick-Morita shows that for a surface Σ_g we have an isomorphism

$$(\Lambda_{\mathbb{Z}} H^1_{\mathbb{R}})_{\mathcal{M}_g} \cong S^2_{\mathbb{Z}}\mathbb{R}$$

induced by the discrete cup product pairing. This means that the extended flux homomorphism induces a surjective homomorphism

$$H_2(\mathrm{Symp}^\delta) \to S^2_{\mathbb{Z}}\mathbb{R}.$$

This induces a map

$$H_{2k}(\mathrm{Symp}^{\delta}) \to S^k(S^2_{\mathbb{Z}}\mathbb{R}),$$

which they show is surjective for $g \geq 3k$. In fact they show an even stronger result. The first Miller-Morita-Mumford class gives a surjective map $H_2(\mathrm{Symp}^{\delta}) \to \mathbb{Z}$, and combining these maps they show that, for $g \geq 3k$, there is a surjective map

$$H_{2k}(\mathrm{Symp}(\Sigma_g)^{\delta}) \to \mathbb{Z} \oplus S^2_{\mathbb{Z}}\mathbb{R} \oplus S^2(S^2_{\mathbb{Z}}\mathbb{R}) \oplus \cdots \oplus S^k(S^2_{\mathbb{Z}}\mathbb{R}).$$

The other part of [21] is concerned with the map

$$H^*(H^{1\,\delta}_{\mathbb{R}}, \mathbb{R}) \to H^*(\mathrm{Symp}^{\delta}_0, \mathbb{R})$$

induced by the flux homomorphism. Here we look at $H^1_{\mathbb{R}}$ as a discrete abelian group, so the cohomology groups $H^k(H^{1\,\delta}_{\mathbb{R}}, \mathbb{R}) = \hom_{\mathbb{Z}}(\Lambda^k_{\mathbb{Z}} H^1_{\mathbb{R}}, \mathbb{R})$ are huge, but by restricting to the continuous cohomology $H^k_{cts}(H^{1\,\delta}_{\mathbb{R}}, \mathbb{R}) \subseteq H^k(H^{1\,\delta}_{\mathbb{R}}, \mathbb{R})$ Kotschick-Morita are able to determine the kernel of this map. Here the continuous cohomology is the subgroup

$$\Lambda^k_{\mathbb{R}} H_1(\Sigma_g, \mathbb{R}) = \hom_{\mathbb{R}}(\Lambda^k_{\mathbb{R}} H^1_{\mathbb{R}}, \mathbb{R}) \subseteq \hom_{\mathbb{Z}}(\Lambda^k_{\mathbb{Z}} H^1_{\mathbb{R}}, \mathbb{R}) = H^k(H^{1\,\delta}_{\mathbb{R}}, \mathbb{R}).$$

Let $\{x_1, y_1, \ldots, x_g, y_g\}$ be a symplectic basis for $H_1(\Sigma_g, \mathbb{R})$ (see fig. 10.1) and let $\omega_0 = \sum x_i \wedge y_i$ be the standard symplectic form on $\Lambda^2_{\mathbb{Z}} H_1(\Sigma_g, \mathbb{R})$. Kotschick-Morita shows that the kernel of the map $\Lambda^*_{\mathbb{R}} H_1(\Sigma_g, \mathbb{R}) \to H^*(\mathrm{Symp}^{\delta}_0, \mathbb{R})$ is generated by $\omega_0 \wedge H_1(\Sigma_g, \mathbb{R})$. This is done by an explicit calculation using the fact that, since Symp_0 is contractible any Symp_0-bundle is trivialisable. The image of $2\omega_0$ in $H^2(\mathrm{Symp}^{\delta}_0, \mathbb{R})$ coincides with the restriction of the class $\alpha \in H^2(\mathrm{Symp}^{\delta}, \mathbb{R})$ which induces the map $H_2(\mathrm{Symp}) \to S^2_{\mathbb{Z}}\mathbb{R} \to \mathbb{R}$.

