Cohomology of Line Bundles

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Contents

Introduction

Let G be a reductive algebraic group over an algebraically closed field k of characteristic $p \geq 0$ and B a Borel subgroup of G. Each character λ of B induces a line bundle $\mathcal{L}(\lambda)$ on the flag variety G/B, and the cohomology group $H^\bullet(G/B, \mathcal{L}(\lambda))$ has a natural G-structure.

In the case where $\mathrm{char}(k) = 0$ the representation theory of G is well understood. Each rational G-module is semi-simple. The simple G-modules are classified by their highest weights, and one has a character formula for these simple modules. Furthermore, the Borel-Weil-Bott theorem [16] (cf. also [19]) gives a complete description of the vanishing behaviour of the cohomology group $H^\bullet(G/B, \mathcal{L}(\lambda))$.

The situation in prime characteristic is much less understood. The simple G-modules are still classified by their highest weights, but a character formula is far from known, and the Borel-Weil-Bott theorem fails completely. In this thesis we collect some well-known results about the vanishing behaviour of the cohomology of line bundles on G/B, and we demonstrate that many of these results can be carried over to the quantum case.

The first result concerns the first cohomology group associated to characters of the quantized Borel subalgebra. We prove that it, if non-zero, contains a unique simple submodule and give a complete description of the vanishing behaviour of this cohomology. Next, we give a fundamental theorem on relations between the cohomology groups in question. It involves the quantum Frobenius homomorphism [29], and it implies an interesting nonvanishing theorem. This result is analogous to the main theorem in [1].

Related to the problem of describing the cohomology of line bundles on G/B is the calculation of the B-cohomology $H^\bullet(B, \lambda) = \mathrm{Ext}^\bullet_B(k, \lambda)$ of 1-dimensional B-modules λ. When $\mathrm{char}(k) = 0$, there is an easy well-known description of this cohomology because we can compare with the corresponding G-cohomology and take advantage of the fact that $H^i(G, -) = 0$ for all $i > 0$ (G is reductive). But when $\mathrm{char}(k) = p > 0$, this approach fails completely. Except for degrees 0 and 1, the problem of determining $H^i(B, \lambda)$ is in this case wide open.

Our contribution in this thesis is to give a couple of general results on the behaviour of $H^\bullet(B, \lambda)$ and to calculate $H^2(B, \lambda)$ and $H^3(B, \lambda)$ explicitly when p is larger than the Coxeter number for G. Our results are based on a combination of several methods, see Section 2.2 below. The main ingredient is the spectral sequence relating B-cohomology to the cohomology for the first Frobenius kernel B_1 of B. We take here advantage of the fact that the cohomology $H^\bullet(B_1, \lambda)$ was completely determined in [9].

Our approach works equally well for quantum groups. Let U_q denote the quantum group corresponding to G with parameter $q \in k^\times$, and let B_q be the Borel subalgebra in U_q corresponding to the negative roots. When q is not a root of unity, we can determine $H^\bullet(B_q, \lambda)$ exactly as in the above characteristic 0 case. So we consider the case where q is an l-th root of unity. Then the problem of describing $H^\bullet(B_q, \lambda)$ is again wide open in general. But our methods allow us to obtain similar results as described above for B.

Moreover, when $\mathrm{char}(k) = 0$, we will also compute $H^4(B_q, \lambda)$ for all characters λ. But this requires a different argument than the one given in the modular case.

Summary

This thesis contains two parts. The first part deals with the modular case, while the second part deals with the quantum case.

Chapter 1. We fix the notation and recall several well-known facts about the cohomology theory of line bundles on G/B. We are especially interested in the vanishing behaviour of the cohomology group $H^\bullet(G/B, \lambda)$. The vanishing behaviour depends on whether k is a field of characteristic 0 or of characteristic $p > 0$. We shall discuss both cases.

Chapter 2. In this chapter we want to compute the B-cohomology $H^\bullet(B, \lambda)$ of 1-dimensional B-modules determined by a Borel character λ. Again, the computations here depend on the characteristic of k. We shall discuss both cases, and in the case where $\mathrm{char}(k) > 0$ we shall for each $\lambda \in X$ determine an upper bound i for the degree in which the cohomology $H^i(B, \lambda)$ can be non-zero and compute all such cohomology in degrees at most 3 when p is larger than the Coxeter number for G. This chapter represents joint work with Henning Haahr Andersen [14].

Chapter 3. We denote by U_q the quantum group with parameter $q \in k^\times$ corresponding to G. By this we mean the specialization at q of the Lusztig integral form of the quantized enveloping algebra attached to the corresponding root system R. Let B_q be the Borel subalgebra in U_q corresponding to the negative roots. In this chapter we will introduce the induction functor from B_q and state some of its properties.

Chapter 4. Just as for B any λ in the corresponding root lattice X defines a Borel character. This chapter deals with the first cohomology groups associated to characters of the Borel subalgebra B_q. We give a complete description of the vanishing behaviour of these cohomology groups. Moreover, we prove that whenever they are non-zero they contain a unique simple submodule. These results are analogous to the classical case [2], and the proofs here follow the same lines as in the classical proofs.

Chapter 5. The vanishing behaviour of $H_q^\bullet(\lambda) = H^\bullet(U_q/B_q, \lambda)$ depend on whether q is a root of unity or not. Andersen, Polo and Kexin proved in [11] that the Borel-Weil-Bott theorem holds for all characters λ when q is not a root of unity, and hence we have a complete description of the vanishing behaviour of $H_q^i(\lambda)$. But when q is a root of unity, the Borel-Weil-Bott is no longer true. In fact, the problem of describing the vanishing behaviour of the cohomology groups associated to characters of the Borel subalgebra B_q is still wide open. In this case we will prove that the main theorem in [1] has an analogue in the quantum case.

Let X^+ be the set of dominant weights. In Section 5.3 we will prove that if $\lambda \in X^+$ lies far away from the walls of the dominant Weyl chamber, then we have for each $w \in W$ that $H_q^\bullet(w \cdot \lambda)$ is non-zero in exactly one degree. Here W is the corresponding Weyl group, and the dot action is given by

$$w \cdot \lambda = w(\lambda + \rho) - \rho \text{ for all } \lambda \in X.$$

As usual ρ denotes half the sum of the positive roots. (see below for the precise definitions). Again, the inspiration comes from the modular case, see [8].

Chapter 6. In this chapter we study the B_q-cohomology $H^\bullet(B_q, -)$ of 1-dimensional B_q-modules determined by a Borel character $\lambda \in X$. Note that $H^0(B_q, -)$ is now the fixed point functor for B_q in the Hopf algebra sense. Since there is an easy well-known description of $H^\bullet(B_q, \lambda)$ of this cohomology, we will in this chapter focus on the root of unity case. We will prove that the results in Chapter 2 have direct analogues for B_q.

Moreover, when $\mathrm{char}(k) = 0$, we will compute $H^4(B_q, \lambda)$ explicitly, and determine a lower bound i for the degree in which the cohomology $H^i(B_q, \lambda)$ can be non-zero. We shall here take advantage of the fact that for any finite dimensional $\bar{B}$-module M we have an isomorphism of vector spaces

$$H^i(\bar{B}, M) \simeq H^{N-i}(\bar{B}, M^* \otimes -2\rho) \text{ for all } i \geq 0.$$

Here $\bar{B}$ is the Borel subgroup in the corresponding complex semisimple group $\bar{G}$, N is the number of positive roots, and M^* is the dual module.

Appendix A. This appendix is a continuation of Chapter 3. Here we derive further consequences of the strong linkage principle. Again, the inspiration comes from the modular case [7], and the proofs run along the same lines. Among these results, we will prove that for any dominant weight λ and for any $w \in W$ the cohomology group $H_q^{l(w)}(w \cdot \lambda)$ is always non-zero. Here $l(w)$ denotes the length of w. This result will be used in Chapter 5.

Appendix B. In this appendix we collect some properties of the Steinberg module which will be needed in Chapter 5.

Appendix C. In [11] Andersen, Polo and Kexin proved some important results on base change for the derived functors of induction. We prove that these results have analogues for the B-cohomology.

Acknowledgments

This thesis presents the results that I have obtained during my graduate studies at University of Aarhus, 2004-2007. Most of the thesis is joint work with my advisor Henning Haahr Andersen, and I would like to thank him for his invaluable help. I also wish to thank Jens Cartsen Jantzen for answering questions from time to time.

During my graduate studies I visited University of Lyon 1. I take here the opportunity to acknowledge the support I received at the mathematical department.

In particular, I would like to thank Olivier Mathieu for informative discussions during my stay in Lyon.

CHAPTER 1

Cohomology of line bundles on G/B

Let B be a Borel subgroup in a reductive algebraic group G over an algebraically closed field k. In this chapter we fix the notation and state well-known results in the cohomology theory of line bundles on the flag variety G/B. Among these results, the vanishing behaviour of the cohomology $H^\bullet(G/B,\lambda)$ of 1-dimensional B-modules λ. In characteristic zero, the Borel-Weil-Bott theorem completely describes this cohomology whereas the corresponding problem in characteristic $p>0$ is still wide open.

1.1. Basic notation

Let k be an algebraically closed field k of characteristic $p \geq 0$, and let G be a reductive algebraic group over k. Let T be a maximal torus in G and B a Borel subgroup containing T. The root system for (G,T) is denoted by R, and we choose a set of positive roots R^+ of R in such way that B corresponds to the negative roots $R^- = -R^+$.

The Euclidean space associated with R will be denoted by $\mathbb{E}$, and the inner product on $\mathbb{E}$ will be denoted by $\langle , \rangle$. Let X be the integral weight space lattice obtained from R. We have a partial order on X given by

$$\lambda \geq \mu \Leftrightarrow \lambda - \mu \text{ can be written as a sum of positive roots.}$$

Let S be the corresponding set of simple roots, and let W be the Weyl group. A weight $\lambda \in X$ is called dominant if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in S$. Here $\alpha^\vee = 2\alpha/\langle \alpha,\alpha\rangle$ is the coroot of α. For each $\alpha \in R$ we let s_α be the reflection associated to α. There are two actions of W on $\mathbb{E}$. The first one is the natural one given by $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee\rangle \alpha$ for all $\alpha \in R$ and $\lambda \in \mathbb{E}$. The second one is the "dot action" given by

$$w \cdot \lambda = w(\lambda+\rho) - \rho \text{ for all } \lambda \in \mathbb{E}, w \in W.$$

Here ρ denotes half the sum of the positive roots. Note that since $\langle \rho, \alpha^\vee\rangle = 1$ for all $\alpha \in S$, we have $s_\alpha \cdot \lambda = s_\alpha(\lambda) - \alpha$ for all $\lambda \in \mathbb{E}$.

The set of dominant weights X^+ parametrizes the simple G-modules via highest weight. For each $\lambda \in X^+$ we let $L(\lambda)$ be the simple G-module of highest weight λ.

1.2. The affine Weyl group

In this section we assume that k is a field of characteristic $p > 0$.

1.2.1. The affine Weyl group. For all $\alpha \in R$ and $n \in \mathbb{Z}$ we let $s_{\alpha,n}$ denote the affine reflection given by

$$s_{\alpha,n} \cdot \lambda = s_\alpha \cdot \lambda + np\alpha \text{ for all } \lambda \in \mathbb{E}.$$

The affine Weyl group W_p is then the group generated by all $s_{\alpha,n}$ with $\alpha \in R$ and $n \in \mathbb{Z}$.

1.2.2. Alcoves. Let h be the Coxeter number for R. We define C_p to be

$$C_p = \{\lambda \in \mathbb{E} \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \text{ for all } \alpha \in R^+\},$$

and its closure

$$\bar{C}_p = \{\lambda \in \mathbb{E} \mid 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq p \text{ for all } \alpha \in R^+\}.$$

The set C_p is called the fundamental alcove, and its closure is a fundamental domain for the action of W_p. Note that $C_p \neq \emptyset$ if and only if $p \geq h$.

An alcove $C \subset \mathbb{E}$ is a subset of the form $C = w \cdot C_p$ for some $w \in W_p$. The closure of C is then $\bar{C} = w \cdot \bar{C}_p$.

1.2.3. Weyl chambers. For any $\alpha \in R$ we let H_α be the reflection hyperplane of s_α for the "dot action" on $\mathbb{E}$. The connected components of the complement of $\cup_{\alpha \in R^+} H_\alpha$ in $\mathbb{E}$ are called the (open) Weyl chambers. The closure of each chamber is a fundamental domain for W.

1.2.4. Finally, for each $r \geq 0$ we let X_r be the set of p^r-restricted weights

$$X_r = \{\lambda \in X \mid 0 \leq \langle \lambda, \alpha^\vee \rangle < p^r \text{ for all } \alpha \in S\}.$$

For each weight $\lambda \in X$ there is a unique decomposition $\lambda = \lambda^0 + p^r\lambda^1$ with $\lambda^0 \in X_r$ and $\lambda^1 \in X$. Later we shall also need the "dot action" of p^r on $\lambda \in X$ given by

$$p^r \cdot \lambda = p^r(\lambda + \rho) - \rho.$$

Note that $p^r \cdot w \cdot \lambda = w \cdot p^r \cdot \lambda$ for all $r \geq 0, \lambda \in X$ and $w \in W$.

1.3. Induced representations

In this section k will be an arbitrary algebraically closed field. Any rational B-module M defines a sheaf $\mathcal{L}(M)$ on G/B whose cohomology group $H^0(G/B, \mathcal{L}(M))$ has a G-structure and $H^0(G/B, \mathcal{L}(M)) \simeq \operatorname{Ind}_B^G M$. The induction functor Ind_B^G from the category of B-modules to the category G-modules is left exact. It turns out that (see e.g. [22, 5.12])

$$H^i(G/B, \mathcal{L}(M)) \simeq R^i \operatorname{Ind}_B^G M \text{ for all } i \geq 0.$$

We shall then write $H^i(M)$ in short for $R^i \operatorname{Ind}_B^G M$. We shall use the convention $H^i(M) = 0$ if $i < 0$.

1.3.1. The generalized tensor identity. For any G-module V we have

$$H^i(M \otimes V) \simeq H^i(M) \otimes V \text{ for all } i \geq 0. \tag{1.1}$$

For more details and for a proof we refer to [1].

1.3.2. Serre duality. Let $N = \dim G/B = \#R^+$. Serre duality [24, III. 7.7] then says that

$$H^i(M)^* \simeq H^{N-i}(M^* \otimes -2\rho) \text{ for all } i \geq 0. \tag{1.2}$$

Here M^* is the dual module of M with the contragredient action. Recall that we have

$$H^i(M) = 0 \text{ for all } i > N. \tag{1.3}$$

This is Grothendieck's vanishing theorem [24, III. 2.7].

1.3.3. Degree zero. We have $B = TU$ where U is the unipotent radical of B. Then any character λ of T extends uniquely to B (by $\lambda(U) = 1$). The 1-dimensional B-module, where B acts via λ, is denoted by λ or sometimes k_λ. In particular, the trivial B-module k may also be written k_0.

It is well-known that $H^0(\lambda)$ is non-zero if and only if λ is dominant. If so, $H^0(\lambda)$ has a unique simple submodule with highest weight λ. Serre duality for line bundles then implies that $H^N(\lambda)$ is non-zero if and only if λ is antidominant, i.e. $\lambda \in -X^+ - 2\rho$, and in this case it has a unique simple quotient with highest weight $-\lambda - 2\rho$.

1.3.4. Degree one. By studying the structure of the cohomology groups for groups of rank one, Andersen proved in [2] that the first cohomology $H^1(\lambda)$, if non-zero, contains a unique simple submodule and gave a complete description of the vanishing behaviour of $H^1(\lambda)$ for all $\lambda \in X$. We shall recover this result in Section 1.5.

1.3.5. Kempf's vanishing theorem. If $\lambda \in X^+$, then

$$H^i(\lambda) = 0 \text{ for all } i > 0. \tag{1.4}$$

In characteristic zero, Kempf's vanishing theorem had been known for a long time, but it was only in 1976 that Kempf proved this theorem in characteristic $p > 0$, see [28]. In 1979 Andersen and Haboush independently found a short proof of this theorem, see [1] and [23].

1.3.6. The Borel-Weil-Bott theorem. Suppose for a second that k is a field of characteristic zero. Let $\lambda \in X$ and choose $w \in W$ such that $w(\lambda + \rho) \in X^+$. The Borel-Weil-Bott theorem [16] (cf. also [19]) then says

$$H^i(\lambda) \simeq \begin{cases} H^0(w \cdot \lambda) & \text{if } i = l(w), \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

Here $l(w)$ denotes the length of w.

This result does not generalise to positive characteristic. In fact, it can be shown that there are cases where $H^i(\lambda)$ is non-zero for several values of i, see Figures 2 and 3.

However, Andersen proved in [8] that if $\lambda \in X^+$ lies far away from the walls of the dominant Weyl chamber, then we have for each $w \in W$ that $H^\bullet(w \cdot \lambda)$ is non-zero in exactly one degree. More precisely, let $\lambda \in X^+$ and write $\lambda = \lambda^0 + p\lambda^1$ for some $\lambda^0 \in X_1$ and $\lambda^1 \in X$. We say that λ is generic if

$$4(h-1) \leq \langle \lambda^1, \alpha \rangle \leq p - 4(h-1) \text{ for all } \alpha \in R^+.$$

Note that generic weights exist only if $p > 8(h-1)$.

Now, if $\lambda \in X^+$ is generic, one can show that for each $w \in W$ we have

$$H^i(w \cdot \lambda) \neq 0 \text{ if and only if } i = l(w). \tag{1.6}$$

For more details and for a proof we refer to [8, Proposition 2.1].

1.4. The strong linkage principle

We assume that k is a field of positive characteristic. Let $\lambda, \mu \in X$. We say that λ is linked to μ if $\lambda \in W_p \cdot \mu$, and λ is strongly linked to μ if $\lambda = \mu$ or if there are reflections $s_1, \dots, s_{r+1} \in W_p$ such that

$$\lambda \leq s_1 \cdot \lambda = \lambda_1 \leq s_2 \cdot \lambda_1 = \lambda_2 \leq \cdots \leq s_r \cdot \lambda_{r-1} = \lambda_r \leq s_{r+1} \cdot \lambda_r = \mu.$$

The following theorem was proved by Andersen in [3]:

Theorem 1.1 (The strong linkage principle)**.** *Let $\lambda \in X^+ - \rho$ and $\mu \in X^+$. If $L(\mu)$ is a composition factor of some $H^i(w \cdot \lambda)$ with $w \in W$ and $i \in \mathbb{N}$, then μ is strongly linked to λ.*

The strong linkage principle implies that the Borel-Weil-Bott theorem holds for small weights, i.e. $\lambda \in \bar{C}_p$ (for details see e.g. [5]). Using some standard homological methods together with the fact that

$$\mathrm{Ext}^1_G(L(\mu), H^0(\lambda)) = 0 \text{ if } \mu \not> \lambda,$$

the strong linkage principle implies (see e.g. [5] or [22])

Proposition 1.2. *Let $\lambda, \mu \in X^+$. If $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \neq 0$, then λ is linked (but not equal) to μ.*

As a direct consequence of the above proposition, we have the following corollary:

Corollary 1.3 (The linkage principle)**.** *Let $\lambda, \mu \in X^+$. If $L(\lambda)$ and $L(\mu)$ occur as composition factors of the same indecomposable module, then λ is linked to μ.*

1.5. The vanishing behaviour

We no longer assume that $\mathrm{char}(k) > 0$. The exact vanishing behaviour of the cohomology $H^\bullet(\lambda)$ is still not known, but there are few known cases. In this section we shall summarize what is known in general.

1.5.1. Rank 1. Let P be a parabolic subgroup containing B. Then any rational B-module M defines a sheaf $\mathcal{L}(M)$ on P/B whose cohomology groups will be denoted by $H^i(P/B, M)$.

Fix $\alpha \in S$ and let $P = P_\alpha$ be the minimal parabolic subgroup corresponding to α. We write $H_\alpha(-)$ in short for $H^i(P_\alpha/B, -)$. The following results completely describe the well-known vanishing behaviour of $H^\bullet_\alpha(\lambda)$, and they were proved by Andersen in [2].

Proposition 1.4. *Let $\lambda \in X$ and let $\lambda_\alpha = \langle \lambda, \alpha^\vee \rangle$ for some $\alpha \in S$. Then $H^0_\alpha(\lambda)$ is non-zero if and only if $\lambda_\alpha \geq 0$. If so, $\dim_k(H^0_\alpha(\lambda)) = \lambda_\alpha + 1$, and the weights of $H^0_\alpha(\lambda)$ are $\lambda, \lambda - \alpha, \dots, s_\alpha \lambda$.*

Proposition 1.5. *Let $\lambda \in X$ and set $\lambda_\alpha = \langle \lambda, \alpha^\vee \rangle$ for $\alpha \in S$. Then $H^i_\alpha(\lambda) = 0$ for all $i > 1$. Moreover, $H^1_\alpha(\lambda)$ is non-zero if and only if $\lambda_\alpha < -1$. If so, we have $\dim_k(H^1_\alpha(\lambda)) = -\lambda_\alpha - 1$, and the weights of $H^1_\alpha(\lambda)$ are $\lambda + \alpha, \lambda + 2\alpha, \ldots, s_\alpha \cdot \lambda$.*

1.5.2. The general case. Set for all $i \geq 0$

$$D_p(i) = \{\lambda \in X \mid H^i(\lambda) \neq 0\}.$$

Independently of p, Grothendieck's vanishing theorem (1.3), Serre duality (1.2) and Kempf's vanishing theorem (1.4) say

$$\begin{aligned} &D_p(i) = \emptyset \text{ for all } i > N, \\ &D_p(i) = -D_p(N-i) - 2\rho \text{ for all } i \geq 0. \\ &D_p(i) \cap X^+ = \emptyset \text{ for all } i > 0. \end{aligned}$$

We further have for all p

$$\begin{aligned} &D_p(0) = X^+, \\ &D_p(N) = -X^+ - 2\rho = w_0 \cdot X^+. \end{aligned}$$

Here w_0 is the longest element in the Weyl group W.

1.5.3. Characteristic zero. Suppose first that $\mathrm{char}(k) = 0$. The Borel-Weil-Bott theorem can be stated as follows:

$$D_0(i) = \bigcup_{w \in W,\, l(w) = i} w \cdot X^+ \text{ for all } i \geq 0.$$

Example 1.6. *Let G be of type B_2, and let α, β be the simple roots with α short. Figure 1 illustrates the Borel-Weil-Bott theorem for groups of type B_2. The number n in the chamber indicates that the weight in this chamber has non-vanishing H^n.*

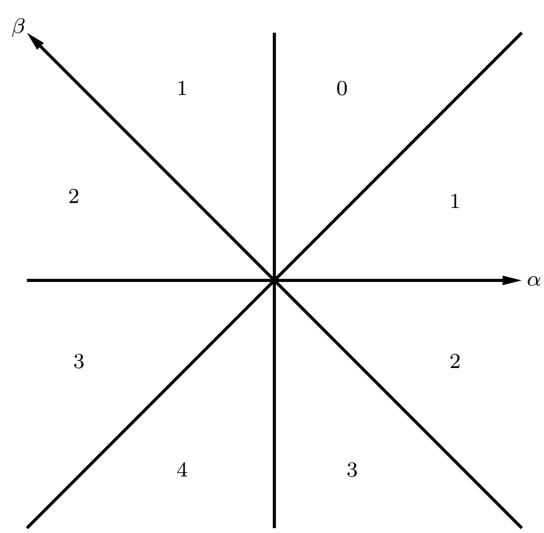

FIGURE 1. The Borel-Weil-Bott theorem for B_2.

1.5.4. Characteristic $p > 0$. Set for all $i \geq 0$

$$E_p(i) = \bigcup_{r \geq 0} \left(p^r \cdot D_0(i) \pm X_r \right).$$

From above, we get that $D_p(0) = E_p(0)$ and $D_p(N) = E_p(N)$.

As mentioned before, Andersen gave in [2] a complete description of the vanishing behaviour of the first cohomology group, and it can be stated as follows

$$D_p(1) = E_p(1). \tag{1.7}$$

Using Serre duality, we further get

$$\begin{aligned} D_p(N-1) &= -D_p(1) - 2\rho \\ &= -E_p(1) - 2\rho \\ &= \bigcup_{r \geq 0} (-p^r \cdot D_0(i) - 2\rho \pm X_r) \\ &= \bigcup_{r \geq 0} (p^r \cdot (-D_0(i) - 2\rho) \pm X_r) \\ &= \bigcup_{r \geq 0} (p^r \cdot D_0(N) \pm X_r) \\ &= E_p(N-1). \end{aligned}$$

The figure below illustrates the vanishing behaviour of H^1 and H^{N-1} for groups of type B_2. We labelled the alcove C with the number n iff $C \subset D_p(n)$. The figure only covers the set of weights $\lambda \in X$ with $|\langle \lambda + \rho, \alpha^\vee \rangle| \leq p^2$ for all $\alpha \in R$.

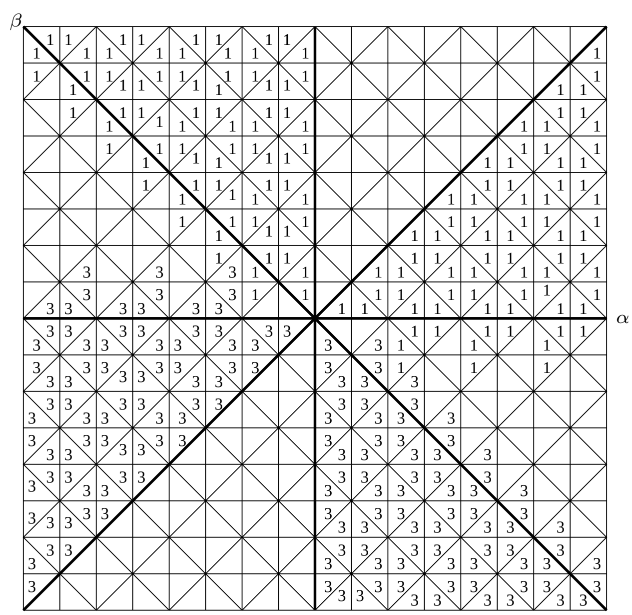

FIGURE 2. $D_p(1)$ and $D_p(N-1)$ for groups of B_2.

Summarizing, we have

$$D_p(i) = E_p(i) \text{ for all } i \in \{0, 1, N-1, N\}. \tag{1.8}$$

Note that the above equality completely describes $D_p(i)$ for groups of type A_2. In [7] Andersen described the vanishing behaviour of all such cohomology for groups of type B_2 and G_2. The description of $D_p(2)$ showed that the above equality does not hold in general.

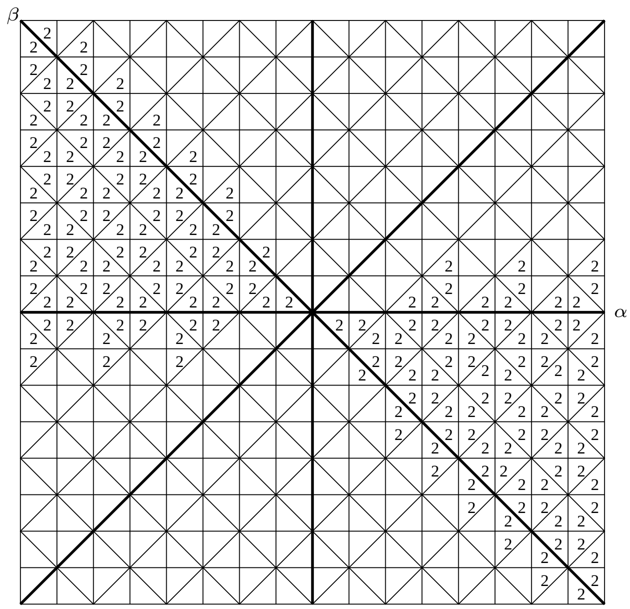

FIGURE 3. $D_p(2)$ for groups of type B_2

1.5.5. We continue to assume that $\mathrm{char}(k) = p > 0$. We shall now recall the main theorem in [1]. We shall later prove the quantum version of this result, and the proof is almost identical. We shall therefore omit the details here. The details can also be found in [1] or [6].

Let $r \geq 0$ and set $\sigma_r = (p^r - 1)\rho$. We shall call this the Steinberg weight. The corresponding simple G-module is called the Steinberg module and denoted St_r. As an easy consequence of the strong linkage principle, we have $\mathrm{St}_r \simeq H^0(\sigma_r)$ (cf. e.g. [22, II. 3.19]).

Let $\mathrm{F}_r : G \to G$ be the r-Frobenius homomorphism. When M is a G-module, we denote by $M^{(r)}$ the Frobenius twist of M, i.e. the same vector space but with action composed with F_r.

$$g \cdot m = \mathrm{F}_r(g)m \text{ for all } g \in G \text{ and } m \in M.$$

Theorem 1.7 ([1, Theorem 2])**.** *Let M be a B-module and let $i, r \geq 0$. Then there is a natural G-isomorphism $H^i(M)^{(r)} \otimes \mathrm{St}_r \simeq H^i(M^{(r)} \otimes (p^r - 1)\rho)$.*

Let M_1, M_2 be B-modules. Using Frobenius reciprocity [22], the evaluation maps $H^0(M_1) \to M_1$ and $H^0(M_2) \to M_2$ give a homomorphism

$$H^0(M_1) \otimes H^0(M_2) \to H^0(M_1 \otimes M_2)$$

which is functorial in both M_1 and M_2. By a simple induction on $i+j$, we obtain a natural homomorphism (the cup-product)

$$\cup_{i,j} : H^i(M_1) \otimes H^j(M_2) \to H^{i+j}(M_1 \otimes M_2).$$

The Frobenius homomorphism F_r clearly gives rise to natural maps

$$\mathrm{F}_r^* : H^i(M_1)^{(r)} \to H^i(M_1^{(r)}) \text{ for all } i \geq 0.$$

The above theorem implies an interesting non-vanishing result:

Proposition 1.8 ([6, Corollary 2.6])**.** *For any $i \geq 0$ we have $E_p(i) \subset D_p(i)$.*

Proof (Sketch). Suppose now that $H^i(\lambda)$ is non-zero for some $\lambda \in X$, and we claim that so is $H^i(p^r\lambda + \mu)$ for all $\mu \in X_r$. Since St_r is simple, the cup-product

$$H^0(\mu) \otimes H^0(\sigma_r - \mu) \to H^0(\sigma_r) \simeq \mathrm{St}_r\,.$$

is surjective. Hence we have the following commutative diagram of G-modules

$$\begin{array}{ccc} H^i(\lambda)^{(r)} \otimes H^0(\mu) \otimes H^0(\sigma_r - \mu) & \twoheadrightarrow & H^i(\lambda)^{(r)} \otimes \mathrm{St}_r \\ \downarrow & & \downarrow \\ H^i(p^r\lambda + \mu) \otimes H^0(\sigma_r - \mu) & \longrightarrow & H^i(p^r \cdot \lambda) \end{array}$$

The theorem above implies that the right vertical map is an isomorphism. Since the top horizontal homomorphism is surjective, then so is the bottom horizontal homomorphism. The claim follows.

By semi continuity [24, III. 12], we get $D_0(i) \subset D_p(i)$ for all i. Serre duality then completes the proof. ■

CHAPTER 2

B-cohomology

In this chapter we shall study the B-cohomology $H^\bullet(B,-) = \mathrm{Ext}_B^\bullet(k,-)$, i.e. the derived functors of the B-fixed point functor $H^0(B,-)$. We are especially interested in the B-cohomology of simple B-modules. In characteristic zero, this is an easy well-known description of this cohomology whereas the corresponding problem in characteristic $p > 0$ is wide open. We shall introduce some new techniques which enable us to compute all such cohomology in degrees at most 3 when p is larger than the Coxeter number h.

In Section (2.5) we determine for each $\lambda \in X$ an upper bound i for the degree in which the cohomology $H^i(B,\lambda)$ can be non-zero. It turns out that this upper bound is the best possible when p is larger than h.

This chapter represents joint work with Henning Haahr Andersen, and it was originally published in [14].

2.1. Known results

In this section we have gathered well-known results about the B-cohomology. Let $\mathrm{ht} : X \to \mathbb{Z}$ be the height function on X which takes 1 on all simple roots.

2.1.1. Characteristic zero. For any B-module M we have the spectral sequence [22, I.4.5]

$$H^r(G, H^s(G/B, M)) \implies H^{r+s}(B, M). \tag{2.1}$$

Suppose now that $\mathrm{char}(k) = 0$. The complete reducibility of G (cf. e.g. [22, II. 5.6]) implies that $H^r(G,-) = 0$ for all $r > 0$. The above spectral sequence (2.1) then degenerates and gives us isomorphisms of B-modules

$$H^r(B, M) \simeq H^0(G, H^r(G/B, M)) \text{ for all } r \geq 0. \tag{2.2}$$

Consider the case where M is the 1-dimensional B-module determined by $\lambda \in X$. Since the only dominant weight μ for which there is a non-trivial G-fixed point in $H^0(G/B, \mu)$ is $\mu = 0$, the Borel-Weil-Bott theorem (1.5) implies that

$$H^r(B, \lambda) \simeq \begin{cases} k & \text{if } \lambda = w \cdot 0 \text{ for some } w \in W \text{ with } l(w) = r, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

For details see [4, Proposition 2.2].

2.1.2. Characteristic $p > 0$. From now on, we assume that $\mathrm{char}(k) = p > 0$.

2.1.3. The linkage principle. The strong linkage principle implies that all composition factors of $H^r(G/B,\lambda)$ have highest weights in $W\cdot\lambda+p\mathbb{Z}R$. Furthermore, it also gives that for each simple G-module $L(\mu)$ we have $H^\bullet(G,L(\mu))=0$ unless $\mu\in W\cdot 0+p\mathbb{Z}R$. Hence the spectral sequence (2.1) shows that

$$H^\bullet(B,\lambda)=0 \text{ unless } \lambda\in W\cdot 0+p\mathbb{Z}R. \tag{2.4}$$

Remark 2.1. As observed in [4], the strong linkage principle also implies that we have the following characteristic p-analogue of (2.3)

$$H^r(B,w\cdot 0)\simeq\begin{cases} k & \text{if } r=l(w),\\ 0 & \text{otherwise.}\end{cases} \tag{2.5}$$

2.1.4. Let $k[U]$ denote the coordinate ring of U. We identify $k[U]$ with the induced module $\mathrm{Ind}_T^B\, k$ of the trivial T-module k. Tensoring the "standard" injective resolution

$$k\to k[U]\to k[U]\otimes k[U]\to\cdots$$

of the trivial B-module k by a weight $\lambda\in X$, we get

$$H^\bullet(B,\lambda)=0 \text{ unless } \lambda\le 0. \tag{2.6}$$

In fact, each term in the resulting resolution of the B-module λ has weights $\ge\lambda$. Hence there are no T-fixed points (and so certainly no B-fixed points either) unless $\lambda\le 0$.

Remark 2.2. A little more careful argument (see e.g. [18, Lemma 2.3]) shows that in fact we have

$$H^i(B,\lambda)=0 \text{ unless } \lambda\le 0 \text{ and } i\le -\mathrm{ht}(\lambda). \tag{2.7}$$

2.1.5. The first cohomology group. It is clear that $H^0(B,k)=k$ and that $H^0(B,\lambda)=0$ for all $\lambda\ne 0$. The first cohomology group $H^1(B,\lambda)$ is also completely known (see [4]).

$$H^1(B,\lambda)\simeq\begin{cases} k & \text{if } \lambda=-p^r\alpha \text{ for some } \alpha\in S, r\ge 0,\\ 0 & \text{otherwise.}\end{cases} \tag{2.8}$$

This may be deduced from the spectral sequence (2.1) by using the fact that the G-socle of $H^1(\lambda)$ is known, see [2]. In particular, $H^0(G,H^1(\lambda))=0$ unless $\lambda=-p^r\alpha$ for some $\alpha\in S$ and $r\ge 0$.

2.1.6. The second cohomology group. One of the main results in [15] is a complete description of $H^2(B,\lambda)$. When $p>h$, we shall recover this result in Section 2.4. One of the features is that for any λ its second B-cohomology group is at most 1-dimensional (as was the case for H^1, see (2.8)).

We emphasize that [15] describes $H^2(B,\lambda)$ for all p whereas we focus in this chapter only on the case $p>h$.

2.1.7. SL_2 and SL_3. The only Borel subgroup B for which the full story about $H^\bullet(B,\lambda)$ is known is the Borel subgroup of SL_2. Since (in general) U is normal in B and T is reductive, we have $H^i(B,\lambda) = H^i(U,k)_{-\lambda}$. Now, when U is 1-dimensional, the cohomology $H^\bullet(U,k)$ is completely described in [32].

In the SL_3 case the cohomology $H^\bullet(B_q,\lambda)$ was calculated in [6]. Here B_q denotes the Borel subalgebra of the quantum group corresponding to SL_3, and q is assumed to be a complex root of unity of odd order at least 3, see Chapter 3. Many of the calculations for this case can easily be carried over to the characteristic p situation giving a start for the determination of B-cohomology for a Borel subgroup B of $SL_3(k)$.

2.2. Methods

Even though the spectral sequence (2.1) is not so effective in characteristic p, it has the following very useful variant.

Note that we may replace G by any parabolic subgroup corresponding to $\alpha \in S$. Fix $\alpha \in S$ and let $P = P_\alpha$ be the minimal parabolic subgroup corresponding to $\alpha \in S$. Writing $H^i_\alpha(-)$ in short for $H^i(P_\alpha/B,-)$, we get in this way for all $i \geq 0$

$$H^i(B,\lambda) \simeq H^i(P_\alpha, H^0_\alpha(\lambda)) \text{ if } \langle\lambda,\alpha^\vee\rangle \geq 0, \tag{2.9}$$

$$H^{i+1}(B,\lambda) \simeq H^i(P_\alpha, H^1_\alpha(\lambda)) \text{ if } \langle\lambda,\alpha^\vee\rangle \leq -2, \tag{2.10}$$

$$H^i(B,\lambda) = 0 \text{ if } \langle\lambda,\alpha^\vee\rangle = -1. \tag{2.11}$$

Note also that $H^i(P_\alpha, M) \simeq H^i(B,M)$ for all $i \geq 0$ when M is a P_α-module. This follows from the same spectral sequence argument by observing that for such M we have $H^0_\alpha(M) \simeq M$ and $H^1_\alpha(M) = 0$.

