The Viterbo Transfer as a Map of Spectra and Twisted Chas-Sullivan products

SOLIDUM PETIT IN PROFUNDIS · UNIVERSITAS ARHUSIENSIS ·

PHD Thesis
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July 26th 2007

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Preface and Acknowledgments

This thesis reflects the scientific research I have done in my four years as a PhD student at the University of Aarhus. As part of my education I spent nine months out of the four years at Stanford University California.

The content of this Thesis is going to be the basis for an article, but there are still a few small kinks that I would like to sort out before that. These kinks are formulated as conjectures in the last section, but I fell confident that they can be resolved within a manageable time frame.

I would like to thank my thesis advisor Marcel Bökstedt for all the tremendous help over the years. I would also like to thank Ralph Cohen, Yasha Eliashberg, Søren Galatius and Eleny Ionel for many insightful consultations while at Stanford.

ABSTRACT. Let L and N be two smooth manifolds of the same dimension. Let $j\colon L \to T^*N$ be an exact Lagrange embedding. We denote the free loop space of X by ΛX. In [Vit97], Claude Viterbo constructed a transfer map $(\Lambda j)^!\colon H^*(\Lambda L) \to H^*(\Lambda N)$. We prove that this transfer map can be realized as a map of Thom spectra $(\Lambda j)_!\colon (\Lambda N)^{-TN} \to (\Lambda L)^{-TL+\eta}$, where η is a virtual bundle defined by the embedding. In [CJ02], John D.S. Jones and Ralph L. Cohen proved that the celebrated Chas-Sullivan product for a manifold N can be realized as a product on the Thom spectrum $(\Lambda N)^{-TN}$, turning it into a ring spectrum. We prove a generalized, "twisted" version of this, proving that the target of $(\Lambda j)_!$ is a Chas-Sullivan type ring spectrum. This leads to the natural conjecture that the Viterbo transfer is a ring spectrum homomorphism. We will describe partial results on this conjecture.

1 Introduction and Statement of Results

Let L and N be closed n-dimensional smooth manifolds, and let $\pi\colon T^*N \to N$ be the projection. Let $j\colon L \to T^*N$ be an exact Lagrange embedding. That is, $j^*\lambda = 0$, where λ is the canonical one form satisfying $d\lambda = \omega$ the canonical symplectic form. A trivial example of this is the zero section of N. This leads to a lot of other trivial examples, because any time-dependent Hamiltonian $H_t : T^*N \to \mathbb{R}$ leads to a Hamiltonian flow φ_t, and $\varphi_t(L)$ is an exact Lagrange embedding for all t. We denote the free loop space of a space X by ΛX. In [Vit97], Viterbo constructed a transfer map $(\Lambda j)^!$ on cohomology, such that

$$\begin{array}{ccc} H^*(\Lambda L) & \xrightarrow{(\Lambda j)^!} & H^*(\Lambda N) \\ {\scriptstyle \mathrm{Ev}_0^*}\big\uparrow\big\downarrow{\scriptstyle i^*} & & {\scriptstyle \mathrm{Ev}_0^*}\big\uparrow\big\downarrow{\scriptstyle i^*} \\ H^*(L) & \xrightarrow{(\pi\circ j)^!} & H^*(M) \end{array}$$

commutes. Here $(\pi \circ j)^!$ is the standard transfer map on cohomology, Ev_0 is the evaluation at base point, and i is the inclusion of constant curves. In this paper we call this map the Viterbo transfer. Viterbo uses this transfer as obstruction to the existence of exact Lagrange embeddings. This is of course related to the classification of exact Lagrange embeddings. It is still not known whether or not any non-trivial examples exist.

Because $j\colon L \to T^*N$ is Lagrange we get a Maslov class in $H^1(L)$. This defines a map $\Lambda L \to \mathbb{Z}$ called the Maslov index, and it turns out that the Viterbo transfer is graded on each component by this Maslov index. We prove the following theorem in this paper.

Theorem 1 *The Viterbo transfer can be realized as a map of spectra*

$$(\Lambda j)_!\colon (\Lambda N)^{-TN} \to (\Lambda L)^{-TL+\eta},$$

where η is a bundle defined by the embedding $j\colon L \to T^*N$, with local dimension the Maslov index.

In the original construction by Viterbo, the Thom isomorphism is used on what turns out to be η in this paper. However, η is not necessarily oriented, but if we assume that $(\pi \circ j)\colon L \to N$ is relative spin it will be.

In [CJ02], the authors construct the Chas-Sullivan product as a map of spectra, making $(\Lambda N)^{-TN}$ into a ring spectrum. We construct a generalized version of this. Let $\mathcal{L}(n)$ be the Grassmannian of Lagrangian subspaces in $\mathbb{R}^{2n}$. It is well-known that $\mathcal{L}(n)$ is homeomorphic to $\mathrm{U}(n)/\mathrm{O}(n)$, and taking the direct limit $\mathcal{L} = \lim_{n\to\infty} \mathcal{L}(n)$, we get the sixth space $\Omega^6\mathrm{O}$ in the eight Bott periodic spaces (see e.g. [Mil63]). Because this is a loop space there is a projection $\pi_\Omega\colon \Lambda\mathcal{L} \to \Omega\mathcal{L}$. We prove the following in this paper.

Theorem 2 *Let M be a closed smooth manifold. For any homotopy class $[f] \in [M, \mathcal{L}] \cong [M, \mathrm{U/O}]$ there is a Chas-Sullivan type ring spectrum structure on*

$$(\Lambda M)^{-T'M+\eta},$$

where η is the virtual bundle induced by the map

$$\Lambda M \to \Lambda \mathcal{L} \to \Omega \mathcal{L} \simeq \mathbb{Z} \times B\mathrm{O}.$$

We note that the virtual bundle in theorem 1 is in fact produced by such an $f \colon L \to \mathcal{L}$. We therefore conjecture that the Viterbo transfer is a ring spectrum homomorphism, and in the last section we outline some progress towards proving this.

Only the last section is devoted to Theorem 2, and in short the construction is the same as the construction of the Chas-Sullivan product, but because the map to $\mathbb{Z} \times B\mathrm{O}$ is a loop map we can extend this product to the Thom spaces given by adding the bundles classified by the map. The majority of the paper consists of the construction of the map in Theorem 1, and since this is a quite extensive construction, we outline the general ideas.

Outline of proof of Theorem 1: We define the space $T^*\Lambda_r N$ as the cotangent space of piecewise geodesics. On these we define functions $A_r : T^*\Lambda_r N \to \mathbb{R}$ depending on a parameter $\mu > 0$ and other parameters. These functions have all their critical values in an interval (a, b), and are essentially approximations of the action integral on finite dimensional manifolds.

In the actual construction, we use the theory of homotopy indices described in section 2, but for the purpose of this overview we assume that A_r is a Morse function. Because the manifold on which it is defined is not closed, we also assume that the unstable manifold of any critical point intersected with the set $A_r^{-1}([a, b])$ is compact. This means that if we define $B = A_r^{-1}(\{a\})$, and take the critical point with the lowest critical value, then the unstable manifold of this critical point intersected with B is a sphere of dimension the Morse index m minus one. If we identify B with a 0-cell, this defines the gluing of an m-cell to this 0-cell. Taking the critical point with the second lowest critical value, we can use the unstable manifold of this to glue a new cell onto the first two.

So by working our way up through the critical values, we construct a CW complex Z associated to A_r, and in section 4 we prove that Z is homotopy equivalent to the Thom space

$$\mathrm{Th}(T\Lambda_r^\mu N),$$

where $\Lambda_r^\mu N$ is the manifold of piecewise geodesics, with r pieces each having length less than μ/r.

Because $j \colon L \to T^*N$ is a Lagrange embedding, by use of the Darboux-Weinstein theorem, we can extend j to a symplectic embedding of a small neighborhood of the zero section in T^*L. Using this neighborhood and the fact that j is exact, we can adjust the definition of A_r, such that all the critical points with critical values above c, for some c with $a < c < b$, are curves inside the neighborhood of L. In fact, if we quotient by the subcomplex Y defined by cells in Z coming from critical points with critical values less than c, we get

$$Z/Y \simeq \mathrm{Th}(T\Lambda_r^{\mu_L} L \oplus \eta). \tag{1}$$

This is the main result of sections 5 through 7. It involves a lot of technical details, but the most important aspect of this calculation is that the function A_r depends on the cotangent space structure in T^*N, and one can define a similar function A_r' on the the cotangent space of L, where the associated

CW-complex is

$$\mathrm{Th}(T\Lambda_r^{\mu_L}L).$$

The two functions A_r and A_r' are different because they depend on the foliation given by vertical directions in the cotangent bundle, and close to L there are two different choices of vertical direction, one coming from T^*L and one from T^*N. The definition of $A_r^{g,S_{\mathcal{L}}}$ in section 5 is a generalization of A_r, where $S_{\mathcal{L}}$ is a generalization of having a choice of vertical directions. Now, one can homotope $S_{\mathcal{L}}$, and relate functions defined by different vertical directions. However, the two choices are not homotopic, because if so, the associated CW complexes of A_r and A_r' would be the same. So instead we stabilize by trivial factors, and homotope the "difference" between the two choices onto the trivial factor. On the trivial factor we can calculate the homotopy type of the associated CW complex, and see that the difference is indeed given by the Thom suspension in equation (1).

The quotient $Z \to Z/Y$ then defines a map

$$\mathrm{Th}(T\Lambda_r^{\mu}N) \to \mathrm{Th}(T\Lambda_r^{\mu_L}L \oplus \eta),$$

and by adding an appropriate Thom bundle on both sides it turns out that this is a map of Thom spectra

$$(\Lambda_r^{\mu}N)^{-TN} \to (\Lambda_r^{\mu_L}L)^{-TL\oplus\eta}.$$

The rest of the construction is checking that this commutes with inclusions into spaces defined by a larger μ, μ_L and r, so that we get a map

$$(\Lambda N)^{-TN} \to (\Lambda L)^{-TL\oplus\eta}.$$

2 The Homotopy Index

Most parts of this section are well-known, and done in more generality in [Con78]. However, we will not need it in such generality, we have altered the theory slightly to suit the purpose of this paper, and it is a vital part of the construction.

Let M be a smooth manifold without boundary, and let $f\colon M \to \mathbb{R}$ be a smooth function. A pseudo-gradient X for f is a smooth vector field on M such that the directional derivative $X(f)$ is positive at non-critical points and $X = 0$ at critical points. Since the convex combination of pseudo-gradients are pseudo-gradients, and since the choice in a single fiber is contractible, the choice of a pseudo-gradient is a contractible choice. We will denote the flow of $-X$ by ψ_t.

Let a and b be regular values of f which are isolated from the critical values of f. We wish to define the *homotopy index* $I_a^b(f,X)$. In [Con78] this is called the Morse index, but we adopt the name homotopy index to avoid confusion.

Definition 2.1 *An index pair (A,B) is a pair of subspaces of M satisfying the following properties*

I1: $B \subset A \subset f^{-1}([a,b])$.

I2: A *and* B *are compact.*

I3: $\mathrm{int}(A)$ *contains all critical points of* f *with critical values in* (a,b).

I4: There is a function $c\colon A \to \mathbb{R} \cup \{\infty\}$ *such that for all* $x \in A$ *we have* $\{t \geq 0 \mid \psi_t(x) \in A\} = [0, c(x)]$ *and* $\{t \mid \psi_t(x) \in B\} = \{c(x)\} \cap \mathbb{R}$.

When such index pairs exist we define the Homotopy index

$$I_a^b(f, X) = [A/B],$$

where $[-]$ *denotes homotopy equivalence class. If* $X = \nabla f$ *we write* $I_a^b(f)$.

This is slightly different from [Con78], but we use this definition for simplicity. Note that it can be shown that the function c in I4 is continuous. B is called the *exit set*.

The following lemma shows that the homotopy index does not depend on the choice of index pair (A, B).

Lemma 2.2 $I_a^b(f, X)$ *is well-defined.*

Proof: If (A, B) and (A', B') is any pair of index pairs, we construct the intersection pair as follows

$$\begin{aligned} B'' &= (B \cup B') \cap (A \cap A') \\ A'' &= (A \cap A'). \end{aligned}$$

This is an index pair with $c''(x) = \min(c(x), c'(x))$, thus reducing to the case where A and B are subsets of A'.

Given any index pair (A, B), we can by using the flow ψ_t create new index pairs

$$\begin{aligned} A_2 &= \psi_t(A) \\ B_2 &= \psi_t(B) \end{aligned}$$

provided we stay within $f^{-1}([a,b])$. Being careful, one can choose t as a function of points in A and still get diffeomorphic pairs. If B is not a subset of B', the function $c'_{|B}$ is not the zero function, and we can use the flow and the function c' to get the index pairs into a position where $B \subset B'$, thus reducing to $A \subset A'$ and $B \subset B'$.

Because $(A \cup B')/B' \simeq A/B$, we can replace (A, B) with $(A \cup B', B')$ which is a new index pair. This further reduces to the case $(A, B) \subset (A', B)$.

For any index pair (A, B) we can define a "flow" $P_t\colon A \to A$ by

$$P_t(x) = \varphi_{\min(t, c(x))}(x).$$

This map flows the set $\{x \mid c(x) \leq t\}$ into B. In particular it fixes B. This gives us a new pair $(P_t(A), B)$ with an equivalent quotient. In fact, the induced map

$$P_t\colon A/B \to A/B$$

is homotopic to the identity. In our case, the flow P_t for the small pair is the restriction of P'_t for the large pair.

Since $\overline{A' - A}$ is compact and $X_{|\overline{A'-A}}$ is non-zero, it has a lower bound, and we can thus find $t_0 > 0$ such that $P'_{t_0}(A') \subset A$. This defines a homotopy right inverse to the inclusion from $A\big/B$ to $A'\big/B$. Doing the same for the pairs $(P'_{t_0}(A'), B) \subset (A, B)$ gives a homotopy right inverse to this map. $\square$

We will need the concept of a *good* index pair (A, B). The definition uses a Riemannian metric on M, but does not depend on it.

Definition 2.3 *An index pair (A, B) is called good if $B \subset f^{-1}(a)$ and if there is an $\varepsilon > 0$ such that when $\sup_{p \in A}\|X_p - X'_p\| < \varepsilon$ then the flow defined by $-X'$ exits A only through B.*

This, together with an assumption on the critical points, will ensure that perturbing the data involved in defining the homotopy index does not change the homotopy index.

Lemma 2.4 *Let $f_s, s \in I$ be a homotopy of smooth functions, and $X_s, s \in I$ a homotopy of vector fields such that X_s is a pseudo-gradient for f_s. Let a, b be regular values isolated from the critical points for all f_s.*

Assume that for all $s_0 \in I$ we have a good index pair (A_{s_0}, B_{s_0}) defining $I_a^b(f_{s_0}, X_{s_0})$ and an $\varepsilon > 0$ such that $\mathrm{int}(A_{s_0})$ contains all critical points of f_s with critical value in (a, b) for $s \in [s_0 - \varepsilon, s_0 + \varepsilon]$. Then

$$I_a^b(f_0, X_0) = I_a^b(f_1, X_1).$$

Proof: Given s_0, we wish to prove that the good index pair $(A, B) = (A_{s_0}, B_{s_0})$ is an index pair for $I_a^b(f_s, X_s)$ when s is sufficiently close to s_0. However, this is not possible because we cannot be certain that I1 is satisfied. Because a and b are isolated critical points, we can replace a and b by $a - \delta$ and $b + \delta$ for some small δ without changing the indices. Now I1 is not a problem.

I2 is obvious and I3 is part of the assumptions in the lemma. Since the good pair assumption makes sure that we only exit A through B, we only need to prove that for any point x in B we have $\{t \geq 0 \mid \psi_t^s(x) \in A\} = \{0\}$ and I4 will follow. This is equivalent to proving that the flow does not return to A when exiting.

Since $-X_{s_0}(f_{s_0})$ restricted to B is negative, the same is true for $-X_s(f_{s_0})$ for s close to s_0, and thus we can find $\delta > 0$ such that $f_{s_0}(\psi_t^s(p))$ is strictly decreasing for $t \in [0, \delta]$ and $p \in B_{s_0}$. Because $f_{s_0}(B) = \{a\}$ and $f_{s_0}(A) \subset [a, b]$ we get for $p \in B$ and $t \in (0, \delta]$ that $\psi_t^s(p)$ is not in A and there is an $\varepsilon > 0$ such that $f_{s_0}(\psi_\delta^s(p)) < a - \varepsilon$. This implies (for s possibly closer to s_0) that $f_s(\psi_\delta^s(p)) < a - \varepsilon/2$. We can similarly assume that $f_s(A_{s_0}) > a - \varepsilon/3$, so the flow for $-X_s$ has exited B_{s_0} and will not return to A_{s_0} because it is a pseudo-gradient for f_s. $\square$

As the following lemma shows, there is a way of producing good index pairs by using what we will call *cut-off* functions.

Lemma 2.5 *Assume that* M *has a Riemannian metric and that there exists non-negative functions* $g_1, g_2, \ldots, g_n \colon M \to \mathbb{R}$ *and constants* $s_1 < t_1, s_2 < t_2, \ldots, s_n < t_n$ *such that*

$$A = f^{-1}([a,b]) \cap \{x \in M \mid g_j(x) \leq s_j + \frac{t_j - s_j}{b-a}(b - f(x))\}$$
$$B = f^{-1}(a) \cap A$$

are compact and the interior of A *contains all the critical points of* $f_{|(a,b)}$*. If*

$$-X_x(g_j) < \frac{t_j - s_j}{b-a} X_x(f) \tag{2}$$

for all $x \in \partial A$ *that satisfy*

$$g_j(x) = s_j + \frac{t_j - s_j}{b-a}(b - f(x)),$$

then (A, B) *is a good index pair.*

We will use this lemma repeatedly. In some cases when $X = \nabla f$ we will prove $\|\nabla g_j\| < \frac{t_j - s_j}{b-a}\|\nabla f\|$, which implies (2). Often (2) will be proven on much larger sets than needed.

Proof: At any point $x \in \partial A$ we must have $f(x) - b \leq 0$ and

$$g_j(x) - s_j - \frac{t_j - s_j}{b-a}(b - f(x)) \leq 0$$

satisfied. The assumptions in the lemma and the fact that X is a pseudo-gradient ensure that for any vector v close to X_x we have: If any of these inequalities is an equality then the directional derivatives of the left hand side in direction $-v$ is negative. So $-v$ points into A, except if the equality $f(x) = a$ is satisfied in which case $-v$ must point out of the set. The boundary is compact so there is an $\varepsilon > 0$ such that this holds for $\|v - X_x\| < \varepsilon$ □

Some very important aspects of homotopy indices are the natural inclusion and quotient maps

$$i \colon I_a^b(f, X) \to I_a^c(f, X)$$
$$q \colon I_a^c(f, X) \to I_b^c(f, x)$$

where $a < b < c$. These maps are constructed as follows. If (A, B) is a good index pair, for $I_a^c(f, X)$ then $(A \cap f^{-1}([a,b]), B)$ is a good index pair for $I_a^b(f, X)$, and the inclusion is the obvious one. Similarly, $(A \cap f^{-1}([b,c]), A \cap f^{-1}(\{b\}))$ is a good index pair for $I_b^c(f, X)$, and the quotient is the map collapsing the set $A \cap f^{-1}([a,b])/B$.

One can create similar constructions for any index pair, not necessarily good, but we will not need that.

3 The Action Integral in Cotangent Bundles

All parts of this section are well-known, but the methods are vital to the construction, and we need to introduce the notation anyway.

Let M be a smooth n-dimensional manifold. We denote points in the cotangent bundle T^*M by (q,p), where q is in M and p is a cotangent vector. Given any coordinate chart $h\colon U \subset M \to U' \subset \mathbb{R}^n$ we define the induced trivialization of the cotangent bundle T^*U by the pullback of cotangent vectors

$$\overline{h}(q,p) = (h(q), p \circ D_{h(q)}(h^{-1})) \in T^*U' \cong U' \times \mathbb{R}^n.$$

We denote points in $T^*U' \subset \mathbb{R}^n \times \mathbb{R}^n$ by (x,y), and the standard symplectic form on $\mathbb{R}^{2n}$ is $\omega_0 = \sum_{i=1}^n dy_i \wedge dx_i$. One can check that the 1-form λ defined below does not depend on the chart h. So

$$\lambda = pdq = \overline{h}^*(\sum_{i=1}^n y_i dx_i)$$
$$\omega = d\lambda = \overline{h}^*\omega_0$$

defines a canonically exact canonical symplectic structure on T^*M.

Given any Hamiltonian $H\colon T^*M \to \mathbb{R}$, we can define the associated Hamiltonian vector field X_H by the formula $dH = \omega(X_H, -)$. This is well-defined because ω is non-degenerate. The flow of X_H will be denoted φ_t and is called the Hamiltonian flow.

Given any metric on M we can induce a metric on T^*M in the following way: At each point (q,p) we split the tangent space $T_{(q,p)}(T^*M)$ in two components, the vertical, which is canonically defined without the metric as the fiber directions, and the horizontal defined by the connection given by the metric on M. This identifies $T_{(q,p)}T^*N$ with $T_qM \times T_q^*M$, on which we use the metric from M to define the inner product on each factor - making this splitting orthogonal. We can also define an almost complex structure J in this splitting by using the isomorphism $\phi_q\colon T_qM \to T_q^*M$ induced by the metric on M

$$J(\delta q, \delta p) = (-\phi^{-1}(\delta p), \phi(\delta q)).$$

This is be compatible with the symplectic structure and the induced metric. The formula for X_H can be rewritten using this metric and almost complex structure as

$$X_H = -J\nabla H. \tag{3}$$

For any smooth manifold X, let ΛX be the space of piecewise smooth curves. The action $A_H\colon \Lambda T^*M \to \mathbb{R}$ is defined by

$$A_H(\gamma) = \int_\gamma \lambda - H(\gamma(t))dt.$$

It is known that the critical points of this integral are precisely the 1-periodic orbits of the Hamiltonian flow (a calculation in the following section proves this).

For a moment we look at the special case in which H only depends on the length of the cotangent vector - that is

$$H(q,p) = h(\|p\|)$$

where $h\colon [0,\infty] \to \mathbb{R}$ is smooth and all derivatives vanish at 0.

In this case we calculate the gradient of H in the orthogonal splitting:

$$\nabla H = (0, h'(\|p\|)p)$$

We get 0 in the first factor because parallel transport does not change the norm of p. Using equation (3) we see that

$$X_H = -J(0, h'(\|p\|)p) = (h'(\|p\|)\phi^{-1}(p), 0) = (h'(\|p\|)p, 0).$$

As hinted, from now on we will suppress ϕ from the notation.