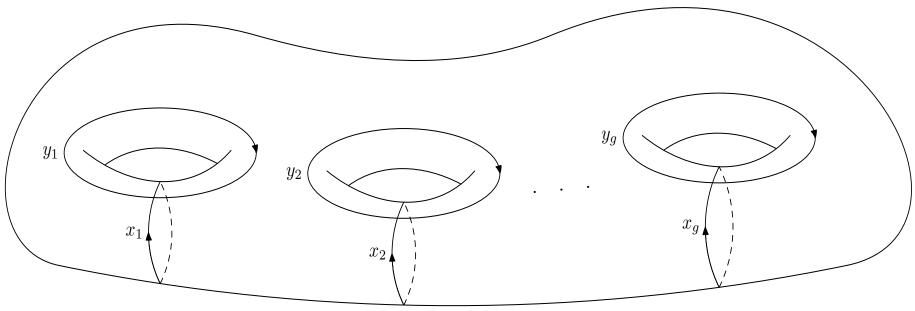


Figure 10.1: The symplectic basis $\{x_1, y_1, \ldots, x_g, y_g\}$.

From the spectral sequence associated to the short exact sequence (10.2) we get, since $H^1(\mathrm{Ham}^{\delta}) = 0$, the exact sequence

$$\begin{aligned} 0 &\to H^2(H^{1\,\delta}_{\mathbb{R}}, \mathbb{R}) \to H^2(\mathrm{Symp}^{\delta}_0, \mathbb{R}) \to \\ &\to H^2(\mathrm{Ham}^{\delta}, \mathbb{R})^{H^1_{\mathbb{R}}} \to H^3(H^{1\,\delta}_{\mathbb{R}}, \mathbb{R}) \to H^3(\mathrm{Symp}^{\delta}_0, \mathbb{R}). \end{aligned}$$

This means that the classes $\omega_0 \wedge H_1(\Sigma_g, \mathbb{R}) \subseteq H^3(H^{1\,\delta}_{\mathbb{R}}, \mathbb{R})$ can be lifted to classes in $H^2(\mathrm{Ham}^\delta, \mathbb{R})^{H^1_{\mathbb{R}}}$. Kotschick-Morita use this fact, together with the corresponding homology spectral sequence, to show that there is an inclusion $H^1_{\mathbb{R}} \subseteq H_2(\mathrm{Ham}^\delta)_{H^1_{\mathbb{R}}}$. We will give an explicit description of these lifted classes below.

10.2 Explicit construction of classes for hamiltonian surface bundles

In this section, we will start by looking at a general Lie group G with Lie algebra $\mathfrak{g}$. Let FG be the homotopy fibre of the canonical map $BG^\delta \to BG$. It is well-known that FG classifies flat G-product bundles. If G is contractible FG is homotopy equivalent to BG^δ.

The homology $H_*(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the homology of the chain complex $\Lambda^*(\mathfrak{g})$ with differential

$$d(v_1 \wedge \cdots \wedge v_k) = \sum_{i<j} (-1)^{i+j} [v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k.$$

There is a natural map $\lambda : H_*(FG) \to H_*(\mathfrak{g})$ from the homology of FG to the Lie algebra homology of $\mathfrak{g}$. We refer to [5] for an explicit, natural construction of this map at the chain level.

In the finite dimensional case, where $H \subseteq G$ is a maximal compact subgroup and G is semi-simple, the short exact sequence of Lie algebras

$$\mathfrak{h} \to \mathfrak{g} \to \mathfrak{h}/\mathfrak{g}$$

splits, so the corresponding spectral sequence is not that interesting. In the case of the infinite dimensional groups $G = \mathrm{Symp}_0$ and $H = \mathrm{Ham}$, the sequence does not, however, split in general, so there might be non-trivial differentials in the spectral sequence.