Recall that when $-1 \leq \langle\lambda,\alpha^\vee\rangle < p$, then $H^0_\alpha(\lambda) \simeq H^1_\alpha(s_\alpha\cdot\lambda)$. Using this together with (2.9)-(2.11), we get for all $i \geq 0$

$$H^i(B,\lambda) \simeq H^{i+1}(B, s_\alpha\cdot\lambda) \quad \text{whenever } -1 \leq \langle\lambda,\alpha^\vee\rangle < p. \tag{2.12}$$

2.2.1. Let B_1 denote the first Frobenius kernel in B. This means that B_1 is the subgroup scheme obtained as the kernel of the Frobenius homomorphism F_1 on B. When M is a B-module, we denote by $M^{(1)}$ the Frobenius twist of M, i.e. the same vector space M but with action composed with F_1. Similarly, if N is a B-module whose restriction to B_1 is trivial, then $N^{(-1)}$ is the B-module such that $(N^{(-1)})^{(1)} = N$.

We have for each B-module M the Lyndon-Hochschild-Serre spectral sequence [22, II. 12.2]

$$H^r(B, H^s(B_1,M)^{(-1)}) \implies H^{r+s}(B,M). \tag{2.13}$$

2.2.2. Consider now the case where $M = \lambda$ for some $\lambda \in X$. If $p > h$, then the cohomology $H^\bullet(B_1,\lambda)$ is completely known for all $\lambda \in X$. According to (2.4), we only need to consider λ's of the form $\lambda = w\cdot 0 + p\mu$ for some $w \in W$ and $\mu \in pR\mathbb{Z}$. Then we have (see [9])

$$H^r(B_1, w\cdot 0 + p\mu)^{(-1)} \simeq S^{(r-l(w))/2}(u^*) \otimes \mu. \tag{2.14}$$

Here u^* denotes the dual of the Lie algebra $u = \mathrm{Lie}(U)$ with the adjoint B-action, S^r denotes the r-th symmetric power, and we interpret S^r to be 0 whenever $r \notin \mathbb{N}$.

2.2.3. When we combine (2.14) and the spectral sequence (2.13), we obtain

Proposition 2.3 ([6, Theorem 4.3.ii])**.** *Suppose $p > h$. Let $w \in W$, $\mu \in X$. Then we have for all i*

$$H^i(B, w \cdot 0 + p\mu) \simeq H^{i-l(w)}(B, p\mu).$$

This result reduces the problem of computing $H^\bullet(B, \lambda)$ to the case where $\lambda \in pX$. Note also that this proposition reproves Remark 2.1 when $p > h$.

2.2.4. In order to effectively take advantage of the spectral sequence (2.13), we need by (2.14) to determine the B-cohomology of $S^n u^* \otimes \lambda$ for $\lambda \in X$. This we don't know how to do in general, but the following short exact sequence will allow us to settle some cases.

Let $\alpha \in S$. Note that the line of weight α in u^* is a B-submodule and that the quotient $V_\alpha = u^*/\alpha$ is a P_α-module. This leads to an exact sequence of B-modules for each $n > 0$

$$0 \to S^{n-1}u^* \otimes \alpha \to S^n u^* \to S^n V_\alpha \to 0. \tag{2.15}$$

Tensoring by a weight $\lambda \in X$, we get

$$0 \to S^{n-1}u^* \otimes (\alpha + \lambda) \to S^n u^* \otimes \lambda \to S^n V_\alpha \otimes \lambda \to 0. \tag{2.16}$$

This gives $H^i(B, S^n u^* \otimes \lambda) = 0$ unless $H^i(B, S^{n-1}u^* \otimes (\lambda + \alpha))$ or $H^i(B, S^n V_\alpha \otimes \lambda)$ is non-zero.

As an easy consequence of (2.9)-(2.11), we get that if λ satisfies $\langle \lambda, \alpha^\vee \rangle \geq -1$, then we have for all i, n

$$H^i(B, S^n V_\alpha \otimes \lambda) \simeq H^i(B, S^n V_\alpha \otimes H^0_\alpha(\lambda)),$$
$$H^{i+1}(B, S^n V_\alpha \otimes s_\alpha \cdot \lambda) \simeq H^i(B, S^n V_\alpha \otimes H^1_\alpha(s_\alpha \cdot \lambda)).$$

If $-1 \leq \langle \lambda, \alpha^\vee \rangle < p$, then $H^0_\alpha(\lambda) \simeq H^1_\alpha(s_\alpha \cdot \lambda)$. Hence we obtain for such λ

$$H^i(B, S^n V_\alpha \otimes \lambda) \simeq H^{i+1}(B, S^n V_\alpha \otimes s_\alpha \cdot \lambda). \tag{2.17}$$

Lemma 2.4 ([14, Lemma 3.2])**.** *Suppose $p > h$ and let $\lambda \in X$. Then we have*

$$H^0(B, V_\alpha \otimes -\lambda) = 0 \textit{ unless } \lambda \in \{R^+ \setminus \{\alpha\} \mid \lambda - \alpha \notin R^+\}.$$

Proof. Let L_α denote the Levi subgroup of P_α. Since V_α is a L_α-module and $p > h$ we get from the linkage principle that $V_\alpha \simeq \oplus L_\alpha(\gamma)$ as L_α-modules. Here γ runs through those roots in $R^+ \setminus \{\alpha\}$ for which $\gamma + \alpha \notin R^+$ and $L_\alpha(\gamma)$ denotes the simple L_α-module of highest weight γ. Note that if $B_\alpha = B \cap L_\alpha$, then

$$H^0(B_\alpha, L_\alpha(\gamma) \otimes -\lambda) = 0 \text{ unless } \lambda = s_\alpha(\gamma).$$

Then the lemma follows. ∎

2.3. B-cohomology of $S^n u^* \otimes \lambda$

In the rest of this chapter we assume that $\mathrm{char}(k) = p > 0$. As mentioned before, in order to calculate $H^2(B, \lambda)$ and $H^3(B, \lambda)$ explicitly, we need to compute some low degree cohomology of $S^n u^* \otimes \lambda$. This is what we do in this section.

2.3.1. Degree zero.

Proposition 2.5 ([14, Proposition 4.1])**.** *Fix $n \in \mathbb{N}$ and $\lambda \in X$. Then*

$$H^0(B, S^n u^* \otimes \lambda) \simeq \begin{cases} k & \text{if } n = -\operatorname{ht}(\lambda) \text{ and } \lambda \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since the weights of $S^n u^*$ are all ≥ 0, we can apply (2.6) to conclude that $H^0(B, S^n u^* \otimes \lambda) = 0$ unless $\lambda \leq 0$. So we may assume λ is not dominant. Choose then $\alpha \in S$ such that $\langle \lambda, \alpha^\vee \rangle < 0$. The exact sequence (2.16) gives

$$H^0(B, S^n u^* \otimes \lambda) \simeq H^0(S^{n-1} u^* \otimes (\alpha + \lambda)).$$

Now an easy induction on n proves the proposition. ∎

Remark 2.6. Proposition 2.5 remains true when $\mathrm{char}(k) = 0$.

2.3.2. Degree 1. For each $\alpha, \beta \in S$ we let

$$a_{\beta\alpha} = \left\langle \beta, \alpha^\vee \right\rangle.$$

Note that for each $\alpha, \beta \in S$ we have

$$\alpha + \beta \in R^+ \text{ if and only if } a_{\beta\alpha} < 0.$$

Proposition 2.7 ([14, Proposition 4.3])**.** *Assume $p > h$ and let $\lambda \in X$. Then*

$$H^1(B, u^* \otimes \lambda) \simeq \begin{cases} k & \text{if } \lambda = -\beta - p^n\alpha \text{ for } \alpha, \beta \in S \text{ and } n > 0, \\ k & \text{if } \lambda = -\beta - \alpha \text{ for } \alpha, \beta \in S \text{ with } a_{\beta\alpha} < 0, \\ k & \text{if } \lambda = -2\alpha \text{ for } \alpha \in S, \\ k^2 & \text{if } \lambda = -\beta - \alpha \text{ for } \alpha, \beta \in S \text{ with } a_{\beta\alpha} = 0, \\ k & \text{if } \lambda = s_\alpha s_\beta \cdot 0 \text{ for } \alpha, \beta \in S \text{ with } a_{\beta\alpha} < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We begin by checking each of the first five cases where the proposition claims that the cohomology is non-zero. So consider first the case where $\lambda = -\beta - p^n\alpha$ for some $\alpha, \beta \in S$ and $n > 0$. We have the following exact sequence

$$0 \to (\beta + \lambda) \to u^* \otimes \lambda \to V_\beta \otimes \lambda \to 0. \tag{2.18}$$

We note that $-\lambda$ is not a weight of V_β and that no weights of $V_\beta \otimes \lambda$ have the form $-p^m\gamma$ with $\gamma \in S$ and $m \geq 0$. Using (2.8), we then have

$$H^0(B, V_\beta \otimes \lambda) = H^1(B, V_\beta \otimes \lambda) = 0.$$

This together with the long exact sequence arising from (2.18) give

$$H^1(B, u^* \otimes \lambda) \simeq H^1(B, -p^n\alpha) \simeq k.$$

Consider now $\lambda = -\beta - \alpha$ for some $\alpha, \beta \in S$ with $\alpha + \beta \in R^+$. In this case we still have that $H^0(B, V_\beta \otimes \lambda) = 0$, see Lemma 2.4. We claim that $H^1(B, V_\beta \otimes -\alpha - \beta) = 0$. To see this, we consider the exact sequence

$$0 \to \alpha \to V_\beta \to Q \to 0. \tag{2.19}$$

Noting that $\alpha + \beta$ is a minimal weight of Q (with multiplicity 1), it follows immediately that $H^0(B, Q \otimes (-\beta - \alpha)) \simeq k$. No weights of $Q \otimes (-\beta - \alpha)$ have the form $-p^m \mu$ with $\mu \in S$ and $m \geq 0$. Therefore we get $H^1(B, Q \otimes (-\beta - \alpha)) = 0$, and hence the long exact sequence coming from (2.19) gives $H^1(B, V_\beta \otimes -\beta - \alpha) = 0$. Combining this claim with the exact sequence (2.18), we get

$$H^1(B, u^* \otimes (-\beta - \alpha)) \simeq H^1(B, -\alpha) \simeq k.$$

Next, consider that $\lambda = -\beta - \alpha$ for some $\alpha, \beta \in S$ with $\alpha + \beta \notin R^+$. Arguing as before, we get that $H^0(B, V_\beta \otimes -\alpha - \beta) = 0$, but this time we also have that $H^0(B, Q \otimes (-\beta - \alpha)) = 0$. Note that if $\beta = \alpha$, then 2α is not a weight of V_α. In this case we get $H^1(B, V_\alpha \otimes -2\alpha) = 0$. Weight considerations as before imply that if $\alpha \neq \beta$, then $H^1(B, V_\beta \otimes -\alpha - \beta) \simeq k$. Inserting in the long exact sequence arising from (2.18), we get the desired conclusions because $H^2(B, -\alpha) \simeq H^1(B, k_0) = 0$.

Finally, consider $\lambda = s_\alpha s_\beta \cdot 0$ for some $\alpha, \beta \in S$ with $a_{\beta\alpha} < 0$. Then $\langle \lambda, \alpha^\vee \rangle = a_{\beta,\alpha} - 2 < 0$. By (2.17), the exact sequence (2.16) gives

$$H^1(B, u^* \otimes \lambda) \simeq H^1(B, V_\alpha \otimes \lambda) \simeq H^0(B, V_\alpha \otimes s_\alpha \cdot \lambda)$$

because $H^1(B, \lambda + \alpha) = H^2(B, \lambda + \alpha) = 0$. Since $s_\alpha \cdot \lambda = -\beta$, we have

$$H^0(B, V_\alpha \otimes s_\alpha \cdot \lambda) \simeq k$$

This settles the last of the non-vanishing cases.

Assume now that $H^1(B, u^* \otimes \lambda) \neq 0$ for some $\lambda \in X$. To finish the proof, we need to show that we are then in one of the above five cases.

Weight considerations show via (2.8) that if $H^1(B, u^* \otimes \lambda)$ is non-zero for some $\lambda \in X$, then $\lambda = -\beta - p^n \alpha$ for some $\beta \in R^+$, $\alpha \in S$, $n \geq 0$. We claim that if $n > 0$, then $\beta \in S$ (i.e. we are in the first case listed in the proposition). If $\beta \notin S$, then (2.8) gives $H^1(B, \lambda + \alpha) = 0$, and hence the exact sequence (2.16) implies

$$H^1(B, u^* \otimes \lambda) \subseteq H^1(B, V_\alpha \otimes \lambda) \simeq H^0(B, V_\alpha \otimes H^1_\alpha(\lambda)).$$

The claimed isomorphism comes from the fact that $\langle \lambda, \alpha^\vee \rangle = -a_{\beta\alpha} - 2p^n < 0$.

Arguing as in the proof of Lemma 2.4 (using the notation from that proof), we get that

$$V_\alpha^* \simeq \bigoplus L_\alpha(-\gamma) \text{ as } L_\alpha\text{-modules.}$$

Here the sum extends over those $\gamma \in R^+$ for which $\gamma - \alpha \notin R^+$. Hence it follows that $H^0(B, V_\alpha \otimes H^1_\alpha(\lambda)) \simeq \mathrm{Hom}_B(V_\alpha^*, H^1_\alpha(\lambda)) = 0$ unless the L_α-socle of $H^1_\alpha(\lambda)$ contains such an $L_\alpha(-\gamma)$. However, we get from [2] that this socle is $L_\alpha(-\beta)$ if $\langle \lambda, \alpha^\vee \rangle \geq -2p^n$ (i.e. if $a_{\beta\alpha} \leq 0$), $L_\alpha(s_\alpha \cdot \lambda)$ if $\langle \lambda, \alpha^\vee \rangle = -2p^n - 1$ (i.e. if $a_{\beta\alpha} = 1$), and $L_\alpha(-\beta + p^n \alpha)$ if $\langle \lambda, \alpha^\vee \rangle < -2p^n - 1$ (i.e. if $a_{\beta\alpha} \geq 1$). In the two last cases the highest weight of the socle is not in $-R^+$. In the first case the above conditions on γ implies $\beta-\alpha\notin R^+$. To investigate this case further, we tensor the sequence (2.15) by $H^1_\alpha(\lambda)$ and obtain the exact sequence

$$H^0(B,u^*\otimes H^1_\alpha(\lambda))\to H^0(B,V_\alpha\otimes H^1_\alpha(\lambda))\to H^1(B,\alpha\otimes H^1_\alpha(\lambda)).$$

By Proposition 1.5, the weights of $H^1_\alpha(\lambda)$ are $\lambda+\alpha,\lambda+2\alpha,\dots,s_\alpha\cdot\lambda$. Then Proposition 2.5 implies that the first term in the above sequence is 0 unless we have $-\beta-p^n\alpha+a\alpha=-\gamma$ for some $a>0$ and $\gamma\in S$. This means $\beta=\gamma+b\alpha$ for some $b\geq 0$. But if $b>0$, then we see that $\beta-\alpha\in R^+$, and since this is not the case, we conclude that $\beta\in S$. Similarly, by (2.8) the third term is 0 unless $\alpha+(-\beta-p^n\alpha)+a\alpha=-p^r\gamma$ for some $a>0,r\geq 0$ and $\gamma\in S$. This again means $\beta=\gamma+b\alpha$ for some $b\geq 0$. As before, we deduce from the fact $\beta-\alpha\notin R^+$ that $b=0$. Our claim follows.

On the other hand, if $n=0$, then we claim that we are in one of the remaining four cases. Since $-\beta-\alpha\notin X^+$, we may choose $\gamma\in S$ such that $\langle\lambda,\gamma^\vee\rangle<0$. As $\langle\lambda,\gamma^\vee\rangle>-p$ we get from (2.17)

$$H^1(B,V_\gamma\otimes\lambda)\simeq H^0(B,V_\gamma\otimes s_\gamma\cdot\lambda). \tag{2.20}$$

Using our assumption that $H^1(B,u^*\otimes\lambda)\neq 0$, the sequence (2.16) relative to γ gives that either $H^1(B,\lambda+\gamma)\neq 0$ or $H^1(B,V_\gamma\otimes\lambda)\neq 0$.

Suppose first that $H^1(B,\lambda+\gamma)\neq 0$. Then $\lambda=-\gamma-p^m\delta$ for some $\delta\in S$ and $m\geq 0$. Since $\lambda=-\beta-\alpha$, we have $m=0$ and $\beta\in\{\gamma,\delta\}\subseteq S$. This means that we are in one of the cases 2, 3 or 4 on the list.

Suppose $H^1(B,V_\gamma\otimes\lambda)\neq 0$. By (2.20), we get $H^0(B,V_\gamma\otimes s_\gamma\cdot\lambda)\neq 0$. Then the sequence

$$H^0(B,u^*\otimes s_\gamma\cdot\lambda)\to H^0(B,V_\gamma\otimes s_\gamma\cdot\lambda)\to H^1(B,\gamma+s_\gamma\cdot\lambda).$$

gives either $H^0(B,u^*\otimes s_\gamma\cdot\lambda)\neq 0$ or $H^1(B,\gamma+s_\gamma\cdot\lambda)\neq 0$. This means that either $s_\gamma\cdot\lambda=-\delta$ or $\gamma+s_\gamma\cdot\lambda=-p^m\delta$ for some $\delta\in S,\ m\geq 0$. The first possibility means that $\lambda=s_\gamma\cdot(-\delta)=s_\gamma s_\delta\cdot 0$, i.e. we are in case 4 or 5 on our list. The second possibility can only occur with $m=0$, and then $s_\gamma\cdot\lambda=-\gamma-\delta$. But in that case

$$H^0(B,V_\gamma\otimes s_\gamma\cdot\lambda)=H^0(B,V_\gamma\otimes-\delta-\gamma),$$

and this is 0 according to Lemma 2.4. ■

The same arguments as in Proposition 2.7 give

Proposition 2.8 ([14, Proposition 4.4]). *Let $\lambda\in X$. If $\operatorname{char}k=0$ then*

$$H^1(B,u^*\otimes\lambda)\simeq\begin{cases} k & \text{if } \lambda=-2\alpha \text{ for } \alpha\in S,\\ k & \text{if } \lambda=-\beta-\alpha \text{ for } \alpha,\beta\in S \text{ with } a_{\beta\alpha}<0,\\ k^2 & \text{if } \lambda=-\beta-\alpha \text{ for } \alpha,\beta\in S \text{ with } a_{\beta\alpha}=0,\\ k & \text{if } \lambda=s_\alpha s_\beta\cdot 0 \text{ for } \alpha,\beta\in S \text{ with } a_{\beta\alpha}<0,\\ 0 & \text{otherwise.}\end{cases}$$

2.4. $H^\bullet(B,\lambda)$ in degrees 2 and 3

In this section we assume $p>h$. We shall compute $H^2(B,\lambda)$ and $H^3(B,\lambda)$ for all $\lambda\in X$.

2.4.1. Degree 2.

Theorem 2.9 ([14, Theorem 5.1])**.** *Let $\lambda \in X$. Then*

$$H^2(B,\lambda) \simeq \begin{cases} k & \text{if } \lambda = p^n(-\alpha) \text{ for } \alpha \in S \text{ and } n > 0, \\ k & \text{if } \lambda = p^n(w \cdot 0) \text{ for } w \in W \text{ with } l(w) = 2, n \geq 0, \\ k & \text{if } \lambda = p^n(-\alpha - p^m\beta) \text{ for } \alpha, \beta \in S, n \geq 0, m > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $\lambda \notin pX$, then we use Proposition 2.3 to reduce to a lower degree cohomology group. These are described in Section 2.1. So suppose $\lambda = p\mu$ for some $\mu \in X$. We then use the spectral sequence (2.13) to compute $H^2(B,\lambda)$. By (2.14), there are only two E_2-terms that may contribute, namely $H^2(B,\mu)$ and $H^0(B,u^* \otimes \mu)$. If $\mu \in -S$, then the first of these terms vanishes (by Proposition 2.3) whereas the second equals k. Hence $H^2(B,-p\alpha) = k$ for all $\alpha \in S$.

On the other hand, if $\mu \notin -S$, then we have that the second term vanishes (according to Proposition 2.5) and $H^2(B,\lambda) \simeq H^2(B,\mu)$. We repeat this argument if $\mu \in pX$ (note that this gives $H^2(B,p\mu) \simeq H^2(B,p^2\mu) \simeq \cdots \simeq H^2(B,p^n\mu)$ for all $\mu \in X$ and all $n > 0$). Otherwise, $H^2(B,\mu)$ identifies with a lower degree cohomology group as before. It is now a matter of bookkeeping to see that this leads to the statement in the theorem. ■

2.4.2. Degree 3.

Theorem 2.10 ([14, Theorem 5.2])**.** *Let $\lambda \in X$. Then*

$$H^3(B,\lambda) \simeq \begin{cases} k & \text{if } \lambda = p^n(-2\alpha) \text{ for } \alpha \in S \text{ and } n > 0, \\ k^2 & \text{if } \lambda = p^n(-\beta - p^m\alpha) \text{ for } \alpha, \beta \in S \text{ and } n, m > 0, \\ k & \text{if } \lambda = p^n(-\beta - \alpha) \text{ for } \alpha, \beta \in S \text{ with} \\ & a_{\beta\alpha} < 0 \text{ and } n > 0, \\ k^2 & \text{if } \lambda = p^n(-\beta - \alpha) \text{ for } \alpha, \beta \in S \text{ with} \\ & a_{\beta\alpha} = 0 \text{ and } n > 0, \\ k & \text{if } \lambda = p^n(s_\alpha s_\beta \cdot 0) \text{ for } \alpha, \beta \in S \text{ with} \\ & a_{\beta\alpha} < 0 \text{ and } n > 0, \\ k & \text{if } \lambda = p^n(w \cdot 0) \text{ for } w \in W \text{ with} \\ & l(w) = 3 \text{ and } n \geq 0, \\ k & \text{if } \lambda = p^n(w \cdot 0 - p^m\alpha) \text{ for } \alpha \in S,\ w \in W \text{ with} \\ & l(w) = 2 \text{ and } n \geq 0, m > 0, \\ k & \text{if } \lambda = p^n(p^m w \cdot 0 - \alpha) \text{ for } \alpha \in S,\ w \in W \text{ with} \\ & l(w) = 2 \text{ and } n \geq 0, m > 0, \\ k & \text{if } \lambda = -\beta - p^n\alpha \text{ for } \alpha, \beta \in S, n > 0, \\ k & \text{if } \lambda = p^n(-\alpha - p^m\beta - p^l\gamma) \text{ for } \alpha, \beta, \gamma \in S \text{ and} \\ & n \geq 0, m > l > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose that $\lambda = p\mu$ for some $\mu \in X$. Consider the spectral sequence (2.13). The only E_2-terms that contribute to $H^3(B,\lambda)$ are $H^3(B,\mu)$ and $H^1(B,u^* \otimes \mu)$.

The latter vanishes if $\mu \in pX$. Hence we get $H^3(B, p\mu) \simeq \cdots \simeq H^3(B, p^n\mu)$ for all $\mu \in X$ and all $n > 0$.

For those μ listed in Proposition 2.7, we have that unless $\mu = -\beta - p^n\alpha$ for some $\alpha, \beta \in S, n > 0$, and in that case we have $H^3(B, \mu) = 0$. Hence

$$H^3(B, p\mu) \simeq H^1(B, u^* \otimes \mu).$$

Suppose now that $\mu = -\beta - p^n\alpha$ with $\alpha, \beta \in S$ and $n > 0$. Proposition 2.7 and Theorem 2.9 (combined with Proposition 2.3) yield that both of the above terms equal k. In this situation we have an exact sequence

$$0 \to H^3(B, \mu) \to H^3(B, p\mu) \to H^1(B, u^* \otimes \mu) \to 0$$

i.e. we have $H^3(B, p\mu) \simeq k^2$.

On the other hand, if μ is not one of those weights listed in Proposition 2.7, then the second term vanishes. In this case we have $H^3(B, p\mu) \simeq H^3(B, \mu)$. Arguing as in Theorem 2.9, the stated results follow. ■

2.5. Upper bound

In this section we determine for each $\lambda \in X$ an upper bound i for the degree in which the cohomology $H^i(B, \lambda)$ can be non-zero.

2.5.1. Upper bound. Suppose now that char $(k) = p > 0$.

Theorem 2.11 (Compare [14, 6])**.** *Let $\lambda \in X$ and $w \in W$. Then*

$$H^i(B, w \cdot 0 + p\lambda) = 0 \textit{ for all } i > l(w) - 2\,\mathrm{ht}(\lambda).$$

Proof. Since $H^i(B, w \cdot 0 + p\lambda) \simeq H^{i-l(w)}(B, p\lambda)$, we see that $H^\bullet(B, w \cdot 0 + p\lambda)$ is zero unless $\lambda \leq 0$. In particular, we may assume that $\mathrm{ht}(\lambda) \leq 0$.

We proceed by induction on $n = l(w) - 2\,\mathrm{ht}(\lambda)$. If $n = 0$, then $w = 1$ and $\lambda = 0$. In this case the claim is true.

Now, assume that $i > n > 0$ and set $\mu = w \cdot 0 + p\lambda$. Since $\mu \notin X^+$, we can choose $\alpha \in S$ with $\langle \mu, \alpha^\vee \rangle < 0$, and then we set

$$a = \max\{j \mid jp \leq \langle s_\alpha \cdot \mu, \alpha^\vee \rangle\} = \begin{cases} -\langle \lambda, \alpha^\vee \rangle & \text{if } l(s_\alpha w) = l(w) - 1, \\ -\langle \lambda, \alpha^\vee \rangle - 1 & \text{if } l(s_\alpha w) = l(w) + 1. \end{cases}$$

We have

$$H^i(B, \mu) \simeq H^{i-1}\left(B, H^1_\alpha(\mu)\right).$$

The weights of $H^1_\alpha(\mu)$ are $\mu + \alpha, \mu + 2\alpha, \ldots, s_\alpha \cdot \mu$. Note that the weights which belong to $W \cdot 0 + p\mathbb{Z}R$ have the form $\upsilon = \mu + jp\alpha$ with $j \in \{1, \ldots, a\}$ or $\upsilon = s_\alpha \cdot \mu - jp\alpha$ with $j \in \{0, \ldots, a\}$.

Consider first $\upsilon = \mu + jp\alpha$ for some $j \in \{1, \ldots, a\}$. Then $\upsilon = w \cdot 0 + p(\lambda + j\alpha)$. Since $i - 1 > n - 1 \geq l(w) - 2\,\mathrm{ht}(\lambda + j\alpha)$, we get by induction that $H^{i-1}(B, \upsilon) = 0$.

Consider now that $\upsilon = s_\alpha \cdot \mu - jp\alpha$ for some $j \in \{0, \ldots, a\}$. Then

$$\upsilon = s_\alpha w \cdot 0 + p(s_\alpha(\lambda) - j\alpha) = s_\alpha w \cdot 0 + p(\lambda - (\langle \lambda, \alpha^\vee \rangle + j)\alpha).$$

Note

$$l(s_\alpha w) - 2\,\mathrm{ht}(\lambda) + 2(\langle \lambda, \alpha^\vee\rangle + j) \le l(s_\alpha w) - 2\,\mathrm{ht}(\lambda) + 2(\langle \lambda, \alpha^\vee\rangle + a)$$
$$= \begin{cases} n-1 & \text{if } l(s_\alpha w) = l(w) - 1, \\ n-1 & \text{if } l(s_\alpha w) = l(w) + 1. \end{cases}$$

Hence by induction $H^{i-1}(B, v) = 0$.

We conclude that $H^{i-1}(B, H^1_\alpha(\mu)) = 0$. This completes the proof. ∎

2.5.2. In [14] we expected this upper bound to be the best possible. As evidence we pointed to the rank 1 computations in [32] and to the quantum case, see [14, Remark 7.1]. This is in fact true when $p > h$. To see this, we need to note that any weight of $S^j u^*$ has height at least j. From Remark 2.2 we can then derive that

$$H^i(B, S^j u^* \otimes \lambda) = 0 \text{ unless } \lambda \le 0 \text{ and } \mathrm{ht}(\lambda) \le -i - j. \tag{2.21}$$

Proposition 2.12. *Suppose $p > h$. Let $\lambda \in X$ with $\lambda \le 0$. Then we have*

$$H^{-2\,\mathrm{ht}(\lambda)}(B, p\lambda) \simeq k.$$

Proof. Let $i, j \in \mathbb{N}$ such that $-2\,\mathrm{ht}(\lambda) = i + 2j$. From (2.21) it follows that

$$H^i(B, S^j u^* \otimes \lambda) = 0 \text{ unless } i = 0 \text{ and } j = -\,\mathrm{ht}(\lambda).$$

Combining this with the spectral sequence (2.13) and Proposition 2.5, we get

$$H^{-2\,\mathrm{ht}(\lambda)}(B, \lambda) \simeq H^0(B, S^{-\,\mathrm{ht}(\lambda)} u^* \otimes \lambda) \simeq k.$$

The proposition is proved. ∎

CHAPTER 3

Quantum groups and their representations

Let U_q be the quantum group corresponding to G, and let B_q be the Borel subalgebra corresponding to the negative roots (see below for the precise definitions). The induction functor H_q^0 from B_q behaves so nicely that some key results in the cohomology theory of line bundles on the flag variety G/B can be carried over to the quantum case. These include analogues of Serre duality, Grothendieck's vanishing theorem, Kempf's vanishing theorem and the strong linkage principle. Moreover, when q is not a root of unity, the Borel-Weil-Bott theorem (1.5) holds for all characters $\lambda \in X$.

In this chapter we shall introduce the quantum group U_q and its Borel subalgebra B_q. We shall also introduce the induction functor H_q^0 from B_q and review some of its properties. The main references are [11] and [13].

3.1. Quantum groups

Let (a_{ij}) denote the Cartan matrix of our root system R of rank n and set $I = \{1, \ldots, n\}$. Since (a_{ij}) is symmetrizable, then we may choose $d_1, \ldots, d_n \in \mathbb{N}$ minimal such that $(d_i a_{ij})$ is symmetric. Hence we have that $d_i \in \{1, 2\}$ for all $i \in I$ unless R has an irreducible component of type G_2. In this case we have $d_i = 3$ for some $i \in I$.

Let $S = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots, and for each $\lambda \in X$ we set

$$\lambda_i = \left\langle \lambda, \alpha_i^\vee \right\rangle \text{ for all } i \in I.$$

3.1.1. Gaussian binomial coefficients. Let v be an indeterminate over $\mathbb{Q}$. Consider the fraction field $\mathbb{Q}(v)$ of the polynomial ring $\mathbb{Q}[v]$. Set for all $i \in I$

$$[a]_i = \frac{v^{d_i a} - v^{-d_i a}}{v^{d_i} - v^{-d_i}} \text{ for } a \in \mathbb{Z},$$
$$[a]_i! = [a]_i \cdots [1]_i \text{ for } a \in \mathbb{N},$$

and the Gaussian binomial coefficients

$$\begin{bmatrix} r \\ a \end{bmatrix}_i = \frac{[r]_i \cdots [r-a+1]_i}{[a]_i!} \quad \text{for } r \in \mathbb{Z}, a \in \mathbb{N}.$$

One can check that all elements defined above belong to the subring $\mathbb{Z}[v, v^{-1}]$ of $\mathbb{Q}(v)$, see [21, Chapter 0].

3.1.2. The first quantum group. The quantum group U_v over $\mathbb{Q}(v)$ associated to (a_{ij}) is the $\mathbb{Q}(v)$-algebra with generators E_i, F_i, K_i and K_i^{-1} (for all $i \in I$) and the following relations (for all $i, j \in I$)

$$K_iK_i^{-1} = K_i^{-1}K_i = 1,\ K_iK_j = K_jK_i, \tag{3.1}$$

$$K_iE_jK_i^{-1} = v^{d_ia_{ij}}E_j,\ K_iF_jK_i^{-1} = v^{-d_ia_{ij}}F_j, \tag{3.2}$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{v^{d_i} - v^{-d_i}}, \tag{3.3}$$

and for $i \neq j$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} a_{ij} - 1 \\ s \end{bmatrix}_i E_i^{1-a_{ij}-s}E_jE_i^s = 0, \tag{3.4}$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} a_{ij} - 1 \\ s \end{bmatrix}_i F_i^{1-a_{ij}-s}F_jF_i^s = 0. \tag{3.5}$$

The quantum group U_v is a Hopf algebra with comultiplication Δ, counit ϵ and antipode ι such that for all $i \in I$ (cf. [21, 4.11]):

$$\begin{aligned} &\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, && \epsilon(E_i) = 0, && \iota(E_i) = -K_i^{-1}E_i, \\ &\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, && \epsilon(F_i) = 0, && \iota(F_i) = -F_iK_i, \\ &\Delta(K_i) = K_i \otimes K_i, && \epsilon(K_i) = 1, && \iota(K_i) = K_i^{-1}. \end{aligned}$$

Lemma 3.1. *There is a unique involutory $\mathbb{Q}(v)$-antiautomorphism τ of U_v with $\tau(E_i) = F_i, \tau(F_i) = E_i$ and $\tau(K_i^{\pm}) = K_i^{\pm}$ for all $i \in I$.*

Proof. One can easily check that the images of the generators satisfy the relations (3.1)-(3.5) in the opposite algebra U_v^{op}. The uniqueness is obvious. Since all generators of U_v are fixed under τ^2, then $\tau^2 = 1$. ■

Example 3.2. *Consider the simplest possible case, namely $(a_{ij}) = (2)$. Then the corresponding quantum group U_v has 4 generators E, F, K and K^{-1} and the following relations*

$$\begin{aligned} KK^{-1} &= 1 = K^{-1}K, \\ KEK^{-1} &= v^2E, KFK^{-1} = v^{-2}F, \\ EF - FE &= \frac{K - K^{-1}}{v - v^{-1}}. \end{aligned}$$

3.1.3. The second quantum group. Consider the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ and define $U_{\mathcal{A}}$ to be the $\mathcal{A}$-subalgebra of U_v generated by all $E_i^{(r)}, F_i^{(r)}, K_i$ and K_i^{-1} with $i \in I$

and $r \in \mathbb{N}$. Here we use the notation

$$\begin{aligned} E_i^{(r)} &= \frac{E_i^r}{[r]_i!} \text{ for } r \in \mathbb{N}, i \in I, \\ F_i^{(r)} &= \frac{F_i^r}{[r]_i!} \text{ for } r \in \mathbb{N}, i \in I. \end{aligned}$$

The quantum group $U_{\mathcal{A}}$ is sometimes called the Lusztig $\mathcal{A}$-form of U_v.

One can show that the restrictions of Δ, ϵ and ι to $U_{\mathcal{A}}$ make $U_{\mathcal{A}}$ into a Hopf algebra. Set now

$$\left[\begin{array}{c} K_i; c \\ t \end{array} \right] = \prod_{s=1}^{t} \frac{K_i v^{d_i(c-s+1)} - K_i^{-1} v^{-d_i(c-s+1)}}{v^{d_i s} - v^{-d_i s}} \text{ for } c \in \mathbb{Z}, t \in \mathbb{N}, i \in I.$$

Then define $U_{\mathcal{A}}^+, U_{\mathcal{A}}^-$ and $U_{\mathcal{A}}^0$ as follows:

$U_{\mathcal{A}}^+$ is the subalgebra generated by all $E_i^{(r)}$ with $i \in I$ and $r \in \mathbb{N}$,

$U_{\mathcal{A}}^-$ is the subalgebra generated by all $F_i^{(r)}$ with $i \in I$ and $r \in \mathbb{N}$,

and

$U_{\mathcal{A}}^0$ is the subalgebra generated by all $K_i^{\pm 1}, \begin{bmatrix} K_i; 0 \\ r \end{bmatrix}$ with $i \in I$ and $r \in \mathbb{N}$.

Hence we have a triangular decomposition of $U_{\mathcal{A}}$, meaning that the multiplication map defines an isomorphism of vector spaces

$$U_{\mathcal{A}} = U_{\mathcal{A}}^- U_{\mathcal{A}}^0 U_{\mathcal{A}}^+. \tag{3.6}$$

This is a consequence of Kac's formula (cf. [29, Lemma 6.5 (a2)]):

$$E_i^{(r)} F_i^{(s)} = \sum_{t=0}^{\min(s,r)} F_i^{(s-t)} \left[\begin{array}{c} K_i; 2t - s - r \\ t \end{array} \right]_i E_i^{(r-t)} \text{ for all } i \in I \text{ and } r, s \in \mathbb{N}.$$

We set $B_{\mathcal{A}} = U_{\mathcal{A}}^- U_{\mathcal{A}}^0$. We will call this the Borel subalgebra of $U_{\mathcal{A}}$.

3.1.4. The third quantum group. Let now $\mathcal{R}$ be a commutative algebra over $\mathcal{A}$. The quantum group over $\mathcal{R}$ associated with the Cartan matrix (a_{ij}) is the $\mathcal{R}$-algebra

$$U_{\mathcal{R}} = U_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{R},$$

which is clearly a Hopf algebra (with comultiplication, counit and antipode induced from $U_{\mathcal{A}}$ and denoted by the same symbols as for U_v).

Similarly, we set

$$U_{\mathcal{R}}^- = U_{\mathcal{A}}^- \otimes_{\mathcal{A}} \mathcal{R},\ U_{\mathcal{R}}^+ = U_{\mathcal{A}}^+ \otimes_{\mathcal{A}} \mathcal{R},\ U_{\mathcal{R}}^0 = U_{\mathcal{A}}^0 \otimes_{\mathcal{A}} \mathcal{R},$$

and

$$B_{\mathcal{R}} = B_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{R}.$$

In this thesis we shall restrict ourselves to the case where $\mathcal{R}$ is a field. So let k be an arbitrary field, and let $q \in k^{\times}$ be a non-zero element. We make k into $\mathcal{A}$-algebra by specializing v to q and define

$$U_q = U_{\mathcal{A}} \otimes_{\mathcal{A}} k.$$

We abuse the notation and write $E_i^{(r)}, F_i^{(r)}$ and $K_i^{\pm 1}$ for the images in U_q for these generators.

