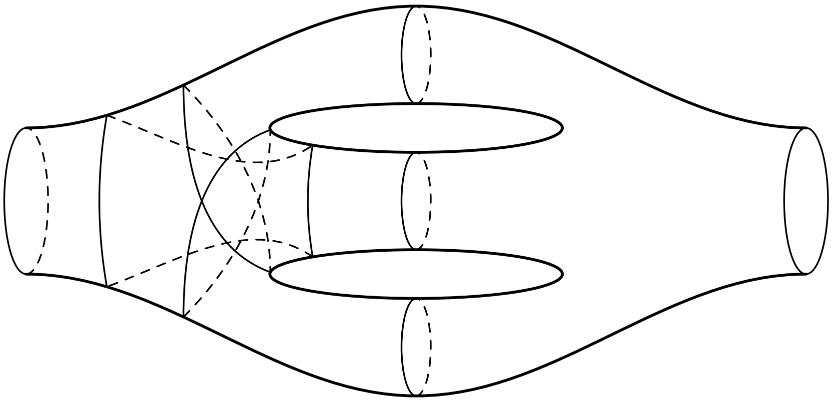
Cohomology of Mapping Class Groups with Coefficients in Functions on Moduli Spaces



Rasmus Villemoes

Colophon

Cohomology of Mapping Class Groups with Coefficients in Functions on Moduli Spaces

—Kohomologi af afbildningsklassegrupper med koefficienter i funktioner på modulirum

A PhD thesis by Rasmus Villemoes. Written under supervision by Jørgen Ellegaard Andersen at the Center for the Topology and Quantization of Moduli Spaces, Faculty of Science, University of Aarhus.

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Cohomology of Mapping Class Groups with Coefficients in Functions on Moduli Spaces

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PhD Dissertation
July 2009

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Introduction

This thesis is an exposition of the results obtained during my four years as a PhD student at the Center for the Topology and Quantization of Moduli Spaces. Some of the results have already appeared in print or on the arXiv, but a few ideas which have not yet made it into separate papers are also presented.

* * *

I will now give an ultrabrief and somewhat informal description of the background for this project. In [1], Andersen considers a family J of Kähler structures on a compact symplectic manifold (M, ω) smoothly parametrized by a manifold $\mathcal{T}$. Under certain topological assumptions, he is able to generalize work by Hitchin [25] to obtain a connection in a certain finite rank bundle over $\mathcal{T}$, a generalized Verlinde bundle. Using Toeplitz operator techniques then allows him to construct a $\mathcal{T}$-parametrized family of star products on (M, ω). If Γ is a group acting on $\mathcal{T}$ and (symplectically) on M, such that J is Γ-equivariant, one may ask if it is possible to turn this family of star products into one Γ-invariant star product on M. This turns out to depend on a number of cohomological conditions. One of these is the vanishing of the first cohomology group

$$H^1(\Gamma, C^\infty(M)) \tag{†}$$

of Γ with coefficients in the module of smooth functions on M.

One situation in which all of the above applies is when $\mathcal{T}$ is the Teichmüller space of a closed surface Σ, Γ the mapping class group of Σ and M is the moduli space of flat $\mathrm{SU}(n)$-connections over $\Sigma - \{*\}$ with fixed central holonomy around the puncture.

One of the original goals of the project was to compute (†) in the case $M = \mathcal{M}^1_{\mathrm{SU}(2)}$. While this has not yet been done, I have, jointly with Andersen, been able to prove the vanishing of $H^1(\Gamma, A)$ for certain modules of functions on certain related moduli spaces. Another motivation for studying these cohomology groups is the question of whether or not the mapping class group has Kazhdan's Property (T). Although this has been answered in the negative by Andersen [2], it may still provide insight to find other counterexamples.

* * *

The thesis is organized as follows: The first chapter serves as an introduction to group cohomology. Since we only need to »recognize a cocycle when we see one«, we provide simple working definitions of cocycles, coboundaries, H^1 and H^0. The last three sections contain a number of well-known results which will be useful in various places.

In Chapter 2, we recall the definition of the mapping class group of a surface, along with the important notion of Dehn twists and known relations between these. We also present the Dehn-Thurston coordinates on the set of multicurves (isotopy classes of 1-submanifolds of Σ). The mapping class group acts on geometric objects associated to Σ, and we use Section 2.4 to describe the action on the set of multicurves and on the first integral homology group of Σ. Speaking of the latter, in Section 2.5 we briefly touch on the kernel of the action, the Torelli group. In the final section we prove that a certain (obvious) obstruction to the vanishing of $H^1(\Gamma, V)$ is satisfied for any unitary representation V, provided $g \geq 3$. An immediate consequence is that counterexamples to Property (T) must be found among the unitary representations which do not restrict to the trivial representation of the Torelli group.

In Chapter 3 we discuss deformation quantizations of a symplectic manifold. The importance of the group (†) becomes apparent in Proposition 3.10. Section 3.4 contains a brief explanation of how geometric quantization of a Kähler manifold induces a star product on the underlying symplectic manifold. This idea is taken one step further in Section 3.5, where we assume the existence of a whole family of Kähler structures, giving a corresponding family of star products.

The so-called moduli space of flat G-connections is the subject of the fourth chapter. There are two quite different, but actually equivalent, definitions. We choose to define it as the space of G-representations of the fundamental group of Σ, modulo conjugation. The alternative definition, from which the space gets its name, is also given, along with a short explanation of the equivalence between the two definitions. (The appendix contains some standard material on principal bundles and connections.) In the following sections we explain how the moduli space is endowed with various structures. Actually, »the« moduli space is slightly misleading, since one is often forced to make small variations in the definition. For example, $\mathcal{M}_G$ is rarely a smooth manifold, but the open and dense subset represented by irreducible representations is. If one needs a manifold which is both smooth and compact, one may restrict to the representations with fixed behaviour on the boundary of Σ. This is what we do in order to obtain our favorite examples, $\mathcal{M}^d_{\mathrm{SU}(n)}$. The last section explains why the theory from Chapter 3 can be applied to these spaces.

In case $G = \mathrm{SL}_2(\mathbb{C})$, it turns out that one may use the set of multicurves on Σ as a basis for the space of regular functions on $\mathcal{M}_G$. This fact is used in Chapter 5 to prove that the cohomology group $H^1(\Gamma, \mathcal{O}(\mathcal{M}_G))$ vanishes. The computations rely on the results from Chapter 2 and on the introduction of the notions of *future* and *past* of *interesting pairs*.

In the sixth chapter, we turn our attention to the abelian moduli space $\mathcal{M}_{\mathrm{U}(1)}$. It is a trivial consequence of the results presented at the end of Chapter 2 that $H^1(\Gamma, L^2) = 0$, where L^2 denotes the space of square-integrable functions on $\mathcal{M}_{\mathrm{U}(1)}$. In [7] this is used to establish that also the cohomology group with coefficients in the space of smooth functions vanishes. However, in Chapter 6 we give an alternative proof of this, which does not rely on the knowledge of $H^1(\Gamma, L^2)$. Instead, we use the same methods as those applied in Chapter 5.

In the final chapter, we explain how a combination of the ideas from Chapters 5 and 6 may be used to prove the vanishing of the cohomology with coefficients in a certain module of »rapidly decreasing« linear combinations of multicurves. We also discuss how these linear combinations give rise to continuous functions on the $SU(2)$ moduli space. Whether or not this association is faithful is one of the open questions with which we end the dissertation.

* * *

I am grateful, first of all, to my supervisor, Jørgen Ellegaard Andersen, for sugggesting this project to me, for patiently answering my many questions, and for believing in my capabilities at times when I did not. I also wish to thank Robert Penner for numerous helpful discussions and Magnus Roed Lauridsen for volunteering to proofread the manuscript with ε-notice.

Enjoy reading!

Chapter 1

Group Cohomology

In this brief chapter, we give a rudimentary introduction to those concepts from the language of group cohomology which we will need later on. It merely serves the purpose of introducing a little notation and adapting known results to the settings we encounter. For a thorough exposition, the reader is referred to Brown's textbook [15].

1.1 Modules over Groups

Let G be a group. A (left) module over G is an abelian group M together with an action of G on M, that is, a homomorphism $G \to \mathrm{Aut}(M)$. Equivalently, A is an ordinary module over the integral group ring $\mathbb{Z}G$. If H is a subgroup of G, any module over G is also a module over H in the obvious way. This is known as restriction of scalars.

If M is a G-module, the group of invariants is the subset M^G of M fixed under the action of G; it is the largest subset of M for which the action of G is trivial. Similarly, the group of co-invariants M_G is the quotient of M by the subgroup generated by elements of the form $m - gm$, $m \in M, g \in G$. This may be thought of as the largest quotient of M on which G acts trivially.

We may define a right G-module to mean a right module over the group ring $\mathbb{Z}G$. Then as usual, if M is a right module and N is a left module over G, we may form the tensor product $M \otimes_{\mathbb{Z}G} N$ (which is only an abelian group) from the tensor product $M \otimes N = M \otimes_{\mathbb{Z}} N$ by introducing the relations $mx \otimes n = m \otimes xn$ for $x \in \mathbb{Z}G$. But since the relations $a(m \otimes n) = ma \otimes n = m \otimes an$ for $a \in \mathbb{Z}$ already hold in $M \otimes N$, we need only add the relations $mg \otimes n = m \otimes gn$ for $g \in G$. As an example, consider $\mathbb{Z}$ as a trivial right $\mathbb{Z}G$-module. Then we have an isomorphism $M_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M$ given by $[m] \mapsto 1 \otimes m$, with inverse given by $a \otimes m \mapsto a[m]$, where $[m]$ denotes the image of $m \in M$ in M_G.

For a commutative ring R, one need not distinguish between left and right modules, because a left module X becomes a right module by defining $x.r = rx$, and vice versa (in fact, this defines an isomorphism between the categories of left and right R-modules). However, for a non-commutative ring, such as the group ring of a non-abelian group, this procedure fails. But if $\varphi\colon R \to R$ is an anti-automorphism, we may turn a left module X into a right module by the formula $x \cdot r = \varphi(r)x$ (the construction above may be seen as a special case, because a ring is commutative if and only if the identity map is an anti-automorphism!).

In particular, for a group ring $\mathbb{Z}G$, the anti-automorphism $g \mapsto g^{-1}$ of G extends to an anti-automorphism of $\mathbb{Z}G$, and in this way we can make sense of the tensor product of two left G-modules M and N. Concretely this means that we obtain $M \otimes_G N$ from the abelian group $M \otimes N$ by introducing the relations $(g^{-1}m) \otimes n = mg \otimes n = m \otimes gn$, and, replacing m by gm this may be written $m \otimes n = gm \otimes gn$. Thus we have $M \otimes_G N = (M \otimes N)_G$, where G acts diagonally on $M \otimes N$: $g(m \otimes n) = gm \otimes gn$. This shows that the usual natural commutativity of the tensor product of abelian groups induces a natural commutativity of the tensor product of left G-modules, $M \otimes_G N \cong N \otimes_G M$.

1.2 Cocycles and Coboundaries

Let M be a G-module. A 1-cocycle (henceforth simply called a cocycle) on G with values in M is a map $u\colon G \to M$ satisfying the *cocycle condition*,

$$u(gh) = u(g) + gu(h) \tag{1.1}$$

for all $g, h \in G$. A cocycle is a coboundary if it is of the form $g \mapsto \delta m(g) = m - gm = (1-g)m$ for some $m \in M$. The sets of all cocycles and coboundaries are denoted by $Z^1(G, M)$ and $B^1(G, M)$, respectively. The first cohomology group of G with coefficients in M is the quotient

$$H^1(G, M) = Z^1(G, M)/B^1(G, M). \tag{1.2}$$

If one wishes to emphasize the given action $\rho\colon G \to \mathrm{Aut}(M)$ of G on M one may also denote this cohomology group by $H^1(G, \rho)$.

The »closed 0-cochains« (or 0-cocycles), the elements $m \in M$ such that $\delta m = 0$, are clearly the same as the invariant elements of M. We put

$$H^0(G, M) = M^G = \{m \in M \mid gm = m \quad \forall g \in G\}.$$

Letting $1 \in G$ denote the identity element, it follows from (1.1) that $u(1) = u(1) + u(1)$, so $u(1) = 0$ for any cocycle u. This in turn implies $0 = u(gg^{-1}) = u(g) + gu(g^{-1})$, which may be rewritten

$$u(g^{-1}) = -g^{-1}u(g). \tag{1.3}$$

Combining (1.1) and (1.3), we obtain

$$\begin{aligned} u(ghg^{-1}) &= u(g) + gu(h) + ghu(g^{-1}) \\ &= gu(h) + (1 - ghg^{-1})u(g). \end{aligned} \tag{1.4}$$

It is clear from (1.1) and (1.3) that a cocycle is determined by its values on a set of generators for G. This observation is particularly useful when the group is finitely generated.

If the action of G on M is trivial, the cocycle condition (1.1) simply means that a cocycle is a group homomorphism $G \to M$. Also, the space of coboundaries vanishes, so we have

$$H^1(G, M) = \mathrm{Hom}(G, M) = \mathrm{Hom}(G_{\mathrm{ab}}, M) \tag{1.5}$$

where G_{ab} denotes the abelianization of G.

1.3 Sums and Products

Let (M_α) be a collection of G-modules. There is an obvious bijection

$$Z^1(G, \prod_\alpha M_\alpha) \to \prod_\alpha Z^1(G, M_\alpha) \tag{1.6}$$

given by mapping a cocycle $u\colon G \to \prod_\alpha M_\alpha$ to the family (u_α) of cocycles given by $u_\alpha(g) = u(g)_\alpha$. This bijection induces an isomorphism

$$H^1(G, \prod_\alpha M_\alpha) \cong \prod_\alpha H^1(G, M_\alpha). \tag{1.7}$$

For direct sums, the situation is a little more subtle. A cocycle $u\colon G \to \bigoplus_\alpha M_\alpha$ need not give rise to an element of $\bigoplus_\alpha Z^1(G, M_\alpha)$. The problem is best explained by introducing a little notation. For $g \in G$, let A_g denote the (finite) set of indices α such that the coordinate $u(g)_\alpha$ is non-zero. Then a necessary and sufficient condition for u to define an element of $\bigoplus_\alpha Z^1(G, M_\alpha)$ is that $\bigcup_{g \in G} A_g$ is finite. One additional assumption that ensures this is that G is finitely generated, since if G is generated by $g_1, \ldots, g_N$, we have that $\bigcup_{g \in G} A_g$ is the finite union of the finite sets A_{g_j}, $j = 1, \ldots, N$. So for finitely generated groups we do have

$$H^1(G, \bigoplus_\alpha M_\alpha) \cong \bigoplus_\alpha H^1(G, M_\alpha). \tag{1.8}$$

1.4 Shapiro's Lemma

Let G be a group, H a subgroup and M a module over H. The coinduced module $\mathrm{Coind}_H^G M$ is the G-module

$$\mathrm{Coind}_H^G M = \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$$

consisting of H-equivariant homomorphisms $\mathbb{Z}G \to M$, ie. maps f satisfying $f(hx) = hf(x)$ for $h \in H$, $x \in \mathbb{Z}G$. The (left) action of G is given by $(g \cdot f)(x) = f(xg)$.

Proposition 1.1. *If G is a group, H a subgroup and M a module over H, then*

$$H^1(H, M) \cong H^1(G, \mathrm{Coind}_H^G M). \tag{1.9}$$

This is a special case of Shapiro's lemma (Proposition III.6.2 in [15]). We are going to need even more specialized versions of (1.9).

Assume G acts on a set S. For any abelian group A, the set $\mathrm{Map}(S, A)$ of all maps $S \to A$ becomes a G-module by setting $(gf)(s) = f(g^{-1}s)$. Writing S as the disjoint union $\bigsqcup_\alpha S_\alpha$ of its G-orbits, clearly

$$\mathrm{Map}(S, A) = \prod_\alpha \mathrm{Map}(S_\alpha, A) \tag{1.10}$$

as G-modules. Choose a set R of representatives for the G-orbits of S. Denote the representative of the orbit S_α by r_α. For each α, let G_α denote the subgroup of G stabilizing r_α. Then there is a bijection $G/G_\alpha \to S_\alpha$ given by $gG_\alpha \mapsto gr_\alpha$. Combining with the usual bijection between the sets of left and right cosets given by $gG_\alpha \mapsto G_\alpha g^{-1}$, we obtain induced bijections

$$\mathrm{Map}(S_\alpha, A) \leftrightarrow \mathrm{Map}(G/G_\alpha, A) \leftrightarrow \mathrm{Map}(G_\alpha\backslash G, A). \tag{1.11}$$

The latter two sets both admit a natural left action of G making them G-modules, and the bijections are then G-isomorphisms.

Considering A as a trivial G_α-module, there is an isomorphism of G-modules

$$\mathrm{Coind}_{G_\alpha}^G A = \mathrm{Hom}_{\mathbb{Z}G_\alpha}(\mathbb{Z}G, A) \to \mathrm{Map}(G_\alpha\backslash G, A) \tag{1.12}$$

given by $f \mapsto (G_\alpha g \mapsto f(g))$.

Theorem 1.2. *With the notation above, we have an isomorphism*

$$H^1(G, \mathrm{Map}(S, A)) \cong \prod_\alpha \mathrm{Hom}(G_\alpha, A). \tag{1.13}$$

Proof. Using (1.10) and (1.7), we obtain

$$H^1(G, \mathrm{Map}(S, A)) \cong \prod_\alpha H^1(G, \mathrm{Map}(S_\alpha, A)).$$

Focusing on the individual factors on the right-hand side, the isomorphisms (1.11) and (1.12) induce an isomorphism

$$H^1(G, \mathrm{Map}(S_\alpha, A)) \cong H^1(G, \mathrm{Coind}_{G_\alpha}^G A).$$

Finally, we may apply Proposition 1.1 and (1.5) to obtain (1.13). □

It will be useful to know an explicit formula for the isomorphism (1.13), and fortunately it is rather easy to describe. In the special case where the action of G on S is transitive, let $r \in S$ be some element and let G_r be the stabilizer of r. Then the isomorphism

$$\varphi\colon H^1(G, \mathrm{Map}(S, A)) \to \mathrm{Hom}(G_r, A)$$

is given by

$$\varphi([u])(g) = u(g)(r). \tag{1.14}$$

In other words, the image of the cohomology class represented by the cocycle u is the homomorphism obtained by restricting u to G_r and post-composing with evaluation in r.

The general case of course has a similar description. The image of the cohomology class $[u]$ under (1.13) is the collection of homomorphisms $u_\alpha\colon G_\alpha \to A$, whose α-coordinate is given by

$$u_\alpha(g) = u(g)(r_\alpha). \tag{1.15}$$

1.5 The Hochschild-Serre Spectral Sequence

A celebrated result relates the cohomology of a group to the cohomology of a normal subgroup and of the quotient.

Theorem 1.3 (Hochschild-Serre). *Let* $1 \to K \to G \to Q \to 1$ *be a short exact sequence of groups, and let* M *be a* G*-module. Then there is a spectral sequence*

$$E^2_{p,q} = H^p(Q, H^q(K, M)) \Longrightarrow H^{p+q}(G, M).$$

In low degrees, this spectral sequence gives an exact sequence of cohomology groups.

Proposition 1.4. *Let* $1 \to K \to G \to Q \to 1$ *be a short exact sequence of groups, and let* M *be a* G*-module. Then there is an exact sequence*

$$0 \to H^1(Q, M^K) \to H^1(G, M) \to H^1(K, M)^G. \tag{1.16}$$

A few remarks are in order: Although M is not necessarily a module over Q, the submodule $M^K = H^0(K, M)$ invariant under K has a natural structure as Q-module. Since G acts on K by group homomorphisms (via conjugation), there is an induced action on $H^1(K, M)$ making it a G-module, and $H^1(K, M)^G = H^0(G, H^1(K, M))$. For a cocycle $u\colon K \to M$ and $g \in G$, the cocycle $g \cdot u$ is given by the commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{c_g} & K \\ {\scriptstyle g\cdot u}\downarrow & & \downarrow{\scriptstyle u} \\ M & \xrightarrow{g} & M \end{array} \tag{1.17}$$

The sequence (1.16) may be continued by two H^2-terms, but we will only need the part shown above. It is not hard to give a direct proof, which does not rely on knowledge of spectral sequences, for the proposition as stated.

Proof of Proposition 1.4. The first map above is given by precomposing a cocycle $u\colon Q \to M^K$ with the projection map $\pi\colon G \to Q$. This clearly maps cocycles to cocycles. If $u \in Z^1(Q, M^K)$ is the coboundary of some element $v \in M^K$, then the cocycle $u \circ \pi \in Z^1(G, M)$ is also the coboundary of v. Hence the first map above is well-defined. Furthermore, if $u \circ \pi$ is a coboundary of some element $v \in M$, then $0 = u(\pi(k)) = (1 - k)v$ for each $k \in K$, so that in fact $v \in M^K$. Then the calculation $u(q) = u(\pi(\tilde{q})) = (1 - \tilde{q})v = (1 - q)v$, where $\tilde{q}$ is any element of G mapping to q under π, shows that u is the coboundary of v. This proves that the first map above is injective, and hence proves exactness at $H^1(Q, M^K)$.

The second map above is given by restricting a cocycle $u\colon G \to M$ to K. Clearly, the restricted map is a cocycle $K \to M$, and coboundaries map to coboundaries. To see that the map takes values in $H^1(K, M)^G$, we compute the action of $g \in G$ using (1.17):

$$\begin{aligned}(g \cdot u)(k) &= g^{-1}u(gkg^{-1}) \\ &= g^{-1}\big((1 - gkg^{-1})u(g) + gu(k)\big) \\ &= u(k) + (1 - k)g^{-1}u(g) \\ &= u(k) - \delta(g^{-1}u(g))(k)\end{aligned}$$

This shows that the cocycles u and $g \cdot u$, when restricted to K, differ by the coboundary $\delta(g^{-1}u(g))$.

Clearly, if u is a cocycle $Q \to M^K$, the composition $K \to G \to Q \to M$ is zero, so the image of the first map is contained in the kernel of second. Conversely, assume that $u\colon G \to M$ is a cocycle which satisfies $u(k) = 0$ for any $k \in K$. Such a cocycle takes values in M^K, because

$$0 = u(gkg^{-1}) = (1 - gkg^{-1})u(g)$$

for any $g \in G$ and $k \in K$. For $q \in Q$, choose some $g \in G$ mapping to q, and put $\widetilde{u}(q) = u(g)$. This is well-defined, as another choice g' of lift would differ from g by an element $k \in K$, and then $u(g') = u(gk) = u(g) + gu(k) = u(g)$. If $q_1, q_2 \in Q$, choose lifts $g_1, g_2 \in G$. Then the product g_1g_2 is a lift of q_1q_2, and we have

$$\widetilde{u}(q_1q_2) = u(g_1g_2) = u(g_1) + g_1u(g_2) = \widetilde{u}(q_1) + q_1\widetilde{u}(q_2),$$

so $\widetilde{u}$ is a cocycle on Q. This proves exactness at $H^1(G, M)$. □

1.6 Kazhdan's Property (T)

Two properties of topological groups, known as Property (T) and Property (FH), respectively, are intimately related to the cohomology of groups with coefficients in real or complex Hilbert spaces. A thorough exposition of these properties and their relationship to group cohomology is far beyond the scope of this thesis. We instead refer the interested reader to the very comprehensive book [11]. In this short section we will simply outline the facts we need.

Definition 1.5. Let G be a topological group and $\pi\colon G \to \mathrm{U}(V)$ be a unitary representation on a Hilbert space V.

(1) Let $\varepsilon > 0$ and $K \subseteq G$ be a compact subset. A unit vector $v \in V$ is called (ε, K)-invariant if

$$\sup_{g\in K} |\pi(g)v - v| < \varepsilon.$$

(2) The representation π is said to have almost invariant vectors if there is an (ε, K)-invariant vector for all such pairs.

Definition 1.6. A topological group G has Kazhdan's Property (T) if any unitary representation of G which has almost invariant vectors has an actual (non-trivial) invariant vector.

Proposition 1.7. *For $g \geq 2$, the discrete group $\mathrm{Sp}(2g, \mathbb{Z})$ has Property (T).*

Proof. By Theorem 1.5.3 of [11], the locally compact group $\mathrm{Sp}(2g, \mathbb{R})$ has Property (T), and by Theorem 1.7.1, Property (T) is inherited by lattices in locally compact groups. Finally, $\mathrm{Sp}(2g, \mathbb{Z})$ is known to be a lattice in $\mathrm{Sp}(2g, \mathbb{R})$. □

When G is a topological group and V is a unitary representation, the space $Z^1(G,V)$ of cocycles is given the topology of uniform convergence over compact subsets. In this topology, $B^1(G,V)$ may or may not be closed in $Z^1(G,V)$; in any case, the quotient

$$\overline{H^1}(G,V) = Z^1(G,V)/\overline{B^1(G,V)} \tag{1.18}$$

is known as the reduced cohomology of G with coefficients in V.

For finitely generated groups, a number of conditions are known to be equivalent to Property (T). The following is quoted from [11], Theorem 3.2.1.

Theorem 1.8. *Let G be a locally compact group which is second countable and compactly generated. The following conditions are equivalent:*

(i) *G has Property (T);*

(ii) *$H^1(G,\pi) = 0$ for every irreducible unitary representation π of G;*

(iii) *$\overline{H^1}(G,\pi) = 0$ for every irreducible unitary representation π of G;*

(iv) *$\overline{H^1}(G,\pi) = 0$ for every unitary representation π of G.*

In fact, one can add a fifth condition to the list.

Lemma 1.9. *Let G be a group satisfying the conditions of Theorem 1.8. Then conditions* (i)–(iv) *are also equivalent to*

(v) *$H^1(G,\pi) = 0$ for every unitary representation π of G.*

Proof. Clearly (v) implies (ii) and hence the other conditions. By the Delorme-Guichardet Theorem (Theorem 2.12.4 in [11]), Property (T) and Property (FH) are equivalent for the class of groups considered, so Property (T) implies that $H^1(G,\pi) = 0$ for any orthogonal representation π. Any unitary representation is in particular an orthogonal representation, so $H^1(G,\pi) = 0$ for any unitary representation as well. □

For a (discrete) set X, we let $\ell^2(X)$ denote the set of square summable functions $X \to \mathbb{C}$, that is, the set

$$\ell^2(X) = \{f\colon X \to \mathbb{C} \mid \sum_{x\in X} |f(x)|^2 < \infty\}. \tag{1.19}$$

We will sometimes write such a function as a formal linear combination $\sum_{x\in X} f(x)x$ or $\sum_{x\in X} f_x x$. If a group G acts on X, it is clear that $\ell^2(X)$ is a unitary representation of G.

Chapter 2

Mapping Class Groups and Multicurves

Let $\Sigma = \Sigma_{g,r}$ denote a compact, oriented surface of genus g with r boundary components, $g, r \geq 0$. The mapping class group $\Gamma = \Gamma(\Sigma) = \Gamma_{g,r}$ of Σ is defined to be the quotient group $\mathrm{Diff}(\Sigma; \partial\Sigma) / \mathrm{Diff}_0(\Sigma; \partial\Sigma)$, where $\mathrm{Diff}(\Sigma; \partial\Sigma)$ is the group of orientation-preserving diffeomorphisms of Σ fixing the boundary point-wise, and $\mathrm{Diff}_0(\Sigma; \partial\Sigma)$ is the subgroup consisting of diffeomorphisms isotopic (smoothly homotopic through diffeomorphisms fixed on the boundary) to the identity. Equivalently, Γ is the group of components $\pi_0 \mathrm{Diff}(\Sigma; \partial\Sigma)$. We will often denote an orientation-preserving diffeomorphism $f \colon \Sigma \to \Sigma$ and its mapping class $f \in \Gamma$ by the same symbol, when there is little or no chance of confusion.

Remark 2.1. Since any homeomorphism of Σ is isotopic (through homeomorphisms) to a diffeomorphism, and since any continuous isotopy between two diffeomorphisms may be smoothed, we could also have defined the mapping class group in terms of the orientation-preserving homeomorphisms of Σ. Thus we may occasionally talk about the mapping class of a homeomorphism or represent elements of Γ by homeomorphisms; this does not cause any ambiguity.

Remark 2.2. Note that in case $r \geq 1$, any diffeomorphism fixing the boundary is automatically orientation-preserving.

The reader is probably aware that different, but related groups are also known as »the« mapping class group. For a survey of the various possibilities we refer to [18].

2.1 Dehn Twists

Let $A \subseteq \mathbb{R}^2$ denote the annulus given in the standard polar coordinates (r, θ) by $1 \leq r \leq 2$. Its boundary components are denoted $\partial_1 A$ and $\partial_2 A$, respectively. Fix a smooth, increasing diffeomorphism $\lambda \colon [1,2] \to [0, 2\pi]$ with vanishing derivatives to all orders at 1 and 2. The *standard (left) twist* of A is the diffeomorphism t given by $(r, \theta) \mapsto (r, \theta + \lambda(r))$. Up to isotopy fixed on the boundary of A, t does not depend on the particular choice of λ.

Next let γ be an oriented simple closed curve on Σ, and let $e\colon A \to \Sigma$ be an embedding such that $e_{|\partial_1 A}$ is an orientation-preserving diffeomorphism onto γ. The geometric Dehn twist associated to e is the diffeomorphism t_e of Σ which on $e(A)$ is given by $e \circ t \circ e^{-1}$, and is the identity on the rest of Σ. The isotopy class of t_e only depends on the isotopy class of γ (as an unoriented curve), and this isotopy class is called the (left) Dehn twist on γ, denoted τ_γ.

2.2 Generators and Relations

A fundamental result is that the mapping class group is generated by Dehn twists. For $g \geq 2$ and $r \leq 1$, Humphries [26] has shown that the minimal number of twists needed is $2g + 1$, and this number is realizable. In fact, Wajnryb [40] gives a complete presentation in terms of generators and relations of the mapping class group $\Gamma_{g,r}$ for $r \leq 1$. Be aware, however, that [40] contains some errors which are corrected in [12]. More recently, Gervais [20] gave a completely general finite (but non-minimal) presentation of $\Gamma_{g,r}$ for $g \geq 1$ and any number of boundary components. The number of generators in [20] is rather large (of the order $(g + r)^2$), but this disadvantage is outweighed by the simplicity and symmetry of the relations.

We will not need a complete presentation of the mapping class group, but we will need the fact that it is generated by Dehn twists, and some simple relations among such twists. For later use we record these well-known results. Proofs can be found in [18] and [27].

Lemma 2.3. *Dehn twists on disjoint curves commute.*

Lemma 2.4. *If α and β are simple closed curves intersecting transversely in a single point, the associated Dehn twists are* braided. *That is, $\tau_\alpha \tau_\beta \tau_\alpha = \tau_\beta \tau_\alpha \tau_\beta$.*

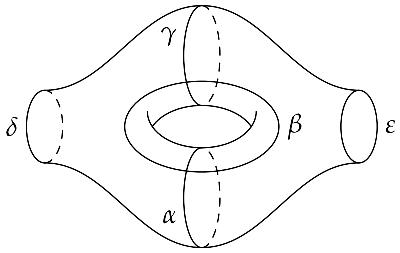
Lemma 2.5. *If α is a simple closed curve on Σ and $f \in \Gamma$, we have $f \circ \tau_\alpha \circ f^{-1} = \tau_{f(\alpha)}$.*

Lemma 2.6 (Chain relation). *Let α, β and γ be simple closed curves in a two-holed torus as in Figure 2.1, and let δ, ε denote curves parallel to the boundary components of the torus. Then $(\tau_\alpha \tau_\beta \tau_\gamma)^4 = \tau_\delta \tau_\varepsilon$.*

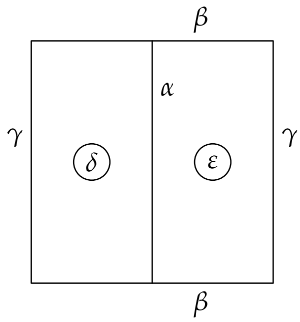
Lemma 2.7 (Lantern relation). *Consider the surface $\Sigma_{0,4}$, ie. a sphere with four holes. Let γ_i denote the i'th boundary component, $0 \leq i \leq 3$, and γ_{ij} a loop enclosing the i'th and j'th boundary components, $1 \leq i < j \leq 3$. Let $\tau_i = \tau_{\gamma_i}$ and $\tau_{ij} = \tau_{\gamma_{ij}}$. Then*

$$\tau_0 \tau_1 \tau_2 \tau_3 = \tau_{12} \tau_{13} \tau_{23}. \tag{2.1}$$

For a picture of the lantern relation, see the left-hand part of Figure 2.4 on page 21.