Both

$$H_1(\mathrm{Ham}^\delta) = \mathrm{Ham}/[\mathrm{Ham}, \mathrm{Ham}]$$

and

$$H_1(\mathrm{Lie}(\mathrm{Ham})) = \mathrm{Lie}(\mathrm{Ham})/[\mathrm{Lie}(\mathrm{Ham}), \mathrm{Lie}(\mathrm{Ham})]$$

vanish so by combining the E^3-term in the spectral sequences for the short exact sequences of groups (10.2) and for the corresponding short exact sequence of Lie algebras

$$\mathrm{Lie}(\mathrm{Ham}) \to \mathrm{Lie}(\mathrm{Symp}) \to H^1_{\mathbb{R}} \tag{10.3}$$

together with the natural map λ, we get the following commutative diagram

$$\begin{array}{ccc} H_3(H^{1\,\delta}_{\mathbb{R}}) & \xrightarrow{d_3} & H_2(\mathrm{Ham}^\delta)_{H^1_{\mathbb{R}}} \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda} \\ H_3(\mathrm{Lie}(H^1_{\mathbb{R}})) & \xrightarrow{d_3} & H_2(\mathrm{Lie}(\mathrm{Ham}))_{H^1_{\mathbb{R}}} \end{array} \tag{10.4}$$

On the left hand side, the map λ is nothing but the natural map

$$H_3((H^1_{\mathbb{R}}{}^\delta)) = \Lambda^3_{\mathbb{Z}} H^1_{\mathbb{R}} \to \Lambda^3_{\mathbb{R}} H^1_{\mathbb{R}} = H_3(\mathrm{Lie}(H^1_{\mathbb{R}})),$$

which is clearly surjective. So one way to see homology in $H_2(\mathrm{Ham}^\delta)_{H^1_{\mathbb{R}}}$ is by trying to calculate the image of $d_3 : H_3(\mathrm{Lie}(H^1_{\mathbb{R}})) \to H_2(\mathrm{Lie}(\mathrm{Ham}))_{H^1_{\mathbb{R}}}$.

Recall from section 8.3 that we have an Ad-invariant inner product

$$\langle f, g\rangle = \int_M fg\omega^n.$$

Now for a closed 1-form $\alpha \in \Omega^1(M)$ we define the 2-cocycle h_α on Lie(Ham) by

$$h_\alpha(f,g) = \langle f, [g,\alpha]\rangle = \int_M fdg \wedge \alpha \wedge \omega^{n-1}.$$

Clearly, this only depends on the cohomology class of α, so we get a linear map $h : H^1_{\mathbb{R}} \to H^2_{cts}(\mathrm{Lie}(\mathrm{Ham}))$, where H^*_{cts} means that we only take cohomology of the complex of continuous cochains. The classes h_α were introduced by Roger in [29] where he also announced that the map h is in fact an isomorphism for any symplectic manifold M. To our knowledge a proof of this has been published, so it should probably be seen as a conjecture rather than a theorem. Below we will see that at least for $M = \Sigma_g$ a closed surface of genus $g \geq 2$ the map is injective.

If we see h_α as a map $H_2(\mathrm{Lie}(\mathrm{Ham}))_{H^1_{\mathbb{R}}} \to \mathbb{R}$, we have

Proposition 10.2.1. *For $g \geq 2$ the map $h_a \circ \lambda : H_2(\mathrm{Ham}(\Sigma_g)^\delta_{H^1_{\mathbb{R}}} \to \mathbb{R}$ is non-trivial for all $a \in H^1_{\mathbb{R}}$.*

Proof. The idea is to show that $h_\alpha \circ d_3$ is non-trivial, since $\lambda : H_3(H^1_{\mathbb{R}}{}^\delta) \to H_3(\mathrm{Lie}(H^1_{\mathbb{R}}))$ is surjective the commutativity of (10.4) will then imply that $h_\alpha \circ \lambda$ is non-trivial. So we have to look closer at the differential $d_3 : E^3_{0,2} \to E^3_{3,0}$ in the spectral sequence for the short exact sequence of Lie algebras (10.3). Recall that the spectral sequence is obtained from a filtration of the chain complex $C_* = \Lambda^*(\mathrm{Lie}(\mathrm{Symp}))$, where

$$F_rC_n = \{x = \sum v_{i1} \wedge \cdots \wedge v_{in} \in \Lambda^n(\mathrm{Lie}(\mathrm{Symp})) \mid \text{For each } i \text{ at least } r \text{ of the } v_{ij}{}'s \text{ lies in Lie(Ham)}\}.$$