Because this vector field is 0 in the last factor, it will parallel transport p and hence this becomes a reparametrization of the geodesic flow. This describes the 1-periodic orbits as closed geodesics with lengths corresponding to $h'(\|p\|)$. In particular we get the formulas:

$$\begin{aligned} \varphi_t(q,p) &= \exp_{(q,p)}(th'(\|p\|)(p,0)) \\ \varphi_t(q,p)_q &= \exp_q(th'(\|p\|)p). \end{aligned}$$

The action of these orbits is calculated to be $\|p\|h'(\|p\|) - h(\|p\|)$. This corresponds to taking minus the intersection of the y-axis with the tangent of h at the point $(x, h(x))$ as in figure 1. This geometric way of calculating the action

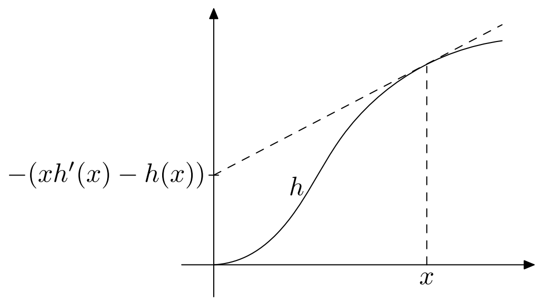

Figure 1: Geometric calculation of critical values.

is very useful for this type of Hamiltonian, and will be used repeatedly.

4 Finite Approximations of the Action Integral in Cotangent Bundles for a Hamiltonian Linear at Infinity

This section is inspired by work in [Cha84] and [Vit97], which use the broken geodesic approach to do what Floer homology later did more generally. We will define finite dimensional approximations of the action integral in cotangent bundles, and compute their homotopy indices. This is the approach used by Viterbo to construct his transfer in [Vit97]. He uses the theory of generating functions, but we will more explicitly construct the functions and obtain more control over the homotopy indices.

We assume that N is a closed manifold with a Riemannian metric and an injective radius $3\varepsilon_0$. On T^*N we have the induced metric. We also assume that $H\colon T^*N \to \mathbb{R}$ is a smooth Hamiltonian with the following property: There exist $R, \mu, c \in \mathbb{R}$, μ not a geodesic length, such that $H(q,p) = \mu\|p\| + c$ when $\|p\| > R$. The Hamiltonian flow of H is denoted φ_t.

We assume that $C_H^1 \geq \mu$ is an upper bound for $\|\nabla H\|$ on all of T^*N, and C_H^2 is an upper bound for the norm of the covariant Hessian $\|\nabla^2 H\|$ on the set $T_R^*N = \{(q,p) \in T^*N \mid \|p\| \leq R\}$. We define the space of piecewise geodesics as

$$\Lambda_r N = \{(q_j)_{j\in\mathbb{Z}_r} \in N^r \mid \operatorname{dist}(q_j, q_{j+1}) < \varepsilon_0\},$$

where $\operatorname{dist}(-,-)$ is the distance in N. This implies

$$T^*\Lambda_r N = \{(q_j, p_j)_{j\in\mathbb{Z}_r} \in (T^*N)^r \mid \operatorname{dist}(q_j, q_{j+1}) < \varepsilon_0\}.$$

We will denote a point in this space by $\overrightarrow{z} = (\overrightarrow{q}, \overrightarrow{p})$, and a single coordinate by $z_j = (q_j, p_j) \in T^*N$. These two spaces are given the product of the Riemannian metrics.

We will define functions resembling A_H on $T^*\Lambda_r N$ having the same critical points (the 1-periodic orbits) with the same critical values, and we will prove that for r large enough these functions admit good index pairs.

Definition 4.1 *For $C_H^1/r < 2\varepsilon_0$ we define*

$$A_r(\overrightarrow{z}) = \sum_{j\in\mathbb{Z}_r} \Big(\int_{\gamma_j} \lambda - H dt\Big) + \sum_{j\in\mathbb{Z}_r} p_j^- \varepsilon_{q_j},$$

where $\gamma_j\colon [0, 1/r] \to T^*N$ is the curve $\gamma_j(t) = \varphi_t(q_j, p_j)$,

$$(q_j^-, p_j^-) = \varphi_{1/r}(q_{j-1}, p_{j-1}) \qquad \text{and} \qquad \varepsilon_{q_j} = \exp_{q_j^-}^{-1}(q_j) \in T_{q_j^-}N,$$

with $\exp\colon TN \to N$ the exponential map.

The term $p_j^- \varepsilon_{q_j}$ is the pairing of cotangent vectors with tangent vectors. This function is well-defined because $C_H^1/r < 2\varepsilon_0$ implies that the distance between q_j^- and q_j is less than $3\varepsilon_0$. In fact, from now on we will assume that $C_H^1/r < \varepsilon_0/3$. Notice that if $\|p_j\| > R$, then p_{j+1}^- is the parallel transport of p_j by a geodesic in the direction of p_j, and thus $\|p_{j+1}^-\| = \|p_j\|$ (see the previous section for details).

When evaluating the function A_r on an r-pieced dissection of a 1-periodic orbit, we see that the last term vanishes because the γ_j's fit together to a closed curve. So A_r equals the action integral on such a curve.

Before the next lemma we need a few more definitions and a few abbreviations. Consider the commutative diagram of isomorphisms

$$\begin{array}{ccc} T_qN & \xrightarrow{P_{q,q'}} & T_{q'}N \\ \Big\downarrow{\scriptstyle\phi} & & \Big\downarrow{\scriptstyle\phi} \\ T_q^*N & \xrightarrow{P_{q,q'}^*} & T_{q'}^*N. \end{array}$$

The isomorphism ϕ is the one induced by the metric, which we suppressed from the notation in the previous section. We will do so again. $P_{q,q'}$ and $P_{q,q'}^*$

are defined by parallel transport along the unique geodesic connecting q and q' when $\mathrm{dist}(q,q') < 3\varepsilon_0$. We use P_{q_j,q_j^-} to define

$$\varepsilon_{p_j} = P_{q_j,q_j^-}(p_j) - p_j^- \in T^*_{q_j^-}N \simeq T_{q_j^-}N$$

both ε_{q_j} and ε_{p_j} are vectors in the (co)tangent space at q_j^-. We can parallel transport any vector in the (co)tangent space at q_j^- to the (co)tangent space at q_j and q_{j-1}. The resulting vectors will by further abuse of notation be denoted the same. So e.g. for ε_{p_j} this means

$$\varepsilon_{p_j} = p_j - p_j^- = p_j - P_{q_j^-,q_j}(p_j^-) \in T_{q_j}N$$

and

$$\varepsilon_{p_j} = p_j - p_j^- = P_{q_j^-,q_{j-1}}\big(p_j - P_{q_j,q_j^-}(p_j^-)\big) \in T^*_{q_{j+1}}N.$$

We also define $P = \max_j(\|p_j\|)$.

Lemma 4.2 *There exists a constant $K' > 0$ which is independent of r, C_H^1 and C_H^2 such that for $K = K'(C_H^2 + C_H^1)$ we have*

$$\begin{aligned}\|\nabla_{q_j}A_r + \varepsilon_{p_j}\| \leq& K\max(R,P)(\|\varepsilon_{q_j}\| + \|\varepsilon_{q_{j+1}}\|)\\ \|\nabla_{p_j}A_r - \varepsilon_{q_{j+1}}\| \leq& \frac{K}{r}\|\varepsilon_{q_{j+1}}\|,\end{aligned}$$

where $\nabla_{q_j}A_r \oplus \nabla_{p_j}A_r = \nabla_{z_j}A_r$ is the gradient with respect to the jth component in $T^*\Lambda_rN$. Furthermore, for $r > K$ the only critical points of A_r are the r-pieced dissections of the 1-periodic flow curves for the Hamiltonian flow.

Note that the r-pieced dissections of the 1-periodic orbits correspond to the points where $\varepsilon_{q_j} = \varepsilon_{p_j} = 0$ for all j.

Proof: We start out by considering one of the integration terms. In the following $\delta\gamma_j$ is a variation of the curve γ_j (a tangent field along γ_j).

$$\begin{aligned}\nabla(&\int_{\gamma_j}\lambda - Hdt)(\delta\gamma_j)\\ &= \int_0^{1/r}\gamma_{jp}(\nabla_t\delta\gamma_{jq}) + (\delta\gamma_{jp})\gamma'_{jq} - \nabla_{\gamma_j(t)}H(\delta\gamma_j(t))dt\\ &=[\gamma_{jp}(t)\delta\gamma_{jq}(t)]_0^{1/r} + \int_0^{1/r} -\gamma'_{jp}(\delta\gamma_{jq}) + (\delta\gamma_{jp})\gamma'_{jq} - \nabla_{\gamma_j(t)}H(\delta\gamma_j(t))dt\\ &= -p_j\delta q_j + p_{j+1}^-\delta q_{j+1}^- - \int_0^{1/r}(J\gamma'_j(t) + \nabla_{\gamma_j(t)}H)(\delta\gamma_j(t))dt \qquad (4)\\ &= -p_j\delta q_j + p_{j+1}^-\delta q_{j+1}^-\end{aligned}$$

The integral vanishes because γ_j is a flow curve, and thus $\gamma_j'(t) = -J\nabla_{\gamma_j(t)}H$. This is a standard calculation, and it is also a proof that the 1-periodic orbits of the flow X_H are the critical points of the action integral.

Our function depends on (q_j, p_j), $j \in \mathbb{Z}_r$, but it is convenient to continue for a while calculating the gradient as if the function depended on both $(q_j, p_j)_{j\in\mathbb{Z}_r}$ and $(q_j^-, p_j^-)_{j\in\mathbb{Z}_r}$ as independent variables. Formally we define

$$T^*\Lambda_r'N = \{(q_j, p_j, q_j^-, p_j^-) \in T^*N^{2r} \mid \operatorname{dist}(q_j, q_j^-) < 2\varepsilon_0, \operatorname{dist}(q_j^-, q_{j+1}) < 2\varepsilon_0\}.$$

We have an embedding

$$\iota\colon T^*\Lambda_r N \to T^*\Lambda_r' N$$

defined by setting $(q_j^-, p_j^-) = \varphi_{1/r}(q_{j-1}, p_{j-1})$ for all j.

We wish to extend the function A_r to a function A_r' defined on $T^*\Lambda_r'N$, and calculate the gradient of A_r' on the image of ι. So, for each point (q, p, q^-, p^-) in T^*N^2 with $\operatorname{dist}(q, q^-) \le 2\varepsilon_0$, we choose a curve connecting (q, p) with (q^-, p^-), such that if $\varphi_{1/r}(q, p) = (q^-, p^-)$ then we chose the flow curve $\varphi_t(q, p)$ used in the definition of A_r. By integrating over the chosen curve, we extend the definition of the integration term of A_r to $T^*\Lambda_r'N$. The calculation above shows that the gradient of this term on the image of ι does *not* depend on the choice of curves, and we have in fact already calculated it.

The second term we extend simply by using the same expression

$$\sum_j p_j^- \exp_{q_j^-}^{-1}(q_j).$$

So we need to calculate the gradient of the function

$$f(q, q^-, p^-) = p^- \exp_{q^-}^{-1}(q) = p^-\varepsilon_q, \qquad q \in N, (q^-, p^-) \in T^*N.$$

One of the components is obvious:

$$\nabla_{p^-} f = \varepsilon_q,$$

however, when moving q and q^- we get some interference from the curvature of N. Assume that $\|p^-\| = 1$. If $\varepsilon_q = 0$ we see that

$$\begin{aligned} \nabla_{q^-} f &= -p^- \\ \nabla_q f &= p^-. \end{aligned}$$

So by compactness we can find $k > 0$ such that

$$\begin{aligned} \|\nabla_{q^-} f + p^-\| &\le k\|\varepsilon_q\| \\ \|\nabla_q f - p^-\| &\le k\|\varepsilon_q\|. \end{aligned}$$

Here we have used the abuse of notation discussed just before the lemma to define p^- as a tangent vector at q. Using that $f(q, q^-, sp^-) = sf(q, q^-, p^-)$ we get a bound in the general case:

$$\begin{aligned} \|\nabla_{q^-} f + p^-\| \le& k\|p^-\|\|\varepsilon_q\| \\ \|\nabla_q f - p^-\| \le& k\|p^-\|\|\varepsilon_q\|. \end{aligned}$$

Adding the two terms of the gradients we obtain

$$\nabla_{q_j, p_j, q_j^-, p_j^-} A_r' = (-\varepsilon_{p_j} + \operatorname{bo}(k\|p_j^-\|\|\varepsilon_{q_j}\|), 0, \operatorname{bo}(k\|p_j^-\|\|\varepsilon_{q_j}\|), \varepsilon_{q_j}), \qquad (5)$$

where the notation bo(c) means some term bounded by c, i.e. $a = b + \text{bo}(c)$ is equivalent to $\|a - b\| \leq c$.

If $H = 0$ we have $(q_j^-, p_j^-) = (q_{j-1}, p_{j-1})$, and the gradient is just the sum of the two components:

$$\nabla_{q_j,p_j} A_r = \big(-\varepsilon_{p_j} + \text{bo}(k(\|p_j^-\|\|\varepsilon_{q_j}\| + \|p_{j+1}^-\|\|\varepsilon_{q_{j+1}}\|)), \varepsilon_{q_{j+1}}\big).$$

However, if H is non-zero we need to understand the differential of the function $\varphi_{1/r}$, and use that for $v \in T_{q_j,p_j}(T^*N)$

$$v(A_r) = (D_{q_j,p_j}(\text{Id} \times \varphi_{1/r})(v))(A_r'). \tag{6}$$

Assume $(q^-, p^-) = \varphi_{1/r}(q, p)$. We would like to compare $D_{q,p}\varphi_{1/r}$ to the parallel transport of (co)tangent vectors from q to q^- used in defining $\varepsilon_{q_{j+1}}$ at $T_{q_j}N$. That is, by using the splittings

$$T_{q,p}T^*N = T_qN \oplus T_q^*N, \quad T_{q^-,p^-}T^*N = T_{q^-}N \oplus T_{q^-}^*N$$

and the parallel transports we define

$$\mathcal{T}_{q,p} = P_{q,q^-} \oplus P_{q,q^-}^* : T_{q,p}T^*N \to T_{q^-,p^-}T^*N.$$

First consider the compact set T_R^*N. On this set we have $\|D_{q,p}\varphi_{1/r} - \mathcal{T}_{q,p}\| < k_2/r$ for some k_2. The question is how k_2 depends on C_H^1 and C_H^2. Take a normal neighborhood of (q, p) (this identifies the tangent spaces locally). In this, $\mathcal{T}_{q,p}$ is $C_H^1 k_3/r$ close to the identity because the length of the flow curve is less than C_H^1/r, and therefore k_3 only depends on the metric on the compact set $\|p\| \leq R$. On the other hand, we have $D_{q,p}\varphi_{1/r}$ is also close to the identity, but how close depends on the first order derivatives of $X_H = -J\nabla H$, which is the vector field defining the flow φ_t. This is bounded in the normal chart by $c_1 C_H^1 + c_2 C_H^2$, for some c_2 depending on the metric and some c_1 depending on how parallel the complex structure is. We conclude that k_2 can be chosen as some constant times $(C_H^1 + C_H^2)$.

On the rest of the cotangent bundle we will get a bound using the action of $\mathbb{R}_+$ given by $t(q, p) = (q, tp)$ and the fact that

$$\mathcal{T}_{q,p} = \mathcal{T}_{q,tp} \qquad \text{and} \qquad \varphi_{1/r}(q, tp) = t\varphi_{1/r}(q, p).$$

The latter only applies when $\|p\| \geq R$ and $\|tp\| \geq R$, because this is where H has the special form $\mu\|p\| + c$. Let $a_t(q, p) = (q, tp)$. Then the differential of a_t in (q, p) splitting is

$$D_{q,p}a_t = \begin{bmatrix} \text{Id} & 0 \\ 0 & t\,\text{Id} \end{bmatrix}.$$

So by decomposing

$$\varphi_{1/r}(q, p) = (\|p\|/R)\varphi_{1/r}(q, (R/\|p\|)p) = a_{\|p\|/R} \circ \varphi_{1/r} \circ a_{R/\|p\|}$$

whenever $\|p\| > R$, and using the bound we already have on the compact set $p \leq R$, we get the bound

$$D_{q,p}\varphi_{1/r} - \mathcal{T}_{q,p} = \begin{bmatrix} \text{bo}(\frac{k_2}{r}) & \text{bo}(\frac{k_2 R}{r \max(R, \|p\|)}) \\ \text{bo}(\frac{k_2 \max(R, \|p\|)}{rR}) & \text{bo}(\frac{k_2}{r}) \end{bmatrix} \tag{7}$$

for all $(q,p) \in T^*N$. Using equations (5), (6) and (7) together with

$$\frac{\|p^-_{j+1}\|}{\max(R, \|p_j\|)} \leq 1,$$

we get the estimates

$$\begin{aligned}
\nabla_{q_j} A_r &= \Big(-\varepsilon_{p_j} + \mathrm{bo}\big(K \max(R,P)(\|\varepsilon_{q_j}\| + \|\varepsilon_{q_{j+1}}\|)\big)\Big) \\
\nabla_{p_j} A_r &= \Big(\varepsilon_{q_{j+1}} + \mathrm{bo}(\frac{K}{r}\|\varepsilon_{q_{j+1}}\|)\Big)
\end{aligned}$$

with K some real number proportional to k_2. This proves the first part of the lemma.

The last part of the lemma is simple: Assume $\frac{K}{r} < 1$, then $\nabla_{p_j} A_r$ is zero if and only if $\varepsilon_{q_{j+1}}$ is zero, and having that for all j we can conclude the same for $\nabla_{q_j} A_r$ and ε_{p_j}. So the critical points are exactly the ones where $\|\varepsilon_{q_j}\| = \|\varepsilon_{p_j}\| = 0$ for all j. □

The function A_r with its gradient does not necessarily have index pairs, but we define a pseudo-gradient X with which it does. On the set where $\max_j \|\varepsilon_{q_j}\| < \varepsilon_0/4$ we use the gradient of A_r, and on the set $\max_j \|\varepsilon_{q_j}\| > \varepsilon_0/3$ we keep the non-zero p-component of the gradient of A_r, but use 0 as the q-component. In between we use some convex combination of them. So

$$X \cdot \nabla A_r \geq \|X\|^2 \geq \sum_j \|\nabla_{p_j} A_r\|^2, \tag{8}$$

and as we only made X different from the gradient on a set where this is non-zero, it is indeed as pseudo-gradient.

Lemma 4.3 *Let a and b be regular values of A_r, and let K be as in the previous lemma. If $r > K$ then (A_r, X) has a good index pair, which will contain all the critical points of any A_r coming from a small C^1-perturbation of H fixing H on the set where $\|p\| > R$. Furthermore, if we do a compactly supported change of X to another pseudo-gradient, it will not change the homotopy index.*

Proof: We will need a global lower bound for $\|X\|$ on the set where $P > 2R$. First we prove this on the set where $X \neq \nabla A_r$. Here we have some $\|\varepsilon_{q_j}\| > \varepsilon_0/4$, and from the previous lemma we have $\|X\| > (1 - K/r)\|\varepsilon_{q_j}\| > (1 - K/r)\varepsilon_0/4$.

So now we look at the case where $X = \nabla A_r$. We will get a lower bound if we show that there exist $k_1, k_2 \in \mathbb{R}_+$ such that if $G_p = \sum_j \|\nabla_{p_j} A_r\| < k_1$, then $G_q = \sum_j \|\nabla_{q_j} A_r\| > k_2$. Define

$$L_q = \sum_j \|\varepsilon_{q_j}\| \qquad \text{and} \qquad L_p = \sum_j \|\varepsilon_{p_j}\|.$$

The statement can, because of the approximation of $\nabla_{p_j} A_r$ in the previous lemma, be reduced to: There exist k_1, k_2 such that $L_q < k_1$ implies $G_q > k_2$. This is the statement we will prove.

Define $\underline{P} = \min_j \|p_j\|$. There are no 1-periodic flow curves on the compact set $R \leq \|p\| \leq 2R$, so there must exist $0 < c < 1$ such that $L_q + L_p > c$ for curves with all z_j's contained in this set. Claim: $k_1 = \min(c/2, \frac{1}{4K}, \frac{c}{8KR})$ works. We will divide the proof of this claim into two cases.

First case: $\underline{P} < P/2$. We know that for some j we have $\|p_j\| = P \geq 2R$ and for some j' we have $\|p_{j'}\| < P/2$. The "curve" $\overrightarrow{z}$ has to move this distance in p-direction and back again. So because $|\|p_j\| - \|p_{j-1}\|| = |\|p_j\| - \|p_j^-\|| < \|\varepsilon_{p_j}\|$ when $\|p_j\| \geq R$, we get that $L_p > P$. With the bound $L_q < 1/(4K)$ we see that this implies

$$
\begin{aligned}
G_q &> \sum_j \|-\varepsilon_{p_j} + \mathrm{bo}\big(KP(\|\varepsilon_{q_j}\| + \|\varepsilon_{q_{j+1}}\|)\big)\| \\
&> \sum_j \|\varepsilon_{p_j}\| - KP(\|\varepsilon_{q_j}\| + \|\varepsilon_{q_{j+1}}\|) > (P - \frac{P}{2}) > R,
\end{aligned}
$$

which is a positive constant.

The second case: $\underline{P} \geq P/2$. In this case we can, because the flow is equivariant with respect to the $\mathbb{R}_+$ action on the set $\|p\| \geq R$, multiply our "piecewise flow curve" with $2R/P$ to obtain a piecewise flow curve on the compact set $R \leq \|p\| \leq 2R$. This does not change any of the ε_{q_j}'s, but it scales the ε_{p_j}'s so we can conclude that the original curve satisfies

$$
\frac{2R}{P} L_p + L_q > c.
$$

Because $L_q < c/2$ this implies that $L_p > \frac{cP}{4R}$, which implies by using the bound $L_q < \frac{c}{8KR}$ that

$$
G_q > \sum_j \|\varepsilon_{p_j}\| - KP(\|\varepsilon_{q_j}\| + \|\varepsilon_{q_{j+1}}\|) > \frac{cP}{4R} - \frac{cP}{8R} > \frac{c}{4}.
$$

This is again a positive constant.

So we have proved that there exists $C > 0$ such that

$$
\|X\| > C
$$

on the non-compact set where $P > 2R$.