3.1.5. Borel characters. Just as for the Borel subgroup B, each $\lambda \in X$ defines a character of B_q as follows: We have a k-algebra homomorphism $\chi_\lambda : U_q^0 \to k$ given by (for all $i \in I$) (see e.g. [11])

$$\chi_\lambda \left(K_i^{\pm 1}\right) = q^{\pm d_i \lambda_i}$$

$$\chi_\lambda \left(\begin{bmatrix} K_i; c \\ t \end{bmatrix} \right) = \begin{bmatrix} \lambda_i + c \\ t \end{bmatrix}_i \text{ for all } c \in \mathbb{Z}, t \in \mathbb{N},$$

The 1-dimensional U_q^0-module, where U_q^0 acts via λ, is denoted by λ or sometimes k_λ. In particular, the trivial U_q^0-module k may also be written k_0. This extends to a B_q-module with the trivial U_q^--action.

3.2. Representations of quantum groups

3.2.1. Weight spaces. Let $U_q^0 \subset U'$ be a subalgebra of U_q and let M be a U'-module. For each $\lambda \in X$ we define the λ-weight space of M by

$$M_\lambda = \{m \in M \mid um = \chi_\lambda(u)m, u \in U_q^0\}.$$

We say that $\lambda \in X$ is a weight of M if $M_\lambda \neq 0$, and a maximal weight of M if we further have

$$E_i^{(r)} M_\lambda = 0 \text{ for all } i \in I \text{ and } r > 0.$$

A U_q-module is said to be a module of highest weight λ if it is generated by an element $v \in M$ with maximal weight λ.

3.2.2. Integrable modules. Let $J \subset I = \{1, \dots, n\}$ and denote by $U_q(J)$ the subalgebra of U_q generated by B_q together with

$$\{E_j^{(r)} \mid j \in J, r \in \mathbb{N}\}.$$

A $U_q(J)$-module M is called integrable if M is the direct sum of its weight spaces and there exists for any $m \in M$ an integer $r_m \in N$ such that

$$E_i^{(r)} m = F_i^{(r)} m = 0 \text{ for all } r > r_m \text{ and } i \in J.$$

The category of integrable $U_q(J)$-modules will be denoted by $\mathscr{C}_q(U_q(J))$. Using some standard homological arguments, one can show that the category $\mathscr{C}_q(U_q(J))$ has enough injective, cf. [11].

Remark 3.3. Clearly, if M is an integrable $U_q(J)$-module M, then any $U_q(J)$-submodule $V \subset M$ is integrable, and we have $V_\lambda = M_\lambda \cap V$ for all $\lambda \in X$.

We are mainly interested in the following categories, for which we introduce a special notation:

$$\mathscr{C}_q^- = \mathscr{C}_q(B_q) \text{ and } \mathscr{C}_q = \mathscr{C}_q(U_q).$$

Lemma 3.4. *For any $M \in \mathscr{C}_q$ the Weyl group W acts on the set of weights of V.*

For a proof we refer to [11, lemma 1.13].

Remark 3.5. If we want to be more precise, the category $\mathscr{C}_q$ is in fact the category of integrable U_q-modules of type **1**. For each $\omega = (\omega_1, \dots, \omega_n)$ with $\omega_i = \pm 1$ for $i \in I$ we define an automorphism $\sigma_\omega : U_q \to U_q$ by

$$\sigma_\epsilon(E_i) = \epsilon_i E_i, \sigma_\epsilon(F_i) = \epsilon_i F_i \text{ and } \sigma_\epsilon(K_i) = \epsilon_i K_i$$

for all $i \in I$.

If M is a U_q-module, then we set

$$M_{\lambda,\omega} = \{m \in M \mid um = \sigma_\omega(u)\chi_\lambda(u)m, u \in U_q^0\}$$

for all ω and λ. So each $M_{\sigma,\omega}$ is a subspace of M that we call a weight space of M.

For any U_q-module M we let M^ω be the U_q-module that is equal to M as a vector space, and where each $u \in U_q$ acts on M^ω as $\sigma_\omega(u)$ acts on M. It is clear that $(M^\omega)^\omega \simeq M$, and M is simple if and only if M^ω is simple. We say that a U_q-module M has type ω if $M = M^\omega$. It is easy to see, however, that this twisting with σ_ω is an equivalence of categories between the category of U_q-modules of type **1** and those of type ω. We shall therefore restrict ourselves from now on to modules of type **1**.

3.2.3. Finally, we define the functor F from the category of $U_q(J)$-modules to the category of integrable $U_q(J)$-modules by

$$F(M) = \left\{ m \in \bigoplus_{\lambda \in X} M_\lambda \mid E_i^{(r)} m = F_i^{(r)} m = 0 \text{ for } r \gg 0,\ i \in J \right\}.$$

This is an integrable $U_q(J)$-submodule of M, see [11]. This is in fact the largest integrable $U_q(J)$-submodule of M.

3.2.4. The small quantum group. Suppose for a second that q is a primitive root of unity, and l is the order of q^2. For each $\alpha_i \in S$ we set $l_i = l/(l, d_i)$. It is well-known that for any root β in R there exist $w \in W$ and $\alpha_i \in S$ such that $\beta = w(\alpha_i)$, and then we let $l_\beta = l_i$.

The small quantum group u_q is the subalgebra generated by all $E_i^{(r)}, F_i^{(r)}$ and $K_i^{\pm}$ with $0 \le r < l_i$. From [29] we get that u_q is a finite dimensional Hopf algebra over k, and we have a triangular decomposition

$$u_q = u_q^- u_q^0 u_q^+.$$

Here the subalgebras u_q^+, u_q^0 and u_q^- are defined in the obvious way.

Let now $\mathscr{C}_q(B_q u_q^+)$ denote the category of $B_q u_q^+$-modules. Note that M belongs to this category if M is a $B_q u_q^+$-module whose restriction to B_q belongs to the category $\mathscr{C}_q^-$.

A $u_q U_q^0$-module M is said to be integrable if it is the direct sum of all M_λ with $\lambda \in X$. The category of integrable $u_q U_q^0$ is denoted by $\mathscr{C}_q(u_q U_q^0)$. Denote also by F the functor which takes any $u_q U_q^0$-modules M to

$$F(M) = \bigoplus_{\lambda \in X} M_\lambda.$$

Similarly we define $\mathscr{C}_q(u_q^- U_q^0)$, $\mathscr{C}_q(U_q^0)$ and F as we did for $\mathscr{C}_q(u_q U_q^0)$.

3.3. Induced representations

In this section we let $U_1 \subset U_2 \subset U_3$ be three algebras among those defined in 3.2.2 an 3.2.4. For each i we let M_i be an integrable U_i-module.

3.3.1. Induction functors. We make

$$\mathrm{Hom}_{U^1}\left(U^2, M_1\right) = \left\{f \in \mathrm{Hom}_k\left(U^2, M_1\right) \mid f(ux) = uf(x) \text{ for } u \in U^1, x \in U^2\right\}$$

into a U^2-module via

$$(yf)(x) = f(xy), \quad x, y \in U^2, f \in \mathrm{Hom}_{U^1}(U^2, M_1).$$

The U^2-module induced by M_1 is

$$H_q^0\left(U^2/U^1, M_1\right) = F\left(\mathrm{Hom}_{U^1}\left(U^2, M_1\right)\right).$$

This is a left exact covariant functor. The induction functor H_q^0 was constructed in [11] and [12].

Choose now an injective resolution $Q_\bullet$ of M_1 in $\mathscr{C}_q(U_1)$. Then we set

$$H_q^i\left(U^2/U^1, M_1\right) = H^i\left(H_q^0\left(U^2/U^1, Q_\bullet\right)\right) \text{ for all } i \geq 0.$$

It turns out that $H_q^i\left(U^2/U^1, M_1\right)$ is independent of the chosen resolution. We shall write H_q^i in short of $H_q^i(U_q/B_q, -)$.

3.3.2. Properties. The induction functor H_q^0 has the following basic properties whose proofs can be found in [11], [12] and [13].

(1) **Frobenius reciprocity:** The map

$$\begin{aligned} \mathrm{Hom}_{U^2}\left(M_2, H_q^0\left(U^2/U^1, M_1\right)\right) &\to \mathrm{Hom}_{U^1}\left(M_2, M_1\right) \\ \varphi &\mapsto Ev \circ \varphi \end{aligned} \tag{3.7}$$

is an isomorphism of vector spaces. Here Ev is the evaluation map

$$\begin{aligned} Ev : H_q^0\left(U^2/U^1, M_1\right) &\to M_1 \\ f &\mapsto f(1). \end{aligned} \tag{3.8}$$

(2) **The tensor identity:** We have an isomorphism of U^2-modules

$$H_q^i\left(U^2/U^1, M_2 \otimes_k M_1\right) \simeq M_2 \otimes_k H_q^i\left(U^2/U^1, M_1\right) \text{ for all } i \geq 0. \tag{3.9}$$

Similarly, one obtains an isomorphism of U^2-modules

$$H_q^i\left(U^2/U^1, M_1 \otimes_k M_2\right) \simeq H_q^i\left(U^2/U^1, M_1\right) \otimes_k M_2 \text{ for all } i \geq 0. \tag{3.10}$$

(3) **Induction is transitive:** We have

$$H_q^0\left(U^3/U^2, H_q^0\left(U^2/U^1, M_1\right)\right) \simeq H_q^0\left(U^3/U^1, M_1\right). \tag{3.11}$$

(4) If M_1 is injective, then so is $H_q^0\left(U^2/U^1, M_1\right)$, and all higher cohomology

$$H_q^j\left(U^2/U^1, M_1\right) \text{ with } j > 0$$

are zero.

(5) Finally, we have the following spectral sequence

$$H_q^i\left(U^3/U^2, H_q^j\left(U^2/U^1, M_1\right)\right) \Rightarrow H^{i+j}\left(U^3/U^1, M_1\right). \tag{3.12}$$

We shall sometimes use these properties in the subsequent chapters without referring to them.

3.3.3. Further properties. Many of the vanishing results of H^0 in the modular case can be carried over to the quantum case, e.g. $H_q^0(\lambda)$ is non-zero if and only if $\lambda \in X^+$, and $H_q^0(\lambda)$ is finite dimensional and contains a unique simple submodule of highest weight λ that we denote by $L_q(\lambda)$. Furthermore, the set of dominant weights parametrizes the simple U_q-modules via highest weight.

One can also prove that Kempf's vanishing theorem (1.4), Serre duality (1.2) and Grothendieck's vanishing theorem (1.3) have analogues for U_q. And when q is not a root of unity, the Borel-Weil-Bott theorem holds for all characters $\lambda \in X$, and $H_q^0(\lambda)$ is simple for each $\lambda \in X^+$. The Borel-Weil-Bott theorem fails in the root of unity case, and the simplicity of H_q^0 also breaks down in general.

Most of these results were proved under some restrictions on k and l in [11] by reduction to the classical case. The mixed case, i.e. the case where the ground field k is of positive characteristic prime to l is investigated in [13] and [34]. These results were proved under some restrictions on l by specialization to the modular case. For the case where one also allows even l, we refer to [10].

However, these methods fail to give a generalization of Kempf's vanishing theorem in the mixed case when $l \leq h$. Ryom-Hansen proved in [33] the quantized analogue of Kempf's vanishing theorem. His proof involves using Kashiwara's crystal bases [27] to analyse the Demazure modules.

3.4. The strong linkage principle

In this section we assume that q is a primitive root of unity, and we set l equal the order of q^2. We present the quantum version of the strong linkage principle and some of its consequences.

3.4.1. The affine Weyl group. The affine Weyl group W_l is the group generated by the affine reflections $s_{\beta,m}$ given by

$$s_{\beta,m} \cdot \lambda = s_\beta \cdot \lambda + kl_\beta \beta, m \in \mathbb{Z}, \beta \in R, \lambda \in X.$$

Note that if $l_i = l$ for all $i \in I$, then W_l is the usual affine Weyl group of R. But in general W_l is the affine Weyl group of the dual root system $R^\vee$, for more details we refer to [13].

Let $\lambda \in X^+ - \rho$ and set

$$\Pi(\lambda) = \{\mu \in X \mid w(\mu) \leq \lambda \text{ for all } w \in W\}.$$

Note that $\Pi(\lambda) = \emptyset$ unless there exists $\mu \in X^+$ such that $\mu \leq \lambda$.

Theorem 3.6 ([5, Theorem 3.9]). *Let $\lambda \in X^+ - \rho$. If $w \in W$ and $j \in \mathbb{N}$, then all weights of $H_q^j(w \cdot \lambda)$ are in $\Pi(\lambda)$, and all weight spaces of these modules are finite dimensional.*

It follows immediately from the above theorem that

Corollary 3.7 ([5, Corollary 3.10]). *If $M \in \mathscr{C}_q^-$ is finite dimensional, then so is $H_q^i(M)$ for all i.*

3.4.2. The linkage. Let $\lambda, \mu \in X$. We say that λ is linked to μ if $\lambda \in W_l \cdot \mu$. Furthermore, λ is said to be strongly linked to μ if $\lambda = \mu$ or if there are reflections $s_1, \dots, s_{r+1} \in W_l$ such that

$$\lambda \leq s_1 \cdot \lambda = \lambda_1 \leq s_2 \cdot \lambda_1 = \lambda_2 \leq \cdots \leq s_r \cdot \lambda_{r-1} = \lambda_r \leq s_{r+1} \cdot \lambda_r = \mu.$$

The following lemma summarizes some basic properties of the strong linkage:

Lemma 3.8. *Let $\lambda \in X$. We have*

(1) $\lambda - l_\alpha \alpha$ is strongly linked to λ for all $\alpha \in R^+$.
(2) If $\lambda \in X^+ - \rho$, then we have that $w \cdot \lambda$ is strongly linked to λ for all $w \in W$.

Proof. (1) Write $\langle \lambda + \rho, \alpha^\vee \rangle = a l_\alpha + b$ where $a, b \in \mathbb{Z}$ and $0 < b \leq l_\alpha$. We have that $s_{\alpha,a} \cdot \lambda = \lambda - b\alpha$, and this shows that $\lambda - b\alpha$ is strongly linked to λ. On the other hand, if $b < l_\alpha$, then we have that $s_{\alpha,(a-1)} \cdot (\lambda - b\alpha) = \lambda - l_\alpha \alpha$. Hence $\lambda - l_\alpha \alpha$ is strongly linked to $\lambda - b\alpha$, and then strongly linked to λ.

(2) This can be proved by a simple induction on $l(w)$. For $l(w) = 0$ there is nothing to prove. Suppose now that $l(w) > 0$, then there exists a simple root $\alpha \in S$ such that $l(s_\alpha w) = l(w) - 1$, and this means that $w^{-1}(\alpha) < 0$. Then $\langle w(\lambda + \rho), \alpha^\vee \rangle = \langle \lambda + \rho, w^{-1}(\alpha)^\vee \rangle \leq 0$. Hence $w \cdot \lambda$ is strongly linked to $(s_\alpha w) \cdot \lambda$. Using induction, we get that $w \cdot \lambda$ is strongly linked to λ. ∎

Theorem 3.9 (The strong linkage principle). *Let $\lambda \in X^+ - \rho$. All composition factors of $H_q^i(w \cdot \lambda), i \in \mathbb{N}, w \in W$, have highest weights strongly linked to λ.*

For more details and for a proof of the strong linkage principle for the quantum case, we refer to [5]. By combining the strong linkage principle and Frobenius reciprocity, one can easily show that

Corollary 3.10 ([5, Corollary 4.4]). *Suppose that M is an indecomposable U_q-module. If $\lambda, \mu \in X^+$ such that $L_q(\lambda)$ and $L_q(\mu)$ both are composition factors of M, then $\mu \in W_l \cdot \lambda$.*

3.4.3. The Euler character. Now, let $M \in \mathscr{C}_q^-$ be a finite dimensional module. We define the formal character of M

$$\mathrm{ch}(M) = \sum_{\lambda \in X} \dim_k (M_\lambda)\, e^\lambda \in \mathbb{Z}[X].$$

Here $\{e^\lambda \mid \lambda \in X\}$ denotes the standard basis of the group ring $\mathbb{Z}[X]$. Since each $H_q^0(\lambda)$ has λ as its unique highest weight, then $\mathrm{ch}\left(H_q^0(\lambda)\right)$ with $\lambda \in X^+$ form a basis of the ring of W-invariants in $\mathbb{Z}[X]$, cf. [22, II. 5].

We denote the Euler character of M by $\chi(M)$

$$\chi(M) = \sum\nolimits_{i\geq 0}(-1)^i \operatorname{ch}\left(H_q^i(M)\right).$$

According to Grothendieck's vanishing theorem, we have $H_q^i(M) = 0$ for all $i > N$, and this shows that χ is well defined.

For any $\lambda \in X$ the Euler character $\chi(\lambda)$ of k_λ is given by Weyl's character formula, see the argument in [22, II.5].

$$\chi(\lambda) = \sum\nolimits_{w\in W}(-1)^{l(w)}e^{w(\lambda+\rho)} \Big/ \sum\nolimits_{w\in W}(-1)^{l(w)}e^{w(\rho)}.$$

Note that Kempf's vanishing theorem implies that

$$\chi(\lambda) = \operatorname{ch}\left(H_q^0(k_\lambda)\right) \text{ for all } \lambda \in X^+. \tag{3.13}$$

This shows that the set of weights of $H_q^0(k_\lambda)$ is independent of l and of k. Using this together with [25, 21.3], we see that this set is saturated in the sense of [25, 13.4], and it is equal to $\Pi(\lambda)$.

CHAPTER 4

The first cohomology of simple $\mathbf{B_q}$-modules

The vanishing behaviour of the cohomology group $H_q^\bullet(\lambda)$ of simple B_q-modules depends on whether q is a root of unity or not. In the case where q is not a root of unity, the Borel-Weil-Bott theorem gives both the vanishing behaviour and the U_q-structure of all such cohomology groups, whereas we know rather little about this cohomology group when q is a root of unity.

In this chapter we assume that $q \in k^\times$ is a primitive root of unity, and we let l be the order of q^2, see Section 3.4. We shall prove that $H_q^1(\lambda)$, if non-zero, contains a unique simple submodule and compute its highest weight. We shall also completely describe the vanishing behaviour of $H_q^1(\lambda)$. Our results depend on whether k is a field of characteristic 0 or of characteristic $p > 0$.

4.1. Rank 1

This section deals with the rank 1 case. Fix $\alpha_i \in S$ and let $U_{q,i}$ be the "parabolic" subalgebra generated by B_q and

$$\{E_i^{(r)} \mid \text{ for all } r \in \mathbb{N}\}.$$

The $U_{q,i}$-module induced by k_λ is denoted by $H_{q,i}^0(\lambda)$. The category of integrable $U_{q,i}$-modules will be denoted by $\mathscr{C}_q(i)$.

Our aim is to prove that $H_{q,i}^1(\lambda)$, if non-zero, contains a unique simple submodule, and then extend this result to the general case.

4.1.1. First we state two results from [11] which completely describe the $U_{q,i}$-structure of $H_{q,i}^\bullet(\lambda)$.

Theorem 4.1. *Let $\lambda \in X$. Then $H_{q,i}^j(\lambda) = 0$ for all $j > 1$. Moreover, $H_{q,i}^0(\lambda)$ is non-zero iff. $\lambda_i \geq 0$. If so, $H_{q,i}^0(\lambda)$ is $(\lambda_i + 1)$-dimensional with a basis $\{e_0, \dots, e_{\lambda_i}\}$. We have for all $j \in \{0, \dots, \lambda_i\}$*

(1) $e_j \in H_{q,i}^0(\lambda)_{\lambda - j\alpha_i}$,

(2) $E_i^{(r)} e_j = \begin{bmatrix} j \\ r \end{bmatrix}_i e_{j-r}$ *for all* $r \in \mathbb{N}$,

(3) $F_i^{(r)} e_j = \begin{bmatrix} \lambda_i - j \\ r \end{bmatrix}_i e_{j+r}$ *for all* $r \in \mathbb{N}$,

(4) $F_t^{(r)} e_j = 0$ *for all* $r \in \mathbb{N}$ *and* $t \neq i$.

We set $e_j = 0$ if $j < 0$ or $j > \lambda_i$.

As mentioned before, $H^0_{q,i}(\lambda)$, if non-zero, contains a unique simple module that we denote by $L_{q,i}(\lambda)$. Furthermore, each simple $U_{q,i}$-module is isomorphic to $L_{q,i}(\lambda)$ for some $\lambda \in X$ with $\lambda_i \geq 0$.

Theorem 4.2. *Let $\lambda \in X$. Then $H^1_{q,i}(\lambda)$ is non-zero iff. $\lambda_i < -1$. If so, $H^1_{q,i}(\lambda)$ is $-(\lambda_i + 1)$-dimensional with a basis $\{f_0, \dots, f_{-\lambda_i-2}\}$. For all $j \in \{0, \dots, -\lambda_i - 2\}$ we have*

(1) $f_j \in H^1_{q,i}(\lambda)_{\lambda+(j+1)\alpha_i}$,

(2) $E_i^{(r)} f_j = \begin{bmatrix} j+r \\ r \end{bmatrix}_i f_{j+r}$ for all $r \in \mathbb{N}$,

(3) $F_i^{(r)} f_j = \begin{bmatrix} -\lambda_i - 2 - j + r \\ r \end{bmatrix}_i f_{j-r}$ for $r \in \mathbb{N}$,

(4) $F_t^{(r)} f_j = 0$ for all $r \in \mathbb{N}$ and $t \neq i$.

We set $f_j = 0$ if $j < 0$ or $j > -\lambda_i - 2$.

4.1.2. Now, let $\lambda \in X$ with $\lambda_i \geq 0$ and set $s_i = s_{\alpha_i}$. By Frobenius reciprocity, we have

$$\mathrm{Hom}_{U_{q,i}}(H^1_{q,i}(s_i \cdot \lambda), H^0_{q,i}(\lambda)) \simeq \mathrm{Hom}_{B_q}(H^1_{q,i}(s_i \cdot \lambda), k_\lambda).$$

From Theorem 4.2 we get that λ is a maximal weight of $H^1_{q,i}(s_i \cdot \lambda)$, and this means that this Hom-space is 1-dimensional and generated by the following $U_{q,i}$-homomorphism

$$f_j \mapsto \begin{bmatrix} \lambda_i \\ j \end{bmatrix}_i e_{\lambda_i - j}. \tag{4.1}$$

Proposition 4.3. *Let $\lambda \in X$ with $\lambda_i \geq 0$. Then*

$$L_{q,i}(\lambda) = \mathrm{span}_k \left\{ e_j \mid \begin{bmatrix} \lambda_i \\ j \end{bmatrix}_i \neq 0 \right\} \subset H^0_{q,i}(\lambda).$$

Proof. Let $L_{q,i}(\lambda)$ be the subspace defined in the proposition. Noting that

$$\begin{bmatrix} \lambda_i \\ j \end{bmatrix}_i = \begin{bmatrix} \lambda_i \\ \lambda_i - j \end{bmatrix}_i \text{ for all } j \leq \lambda_i,$$

we get from (4.1) that $L_{q,i}(\lambda)$ is a $U_{q,i}$-submodule of $H^0_{q,i}(\lambda)$. Clearly, it would be enough to show that each submodule $0 \neq L$ of $H^0_{q,i}(\lambda)$ contains $L_{q,i}(\lambda)$.

Let μ be minimal among weights of L. Then

$$F_i^{(r)} L_\mu = 0 \text{ for all } r > 0.$$

Combining this with Theorem 4.1, it follows immediately that $L_\mu = L_{s_i(\lambda)} = k e_{\lambda_i}$. Since

$$E_i^{(r)} e_{\lambda_i} = \begin{bmatrix} \lambda_i \\ r \end{bmatrix}_i e_{\lambda_i - r} = \begin{bmatrix} \lambda_i \\ \lambda_i - r \end{bmatrix}_i e_{\lambda_i - r} \text{ for all } r \leq \lambda_i,$$

the proposition follows. ∎

4.1.3. Exactly as in [30, Proposition 3.2], we get

Lemma 4.4. *Let $m, n \in \mathbb{N}$ with $m \geq n$. Write $m = m_2 + l_i m_1$ and $n = n_2 + l_i n_1$ with $0 \leq m_2, n_2 \leq l_i - 1$. Then we have*

$$\begin{bmatrix} m \\ n \end{bmatrix}_i = \binom{m_1}{n_1} \begin{bmatrix} m_2 \\ n_2 \end{bmatrix}_i .$$

Note that the above lemma also holds when $l_i = 1$. In this case we have

$$\begin{bmatrix} m \\ n \end{bmatrix}_i = \binom{m}{n}.$$

Remark 4.5. Suppose that $l_i > 1$ and $-(a+1)l_i \leq \lambda_i \leq -al_i - 2$ for some $a \in \mathbb{N}$. Hence $al_i \leq -\lambda_i - 2 \leq (a+1)l_i - 2$. This implies that $f_{al_i - 1 + r} = 0$ for all $r \geq l_i$. We get from Lemma 4.4

$$\begin{bmatrix} al_i - 1 + r \\ r \end{bmatrix}_i = \binom{a}{0} \begin{bmatrix} r - 1 \\ r \end{bmatrix}_i = 0 \text{ for all } r = \{1, \dots, l_i - 1\}.$$

Using this together with Theorem 4.2, it follows immediately that $\lambda + al_i\alpha_i$ is a maximal weight of $H^1_{q,i}(\lambda)$.

$$E_i^{(r)} f_{al_i - 1} = \begin{bmatrix} al_i - 1 + r \\ r \end{bmatrix}_i f_{al_i - 1 + r} = 0 \text{ for all } r > 0.$$

Therefore we conclude that $H^1_{q,i}(\lambda)$ is not simple for such λ since $s_i \cdot \lambda \neq \lambda + al_i\alpha_i$.

4.1.4. Characteristic zero. Let k be an arbitrary field of characteristic 0.

Theorem 4.6. *Let $\lambda \in X$ with $\lambda_i < -1$. Then $H^1_{q,i}(\lambda)$ contains a unique simple submodule M. The highest weight μ of M is*

$$\mu = \begin{cases} s_i \cdot \lambda & \text{if } \lambda_i \geq -l_i \text{ or } \lambda_i \equiv -1 \bmod l_i, \\ \lambda + al_i\alpha_i & \text{if } -(a+1)l_i \leq \lambda_i \leq -al_i - 2 \text{ for some } a \in \mathbb{N}. \end{cases}$$

Note that the second case listed in Theorem 4.6 does not make sense unless $l_i > 1$, and if $l_i = 1$, then we are in the first case, namely $\lambda_i \equiv -1 \bmod l_i$.

Proof. (1) Suppose first that $\lambda_i \geq -l_i$ or $\lambda_i \equiv -1 \bmod l_i$. Using Lemma 4.4, we get that

$$\begin{bmatrix} -\lambda_i - 2 \\ j \end{bmatrix}_i \neq 0 \text{ for all } j \in \{0, \dots, -\lambda_i - 2\}. \tag{4.2}$$

When we combine Proposition 4.3 and (4.1), we get that $H^1_{q,i}(\lambda)$ is simple with highest weight $s_i \cdot \lambda$.

(2) Suppose now that $-(a+1)l_i \leq \lambda_i \leq -al_i - 2$ for some $a \geq 1$. Remark 4.5 implies in this case that $H^1_{q,i}(\lambda)$ is not simple and that $\lambda + al_i\alpha_i$ is a maximal weight of $H^1_{q,i}(\lambda)$.

Let M be a simple submodule of $H^1_{q,i}(\lambda)$ with highest weight μ. It follows immediately from Theorem 4.2 that $\mu \neq s_i \cdot \lambda$. We want to prove that $\mu = \lambda + al_i\alpha_i$.

Using the notation from Theorem 4.2, we let $f_j \in M_\mu$ for some $j < -\lambda_i - 2$. So

$$E_i^{(r)} f_j = \begin{bmatrix} j+r \\ r \end{bmatrix}_i f_{j+r} = 0 \text{ for all } r > 0.$$

In particular, we have that $[j+1]_i = 0$ since $f_{j+1} \neq 0$. Hence $j \equiv -1 \bmod l_i$, and this implies that $j = bl_i - 1$ for some $b \in \mathbb{N}$ with $1 \leq b \leq a$. We want to prove that $b = a$. We prove this by contradiction. If $b < a$, then $f_{(b+1)l_i-1}$ is non-zero, and

$$\begin{aligned} E_i^{(l_i)} f_{bl_i-1} &= \begin{bmatrix} bl_i - 1 + l_i \\ l_i \end{bmatrix}_i f_{(b+1)l_i-1} \\ &= \begin{bmatrix} l_i - 1 \\ 0 \end{bmatrix}_i \binom{b}{1} f_{(b+1)l_i-1} = bf_{(b+1)l_i-1} \neq 0. \end{aligned}$$

This shows that $\lambda + bl_i\alpha_i$ is not a maximal weight of M unless $b = a$.

It remains to show that M is unique. Suppose that M_1 and M_2 are simple submodules of M, then

$$0 \neq H^1_{q,i}(\lambda)_{\lambda + al_i\alpha_i} \subset M_1 \cap M_2.$$

Hence we conclude that M is unique, and this completes the proof. ∎

Remark 4.7. When $\lambda_i < -1$, it follows immediately from the proof of the above theorem that $H^1_{q,i}(\lambda)$ is simple with highest weight $s_i \cdot \lambda$ if and only if $\lambda_i \geq -l_i$ or $\lambda_i \equiv -1 \bmod l_i$.

4.1.5. Characteristic p. Let k be an arbitrary field of characteristic $p > 0$. We shall prove that Theorem 4.6 has an analogue in positive characteristic.

Lemma 4.8. *Let a and b be non-negative integers. If $a = a_1 + a_2 p$ with $0 \leq a_1 < p$ and $b = b_1 + b_2 p$ with $0 \leq b_1 < p$, then*

$$\binom{a}{b} \equiv \binom{a_1}{b_1}\binom{a_2}{b_2} \bmod p.$$

Proof. In characteristic p, we have

$$(X+Y)^a = (X+Y)^{a_1}(X^p + Y^p)^{a_2}.$$

By looking at the coefficient of $X^a Y^{a-b}$ in both sides, the lemma follows. ∎

Note that if $a = \sum_{k\geq 0} a_k p^k$ and $b = \sum_{k \geq 0} b_k p^k$ are the p-adic expansions of a and b, then

$$\binom{a}{b} \equiv \prod_{k\geq 0} \binom{a_k}{b_k} \bmod p.$$

Hence $\binom{a}{b} \equiv 0 \bmod p$ if and only if there exists $k \geq 0$ such that $a_k < b_k$.

Let $a \in \mathbb{Z}$ and $d \in \mathbb{N}$. Write $a = a_1 + da_2$ with $0 \leq a_1 < d$. Then we set $\langle a \rangle_d = a_2$. Note that $\langle - \rangle_d$ is well defined, and $\langle a \rangle_1 = a$ for all $a \in \mathbb{Z}$.

Theorem 4.9. *Suppose that $\lambda_i < -1$. Then $H^1_{q,i}(\lambda)$ contains a unique simple submodule M. The highest weight μ of M is*

$$\mu = \begin{cases} s_i \cdot \lambda & \text{if } \lambda_i \geq -l_i, \\ s_i \cdot \lambda & \text{if } \lambda_i \equiv -1 \bmod l_i \text{ and } \langle -\lambda_i - 2\rangle_{l_i} < p, \\ s_i \cdot \lambda & \text{if } \lambda_i \equiv -1 \bmod l_i \text{ and } \langle -\lambda_i - 2\rangle_{l_i} = p^m - 1 \text{ for some } m \in \mathbb{N}, \\ \lambda + ap^m l_i\alpha_i & \text{if } \lambda_i \equiv -1 \bmod l_i \text{ and } ap^m \leq \langle -\lambda_i - 2\rangle_{l_i} \leq (a+1)p^m - 2 \\ & \text{for some } a, m \in \mathbb{N} \text{ with } a < p, \\ \lambda + bp^m l_i\alpha_i & \text{if } -(a+1)l_i \leq \lambda_i \leq -al_i - 2 \text{ for some } a \in \mathbb{N} \text{ and} \\ & bp^m \leq a < (b+1)p^m \text{ where } m \geq 0 \text{ and } b \in \mathbb{N} \text{ with } b < p. \end{cases}$$

To prove this theorem, we need the following lemma:

Lemma 4.10. *Let $\langle -\lambda_i - 2\rangle_{l_i} = \sum_{k=0}^{m} a_k p^k$ be the p-adic expansion of $\langle -\lambda_i - 2\rangle_{l_i}$.*

(1) If λ_i satisfies condition (4) listed in Theorem 4.9, then $\mu \neq s_i \cdot \lambda$ is a maximal weight of $H^1_{q,i}(\lambda)$ if and only if μ is of the form

$$\lambda + \left(\sum_{k=t}^{m} a_k p^k\right) l_i\alpha_i,$$

for some t with $m \geq t > s = \min\{k \mid a_k < p-1\} \geq 0$.

(2) If λ_i satisfies condition (5) listed in Theorem 4.9, then $\mu \neq s_i \cdot \lambda$ is a maximal weight of $H^1_{q,i}(\lambda)$ if and only if μ is of the form

$$\lambda + \left(\sum_{k=t}^{m} a_k p^k\right) l_i\alpha_i,$$

for some t with $m \geq t \geq 0$.

Proof. (1) First, note that s is well defined. Let $\mu \neq s_i \cdot \lambda$ be a maximal weight of $H^1_{q,i}(\lambda)$. Using the notation from Theorem 4.2, we let $f_j \in H^1_{q,i}(\lambda)_\mu$ for some $j < -\lambda_i - 2$. Then

$$E_i^{(r)} f_j = \begin{bmatrix} j+r \\ r \end{bmatrix}_i f_{j+r} = 0 \text{ for all } r > 0.$$

In particular, we have that $[j+1]_i = 0$ because $f_{j+1} \neq 0$. This implies that $j \equiv -1 \bmod l_i$. Hence $j = bl_i - 1$ for some $b \in \mathbb{N}$ with $1 \leq b \leq \langle -\lambda_i - 2\rangle_{l_i} + 1$. Since $\mu \neq s_i \cdot \lambda$, we have that $b \neq \langle -\lambda_i - 2\rangle_{l_i} + 1$. Therefore $j = bl_i - 1$ for some $b \in \mathbb{N}$ with $1 \leq b \leq \langle -\lambda_i - 2\rangle_{l_i}$. We now want to show that

$$b = \langle -\lambda_i - 2\rangle_{l_i} = \sum_{k=t}^{m} a_k p^k$$

for some $t > s$. So let

$$b = \sum_{k=0}^{m} b_k p^k$$

be the p-adic expansion of b and set $t = \min\{k \mid b_k \neq 0\}$. We claim that $a_k = b_k$ for all $k \geq t$. Assume by contradiction that there exists $r \in \mathbb{N}$ with $t \leq r \leq m$ and $b_r < a_r$. Hence $b + p^t \leq \langle -\lambda_i - 2\rangle_{l_i}$, and this means that $f_{(b+p^t)l_i-1} \neq 0$.

On the other hand, we get from Lemma 4.8 that

$$\begin{bmatrix} bl_i - 1 + p^t l_i \\ p^t l_i \end{bmatrix}_i = \begin{bmatrix} l_i - 1 \\ 0 \end{bmatrix}_i \binom{b + (p^t - 1)}{p^t} \neq 0. \tag{4.3}$$

So

$$E_i^{(p^t l_i)} f_j = \begin{bmatrix} bl_i - 1 + p^t l_i \\ p^t l_i \end{bmatrix}_i f_{(b+p^t)l_i-1} \neq 0. \tag{4.4}$$

This contradicts the assumption that μ is a maximal weight of $H^1_{q,i}(\lambda)$. Therefore we have that $b_k \geq a_k$ for all $k \geq t$.

Now, assume that there exists $r \in \mathbb{N}$ with $b_r > a_r$. Since $b \leq \langle -\lambda_i - 2\rangle_{l_i}$, it follows immediately that there exists another $r' \in \mathbb{N}$ with $r < r' \leq m$ and $b_{r'} < a_{r'}$. But this is impossible. Hence

$$b = \sum_{k=t}^{m} a_k p^k.$$

It remains to prove that $t > s$. Again, we assume by contradiction that $t \leq s$. Then

$$\begin{aligned} b + p^t &= \left(\sum_{k=t}^{m} a_k p^k\right) + p^t \\ &= \left(\sum_{k=t}^{m} a_k p^k\right) + 1 + \left(\sum_{k=0}^{t-1} (p-1) p^k\right) \\ &= 1 + \sum_{k=0}^{m} a_k p^k \text{ (because } t \leq s) \\ &= 1 + \langle -\lambda_i - 2\rangle_{l_i}. \end{aligned}$$

This implies that $f_{(b+p^t)l_i-1} \neq 0$. So

$$\begin{aligned} E_i^{(p^t l_i)} f_j &= \begin{bmatrix} bl_i - 1 + p^t l_i \\ p^t l_i \end{bmatrix}_i f_{(b+p^t)l_i-1} \\ &= \begin{bmatrix} l_i - 1 \\ 0 \end{bmatrix}_i \binom{b + (p^t - 1)}{p^t} f_{(b+p^t)l_i-1} \neq 0. \end{aligned}$$

Thus we conclude that j has to be of the form

$$\left(\sum_{k=t}^{m} a_k p^k\right) l_i - 1$$

for some $t > s$.

Conversely, suppose that j is of the form

$$\left(\sum_{k=t}^{m} a_k p^k\right) l_i - 1$$

for some t with $s < t$. We shall use Lemma 4.8 to prove that

$$E_i^{(r)} f_j = \begin{bmatrix} \left(\sum_{k=t}^{m} a_k p^k\right) l_i - 1 + r \\ r \end{bmatrix}_i f_{j+r} = 0 \text{ for all } r > 0.$$

Write $r = r_1 + r_2 l_i$ with $0 \leq r_1 < l_i$. If $r_1 \neq 0$, then

$$\begin{bmatrix} \left(\sum_{k=t}^{m} a_k p^k\right) l_i - 1 + r \\ r \end{bmatrix}_i = \begin{bmatrix} r_1 - 1 \\ r_1 \end{bmatrix}_i \begin{pmatrix} \sum_{k=t}^{m} a_k p^k + r_2 \\ r_2 \end{pmatrix} = 0.$$

Suppose now that $r_1 = 0$ and let

$$r_2 = \sum_{k=0}^{m} r_k p^k \neq 0$$

be the p-adic expansion of r_2. Set $h = \min\{k \mid r_k \neq 0\}$. If $h < t$, then we have

$$\begin{bmatrix} \left(\sum_{k=t}^{m} a_k p^k\right) l_i - 1 + r \\ r \end{bmatrix}_i = \begin{pmatrix} \sum_{k=t}^{m} a_k p^k + \sum_{k=h+1}^{m} r_k p^k + (r_h p^h - 1) \\ \sum_{k=h+1}^{m} r_k p^k + r_h p^h \end{pmatrix} = 0.$$

On the other hand, if $h \geq t$, then we have

$$\begin{aligned} \sum_{k=t}^{m} a_k p^k + r_2 &\geq \sum_{k=t}^{m} a_k p^k + p^t \\ &= \sum_{k=t}^{m} a_k p^k + \sum_{k=0}^{t-1} (p-1) p^k + 1 > \langle -\lambda_i - 2 \rangle_{l_i} + 1. \end{aligned}$$

The last inequality comes from the assumption that $s < t$. In this case we have that $f_{j+r} = 0$, and this settles the first case.