If we let $Z^r_{pq} = \{x \in F_pC_{p+q} \mid dx \in F_{p-r}C_{p+q-1}\}$ then the E^r-term in the spectral sequence is given by

$$E^r_{pq} = Z^r_{pq}/dZ^{r-1}_{p+r-1,q-r+2} + Z^{r-1}_{p-1,q+1}.$$

In order to see what the differential d_3 does, take an element $a \wedge b \wedge c \in \Lambda^3 H^1_{\mathbb{R}}$ and representatives α, β and $\gamma \in \Omega^1(\Sigma_g)$ for a, b and c respectively. We have

$$d(\alpha \wedge \beta \wedge \gamma) = -[\alpha,\beta] \wedge \gamma + [\alpha,\gamma] \wedge \beta - [\beta,\gamma] \wedge \alpha,$$

and since $[\mathrm{Lie}(\mathrm{Symp}), \mathrm{Lie}(\mathrm{Symp})] \subseteq \mathrm{Lie}(\mathrm{Ham})$ we have $d(\alpha \wedge \beta \wedge \gamma) \in F_1C_2$.

This is the differential on the E^2-page of the spectral sequence. We know that $\mathrm{Lie}(\mathrm{Ham}) = [\mathrm{Lie}(\mathrm{Ham}), \mathrm{Lie}(\mathrm{Ham})]$ so the d_2-differential vanishes. To calculate the next differential we have to add something in F_2C_3 to $\alpha \wedge \beta \wedge \gamma$ so that the differential maps it into F_2C_2. To make the calculations more transparent, we will identify Lie(Ham) with the exact 1-forms $B^1(\Sigma_g)$ and first switch to functions at the end. Pick $df_\gamma, dg_\gamma \in \mathrm{Lie}(\mathrm{Ham})$ such that $[\alpha, \beta] = [df_\gamma, dg_\gamma]$ and similarly for f_β, g_β and f_α, g_α. Then if we set

$$x = \alpha \wedge \beta \wedge \gamma - df_\gamma \wedge dg_\gamma \wedge \gamma + df_\beta \wedge dg_\beta \wedge \beta - df_\alpha \wedge dg_\alpha \wedge \alpha,$$

we see that $x \equiv \alpha \wedge \beta \wedge \gamma \mod Z^2_{2,1}$, so both elements represent $a \wedge b \wedge c \in E^3_{3,0} = E^2_{3,0} = H_3(\mathrm{Lie}(H^1_{\mathbb{R}}))$.

We have

$$\begin{aligned} dx = &-[df_\gamma, \gamma] \wedge dg_\gamma + [dg_\gamma, \gamma] \wedge df_\gamma + \\ &+ [df_\beta, \beta] \wedge dg_\beta - [dg_\beta, \beta] \wedge df_\beta - \\ &- [f_\alpha, \alpha] \wedge dg_\alpha + [dg_\alpha, \alpha] \wedge df_\alpha. \end{aligned}$$

Now let $\tau \in Z^1(\Sigma_g)$ be a closed 1-form then

$$\begin{aligned} h_{[\tau]}(dx) = &-\langle [df_\gamma, \gamma], [dg_\gamma, \tau] \rangle + \langle [dg_\gamma, \gamma], [df_\gamma, \tau] \rangle + \\ &+ \langle [df_\beta, \beta], [dg_\beta, \tau] \rangle - \langle [dg_\beta, \beta], [df_\beta, \tau] \rangle - \\ &- \langle [df_\alpha, \alpha], [dg_\alpha, \tau] \rangle + \langle [dg_\alpha, \alpha], [df_\alpha, \tau] \rangle \end{aligned}$$

and here

$$\begin{aligned} \langle [df_\gamma, \gamma], [dg_\gamma, \tau] \rangle - \langle [dg_\gamma, \gamma], [df_\gamma, \tau] \rangle &= \langle [\tau, [df_\gamma, \gamma]], dg_\gamma \rangle - \langle [\gamma, [df_\gamma, \tau]], dg_\gamma \rangle \\ &= \langle [df_\gamma, dg_\gamma], [\gamma, \tau] \rangle \\ &= \langle [\alpha, \beta], [\gamma, \tau] \rangle. \end{aligned}$$

Similarly in the other two cases, so we get that

$$h_{[\tau]}(dx) = -\langle [\alpha, \beta], [\gamma, \tau] \rangle + \langle [\alpha, \gamma], [\beta, \tau] \rangle - \langle [\beta, \gamma], [\alpha, \tau] \rangle.$$

Now choose a symplectic basis $\{x_1, y_1, \cdots, x_g, y_g\}$ for $H_1(\Sigma_g, \mathbb{R})$ (see fig. 10.1) and let x_i^* and y_i^* be the corresponding Poincaré duals. Pick 1-forms α_i and β_i representing x_i^* and y_i^* respectively, such that they have support near y_i and x_i respectively.

