Spectral and Scattering Theory for Translation Invariant Models in Quantum Field Theory

Morten Grud Rasmussen

Colophon

Spectral and Scattering Theory for Translation Invariant Models in Quantum Field Theory

A PhD thesis by Morten Grud Rasmussen. Written under the supervision of Jacob Schach Møller at Department of Mathematical Sciences, Faculty of Science, Aarhus University.

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Contents

Introduction

This Ph.D. thesis consists of the three papers, "The Translation Invariant Massive Nelson Model: II. The Lower Part of the Essential Spectrum," "Asymptotic Completeness in Quantum Field Theory: Translation Invariant Nelson Type Models Restricted to the Vacuum and One-Particle Sectors" and "A Taylor-like Expansion of a Commutator with a Function of Self-adjoint, Pairwise Commuting Operators" as well as an overview, Chapter 1. We will refer to the papers as [MR], [GMR] and [Ras], respectively. The aim of the overview is to explain, mostly without proof, the role of some of the techniques used in these papers. We do not intend to present optimal results, instead we try to make it simple and informative.

* * *

We will now give a brief description of the overview. A natural first step in the spectral analysis of an operator is to determine the structure of the spectrum, i.e. where is the discrete spectrum, how is it distributed and where is the essential spectrum? The expected picture for many physically motivated operators including those we consider in [MR] and [GMR] is that they satisfy a so-called HVZ[1] theorem, which states that the spectrum below a certain point Σ_{ess} is discrete and can only accumulate at Σ_{ess}, and everything above Σ_{ess} belongs to the essential spectrum, $\sigma_{\mathrm{ess}} = [\Sigma_{\mathrm{ess}}, \infty)$. A common way to prove the first part of this statement is to make a "smooth energy cut-off" below the point Σ_{ess} of the Hamiltonian operator H, i.e. look at $f(H)$ for an arbitrary, compactly supported smooth function with support below Σ_{ess}. If H is known to be bounded from below, we are done if we can prove that $f(H)$ is a compact operator. This is one of many reasons for introducing the functional calculus of almost analytic extensions known as the Helffer-Sjöstrand formula, which is done in Section 1.

In all three papers, commutators between (possibly) unbounded operators play an important role. One has to pay attention to domain questions when dealing with unbounded operators in general and in particular

[1] for Hunziker-van Winter-Жислин

when defining the commutator. Thankfully, the concept of an operator being of class $C^k(A)$ for $k \geq 1$ and A some self-adjoint operator, often provide rigorous justification for calculations which would be trivial, were it not for the domain questions. Section 2 is devoted to this subject.

Now with firm ground under our feet, we proceed to see how one can use commutators to answer questions of the following kind: What does the essential spectrum look like, in particular how are the embedded eigenvalues, if they exist, distributed? Can they accumulate, and if so, where? How about their multiplicities? Is there any singular continuous spectrum? The keyword for answering these questions is Mourre theory, and this is the topic of Section 3.

In order to formulate the models we investigate in [MR] and [GMR], we need to introduce the concept of second quantisation. Second quantisation is a method for constructing a quantum field theory based on the Hamiltonian formulation of quantum mechanics; several other approaches exist, but we will not elaborate on that here. The term "second quantisation" comes from the fact that it can be seen as the process of quantising the classical theory of a fixed number of particles, which again is quantised to deal with the theory of an arbitrary number of particles. In the resulting theory, the dynamics is determined by a self-adjoint operator, the Hamiltonian, acting on a Hilbert space with a certain structure, called a Fock space. The basic constructions in Bosonic Fock spaces are treated in Section 4.

The translation invariance of a model implies that its Hamiltonian commutes with the operator of total momentum. This again implies that the Hamiltonian and the operator of total momentum are simultaneously diagonalisable, i.e. there exists a direct integral representation of the Hamiltonian with respect to total momentum. In Section 5, we will briefly discuss the connection between the Hamiltonian and the fibers in the direct integral.

In [GMR] a series of so-called weak propagation estimates is shown. Weak propagation estimates are statements of the form

$$\int_0^\infty \|B(t)e^{-itH}\varphi\|^2 \frac{\mathrm{d}t}{t} \leq C\|\varphi\|^2,$$

where $B(t)$ is some time-dependent observable. The proofs of all of these propagation estimates depend heavily on pseudo-differential calculus, so before going into the details of the role of the different propagation estimates, in Section 6 we will present a useful lemma from pseudo-differential calculus and briefly discuss its relation to [Ras]. As a warm-up for the next section, we will also briefly discuss two lemmas related to the propagation estimates.