(2) Suppose now that λ satisfies condition (5) listed in Theorem 4.9. This implies that $a l_i \leq -\lambda_i - 2 \leq (a+1) l_i - 2$, and hence $\langle -\lambda_i - 2 \rangle_{l_i} = a$.

Let $\mu \neq s_i \cdot \lambda$ be a maximal weight of $H^1_{q,i}(\lambda)$ and let $f_j \in H^1_{q,i}(\lambda)_\mu$ for some j. The same argument as before gives that μ is of the form

$$\lambda + \left(\sum_{k=t}^{m} a_k p^k\right) l_i \alpha_i \text{ for some } t \geq 0.$$

Conversely, suppose that j is of the form

$$\left(\sum_{k=t}^{m} a_k p^k\right) l_i - 1 \text{ for some } t \geq 0.$$

We shall use Lemma 4.8 to prove that

$$E_i^{(r)} f_j = \begin{bmatrix} \left(\sum_{k=t}^{m} a_k p^k\right) l_i - 1 + r \\ r \end{bmatrix}_i f_{j+r} = 0 \text{ for all } r > 0.$$

Write $r = r_1 + r_2 l_i$ with $0 \le r_1 < l_i$. If $r_1 \ne 0$, then

$$\begin{bmatrix} \left(\sum\limits_{k=t}^{m} a_k p^k\right) l_i - 1 + r \\ r \end{bmatrix}_i = \begin{bmatrix} r_1 - 1 \\ r_1 \end{bmatrix}_i \begin{pmatrix} \sum\limits_{k=t}^{m} a_k p^k + r_2 \\ r_2 \end{pmatrix} = 0.$$

Suppose now that $r_1 = 0$ and let

$$r_2 = \sum_{k=0}^{m} r_k p^k \ne 0$$

be the p-adic expansion of r_2. Set $h = \min\{k \mid r_k \ne 0\}$. If $h < t$, then we have

$$\begin{bmatrix} \left(\sum\limits_{k=t}^{m} a_k p^k\right) l_i - 1 + r \\ r \end{bmatrix}_i = \begin{pmatrix} \sum\limits_{k=t}^{m} a_k p^k + \sum\limits_{k=h+1}^{m} r_k p^k + (r_h p^h - 1) \\ \sum\limits_{k=h+1}^{m} r_k p^k + r_h p^h \end{pmatrix} = 0.$$

If $h \ge t$, then we have

$$\sum_{k=t}^{m} a_k p^k + r_2 \ge \sum_{k=t}^{m} a_k p^k + p^t \ge \langle -\lambda_i - 2\rangle_{l_i} + 1.$$

Therefore we get that $f_{j+r} = 0$ since $-\lambda_i - 2 \not\equiv -1 \bmod l_i$. The lemma is proved. ■

Proof of Theorem 4.9. The uniqueness of M is obvious since all weights of $H^1_{q,i}(\lambda)$ occur with multiplicity 1, see the argument given at the end of the proof of Theorem 4.6.

(1) Suppose that $\lambda_i \ge -l_i$. Hence $-\lambda_i - 2 \le l_i - 2$. We then get that

$$\begin{bmatrix} -\lambda_i - 2 \\ j \end{bmatrix}_i \ne 0 \text{ for all } j \in \{0, \dots, -\lambda_i - 2\}.$$

This shows that $H^1_{q,i}(\lambda)$ is simple with highest weight $s_i \cdot \lambda$, cf. the argument in the proof of Theorem 4.6.

(2) Suppose that $\lambda_i \equiv -1 \bmod l_i$ and $\langle -\lambda_i - 2\rangle_{l_i} < p$. When we combine Lemma 4.4 and Lemma 4.8, we get that

$$\begin{bmatrix} -\lambda_i - 2 \\ j \end{bmatrix}_i = \begin{bmatrix} l_i - 1 \\ j_1 \end{bmatrix}_i \begin{pmatrix} \langle -\lambda_i - 2\rangle_{l_i} \\ j_2 \end{pmatrix} \ne 0 \text{ for all } j \in \{0, \dots, -\lambda_i - 2\}$$

where $j = j_1 + j_2 l_i$. Note that if $j \le -\lambda_i - 2$, then $j_2 \le \langle -\lambda_i - 2\rangle_{l_i}$. Therefore $H^1_{q,i}(\lambda)$ is simple with highest weight $s_i \cdot \lambda$.

(3) Suppose that $\lambda_i \equiv -1 \bmod l_i$ and $\langle -\lambda_i - 2\rangle_{l_i} = p^m - 1$ for some $m \in \mathbb{N}$. The same argument as before gives that $H^1_{q,i}(\lambda)$ is simple with highest weight $s_i \cdot \lambda$.

Now, let

$$\langle -\lambda_i - 2\rangle_{l_i} = \sum_{k=0}^{m} a_k p^k$$

be the p-adic expansion of $\langle -\lambda_i - 2\rangle_{l_i}$.

(4) Lemma 4.10 shows that $H^1_{q,i}(\lambda)$ is not simple. So let M be a simple submodule of $H^1_{q,i}(\lambda)$ with highest weight μ. It follows immediately from Theorem 4.2 that $\mu \neq s_i \cdot \lambda$. By Lemma 4.10, we get that μ is of the form of

$$\lambda + \left(\sum_{k=t}^{m} a_k p^k\right) l_i \alpha_i,$$

for some t with $t > s = \min\{k \mid a_k < p-1\}$. Clearly, we are done if we can prove that $\lambda + ap^m l_i \alpha_i$ is already a weight of M.

We have

$$\begin{aligned}
&F_i^{\left(\sum_{k=t}^{m-1} a_k p^k\right) l_i} f_{\left(\sum_{k=t}^{m} a_k p^k\right) l_i - 1} \\
&= \begin{bmatrix} (-\lambda_i - 2) - \left(\left(\sum_{k=t}^{m} a_k p^k\right) l_i - 1\right) + \left(\sum_{k=t}^{m-1} a_k p^k\right) l_i \\ \left(\sum_{k=t}^{m-1} a_k p^k\right) l_i \end{bmatrix}_i f_{ap^m l_i - 1} \\
&= \begin{bmatrix} (l_i - 1) + \langle -\lambda_i - 2\rangle_{l_i} l_i - \left(\sum_{k=t}^{m} a_k p^k\right) l_i + 1 + \left(\sum_{k=t}^{m-1} a_k p^k\right) l_i \\ \left(\sum_{k=t}^{m-1} a_k p^k\right) l_i \end{bmatrix}_i f_{ap^m l_i - 1} \\
&= \begin{bmatrix} l_i + \left(\sum_{k=0}^{m} a_k p^k\right) l_i - \left(\sum_{k=t}^{m} a_k p^k\right) l_i + \left(\sum_{k=t}^{m-1} a_k p^k\right) l_i \\ \left(\sum_{k=t}^{m-1} a_k p^k\right) l_i \end{bmatrix}_i f_{ap^m l_i - 1} \\
&= \begin{bmatrix} l_i + \left(\sum_{k=0}^{m-1} a_k p^k\right) l_i \\ \left(\sum_{k=t}^{m-1} a_k p^k\right) l_i \end{bmatrix}_i f_{ap^m l_i - 1} \\
&= \begin{pmatrix} \sum_{k=0}^{m-1} a_k p^k + 1 \\ \sum_{k=t}^{m-1} a_k p^k \end{pmatrix} f_{ap^m l_i - 1} \neq 0 \text{ (because } s < t).
\end{aligned}$$

This shows that $\lambda + ap^m l_i \alpha_i$ is a weight of M.

Using this together with the assumption that M is simple, we get $\mu = \lambda + ap^m l_i \alpha_i$.

(5) If $-(a+1)l_i \leq \lambda_i \leq -al_i - 2$, then $al_i \leq -\lambda_i - 2 \leq (a+1)l_i - 2$. There exist $b \in \mathbb{N}$ and $m \in \mathbb{N}_0$ such that $bp^m \leq a < (b+1)p^m$ with $b < p$. Again, Lemma 4.10 shows that $H^1_{q,i}(\lambda)$ is not simple. Let M be a simple submodule of $H^1_{q,i}(\lambda)$ with highest weight μ. As before, we get from Theorem 4.2 that $\mu \neq s_i \cdot \lambda$. Lemma 4.10

then gives that μ is of the form of

$$\lambda + \left(\sum_{k=t}^{m} a_k p^k\right) l_i \alpha_i$$

for some $t \geq 0$. Similarly, it will be enough to show that $\lambda + bp^m l_i \alpha_i$ is already weight of M. Set $r = (-\lambda_i - 2) - \langle -\lambda_i - 2 \rangle_{l_i} < l_i - 1$, and then

$$\begin{aligned}
&F_i^{\left(\sum_{k=t}^{m-1} a_k p^k\right) l_i} f_{\left(\sum_{k=t}^{m} a_k p^k\right) l_i - 1} \\
&= \begin{bmatrix} (-\lambda_i - 2) - \left(\left(\sum_{k=t}^{m} a_k p^k\right) l_i - 1\right) + \left(\sum_{k=t}^{m-1} a_k p^k\right) l_i f_{ap^m l_i - 1} \\ \left(\sum_{k=t}^{m-1} a_k p^k\right) l_i \end{bmatrix}_i f_{ap^m l_i - 1} \\
&= \begin{bmatrix} r + \left(\sum_{k=0}^{m} a_k p^k\right) l_i - \left(\sum_{k=t}^{m} a_k p^k\right) l_i + 1 + \left(\sum_{k=t}^{m-1} a_k p^k\right) l_i f_{ap^m l_i - 1} \\ \left(\sum_{k=t}^{m-1} a_k p^k\right) l_i \end{bmatrix}_i f_{ap^m l_i - 1} \\
&= \begin{bmatrix} r + \left(\sum_{k=0}^{m-1} a_k p^k\right) l_i + 1 f_{ap^m l_i - 1} \\ \left(\sum_{k=t}^{m-1} a_k p^k\right) l_i \end{bmatrix}_i f_{ap^m l_i - 1} \\
&= \begin{pmatrix} \sum_{k=0}^{m-1} a_k p^k \\ \sum_{k=t}^{m-1} a_k p^k \end{pmatrix} \begin{bmatrix} r+1 \\ 0 \end{bmatrix}_i f_{ap^m l_i - 1} \neq 0.
\end{aligned}$$

Arguing as before, we conclude that $\mu = \lambda + bp^m l_i \alpha_i$, and this finishes the proof. ∎

Remark 4.11. It follows immediately from the proof of the above theorem that $H^1_{q,i}(\lambda)$ is simple with highest weight $s_i \cdot \lambda$ if and only if one of the following conditions is satisfied

(1) $\lambda_i \geq -l_i$,
(2) $\lambda_i \equiv -1 \bmod l_i$ and $\langle -\lambda_i - 2 \rangle_{l_i} < p$,
(3) $\lambda_i \equiv -1 \bmod l_i$ and $\langle -\lambda_i - 2 \rangle_{l_i} = p^m - 1$ for some $m \in \mathbb{N}$.

4.2. The general case

We now return to the case of an arbitrary Cartan matrix. In the following we shall write $H^j_q(n/i, -)$ instead of $H^j_q(U_q/U_{q,i}, -)$.

4.2.1. For any $U_{q,i}$-module M we get from the spectral sequence (3.12)

$$H^j_q(M) \simeq H^j_{q,i}(U_q/U_{q,i}, M) \text{ for all } j \geq 0. \tag{4.5}$$

This follows by observing that for such M we have $H^0_{q,i}(M) \simeq M$ and $H^1_{q,i}(M) = 0$.

When we apply Theorem 4.1 and Theorem 4.2 to the spectral sequence (3.12), it follows easily that

(1) If $\lambda_i \leq -1$ for some $i \in I$, then we have an isomorphism of U_q-modules
$$H^j_q(n/i, H^1_{q,i}(\lambda)) \simeq H^{j+1}_q(k_\lambda) \text{ for all } j \geq 0. \tag{4.6}$$
(2) If $\lambda_i \geq 0$ for some $i \in I$, then we have an isomorphism of U_q-modules
$$H^j_q(n/i, H^0_{q,i}(\lambda)) \simeq H^j_q(\lambda) \text{ for all } j \geq 0. \tag{4.7}$$
(3) If $\lambda_i = -1$ for some $i \in I$, then
$$H^j_q(\lambda) = 0 \text{ for all } j \geq 0. \tag{4.8}$$

Corollary 4.12. *Let $i \in I$. If $H^1_{q,i}(\lambda)$ is simple, then*
$$H^{j+1}_q(\lambda) \simeq H^j_q(s_i \cdot \lambda) \textit{ for all } j \geq 0.$$

Proof. From Remark 4.7 and Remark 4.11, we get that
$$\begin{aligned} H^{j+1}_q(\lambda) &\simeq H^j_q(n/i, H^1_{q,i}(\lambda)) \text{ (see (4.6))} \\ &\simeq H^j_q(n/i, H^0_q(s_i \cdot \lambda)) \\ &\simeq H^j_q(s_i \cdot \lambda) \text{ (see (4.7))}. \end{aligned}$$
The corollary then follows. ■

4.2.2. Let $\lambda \in X$ and suppose that $\lambda_i \leq -1$ for some $i \in I$. We define Ev_i to be the evaluation map
$$Ev_i : H^1_q(\lambda) \simeq H^0_q(n/i, H^1_{q,i}(\lambda)) \to H^1_{q,i}(\lambda)$$
given by
$$Ev_i(f) = f(1) \text{ for all } f \in H^0_q(n/i, H^1_{q,i}(\lambda)).$$
It is a U_i-homomorphism.

Proposition 4.13. *Let $\lambda \in X$ and suppose that $\lambda_i \leq -1$. Then Ev_i restricted to the subspace consisting of U^+_q-invariants*
$$H^1_q(\lambda)^{U^+_q} = \left\{ x \in H^1_q(\lambda) \mid ux = \varepsilon(u)x, u \in U^+_q \right\}$$
is injective.

Proof. Let $\varphi \in H^1_q(\lambda)$ be a non-zero U^+_q-invariant. Choose $x \in U_q$ such that $\varphi(x) \neq 0$. We then get from (3.6) that $x = bu$ for some $b \in B_q$ and $u \in U^+_q$. Hence
$$\varphi(x) = \varphi(bu) = b\varphi(u) = b\varphi(1) \neq 0.$$
Therefore $Ev_i(\varphi) = \varphi(1) \neq 0$. ■

As an easy consequence of the above proposition, it follows that if $H^1_q(\lambda) \neq 0$ for some $\lambda \in X$, then there exists a unique simple root $\alpha_i \in S$ with $\lambda_i = \langle \lambda, \alpha_i^\vee \rangle < -1$. We get the existence of α_i from Kempf's vanishing theorem because if $\lambda \in X^+$, then $H^1_q(\lambda) = 0$. Choose now $i \in I$ such that $\lambda_i \leq -1$. If $Ev_i(H^1_q(\lambda)_\mu) \neq 0$, then μ has to be a weight of $H^1_{q,i}(\lambda)$. Therefore the uniqueness of α_i follows easily from Proposition 4.13 and Theorem 4.2.

4.2.3. Characteristic zero. Let k be an arbitrary field of characteristic 0.

Theorem 4.14. *Suppose that $H^1_q(\lambda) \neq 0$ for some $\lambda \in X$ and let $i \in I$ such that $\lambda_i < -1$. Then $H^1_q(\lambda)$ contains a unique simple submodule M. The highest weight μ of M is*

$$\mu = \begin{cases} s_i \cdot \lambda & \text{if } \lambda_i \geq -l_i \text{ or } \lambda_i \equiv -1 \bmod l_i, \\ \lambda + a l_i \alpha_i & \text{if } -(a+1)l_i \leq \lambda_i \leq -a l_i - 2 \text{ for some } a \in \mathbb{N}. \end{cases}$$

Proof. The uniqueness of M follows immediately from Theorem 4.6 and Proposition 4.13.

Let M be a simple submodule of $H^1_q(\lambda)$, and pick a maximal weight μ of M. This is possible because $H^1_q(\lambda)$ is finite dimensional, see Corollary 3.7. Let $\varphi \in M_\mu$ be non-zero. Proposition 4.13 then implies that $Ev_i(\varphi)$ is a non-zero $U^+_{q,i}$-invariant. Using this together with Theorem 4.6, we get that

$$\mu = \begin{cases} s_i \cdot \lambda & \text{if } \lambda_i \geq -l_i \text{ or } \lambda_i \equiv -1 \bmod l_i, \\ \lambda + a l_i \alpha_i \text{ or } s_i \cdot \lambda & \text{if } -(a+1)l_i \leq \lambda_i \leq -a l_i - 2 \text{ for some } a \in \mathbb{N}. \end{cases}$$

Suppose that $-(a+1)l_i \leq \lambda_i \leq -a l_i - 2$ for some $a \in \mathbb{N}$. We want to prove that $\mu = \lambda + a l_i \alpha_i$. We prove this by contradiction. Assume that $\mu = s_i \cdot \lambda$ and set $\nu = \lambda + a l_i \alpha_i$.

First, we show that ν is already a weight of M. Let $r \in \mathbb{N}$ such that $r\alpha_i = \mu - \nu$. We get from Theorem 4.2 that

$$Ev_i\left(F_i^{(r)}\varphi\right) = F_i^{(r)} Ev_i(\varphi) \neq 0 \Rightarrow F_i^{(r)}\varphi \in M_\nu.$$

Next, let $\psi \in M_\nu$ be non-zero. We claim that $E_i^{(r)}\psi = 0$. Since ν is a maximal weight of $H^1_{q,i}(\lambda)$, we see that

$$Ev_i(E_i^{(r)}\psi) = E_i^{(r)} Ev_i(\psi) = 0.$$

This implies that $E_i^{(r)}\psi = 0$.

Finally, set $N = U_q\psi$. This is a U_q-submodule of M. Since M is simple, we have that $M = N$. But this is impossible. To see this, it suffices to show that μ is not a weight of N. Assume by contradiction that μ is a weight of N. Hence there exists $x \in U_q$ such that $x\psi \in N_\mu$. We get from the triangular decomposition (3.6) of U_q that $x = bu$ for some $b \in B_q$ and $u \in U_q^+$. Note that the weight of $u\psi$ has the form

$$\nu + \sum_{j \in I} m_j \alpha_j.$$

Therefore, if $u\psi \neq 0$, then $u = E_i^{(r)}$. But $E_i^{(r)}\varphi = 0$.

In this case we conclude that $H^1_q(\lambda)$ contains a unique simple submodule with highest weight $\lambda + a l_i \alpha_i$. ∎

We shall now describe the vanishing behaviour of $H_q^1(\lambda)$.

Theorem 4.15. *$H_q^1(\lambda) \neq 0$ if and only if there exists $\alpha_i \in S$ such that one of the following conditions is satisfied*

(1) $-l_i \leq \lambda_i \leq -2$ and $s_i \cdot \lambda$ is dominant,
(2) $\lambda_i \equiv -1 \bmod l_i$ and $s_i \cdot \lambda$ is dominant,
(3) $-(a+1)l_i \leq \lambda_i \leq -al_i - 2$ for some $a \in \mathbb{N}$ and $\lambda + al_i\alpha_i$ is dominant.

Proof. It follows immediately from Theorem 4.14 that if $H_q^1(\lambda) \neq 0$, then there exists a unique simple root α_i such that one of the above conditions is satisfied.

Conversely, if $-l_i \leq \lambda_i \leq -2$ or $\lambda_i \equiv -1 \bmod l_i$, then we get from Corollary 4.12 that

$$H_q^1(\lambda) \simeq H_q^0(s_i \cdot \lambda) \neq 0$$

because $s_i \cdot \lambda$ is dominant.

Now, suppose that $-(a+1)l_i \leq \lambda_i \leq -al_i - 2$ for some $a \in \mathbb{N}$ and $\lambda + al_i\alpha_i$ is dominant. By Theorem 4.6, we know that $H_{q,i}^1(\lambda)$ contains the unique simple submodule $L_{q,i}(\lambda + al_i\alpha_i)$. So

$$\begin{aligned} H_q^0(L_{q,i}(\lambda + al_i\alpha_i)) &\simeq H_q^0(n/i, L_{q,i}(\lambda + al_i\alpha_i)) \ (4.5) \\ &\subset H_q^0(n/i, H_{q,i}^1(\lambda)) \\ &\simeq H_q^1(\lambda). \end{aligned}$$

Then it is enough to prove that $H_q^0(L_{q,i}(\lambda + al_i\alpha_i)) \neq 0$. For this we need the following exact sequence

$$0 \to K \to L_{q,i}(\lambda + al_i\alpha_i) \to k_{\lambda + al_i\alpha_i} \to 0.$$

Here the map $L_{q,i}(\lambda + al_i\alpha_i) \to k_{\lambda+al_i\alpha_i}$ is the projection map which is clearly a B_q-homomorphism.

By assumption, we have that $H_q^0(\lambda + al_i\alpha_i)$ is non-zero. Therefore we are done if we can prove that $\lambda + al_i\alpha_i$ is not a weight of $H_q^1(K)$. By construction, we see that the weights of K have the form $\lambda + al_i\alpha_i - j\alpha_i$ where $j \in \{1, \ldots, \lambda_i + 2al_i\}$. The long exact sequences arising from taking full B_q-filtrations of K imply that if $\lambda + al_i\alpha_i$ were a weight of $H_q^1(K)$, then it would also be a weight of $H_q^1(\lambda + al_i\alpha_i - j\alpha_i)$ for some $j \in \{1, \ldots, \lambda_i + 2al_i\}$. But this is impossible:

(1) If $\langle \lambda + al_i\alpha_i - j\alpha_i, \alpha_i^\vee \rangle \geq 0$, then $\lambda + al_i\alpha_i - j\alpha_i$ is dominant. By Kempf's vanishing theorem, we get that $H_q^1(\lambda + al_i\alpha_i - j\alpha_i) = 0$.
(2) If $\langle \lambda + al_i\alpha_i - j\alpha_i, \alpha_i^\vee \rangle < 0$, then it follows immediately from Proposition 4.13 that the highest weight of $H_q^1(\lambda + al_i\alpha_i - j\alpha_i)$ is not bigger than $s_i \cdot (\lambda + al_i\alpha_i - j\alpha_i)$.

$$\begin{aligned} s_i \cdot (\lambda + al_i\alpha_i - j\alpha_i) &= s_i(\lambda + al_i\alpha_i - j\alpha_i) - \alpha_i \\ &= \lambda + al_i\alpha_i - j\alpha_i - \left\langle \lambda + al_i\alpha_i - j\alpha_i, \alpha_i^\vee \right\rangle \alpha_i - \alpha_i. \end{aligned}$$

Hence we have that $s_i \cdot (\lambda + al_i\alpha_i - j\alpha_i)$ is strictly smaller than $\lambda + al_i\alpha_i$.

The theorem follows. ■

Remark 4.16. In case 1 and 2 we have $H^1_q(\lambda) \simeq H^0_q(s_i \cdot \lambda)$.

Example 4.17. *Suppose $l = l_i$ for all $i \in I$. As in the modular case , the figure below illustrates the vanishing behaviour of H^1_q for groups of type B_2.*

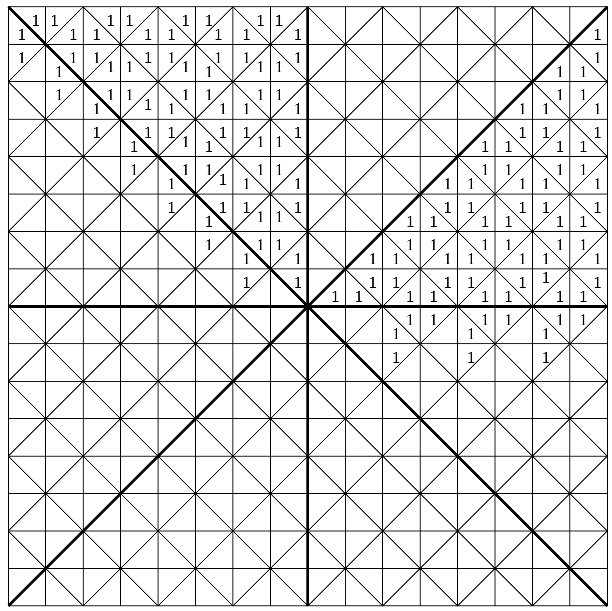

FIGURE 1. The vanishing behaviour of $H^1_q(\lambda)$ for groups of B_2.

4.2.4. Characteristic p>0. Let k be an arbitrary field of characteristic $p > 0$.

Theorem 4.18. *Suppose that $H^1_q(\lambda) \neq 0$ for some $\lambda \in X$ and let $i \in I$ such that $\lambda_i < -1$. Then $H^1_q(\lambda)$ contains a unique simple submodule M. The highest weight μ of M is*

$$\mu = \begin{cases} s_i \cdot \lambda & \text{if } \lambda_i \geq -l_i, \\ s_i \cdot \lambda & \text{if } \lambda_i \equiv -1 \operatorname{mod} l_i \text{ and } \langle -\lambda_i - 2 \rangle_{l_i} < p, \\ s_i \cdot \lambda & \text{if } \lambda_i \equiv -1 \operatorname{mod} l_i \text{ and } \langle -\lambda_i - 2 \rangle_{l_i} = p^m - 1 \text{ for some } m \in \mathbb{N}, \\ \lambda + ap^m l_i \alpha_i & \text{if } \lambda_i \equiv -1 \operatorname{mod} l_i \text{ and } ap^m \leq \langle -\lambda_i - 2 \rangle_{l_i} \leq (a+1)p^m - 2 \\ & \text{for some } a, m \in \mathbb{N} \text{ with } a < p, \\ \lambda + bp^m l_i \alpha_i & \text{if } -(a+1)l_i \leq \lambda_i \leq -al_i - 2 \text{ for some } a \in \mathbb{N} \text{ and} \\ & bp^m \leq a < (b+1)p^m \text{ where } m \geq 0 \text{ and } b \in \mathbb{N} \text{ with } b < p. \end{cases}$$

Proof. The proof is very similar to the one given in Theorem 4.14. ■

Theorem 4.19. *$H_q^1(\lambda) \neq 0$ if and only if there exists $\alpha_i \in S$ such that one of the following conditions is satisfied*

(1) $-l_i \leq \lambda_i \leq -2$ and $s_i \cdot \lambda$ is dominant,
(2) $\lambda_i \equiv -1 \bmod l_i$ with $\langle -\lambda_i - 2 \rangle_{l_i} < p$ and $s_i \cdot \lambda$ is dominant,
(3) $\lambda_i \equiv -1 \bmod l_i$ with $\langle -\lambda_i - 2 \rangle_{l_i} = p^m - 1$ for some $m \in \mathbb{N}$ and $s_i \cdot \lambda$ is dominant,
(4) $\lambda_i \equiv -1 \bmod l_i$ with $ap^m \leq \langle -\lambda_i - 2 \rangle_{l_i} \leq (a+1)p^m - 2$ for some $a, m \in \mathbb{N}$ with $a < p$ and $\lambda + ap^m l_i \alpha_i$ is dominant,
(5) $-(a+1)l_i \leq \lambda_i \leq -al_i - 2$ for some $a \in \mathbb{N}$ such that $bp^m \leq a < (b+1)p^m$ where $m \geq 0, 0 < b < p$ and $\lambda + bp^m l_i \alpha_i$ is dominant.

Proof. Again, we omit the details. We refer to Theorem 4.15. ■

Remark 4.20. In case 1, 2, and 3 we have $H_q^1(\lambda) \simeq H_q^0(s_i \cdot \lambda)$.

CHAPTER 5

Vanishing behaviour

The exact vanishing behaviour of the cohomology group $H_q^\bullet(\lambda)$ is still not known. In this chapter we shall summarize what we know in general.

5.1. The Frobenius twist

In this section we introduce the quantum Frobenius homomorphism. First, we make a brief review of some basic constructions in [31].

5.1.1. Root datum. Fix an integer $l \geq 1$ and let l' be l or $2l$ if l is odd, and $2l$ if l is even.

Definition 5.1. A Cartan datum is a pair $(I, \cdot)$ consisting of a finite set with a symmetric bilinear $\mathbb{Z}$-valued form $i, j \mapsto i \cdot j$ on the free abelian group $\mathbb{Z}[I]$ with I as basis. It is assumed that

(1) $i \cdot i \in \{2, 4, \ldots\}$ for all $i \in I$,
(2) $2\frac{i \cdot j}{i \cdot i} \in \{0, -1, -2, \ldots\}$ for all $i \neq j$ in I.

If $(I, \cdot)$ is a Cartan datum, we can define a new Cartan datum $(I, \circ)$ with

$$i \circ j = l_i l_j (i \cdot j) \text{ for } i, j \in I.$$

Here l_i denotes the smallest integer ≥ 1 such that

$$l_i (i \cdot i)/2 \in l\mathbb{Z}.$$

Note that l_i divides l.

Definition 5.2. A root datum is a quadruple $((I, \cdot), X, Y, \langle , \rangle)$ consisting of

(1) A Cartan datum $(I, \cdot)$.
(2) Two finitely generated free abelian groups X, Y with a perfect bilinear pairing
$$\langle , \rangle : Y \times X \to \mathbb{Z}.$$
(3) An embedding $I \subset X$ $(i \mapsto i')$ and an embedding $I \subset Y$ $(i \mapsto i)$ such that
$$\langle i, j' \rangle = 2\frac{i \cdot j}{i \cdot i} \text{ for all } i, j \in I.$$

Given a root datum $((I, \cdot), X, Y, \langle , \rangle)$, we define a new one $((I, \circ), X^*, Y^*, \langle , \rangle^*)$ as follows: We set

$$X^* = \{\zeta \in X \,|\langle i, \zeta \rangle \in l_i\mathbb{Z} \text{ for all } i \in I\} \subset X,$$
$$Y^* = \mathrm{Hom}_{\mathbb{Z}}(X^*, \mathbb{Z}).$$

The pairing $Y^* \times X^* \to \mathbb{Z}$ is given by the evaluation map. We include I in X^* by $i \mapsto i'^* = l_i i'$, and we include I in Y^* by sending each i to the element $i^* \in Y^*$ whose value at each $\zeta \in X^*$ is given by $\langle i, \zeta \rangle / l_i$.

5.1.2. The algebras U_q and U_q^*. Let (a_{ij}) be the Cartan matrix of our root system R of rank n and set $I = \{1, \dots n\}$. Choose $d_i \in \mathbb{N}$ minimal such that

$$d_i a_{ij} = d_j a_{ji} \text{ for all } i, j \in I.$$

Throughout, we shall restrict ourselves to the adjoint root datum. More precisely, take $X = \mathbb{Z}[I]$ with the obvious embedding $I \to X$ and let $Y = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$. For any $i, j \in I$ we let $i \cdot j = d_i a_{ij}$, and the pairing $\langle , \rangle : Y \times X \to \mathbb{Z}$ is given by the evaluation map. We include I in Y by sending i to an element in Y whose value at each $j \in X$ is given by $2\frac{i \cdot j}{i \cdot i}$. With these choices, there is a canonical monomorphism $\psi : X \to X^*$ sending i' to i'^*.

Remark 5.3. Clearly, the corresponding Cartan matrices $(\langle i, j' \rangle)$ and $(\langle i^*, j'^* \rangle)$ are equal when $l = l_i$ for all $i \in I$.

Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ be the Laurent ring of polynomials over $\mathbb{Z}$, and let $\mathcal{A}'$ be the quotient ring of $\mathcal{A}$ by the ideal generated by the l'-th cyclotomic polynomial. Note $v^2 \in \mathcal{A}'$ has order l.

In [31] Lusztig defined a quantum group associated to each root datum, i.e. the associative $\mathbb{Q}(v)$-algebra with 1 and generated by all E_i, F_i and $K_i^{\pm}$ $(i \in I)$ together with a list of relations. The quantum group associated to the adjoint root datum $((I, \cdot), X, Y, \langle , \rangle)$ is denoted by U_v, and $U_{\mathcal{A}}$ will be its $\mathcal{A}$-form.

Let k be an arbitrary field and $q \in k^{\times}$. As in Section 3.1, k becomes an $\mathcal{A}$-algebra by specializing v to q. By tensoring the $\mathcal{A}$-form with k, we obtain a new algebra U_q which is a Hopf algebra. In addition to being a field, we shall also assume that k is an $\mathcal{A}'$-algebra, i.e. l is the order of q^2. This new algebra is in fact the one we defined in Chapter 3, and we will therefore change our notation a bit, and adopt the notation of the previous chapters.

The quantum group associated to $((I, \cdot), X^*, Y^*, \langle , \rangle^*)$ will be denoted by U_q^*. We have similar notations attached to U_q^*. The corresponding Borel subalgebra will be denoted by B_q^*, and $\mathscr{C}_q^*$ will be the category of integrable U_q^*-modules.

5.1.3. The algebras $\dot{U}_q$ and $\dot{U}_q^*$. To introduce the quantum Frobenius homomorphism we have to work over a modification of the quantum group U_v.

Let $\lambda, \lambda' \in X$ and consider the projections

$$\pi_{\lambda, \lambda'} : U_v \mapsto U_v \bigg/ \left(\sum_{\mu \in Y} \left(K_\mu - v^{\langle \mu, \lambda \rangle} \right) U_v + \sum_{\mu \in Y} \left(K_\mu - v^{\langle \mu, \lambda' \rangle} \right) U_v \right).$$

Then we set

$$\dot{U}_v = \bigoplus_{\lambda, \lambda' \in X} \pi_{\lambda, \lambda'}(U_v).$$

This is an associative $\mathbb{Q}(v)$-algebra without 1, and $1_\lambda = \pi_{\lambda,\lambda}(1) \in \dot{U}_v$ ($\lambda \in X$) satisfy

$$1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'} 1_\lambda \text{ and } \pi_{\lambda,\lambda'}(U_v) = 1_\lambda \dot{U}_v 1_{\lambda'}.$$

We omit the details which can be found in [31, Chapter 23].

In $\dot{U}_v$ there is an $\mathcal{A}$-form $\dot{U}_{\mathcal{A}}$ that comes from the $\mathcal{A}$-form of U_v, see [31, 23.2]. As usual, when we tensor the $\mathcal{A}$-form with k, we obtain a new algebra $\dot{U}_q$ over k.

5.1.4. A $\dot{U}_q$-module is said to be unital if for any $m \in M$ we have $1_\lambda m = 0$ for all but finitely many $\lambda \in X$ and

$$\sum_{\lambda \in X} 1_\lambda m = m.$$

A unital $\dot{U}_q$-module is integrable if for all $m \in M$ we have that

$$E_i^{(r)} m = F_i^{(r)} m = 0 \text{ for all } i \in I \text{ and } r \gg 0.$$

We let $\dot{\mathscr{C}}_q$ be the category of integrable $\dot{U}_q$-modules.

Proposition 5.4 ([31, 23.1.4 and 31.1.6-7])**.** *The categories $\mathscr{C}_q$ and $\dot{\mathscr{C}}_q$ are equivalent.*

The quantum group associated to $((I,\cdot), X^*, Y^*, \langle,\rangle^*)$ will be denoted by $\dot{U}_v^*$. We of course have similar notations attached to $\dot{U}_v$ and $\dot{U}_v^*$. We let $\dot{\mathscr{C}}_q$ be the category of integrable $\dot{U}_q$-modules and $\dot{\mathscr{C}}_q^*$ the category of integrable $\dot{U}_q^*$-modules.

5.1.5. Let $\bar{U}$ be the specialisation at k of the Kostant $\mathbb{Z}$-form of the enveloping algebra of the Lie algebra for the semisimple algebraic $\overline{k}$-group $\bar{G}$ corresponding to the Cartan matrix $(\langle i^*, j'^* \rangle)$. Moreover, we take $\bar{G}$ to be defined and split over k. The category of locally finite $\bar{U}$-modules identifies with the category of rational $\bar{G}$-modules. We shall also need the category of locally finite $\bar{B}$-modules $\bar{\mathscr{C}}(\bar{B})$ where $\bar{B}$ is the Borel subgroup of $\bar{G}$.

Proposition 5.5. *We have an isomorphism of categories $i : \dot{\mathscr{C}}_q^* \to \overline{\mathscr{C}}$.*

For a proof we refer to [10].

5.1.6. The quantum Frobenius homomorphism. We are now able to introduce the quantum Frobenius homomorphism.

Theorem 5.6 ([31, 35.1.9])**.** *There is a unique k-algebra homomorphism* $\mathrm{Fr} : \dot{U}_q \to \dot{U}_q^*$ *given by*

$$\mathrm{Fr}\left(E_i^{(r)} 1_\zeta\right) = \begin{cases} E_i^{(r/l_i)} 1_\zeta & \text{if } r \in l_i\mathbb{Z} \text{ and } \zeta \in X^*, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{Fr}\left(F_i^{(r)} 1_\zeta\right) = \begin{cases} F_i^{(r/l_i)} 1_\zeta & \text{If } r \in l_i\mathbb{Z} \text{ and } \zeta \in X^*, \\ 0 & \text{otherwise,} \end{cases}$$

for all $i \in I, \zeta \in X$ and $r \in \mathbb{N}$.

When we compose this with the canonical monomorphism $\psi : X \to X^*$, we get an algebra homomorphism $\mathrm{Fr}_l : \dot{U}_q \to \dot{U}_q^*$ given by

$$\mathrm{Fr}_l\left(E_i^{(r)} 1_\zeta\right) = \mathrm{Fr}\left(E_i^{(r)} 1_{\psi(\zeta)}\right) \text{ and } \mathrm{Fr}_l\left(F_i^{(r)} 1_\zeta\right) = \mathrm{Fr}\left(F_i^{(r)} 1_{\psi(\zeta)}\right)$$

for all $i \in I, r \geq 0$ and $\zeta \in X$, cf. [31, 23.2.5].