Then we have

$$[\alpha_i, \alpha_j] = [\beta_i, \beta_j] = 0$$

and for $i \neq j$

$$[\alpha_i, \beta_j] = 0,$$

simply because the 1-forms have disjoint support. We then get from the calculation above that for $i \neq j$

$$\begin{aligned} h_{y_j^*}(d(x_i^* \wedge y_i^* \wedge x_j^*)) &= -\langle [\alpha_i, \beta_i], [\alpha_j, \beta_j] \rangle + \langle [\alpha_i, \alpha_j], [\beta_i, \beta_j] \rangle - \langle [\beta_i, \alpha_j], [\alpha_i, \beta_j] \rangle \\ &= -\langle [\alpha_i, \beta_i], [\alpha_j, \beta_j] \rangle. \end{aligned}$$

If we let X_i and Y_i be the symplectic vector fields corresponding to α_i and β_i we get that the hamiltonian function that corresponds to the exact form $[\alpha_i, \beta_i]$ is given by $\omega(X_i, Y_i) + \text{const}$, where the constant is chosen such that the function has zero integral. That is

$$\text{const} = -\int_{\Sigma_g} \omega(X_i, Y_i)\omega = -\int_{\Sigma_g} \iota_{X_i}\omega \wedge \iota_{Y_i}\omega = -\int_{\Sigma_g} \alpha_i \wedge \beta_i = -1.$$

This gives

$$\begin{aligned} h_{y_j^*}(d(x_i^* \wedge y_i^* \wedge x_j^*)) &= -\int_{\Sigma_g} (\omega(X_i, Y_i) - 1)\,(\omega(X_j, Y_j) - 1)\,\omega \\ &= -2 - \text{vol} = -2g \neq 0 \end{aligned}$$

in which we have used that $\omega(X_i, Y_i)$ and $\omega(X_j, Y_j)$ have disjoint support. □

From the proof above we get:

Theorem 10.2.2. *The classes*

$$h_a \in \hom(H_2(\text{Ham}(\Sigma_g)^\delta)_{H^1_{\mathbb{R}}}, \mathbb{R}) = H^2(\text{Ham}(\Sigma_g)^\delta, \mathbb{R})^{H^1_{\mathbb{R}}}$$

are the explicit representatives sought after by Kotschick-Morita in [21].

After the above work had been finished we realised that in [31], Vizman have also calculated the differential d_3, but in a different context than ours. Her goal is to show that for a general symplectic manifold M, h_α will in many cases lie in the kernel of the differential $d_3 : H^2_{cts}(\text{Lie}(\text{Ham}))^{H^1_{\mathbb{R}}} \to H^3(\text{Lie}(H^1_{\mathbb{R}}))$, so it is possible to extend it to a class in $H^2_{cts}(\text{Lie}(\text{Symp}))$. She shows that

$$\begin{aligned} d_3(h_\alpha)([\beta_1] \wedge [\beta_2] \wedge [\beta_3]) = n(n-1) &\int_M \alpha \wedge \beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \omega^{n-2} - \\ - n^2 \sum_{\substack{\text{cyclic perm} \\ \text{of the } \beta_i\text{'s}}} &\int_M \alpha \wedge \beta_1 \wedge \omega^{n-1} \cdot \int_M \beta_2 \wedge \beta_3 \wedge \omega^{n-1}, \end{aligned}$$

which is consistent with our result for $M = \Sigma_g$. This more general formula should make it possible to find cohomology classes in $H^2(F\text{Ham}(M), \mathbb{R})^{H^1_{\mathbb{R}}}$ for more general symplectic manifolds in the same way as above, but since $\text{Ham}(M)$ is generally not contractible, it is probably hard to say whether these classes are coming from $H^2(B\text{Ham}(M)^\delta, \mathbb{R})^{H^1_{\mathbb{R}}}$.