(a) A two-holed torus.



(b) A more schematic picture.

Figure 2.1: The chain relation.

Corollary 2.8. *If $g \geq 2$, the Dehn twist on a boundary component of $\Sigma_{g,r}$ can be written in terms of Dehn twists on non-separating curves.*

Proof. The assumption on the genus implies that we may find an embedding of $\Sigma_{0,4} \to \Sigma_{g,r}$ such that γ_0 is mapped to the boundary component in question and the remaining six curves involved in the lantern relation are mapped to non-separating curves (think of $\Sigma_{g,r}$ as being obtained by gluing three boundary components of $\Sigma_{g-2,r+2}$ to γ_1, γ_2 and γ_3, respectively). Then the relation $\tau_0 = \tau_{12}\tau_{13}\tau_{23}\tau_3^{-1}\tau_2^{-1}\tau_1^{-1}$ also holds in $\Gamma_{g,r}$. □

Proposition 2.9. *If $g \geq 2$, $\Gamma_{g,r}$ is generated by Dehn twists on* non-separating *curves.*

Proof. We already know that the mapping class group is generated by Dehn twists. If $g \geq 3$ and γ is a separating curve in Σ, cut Σ along γ and apply Corollary 2.8 to the component which has genus ≥ 2, showing that τ_γ can be written in terms of twists on non-separating curves in Σ.

Now assume $g = 2$ and that γ is a separating curve. The above argument still holds if the two components of the cut surface Σ_γ has genera 0 and 2, so assume that γ cuts Σ into two genus 1 surfaces, Σ_1 and Σ_2. In Σ_1, we may find a separating simple closed curve η such that cutting Σ_1 along η yields a genus 0 surface and a two-holed torus, whose other boundary component is γ (η may, if necessary, be chosen to be null-homotopic). Then in the surface Σ, the twist in η can be written in terms of non-separating curves. But then the chain relation (Lemma 2.6) shows that τ_γ can be written in terms of τ_η and twists in three non-separating curves in Σ_1. □

Corollary 2.10. *When $g \geq 3$, the abelianization $H_1(\Gamma, \mathbb{Z})$ of the mapping class group vanishes, and when $g = 2$ the group $H_1(\Gamma, \mathbb{Z})$ is cyclic of order dividing* 10.

Proof. By the preceding proposition, $\Gamma_{g,r}$ is generated by twists in non-separating curves. By the classification of surfaces and Lemma 2.5, these generators are all conjugate in Γ, so they represent the same element τ in $H_1(\Gamma, \mathbb{Z})$. Hence this group is cyclic.

When $g \geq 3$, one may embed the lantern relation in Σ in such a way that all seven curves are non-separating (see Figure 2.4 on page 21). From this it follows that the generator represented by a non-separating twist satisfies the relation $4\tau = 3\tau$, so $H_1(\Gamma, \mathbb{Z}) = 0$. When $g = 2$, we may embed the chain relation in Σ in such a way that all five curves are non-separating. In this case, we get that the generator τ satisfies $12\tau = 2\tau$, so τ has order dividing 10. □

It can in fact be shown that $H_1(\Gamma_{2,r}, \mathbb{Z}) \cong \mathbb{Z}/10\mathbb{Z}$, but we will not need this fact. We will, however, need the fact that $H_1(\Gamma, \mathbb{Q}) = 0$, so that also

$$\mathrm{Hom}(\Gamma, \mathbb{C}) = 0 \tag{2.2}$$

whenever $g \geq 2$.

2.3 Multicurves

A *multicurve* on Σ is the isotopy class of an unoriented, compact, closed 1-submanifold, such that no component is homotopically trivial. We will often think of a multicurve as a collection of circles disjointly embedded in Σ. In this definition, parallel copies of the same isotopy class of a simple closed curve are allowed. Depending on the context, one may or may not allow components parallel to a boundary component of Σ. We take the liberal viewpoint of allowing boundary-parallel components, and denote the set of all multicurves by $\mathcal{S} = \mathcal{S}(\Sigma)$. The subset without boundary parallel components is denoted $\mathcal{S}'$.

2.3.1 Dehn-Thurston Coordinates

Surfaces of negative Euler characteristic admit decompositions into pairs of pants. Using this fact, Dehn found a way to parametrize the set of all multicurves, which was later rediscovered and generalized (to encompass the notion of *measured foliations*) by Thurston. A detailed exposition can be found in [32]; presently we will give an informal presentation of the ideas involved.

Let $\mathcal{B} = \{\beta_1, \beta_2, \ldots, \beta_r\}$ denote the set of boundary components of Σ. A *pants decomposition* of Σ is a choice of $3g + r - 3$ simple, closed, disjoint curves $\mathcal{P} = \{\pi_1, \ldots, \pi_{3g+r-3}\}$ such that the surface $\Sigma_{\mathcal{P}}$ obtained by cutting along each π_j is a disjoint union of $-\chi(\Sigma) = 2g + r - 2$ *pairs of pants*. A pair of pants is simply a three-holed sphere.

In the language of [32], a *basis* for the set of multicurves consists of a choice of pants decomposition along with a choice of characteristic maps identifying each pair of pants with a standard copy, and also an identification of a closed regular neighbourhood of each pants curve with a standard annulus. In order to simplify terminology and notation, we will implicitly assume that these additional choices have been made whenever we use a pants decomposition to give coordinates on $\mathcal{S}$.

A pants decomposition gives rise to a map

$$m = m^{\mathcal{P}} \colon \mathcal{S} \to \mathbb{N}^{\mathcal{P}}. \tag{2.3}$$

The coordinates of this maps will be denoted by $m_\gamma \colon \mathcal{S} \to \mathbb{N}$ for $\gamma \in \mathcal{P}$. For a multicurve κ, $m_\gamma(\kappa)$ is the geometric intersection number between γ and κ.

Remark 2.11. It is convenient to adopt the convention that $m_\gamma(\kappa) = 0$ for every boundary curve $\gamma \in \mathcal{B}$; equivalently, we sometimes think of m as a map into $\mathbb{N}^{\mathcal{P}\cup\mathcal{B}}$ with all $\mathcal{B}$-coordinates identically 0.

There is also a map

$$t = t^{\mathcal{P}} : \mathcal{M} \to \mathbb{Z}^{\mathcal{P}\cup\mathcal{B}}, \tag{2.4}$$

which is somewhat harder to describe, since it depends on the above-mentioned additional choices. Essentially, t measures how much a given multicurve »twists« with respect to a set of six model arcs connecting the boundary components of the standard pair of pants. The coordinates of (2.4) will be denoted by t_γ.

Theorem 2.12 (Dehn). *The map*

$$(m, t) : \mathcal{S} \to \mathbb{N}^{\mathcal{P}} \times \mathbb{Z}^{\mathcal{P}\cup\mathcal{B}} \tag{2.5}$$

is a bijection onto the subset $D_{\mathcal{P}} \subseteq \mathbb{N}^{\mathcal{P}} \times \mathbb{Z}^{\mathcal{P}\cup\mathcal{B}}$ satisfying:

(a) *For each $\gamma \in \mathcal{P} \cup \mathcal{B}$, if $m_\gamma = 0$, then $t_\gamma \geq 0$.*

(b) *If $\gamma_1, \gamma_2, \gamma_3$ are the three boundary curves of a pair of pants, then $m_{\gamma_1} + m_{\gamma_2} + m_{\gamma_3}$ is even.*

The subset $D'_{\mathcal{P}} \subseteq D_{\mathcal{P}}$ corresponding to elements of $\mathcal{S}'$ are those parameters which additionally satisfy

(c) *For each $\gamma \in \mathcal{B}$, $t_\gamma = 0$.*

Clearly, if a multicurve κ contains a parallel copy of a curve γ, the intersection number $m_\gamma(\kappa)$ is 0. In this case, the non-negative number $t_\gamma(\kappa)$ is used to record the number of copies of γ occuring in κ.

2.3.2 Change of Coordinates

In order to define intrinsic properties of multicurves in terms of the Dehn-Thurston coordinates, we will need to know to what extent these properties depend on the chosen basis.

For a fixed pants decomposition, different choices of model arcs give rise to twist coordinates differing by a linear map for each pants curve. More precisely, if t_γ and t'_γ are the twist coordinates associated to the pants curve γ and two different choices of model arcs in a pair of pants bounded by γ, then there is a constant d_γ such that

$$t_\gamma = t'_\gamma + d_\gamma m_\gamma \tag{2.6}$$

as maps $\mathcal{S} \to \mathbb{Z}$.

As proved by Hatcher and Thurston [24], any two pants decompositions of Σ differ by a finite sequence of so-called *elementary moves*.

Theorem 2.13. *If $\mathcal{P}$ and $\mathcal{P}'$ are the pants decompositions underlying two bases differing by an elementary move, the coordinate transformation*

$$(m^{\mathcal{P}'}, t^{\mathcal{P}'}) \circ (m^{\mathcal{P}}, t^{\mathcal{P}})^{-1} : D_{\mathcal{P}} \to \mathcal{S} \to D_{\mathcal{P}'} \tag{2.7}$$

is given by piecewise integral linear expressions. Consequently, for any two given bases, the corresponding coordinate transformation is also given by such expressions.

Proof. The first part of this theorem follows from the explicit piecewise linear expressions given in [32]. Equation (2.6) shows that the coordinate transformation from one basis to another with the same underlying pants decomposition is given by piecewise integral linear maps. □

2.4 Actions on Geometric Objects

The action of the mapping class group on the first homology group $H_1(\Sigma, \mathbb{Z})$ and on the set of multicurves will play a vital role in this thesis. It is therefore convenient to give a description of the action of a Dehn twist on these objects.

2.4.1 Homology

Lemma 2.14. *Let γ be a simple closed curve on Σ, and let m be an element of $H_1(\Sigma, \mathbb{Z})$. Then*

$$\tau_\gamma m = m + \omega(m, [\hat{\gamma}])[\hat{\gamma}], \tag{2.8}$$

where $\hat{\gamma}$ denotes any of the oriented versions of γ, $[\hat{\gamma}]$ the homology class it represents in $H_1(\Sigma, \mathbb{Z})$ and $\omega(\bullet, \bullet)$ is the intersection pairing.

It is clear that the formula (2.8) is independent of the choice of $\hat{\gamma}$.

Corollary 2.15. *If τ_γ acts non-trivially on m, the orbit $\{\tau_\gamma^n m \mid n \in \mathbb{Z}\}$ is infinite.*

For convenience, we recall this well-known fact.

Lemma 2.16. *Let $m \in H_1(\Sigma, \mathbb{Z})$ be an element which is not divisible by any positive integer. Then m is part of some symplectic basis for $H_1(\Sigma, \mathbb{Z})$.*

Proof. Choose $2g$ oriented, simple closed curves α_j, β_j representing a symplectic basis for $H_1(\Sigma)$, such that α_j and β_j intersect transversely in a single point, $j = 1, \ldots, g$. Let a_j, b_j be the coordinates of m with respect to this basis. That m is indivisible precisely means that the greatest common divisor of these $2g$ numbers is 1. Hence, there exists another $2g$-tuple a'_j, b'_j of integers such that

$$a_1 b'_1 - b_1 a'_1 + \cdots + a_g b'_g - b_g a'_g = 1.$$

Let m' denote the homology element whose coordinates are the a'_j, b'_j. Then $\omega(m, m') = 1$, so $W = \operatorname{span}_{\mathbb{Z}}(m, m')$ is a symplectic subspace of $H_1(\Sigma, \mathbb{Z})$, and the restriction of the intersection pairing $\omega(\bullet, \bullet)$ to the symplectic complement

$$W^\perp = \{x \in H_1(\Sigma, \mathbb{Z}) \mid \omega(m, x) = \omega(m', x) = 0\}$$

is non-degenerate. A symplectic basis for $W^\perp$ together with (m, m') constitute a symplectic basis for the whole space. □

Corollary 2.17. *Any element in $H_1(\Sigma, \mathbb{Z})$ can be represented by a cycle consisting of parallel copies of a single simple, closed, oriented curve.*

Proof. The element 0 is represented by the empty sum. If m is non-zero, let d be the largest integer for which there exists m_d such that $dm_d = m$. Then m_d is indivisible, and hence part of some symplectic basis. Since $\mathrm{Sp}(H_1(\Sigma, \mathbb{Z}))$ acts transitively on the set of symplectic bases, and since Γ surjects onto this group, there is an element $\varphi \in \Gamma$ such that $\varphi[\alpha_1] = m_d$. But then the oriented, simple closed curve $\varphi(\alpha_1)$ represents m_d, and m is represented by d parallel copies of this curve. □

Lemma 2.18. *Let $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ be simple, closed, oriented curves representing a symplectic basis for $H_1(\Sigma)$. Given any any non-zero homology element m, there exists a curve γ such that at least one of the sequences $\|\tau_\gamma^n m\|$, $\|\tau_\gamma^{-n} m\|$, $n = 0, 1, 2, \ldots$, is strictly increasing.*

Here and elsewhere, the norm $\|m\|$ of a homology element m with respect to a given symplectic basis is the sum of the absolute values of the coordinates of m with respect to the basis, ie.

$$\|m\| = |a_1| + |b_1| + \cdots + |a_g| + |b_g|. \tag{2.9}$$

for $m = a_1[\alpha_1] + b_1[\beta_1] + \cdots + a_g[\alpha_g] + b_g[\beta_g]$.

Proof. At least one of the coordinates of m is non-zero. Assume without loss of generality that $a_1 \neq 0$ and put $\gamma = \beta_1$. Then, for any $n \in \mathbb{Z}$, the coordinates of $\tau_\gamma^n m$ are

$$(a_1, b_1 + na_1, a_2, b_2, \ldots, a_g, b_g)$$

by (2.8) above. Then clearly if a_1 and b_1 have the same sign, the sequence $\|\tau_\gamma^n m\|$ is increasing, while if a_1 and b_1 have opposite signs the sequence $\|\tau_\gamma^{-n} m\|$ is increasing. In the case where $b_1 = 0$ both sequences are increasing. □

Note that we may in fact in all cases choose the Dehn twist from a finite collection, namely, the twists in the curves representing the symplectic basis.

2.4.2 Multicurves

A formula similar to (2.8) holds for the action of a Dehn twist on a multicurve, except in this case, only for twists in curves which are part of the chosen pants decomposition.

Proposition 2.19. *Let $\mathcal{P}$ be a pants decomposition of Σ, $\kappa \in \mathcal{S}$ a multicurve and $\gamma \in \mathcal{P}$ a pants curve. Then the Dehn-Thurston coordinates $(m^{\mathcal{P}}, t^{\mathcal{P}})(\tau_\gamma \kappa)$ are given by*

$$t_\gamma^{\mathcal{P}}(\tau_\gamma \kappa) = t_\gamma^{\mathcal{P}}(\kappa) + m_\gamma^{\mathcal{P}}(\kappa), \tag{2.10}$$

with all other coordinates unchanged.

Since the action of a twist in a boundary component on a multicurve is trivial, the above formula also trivially holds for $\gamma \in \mathcal{B}$, cf. Remark 2.11.

This proposition allows us to prove a number of important facts.

Proposition 2.20. *Let γ be a simple closed curve and κ a multicurve. Then the following are equivalent:*

(1) *The twist τ_γ acts trivially on κ.*

(2) *The twist τ_γ acts trivially on each component of κ.*

(3) *The geometric intersection number between γ and κ is zero.*

(4) *One may realize γ and κ disjointly.*

Conversely, if τ_γ acts non-trivially on κ, the orbit $\{\tau_\gamma^n \kappa \mid n \in \mathbb{Z}\}$ is infinite.

Proof. The equivalence of (3) and (4) follows from the definition of geometric intersection number. If γ is not parallel to a boundary component, it is part of some pants decomposition $\mathcal{P}$. Then the equivalence between (1) and (3) follows from (2.10). Clearly (2) implies (1). Since the geometric intersection number between γ and κ is the sum of the intersection numbers between γ and the components of κ, we see that (3) implies (2).

If γ is parallel to a boundary component, (1)–(4) are trivially satisfied. Finally, the last assertion follows from (2.10). □

Remark 2.21. Let $\mathcal{S}_\partial \subseteq \mathcal{S}$ denote the set of multicurves consisting entirely of components parallel to boundary components of Σ. Then any multicurve $\kappa \in \mathcal{S}$ admits a unique decomposition $\kappa = \kappa' \cup \kappa_\partial$ with $\kappa' \in \mathcal{S}'$ and $\kappa_\partial \in \mathcal{S}_\partial$ (the empty multicurve is the sole element of $\mathcal{S}' \cap \mathcal{S}_\partial$). The mapping class group orbit of κ is infinite if and only if κ' is not the empty multicurve; otherwise the orbit is trivial.

To any pants decomposition $\mathcal{P}$, we associate an integer-valued norm on the set $\mathcal{S}$ by putting

$$\|\kappa\|_{\mathcal{P}} = \sum_{\gamma \in \mathcal{P}} m_\gamma(\kappa) + \sum_{\gamma \in \mathcal{P} \cup \mathcal{B}} |t_\gamma(\kappa)|. \tag{2.11}$$

It is clear that the cardinalities of

$$\{\kappa \in \mathcal{S} \mid \|\kappa\|_{\mathcal{P}} \leq N\} \tag{2.12}$$

$$\{\kappa \in \mathcal{S} \mid \|\kappa\|_{\mathcal{P}} = N\} \tag{2.13}$$

are bounded above by polynomials in N of degrees $2|\mathcal{P}| + |\mathcal{B}| = 6g + 3r - 6$ and $6g + 3r - 7$, respectively. The multicurve analogue of Lemma 2.18 is:

Theorem 2.22. *Given any pants decomposition $\mathcal{P}$, there exists a finite set T of Dehn twists with the following property: For any multicurve $\kappa \in \mathcal{S}$ with $\kappa' \neq \emptyset$, there exists an element $\tau \in T$ and a sign $\varepsilon = \pm 1$ such that the sequence*

$$\|\tau^{\varepsilon n} \kappa\|_{\mathcal{P}}, \quad n = 0, 1, 2, \ldots \tag{2.14}$$

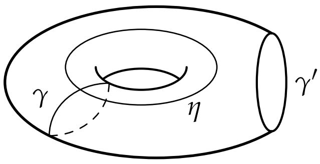
is strictly increasing.

Proof. Let T_1 denote the set of twists in the pants curves $\mathcal{P}$. Then if κ is a multicurve for which at least one coordinate $m_\gamma(\kappa)$ is non-zero, let ε be the sign of $t_\gamma(\kappa)$ (if $t_\gamma(\kappa) = 0$, ε may be chosen arbitrarily). Then from the formula (2.10) and the definition (2.11) we have

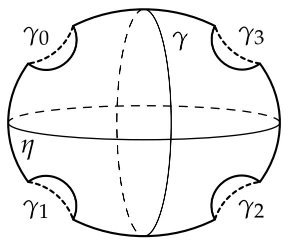
$$\|\tau_\gamma^{\varepsilon n}\kappa\|_{\mathcal{P}} = \|\kappa\|_{\mathcal{P}} + nm_\gamma(\kappa). \tag{2.15}$$

Hence the set T_1 suffices for all multicurves except those consisting entirely of components parallel to the pants curves (and the boundary components).

For each pants curve γ, there are only two possible configurations of the pants decomposition near γ. Either (a) the two sides of γ belong to the same pair of pants, or (b) γ bounds two different pairs of pants; see Figure 2.2. In 2.2(a), γ' denotes the pants curve separating the torus from the rest of the surface, whereas in 2.2(b) the neighbouring pants curves of γ are denoted γ_j (some of these may actually be the same curve in Σ; this is not important for us). For each pants curve γ, consider the twist in the curve η shown in the relevant part of the figure, and let T_2 be the set of these twists. We put $T = T_1 \cup T_2$.



(a) A one-holed torus.



(b) A four-holed sphere.

Figure 2.2: Two local configurations of a pants decomposition.

Now, let κ be a multicurve with at least one component which is parallel to some pants curve γ. Let η denote the curve corresponding to γ. It is then clear that all coordinates of κ are invariant under τ_η except t_γ and m_γ. Let $c = t_\gamma(\kappa) > 0$ denote the number of parallel copies of γ occuring in κ. We will derive formulas for $m_\gamma(\tau_\eta^n\kappa)$ and $t_\gamma(\tau_\eta^n\kappa)$. Let $\mathcal{P}'$ denote the pants decomposition obtained by exchanging γ with η. Fortunately, this precisely corresponds to one of the elementary moves, and the coordinate transformations are easily computed using the formulas given in [32].

First consider case (a). In this case,

$$m_\eta^{\mathcal{P}'}(\kappa) = c$$
$$t_\eta^{\mathcal{P}'}(\kappa) = 0$$

so

$$t_\eta^{\mathcal{P}'}(\tau_\eta^n\kappa) = nc$$

by (2.10). Transforming back to the $\mathcal{P}$-coordinates, we get

$$m_\gamma^{\mathcal{P}}(\tau_\eta^n\kappa) = |n|c \tag{2.16}$$
$$t_\gamma^{\mathcal{P}}(\tau_\eta^n\kappa) = -\operatorname{sgn}(n)c \tag{2.17}$$

with $\operatorname{sgn}(0) = -1$. From (2.16) and (2.17), it follows that

$$\begin{aligned}\|\tau_\eta^n \kappa\|_{\mathcal{P}} &= \|\kappa\|_{\mathcal{P}} - |t_\gamma^{\mathcal{P}}(\kappa)| + m_\gamma^{\mathcal{P}}(\tau_\eta^n \kappa) + |t_\gamma^{\mathcal{P}}(\tau_\eta^n \kappa)| \\ &= \|\kappa\|_{\mathcal{P}} + |n|c\end{aligned}$$

so, in fact, both sequences $\|\tau_\eta^{\pm n}\kappa\|$, $n \geq 0$, are increasing.

The second case is treated similarly. We now have

$$\begin{aligned} m_\eta^{\mathcal{P}'}(\kappa) &= 2c \\ t_\eta^{\mathcal{P}'}(\kappa) &= -c. \end{aligned}$$

From this we obtain $t_\eta^{\mathcal{P}'}(\tau_\eta^n \kappa) = (2n-1)c$, and transforming back yields

$$\begin{aligned} m_\gamma^{\mathcal{P}}(\tau_\eta^n \kappa) &= 4|n|c \\ t_\gamma^{\mathcal{P}}(\tau_\eta^n \kappa) &= -4nc + \operatorname{sgn}(n) \cdot (2n-1)c \end{aligned}$$

again with $\operatorname{sgn}(0) = -1$. From this we get for $n \geq 0$ that

$$\|\tau_\eta^n \kappa\|_{\mathcal{P}} = \|\kappa\|_{\mathcal{P}} + 6nc. \qquad \square$$

2.5 The Torelli Group

The mapping class group action on $H_1(\Sigma, \mathbb{Z})$ preserves the intersection pairing on homology, so there is a homomorphism $\Gamma \to \operatorname{Sp}(H_1(\Sigma, \mathbb{Z}))$. It is well-known that this is a surjection, so we have a short exact sequence

$$1 \longrightarrow \mathcal{T} \longrightarrow \Gamma \longrightarrow \operatorname{Sp}(H_1(\Sigma, \mathbb{Z})) \longrightarrow 1. \tag{2.18}$$

The kernel $\mathcal{T}$ is known as the *Torelli group* of Σ. For later use, we record a few facts about $\mathcal{T}$.

It follows immediately from (2.8) that if γ is a separating curve, $\tau_\gamma \in \mathcal{T}$. From the same equation, if γ and η are homologous curves, $\tau_\gamma \tau_\eta^{-1} \in \mathcal{T}$. This applies in particular to the situation where γ, η is a pair of non-separating curves such that $\gamma \cup \eta$ separates the surface. If one of the connected components of the cut surface $\Sigma_{\gamma \cup \eta}$ is a torus with two holes, we call (γ, η) a *genus* 1 *bounding pair*, and the associated map $\tau_\gamma \tau_\eta^{-1}$ a *genus* 1 *bounding pair map*.

Theorem 2.23 (Johnson [28]). *When $g \geq 3$, $r \leq 1$, the Torelli group of $\Sigma_{g,r}$ is generated by genus 1 bounding pair maps.*

Corollary 2.17 shows that an irreducible element of $H_1(\Sigma, \mathbb{Z})$ can be represented by a single, oriented, simple closed curve. It is an interesting and non-trivial fact that the Torelli group acts transitively on the set of possible choices.

Lemma 2.24. *Assume γ and δ are oriented, non-separating, simple closed curves on Σ representing the same element in $H_1(\Sigma, \mathbb{Z})$. Then there is an element $t \in \mathcal{T}$ such that $t(\gamma) = \delta$.*

This is a special case of Lemma 6.2 of [33].

2.6 Unitary Representations

The purpose of this section is to prove a result which applies to any unitary representation of the mapping class group, provided the genus of the surface is at least 3. Throughout this section, let V be a real or complex Hilbert space endowed with an action of Γ preserving the inner product. For a simple closed curve γ, we let $V_\gamma = V^{\tau_\gamma}$ denote the set of vectors fixed under the action of the twist τ_γ, and we let $p_\gamma \colon V \to V_\gamma$ denote the orthogonal projection onto the (obviously closed) subspace V_γ. The main theorem of this section is

Theorem 2.25. *Assume the genus of Σ is at least 3. For any cocycle $u \colon \Gamma \to V$ and any simple closed curve α we have $p_\alpha u(\tau_\alpha) = 0$.*

The proof of this theorem only requires the simple relations in the mapping class group mentioned in Section 2.2.

Throughout the rest of the section, let u denote a fixed cocycle $\Gamma \to V$. The motivation for the above theorem comes from the following observation: If $u(\varphi) = (1-\varphi)v$ for some vector $v \in V$, it is clear that $u(\varphi)$ is killed by the projection onto the subspace V^φ fixed by φ. Hence, vanishing of the entity $p_\alpha u(\tau_\alpha)$ for each simple closed curve α is a necessary condition for the vanishing of the cohomology group $H^1(\Gamma, V)$.

If α and γ are disjoint simple closed curves, the unitary actions τ_α and τ_γ on V commute. Hence the associated projections p_α and p_γ commute with each other and with τ_α, τ_γ. If $\varphi\tau_\alpha\varphi^{-1} = \tau_\beta$, then $\varphi p_\alpha \varphi^{-1} = p_\beta$ for $\varphi \in \Gamma$.

We will use the shorthand notation s_α for $p_\alpha u(\tau_\alpha)$.

Lemma 2.26. *The entity s is natural in the sense that $s_{\varphi(\alpha)} = \varphi s_\alpha$ for $\varphi \in \Gamma$ and any simple closed curve α.*

Proof. Since $\tau_{\varphi(\alpha)} = \varphi\tau_\alpha\varphi^{-1}$, it is easy to see that $p_{\varphi(\alpha)} = \varphi p_\alpha \varphi^{-1}$. Hence

$$\begin{aligned} s_{\varphi(\alpha)} &= p_{\varphi(\alpha)} u(\tau_{\varphi(\alpha)}) \\ &= \varphi p_\alpha \varphi^{-1} u(\varphi\tau_\alpha\varphi^{-1}) \\ &= \varphi p_\alpha \varphi^{-1}\big((1-\varphi\tau_\alpha\varphi^{-1})u(\varphi) + \varphi u(\tau_\alpha)\big) \\ &= \varphi p_\alpha u(\tau_\alpha) \\ &= \varphi s_\alpha \end{aligned}$$

as claimed. □

Lemma 2.27. *Let α be a simple closed curve, and let $\varphi \in \Gamma$ be any element commuting with τ_α. Then $\varphi s_\alpha = s_\alpha$.*

Proof. We have $\varphi\tau_\alpha = \tau_\alpha\varphi$. Applying u and the cocycle condition we obtain the equation $u(\varphi) + \varphi u(\tau_\alpha) = u(\tau_\alpha) + \tau_\alpha u(\varphi)$. Applying p_α on both sides, the terms involving $u(\varphi)$ cancel (since obviously $p_\alpha\tau_\alpha = p_\alpha$), so we obtain $p_\alpha\varphi u(\tau_\alpha) = s_\alpha$. The claim then follows from the fact that p_α and φ also commute. □

Assume α and β are two non-separating simple closed curves such that $\alpha \cup \beta$ is non-separating, and consider the number $c_{\alpha\beta} = \langle s_\alpha, s_\beta \rangle$.

Lemma 2.28. *The number $c_{\alpha\beta}$ only depends on the cocycle u, not on the pair (α, β) used to compute it.*

Proof. Let (α', β') be any other pair such that $\alpha' \cup \beta'$ does not separate Σ. Then, by the classification of surfaces, there is a diffeomorphism $\varphi \in \Gamma$ such that $\varphi(\alpha) = \alpha'$ and $\varphi(\beta) = \beta'$. By the naturality from Lemma 2.26 we have

$$\langle s_{\alpha'}, s_{\beta'} \rangle = \langle s_{\varphi(\alpha)}, s_{\varphi(\beta)} \rangle = \langle \varphi s_\alpha, \varphi s_\beta \rangle = \langle s_\alpha, s_\beta \rangle$$

since φ acts unitarily. □

The vector $s_\alpha = p_\alpha u(\tau_\alpha) \in V$ obviously only depends on the cohomology class $[u] \in H^1(\Gamma, V)$ of u. Hence, we have essentially proved that there exists a well-defined map $c\colon H^1(\Gamma, V) \to \mathbb{C}$, whose value on $[u]$ is given by picking any two jointly non-separating simple closed curves α, β and computing the number $c([u]) = \langle p_\alpha u(\tau_\alpha), p_\beta u(\tau_\beta) \rangle$.

Lemma 2.29. *When $g \geq 3$, the map c is identically* 0*.*

Proof. In any surface of genus at least 2, one may embed the two-holed torus relation (Lemma 2.6) in such a way that δ and ε are non-separating (the curves α, β, γ occuring in the two-holed torus relation are always non-separating). If the genus of the surface is at least 3, the complement of the two-holed torus is a surface of genus at least 1. Hence, in that subsurface we may find a sixth non-separating curve η. Observe that η makes a non-separating pair with each of the other five curves. See Figure 2.3.

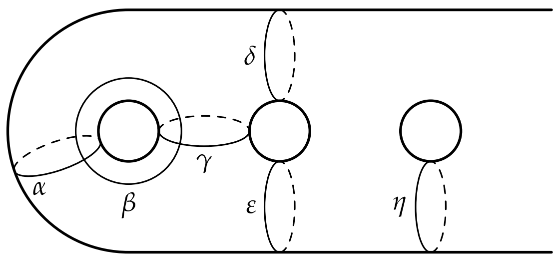


Figure 2.3: A two-holed torus embedded in a surface of genus at least 3.

Applying u and the cocycle condition repeatedly to the two-holed torus relation yields the equation

$$u(\tau_\alpha) + \tau_\alpha u(\tau_\beta) + \cdots = u(\tau_\delta) + \tau_\delta u(\tau_\varepsilon). \tag{2.19}$$

The dots on the left-hand side represent 10 terms involving various actions of $\tau_\alpha, \tau_\beta, \tau_\gamma$ on the values of u on these twists. Since each of the five curves is disjoint from η, we have $\tau_\alpha^{\pm 1} s_\eta = s_\eta$, and similarly for $\beta, \gamma, \delta, \varepsilon$. Now we take the inner product of (2.19) with s_η to obtain

$$4\langle u(\tau_\alpha), s_\eta \rangle + 4\langle u(\tau_\beta), s_\eta \rangle + 4\langle u(\tau_\gamma), s_\eta \rangle = \langle u(\tau_\delta), s_\eta \rangle + \langle u(\tau_\varepsilon), s_\eta \rangle \tag{2.20}$$

using the fact that $\langle \varphi \bullet, \bullet \rangle = \langle \bullet, \varphi^{-1} \bullet \rangle$. But since $\tau_\alpha s_\eta = s_\eta$, we also have $p_\alpha s_\eta = s_\eta$, and since the projection p_α is self-adjoint, the first term in (2.20) is equal to $4\langle s_\alpha, s_\eta \rangle = 4c$. Similar remarks apply to the other terms, so (2.20) reduces to $12c = 2c$, and $c = 0$. □

Now we are ready to prove the main result of this section.