5.1.7. Given an integrable B_q^*-module M, resp. $\dot{U}_q^*$-module. We use the quantum Frobenius homomorphism Fr_l to make M into an integrable $\dot{B}_q$, resp. $\dot{U}_q$-module, and hence into an integrable B_q-module, resp. U_q, that we denote by $M^{(1)}$.

Proposition 5.7. *For any $M \in \bar{\mathscr{C}}(\bar{B})$ we have a natural isomorphism of U_q-modules*

$$H_q^0\left(U_q/B_q u_q^+, (i^{-1}M)^{(1)}\right) \simeq \left(i^{-1} H^0\left(M\right)\right)^{(1)}.$$

Proof. Note that $\mathrm{Fr}_l(\dot{B}_q \dot{u}_q^+) \subset \dot{B}_q^*$. We can therefore regard $(i^{-1}M)^{(1)}$ as an integrable $B_q u_q^+$-module.

We first prove that

$$\mathrm{Hom}_{\dot{B}_q^*}\left(\dot{U}_q^*, i^{-1}M\right) \simeq \mathrm{Hom}_{\dot{B}_q \dot{u}_q^+}\left(\dot{U}_q, \left(i^{-1}M\right)^{(1)}\right) \text{ (as vector spaces).}$$

So let φ be in the left side, and we need to prove that $\varphi \circ \mathrm{Fr}$ is in the right side. For any $x \in \dot{U}_q$ and $u \in \dot{B}_q \dot{u}_q^+$ we have

$$\begin{aligned} (\varphi \circ \mathrm{Fr}_l)(ux) &= \varphi\left(\mathrm{Fr}_l(u)\, \mathrm{Fr}_l(x)\right) \\ &= \mathrm{Fr}_l(u)\varphi\left(\mathrm{Fr}_l(x)\right) \\ &= u\varphi\left(\mathrm{Fr}_l(x)\right). \end{aligned}$$

Hence we have a homomorphism of vector spaces

$$\begin{aligned} \phi : \mathrm{Hom}_{\dot{B}_q^*}\left(\dot{U}_q^*, i^{-1}M\right) &\to \mathrm{Hom}_{\dot{B}_q \dot{u}_q^+}\left(\dot{U}_q, \left(i^{-1}M\right)^{(1)}\right) \\ \varphi &\mapsto \varphi \circ \mathrm{Fr}_l\,. \end{aligned}$$

Conversely, let φ be in the right side and let $\mathfrak{I}$ be the kernel of Fr_l . Then Fr_l induces an isomorphism F

$$\dot{U}_q \xrightarrow{\pi} \dot{U}_q/\mathfrak{I} \xrightarrow{\mathrm{F}} \dot{U}_q^*.$$

For any $x \in \dot{U}_q^*$ we define

$$\phi'(\varphi)(x) = \varphi(y)$$

where $\pi(y) = \mathrm{F}^{-1}(x)$. Since $\mathfrak{I}$ annihilates $(i^{-1}M)^{(1)}$, it follows that $\phi'(\varphi)$ is well defined.

We now want to prove that $\phi'(\varphi)$ is in the left side. So let $x \in \dot{U}_q^*$ and let $u \in \dot{B}_q^*$

$$\begin{aligned}\phi'(\varphi)(ux) &= \varphi\left(u_1 x_1\right) \text{ (where } \pi(u_1) = \mathrm{F}^{-1}(u) \text{ and } \pi(x_1) = \mathrm{F}^{-1}(x)) \\ &= u_1\varphi(x_1) \\ &= \mathrm{Fr}_l\left(u_1\right)\varphi(x_1) \\ &= u\phi'(\varphi)(x).\end{aligned}$$

Clearly, we get an isomorphism of vector spaces

$$\phi : \left(i^{-1}H^0\left(M\right)\right)^{(1)} \to H_q^0\left(U_q/B_q u_q^+, (i^{-1}M)^{(1)}\right).$$

It remains to show that ϕ is a homomorphism of U_q-modules. Let $x, y \in U_q$

$$\begin{aligned}\left(y\phi(\varphi)\right)(x) &= \phi(\varphi)(xy) \\ &= \varphi\left(\mathrm{Fr}_l(xy)\right) \\ &= \varphi\left(\mathrm{Fr}_l(x)\,\mathrm{Fr}_l(y)\right) \\ &= \left(\mathrm{Fr}_l(y)\varphi\right)\left(\mathrm{Fr}_l(x)\right) \\ &= \left(y\varphi\right)\left(\mathrm{Fr}_l(x)\right) \\ &= \phi(y\varphi)(x).\end{aligned}$$

This finishes the proof. ■

5.2. The vanishing behaviour

In this section we prove the quantum version of the main theorem in [1] and then derive some of its consequences. We denote by X_l the set of restricted weights

$$X_l = \left\{\lambda \in X \mid 0 \le \left\langle \lambda, \alpha_i^\vee \right\rangle < l_i \text{ for all } \alpha_i \in S\right\}.$$

Recall the canonical monomorphism $\psi : X \to X^*$. Each $\lambda \in X$ can be decomposed uniquely $\lambda = \lambda_1 + \psi(\lambda_2)$ where $\lambda_1 \in X_l$ and $\lambda_2 \in X$.

5.2.1. First, we define the category $\mathscr{C}_q(u_q)$ to be the category of u_q-modules M such that

$$M = \bigoplus_{\lambda \in X_l} M_\lambda$$

where

$$M_\lambda = \{m \in M \mid um = \chi_\lambda(u)m \text{ for all } u \in u_q^0\}.$$

Here χ_λ is the restriction to u_q^0 of the k-algebra homomorphism χ_λ given in Section 3.1. We also denote by F the functor which takes any u_q-modules M to

$$F(M) = \bigoplus_{\lambda \in X_l} M_\lambda.$$

For more details we refer to [12]. Similarly, we define the induction functor H_q^0 from u_q as we did in Chapter 3.

5.2.2. Let $\hat{u}_q = B_q u_q^+$ and let

$$\hat{Z}_q = H_q^0(\hat{u}_q/B_q, -).$$

The functor $\hat{Z}_q$ is exact, see Proposition B.10. For each $\lambda \in X$ we have that $\hat{Z}_q(k_\lambda)$ contains a unique simple submodule of highest weight λ. We denote this submodule by $\hat{L}_q(\lambda)$, see Proposition B.1. For any $\lambda \in X$ we shall write $\hat{Z}_q(\lambda)$ instead of $\hat{Z}_q(k_\lambda)$.

Proposition 5.8 ([10, Proposition 3.15])**.** *For any $M \in \bar{\mathscr{C}}(\bar{B})$ we have a natural isomorphism of U_q-modules*

$$H_q^j\left(U_q/\hat{u}_q, (i^{-1}M)^{(1)}\right) \simeq \left(i^{-1}H^j\left(M\right)\right)^{(1)} \text{ for all } j \geq 0.$$

Proof. The proposition holds for $j = 0$. By a degree shift argument, it is enough to show that if I is an injective locally finite $\bar{B}$-module, then $H_q^j\left(U_q/\hat{u}_q, (i^{-1}I)^{(1)}\right) = 0$ for $j > 0$. Here we may quickly reduce to the case $I = k[\bar{B}]$, the coordinate ring of $\bar{B}$, cf. [22, I. 3.9].

As we did in Proposition 5.7, we can show that

$$\left(i^{-1}k[\bar{B}]\right)^{(1)} \simeq H_q^0\left(\hat{u}_q/u_q, k\right).$$

Corollary B.16 then implies

$$\begin{aligned} H_q^j\left(U_q/\hat{u}_q, \left(i^{-1}k[\bar{B}]\right)^{(1)}\right) &\simeq H_q^j\left(U_q/\hat{u}_q, H_q^0\left(\hat{u}_q/u_q, k\right)\right) \\ &\simeq H_q^j\left(U_q/u_q, k\right) \\ &= 0 \text{ for all } j > 0. \end{aligned}$$

This finishes the proof. ■

5.2.3. Set $\sigma_l = (1/2)\sum_{\alpha \in R^+}(l_\alpha - 1)\alpha$. We call this the Steinberg weight. The corresponding simple U_q-module $L_q(\sigma_l)$ is called the Steinberg module and denoted by St_l, for details we refer to Appendix B.

Theorem 5.9. *For any $M \in \bar{\mathscr{C}}(\bar{B})$ we have a natural isomorphism of U_q-modules*

$$H_q^j\left((i^{-1}M)^{(1)} \otimes_k \sigma_l\right) \simeq \left(i^{-1}H^j\left(M\right)\right)^{(1)} \otimes_k \mathrm{St}_l \text{ for all } j \geq 0.$$

Proof. As a $\hat{u}_q$-module, we get from Corollary B.7 that St_l is isomorphic to $\hat{Z}_q(\sigma_l)$. Since the functor $\hat{Z}_q$ is exact, then we have for all $j \geq 0$

$$\begin{aligned} \left(i^{-1}H^j\left(M\right)\right)^{(1)} \otimes_k \mathrm{St}_l &\simeq H_q^j\left(U_q/\hat{u}_q, (i^{-1}M)^{(1)}\right) \otimes_k \mathrm{St}_l \\ &\simeq H_q^j\left(U_q/\hat{u}_q, (i^{-1}M)^{(1)} \otimes_k \mathrm{St}_l\right) \\ &\simeq H_q^j\left(U_q/\hat{u}_q, (i^{-1}M)^{(1)} \otimes_k \hat{Z}_q(\sigma_l)\right) \\ &\simeq H_q^j\left(U_q/\hat{u}_q, \hat{Z}_q\left((i^{-1}M)^{(1)} \otimes_k \sigma_l\right)\right) \\ &\simeq H_q^j\left((i^{-1}M)^{(1)} \otimes_k \sigma_l\right). \end{aligned}$$

The theorem follows. ■

5.2.4. Let $M_1, M_2 \in \mathscr{C}_q^-$. As an easy consequence of Frobenius reciprocity, the evaluation maps $H_q^0(M_1) \to M_1$ and $H_q^0(M_2) \to M_2$ give a homomorphism

$$H_q^0(M_1) \otimes_k H_q^0(M_2) \to H_q^0(M_1 \otimes_k M_2)$$

which is functorial in both M_1 and M_2. By a simple induction on $s+t$, we then obtain a natural homomorphism (the cup-product)

$$\cup_{s,t} : H_q^s(M_1) \otimes_k H_q^t(M_2) \to H_q^{s+t}(M_1 \otimes_k M_2).$$

The Frobenius homomorphism Fr_l clearly gives rise to a natural homomorphism

$$\mathrm{Fr}_l^* : \left(i^{-1} H^t\left(M_1\right)\right)^{(1)} \to H_q^t\left((i^{-1} M_1)^{(1)}\right) \text{ for all } t \geq 0.$$

Set

$$\begin{aligned} D_0(j) &= \bigcup_{w \in W \,:\, l(w)=j} w \cdot X^+, \\ D^p(j) &= \left\{\lambda \in X \,\middle|\, H^j(\lambda) \neq 0\right\}, \\ D_l^p(j) &= \left\{\lambda \in X \,\middle|\, H_q^j(\lambda) \neq 0\right\}, \end{aligned}$$

and

$$E_l^p(j) = D_0(j) \cup \left(\psi\left(D^p(j)\right) \pm X_l\right).$$

Proposition 5.10. *For any j we have*

$$\psi\left(D^p(j)\right) + X_l \subseteq D_l^p(j).$$

Proof. Suppose that $H^j(\lambda)$ is non-zero for some $\lambda \in X$ and $j \in \mathbb{N}$. We shall show that so is $H_q^j(\psi(\lambda)+\mu)$ for all $\mu \in X_l$. Since the Steinberg module St_l is simple, the cup-product

$$H_q^0(\mu) \otimes_k H_q^0(\sigma_l - \mu) \to H_q^0(\sigma_l) = \mathrm{St}_l$$

is surjective. We then have the following commutative diagram of U_q-modules

$$\begin{array}{ccc} H_q^j(\psi(\lambda)) \otimes_k H_q^0(\mu) \otimes_k H_q^0(\sigma_l - \mu) & \longrightarrow & H_q^j(\psi(\lambda)+\mu) \otimes_k H_q^0(\sigma_l - \mu) \\ \downarrow & & \downarrow \\ H_q^j(\psi(\lambda)) \otimes_k \mathrm{St}_l & \longrightarrow & H_q^j(\psi(\lambda)+\sigma_l) \simeq (i^{-1}H^j(\lambda))^{(1)} \otimes_k \mathrm{St}_l \end{array}$$

Using the above corollary, the bottom horizontal homomorphism is surjective. Then so is the right vertical homomorphism. The proposition follows. ■

By Serre duality, we have

Corollary 5.11. *For any $j \geq 0$ we have*

$$\psi\left(D^p(j)\right) - X_l \subseteq D_l^p(j).$$

Proof. Suppose that $\lambda \in D^p(j)$, then

$$\begin{aligned} -\lambda - 2\rho \in D^p(N-j) &\Rightarrow \psi(-\lambda-2\rho) + X_l \subseteq D_l^p(N-j) \\ &\Rightarrow \psi(\lambda+2\rho) - X_l - 2\rho \subseteq D_l^p(j) \\ &\Rightarrow \psi(\lambda) - X_l + 2\left(\psi(\rho) - \rho\right) \subseteq D_l^p(j) \\ &\Rightarrow \psi(\lambda) - X_l + 2\sigma_l \subseteq D_l^p(j) \\ &\Rightarrow \psi(\lambda) - X_l \subseteq D_l^p(j) \end{aligned}$$

This completes the proof. ■

Theorem 5.12. *For any $j \geq 0$ we have*

$$E_l^p(j) \subseteq D_l^p(j).$$

Proof. Use Theorem A.6, Proposition 5.10 and Corollary 5.11. ■

5.2.5. Now, if k is a field of characteristic 0, we have $D^0(j) = D_0(j)$. Hence

$$D_0(j) \cup \left(\psi\left(D_0(j)\right) \pm X_l\right) \subseteq D_l^0(j).$$

If k is a field of characteristic $p > 0$, we have

$$\bigcup_{m \geq 0} p^m \cdot D_0(j) \pm X_{p^m} \subseteq D_l^p(j),$$

cf. [3, Corollary 3.4]. Here the "dot action" of p^m on X is given by

$$p^m \cdot \lambda = p^m(\lambda + \rho) - \rho.$$

Hence

$$D_0(j) \cup \left(\psi\left(\bigcup_{m\geq 0} p^m \cdot D_0(j) \pm X_{p^m}\right) \pm X_l\right) \cup \left(\psi\left(\bigcup_{m\geq 0} p^m \cdot D_0(j) \mp X_{p^m}\right) \pm X_l\right) \subseteq D_l^p(j).$$

The exact vanishing behaviour of H_q^j is still not known, but there are a few cases where we can completely describe the subset $D_l^p(i)$: We have that

$$D_l(0) = X^+ = E_l^p(0).$$

When we combine Theorem 4.15 and Theorem 4.19, we get that

$$E_l^p(1) = D_l^p(1).$$

Using Serre duality, we further have that

$$D_l(N) = -D_l(0) - 2\rho = E_l^p(N) \text{ and } D_l(N-1) = -D_l(1) - 2\rho = E_l^p(N-1).$$

The equality in Theorem 5.12 does not hold in general. It already fails for type B_2 and $j = 2$. The argument given in the modular case [7] will also work in the quantum case.

5.3. Generic weights

We assume that q is a root of unity and $l = l_i$ for all $i \in I$, hence the Steinberg weight $\sigma_l = (l-1)\rho$.

5.3.1. If the root system R is indecomposable, we let α_0 be the highest short root. We then define h to be the Coxeter number $h = \langle \rho, \alpha_0^\vee \rangle + 1$. In general we let h be the maximum of the Coxeter numbers of the indecomposable components of R.

Let C_l denote the bottom alcove in X^+

$$C_l = \left\{ \lambda \in X \mid 0 < \left\langle \lambda + \rho, \alpha^\vee \right\rangle < l \text{ for all } \alpha \in R^+ \right\},$$

and its closure

$$\overline{C}_l = \left\{ \lambda \in X \mid 0 \leq \left\langle \lambda + \rho, \alpha^\vee \right\rangle \leq l \text{ for all } \alpha \in R^+ \right\}$$

is a fundamental domain for the action of the affine Weyl group W_l on X. Note that C_l is not empty if and only if $l \geq h$ for all $\alpha \in R^+$.

An alcove C in X is a subset of the form $C = w \cdot C_l$ for some $w \in W_l$. The closure $\overline{C}$ of C is then $\overline{C} = w \cdot \overline{C}_l$.

5.3.2. We are mainly interested in the set of the weights which satisfy the Borel-Weil-Bott theorem. We already know that the Borel-Weil-Bott theorem holds for small weights, i.e. all weights in $\overline{C}_l$. This is a direct consequence of the strong linkage principle, see e.g. [5].

In this section we shall prove that (1.6) has an analogue for U_q. We say that $\lambda \in X^+$ is generic if for each $w \in W$ we have

$$H^i(w \cdot \lambda) \neq 0 \text{ if and only if } i = l(w).$$

Proposition 5.13. *For any $\lambda \in X$ we have*

$$\operatorname{ch}\left(\hat{Z}_q(\lambda)\right) = e^{\lambda - (l-1)\rho} \operatorname{ch}\left(\mathrm{St}_l\right).$$

Proof. By Proposition B.9 , we have an isomorphism of $U_q^0 u_q^+$-modules

$$\hat{Z}_q(\lambda) \simeq \operatorname{Hom}_k(u_q^+, \lambda).$$

According to [31, Theorem 8.3], the elements (taking in a suitable order [31, (4.3)])

$$\left\{ \prod_{\alpha > 0} E_\alpha^{n_\alpha} \,\middle|\, n_\alpha \in \{0, \ldots, l-1\} \right\}$$

form a basis of u_q^+. Hence

$$\begin{aligned} \operatorname{ch}\left(\hat{Z}_q(\lambda)\right) &= e^\lambda \prod_{\alpha > 0} (1 + e^{-\alpha} + e^{-2\alpha} + \cdots + e^{-(l-1)\alpha}) \\ &= e^\lambda \prod_{\alpha > 0} \frac{(1 - e^{-l\alpha})}{(1 - e^{-\alpha})}. \end{aligned}$$

We know that St_l is isomorphic to $\hat{Z}_q((l-1)\rho)$ as a $B_q u_q^+$-module. Therefore

$$\begin{aligned}\mathrm{ch}\left(\hat{Z}_q(\lambda)\right) &= e^{\lambda}\prod_{\alpha>0}\frac{(1-e^{-l\alpha})}{(1-e^{-\alpha})}\\ &= e^{\lambda-(l-1)\rho}e^{(l-1)\rho}\prod_{\alpha>0}\frac{(1-e^{-l\alpha})}{(1-e^{-\alpha})}\\ &= e^{\lambda-(l-1)\rho}\,\mathrm{ch}\left(\mathrm{St}_l\right).\end{aligned}$$

The proposition follows. ∎

In particular, we have that $\mu \in X$ is a weight of $\hat{Z}_q(\lambda)$ if and only if $\mu-\lambda+(l-1)\rho$ is a weight of St_l.

Let $\lambda \in X$ and write $\lambda_1 + l\lambda_2$ where $\lambda_1 \in X_l$ and $\lambda_2 \in X$. Throughout, we let λ_1 and λ_2 refer to this decomposition.

Lemma 5.14. *Let μ be a weight of $\hat{Z}_q(\lambda)$ for some $\lambda \in X$. Then*

$$\left|\left\langle \lambda_2 - (\mu_2+\rho), \alpha^\vee\right\rangle\right| < 2(h-1) \text{ for all } \alpha \in R.$$

Proof. Suppose that R is indecomposable, and let α_0 be the highest short root. For each $\alpha \in R$ we have that

$$\begin{aligned} l\left|\left\langle \lambda_2-(\mu_2+\rho),\alpha^\vee\right\rangle\right| &= \left|\left\langle l\mu_2 - l\lambda_2 + l\rho, \alpha^\vee\right\rangle\right|\\ &= \left|\left\langle l\mu_2 - l\lambda_2 + l\rho + (\mu_1-\lambda_1-\rho) - (\mu_1-\lambda_1-\rho), \alpha^\vee\right\rangle\right|\\ &= \left|\left\langle (\mu_1+l\mu_2) - (\lambda_1+l\lambda_2) + (l-1)\rho + (\lambda_1-\mu_1) + \rho, \alpha^\vee\right\rangle\right|\\ &= \left|\left\langle \mu-\lambda+(l-1)\rho+(\lambda_1-\mu_1)+\rho,\alpha^\vee\right\rangle\right|\\ &\leq \left|\left\langle \mu-\lambda+(l-1)\rho,\alpha^\vee\right\rangle\right| + \left|\left\langle \lambda_1-\mu_1,\alpha^\vee\right\rangle\right| + \left|\left\langle \rho,\alpha^\vee\right\rangle\right|.\end{aligned}$$

Set $\nu = \mu-\lambda+(l-1)\rho$. Then

$$l\left|\left\langle \lambda_2-(\mu_2+\rho),\alpha^\vee\right\rangle\right| \leq \left|\left\langle \nu,\alpha^\vee\right\rangle\right| + \left|\left\langle \lambda_1-\mu_1,\alpha^\vee\right\rangle\right| + \left|\left\langle \rho,\alpha^\vee\right\rangle\right|. \tag{5.1}$$

We have

$$\left|\left\langle \rho,\alpha^\vee\right\rangle\right| \leq \left\langle \rho,\alpha_0^\vee\right\rangle = h-1, \tag{5.2}$$

and

$$\left|\left\langle \lambda_1-\mu_1,\alpha^\vee\right\rangle\right| \leq \left\langle (l-1)\rho,\alpha_0^\vee\right\rangle = (l-1)\left\langle \rho,\alpha_0^\vee\right\rangle = (l-1)(h-1). \tag{5.3}$$

Using the above proposition, we see that ν is a weight of the Steinberg module. Pick $w \in W$ such that $w(\nu)$ is dominant. Since the Weyl group W acts on the weights of St_l, then $w(\nu)$ is still a weight of St_l. Hence

$$\begin{aligned}\left|\left\langle \nu, \alpha^\vee \right\rangle\right| &\leq \max_{\beta\in R}\left|\left\langle \nu, \beta^\vee \right\rangle\right| \\ &= \max_{\beta\in R}\left|\left\langle w(\nu), \beta^\vee \right\rangle\right| \\ &= \left\langle w(\nu), \alpha_0^\vee \right\rangle \\ &\leq \left\langle (l-1)\rho, \alpha_0^\vee \right\rangle \\ &= (l-1)(h-1). \end{aligned} \tag{5.4}$$

Using (5.1), (5.2), (5.3) and (5.4), we get

$$\begin{aligned} l\left|\left\langle \lambda_2 - (\mu_2+\rho), \alpha^\vee \right\rangle\right| &\leq \left|\left\langle \nu, \alpha^\vee \right\rangle\right| + \left|\left\langle \lambda_1 - \mu_1, \alpha^\vee \right\rangle\right| + \left|\left\langle \rho, \alpha^\vee \right\rangle\right| \\ &\leq (l-1)(h-1) + (l-1)(h-1) + (h-1) \\ &= (2l-1)(h-1) \\ &< 2l(h-1). \end{aligned}$$

This finishes the proof. ■

For any $\lambda \in X$ and $w \in W$ we let $\lambda^w \in X$ such that $l\lambda^w + w\cdot\lambda \in X_l$.

Lemma 5.15. *Let $\lambda \in X$ and $w \in W$. Then*

$$\left|\left\langle (\lambda_1)^w, \alpha^\vee \right\rangle\right| \leq 2(h-1) \text{ for all } \alpha \in R.$$

Proof. Again, we assume that R is indecomposable. For all $\alpha \in R$ we have

$$\begin{aligned} l\left|\left\langle (\lambda_1)^w, \alpha^\vee \right\rangle\right| &= \left|\left\langle l(\lambda_1)^w + w(\lambda_1+\rho) - w(\lambda_1+\rho), \alpha^\vee \right\rangle\right| \\ &\leq \left|\left\langle l(\lambda_1)^w + w(\lambda_1+\rho), \alpha^\vee \right\rangle\right| + \left|\left\langle w(\lambda_1+\rho), \alpha^\vee \right\rangle\right|. \end{aligned} \tag{5.5}$$

We have

$$\begin{aligned} \left|\left\langle w(\lambda_1+\rho), \alpha^\vee \right\rangle\right| &= \left|\left\langle \lambda_1+\rho, w^{-1}\left(\alpha^\vee\right) \right\rangle\right| \\ &\leq \left\langle \lambda_1+\rho, \alpha_0^\vee \right\rangle \\ &\leq \left\langle l\rho, \alpha_0^\vee \right\rangle \\ &= l(h-1). \end{aligned} \tag{5.6}$$

$$\begin{aligned} \left|\left\langle l(\lambda_1)^w + w(\lambda_1+\rho), \alpha^\vee \right\rangle\right| &= \left|\left\langle l(\lambda_1^w) + w\cdot\lambda_1 + \rho, \alpha^\vee \right\rangle\right| \\ &= \left\langle l(\lambda_1)^w + w\cdot\lambda_1 + \rho, \alpha_0^\vee \right\rangle \\ &\leq \left\langle l\rho, \alpha_0^\vee \right\rangle \\ &= l(h-1). \end{aligned} \tag{5.7}$$

Hence

$$\begin{aligned} l\left|\left\langle (\lambda_1)^w, \alpha^\vee \right\rangle\right| &\leq \left|\left\langle l(\lambda_1)^w + w(\lambda_1+\rho), \alpha^\vee \right\rangle\right| + \left|\left\langle w(\lambda_1+\rho), \alpha^\vee \right\rangle\right| \\ &\leq 2l(h-1). \end{aligned}$$

We are done. ■

Proposition 5.16. *Suppose that* $\operatorname{char} k = 0$. *Then* $\lambda \in X^+$ *is generic if*

$$4(h-1) \leq \left\langle \lambda_2, \alpha^\vee \right\rangle \text{ for all } \alpha \in R^+.$$

Let us first make an observation. Using the tensor identity, we get that for all $i \geq 0$

$$\begin{aligned} H_q^i\left(U_q/\hat{u}_q, \hat{L}_q(\mu)\right) &\simeq H_q^i\left(U_q/\hat{u}_q, L_q(\mu_1) \otimes_k l\mu_2\right) \text{(cf. Theorem B.6)} \\ &\simeq L_q(\mu_1) \otimes_k H_q^i(U_q/\hat{u}_q, l\mu_2) \\ &\simeq L_q(\mu_1) \otimes_k \left(i^{-1} H^i(\mu_2)\right)^{(1)}. \end{aligned}$$

Since $\hat{Z}_q$ is exact, we see that

$$H_q^i\left(U_q/\hat{u}_q, \hat{Z}_q(w\cdot\lambda)\right) \simeq H_q^i(w\cdot\lambda),$$

and this implies that $H_q^i(w\cdot\lambda) = 0$ if all the composition factors $\hat{L}_q(\mu)$ of $\hat{Z}_q(w\cdot\lambda)$ satisfy $H^i(\mu_2) = 0$.

Proof of Proposition 5.16. Let $w \in W$ and $i \in \mathbb{N}$ with $i \neq l(w)$. Suppose that $\hat{L}_q(\mu)$ is a composition factor of $\hat{Z}_q(w\cdot\lambda)$. We are done if we can prove that $\mu_2 + \rho \in w \cdot X^+$. By definition, we see that

$$w\cdot\lambda = w\cdot\lambda_1 + lw(\lambda_2) = (w\cdot\lambda_1 + l(\lambda_1)^w) + l(w(\lambda_2) - (\lambda_1)^w),$$

and hence $(w\cdot\lambda)_2 = w(\lambda_2) - (\lambda_1)^w$.

For all $\alpha \in R^+$ we have

$$\begin{aligned} \left\langle w^{-1}(\mu_2+\rho), \alpha^\vee \right\rangle &= \left\langle \mu_2 + \rho, w\left(\alpha^\vee\right) \right\rangle \\ &= \left\langle \mu_2 + \rho - (w\cdot\lambda)_2 + (w\cdot\lambda)_2, w\left(\alpha^\vee\right) \right\rangle \\ &= \left\langle \mu_2 + \rho - (w\cdot\lambda)_2, w\left(\alpha^\vee\right) \right\rangle + \left\langle (w\cdot\lambda)_2, w\left(\alpha^\vee\right) \right\rangle \\ &= \left\langle \mu_2 + \rho - (w\cdot\lambda)_2, w\left(\alpha^\vee\right) \right\rangle + \left\langle w(\lambda_2) - (\lambda_1)^w, w\left(\alpha^\vee\right) \right\rangle \\ &= \left\langle \mu_2 + \rho - (w\cdot\lambda)_2, w\left(\alpha^\vee\right) \right\rangle + \left\langle w(\lambda_2), w\left(\alpha^\vee\right) \right\rangle \\ &+ \left\langle -(\lambda_1)^w, w\left(\alpha^\vee\right) \right\rangle. \end{aligned}$$

Since μ is a weight of $\hat{Z}_q(w\cdot\lambda)$, Lemma 5.14 implies that

$$\left|\left\langle (w\cdot\lambda)_2 - (\mu_2+\rho), w\left(\alpha^\vee\right) \right\rangle\right| < 2(h-1). \tag{5.8}$$

We also have

$$4(h-1) \leq \left\langle \lambda_2, \alpha^\vee \right\rangle,$$

and

$$\left|\left\langle (\lambda_1)^w, w\left(\alpha^\vee\right) \right\rangle\right| \leq 2(h-1), \tag{5.9}$$

see Lemma 5.15. Therefore we get that

$$\begin{aligned} \left\langle w^{-1}(\mu_2+\rho), \alpha^\vee \right\rangle &= \left\langle \mu_2 + \rho - (w\cdot\lambda)_2, w\left(\alpha^\vee\right) \right\rangle + \left\langle \lambda_2, \alpha^\vee \right\rangle + \left\langle -(\lambda_1)^w, w\left(\alpha^\vee\right) \right\rangle \\ &> -2(h-1) + 4(h-1) - 2(h-1) \\ &= 0. \end{aligned}$$

This completes the proof. ■

Proposition 5.17. *Suppose that* $\operatorname{char} k = p \geq 0$. *Then* $\lambda \in X^+$ *is generic if*

$$4(h-1) \leq \langle \lambda_2, \alpha^\vee \rangle \leq l - 4(h-1) \text{ for all } \alpha \in R^+.$$

It is assumed that $l > 8(h-1)$.

Proof. Let $w \in W$ and $i \in \mathbb{N}$ with $i \neq l(w)$. Suppose that $\hat{L}_q(\mu)$ is a composition factor of $\hat{Z}_q(w \cdot \lambda)$. We are done if we can prove that

$$\left|\langle w^{-1}(\mu_2 + \rho), \alpha^\vee \rangle\right| < l \text{ for all } \alpha \in R^+$$

which means that $\mu_2 \in w \cdot \overline{C_l}$.

By assumption, we have

$$\left|\langle \lambda_2, \alpha^\vee \rangle\right| \leq l - 4(h-1) \text{ for all } \alpha \in R^+.$$

Using this together with (5.8) and (5.9), we get for any $\alpha \in R^+$

$$\begin{aligned}
&\left|\langle w^{-1}(\mu_2 + \rho), \alpha^\vee \rangle\right| \\
&= \left|\langle \mu_2 + \rho, w\left(\alpha^\vee\right) \rangle\right| \\
&= \left|\langle -(\mu_2 + \rho), w\left(\alpha^\vee\right) \rangle\right| \\
&= \left|\langle (w \cdot \lambda)_2 - (w \cdot \lambda)_2 - (\mu_2 + \rho), w\left(\alpha^\vee\right) \rangle\right| \\
&\leq \left|\langle (w \cdot \lambda)_2 - (\mu_2 + \rho), w\left(\alpha^\vee\right) \rangle\right| + \left|\langle (w \cdot \lambda)_2, w\left(\alpha^\vee\right) \rangle\right| \\
&\leq \left|\langle (w \cdot \lambda)_2 - (\mu_2 + \rho), w\left(\alpha^\vee\right) \rangle\right| + \left|\langle w(\lambda_2), w\left(\alpha^\vee\right) \rangle\right| + \left|\langle (\lambda_1)^w, w\left(\alpha^\vee\right) \rangle\right| \\
&\leq \left|\langle (w \cdot \lambda)_2 - (\mu_2 + \rho), w\left(\alpha^\vee\right) \rangle\right| + \left|\langle \lambda_2, \alpha^\vee \rangle\right| + \left|\langle (\lambda_1)^w, w\left(\alpha^\vee\right) \rangle\right| \\
&< 2(h-1) + l - 4(h-1) + 2(h-1) \\
&= l.
\end{aligned}$$

The proposition follows. ■

CHAPTER 6

$\mathbf{B_q}$-cohomology

In this chapter the field k will be arbitrary, and we consider $q \in k^\times$. We shall demonstrate that the results in Chapter 2 have direct analogues for B_q. The proofs are almost identical, and we therefore omit the details.

Moreover, when $\mathrm{char}(k) = 0$, we shall compute $H^4(B_q, \lambda)$ for all $\lambda \in X$ and determine a lower bound i for the degree in which the cohomology $H^i(B_q, \lambda)$ can be non-zero. All modules we consider in this chapter are finite dimensional unless otherwise specified.

6.1. Analogues of B-cohomology

For any $M \in \mathscr{C}_q^-$ we have

$$\mathrm{Hom}_{\mathscr{C}_q^-}(k, M) = \{m \in M \mid um = \epsilon(b)m \text{ for all } b \in B_q\} = M^{B_q}.$$

As usual ϵ denotes the counit of the Hopf algebra B_q. Note that the functor

$$\mathrm{Hom}_{\mathscr{C}_q^-}(k, -) : \mathscr{C}_q^- \to \{ \text{ Vector spaces over } k \ \}$$

is a left exact functor. The right derived functors are denoted

$$H^i(B_q, M) = \mathrm{Ext}^i_{\mathscr{C}_q^-}(k, M) \text{ for all } i \geq 0.$$

This is the Hochschild cohomology of M. Note that we may replace B_q with any parabolic subalgebra containing B_q.

In order to simplify our notation, we let

$$a_{\alpha\beta} = \langle \alpha, \beta^\vee \rangle \text{ for all } \alpha, \beta \in S.$$

For each $\alpha \in S$ and $r \in \mathbb{N}$ we further define

$$\begin{aligned} W_\alpha &= \{w \in W \mid \langle w \cdot 0, \alpha^\vee \rangle \geq 0\}, \\ W(r) &= \{w \in W \mid l(w) = r\}, \\ W_\alpha(r) &= W(r) \cap W_\alpha = \{w \in W(r) \mid \langle w \cdot 0, \alpha^\vee \rangle \geq 0\}. \end{aligned}$$

6.1.1. When q is not a root of unity, then we can argue as in Section 2.1 using this time the quantized Borel-Weil-Bott theorem and the complete reducibility of U_q [11, Corollary 7.7]. In this way we then obtain the following complete description of $H^\bullet(B_q, \lambda)$ (in analogy with (2.3)):

$$H^r(B_q, \lambda) \simeq \begin{cases} k & \text{if } \lambda = w \cdot 0 \text{ for some } w \in W(r), \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

6.1.2. We let from now on $q \in k^\times$ denote a primitive l-th root of unity. We assume that l is odd, larger than the Coxeter number h and prime to 3 if the root system R contains a component of type G_2.

For each $\alpha \in S$ we let $E_\alpha, F_\alpha, K_\alpha^{\pm 1}$ denote the standard generators. The small quantum group u_q is the subalgebra of U_q generated by all $E_\alpha, F_\alpha, K_\alpha^{\pm 1}$ modulo the ideal generated by $K_\alpha^l - 1$. Moreover, b_q will denote the small quantum Borel subalgebra of u_q corresponding to B_q.

We have a quantum Frobenius homomorphism $\mathrm{Fr}_l : U_q \to \bar{U}$, see [13]. Here $\bar{U}$ denotes the specialisation at k of the Kostant $\mathbb{Z}$-form of the enveloping algebra of the Lie algebra for the semisimple group $\bar{G}$ corresponding to R. We identify the category of finite dimensional $\bar{U}$-modules with the category of finite dimensional rational $\bar{G}$-modules. We shall also need the restriction of Fr_l to B_q mapping into the enveloping algebra associated with the Borel subgroup $\bar{B}$ in $\bar{G}$.

6.1.3. We limit ourselves to finite dimensional modules for U_q and B_q of type **1**. So if M is a U_q (resp. B_q)-module whose restriction to u_q (resp. b_q) is trivial, then we use the quantum Frobenius homomorphism Fr_l to make M into a $\bar{G}$ (resp. $\bar{B}$)-module that we denote by $M^{(-1)}$ in analogy with the notation in Section 2.2. Similarly, if N is a $\bar{G}$ (resp. $\bar{B}$)-module then $N^{(1)}$ denotes the U_q (resp. B_q)-module obtained via Fr_l.

As in Section 2.2 we have for each B_q-module M the Lyndon-Hochschild-Serre spectral sequence

$$H^r(\bar{B}, H^s(b_q, M)^{(-1)}) \implies H^{r+s}(B_q, M). \tag{6.2}$$

The cohomology $H^r(b_q, \lambda)$ is completely known, see [20]

$$H^r(b_q, \lambda) = 0 \text{ for all r} \geq 0 \text{ unless } \lambda \in W \cdot 0 + l\mathbb{Z}R. \tag{6.3}$$

$$H^r(b_q, w \cdot 0 + l\lambda)^{(-1)} \simeq S^{(r-l(w))/2}\bar{u}^* \otimes \lambda \tag{6.4}$$

where $\bar{u}$ is the Lie algebra of the unipotent radical of $\bar{B}$.

The same arguments as before (see (2.4), Proposition 2.3, Theorem 2.11 and Proposition 2.12) give

$$H^r(B_q, \lambda) = 0 \text{ for all r} \geq 0 \text{ unless } \lambda \in W \cdot 0 + l\mathbb{Z}R, \tag{6.5}$$

$$H^r(B_q, w \cdot 0 + l\lambda) \simeq H^{r-l(w)}(B_q, l\lambda) \text{ for all } w \in W \text{ and } r \in \mathbb{N}, \tag{6.6}$$

$$H^r(B_q, w \cdot 0 + l\lambda) = 0 \text{ for all r} > l(w) - 2\,\mathrm{ht}(\lambda), \tag{6.7}$$

$$H^{l(w)-2\,\mathrm{ht}(\lambda)}(B_q, w \cdot 0 + l\lambda) \simeq k. \tag{6.8}$$

Remark 6.1. Note that the upper bound in (6.7) is independent of l, see the argument given in the proof of Theorem 2.11.