Because of equation (8) we obtain

$$
\big|X \cdot \nabla \|p_j\|\big| \leq \|X\| \leq C^{-1}\|X\|^2 \leq C^{-1} X \cdot \nabla A_r.
$$

So we can use $\|p_j\|$ as a cut-off function with $s_j > 2R$ and $t_j - s_j > C^{-1}(b - a)$ (see lemma 2.5).

Define $f_j(\overrightarrow{z}) = \mathrm{dist}(q_j, q_{j+1})$. If $f_j < 2\varepsilon_0/3$ we will, because of the assumption $C_H^1/r < \varepsilon_0/3$, get $\|\varepsilon_{q_j}\| > \varepsilon_0/3$. The way we defined X is such that at a point like this we have $X \cdot \nabla f_j = 0$. So we can use f_j as a cut-off function with any $\varepsilon_0 > t_j > s_j > 2\varepsilon_0/3$. We now have enough cut-off functions to ensure that when using lemma 2.5 we get a compact index pair inside our open manifold.

The fact that our pair contains the critical points of A_r when perturbed in C^1 follows from: As long as C_H^1/r is less that $\varepsilon_0/3$ we can only get critical points when $f_j < \varepsilon_0/3$, and we get no critical points with $p_j > R$ because of the way we defined H on $\|p\| > R$.

The last statement follows because any compact subset can be contained in an index pair constructed in this way, and changing the pseudo-gradient on the interior of an index pair will not change the homotopy index. □

Because all critical points of A_r lie in the compact set $P < R$, the set of critical values must be compact. So the total index $I(A_r, X)$ is well-defined. The lemma above tells us that changing the Hamiltonian within the proper bounds and conditions does not change the total index. So we define a specific Hamiltonian and calculate the total index. To do this we first need part of the small lemma (we will use the rest later):

Lemma 4.4 *The set of lengths of geodesics in N is closed and with measure zero.*

Proof: For any c there is a compact manifold inside the space of curves containing all geodesics with energy less than c. On this manifold the energy is smooth, and has the geodesics as critical points, so by Sards theorem the energy spectrum below c of geodesics is closed and with measure zero. But since c is arbitrary this is true for the entire spectrum. For geodesics the length is easily related to the energy. So we conclude that the length spectrum is closed and with measure zero. □

Define $H_0(q,p) = h(\|p\|)$ where $h(t) = \frac{\mu}{2}t^2$ when $t < \frac{\mu-\varepsilon}{\mu}$ for some $\varepsilon > 0$ such that $[\mu - \varepsilon, \mu]$ does not contain any geodesic lengths.

We still need $h(t) = \mu t + c$ outside a compact set, but we also want h to be convex so that all the 1-periodic orbits will lie in the set where h is quadratic. In fact we want h'' to be a bump function constantly equal to μ on the set $\|p\| < \frac{\mu-\varepsilon}{\mu}$ and zero on the set $\|p\| > \frac{\mu+\varepsilon}{\mu}$ such that it integrates to μ. Notice that $\frac{\mu+\varepsilon}{\mu} < 2$, so for this Hamiltonian we can choose $R = 2$.

Lemma 4.5 *There exists a constant $C > 0$ such that for any Hamiltonian H_0 described above we have that $r > C\mu$ implies existence of index pairs and*

$$I(A_r, X) = \mathrm{Th}(T\Lambda_r^{\mu} N),$$

where $\Lambda_r^{\mu} N$ is the manifold of piecewise geodesic curves in N, each piece having length less than μ/r.

We define $\overline{\Lambda}_r^a N$ to be piecewise geodesics, each piece having length less than or equal to a/r.

Proof: We start by defining an embedding

$$i\colon \overline{\Lambda}_r^{(\mu-\varepsilon)} N \to T^*\Lambda_r N$$

(the ε and μ coming from the definition of H_0 above) by

$$(i(\overrightarrow{q}))_j = \big(q_j, \mu^{-1} r \exp_{q_j}^{-1}(q_{j+1})\big).$$

This is a section in the bundle $T^*\Lambda_r N \to \Lambda_r N$ restricted to $\overline{\Lambda}_r^{(\mu-\varepsilon)} N$. Because $\|\mu^{-1} r \exp_{q_j}^{-1}(q_{j+1})\| < \frac{\mu-\varepsilon}{\mu}$, the point $(i(\overrightarrow{q}))_j$ will lie in the set where h is quadratic. This means that the flow curve γ_j is easy to understand, and in fact we have chosen p_j in such a way that $q_{j+1}^- = q_{j+1}$. This implies that on the image of i all ε_{q_j} are 0. In fact, this is the unique point in the fiber over q_j that flows to the fiber over q_{j+1}. The image of this embedding contains all the critical points of A_r, because it contains all the curves with $\|p_j\| \leq \mu - \varepsilon$ and $\varepsilon_{q_j} = 0$.

We will use the fiber directions ($\overrightarrow{p}$ directions) as a normal bundle. In fact because $\varepsilon_{q_j} = 0$ for all j, lemma 4.2 tells us that $\nabla_{p_j} A_r = 0$, and because this is the only point in the fiber that flows to a point over q_{j+1}, this is the only critical point when restricting A_r to the fiber.

Claim: Let $\overrightarrow{q} \in \overline{\Lambda}_r^{\mu-\varepsilon} N$ be fixed. The function $A_r(\overrightarrow{q}, \overrightarrow{p})$ goes to $-\infty$ if $\|\overrightarrow{p}\|$ goes to ∞.

This makes the point of the embedding the global maximum in the fiber.

Proof of claim: The condition $\|\overrightarrow{p}\| \to \infty$ is equivalent to $\|p_j\| \to \infty$ for some j. So we look at the terms in the definition of our finite approximation that involves p_j:

$$f(p_j) = \int_{\gamma_j} (\lambda - H dt) + p_{j+1}^- \varepsilon_{q_{j+}}.$$

Assume that $\|p_j\| > 2$. The integration part was already calculated in the previous section and is $(\|p_j\| h'(\|p_j\|) - h(\|p_j\|))/r$, which is constant on the set $\|p_j\| > 2$. Because $\operatorname{dist}(q_j, q_{j+1}) \leq (\mu - \varepsilon)/r$ and $\operatorname{dist}(q_j, q_{j+1}^-) = \mu/r$, we are in the situation depicted in figure 2. Take the metric we have on N and multiply

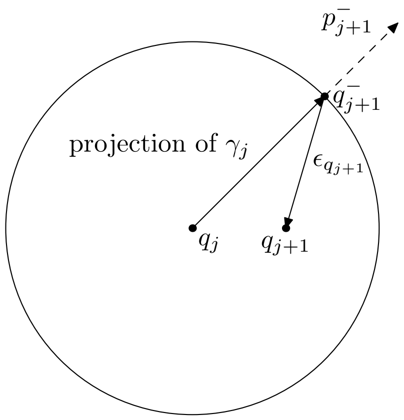

Figure 2: Position of points when the norm of p_j is larger than 2.

it with r/μ, and take a normal chart around q_j in this new metric. Then the circle in the picture is mapped to the unit circle. If the metric on the unit disc is "flat enough" then the paring $p_j^- \varepsilon_{q_{j+1}}$ will be negative, so by making μ/r smaller than some constant depending on the metric, we know that the second term is negative when $p_j > 2$. However, multiplying p_j with a constant larger than 1 scales this term by the same constant. So, A_r goes to $-\infty$ if $\|p_j\|$ goes to ∞.

On the image of the embedding the last term vanishes, and

$$A_r(i(\overrightarrow{q})) = \sum_j \int_{\gamma_j} (pdq - Hdt) = \frac{1}{2\mu} \sum_j r \|\exp_{q_j}^{-1}(q_{j+1})\|^2.$$

This is μ^{-1} times the energy functional

$$e(\gamma) = \frac{1}{2} \int_0^1 \|\gamma'(t)\| dt$$

evaluated on the piecewise geodesic $\overrightarrow{q}$. This is positive and we conclude that if we look at the set defined by $A_r \geq -1$ intersected with one of the fibers, we get a bounded set diffeomorphic to a closed disc. This is true over every point in the compact set $\overline{\Lambda}_r^{(\mu-\varepsilon)}$, so the set

$$A = \{(\overrightarrow{q}, \overrightarrow{p}) \mid \overrightarrow{q} \in \overline{\Lambda}_r^{(\mu-\varepsilon)} N, A_r(\overrightarrow{q}, \overrightarrow{p}) \geq -1\}$$

is compact and has points in each fiber.

We now change the pseudo-gradient X to another X'. We do this on a neighborhood of A - say the interior of the compact set

$$A' = \{(\overrightarrow{q}, \overrightarrow{p}) \mid \overrightarrow{q} \in \Lambda_r^{(\mu-\varepsilon/2)} N, A_r(\overrightarrow{q}, \overrightarrow{p}) < -2\}.$$

Lemma 4.3 tells us that this does not change the homotopy index. We will only specify X' on A, because the choice of a pseudo-gradient is contractible, and so it is easy to extend and interpolate. As we noted before, the $\overrightarrow{p}$ part of the gradient of A_r is non-zero except on the embedding, so we will use this as X' except in a small neighborhood of the embedding. On this set we will use the gradient of the energy functional, which we proved coincided with A_r on the embedding. Since minus the gradient of the energy functional on piecewise geodesics defines a flow that flows in a direction in which the longest geodesic piece gets shorter or stays the same length, we know that the gradient of the energy will preserve $\overline{\Lambda}_r^{(\varepsilon-\mu)} N$. This implies that the pseudo-gradient X' will point into or be parallel to the boundary of A close to the embedding. So the points at which the flow of $-X'$ exits A is precisely the points where $A_r = -1$. In each fiber this is the boundary of the disc defined by $A_r \geq -1$. So if we define

$$B = \{(\overrightarrow{q}, \overrightarrow{p}) \mid \overrightarrow{q} \in \overline{\Lambda}_r^{(\mu-\varepsilon)} N, A_r(\overrightarrow{q}, \overrightarrow{p}) = -1\},$$

then (A, B) is an index pair for $I(A_r, X')$, and they are the appropriate disc and sphere bundles needed to prove the lemma. □

This proof can be slightly modified to prove that the homotopy index of this specific Hamiltonian with respect to an interval (a, b) (simply by using the same argument for the embedding restricted to an appropriate subspace) is

$$I_a^b(A_r, X) = \mathrm{Th}(T\Lambda_r^{\min(\sqrt{2\mu a},\mu)} N) \Big/ \mathrm{Th}(\Lambda_r^{\min(\sqrt{2\mu b},\mu)} N).$$

Here the $x \mapsto \sqrt{2\mu x}$ is the conversion from μ^{-1} times energy to length. This is needed because the critical value corresponding to a geodesic was calculated in the proof to be μ^{-1} times the energy.

It seems that increasing r by 1 gives a Thom suspension of the total homotopy index. The next lemma proves this for any a, b and H.

Lemma 4.6 *Assume that $r > K$ (from lemma 4.3). The index $I_a^b(A_{r+1}, X_{r+1})$ is the relative Thom suspension of $I_a^b(A_r, X_r)$ by the bundle*

$$\mathrm{Ev}_{z_0}^* TN.$$

Note that the relative Thom suspension of A/B with a bundle ζ over A is defined to be $D\zeta/(S\zeta \cup D\zeta_{|B})$.

Proof: Previously we indexed the points in $\Lambda_{r+1}T^*N$ by $j \in \mathbb{Z}_r$, but for the purpose of this lemma, we index them by $0, 1, \dots, r$ in $\mathbb{Z}$, and the points in $\Lambda_r T^*N$ by $0, 1, \dots, r-1$. So we think of z_r as the extra point, and define the projection

$$\pi\colon T^*\Lambda_{r+1}N \to T^*\Lambda_r N$$

by forgetting z_r.

Recall the definition of A_{r+1}: We defined γ_j as the flow curve $\varphi_t(z_j), t \in [0, 1/(r+1)]$, but we could just as well have defined $\gamma_j = \varphi_t(z_j), t \in [0, t_j]$ where $\sum_j t_j = 1$ (and of course $z_j^- = \varphi_{t_j}(z_j)$). This would mean that instead of bounding $D\varphi_{1/(r+1)}$ we would have to bound $D\varphi_{\max_j t_j}$ in lemma 4.2, but this is no problem if we assume something like $t_j < 2/r$ for all j. Similarly, all other lemmas are also true in this slightly more general case. Because $\sum_j t_j = 1$, the critical points will still be the 1-periodic orbits, but they are dissected differently. The critical values of these points also stay the same. This modified construction is needed because we wish to define an alternate A_{r+1} by $t_j = 1/r$ for $j \neq r$ and $t_r = 0$. Effectively this means that $z_0^- = z_r$. We can always change the A_{r+1} back to the standard one by homotoping the t_j's, and we get the same homotopy index because we have good index pairs during the homotopy for any interval (a, b).

Fixing all points $z_0, \dots, z_{r-1}$ and q_r, but taking two different values of p_r, say p_{r1} and p_{r2}, we get

$$A_{r+1}(p_{r1}) - A_{r+1}(p_{r2}) = (p_{r1} - p_{r2})\varepsilon_{q_0} = (p_{r1} - p_{r2}) \exp_{q_r}^{-1}(q_0), \qquad (9)$$

which implies that for all parameters except p_r fixed and $q_r = q_0$, the function A_{r+1} is constant, and the gradient $\nabla_{q_r} A_{r+1}$ as a function of p_r has a unique zero. We use this to define an embedding $i_r\colon T^*\Lambda_r N \to T^*\Lambda_{r+1}N$ by putting the new point $q_r = q_0$ and p_r equal to this unique critical point. Checking the terms in A_{r+1} and A_r we see that we have a commutative diagram

$$\begin{array}{ccc} T^*\Lambda_r N & \xrightarrow{\;i_r\;} & T^*\Lambda_{r+1}N \\ & {\scriptstyle A_r}\searrow \quad \swarrow{\scriptstyle A_{r+1}} & \\ & \mathbb{R} & \end{array}$$

We defined i_r such that on the image we have $\nabla_{q_r} A_{r+1} = 0$, but from lemma 4.2 we see that also $\nabla_{p_r} A_{r+1} = 0$. Also, the embedding i_r maps critical points bijectively to critical points.

We would like to understand the Hessian with respect to variations of z_r on the embedding. This would give an understanding of how A_r behaves on a small normal bundle of the embedding. Equation (9) tells us that by changing

p_r changes the gradient $\nabla_{q_r} A_{r+1}$ by $-p_r$, and the expression in lemma 4.2 tells us that by changing q_r we approximately change ∇_{p_j} by $-q_r$, so in a normal chart and (q,p) splitting we approximately have the Hessian

$$\begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix},$$

which is similar to

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

So the negative eigenspace and positive eigenspace of this as a bundle over the embedding are both isomorphic to $\mathrm{Ev}_{q_0}^* TN$. This is a very good indication that the lemma is correct, but formally we need to define an index pair and a pseudo-gradient showing this. These can be constructed in much the same way as we constructed the index pair and pseudo-gradient in the previous proof. This case, however, is a bit more technical, because in the other case we had only negative directions in the normal bundle, but the essential ideas are the same. □

We summarize the most important facts of this section in the following proposition.

Proposition 4.7 *There exist constants $K' > 0$ and $r_0 > 0$ such that: If H is any Hamiltonian with the properties and bounds described in the beginning of this section and $r > \max(K'(C_H^1 + C_H^2), r_0\mu)$ then*

$$I(A_r, X) = \mathrm{Th}(T\Lambda_r^\mu N).$$

Proof: We will prove that the K' from lemma 4.2 also works as K' in this case. Let H_0 be as in lemma 4.5, and choose $r_0 > C$ from that lemma such that we can use the result. Also, choose r_0 large enough for us to use lemma 4.3 on H_0. For this we need that $r_0 > K'(C_{H_0}^1 + C_{H_0}^2)$, but the bounds on H_0 can be chosen to be $C_{H_0}^1 = \mu$ and $C_H^2 = C'\mu$, where C' is a constant depending only on the metric. So this choice is also linear in μ as the lemma requires. We define $H_t = (1-t)H_0 + tH$, and then by the assumption on r we can use lemma 4.3 on H_t for any t (we can use convex combinations of the bounds on H and H_0 as bounds on H_t). Thus if we chose a and b large enough to contain all the critical points of A_r^t for all $t \in I$, then by lemma 2.4 we see that the total index is defined for all t and is constant as a function of t. □

5 Generalized Approximations for Hamiltonians Defined Near the Zero Section

In this section we introduce a more general family of finite dimensional approximations to the action integral. For now we still assume that T^*N has an induced Riemannian metric. The Hamiltonians we will focus on are of the specific type

$$H\colon T^*_{\varepsilon_1} N \to \mathbb{R},$$

where $T^*_{\varepsilon_1}N = \{(q,p) \in T^*N \mid \|p\| < \varepsilon_1\}$ and $H(q,p) = h(\|p\|)$. We use two parameters $\mu > 0$ and $\delta > 0$ to define h. We want $h(0) = h'(0) = 0$, and we want h'' to be a positive function with upper bound $2\mu\delta^{-1}$ and support in $[0,\delta]$, such that h' is constantly equal to μ on $[\delta, \varepsilon_1]$. Note that any tangent to h intersects the y-axis above $-\mu\delta$ and below 0. This means that the action integral for this Hamiltonian has critical values in $[0, \delta\mu]$ (see section 3). Because of this we will assume that $a = -\mu\delta$ and $b = 2\mu\delta$ when calculating homotopy indices. We will prove that for any $\mu > 0$ not a geodesic length, we can find $\delta_0 > 0$ small enough such that when $\delta_0 > \delta > 0$, we get cut-off functions (for appropriate r's). This will imply that the homotopy index is constant during continuous variations of the not yet defined parameters needed to define these more general finite dimensional approximations. Of course the homotopy index will also be constant when perturbing H within the specifications above.

The first parameter we need in the definition of the approximations is a symplectic compatible and complete Riemannian metric g on T^*N. Note that we want the metric defined on the entire cotangent bundle. This is to make it possible to use compactness arguments on $T^*_{\varepsilon_1}N$, e.g. we get an injective radius on $T^*_{\varepsilon_1}N$, although the geodesics may exit the set. The metrics we will look at later are, however, not all defined on more than a neighborhood of $T^*_{\varepsilon_1}N$, but it is easy to extend them. This metric may be different from the induced metric we have on T^*N, which we will no longer use unless specified.

For us to get good index pairs, we will in this section and the next restrict our attention to a specific subspace of the previously defined loop space $T^*\Lambda_r N$. We define

$$\Lambda_{r,\beta}T^*_{\varepsilon_1}N = \Lambda^g_{r,\beta}T^*_{\varepsilon_1}N = \{\overrightarrow{z} \in (T^*_{\varepsilon_1}N)^r \mid e_g(\overrightarrow{z}) < \beta\},$$

where $e_g(\overrightarrow{z}) = r\sum_j \operatorname{dist}_g(z_j, z_{j+1})^2$ is the energy of the closed piecewise geodesic curve connecting the z_j's in cyclic order.

We need another parameter, but this requires some definitions: For any symplectic bundle $\xi \to M$ denote by $\mathcal{L}(\xi) \to M$ the fiber bundle with fiber $\mathcal{L}(\xi)_m$ the Grassmannian of Lagrangian subspaces of ξ_m. If ξ has a metric and the manifold has a Riemannian metric, we can induce a Riemannian metric on $\mathcal{L}(\xi)$: Each fiber is a Grassmannian of Lagrangian subspaces of a vector space with a metric, which means it has an induced metric. We define the orthogonal complement to the fiber by parallel transport of the Lagrangian subspaces, and use the metric on M to define the inner product on this complement.

The second parameter we need is a section $S_{\mathcal{L}}$ in the bundle

$$\Lambda\mathcal{L}(T(T^*_{\varepsilon_1}N)) \to \Lambda T^*_{\varepsilon_1}N. \tag{10}$$

This section should have the property that there exists a constant $C_{\mathcal{L}}$ such that for any curve $\gamma \in \Lambda T^*_{\varepsilon_1}N$ with energy less than β we have

$$\begin{aligned} e(S_{\mathcal{L}}(\gamma)) &< C_{\mathcal{L}} \\ \sup_t \|(D_\gamma S_{\mathcal{L}}(\delta\gamma))(t)\| &< C_{\mathcal{L}} \sup_t \|\delta\gamma(t)\|, \end{aligned} \tag{11}$$

where e is the energy functional $e(\gamma) = \int_0^1 \|\gamma'(t)\|^2 dt$ on the total space in the induced metric on $\mathcal{L}(T(T^*_{\varepsilon_1}N))$, and $\delta\gamma$ is a variation of γ. Note that this definition depends on β, and later when we choose β we will make sure that the choice is independent of $C_{\mathcal{L}}$.

In the bundle

$$\mathcal{L}(T(T^*_{\varepsilon_1}N)) \to T^*_{\varepsilon_1}N,$$

we have a canonical section s given by the vertical directions (the p-directions). We get a section in the bundle in equation (10) by looping s. We denote this canonical section $S_{\mathcal{L}0}$, and it has the bounds from equation (11) because it is the loop of a smooth map.

Given a Lagrangian subspace $L \subset T_z(T^*N)$ we define the *wedge* map

$$v_L\colon T_z(T^*N) \to T^*N$$

by decomposing $u \in T_z(T^*N)$ as $u = u_L + u_{L^\perp}$ ($L^\perp = JL$) and then exponentiating u_L to a point in T^*N say z'. Now parallel transport $u_{L^\perp}$ to z' using the exponential curve and exponentiate that to get z'' (see figure 3). We define

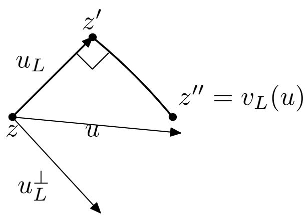

Figure 3: The wedge map.

$v_L(u) = z''$. This is a diffeomorphism at $u = 0$, so by compactness of $T^*_{\varepsilon_1}N$ there is an $\varepsilon > 0$ such that: For any $z, z'' \in T^*_{\varepsilon_1}N$ with $\mathrm{dist}(z, z'') < \varepsilon$ and L a Lagrangian in $T_z(T^*N)$ there is a unique $u \in T_z(T^*N)$ close to 0 with $v_L(u) = z''$. In this case, we define the *L-curve* $\gamma^{\llcorner}(z, z'', L)$ by the curve starting at z'', going to z' by the exponential curve, and then continuing to z by the other exponential curve. We do not care about parametrization. Notice that this curve may not be fully contained in $T^*_{\varepsilon_1}N$.

We will use these L-curves to define our approximations. Figure 4 illustrates many of the aspects of the definition.