Proof of Theorem 2.25. We first treat the case where α is non-separating. We cannot simply put $\alpha = \beta$ in the computation of c, since (α, α) is not a non-separating pair. But when the surface has genus at least 3, we may embed the lantern relation (Lemma 2.7) in such a way that all seven curves are non-separating. Furthermore, it can be done in such a way that γ_0 makes a non-separating pair with each of the other six curves. On Figure 2.4 this is shown for a genus 3 surface; note that the shown surface has been cut along γ_0. The right-hand part of the cut surface (a sphere with four holes) could be replaced by a surface with any genus and (at least) four boundary components. Now the cocycle condition applied to the lantern relation gives

$$u(\tau_0) + \tau_0 u(\tau_1 \tau_2 \tau_3) = u(\tau_{12}\tau_{13}\tau_{23}). \tag{2.21}$$

Finally, taking the inner product with s_{γ_0} on both sides and applying computations similar to those above, we get $\langle s_{\gamma_0}, s_{\gamma_0} \rangle = \langle u(\tau_0), s_{\gamma_0} \rangle = 0$. Hence $s_{\gamma_0} = 0$, and by naturality (Lemma 2.26) this holds for any non-separating curve.

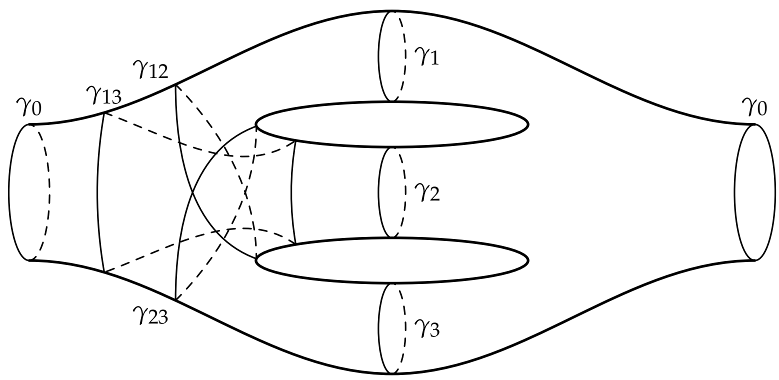


Figure 2.4: An embedding of the lantern relation such that all seven curves are non-separating. The γ_0 on the left is identified with that on the right.

If α is separating, we use the fact that one of the sides of α has genus at least 2 and Corollary 2.8 to write τ_α as a product of twists in six non-separating curves. For some appropriate choice of signs ε_j, we thus have $\tau_\alpha = \prod_{j=1}^{6} \tau_j^{\varepsilon_j}$, where the τ_j are the twists in the appropriate non-separating curves disjoint from α. Now apply the cocycle condition and take the inner product with s_α to obtain

$$\langle s_\alpha, s_\alpha \rangle = \langle u(\tau_\alpha), s_\alpha \rangle = \langle u(\tau_1^{\varepsilon_1}), s_\alpha \rangle + \cdots + \langle \tau_1^{\varepsilon_1}\tau_2^{\varepsilon_2}\tau_3^{\varepsilon_3}\tau_4^{\varepsilon_4}\tau_5^{\varepsilon_5} u(\tau_6^{\varepsilon_6}), s_\alpha \rangle.$$

By Lemma 2.27, $\tau_j^{\pm 1} s_\alpha = s_\alpha$, so using the unitarity of the action this reduces to

$$\langle s_\alpha, s_\alpha \rangle = \sum_{j=1}^{6} \langle u(\tau_j^{\varepsilon_j}), s_\alpha \rangle.$$

Finally, we conclude that each term on the right-hand side vanishes by writing s_α as $p_j s_\alpha$, moving the self-adjoint projection p_j to $u(\tau_j^{\varepsilon_j})$ and using that $s_\beta = 0$ for non-separating curves β. ☐

Corollary 2.30. *Assume $\Gamma \to \mathrm{U}(V)$ is a unitary representation of the mapping class group which restricts to the trivial representation of the Torelli group. Then any cocycle $\Gamma \to V$ restricts to the zero map on $\mathcal{T}$.*

Proof. Let $u \colon \Gamma \to V$ be a cocycle. It suffices to prove the claim for a set of generators of $\mathcal{T}$. To this end, we use the fact that $\mathcal{T}$ is generated by genus 1 bounding pair maps. Let $t = \tau_\gamma \tau_\delta^{-1}$ be such a generator. Obviously, $t = \tau_\gamma t \tau_\gamma^{-1}$, so

$$u(t) = u(\tau_\gamma t \tau_\gamma^{-1}) = (1 - t)u(\tau_\gamma) + \tau_\gamma u(t) = \tau_\gamma u(t)$$

since t acts trivially on V. From this, we infer that $p_\gamma u(t) = u(t)$. On the other hand, $\tau_\gamma = \tau_\delta$ as operators on V, so $p_\gamma = p_\delta$. Hence,

$$p_\gamma u(t) = p_\gamma(u(\tau_\gamma) - \tau_\gamma \tau_\delta^{-1} u(\tau_\delta)) = p_\gamma u(\tau_\gamma) - t p_\delta u(\tau_\delta) = 0$$

by Theorem 2.25. ☐

We note that this immediately proves

Theorem 2.31. *Assume $\Gamma \to \mathrm{U}(V)$ is a unitary representation of the mapping class group which restricts to the trivial representation of the Torelli group. Then $H^1(\Gamma, V) = 0$.*

Proof. Combine the previous corollary with the exact sequence (1.16) and the fact that $\mathrm{Sp}(2g, \mathbb{Z})$ has Property (T) (Propositions 1.4 and 1.7). ☐

Chapter 3

Quantization

In this chapter we will introduce two notions of »quantization«, deformation quantization, also known as star products, and geometric quantization. Under some general assumptions, one can obtain a star product on so-called pre-quantizable compact Kähler manifolds using Berezin-Toeplitz operators and geometric quantization.

3.1 Poisson Algebras and Manifolds

Let A be an algebra over $\mathbb{R}$ or $\mathbb{C}$. A *Poisson bracket* on A is a linear map $\{\bullet, \bullet\}\colon A \otimes A \to A$ making A into a Lie algebra, with the further requirement that the bracket is a derivation in the first variable (thus, by antisymmetry, in both). This means that for $x, y, z \in A$ we have $\{xy, z\} = x\{y, z\} + \{x, z\}y$ along with the usual Lie algebra axioms. A *Poisson algebra* is, of course, an algebra equipped with a Poisson bracket. A simple example of a Poisson algebra is the usual way of turning an associative algebra into a Lie algebra; one easily checks that the commutator $[x, y] = xy - yx$ in fact is a derivation in each variable. However, since the algebras we will consider are commutative, this does not yield an interesting structure.

A special case is when A is the algebra $C^\infty(M)$ of smooth functions on a manifold M. A *Poisson manifold* is a smooth manifold M equipped with a Poisson bracket on $C^\infty(M)$. The most common source of Poisson manifolds are symplectic manifolds. Recall that a symplectic form on a manifold M is a closed, non-degenerate 2-form $\omega \in \Omega^2(M)$. Here non-degenerate means that at each point $p \in M$, ω_p is non-degenerate, and hence a symplectic form ω defines an isomorphism $\widetilde{\omega}\colon TM \to T^*M$ between the tangent and cotangent bundles, given by $X \mapsto i_X\omega$. A manifold equipped with a symplectic form is called a symplectic manifold. It is easy to see that a manifold admitting a symplectic form is even-dimensional. Letting $\dim M = 2m$, one may show that $\omega^m = \omega^{\wedge m}$ is an orientation form on M, so symplectic manifolds are orientable.

For any smooth function $f \in C^\infty(M)$, the differential df is a smooth section of the cotangent bundle, so the composition with the inverse of the isomorphism $\widetilde{\omega}$ gives a smooth section of the tangent bundle, ie. a vector field. This vector field is denoted X_f, and is called the Hamiltonian of f. It is characterized by $i_{X_f}\omega = df$, or equivalently $\omega(X_f, Y) = df(Y) = Yf$ for any smooth vector field Y on M. Conversely, if for a vector field X the 1-form $i_X\omega$ is exact, we call X a *Hamiltonian* vector field. If $i_X\omega$ is closed, we call X a *symplectic* vector field. A symplectic vector field is characterized by preserving the symplectic form; we have $\mathcal{L}_X\omega = di_X\omega$ since ω is closed, so X is symplectic if and only if the Lie derivative of ω along X vanishes. By the Poincaré lemma, a symplectic vector field is locally the Hamiltonian vector field for some smooth function, so symplectic vector fields are also called *locally Hamiltonian*.

We note that for symplectic vector fields X, Y, we have

$$i_{[X,Y]}\omega = \mathcal{L}_X i_Y\omega - i_Y\mathcal{L}_X\omega = i_X di_Y\omega + di_X i_Y\omega = d(\omega(Y, X)) \tag{3.1}$$

showing that the Lie bracket of symplectic vector fields is Hamiltonian.

Given two smooth functions f, g, the pairing $\omega(X_f, X_g)$ is a smooth function on M.

Lemma 3.1. *The assignment $\{f, g\} = \omega(X_f, X_g)$ determines a Poisson structure on $C^\infty(M)$.*

Proof. Clearly $\{\bullet, \bullet\}$ is bilinear and anti-symmetric. To see that it is a derivation, we use the interpretation of the Hamiltonian as a directional derivative:

$$\begin{aligned}\{fg, h\} = \omega(X_{fg}, X_h) = X_h(fg) = X_h(f)\cdot g + f\cdot X_h(g)\\ = \omega(X_f, X_h)\cdot g + f\cdot\omega(X_g, X_h) = \{f, h\}g + f\{g, h\}.\end{aligned}$$

Before proving that is satisifes the Jacobi identity, we first observe that for two functions f, g we have

$$i_{[X_f, X_g]}\omega = d(\omega(X_g, X_f)) = d\{g, f\}$$

by (3.1), since Hamiltonian vector fields are in particular symplectic. This shows that the vector field $[X_f, X_g]$ is the Hamiltonian vector field associated to $\{g, f\}$. Next, let $X_j = X_{f_j}$ be the Hamiltonians of the smooth function f_j, $j = 1, 2, 3$. Then

$$\begin{aligned}X_1\omega(X_2, X_3) = X_1\{f_2, f_3\} = \omega(X_{\{f_2, f_3\}}, X_1)\\ = \{\{f_2, f_3\}, f_1\} = \omega([X_3, X_2], X_1) = \omega(X_1, [X_2, X_3]).\end{aligned} \tag{3.2}$$

Since ω is closed, we have

$$\begin{aligned}0 &= d\omega(X_1, X_2, X_3)\\ &= X_1\omega(X_2, X_3) - X_2\omega(X_1, X_3) + X_3\omega(X_1, X_2)\\ &\quad - \omega([X_1, X_2], X_3) + \omega([X_1, X_3], X_2) - \omega([X_2, X_3], X_1).\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) (and cyclic permutations of the former) we obtain

$$\{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\} + \{\{f_1, f_2\}, f_3\} = 0,$$

which is exactly the Jacobi identity. □

Note that the proof in particular shows that, up to a sign, the map $f \mapsto X_f$ is a homomorphism of Lie algebras $C^\infty(M) \to \mathcal{X}(M)$. The kernel consists of the locally constant functions on M.

We used the closedness of the symplectic form to deduce the Jacobi identity. One may in fact show that for any non-degenerate 2-form α, the pairing $(f,g) = \alpha(\widetilde{\alpha}^{-1}df, \widetilde{\alpha}^{-1}dg)$ is a Lie bracket on $C^\infty(M)$ if and only if α is closed.

Remark 3.2. A Poisson structure on the algebra $C^\infty(M,\mathbb{R})$ may be uniquely extended to a Poisson structure on the algebra of complex-valued functions $C^\infty(M,\mathbb{C}) \cong C^\infty(M,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ by complex linearity. Hence the above construction of a Poisson structure via a symplectic structure also applies to $C^\infty(M,\mathbb{C})$. We shall primarily be interested in the complex valued functions, but our Poisson brackets are usually a priori only defined on the real functions. From this remark on, it is understood that such a Poisson bracket is extended to the complex valued functions.

A Poisson bracket defined on $C^\infty(M,\mathbb{C})$ need not in general restrict to a Poisson bracket on $C^\infty(M,\mathbb{R})$, since nothing in the axioms prevents the bracket of two real-valued functions from taking non-real values.

3.2 Deformation Quantization

Given a commutative complex Poisson algebra A, one may wish to »deform« the product in A to be »less commutative«. This is formalized by the notion of a deformation quantization. We let $\mathbb{C}_h = \mathbb{C}[[h]]$ denote the ring of formal power series with complex coefficients, and similarly $A_h = A[[h]]$ the algebra of formal power series with coefficients in A. Clearly A_h is a $\mathbb{C}_h$-module and we think of A_h as a filtered module using the h-filtration $A_h \supseteq hA_h \supseteq \cdots$. By extending the Poisson bracket on A $\mathbb{C}_h$-linearly it also becomes a Poisson algebra.

Definition 3.3. A *deformation quantization* of (or *star product* on) A is an associative product $*\colon A_h \otimes_{\mathbb{C}_h} A_h \to A_h$ such that

$$a * b = ab \pmod h \quad \text{and} \tag{3.4a}$$

$$a * b - b * a = \{a,b\}h \pmod{h^2} \tag{3.4b}$$

for all $a, b \in A \subseteq A_h$.

The condition (3.4a) states that the zeroth order term of the star product is the usual product in A, whereas (3.4b) states that, to first order, the failure of $*$ to be commutative is measured by the Poisson bracket. Often, for normalization reasons, the right-hand side of (3.4b) is replaced by some complex constant times $\{a,b\}h$ such as i or $\pm\frac{1}{2}$. If the algebra A has a unit, this is usually also required to be a two-sided unit for $*$; whether or not it is required, it will be the case in all of the examples we shall encounter.

A star product is uniquely determined by its *coefficients*, which are the bilinear maps $c_r\colon A \times A \to A$ defined by

$$a * b = \sum_{r=0}^{\infty} c_r(a,b)h^r.$$

In terms of the coefficients, the axioms (3.4) read

$$c_0(a,b) = ab \tag{3.5a}$$
$$c_1(a,b) - c_1(b,a) = \{a,b\} \tag{3.5b}$$

along with a lot of relations arising from the associativity of $*$.

3.2.1 Equivalence of Star Products

Intuitively, an equivalence between two star products $*, *'$ on A should be a vector space automorphism of A_h intertwining the two star products, ie. a map $(A_h, *) \to (A_h, *')$ which is an isomorphism of $\mathbb{C}_h$-algebras. Pursuing the idea that a star product is a deformation of the ordinary product in A, it is natural to demand that this automorphism is the identity on A. Using that for any two complex vector spaces V, W, there is an isomorphism $\mathrm{Hom}_{\mathbb{C}}(V,W)[[h]] \cong \mathrm{Hom}_{\mathbb{C}_h}(V_h, W_h)$ of $\mathbb{C}_h$-modules, which is even an isomorphism of $\mathbb{C}_h$-algebras if $V = W$, we arrive at the following definition.

Definition 3.4. Two star products $*, *'$ are said to be *equivalent* if there is a sequence $T_j \in \mathrm{Hom}_{\mathbb{C}}(A,A)$ of linear maps, $j \in \mathbb{N}$, such that $T_0 = \mathrm{id}_A$ and such that the map $T = \sum T_j h^j \colon A_h \to A_h$ satisfies

$$T(x * y) = T(x) *' T(y) \tag{3.6}$$

for all $x, y \in A_h$.

3.3 Star Products on Symplectic Manifolds

In the special case of a symplectic manifold (M, ω), we may impose further conditions on star products. Throughout this section, A denotes the algebra $C^\infty(M) = C^\infty(M, \mathbb{C})$ equipped with the Poisson structure determined by ω, and $A_h = C^\infty(M)[[h]]$. With a slight abuse of terminology, we will also refer to a star product on A as a star product on M.

3.3.1 Differential Operators

Recall that on a manifold M, a differential operator D is a linear map from the space of smooth functions to itself, such that in a chart (U, x), D may be written as an element in $C^\infty(U, \mathbb{R})[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}]$. More precisely, the action of D on a smooth function f can be written

$$Df_{|U} = \sum_J D_J \frac{\partial^{|J|}}{\partial x^J} f$$

where the sum is over some finite set of multiindices $J = (j_1, \dots, j_n)$, D_J are smooth functions on U, $|J| = j_1 + \dots + j_n$ and $\frac{\partial^{|J|}}{\partial x^J} = \frac{\partial^{j_1}}{\partial x_1^{j_1}} \cdots \frac{\partial^{j_n}}{\partial x_n^{j_n}}$. The usual notion of order of a differential operator on $\mathbb{R}^n$ as the maximal number of partial derivatives occuring does not carry over directly to the manifold setting because a local chart is involved. Instead, we define the local order of D at $p \in M$ to be the minimal $k \in \mathbb{N}$ such that for all $f \in C^\infty(M)$ with $f(p) = 0$, we have $D(f^{k+1})(p) = 0$. The order of D is then defined to be the maximum of the local orders over all $p \in M$. For non-compact manifolds, this may be infinite. A differential operator of order 0 is simply multiplication by a smooth function, and a differential operator of (constant) order 1 is a vector field on M. A theorem of Peetre states that a differential operator is the same as a local operator, ie. an operator such that for every point $p \in M$, $(Df)(p)$ only depends on the germ of f at p. The space of all differential operators on M is denoted $\mathcal{D}(M)$.

3.3.2 Differential Star Products

Returning to deformation quantizations, a star product is said to be *differential* if the coefficients $c_r \colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ are bidifferential operators, meaning that for fixed $f \in C^\infty(M)$, $c(f, \bullet)$ and $c(\bullet, f)$ are differential operators. In particular, a differential star product is computable locally, which need not be the case for a general star product.

An equivalence $T = \mathrm{id} + \sum_{j \geq 1} T_j h^j$ of star products is said to be a *differential equivalence* if the T_j are differential operators. Fortunately, for differential star products, there is no difference between being equivalent and differentially equivalent:

Theorem 3.5. *Assume that T is an equivalence between the differential star products $*$ and $*'$. Then T is a differential equivalence.*

This is Theorem 2.22 in [23]. In fact Gutt and Rawnsley gives a complete classification of the equivalence classes of differential star products on a symplectic manifold: It turns out that the equivalence classes of differential star products are in bijective correspondence with the elements of the affine vector space $-[\omega] \cdot h^{-1} + H^2(M, \mathbb{R})[[h]]$, so in particular if $H^2(M, \mathbb{R}) = 0$, there is a unique differential star product up to equivalence. However, for compact symplectic manifolds, $H^2(M, \mathbb{R})$ never vanishes (the symplectic form represents a non-zero class), so informally speaking there are »many« equivalence classes of differential star products.

We will need a few more results and definitions from [23]. First, the associativity of a star product allows us to introduce the $*$-commutator, $[a, b]_* = a * b - b * a$ for $a, b \in A_h$, making A_h into a Lie algebra. By (3.4b), this Lie algebra structure is to first order h times the one obtained by extending the Poisson bracket on A $\mathbb{C}_h$-linearly to A_h. The same equation shows that $[a, b]_*$ is a multiple of h, so that taking iterated $*$-brackets leads to formal series with higher and higher initial terms. This we express by saying that $(A_h, [\bullet, \bullet]_*)$ is a *pro-nilpotent* Lie-algebra. The adjoint representation of this Lie algebra on itself is denoted ad_*, so for any element a we have an endomorphism $\mathrm{ad}_* a$ of A_h defined by $\mathrm{ad}_* a(b) = [a, b]_*$ for $b \in A_h$. That $[\bullet, \bullet]_*$ is pro-nilpotent implies that the exponential of $\mathrm{ad}_* a$ is a well-defined automorphism of A_h, given by the usual formula

$$\exp(\mathrm{ad}_* a) = \sum_{j=0}^{\infty} \tfrac{1}{j!} (\mathrm{ad}_* a)^{\circ j} = \mathrm{id} + \mathrm{ad}_* a + \tfrac{1}{2} \mathrm{ad}_* a \circ \mathrm{ad}_* a + \cdots$$

since applying this to any element $b \in A_h$ yields an infinite (formal) series with only finitely many terms in each degree.

Lemma 3.6. *If M is connected, $\exp(\operatorname{ad}_* a) = \operatorname{id}$ if and only if $a = \sum_j a_j h^j \in \mathbb{C}[[h]]$, ie. each function a_j is constant.*

Another important fact is that the composition of two such automorphisms may be described as the exponential of another Lie algebra element. Namely, define $a \circ_* b$ as the usual Baker-Campbell-Hausdorff composition

$$a \circ_* b = a + b + \tfrac{1}{2}[a,b]_* + \tfrac{1}{12}([a,[a,b]_*]_* + [b,[b,a]_*]_*) + \cdots .$$

Again quoting from [23], we have

Lemma 3.7. *The composition $\circ_*$ is associative and satisfies*

$$\exp \operatorname{ad}_*(a \circ_* b) = (\exp \operatorname{ad}_* a) \circ (\exp \operatorname{ad}_* b) \tag{3.7}$$

$$a \circ_* b \circ_* (-a) = \exp(\operatorname{ad}_* a)(b). \tag{3.8}$$

In studying the equivalence classes of differential star products, it is useful to know the self-equivalences of a star product. For manifolds with vanishing first de Rham cohomology, these are easy to describe.

Theorem 3.8. *Let $*$ be a star product on the symplectic manifold (M, ω), and assume $H^1(M, \mathbb{R}) = 0$. Then any automorphism $T = \operatorname{id} + \sum_{j\geq 1} T_j h^j$ of $*$ is inner, ie. of the form $T = \exp(\operatorname{ad}_* a)$ for some $a \in A_h$.*

As we will need to know how this a looks like, we recall the proof from [23], Proposition 3.3. However, we first need a lemma describing the derivations of the Poisson bracket.

Lemma 3.9. *Let X be a vector field on (M, ω) which is a derivation of the Poisson bracket. Then X is a symplectic vector field. If in addition $H^1(M) = 0$, X is an inner derivation, meaning that $Xf = \{a, f\}$ for some smooth function a.*

Proof. The derivation assumption means that $X\{f,g\} = \{Xf, g\} + \{f, Xg\}$ for all $f, g \in C^\infty(M)$. This may be rewritten $\omega(X_{\{f,g\}}, X) = X_g \omega(X_f, X) - X_f \omega(X_g, X)$, which in turn can be written

$$d(i_X \omega)(X_f, X_g) = 0 \tag{3.9}$$

for all $f, g \in C^\infty(M)$. Now, for any given tangent vectors $v, w \in T_p M$ one may find smooth functions f, g such that $(X_f)_p = v$ and $(X_g)_p = w$, so (3.9) implies that $i_X \omega$ is closed, so X is symplectic.

The assumption $H^1(M) = 0$ then implies that $i_X \omega = d(-a)$ for some smooth function a, so X is the Hamiltonian vector field associated to $-a$. But then $Xf = X_{-a} f = \omega(X_f, X_{-a}) = \{a, f\}$. □

Proof of Theorem 3.8. We will build $a = \sum_{j\geq 0} a_j h^j$ recursively. Writing out the assumption $T(f * g) = T(f) * T(g)$ and equating the coefficients of h gives

$$c_1(f,g) + T_1(fg) = c_1(f,g) + T_1(f)g + fT_1(g)$$

implying that T_1 is a vector field. Equating the coefficients of h^2 we obtain (omitting the term $c_2(f,g)$ on both sides)

$$T_1(c_1(f,g)) + T_2(fg) = c_1(T_1(f),g) + c_1(f,T_1(g)) + T_2(f)g + fT_2(g).$$

Skew-symmetrization of this relation (ie. subtracting the same formula with f and g interchanged) combined with (3.5b) yields

$$T_1(\{f,g\}) = \{T_1(f),g\} + \{f,T_1(g)\}$$

so we see that, by Lemma 3.9, T_1 is a Hamiltonian vector field. Write $T_1 f = \{a_0, f\}$ for some smooth function a_0. Then the composition $\exp(-\mathrm{ad}_* a_0) \circ T$ is of the form $\mathrm{id} + O(h^2)$ as is easily seen by applying it to some smooth function f and using that $[a_0, f]_* = \{a_0, f\}h + O(h^2)$.

These considerations both form the start of an induction and the idea for the inductive step. Namely, assume that we have found $a^{(k-1)} = a_0 + a_1 h + \cdots + a_{k-1}h^{k-1}$ such that

$$T' = \exp(-\mathrm{ad}_* a^{(k-1)}) \circ T = \mathrm{id} + h^{k+1}T'_{k+1} + O(h^{k+2}).$$

Then, since T' is also an automorphism of $*$, we may repeat the above argument: Taking the coefficient of h^{k+1} in the equation $T'(f*g) = T'(f) * T'(g)$ gives

$$c_{k+1}(f,g) + T'_{k+1}(fg) = c_{k+1}(f,g) + T'_{k+1}(f)g + fT'_{k+1}(g)$$

showing that T'_{k+1} is a vector field. Similarly, skew-symmetrizing the expressions for the coefficients of h^{k+2} we obtain

$$T'_{k+1}(\{f,g\}) = \{T'_{k+1}(f),g\} + \{f,T'_{k+1}(g)\}$$

so T'_{k+1} is a Hamiltonian vector field. Choosing a_k such that $T'_{k+1}f = \{a_k, f\}$, we may put $a^{(k)} = a^{(k-1)} + a_k h^k$. Then $\exp(-\mathrm{ad}_* a^{(k)}) \circ T = \mathrm{id} + O(h^{k+2})$, completing the inductive step, and $a = \lim_{k\to\infty} a^{(k)}$ is the desired a. □

3.3.3 Invariant Star Products

Assume a group Γ acts on the *compact* symplectic manifold (M,ω) via symplectomorphisms, ie. $\gamma\colon M \to M$ is a diffeomorphism and $\gamma^*\omega = \omega$ for all $\gamma \in \Gamma$. This action clearly induces an action on $C^\infty(M)$ by $(\gamma \cdot f)(x) = f(\gamma^{-1}x)$, which we may extend h-linearly to $C^\infty(M)[[h]]$. We say that a star product $*$ is Γ-invariant if $\gamma \cdot (f*g) = (\gamma\cdot f) * (\gamma \cdot g)$ for all $f,g \in C^\infty(M)$.

Assume $*$ and $*'$ are equivalent differential Γ-invariant star products on M. Thus, by Theorem 3.5, we have an equivalence $T = \mathrm{id} + \sum_{j\geq 1} T_j h^j$, where $T_j\colon C^\infty(M) \to C^\infty(M)$ are differential operators, such that

$$T(f*g) = T(f) *' T(g) \tag{3.10}$$

for all $f,g \in C^\infty(M)$. We may now ask if we can find another equivalence T' which is Γ-invariant, in the sense that $\gamma \circ T' = T' \circ \gamma$ for all $\gamma \in \Gamma$. Under certain cohomological assumptions, this is indeed the case. We quote Proposition 6 from [1].

Proposition 3.10 (Andersen). *Assume that the group cohomology vector space $H^1(\Gamma, C_0^\infty(M))$ and the first de Rham cohomology space $H^1_{\mathrm{dR}}(M,\mathbb{R})$ both vanish. Then there is a Γ-invariant equivalence between $*$ and $*'$.*

Here $C_0^\infty(M)$ denotes the subspace of $C^\infty(M)$ consisting of smooth functions with mean value 0, ie. functions for which

$$\frac{1}{m!}\int_M f\omega^m = 0.$$

Since Γ acts by symplectomorphisms, the action on $C^\infty(M)$ preserves this subspace, so $C_0^\infty(M)$ is a Γ-module.

Proof. Let T be any equivalence. Then (3.10) implies that for $f, g \in C^\infty(M)$ we have $f * g = T^{-1}(T(f) *' T(g))$, and replacing f and g by $T^{-1}(f)$ and $T^{-1}(g)$, respectively, we obtain $T^{-1}(f) * T^{-1}(g) = T^{-1}(f *' g)$ (that is, T^{-1} is an equivalence from $*'$ to $*$). Then using the Γ-invariance of $*$ and $*'$ we have

$$\begin{aligned} T^{-1}\circ\gamma\circ T(f*g) &= T^{-1}\circ\gamma\big(T(f)*'T(g)\big) \\ &= T^{-1}\big(\gamma T(f)*'\gamma T(g)\big) \\ &= T^{-1}\gamma T(f) * T^{-1}\gamma T(g) \end{aligned}$$

showing that $T^{-1}\circ\gamma\circ T$ is an automorphism of $*$, except for the fact that it is of the form $\gamma + \sum_{j=1}^\infty S_j h^j$. Precomposing with γ^{-1} we obtain a genuine automorphism of $*$, so that, by Theorem 3.8, we have

$$T^{-1}\circ\gamma\circ T = \exp(\mathrm{ad}_* a_\gamma)\circ\gamma \tag{3.11}$$

for some $a_\gamma \in C^\infty(M)[[h]]$. Writing $a_\gamma = \sum_{j=0}^\infty a_\gamma^{(j)} h^j$, we may without loss of generality assume that each $a_\gamma^{(j)} \in C_0^\infty(M)$, since by Lemma 3.6 we may replace $a_\gamma^{(j)}$ by $a_\gamma^{(j)} - c_\gamma^{(j)}$, where $c_\gamma^{(j)} \in \mathbb{C}$ is the mean value of $a_\gamma^{(j)}$. By the same lemma, a_γ is then uniquely determined by (3.11). Clearly we have $a_\gamma^{(0)} = 0$ for all $\gamma \in \Gamma$.

Assume inductively that T is an equivalence for which a_γ, determined by (3.11) and the requirement $a_\gamma \in C_0^\infty(M)[[h]]$, vanishes modulo h^{k-1} for all $\gamma \in \Gamma$. We observe that

$$\exp(\mathrm{ad}_* a_{\gamma\eta})\circ\gamma\eta = \exp(\mathrm{ad}_* a_\gamma)\circ\gamma\circ\exp(\mathrm{ad}_* a_\eta)\circ\eta,$$

implying that

$$\begin{aligned} \exp(\mathrm{ad}_* a_{\gamma\eta}) &= \exp(\mathrm{ad}_* a_\gamma)\circ\exp(\mathrm{ad}_*(\gamma a_\eta)) \\ &= \exp(\mathrm{ad}_* a_\gamma \circ_* \gamma(a_\eta)) \end{aligned} \tag{3.12}$$

for all $\gamma, \eta \in \Gamma$. Now since $a_\gamma^{(j)} = 0$ for all $j < k$, we have by the properties of $\circ_*$ that $a_\gamma \circ_* \gamma(a_\eta) = (a_\gamma^{(k)} + \gamma a_\eta^{(k)})h^k + O(h^{k+1})$, so we see that (3.12) implies

$$a_{\gamma\eta}^{(k)} = a_\gamma^{(k)} + \gamma a_\eta^{(k)}$$

for all $\gamma, \eta \in \Gamma$. But this precisely means that $\{a_\gamma^{(k)}\}_{\gamma\in\Gamma}$ is a 1-cocycle $\Gamma \to C_0^\infty(M)$ (cf. (1.1)), so that by assumption it is a coboundary $a_\gamma^{(k)} = f^{(k)} - \gamma f^{(k)}$ for some smooth function $f^{(k)} \in C_0^\infty(M)$. Now replacing T by $T \circ \exp(\mathrm{ad}_* f^{(k)} h^k)$ in (3.11), we obtain

$$\begin{aligned}
&\exp(-\mathrm{ad}_* f^{(k)} h^k) \circ T^{-1} \circ \gamma \circ T \circ \exp(\mathrm{ad}_* f^{(k)} h^k)\\
&\quad = \exp(-\mathrm{ad}_* f^{(k)} h^k) \circ \exp(\mathrm{ad}_* a_\gamma) \circ \gamma \circ \exp(\mathrm{ad}_* f^{(k)} h^k) \circ \gamma^{-1} \circ \gamma\\
&\quad = \exp(-\mathrm{ad}_* f^{(k)} h^k) \circ \exp(\mathrm{ad}_* a_\gamma) \circ \exp(\mathrm{ad}_* \gamma f^{(k)} h^k) \circ \gamma\\
&\quad = \exp(\mathrm{ad}_*(f^{(k)} h^k \circ_* a_\gamma \circ_* \gamma f^{(k)} h^k)) \circ \gamma
\end{aligned}$$

so we see that $T \circ \exp(\mathrm{ad}_* f^{(k)} h^k)$ is a new equivalence between $*$ and $*'$ whose corresponding »a_γ« is $f^{(k)} h^k \circ_* a_\gamma \circ_* \gamma f^{(k)} h^k$, which by construction vanishes modulo h^k.

Inductively, we obtain an equivalence T such that $T^{-1} \circ \gamma \circ T = \gamma$ for all $\gamma \in \Gamma$, ie. an invariant equivalence. □

3.4 Geometric Quantization

Until now we have only discussed abstract properties of star products on symplectic manifolds. In this section we will present a general method to obtain a star product on a compact symplectic manifold (M, ω). For more details, see e.g. [34] and [29]. The method requires two additional assumptions: The existence of a pre-quantum line bundle and a Kähler structure on (M, ω). These concepts are the topics of the following two subsections.