10.3 Unsolved problems

In this last section, we will describe some ideas about how to generalise the work of Kotschick-Morita [21]. There are no final results in this section, but the ideas might be interesting in the sense that they present new angles on known results.

We had hoped that we would be able to generalise the result of Kotschick-Morita mentioned in section 10.1, i.e. that there is a surjective map

$$H_{2k}(\mathrm{Symp}) \to S^k(\Lambda^2_{\mathbb{Z}} H^1_{\mathbb{R}}).$$

In the case $k = 1$, this is true for any symplectic manifold M, since it only builds on the fact that $\mathrm{Ham}(M)$ is perfect, which is always true. We had hoped that we would be able to generalise this under some mild restrictions by looking at a cover of $\mathrm{Symp}(M)$ where the extended flux homomorphism is always defined. However the proof of Kotschick-Morita's result relies heavily on techniques from the theory of surfaces, and we were not able find an alternative proof.

We will still give the definition of the extended flux homomorphism on a cover of Symp. Let (M, ω) be a symplectic manifold such that ω has integral periods and let (L, α) be a prequantum line bundle. Furthermore, let $\mathrm{Symp}_L \subseteq \mathrm{Symp}$ be the subgroup of symplectomorphisms that fixes the isomorphism class of the line bundle L, so if e.g. $H^2(M, \mathbb{Z})$ is torsion-free then $\mathrm{Symp}_L = \mathrm{Symp}$. Then if we denote the group of bundle maps of L lying over a symplectomorphism by $\mathcal{G}_{\mathrm{Symp}}$ we have the following short exact sequence of groups

$$1 \to \mathcal{G} \to \mathcal{G}_{\mathrm{Symp}} \to \mathrm{Symp}_L \to 1,$$

where $\mathcal{G} = \mathrm{Map}(M, \mathbb{R}/\mathbb{Z})$ is the gauge group. Now we can define a map

$$F : \mathcal{G}_{\mathrm{Symp}} \to H^1_{\mathbb{R}}$$

by

$$F(\phi) = \phi^*\alpha - \alpha.$$

We see that the identity component of the gauge group $\mathcal{G}_0 = \mathrm{Map}(M, \mathbb{R})/\mathbb{Z}$ lies in the kernel of F, so if we set $\widetilde{\mathrm{Symp}} = \mathcal{G}_{\mathrm{Symp}}/\mathcal{G}_0$, we get an induced map

$$\widetilde{\mathrm{Flux}} : \widetilde{\mathrm{Symp}} \to H^1_{\mathbb{R}}.$$

We have that $\widetilde{\mathrm{Symp}}$ is a cover of Symp with fibre $H^1_{\mathbb{Z}}$. If we restrict to Symp_0, we see that $\widetilde{\mathrm{Symp}}_0$ has a natural trivialisation by identifying Symp_0 with $(\mathcal{G}_{\mathrm{Symp}})_0/\mathcal{G}_0$, and that $\widetilde{\mathrm{Flux}}$ and Flux coincide here as maps $\mathrm{Symp}_0 \to H^1_{\mathbb{R}}/\Gamma$.

The above fits nicely with the results of McDuff [24], where it is shown that there is always an extension of flux $\widehat{\mathrm{Flux}} : \mathrm{Symp} \to H^1_{\mathbb{R}}/H^1_{\mathbb{Z}}$. This also suggests that the above construction could make it possible to say maore about when Flux extends besides what is shown in [24].

In section 10.1, we noted that the kernel of the map

$$\Lambda^*_{\mathbb{R}} H_1(M, \mathbb{R}) \to H^*(\mathrm{Symp}, \mathbb{R})$$

was generated by $\omega_0 \wedge H_1(M,\mathbb{R})$. This was shown in [21], first by showing that the ideal generated by $\omega_0 \wedge H_1(M,\mathbb{R})$ lies in the kernel and then, by evaluating on certain homology classes, it is seen that this is in fact all of the kernel.