Definition 5.1 *For any $\overrightarrow{z} \in \Lambda_{r,\beta}T^*_{\varepsilon_1}N$ let $S_{\mathcal{L}}(\overrightarrow{z})$ be the section evaluated at the piecewise geodesic connecting the z_j's in cyclic order. For large r we define*

$$\begin{aligned}\gamma_j(t) &= \varphi_t(z_j), \quad t \in [0, 1/r] \\ z_j^- &= \gamma_{j-1}(1/r) \\ \gamma_j^{\llcorner} &= \gamma^{\llcorner}(z_j, z_j^-, S_{\mathcal{L}}(\overrightarrow{z})(j/r))\end{aligned}$$

and

$$A_r^{g,S_{\mathcal{L}}}(\overrightarrow{z}) = \sum_{j=1}^{r}\Big(\int_{\gamma_j}(\lambda - H dt) + \int_{\gamma_j^{\llcorner}}\lambda\Big).$$

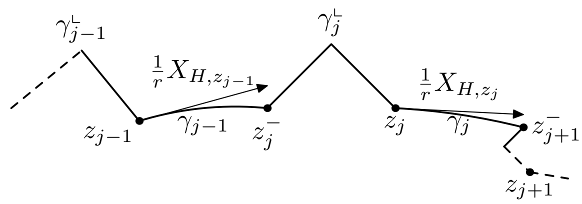

Figure 4: Curves involved in definition of $A_r^{g,s_{\mathcal{L}}}$.

If $S_{\mathcal{L}} = S_{\mathcal{L}0}$ and the metric is induced from a metric on N, we compare this to definition 4.1: In this case the curve $\gamma_j^{\llcorner}$ will be the curve first going in direction $\exp^{-1}_{q_j^-}(q_j)$, parallel transporting p_j^-, and when reaching the fiber over q_j it is a line in the fiber down to p_j. Integrating this over λ, one gets the term $p_j^- \varepsilon_{q_j}$. So this is indeed a generalization albeit only on a subset. From now on we will simply denote $A_r^{g,S_{\mathcal{L}}}$ by A_r.

We use the gradient of A_r and not a pseudo-gradient. We will need cut-off functions, and so we define an energy type functional

$$E(\overrightarrow{z}) = \sum_j \operatorname{dist}(z_j, z_j^-)^2.$$

This is zero if and only if $\overrightarrow{z}$ is a 1-periodic orbit.

Lemma 5.2 *There exist constants $K > 0$ (only dependent on the metric) and $\delta_0 > 0$ such that: If $0 < \delta < \delta_0$ and $r > K\delta^{-1}$ then*

$$\|\nabla E\|^2 \leq 5E \leq 15\|\nabla A_r\|^2 \leq 45E,$$

equalities only if $E = 0$

This lemma has a very interesting implication: The critical points of A_r are the 1-periodic orbits regardless of the metric and $S_{\mathcal{L}}$. We will, however, later see that in the case of a non-degenerate critical point, the Morse index will depend on $S_{\mathcal{L}}$.

Proof: We start by proving the second and the third inequality. To do this we need a bound on E, so we calculate

$$E(\overrightarrow{z}) < \sum_j \left(\tfrac{\mu}{r} + \operatorname{dist}(z_j, z_{j+1})\right)^2 < \sum_j \left(\tfrac{2\mu^2}{r^2} + 2\operatorname{dist}(z_j, z_{j+1})^2\right) < \frac{2\mu^2 + 2\beta}{r}.$$

Like in the proof of lemma 4.2, we need the gradient of A_r with respect to z_j. First we prove a special case: Assume that all the Lagrangian subspaces $S_{\mathcal{L}}(\overrightarrow{z})(j'/r) \subset T_{z_{j'}} T^*_{\varepsilon_1} N$ for any $j' \neq j$ is constant to the first order in z_j.

We can now look at the problem locally, since the gradient of A_r with respect to z_j only depends on H and the positions of z_{j-1} and z_{j+1}. Because $T^*_{\varepsilon_1} N$ is compact, we can choose ball shaped Darboux charts h for all points z such that $h(z) = 0$, and assume that they have a radius bounded from below by some small positive number. This is not a continuous family of charts, but just a choice for all points with this common bound. We can pick

r large enough such that the points z_{j-1}, z_j^-, z_j , z_{j+1}^- and z_{j+1} lie inside the chart centered at z_j. By composing the chart at z_j with a symplectic linear map we may assume that the differential $D_{z_j}h$ is an isometry. We can, by further composing with an element of $U(n)$, assume that the Lagrangian $L_j = (D_{z_j}h)(S_{\mathcal{L}}(\overrightarrow{q})(j/r)) \subset T_0\mathbb{R}^{2n}$ is equal to the imaginary part $i\mathbb{R}^n$ of $\mathbb{R}^{2n} \simeq \mathbb{C}^n$.

To further simplify the case, we assume for the moment that the chart is in fact also a Gaussian coordinate chart, and that the Lagrangian $S_{\mathcal{L}}(\overrightarrow{z})(j/r) \in T_{z_j}(T^*_{\varepsilon_1}N)$ is the parallel transport to the first order in z_j. So in fact, to the first order in z_j the Lagrangian $(D_{z_j}h)S_{\mathcal{L}}(\overrightarrow{q})(j/r)) \subset T_0\mathbb{R}^{2n}$ is constant.

Fixing all $z_{j'}$'s except z_j we replace the function A_r by

$$f(z_j) = \int_{\gamma_{j-1}} (\lambda_0 - Hdt) + \int_{\gamma_j^{\llcorner}} \lambda_0 + \int_{\gamma_j} (\lambda_0 - Hdt) + \int_{\gamma_{j+1}^{\llcorner}} \lambda_0,$$

where we think of the points and curves as lying in $\mathbb{R}^{2n}$ and $\lambda_0 = \sum_j y_j dx_j$ is the standard 1-form in $\mathbb{R}^{2n}$, with $d\lambda_0 = \omega_0$ the standard symplectic form. The function f is just A_r as a function of z_j plus a constant.

As in the proof of lemma 4.2, we now assume that f depends on the *independent* coordinates $z_j = (x_j, y_j)$ and $z_{j+1}^- = (x_{j+1}^-, y_{j+1}^-)$, and equation (4) used on the cotangent bundle $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ calculates the gradient of the integral along γ_j to be

$$\nabla_{x_j, y_j, x_{j+1}^-, y_{j+1}^-}(\int_{\gamma_j} \lambda_0 - Hdt) = (0, 0, y_{j+1}^-, 0). \tag{12}$$

We will only need the fact that the first two factors are zero, because we will deal with the gradient with respect to z_j^- later. Because we are in $\mathbb{R}^{2n}$ and the metric at 0 agrees with the standard metric, we can, by making r larger, assume that the tangents of the four geodesics in the two relevant L-curves $\gamma_j^{\llcorner}$ and $\gamma_{j+1}^{\llcorner}$ are close to being parallel to either $\mathbb{R}^n$ or $i\mathbb{R}^n$. This uses the first of the two bounds on the section $S_{\mathcal{L}}$ because we want $L_{j+1} = S_{\mathcal{L}}(\overrightarrow{z})((j+1)/r) \subset T_{z_{j+1}}\mathbb{R}^{2n}$ to be close to L_j. This is possible because as r grows, the points z_j and z_{j+1}, at which L_j and L_{j+1} are Lagrangians, move closer to each other, and we know that the energy of the curve defined by the section is bounded by $C_{\mathcal{L}}$. So increasing r will force L_j and L_{j+1} closer. We can also, by increasing r, assume that each of the four geodesic pieces are close to being linear in $\mathbb{R}^{2n}$, meaning: The tangent vectors in $\mathbb{R}^{2n}$ along each of the geodesic curves do not vary much along the curve compared to its length.

If the metric were flat, then ε_{x_j} and ε_{y_j} would both be linear and parallel to $\mathbb{R}^n$ and $i\mathbb{R}^n$ respectively, and we would get

$$\nabla_{x_j, y_j} \int_{\gamma_j^{\llcorner}} \lambda_0 = \nabla_{z_j} \omega_0(\varepsilon_{x_j}, \varepsilon_{y_j}) = (\varepsilon_{y_j}, 0).$$

Since the geodesics ε_{y_j} and ε_{x_j} are close to being linear and close to being parallel to the real or the imaginary part, and since the chart is Gaussian at z_j, we get that

$$\nabla_{z_j} \int_{\gamma_j^{\llcorner}} \lambda_0 = A(\varepsilon_{y_j}, 0),$$

where A is close to the identity.

To calculate the other part of the gradient we linear translate the coordinate such that $z_{j+1}^- = 0$. This changes the gradient of the individual integration parts, but not the overall gradient of f at the point z_{j+1}^-. Even though this new chart is not necessarily Gaussian, for now we assume it is and by similar arguments we get that

$$\nabla_{z_{j+1}^-} \int_{\gamma_{j+1}^{\llcorner}} \lambda_0 = B(\varepsilon_{y_{j+1}}, 0),$$

where B is close to the identity. In these coordinates, the gradient of the integral is

$$\nabla_{z_j, z_{j+1}^-} (\int_{\gamma_j} \lambda_0 - H dt) = (-y_{j+1}, 0, 0, 0). \tag{13}$$

Thus the actual gradient of f as a function of z_j is

$$\nabla_{z_j} f = A(\varepsilon_{y_j}, 0) - \Phi_r(B(0, \varepsilon_{x_{j+1}}))$$

where Φ_r is the differential $D_{z_j}\varphi_{1/r}$. By increasing r this will be close to the identity in our coordinates. This is where we choose K, because if we change H, we also need to make r larger for Φ_r to be close to the identity, and it is not difficult to prove that $r > K\delta^{-1}$ is good enough for some K depending only on the metric g. All other bounds on r has not depended on H, and can now be taken care of by decreasing δ_0.

We conclude in this special case that $\|\nabla_{z_j} A_r\|^2 \approx (\varepsilon_{y_j}^2 + \varepsilon_{x_{j+1}}^2)$. In fact we see that $E/2 \leq \|\nabla A_r\| \leq 2E$, equality only when $E = 0$.

If the Lagrangian at any of the points did move as a function of z_j, we would get an additional summand in the gradient of A_r. This summand, however, is negligible: The second bound on the section $S_{\mathcal{L}}$ implies that when differentiating at z_j in a unit direction, the corresponding variation in any of the Lagrangians is bounded in norm by $C_{\mathcal{L}}$. Changing the Lagrangian by an "angle" no bigger than $C_{\mathcal{L}}$ at a point $z_{j'}$ changes the overall integral over $\gamma_{j'}^{\llcorner}$ of λ_0 by a maximum of $C_{\mathcal{L}}(\varepsilon_{x_{j'}}^2 + \varepsilon_{y_{j'}}^2)$. So the overall norm of this summand in the gradient is bounded by $2C_{\mathcal{L}}E$. If E is small then this is small compared to the length of the already computed summand where $\|A_r\| \geq \sqrt{E/2}$.

In much the same way we need to see what happens if the two charts are not Gaussian. This means that the metric changes to the first order as a function of z_j (in $\mathbb{R}^{2n}$). But once again, a variation of the metric δg can change the integral over $\gamma_j^{\llcorner}$ by no more than

$$C_g(\varepsilon_{y_j}^2 + \varepsilon_{x_j}^2 + \varepsilon_{y_{j+1}}^2 + \varepsilon_{x_{j+1}}^2) \leq 2C_g E,$$

where C_g comes from a compactness argument involving g. This terms is also negligible.

We use this to realize that if E is small enough compared to the constants C_g and $C_{\mathcal{L}}$, we get that $E/3 \leq \|\nabla A_r\|^2 \leq 3E$, equality only when $E = 0$.

The first inequality in the lemma follows from the fact that we can easily calculate the gradient of E with respect to the coordinates z_j and z_j^-, again thought of as independent variables, and get

$$\nabla_{z_j, z_j^-} E = (-\exp_{z_j}(z_j^-), \exp_{z_j^-}(z_j)),$$

which implies that

$$\nabla_{z_j} E = -\exp_{z_j}(z_j^-) + \Phi_r(\exp_{z_{j+1}^-}(z_{j+1})).$$

Since Φ_r is close to the identity we get $\|\nabla E\|^2 \leq 5E$, equality only when $E = 0$. □

We also need cut-off functions that keep us away from the boundary defined by $\|p_j\| = \varepsilon_1$. The next lemma provides just that.

Lemma 5.3 *Let K be as in the previous lemma. Given any $k > 0$ we can find $\delta_0 > 0$ such that: If $0 < \delta < \delta_0$ and $r > K\delta^{-1}$, then $\|p_j\| \geq \varepsilon_1/2$ for any j implies*

$$\nabla A_r \cdot \nabla \|p_j\| < rk\|\nabla A_r\|^2,$$

where the function $\|p_j\|$ is defined using the original Riemannian metric on N.

Proof: $\|(\nabla\|p_j\|)\| = \|(\nabla_{z_j}\|p_j\|)\|$ has an upper bound $c_1 > 0$ on the set $\|p_j\| \geq \varepsilon_1/2$. If we have a lower bound of the type

$$\|\nabla A_r\| > \frac{c_2}{\sqrt{r}} \tag{14}$$

on the same set, then we can divide the proof into two cases:

First case: If $\|\nabla_{z_j} A_r\| > \frac{c_1}{rk}$ then

$$\nabla A_r \cdot \nabla\|p_j\| \leq \|\nabla_{z_j} A_r\|\|(\nabla\|p_j\|)\| \leq c_1\|\nabla_{z_j} A_r\| < rk\|\nabla_{z_j} A_r\|^2 \leq rk\|\nabla A_r\|^2.$$

Second case: If $\|\nabla_{z_j} A_r\| \leq \frac{c_2}{rk}$ then

$$\nabla A_r \cdot \nabla\|p_j\| \leq \|\nabla_{z_j} A_r\|\|(\nabla\|p_j\|)\| \leq c_1\|\nabla_{z_j} A_r\| \leq \frac{c_1^2}{rk} < kc_2 < rk\|\nabla A_r\|^2.$$

The second-to-last inequality is true for r large enough, and this can be accomplished by decreasing δ_0.

So we need a bound like (14). If we pick δ_0 smaller than the above lemma needs then $3\|\nabla A_r\|^2 \geq E$. So a lower bound on E of the type c/r will do the trick. This follows if we prove that $\sum_j \operatorname{dist}(z_j^-, z_j) < \sqrt{c}$. However, because we are on a compact set we can do this in any metric. So we choose to do this in the original, induced metric.

There are two parts of this argument, similar to the two parts of the proof in lemma 4.3, stating that $\|p_j\|$ could be used as a cut-off function.

First case: Assume that $\|p_{j'}\| < \varepsilon_1/4$ for some j'. By picking $\delta_0 < \varepsilon_1/4$ we get that the Hamiltonian flow preserves $\|p\|$ if $\|p\| > \varepsilon_1/4$. Since the curve has to go from $\|p_j\| > \varepsilon_1/2$ to $\|p_{j'}\| < \varepsilon_1/4$ and back, we can conclude that the sum of the lengths of the geodesics connecting the z_j's with the z_j^-'s is more than $\varepsilon_1/2$.

Second case: Assume that $\|p_{j'}\| \geq \varepsilon_1/4$ for all j'. Now the situation is this: We have a vector field $X = X_H$ on a compact Riemannian manifold $M = \{(q,p) \in T^*L \mid \varepsilon_1/4 \leq \|p\| \leq \varepsilon_1\}$, the flow ψ_t of X has no 1-periodic orbits, and we want a lower bound for the the sum of the lengths of r geodesics needed to close an r-piecewise flow curve. Consider the functional

$$Q(\gamma) = \int_0^1 \|\gamma'(t) - X_{\gamma(t)}\| dt$$

on piecewise smooth curves. We can approximate the sum of the lengths of the geodesics using this integral in the following way: If $\overrightarrow{z}$ is a piecewise flow curve, define γ_k as the piecewise smooth curve, which on $[j/r, (j+1-1/k)]$ is the flow curve re-parametrized and on $[(j+1-1/k)/r, (j+1)/r]$ is the geodesic. For k going to infinity the parametrization of the flow curve goes to the standard parametrization and thus the integral goes to zero on this part, and on the geodesics the integral goes to the length of the geodesic. So all we need is a lower bound on Q.

This is obtained by relating it to a more well-known functional. Let $\beta(t) = \psi_{-t}(\gamma(t))$. Then

$$Q(\gamma) > C \int_0^1 \|D_{\gamma(t)}\psi_{-t}(\gamma'(t) - X_{\gamma(t)})\| dt = C \int_0^1 \|\beta'(t)\| dt > c'$$

where $C < \|D_z\psi_t\|$ for all $z \in M$. The last inequality comes from the fact that for γ to be closed we need $\beta(1) = \psi_{-1}(\beta(0))$, which means that β is a curve starting at some point x and ending at the point $x \neq \varphi_1(x)$. However, x is in a compact set, so the distance from x to $\varphi_1(x)$ must be bounded from below by a positive constant. □

Lemma 5.4 *For $K > 0$ large enough (only dependent on the metric) there exists $\delta_0 > 0$ small enough and $\beta > 0$ large enough (β only depends on K and μ) such that: If $0 < \delta < \delta_0$ and $r \in [K\delta^{-1}, 2K\delta^{-1}]$, then we have a good index pair for the total index of $A_r\colon \Lambda_{r,\beta}T^*_{\varepsilon_1}N \to \mathbb{R}$. Furthermore, for all small C^1-perturbations of H within the provided properties, the good index pair will contain all of A_r's critical points.*

Proof: If we choose K larger and δ_0 smaller than needed in lemma 5.2, we see that $\nabla A_r \cdot \nabla E < 15\nabla A_r \cdot \nabla A_r$, and that $E = 0$ for critical points. So we can use E as a cut-off function with $0 < s$ and $t - s > 45\mu\delta = 15 \cdot (b - a)$.

We wish to use $\|p_j\|$ as a cut-off function with $s_j = \varepsilon_1/2$ and $t_j = 3\varepsilon_1/4$. Looking at lemma 2.5 again, we see that we need

$$\nabla A_j \cdot \nabla\|p_j\| < \tfrac{t_j - s_j}{b-a}\|\nabla A_r\|^2 = \tfrac{\varepsilon_1}{12\mu\delta}\|\nabla A_r\|^2.$$

The assumptions in the lemma tells us that $\frac{\varepsilon_1}{12\mu\delta} > \frac{\varepsilon_1 r}{24\mu K}$, so putting $k = \frac{\varepsilon_1}{24\mu K}$ and using the previous lemma, we get what we want, provided that δ_0 is small enough.

When using the formula in lemma 2.5 to try and create good index pairs using E and the functions $\|p_j\|$ as cut-off functions (with the prescribed s's and t's), it is not clear that one gets a compact subset of $\Lambda_{r,\beta}M_{\varepsilon_1}$. This is because we need to prove that the functions we chose separate the pair we create from the "boundary". The functions $\|p_j\|$ take care of one type of boundary, but we need to see that E takes care of the other type. That is, the one coming from

$e(\overrightarrow{z}) < \beta/r$. This is proven by calculating e on the boundary of the index pair coming from using E as a cut-off function

$$e(\overrightarrow{z}) < r \sum_j (\tfrac{\mu}{r} + \mathrm{dist}(z_j^-, z_j))^2 < 2\tfrac{\mu^2}{r} + 2E(\overrightarrow{z}) \leq 2\tfrac{\mu^2}{r} + 2t.$$

We were able to choose t as close to $45\mu\delta$ as possible, say $46\mu\delta$. This is smaller than $92K\mu/r$, so if β is larger than $(92K + \mu)2\mu$ then E will keep us away from the boundary, and our index pair is compact. The last statement in the lemma follows from the fact that E is continuously dependent on $H \in C^1$ and for critical points we know $E = 0$ and $\|p_j\| < \delta$. □

6 Stabilization of Approximations Defined Near the Zero Section

The setting is as in the previous section, except that now we look at $N \times D^k$ for some k. We define $H\colon T^*_{\varepsilon_1}N \times (D^{2k}, \omega_0) \to \mathbb{R}$ by $H(z_1, z_2) = H_N(z_1) + H_D(z_2)$, where H_N is as in the previous section (dependent on a δ), ω_0 is the standard symplectic form on D^{2n} and

$$H_D(z_2) = \|z_2\|^2.$$

The Hamiltonian flow for H_D is circular around 0 with revolution time 2π, but we only flow for a time period of 1, so the only orbit is 0, and this orbit has action 0. So the 1-periodic orbits for H are the same as in the previous section on the first factor and constantly equal to 0 on the second factor, and they have the same action.

There are no difficulties in defining finite approximations just as in the previous section. Note that this requires a section

$$S_{\mathcal{L}}\colon \Lambda(T^*_{\varepsilon_1}N \times D^{2k}) \to \Lambda\mathcal{L}(T(T^*_{\varepsilon_1}N) \times (R^{2k}, \omega_0))$$

with the same type of bounds as before on curves with energy less than β, and it requires a compatible metric g on the space $T^*(N \times D^k)$. However, in this case we assume that this metric is the product of a compatible metric with the standard metric on $\mathbb{R}^{2k}$. The corresponding finite version of the loop space $\Lambda_{r,\beta}(T^*_{\varepsilon_2}N \times D^{2k})$ will consist of curves denoted by $\overrightarrow{z} = (\overrightarrow{z_1}, \overrightarrow{z_2})$.

The lemmas (with proofs) from the previous section still hold, except we need more cut-off functions to keep the index pairs away from the boundary of D^{2k}. However, the techniques used in lemma 5.3 can be copied directly to prove that we get a similar result for the function $\|(z_2)_j\|$. So we conclude that the lemma at the end of the previous section still holds in this slightly more general case.

Definition 6.1 *A section $S_{\mathcal{L}}$ is said to be of* product type *if it factors through*

$$\Lambda(T^*_{\varepsilon_1}N) \quad \textit{and} \quad \Lambda(\mathcal{L}(T(T^*_{\varepsilon_1}N)) \times \mathcal{L}(k)),$$

where $\mathcal{L}(k)$ is the Grassmannian of Lagrangian subspaces of $\mathbb{R}^{2k}$.

Factoring through the first of the two spaces is equivalent to the section not depending on the second (contractible) factor ΛD^{2k}. Factoring through the second space is equivalent to all Lagrangians defined by the section split as direct sums of two Lagrangians, one in each factor. On the level of fibers this corresponds to the section being in the subspace similar to $(\Lambda\mathcal{L}(n))\times(\Lambda\mathcal{L}(k)) \subset \Lambda\mathcal{L}(n+k)$.