3.4.1 Pre-quantization

Definition 3.11. A *prequantum line bundle* over (M, ω) is a triple $(\mathcal{L}, (\cdot,\cdot), \nabla)$ consisting of a complex line bundle $\mathcal{L}$ over M, an Hermitian structure $(\cdot,\cdot)$ on $\mathcal{L}$ and a compatible connection ∇ in $\mathcal{L}$ whose curvature F_∇ is $-i\omega$. If there exists a pre-quantum line bundle over (M, ω) we call (M, ω) *pre-quantizable*.

That ∇ is compatible with the Hermitian structure means that for any vector field X on M and sections s_1, s_2 of $\mathcal{L}$, we have $X(s_1, s_2) = (\nabla_X s_1, s_2) + (s_1, \nabla_X s_2)$ as complex-valued functions on M. The curvature of a connection in a line bundle (or more generally a vector bundle) is by definition the 2-form on M with values in the endomorphism bundle $\mathrm{End}(\mathcal{L})$ defined by $F_\nabla(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$ for vector fields X, Y on M and a section s of $\mathcal{L}$. In the case of a line bundle, the endomorphism bundle is simply the trivial line bundle $M \times \mathbb{C}$, so we may identify the curvature with a (complex-valued) 2-form on M.

It turns out that the existence of a pre-quantum line bundle depends on the symplectic form ω. In fact, identifying de Rham cohomology with Čech cohomology via the canonical isomorphism $H^*_{\mathrm{dR}}(M) \cong \check{H}^*(M, \mathbb{R})$, (M, ω) is pre-quantizable if and only if $[\frac{\omega}{2\pi}]$ is an element of the image of

$\check{H}^2(M,\mathbb{Z}) \to \check{H}^2(M,\mathbb{R})$. Also, the inequivalent choices of a pre-quantum line bundle, when they exist, are parametrized by $\check{H}^1(M,\mathbb{R}/\mathbb{Z})$ (see [42] for details).

Assume (M,ω) is pre-quantizable and fix a pre-quantum line bundle $(\mathcal{L},(\cdot,\cdot),\nabla)$. For any positive integer k, we denote by $\mathcal{L}^k$ the k'th iterated tensor product of $\mathcal{L}$ with itself. It is again a complex line bundle over M with a Hermitian structure $(\cdot,\cdot)^k$ and connection ∇^k, whose curvature is $-ki\omega$. Thus $\mathcal{L}^k$ is a pre-quantum line bundle over the symplectic manifold $(M,k\omega)$. If $f\colon U \to \mathcal{L}$ is a local orthonormal frame for $\mathcal{L}$ (ie. $(f,f)=1$), any section of $\mathcal{L}$ over U may be written $s=\sigma f$ for a unique complex-valued function σ on U. Then $(s,t)=(\sigma f,\tau f)=\sigma\overline{\tau}$, and given $2k$ sections $s_1,\dots,s_k,t_1,\dots,t_k$ of $\mathcal{L}$ over U, we may describe $(\cdot,\cdot)^k$ and ∇^k by

$$(s_1\otimes\cdots\otimes s_k, t_1\otimes\cdots\otimes t_k) = \sigma_1\cdots\sigma_k\overline{\tau_1\cdots\tau_k}$$
$$\nabla_X(s_1\otimes\cdots\otimes s_k) = \sum_{j=1}^{k} s_1\otimes\cdots\otimes\nabla_X s_j\otimes\cdots\otimes s_k.$$

It is easy to see that the Hermitian structure is independent of the chosen local frame, and that the connection is compatible with the Hermitian structure.

3.4.2 Kähler Structure

In order to be able to make sense of »holomorphic sections« of a complex vector bundle, the base space needs to have some sort of complex structure. A particularly nice way of formalizing this is to choose an *almost complex structure* J on M, making (M,ω,J,g) into an *almost Kähler manifold*. This means that $J\in C^\infty(M,\mathrm{End}(TM))$ is a smooth section of the endomorphism bundle of TM such that $J^2=-\mathrm{id}$, and such that the assignment $g(X,Y)=\omega(X,JY)$ for X,Y vector fields on M is a Riemannian metric, called the Kähler metric. The symmetry of g is equivalent to $\omega(X,Y)=\omega(JX,JY)$ for all vector fields X,Y.

Extending the almost complex structure J complex linearly to the complexified tangent bundle $TM_{\mathbb{C}} = TM\otimes\mathbb{C} = TM\oplus iTM$, we obtain a splitting $TM_{\mathbb{C}} = T'M\oplus T''M$ into the *holomorphic* and *anti-holomorphic* parts, respectively, given by the i and $-i$ eigenspaces

$$T'M = \ker(J-i\mathrm{Id}) = \mathrm{im}(\mathrm{Id}-iJ) \tag{3.13a}$$
$$T''M = \ker(J+i\mathrm{Id}) = \mathrm{im}(\mathrm{Id}+iJ). \tag{3.13b}$$

For a tangent vector X, we denote by $X=X'+X''$ its splitting into its holomorphic and antiholomorphic parts; it is easy to see that these are given by $X'=\frac{1}{2}(X-iJX)$ and $X''=\frac{1}{2}(X+iJX)$. Similarly, we let $\Lambda^1_{\mathbb{C}}M$ denote the complexified cotangent bundle, and define the subbundles

$$T^{(1,0)}M = \Lambda^{1,0}M = \{\xi\in\Lambda^1_{\mathbb{C}}M \mid \xi(X)=0\ \forall X\in T''M\} \tag{3.14a}$$
$$T^{(0,1)}M = \Lambda^{0,1}M = \{\xi\in\Lambda^1_{\mathbb{C}}M \mid \xi(X)=0\ \forall X\in T'M\}. \tag{3.14b}$$

We let $\Lambda^{p,0}M$ and $\Lambda^{0,p}M$ denote the p'th exterior power of $\Lambda^{1,0}M$ and $\Lambda^{0,1}M$, respectively, and $\Lambda^{p,q}M = \Lambda^{p,0}M\otimes\Lambda^{0,q}M$. Then we have a splitting of the bundle $\Lambda^k_{\mathbb{C}} M$

$$\Lambda^k_{\mathbb{C}} M = \bigoplus_{p+q=k} \Lambda^{p,q} M. \tag{3.15}$$

The sections of $\Lambda^{p,q} M$ are called the forms of *type* (p,q), and are denoted $\Omega^{p,q}(M)$. Clearly (3.15) implies that $\Omega^k(M,\mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(M)$. Let $\pi_{p,q}$ denote the projection onto the forms of type (p,q). For a fixed (p,q), we define the operators $\partial, \bar{\partial}$ by $\partial = \pi_{p+1,q} \circ d$ and $\bar{\partial} = \pi_{p,q+1} \circ d$. Restricting the exterior derivative d to $\Omega^{p,q}$, we may write it as the sum $\sum_{r+s=p+q+1} \pi_{r,s} \circ d = \partial + \bar{\partial} + \cdots$.

Proposition 3.12. *Let J be an almost complex structure on a manifold M. Then the following are equivalent.*

(1) *The Lie bracket of two holomorphic vector fields is a holomorphic vector field.*
(2) *The exterior derivative of a form of type (p,q) only has components of type $(p+1,q)$ and $(p,q+1)$, ie. $d = \partial + \bar{\partial}$.*
(3) *The* Nijenhuis tensor, *given by $N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$ for smooth vector fields X, Y, vanishes identically.*
(4) *The almost complex structure is induced by a complex structure, ie. a holomorphic atlas making M a complex manifold.*

If J satisfies any of these conditions, it is called *integrable*. All the almost complex structures we shall consider are assumed to be integrable, so that (M,ω,J,g) is a Kähler manifold. The symplectic form of a Kähler manifold (or rather, its extension to the complexified tangent bundle) is of type $(1,1)$, because if X and Y are both (anti-)holomorphic, we have $\omega(X,Y) = \omega(JX,JY) = \omega(\pm iX, \pm iY) = (\pm i)^2 \omega(X,Y) = -\omega(X,Y)$.

The Riemannian metric g on a Kähler manifold gives rise to the Ricci curvature R, which is a symmetric 2-tensor on the tangent bundle; as usual we implicitly extend it to the complexified tangent bundle. The *Ricci form* of the Kähler manifold is the $(1,1)$-form Ric defined by $\mathrm{Ric}(X,Y) = R(JX,Y)$. This is a closed form, and Hodge decomposition allows us to write it as $\mathrm{Ric} = \mathrm{Ric}^H + 2i\partial\bar{\partial}F$ where Ric^H is the harmonic part, and F is some smooth function on M. The unique function satisfying this equation with average zero over M, ie. $\int_M F\omega^m = 0$, is called the *Ricci potential*.

3.4.3 Toeplitz Operators

We now fix the following data: A pre-quantizable symplectic manifold (M,ω), a pre-quantum line bundle $(\mathcal{L}, (\cdot,\cdot), \nabla)$ over M, and a Kähler strucure J on M. Also, fix some integer k, and consider the line bundle $\mathcal{L}^k$ with the Hermitian metric $(\cdot,\cdot)^k$ and compatible connection ∇^k. For simplicity, we will denote these by $(\cdot,\cdot)$ and ∇, respectively. This should not cause any confusion. We denote by $\mathcal{H}^{(k)}$ the complex vector space of smooth sections $M \to \mathcal{L}^k$. The connection ∇ (or rather its extension to the complexified tangent bundle of M) splits into $\nabla^{1,0} + \nabla^{0,1}$, where $\nabla^{1,0}_X = \nabla_{X'}$ and $\nabla^{0,1}_X = \nabla_{X''}$, according to the splitting of tangent vectors into holomorphic and antiholomorphic parts determined by J. So for a smooth section s of $\mathcal{L}^k$, $\nabla^{1,0}s$ (respectively $\nabla^{0,1}s$) measures the derivative of s in the holomorphic (respectively, antiholomorphic) directions. We define

$$H^{(k)} = \{s \in \mathcal{H}^{(k)} \mid \nabla^{0,1}s = 0\} \tag{3.16}$$

and call $H^{(k)}$ the space of *holomorphic sections* of $\mathcal{L}^k$. By the general theory for elliptic operators, $H^{(k)}$ is a finite-dimensional subspace of $\mathcal{H}^{(k)}$; see e.g. [41].

There is a Hermitian inner product on $\mathcal{H}^{(k)}$ defined by integrating the fibre-wise inner product of two sections with respect to the volume form determined by the symplectic form, ie.

$$\langle s_1, s_2 \rangle = \frac{1}{m!} \int_M (s_1, s_2)\omega^m$$

This is not a Hilbert space structure, since $\mathcal{H}^{(k)}$ is not complete. However, the subspace $H^{(k)}$ inherits this inner product and is, of course, both closed and complete, since it is finite-dimensional. We also have an orthogonal projection $\pi^{(k)} \colon \mathcal{H}^{(k)} \to H^{(k)}$, and the inner product induces an operator norm $\|\bullet\|$ in $\mathrm{End}(H^{(k)})$.

In general, multiplying a holomorphic section of $\mathcal{L}^k$ by a smooth function does not give a holomorphic section. We may however project it back to the holomorphic sections.

Definition 3.13. Let $f \in C^\infty(M, \mathbb{C})$ be a smooth complex-valued function on M. The *Toeplitz operator* $T_f^{(k)}$ is the map $\mathcal{H}^{(k)} \to H^{(k)}$ defined by

$$T_f^{(k)}(s) = \pi^{(k)}(fs) \tag{3.17}$$

for a smooth section s of $\mathcal{L}^k$. Restricting it to $H^{(k)}$, we may also think of $T_f^{(k)}$ as an endomorphism of $H^{(k)}$.

As an endomorphism of $H^{(k)}$, there is a simple expression, due to Bordemann, Meinrenken and Schlichenmaier [14], for the operator norm of $T_f^{(k)}$ as $k \to \infty$.

Lemma 3.14. $\lim_{k\to\infty} \|T_f^{(k)}\| = \sup_{x\in M} |f(x)|$.

There is no reason to expect that the composition of two Toeplitz operators is again a Toeplitz operator. But by the results of Schlichenmaier [34], there is an asymptotic expansion in terms of Toeplitz operators.

Theorem 3.15. *Let $f, g \in C^\infty(M, \mathbb{C})$. Then there is an asymptotic expansion*

$$T_f^{(k)} T_g^{(k)} \sim \sum_{\ell=0}^{\infty} T_{c_\ell(f,g)}^{(k)} k^{-\ell} \tag{3.18}$$

for uniquely determined $c_\ell(f,g) \in C^\infty(M, \mathbb{C})$. Moreover, $c_0(f,g) = fg$.

In the statement above, $\sim$ means that

$$\|T_f^{(k)}T_g^{(k)} - \sum_{\ell=0}^{L} T^{(k)}_{c_\ell(f,g)}k^{-\ell}\| \in O(k^{-(L+1)}) \tag{3.19}$$

for all positive integers L. The usefulness of this asymptotic expansion, for our purposes, stems from the fact that these c_ℓ are actually (almost) the coefficients of a differentiable star product on M, as proved by Karabegov and Schlichenmaier [29].

Theorem 3.16. *The product $\star$ on $C^\infty(M,\mathbb{C})[[h]]$ defined by*

$$f \star g = \sum_{\ell=0}^{\infty}(-1)^\ell c_\ell(f,g)h^\ell$$

for $f,g \in C^\infty(M,\mathbb{C})$ is a differentiable star product on M.

We call $\star$ the *Berezin-Toeplitz* star product on (M,ω).

3.5 The Formal Hitchin Connection

In the previous section, we saw how the choice of an integrable complex structure on M and a pre-quantum line bundle induces a star product on M. In the presence of a symmetry group Γ acting on M by symplectomorphisms, we have also discussed a group cohomological obstruction to the uniqueness of a Γ-invariant star product. To show the *existence* of a Γ-invariant star product, we could either try to single out a specific complex structure with sufficiently nice properties, or we could consider a whole family of complex structures and try to patch the corresponding family of star products together to a single, Γ-invariant star product on M. The latter strategy is used in [1], and we will recall the most important results.

Remark 3.17. For a smooth vector bundle W over M, a smooth map $f\colon N \to C^\infty(M,W)$ from a smooth manifold N to the space of sections of W is by definition a smooth section of the pullback bundle π_M^*W over $N \times M$. This is motivated by observing that for $x \in N$, $y \in M$, $f(x)(y)$ should be an element of W_y, the fibre of W over y. Similarly, a k-form on N with values in $C^\infty(M,W)$ means a smooth section of $\pi_N^*(\Lambda^k T^*N) \otimes \pi_M^*W$ over $N \times M$.

We assume that $\mathcal{T}$ is a manifold which smoothly parametrizes Kähler structures on (M,ω), which means that there is a smooth map $J\colon \mathcal{T} \to C^\infty(M,\mathrm{End}(TM))$ such that $J(\sigma) = J_\sigma$ is an integrable complex structure on M making $(M,\omega,J_\sigma,g_\sigma)$ a Kähler manifold. We will impose further assumptions along the way. The various constructions on M in Section 3.4 above which depend on the choice of almost complex structure now makes sense for each $\sigma \in \mathcal{T}$, and we will denote the dependence on σ by adding it as a subscript. For example, the holomorphic tangent bundle with respect to J_σ will be denoted $T'_\sigma M$.

For each $\sigma \in \mathcal{T}$ the construction in section 3.4.3 gives a finite dimensional vector space $H_\sigma^{(k)} \subseteq C^\infty(M,\mathcal{L}^k)$. Considering the trivial bundle $\mathcal{H}^k = \mathcal{T} \times C^\infty(M,\mathcal{L}^k)$ over $\mathcal{T}$, we shall assume that the $H_\sigma^{(k)}$ form a (finite rank) subbundle of $\mathcal{H}^{(k)}$, denoted $H^{(k)}$.

3.5.1 The Hitchin Connection

We wish to find a connection in $\mathcal{H}^{(k)}$ which preserves the subbundle $H^{(k)}$. Denoting the trivial connection in $\mathcal{H}^{(k)}$ by $\hat{\nabla}^t$ and the vector space of differential operators in $C^\infty(M, \mathcal{L}^k)$ by $\mathcal{D}(M, \mathcal{L}^k)$, it turns out that there is a 1-form u on $\mathcal{T}$ with values in $\mathcal{D}(M, \mathcal{L}^k)$, such that the connection $\hat{\nabla}$ in $\mathcal{H}^{(k)}$ defined by

$$\hat{\nabla}_V = \hat{\nabla}^t_V - u(V)$$

preserves the subbundle $H^{(k)}$, provided J and M satisfy a few additional conditions, one of which is that $H^1(M) = 0$. Another assumption is that the first Chern class of the symplectic manifold (M, ω) is $n[\frac{\omega}{2\pi}] \in H^2(M, \mathbb{Z})$ for some $n \in \mathbb{Z}$. This connection is called the *Hitchin connection*. For details we refer to [1] and [4].

3.5.2 Formal Connections and Formal Trivializations

The projections $\pi^{(k)}_\sigma \colon C^\infty(M, \mathcal{L}^k) \to H^{(k)}_\sigma$ form a smooth map $\pi^{(k)}$ from $\mathcal{T}$ to the space of bounded operators on (the L^2-completion of) $C^\infty(M, \mathcal{L}^k)$. The Toeplitz operator $T^{(k)}_f$ associated to a smooth function f on M is then a section of the bundle $\operatorname{Hom}(\mathcal{H}^{(k)}, H^{(k)})$, defined at $\sigma \in \mathcal{T}$ by

$$T^{(k)}_{f,\sigma}(s) = \pi^{(k)}_\sigma(fs)$$

for $s \in \mathcal{H}^{(k)}_\sigma$. We may also regard these operators as sections of the bundle $\operatorname{End}(H^{(k)})$.

Theorem 3.15 gives a sequence of coefficients c^σ_ℓ for each $\sigma \in \mathcal{T}$. It may be shown that the estimate (3.19) holds uniformly over compact subsets of $\mathcal{T}$. Hence we get a $\mathcal{T}$-parametrized family $\star_\sigma$ of star products on M.

In order to glue these star products together to determine a single star product on M, we consider the trivial bundle $C_h = \mathcal{T} \times C^\infty(M)[[h]]$ over $\mathcal{T}$. The space of sections of this bundle carries a product $\star$ defined by $(f \star g)_\sigma = f_\sigma \star_\sigma g_\sigma$.

Definition 3.18. A formal connection D is a connection in C_h over $\mathcal{T}$ of the form

$$D_V f = V[f] + \widetilde{D}(V)(f)$$

for smooth vector fields V on $\mathcal{T}$ and a smooth section f of C_h. Here $\widetilde{D}$ is a smooth 1-form on $\mathcal{T}$ with values in $\mathcal{D}(M)[[h]]$.

We may think of a formal connection as the trivial connection in the bundle C_h, plus a correction term $\widetilde{D}$, given by a formal series of differential operators

$$\widetilde{D}(V) = \sum_{\ell=0}^{\infty} \widetilde{D}^{(\ell)}(V) h^\ell.$$

We wish to use a formal connection to connect the different fibres of C_h and thus relate the different star products. Theorem 12 in [1] states that there is a unique formal connection which is well-suited for our purpose. Before stating the theorem, note that the Hitchin connection $\hat{\nabla}$ in $H^{(k)}$ induces a connection $\hat{\nabla}^e$ in the endomorphism bundle. Since the star products are defined in terms of the Toeplitz operators, which are sections of the bundle $H^{(k)}$, it is natural that the desired connection behaves nicely with respect to the Toeplitz operators and the connection $\hat{\nabla}^e$.

Theorem 3.19. *There is a unique formal connection D which satisfies that*

$$\hat{\nabla}^e_V T^{(k)}_f \sim T^{(k)}_{(D_V f)(1/(2k+n))}$$

for all smooth sections f of C_h and smooth vector fields V on $\mathcal{T}$. Moreover,

$$\widetilde{D}^{(0)} = 0.$$

In the statement above, $\sim$ means that

$$\left\| \hat{\nabla}^e_V T^{(k)}_f - \left(T^{(k)}_{V[f]} + \sum_{\ell=1}^{L} T^{(k)}_{\widetilde{D}^{(\ell)}_V f} \frac{1}{(2k+n)^\ell} \right) \right\| = O(k^{-(L+1)})$$

uniformly over compact subsets of $\mathcal{T}$, for all smooth sections f of C_h. This connection is called the *formal Hitchin connection*. One of the nice properties of this connection is that it is a derivation of the product $\star$ of sections:

Lemma 3.20. *For any sections f, g of C_h and vector field V on $\mathcal{T}$, we have*

$$D_V(f \star g) = D_V(f) \star g + f \star D_V(g).$$

Another nice property is that D is flat if Hitchin's connection $\hat{\nabla}$ is projectively flat, and this is indeed the case in the moduli space setting, where Hitchin originally constructed his connection.

The final ingredient we need to construct an invariant star product is the notion of a formal trivialization of a formal connection.

Definition 3.21. A formal trivialization of a formal connection D is a smooth map $P\colon \mathcal{T} \to \mathcal{D}(M)[[h]]$, written $P_\sigma = \sum_{\ell=0}^\infty P^\ell_\sigma h^\ell$, such that P^0_σ is an isomorphism $C^\infty(M) \to C^\infty(M)$ for all $\sigma \in \mathcal{T}$, and such that

$$D_V(P_\sigma(f)) = 0$$

for all vector fields V on $\mathcal{T}$ and all $f \in C^\infty(M)[[h]]$.

Under sufficiently strong assumptions, we may find a Γ-equivariant formal trivialization.

Proposition 3.22. *Assume the formal connection D is flat and that $\widetilde{D}^{(0)} = 0$. Then locally there exists a formal trivialization of C_h. If $H^1(\mathcal{T}, \mathbb{R}) = 0$, there exists a globally defined formal trivialization. If, moreover, $H^1_\Gamma(\mathcal{T}, \mathcal{D}(M)) = 0$, the formal trivialization can be chosen to be Γ-equivariant.*

Here $H^1_\Gamma(\mathcal{T}, \mathcal{D}(M))$ denotes the first Γ-equivariant de Rham-cohomology group of $\mathcal{T}$ with values in the real vector space of all differential operators on M. Using this proposition for the formal Hitchin connection from Theorem 3.19, we may define a new star product $*$ on $C^\infty(M)[[h]]$ by

$$f * g = P_\sigma^{-1}(P_\sigma(f) \star_\sigma P_\sigma(g)). \tag{3.20}$$

On the left, f and g are simply formal functions on M, ie. elements of $C^\infty(M)[[h]]$, whereas on the right we consider them as (constant) sections of the trivial bundle C_h, evaluated at $\sigma \in \mathcal{T}$. This definition of $\star$ is actually independent of the $\sigma \in \mathcal{T}$ used to compute it, as can be seen by differentiating the right-hand side along a vector field on $\mathcal{T}$ and using the properties of the formal Hitchin connection and the formal trivialization (see Proposition 5 of [1]). If P is Γ-equivariant, (3.20) defines a Γ-invariant star product on M.

In [4], Andersen and Gammelgaard have proved that a mapping class group equivariant trivialization exists to first order by giving an explicit formula for it. Their construction does not use any of the cohomological assumptions from Proposition 3.22.

Chapter **4**

Moduli Spaces

The purpose of this chapter is to give an introduction to the moduli space of flat connections over a surface, and the various structures it carries.

Let Σ be a compact surface and let $p \in \Sigma$ be a fixed basepoint. If the surface has boundary, we will assume that $p \in \partial\Sigma$. We let $\pi_1 = \pi_1(\Sigma, p)$ denote the fundamental group of Σ. Also, let G be a connected Lie group.

Definition 4.1. The *moduli space of flat G-connections on Σ* is the set

$$\mathcal{M}_G = \mathrm{Hom}(\pi_1, G)/G \tag{4.1}$$

of G-valued representations of π_1, modulo conjugation in G.

This will be our working definition of $\mathcal{M}_G$.

4.1 Gauge Theoretic Definition

The name comes from the well-known fact that $\mathcal{M}_G$ may also be defined as follows: One considers the class $\mathcal{F}$ of pairs (P, A) where P is a principal G-bundle over Σ, and A is a flat connection in P. Two such pairs (P_j, A_j), $j = 1, 2$, are equivalent if there exists a bundle isomorphism $\psi \colon P_1 \to P_2$ such that $\psi^* A_2 = A_1$. In particular, (P, A) and (P, A') are equivalent if and only if A and A' are gauge equivalent connections in P. One may then define the moduli space of flat connections to be the set $\mathcal{F}/\!\sim$ of equivalence classes of flat G-bundles.

If G is simply-connected, there is only one isomorphism class of principal G-bundles over a 2-manifold, the trivializable bundles. In this case we would obtain the same space by only considering the space of flat connections in a fixed principal bundle, which we might as well choose to be the trivial bundle $\Sigma \times G$.

Let us briefly recall why these definitions agree. Let P be a principal bundle over Σ, and A a flat connection in P. Trivializing P over $p \in \Sigma$ induces a homomorphism $\pi_1(\Sigma, p) \to G$ by taking the holonomy with respect to A along loops based at p. The conjugacy class of this homomorphism is independent of the chosen trivialization, so we have a well-defined map

$$\mathrm{Hol}\colon \mathcal{F}/\sim \to \mathcal{M}_G. \tag{4.2}$$

Conversely, if $\rho\colon \pi_1 \to G$ is a given homomorphism, one considers the trivial G-bundle $\widetilde{P} = \widetilde{\Sigma} \times G$ over the universal covering space $\widetilde{\Sigma}$ of Σ. The fundamental group acts on the right on this space: For $\gamma \in \pi_1$ and $(y, g) \in \widetilde{P}$, let $(y, g) \cdot \gamma = (y \cdot \gamma, \rho(\gamma)^{-1} g)$, where $y \cdot \gamma$ denotes the natural action of $\pi_1(\Sigma)$ on the covering space. This action is free, and it is easy to see that the quotient $P = \widetilde{P}/\pi_1$ is a principal G-bundle over Σ. The trivial connection on $\widetilde{P}$, which is the pull-back of the Maurer-Cartan form on G, is invariant under the action of π_1, so it descends to a flat connection on P. This gives a well-defined map $\mathcal{M}_G \to \mathcal{F}/\sim$, which is the inverse of Hol.

Henceforth, $\mathcal{M}_G$ may denote either model for the moduli space; which model we have in mind will be clear from the context.

4.2 Mapping Class Group Action

Let φ be a diffeomorphism of Σ. Clearly φ induces an isomorphism $\varphi_*\colon \pi_1(\Sigma, p) \to \pi_1(\Sigma, \varphi(p))$. Postcomposing with the »lasso isomorphism« induced by any path from p to $\varphi(p)$, we obtain an automorphism of $\pi_1(\Sigma, p)$. Different choices of path from p to $\varphi(p)$ gives rise to conjugate automorphisms of $\pi_1(\Sigma, p)$, as does isotopic diffeomorphisms. These facts show that there is a well-defined action of the mapping class group on $\mathcal{M}_G$.

In case Σ has boundary, this discussion becomes somewhat simpler because of our assumption that $p \in \partial\Sigma$. In fact, a diffeomorphism fixing the boundary induces an automorphism of $\pi_1(\Sigma, p)$ which only depends on the isotopy class of the diffeomorphism. Hence, the mapping class group acts on $\mathrm{Hom}(\pi_1(\Sigma, p), G)$ itself, and this action commutes with the conjugation action of G.

Given any flat bundle (P, A) and diffeomorphism φ of Σ, one gets another flat bundle $(\varphi^* P, \varphi^* A)$ by the usual pull-back of bundles and connections. The proof that isotopic diffeomorphisms produces isomorphic bundles can easily be extended to produce an isomorphism of bundles-with-connection. Hence the mapping class group acts on $\mathcal{F}/\sim$, and clearly the map (4.2) is equivariant with respect to these actions.

The various structures on the moduli space described in the following sections are all preserved by the action of the mapping class group.

4.3 Algebraic Structure

Assume G is a linear algebraic group over the complex numbers. If $P = \langle g_\lambda,\ \lambda \in \Lambda \mid r_\mu,\ \mu \in M \rangle$ is a presentation of $\pi_1(\Sigma)$, one may identify $\mathrm{Hom}(\pi_1, G)$ with the closed subset V_P of G^Λ consisting of elements $(A_\lambda)_{\lambda \in \Lambda}$ satisfying

$$r_\mu(A_\lambda) = 1 \tag{4.3}$$

for each $\mu \in M$, where r_μ is considered as a word in the A_λ. Clearly the equations (4.3) are polynomial, so V_P has the structure of an affine algebraic set.

Given any other presentation P', one obtains an isomorphism of affine algebraic sets $V_P \to V_{P'}$ by simply writing the generators from P' in terms of the generators from P. This implies that $\mathrm{Hom}(\pi_1, G)$ itself has a well-defined structure as an affine algebraic set, and we can make sense of the space

$$\mathcal{O}(\mathrm{Hom}(\pi_1, G)) \tag{4.4}$$

of regular functions on $\mathrm{Hom}(\pi_1, G)$. Since the diagonal action of G is algebraic, this action preserves the property of being a regular function. We define the space of *regular functions on the moduli space* by

$$\mathcal{O}(\mathcal{M}_G) = \mathcal{O}(\mathrm{Hom}(\pi_1, G))^G, \tag{4.5}$$

the space of regular functions on $\mathrm{Hom}(\pi_1, G)$ invariant under G. It is not claimed that $\mathcal{M}_G$ itself has the structure of an affine algebraic set.

4.4 Smoothness

When trying to endow $\mathcal{M}_G$ with a smooth structure, one encounters two problems. If Σ is a closed surface, $\pi_1(\Sigma)$ has the well-known presentation

$$\pi_1(\Sigma) \cong \langle \alpha_j, \beta_j \mid \prod_{j=1}^{g} [\alpha_j, \beta_j] = 1 \rangle. \tag{4.6}$$

Letting $Q\colon G^{2g} \to G$ denote the map

$$Q(A_1, B_1, \ldots, A_g, B_g) = \prod_{j=1}^{g} [A_j, B_j], \tag{4.7}$$

we may identify $\mathrm{Hom}(\pi_1(\Sigma), G)$ with $Q^{-1}(1)$. The first problem is that this set is in general singular since 1 is (in general) not a regular value of Q.

Even when dealing with a surface with boundary, so that $\pi_1(\Sigma)$ is a free group and $\mathrm{Hom}(\pi_1, G)$ is identified with a product of copies of G, there are points at which the conjugation action is not principal. These problems can be handled by only considering the *irreducible* representations.

To be concrete, we will consider the case $G = \mathrm{SU}(n)$. If $\rho\colon \pi_1 \to G$ is an irreducible representation, the stabilizer $G_\rho = Z(\operatorname{im}\rho)$ of ρ coincides with the center of G. This follows from the simple observation that a non-central element of G has at least two distinct eigenvalues. More precisely, if $A \in G_\rho$ has eigenvalue λ, it is clear that the λ-eigenspace of A is preserved by $\rho(x)$ for each $x \in \pi_1$. If A is not central, the λ-eigenspace is a proper subspace and ρ is reducible. Conversely, if ρ is reducible, there are non-central elements of G commuting with ρ.

By Proposition I.2.5 of [10], we have proved

Lemma 4.2. *Let* $G = \mathrm{SU}(n)$*. The set* $\mathrm{Hom}^{\mathrm{irr}}(\pi_1, G)$ *of irreducible representations is dense and open, and the quotient*

$$\mathcal{M}_G^\circ = \mathrm{Hom}^{\mathrm{irr}}(\pi_1, G)/G \subseteq \mathcal{M}_G \tag{4.8}$$

is a smooth manifold.

If Σ has a single boundary component, one obtains a smooth compact manifold by fixing the holonomy along the boundary to be a central element: Let $d \in \mathbb{Z}/n\mathbb{Z}$ be relatively prime to n, and let $D = e^{2\pi i d/n} I$ be the corresponding central element of $\mathrm{SU}(n)$. The element $\gamma = \prod_{j=1}^{g}[\alpha_j, \beta_j]$ corresponds to a loop going once around the boundary. Let

$$\mathrm{Hom}_d(\pi_1, \mathrm{SU}(n)) = \{\rho \in \mathrm{Hom}(\pi_1, \mathrm{SU}(n)) \mid \rho(\gamma) = D\}. \tag{4.9}$$

Theorem 4.3. *The elements of* $\mathrm{Hom}_d(\pi_1, \mathrm{SU}(n))$ *are irreducible, so that*

$$\mathcal{M}^d_{\mathrm{SU}(n)} = \mathrm{Hom}_d(\pi_1, \mathrm{SU}(n))/\mathrm{SU}(n) \tag{4.10}$$

is a smooth, compact manifold.