For the reader who is not familiar with propagation estimates, the roles played by the four propagation estimates might be a bit blurred by their somewhat technical appearance and equally technical proofs. To make up for this, in Section 7 we will go through the propagation estimates, their mutual relations, their consequences and the connection to the Mourre estimate.

Scattering theory is the study of an interacting system on a scale of time (or distance) which is large compared to the scale of the interaction. Scattering normally involves a comparison of two dynamics for the same system, the given dynamics and a "free" dynamics. This gives rise to the concept of the so-called wave operators. Asymptotic completeness in quantum mechanics is the statement that the state space splits into a direct sum of bound and scattering states. The definition of bound states depend on the concrete model while scattering states are states that in the far distant past and/or future appear to be "asymptotically free." As the interaction we consider in [GMR] is a so-called short-range interaction, one would intuitively expect that for scattering states, the particles – at least for large times – are far apart. This idea is exploited to prove an intermediate result called geometric asymptotic completeness. Section 8 is devoted to the discussion of these subjects.

* * *

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Morten Grud Rasmussen
Århus, 7 May 2010

Chapter 1

Overview

The purpose of this overview is to expose some of the main tools, ideas and concepts used in the papers [MR], [GMR] and [Ras]. However, before performing this task we will briefly outline the contents of the three papers and discuss how they relate to each other.

In [MR], the two authors study a large class of linearly coupled translation invariant massive models which include both the Nelson model and the Fröhlich polaron model. Many results related to the energy-momentum spectrum on this class models valid for all values of the coupling constant have been obtained in previous work by Møller, see e.g. [Møl05] and [Møl06], including an HVZ theorem and non-generacy of ground states. In [MR], we focus our attention to the lower part of the essential spectrum, i.e. the range between the bottom of the essential spectrum and either the two-boson threshold, if there are no exited isolated mass shells, or the one-boson threshold pertaining to the first exited isolated mass shell, if it exists. In this region we prove a Mourre estimate and C^2 regularity with respect to a suitably chosen conjugate operator, implying that this region contains no singular continuous spectrum. Going further up in the spectrum is a difficult task for several reasons, see the introduction in [MR] for a detailed explanation.

A natural next step would be to prove asymptotic completeness in the considered region. As the model in this region is expected to resemble the model with at most one field particle in many aspects, the authors of [GMR] set out to handle this simplified model. The class of models studied in [GMR] is basically the class of models from [MR] restricted to the vacuum and one-particle sectors, although strictly speaking, in [GMR] they are not assumed to be massive. However, the positive mass assumption for the field particles in [MR] is primarily used to avoid infrared problems, and in [GMR], due to the finite particle assumption, we have already avoided infrared problems. Another consequence of restricting to the vacuum and one-particle sectors is that the method for proving the Mourre estimate in [MR] now works for the whole of the essential energy-momentum spectrum. The asymptotic completeness statement proved in [GMR] is therefore not restricted to a limited region. We hope to be able to combine the methods used in [GMR] with the Mourre estimate of [MR] to prove the aforementioned partial asymptotic completeness for the full model.

An important part of many of the proofs in both [MR] and [GMR] is to be able to compute commutators and to give estimates on their norms. When computing commutators of functions of the momentum coordinates with functions of the position coordinates, this is usually performed using pseudo-differential calculus. However, for computing commutators of functions of second quantisations of momentum operators with the second quantisation functor applied to functions of the position, one cannot use pseudo-differential calculus directly. In [Møl05], Møller computes such a commutator in some special cases where the function acting on the second quantised momentum operators is of one of three certain types. This leads to a mathematically unnatural assumption on one of the dispersion relations. To avoid this assumption, in [Ras], the author develops "an abstract pseudo-differential calculus," that also works for these more complicated momentum-position related commutators. We stress that it is not an extension of the already existing pseudo-differential calculus, as one e.g. is not able to use this method to say anything about commutators of functions that depend simultaneously on both position and momentum, at least not directly. However, as the results of [Møl05] have been extended to hold for a larger class of models using the method developed in [Ras] – see the section on localisation errors in [MR] – and as it plays a central role in computing a commutator in [MR], this method has proven to be useful.