Remark 6.2. Suppose for a second that $\mathrm{char}(k) = 0$. Using (6.1) together with the fact that $\mathrm{ht}(w \cdot 0) \leq -l(w)$ for all $w \in W$, we get $H^i(\bar{B}, \lambda) = 0$ unless $\mathrm{ht}(\lambda) \leq -i$. From this we can then derive that

$$H^i(\bar{B}, S^j\bar{u}^* \otimes \lambda) = 0 \text{ unless } \lambda \leq 0 \text{ and } \mathrm{ht}(\lambda) \leq -i - j. \tag{6.9}$$

Combining this with (6.4), the Lyndon-Hochschild-Serre spectral sequence (6.2) reproves (6.7).

6.1.4. Let $M \in \mathscr{C}_q^-$. Clearly, M is a B_q-submodule of $Q_0 = H_q^0(B_q/U_q^0, M)$. The same is true for Q_0/M and $H^0(B_q/U_q^0, Q_0/M)$, etc. Then

$$0 \to M \to Q_0 \to Q_1 \to Q_2 \to \cdots \tag{6.10}$$

is an injective resolution of M in $\mathscr{C}_q^-$. We call this the "standard" resolution of M. The weights of each term in the resolution have the form $\lambda + \mu$ where λ is a weight of M and $\mu \geq 0$. Hence when we apply $H^0(B_q, -)$ to this resolution, then all terms vanish unless M has a weight which is ≤ 0. In particular, we get

$$H^\bullet(B_q, \lambda) = 0 \text{ unless } \lambda \leq 0. \tag{6.11}$$

6.1.5. Degrees 0 and 1. Using the Lyndon-Hochschild-Serre spectral sequence (6.2), the cohomology for B_q can be related to that for $\bar{B}$. Combining this with the results in Chapter 2, we are now able to completely determine some of the Hochschild cohomology of 1-dimensional B_q-modules.

It is clear that

$$H^0(B_q, k) \simeq k \text{ and } H^0(B_q, \lambda) \neq 0 \text{ if and only if } \lambda = 0.$$

Noting that the only E_2-term in (6.2) that contributes to $H^1(B_q, l\lambda)$ is $H^1(\bar{B}, \lambda)$, we have

$$H^1(B_q, l\lambda) \simeq H^1(\bar{B}, \lambda).$$

Therefore the description of the first cohomology $H^1(B_q, \lambda)$ depends on whether k is a field of characteristic 0 or of characteristic $p > 0$. If $\mathrm{char}(k) = 0$, then we obtain from (2.3)

$$H^1(B_q, \lambda) \simeq \begin{cases} k & \text{if } \lambda = -\alpha \text{ or } -l\alpha \text{ for } \alpha \in S, \\ 0 & \text{otherwise.} \end{cases} \tag{6.12}$$

On the other hand, if $\mathrm{char}(k) = p > 0$, then we have (using this time (2.8))

$$H^1(B_q, \lambda) \simeq \begin{cases} k & \text{if } \lambda = -p^n\alpha \text{ or } -lp^n\alpha \text{ for } \alpha \in S, n \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.13}$$

Remark 6.3. Both (6.12) and (6.13) remain true when $l \leq h$, and the argument given in the modular case will also work in the quantum case, see [4] or Subsection 2.1.5. But this time we need to use Theorem 4.15 and Theorem 4.19 instead of [2].

6.1.6. Degree 2. The only terms in (6.2) that may contribute to $H^2(B_q, l\lambda)$ are $H^2(\bar{B}, \lambda)$ and $H^0(\bar{B}, \bar{u}^* \otimes \lambda)$. Hence by (2.3) and Proposition 2.5 we get

Theorem 6.4 ([14, Theorem 7.2])**.** *Let $\lambda \in X$. If $\mathrm{char}\, k = 0$, then*

$$H^2(B_q, \lambda) \simeq \begin{cases} k & \text{if } \lambda = -l\alpha \text{ for some } \alpha \in S, \\ k & \text{if } \lambda = lw \cdot 0 \text{ for some } w \in W(2), \\ k & \text{if } \lambda = -\beta - l\alpha \text{ for some } \alpha, \beta \in S, \\ k & \text{if } \lambda = w \cdot 0 \text{ for some } w \in W(2), \\ 0 & \text{otherwise.} \end{cases}$$

When $p > 0$ we replace (2.3) in the above argument by Theorem 2.9. Then we find

Theorem 6.5 ([14, Theorem 7.3]). *Let $\lambda \in X$. If* $\operatorname{char} k = p > 0$, *then*

$$H^2(B_q, \lambda) \simeq \begin{cases} k & \text{if } \lambda = lp^n(-\alpha) \text{ for } \alpha \in S, n \geq 0, \\ k & \text{if } \lambda = lp^n(w \cdot 0) \text{ for } w \in W(2) \text{ and } n \geq 0, \\ k & \text{if } \lambda = lp^n(-\alpha - p^m\beta) \text{ for } \alpha, \beta \in S, n \geq 0, m > 0 \\ k & \text{if } \lambda = w \cdot 0 \text{ for } w \in W(2), \\ k & \text{if } \lambda = -\beta - lp^n\alpha \text{ for } \alpha, \beta \in S, n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

6.1.7. Degree 3. We now turn to $H^3(B_q, \lambda)$. The only E_2-terms in (6.2) that contribute to $H^3(B_q, l\lambda)$ are $H^3(\bar{B}, \lambda)$ and $H^1(\bar{B}, \bar{u}^* \otimes \lambda)$. As in the modular case we get

Theorem 6.6 ([14, Theorem 7.5]). *Suppose that* $\operatorname{char}(k) = p > h$. *If $\lambda \in X$, then*

$$H^3(B_q, \lambda) \simeq \begin{cases} k & \text{if } \lambda = lp^n(-2\alpha) \text{ for } \alpha \in S \text{ and } n > 0, \\ k^2 & \text{if } \lambda = lp^n(-\beta - p^m\alpha) \text{ for } \alpha, \beta \in S \text{ and} \\ & n, m > 0, \\ k & \text{if } \lambda = lp^n(-\beta - \alpha) \text{ for } \alpha, \beta \in S \text{ with} \\ & a_{\beta\alpha} < 0 \text{ and } n > 0, \\ k^2 & \text{if } \lambda = lp^n(-\beta - \alpha) \text{ for } \alpha, \beta \in S \text{ with} \\ & a_{\beta\alpha} = 0 \text{ and } n > 0, \\ k & \text{if } \lambda = lp^n(s_\alpha s_\beta \cdot 0) \text{ for } \alpha, \beta \in S \text{ with} \\ & a_{\beta\alpha} \neq 0 \text{ and } n > 0, \\ k & \text{if } \lambda = lp^n(w \cdot 0) \text{ for } w \in W(3) \text{ and } n \geq 0, \\ k & \text{if } \lambda = lp^n(w \cdot 0 - p^m\alpha) \text{ for } \alpha \in S,\ w \in W(2) \text{ and} \\ & n \geq 0, m > 0 \\ k & \text{if } \lambda = lp^n(p^m w \cdot 0 - \alpha) \text{ for } \alpha \in S,\ w \in W(2) \text{ and} \\ & n \geq 0, m > 0 \\ k & \text{if } \lambda = lp^n(-\alpha - p^m\beta - p^v\gamma) \text{ for } \alpha, \beta, \gamma \in S \\ & \text{and } n \geq 0, m > v > 0, \\ k & \text{if } \lambda = l(-\beta - p^n\alpha) \text{ for } \alpha, \beta \in S \text{ and } n > 0, \\ k & \text{if } \lambda = w \cdot 0 \text{ for } w \in W(3), \\ k & \text{if } \lambda = w \cdot 0 - lp^n\alpha \text{ for } \alpha \in S,\ w \in W(2) \text{ and } n \geq 0, \\ k & \text{if } \lambda = -\beta - lp^n\alpha \text{ for } \alpha, \beta \in S \text{ and } n \geq 0, \\ k & \text{if } \lambda = -\beta - lp^n w \cdot 0 \text{ for } \alpha \in S,\ w \in W(2) \text{ and } n \geq 0, \\ k & \text{if } \lambda = -\alpha + lp^n(-\beta - p^m\gamma) \text{ for } \alpha, \beta, \gamma \in S \\ & \text{and } n \geq 0, m > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6.7 ([14, Theorem 7.4]). *Let $\lambda \in X$. If* $\operatorname{char}(k) = 0$ *then*

$$H^3(B_q, \lambda) \simeq \begin{cases} k & \text{if } \lambda = l(-2\alpha) \text{ for } \alpha \in S, \\ k & \text{if } \lambda = l(-\beta-\alpha) \text{ for } \alpha,\beta \in S \text{ with } a_{\beta\alpha} < 0, \\ k^2 & \text{if } \lambda = l(-\beta-\alpha) \text{ for } \alpha,\beta \in S \text{ with } a_{\beta\alpha} = 0, \\ k & \text{if } \lambda = l(s_\alpha s_\beta \cdot 0) \text{ for } \alpha,\beta \in S \text{ with } a_{\beta\alpha} \neq 0, \\ k & \text{if } \lambda = l(w \cdot 0) \text{ for } w \in W(3), \\ k & \text{if } \lambda = w \cdot 0 \text{ for } w \in W(3), \\ k & \text{if } \lambda = w \cdot 0 - l\alpha \text{ for } \alpha \in S \text{ and } w \in W(2), \\ k & \text{if } \lambda = lw \cdot 0 - \alpha \text{ for } \alpha \in S \text{ and } w \in W(2), \\ k & \text{if } \lambda = -\beta - l\alpha \text{ for } \alpha,\beta \in S, \\ 0 & \text{otherwise.} \end{cases}$$

6.2. Lower bound

We assume from now on that $\operatorname{char}(k) = 0$.

6.2.1. Let M be a $\bar{B}$-module. In characteristic zero, the spectral sequence (2.1) degenerates and then gives isomorphisms of $\bar{B}$-modules

$$H^i\left(\bar{B}, M\right) \simeq H^0\left(\bar{G}, H^i(\bar{G}/\bar{B}, M)\right) \text{ for all } i \geq 0. \tag{6.14}$$

Combined with Serre duality and the complete reducibility of finite dimensional $\bar{G}$-modules, this gives us isomorphisms of vector spaces

$$H^i\left(\bar{B}, M\right) \simeq H^{N-i}\left(\bar{B}, M^* \otimes -2\rho\right) \text{ for all } i \geq 0. \tag{6.15}$$

Here M^* is the dual module. Hence we get for each $\lambda \in X$

$$H^i\left(\bar{B}, S^n\bar{u}^* \otimes \lambda\right) \simeq H^{N-i}\left(\bar{B}, S^n\bar{u} \otimes -\lambda - 2\rho\right) \text{ for all } i \geq 0. \tag{6.16}$$

Let σ be the maximal long root in the corresponding semisimple Lie algebra. From (6.16) and (6.9) we can then derive that

$$H^i(\bar{B}, S^n\bar{u}^* \otimes \lambda) = 0 \text{ if } -\operatorname{ht}(\lambda) - \operatorname{ht}(2\rho) - n\operatorname{ht}(\sigma) > i - N.$$

Suppose that the corresponding Lie algebra has rank larger than 1. Then

$$H^i(\bar{B}, S^n\bar{u}^* \otimes \lambda) = 0 \text{ if } -\operatorname{ht}(\lambda) - \operatorname{ht}(2\rho) + N > i + n\operatorname{ht}(\sigma) \geq i + 2n.$$

Using this together with the spectral sequence (6.2), we get for each $w \in W, \lambda \leq 0$

$$H^i(B_q, w \cdot 0 + l\lambda) = 0 \text{ for } i < l(w) - \operatorname{ht}(\lambda) - \operatorname{ht}(2\rho) + N. \tag{6.17}$$

Suppose now that B_q is the Borel subalgebra in the quantum group of type SL_2, and α is the simple root. In this case we have for each $m \geq 1$ and $j > 0$ (cf. [6])

$$H^j(B_q, -lm\alpha) \simeq \begin{cases} k & \text{if } j = 2m, 2m-1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.18}$$

and

$$H^j(B_q, -lm\alpha - \alpha) \simeq \begin{cases} k & \text{if } j = 2m, 2m+1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.19}$$

When we combine (6.17), (6.18) and (6.19), we have in general

Proposition 6.8. *Let $\lambda \in X$ and $w \in W$. Then*

$$H^i(B_q, w\cdot 0 + l\lambda) = 0 \quad \textit{for} \quad i < l(w) - \mathrm{ht}(\lambda) - \mathrm{ht}(2\rho) + N.$$

6.2.2. Let B_q be the Borel subalgebra in the quantum group of type SL_3. We have $-\mathrm{ht}(2\rho) + N = -1$ and hence

$$H^i(B, w\cdot 0 + l\lambda) = 0 \;\text{ for }\; i < l(w) - \mathrm{ht}(\lambda) - 1.$$

Weight considerations (cf. [6]) give for each $m > 2$

$$H^r(B_q, -ml\rho) \simeq \begin{cases} k & \text{if } r = 2m-1, 2m, 4m-4, 4m-3, 4m-1, \\ 0 & \text{otherwise.} \end{cases}$$

This shows that there are cases where $H^{l(w)-\mathrm{ht}(\lambda)-1}(B_q, w\cdot 0 + l\lambda)$ is non-zero.

As with the upper bound, we could hope that the lower bound described in Proposition 6.8 is the best possible. Unfortunately, this already fails in the SL_2 case, see e.g. (6.18).

6.3. Methods

We continue to assume that $\mathrm{char}(k) = 0$. We want to compute the fourth cohomology group $H^4(B_q, \lambda)$ explicitly. Since we don't know how to compute the B_q-cohomology of $S^n u^* \otimes \lambda$ in general, we need a different argument.

6.3.1. With Section 4.1 in mind, we fix a simple root α and let P_α $(=U_{q,\alpha})$ be the minimal parabolic subalgebra of U_q corresponding to α. To simplify our notation, we shall also write $H_\alpha(-)$ and $L_\alpha(\lambda)$ in short of $H_{q,\alpha}(-)$ and $L_{q,\alpha}(-)$.

As with the modular case, when M is a P_α-module, we have

$$H^i(P_\alpha, M) \simeq H^i(B_q, M) \text{ for all } i \geq 0.$$

Using this together with the spectral sequence (3.12) we get for all $i \geq 0$

$$H^i(B_q, \mu) \simeq H^i(P_\alpha, H^0_\alpha(\mu)) \text{ if } \langle \mu, \alpha^\vee \rangle \geq 0, \tag{6.20}$$

$$H^{i+1}(B_q, \mu) \simeq H^i(P_\alpha, H^1_\alpha(\mu)) \text{ if } \langle \mu, \alpha^\vee \rangle \leq -2, \tag{6.21}$$

$$H^i(B_q, \mu) = 0 \text{ if } \langle \mu, \alpha^\vee \rangle = -1. \tag{6.22}$$

6.3.2. Suppose first that $\mu \in X$ with $-l \leq \langle \mu, \alpha^\vee \rangle < -1$ or $\langle \mu, \alpha^\vee \rangle \equiv -1 \bmod l$ for some $\alpha \in S$. By Remark 4.7, $H^1_\alpha(\mu)$ is simple and isomorphic to $H^0_\alpha(s_\alpha \cdot \mu)$. Hence we obtain for such μ

$$H^{i+1}(B_q, \mu) \simeq H^i(B_q, s_\alpha \cdot \mu) \text{ for all } i \geq 0. \tag{6.23}$$

In particular, we have $H^4(B_q, \mu)) \simeq H^3(B_q, s_\alpha \cdot \mu)$ which in this case completely describes $H^4(B_q, \mu)$ for all μ, see Theorem 6.7.

6.3.3. Now, let $\mu \in X$ with $\langle \mu, \alpha^\vee \rangle < 0$ for some $\alpha \in S$. Suppose that $\langle s_\alpha \cdot \mu, \alpha^\vee \rangle = al + d$ for some $a \geq 0$ and $0 \leq d < l - 1$. Set $\lambda = s_\alpha \cdot \mu$ and $\lambda' = \lambda - (d+1)\alpha$.

Recall that $H^1_\alpha(\mu)$ has a unique simple quotient with highest weight λ. Combining this with Theorem 4.6, we have the following exact sequence

$$0 \to L_\alpha\left(\lambda'\right) \to H^1_\alpha(\mu) \to L_\alpha(\lambda) \to 0. \tag{6.24}$$

In order to effectively take advantage of the long exact sequence resulting from (6.24), we need to compute some low degree B_q-cohomology of the simple P_α-module $L_\alpha(\lambda)$. As was the case in Chapter 2, we don't know how to do that in general. However, the following exact sequence will allow us to compute $H^\bullet(B_q, L_q(\lambda))$ for some $\lambda \in X$ in degrees at most 3.

By dualizing (6.24), we obtain an exact sequence

$$0 \to L_\alpha(\lambda) \to H^0_\alpha(\lambda) \to L_\alpha(\lambda') \to 0$$

from which we get the long exact sequence

$$\cdots \to H^i\left(B_q, L_\alpha\left(\lambda'\right)\right) \to H^{i+1}\left(B_q, L_\alpha\left(\lambda\right)\right) \to H^{i+1}\left(B_q, \lambda\right) \to \cdots . \tag{6.25}$$

This gives $H^{i+1}(B_q, L_q(\lambda)) = 0$ unless $H^i(B_q, L_q(\lambda')) \neq 0$ or $H^{i+1}(B_q, \lambda) \neq 0$.

Unfortunately, we will not be able to compute $H^2(B_q, L_q(\lambda))$ and $H^3(B_q, L_q(\lambda))$ explicitly in all cases because there will be some few cases where both the first and the third term in (6.25) are non-zero at the same time.

6.3.4. The following result will turn out to be useful in connection with the above.

Lemma 6.9. *Let $w \in W$. Then*

$$w \cdot 0 \pm \gamma \notin W \cdot 0 \textit{ for } \gamma \in R^+ \textit{ unless } \pm w^{-1}(\gamma) \in S.$$

If so, we have $w \cdot 0 \pm \gamma = s_\gamma w \cdot 0$.

Proof. Recall that we denote by $\langle , \rangle$ the inner product on $\mathbb{E}$ and $\alpha^\vee = 2/\langle \alpha, \alpha \rangle$ the coroot of $\alpha \in S$. We have

$$\begin{aligned} w \cdot 0 - \gamma \in W \cdot 0 &\Leftrightarrow w(\rho) - \gamma \in W(\rho) \\ &\Rightarrow \langle w(\rho) - \gamma, w(\rho) - \gamma \rangle = \langle \rho, \rho \rangle \\ &\Rightarrow \langle \rho, \rho \rangle - 2\langle w(\rho), \gamma \rangle + \langle \gamma, \gamma \rangle = \langle \rho, \rho \rangle \\ &\Rightarrow -2\langle w(\rho), \gamma \rangle + \langle \gamma, \gamma \rangle = 0 \\ &\Rightarrow \langle \rho, w^{-1}(\gamma)^\vee \rangle = 1. \end{aligned}$$

A similar argument works for $w \cdot 0 + \gamma$. The proposition is proved. ■

Remark 6.10. Using the same argument, one can show that we have for all $w \in W, \alpha \in S$ and $j \in \mathbb{N}$ that

$$w \cdot 0 \pm j\alpha \in W \cdot 0 \Leftrightarrow \langle w \cdot 0, \alpha^\vee \rangle = \mp j - 1 \Leftrightarrow w \cdot 0 \pm j\alpha = s_\alpha w \cdot 0.$$

6.4. B_q-cohomology of $L_\alpha(\lambda)$

In this section we compute some low degree cohomology of $L_\alpha(\lambda)$.

6.4.1. Degrees 0 and 1.

Proposition 6.11. *Let $\lambda \in X$ and assume that $\langle \lambda, \alpha^\vee \rangle \geq 0$ for some $\alpha \in S$. Then*

$$H^0\left(B_q, L_\alpha(\lambda)\right) \simeq \begin{cases} k & \text{if } \lambda = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have

$$H^0_\alpha(B_q, L_\alpha(\lambda)) \subset H^0_\alpha(B_q, H^0_\alpha(\lambda)) \simeq H^0_q(B_q, \lambda).$$

Hence $H^0_\alpha(B_q, L_\alpha(\lambda))$ is non-zero if and only if $\lambda = 0$. The proposition is proved. ■

Proposition 6.12. *Let $\lambda \in X$ and assume that $\langle \lambda, \alpha^\vee \rangle \geq 0$ for some $\alpha \in S$. Then*

$$H^1\left(B_q, L_\alpha(\lambda)\right) \simeq \begin{cases} k & \text{if } \lambda = (l-1)\alpha, \\ k & \text{if } \lambda = -\beta \text{ or } \lambda = -l\beta \text{ for some } \beta \in S\backslash\{\alpha\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose that $\langle \lambda, \alpha^\vee \rangle < l$ or $\langle \lambda, \alpha^\vee \rangle \equiv -1 \bmod l$, then $H^0_\alpha(\lambda)$ is simple and isomorphic to $L_\alpha(\lambda)$. Hence

$$H^1\left(B_q, L_\alpha(\lambda)\right) \simeq H^1(B_q, \lambda) \simeq \begin{cases} k & \text{if } \exists \beta \in S\backslash\{\alpha\} : \lambda = -\beta, \\ k & \text{if } \exists \beta \in S\backslash\{\alpha\} : \lambda = -l\beta \text{ and } a_{\beta\alpha} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose now that $\langle \lambda, \alpha^\vee \rangle = al + d$ for some $a \geq 1, 0 \leq d < l-1$ and set $\lambda' = \lambda - (d+1)\alpha$. Then we have a short exact sequence

$$0 \to L_\alpha(\lambda) \to H^0_\alpha(\lambda) \to L_\alpha(\lambda') \to 0$$

which gives the following exact sequence

$$0 \to H^0\left(B_q, L_\alpha(\lambda')\right) \to H^1\left(B_q, L_\alpha(\lambda)\right) \to H^1\left(B_q, \lambda\right) \to \cdots . \tag{6.26}$$

Note that $\lambda' = 0$ if and only if $\lambda = (l-1)\alpha$. This gives the desired result for $\lambda = (l-1)\alpha$.

Suppose now that $\lambda \neq (l-1)\alpha$. Using (6.26) together with (6.12), we get that $H^1\left(B_q, L_\alpha(\lambda)\right) = 0$ unless $\lambda = -l\beta$ for some $\beta \in S$ with $a_{\beta\alpha} < 0$. In this case the same argument applied to $\lambda' = -l\beta - \alpha$ gives that

$$0 \to H^1\left(B_q, L_\alpha(\lambda')\right) \hookrightarrow H^1\left(B_q, \lambda'\right) = 0.$$

This finishes the proof. ■

6.4.2. Degree 2.

Proposition 6.13. *Let $\lambda \in X$ and assume that $\langle \lambda, \alpha^\vee \rangle \geq 0$ for some $\alpha \in S$. Suppose that $\lambda \neq -l\beta$ for all $\beta \in S$. Then we have*

$$H^2(B_q, L_\alpha(\lambda)) \simeq \begin{cases} k & \text{if } \lambda = l\alpha, \\ k & \text{if } \lambda = s_\alpha s_\beta \cdot 0 + l\alpha \text{ for some } \beta \in S\backslash\{\alpha\}, \\ k & \text{if } \lambda = lw \cdot 0 \text{ for some } w \in W_\alpha(2), \\ k & \text{if } \lambda = w \cdot 0 \text{ for some } w \in W_\alpha(2), \\ k & \text{if } \lambda = -l\beta + (l-1)\alpha \text{ for some } \beta \in S\backslash\{\alpha\}, \\ k & \text{if } \lambda = -\alpha - l\beta \text{ for some } \beta \in S \text{ with } a_{\beta\alpha} < 0, \\ k & \text{if } \lambda = -\gamma - l\beta \text{ for some } \beta, \gamma \in S\backslash\{\alpha\}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $\lambda = -l\beta$ for some $\beta \in S$ with $a_{\beta\alpha} < 0$, then $\lambda' = -l\beta - \alpha$. Proposition 6.12 and Theorem 6.4 yield that both the first and the third term in the sequence (6.25) equal k.

Proof. Suppose that $\langle \lambda, \alpha^\vee \rangle < l$ or $\langle \lambda, \alpha^\vee \rangle \equiv -1 \bmod l$. Since $\lambda \neq -l\beta$ for all $\beta \in S$, Theorem 6.4 gives in this case that

$$\begin{aligned} H^2(B_q, L_\alpha(\lambda)) &\simeq H^2(B_q, \lambda) \\ &\simeq \begin{cases} k & \text{if } \lambda = w \cdot 0 \text{ for } w \in W_\alpha(2), \\ k & \text{if } \lambda = lw \cdot 0 \text{ for } w \in W(2) \text{ with } \langle w \cdot 0, \alpha^\vee \rangle = 0, \\ k & \text{if } \lambda = -\alpha - l\beta \text{ for } \beta \in S \text{ with } a_{\beta\alpha} = -1, \\ k & \text{if } \lambda = -\gamma - l\beta \text{ for } \beta, \gamma \in S\backslash\{\alpha\} \text{ with } a_{\beta\alpha} = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Suppose now that $\langle \lambda, \alpha^\vee \rangle = al + d$ for some $a \geq 1, 0 \leq d < l - 1$ and set $\lambda' = \lambda - (d+1)\alpha$. Then we have

$$0 \to L_\alpha(\lambda) \to H^0_\alpha(\lambda) \to L_\alpha(\lambda') \to 0,$$

which give rise to the following exact sequence

$$\cdots \to H^1(B_q, L_\alpha(\lambda')) \to H^2(B_q, L_\alpha(\lambda)) \to H^2(B_q, \lambda) \to \cdots. \tag{6.27}$$

Consider first the case where $H^1(B_q, L_\alpha(\lambda')) \neq 0$. Then we get from Proposition 6.12 that $\lambda' \in \{(l-1)\alpha, -\beta, -l\beta\}$ for some $\beta \in S\backslash\{\alpha\}$. We consider each of these cases.

(1) If $\lambda' = (l-1)\alpha$, then $\lambda = l\alpha$. Using (6.27) together with (6.11), we get

$$H^2(B_q, L_\alpha(l\alpha)) \simeq H^1(B_q, L_\alpha((l-1)\alpha)) \simeq k$$

because $H^1(B_q, l\alpha) = H^2(B_q, l\alpha) = 0$.

(2) If $\lambda' = -\beta$ for some $\beta \in S\backslash\{\alpha\}$, then $\lambda = s_\alpha s_\beta \cdot 0 + l\alpha$. Arguing as before, we get

$$H^2(B_q, L_\alpha(s_\alpha s_\beta \cdot 0 + l\alpha)) \simeq H^1(B_q, L_\alpha(-\beta)) \simeq k.$$

(3) If $\lambda' = -l\beta$ for some $\beta \in S\backslash\{\alpha\}$, then $\lambda = -l\beta + (l-1)\alpha$. Hence

$$H^2\left(B_q, L_\alpha\left(-l\beta + (l-1)\alpha\right)\right) \simeq H^1\left(B_q, L_\alpha\left(-l\beta\right)\right) \simeq k.$$

Let us now look at the case where $H^1\left(B_q, L_\alpha\left(\lambda'\right)\right) = 0$. Using Theorem 6.4 together with (6.27), we get that $H^2\left(B_q, L_\alpha\left(\lambda\right)\right) = 0$ unless we are in one of the first four cases listed in Theorem 6.4. Since $\lambda \neq -l\beta$ for all $\beta \in S$ and $\lambda \neq w \cdot 0$ for all $w \in W_\alpha(2)$, then there are only two cases left to consider. To investigate this, we need the following exact sequence

$$0 \to L_\alpha\left(\lambda'\right) \to H^0_\alpha\left(\lambda'\right) \to L_\alpha\left(\lambda''\right) \to 0. \tag{6.28}$$

Here we define λ'' to be $(\lambda')'$.

(1) If $\lambda = lw \cdot 0$ for some $w \in W(2)$ with $\langle w \cdot 0, \alpha^\vee \rangle > 0$, then $\lambda' = lw \cdot 0 - \alpha$ and $\lambda'' = l(w \cdot 0 - \alpha)$. When we combine the long exact sequence coming from (6.28) with Proposition 6.12 and Theorem 6.4, we get

$$0 \to H^2\left(B_q, L_\alpha\left(lw \cdot 0 - \alpha\right)\right) \to H^2\left(B_q, lw \cdot 0 - \alpha\right) = 0.$$

Using this together with Proposition 6.12 and Theorem 6.4, the sequence (6.27) gives that

$$H^2\left(B_q, L_\alpha\left(lw \cdot 0\right)\right) \simeq H^2\left(B_q, lw \cdot 0\right) \simeq k.$$

(2) If $\lambda = -\gamma - l\beta$ for some $\beta, \gamma \in S\backslash\{\alpha\}$ with $a_{\beta\alpha} < 0$, then $\lambda' = s_\alpha s_\gamma \cdot 0 - l\beta$ and $\lambda'' = -\gamma - l\beta - l\alpha$. As before, we get $H^2\left(B_q, L_\alpha\left(s_\alpha s_\gamma \cdot 0 - l\beta\right)\right) = 0$, and hence

$$H^2\left(B_q, L_\alpha\left(-\gamma - l\beta\right)\right) \simeq H^2\left(B_q, -\gamma - l\beta\right) \simeq k.$$

Finally, if $\lambda = -\alpha - l\beta$ for some $\beta \in S$ with $a_{\beta\alpha} < -1$, then $\lambda' = -l\beta - l\alpha$ and $\lambda'' = -l\beta - l\alpha - \alpha$. In this case the same argument applied to $-l\beta - l\alpha$ gives that $H^2\left(B_q, L_\alpha\left(-l\beta - l\alpha\right)\right) = 0$, and hence

$$H^2\left(B_q, L_\alpha\left(-\alpha - l\beta\right)\right) \simeq H^2\left(B_q, -\alpha - l\beta\right) \simeq k.$$

The proposition is proved ■

6.4.3. Degree 3. As was the case with the second cohomology group, we will not be able to compute $H^3(B_q, L_\alpha(\lambda))$ for all $\lambda \in X$, either.

Proposition 6.14. *Let $\lambda \in X$ and assume that $\langle \lambda, \alpha^\vee \rangle \geq 0$ for some $\alpha \in S$. Suppose that*

(1) $\lambda \neq -2l\beta$ for some $\beta \in S$ with $a_{\alpha\beta} = -1$,
(2) $\lambda \neq -l\beta$ for all $\beta \in S$,
(3) $\lambda \neq lw \cdot 0$ for all $w \in W(2)$,
(4) $\lambda \neq -l\beta - \gamma$ for all $\beta, \gamma \in S$,
(5) $\lambda \neq -l\beta + (l-1)\alpha$ for all $\beta \in S\backslash\{\alpha\}$.

Then we have

$$H^3(B_q, L_\alpha(\lambda)) \simeq \begin{cases} k & \textit{if } \lambda = l\alpha + (l-1)\alpha, \\ k & \textit{if } \lambda = -\beta + l\alpha \textit{ for some } \beta \in S\backslash\{\alpha\}, \\ k & \textit{if } \lambda = -l\beta + l\alpha \textit{ for } \beta \in S\backslash\{\alpha\}, \\ k & \textit{if } \lambda = s_\alpha s_\gamma \cdot 0 - l\beta + l\alpha \textit{ for } \gamma, \beta \in S\backslash\{\alpha\}, \\ k & \textit{if } \lambda = s_\alpha w \cdot 0 + l\alpha \textit{ for } w \in W_\alpha(2), \\ k & \textit{if } \lambda = lw \cdot 0 + (l-1)\alpha \textit{ for } w \in W_\alpha(2), \\ k & \textit{if } \lambda = l(-2\beta) \textit{ for } \beta \in S\backslash\{\alpha\} \textit{ with } a_{\beta\alpha} \neq -1, \\ k & \textit{if } \lambda = l(-\gamma - \beta) \textit{ for } \gamma, \beta \in S \textit{ with } a_{\beta\gamma} < 0, \\ k & \textit{if } \lambda = l(-\alpha - \beta) \textit{ for } \beta \in S \textit{ with } a_{\beta\alpha} \leq -2, \\ k & \textit{if } \lambda = w \cdot 0 \textit{ for } w \in W_\alpha(3), \\ k & \textit{if } \lambda = lw \cdot 0 \textit{ for } w \in W_\alpha(3), \\ k & \textit{if } \lambda = lw \cdot 0 - \beta \textit{ for } w \in W_\alpha(2) \textit{ and } \beta \in S\backslash\{\alpha\}, \\ k & \textit{if } \lambda = lw \cdot 0 - \alpha \textit{ for } w \in W(2) \textit{ and } \langle w \cdot 0, \alpha^\vee \rangle \geq 1, \\ k & \textit{if } \lambda = w \cdot 0 - l\beta \textit{ for } w \in W(2) \textit{ and } \beta \in S\backslash\{\alpha\}, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The proof here follows the same lines as the proof of Proposition 6.13. So suppose first that $\langle \lambda, \alpha^\vee \rangle < l$ or $\langle \lambda, \alpha^\vee \rangle \equiv -1 \bmod l$. Then arguing as before using this time Theorem 6.7, we get

$$H^3(B_q, L_\alpha(\lambda)) \simeq H^3(B_q, \lambda)$$

$$\simeq \begin{cases} k & \text{if } \lambda = l(-2\beta) \text{ for } \beta \in S \text{ with } a_{\beta\alpha} = 0, \\ k & \text{if } \lambda = l(-\gamma - \beta) \text{ for } \gamma, \beta \in S \text{ with } a_{\beta\gamma} < 0 \text{ and} \\ & a_{\gamma\alpha} = a_{\beta\alpha} = 0, \\ k & \text{if } \lambda = l(-\alpha - \beta) \text{ for } \beta \in S \text{ with } a_{\beta\alpha} = -2, \\ k & \text{if } \lambda = lw \cdot 0 \text{ for } w \in W(3) \text{ with } \langle w \cdot 0, \alpha^\vee \rangle = 0, \\ k & \text{if } \lambda = w \cdot 0 \text{ for } w \in W_\alpha(3), \\ k & \text{if } \lambda = lw \cdot 0 - \beta \text{ for } \beta \in S\backslash\{\alpha\} \text{ and } w \in W(2) \text{ with} \\ & \langle w \cdot 0, \alpha^\vee \rangle = 0, \\ k & \text{if } \lambda = lw \cdot 0 - \alpha \text{ for } w \in W(2) \text{ with } \langle w \cdot 0, \alpha^\vee \rangle = 1, \\ k & \text{if } \lambda = w \cdot 0 - l\beta \text{ for } \beta \in S\backslash\{\alpha\} \text{ and } w \in W(2) \text{ such that} \\ & a_{\beta\alpha} = 0 \text{ if } \langle w \cdot 0, \alpha^\vee \rangle \geq 0, \text{ and } a_{\beta\alpha} = -1 \text{ if } \langle w \cdot 0, \alpha^\vee \rangle < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose now that $\langle \lambda, \alpha^\vee \rangle = al + d$ for some $a \geq 1, 0 \leq d < l - 1$ and set $\lambda' = \lambda - (d+1)\alpha$. Similarly, we define λ'' to be $(\lambda')'$. Then we have

$$0 \to L_\alpha(\lambda) \to H^0_\alpha(\lambda) \to L_\alpha(\lambda') \to 0,$$

$$0 \to L_\alpha(\lambda') \to H^0_\alpha(\lambda') \to L_\alpha(\lambda'') \to 0,$$

which give rise to the following long exact sequences

$$\cdots \to H^2(B_q, L_\alpha(\lambda')) \to H^3(B_q, L_\alpha(\lambda)) \to H^3(B_q, \lambda) \to \cdots, \tag{6.29}$$

$$\cdots \to H^2(B_q, L_\alpha(\lambda'')) \to H^3(B_q, L_\alpha(\lambda')) \to H^3(B_q, \lambda') \to \cdots. \tag{6.30}$$

Consider first the case where $H^2(B_q, L_\alpha(\lambda')) \neq 0$. Note that if $\lambda' = -l\beta$ or $\lambda' = -\alpha - l\beta$ for some $\beta \in S\backslash\{\alpha\}$, then $\lambda = -l\beta + (l-1)\alpha$ or $\lambda = -l\beta$. By assumption, this means that there are 6 cases in Proposition 6.13 to consider.

(1) If $\lambda' = l\alpha$, then $\lambda = l\alpha + (l-1)\alpha$. Since $H^2(B_q, l\lambda) = H^3(B_q, l\lambda) = 0$, we get from (6.29) that

$$H^3(B_q, L_\alpha(l\alpha + (l-1)\alpha)) \simeq H^2(B_q, L_\alpha(l\alpha)) \simeq k.$$

(2) If $\lambda' = s_\alpha s_\beta \cdot 0 + l\alpha$ for $\beta \in S\backslash\{\alpha\}$, then $\lambda = -\beta + l\alpha$. As before, we have that

$$H^3(B_q, L_\alpha(-\beta + l\alpha)) \simeq H^2(B_q, L_\alpha(s_\alpha s_\beta \cdot 0 + l\alpha)) \simeq k.$$

(3) If $\lambda' = lw \cdot 0$ for $w \in W_\alpha(2)$, then $\lambda = lw \cdot 0 + (l-1)\alpha$. By Remark 6.10, Theorem 6.4 and (6.12), it follows immediately that

$$H^3(B_q, lw \cdot 0 + (l-1)\alpha) \simeq H^2(B_q, l(w \cdot 0 + \alpha)) = 0,$$
$$H^2(B_q, lw \cdot 0 + (l-1)\alpha) \simeq H^1(B_q, l(w \cdot 0 + \alpha)) = 0.$$

Then

$$H^3(B_q, L_\alpha(lw \cdot 0 + (l-1)\alpha)) \simeq H^2(B_q, L_\alpha(lw \cdot 0)) \simeq k.$$

(4) If $\lambda' = w \cdot 0$ for $w \in W_\alpha(2)$, then $\lambda = s_\alpha w \cdot 0 + l\alpha$. Hence

$$H^3(B_q, L_\alpha(s_\alpha w \cdot 0 + l\alpha)) \simeq H^2(B_q, L_\alpha(w \cdot 0)) \simeq k.$$

(5) If $\lambda' = -l\beta + (l-1)\alpha$ for $\beta \in S\backslash\{\alpha\}$, then $\lambda = -l\beta + l\alpha$. So

$$H^3(B_q, L_\alpha(-l\beta + l\alpha)) \simeq H^2(B_q, L_\alpha(-l\beta + (l-1)\alpha)) \simeq k.$$

(6) Finally, if $\lambda' = -\gamma - l\beta$ for $\beta, \gamma \in S\backslash\{\alpha\}$, then $\lambda = s_\alpha s_\gamma \cdot 0 - l\beta + l\alpha$. Hence

$$H^3(B_q, L_\alpha(s_\alpha s_\gamma \cdot 0 - l\beta + l\alpha)) \simeq H^2(B_q, L_\alpha(-\gamma - l\beta)) \simeq k.$$

Consider next the case where $H^2(B_q, L_\alpha(\lambda')) = 0$. Then (6.29) implies that

$$H^3(B_q, L_\alpha(\lambda)) = 0 \text{ unless } H^3(B_q, \lambda) \neq 0.$$

By assumption, there are seven cases left in Theorem 6.7 to consider. We consider each of these cases.