We will try to put this into a slightly different context, where one can see that these classes arise in the same way as in the case of finite dimensional Lie groups. This might help to understand the nature of these classes, but the underlying argument that ensures the non-triviality of the classes is really the same as the one Kotschick-Morita use.

For an ordinary Lie group G with maximal compact subgroup K the quotient G/K is contractible, and one way to construct characteristic classes in $H^*(BG^\delta,\mathbb{R})$ is as follows: Take a G-invariant closed form $\tau \in \Omega^k(G/K)^G$, this induces a form on the total space of the associated bundle $EG^\delta \times_{G^\delta} G/K \to BG^\delta$, and since G/K is contractible, the map

$$H^*(BG^\delta) \to H^*(EG^\delta \times_{G^\delta} G/K)$$

is an isomorphism, so we get a map

$$H^k(\Omega^*(G/K)^G) \to H^k(BG^\delta).$$

In the finite dimensional case, this map is injective. This is true because there exists a discrete, torsion-free subgroup $\Lambda \subseteq G$ such that $\Lambda\backslash G$ is a closed manifold. Then one can show that the composite map

$$H^k(\Omega^*(G/K)^G) \to H^k(BG^\delta) \to H^k(B\Lambda) \to H^k(\Lambda\backslash(G/K))$$

is injective.

In the infinite dimensional case with $G = \mathrm{Symp}_0$ and $K = \mathrm{Ham}$, we have $G/K = H^1_\mathbb{R}$ and $\Omega^k(G/K)^G = \Lambda^k H^{1*}_\mathbb{R}$, since Symp_0 acts on $H^1_\mathbb{R}$ by translations. Furthermore, the differential vanishes, since the forms are translation invariant, so in this case we have $H^k(\Omega^*(G/K)^G) = \Lambda^k(H^{1*}_\mathbb{R}) = \Lambda^k H_1(M,\mathbb{R})$.

If we could find a discrete subgroup in Symp_0 lying over the lattice $H^1_\mathbb{Z} \subseteq H^1_\mathbb{R}$, we would have $\Lambda\backslash(G/K) = H^1_\mathbb{R}/H^1_\mathbb{Z}$, and we could imitate the proof of the finite dimensional case to see that the map $\Lambda^k H_1(\Sigma_g,\mathbb{R}) \to H^k(B\mathrm{Symp}_0^\delta,\mathbb{R})$ is injective.

This is of course not possible, since Kotschick-Morita has shown that the kernel of this map is generated by $\omega_0 \wedge H_1(M,\mathbb{R})$. There are, however, many lattices in $H^1_\mathbb{Z} \subseteq H^1_\mathbb{R}$ of rank g, with the property that we can choose a discrete subgroup $\Lambda \subseteq \mathrm{Symp}_0$ over this lattice. One can e.g. pick symplectomorphisms $\phi_1,\dots,\phi_g$ with support near $x_1,\dots,x_g$ in fig. 10.1, and such that the $\mathrm{Flux}(\phi_i)$'s generate a lattice of rank g in $H^1_\mathbb{Z}$.

If instead we set $G = \mathrm{Symp}(\Sigma_g)$ the whole symplectic group and let K be the kernel of the extended flux homomorphism $\widehat{\mathrm{Flux}} : \mathrm{Symp} \to H^1_\mathbb{R}$, then we have that $\Omega^k(G/K)^G = (\Lambda^k H^1_\mathbb{R})^{\mathcal{M}_g}$ and again the differentials vanish. We see here that $\omega_0^g \in (\Lambda^{2g} H^1_\mathbb{R})^{\mathcal{M}_g}$, and if, in this case, we could find a discrete group $\Lambda \subseteq \mathrm{Symp}$ lying over $H^1_\mathbb{Z}$, this would imply that the corresponding class in $H^{2g}(B\mathrm{Symp}(\Sigma_g),\mathbb{R})$ is non-zero. This is quite conjectural, but if this approach is successful it would generalise the result from [21] that $\omega_0^k \neq 0$ for $g \geq 3k$. It would also give a direct explanation of why the class ω_0^2 is non-zero in $H^4(B\mathrm{Symp},\mathbb{R})$ but ω_0^2 vanishes in $H^4(B\mathrm{Symp}_0,\mathbb{R})$.

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