1 The Helffer-Sjöstrand Formula

The functional calculus of almost analytic extensions known as the Helffer-Sjöstrand formula is a useful tool in the computation of commutators. In both [MR], [GMR] and [Ras], it is used in several crucial steps. The usefulness comes from the fact that it reduces the task of computing commutators with functions of a self-adjoint operator to computing the commutator with its resolvent, and in the case of several pairwise commuting, self-adjoint operators, it also reduces the task, although to a somewhat more complicated one than in the case with only one self-adjoint operator. The idea is to replace the function by an integral representation based on the Cauchy integral formula using an "almost analytic extension," see below. We will now present the formula without proof. For details, see the monographs [Dav95] and [DS99].

In the following, we write $\bar{\partial} = (\bar{\partial}_1, \ldots, \bar{\partial}_\nu)$ where $\bar{\partial}_j = \frac{1}{2}(\partial_{u_j} + i\partial_{v_j})$ and u_j and v_j are the real and imaginary parts of $z_j \in \mathbb{C}$, respectively, and $z = (z_1, \ldots, z_\nu) \in \mathbb{C}^\nu$. Let $f \in C^\infty(\mathbb{R}^\nu)$ and assume that there exists an $s \in \mathbb{R}$ with the property that for any multi-index α there exists a constant $C_\alpha > 0$ such that $|\partial^\alpha f(x)| \leq C_\alpha \langle x \rangle^{s-|\alpha|}$. Then there exists an *almost analytic extension* $\tilde{f} \in C^\infty(\mathbb{C}^\nu)$ of f satisfying:

(i) $\operatorname{supp}(\tilde{f}) \subset \{u + iv \,|\, u \operatorname{supp}(f), |v| \leq C\langle u \rangle\}$.

(ii) For any $\ell \geq 0$ there is a $C_\ell > 0$ such that $|\bar{\partial}\tilde{f}(z)| \leq C_\ell \langle z \rangle^{s-\ell-1} |\operatorname{Im} z|^\ell$.

Let $A = (A_1, \ldots, A_\nu)$ be a vector of pairwise commuting, self-adjoint operators. If $s < 0$, we now get the representation

$$f(A) = 2|S^{2\nu-1}|^{-1} \sum_{j=1}^{\nu} \int_{\mathbb{C}^\nu} \bar{\partial}_j \tilde{f}(z)(A_j - \bar{z}_j)|A - z|^{-2\nu} \mathrm{d}z,$$

where $|S^{2\nu-1}|$ is the area of the unit sphere in $\mathbb{R}^{2\nu}$. For $\nu = 1$, this reduces to

$$f(A) = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)(A - z)^{-1} \mathrm{d}z.$$

A commutator of the form $[B, f(A)]$ can thus be written as

$$[B, f(A)] = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)[B, (A - z)^{-1}] \mathrm{d}z,$$

or, using $[B, (A - z)^{-1}] = -(A - z)^{-1}[B, A](A - z)^{-1}$,

$$[B, f(A)] = -\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)(A - z)^{-1}[B, A](A - z)^{-1} \mathrm{d}z.$$

2 The C^k Regularity Classes

We begin by reviewing what a commutator is. For this purpose, let T and A be (possibly unbounded) operators with domains $\mathcal{D}(T)$ and $\mathcal{D}(A)$ respectively, acting on a complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ conjugate linear in the first variable. A priori, one might think that the commutator $[T, iA]$ of T and A – where the i is to insure that for self-adjoint T and A, the commutator is symmetric – should be defined as $[T, iA] = i(TA - AT)$ on some appropriate domain. However, as always the case when dealing with unbounded operators, one should be careful regarding domain issues; for an arbitrary pair of operators T, A, we have no reason to expect that T maps anything from its domain $\mathcal{D}(T)$ except the zero vector into the domain of A, $\mathcal{D}(A)$. To solve this issue, we instead view $[T, iA]$ as a form on $\mathcal{D}(T^*) \cap \mathcal{D}(A^*) \times \mathcal{D}(T) \cap \mathcal{D}(A)$ given by

$$\langle \psi, [T, iA]\varphi \rangle = i(\langle T^*\psi, A\varphi \rangle - \langle A^*\psi, T\varphi \rangle).$$

To simplify matters, in the following we will assume that T is either bounded or self-adjoint, and that A is self-adjoint. For a treatment of more general operators, see [GGM04]. When T is a bounded operator, we will denote it by B, and when it is self-adjoint, we will denote it by H. This ensures that $[T, iA]$ is a form on $\mathcal{D}(T) \cap \mathcal{D}(A) (= \mathcal{D}(A)$ if $T = B$).