(1) If $\lambda = -2l\beta$ for $\beta \in S$ with $a_{\alpha\beta} < -1$, then $\lambda' = -2l\beta - \alpha$ and $\lambda'' = -2l\beta - l\alpha$. When we apply Proposition 6.13 and Theorem 6.7 to (6.30), we get

$$0 \to H^3(B_q, L_\alpha(-2l\beta - \alpha)) \to H^3(B_q, -2l\beta - \alpha) = 0.$$

Combined with Theorem 6.7, the sequence (6.29) gives

$$H^3(B_q, L_\alpha(-2l\beta)) \simeq H^3_q(B_q, -2l\beta) \simeq k.$$

(2) If $\lambda = l(-\gamma - \beta)$ for $\gamma, \beta \in S\backslash\{\alpha\}$ with $a_{\beta\gamma} < 0$ such that $a_{\gamma\alpha} < 0$ or $a_{\beta\alpha} < 0$, then $\lambda' = l(-\gamma - \beta) - \alpha$ and $\lambda'' = l(-\alpha - \beta - \gamma)$. Arguing as before, we get

$$H^3(B_q, L_\alpha(l(-\gamma - \beta))) \simeq H^3(B_q, l(-\gamma - \beta)) \simeq k.$$

(3) If $\lambda = l(-\alpha-\beta)$ for $\beta \in S$ with $a_{\beta\alpha} = -3$, then $\lambda' = l(-\alpha-\beta)-\alpha$ and $\lambda'' = l(-2\alpha-\beta)$. Again, we get that $H^3(B_q, L_\alpha(l(-\alpha-\beta)-\alpha)) = 0$ which implies

$$H^3\left(B_q, L_\alpha(l(-\alpha-\beta))\right) \simeq H^3(B_q, l(-\alpha-\beta)) \simeq k.$$

(4) If $\lambda = lw\cdot 0$ for $w \in W(3)$ with $\langle w\cdot 0, \alpha^\vee\rangle > 0$, then $\lambda' = lw\cdot 0 - \alpha$ and $\lambda'' = l(w\cdot 0 - \alpha)$. By Remark 6.10 and Proposition 6.13, it follows that $H^2(B_q, L_\alpha(lw\cdot 0 - l\alpha)) = 0$ because $w\cdot 0 - \alpha \notin W\cdot 0$. Using this together with Theorem 6.7 and (6.30), we get

$$0 \to H^3\left(B_q, L_\alpha(lw\cdot 0-\alpha)\right) \to H^3\left(B_q, (lw\cdot 0 - \alpha)\right) = 0,$$

and hence

$$H^3\left(B_q, L_\alpha(lw\cdot 0)\right) \simeq H^3(B_q, lw\cdot 0) \simeq k.$$

(5) If $\lambda = lw\cdot 0 - \beta$ for $\beta \in S\backslash\{\alpha\}$ and $w \in W(2)$ and $\langle w\cdot 0, \alpha^\vee\rangle > 0$, then $\lambda' = lw\cdot 0 + s_\alpha s_\beta \cdot 0$ and $\lambda'' = l(w\cdot 0 - \alpha) - \beta$. Similarly, we obtain in this case that $H^3\left(B_q, L_\alpha(lw\cdot 0 + s_\alpha s_\beta\cdot 0)\right) = 0$ and then

$$H^3\left(B_q, L_\alpha(lw\cdot 0 - \beta)\right) \simeq H^3(B_q, lw\cdot 0 - \beta) \simeq k.$$

(6) If $\lambda = lw\cdot 0 - \alpha$ for $w \in W(2)$ and $\langle w\cdot 0, \alpha^\vee\rangle > 1$, then $\lambda' = lw\cdot 0 - l\alpha$ and $\lambda'' = l(w\cdot 0 - \alpha) - \alpha$. Hence

$$H^3\left(B_q, L_\alpha(lw\cdot 0 - \alpha)\right) \simeq H^3(B_q, lw\cdot 0 - \alpha) \simeq k.$$

(7) If $\lambda = w\cdot 0 - l\beta$ for $w \in W(2), \beta \in S\backslash\{\alpha\}$ with $a_{\beta\alpha} < 0$. If $\langle w\cdot 0, \alpha^\vee\rangle \geq 0$, then $\lambda' = s_\alpha w\cdot 0 - l\beta$ and $\lambda'' = w\cdot 0 - l\alpha - l\beta$, and if $\langle w\cdot 0, \alpha^\vee\rangle < 0$, then $\lambda' = s_\alpha w\cdot 0 - l\beta - l\alpha$ and $\lambda'' = w\cdot 0 - l\alpha - l\beta$. When we combine (6.29) and (6.30), we clearly get in both cases that

$$H^3\left(B_q, L_\alpha(w\cdot 0 - l\beta)\right) \simeq k.$$

This completes the proof. ■

6.5. Degree 4

Since we couldn't compute $H^2(B_q, L_\alpha(\mu))$ and $H^3(B_q, L_\alpha(\mu))$ explicitly in all cases, we can't compute $H^4(B_q, \lambda)$ for all $\lambda \in X$ by only using the exact sequence (6.24). Therefore there are some cases which need to be handled differently. Our first step will then be to compute $H^4\left(B_q, \lambda\right)$ for some special weights. We shall perform these computations using the spectral sequence (6.2).

6.5.1. Case 1. By (6.7) and (6.8), we have for each $\alpha \in S$

$$H^4\left(B_q, -l\alpha\right) \simeq 0, \tag{6.31}$$

$$H^4\left(B_q, -2l\alpha\right) \simeq k. \tag{6.32}$$

6.5.2. Case 2. Suppose that $\lambda = ls_\alpha(-\beta)$ for some $\alpha, \beta \in S$ such that $\alpha \neq \beta$. The only terms in the spectral sequence (6.2) that may contribute to $H^4(B_q, ls_\alpha(-\beta))$ are $H^4(\bar{B}, s_\alpha(-\beta))$, $H^2(\bar{B}, \bar{u}^* \otimes s_\alpha(-\beta))$ and $H^0(\bar{B}, S^2\bar{u}^* \otimes s_\alpha(-\beta))$.

We claim first that $H^2(\bar{B}, \bar{u}^* \otimes s_\alpha(-\beta)) = 0$. We prove this by contradiction. So suppose that there exist $\mu \in R^+$ and $w \in W(2)$ such that $\mu + s_\alpha(-\beta) = w \cdot 0$. Clearly, we have $w = s_\alpha s_\beta$ or $w = s_\beta s_\alpha$. Since $s_\alpha s_\beta \cdot 0 < s_\alpha(-\beta)$, we must have $w = s_\beta s_\alpha$ and hence

$$\mu = s_\beta s_\alpha \cdot 0 - s_\alpha(-\beta) = -(a_{\beta\alpha} + 1)\alpha + a_{\alpha\beta}\beta \notin R^+.$$

By Lemma 6.9, we have $H^4\left(\bar{B}, s_\alpha(-\beta)\right) = 0$. Proposition 2.5 then implies

$$H^4\left(B_q, ls_\alpha(-\beta)\right) \simeq H^0\left(\bar{B}, S^2\bar{u}^* \otimes s_\alpha(-\beta)\right) \simeq \begin{cases} k & \text{if } a_{\beta\alpha} = -1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.33}$$

6.5.3. Case 3. Our next step will be to treat the weight $\lambda = ls_\alpha s_\beta \cdot 0$ for some $\alpha, \beta \in S$ such that $\alpha \neq \beta$. As before, we have

$$H^4\left(\bar{B}, s_\alpha s_\beta \cdot 0\right) = H^2\left(\bar{B}, \bar{u}^* \otimes s_\alpha s_\beta \cdot 0\right) = 0.$$

By Proposition 2.5, we then obtain

$$H^4\left(B_q, ls_\alpha s_\beta \cdot 0\right) \simeq H^0\left(\bar{B}, S^2\bar{u}^* \otimes s_\alpha s_\beta \cdot 0\right) \simeq \begin{cases} k & \text{if } a_{\beta\alpha} = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.34}$$

6.5.4. Case 4. The next step is to set $\lambda = 2ls_\alpha s_\beta \cdot 0$ for some $\alpha, \beta \in S$ with $a_{\alpha\beta} = -1$. We claim first that $H^2(\bar{B}, \bar{u}^* \otimes 2s_\alpha s_\beta \cdot 0) = 0$. We prove this by contradiction. So suppose that we can find $\mu \in R^+$ such that $\mu + 2s_\alpha s_\beta \cdot 0 = s_\beta s_\alpha \cdot 0$. Then

$$\begin{aligned} \mu &= -\alpha - (1 - a_{\alpha\beta})\beta + 2\beta + 2(1 - a_{\beta\alpha})\alpha \\ &= (1 - 2a_{\beta\alpha})\alpha + (1 + a_{\alpha\beta})\beta \\ &= (1 - 2a_{\beta\alpha})\alpha \notin R^+. \end{aligned}$$

Next, we claim that $H^4(\bar{B}, 2s_\alpha s_\beta \cdot 0) = 0$. We assume by contradiction that $2s_\alpha s_\beta \cdot 0 = w \cdot 0$ for some $w \in W(4)$. Clearly, $w = s_\alpha s_\beta s_\alpha s_\beta$ or $w = s_\beta s_\alpha s_\beta s_\alpha$. So suppose first that $2s_\alpha s_\beta \cdot 0 = s_\alpha s_\beta s_\beta s_\alpha \cdot 0$. Then

$$\begin{aligned} \text{LHS} &= 2s_\alpha s_\beta \cdot 0 = -2\beta - 2(1 - a_{\beta\alpha})\alpha. \\ \text{RHS} &= s_\alpha s_\beta s_\alpha s_\beta \cdot 0 \\ &= s_\alpha s_\beta \cdot (-\beta - (1 - a_{\beta\alpha})\alpha) \\ &= s_\alpha \cdot (-(1 - a_{\beta\alpha})\alpha - (1 - a_{\beta\alpha})\beta) \\ &= -a_{\beta\alpha}\alpha - (1 - a_{\beta\alpha})\beta + (1 - a_{\beta\alpha})a_{\beta\alpha}\alpha \\ &= (-a_{\beta\alpha} + (1 - a_{\beta\alpha})a_{\beta\alpha})\alpha - (1 - a_{\beta\alpha})\beta. \end{aligned}$$

Looking at the coefficient of α and β in both sides, we get

$$\begin{cases} \text{Coeff of } \alpha : -a_{\beta\alpha} + (1 - a_{\beta\alpha})a_{\beta\alpha} = -2(1 - a_{\beta\alpha}), \\ \text{Coeff of } \beta : -1 + a_{\beta\alpha} = -2. \end{cases}$$

Then

$$\begin{cases} \text{Coeff of } \alpha : -a_{\beta\alpha} + (1 - a_{\beta\alpha})a_{\beta\alpha} = -2(1 - a_{\beta\alpha}), \\ \text{Coeff of } \beta : a_{\beta\alpha} = -1. \end{cases}$$

This is clearly a contradiction. Suppose now that $2s_\alpha s_\beta \cdot 0 = s_\beta s_\alpha s_\beta s_\alpha \cdot 0$.

$$\begin{aligned} \text{LHS} &= -2\beta - 2(1 - a_{\beta\alpha})\alpha. \\ \text{RHS} &= s_\beta s_\alpha s_\beta s_\alpha \cdot 0 \\ &= s_\beta s_\alpha \cdot (-\alpha - 2\beta) \\ &= s_\beta \cdot (-2\beta + 2a_{\beta\alpha}\alpha) \\ &= (1 + 2a_{\beta\alpha})\beta + 2a_{\beta\alpha}\alpha. \end{aligned}$$

Hence

$$\begin{cases} \text{Coeff of } \alpha : 2a_{\beta\alpha} = -2(1 - a_{\beta\alpha}), \\ \text{Coeff of } \beta : (1 + 2a_{\beta\alpha}) = -2. \end{cases}$$

This is also impossible.

Using Proposition 2.5, we then conclude that

$$H^4\left(B_q, 2ls_\beta s_\alpha \cdot 0\right) = H^0\left(\bar{B}, S^2\bar{u}^* \otimes 2s_\beta s_\alpha \cdot 0\right) = 0. \tag{6.35}$$

6.5.5. Case 5. Let $w = s_\alpha s_\beta s_\gamma$ for $\alpha, \beta, \gamma \in S$ and assume that $\langle s_\beta s_\gamma \cdot 0, \alpha^\vee\rangle \geq 0$. Proposition 2.5 together with the observation that $-\operatorname{ht}(w \cdot 0) \geq l(w) = 3$ give

$$H^4(\bar{B}, s_\alpha s_\beta s_\gamma \cdot 0) = H^0\left(\bar{B}, S^2\bar{u}^* \otimes s_\alpha s_\beta s_\gamma \cdot 0\right) = 0.$$

Hence we get

$$H^4(B_q, ls_\alpha s_\beta s_\gamma \cdot 0) \simeq H^2\left(\bar{B}, \bar{u}^* \otimes s_\alpha s_\beta s_\gamma \cdot 0\right). \tag{6.36}$$

Recall that the line of weight α in $\bar{u}^*$ is a $\bar{B}$-module and that the quotient $\bar{V}_\alpha$ is a P_α-module. This gives the following short exact sequence

$$0 \to \alpha \to \bar{u}^* \to \bar{V}_\alpha \to 0. \tag{6.37}$$

When we apply Proposition 2.8 and Lemma 6.9 to (6.37), we obtain

$$\begin{aligned} 0 &\to H^1\left(\bar{B}, \bar{V}_\alpha \otimes s_\alpha s_\beta s_\gamma \cdot 0\right) \to H^2\left(\bar{B}, s_\alpha s_\beta s_\gamma \cdot 0 + \alpha\right) \\ &\to H^2\left(\bar{B}, \bar{u}^* \otimes s_\alpha s_\beta s_\gamma \cdot 0\right) \to H^2\left(\bar{B}, \bar{V}_\alpha \otimes s_\alpha s_\beta s_\gamma \cdot 0\right) \to 0. \end{aligned} \tag{6.38}$$

In order to compute $H^2\left(\bar{B}, \bar{u}^* \otimes s_\alpha s_\beta s_\gamma \cdot 0\right)$, we need to compute the remaining terms in the above sequence.

From Remark 6.10 we have

$$H^2(\bar{B}, s_\alpha s_\beta s_\gamma \cdot 0 + \alpha) = 0 \text{ unless } \langle s_\beta s_\gamma \cdot 0, \alpha^\vee\rangle = 0.$$

And when $\langle s_\beta s_\gamma \cdot 0, \alpha^\vee\rangle = 0$, we have $s_\alpha s_\beta s_\gamma \cdot 0 + \alpha = s_\beta s_\gamma \cdot 0$, and in this case we get $H^2(\bar{B}, s_\alpha s_\beta s_\gamma \cdot 0 + \alpha) \simeq k$.

Suppose first that $\langle s_\beta s_\gamma \cdot 0, \alpha^\vee\rangle = 0$. So

$$\langle s_\alpha s_\beta s_\gamma \cdot 0 + \rho, \alpha^\vee\rangle = -\langle s_\beta s_\gamma \cdot 0, \alpha^\vee\rangle - 1 = -1 < 0.$$

The same argument given in (2.17) gives

$$H^1\left(\bar{B}, \bar{V}_\alpha \otimes s_\alpha s_\beta s_\gamma \cdot 0\right) \simeq H^0\left(\bar{B}, \bar{V}_\alpha \otimes s_\beta s_\gamma \cdot 0\right) = 0, \tag{6.39}$$

$$H^2\left(\bar{B}, \bar{V}_\alpha \otimes s_\alpha s_\beta s_\gamma \cdot 0\right) \simeq H^1\left(\bar{B}, \bar{V}_\alpha \otimes s_\beta s_\gamma \cdot 0\right). \tag{6.40}$$

By Lemma 6.9, we have $H^2\left(\bar{B}, s_\beta s_\gamma \cdot 0 + \alpha\right) = H^0\left(\bar{B}, \bar{V}_\alpha \otimes s_\beta s_\gamma \cdot 0\right) = 0$, and hence (6.37) gives the following exact sequence

$$\begin{aligned} 0 \to H^1\left(\bar{B}, s_\beta s_\gamma \cdot 0 + \alpha\right) &\to H^1\left(\bar{B}, \bar{u}^* \otimes s_\beta s_\gamma \cdot 0\right) \\ &\to H^1\left(\bar{B}, \bar{V}_\alpha \otimes s_\beta s_\gamma \cdot 0\right) \to 0. \end{aligned} \tag{6.41}$$

Using this together with Proposition 2.8, we get

$$H^1\left(\bar{B}, \bar{V}_\alpha \otimes s_\beta s_\gamma \cdot 0\right) \simeq \begin{cases} k^2 & \text{if } \alpha \neq \gamma \text{ and } a_{\beta\gamma} = 0, \\ k & \text{otherwise.} \end{cases}$$

Combining this with sequence (6.38), it follows

$$H^4(B_q, l s_\alpha s_\beta s_\gamma \cdot 0) \simeq H^2\left(\bar{B}, \bar{u}^* \otimes s_\alpha s_\beta s_\gamma \cdot 0\right) \simeq \begin{cases} k^3 & \text{if } \alpha \neq \gamma \text{ and } a_{\beta\gamma} = 0, \\ k^2 & \text{otherwise.} \end{cases}$$

Suppose now that $\langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle \neq 0$, and hence

$$H^2\left(\bar{B}, s_\alpha s_\beta s_\gamma \cdot 0 + \alpha\right) = 0.$$

In this case the exact sequence (6.38) implies that

$$H^2\left(\bar{B}, \bar{u}^* \otimes s_\alpha s_\beta s_\gamma \cdot 0\right) \simeq H^2\left(\bar{B}, \bar{V}_\alpha \otimes s_\alpha s_\beta s_\gamma \cdot 0\right) \simeq H^1\left(\bar{B}, \bar{V}_\alpha \otimes s_\beta s_\gamma \cdot 0\right).$$

Here the last isomorphism comes from the fact that $\langle s_\alpha s_\beta s_\gamma \cdot 0 + \rho, \alpha^\vee \rangle < 0$. Arguing as before, we get

$$H^4(B_q, l s_\alpha s_\beta s_\gamma \cdot 0) \simeq H^2\left(\bar{B}, \bar{u}^* \otimes s_\alpha s_\beta s_\gamma \cdot 0\right) \simeq \begin{cases} k^2 & \text{if } \alpha \neq \gamma \text{ and } a_{\beta\gamma} = 0, \\ k & \text{otherwise.} \end{cases}$$

Summarizing, we obtain

$$H^4\left(B_q, l s_\alpha s_\beta s_\gamma \cdot 0\right) \simeq \begin{cases} k^3 & \text{if } \langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle = 0,\ \alpha \neq \gamma \text{ and } a_{\beta\gamma} = 0, \\ k^2 & \text{if } \langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle = 0 \text{ and } \alpha = \gamma \text{ or } a_{\beta\gamma} \neq 0, \\ k^2 & \text{if } \langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle \neq 0,\ \alpha \neq \gamma \text{ and } a_{\beta\gamma} = 0, \\ k & \text{if } \langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle \neq 0, \text{ and } \alpha = \gamma \text{ or } a_{\beta\gamma} \neq 0. \end{cases} \tag{6.42}$$

6.5.6. We are now able to compute the fourth cohomology $H^4(B_q, \mu)$. According to (6.6), we only need to consider μ's of the form $\mu = l\lambda$ for some $\lambda \in X$.

Proposition 6.15. *Let $\lambda \in X$ and assume that $\lambda \neq s_\alpha s_\beta s_\gamma \cdot 0 + \alpha$ for all $\alpha, \beta, \gamma \in S$ with $l(s_\alpha s_\beta s_\gamma) = 3$ and $\langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle \geq 1$. Then we have*

$$H^4(B_q, l\lambda) \simeq \begin{cases} k & \text{if } \lambda = -2\alpha \text{ for } \alpha \in S, \\ k & \text{if } \lambda = -\alpha - \beta \text{ for } \alpha, \beta \in S \text{ with } a_{\alpha\beta} = 0, -1, \\ k & \text{if } \lambda = s_\alpha s_\beta \cdot 0 - \alpha \text{ for } \alpha \in S, \beta \in S \backslash \{\alpha\}, \\ k & \text{if } \lambda = 2 s_\alpha s_\beta \cdot 0 \text{ for } \alpha \in S, \beta \in S \backslash \{\alpha\} \text{ with } a_{\alpha\beta} \neq -1, \\ k & \text{if } \lambda = s_\alpha \cdot (-\beta - \gamma) \text{ for } \alpha \in S, \beta, \gamma \in S \backslash \{\alpha\} \text{ with } a_{\gamma\beta} < 0, \\ k^3 & \text{if } \lambda = s_\alpha s_\beta s_\gamma \cdot 0 \text{ for } \alpha, \beta, \gamma \in S, \\ & \text{such that } \langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle = 0, \ \alpha \neq \gamma \text{ and } a_{\beta\gamma} = 0, \\ k^2 & \text{if } \lambda = s_\alpha s_\beta s_\gamma \cdot 0 \text{ for some } \alpha, \beta, \gamma \in S \\ & \text{such that } \langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle = 0 \text{ and } \alpha = \gamma \text{ or } a_{\beta\gamma} \neq 0, \\ k^2 & \text{if } \lambda = s_\alpha s_\beta s_\gamma \cdot 0 \text{ for } \alpha, \beta, \gamma \in S \\ & \text{such that } \langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle > 0, \ \alpha \neq \gamma \text{ and } a_{\beta\gamma} = 0, \\ k & \text{if } \lambda = s_\alpha s_\beta s_\gamma \cdot 0 \text{ for } \alpha, \beta, \gamma \in S \\ & \text{such that } \langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle > 0, \text{ and } \alpha = \gamma \text{ or } a_{\beta\gamma} \neq 0, \\ k & \text{if } \lambda = w \cdot 0 \text{ for } w \in W(4), \\ 0 & \text{otherwise.} \end{cases}$$

We return later to the case where $\lambda = s_\alpha s_\beta s_\gamma \cdot 0 + \alpha$.

Proof. The proof follows the strategy described in Section 6.3. So choose $\alpha \in S$ with $\langle \lambda, \alpha^\vee \rangle < 0$ and set

$$\mu = s_\alpha \cdot l\lambda = l(s_\alpha \lambda) - \alpha \in lX - \alpha.$$

Note that $\langle \mu, \alpha^\vee \rangle = -l \langle \lambda, \alpha^\vee \rangle - 2$.

If $\langle \lambda, \alpha^\vee \rangle = -1$, then $\langle \mu, \alpha^\vee \rangle = l - 2$. In this case we obtain from Theorem 6.7 that $H^3(B_q, \mu) = 0$ unless $\mu = lw \cdot 0 - \alpha$ for $w \in W(2)$ with $\langle w \cdot 0, \alpha^\vee \rangle = 1$ or $\mu = -l\beta - \alpha$ for $\beta \in S$ with $a_{\beta\alpha} = -1$. By assumption, the first possibility does not occur. So

$$H^4(B_q, l\lambda) \simeq H^3(B_q, \mu) \simeq \begin{cases} k & \text{if } \mu = -l\beta - \alpha \text{ for } \beta \in S \text{ with } a_{\beta\alpha} = -1, \\ 0 & \text{otherwise.} \end{cases}$$

This gives

$$H^4(B_q, l\lambda) \simeq \begin{cases} k & \text{if } \lambda = -\beta - \alpha \text{ for } \beta \in S \text{ with } a_{\beta\alpha} = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose now that $\langle \lambda, \alpha^\vee \rangle < -1$ and set $\mu' = \mu - (l-1)\alpha \in lX$. Then we have a short exact sequence

$$0 \to L_\alpha(\mu') \to H^1_\alpha(l\lambda) \to L_\alpha(\mu) \to 0,$$

which gives rise to the following exact sequence

$$\cdots \to H^3(B_q, L_\alpha(\mu')) \to H^4(B_q, l\lambda) \to H^3(B_q, L_\alpha(\mu)) \to \cdots . \tag{6.43}$$

Consider first the case where $H^3(B_q, L_\alpha(\mu')) \neq 0$. Since $\mu' \in lX$, there are by Proposition 6.14 seven cases to consider.

(1) If $\mu' = -2l\beta$ for $\beta \in S\backslash\{\alpha\}$, then $\mu = -2l\beta + (l-1)\alpha$ and $l\lambda = 2ls_\alpha s_\beta \cdot 0$. When we combine Proposition 6.13 and Proposition 6.14, we get
$$H^3(B_q, L_\alpha(-2l\beta + (l-1)\alpha)) = H^2(B_q, L_\alpha(-2l\beta + (l-1)\alpha)) = 0.$$
In this case (6.43) implies
$$H^4\left(B_q, 2ls_\alpha s_\beta \cdot 0\right) \simeq H^3\left(B_q, L_\alpha(-2l\beta)\right) \simeq \begin{cases} 0 & \text{if } a_{\alpha\beta} = -1, \\ k & \text{otherwise.} \end{cases}$$
The last isomorphism follows from Proposition 6.14 and (6.35).

(2) If $\mu' = -l\beta$ for $\beta \in S\backslash\{\alpha\}$, then $\mu = -l\beta + (l-1)\alpha$ and $l\lambda = ls_\alpha s_\beta \cdot 0$. Hence we get the desired result from (6.34).

(3) If $\mu' = lw \cdot 0$ for $w \in W_\alpha(2)$, then $\mu = lw \cdot 0 + (l-1)\alpha$ and $l\lambda = ls_\alpha w \cdot 0$. Clearly, $l(s_\alpha w) = 3$, otherwise we have $l(s_\alpha w) = 1$ which clearly contradicts the assumption that $w \in W_\alpha(2)$. We now refer to (6.42).

(4) If $\mu' = -l\beta + l\alpha$ for $\beta \in S\backslash\{\alpha\}$, then $\mu = -l\beta + 2l\alpha - \alpha$ and $l\lambda = l(s_\alpha s_\beta \cdot 0 - \alpha)$. By Proposition 6.13 and Proposition 6.14, we get
$$H^3(B_q, L_\alpha(-l\beta + 2l\alpha - \alpha)) \simeq H^2(B_q, L_\alpha(-l\beta + 2l\alpha - \alpha)) = 0.$$
Using this together with (6.43), it follows
$$H^4\left(B_q, l(s_\alpha s_\beta \cdot 0 - \alpha)\right) \simeq H^3\left(B_q, L_\alpha(-l\beta + l\alpha)\right) \simeq k.$$

(5) If $\mu' = l(-\alpha - \beta)$ for $\beta \in S$ with $a_{\beta\alpha} \leq -2$, then $\mu = l(-\beta) - \alpha$ and $l\lambda = ls_\alpha(-\beta)$. In this case we refer to (6.33).

(6) If $\mu' = l(-\gamma-\beta)$ for $\gamma, \beta \in S\backslash\{\alpha\}$ with $a_{\gamma\beta} < 0$, then $\mu = l(-\gamma-\beta)+(l-1)\alpha$ and $l\lambda = ls_\alpha(-\gamma - \beta) - l\alpha$. Arguing as before, we have
$$\begin{aligned} H^3(B_q, L_\alpha(l(-\gamma - \beta) + (l-1)\alpha)) &= H^2(B_q, L_\alpha(l(-\gamma - \beta) + (l-1)\alpha)) \\ &= 0 \end{aligned}$$
which implies
$$H^4\left(B_q, ls_\alpha(-\gamma - \beta) - l\alpha\right) \simeq H^3\left(B_q, L_\alpha(-l\gamma - l\beta)\right) \simeq k.$$

(7) Finally, if $\mu' = lw \cdot 0$ for $w \in W_\alpha(3)$, then $\mu = lw \cdot 0 + (l-1)\alpha$ and $l\lambda = ls_\alpha w \cdot 0$. If $l(s_\alpha w) = 4$, then (6.43) implies in this case that
$$H^4(B_q, ls_\alpha w \cdot 0) \simeq H^3(B_q, L_\alpha(s_\alpha w \cdot 0)) \simeq k$$
because $H^3(B_q, L_\alpha(lw \cdot 0 + (l-1)\alpha)) = H^2(B_q, L_\alpha(lw \cdot 0 + (l-1)\alpha) = 0$. For the case where $l(s_\alpha w) = 2$, see (6.34).

Suppose now that $H^3(B_q, L_\alpha(\mu')) = 0$ and $H^3(B_q, L_\alpha(\mu)) \neq 0$. Since $\mu \in lX - \alpha$, there are 5 cases in Proposition 6.14 to consider.

(1) If $\mu = -\alpha - \beta l$ for $\beta \in S$ with $a_{\beta\alpha} < -1$, then $l\lambda = -ls_\alpha(\beta)$. The desired result follows from (6.33).

(2) If $\mu = -l\beta + (l-1)\alpha$ for $\beta \in S\backslash\{\alpha\}$, then $l\lambda = ls_\alpha(-\beta + \alpha) = ls_\alpha s_\beta \cdot 0$. This case was treated in (6.34).

(3) If $\mu = l\alpha + (l-1)\alpha$, then $l\lambda = -2l\alpha$. In this case we refer to (6.32).

(4) If $\mu = lw \cdot 0 + (l-1)\alpha$ for $w \in W_\alpha(2)$, then $l\lambda = ls_\alpha w \cdot 0$. As before, we have $l(s_\alpha w) = 3$, and now the desired result for $\lambda = s_\alpha w$ follows from (6.42).

(5) Finally, if $\mu = lw \cdot 0 - \alpha$ for some $w \in W(2)$ with $\langle w \cdot 0, \alpha^\vee \rangle > 1$, then $l\lambda = ls_\alpha(w \cdot 0) = l(s_\alpha w \cdot 0 + \alpha)$. By assumption, $l(s_\alpha w) = 1$, and in this case we get that $\lambda = -\beta + \alpha \not< 0$ for some $\beta \in S \backslash \{\alpha\}$. Now (6.11) settles this case.

The proposition is proved. ■

6.5.7. It remains to compute $H^4(B_q, l\lambda)$ when $\lambda = s_\alpha s_\beta s_\gamma \cdot 0 + \alpha$ for some $\alpha, \beta, \gamma \in S$ with $l(s_\alpha s_\beta s_\gamma) = 3$ and $\langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle \geq 1$. Note first

$$\begin{aligned} \langle s_\alpha s_\beta s_\gamma \cdot 0 + \alpha, \alpha^\vee \rangle &= \langle s_\alpha s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle + 2 \\ &= -\langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle. \end{aligned}$$

If $\langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle = 1$, then we get via (6.23) that

$$\begin{aligned} H^4(B_q, l(s_\alpha s_\beta s_\gamma \cdot 0 + \alpha)) &\simeq H^3(B_q, l(s_\beta s_\gamma \cdot 0 - \alpha)) \\ &\simeq H^2(B_q, l(s_\beta s_\gamma \cdot 0)) \\ &\simeq k. \end{aligned}$$

Suppose now that $\langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle > 1$ and consider the spectral sequence (6.2). The only terms that may contribute are $H^4(\bar{B}, s_\alpha s_\beta s_\gamma \cdot 0 + \alpha)$, $H^2(\bar{B}, \bar{u}^* \otimes s_\alpha s_\beta s_\gamma \cdot 0 + \alpha)$ and $H^0(\bar{B}, S^2\bar{u}^* \otimes s_\alpha s_\beta s_\gamma \cdot 0 + \alpha)$. From Lemma 6.9, we get

$$H^4\left(\bar{B}, s_\alpha s_\beta s_\gamma \cdot 0 + \alpha\right) = 0. \tag{6.44}$$

On the other hand, since $\lambda = s_\alpha s_\beta s_\gamma \cdot 0 + \alpha = s_\beta s_\gamma \cdot 0 - \langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle \alpha$, one can easily show that in this case we have $-\operatorname{ht}(\lambda) > 2$. By Proposition 2.5, this means that

$$H^0\left(\bar{B}, S^2\bar{u}^* \otimes s_\alpha s_\beta s_\gamma \cdot 0 + \alpha\right) = 0.$$

Therefore

$$H^4(B_q, l(s_\alpha s_\beta s_\gamma \cdot 0 + \alpha)) \simeq H^2(\bar{B}, \bar{u}^* \otimes s_\alpha s_\beta s_\gamma \cdot 0 + \alpha). \tag{6.45}$$

We now want to compute $H^2(\bar{B}, \bar{u}^* \otimes s_\alpha s_\beta s_\gamma \cdot 0 + \alpha)$. The sequence (6.37) gives the following exact sequence

$$\begin{aligned} &H^2\left(\bar{B}, s_\alpha s_\beta s_\gamma \cdot 0 + 2\alpha\right) \to H^2\left(\bar{B}, \bar{u}^* \otimes s_\alpha s_\beta s_\gamma \cdot 0 + \alpha\right) \to \\ &H^2\left(\bar{B}, \bar{V}_\alpha \otimes s_\alpha s_\beta s_\gamma \cdot 0 + \alpha\right) \to H^3\left(\bar{B}, s_\alpha s_\beta s_\gamma \cdot 0 + 2\alpha\right). \end{aligned} \tag{6.46}$$

Via Remark 6.10, we get

$$H^3\left(\bar{B}, s_\alpha s_\beta s_\gamma \cdot 0 + 2\alpha\right) = 0,$$

and

$$s_\alpha s_\beta s_\gamma \cdot 0 + 2\alpha \in W \cdot 0 \Leftrightarrow \langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle = 1.$$

By assumption, we have $\langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle > 1$. Then (6.46) gives that

$$\begin{aligned} H^4(B_q, l(s_\alpha s_\beta s_\gamma \cdot 0 + \alpha)) &\simeq H^2\left(\bar{B}, \bar{u}^* \otimes s_\alpha s_\beta s_\gamma \cdot 0 + \alpha\right) \\ &\simeq H^2\left(\bar{B}, \bar{V}_\alpha \otimes s_\alpha s_\beta s_\gamma \cdot 0 + \alpha\right) \\ &\simeq H^1\left(\bar{B}, \bar{V}_\alpha \otimes s_\beta s_\gamma \cdot 0 - \alpha\right). \end{aligned}$$

When we combine the short exact sequence (6.37) and Proposition 2.8, we get the following sequence

$$0 \to H^1\left(\bar{B}, \bar{V}_\alpha \otimes s_\beta s_\gamma \cdot 0 - \alpha\right) \to k \to H^2\left(\bar{B}, \bar{u}^* \otimes s_\beta s_\gamma \cdot 0 - \alpha\right). \tag{6.47}$$

We now claim that $H^2\left(\bar{B}, \bar{u}^* \otimes s_\beta s_\gamma \cdot 0 - \alpha\right) = 0$. We have

$$H^4(\bar{B}, s_\beta s_\gamma \cdot 0 - \alpha) = H^0(\bar{B}, S^2\bar{u}^* \otimes s_\beta s_\gamma \cdot 0 - \alpha) = 0,$$

see Proposition 2.5 and Lemma 6.9. By Proposition 6.15, it follows that

$$H^4(B_q, l(s_\beta s_\gamma \cdot 0 - \alpha)) = 0$$

which is clearly implies that $H^2(\bar{B}, \bar{u}^* \otimes s_\beta s_\gamma \cdot 0 - \alpha) = 0$. We conclude in this case

$$H^4(B_q, l(s_\alpha s_\beta s_\gamma \cdot 0 + \alpha)) \simeq H^1\left(\bar{B}, \bar{V}_\alpha \otimes s_\beta s_\gamma \cdot 0 - \alpha\right) \simeq k.$$

Combined with Proposition 6.15, this describes completely this cohomology group $H^4(B_q, l\lambda)$ for all $\lambda \in X$.

Theorem 6.16. *Let $\lambda \in X$. Then we have*

$$H^4(B_q, l\lambda) \simeq \begin{cases} k & \text{if } \lambda = -2\alpha \text{ for } \alpha \in S, \\ k & \text{if } \lambda = -\alpha - \beta \text{ for } \alpha, \beta \in S \text{ with } a_{\alpha\beta} = 0, -1, \\ k & \text{if } \lambda = s_\alpha s_\beta \cdot 0 - \alpha \text{ for } \alpha \in S, \beta \in S \backslash \{\alpha\}, \\ k & \text{if } \lambda = 2 s_\alpha s_\beta \cdot 0 \text{ for } \alpha \in S, \beta \in S\backslash\{\alpha\} \text{ with } a_{\alpha\beta} \neq -1, \\ k & \text{if } \lambda = s_\alpha \cdot (-\beta - \gamma) \text{ for } \alpha \in S, \beta, \gamma \in S\backslash\{\alpha\} \text{ with } a_{\gamma\beta} < 0, \\ k^3 & \text{if } \lambda = s_\alpha s_\beta s_\gamma \cdot 0 \text{ for } \alpha, \beta, \gamma \in S, \\ & \text{such that } \langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle = 0,\ \alpha \neq \gamma \text{ and } a_{\beta\gamma} = 0, \\ k^2 & \text{if } \lambda = s_\alpha s_\beta s_\gamma \cdot 0 \text{ for some } \alpha, \beta, \gamma \in S \\ & \text{such that } \langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle = 0 \text{ and } \alpha = \gamma \text{ or } a_{\beta\gamma} \neq 0, \\ k^2 & \text{if } \lambda = s_\alpha s_\beta s_\gamma \cdot 0 \text{ for } \alpha, \beta, \gamma \in S \\ & \text{such that } \langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle > 0,\ \alpha \neq \gamma \text{ and } a_{\beta\gamma} = 0, \\ k & \text{if } \lambda = s_\alpha s_\beta s_\gamma \cdot 0 \text{ for } \alpha, \beta, \gamma \in S \\ & \text{such that } \langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle > 0, \text{ and } \alpha = \gamma \text{ or } a_{\beta\gamma} \neq 0, \\ k & \text{if } \lambda = w \cdot 0 \text{ for } w \in W(4), \\ k & \text{if } \lambda = s_\alpha s_\beta s_\gamma \cdot 0 + \alpha \text{ for } \alpha, \beta, \gamma \in S \\ & \text{such that } l(s_\alpha s_\beta s_\gamma) = 3 \text{ and } \langle s_\beta s_\gamma \cdot 0, \alpha^\vee \rangle \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

APPENDIX A

Further consequences of the strong linkage principle

This appendix is a continuation of Section 3.4. We shall improve the statement in Theorem 3.6 for $i \neq l(w)$.