Proof. If ρ can be decomposed as a sum of representations of dimensions k and $n-k$ for some $1 \le k < n$, the matrices $A_j = \rho(\alpha_j)$, $B_j = \rho(\beta_j)$ may be assumed (possibly after conjugation) to be block diagonal,

$$A_j = \begin{pmatrix} A_j' & 0 \\ 0 & A_j'' \end{pmatrix} \qquad B_j = \begin{pmatrix} B_j' & 0 \\ 0 & B_j'' \end{pmatrix}$$

for some matrices $A_j', B_j' \in \mathrm{SU}(k)$, $A_j'', B_j'' \in \mathrm{SU}(n-k)$. Both matrices

$$D' = \prod_{j=1}^{g}[A_j', B_j'] \qquad D'' = \prod_{j=1}^{g}[A_j'', B_j'']$$

are then diagonal matrices with $e^{2\pi i d/n}$ on the diagonal. On the one hand, the determinant of D' is clearly 1 since it is a product of commutators, but on the other hand it is also equal to $e^{2\pi i dk/n}$, which can not equal 1 since d and n are relatively prime. This proves that ρ is irreducible. □

There is no problem in defining the space of smooth functions on $\mathcal{M}_G^\circ$ or $\mathcal{M}^d_{\mathrm{SU}(n)}$. When $\mathrm{Hom}(\pi_1, G)$ is smooth (that is, when Σ has boundary), we let

$$C^\infty(\mathcal{M}_G) = C^\infty(\mathrm{Hom}(\pi_1, G))^G, \tag{4.11}$$

the space of smooth G-invariant functions. In the case where Σ is closed, we define $C^\infty(\mathrm{Hom}(\pi_1, G))$ to be the space of all smooth functions on G^{2g} modulo the ideal of functions vanishing on $Q^{-1}(1)$, and then again define $C^\infty(\mathcal{M}_G)$ by (4.11).

4.5 Tangent Spaces

The tangent space to $\mathcal{M}_G$ at a conjugacy class $[\rho]$ can naturally be identified with the the tangent space of $\operatorname{Hom}(\pi_1, G)$ at ρ, modulo the tangent space to the G-orbit through ρ. Following Goldman [21] we will now see how this gives rise to a group cohomology description of the tangent space. Let ρ_t be a smooth one-parameter family of representations with $\rho_0 = \rho$. Then $\rho_t(x) = A_t(x)\rho(x)$ for some smooth family $A_t\colon \pi_1 \to G$ satisfying $A_0(x) = 1$ for all $x \in \pi_1$. Let $u\colon \pi_1 \to \mathfrak{g}$ denote the map from π_1 to the Lie algebra of G obtained by differenting A_t and evaluating at $t = 0$. Differentiating the homomorphism condition

$$\rho_t(xy) = \rho_t(x)\rho_t(y) = A_t(xy)\rho(xy) = A_t(x)\rho(x)A_t(y)\rho(y)$$

and evalutating at $t = 0$ then yields

$$u(xy)\rho(xy) = u(x)\rho(x)\rho(y) + \rho(x)u(y)\rho(y)$$

which can be written as

$$u(xy) = u(x) + \operatorname{Ad}(\rho(x))(u(y)).$$

This equation precisely means that u is a cocycle when the Lie algebra $\mathfrak{g}$ is considered as a π_1-module via the composition $\operatorname{Ad} \circ \rho$. As such, we denote it by $\mathfrak{g}_\rho$. Since any such cocycle gives rise to a one-parameter family of homomorphisms, we have the identification

$$T_\rho \operatorname{Hom}(\pi_1, G) \cong Z^1(\pi_1, \mathfrak{g}_\rho).$$

Next, let $\rho_t(x) = a_t\rho(x)a_t^{-1}$ be a deformation of ρ along the orbit through ρ, where a_t is a curve through $1 \in G$. Let $X \in \mathfrak{g}$ be the derivative of this family at $t = 0$. Writing $A_t(x) = \rho_t(x)\rho(x)^{-1} = a_t\rho(x)a_t^{-1}\rho(x)^{-1}$, we see that the tangent vector represented be the family ρ_t is the cocycle u_X given by

$$u_X(x) = X - \rho(x)X\rho(x)^{-1} = X - \operatorname{Ad}(\rho(x))X,$$

that is, precisely the coboundary of X.

Proposition 4.4. *The tangent space $T_{[\rho]}\mathcal{M}_G$ is the cohomology group*

$$H^1(\pi_1, \mathfrak{g}_\rho). \tag{4.12}$$

This description is of course only valid at the smooth points.

Remark 4.5. Differentiating the equation (4.9) defining the elements of $\operatorname{Hom}_d(\pi_1, \mathrm{SU}(n))$, we see that the tangent space to $\mathcal{M}^d_{\mathrm{SU}(n)}$ at $[\rho]$ is the subspace of (4.12) represented by cocycles satisfying $u(\gamma) = 0$.

It is also possible to give a description of the tangent space in the gauge theoretic setting in terms of de Rham-cohomology of Σ. Let $A \in \mathcal{F}$ be a flat connection in a fixed principal G-bundle P over Σ. By Remark A.7, the tangent space to the space of all connections at A is identified with the space $\Omega^1(\Sigma, \operatorname{Ad} P)$ of $\operatorname{Ad} P$-valued 1-forms on Σ. In order to determine $T_A\mathcal{F}$, we will derive a condition for $a \in \Omega^1(\Sigma, \operatorname{Ad} P)$ to be tangent to $\mathcal{F}$. Differentiating the equation $F_{A+ta} = 0$ with respect to t gives

$$\begin{aligned}\frac{d}{dt}F_{A+ta} &= \frac{d}{dt}\big(d(A+ta) + \tfrac{1}{2}[(A+ta)\wedge(A+ta)]\big)\\ &= \frac{d}{dt}\big(F_A + tda + \tfrac{t^2}{2}[a\wedge a] + t[A\wedge a]\big)\\ &= da + [A\wedge a] + t[a\wedge a]\\ &= 0,\end{aligned}$$

and evaluting at $t = 0$ gives the condition $da + [A \wedge a] = d_A a = 0$ for a to be tangent to $\mathcal{F}$ at A. Thus we identify the tangent space $T_A\mathcal{F}$ with the vector space of closed 1-forms on Σ with values in the adjoint bundle $\operatorname{Ad} P$, with respect to the differential d_A:

$$T_A\mathcal{F} = Z^1(\Sigma, \operatorname{Ad} P; d_A) \subseteq \Omega^1(\Sigma, \operatorname{Ad} P) \tag{4.13}$$

Now we have to determine the subspace tangent to the action of the gauge group. Let φ_t be a one-parameter family of gauge transformations with $\varphi_0 = \mathrm{id}$, and let $g_t\colon P \to G$ be the corresponding family of G-equivariant maps (ie. maps satisfying (A.8)) where $g_0 = e$. For a fixed $p \in P$, $g_t(p)$ is a curve through $e \in G$, and hence $\frac{d}{dt}\big|_{t=0} g_t$ is a map $f\colon P \to \mathfrak{g}$. We note that the G-equivariance of each g_t implies that f is G-equivariant:

$$\begin{aligned}f(pg) = \frac{d}{dt}\Big|_{t=0} g_t(pg) &= \frac{d}{dt}\Big|_{t=0} g^{-1}g_t(p)g\\ &= \frac{d}{dt}\Big|_{t=0}(c(g^{-1})\circ g_t)(p) = \operatorname{Ad}(g^{-1})(f(p)),\end{aligned}$$

so f is in fact an element of $\Omega^0(\Sigma, \operatorname{Ad} P) = \Omega^0_b(P, \mathfrak{g})$.

We have $\varphi_t^* A = \operatorname{Ad}(g_t^{-1}) \circ A + g_t^*\theta$ by (A.9), and if we differentiate this equation with respect to t we obtain

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\varphi_t^*A &= \frac{d}{dt}\Big|_{t=0}\operatorname{Ad}(g_t^{-1})\circ A + \frac{d}{dt}\Big|_{t=0} g_t^*\theta\\ &= \operatorname{ad}(-f)\circ A + \frac{d}{dt}\Big|_{t=0} g_t^*\theta\end{aligned} \tag{4.14}$$

The last term is in fact equal to $df \in \Omega^1(P, \mathfrak{g})$, but it takes some effort to see this. Let α be a curve in P with $\alpha(0) = p$ and $\alpha'(0) = X \in T_pP$. Then

$$\begin{aligned}g_t^*\theta(X) &= \theta_{g_t(p)}(D_p g_t(X))\\ &= \theta\Big(\frac{d}{ds}\Big|_{s=0} g_t(\alpha(s))\Big)\\ &= D_{g_t(p)}L_{g_t(p)^{-1}}\Big(\frac{d}{ds}\Big|_{s=0} g_t(\alpha(s))\Big)\\ &= \frac{d}{ds}\Big|_{s=0}\Big(g_t(p)^{-1}\cdot g_t(\alpha(s))\Big)\end{aligned}$$

so that

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} g_t^*\theta(X) &= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\Big(g_t(p)^{-1}\cdot g_t(\alpha(s))\Big)\\ &= \frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0}\Big(g_t(p)^{-1}\cdot g_t(\alpha(s))\Big)\\ &= \frac{d}{ds}\Big|_{s=0}\big(-f(p)+f(\alpha(s))\big)\\ &= df(X).\end{aligned}$$

Now continuing (4.14), we have

$$\frac{d}{dt}\Big|_{t=0}\varphi_t^* A = [A\wedge f] + df = d_A f$$

so we see that a vector tangent to the gauge group action at A is an exact 1-form in $\Omega^1(\Sigma, \operatorname{Ad} P; d_A)$. On the other hand, given any G-equivariant map $f\colon G \to P$, $g_t(p) = \exp(tf(p))$ defines a 1-parameter family of gauge transformations which induces the tangent vector $d_A f$ at A. Thus $T_A(A\mathcal{G}) = B^1(\Sigma, \operatorname{Ad} P; d_A)$, and we obtain the important identification of the tangent space to the moduli space with a twisted cohomology group of Σ.

Theorem 4.6. *Let A be a flat connection in a principal G-bundle $P \to \Sigma$. Then*

$$T_{[A]}\mathcal{M}_G \cong H^1(\Sigma, \operatorname{Ad} P; d_A). \tag{4.15}$$

That is, the tangent space $T_{[A]}\mathcal{M}_G$ at the gauge equivalence class $[A]$ of the moduli space of flat connections is identified with the first (de Rham) cohomology group of Σ, with coefficients in the adjoint bundle Ad P *and differential d_A induced by A.*

4.6 Symplectic Structure

Assume that the Lie algebra $\mathfrak{g}$ admits a non-degenerate, symmetric, bilinear form $B\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ invariant under the adjoint action of G. For example, if G is compact we may find an invariant inner product on $\mathfrak{g}$. An explicit example for $G = \mathrm{U}(n)$ (or $G = \mathrm{SU}(n)$) is given by $B(X,Y) = \operatorname{tr}(X^*Y) = -\operatorname{tr}(XY)$ for skew-hermitian (and trace-less) matrices X, Y.

Under this fairly general assumptions on the group G, Goldman [21] observed that (the smooth part of) the moduli space over a closed surface admits a symplectic structure.

The pairing B induces a map B_* from the tensor bundle $B_*\colon \operatorname{Ad} P \otimes \operatorname{Ad} P$ to the trivial $\mathbb{R}$-bundle over Σ. For d_A-closed 1-forms $\varphi, \psi \in \Omega^1(\Sigma, \operatorname{Ad} P; d_A)$ representing tangent vectors at a point $[A] \in \mathcal{M}_G$, we see that $B_*(\varphi\wedge\psi)$ is an ordinary 2-form on Σ, and we put

$$\omega([\varphi],[\psi]) = \int_\Sigma B_*(\varphi\wedge\psi). \tag{4.16}$$

That this does in fact define a symplectic form on $\mathcal{M}_G^\circ$ is, in the case $G = \mathrm{U}(n)$, originally due to Atiyah and Bott [9].

The corresponding form in the algebraic setting is defined using cup product in group cohomology and the π_1-homomorphism $B_*\colon \mathfrak{g}_\rho \otimes \mathfrak{g}_\rho \to \mathbb{R}$ to obtain a map

$$H^1(\pi_1, \mathfrak{g}_\rho) \times H^1(\pi_1, \mathfrak{g}_\rho) \xrightarrow{B_*} H^2(\pi_1, \mathbb{R}), \tag{4.17}$$

and then composing with a canonical isomorphism from the latter vector space to $\mathbb{R}$. In [30], Karshon showed in a purely algebraic way that this form is closed.

For a surface with boundary, the fundamental group is free, so $H^k(\pi_1, \bullet)$ and $H_k(\pi_1, \bullet)$ vanish for any $k \geq 2$, since π_1 admits a 1-dimensional CW-complex as an Eilenberg-MacLane space $K(\pi_1, 1)$. Hence we can not directly use (4.17) to define a symplectic pairing. However, one may still use (4.16) in order to obtain a Poisson structure, the symplectic leaves of which are obtained by fixing the holonomy along the boundary components of Σ. For details, see [10].

In the case of our favorite example, $\mathcal{M}^d_{\mathrm{SU}(n)}$, a simple modification of (4.17) does provide a group cohomological description of the symplectic form. Letting $G = \mathrm{SU}(n)$ and $G' = G/Z(G)$, the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ factors through G'. Let $\rho\colon \pi_1 \to G$ be a homomorphism representing an element of $\mathcal{M}^d_G$, and let ρ' denote the induced representation $\pi_1 \to G'$. The condition (4.9) defining the space $\mathcal{M}^d_G$ then implies that ρ' induces a representation $\overline{\rho}\colon \overline{\pi}_1 \to G'$ of the fundamental group of the closed surface obtained by gluing a disc to the boundary component.

The groups G and G' both has $\mathfrak{g}$ as Lie algebra. Now, the π_1-module obtained from $\mathfrak{g}_{\overline{\rho}}$ via the surjection $\pi_1 \to \overline{\pi}_1$ is the same as the π_1-module $\mathfrak{g}_\rho$ itself. This means we have an induced map

$$H^1(\overline{\pi}_1, \mathfrak{g}_{\overline{\rho}}) \to H^1(\pi_1, \mathfrak{g}_\rho). \tag{4.18}$$

It is easy to see that this map is injective, and the image is precisely the elements represented by cocycles u satisfying $u(\gamma) = 0$, which we recognize as the tangent space $T_{[\rho]}\mathcal{M}^d_G$, cf. Remark 4.5. This allows us to identify this tangent space with $H^1(\overline{\pi}_1, \mathfrak{g}_{\overline{\rho}})$, and we can use (4.17) to define a non-degenerate pairing on this space.

A much more general treatment is given by Biswas and Guruprasad [13]. For a surface with n boundary circles $\gamma_1, \dots, \gamma_n$, they consider n conjugacy classes $C_1, \dots, C_n$ in G. The moduli space $\mathcal{P}$ of representation $\rho\colon \pi_1 \to G$ satisfying $\rho(\gamma_j) \in C_j$ is called the space of *parabolic representations*. The condition for a cocycle $u\colon \pi_1 \to \mathfrak{g}$ to be tangent to $\mathcal{P}$ is that there exists elements $\mu_j \in \mathfrak{g}$ such that

$$u(\gamma_j) = \mu_j - \mathrm{Ad}\,\rho(\gamma_j)\mu_j.$$

Such cocycles are called parabolic, and the tangent space to $\mathcal{P}$ at ρ is thus identified with the *parabolic group cohomology* $H^1_{\mathrm{par}}(\pi_1, \mathfrak{g}_\rho)$. This identification is used to define a symplectic structure on $\mathcal{P}$.

4.7 Holonomy Functions

A rich source of functions on the moduli space is the notion of holonomy functions. Let $h\colon G \to \mathbb{C}$ be a class function, a function invariant under conjugation. An oriented closed curve γ determines a conjugacy class in π_1, so for any $[\rho] \in \mathcal{M}_G$ we have that $\rho(\gamma)$ is a well-defined conjugacy class in G. This allows us to define a function $h_\gamma \colon \mathcal{M}_G \to \mathbb{C}$ by $h_\gamma([\rho]) = h(\rho(\gamma))$. It is clear that this only depends on the free homotopy class of γ.

Note that the mapping class group acts on the set of free homotopy classes of oriented loops, which may be identified with the set $\hat{\pi}_1$ of conjugacy classes in π_1.

Lemma 4.7. *The association $\gamma \mapsto h_\gamma$ is equivariant with respect to the mapping class group actions on $\hat{\pi}_1$ and $\mathrm{Fun}(\mathcal{M}_G)$.*

Proof. Let $\varphi \in \Gamma$, γ a closed curve on Σ and $[\rho] \in \mathcal{M}_G$. Then

$$(\varphi \cdot h_\gamma)([\rho]) = h_\gamma(\varphi^{-1}[\rho]) = h_\gamma([\rho \circ \varphi]) = h(\rho(\varphi(\gamma))) = h_{\varphi(\gamma)}([\rho]). \quad \square$$

In the case where h is the trace functional and G is a classical matrix group, Goldman [22] has computed formulas for the Poisson bracket $\{h_\alpha, h_\beta\}$ of two holonomy functions (see [22] for the list of Lie groups). For example, if $G = \mathrm{GL}_n(\mathbb{R})$ or $G = \mathrm{GL}_n(\mathbb{C})$, the formula is

$$\{h_\alpha, h_\beta\} = \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta) h_{\alpha_p \beta_p}. \tag{4.19}$$

Here α and β are assumed to be in general position; ie. the images are disjoint except for a finite number of transverse intersection points $\alpha \# \beta$. At such an intersection point p, $\varepsilon(p; \alpha, \beta) = \pm 1$ is the oriented intersection number (defined in terms of the orientation of Σ), and $\alpha_p \beta_p$ denotes the curve obtained by starting at p and traversing α followed by β. For other groups, similar (but different) formulae were obtained.

This leads one to consider the free module $\mathbb{Z}\hat{\pi}_1$ on the set of free homotopy classes of loops in Σ. Equipping this module with the bracket determined by

$$[\alpha, \beta] = \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta) \alpha_p \beta_p, \tag{4.20}$$

for (homotopy classes of) loops α, β, the equation (4.19) shows that the map $\alpha \mapsto f_\alpha$ induces a homomorphism of Lie algebras $\mathbb{Z}\hat{\pi}_1 \to \mathcal{O}(\mathcal{M}_G)$ if G is $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$. It is also clearly equivariant with respect to the action of the mapping class group. However, the map is not injective, and the discussion in [22] shows that it is hard to give a nice description of the kernel. Moreover, it would be nice to have a geometrically defined (that is, in terms of loops on the surface) Γ-module with a Lie algebra structure defined without reference to the particular Lie group in question.

4.7.1 Chord Diagrams

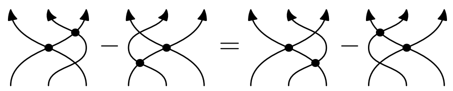
Following the same idea of assigning functions on the moduli space to loops on the surface, Andersen, Mattes and Reshetikhin [5] generalized Goldman's approach by introducing the algebra of chord diagrams on a surface. This algebra carries a Poisson bracket defined by a formula similar to (4.20) above. Coloring the chord diagrams by finite-dimensional representations of G, one obtains a Poisson algebra which in many cases maps surjectively to $\mathcal{O}(\mathcal{M}_G)$, the space of algebraic functions on the moduli space. We recall the definitions and main results.

A *chord diagram* is a finite collection of oriented circles and a finite collection of chords (unoriented line segments) whose endpoints lie on the circles. The chords are assumed to be disjoint, and in particular no two endpoints coincide. The circles of a chord diagram are called the *core components*, and together they constitute the *skeleton* of the diagram.

A *geometric chord diagram on* Σ is a smooth map from a chord diagram to Σ, mapping chords to points. When drawing (parts of) geometric chord diagrams, images of chords will be drawn as fat dots. A *chord diagram on* Σ is a homotopy class of geometric chord diagrams. Every chord diagram contains a *generic* chord diagram: a geometric chord diagram whose skeleton is immersed in Σ and which has only transverse double points.

We denote by $\mathcal{D}(\Sigma)$ the complex vector space with basis the set of chord diagrams on Σ. This vector space is graded by the number of chords, and union of diagrams makes $\mathcal{D}(\Sigma)$ a commutative graded algebra with unit the empty diagram.

Consider the (local) relation



(4.21)

as well as the relations obtained from this one by reversing orientation of core components and changing the sign according to the rule: For every chord intersecting a core component with reversed orientation, the diagram is multiplied by -1. These relations are called the 4T-*relations*, and the subspace spanned by them is denoted $4\mathrm{T}(\Sigma)$. This is an ideal and homogenous, so the quotient

$$\mathcal{C}(\Sigma) = \mathcal{D}(\Sigma)/4\mathrm{T}(\Sigma) \tag{4.22}$$

is also a commutative, graded algebra. Furthermore, there is a Poisson structure on $\mathcal{C}(\Sigma)$ defined as follows: Given two chord diagrams on Σ, choose geometric chord diagram $i_j \colon D_j \to \Sigma$, $j = 1, 2$ representing them such that their product is a generic chord diagram. For $p \in D_1 \# D_2$ we define $D_1 \cup_p D_2$ to be the chord diagram on Σ obtained by joining $i_1^{-1}(p)$ and $i_2^{-1}(p)$ by a chord mapped to p.

Proposition 4.8 (Andersen, Mattes, Reshetikhin). *The bracket*

$$\{[D_1], [D_2]\} = \sum_{p \in D_1 \# D_2} \varepsilon(p; D_1, D_2)[D_1 \cup_p D_2] \tag{4.23}$$

is well-defined and determines a Poisson structure on $\mathcal{C}(\Sigma)$.

The algebra $\mathcal{C}(\Sigma)$ equipped with the bracket (4.23) is called the *Poisson algebra of chord diagrams on* Σ. Requiring that every core component of a diagram be colored by a finite dimensional representation of a Lie group G yields the Poisson algebra $\mathcal{C}(\Sigma; G)$. In [5], the authors also construct a

Poisson homomorphism $\Psi\colon \mathcal{C}(\Sigma; G) \to \mathcal{O}(\mathcal{M}_G)$. If G is a matrix group, we may color all core components by the standard representation, and hence Ψ defines a Poisson homomorphism

$$\Psi\colon \mathcal{C}(\Sigma) \to \mathcal{O}(\mathcal{M}_G). \tag{4.24}$$

This map is surjective for the groups $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$. Furthermore, a diffeomorphism of Σ clearly preserves the 4T-relations, hence gives an action of the mapping class group on $\mathcal{C}(\Sigma)$, and an easy computation similar to Lemma 4.7 shows that Ψ is Γ-equivariant (see e.g. Theorem 2.22 of [36]).

Understanding the kernel of Ψ thus provides one with a »geometric« model for $\mathcal{O}(\mathcal{M}_G)$, where the action of the mapping class group is simply given by its action on chord diagrams (modulo $\ker\Psi$). In the special case $G = \mathrm{SL}_2(\mathbb{C})$, there is a particularly simple geometric model, which we will present in the next chapter.

4.8 Quantizing the Moduli Space

Let us briefly review how the contents of Chapter 3, and in particular Section 3.4, apply to the moduli space $\mathcal{M} = \mathcal{M}^d_{\mathrm{SU}(n)}$. The symplectic structure on $\mathcal{M}$ is given by combining (4.17) and (4.18) as explained above. The line bundle $\mathcal{L}$ together with the Hermitian structure and compatible connection ∇ was constructed in [19]. Freed also proves that the curvature of ∇ satisfies the prequantum condition. Atiyah and Bott [9] have proved that the moduli space is simply connected and that $H^2(\mathcal{M}, \mathbb{Z}) \cong \mathbb{Z}$. For $n = 2, d = 1$, Thaddeus [38] gives an elementary proof that the holonomy function associated to a simple closed curve is a perfect Bott-Morse function. Using this fact, he finds the Poincaré polynomial and thus the Betti numbers.

The Teichmüller space $\mathcal{T}$ parametrizes, by definition, complex structures on Σ. A point $\sigma \in \mathcal{T}$ induces a Hodge star operator on ($\mathrm{Ad}\,P$-valued) 1-forms on Σ, and this can be used to obtain the identification

$$T_{[A]}\mathcal{M} = H^1(\Sigma, \mathrm{Ad}\,P) \cong \ker(d_A) \cap \ker(*d_A*) \tag{4.25}$$

of the tangent space to $\mathcal{M}$ with the harmonic $\mathrm{Ad}\,P$-valued forms. The square of $*\colon H^1(\Sigma, \mathrm{Ad}\,P) \to H^1(\Sigma, \mathrm{Ad}\,P)$ is -1, so we get an almost complex structure on $\mathcal{M}$ by putting $J_\sigma = -*$. Narasimhan and Seshadri [31] have proved that this almost complex structure is integrable, so that $(\mathcal{M}, \omega, J_\sigma)$ is a smooth, compact Kähler manifold, which is denoted $\mathcal{M}_\sigma$. This also gives $\mathcal{L}$ the structure of a holomorphic line bundle. The $(0,1)$-part of the connection ∇ allows us to define a section $s \in C^\infty(\mathcal{M}, \mathcal{L}^k)$ to be holomorphic if $\nabla^{0,1}s = 0$.

For each integer k, one gets a vector bundle $\mathcal{V}_k$ over $\mathcal{T}$, whose fibre at σ is the space $H^0(\mathcal{M}_\sigma, \mathcal{L}^k_\sigma)$ of holomorphic sections of $\mathcal{L}^k$. This bundle is usually called the Verlinde bundle (at level k), and is the original setting for the Hitchin connection [25].

Chapter 5

Regular Functions on the $\mathrm{SL}_2(\mathbb{C})$ Moduli Space

In this chapter, our primary object of study is the moduli space $\mathcal{M} = \mathcal{M}_{\mathrm{SL}_2(\mathbb{C})}$ of flat $\mathrm{SL}_2(\mathbb{C})$-connections over Σ. Most of the material has been published in [8], and before that it appeared in two separate papers on the arXiv, [39] and [6].

Throughout the chapter, we assume that Σ is a surface of genus at least 2, with any number of boundary components. Let $\mathcal{O} = \mathcal{O}(\mathcal{M})$ denote the space of regular functions on $\mathcal{M}$. For the precise definition of these, see Section 4.3.

Theorem 5.1. *The cohomology group $H^1(\Gamma, \mathcal{O})$ vanishes.*

The proof of this theorem is the main goal of the chapter. Along the way we will introduce ideas which happen to be useful for the computation of the cohomology with coefficients in other modules of functions on other moduli spaces.

5.1 Multicurves as Functions

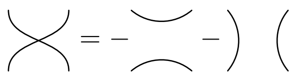
A matrix $A \in \mathrm{SL}_2(\mathbb{C})$ has the property that $\operatorname{tr} A = \operatorname{tr}(A^{-1})$. Letting h denote the trace function $\mathrm{SL}_2(\mathbb{C}) \to \mathbb{C}$, we may thus associate a function h_γ on $\mathcal{M}$ to any unoriented closed curve γ by orienting γ arbitrarily. For a multicurve κ on Σ with components κ_j, we associate a function ν_κ on $\mathcal{M}$ by

$$\nu_\kappa = \prod_j (-h_{\kappa_j}). \tag{5.1}$$

Let $\mathbb{C}\mathcal{S}$ denote the complex vector space freely spanned by the set of multicurves on Σ.

Theorem 5.2. *The linear map $\mathbb{C}\mathcal{S} \to \mathrm{Fun}(\mathcal{M})$ given by $\kappa \mapsto \nu_\kappa$ is a mapping class group equivariant isomorphism onto the space of regular functions on $\mathcal{M}$.*

This theorem is due to Bullock, Frohman, and Kania-Bartoszyńska [16]. In fact, one may define a natural algebra structure on $\mathbb{C}\mathcal{S}$ making the above map an isomorphism of algebras. The multiplication of two multicurves is defined by taking their disjoint union and then applying the *Kauffman bracket procedure*: Remove all crossings via the rule



(5.2)

and, for each resulting multicurve, remove trivial loops at the cost of a factor of -2 to the multicurve (corresponding to minus the trace of the holonomy along the trivial loop, viz. the identity matrix). See [16] and [36] for details.

5.2 Splitting the Cohomology

Theorem 5.2 suggests that a proof of Theorem 5.1 can be obtained by studying the action of the mapping class group on the set of multicurves, and this is indeed our strategy. Let us partition $\mathcal{S}$ into mapping class group orbits

$$\mathcal{S} = \bigsqcup_{\alpha} \mathcal{S}_\alpha.$$

Lemma 5.3. *There is an isomorphism of* Γ*-modules*

$$\mathbb{C}\mathcal{S} \cong \bigoplus_{\alpha} \mathbb{C}\mathcal{S}_\alpha \tag{5.3}$$

which induces an isomorphism on cohomology

$$H^1(\Gamma, \mathcal{O}) \cong H^1(\Gamma, \mathbb{C}\mathcal{S}) \cong \bigoplus_{\alpha} H^1(\Gamma, \mathbb{C}\mathcal{S}_\alpha). \tag{5.4}$$

Proof. The isomorphism (5.3) is clear since each $\mathbb{C}\mathcal{S}_\alpha$ is obviously a Γ-invariant subspace of $\mathbb{C}\mathcal{S}$. The first isomorphism in (5.4) is due to Theorem 5.2, whereas the second comes from (1.8), using the fact that the mapping class group is finitely generated. □

To prove Theorem 5.1 it suffices to consider each summand on the right-hand side of (5.4). By Remark 2.21, each orbit S_α is either infinite or consists of a single point. In the latter case, $\mathbb{C}\mathcal{S}_\alpha$ is simply a copy of $\mathbb{C}$ with trivial Γ-action, so in this case

$$H^1(\Gamma, \mathbb{C}\mathcal{S}_\alpha) = H^1(\Gamma, \mathbb{C}) = \mathrm{Hom}(\Gamma, \mathbb{C}) = 0 \tag{5.5}$$

by (1.5) and (2.2). Henceforth we assume that the orbit $\mathcal{S}_\alpha$ under consideration is infinite.

Proposition 5.4. *Let* $u\colon \Gamma \to \mathbb{C}\mathcal{S}_\alpha$ *be a cocycle and* ε *a simple closed curve on* Σ. *For any multicurve* $\kappa \in \mathcal{S}_\alpha$ *with* $\tau_\varepsilon \kappa = \kappa$, *the coefficient of* κ *in* $u(\tau_\varepsilon)$ *is* 0.

Proof. Let κ be a multicurve fixed by τ_ε. First assume that κ contains at least one component which is not parallel to a boundary component of Σ nor to ε. In this case, one may find a simple closed curve α disjoint from ε such that τ_α acts non-trivially on κ. Hence, since τ_α and τ_ε commute, we get

$$(1-\tau_\alpha)u(\tau_\varepsilon) = (1-\tau_\varepsilon)u(\tau_\alpha). \tag{5.6}$$

Since $\tau_\varepsilon\kappa = \kappa$, the coefficient of κ on the right-hand side of (5.6) is 0. But then the coefficients of κ and $\tau_\alpha^{-1}\kappa$ in $u(\tau_\varepsilon)$ are identical. This argument can be repeated with $\tau_\alpha^{-1}\kappa$ instead of κ, and this way we see that the coefficients of all multicurves $\tau_\alpha^{-n}\kappa$, $n \geq 0$, are identical. Since these multicurves are all distinct, and since $u(\tau_\varepsilon)$ contains at most finitely many non-zero terms, the common coefficient must be 0.