For the rest of this section, $S_{\mathcal{L}}$ will be of product type. In this case A_r splits into two factors

$$A_r(\overrightarrow{z_1},\overrightarrow{z_2}) = A_r^N(\overrightarrow{z_1}) + A_r^D(\overrightarrow{z_1},\overrightarrow{z_2})$$

where A_r^N is the function defined in the previous section by restricting our section to the first factor, and $A_r^D(\overrightarrow{z_1},-)$ is the finite approximation on D^{2k} defined by the constant section, which is the restriction of $S_{\mathcal{L}}(\overrightarrow{z_1})$ to the second factor.

This gives us

$$\begin{aligned} \nabla_{\overrightarrow{z_1}} A_r &= \nabla A_r^N + \nabla_{\overrightarrow{z_1}} A_r^D \\ \nabla_{\overrightarrow{z_2}} A_r &= \nabla_{\overrightarrow{z_2}} A_r^D \end{aligned} \tag{15}$$

It would be nicer if we did not have the second term in the first line, but looking back at the proof of lemma 5.2, one can check that this is one of the parts of the gradient that were negligible (this will be proven later). We will show that changing the gradient to the pseudo-gradient

$$X = X_1 \oplus X_2 = \nabla A_r^l \oplus \nabla_{\overrightarrow{z_2}} A_r^k,$$

will not change the validity of the lemmas in the previous section. This is made more precise and more general in the following technical lemma, which is a small generalization of lemma 5.4.

Lemma 6.2 *For $K > 0$ large enough (only dependent on the metric) there exist $\delta_0 > 0$ small enough and $\beta > 0$ large enough (β only dependent on K and μ) such that: If $0 \le t \le 1$, $0 < s \le 1$, $0 < \delta < \delta_0$ and $r \in [K\delta^{-1}, 2K\delta^{-1}]$, then the vector field*

$$X = (s\nabla A_r^N + t\nabla_{\overrightarrow{z_1}} A_r^D) \oplus \nabla_{\overrightarrow{z_2}} A_r^D$$

is a pseudo-gradient for $A_r\colon \Lambda_{r,\beta}T^*_{\varepsilon_1}N \to \mathbb{R}$ and (A_r, X) has good index pairs.

Proof: First we notice that the energy splits

$$E(\overrightarrow{z}) = E_1(\overrightarrow{z}) + E_2(\overrightarrow{z}),$$

where $E_1(\overrightarrow{z}) = E(\overrightarrow{z_1}, 0)$ and $E_2(\overrightarrow{z}) = E(0, \overrightarrow{z_2})$. So the gradient of E is

$$\nabla E = \nabla E_1 \oplus \nabla E_2.$$

Because of the bounds in equation (11), taking a variation of $\overrightarrow{z_1}$ in a unit direction will not change the angles of the Lagrangians on the second factor by more than $C_{\mathcal{L}}$. This changes A_r by no more than $C_{\mathcal{L}}E_2$, so the norm of the gradient $\nabla_{\overrightarrow{z_1}} A_r^D$ is less than $C_{\mathcal{L}} E_2$. Using this, equation (15) and lemma 5.2 on both E_1, A_r^N and E_2, A_r^D we get

$$\begin{aligned} X \cdot \nabla E \leq & \big(s\|\nabla_{\overrightarrow{z_1}} A_r^N\| + t\|\nabla_{\overrightarrow{z_1}} A_r^D\|\big)\|\nabla E_1\| + \|\nabla_{\overrightarrow{z_2}} A_r^D\|\|\nabla E_2\| \\ \leq & \big(s\sqrt{3E_1} + tC_{\mathcal{L}}E_2\big)\sqrt{5E_1} + \sqrt{3E_2}\sqrt{5E_2} \\ \leq & \ 4(sE_1 + tC_{\mathcal{L}}E_2\sqrt{E_1} + E_2) \end{aligned}$$

and

$$\begin{aligned} X \cdot \nabla A_r \quad & \geq (s\nabla_{\overrightarrow{z_1}} A_r^N + t\nabla_{\overrightarrow{z_1}} A_r^D) \cdot (\nabla_{\overrightarrow{z_1}} A_r^N + \nabla_{\overrightarrow{z_1}} A_r^D) + \|\nabla_{\overrightarrow{z_2}} A_r^D\|^2 \\ & \geq sE_1/3 - (t+s)\sqrt{E_1/3}C_{\mathcal{L}}E_2 - t(C_{\mathcal{L}}E_2)^2 + E_2/3. \end{aligned} \tag{16}$$

By making δ_0 smaller we make r larger and this makes $E = E_1 + E_2$ smaller. So for small δ_0 we can assume

$$X \cdot \nabla E \leq 5(sE_1 + E_2) \leq 20 X \cdot \nabla A_r.$$

This proves that X is a pseudo-gradient, because at non-critical points we have $E_1 + E_2 > 0$. We also see that we can still use E as a cut-off function in the same way we did in the proof of lemma 5.4.

Similarly, the functions $\|p_j\|$ only depend on $\overrightarrow{z_1}$ and so by using lemma 5.3 on A_r^N we get on the set $\|p_j\| > \varepsilon_1/2$ that

$$\begin{aligned} X \cdot \nabla\|p_j\| & < s\nabla_{\overrightarrow{z_1}} A_r^N \cdot \nabla\|p_j\| + tC_{\mathcal{L}}E_2\|\nabla p_j\| \\ & < srk\|\nabla_{\overrightarrow{z_1}} A_r^N\|^2 + 2tC_{\mathcal{L}}E_2 c \\ & < rk(X \cdot \nabla A_r). \end{aligned}$$

To get the last inequality we may have to further increase r.

The result for the cut-off functions $\|z_{2,j}\|$ is easier: We bound $X \cdot \nabla\|z_j^2\|$ by rkE_2 using the same argument as in the proof of lemma 5.3. Now we have enough cut-off functions to get compact sets, just as we did in the proof of lemma 5.4. □

Because $\nabla_{uz_2} H_D = u\nabla_{z_2} H_D$, we see that flow curves for the Hamiltonian flow of H_D is preserved under scaling: If γ is a flow curve then $u\gamma$ is a flow curve. This means that the curve over which we integrate λ_0 scales proportionally with $\overrightarrow{z_2}$, so the integral scales quadratically. Also, H_k is quadratic so we get

$$A_r^D(\overrightarrow{z_1}, u\overrightarrow{z_2}) = u^2 A_r^D(\overrightarrow{z_1}, \overrightarrow{z_2}). \tag{17}$$

This is a smooth function, so it must be equal to its Hessian at 0. So $A_r^D(\overrightarrow{z_1}, -)$ is in fact a quadratic form in $\overrightarrow{z_2}$. If the critical point 0 were degenerate then it would not be an isolated critical point. So it is in fact non-degenerate. We define ζ over $\Lambda_{r,\beta} T^*_{\varepsilon_1} N$ by taking for each point $\overrightarrow{z_1}$ the bundle of negative eigenspaces for the Hessian of $A_r^D(\overrightarrow{z_1}, -)$ at the point 0 in D^{2kr} (recall that we gave D^{2k} the standard metric and the finite loop space the product metric).

Lemma 6.3 *Assume that β, δ and r satisfy the conditions in the previous lemma. Then the total homotopy index of A_r is the relative Thom suspension of ζ (ζ defined above) of the homotopy index of A_r^N.*

Note that the relative Thom suspension of A/B of a bundle ζ over A is defined to be $D\zeta/(S\zeta \cup D\zeta_{|B})$.

Proof: First choose an index pair (A,B) for A_r^N, but with $a = -\mu\delta/2$ and $b = 3\mu\delta/4$. We will extend this to an index pair for A_r. Let $E^{\pm}_{\overrightarrow{z_1}}$ be the negative/positive eigenbundle of $A_r^D(\overrightarrow{z_1}, -)$. It is easy to construct index pairs very close to zero for a non-degenerate quadratic form on D^{2kr}, so we do this fiber-wise

$$A_{\overrightarrow{z_1}} = D_\varepsilon E^-_{\overrightarrow{z_1}} \times D_\varepsilon E^+_{\overrightarrow{z_1}}$$
$$B_{\overrightarrow{z_1}} = S_\varepsilon E^-_{\overrightarrow{z_1}} \times D_\varepsilon E^+_{\overrightarrow{z_1}}.$$

Since $e(A) \in [0,\beta)$ and A is compact we have $e(A) \in [0, \beta - c]$, so we can find an $\varepsilon > 0$ such that $A_{\overrightarrow{z_1}}$ is contained in $\Lambda_{r,\beta}(T^*_{\varepsilon_1}N \times D^{2k})$ for all $\overrightarrow{z_1} \in A$. Also, because the index pair was made with a more narrow choice of a and b, we can assume that A_r on $A_{\overrightarrow{z_1}}$ is in the interval $[-\mu\delta, 2\mu\delta]$. Define

$$A' = \bigcup_{\overrightarrow{z_1} \in A} A_{\overrightarrow{z_1}}$$
$$B' = (\bigcup_{\overrightarrow{z_1} \in B} A_{\overrightarrow{z_1}}) \cup (\bigcup_{\overrightarrow{z_1} \in A} B_{\overrightarrow{z_1}})$$

for such an ε. Claim: We can find $0 < s \leq 1$ and put $t = 0$ in the previous lemma such that (A_r, X) has this as an index pair. I1 and I2 from the definition of index pair have been taken care of. I3 is as noted before because critical points of A_r are of the form $(\overrightarrow{z_1}, 0)$, where $\overrightarrow{z_1}$ is a critical point for A_r^N. To get I4 we choose s, because s controls the speed of the flow on the base compared to the flow in the fiber. For s very small, any point in $B_{\overrightarrow{z_1}}$ will flow entirely out of $\Lambda_{r,\beta}(T^*N \times D^{2k})$ before the quadratic form $A_r(\overrightarrow{z_1}, -)$ has a chance to change much. □

For us to use this lemma we would like to be able to compute the isomorphism class of the negative eigenbundle, and the next lemma helps us do just that.

Lemma 6.4 *Let $B_r^D(\overrightarrow{z_1}, -)$ be the quadratic form defined like $A_r^D(\overrightarrow{z_1}, -)$, but with $H_D = 0$. Let ζ_B be a choice of negative bundle over $\Lambda_{r,\beta}T^*N$ for B_r^D then*

$$\zeta \cong \zeta_B \oplus \mathbb{R}^{2k}.$$

Note that B_r^D is degenerate in each fiber, because in the fiber over $\overrightarrow{z_1}$, the quadratic form $B_r^D(\overrightarrow{z_1}, -)$ is a finite approximation, which has the $2k$ dimensional subspace of the constant curves as degenerate points (the 1-periodic orbits). We did not use this Hamiltonian in the definition of our finite approximation because we wanted index pairs, but the negative eigenbundle is easier to calculate for the quadratic form B_r^D.

Proof: We define a continuous family A^s of quadratic forms by finite dimensional approximation of the Hamiltonians $H_s = H_N + H_D^s$, where $H_D^s = s\|z_2\|^2$. Then $B_r^D = A^0$ and $A_r^D = A^1$. The same argument as before shows us that these are quadratic forms. Since the Hamiltonian flow for these only has the trivial 1-periodic orbit for $0 < s < 2\pi$, the negative eigenbundles are isomorphic, but we need to see what happens at $s = 0$. The critical points of A^0 are precisely the constant curves, so the Hessian is degenerate, and the kernel as a bundle over $\Lambda_{r,\beta}T^*N$ is the trivial bundle of dimension $2k$. We prove the lemma by proving that for a small perturbation of $s = 0$ in positive direction, this kernel becomes part of the negative eigenspace. This is a point-wise calculation, so assume $\overrightarrow{z_1}$ is fixed.

Denote by E_-, E_0 and E_+ the negative, zero and positive eigenspace of a representing matrix for A^0. It is enough to prove that the first order change in s at $s = 0$ of A_r^s is negative definite on the kernel E_0. Indeed, if so we can restrict A_r^0 to the sphere of $E_0 \oplus E_-$ and what we see is a non-positive function on a closed manifold, which is then perturbed to the first order to be negative on the set where it is zero. This will imply that the function is in fact going to be negative on the entire sphere for very small s, and thus A_r^s is negative definite on $E_0 \oplus E_-$ for small $s > 0$.

So to prove this negativity on E_0, we look at A_r^s on E_0 for s close to zero. The kernel E_0 is the set of constant curves, so we assume that $z_j = z_{j+1}$ for all $j \in \mathbb{Z}_r$. We need to take a look at the precise definition of

$$A_r^s = \sum_j \left(\int_{\gamma_j} \lambda_0 - H_s dt + \int_{\gamma_j^\llcorner} \lambda_0\right)$$

For $s = 0$ all of this is zero (on E_0) because γ_j and $\gamma_j^\llcorner$ are constant, and H_s is zero. We want to prove that the dominating term when perturbing to positive s is $-H_s$, which is negative.

Because any time independent Hamiltonian is constant on its flow curves, we can rewrite this as

$$\begin{aligned} A_r^s &= \int_{\sum_j(\gamma_j+\gamma_j^\llcorner)} \lambda_0 + \frac{1}{r}\sum_j H_s(z_j) = \\ &= \int_{\sum_j(\gamma_j+\gamma_j^\llcorner)} \lambda_0 - H_s(z_0). \end{aligned}$$

The curves γ_j are the $1/r$ time flow curves of H_s, so they have lengths of order $\|\nabla H_s\|/r$ which is of order $s\|z_0\|/r$, and since $z_j = z_{j+1}$, and $\gamma_j^\llcorner$ connects the endpoint of γ_j with z_{j+1}, the same is true for $\gamma_j^\llcorner$. This means that the integral, which is the symplectic area enclosed by the closed curve obtained by concatenating γ_j and $\gamma_j^\llcorner$, is of order $(s\|z_0\|/r)^2$. We have r of these terms summed, but this is still of order $(s\|z_0\|)^2/r$. The term $H_s(z_0)$ is equal to $s\|z_0\|^2$, so this is the dominating term in s and the lemma follows. □

7 The Maslov Bundle and Index of Related Finite Actions

Let $j\colon L \to T^*N$ be a Lagrangian embedding. We will define the Maslov bundle relative to this embedding. It is a generalization of the Maslov index related to curves of Lagrangian subspaces in $\mathbb{R}^{2n}$ (see e.g. [MS98]). In fact the bundle is a canonical virtual vector bundle over ΛL, such that the dimension of this bundle on each component is precisely the Maslov index.

The projection $T^*N \to N$ will be denoted π. For any point $q \in L$ the tangent space T_qL is mapped by j_* to a Lagrangian subspace of $T_{j(q)}(T^*N)$, and by abuse of notation we define

$$j_*\colon L \to \mathcal{L}(T(T^*N)).$$

See the previous section for definition of $\mathcal{L}(T(T^*N))$. A stabilization of this map with a vector bundle $\zeta \to N$ will be denoted by $j_* \oplus \zeta$ and is defined by

$$(j_* \oplus \zeta)(q) = j_*(T_qL) \oplus (\pi^*\zeta) \subset T_{j(q)}(T^*N) \oplus (\pi^*\zeta) \oplus (\pi^*\zeta)^*,$$

which is also Lagrangian (in the obvious symplectic structure).

If ζ is the normal bundle of N for some embedding $i_N\colon N \to \mathbb{R}^n$, we get a canonical symplectic trivialization

$$\eta_i\colon T(T^*N) \oplus (\pi^*\zeta) \oplus (\pi^*\zeta)^* \to T^*N \times (\mathbb{R}^{2n}, \omega_0).$$

This is defined by using the Riemannian metric induced from i_N to split the tangent space of T^*N at z into $T_{\pi(z)}N \oplus T^*_{\pi(z)}N$, then mapping $V_z = T_{\pi(z)}N \oplus \zeta_{\pi(z)}$ isomorphically to $\mathbb{R}^n$ by the obvious map, and mapping $V_z^* = T^*_{\pi(z)}N \oplus \zeta^*_{\pi(z)}$, by the inverse of the dual to this map, to $i\mathbb{R}^n$. If we compose this map with $j_* \oplus \zeta$ we get a map from L to $\mathcal{L}(n)$ (the Grassmannian of Lagrangian subspaces in R^{2n}), and since all embeddings are isotopic for n sufficiently large, we have a map unique up to homotopy

$$J_*\colon L \to \mathcal{L}(n).$$

If we stabilize this map to get a map to $\mathcal{L}(2n)$, then we can homotope it to be a map into $\mathcal{L}(n) \times \mathcal{L}(n) \subset \mathcal{L}(2n)$ which is constant on the first factor. This means that if we further stabilize this map by the tangent space of T^*N and choose to trivialize this copy of T^*N with its normal bundle, we have a different interpretation of the map J_*: For any point $l \in L$ we have two different Lagrangian subspaces of $T_l(T^*N)$, the tangent space of L and the horizontal part of $T(T^*N)$. So we have two different sections in the bundle $\mathcal{L}(T(T^*N))_{|L}$. If we stabilize both sections with $\mathbb{R}^n \subset \mathbb{R}^{2n}$ to make them bundles of Lagrangian subspaces in a higher dimensional symplectic bundle over L, we can - if n is large enough - homotope their "difference" through Lagrangian subspaces to the $\mathbb{R}^{2n}$ component. It is this difference that J_* measures.

Because $\lim_{n\to\infty} \mathcal{L}(n) = \mathcal{L} \simeq \mathrm{U/O} \simeq \Omega^6\mathrm{O}$ (see e.g. [MS98] for the first homotopy equivalence and [Mil63] for the last) is an H-space we have a homotopy equivalence

$$\mathrm{Ev}_0 \times \pi_\Omega\colon \Lambda\mathcal{L} \to \mathcal{L} \times \Omega\mathcal{L}, \tag{18}$$

where Ev_0 is evaluation at the base point, and π_Ω is homotopic to point-wise multiplication with the homotopy inverse of Ev_0.

Definition 7.1 *The Maslov bundle η is the virtual bundle defined by the map*

$$\Lambda L \xrightarrow{\Lambda J_*} \Lambda\mathcal{L}(n) \longrightarrow \Lambda\mathcal{L} \xrightarrow{\pi_\Omega} \Omega\mathcal{L} \xrightarrow{\simeq} \mathbb{Z} \times B\mathrm{O}.$$

The latter equivalence comes from Bott periodicity (see e.g. [Mil63]). The $\mathbb{Z}$ corresponds to the dimension of the virtual bundle or $\pi_1(\mathcal{L})$, the latter being one of the many definitions of the Maslov index (see [MS98]).

We want to use this to calculate the type of homotopy indices defined in the previous section. To do this we need to put the loop of J_* on a standard form.

For any $V_+ \subset \mathbb{R}^n$, $V_- \subset \mathbb{R}^n$ and $V_0 \subset \mathbb{R}^n$ pairwise orthogonal and $V_+ \oplus V_- \oplus V_0 = \mathbb{R}^n \subset \mathbb{R}^{2n}$, we define the curve $\gamma_{(V_+,V_-,V_0)} \in \Omega\mathcal{L}(n)$ by

$$\gamma_{(V_+,V_-,V_0)}(t) = e^{i\pi t}V_+ \oplus e^{-i\pi t}V_- \oplus V_0 \in \mathcal{L}(n),$$

for $t \in [0,1]$. The space of such curves will be denoted $\Omega_s\mathcal{L}(n)$ (s for standard form). Over this space we have the three canonical bundles V_+, V_- and V_0.

Lemma 7.2 *Any map $f\colon K \to \Lambda\mathcal{L}$, where K is of compact homotopy type, can be homotoped to factor through*

$$\mathcal{L}(n_1) \times \Omega_s\mathcal{L}(n_2) \subset \Lambda\mathcal{L}(n_1+n_2) \to \Lambda\mathcal{L}$$

for large enough n_1 and n_2. Furthermore, the virtual bundle defined by the map f over K by composing it with π_Ω is the pullback of $V_+ - V_-$.

Proof: The map factors through $\mathcal{L}(n_1) \times \Omega\mathcal{L}(n)$ because the map in equation (18) is a homotopy equivalence and because K is of compact homotopy type. So we can consider a map f' to $\Omega\mathcal{L}(n)$ and show that it factors through $\Omega_s\mathcal{L}(n)$ for large n. We may have to stabilize by a number of Ωi_n's, where

$$i_n\colon \mathcal{L}(n) \to \mathcal{L}(n+1)$$

is the standard stabilization which gives the limit $\mathcal{L}$.

This part of the proof is standard Morse theory as in the proof of Bott periodicity (see e.g. [Mil63]): Multiplication with $e^{-i\pi t/2}$ on the curves in $\mathcal{L}(n)$ gives a homeomorphism of $\Omega\mathcal{L}(n) = \Omega(\mathcal{L}(n),\mathbb{R}^n,\mathbb{R}^n)$ to $\Omega(\mathcal{L}(n),\mathbb{R}^n,i\mathbb{R}^n)$. We now think of f' as mapping into this space. In [Mil63] part IV paragraph 24 the space of minimal geodesics for this space is computed to be (with some interpretation into the current context)

$$\Omega_m(n) = \Omega_m(\mathcal{L}(n),\mathbb{R}^n,i\mathbb{R}^n) = \{\gamma \mid \gamma(t) = e^{i\pi t/2}W \oplus e^{-i\pi t/2}W^\perp, W \subset \mathbb{R}^n\}.$$

The embedding of this space into $\Omega(\mathcal{L}(n),\mathbb{R}^n,i\mathbb{R}^n)$ has high connectivity on the components where $\dim(W)$ and $\dim(W^\perp)$ are both high.

To get high connectivity we stabilize f' by

$$\gamma(t) = e^{i\pi t/2}\mathbb{R} \oplus e^{-i\pi t/2}\mathbb{R} \subset \mathbb{R}^4.$$

That is, we compose with the map

$$\oplus\gamma\colon \Omega\mathcal{L}(n,\mathbb{R}^n,i\mathbb{R}^n) \to \Omega\mathcal{L}(n+2,\mathbb{R}^{n+2},i\mathbb{R}^{n+2})$$

given by direct sum with γ. If we do this k times, we can homotope the map $(\oplus\gamma)^{\circ k} \circ f'$ to factoring through $\Omega_m(n+2k)$.

Going back with the homeomorphism to $\Omega\mathcal{L}(n)$ we see that the stabilization we did corresponds to having stabilized with

$$e^{i\pi t/2}\gamma(t) = e^{i\pi t}\mathbb{R} \oplus \mathbb{R}$$

k times. We have argued that we can homotope the map after such a stabilization to the following subspace

$$e^{i\pi t/2}\Omega_m = \{\gamma \mid \gamma(t) = e^{i\pi t}W \oplus W^{\perp}\},$$

so by further stabilizing with

$$\gamma_2(t) = (e^{-i\pi t}\mathbb{R})^{\oplus k},$$

one has in total stabilized f with something homotopic to a standard stabilization, because it is easy to get the two "twistings" to cancel out. We have now homotoped the map f, stabilized in the standard way, to a map into $\Omega_s(\mathcal{L}(n+3k))$. The last statement follows from the fact that these highly connected inclusions of Grassmannians into $\Omega\mathcal{L} \simeq \mathbb{Z} \times BO$ used in the proof, are the standard way of identifying the stable bundle with the difference of two actual bundles. □

What we in fact proved was that the map is homotopic to a map factoring through $\Omega_s\mathcal{L}(n_2)$, where V_- is the trivial bundle of dimension k. This is equivalent to the fact that any stable bundle over a compact space can be written as the difference between a bundle and a trivial bundle.