Even though we now have a good definition of a commutator, one should still be careful with formal manipulations. For an example of how bad things can go, even with $\mathcal{D}(H) \cap \mathcal{D}(A)$ being a dense set in $\mathcal{D}(H)$, we refer the reader to [GG99]. To avoid such trouble, we introduce the concept of T being *of class* $C^k(A)$ for some $k \geq 1$. As mentioned in the introduction, the condition that T is of class $C^k(A)$ often provides rigorous justification for calculations that would be trivial without the domain issues. We will only consider $k \in \mathbb{N}$, for the definition of $C^\sigma(A)$ for $\sigma \in \mathbb{R}_+ \setminus \mathbb{N}$, we refer to the paper [Sah97] or the monographs [ABG96] and [GŁ02].

Definition 2.1 (The $C^k(A)$ property of bounded operators). Let B be a bounded operator and $k \in \mathbb{N}$. We say that $B \in C^k(A)$ if, for all $\varphi \in \mathcal{H}$, the map $\mathbb{R} \ni s \mapsto e^{-isA}Be^{isA}\varphi \in \mathcal{H}$ is k times continuously differentiable. If $B \in C^k(A)$, B is said to be *of class* $C^k(A)$.

For the case $k = 1$, the following alternative characterisations are often useful.

Proposition 2.2. *Let $B \in \mathcal{B}(\mathcal{H})$. The following are equivalent.*

(i) $B \in C^1(A)$.

(ii) It holds that $\limsup_{s \to 0^+} \frac{1}{s} \|e^{-isA}Be^{isA} - B\| < \infty$.

(iii) There is a constant C such that for all $\psi, \varphi \in \mathcal{D}(A)$,

$$|\langle A\psi, B\varphi \rangle - \langle \psi, BA\varphi \rangle| \leq C\|\varphi\|\|\psi\|.$$

(iv) *B maps $\mathcal{D}(A)$ into itself and $AB - BA \colon \mathcal{D}(A) \to \mathcal{H}$ extends to a bounded operator on $\mathcal{H}$.*

(v) *There exists a core $\mathcal{C}$ for A such that $B\mathcal{C} \subset \mathcal{D}(A)$ and $AB - BA$ extends from $\mathcal{C}$ to a bounded operator on $\mathcal{H}$.*

These equivalences are well-known, see e.g. [ABG96], [GGM04] and [FGS08]. For $B \in C^1(A)$ the commutator $[B, iA]$, which a priori only is defined as a form on $\mathcal{D}(A)$, extends uniquely to a bounded operator on $\mathcal{H}$ by (iv) (or (v)). We denote this extension by $[B, iA]^\circ$. If $B \in C^k(A)$, we will also write $\mathrm{ad}^k_{iA}(B)$ for the iterated commutator defined recursively by $\mathrm{ad}^k_{iA}(B) = [\mathrm{ad}^{k-1}_{iA}(B), iA]^\circ$. (Note that $\mathrm{ad}^j_{iA}(B) \in C^{k-j}(A)$ for $j < k$ if and only if $B \in C^k(A)$).

Proposition 2.3. *Let A be a self-adjoint operator. Then:*

(i) *The linear map $\mathcal{A} \colon C^1(A) \to \mathcal{B}(\mathcal{H})$ defined by $B \mapsto [B, iA]^\circ$ is closed for the weak operator topology.*

(ii) *The space $C^1(A)$ is a sub-algebra of $\mathcal{B}(\mathcal{H})$ and $\mathcal{A}$ is a derivation on it.*

(iii) *If $B \in C^1(A)$ and $z \in \rho(B)$, then $R(z) = (B - z)^{-1} \in C^1(A)$ and $[R(z), iA] = -R(z)[B, iA]R(z)$.*

We now extend the property of being of class $C^k(A)$ to self-adjoint operators. We note that it is possible to extend this property even further to include merely closed and densely defined operators, see [GGM04] for the case $k = 1$.