A.1. Some exact sequences

A.1.1. Let $\lambda \in X^+$ and set $s_i = s_{\alpha_i}$ for some $\alpha_i \in S$. Then we have the following short exact sequence

$$0 \to K_i(\lambda) \to H^0_{q,i}(\lambda) \to k_\lambda \to 0. \tag{A.1}$$

Here the homomorphism $H^0_{q,i}(\lambda) \to k_\lambda$ is the evaluation map.

According to Theorem 4.1, $H^0_{q,i}(\lambda)$ has dimension $\langle \lambda, \alpha_i^\vee \rangle + 1$, and the weights of $H^0_{q,i}(\lambda)$ are $\lambda, \lambda - \alpha_i, \dots, s_i(\lambda) = \lambda - \lambda_i \alpha_i$. This shows that if $\langle \lambda, \alpha_i^\vee \rangle > 0$, then the kernel $K_i(\lambda)$ contains a B_q-invariant line (in the Hopf algebra sense) whose weight is $s_i(\lambda)$. Therefore we get the short exact sequence

$$0 \to k_{s_i(\lambda)} \to K_i(\lambda) \to Q_i(\lambda) \to 0. \tag{A.2}$$

Using the notation from Theorem 4.1, we let $\{e_0, \dots, e_{\lambda_i}\}$ be a basis of $H^0_{q,i}(\lambda)$. Suppose further that $\lambda_i > 1$, and let $\{e'_0, \dots, e'_{\lambda_i - 2}\}$ be a basis of $H^0_{q,i}(\lambda - \alpha_i)$. Moreover, let $\overline{e}_1, \dots, \overline{e}_{\lambda_i - 1}$ be the images of the basis elements $e_1, \dots, e_{\lambda_i - 1}$ in $Q_i(\lambda)$. Then we can define a non-zero B_q-homomorphism $\phi : Q_i(\lambda) \to H^0_{q,i}(\lambda - \alpha_i)$ via

$$\overline{e}_j \mapsto [\lambda_i - j]_i \, e'_{j-1} \text{ for all } j \in \{1, \dots, \lambda_i - 1\}.$$

Set $I_i(\lambda) = \operatorname{im} \phi, C_i(\lambda) = \ker \phi$ and $N_i(\lambda) = \operatorname{coker} \phi$. By Theorem 4.1, we see that the set of weights of both $C_i(\lambda)$ and $N_i(\lambda)$ is

$$\left\{ s_i(\lambda) + a l_i \alpha_i : 0 < a l_i < \langle \lambda + \rho, \alpha_i^\vee \rangle \right\},$$

and all weights occur with multiplicity 1.

If we now write $\tilde{K}_i(\lambda) = K_i(\lambda + \rho) \otimes k_{-\rho}$ and similarly for the other modules, we obtain four short exact sequences in $\mathscr{C}_q^-$

$$0 \to \tilde{K}_i(\lambda) \to H^0_{q,i}(\lambda + \rho) \otimes_k k_{-\rho} \to k_\lambda \to 0, \tag{A.3}$$

$$0 \to k_{s_i \cdot \lambda} \to \tilde{K}_i(\lambda) \to \tilde{Q}_i(\lambda) \to 0, \tag{A.4}$$

$$0 \to \tilde{C}_i(\lambda) \to \tilde{Q}_i(\lambda) \to \tilde{I}_i(\lambda) \to 0, \tag{A.5}$$

$$0 \to \tilde{I}_i(\lambda) \to H^0_{q,i}(\lambda + \rho - \alpha_i) \otimes_k k_{-\rho} \to \tilde{N}_i(\lambda) \to 0, \tag{A.6}$$

cf. [5]. The modules $\tilde{C}_i(\lambda)$ and $\tilde{N}_i(\lambda)$ both have weights

$$\left\{ s_i \cdot \lambda + a l_i \alpha_i : 0 < a l_i < \langle \lambda + \rho, \alpha_i^\vee \rangle \right\}.$$

When we combine the tensor identity with the spectral sequence (3.12), we get that for any integrable $U_{q,i}$-module V we have $H_q^j(V \otimes_k k_{-\rho}) = 0$ for all j. Applying this to the long exact sequences arising from (A.3) and (A.6), we get that

$$H_q^0\left(\tilde{I}_i(\lambda)\right) = 0, H_q^j\left(\tilde{N}_i(\lambda)\right) \simeq H_q^{j+1}\left(\tilde{I}_i(\lambda)\right) \text{ and } H_q^j(\lambda) \simeq H_q^{j+1}\left(\tilde{K}_i(\lambda)\right) \quad \text{(A.7)}$$

for all $j \geq 0$. The long exact sequences coming from (A.4) and (A.5) then imply

Proposition A.1. *Let $\lambda \in X$ and suppose that $\langle \lambda, \alpha_i^\vee \rangle \geq -1$ for some $\alpha_i \in S$. Then we have long exact sequences in $\mathscr{C}_q$*

$$\cdots \to H_q^{j+1}(s_i \cdot \lambda) \to H_q^j(\lambda) \to H_q^{j+1}\left(\tilde{Q}_i(\lambda)\right) \to \cdots$$

and

$$\cdots \to H_q^{j+1}\left(\tilde{C}_i(\lambda)\right) \to H_q^{j+1}\left(\tilde{Q}_i(\lambda)\right) \to H_q^j\left(\tilde{N}_i(\lambda)\right) \to \cdots.$$

A.1.2. Suppose now that $\lambda \in X^+ - \rho$ and let $w_0 = s_1 s_2 \cdots s_N$ be a reduced expression for the longest element $w_0 \in W$ where $s_i = s_{\alpha_i}$ for $\alpha_i \in S$. If we define λ_m to be $(s_m \cdots s_1) \cdot \lambda$ for $m \in \{1, \ldots, N\}$, then we get from [5, Lemma 3.11] that if $\lambda \in X^+$, there is a unique, up to a scalar, non-zero homomorphism in $\mathscr{C}_q$

$$H_q^N(\lambda_N) \to H_q^0(\lambda).$$

The image of this homomorphism is $L_q(\lambda)$.

Proposition A.2. *Let $\lambda \in X^+ - \rho$ and suppose that $L_q(\mu)$ is a composition factor of $H_q^i(\lambda_m)$ for some $i \in \{0, \ldots, N\}$ and $m \in \{0, \ldots, N\}$, then one of the following conditions is satisfied*

(1) $L_q(\mu)$ occurs as a composition factor of the image of the composite

$$H_q^{N+i-m}(\lambda_N) \to H_q^{N+i-m-1}(\lambda_{N-1}) \to \cdots \to H_q^i(\lambda_m),$$

(2) $L_q(\mu)$ is a composition factor of $H_q^r(\lambda_{j+1} + al_{j+1}\alpha_{j+1})$ for some $r \geq i + j - m \geq i$ and $0 < al_{j+1} < \langle \lambda_j + \rho, (\alpha_{j+1})^\vee \rangle$.

Proof. Repeated use of Proposition A.1 shows that we have either $L_q(\mu)$ is a composition factor of the image of the composed homomorphism

$$H_q^{N+i-m}(\lambda_N) \to H_q^i(\lambda_m),$$

or $L_q(\mu)$ is a composition factor of $H_q^{i+j-m+1}(\tilde{C}_{j+1}(\lambda_j))$ or $H_q^{i+j-m}(\tilde{N}_{j+1}(\lambda_j))$ for some $j \in \mathbb{N}$ with $j \geq m$.

Since we know all the weights of $\tilde{C}_{j+1}(\lambda_j)$ and $\tilde{N}_{j+1}(\lambda_j)$, then by looking at the long exact sequences arising from taking full filtrations of these two B_q-modules, we see that the last possibility implies that $L_q(\mu)$ is a composition factor of $H_q^r(\lambda_{j+1} + al_{j+1}\alpha_{j+1})$ for some $r \geq i + j - m \geq i$ and $0 < al_{j+1} < \langle \lambda_j + \rho, (\alpha_{j+1})^\vee \rangle$. ■

Similarly, one can show

Proposition A.3. *Let $\lambda \in X^+ - \rho$ and suppose that $L_q(\mu)$ is a composition factor of $H_q^i(\lambda_m)$ for some $m \in \{0, \dots, N\}$ and $i \in \{0, \dots, m\}$, then one of the following conditions is satisfied*

(1) *$L_q(\mu)$ occurs as a composition factor of the image of the composite*

$$H_q^i(\lambda_m) \to H_q^{i-1}(\lambda_{m-1}) \to \cdots \to H_q^0(\lambda_{m-i}),$$

(2) *$L_q(\mu)$ is a composition factor of $H_q^r(\lambda_{j+1} + al_{j+1}\alpha_{j+1})$ for some $r \geq i + j - m - 1 \geq 0$ and $0 < al_{j+1} < \langle \lambda_j + \rho, \alpha_{j+1}{}^\vee \rangle$.*

A.2. Some results on weights

Let $\lambda \in X^+ - \rho$. By Theorem 3.6, if $w \in W$ and $j \in \mathbb{N}$, then all weights of $H_q^j(w \cdot \lambda)$ are in

$$\Pi(\lambda) = \{\mu \in X \mid w(\mu) \leq \lambda \text{ for all } w \in W\}.$$

We shall now show that we further have

Proposition A.4. *Let $\lambda \in X^+ - \rho$ and let $w \in W$. Suppose that $i \neq l(w)$, then all the weights of $H_q^i(w \cdot \lambda)$ are strictly less than λ.*

To prove this result, we need the following lemma:

Lemma A.5. *Suppose that $\lambda \in X^+ - \rho$. Let $w \in W$ and $i \in \{1, \dots, n\}$ such that $s_i w > w$. If $a \in \mathbb{N}$ satisfies $0 < al_i < \langle w(\lambda + \rho), \alpha_i^\vee \rangle$, then $\mu = y \cdot (s_i w \cdot \lambda + al_i \alpha_i)$ is strongly linked to and strictly less than λ for all $y \in W_l$.*

Proof. We use Lemma 3.8. Suppose that $y(\alpha_i) < 0$, then we have $\mu = ys_i w \cdot \lambda + al_i y(\alpha_i)$ is strongly linked to $ys_i w \cdot \lambda$ and hence to λ. By assumption, we have that if $y(\alpha_i) > 0$, then $\mu = s_{y(\alpha_i)} yw \cdot \lambda + al_i y(\alpha_i)$ is strongly linked to $yw \cdot \lambda$ and hence strongly linked to λ. We have in each case that μ is strictly less than λ. ■

Proof of Proposition A.4. Let $L_q(\mu)$ be a composition factor of $H_q^i(w \cdot \lambda)$. We claim that μ is strictly less than λ when $i \neq l(w)$. Let $w_0 = s_1 \cdots s_N$ be a reduced expression for w_0 such that $w = s_m \cdots s_1$. Hence $\lambda_m = (s_m \cdots s_1) \cdot \lambda = w \cdot \lambda$.

Suppose that $i > l(w) = m$. We then have $N + i - m > N$, and hence $H_q^{N+i-m}(\lambda_N) = 0$. Using this together with Proposition A.2, we see that $L_q(\mu)$ is a composition factor of $H_q^r(\lambda_{j+1} + al_{j+1}\alpha_{j+1})$ for some $r \geq i + j - m \geq i$ and $0 < al_{j+1} < \langle \lambda_j + \rho, (\alpha_{j+1})^\vee \rangle$. Therefore we get from the strong linkage principle that μ is strongly linked to $y \cdot (\lambda_{j+1} + al_{j+1}\alpha_{j+1})$ for some $y \in W$ such that $y \cdot (\lambda_{j+1} + al_{j+1}\alpha_{j+1}) \in X^+ - \rho$. Lemma A.5 then implies that μ is strictly less than λ. This settles the case $i > l(w)$.

A similar argument works for the case $i < l(w)$. Use Proposition A.3 instead of Proposition A.2. ■

Theorem A.6. *Let $\lambda \in X^+$ and $w \in W$. Then λ is the unique highest weight of $H_q^{l(w)}(w \cdot \lambda)$, and it occurs with multiplicity* 1.

To prove this theorem, we need the following lemma:

Lemma A.7. *Suppose that $\lambda \in X^+ - \rho$. Let $w \in W$ such that $\langle w(\lambda+\rho), \alpha_i^\vee \rangle \geq 0$ for some $\alpha_i \in S$. If $L_q(\mu)$ is a composition factor of some $H_q^j(\tilde{Q}_i(w \cdot \lambda))$, then μ is strongly linked to and strictly less than λ.*

Proof. It is enough to prove the result for $H_q^j(\tilde{N}_i(w\cdot\lambda))$ and $H_q^j(\tilde{C}_i(w\cdot\lambda))$. The lemma then follows immediately from the strong linkage principle and Lemma A.5, cf. the argument given in Proposition A.4. ■

Proof of Theorem A.6. Let $w_0 = s_1 \cdots s_N$ be a reduced expression for w_0 such that $w = s_m \cdots s_1$. Hence $\lambda_m = (s_m \cdots s_1)\cdot\lambda = w\cdot\lambda$. We get from Proposition A.1 an exact sequence

$$\cdots \to H_q^j\left(\tilde{Q}_j(\lambda_{j+1})\right) \to H_q^{j+1}(\lambda_{j+1}) \to H_q^j(\lambda_j) \to H_q^{j+1}\left(\tilde{Q}_i(\lambda_{j+1})\right) \to \cdots .$$

Lemma A.7 gives that $L_q(\lambda)$ does not occur as a composition factor of the kernel or the cokernel of these homomorphisms

$$H_q^{j+1}(\lambda_{j+1}) \to H_q^j(\lambda_j).$$

This implies that $L_q(\lambda)$ occurs in each $H_q^j(\lambda_j)$ with the same multiplicity as in the image M of the composed homomorphism

$$H_q^N(\lambda_N) \to H_q^0(\lambda).$$

Since $L_q(\lambda)$ is a composition factor of $H_q^0(\lambda)$ with multiplicity 1, it is also a composition factor of M with the same multiplicity. This completes the proof. ■

Corollary A.8. *Let λ, w as in Theorem A.6. Then $H_q^{l(w)}(w\cdot\lambda) \neq 0$.*

APPENDIX B

The Steinberg module

Set $\sigma_l = (1/2)\sum_{\alpha\in R^+}(l_\alpha - 1)\alpha$. We call this the Steinberg weight. The corresponding simple U_q-module $L_q(\sigma_l)$ is called the Steinberg module and denoted by St_l. In this appendix we collect some properties of the Steinberg module. From [5, Corollary 4.7] we get that

$$\mathrm{St}_l = L_q(\sigma_l) \simeq H^0_q(\sigma_l).$$

Note that $\langle \sigma_l, \alpha_i^\vee\rangle = l_i - 1$ for all $i \in I$.

Set $\hat{u}_q = B_q u_q^+$ and $\tilde{u}_q = u_q U_q^0$. We will be mainly interested in the following induction functors

$$\hat{Z}_q = H^0_q(\hat{u}_q/B_q, -),$$
$$\tilde{Z}_q = H^0_q(\tilde{u}_q/u_q^- U_q^0, -).$$

For any $\lambda \in X$ we shall write $\hat{Z}_q(\lambda)$, respectively $\tilde{Z}_q(\lambda)$, instead of $\hat{Z}_q(k_\lambda)$, respectively $\tilde{Z}_q(k_\lambda)$. We denote by X_l the set of restricted weights

$$X_l = \left\{\lambda \in X \mid 0 \le \left\langle \lambda, \alpha_i^\vee \right\rangle < l_i \text{ for all } \alpha_i \in S\right\}.$$

Recall the canonical monomorphism $\psi : X \to X^*$, see Subsection 5.1.1. Each $\lambda \in X$ can be decomposed uniquely $\lambda = \lambda_1 + \psi(\lambda_2)$ where $\lambda_1 \in X_l$ and $\lambda_2 \in X$.

Proposition B.1. *For any $\lambda \in X$ we have*

(1) $\hat{Z}_q(\lambda)$ has a unique u_q^+-stable line whose weight is λ.
(2) Conversely, if V is an integrable $\hat{u}_q$-module having a unique u_q^+-stable line whose weight is λ, then V is isomorphic to a submodule of $\hat{Z}_q(\lambda)$.

In particular, we have that $\hat{Z}_q(\lambda)$ contains a unique simple submodule of highest weight λ. We denote this submodule by $\hat{L}_q(\lambda)$. Furthermore, each simple module $\hat{u}_q$-module in $\mathscr{C}_q(\hat{u}_q)$ is isomorphic to $\hat{L}_q(\lambda)$ for some $\lambda \in X$.

Proof. Exactly as in [11, Proposition 5.2.2], we see that

$$\hat{Z}_q(\lambda)_\lambda = \hat{Z}_q(\lambda)^{u_q^+} \simeq k.$$

This settles the first claim. Clearly, λ is a maximal weight of V, then the projection $\varphi : V \to k_\lambda$ is a non-zero B_q-homomorphism. We can assume that φ is non-zero on the socle of V since the one-dimensional space of weight λ is in the socle of V. Using Frobenius reciprocity, we obtain a non-zero $\hat{u}_q$-homomorphism $\hat{\varphi} : V \to \hat{Z}_q(\lambda)$. The one-dimensional space of weight λ in the socle of V is not in the kernel of $\hat{\varphi}$. The second claim follows. ■

Similarly, one can show that $\tilde{Z}_q(\lambda)$ contains a unique simple submodule of highest weight λ. We denote this simple module by $\tilde{L}_q(\lambda)$. We also have that each simple module in $\mathscr{C}_q(\tilde{u}_q)$ is isomorphic to $\tilde{L}_q(\lambda)$ for some $\lambda \in X$.

Remark B.2. From Lemma 3.1 we see immediately that there is a unique involutory antiautomorphism τ of U_q given by

$$\tau(E_i^{(r)}) = F_i^{(r)}, \tau(F_i^{(r)}) = E_i^{(r)} \text{ and } \tau(K_i^{\pm}) = K_i^{\pm}$$

for all $i \in I$ and for all $r \in \mathbb{N}$. The restriction of τ to $\tilde{u}_q$ is still an antiautomorphism of $\tilde{u}_q$. Therefore, for any $\tilde{u}_q$-module N we get a new $\tilde{u}_q$-module N^τ, the contravariant dual of N, by setting $N^\tau = \mathrm{Hom}_k(N, k)$, and $\tilde{u}_q$ acts on N^τ as follows: Let $u \in \tilde{u}_q$

$$(uf)(n) = f(\tau(u)n) \text{ for all } f \in N^\tau \text{ and } n \in N.$$

If N is an integrable $\tilde{u}_q$-module, then so is N^τ with

$$N^\tau_\lambda = \{f \in N^\tau \mid f(N_\mu) = 0 \text{ for all } \mu \neq \lambda\}.$$

For any $\lambda, \mu \in X$ we clearly have

$$\tilde{L}_q(\lambda)^\tau \simeq \tilde{L}_q(\lambda) \text{ and } \mathrm{Ext}^1_{\tilde{u}_q}\left(\tilde{L}_q(\lambda), \tilde{L}_q(\mu)\right) \simeq \mathrm{Ext}^1_{\tilde{u}_q}\left(\tilde{L}_q(\mu), \tilde{L}_q(\lambda)\right). \tag{B.1}$$

Next, we shall show that if λ is restricted, then $L_q(\lambda)$ remains simple as a $\tilde{u}_q$-module. To prove this, we need the following proposition [13]:

Proposition B.3. *Let $\lambda \in X_l$. Then $H^0_q(\lambda)^{u_q^+} = H^0_q(\lambda)_\lambda \simeq k$.*

This result remains true when l is even, and the proof given in [13] will also work for the case of an arbitrary l. Proposition B.3 implies that

$$L_q(\lambda)^{u_q^+} = L_q(\lambda)_\lambda \simeq k.$$

Hence we have

Proposition B.4. *Let $\lambda \in X_l$. As a $\tilde{u}_q$-module, $L_q(\lambda)$ contains a unique simple submodule with highest weight λ.*

Proposition B.5. *Let $\lambda \in X_l$. Then $L_q(\lambda)$ is simple as a $\tilde{u}_q$-module.*

Proof. The above proposition shows that $L_q(\lambda)$ has $\tilde{L}_q(\lambda)$ as the unique simple $\tilde{u}_q$-submodule. On the other hand, we get from Remark B.2 that $L_q(\lambda)$ has $\tilde{L}_q(\lambda)$ as the unique simple quotient. Since $\dim_k L_q(\lambda)_\lambda = 1$, the proposition follows. ■

Then we have the following result:

Theorem B.6. *Let $\lambda \in X$ and write $\lambda = \lambda_1 + \psi(\lambda_2)$ with $\lambda_1 \in X_l$. Then $L_q(\lambda_1)$ is simple as a $\tilde{u}_q$-module, and therefore simple as a $\hat{u}_q$-module. We have that*

$$\tilde{L}_q(\lambda) \simeq L_q(\lambda_1) \otimes_k \psi(\lambda_2) \text{ and } \hat{L}_q(\lambda) \simeq L_q(\lambda_1) \otimes_k \psi(\lambda_2).$$

Corollary B.7. *The Steinberg module is simple as $\tilde{u}_q$-module, and therefore simple as $\hat{u}_q$-module.*

Remark B.8. Looking closely at the action of U_q^0 on the one-dimensional module $\psi(\lambda_2)^*$, we get

$$\begin{aligned}\tilde{L}_q(\lambda)^* &\simeq (L_q(\lambda_1) \otimes_k \psi(\lambda_2))^* \simeq L_q(\lambda_1)^* \otimes_k -\psi(\lambda_2)\\ &\simeq L_q(-w_0\lambda_1) \otimes_k -\psi(\lambda_2)\\ &\simeq \tilde{L}_q\left(-w_0\lambda_1 - \psi(\lambda_2)\right)\end{aligned}$$

Similarly, one gets $\hat{L}_q(\lambda)^* \simeq \hat{L}_q(-w_0\lambda_1 - \psi(\lambda_2))$.

Proposition B.9. *For any integrable $u_q^- U_q^0$-module M we have*

$$\tilde{Z}_q(M) \simeq \mathrm{Hom}_k(u_q^+, M)$$

as vector spaces. Similarly, for any integrable B_q-module M we have

$$\hat{Z}_q(M) \simeq \mathrm{Hom}_k(u_q^+, M)$$

as vector spaces.

We even have an isomorphism of $U_q^0 u_q^+$-modules. Exactly as in [12, (1.1) and (1.2)], we can show that this result remains true when l is even. The proof only relies on [29, Theorem 8.3] and [30, 2.3(h) and Lemma 2.5 (d) (e)] which are valid for all $l \in \mathbb{N}$.

By looking on the action of u_q^+ on $\hat{Z}_q(k_\lambda)$ and $\tilde{Z}_q(k_\lambda)$, we get that $\lambda - 2\sigma_l$ is the unique minimal weight of $\tilde{Z}_q(\lambda)$ and $\hat{Z}_q(\lambda)$. Moreover, since the subalgebra u_q^+ is finite dimensional, we have

Proposition B.10. *(1) The functors $\hat{Z}_q$ and $\tilde{Z}_q$ are exact.*
(2) For any $\lambda \in X$ we have an isomorphism of $\tilde{u}_q$-modules $\hat{Z}_q(\lambda) \simeq \tilde{Z}_q(\lambda)$.

Proposition B.11. *We have isomorphism of $\hat{u}_q$-modules, resp. of $\tilde{u}_q$-modules, $\hat{Z}_q(\lambda)^* \simeq \hat{Z}_q(2\sigma_l - \lambda)$, resp. $\tilde{Z}_q(\lambda)^* \simeq \tilde{Z}_q(2\sigma_l - \lambda)$ for all $\lambda \in X$.*

Proof. As a u_q^+-module, $\hat{Z}_q(\lambda)$ is the injective hull, and the projective cover of the trivial module. This is a direct consequence of Proposition B.9. Hence, as a $U_q^0 u_q^+$-module, $\hat{Z}_q(\lambda)$ is the projective cover of the one-dimensional module $\lambda - 2\sigma_l$ since $\lambda - 2\sigma_l$ is the unique minimal weight of $\hat{Z}_q(\lambda)$. This means that $\hat{Z}_q(\lambda)$ has a simple head $2\sigma_l - \lambda$, and then $\hat{Z}_q(\lambda)^*$ (defined via the antipode ι) has a simple socle $2\sigma_l - \lambda$. Therefore we have a non-zero $u_q^- U_q^0$-homomorphism $\hat{Z}_q(\lambda)^* \to 2\sigma_l - \lambda$. Using Frobenius reciprocity, we obtain a non-zero $\hat{u}_q$-homomorphism $\hat{Z}_q(\lambda)^* \to \hat{Z}_q(2\sigma_l - \lambda)$.

Since $\hat{Z}_q(\lambda)^*$ has a simple socle $2\sigma_l - \lambda$ as a $U_q^0 u_q^+$-module, this homomorphism has to be injective, and then an isomorphism because of the dimension. The second claim follows immediately from Proposition B.10. ■

Proposition B.12. *$\tilde{Z}_q(\sigma_l)$ and $\hat{Z}_q(\sigma_l)$ are simple.*

Proof. By Proposition B.1, $\hat{L}_q(\sigma_l)$ is the unique simple submodule of $\hat{Z}_q(\sigma_l)$. Since $\hat{L}_q(\sigma_l)$ is self dual, then Proposition B.11 shows that $\hat{Z}_q(\sigma_l)$ has $\hat{L}_q(\sigma_l)$ as the unique simple quotient. We have $\dim_k \hat{Z}_q(\sigma_l)_{\sigma_l} = 1$, hence $\hat{Z}_q(\sigma_l)$ must be simple. Theorem B.6 implies that $\hat{Z}_q(\sigma_l)$ is still simple as a $\tilde{u}_q$-module. ■

Corollary B.13. *The Steinberg module St_l is isomorphic to $\hat{Z}_q(\sigma_l)$, respectively to $\tilde{Z}_q(\sigma_l)$, as a $\hat{u}_q$-module, respectively as a $\tilde{u}_q$-module.*

Proof. Using Frobenius reciprocity, we obtain a $\hat{u}_q$-homomorphism $\mathrm{St}_l \to \hat{Z}_q(\sigma_l)$. Since both modules are simple, we obtain the first isomorphism. Similarly, one get that St_l is isomorphic to $\tilde{Z}_q(\sigma_l)$. ■

We will now show that St_l is injective in $\mathscr{C}_q(u_q)$. Clearly, it suffices to show that St_l is injective in $\mathscr{C}_k(\tilde{u}_q)$, cf. the argument in [12, 4.1].

Proposition B.14. *The Steinberg module St_l is injective in $\mathscr{C}_q(\tilde{u}_q)$.*

We need the following lemma:

Lemma B.15. *Let $\lambda, \mu \in X$. If $\mathrm{Ext}^1_{\tilde{u}_q}\left(\tilde{L}_q(\lambda), \tilde{Z}_q(\mu)\right) \neq 0$, then $\lambda > \mu$.*

Proof. The argument used in [2, Proposition 4.2] will also work for the case of $\tilde{u}_q$. One only has to replace B_q with $u_q^- U_q^0$. ■

Proof of Proposition B.14. Since $\mathrm{St}_l \simeq \tilde{Z}_q(\sigma_l)$, we get from Lemma B.15 that

$$\mathrm{Ext}^1_{\tilde{u}_q}\left(\tilde{L}_q(\lambda), \mathrm{St}_l\right) \neq 0 \Rightarrow \lambda > \sigma_l.$$

Suppose now that $\lambda > \sigma_l$. We have the following short exact sequence in $\mathscr{C}_q(\tilde{u}_q)$

$$0 \to \tilde{L}_q(\lambda) \to \tilde{Z}_q(\lambda) \to Q \to 0.$$

Any homomorphism $\varphi : \mathrm{St}_l \to Q$ has to be injective. Since $\dim_k Q < \dim \mathrm{St}_l$, it follows that $\mathrm{Hom}_{\tilde{u}_q}(\mathrm{St}_l, Q) = 0$. Hence

$$\mathrm{Ext}^1_{\tilde{u}_q}\left(\tilde{L}_q(\lambda), \mathrm{St}_l\right) \simeq \mathrm{Ext}^1_{\tilde{u}_q}\left(\mathrm{St}_l, \tilde{L}_q(\lambda)\right) \hookrightarrow \mathrm{Ext}^1_{\tilde{u}_q}\left(\mathrm{St}_l, \tilde{Z}_q(\lambda)\right) = 0.$$

This finishes the proof. ■

Similarly, one can show that St_l is injective in $\mathscr{C}_q(\hat{u}_q)$.

Corollary B.16. *The induction functors $H_q^0\left(U_q/u_q, -\right)$ and $H_q^0\left(\hat{u}_q/u_q, -\right)$ are exact.*

Proof. Let $M \in \mathscr{C}_q(u_q)$. Using the tensor identity, we get that

$$\mathrm{St}_l \otimes_k H_q^j\,(U_q/u_q, M) \simeq H_q^j\,(U_q/u_q, \mathrm{St}_l \otimes_k M) = 0 \text{ for all } j > 0,$$

because $\mathrm{St}_l \otimes_k M$ is injective in $\mathscr{C}_q(u_q)$, see [25, Appendix T]. A similar argument works for $H_q^0\,(\tilde{u}_q/u_q, -)$. ■

APPENDIX C

Base change

In [11] Andersen, Polo and Kexin proved some important results on base change for the derived functors of induction. In this appendix we shall demonstrate that these results have analogues for the B-cohomology.

C.1. Let v be an indeterminate, and let $\mathfrak{m}$ be the ideal in $\mathbb{Z}[v]$ generated by $v-1$ and an odd prime p. We assume that $p > 3$ if the root system R contains a component of type G_2. Set now

$$\phi_p = \frac{v^p - 1}{v - 1},$$

and then we define $\mathcal{A}'$ to be

$$\mathcal{A}' = \mathbb{Z}[v]_{\mathfrak{m}}/(\phi_p) = \mathbb{Z}_p[\zeta]_{(\zeta-1)}$$

where ζ is a p-th root of unity.

Let $B_\Gamma = B_{\mathcal{A}'} \otimes \Gamma$ and $U^0_\Gamma = U^0_{\mathcal{A}'} \otimes \Gamma$ for some $\mathcal{A}'$-algebra Γ. If M is a $B_{\mathcal{A}'}$-module, we shall write M_Γ for the B_Γ-module $M \otimes \Gamma$. Our aim is to study the relations between the $B_{\mathcal{A}'}$-cohomology and B_Γ-cohomology. Note that $H^0(B_\Gamma, -)$ is now the fixed point functor for B_Γ in the Hopf algebra sense.

C.2. Let M be an integrable $B_{\mathcal{A}'}$-module and assume that M is free as $\mathcal{A}'$-module. Clearly, M is a $B_{\mathcal{A}'}$-submodule of $Q^0 = H^0\left(B_{\mathcal{A}'}/U^0_{\mathcal{A}'}, M\right)$. Set $I^0 = Q^0/M$ and define $Q^1 = H^0\left(B_{\mathcal{A}'}/U^0_{\mathcal{A}'}, Q^0/M\right)$, etc. Since $\mathcal{A}'$ is a local ring and M is free, we obtain by construction a resolution of M consisting of free $\mathcal{A}'$-modules

$$0 \to M \to Q^0 \to Q^1 \to Q^2 \to Q^3 \to \cdots, \tag{C.1}$$

where $Q^i = H^0(B_{\mathcal{A}'}/U^0_{\mathcal{A}'}, V^i)$ for some $\mathcal{A}'$-free module V^i, cf. [11, Lemma 2.18].

C.3. Set $M^i = H^0\left(B_{\mathcal{A}'}, Q^i\right)$ and let $d^i : M^i \to M^{i+1}$ be the differential in the complex $M^\bullet$. Set further $R^{-1} = M^0$, $B^i = \mathrm{Im}(d^i)$ and $R^i = \mathrm{Coker}(d^i)$ for all $i \geq 0$. Then we obtain for each $i \geq 0$ the following exact sequences

$$0 \to B^i \to M^{i+1} \to R^i \to 0. \tag{C.2}$$

$$0 \to H^i\left(B_{\mathcal{A}'}, M\right) \to R^{i-1} \to B^i \to 0. \tag{C.3}$$

Noting that $\mathcal{A}'$ is a principal ideal domain, it follows that M^i is a free $\mathcal{A}'$-submodule of Q^i. Hence (C.2) gives

$$\mathrm{Tor}_j^{\mathcal{A}'}(B^i, \Gamma) \simeq \mathrm{Tor}_{j+1}^{\mathcal{A}'}(R^i, \Gamma) = 0 \text{ for all } i \geq 0 \text{ and } j \geq 1.$$

Therefore, the long exact sequences coming from (C.2) and (C.3) give

$$0 \to \operatorname{Tor}_1^{\mathcal{A}'}(R^i, \Gamma) \to B^i_\Gamma \to M^{i+1}_\Gamma \to R^i_\Gamma \to 0. \tag{C.4}$$

$$0 \to H^i\left(B_{\mathcal{A}'}, M\right)_\Gamma \to R^{i-1}_\Gamma \to B^i_\Gamma \to 0. \tag{C.5}$$

By Frobenius reciprocity, we get

$$\begin{aligned} H^0\left(B_{\mathcal{A}'}, Q^i\right)_\Gamma &= \operatorname{Hom}_{B_{\mathcal{A}'}}\left(\mathcal{A}', Q^i\right) \otimes \Gamma \\ &= \operatorname{Hom}_{U^0_{\mathcal{A}'}}\left(\mathcal{A}', V^i\right) \otimes \Gamma \\ &= (V^i)_0 \otimes \Gamma. \end{aligned}$$

On the other hand, we have for all $i \geq 0$

$$M^i_\Gamma = Q^i_\Gamma \simeq H^0\left(B_\Gamma/U^0_\Gamma, V^i \otimes \Gamma\right),$$

cf. [11, Lemma 3.1]. Therefore

$$\begin{aligned} H^0\left(B_\Gamma, Q^i_\Gamma\right) &= \operatorname{Hom}_{B_\Gamma}\left(\Gamma, Q^i_\Gamma\right) \\ &= \operatorname{Hom}_{U^0_\Gamma}\left(\Gamma, V^i_\Gamma\right) \\ &= (V^i_\Gamma)_0 \\ &= H^0\left(B_{\mathcal{A}'}, Q^i\right)_\Gamma. \end{aligned}$$

This implies that $H^i\left(B_\Gamma, M_\Gamma\right)$ is the kernel of $R^{i-1}_\Gamma \to M^{i+1}_\Gamma$. When we combine the sequences (C.4) and (C.5), we therefore get a commutative diagram:

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & \operatorname{Tor}_1^{\mathcal{A}'}(R^i, \Gamma) & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & H^i(B_{\mathcal{A}'}, M)_\Gamma & \longrightarrow & R^{i-1}_\Gamma & \longrightarrow & B^i_\Gamma & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & H^i(B_\Gamma, M_\Gamma) & \longrightarrow & R^{i-1}_\Gamma & \longrightarrow & M^{i+1}_\Gamma & & \end{array}$$

By the snake lemma, $\operatorname{Tor}_1^{\mathcal{A}'}(R^i, \Gamma)$ is the cokernel of $H^i\left(B_{\mathcal{A}'}, M\right)_\Gamma \to H^i\left(B_\Gamma, M_\Gamma\right)$, and then

Theorem C.1. *Let Γ be an $\mathcal{A}'$-algebra. If M is an integrable $B_{\mathcal{A}'}$-module and free as an $\mathcal{A}'$-module, then*

$$0 \to H^i\left(B_{\mathcal{A}'}, M\right)_\Gamma \to H^i\left(B_\Gamma, M_\Gamma\right) \to \operatorname{Tor}_1^{\mathcal{A}'}(R^i, \Gamma) \to 0.$$

C.4. Take $\Gamma = \mathbb{Q}(\zeta)$, the fraction field of $\mathcal{A}'$, and let Γ' be a field of $\mathrm{char}(k) = p > 0$. We make k into an $\mathcal{A}'$-algebra by a choice of $q \in k^{\times}$ where q is an l-th for some $l > 1$ odd, and l is prime to 3 if the root system R contains a component of type G_2. Then we get from Theorem C.1 the following exact sequences

$$0 \to H^i\left(B_{\mathcal{A}'}, M\right)_{\Gamma} \to H^i\left(B_{\Gamma}, M_{\Gamma}\right) \to \mathrm{Tor}_1^{\mathcal{A}'}(R^i, \Gamma) \to 0,$$

$$0 \to H^i\left(B_{\mathcal{A}'}, M\right)_{\Gamma'} \to H^i\left(B_{\Gamma'}, M_{\Gamma'}\right) \to \mathrm{Tor}_1^{\mathcal{A}'}(R^i, \Gamma') \to 0.$$

Since $\mathcal{A}'$ is a principal ideal domain, it follows immediately that if $H^i\left(B_{\Gamma}, M_{\Gamma}\right)$ is non-zero, then so is $H^i\left(B_{\Gamma'}, M_{\Gamma'}\right)$. In particular, we get $H^4\left(B_{\Gamma}, M_{\Gamma}\right)$ is non-zero, then so is $H^4\left(B_{\Gamma'}, M_{\Gamma'}\right)$. Recall now Theorem 6.16 which completely describes the cohomology group $H^4\left(B_{\Gamma}, M_{\Gamma}\right)$.

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