If we want to use this together with lemmas 6.3 and 6.4, we need to calculate negative bundles for the quadratic form B_r^D whenever $S_{\mathcal{L}}$ is of product type and on standard form on the factor ΛD^{2k}.

Lemma 7.3 *Assume that r is odd and large enough. For any point γ in $X = \mathcal{L}(n_1) \times \Omega_s\mathcal{L}(n_2) \subset \Lambda\mathcal{L}(n_1+n_2)$ we have an associated quadratic form $B_r^{\gamma} \colon \mathbb{C}^{(n_1+n_2)r} \to \mathbb{R}$ defined by the Hessian of the finite approximation for $H_D = 0$. The negative eigenbundle over X of this quadratic form is isomorphic to*

$$\mathbb{R}^{(n_1+n_2)(r-2)+n_1} \oplus V_+ \oplus V_-^{\perp},$$

where $V_-^{\perp} = V_+ \oplus V_0$.

We use the notation $V_-^{\perp}$ to emphasize that it is a complement bundle to V_-, making the bundle in the lemma isomorphic to $V_+ - V_-$ plus a trivial bundle of dimension $(n_1 + n_2)$.

Note that even though these quadratic forms have kernels, there is indeed a negative eigenbundle, because the kernel is the same for all the forms.

Proof: Given any γ we will describe a choice of subspace on which B_r^{γ} is negative, and argue that it has maximal dimension. It will be obvious that this choice is continuous as a function of X.

Because $\gamma(t) = L \oplus \gamma_1(t)$ with $L \subset \mathbb{C}^{n_1}$ and $\gamma_1(t) \subset \mathbb{C}^{n_2}$ we see that B_r^{γ} splits as a sum of $B_r^L \colon \mathbb{C}^{n_1 r} \to \mathbb{R}$ and $B_r^{\gamma_1} \colon \mathbb{C}^{n_2 r} \to \mathbb{R}$.

For now we restrict our attention to B_r^L. Note that for any quadratic form $x^T Ax$, the set of critical points is precisely the kernel of A. We wish to define real vector spaces E_-, E_0 and E_+ on which B_r^L is negative, zero (to the second order) and positive respectively. Because $E_- \oplus E_0 \oplus E_+$ will be all of $\mathbb{C}^{n_1 r}$ we conclude that $\dim(E_-)$ is maximal.

Let $\rho = e^{i2\pi/r}$ be the standard r'th root of unity. Use this to define the real vector spaces E_m by

$$E_m = \{(b\rho^{mj})_{j\in\mathbb{Z}_r} \mid b \in \mathbb{C}^{n_1}\} \subset \mathbb{C}^{n_1 r},$$

for any $m \in \mathbb{Z}_r$. The action of $\mathrm{U}(n_1)$ preserves E_m.

Let M_- be the subset of $\mathbb{Z}_r$ containing the classes $[1], [2], \dots, [(r-1)/2]$, and $M_+ = -M_-$. These are disjoint and $\mathbb{Z}_r = M_- \cup M_+ \cup \{0\}$. Use this to define

$$E_- = \bigoplus_{m\in M_-} E_m$$
$$E_+ = \bigoplus_{m\in M_+} E_m.$$

We have already defined E_0 as the constant curves, which we know to be critical points of B_r^L, because for r large enough the only critical points of B_r^L are the 1-periodic orbits.

Since $H = 0$ we get no flow and the first term in the definition of the finite dimensional approximation vanishes (see the definition of the finite approximations in section 5). So B_r^L is in fact just a calculation of the symplectic area for the concatenation of the L-curves connecting z_j to z_{j+1} defined by L. The action of $\mathrm{U}(n_1)$ just rotates the curves and preserves the symplectic area and the spaces E_m, so we can assume that $L = i\mathbb{R}^{n_1}$. This means that B_r^L splits as a sum $A_1 + A_2 \cdots + A_{n_1}$, where A_k only depends on the k'th complex coordinate of the z_j's. We now look at one A_k at a time. In fact we assume for the time being that $n_1 = 1$, so that we do not have to redefine E_m. This means that symplectic volume is now the normal area in $\mathbb{C}$ (with sign).

We will need the following facts about the spaces E_m, which because of the assumption $n_1 = 1$ are subspaces of $\mathbb{C}^r$. For $(z_j)_{j\in\mathbb{Z}_r} \in E_m, (w_j)_{j\in\mathbb{Z}_r} \in E_{m;}$ we have

$$\begin{aligned}(z_{j+1})_{j\in\mathbb{Z}_r} &\in E_m\\ (\overline{z_j})_{j\in\mathbb{Z}_r} &\in E_{-m}\\ \mathrm{Re}(z_j)_{j\in\mathbb{Z}_r} &\in E_m \oplus E_{-m}\\ \mathrm{Im}(z_j)_{j\in\mathbb{Z}_r} &\in E_m \oplus E_{-m}\\ (z_j \cdot w_j)_{j\in\mathbb{Z}_r} &\in E_{m+m'}.\end{aligned}$$

and if $m \neq 0$

$$\sum_j z_j = 0.$$

Notice that the second to last fact makes sense only because we have $n_1 = 1$.

Any vector $\overrightarrow{z} = (z_j)_{j\in\mathbb{Z}_r} \in E_-$ can be written as

$$z_j = \sum_{m\in M_-} \alpha_m \rho^{mj}$$

with $\alpha_m \in \mathbb{C}$. Since $L = i\mathbb{R}$, the L-curves are parallel to the real axis on the first geodesic piece and parallel to the imaginary axis on the second. So we have

$$B_r^L(\overrightarrow{z}) = \sum_j y_j(x_{j+1} - x_j),$$

where $z_j = x_j + iy_j$. Rewriting this we get:

$$\begin{aligned} 4B_r^L(\overrightarrow{z}) &= 2\sum_j((y_{j+1} + y_j)(x_{j+1} - x_j) - (y_{j+1} - y_j)(x_{j+1} - x_j)) \\ &= \sum_j(2(y_{j+1} + y_j)(x_{j+1} - x_j) - \operatorname{Im}((z_{j+1} - z_j)^2)) \qquad (19) \\ &= \sum_j 2(y_{j+1} + y_j)(x_{j+1} - x_j). \end{aligned}$$

The last equality holds because of the facts stated for the spaces E_m and because we restricted $\overrightarrow{z}$ to E_-, so that no term involving E_0 occurs in the expression.

This splitting of the sum into two terms has a geometric interpretation: Take the area enclosed by the curve defined by just connecting the points z_j by straight lines, but then subtract the area defined by the triangles that are defined by the L-curves and this straight line. So what we just argued was that the areas of the triangles cancel each other out (on E_-). Using our expression for $\overrightarrow{z}$ we also see that all terms involving products of terms with different values of m cancel out. This actually means that B_r^L *restricted to* E_- splits orthogonally on the subspaces E_m, $m \in M_-$. All we need now is to calculate B_r^L on each of the E_m's, but here the geometrical interpretation tells us that the area is simply

$$2B_r^L((\alpha_m\rho^{mj})_{j\in\mathbb{Z}_r}) = -\sum_j \|\alpha_m\|^2 \sin(2\pi m/r) = -r\|\alpha_m\|^2 \sin(2\pi m/r),$$

which is negative because $m \in M_-$.

The same arguments show that B_r^L is positive on E_+, but be warned: The spaces E_m is not an orthogonal splitting of B_r^L. This is because E_- and E_+ are not orthogonal, since canceling the terms really used the assumptions on M_- and M_+.

If n_1 is not 1 it is easy to see that the E_m's are just direct sums of the E_m^i's and thus see that a E_- is a maximal negative subspace for B_r^L.

We now turn our attention to $A = B_r^{\gamma_1}$. It is easy to see that because of the form of γ_1, we can again split A into a sum of

$$\begin{aligned} A_+ &: (\mathbb{C} \otimes V_+)^r \to \mathbb{R} \\ A_0 &: (\mathbb{C} \otimes V_0)^r \to \mathbb{R} \\ A_- &: (\mathbb{C} \otimes V_-)^r \to \mathbb{R}, \end{aligned}$$

where $(\mathbb{C} \otimes V_+) \oplus (\mathbb{C} \otimes V_0) \oplus (\mathbb{C} \otimes V_-) = \mathbb{C}^{n_2}$ since $V_+ \oplus V_0 \oplus V_- = \mathbb{R}^{n_2}$.

In the case of A_0 we see that this is very similar to the previous case of B_r^L. In fact, using an isometry $V_0 \cong \mathbb{R}^k$, we see that A_0 is actually the same as B_r^L.

So a choice of maximal negative subspace for A_0 is

$$E^0_- = \bigoplus_{m\in M_-} E^0_m$$
$$E^0_m = \{(b\rho^{mj})_{j\in\mathbb{Z}_r} \mid b \in \mathbb{C}\otimes V_0\}.$$

Similarly, we take an isometry $V_+ \simeq \mathbb{R}^n$. We now split this as a sum and assume again that $n = 1$, and find a maximal negative subspace.

Recall the geometric interpretation of splitting B^L_r into two terms in equation (19). We see that the difference between having L constantly equal to $i\mathbb{R}$ and having the Lagrangian curve $t \mapsto e^{i\pi t}$, is only that the triangles, whose areas we have to subtract, are different. The area of the triangle defined by z_{j+1} and z_j and some Lagrangian L is given by

$$4\,\text{area} = \operatorname{Im}(u^2(z_{j+1} - z_j)^2),$$

where u^{-1} is a unit vector in $\mathbb{C}$ with real span L. This means that the expression for A_+ is

$$4A_+(\overrightarrow{z}) = \sum_j (2(y_{j+1} + y_j)(x_{j+1} - x_j) - \operatorname{Im}(\rho^{-j}(z_{j+1} - z_j)^2)),$$

and this time the last sum does not sum to zero, and there is no symmetry between the previously defined positive and negative spaces. This is because multiplication by ρ^{-j} takes us from E_m to E_{m-1}.

We first prove that the spaces E_m are in fact orthogonal with respect to the bilinear form defined by the first part of the sum - that is, define:

$$Q(\overrightarrow{z}) = \sum_j (y_{j+1} + y_j)(x_{j+1} - x_j)$$
$$q(\overrightarrow{z_1}, \overrightarrow{z_2}) = Q(\overrightarrow{z_1} + \overrightarrow{z_2}) - Q(\overrightarrow{z_1}) - Q(\overrightarrow{z_2}).$$

By once again using the properties of the spaces E_m, we immediately see that E_m is orthogonal to $E_{m'}$ if $m \neq \pm m'$, so all we need to consider is $\overrightarrow{z_1} = \alpha_m\rho^{mj}$ and $\overrightarrow{z_2} = \alpha_{-m}\rho^{-mj}$ and prove that $q(\overrightarrow{z_1}, \overrightarrow{z_2}) = 0$. By expanding the expression for $Q(\overrightarrow{z_1} + \overrightarrow{z_2})$ and comparing with $Q(\overrightarrow{z_2}) + Q(\overrightarrow{z_1})$, we are left with two terms that are symmetric in changing the sign of m. Call these T_1 and T_2. We reduce one of these terms by

$$T_1 = \frac{1}{4i}\sum_j \Big(\alpha_m\rho^{mj}(\rho^m + 1) - \overline{\alpha_m}\rho^{-mj}(\rho^{-m} + 1)\Big)\cdot$$
$$\Big(\alpha_{-m}\rho^{-mj}(\rho^{-m} - 1) + \overline{\alpha_{-m}}\rho^{mj}(\rho^m - 1)\Big),$$

and by using the properties we get

$$T_1 = \frac{r}{4i}\Big(\alpha_m\alpha_{-m}(\rho^m + 1)(\rho^{-m} - 1) - \overline{\alpha_m\alpha_{-m}}(\rho^{-m} + 1)(\rho^m - 1)\Big)$$
$$= \frac{r}{2}\operatorname{Im}\big(\alpha_m\alpha_{-m}(\rho^m + 1)(\rho^{-m} - 1)\big) = \frac{r}{2}\operatorname{Im}\big(\alpha_m\alpha_{-m}(\rho^{-m} - \rho^m)\big).$$

Since T_2 was the same but with $-m$ and m interchanged we see that

$$T_1 + T_2 = 0.$$

Now we look at what happens on the previously positive subspace - that is, assume that $\overrightarrow{z} \in E_+$. The first part of the sum is the same as before. Looking at the last part we see that when summing we only get a contribution from the E_0 part of $\rho^{-j}(z_{j+1} - z_j)^2$. This means that we only get a contribution from the $E_1 = E_{-(r-1)}$ part of $(z_{j+1} - z_j)^2$. Since $(z_{j+1} - z_j)$ is in E_+, we only get a contribution from the $E_{-(r-1)/2}$ part of it. Define $m_1 = [(r+1)/2] = [-(r-1)/2] \in \mathbb{Z}_r$. From all this we get

$$2A_+(\overrightarrow{z}) = \Big(-\sum_{m\in M_+} r\|\alpha_m\|^2 \sin(2\pi m/r)\Big) - r\,\mathrm{Im}\Big((\alpha_{m_1}(\rho^{m_1} - 1))^2\Big)$$

for $\overrightarrow{z} \in E_+$. For r large enough we will have ρ^{m_1} approximately -1, so we get

$$2A_+(\overrightarrow{z}) \approx \Big(-\sum_{m\in M_+} r\|\alpha_m\|^2 \sin(2\pi m/r)\Big) - 4r\,\mathrm{Im}\Big((\alpha_{m_1})^2\Big).$$

Denote by $Q \subset E_+$ the one-dimensional subspace where $\mathrm{Im}(\alpha_{m_1}^2) = \|\alpha_{m_1}\|^2$ and $\alpha_m = 0$, $m \neq m_1$. Also denote by $Q' \subset E_+$ the $(r-2)$-dimensional subspace given by $\mathrm{Im}(\alpha_{m_1}^2) = -\|\alpha_{m_1}\|^2$. We now have $Q \oplus Q' = E_+$, and because the positive term coming from the sum is much smaller numerically than the "new" term on Q, we see that A_+ is negative on Q and positive on Q'.

The kernel of A_+ is the set of constant curves because it is still a finite approximation for $H_D = 0$.

Now assume that $\overrightarrow{z}$ is in the space $E_- \oplus Q$. Because the E_m's were orthogonal with respect to the first part of the expression we get the same first term. We also notice that the second part of the sum is again non-vanishing only when dealing with products from $E_{-(r-1)/2}$, so in fact we get a very similar expression:

$$2A_+(\overrightarrow{z}) \approx \Big(-\sum_{m\in M_-\cup\{m_1\}} r\|\alpha_m\|^2 \sin(2\pi m/r)\Big) - 4r\,\mathrm{Im}\Big((\alpha_{m_1})^2\Big).$$

Again, because of the assumption that $\|\alpha_{m_1}\|^2 = \mathrm{Im}(\alpha_{m_1}^2)$, we see that this is negative. There is one positive term in the sum, but this as before is smaller numerically than the "new" negative term.

This leads us to define:

$$\begin{aligned}
E_-^+ &= \Big(\bigoplus_{m\in M_-} E_m^+\Big) \oplus Q^+ \\
E_m^+ &= \{(b\rho^{mj})_{j\in\mathbb{Z}_r} \mid b \in \mathbb{C}\otimes V_+\} \\
Q^+ &= \{(b\rho^{m_1 j})_{j\in\mathbb{Z}_r} \mid b \in \mathbb{C}\otimes V_+, \|b\|^2 = \mathrm{Im}(b^2)\}.
\end{aligned}$$

From the above arguments it is clear that E_-^+ is a maximal negative subspace for A_+. Note that Q^+ is canonically isomorphic to V_+.

We also define:

$$\begin{aligned}
E_-^- &= \Big(\bigoplus_{m\in M_- - \{m_1\}} E_m^-\Big) \oplus Q^- \\
E_m^- &= \{(b\rho^{mj})_{j\in\mathbb{Z}_r} \mid b \in \mathbb{C}\otimes V_-\} \\
Q^- &= \{(b\rho^{(m_1-1)j})_{j\in\mathbb{Z}_r} \mid b \in \mathbb{C}\otimes V_-, \|b\|^2 = -\mathrm{Im}(b^2)\}.
\end{aligned}$$

A similar calculation shows us that E_-^- is a maximal negative subspace for A_-. Note that as before Q^- is isomorphic to V_-, but in this case it is more important that $E_-^- \oplus V_-$ is canonically isomorphic to

$$\bigoplus_{m \in M_-} E_m^-,$$

which leads us to introduce the notation $E_-^- \cong (\oplus_{m \in M_-} E_m^-) \ominus V_-$.

Putting all this together we see that

$$W = E_- \oplus E_-^+ \oplus E_-^0 \oplus E_-^- \cong (\bigoplus_{m \in M_-} W_m) \oplus V_+ \ominus V_-$$
$$W_m = \{b\rho^{mj} \mid b \in \mathbb{C}^{n_1+n_2}\}$$

is a choice of maximal negative subspace. □

8 Construction of the Viterbo Transfer as a Map of Spectra

Let L and N be closed smooth n-manifolds and let $j \colon L \to T^*N$ be an exact Lagrange embedding, i.e. the canonical one form λ_N on T^*N is pulled back to zero by j. In this section we construct the Viterbo transfer (see [Vit97]) as a map of spectra

$$(\Lambda j)_! \colon \Lambda N^{-T'N} \to \Lambda L^{-T'L+\eta},$$

where η is the Maslov bundle defined in section 7, and $\Lambda N^{-T'N}$ is notation for the Thom-spectrum defined by a complement bundle to $T'N = \mathrm{Ev}_0^* TN$. The construction is similar to that of Viterbo, but the lemmas in the previous sections make it much easier to control and understand the actual Thom spaces, and thus get the map as a spectrum map.

Start by giving both N and L Riemannian metrics. The Darboux-Weinstein Theorem tells us (see e.g. [MS98]) that for a small $\varepsilon_1 > 0$, we can symplectically extend j to a map $j \colon T^*_{2\varepsilon_1} L \to T^*N$. To distinguish between coordinates in T^*L and T^*N, we denote them by (q_N, p_N) and (q_L, p_L). It is very important for the construction that exactness of the embedding implies that $p_N dq_N - p_L dq_L = \lambda_N - \lambda_L$ defined on $T^*_{2\varepsilon_1} L$ is exact. It is closed but not exact for a non-exact Lagrange embedding. This implies that the two action integrals $\int_\gamma \lambda_N - Hdt$ and $\int_\gamma \lambda_L - Hdt$ are equal on closed curves in $T^*_{2\varepsilon_1} L$. This means that if we have a Hamiltonian on T^*N, which restricted to our neighborhood of L depends only on $\|p_L\|$, then we can use the method at the very end of section 3 to calculate the action integral on closed curves. In the following definition it is important to keep this method in mind.

First we define H near L. Here we define the Hamiltonian in terms of the coordinates (q_L, p_L). In fact we want it to be like the ones we looked at in section 5. So we want it to be a function of $\|p_L\|$, convex and linear with slope μ_L outside some small δ-neighborhood of the zero section. As before, we assume that μ_L is not the length of any geodesic in L. This is only the definition of H on $T^*_{\varepsilon_1} L$. On the rest of $T^*_{2\varepsilon_1} L$ we want it to be concave and "level off" to be a constant c near the boundary (see figure 5). We want this leveling off to happen so fast that any 1-periodic orbit in $T^*_{2\varepsilon_1}L - T^*_{\varepsilon_1}L$ has action below $-\delta\mu_L$. By using lemma 4.4, this can be done if δ is small enough, as seen in figure 5, where $\mu'_L < \mu_L$ is the maximum of geodesic lengths less

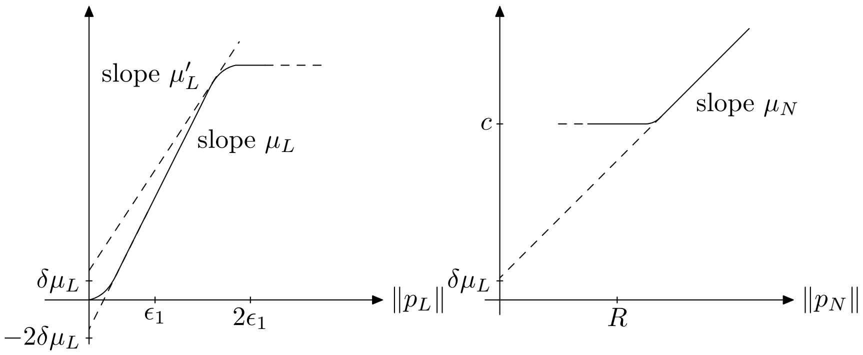

Figure 5: Definition of H.

than μ_L. It is clear that the larger μ_L is, the larger c becomes.

Outside $T^*_{2\varepsilon_1}L$, we define H to be the constant c on $T^*_RN - T^*_{2\varepsilon_1}L$ for a fixed $R > 0$, such that the embedding of $T^*_{2\varepsilon_1}L$ is inside T^*_RN. Outside T^*_RN, we define it to be a function of $\|p_N\|$, convex and linear outside some $R' > R$ with slope μ_N. Again we want the action of the 1-periodic orbits coming from this part to be less than $-\delta\mu_L$. So the larger c is, the larger we can choose μ_N (see figure 5).

We want to pick all the parameters such that we can use the lemmas in all of the preceding sections, but first we concentrate on section 4, and assume that T^*N has the induced metric. Notice that $C^1_H = \max(k\mu_L, \mu_N)$ works as a bound on the gradient of H. The k is there because we used the induced metric from L to define H near L. For small δ we can assume that δ^{-1} times some constant is an upper bound C^2_H on the covariant Hessian of H. For small δ we can further assume that $C^2_H > C^1_H$. So using proposition 4.7, we conclude that there exist a $K > 0$ and $\delta_0 > 0$ such that: If $0 < \delta < \delta_0$ and $r > K\delta^{-1}$ then

$$I(A_r, X) = \mathrm{Th}(T\Lambda^{\mu_N}_r N),$$

where X is defined in the section.