Since we are assuming that the orbit $\mathcal{S}_\alpha$ is infinite, the only remaining case is when ε is not boundary parallel and κ consists of parallel copies of ε (and possibly some boundary parallel components). In this case, we embed the chain relation (Lemma 2.6) in such a way that our ε is the ε occuring in the chain relation. The cocycle condition then yields

$$u((\tau_\alpha\tau_\beta\tau_\gamma)^4) = u(\tau_\delta) + \tau_\delta u(\tau_\varepsilon).$$

Clearly all four curves $\alpha, \beta, \gamma, \delta$ are disjoint from κ, so the first case considered shows that the coefficient of κ in all terms but the last are 0. This concludes the proof. □

When $g \geq 3$, this proof can be reduced to Theorem 2.25 as follows: The space $\mathbb{C}\mathcal{S}$ is a subspace of the unitary representation $\ell^2(\mathcal{S})$. In the language from Section 2.6, $\tau_\varepsilon\kappa = \kappa$ implies $p_\varepsilon\kappa = \kappa$, so

$$\langle u(\tau_\varepsilon), \kappa\rangle = \langle u(\tau_\varepsilon), p_\varepsilon\kappa\rangle = \langle p_\varepsilon u(\tau_\varepsilon), \kappa\rangle = 0.$$

5.3 The Dual Module

The algebraic dual $(\mathbb{C}\mathcal{S}_\alpha)^* = \mathrm{Hom}(\mathbb{C}\mathcal{S}_\alpha, \mathbb{C})$ may be naturally identified with the space $\mathrm{Map}(\mathcal{S}_\alpha, \mathbb{C})$ of *all* maps from $\mathcal{S}_\alpha$ to $\mathbb{C}$. Since $\mathbb{C}\mathcal{S}_\alpha$ can be thought of as the space of finitely supported functions $\mathcal{S}_\alpha \to \mathbb{C}$, there is a mapping class group equivariant inclusion

$$\mathbb{C}\mathcal{S}_\alpha \hookrightarrow \mathrm{Map}(\mathcal{S}_\alpha, \mathbb{C}). \tag{5.7}$$

Proposition 5.5. *The induced map*

$$H^1(\Gamma, \mathbb{C}\mathcal{S}_\alpha) \to H^1(\Gamma, \mathrm{Map}(\mathcal{S}_\alpha, \mathbb{C})) \tag{5.8}$$

is identically 0.

Proof. Pick any representative $\kappa \in \mathcal{S}_\alpha$ for the orbit. Then by Theorem 1.2 the right-hand side of (5.8) is isomorphic to

$$\mathrm{Hom}(\Gamma_\kappa, \mathbb{C})$$

where Γ_κ is the stabilizer of κ in Γ.

Let $u\colon \Gamma \to \mathbb{C}\mathcal{S}_\alpha$ be a cocycle and $\varphi \in \Gamma_\kappa$. By the formula (1.14), we must prove that $u(\varphi)(\kappa) = 0$, but since $u(\bullet)(\kappa)\colon \Gamma_\kappa \to \mathbb{C}$ is a homomorphism, it suffices to consider any positive power of φ. Hence we may without loss of generality assume that φ fixes each component and each side of the components of κ. This means that φ can be written as a product $\tau_k^{\pm 1} \cdots \tau_1^{\pm 1}$ of twists in curves not intersecting κ. By Proposition 5.4, $u(\tau_j^{\pm 1})(\kappa) = 0$ for each j, which clearly implies that $u(\varphi)(\kappa) = 0$. □

5.4 Almost Invariant Colorings

Let us summarize what we know by now. Any cocycle $u\colon \Gamma \to \mathbb{C}\mathcal{S}$ splits as a sum of (finitely many) cocycles $u_\alpha\colon \Gamma \to \mathbb{C}\mathcal{S}_\alpha$. For each of these cocycles, there exists a map $U_\alpha\colon \mathcal{S}_\alpha \to \mathbb{C}$ such that

$$u(f) = (1 - f)U_\alpha \tag{5.9}$$

for each $f \in \Gamma$. The proof of Theorem 5.1 is complete if we can prove that U_α can be modified in such a way that (5.9) still holds, and such that U_α has finite support. It turns out that a nice way to handle this problem is by introducing a little terminology.

Let G be a group and X a set on which G acts. We define a *coloring* (or C-coloring) of X to be any map $c\colon X \to C$ into some set C of »colors«. We will use the following terminology:

- ▷ A coloring c is *invariant* if $c(gx) = c(x)$ for each $g \in G$ and $x \in X$.
- ▷ A coloring is *almost invariant* if, for each $g \in G$, the identity $c(gx) = c(x)$ fails for only finitely many $x \in X$.
- ▷ Two colorings are *equivalent* if they assign different colors to only finitely many elements of X; this is clearly an equivalence relation on the set of C-colorings.
- ▷ A coloring is *trivial* if it is equivalent to a coloring which is constant on each orbit of X, that is, an invariant coloring.

Notice that an almost invariant coloring is not the same as a trivial coloring. For example, in the simple case of $\mathbb{Z}$ acting on itself, $c(n) = 1$ for $n \geq 0$ and $c(n) = 0$ for $n < 0$ defines an almost invariant $\{0,1\}$-coloring, but this coloring is not equivalent to a constant coloring.

If two colorings are equivalent and one is almost invariant, so is the other.

A *simplification* of a coloring c is a coloring obtained by post-composing c with some map $i\colon C \to C'$ (one »identifies« some of the colors). Clearly a simplification of an almost invariant coloring is almost invariant.

Remark 5.6. If there exists an almost invariant, non-trivial C-coloring c, there also exists an almost invariant coloring where exactly two colors are used. To see this, partition C into $C_0 \sqcup C_1$ such that $c^{-1}(C_k)$, $k = 0, 1$, are both infinite, and define a $\{0,1\}$-coloring by composing c with the map $i\colon C \to \{0,1\}$ determined by $z \in C_{i(z)}$. Hence, if one wants to prove the non-existence of almost invariant, non-trivial colorings, it suffices to consider colorings where two colors are used.

5.4.1 Finitely Generated Groups

If $S \subset G$ is a set of generators for G, a coloring is almost invariant if and only if for each $g \in S$ we have $c(gx) = c(x)$ for all but finitely many $x \in X$. This observation is of course particularly useful when G is finitely generated.

In this case, we also observe that any almost invariant coloring is actually invariant on all but finitely many orbits. To see this, choose a finite generating set S. Partition X into the G-orbits $X = \bigsqcup_\alpha X_\alpha$. For each index α corresponding to an orbit with non-constant coloring, we may choose an element $s(\alpha) \in S$ such that there exists at least one element $x \in X_\alpha$ for which the colors of x and $s(\alpha)x$ differ. Since each element in S can be chosen at most a finite number of times, there are only finitely many orbits where the coloring is not constant.

Conversely, given G-sets X_α together with almost invariant colorings c_α, all but finitely many of which are constant, we get an almost invariant coloring on $X = \bigsqcup_\alpha X_\alpha$ by putting $c(x) = c_\alpha(x)$ for $x \in X_\alpha$. These observations suggest that for finitely generated groups, it suffices to consider almost invariant colorings of orbits.

5.4.2 The Orbit of a Multicurve

Let $X = \mathcal{S}_\alpha$ be the mapping class group orbit of some multicurve.

Theorem 5.7. *When $g \geq 2$, r arbitrary, there are no non-trivial almost invariant colorings of X.*

Any coloring of a finite set is trivial, so this theorem is only interesting when X is infinite. Let c be some fixed, almost invariant coloring of X.

The proof of Theorem 5.7 consists of a series of relatively simple observations. The key notion is that of an *interesting pair*, which is a pair (τ_γ, κ) consisting of a Dehn twist τ_γ and a multicurve $\kappa \in X$ such that $\tau_\gamma \kappa \neq \kappa$.

Since τ_γ changes the color of only finitely many multicurves, the elements $\tau_\gamma^n \kappa$ all have the same color for all sufficiently large values of n. This color is called the *future* of the interesting pair (τ_γ, κ), denoted $\mathrm{fut}(\tau_\gamma, \kappa)$. Similarly, we may consider the *past* $\mathrm{pas}(\tau_\gamma, \kappa)$ of an interesting pair; the common color of all multicurves $\tau_\gamma^{-n}\kappa$ for sufficiently large n. We will also need to consider pairs of the form $(\tau_\gamma^{-1}, \kappa)$; the same definition of future and past applies to these, and clearly $\mathrm{fut}(\tau_\gamma^{\pm 1}, \kappa) = \mathrm{pas}(\tau_\gamma^{\mp 1}, \kappa)$.

Lemma 5.8. *For any interesting pair (τ_α, κ), we have*

$$\mathrm{pas}(\tau_\alpha^{-1}, \kappa) = \mathrm{fut}(\tau_\alpha, \kappa) = \mathrm{pas}(\tau_\alpha, \kappa) = \mathrm{fut}(\tau_\alpha^{-1}, \kappa).$$

Proof. It suffices to prove the middle identity. We may find a non-separating simple closed curve β different and disjoint from α such that (τ_β, κ) is also interesting. To see this, let δ be a component of κ for which $\tau_\alpha \delta \neq \delta$, and assume that α and δ are represented by geodesics with respect to some choice of hyperbolic metric. Cutting Σ along α then yields a (possibly non-connected) surface with geodesic boundary, in which δ is a number of properly embedded hyperbolic arcs. At least one of the connected components of the cut surface has genus at least 1, so in this component we may find a closed geodesic β, not parallel to a boundary component, intersecting one of the δ-arcs. In the original surface, β is still a geodesic intersecting the geodesic δ; hence $\tau_\beta\delta \neq \delta$ and (τ_β, κ) is interesting.

Next, since τ_α and τ_β commute, we see that $\tau_\alpha^n \tau_\beta^m \kappa$ is an $\mathbb{Z} \times \mathbb{Z}$-indexed family of distinct multicurves. By assumption, both τ_α and τ_β change the color of finitely many multicurves. Hence, outside some bounded region in $\mathbb{Z} \times \mathbb{Z}$, moving from one diagram to a neighbour does not change the color, and since we can connect the future of (τ_α, κ) to its past using such moves, the claim follows. □

From now on, we will only consider the future.

Lemma 5.9. *Assume that α and β are simple closed curves with $i(\alpha, \beta) \leq 1$, and that κ is a multicurve such that (τ_α, κ), (τ_β, κ) are interesting pairs. Then* $\operatorname{fut}(\tau_\alpha, \kappa) = \operatorname{fut}(\tau_\beta, \kappa)$.

Proof. If $i(\alpha, \beta) = 0$ the result follows from the proof of Lemma 5.8.

Now assume $i(\alpha, \beta) = 1$. Then $\alpha \cup \beta$ is contained in a subsurface Σ' of genus 1 with one boundary component γ. If κ can not be isotoped to be contained entirely in Σ', either some component of κ intersects γ essentially, or some component of κ lives in the complement of Σ'. In the former case, it is clear that (τ_γ, κ) is interesting, so the $i = 0$ case implies $\operatorname{fut}(\tau_\alpha, \kappa) = \operatorname{fut}(\tau_\gamma, \kappa) = \operatorname{fut}(\tau_\beta, \kappa)$. In the latter case, use the fact that the complement of Σ' has genus at least 1 to find a simple closed curve intersecting κ essentially.

Otherwise, κ lives entirely in Σ'. Let κ_0 denote any component of κ on which τ_α acts non-trivially. Then κ_0 is a simple closed curve in a torus with one boundary component. Since κ_0 is not a parallel copy of the boundary component, it must be a non-separating curve not parallel to α. Choose orientations of α, β and κ_0. Then, with respect to the basis for $H_1(\Sigma')$ represented by α and β, κ_0 must have coordinates (p, q) with $\gcd(p, q) = 1$ and $(p, q) \neq \pm(1, 0)$. Any other component of κ is forced to be either parallel to the boundary component of Σ' or to κ_0. The only way that τ_β can act on some component of κ is then that τ_β acts on κ_0; hence also $(p, q) \neq \pm(0, 1)$.

Consider the schematic picture of Σ' on Figure 5.1 on the facing page, where the boundary component is the circle in the center and α and β are the sides of the square. We construct two disjoint simple closed curves γ_1, γ_2 as follows: Draw two essential, disjoint arcs in Σ' with the endpoints on the boundary component, and use the fact that the complement of Σ' has genus at least 1 to close them up in such a way that they are disjoint and not homotopic to a curve contained in Σ'. By the above description of κ_0, $(\tau_{\gamma_j}, \kappa)$ are both interesting pairs. Now the $i = 0$ case implies that

$$\operatorname{fut}(\tau_\alpha, \kappa) = \operatorname{fut}(\tau_{\gamma_1}, \kappa) = \operatorname{fut}(\tau_{\gamma_2}, \kappa) = \operatorname{fut}(\tau_\beta, \kappa). \qquad \square$$

The next proposition extends the above lemma to $i(\alpha, \beta) \leq 2$, but its proof is rather technical. Also, as explained in the comments following the proof, it is in fact not needed when one is only interested in surfaces with at most one boundary component.

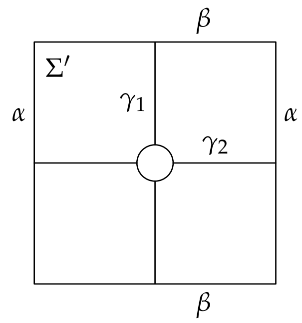


Figure 5.1: A torus with one boundary component.

Proposition 5.10. *Assume that α and β are simple closed curves with $i(\alpha, \beta) = 2$, and that κ is a multicurve such that (τ_α, κ) and (τ_β, κ) are interesting. Then* $\mathrm{fut}(\tau_\alpha, \kappa) = \mathrm{fut}(\tau_\beta, \kappa)$.

Proof. Let N be a regular neighbourhood of $\alpha \cup \beta$. We distinguish these four cases.

(1) At least one of α and β is non-separating in N.

(2) Both α and β are separating in N, but non-separating in Σ.

(3) Both α and β are separating in N, but one is non-separating in Σ.

(4) Both α and β are separating in Σ.

In case (1), assume without loss of generality that α is non-separating. This means that when cutting N along α, there is at least one arc b of β connecting the two sides of α. Now construct two curves γ_1, γ_2 as follows: Make two parallel copies of b and close them up using arcs going in opposite directions along α. Applying small isotopies in a tubular neighbourhood of α we obtain a situation as depicted in Figure 5.2. We observe that each γ_j intersects α in exactly one point, and also they intersect each other in exactly one point p. Furthermore, since $i(\alpha, \beta) = 2$, the arc b does not start and end at the same point of α, so we have $i(\gamma_j, \beta) = 1$ for $j = 1, 2$.

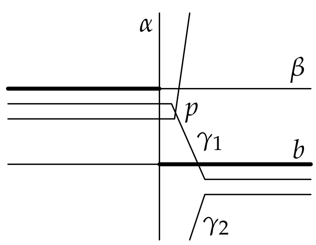


Figure 5.2: When α is non-separating in N, the two sides of α are connected by an arc of β.

Now let κ_0 be some component of κ on which τ_α acts non-trivially. We claim that at least one of γ_1 and γ_2 intersects κ_0 essentially. Assume the contrary, and orient γ_1 and γ_2 oppositely along b. Choose geodesic representatives γ_1', γ_2' and κ_0' of these three curves. Then γ_j' is disjoint from κ_0', and necessarily γ_1' and γ_2' intersect transversally in a single point p'. But then $(\gamma_1'\gamma_2')_{p'} \in \pi_1(\Sigma, p')$ is a representative of the free homotopy class of (an oriented version of) α which does not intersect κ_0', implying that $i(\kappa_0, \alpha) = 0$. This contradicts the choice of κ_0.

So one of the pairs $(\tau_{\gamma_j}, \kappa)$ is interesting, and by Lemma 5.9 we have

$$\mathrm{fut}(\tau_\alpha, \kappa) = \mathrm{fut}(\tau_{\gamma_j}, \kappa) = \mathrm{fut}(\tau_\beta, \kappa).$$

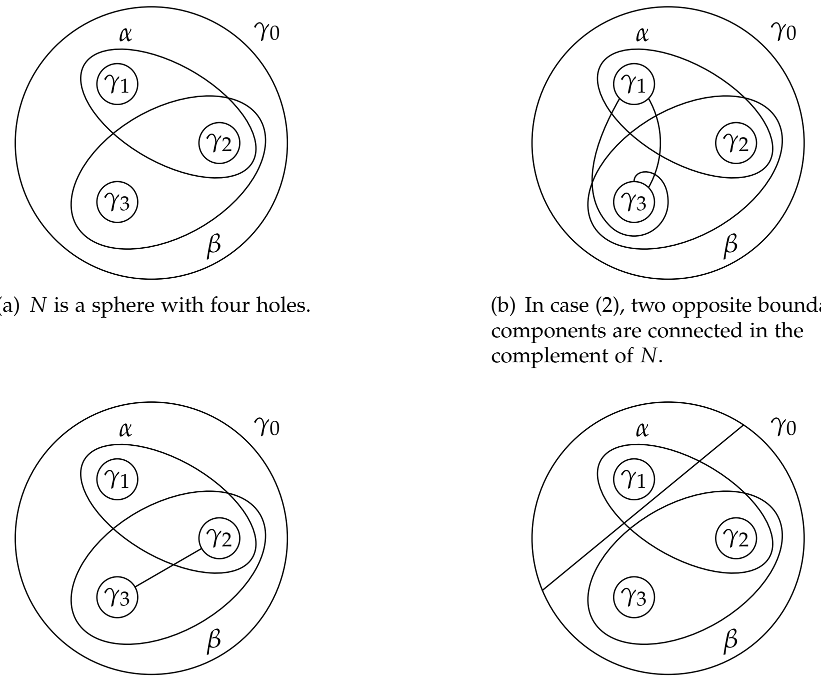
This ends case (1).

In cases (2)–(4), notice that N is necessarily a sphere with four holes, and α and β divide N into two pairs of pants in two different ways. Denote the boundary components of N by γ_j, $j = 0, 1, 2, 3$, such that γ_1, γ_2 are on one side of α and γ_0, γ_3 on the other, and such that γ_0, γ_1 are on one side of β and γ_2, γ_3 on the other. Schematically we have Figure 5.3(a) on the facing page.

Throughout the rest of the proof, we assume that α, β, γ_j, $j = 0, 1, 2, 3$, denote geodesic representatives for their isotopy classes. Also, we let δ be the geodesic representative of some component of κ on which τ_α acts non-trivially. If δ does not live entirely in N, a twist in one of the boundary components acts non-trivially on δ, and since this boundary component is disjoint from α and β we are done by Lemma 5.9. Otherwise, δ is a separating curve in N which is not parallel to a boundary component. Clearly δ can not be parallel to β, since in that case κ could not consist of any component on which τ_β acts non-trivially. Hence δ is different from both α and β.

In case (2), it is not hard to see that at least one of the »opposite« pairs γ_1, γ_3 and γ_0, γ_2 can be connected by an arc in the complement of N. Take two parallel copies of this arc, and close them up by arcs intersecting each other, α and β exactly once as in Figure 5.3(b) on the next page (the two connecting arcs shown are related by a twist in γ_3). We may then argue exactly as in case (1) to see that the twist in at least one of these simple closed curves acts non-trivially on the multicurve in question.

In case (3), assume without loss of generality that β is separating and α is nonseparating. This means that it is impossible to connect any of γ_0 and γ_1 to any of γ_2 and γ_3 in the complement of N. But then, since α is non-separating, one may connect either γ_0 to γ_1 or γ_2 to γ_3 in the complement of N. Assume without loss of generality that the latter is the case, and construct a simple closed curve γ disjoint from β intersecting γ_2, α and γ_3 exactly once each by composing the arc in the complement of N with an arc in N, as in Figure 5.3(c) on the facing page. Observe that the geodesic representative of γ necessarily intersects γ_2, α and γ_3 exactly once and is disjoint from β, so this representative contains a subarc in N starting at γ_2 and ending at γ_3. We now claim that this arc intersects δ (recall that δ has been chosen to be a geodesic). Assume the contrary. Then δ is a simple closed curve in the surface obtained by cutting N along this arc, which is a pair of pants. The »legs« are γ_0 and γ_1, whereas the »waist« is composed of four segments; two copies of the connecting arc and the remaining boundary components (cut open). Since δ is simple, it is parallel



(a) N is a sphere with four holes.

(b) In case (2), two opposite boundary components are connected in the complement of N.

(c) In case (3), two »neighbouring« boundary components are connected in the complement of N.

(d) In case (4), there exists an essential arc in the complement of N starting and ending at the same boundary component.

Figure 5.3: There are four different topological cases when two curves intersect in two points.

to one of the boundary components of the pair of pants. But δ is certainly not parallel to any of the original boundary components, nor is it parallel to the »waist«, since the latter is parallel to β. This contradiction implies that (τ_γ, κ) is an interesting pair, and since γ is disjoint from β and intersects α in a single point, Lemma 5.9 yields the desired result,

$$\operatorname{fut}(\tau_\alpha, \kappa) = \operatorname{fut}(\tau_\gamma, \kappa) = \operatorname{fut}(\tau_\beta, \kappa).$$

Finally, in case (4), none of the four boundary components of N can be connected in the complement of N. This means that at least one of the connected components of $\Sigma - N$ must have positive genus. Assume without loss of generality that the component Σ_0 bounded by γ_0 has positive genus. Now take some non-separating, essential arc in Σ_0 with its endpoints on γ_0 and compose it with some essential arc in N disjoint from β and intersecting α in exactly two points (cf. Figure 5.3(d)) to obtain a non-separating curve γ in Σ. We claim that τ_γ acts non-trivially on δ, ie. that the arc in N intersects δ essentially. To see this, we argue as in case (3) above. Observe that γ has geometric intersection number 2 with α and γ_0. Hence, the geodesic representative of γ intersects α and γ_0 exactly twice, so this geodesic contains a subarc in N looking like the one depicted in

Figure 5.3(d). We claim that this arc intersects δ. If this were not the case, we may cut N along this arc to obtain a cylinder (bounded by one of the original boundary components and a curve coming from the cut) and a pair of pants (bounded by two of the original boundary components and a curve from the cut), and δ lives completely in one of these. Since δ is not parallel to any of the boundary components of N, we conclude that δ is parallel to the third boundary component of the pair of pants. But this third boundary component is clearly parallel to β, which contradicts the fact that κ does not contain any component parallel to β. Hence (τ_γ, κ) is interesting, and since γ is non-separating and intersects α in two points, by case (3) and Lemma 5.9 we have

$$\mathrm{fut}(\tau_\alpha, \kappa) = \mathrm{fut}(\tau_\gamma, \kappa) = \mathrm{fut}(\tau_\beta, \kappa),$$

which finishes the last case. □

Now we turn to the (finite) presentation of the mapping class group given by Gervais in [20], where the generators are twists in certain curves. A key property of this presentation is that any two curves involved intersect each other in at most two points. It should be pointed out, however, that if one is only interested in surfaces with at most one boundary component, a much earlier result by Wajnryb [40] yields a presentation where each pair of curves intersect in at most one point. In this case, one does not need the rather technical Proposition 5.10 above in the following (simply replace all references to [20] by [40] and all occurences of »at most two points« by »at most one point«).

Proposition 5.11. *Let C denote the set of curves from [20] such that $\{\tau_\eta \mid \eta \in C\}$ generate Γ. Let $\alpha, \beta \in C$ be two of these curves, and let $\kappa_1, \kappa_2 \in X$ be multicurves such that (τ_α, κ_1) and (τ_β, κ_2) are interesting. Then*

$$\mathrm{fut}(\tau_\alpha, \kappa_1) = \mathrm{fut}(\tau_\beta, \kappa_2).$$

Proof. We may find a sequence of curves $\eta_1, \eta_2, \ldots, \eta_n \in C$ and exponents $\varepsilon_j = \pm 1$ such that, writing $\tau_j = \tau_{\eta_j}^{\varepsilon_j}$, $\tau_n \cdots \tau_2 \tau_1 \kappa_1 = \kappa_2$. For each $1 \leq j \leq n$ we may assume that $(\tau_j, \tau_{j-1} \cdots \tau_1 \kappa_1)$ is interesting; otherwise we may simply omit the corresponding τ_j. Now using alternately the fact that η_j and η_{j+1} intersect in at most two points and the obvious fact that $\mathrm{fut}(\tau_\gamma, \kappa) = \mathrm{fut}(\tau_\gamma, \tau_\gamma \kappa)$ for any interesting pair (τ_γ, κ), we obtain a sequence of identities

$$\begin{aligned}
\mathrm{fut}(\tau_1, \kappa_1) &= \mathrm{fut}(\tau_1, \tau_1\kappa_1) = \mathrm{fut}(\tau_2, \tau_1\kappa_1) \\
&= \mathrm{fut}(\tau_2, \tau_2\tau_1\kappa_1) = \mathrm{fut}(\tau_3, \tau_2\tau_1\kappa_1) \\
&\vdots \\
&= \mathrm{fut}(\tau_{n-1}, \tau_{n-1}\cdots\tau_2\tau_1\kappa_1) = \mathrm{fut}(\tau_n, \tau_{n-1}\cdots\tau_2\tau_1\kappa_1) \\
&= \mathrm{fut}(\tau_n, \tau_n\cdots\tau_2\tau_1\kappa_1) = \mathrm{fut}(\tau_n, \kappa_2)
\end{aligned}$$

which may be augmented by the identities $\mathrm{fut}(\tau_\alpha, \kappa_1) = \mathrm{fut}(\tau_1, \kappa_1)$ and $\mathrm{fut}(\tau_n, \kappa_2) = \mathrm{fut}(\tau_\beta, \kappa_2)$ to obtain the desired result. □

Lemma 5.12. *Let $\varphi \in \Gamma$ be any diffeomorphism, and (τ_α, κ) an interesting pair. Then $(\tau_{\varphi(\alpha)}, \varphi\kappa)$ is also interesting and* $\mathrm{fut}(\tau_\alpha, \kappa) = \mathrm{fut}(\tau_{\varphi(\alpha)}, \varphi\kappa)$.

Proof. Recall that $\varphi \circ \tau_\alpha \circ \varphi^{-1} = \tau_{\varphi(\alpha)}$. Hence $\tau_{\varphi(\alpha)}(\varphi\kappa) = \varphi(\tau_\alpha\kappa) \neq \varphi\kappa$, so $(\tau_{\varphi(\alpha)}, \varphi\kappa)$ is interesting. Also we have

$$\tau^n_{\varphi(\alpha)} = \varphi \circ \tau^n_\alpha \circ \varphi^{-1},$$

so $\tau^n_{\varphi(\alpha)}(\varphi\kappa) = \varphi(\tau^n_\alpha\kappa)$. Since the different multicurves $\tau^n_\alpha\kappa$ have the same color for all sufficiently large n, and since φ changes the color of only finitely many multicurves, the result follows. □

Proposition 5.13. *All interesting pairs (τ_γ, κ) where γ is a non-separating curve, have the same future.*

Proof. Let τ_α be a twist on a non-separating curve which is part of the generating set for Γ from [20]. Then Proposition 5.11, with $\alpha = \beta$, implies that the future is a property of τ_α alone, and not of the particular multicurve on which τ_α acts. If γ is any non-separating curve, choose a diffeomorphism of Σ carrying γ to α and apply Lemma 5.12. □

We are now ready to prove the non-existence of almost invariant colorings.

Proof of Theorem 5.7. By Remark 5.6, it suffices to consider a $\{0,1\}$-coloring of X. Assume that $c\colon X \to \{0,1\}$ is almost invariant. Choose a finite set $\alpha_1, \dots, \alpha_N$ of non-separating curves such that the twists in these curves generate Γ (we do not require that these intersect pairwise in at most two points). To be concrete, assume that the common future (cf. Proposition 5.13) of all interesting pairs (τ_γ, κ) with γ non-separating is 0. We must then prove that the set $B = c^{-1}(1) \subset X$ is finite. For each element $\kappa \in B$, choose a generator τ_{α_j} such that $(\tau_{\alpha_j}, \kappa)$ is interesting (this must be possible since the action is transitive and the τ_{α_j} generate Γ). This defines a map $f\colon B \to \{1, 2, \dots, N\}$. We claim that for each $j \in \{1, \dots, N\}$, the preimage $f^{-1}(j)$ is finite.

To see this, for each $\kappa \in f^{-1}(j)$ consider the τ_{α_j}-orbit through κ, ie. the set $S_j(\kappa) = \{\tau^n_{\alpha_j}\kappa \mid n \in \mathbb{Z}\}$. Let B_j be the union of the 1-colored multicurves occuring in these orbits, ie.

$$B_j = \bigcup_{\kappa \in f^{-1}(j)} (S_j(\kappa) \cap B),$$

so that $f^{-1}(j) \subseteq B_j$. There are only finitely many 1-colored multicurves in each $S_j(\kappa)$ by Proposition 5.13 and Lemma 5.8. Since τ_{α_j} changes the color of at least one multicurve in each $S_j(\kappa)$, namely κ, there can be only finitely many distinct sets by the almost invariance of the coloring. This proves that B_j is finite for each j, so also $B = \bigcup_j B_j$ is finite. □

5.5 Injectivity

Proposition 5.14. *The map* (5.8) *is injective.*

Proof. Let $u\colon \Gamma \to \mathbb{C}\mathcal{S}_\alpha$ be a cocycle mapping to 0 under (5.8). This means there exists $U\colon \mathcal{S}_\alpha \to \mathbb{C}$ such that $u(f)(\kappa) = (1-f)U(\kappa) = U(\kappa) - U(f^{-1}\kappa)$ for each $f \in \Gamma$ and $\kappa \in \mathcal{S}_\alpha$. Since $u(f)$ is a finitely supported map, this means that U is an almost invariant $\mathbb{C}$-coloring of $\mathcal{S}_\alpha$. By Theorem 5.7 we then get that there is a number z such that $U(\kappa) = z$ for all but finitely many κ. Putting $U'(\kappa) = U(\kappa) - z$, we get an element $U' \in \mathbb{C}\mathcal{S}_\alpha$ such that u is the coboundary of U'. □

This is the last piece of the puzzle.

Proof of Theorem 5.1. Combining Theorem 5.2 with the isomorphisms (5.4), it suffices to prove the vanishing of $H^1(\Gamma, \mathbb{C}\mathcal{S}_\alpha)$ for each mapping class group orbit $\mathcal{S}_\alpha$. This follows from Propositions 5.5 and 5.14. □

Chapter **6**

Smooth Functions on the Abelian Moduli Space

In this chapter, we assume that the genus of Σ is at least 3 and that Σ has at most one boundary component. Consider the abelian moduli space

$$\mathcal{M} = \mathcal{M}_{\mathrm{U}(1)} = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{U}(1)) = \mathrm{Hom}(H_1(\Sigma), \mathrm{U}(1)).$$

As a smooth manifold, $\mathcal{M}$ is diffeomorphic to a $2g$-torus $\mathrm{U}(1)^{2g}$; an explicit diffeomorphism is given by choosing a symplectic basis $(x_1, y_1, \ldots, x_g, y_g)$ for $H_1(\Sigma, \mathbb{Z})$ and mapping $\rho \in \mathcal{M}$ to

$$(\rho(x_1), \rho(y_1), \ldots, \rho(x_g), \rho(y_g)) \in \mathrm{U}(1)^{2g}. \tag{6.1}$$

The usual symplectic structure on $\mathrm{U}(1)^{2g}$ induces a well-defined symplectic structure on $\mathcal{M}$, since any two identifications differ by an element of $\mathrm{Sp}(2g, \mathbb{Z})$. Hence $\mathcal{M}$ is a smooth symplectic manifold. The mapping class group acts by symplectomorphisms, so both $L^2(\mathcal{M})$ and $C^\infty(\mathcal{M})$ are Γ-modules. The subspaces consisting of functions with mean value 0, which is the same as the orthogonal complement of the constant functions, are denoted by $L^2_0(\mathcal{M})$ and $C^\infty_0(\mathcal{M})$, respectively.

In [7], Andersen and the author proved the following two theorems.

Theorem 6.1. *The cohomology group $H^1(\Gamma, L^2(\mathcal{M}))$ vanishes.*

Theorem 6.2. *The cohomology group $H^1(\Gamma, C^\infty(\mathcal{M}))$ vanishes.*

The proof of Theorem 6.1 given in [7] relies on the fact that for $g \geq 2$, the integral symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ has Kazhdan's Property (T), along with applying the Hochschild-Serre spectral sequence to the exact sequence (2.18). In fact, we have already explained how to prove Theorem 6.1, since it is obviously a special case of Theorem 2.31. The proof of Theorem 6.2 in [7] in turn relies on Theorem 6.1.

Presently, we will give an alternative proof of Theorem 6.2 which does not use »expensive« tools such as Property (T) and spectral sequences. Instead, the proof is inspired by the ideas underlying the proof of Theorem 5.1.

6.1 Pure Phase Functions

There is a natural orthonormal basis for $L^2(\mathcal{M})$ parametrized by $H_1(\Sigma)$, which can be described in several different ways.

The intrinsic definition is rather simple. To a homology element $m \in H_1(\Sigma)$, we associate the function $\widetilde{m}$ on $\mathcal{M}$ given by evaluation in m, ie. we put

$$\widetilde{m}(\rho) = \rho(m) \in \mathrm{U}(1) \subset \mathbb{C}$$

for $\rho \in \mathcal{M} = \mathrm{Hom}(H_1(\Sigma), \mathrm{U}(1))$.