Because of the way we defined H, we see that the natural quotient map, defined in the end of section 2, gives a map from the total index to the index $I_\delta = I^{2\delta\mu_L}_{-\delta\mu_L}(A_r, X)$. Recall that the action of H restricted to $T^*_{\varepsilon_1}L$ has all of its critical values in this interval, and all other critical values are below $-\delta\mu_L$.

Lemma 5.4 tells us that by possibly making K larger and δ_0 smaller, we can find β such that we get a good index pair for the index I_δ on the subset $\Lambda_{r,\beta}T^*_{\varepsilon_1}L$, provided $r \in [K\delta^{-1}, 2K\delta^{-1}]$ and $0 < \delta < \delta_0$. For this to make sense, we recall that X in section 4 was defined to be the gradient of A_r when the $\max_j \|\varepsilon_{q_j}\|$ were small, so we may assume that X is the gradient of A_r on the subset $\Lambda_{r,\beta}T^*_{\varepsilon_1}L$, which was the assumption in section 5. Furthermore, a good pair defined inside a subset is also a good pair on the entire set, provided that it contains all the critical points, and we already accounted for that.

We have two metrics on $T^*_{\varepsilon_1}L$: One from T^*N, which we will denote g_N, and one from T^*L, which we will denote g_L. We also have two canonical sections in the bundle in equation (10) (with N replaced by L), which come from taking the vertical directions with respect to T^*N and T^*L. We denote these $S^N_{\mathcal{L}}$ and $S^L_{\mathcal{L}}$ respectively. This data produces two finite approximations

$$A^1_r = A^{g_N,S^N_{\mathcal{L}}}_r : \Lambda^{g_N}_{r,\beta}T^*_{\varepsilon_1}L \to \mathbb{R}$$
$$A^2_r = A^{g_L,S^L_{\mathcal{L}}}_r : \Lambda^{g_L}_{r,\beta}T^*_{\varepsilon_1}L \to \mathbb{R}.$$

We assume that K and β are large enough for all of our lemmas to work for both approximations. We will use most of the lemmas in the previous sections to relate them.

Proposition 8.1 *Assume that r is odd and large enough. The total index of A^1_r, which we denoted I_δ, is stably equivalent to*

$$\mathrm{Th}((T\Lambda^{\mu_L}_r L) \oplus \eta),$$

where η is the Maslov bundle defined in section 7.

Proof: First we argue that the homotopy index does not depend on the metric: Use the convex change to get a smooth family g_t of metrics relating one with the other. Make sure that $K > 0$ and $\beta > 0$ are large enough for lemma 5.4 to work for all metrics g_t. Then fix δ_0 such that the lemma works for this K, β and all metrics g_t. We now have good pairs for all t, but it is a problem that the spaces $\Lambda^{g_t}_{r,\beta}T^*_{\varepsilon_1}L$ depend on t. However, this can be resolved locally: Given any $t \in [0,1]$ we want to argue that the homotopy index is constant in a small neighborhood of t even though the manifold changes. This is because the boundary of the manifold is given by a continuous function on a bigger manifold, so by going to a slightly smaller manifold still containing our good index pair, we can use lemma 2.4.

Let s_N and s_L be the sections in the bundle

$$\mathcal{L}(T(T^*_{\varepsilon_1}L)) \to T^*_{\varepsilon_1}L$$

given by the vertical foliations from of T^*N and T^*L respectively. Then $S^N_{\mathcal{L}} = \Lambda s_N$ and $S^L_{\mathcal{L}} = \Lambda s_L$.

Redefine A^1_r as the finite approximation defined by $S^N_{\mathcal{L}}$ but using the metric induced from L. As we just argued, this does not change the index.

We stabilize the section $S^N_{\mathcal{L}}$ to the section of product type (see definition 6.1)

$$S^N_{\mathcal{L},k} = S^N_{\mathcal{L}} \oplus i\mathbb{R}^k$$

and get a new finite dimensional approximation from section 6. Lemma 6.3 tells us that since we chose this section to be independent of the point in $\Lambda_{r,\beta}T^*_{\varepsilon_1}L$ on the second factor, this new approximation has as its total index the m-fold suspension of I_δ for some m.

If we chose k large enough we can as in the beginning of section 7 homotope the difference of s_N and s_L into $\mathcal{L}(k)$ - that is, we can homotope s_N through sections in $\mathcal{L}T(T^*_{\varepsilon_1}L \times D^{2k}) \to T^*_{\varepsilon_1}L \times D^{2k}$ to

$$s_L \oplus J_*,$$

where s_L and J_* do not depend on the point in D^{2k}. So when looping this we get a section of product type.

Because we can assume that this homotopy is smooth, by looping it we get a homotopy of sections in the bundle (10) with the bounds in equation (11). However, we need lemma 7.3 to identify the negative eigenbundle of the quadratic form given the finite approximations defined by stabilizing s_L with J_*. For this we need to homotope ΛJ_* to the subset of curves in lemma 7.3, and this can be done by lemma 7.2 if k was chosen large enough. The catch is that we need this homotopy to fulfill the bounds in equation (11). If we do not have this bound, we cannot argue that the homotopy index is constant during the homotopy, because we do not know if good index pairs exist. The map, we are homotoping ΛJ_* to, is not the loop of a differentiable map. So we must argue in some other way that this can be done within the bounds. Indeed, there exists a deformation retraction $f_t, t \in [0,1]$ of the set of curves with energy less than β onto a finite dimensional compact manifold satisfying a bound like the second one in (11)[1]. The homotopy $f_t \circ \Lambda J_*$ fulfills the bounds, because ΛJ_* does and f_t fulfills the last of them, and maps the set with energy less than β to itself. The image of f_1 is a compact manifold, so homotoping ΛJ_* only on this subspace into the set we need it to lie in, will ensure the needed bounds for the composition.

This homotopy is a compact family of sections, so there is a $\delta_0 > 0$ small enough to make lemma 5.4 and lemma 6.2 work for all of them simultaneously. So we use lemma 6.3 to conclude that the index I_δ is the relative Thom suspension of the total index of A_r^2 by the negative eigenbundle of the quadratic form added. We then use lemma 6.4, lemma 7.3 and lemma 7.2 to conclude that this bundle is a trivial bundle of dimension m plus the Maslov bundle. □

Corollary 8.2 *The index quotient map to I_δ induces a map of spectra*

$$(\Lambda^{\mu_N} N)^{-T'N} \to (\Lambda^{\mu_L} L)^{-T'L \oplus \eta},$$

where $\Lambda^\mu X$ is the loop space of curves with lengths less than μ in the Riemannian metric on X.

Proof: First we note that the space

$$\mathrm{Th}(T\Lambda_r^{\mu_N} N)$$

can by parallel transport along the curve be identified with

$$(\Lambda_r^{\mu_n} N)^{\mathrm{Ev}_{z_0}^* TN^{\oplus r}}.$$

Here we switch notation for Thom bundles, because the name of the base space is not included in the bundle name. This identification makes sense in light of lemma 4.6.

Let $\psi \to A$ be any bundle over A where (A, B) is an index pair defining the total index of $A_r\colon T^*\Lambda_r N \to \mathbb{R}$. Then any quotient map $f\colon A/B \to A/B'$ has

[1] just pick an s such that $\beta/\sqrt{s}$ is smaller than the injective radius and contract onto piecewise geodesics

a relative Thom suspension using ψ defined by

$$D\psi/(D\psi_{|B} \cup S\psi) \to D\psi/(D\psi_{|B'} \cup S\psi).$$

If A/B is already a Thom space of some bundle $\psi' \to Y$, then the left hand side is the Thom space of $\psi' \oplus \psi_{|Y} \to Y$. Similarly for the right hand side. So using lemma 4.6 but adding on top of the index quotient a copy of a complement to the bundle $\mathrm{Ev}_{q_0}^* TN$, we do a standard suspension of the indices instead of the Thom suspension by the bundle. So the new quotient map obtained is just the suspension of the old one. This, however, looks a bit confusing considering the previous lemma: When increasing r by 2 we Thom suspended the index I_δ by the bundle $2\,\mathrm{Ev}_{q_0}^* TL$, but we just added two times a complement to $\mathrm{Ev}_{q_0}^* TN$, so why should this be a suspension? This can be explained by the fact that since $T_{\varepsilon_1}^* L$ is embedded into T^*N, they have isomorphic tangent bundles and two copies of TL is thus isomorphic to two copies of the pullback of TN.

Since the previous lemma worked only for r odd we see that removing two copies of $\mathrm{Ev}_{q_0}^* TN$ and two copies of $\mathrm{Ev}_{q_0}^* TL$ on each side of the map

$$\mathrm{Th}(T\Lambda_r^{\mu_N} N) \to \mathrm{Th}(T\Lambda_r^{\mu_L} L \oplus \eta)$$

cannot fully convert all the tangent spaces to suspensions on both sides simultaneously, so we choose the convention of removing an extra on each side, making both sides into spaces representing the wanted spectra. $\square$

As noted earlier, the larger μ_L is, the larger we can choose μ_N, which is a good indication that these maps glue together into something non-trivial.

Theorem 1 *The Viterbo transfer can be realized as a map of spectra*

$$(\Lambda j)_! \colon (\Lambda N)^{-TN} \to (\Lambda L)^{-TL+\eta}$$

Proof: The map is the same as the Viterbo transfer because the construction is the same. The difference is that in our construction we actually calculate the homotopy type of I_δ (not the same name as in [Vit97]) and add appropriate bundles to both sides.

We need to argue that the maps from the previous corollary are compatible with inclusions when increasing either μ_L or μ_N. Since we already proved that the quotients induce the maps, we only need to prove that they are compatible with inclusions.

First we establish two facts about the inclusion maps for Hamiltonians with two different slopes $\mu_1 < \mu_2$ not geodesics lengths. For the Hamiltonian H_1 in figure 6 we can use lemma 4.7, lemma 4.5 and the note right after the proof of lemma 4.5 to conclude that the inclusion of homotopy indices $I_{-1}^{a}(A_r, X)$ into the total index induces the Thom suspension of the inclusions defined by including curves with length less than μ_1 into the set of curves with length less than μ_2.

For the approximation associated to the Hamiltonian H_2 in figure 7 we have a natural quotient map from the total index to $I_{-\mu\delta}^{2\mu\delta}(A_r, X)$. We show that this is homotopic to the inclusion as above. In the case of H_2' we can, by making the part with slope μ_1 long enough, assume that the new bend has

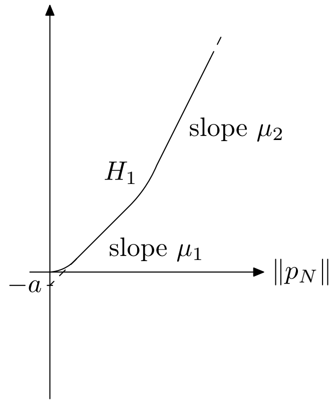

Figure 6: The Hamiltonian H_1.

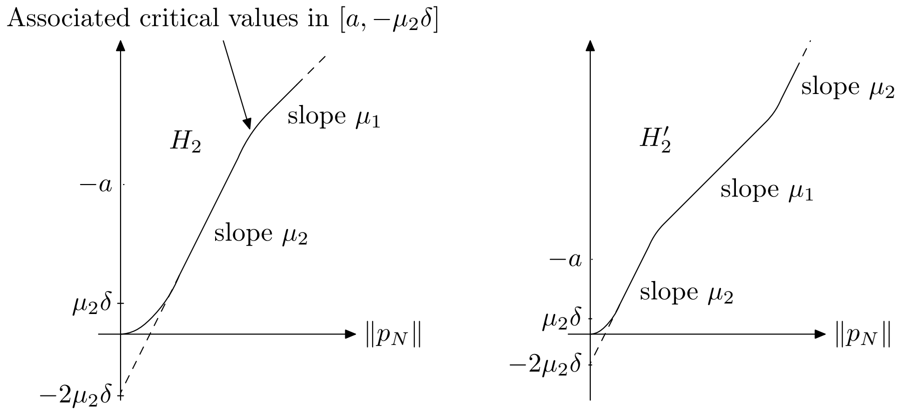

Figure 7: Hamiltonians H_2 and H_2'.

its associated critical points above $2\mu_2\delta$. The inclusion of $I_a^{2\mu_2\delta}(A_r, X)$ into the total index is the same as the previous inclusion. Now by shortening the linear part with slope μ_1, and then letting the bends cancel out, we remove the critical points with critical value less than $-\mu_2\delta$. If we think of this in terms of Morse theory and CW-complexes, we see that the new cells glued on in the inclusion effectively kill of the subcomplex $I_a^{-\mu_2\delta}(A_r, X)$, so collapsing this subcomplex is homotopic to the inclusion. This argument can be made precise by doing a small perturbation of A_r to make it Morse. So the two maps are indeed homotopic.

Now look at the problem of increasing μ_L. This can be done by a homotopy that simply multiplies the part of H close to L by $a_t \in \mathbb{R}$ increasing in t where $a_0 = 1$ and a_1 is the ratio between the new and the old μ_L. Outside $T^*_{2\varepsilon_1}L$ we simply translate H upwards so we do not change the slope μ_N. During this process we move the critical points, and when $a_t\mu_L$ is a geodesic length we create new critical points. We create two critical points per geodesic with this length: One down by the bottom bend very close to the zero section of

L, and one up in the concave part of H. At their creation they have the same critical value, which is larger than all the other critical values. We assume that our bounds for the index pair are chosen such that this is within I_δ. One of these new critical values stays above zero, but the other (from the concave part) moves below zero and through $-\mu_L\delta$, effectively collapsing this part of the homotopy index, but as we saw, this collapse induces the inclusion.

The case of changing μ_N is much easier, because by looking at the Hamiltonian on figure 8, taking the quotient from the total index to I_δ is the compo-

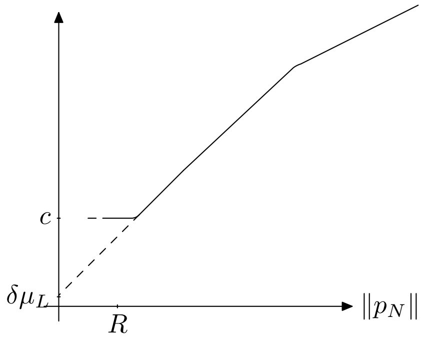

Figure 8: Lowering μ_N.

sition of two quotients where one is the inclusion and the other is the quotient for the larger μ_N. □

9 Twisted Chas-Sullivan Products

Let $\mathrm{Ev}_0 \colon \Lambda M \to M$ be the map defined by evaluation at the base point. Define $T'M = \mathrm{Ev}_0^* TM$. For any virtual bundle $\eta \to X$ we denote the Thom spectra of η by X^η. In the case where η is an actual bundle we use the same notation for the Thom space.

In [CJ02], the authors prove that the Chas-Sullivan product on an n-manifold M can be realized as the product on a ring spectrum

$$(\Lambda M)^{-T'M} \wedge (\Lambda M)^{-T'M} \to (\Lambda M)^{-T'M}.$$

We will create a slightly more general construction of this, and thereby prove that both source and target of the Viterbo Transfer is a ring spectrum.

Theorem 2 *Let M be a closed smooth manifold. For any homotopy class $[f] \in [M, \mathcal{L}] \cong [M, \mathrm{U/O}]$ there is a Chas-Sullivan type ring spectrum structure on*

$$(\Lambda M)^{-T'M+\eta},$$

where η is the virtual bundle induced by the map

$$\Lambda M \to \Lambda\mathcal{L} \to \Omega\mathcal{L} \simeq \mathbb{Z} \times B\mathrm{O}.$$

Note that the case in which f is null homotopic produces the standard Chas-Sullivan product.

Proof: Define

$$\Lambda X \times_X \Lambda X = \{(\gamma_1, \gamma_2) \in \Lambda X \times \Lambda X \mid \gamma_1(0) = \gamma_2(0)\}.$$

This has a concatenation map ρ_X to ΛX, defined by

$$\rho_X(\gamma_1, \gamma_2)(t) = \begin{cases} \gamma_1(2t) & t \in [0, 1/2] \\ \gamma_1(2t-1) & t \in [1/2, 1]. \end{cases}$$

For bundles η_1 and η_2 over ΛX, we define $\eta_1 \times_X \eta_2$ to be the restriction of $\eta_1 \times \eta_2$ to $\Lambda X \times_X \Lambda X$.

We will need a lot of structure on M, and a very natural way of getting all the structure we need is to choose an embedding $i\colon M \to \mathbb{R}^l$. This induces a Riemannian metric on M. Let $\varepsilon \leq 0$ be small enough to make any ball with radius ε geodesic convex. So for any points $m_1, m_2 \in M$ with $\mathrm{dist}(m_1, m_2) \leq \varepsilon$, we define $tm_1 + (1-t)m_2$ by using the unique geodesic connecting m_1 and m_2. Define

$$\Lambda M \times_M^\varepsilon \Lambda M = \{(\gamma_1, \gamma_2) \in \Lambda M \times \Lambda M \mid \mathrm{dist}(\gamma_1(0), \gamma_2(0)) \leq \varepsilon\}$$

and

$$\Lambda M \times_M^\varepsilon M = \{(\gamma, m) \in \Lambda M \times M \mid \mathrm{dist}(\gamma(0), m) \leq \varepsilon\} \simeq \Lambda M.$$

We have the obvious projection $P_0\colon \Lambda M \times_M^\varepsilon M \to \Lambda M$. However, we define $P_t(\gamma, m)$ by taking the point $m' = (1-t)\gamma(0) + tm$ and conjugating γ with the unique geodesic curve, parametrized by arc length, connecting m' and $\gamma(0)$, and then reparametrizing in the obvious way. So $P_1(\gamma, m)$ is a closed curve in M starting at m.

Regardless of base we will denote the trivial bundle, with the standard metric, of dimension k by $[k]$. Let $\nu \to M$ be the orthogonal complement bundle to TM in $\mathbb{R}^l$ and let $\phi\colon \nu \oplus TM \to [l]$ be the induced isometry of bundles. We define $\nu' = \mathrm{Ev}_0^* \nu$. On the space $\Lambda M \times_M \Lambda M$ the two base points are the same, so ν' and $T'M$ define bundles here as well.

In [CJ02], the authors appeal to Hilbert space versions of the loop spaces to get a Pontrjagin-Thom collapse map. We will do a more explicit construction using a little trick. Simply define a map

$$\Lambda M \times_M^\varepsilon \Lambda M \to T'M,$$

where $T'M$ is the bundle over $\Lambda M \times_M \Lambda M$, by

$$(\gamma_1, \gamma_2) \mapsto \Big(p\big(\gamma_1, \gamma_2\big), \exp^{-1}_{\gamma_1(0)}(\gamma_2(0))\Big),$$

where $p(\gamma_1, \gamma_2) = (P_1(\gamma_1, \gamma_2(0)), \gamma_2)$. We see that the boundary of $\Lambda M \times_M^\varepsilon \Lambda M$ is mapped to vectors in $T'M$ with length ε, so if we multiply the last factor by ε^{-1}, this is mapped to the sphere bundle. Thus we can extend the map to all of $\Lambda M \times \Lambda M$, provided that we map to the Thom space of $T'M$ and map the complement of $\Lambda M \times_M^\varepsilon \Lambda M$ to the base point. Because this is not surjective, it is not a Pontrjagin-Thom collapse map, but it is homotopic to one. We will use this construction with two copies of ν' added on both sides to get a map

$$\tau\colon (\Lambda M \times \Lambda M)^{\nu' \times \nu'} \to (\Lambda M \times_M \Lambda M)^{[l] \oplus \nu'}.$$

Explicitly we define τ by

$$\tau(\gamma_1, \gamma_2, v_1, v_2) = \Big(p(\gamma_1, \gamma_2), \phi\big(\varepsilon^{-1} \exp^{-1}_{\gamma_1(0)}(\gamma_2(0)) \oplus v_1\big), v_2\Big).$$

On the left hand side we use the Thom space defined by the bundle $D\nu' \times D\nu'$, and on the right hand side we use $D([l] \oplus \nu')$, where all discs have radius one. The fact that p is not defined on the entire space is not a problem, because we map this part to the base point.

Since the diagram

$$\begin{array}{ccc} [l] \oplus \nu' & \xrightarrow{\overline{\rho_M}} & [l] \oplus \nu' \\ \downarrow & & \downarrow \\ \Lambda M \times_M \Lambda M & \xrightarrow{\rho_M} & \Lambda M \end{array}$$

is pullback we can compose τ with

$$\rho_M^{[l]\oplus\nu'}\colon (\Lambda M \times_M \Lambda M)^{[l]\oplus\nu'} \to (\Lambda M)^{[l]\oplus\nu'}$$

to get a map

$$(\Lambda M)^{\nu'} \wedge (\Lambda M)^{\nu'} \cong (\Lambda M \times \Lambda M)^{\nu' \times \nu'} \to (\Lambda M)^{[l]\oplus\nu'} \cong \Sigma^l (\Lambda M)^{\nu'}.$$

This is the Chas-Sullivan product on the $(2l)$-fold suspension of $(\Lambda M)^{-T'M}$.

Now we need some technical tools for the case in which f is not null homotopic. Define

$$X = \coprod_{k \in \mathbb{N} \cup \{0\}} \mathrm{Gr}_k(2k),$$

where $\mathrm{Gr}_k(2k)$ is the Grassmannian of k-dimensional subspaces in $\mathbb{R}^{2k}$. This is a strictly associative monoid with product

$$\mathrm{Gr}_{k_1}(2k_1) \times \mathrm{Gr}_{k_2}(2k_2) \to \mathrm{Gr}_{(k_1+k_2)}(2(k_1+k_2))$$

given by direct sum. In [May77], it is proved that the loop structures on $\mathbb{Z} \times B\mathrm{O}$ coming from the two homotopy equivalences

$$\Omega BX \simeq \mathbb{Z} \times B\mathrm{O} \simeq \Omega \mathcal{L}$$

are equivalent. This implies the homotopy equivalence

$$BX \simeq \mathcal{L},$$

so we might as well assume that f maps to BX. We use the H-space structure of BX to define the projection

$$\pi_\Omega\colon \Lambda BX \to \Omega BX.$$

It is possible to construct π_Ω such that the diagram

$$\begin{array}{ccc} \Lambda BX \times_{BX} \Lambda BX & \xrightarrow{\rho_{BX}} & \Lambda BX \\ \downarrow{\scriptstyle \pi_\Omega \times \pi_\Omega} & & \downarrow{\scriptstyle \pi_\Omega} \\ \Omega BX \times \Omega BX & \xrightarrow{\rho} & \Omega BX \end{array}$$

commutes, and such that π_Ω of any constant curve is the constant curve in ΩBX. We define $\Lambda_\Omega f = \pi_\Omega \circ \Lambda f \colon \Lambda M \to \Omega BX$.

Recall the definition of BX as

$$BX = \Big(\coprod_{p \in \mathbb{N} \cup \{0\}} \Delta_p \times X^p \Big) /_\sim .$$

The space X embeds into ΩBX by the adjoint of the map $\Sigma X \to \Delta_1 \times X/_\sim \subset BX$. This maps $\mathrm{Gr}_k(2k)$ into the component identified with $\{k\} \times B\mathrm{O} \subset \mathbb{Z} \times B\mathrm{O}$. The embedding has high connectivity on the components for large k, because up to homotopy it is the standard embedding $\mathrm{Gr}_k(2k) \to B\mathrm{O}$. We now embed $X \times X$ into ΩBX by the composition $X \times X \to \Omega BX \times \Omega BX \to \Omega BX$. We define the curve $e \in \Omega BX$ by the inclusion of the point $(\mathbb{R} \subset \mathbb{R}^2)$ in $\mathrm{Gr}_1(2) \subset \Omega BX$. Concatenations with e on either side of a curve defines a map $\Omega BX \to \Omega BX$ that up to homotopy is the map that adds one to the $\mathbb{Z}$ component of $\mathbb{Z} \times B\mathrm{O}$. For any map h into ΩBX we will denote the map H composed with k_1 pre-concatenations of e and k_2 post-concatenations by $k_1 + h + k_2$.

The space of curves in M with lengths less than μ is denoted $\Lambda^\mu M$. This has compact homotopy type and so $k_1 + (\Lambda_\Omega f)_{|\Lambda^\mu M}$ can be homotoped to factor through X for large k_1. Let G_t^μ be such a homotopy with $G_0^\mu = k_1 + (\Lambda_\Omega f)_{|\Lambda^\mu M}$. By pulling back the canonical bundle over X with G_0^μ we define a bundle over $\Lambda^\mu M$, which stably represents the virtual bundle η. We will denote this by $([k_1] + \eta)$. We use this to define the spectrum $(\Lambda M)^{-T'M + \eta}$ by letting

$$(\Lambda^\mu M)^{\nu' \oplus ([k_1] + \eta)}$$

be the $(l + k_1)$th space. By increasing k_1 we get a suspension, after which we can increase μ and extend the homotopy G_t^μ and the limit of this defines the spectrum. Different choices of the homotopy G_t^μ gives different identifications of this part of the spectrum with a Thom space. In fact, if G_1^μ had factored through $X \times X$ instead of X we would have an identification of the bundle with a direct sum of bundles. Let F_t^μ be a homotopy of $F_0^\mu = (\Lambda_\Omega f)_{\Lambda^\mu M} + k_2$, such that F_1^μ factors through X, and use this to define the bundle $(\eta + [k_2])$. Since the diagram

$$\begin{array}{ccc} \Lambda M \times_M \Lambda M & \xrightarrow{\rho_M} & \Lambda M \\ {\scriptstyle (k_1 + \Lambda_\Omega f) \times_M (\Lambda_\Omega f + k_2)} \downarrow & & \downarrow {\scriptstyle k_1 + \Lambda_\Omega f + k_2} \\ \Omega BX \times \Omega BX & \xrightarrow{\rho} & \Omega BX \end{array}$$

commutes and ρ_M is a cofibration, we get that

$$H_t^\mu \colon \rho_M \circ (G_t^\mu \times_M F_t^\mu) \circ \rho_M^{-1}$$

is a homotopy on the image of ρ_M from $H_0 = k_1 + \Lambda_\Omega f + k_2$ to a map H_1^μ that factors through $X \times X$. This defines a representative of the bundle η over the image of ρ_M which we will denote $([k_1] + \eta + [k_2])$. We constructed H_t^μ such that the diagram

$$\begin{array}{ccc} \Lambda M \times_M \Lambda M & \xrightarrow{\rho_M} & \Lambda M \\ {\scriptstyle G_t^\mu \times_M F_t^\mu}\downarrow & & \downarrow{\scriptstyle H_t^\mu} \\ X \times X & \longrightarrow & X \times X \end{array}$$

commutes. This gives an identification of $([k_1]+\eta) \times_M (\eta + [k_2])$ as the pullback $\rho_M^*([k_1] + \eta + [k_2])$. Adding this bundle to the map $\rho_M^{[l]\oplus\nu'}$ we get the map

$$(\Lambda^\mu M \times_M \Lambda^\mu M)^{[l]\oplus\nu\oplus(([k_1]+\eta)\times_M(\eta+[k_2]))} \to (\Lambda^{2\mu} M)^{[l]\oplus\nu\oplus([k_1]+\eta+[k_2])}. \quad (20)$$

If we choose other homotopies G_t^μ and F_t^μ we get another identification of the Thom space on the right hand side, but this cancels out due to the fact that we also get another identification on the left hand side.

In the definition of τ we used the map p to map $\Lambda M \times_M^\varepsilon \Lambda M$ to $\Lambda M \times \Lambda M$. This map was canonically homotopic to the identity using P_t, so inserting this homotopy before $(G_1^\mu \times f_1^\mu)_{|\Lambda M \times_M^\varepsilon \Lambda M}$, we get an identification of the bundles $([k_1]+\eta) \times (\eta + [k_2])_{|\Lambda^{\mu-2\varepsilon} M \times_M^\varepsilon \Lambda^\mu M}$ and the pullback $p_|^*([k_1]+\eta) \times_M (\eta + [k_2])$. With this added on top of τ we get the map

$$\begin{aligned} \tau^{([k_1]+\eta)\times(\eta+[k_2])} \colon (\Lambda^{\mu-2\varepsilon} M \times \Lambda^\mu M)^{(\nu\oplus([k_1]+\eta))\times(\nu\oplus(\eta+[k_2]))} \to \\ \to (\Lambda^\mu M \times_M \Lambda^\mu M)^{[l]\oplus\nu\oplus(([k_1]+\eta)\times_M(\eta+[k_2]))}. \end{aligned}$$

Composing this with the map in equation (20) we define the map

$$\tau_{2l+2k} \colon \Sigma^{l+k}(\Lambda^\mu M)^{-T'M+\eta} \wedge \Sigma^{l+k}(\Lambda^\mu M)^{-T'M+\eta} \to \Sigma^{2l+2k}(\Lambda^\mu M)^{-T'M+\eta},$$

which is the definition of the product on a subset of the spectrum. If we make k larger we get a suspension of the same map, which we can extend to a larger subset, so this indeed defines a map of spectra. There are some subtleties involving reordering of suspensions, but this is a well studied aspect related to the construction of the spectra and its smash product with itself.

The unit of this ring spectrum is given by the map from the sphere spectrum induced by the Pontrjagin collapse map $S^l \to M^\nu \to (\Lambda M)^{\nu'+\eta}$. The latter inclusion is the standard inclusion using the fact that the bundle η restricted to constant curves is trivial of dimension zero. Indeed, this is because the map π_Ω projects constant curves to the constant curve. It is easy to check that inserting this on either side of the product produces a map homotopic to the identity. □

The following lemma is well-known, but the construction we present is interesting in light of our construction of the product. It also provides insight into the conjecture following it.

Lemma 9.1 *The Chas-Sullivan product is A_∞.*

Proof: We start by describing a homotopy from $\tau \circ (\tau \wedge \mathrm{Id})$ to $\tau \circ (\mathrm{Id} \wedge \tau)$. On the level of base spaces we look at the map $p\colon \Lambda M \times_M^{\varepsilon} \Lambda M \to \Lambda M \times_M \Lambda M \to \Lambda M$. The compositions $p \circ (p \times \mathrm{Id})$ and $p \circ (\mathrm{Id} \times p)$ are not well-defined, because we only defined p on a subset of $\Lambda M \times \Lambda M$. However, on the set $\Lambda M^{\times_\varepsilon 3} = \{\gamma_1, \gamma_2, \gamma_3 \in \Lambda M \mid \mathrm{dist}(\gamma_i(0), \gamma_j(0)) < \varepsilon/2\}$, both compositions are well-defined. The first picture in Figure 9 shows three curves in $\Lambda M^{\times_\varepsilon 3}$, and

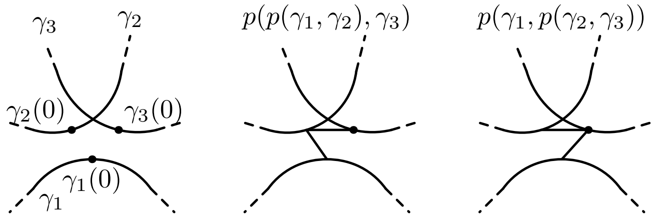

Figure 9: Connecting curves in different order.

the two others illustrate the difference between the two different compositions. After a reparametrization, the two curves differ only on the inserted geodesic pieces, and it is easy to define a homotopy between them using a parameter $s \in [0, 1]$ that specifies at which point on the geodesic, which is horizontal in the figure, the other geodesic starts. Recall that the curve runs through both geodesics in each direction. Figure 10 illustrates how to define the curve for $s = 1/2$.

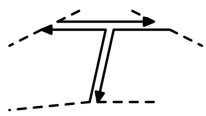

Figure 10: Geodesic overlap.

Adding the bundles ν' on the factors, we see that the two maps we wish to compare are

$$\begin{aligned}
&\tau(\gamma_1, \tau(\gamma_2, \gamma_3, v_1, v_2), v_3) = \\
&\quad (p(\gamma_1, p(\gamma_2, \gamma_3)), \varepsilon^{-1} \exp^{-1}_{\gamma_1(0)}(\gamma_3(0)) \oplus v_1, \varepsilon^{-1} \exp^{-1}_{\gamma_2(0)}(\gamma_3(0)) \oplus v_2, v_3) \\
&\tau(\tau(\gamma_1, \gamma_2, v_1, v_2), \gamma_3, v_3) = \\
&\quad (p(p(\gamma_1, \gamma_2), \gamma_3)), \varepsilon^{-1} \exp^{-1}_{\gamma_1(0)}(\gamma_2(0)) \oplus v_1, \varepsilon^{-1} \exp^{-1}_{\gamma_2(0)}(\gamma_3(0)) \oplus v_2, v_3),
\end{aligned}$$

where the outer τ is the identity on the trivial components produced by the inner τ. Indeed, this corresponds to identifying the outer τ with a suspension of the original τ. We only looked at the compositions of p on a subset, but this is easily remedied by inserting a homotopy that shrinks the part of the space which is not sent to the base point. We do this by using a homotopy that increases the ε^{-1} factor on the inverse exponentials to $4\varepsilon^{-1}$. This has the result that everything outside $(\Lambda M)^{\times_\varepsilon 3}$ is sent to the base point by either map. Now it is easy to use the homotopy we constructed for the p's, and at the same time homotope the factor $4\varepsilon^{-1} \exp^{-1}_{\gamma_1(0)}(x)$, by letting x be the point on the one geodesic at which the other geodesic starts. This does not change the fact that the complement of $(\Lambda M)^{\times_\varepsilon 3}$ is sent to the base point.

We conclude that the Chas-Sullivan product is homotopy associative. But what about higher associativity homotopies? We generalize the above construction to i curves $\gamma_1, \ldots, \gamma_i$. For the purpose of doing this we define spaces K_i, for $i \geq 2$, homeomorphic to and with the same boundary maps as the spaces Stasheff defines in [Sta63]. Define

$$K_i = \{(s_{i-2}, s_{i-3}, \ldots, s_1) \in \mathbb{R}^{i-2} \mid s_{i-2} \in [0,1], s_j \in [0, 1 + s_{j+1}], j < i-2\}.$$

This space has a useful geometric interpretation. Place i distinct clockwise ordered points $p_1, \ldots, p_i$ on the unit circle. Now draw the line segment from p_i to p_{i-1} parametrized by the interval $[0,1]$. Then draw a new line segment parametrized by the interval $[s_{i-2}, 1 + s_{i-2}]$, starting at the point on the first segment corresponding to s_{i-2} and ending at p_{i-2} (see figure 11). We wish to

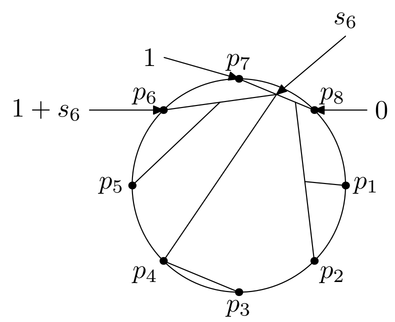

Figure 11: Line segments for $i = 8$ and parametrizations of the first two segments.

continue by drawing a line segment starting on either of the two existing line segments and ending at p_{i-3}, but it should only intersect the other segments at its starting point. We see that because of the way we parametrized the second line segment, the parameter s_{i-3} uniquely defines a starting point for such a curve. Furthermore, by parametrizing this third segment by the interval $[s_{i-3}, 1+s_{i-3}]$, we see that the next parameter s_{i-4} uniquely defines a new line segment starting at a point on the first three segments and ending at p_{i-4}, not intersecting the other segments except at its starting point. Continuing this down to and including p_1 we identify K_i with a space of diagrams. Figure 11 illustrates a point in K_8. This geometrical interpretation is valid no matter how we space the points around the circle.

The points in the space K_i are generalizations of choices of parentheses on a product of i terms, and the points where the line segments both start and end at the points p_j correspond to actual choices of parentheses. For instance, the diagram where all the segments start at p_i corresponds to $(\gamma_1(\gamma_2(\cdots(\gamma_{i-1}\gamma_i))\cdots)$, and the one where they join neighboring points all the way around the circle corresponds to $((\cdots(\gamma_1\gamma_2)\gamma_3)\cdots\gamma_i)$. The boundary of K_i divides up into natural subsets, each related to fixing one set of parentheses. In the geometrical interpretation the boundary component given by $\gamma_1\cdots\gamma_{j-1}(\gamma_j\cdots\gamma_{j+r})\gamma_{j+r+1}\cdots\gamma_i$ is defined as follows: The subset of K_i where the sub-diagram defined by points p_j through p_{j+r} and the segments ending at the points p_j through p_{j+r-1} is in fact a viable diagram representing a point in K_r, and the complement of this sub-diagram together with the point p_{j+r} is also a viable diagram representing a point in K_{i-r+1}. It is easy to check that the union of all these components is the entire boundary of K_i, because the boundary consists of the points where there is a segment either going from p_{j+1} to p_j or from p_i to p_j, and all such can be divided. Conversely, if such a subdivision is possible then one of the segments must be in such a position. Also if there are two possible subdivisions, then we can in fact divide it into three diagrams d_1, d_2 and d_3 where d_1 and d_2 have a point in common and as do d_2 and d_3. This means that these boundary components only intersect at their own boundary, and so we have a system of inclusions satisfying the same relations as the spaces defined by Stasheff.

To prove A_n we need to define maps

$$M_i \colon K_i \times ((\Lambda M)^{-T'M})^{\wedge i} \to (\Lambda M)^{-T'M}$$

for $i \leq n$, which are compatible with these identifications of the boundary of K_i as a union of products of the form $K_r \times K_{i-r+1}$. That is,

$$M_{i-r+1}(k, x_1, \dots, x_j, M_r(k', x_{j+1}, \dots, x_{j+r}), x_{j+r+1}, \dots, x_i) = \\ M_i(g(k,k'), x_1, \dots, x_i),$$

where $g \colon M_r \times M_{i-r+1} \to M_i$ is the inclusion of this particular face.

First homotope the factor ε^{-1} in the definition of τ to be $n^2\varepsilon^{-1}$. This will imply that the maps constructed below send all but the interior of

$$(\Lambda M)^{\times^n_\varepsilon i} = \{(\gamma_1, \cdots, \gamma_i) \in (\Lambda M)^{\times i} \mid \operatorname{dist}(\gamma_j(0), \gamma_k(0)) < \varepsilon/n\}$$

to the base point, for any $i \leq n$. So we restrict our attention to these spaces. The geometrical interpretation of the spaces K_i makes it easy to define the maps on the level of base spaces, simply by using the same idea as before. That is, given a point in K_i and i curves $\gamma_1, \dots \gamma_i$, we identify the first segment between p_{i-1} and p_i with the geodesic connecting $\gamma_{i-1}(0)$ and $\gamma_i(0)$. Then we identify the second segment with the geodesic starting at the point corresponding to s_{i-2} on the first geodesic and ending at $\gamma_{i-2}(0)$. We continue this pattern and identify each segment with a geodesic. This defines a map from the union of the segments to M. Now imagine a small line segment emanating from the circle at each point p_j as in figure 12. We then define a curve in M by defining a curve

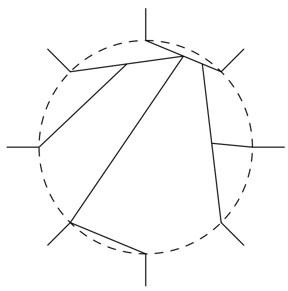

Figure 12: Emanating line segments.

following the line segments in the diagram and mapping it to M. Starting at the point p_i corresponding to $\gamma_i(0)$ we follow the segments, and when hitting a fork of segments we always take a left turn. We start by assuming that we just came in using the emanating segment at p_i. When exiting the figure at an emanating line segment at a point p_j we do not have a map to M, but instead we run once around the curve γ_j, and then we reenter using the same emanating line segment. Eventually we return to the point p_i and exit via the emanating segment and run once around γ_i. We end up having followed each segment exactly once in each direction at all generic points. From this description it is clear that we run through the curves $\gamma_1, \dots \gamma_i$ in order, but with a number of various geodesic pieces put in at their starting points to connect them. The choice of parametrization is a contractible choice, and we have already defined it on K_2, so it is not difficult to inductively choose parametrizations of the curve we just described on all the spaces K_i up to K_n. We have thus defined maps $m_i \colon K_i \times (\Lambda M)^{\times_\varepsilon^n i} \to \Lambda M$. As before, for each curve γ_j we can take the starting point of the geodesic associated with the segment ending at p_j, and use that to define the map on the level of Thom spaces, such that we get the inverse exponential of the correct points at the points actually corresponding to a choice of parentheses. □

Conjecture 3 *The twisted Chas-Sullivan product is A_∞.*

To prove this we need to insert the bundles, used in defining the product, over the construction in the lemma above, but for now the details of this construction seems obscure.

The construction of these twisted products and theorem 1 also motivates the following conjecture.

Conjecture 4 *The Viterbo transfer is a ring spectrum homomorphism using twisted Chas-Sullivan products.*

We have yet to discover a complete proof of this statement. There is, however, a construction of the Chas-Sullivan product using the methods from section 4 that will commute with the collapse map used in defining the Viterbo transfer. We will use the rest of this section to outline the construction.

In section 4 we defined the function A_r and the pseudo-gradient X for any Hamiltonian, and in section 8 we defined the Hamiltonian H depending on $\mu_L > 0, \mu_N > 0$ and $c > 0$ for which we took the total index $I_{-k}^{2\delta\mu}(A_r, X)$, for some large k, and collapsed to an index I_δ. This was the construction of the Viterbo transfer. Any good index pair (A, B) for the first of these indices would have $B \subset A_r^{-1}(-k)$, and we know from section 4 that such exist. We also define the similar map A_{2r+1}. However, unlike in section 4, we define A_{2r+1} to approximate flow curves parametrized by the interval $[0, 2]$ instead of $[0, 1]$. This we do by using the same construction as in the proof of lemma 4.6, defining $t_j = 1/n$ if $j \neq n$ and $t_n = 0$. This way we have $\sum_j t_j = 2$ instead of 1, and the critical points of A_{2r+1} will then be 2-periodic orbits. The goal is now to construct a good index pair (A_2, B_2) for this function and construct a map $f \colon A \times A \to A_2$ which will induces a map of quotients

$$A/B \wedge A/B \to A_2/B_2.$$

Lemma 4.7 tells us that the right hand side is homotopic to $\mathrm{Th}(T\Lambda_r^{\mu_N} N) \wedge \mathrm{Th}(T\Lambda_r^{\mu_N} N)$ and the left hand side is homotopic to $\mathrm{Th}(T\Lambda_{2r+1}^{\mu} N)$. In the construction of the Viterbo map, we formally inverted the bundles on the quotients by adding inverse bundles. Doing this for the above map we get a map

$$\Sigma^l(\Lambda N)^{-T'N} \wedge \Sigma^l(\Lambda N)^{-T'N} \to \Sigma^{2l}(\Lambda N)^{-T'N},$$

which turns out to be the Chas-Sullivan product on curves of length less than μ_L.

The construction of f on points where the two base points are near each other is as follows: Take two curves $\overrightarrow{z_1}, \overrightarrow{z_2}$ in $T^*\Lambda_r M$. Then the first n factors of $f(\overrightarrow{z_1}, \overrightarrow{z_2})$ are the factors of $\overrightarrow{z_1}$, and similarly the last n factors are the factors of $\overrightarrow{z_2}$. The remaining middle factor (q_n, p_n) is defined by letting $q_n = q_0$ and letting p_n be the parallel transport of p_{2n} from q_{2n} to $q_n = q_0$. One can check that this approximately produces the identity

$$A_r(\overrightarrow{z_1}) + A_r(\overrightarrow{z_2}) = A_{2r+1}(f(\overrightarrow{z_1}, \overrightarrow{z_2})).$$

This is why we need the extra point. In the proof of lemma 4.3 we saw that we can create good index pairs containing any compact subset of $T^*\Lambda_r N$, and by choosing the value k very large he get that $(A_{2r+1} \circ f)_{|A\times B \cup B \times A} < -k + 2\delta\mu$ is below the lowest critical value for A_{2r+1}, so this induces a map of the quotients.

The only problem with this map so far is that we did not really define it if the points q_n and q_{2n} are to far apart. These points are the base points of $\overrightarrow{z_1}$ and $\overrightarrow{z_2}$, so this is similar to the problem defining the Chas-Sullivan product. However, in section 4 when we defined the pseudo-gradient for A_{2r+1}, we defined it such that if these points get too far apart, the flow of X will flow to B_2, and thus by using the flow of $-X$ we can get the space, where $\mathrm{dist}(q_n, q_{2n})$ is large, mapped to B_2 which is our base point on the left hand side. In fact, we can do this by only flowing the point p_n and thereby mapping these points into the disc bundle of the copy of $\mathrm{Ev}_0^* TN$ that comes from adding the extra point (q_n, p_n), proving that this is indeed a realization of the Chas-Sullivan product.

So this is an A_∞ product on the spectrum, and by being careful when taking the quotient to I_δ, one can retain this fact and get an A_∞ product on the left hand side of the Viterbo transfer. However, we still need to relate this construction on the index I_δ to the construction of the twisted Chas-Sullivan product. This would also prove that in such a case the twisted product is in fact A_∞.

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