Under the identification (6.1), the function corresponding to the homology element $m = a_1 x_1 + b_1 y_1 + \cdots a_g x_g + b_g y_g$ is simply the trigonometric monomial

$$(z_1, w_1, \ldots, z_g, w_g) \mapsto z_1^{a_1} w_1^{b_1} \cdots z_g^{a_g} w_g^{b_g}$$

on $\mathrm{U}(1)^{2g}$. From this description, it is clear that the family $\{\widetilde{m} \mid m \in H_1(\Sigma)\}$ constitutes an orthonormal basis for $L^2(\mathcal{M})$.

Lemma 6.3. *There is a mapping class group equivariant isomorphism*

$$L^2(\mathcal{M}) \cong \ell^2(H_1(\Sigma)) \tag{6.2}$$

where $H_1(\Sigma)$ is considered as a discrete set.

Recall from (1.19) that $\ell^2(H_1(\Sigma))$ denotes the set of all maps $f\colon H_1(\Sigma) \to \mathbb{C}$ such that $\sum_{m \in H_1(\Sigma)} |f(m)|^2 < \infty$.

Proof. We compute

$$(\varphi \cdot \widetilde{m})(\rho) = \widetilde{m}(\varphi^{-1} \cdot \rho) = (\varphi^{-1} \cdot \rho)(m) = \rho(\varphi \cdot m) = \widetilde{\varphi \cdot m}(\rho),$$

proving the equivariance claim. □

Since the element $0 \in H_1(\Sigma)$ clearly corresponds to the constant function 1 on $\mathcal{M}$, we immediately obtain

Lemma 6.4. *Put $H' = H_1(\Sigma) - \{0\}$, considered as a discrete set. Then there is a mapping class group equivariant isomorphism*

$$L_0^2(\mathcal{M}) \cong \ell^2(H'). \tag{6.3}$$

It is very convenient that the action of the mapping class group can be described by a permutation action on an orthonormal basis.

6.2 Smooth Functions

Elements of $L_0^2(\mathcal{M})$ can be thought of as formal linear combinations $\sum_{m \in H'} f_m m$ with $\sum_{m \in H'} |f_m|^2 < \infty$. We will also need to know under which conditions a collection of coefficients (f_m) defines a smooth function. Choose a basis for $H_1(\Sigma)$, and define the norm of a homology element as in equation (2.9). A classical result from harmonic analysis (see [37]) on $\mathrm{U}(1)$ gives the following characterization of the smooth functions.

Proposition 6.5. *The formal sum $\sum_{m \in H_1(\Sigma)} f_m m$ defines a smooth function on $\mathcal{M}$ if and only if $|f_m|$ approaches 0 faster than any polynomial in $\|m\|^{-1}$, or equivalently, if and only if for each $k \in \mathbb{N}$, there is a constant F_k such that*

$$\|m\|^k |f_m| \le F_k \tag{6.4}$$

for all $m \in H_1(\Sigma)$.

These conditions are independent of the chosen basis for $H_1(\Sigma)$. A map $f\colon H_1(\Sigma) \to \mathbb{C}$ satisfying the above condition is called *rapidly decreasing*.

6.3 Proof of Theorem 6.2

There is a mapping class group equivariant inclusion of $C_0^\infty(\mathcal{M})$ into the set $\operatorname{Map}(H', \mathbb{C})$ of all maps from H' to $\mathbb{C}$. Note that this map factors through the unitary representation $\ell^2(H')$.

Lemma 6.6. *The induced map*

$$H^1(\Gamma, C_0^\infty(\mathcal{M})) \to H^1(\Gamma, \operatorname{Map}(H', \mathbb{C})) \tag{6.5}$$

is identically 0.

Proof. Fix some oriented, non-separating, simple closed curve μ in Σ, and let $m = [\mu]$ denote its homology class. Put $m_n = nm \in H'$ for $n \in \mathbb{Z}_+$. Then by Theorem 2.17 $\{m_n\}$ is a set of representatives of the Γ-orbits in H'. Hence, by Theorem 1.2, the right-hand side of (6.5) decomposes as a countable direct product

$$\prod_{n \in \mathbb{Z}_+} \operatorname{Hom}(\Gamma_{m_n}, \mathbb{C}).$$

Clearly, all the stabilizer subgroups are equal to Γ_m. Now let $[u] \in H^1(\Gamma, C_0^\infty(\mathcal{M}))$ and let $f \in \Gamma_m$. We must prove that the coefficient of m_n in $u(f)$ is equal to zero. Note that $f(\mu)$ is some simple closed curve representing the same element in $H_1(\Sigma)$ as μ, so by Lemma 2.24 there exists an element $t \in \mathcal{T}$ such that $tf(\mu) = \mu$. We have $u(tf) = u(t) + tu(f) = u(f)$ by Corollary 2.30. Since tf preserves the homotopy class of μ, tf is induced by an element of the surface obtained by cutting Σ along μ; this implies that tf can be written as a product $\tau_k^{\pm 1} \cdots \tau_2^{\pm 1} \tau_1^{\pm 1}$ of Dehn twists commuting with τ_μ. Each of these twists fixes μ and hence m_n, but this implies that

$$\langle u(tf), m_n \rangle = \sum_{j=1}^{k} \langle u(\tau_j^{\pm 1}), m_n \rangle = \sum_{j=1}^{k} \langle u(\tau_j^{\pm 1}), p_j m_n \rangle = 0$$

by Theorem 2.25. □

The above lemma implies that for any cocycle $u\colon \Gamma \to C_0^\infty(\mathcal{M})$, there exists a map $F\colon H' \to \mathbb{C}$ such that for each $g \in \Gamma$, the map $(1-g)F\colon H' \to \mathbb{C}$ given by $(1-g)F(m) = F(m) - F(g^{-1}m)$ corresponds to the smooth function $u(g)$. We must prove that F may be modified in such a way that it itself represents a smooth function on $\mathcal{M}$, ie., is rapidly decreasing.

For this, we adapt the notions of almost invariant colorings and future and past of interesting pairs from Chapter 5 to this setting. By an interesting pair, we now mean a pair (τ, m) consisting of a left or right Dehn twist τ and a homology element m such that $\tau m \neq m$ (necessarily the twist is on a non-separating curve). From now on, assume that $F\colon H' \to \mathbb{C}$ is a map satisfying that for each $g \in \Gamma$, the map $(1-g)F\colon H' \to \mathbb{C}$ is rapidly decreasing. Also, fix some symplectic basis $(x_1, y_1, \ldots, x_g, y_g)$ for $H_1(\Sigma)$, and let $\|\bullet\|$ denote the associated norm on $H_1(\Sigma)$ as in (2.9).

Lemma 6.7. *For each interesting pair (τ, m), both limits*

$$\operatorname{fut}(\tau, m) = \lim_{n\to\infty} F(\tau^n m) \qquad \operatorname{pas}(\tau, m) = \lim_{n\to\infty} F(\tau^{-n} m)$$

exist.

Proof. The collection $F(\tau^n m) - F(\tau^{n-1} m)$, $n \in \mathbb{Z}$ is absolutely summable, since $(1-\tau)F$ is rapidly decreasing. In particular, both sums

$$\sum_{n=1}^{\infty} \big(F(\tau^n m) - F(\tau^{n-1} m)\big)$$
$$\sum_{n=1}^{\infty} \big(F(\tau^{-n} m) - F(\tau^{-(n-1)} m)\big)$$

exist. But this precisely means that the expressions

$$F(\tau^N m) = F(m) + \sum_{n=1}^{N} \big(F(\tau^n m) - F(\tau^{n-1} m)\big)$$
$$F(\tau^{-N} m) = F(m) + \sum_{n=1}^{N} \big(F(\tau^{-n} m) - F(\tau^{-(n-1)} m)\big)$$

have limits as $N \to \infty$. □

The identity

$$\operatorname{fut}(\tau, m) = \operatorname{pas}(\tau^{-1}, m) \tag{6.6}$$

follows directly from the definition.

In order to compare the futures of commuting Dehn twists, we need a little technical result. In this lemma, the norm refers to the usual Euclidean structure on $\mathbb{R}^n$.

Lemma 6.8. *Let $a, b, c \in \mathbb{R}^n$ with $\|a\| = \|c\| = 1$, a and c not parallel. For each $t \in \mathbb{R}$, let $L(t)$ be the line through $ta + b$ in the direction of c. Let $P(t)$ denote the point on $L(t)$ closest to the origin. Then there exists a constant $k > 0$ such that for all $|t|$ large enough, we have $\|P(t)\| \geq k|t|$.*

Proof. The line $L(t)$ is parametrized by $ta + b + sc$, $s \in \mathbb{R}$. It is easy to obtain an expression for the value of s for which $\|ta + b + sc\|$ attains its minimum. We have

$$\frac{d}{ds}\|ta + b + sc\|^2 = 2s + 2t\langle a, c\rangle + 2\langle b, c\rangle$$

so the minimum is attained for $s = -t\langle a, c\rangle - \langle b, c\rangle$. The value of $\|P(t)\|^2$ is then

$$\|b\|^2 - \langle b, c\rangle^2 + 2t(\langle a, b\rangle - \langle a, c\rangle\langle b, c\rangle) + t^2(1 - \langle a, c\rangle^2).$$

Since a and c are non-parallel unit vectors, the coefficient of t^2 in this expression is positive. Letting $0 < \theta < \pi$ denote the angle between a and c, we have $1 - \langle a, c\rangle^2 = \sin^2\theta$, so $\|P(t)\|$ is asymptotically equal to $|t| \sin\theta$.□

Remark 6.9. Although the above lemma is formulated and proved in terms of the ordinary Euclidean distance, the fact that all norms on $\mathbb{R}^n$ are equivalent immediately shows that the same conclusion holds (with another constant) for any other norm. The assumption that a and c are unit vectors is of course not important; it is only important that they are non-zero and non-parallel.

Lemma 6.10. *Let α and β be distinct and disjoint simple closed curves, such that both (τ_α, m) and (τ_β, m) are interesting pairs. Then*

$$\operatorname{fut}(\tau_\alpha, m) = \operatorname{fut}(\tau_\beta, m). \tag{6.7}$$

Proof. It is convenient to orient α and β in such a way that $\omega(m, [\alpha])$ and $\omega(m, [\beta])$ are positive. If α and β are homologous, τ_α and τ_β act identically on $H_1(\Sigma)$, in which case the claim is trivial.

Let $\varepsilon > 0$. We may find an $N \in \mathbb{Z}_+$ so large that both $F(\tau_\alpha^n m)$ and $F(\tau_\beta^n m)$ differ from the respective futures by at most $\varepsilon/3$ for all $n \geq N$. We wish to find an, if necessary, even larger N so that these two numbers differ by at most $\varepsilon/3$ for $n \geq N$. To this end, use the fact that $(1 - \tau_\alpha^{-1}\tau_\beta)F$ is rapidly decreasing to find a constant C_2, such that

$$|(1 - \tau_\alpha^{-1}\tau_\beta)F(x)| \leq C_2\|x\|^{-2} \tag{6.8}$$

for each $x \in H'$. Applying Lemma 6.8 with $a = [\alpha]$, $b = m$ and $c = [\beta] - [\alpha]$, we find a constant k and an N so that for all $n \geq N$, all homology elements $(\tau_\alpha^{-1}\tau_\beta)^r\tau_\alpha^n m$, $r \in \mathbb{Z}$, have norm at least kn. Since

$$F(\tau_\beta^n m) - F(\tau_\alpha^n m) = \sum_{r=1}^{n} \big(F((\tau_\alpha^{-1}\tau_\beta)^r\tau_\alpha^n m) - F((\tau_\alpha^{-1}\tau_\beta)^{r-1}\tau_\alpha^n m)\big) \tag{6.9}$$

we may estimate

$$\begin{aligned}
|F(\tau_\beta^n m) - F(\tau_\alpha^n m)| &\leq \sum_{r=1}^{n} |F((\tau_\alpha^{-1}\tau_\beta)^r\tau_\alpha^n m) - F((\tau_\alpha^{-1}\tau_\beta)^{r-1}\tau_\alpha^n m)| \\
&= \sum_{r=1}^{n} |(1 - \tau_\alpha^{-1}\tau_\beta)F(\tau_\alpha^{n-r}\tau_\beta^r m)| \\
&\leq \sum_{r=1}^{n} C_2\|\tau_\alpha^{n-r}\tau_\beta^r m\|^{-2} \\
&\leq \sum_{r=1}^{n} C_2(kn)^{-2} \\
&\leq \frac{C_2}{k^2 n}.
\end{aligned}$$

By choosing n sufficiently large, this quantity can be made smaller than $\varepsilon/3$. Hence $\mathrm{fut}(\tau_\alpha, m)$ and $\mathrm{fut}(\tau_\beta, m)$ differ by at most ε, and since this holds for any $\varepsilon > 0$, they are equal. ☐

Corollary 6.11. *The future is equal to the past.*

Proof. If (τ_α, m) is an interesting pair, it is always possible to find another simple closed curve β such that (τ_β, m) is also interesting, and such that $[\beta] \neq \pm[\alpha]$. Then Lemma 6.10 yields $\mathrm{fut}(\tau_\alpha, m) = \mathrm{fut}(\tau_\beta, m)$, and a completely similar proof shows that $\mathrm{fut}(\tau_\beta, m) = \mathrm{pas}(\tau_\alpha, m)$. ☐

The next result is analogous to Lemma 5.9.

Lemma 6.12. *Assume that α and β are simple closed curves intersecting in exactly one point, and that m is a homology element such that (τ_α, m), (τ_β, m) are interesting pairs. Then* $\mathrm{fut}(\tau_\alpha, m) = \mathrm{fut}(\tau_\beta, m)$.

Proof. Let Σ' denote a regular neighbourhood of $\alpha \cup \beta$. If m lies in the image of $H_1(\Sigma') \to H_1(\Sigma)$, it can be represented by (parallel copies of) a simple closed curve in Σ', and the coordinates of m with respect to the basis for $H_1(\Sigma')$ represented by (oriented versions of) α and β is a pair (p, q) with $p \neq 0$ and $q \neq 0$. Hence, we may find simple closed curves γ_j, $j = 1, 2$, exactly as in the proof of Lemma 5.9 disjoint from each other and from α and β such that (τ_{γ_j}, m) are interesting. Then by Lemma 6.10 we have

$$\mathrm{fut}(\tau_\alpha, m) = \mathrm{fut}(\tau_{\gamma_1}, m) = \mathrm{fut}(\tau_{\gamma_2}, m) = \mathrm{fut}(\tau_\beta, m).$$

If m does not lie in the image of $H_1(\Sigma') \to H_1(\Sigma)$, choose $2g - 2$ simple closed curves $\alpha_2, \beta_2, \ldots, \alpha_g, \beta_g$ in the complemt $\Sigma - \Sigma'$ extending (α, β) to a symplectic basis for $H_1(\Sigma)$. Since at least one of the coordinates of m with respect to these $2g - 2$ curves is non-zero, m makes an interesting pair with some α_j or β_j; again the claim follows from Lemma 6.10. ☐

Since we assume that the surface Σ has at most one boundary component, we do not need an equivalent of the technical Proposition 5.10. Instead, we proceed to the equivalent of Proposition 5.11.

Proposition 6.13. *Let C denote the set of curves from [40] such that $\{\tau_\eta \mid \eta \in C\}$ generate Γ. Let $\alpha, \beta \in C$ be two of these curves, and let $m_1, m_2 \in H'$ be homology elements in the same mapping class group orbit, such that (τ_α, m_1) and (τ_β, m_2) are interesting. Then*

$$\mathrm{fut}(\tau_\alpha, m_1) = \mathrm{fut}(\tau_\beta, m_2).$$

The proof of Proposition 5.11 can be repeated almost verbatim.

Lemma 6.14. *Let $\varphi \in \Gamma$ be a diffeomorphism and (τ_γ, m) an interesting pair. Then $(\tau_{\varphi(\gamma)}, \varphi m)$ is also interesting, and* $\mathrm{fut}(\tau_\gamma, m) = \mathrm{fut}(\tau_{\varphi(\gamma)}, \varphi m)$.

Proof. For any $\varepsilon > 0$ there are only finitely many elements $x \in H'$ such that $|F(x) - F(\varphi x)| > \varepsilon$, since $(1 - \varphi^{-1})F\colon H' \to \mathbb{C}$ is rapidly decreasing, Hence we may choose an n so large that all three inequalities

$$\begin{aligned} |F(\tau_\gamma^n m) - \mathrm{fut}(\tau_\gamma, m)| &< \varepsilon \\ |F(\tau_{\varphi(\gamma)}^n(\varphi m)) - \mathrm{fut}(\tau_{\varphi(\gamma)}, \varphi m)| &< \varepsilon \\ |F(\tau_\gamma^n m) - (F\varphi\tau_\gamma^n m)| &< \varepsilon \end{aligned}$$

are satisfied. Since this holds for any $\varepsilon > 0$ and $\tau_{\varphi(\gamma)}^n(\varphi m) = \varphi\tau_\gamma^n m$ the claim follows. □

Proposition 6.15. *The future of an interesting pair (τ_γ, m) only depends on the mapping class group orbit of m.*

The proof of Proposition 5.13 applies verbatim.

Theorem 6.16. *Let $F\colon H' \to \mathbb{C}$ be any map such that $u(\varphi) = (1-\varphi)F\colon H' \to \mathbb{C}$ is rapidly decreasing for every $\varphi \in \Gamma$. Then there exists a rapidly decreasing map $f\colon H' \to \mathbb{C}$ such that $u(\varphi) = (1-\varphi)f$.*

Proof. Proposition 6.15 shows that we may modify F by a constant on each mapping class group orbit in H' such that the future (and past) of any interesting pair is 0. Let f be the result of this modification. Hence for any $x \in H'$ and any twist τ_γ such that $\tau_\gamma x \neq x$ we have

$$\lim_{n\to\infty} f(\tau_\gamma^n x) = \lim_{n\to\infty} f(\tau_\gamma^{-n} x) = 0. \tag{6.10}$$

We claim that f is rapidly decreasing, so we must prove that the condition from Proposition 6.5 is satisfied. Fix $2g$ simple closed curves α_j, β_j, $j = 1, \ldots, g$, representing a symplectic basis for $H_1(\Sigma)$, and let $k \geq 2$ be given. Write $\tau_j = \tau_{\alpha_j}$ and $\tau_{g+j} = \tau_{\beta_j}$ for $1 \leq j \leq g$. We must find a constant c_k such that $|f(m)|\|m\|^k \leq c_k$ for all $m \in H'$.

By assumption, $(1 - \tau_j^{\pm 1})F = (1 - \tau_j^{\pm 1})f$ is rapidly decreasing for $j = 1, \ldots, 2g$. Hence there are constants $C_{k+1}^{j,\pm}$ such that

$$|f(x) - f(\tau_j^{\mp 1}x)|\|x\|^{k+1} \leq C_{k+1}^{j,\pm} \leq C_{k+1} \tag{6.11}$$

for all $x \in H'$. Here C_{k+1} denotes the largest of the $4g$ numbers $C_{k+1}^{j,\pm}$. We now claim that we may put $c_k = C_{k+1}/k$.

Let $m \in H'$ be any given element. In order to estimate $|f(m)|\|m\|^k$, choose by Lemma 2.18 an index $1 \leq j \leq 2g$ and a sign $\varepsilon = \pm 1$ such that $\|\tau_j^{\varepsilon n} m\|$, $n = 0, 1, 2 \ldots$, is strictly increasing. Assume $\varepsilon = +1$. For each $R \geq 1$, we have the telescoping sum

$$f(\tau_j^R m) - f(m) = \sum_{r=1}^{R} f(\tau_j^r m) - f(\tau_j^{r-1} m)$$

and hence, since $f(\tau_j^R m) \to 0$ for $R \to \infty$, we obtain

$$\begin{aligned} |f(m)| &= \Big|\sum_{r=1}^{\infty}(f(\tau_j^r m) - f(\tau_j^{r-1} m))\Big| \\ &\leq \sum_{r=1}^{\infty}|f(\tau_j^r m) - f(\tau_j^{r-1} m)|. \end{aligned}$$

Each term in this sum can be estimated using (6.11) (with $x = \tau_j^r m$), so we obtain

$$\begin{aligned} |f(m)| &\leq C_{k+1}\sum_{r=1}^{\infty}\frac{1}{\|\tau_j^r m\|^{k+1}} \\ &\leq C_{k+1}\sum_{r=\|m\|+1}^{\infty}\frac{1}{r^{k+1}} \\ &< C_{k+1}\int_{\|m\|}^{\infty}\frac{1}{r^{k+1}}dr \\ &= \frac{C_{k+1}}{k\|m\|^k} \end{aligned}$$

using the fact that $\|\tau_j^r m\|$ is a strictly increasing sequence of integers and elementary estimates. □

Chapter 7

Ideas and Conjectures

The contents of this chapter are of a more speculative nature than the rest of the thesis. We begin by combining the ideas from the previous two chapters in order to prove a theorem about the cohomology of Γ with coefficients in a certain (formal) space of linear combinations of multicurves. We then explain how these linear combinations give rise to continuous functions on the $\mathrm{SU}(2)$ moduli space, though not necessarily faithfully. In the last section, we present a few conjectures, whose solution seem within reach, and also a few more vaguely phrased problems.

7.1 Rapidly Decreasing Coefficients

Let $\mathcal{P}$ be a pants decomposition of Σ and consider the associated norm $\|\bullet\|_{\mathcal{P}}$ on $\mathcal{S}$. We shall call a function $f\colon \mathcal{S} \to \mathbb{C}$ *rapidly decreasing* (with respect to $\mathcal{P}$) if f satisfies the equivalent of (6.4), that is, for each $k \in \mathbb{N}$ there exists a constant F_k such that for all $\kappa \in \mathcal{S}$ we have

$$|f(\kappa)|\|\kappa\|^k \leq F_k. \tag{7.1}$$

Lemma 7.1. *The property of being rapidly decreasing does not depend on the chosen pants decomposition.*

Proof. The norms $\|\bullet\|, \|\bullet\|'$ associated to two different pants decomposition are equivalent in the usual sense that there exists constants $c, C > 0$ such that

$$c\|\bullet\| \leq \|\bullet\|' \leq C\|\bullet\|, \tag{7.2}$$

since by Theorem 2.13, the norms are related via piecewise integral linear expressions. □

We denote the space of all rapidly decreasing functions $\mathcal{S} \to \mathbb{C}$ by $\mathcal{R}$.

Lemma 7.2. *The mapping class group action on* $\mathrm{Map}(\mathcal{S}, \mathbb{C})$ *preserves the subset* $\mathcal{R}$.

Proof. It suffices to prove that every Dehn twist preserves $\mathcal{R}$. Let $f \in \mathcal{R}$, and let γ be a simple closed curve on Σ. Choose a pants decomposition of Σ containing γ. Since f is rapidly decreasing, there are constants F_k satisfying (7.1).

The estimate

$$\begin{aligned} \|\tau_\gamma^{\pm 1}\kappa\| &= \|\kappa\| - |t_\gamma(\kappa)| + |t_\gamma(\kappa) \pm m_\gamma(\kappa)| \\ &\leq \|\kappa\| + |t_\gamma(\kappa)| + m_\gamma(\kappa) \\ &\leq 2\|\kappa\| \end{aligned}$$

shows that $\frac{1}{2}\|\kappa\| \leq \|\tau_\gamma^{\pm 1}\kappa\| \leq 2\|\kappa\|$ for any multicurve κ. Hence

$$\begin{aligned} |(\tau_\gamma \cdot f)(\kappa)|\|\kappa\|^k &= |f(\tau_\gamma^{-1}\kappa)|\|\kappa\|^k \\ &\leq F_k \|\tau_\gamma^{-1}\kappa\|^{-k}\|\kappa\|^k \\ &\leq 2^k F_k \end{aligned}$$

for any $k \in \mathbb{N}$ and any multicurve κ. □

7.2 Cohomology

The next result, and its proof, can be seen as a hybrid of Theorem 5.1 and Theorem 6.2.

Theorem 7.3. *The cohomology group $H^1(\Gamma, \mathcal{R})$ vanishes.*

We state the necessary adaptations of the results from Chapters 5 and 6.

Lemma 7.4. *The map $H^1(\Gamma, \mathcal{R}) \to H^1(\Gamma, \mathrm{Map}(\mathcal{S}, \mathbb{C}))$ is zero.*

Proof. The target is the direct product

$$\prod_\alpha H^1(\Gamma, \mathrm{Map}(\mathcal{S}_\alpha, \mathbb{C})),$$

where $\mathcal{S} = \bigsqcup \mathcal{S}_\alpha$ is the splitting of $\mathcal{S}$ into mapping class group orbits. Choosing a representative κ_α for each $\mathcal{S}$ and letting $\Gamma_\alpha = \Gamma_{\kappa_\alpha}$ denote the stabilizer of κ_α, we obtain an isomorphism

$$H^1(\Gamma, \mathrm{Map}(\mathcal{S}_\alpha, \mathbb{C})) \to \mathrm{Hom}(\Gamma_\alpha, \mathbb{C}).$$

The proof of Proposition 5.5 shows that the restriction of a cocycle $u \colon \Gamma \to \mathcal{R}$ to Γ_α followed by evaluation in κ_α is identically 0. □

This theorem can be rephrased as follows: For any cocycle $u \colon \Gamma \to \mathcal{R}$, there exists an map $U \colon \mathcal{R} \to \mathbb{C}$ such that $u(\varphi) = (1 - \varphi)U$ for every $\varphi \in \Gamma$. The proof of Theorem 7.3 is complete once we prove:

Theorem 7.5. *Let $U \colon \mathcal{S} \to \mathbb{C}$ be a map such that $(1 - \varphi)U$ is rapidly decreasing for each $\varphi \in \Gamma$. Then there exists a mapping class group invariant map $C \colon \mathcal{S} \to \mathbb{C}$ such that $U - C$ is rapidly decreasing.*

Let U be a map satisfying the above hypothesis. By an interesting pair we now (again) mean a pair (τ, κ) consisting of a right or left Dehn twist τ and a multicurve κ such that $\tau\kappa \neq \kappa$.

Lemma 7.6. *For each interesting pair (τ, κ), both limits*

$$\operatorname{fut}(\tau, m) = \lim_{n\to\infty} F(\tau^n m) \qquad \operatorname{pas}(\tau, m) = \lim_{n\to\infty} F(\tau^{-n} m) \tag{7.3}$$

exist.

The proof of Lemma 6.7 can be used verbatim. Also, since the Dehn-Thurston coordinates parametrize $\mathcal{S}$ as a subset of Euclidean space (but equipped with a norm which is easier to work with), Lemma 6.8 can be used to rephrase Lemma 6.10 and Corollary 6.11:

Lemma 7.7. *Let α and β be distinct and disjoint simple closed curves, such that both (τ_α, κ) and (τ_β, κ) are interesting pairs. Then*

$$\operatorname{fut}(\tau_\alpha, \kappa) = \operatorname{fut}(\tau_\beta, \kappa). \tag{7.4}$$

Lemma 7.8. *For any interesting pair (τ, κ), $\operatorname{fut}(\tau, \kappa) = \operatorname{pas}(\tau, \kappa)$.*

The statements and proofs of Lemma 5.9, Proposition 5.10 and Proposition 5.11 need not be modified to hold in this new context.

Lemma 7.9. *Let $\varphi \in \Gamma$ be any diffeomorphism, and (τ_α, κ) an interesting pair. Then $(\tau_{\varphi(\alpha)}, \varphi\kappa)$ is also interesting and $\operatorname{fut}(\tau_\alpha, \kappa) = \operatorname{fut}(\tau_{\varphi(\alpha)}, \varphi\kappa)$.*

Proof. Use the idea from the proof of Lemma 6.14. □

Proposition 7.10. *The future of an interesting pair (τ_γ, κ) depends only on the mapping class group the orbit of κ, and is independent of the twist τ_γ used to compute it.*

Proof. Let α be a non-separating curve which is part of the generating set from [20]. Then Proposition 5.13 with $\alpha = \beta$ shows that $\operatorname{fut}(\tau_\alpha, \kappa)$ only depends on the mapping class group orbit of κ. But then, for any interesting pair (τ_γ, κ) with γ non-separating, we may find a diffeomorphism $\varphi \in \Gamma$ taking γ to α and apply Lemma 7.9. Finally, if γ is a separating curve, we may find a non-separating curve β disjoint from γ such that (τ_β, κ) is interesting. Then Lemma 7.7 shows that $\operatorname{fut}(\tau_\gamma, \kappa) = \operatorname{fut}(\tau_\beta, \kappa)$. □

Proof of Theorem 7.5. We define the map $C\colon \mathcal{S} \to \mathbb{C}$ as follows: If the mapping class group orbit of κ is trivial, put $C(\kappa) = U(\kappa)$. Otherwise, choose a Dehn twist τ acting non-trivially on κ, and put $C(\kappa) = \operatorname{fut}(\tau, \kappa)$. Proposition 7.10 shows that this gives a well-defined, mapping class group invariant map. Let $V = U - C$. Clearly $(1 - \varphi)V = (1 - \varphi)U$, so $(1 - \varphi)V$ is rapidly decreasing for every $\varphi \in \Gamma$.

To see that V is rapidly decreasing, we proceed as in the proof of Theorem 6.16. Let $k \in \mathbb{N}$. Fix some pants decomposion $\mathcal{P}$ of Σ, and let T denote the corresponding set of Dehn twists from Theorem 2.22. Since

$(1-\tau)V$ is rapidly decreasing for each of the finitely many $\tau \in T$, we may find a constant K_{k+1} such that

$$|V(\kappa) - V(\tau^{\pm 1}\kappa)| \|\kappa\|^{k+1} \leq K_{k+1} \tag{7.5}$$

for all $\tau \in T$ and all $\kappa \in \mathcal{S}$.

We now wish to estimate $|V(\kappa)| \|\kappa\|^k$ for any $\kappa \in \mathcal{S}$. By construction, $V(\kappa) = 0$ for boundary parallel multicurves κ. Assume $\kappa' \neq \emptyset$ and choose an element $\tau \in T$ and an exponent $\varepsilon = \pm 1$ such that $\|\tau^{\varepsilon n}\kappa\|$, $n \geq 0$, is strictly increasing. Assume $\varepsilon = -1$. Then

$$V(\tau^{-R}\kappa) - V(\kappa) = \sum_{r=1}^{R} V(\tau^{-r}\kappa) - V(\tau^{-r+1}\kappa)$$

implies that

$$|V(\kappa)| \leq \Big| \sum_{r=1}^{R} V(\tau^{-r}\kappa) - V(\tau^{-r+1}\kappa) \Big| + |V(\tau^{-R}\kappa)| \tag{7.6}$$

for all $R > 0$. By construction, $|V(\tau^{-R}\kappa)| \to 0$ as $R \to \infty$, and the terms in the sum can be estimated using (7.5), so we get

$$\begin{aligned} |V(\kappa)| &\leq \sum_{r=1}^{\infty} \frac{K_{k+1}}{\|\tau^{-r}\kappa\|^{k+1}} \\ &\leq \sum_{r=\|\kappa\|+1}^{\infty} \frac{K_{k+1}}{r^{k+1}} \\ &\leq K_{k+1} \int_{\|\kappa\|}^{\infty} \frac{1}{r^{k+1}} dr \\ &= \frac{K_{k+1}}{k\|\kappa\|^k}. \end{aligned}$$

This shows that a constant satisfying the requirement (7.1) is K_{k+1}/k. □

7.3 Normalized Holonomy Functions

The inclusion $\mathrm{SU}(2) \hookrightarrow \mathrm{SL}_2(\mathbb{C})$ induces an embedding

$$\mathrm{Hom}(\pi_1, \mathrm{SU}(2)) \hookrightarrow \mathrm{Hom}(\pi_1, \mathrm{SL}_2(\mathbb{C})) \tag{7.7}$$

which in turn induces a map

$$j\colon \mathcal{M} = \mathcal{M}_{\mathrm{SU}(2)} \to \mathcal{M}_{\mathrm{SL}_2(\mathbb{C})}.$$

Since (7.7) is an embedding of a real slice, the restriction of regular functions

$$\mathcal{O}(\mathrm{Hom}(\pi_1, \mathrm{SL}_2(\mathbb{C}))) \to \mathrm{Fun}(\mathrm{Hom}(\pi_1, \mathrm{SU}(2)))$$

is injective. The regular functions on $\mathcal{M}_{\mathrm{SL}_2(\mathbb{C})}$ are by definition a subset of $\mathcal{O}(\mathrm{Hom}(\pi_1, \mathrm{SL}_2(\mathbb{C})))$, so we get that the restriction map

$$j^*\colon \mathcal{O}(\mathcal{M}_{\mathrm{SL}_2(\mathbb{C})}) \to \mathrm{Fun}(\mathcal{M}_{\mathrm{SU}(2)}) \tag{7.8}$$

is also injective.

Compactness of $\mathcal{M}$ allows us to equip the space $C(\mathcal{M})$ of continuous functions with the uniform norm $\|\bullet\|_\infty$. It is obvious that the regular functions are continuous. We normalize the multicurve functions from Chapter 5 by putting

$$h_\kappa = \frac{\nu_\kappa}{\|\nu_\kappa\|_\infty}.$$

Incidentally, $\|\nu_\kappa\|_\infty$ is a power of 2, since $|\nu_\kappa(\rho)|$ attains its maximal value at the trivial representation, and the maximum is 2 to the number of components of κ.

Theorem 7.11. *For any element $f \in \mathcal{R}$, the sum*

$$\sum_{\kappa \in \mathcal{S}} f(\kappa) h_\kappa \tag{7.9}$$

defines a continuous function on $\mathcal{M}$.

Proof. Fix some pants decomposition of Σ, and consider the associated norm on $\mathcal{S}$. There is a constant C such that

$$|\{\kappa \in \mathcal{S} \mid \|\kappa\| = n\}| \leq C n^{6g+3r-7}.$$

Now choose a constant K such that

$$|f(\kappa)| \|\kappa\|^{6g+3r-5} \leq K$$

for all $\kappa \in \mathcal{S}$. Then for any $x \in \mathcal{M}$ we have

$$\begin{aligned} \sum_{\substack{\kappa \in \mathcal{S} \\ \kappa \neq \emptyset}} |f(\kappa) h_\kappa(x)| &= \sum_{n \in \mathbb{Z}_+} \sum_{\|\kappa\| = n} |f(\kappa) h_\kappa(x)| \\ &\leq \sum_{n \in \mathbb{Z}_+} \sum_{\|\kappa\| = n} \frac{K}{n^{6g+3r-5}} \\ &\leq \sum_{n \in \mathbb{Z}_+} \frac{CK}{n^2} < \infty \end{aligned}$$

since $|h_\kappa(x)| \leq 1$. Thus (7.9) is absolutely and uniformly convergent on $\mathcal{M}$, and the limit is a continuous function. □

A slight variation of the above theme is to consider other notions of »rapidly decreasing« linear combinations. For example, exponentially decaying functions $\mathcal{S} \to \mathbb{C}$ give rise to functions on $\mathcal{M}_{\mathrm{SU}(2)}$ which may be extended to an open neighbourhood in $\mathcal{M}_{\mathrm{SL}_2(\mathbb{C})}$.

7.4 Faithfulness

Injectivity of (7.8) clearly implies that the normalized holonomy functions h_κ, $\kappa \in \mathcal{S}$, are linearly independent as functions on $\mathcal{M}$. However, this does not automatically imply that the map $\Phi \colon \mathcal{R} \to C(\mathcal{M})$ given by (7.9) is injective. It is at present unknown to the author whether or not Φ is injective, but if it is, Theorem 7.3 of course translates into a statement about some subspace of the set of continuous functions on $\mathcal{M}_{\mathrm{SU}(2)}$.

When we restrict further to the compact manifold $\mathcal{M}^1_{\mathrm{SU}(2)}$, the space of SU(2)-valued representations with holonomy $-I$ around the single boundary component, it is no longer true that the holonomy functions $\{\nu_\kappa \mid \kappa \in \mathcal{S}\}$ are linearly independent: Each boundary parallel component contributes a factor of 2 to the holonomy function. That is, if we write a multicurve κ as $\kappa_\partial \cup \kappa'$ (cf. Remark 2.21), we have

$$\nu_\kappa = 2^{|\pi_0 \kappa_\partial|} \nu_{\kappa'}$$

as functions on $\mathcal{M}^1_{\mathrm{SU}(2)}$, where $|\pi_0\kappa_\partial|$ is the number of components of κ_∂. This relation is, however, the only additional relation needed [3], so if we only allow multicurves without boundary parallel components we do get that $\mathbb{C}\mathcal{S}'$ injects into $C(\mathcal{M}^1_{\mathrm{SU}(2)})$. Letting $\mathcal{R}'$ denote the space of rapidly decreasing functions on $\mathcal{S}'$ it is clear that we also have

Theorem 7.12. *The cohomology group $H^1(\Gamma, \mathcal{R}')$ vanishes.*

By normalizing the holonomy functions ν_κ, $\kappa \in \mathcal{S}'$, over $\mathcal{M}^1_{\mathrm{SU}(2)}$ we clearly obtain a map

$$\Phi' \colon \mathcal{R}' \to C(\mathcal{M}^1_{\mathrm{SU}(2)}). \tag{7.10}$$

7.5 Open Questions

We end the dissertation with stating a few problems which may be seen as natural continuations of the work presented.

The first question is whether the maps Φ and Φ' are injective. It is also an interesting problem to try to describe which continuous functions one can obtain as rapidly decreasing linear combinations of holonomy functions.

Conjecture 7.13. *The map Φ' is injective, and it takes values in the smooth functions on $\mathcal{M}^1_{\mathrm{SU}(2)}$.*

The author and Andersen are currently working on a proof of this conjecture. Our approach uses Fourier analysis along a torus fibration; a strategy which was recently used by Charles and Marche [17] to give an alternative proof of the linear independence of the holonomy functions associated to multicurves when considered as functions on the SU(2) moduli space.

A proof of the above conjecture would be a major step towards answering the question initiating the project. With Theorems 5.1 and 6.2 (and also 7.3 and 7.12) as motivation, I claim:

Conjecture 7.14. *The cohomology group*

$$H^1(\Gamma, C^\infty(\mathcal{M}^1_{\mathrm{SU}(2)}))$$

vanishes.

It is obvious to ask if $H^1(\Gamma, \mathcal{O}(\mathcal{M}_G))$ vanishes for algebraic groups other $\mathrm{SL}_2(\mathbb{C})$. For $G = \mathrm{SL}_n(\mathbb{C})$, Sikora [35] has constructed a geometric model: He defines the n'th skein algebra $\mathbb{A}_n$ of Σ to be the algebra generated by so-called n-graphs on Σ modulo certain simple local relations. He then proves that $\mathcal{O}(\mathcal{M}_{\mathrm{SL}_n(\mathbb{C})})$ is isomorphic to $\mathbb{A}_n$ modulo its nilradical. The SL_n case is of particular interest because of the inclusion $\mathcal{M}^d_{\mathrm{SU}(n)} \subseteq \mathcal{M}_{\mathrm{SL}_n(\mathbb{C})}$.

Problem 7.15. Find a relationship between the regular functions on the $\mathrm{SL}_2(\mathbb{C})$ moduli space and the smooth functions on $\mathcal{M}^d_{\mathrm{SU}(n)}$.

Adapting Sikora's methods, Skovborg [36] shows how to present $\mathcal{O}(\mathcal{M}_G)$ as an explicit quotient of the algebra $\mathcal{C}(\Sigma)$ of chords diagrams, for $G = \mathrm{GL}_n(\mathbb{C})$ and $G = \mathrm{SL}_n(\mathbb{C})$. Despite of these results, a calculation of $H^1(\Gamma, \mathcal{O}(\mathcal{M}_G))$ seems rather hard, since the most obvious way to compute a cohomology group $H^1(G, M/N)$ is to compute the adjacent terms $H^1(G, M)$ and $H^2(G, N)$ in the long exact cohomology sequence. Regarding the former of these, understanding the cohomology with coefficients in the algebra of chord diagrams may also be interesting in itself.

Problem 7.16. Compute $H^1(\Gamma, \mathcal{C}(\Sigma))$.

However, $\mathcal{C}(\Sigma)$ is itself a quotient (4.22).

Although the inspiration for considering the cohomology of the mapping class group with various twisted coefficients came from the study of moduli spaces and their quantization, the topology and geometry of these spaces did not appear at all in the proofs of Theorems 5.1, 6.2 and 7.3.

Problem 7.17. Let G be a group acting on a (discrete) set X, and consider some class $\mathcal{V}$ of functions on X preserved by the G-action. Formulate conditions on G, X and $\mathcal{V}$ which are sufficient to deduce $H^1(G, \mathcal{V}) = 0$.

Some of the key ingredients in the proofs seem to be that G is generated by a single conjugacy class (that of a twist in a non-separating curve), that the orbit under such a generator is either trivial or infinite, the existence of sufficiently many free abelian subgroups of G of rank 2, and, of course, the particular nature of the class of functions considered. Without additional structure on X it seems hard to think of other classes of functions than the permutation modules $\ell^p(X)$, $\mathbb{C}X$ and $\mathrm{Map}(X, \mathbb{C})$. In our cases we used an embedding of X into Euclidean space to obtain a norm, which had certain good properties with respect to the group action.

Appendix **A**

Principal Bundles and Connections

Let G be a Lie group and B a manifold. Recall that a principal G-bundle over B consists of a smooth map $\pi\colon P \to B$, where P is a smooth manifold equipped with a smooth right action of G, such that the G-orbits are exactly the fibres of π. Moreover, P is locally trivial in the sense that there exists an open covering $\{U_\alpha\}$ of B and G-equivariant diffeomorphisms $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$ mapping the fibre over x to $\{x\} \times G$ for all $x \in U_\alpha$.

Fixing a point $p \in P$, the differential of the right action $g \mapsto p \cdot g$ at the identity in G is a linear map $v_p\colon \mathfrak{g} \to T_pP$. It is easy to see that the sequence

$$0 \longrightarrow \mathfrak{g} \xrightarrow{v_p} T_pP \xrightarrow{\pi_*} T_{\pi(p)}B \longrightarrow 0 \tag{A.1}$$

is exact. Vectors in the subspace $V_p = v_p(\mathfrak{g})$ are called the *vertical* vectors at p. A *connection* in P is a 1-form $A \in \Omega^1(P, \mathfrak{g})$ on P with values in the vector space $\mathfrak{g}$, such that $A_p \circ v_p = \mathrm{id}_{\mathfrak{g}}$ for all $p \in P$, and such that $R_g^*A = \mathrm{Ad}(g^{-1}) \circ A$, where R_g denotes the right multiplication by $g \in G$ on P, and $\mathrm{Ad}(g^{-1})$ is the adjoint action of g^{-1} on $\mathfrak{g}$. The latter requirement implies (and is in fact equivalent to) that the *horizontal* subspaces $H_p = \ker A_p$ are permuted by the right action, ie. $R_{g_*}H_p = H_{pg}$. An element $a \in \mathfrak{g}$ gives rise to a vectorfield a^* on P given by $a^*_p = v_p(a)$, called the fundamental vector field associated to a. The fundamental vector fields satisfy $[a^*, b^*] = [a, b]^*$, where the left-hand side is the bracket of vector fields on P, and the right-hand side is the Lie bracket in $\mathfrak{g}$. Thus the *vertical bundle* $VP \subseteq TP$ is integrable in the Frobenius sense. A connection can also be specified by a giving a complementary subbundle $HP \subseteq TP$ invariant under the right action of G; ie. a smooth distribution such that at each point $p \in P$, $T_pP = V_pP \oplus H_pP$ and $H_{pg} = R_{g*}(H_p)$. For a vector field Z on P, let $Z = X + Y$ be the decomposition in horizontal and vertical vector fields determined by HP. Then the form A is reconstructed as $A_p(Z) = v_p^{-1}(Y_p)$.

The form $\theta \in \Omega^1(G, \mathfrak{g})$ defined by $\theta_g = (L_{g^{-1}})_*$ is called the Maurer-Cartan form on G. In a trivial bundle $P = B \times G$, there is a connection defined by $A = \pi_2^*\theta$, where $\pi_2\colon B \times G \to G$ is projection on the second factor. The horizontal bundle is simply the tangent spaces $T_x B \subseteq T_{(x,g)}(B \times G)$. Thus connections always exist locally, and since the two defining conditions are affine, we see that we may use a partition of unity to glue local connections together to obtain a global connection. Hence any principal G-bundle admits a connection. We denote the set of all connections by $\mathcal{A}$.

Given a connection A, the wedge product $A \wedge A \in \Omega^2(P, \mathfrak{g} \otimes \mathfrak{g})$ is a 2-form on P with values in the vector space $\mathfrak{g} \otimes \mathfrak{g}$. The Lie bracket $[\bullet, \bullet]$ is a linear map from $\mathfrak{g} \otimes \mathfrak{g}$ to $\mathfrak{g}$, and we denote by $[A \wedge A] \in \Omega^2(P, \mathfrak{g})$ the composition of $A \wedge A$ with this linear map. For tangent vectors $X, Y \in T_p P$, we have $[A \wedge A](X, Y) = [A(X) \otimes A(Y) - A(Y) \otimes A(X)] = 2[A(X), A(Y)]$.

The curvature form of A is the $\mathfrak{g}$-valued 2-form F_A defined by

$$F_A = dA + \tfrac{1}{2}[A \wedge A]. \tag{A.2}$$

If the form F_A is identically 0, A is called a *flat* connection. We denote the subspace of $\mathcal{A}$ consisting of flat connections by $\mathcal{F}$.

A.1 Associated Bundles

Given a representation $\rho\colon G \to \mathrm{Aut}(V)$ of the Lie group G on a vector space V, we may construct a vector bundle $E = E_\rho = \rho P$ over B with standard fibre V as follows: As a set, put $E = P \times_G V$, where G acts diagonally, $(p, v) \cdot g = (pg, \rho(g^{-1})v)$. Clearly the map π induces a map $\pi_E\colon E \to B$. A local trivialization $\varphi\colon \pi^{-1}(U) \to U \times G$ induces a trivialization $\widetilde{\varphi}\colon \pi_E^{-1}(U) \to U \times V$ given by

$$\widetilde{\varphi}([p, v]) = (\pi(p), \rho(\pi_2 \varphi(p))v) \tag{A.3}$$

where $\pi_2\colon U \times G \to G$ is the projection on the second factor. This is well-defined, because if $[p, v] = [q, w]$, we have $q = pg$ for some (uniquely determined) $g \in G$, and then $w = \rho(g^{-1})v$, whence

$$\widetilde{\varphi}([q, w]) = (\pi(pg), \rho(\pi_2 \varphi(pg))w = (\pi(p), \rho(\pi_2 \varphi(p)g)\rho(g^{-1})v = \widetilde{\varphi}([p, v]).$$

A V-valued form $\alpha \in \Omega^k(P, V)$ is called ρ-equivariant, if $R_g^* \alpha = \rho(g^{-1}) \circ \alpha$ for all $g \in G$. In this language, a connection on P is Ad-equivariant. Furthermore, α is called horizontal, if the contraction $i_Y \alpha$ of α with any vertical vector Y vanishes. A form on P with values in V which is both horizontal and equivariant is called basic. We denote the space of basic k-forms by $\Omega_b^k(P, V)$.

Lemma A.1. *There is an isomorphism between the space of basic k-forms on P and the space of E_ρ-valued k-forms on the base B.*

Proof. Given a basic k-form $\alpha \in \Omega_b^k(P, V)$, a point $x \in B$ and k tangent vektors $X_1, \ldots, X_k \in T_x B$, choose a point $p \in \pi^{-1}(x)$, and lifts $\widetilde{X}_i$ of X_i (ie., $\pi_* \widetilde{X}_i = X_i$). Then define $\alpha_\#(X_1, \ldots, X_k)$ to be the equivalence class of the point $(p, \alpha(\widetilde{X}_1, \ldots, \widetilde{X}_k))$ in E. This is independent of the choice of lifts $\widetilde{X}_i$ because α is horizontal (and any two lifts differ by a vertical vector), and it is independent of the choice of p because of ρ-equivariance. Using a local trivialization of P, smooth vector fields on B lift to smooth vector fields on P, so $\alpha_\#$ is a smooth k-form on B with values in E.

Conversely, given such a form β, we simply proceed as above and let $p \in P$ be some point, and $X_1, \dots, X_k$ tangent vectors at p. Then $\beta(\pi_* X_1, \dots, \pi_* X_k)$ is a point in the fibre of E over $\pi(p)$, so it has a unique representative of the form (p, v). Put $\beta^\#(X_1, \dots, X_k) = v$. Clearly $\beta^\#$ is horizontal, because π_* maps vertical vectors to 0. It is also equivariant, because $\pi_*(R_{g_*} X_i) = \pi_*(X_i)$, and the representative of (p, v) with first coordinate pg is $(pg, \rho(g^{-1})v)$.

The two operations just described are easily seen to be inverse to each other. ☐

A.2 Covariant Derivatives

The ordinary exterior differential $d\colon \Omega^*(P, V) \to \Omega^{*+1}(P, V)$ restricts to a map on the equivariant forms, because d commutes with pull-back and linear maps. However, the exterior differential of a horizontal form need not be horizontal, so d does not restrict to a map on the basic forms. But given a connection A in P, we may consider the projection operator $h\colon TP \to HP$ which to a tangent vector X associates the horizontal part hX of X (with respect to A). Then define $h^*\colon \Omega^k(P, V) \to \Omega^k(P, V)$ by the formula

$$h^*\alpha(X_1, \dots, X_k) = \alpha(hX_1, \dots, hX_k).$$

Evidently h^* is a projection onto the horizontal forms on P. Define the operator d_A as $h^* \circ d$, ie. $d_A\alpha(X_0, \dots, X_k) = (d\alpha)(hX_0, \dots, hX_k)$.

Theorem A.2. *The operator d_A maps equivariant forms to equivariant forms, and hence basic forms to basic forms.*

Proof. If α is equivariant, the equivariance of $d_A\alpha$ follows basically because the horizontal subspaces are permuted by R_{g_*}: It is easy to see that R_{g_*} commutes with h, so R_g^* commutes with h^*. Then an easy calculation shows that $R_g^* d_A\alpha = R_g^* h^* d\alpha = h^* d R_g^* \alpha = h^* d(\mathrm{Ad}(g^{-1}) \circ A) = \mathrm{Ad}(g^{-1}) \circ d_A\alpha$. ☐

On the basic forms, we have another expression for d_A.

Lemma A.3. *For a basic form α, the operator d_A is given by*

$$d_A\alpha = d\alpha + \dot\rho(A) \wedge \alpha. \tag{A.4}$$

This requires an explanation: The representation ρ induces a linear map $\dot\rho\colon \mathfrak{g} \to \mathrm{End}(V)$, and $\dot\rho(A)$ is a 1-form on P with values in the vector space $\mathrm{End}(V)$. Then $\dot\rho(A) \wedge \alpha$ is a $k+1$-form on P with values in the vector space $\mathrm{End}(V) \otimes V$, and we apply the canonical contraction to obtain a $k+1$-form with values in V, still denoted $\dot\rho(A) \wedge \alpha$. Before we can prove Lemma A.3, we need to linearize the equivariance condition on forms.

Lemma A.4. *For a fundamental vector field a^* on P, the Lie derivative $\mathcal{L}_{a^*}\alpha$ of an equivariant form α is given by*

$$\mathcal{L}_{a^*}\alpha = \dot\rho(-a) \circ \alpha.$$

Proof. Putting $g_t = \exp(ta)$, R_{g_t} is precisely the flow of a^* at time t. So by definition of the Lie derivative

$$\mathcal{L}_{a^*}\alpha = \tfrac{d}{dt}\big|_{t=0} R_{g_t}^*\alpha = \tfrac{d}{dt}\big|_{t=0} \rho(\exp(-ta)) \circ \alpha = \dot{\rho}(-a) \circ \alpha. \qquad \square$$

Proof (of Lemma A.3). Clearly the left-hand side of (A.4) is a horizontal form. To show that the right-hand side is horizontal, it is clearly enough to show that the contraction with a fundamental vector field is 0. So let $a \in \mathfrak{g}$, and then by the properties of insertion and Lie derivative, we obtain

$$i_{a^*}d\alpha + i_{a^*}(\dot{\rho}(A) \wedge \alpha) = \mathcal{L}_{a^*}\alpha - di_{a^*}\alpha + i_{a^*}\dot{\rho}(A) \wedge \alpha - \dot{\rho}(A) \wedge i_{a^*}\alpha$$

The second and last term vanish because α is horizontal. Now,

$$i_{a^*}\dot{\rho}(A) \wedge \alpha = \dot{\rho}(A(a^*)) \wedge \alpha = \dot{\rho}(a) \circ \alpha,$$

so the remaining two terms cancel by Lemma A.4. Hence the right-hand side of (A.4) is also horizontal.

Now let $X_0, \ldots, X_k$ be $k+1$ horizontal vector fields on P. Then the left-hand side of (A.4) is

$$d_A\alpha(X_0, \ldots, X_k) = (h^*d\alpha)(X_0, \ldots, X_k) = d\alpha(X_0, \ldots, X_k)$$

whereas the right-hand side is

$$(d\alpha + \dot{\rho}(A) \wedge \alpha)(X_0, \ldots, X_k) = d\alpha(X_0, \ldots, X_k) + (\dot{\rho}(A) \wedge \alpha)(X_0, \ldots, X_k).$$

Expanding the wedge product as a sum over permutations, we see that each term is 0, since $A(X_i) = 0$ by definition of horizontal vectors. $\square$

Although d_A can now be viewed as a map $\Omega_b^*(P, V) \to \Omega_b^{*+1}(P, V)$, it is not necessarily a differential, ie. d_A^2 need not be 0. In fact, the curvature of A is an obstruction to $d_A^2 = 0$:

Theorem A.5. *For $\alpha \in \Omega_b^k(P, V)$, we have $d_A^2\alpha = \dot{\rho}(F_A) \wedge \alpha$.*

Proof. We calculate (using Lemma A.3)

$$\begin{aligned} d_A d_A \alpha &= d_A(d\alpha + \dot{\rho}(A) \wedge \alpha) \\ &= dd\alpha + \dot{\rho}(dA) \wedge \alpha - \dot{\rho}(A) \wedge d\alpha + \dot{\rho}(A) \wedge d\alpha + \dot{\rho}(A) \wedge (\dot{\rho}(A) \wedge \alpha) \\ &= \dot{\rho}(dA) \wedge \alpha + (\dot{\rho}(A) \wedge \dot{\rho}(A)) \wedge \alpha. \end{aligned}$$

Comparing this with the definition of the curvature $F_A = dA + \frac{1}{2}[A \wedge A]$, we need to prove that

$$\tfrac{1}{2}\dot{\rho}[A \wedge A] = \dot{\rho}(A) \wedge \dot{\rho}(A). \tag{A.5}$$

Locally, we may write $A = \sum dx_i \otimes a_i$, where (x_i) is a local coordinate system on P, and a_i are elements in $\mathfrak{g}$. Then $\dot{\rho}(A) = \sum dx_i \otimes \dot{\rho}(a_i)$, so the right-hand side of (A.5) may be written

$$\begin{aligned} \dot{\rho}(A) \wedge \dot{\rho}(A) &= \sum dx_i \wedge dx_j \otimes \dot{\rho}(a_i)\dot{\rho}(a_j) \\ &= \sum_{i<j} dx_i \wedge dx_j \otimes (\dot{\rho}(a_i)\dot{\rho}(a_j) - \dot{\rho}(a_j)\dot{\rho}(a_i)) \\ &= \sum_{i<j} dx_i \wedge dx_j \otimes \dot{\rho}[a_i, a_j] \end{aligned} \tag{A.6}$$

since $\dot{\rho}$ is a Lie algebra homomorphism. Turning to the left-hand side of (A.5), we have $A \wedge A = \sum dx_i \wedge dx_j \otimes a_i \otimes a_j$, so $[A \wedge A] = \sum dx_i \wedge dx_j \otimes [a_i, a_j]$. Thus

$$\dot{\rho}[A \wedge A] = \sum dx_i \wedge dx_j \otimes \dot{\rho}[a_i, a_j] = 2 \sum_{i<j} dx_i \wedge dx_j \otimes \dot{\rho}[a_i, a_j] \qquad \text{(A.7)}$$

since $dx_i \wedge dx_j = -dx_j \wedge dx_i$ and $[a_i, a_j] = -[a_j, a_i]$. Comparing (A.6) and (A.7) with (A.5), this proves the theorem. □

We note that this in particular implies that $d_A^2 = 0$ if A is flat.

A.3 The Adjoint Bundle

An important example of an associated bundle is the adjoint bundle $\operatorname{Ad} P = E_{\operatorname{Ad}}$ defined via the adjoint representation $\operatorname{Ad}\colon G \to \operatorname{Aut}(\mathfrak{g})$ of G on its Lie algebra. Until now we have only shown that there exists at least one connection, but they are in fact in rich supply.

Theorem A.6. *The set of connections is an affine space for the space $\Omega_b^1(P, \mathfrak{g})$ of basic 1-forms on P: Given any connection A and basic 1-form α, $A + \alpha$ is also a connection, and all connections are obtained this way.*

Proof. Since both A and α are Ad-equivariant, so is $A + \alpha$. Since α is horizontal, $(A + \alpha)_p \circ v_p = A_p \circ v_p = \mathrm{id}$, so $A + \alpha$ is a connection. Conversely, given two connections A_0, A_1, their difference is clearly Ad-equivariant, and since A_0 and A_1 agree on vertical vectors, $A_0 - A_1$ is horizontal. □

Remark A.7. Using the identification given in Lemma A.1 of the $\operatorname{Ad} P$-valued 1-forms on B and the basic 1-forms on P, the tangent space $T_A \mathcal{A}$ to the space of all connections is naturally identified with $\Omega^1(B, \operatorname{Ad} P)$.

The sections of $\operatorname{Ad} P$ are, again by Lemma A.1, the same as the Ad-equivariant maps $P \to \mathfrak{g}$, ie. maps f satisfying $f(pg) = \operatorname{Ad}(g^{-1}) f(p)$ for all $p \in P$, $g \in G$. Since Ad acts on $\mathfrak{g}$ by Lie automorphisms, the point-wise bracket of two such functions is again Ad-equivariant. Hence $\operatorname{Ad} P$ is a bundle of Lie algebras over B.

We have already seen a couple of interpretations of the curvature of a connection. Here are some more useful properties:

Theorem A.8. *Let A be a connection in P, and F_A its curvature form.*

(1) *The form $F_A \in \Omega^2(P, \mathfrak{g})$ is horizontal and* Ad*-equivariant, thus basic, and hence defines a 2-form also denoted $F_A \in \Omega^2(B, \operatorname{Ad} P)$.*
(2) *We have $d_A A = F_A$.*
(3) *The covariant derivative of F_A vanishes, $d_A F_A = 0$ (Bianchi's identity).*

Proof. We first prove that the curvature is horizontal, then we use this to prove (2), from which it follows from Theorem A.2 that F_A is Ad-equivariant (since the connection is). Now, to prove that F_A is horizontal, we proceed as in the proof of Lemma A.3. Let $a \in \mathfrak{g}$, and consider the contraction of F_A with a^*:

$$\begin{aligned} i_{a^*}F_A &= i_{a^*}dA + \tfrac{1}{2}i_{a^*}[A \wedge A] \\ &= \mathcal{L}_{a^*}A - di_{a^*}A + \tfrac{1}{2}i_{a^*}[A \wedge A] \\ &= \mathrm{ad}(-a) \circ A + \tfrac{1}{2}i_{a^*}[A \wedge A] \end{aligned}$$

by Lemma A.4 and the fact that $A(a^*) = a$ is constant. Now it is easy to see that the remaining two terms cancel (for instance by contracting with an arbitrary vector field).

Now, both sides of the equation in (2) are horizontal forms, so let X, Y be arbitrary horizontal vector fields. Then $d_A A(X,Y) = dA(hX, hY) = dA(X,Y)$. On the other hand, $F_A(X,Y) = dA(X,Y) + \frac{1}{2}[A \wedge A](X,Y) = dA(X,Y)$. This proves (2) and (1).

Finally, to prove (3), we calculate

$$\begin{aligned} d_A F_A &= h^* dF_A = h^* ddA + \tfrac{1}{2}h^* d[A \wedge A] \\ &= \tfrac{1}{2}h^*([dA \wedge A] - [A \wedge dA]) = h^*[dA \wedge A] = [h^* dA \wedge h^* A] = 0 \end{aligned}$$

since $h^* A = 0$. □

Note that by (1) and Lemma A.3, the Bianchi identity may also be written $dF_A + \mathrm{ad}(A) \wedge F_A = 0$, or equivalently $dF_A = [F_A \wedge A]$. Yet another interpretation of the curvature is this: From the proof, for horizontal vector fields X, Y, we have $F_A(X,Y) = dA(X,Y) = XA(Y) - YA(X) - A([X,Y]) = -A([X,Y])$. From this equation and the fact that F_A is horizontal, we see that the connection A is flat if and only if the horizontal bundle HP is integrable.

A.4 Gauge Transformations

A gauge transformation of P is a bundle automorphism of P, ie. a G-equivariant bundle map $\varphi \colon P \to P$ covering the identity on B. Because of the G-equivariance, a gauge transformation is completely determined by its action on one point in each fibre. In fact, we may identify the group of gauge transformations $\mathcal{G}$ with the set of maps $g \colon P \to G$ satisfying

$$g(ph) = h^{-1}g(p)h. \tag{A.8}$$

To a gauge transformation φ, we associate the map g_φ defined by the equation $\varphi(p) = pg_\varphi(p)$. Then for $h \in G$, we have $\varphi(ph) = phg_\varphi(ph)$, but on the other hand $\varphi(ph) = \varphi(p)h = pg_\varphi(p)h$. This implies that $hg_\varphi(ph) = g_\varphi(p)h$, so g_φ satisfies the condition (A.8). On the other hand, given any such map, it is easy to see that the map $p \mapsto pg(p)$ is a gauge transformation of P.

Letting $c(h)$ denote the conjugation by h, $c(h)(a) = hah^{-1}$ for $a \in G$, the set of maps satisfying (A.8) can also be described as the set of c-equivariant maps $P \to G$, analogous to the case of associated vector bundles. Following this analogy further, we obtain a bijection between $\mathcal{G}$ and the set of sections of the fibre bundle $P_c = (P \times G)/G$ over B with standard fiber G, where the action of g on $P \times G$ is given by $(p,a) \cdot g = (pg, g^{-1}ag)$. The fibre over a point x of this bundle may be described as the Lie group of diffeomorphisms of the fibre $\pi^{-1}(x)$ that commute with the action of G on $\pi^{-1}(x)$, so P_c is a bundle of Lie groups over B.

In general, the right multiplication $R_g \colon P \to P$ by an element $g \in G$ is not a gauge transformation; in fact, this is the case if and only if g is central in G. However, in case $P = B \times G$ is the trivial bundle, left multiplication $L_g \colon P \to P$ does commute with the right action of G on P. So in this case we may also view the gauge group as the group of smooth maps $f \colon B \to G$ under point-wise multiplication, where the associated gauge transformation is given by $(x,g) \mapsto (x, f(x)g)$

The pull-back of a connection under a gauge transformation is again a connection, and we denote $\varphi^* A$ by A^φ. Clearly this defines a right action of $\mathcal{G}$ on $\mathcal{A}$. A gauge transformation φ may be written as the composition

$$P \xrightarrow{\Delta} P \times P \xrightarrow{\mathrm{id} \times g_\varphi} P \times G \xrightarrow{\mu} P$$

where Δ is the diagonal and μ is the action of G on P. One may calculate that the differential of μ at a point (p,g) is given by

$$\mu_*(X,Y) = R_{g_*} X + v_{pg} \theta Y \in T_{pg} P$$

for $X \in T_p P$, $Y \in T_g G$, where v_{pg} is the injection of vertical vectors $\mathfrak{g} \to T_{pg} P$ and θ is the Maurer-Cartan form on G. Using this, we may calculate the pull-back A^φ of a connection under a gauge transformation:

$$\begin{aligned}
(A^\varphi)_p(X) &= A_{\varphi(p)}(\varphi_* X) \\
&= A_{\varphi(p)}\big(R_{g_\varphi(p)_*} X + v_{\varphi(p)}(\theta g_{\varphi_*} X)\big) \\
&= (R^*_{g_\varphi(p)} A)_p(X) + \theta(g_{\varphi_*} X) \\
&= (\mathrm{Ad}(g_\varphi(p)^{-1}) \circ A)_p(X) + g_\varphi^* \theta(X)
\end{aligned}$$

from which we deduce that

$$A^\varphi = \mathrm{Ad}(g_\varphi^{-1}) \circ A + g_\varphi^* \theta. \tag{A.9}$$

Since pull-back commutes with the various operations involved in the definition (A.2) of the curvature (exterior differentiation, wedge product and the linear map induced by the Lie bracket), the action of $\mathcal{G}$ on $\mathcal{A}$ restricts to an action on $\mathcal{F}$. In fact, a formula similar to (A.9) reads

$$\varphi^* F_A = \mathrm{Ad}(g_\varphi^{-1}) \circ F_A, \tag{A.10}$$

and clearly $\varphi^* F_A$ is the curvature F_{A^φ} associated to the connection A^φ.

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