Definition 2.4 (The $C^k(A)$ property of self-adjoint operators). Let H be a self-adjoint operator on $\mathcal{H}$ and $k \in \mathbb{N}$. We say that H is of class $C^k(A)$ if $(H - z_0)^{-1} \in C^k(A)$ for some $z_0 \in \rho(H)$.

Note that we do not extend the *set* $C^k(A)$ but only the property of being *of class* $C^k(A)$. This ensures that Proposition 2.3 is still valid. However, while (i) and (ii) clearly fail to extend to unbounded, self-adjoint operators, (iii) remains true. Note also that (iii) ensures that for bounded, self-adjoint operators, the two definitions coincide.

Let H be a self-adjoint operator of class $C^1(A)$. From Definition 2.4 and the form identity $[(H - z_0)^{-1}, A] = -(H - z_0)^{-1}[H, A](H - z_0)^{-1}$ which holds on $\mathcal{D}(A)$, it follows that $[H, A]$ extends to an operator $[H, A]^\circ$ in $\mathcal{B}(\mathcal{D}(H), \mathcal{D}(H)^*)$. By sandwiching $[H, A]^\circ$ with $R(z) := (H - z)^{-1}$, we see that $R(z) \in C^1(A)$ for any $z \in \rho(H)$. Hence the $C^1(A)$ property is independent of the choice of z_0. In fact, this is also true for the $C^k(A)$ property for $k > 1$. We will now outline a proof of this fact.

Proposition 2.5. *Let A and H be self-adjoint operators with H of class $C^k(A)$ and $z_0 \in \rho(H)$ be such that $(H - z_0)^{-1} \in C^k(A)$. Let $z \in \rho(H)$ and write $R_0 = (H - z_0)^{-1}$, $R = (H - z)^{-1}$ and $\alpha = z - z_0$. Then $R \in C^k(A)$ and for $\ell \leq k$,*

$$\mathrm{ad}_{iA}^{\ell}(R) = \sum_{n=1}^{\ell} \sum_{\sum_{j=1}^{n} a_j = \ell} \frac{\ell!}{a_1! \cdots a_n!} \alpha^{n-1}(1 + \alpha R) \prod_{j=1}^{n} \mathrm{ad}_{iA}^{a_j}(R_0)(1 + \alpha R).$$

In particular, the $C^k(A)$ property is independent of the choice of $z \in \rho(H)$.

We note that the quotient in front of the terms of the sum is in fact a multinomial coefficient and that the sum may actually be seen as the sum of all decompositions of k, counted with multiplicity. The number n is then interpreted as the number of terms in the decomposition.

Proof. Assume that $R \in C^{m-1}(A)$ and the formula holds for $\ell = m - 1$. The form identity

$$[\mathrm{ad}_{iA}^{m-1}(R), iA] = \sum_{n=1}^{m} \sum_{\sum_{j=1}^{n} a_j = m} \frac{m!}{a_1! \cdots a_n!} \alpha^{n-1}(1+\alpha R) \prod_{j=1}^{n} \mathrm{ad}_{iA}^{a_j}(R_0)(1+\alpha R)$$

now holds by induction, and hence $[\mathrm{ad}_{iA}^{m-1}(R), iA]$ extends to a bounded operator $\mathrm{ad}_{iA}^{m}(R)$ satisfying the identity in the proposition. $\square$

We finish this section by stating a few results that would be trivial without the domain issues. For a proof of those not treated in the text, we refer the reader to [ABG96] and [GŁ02].

Proposition 2.6. *Let H, A be self-adjoint operators such that H is of class $C^1(A)$. Then:*

(i) The form $[H, iA]$ on $\mathcal{D}(H) \cap \mathcal{D}(A)$ extends to a bounded form $[H, iA]^\circ$ on $\mathcal{D}(H)$.

(ii) The virial relation holds: $\mathbb{1}_{\{\lambda\}}(H)[H, iA]^\circ \mathbb{1}_{\{\lambda\}}(H) = 0$.

(iii) For $z \in \rho(H)$, $[(H - z)^{-1}, iA] = (H - z)^{-1}[H, iA]^\circ (H - z)^{-1}$.

(iv) For $f \in C_0^\infty(\mathbb{R})$, $[f(H), iA]$ is bounded.

(v) For $z \in \rho(H)$, $(H - z)^{-1}$ preserves $\mathcal{D}(A)$.

3 Mourre Theory

At the end of the previous section, the virial relation was stated (Proposition 2.6 (ii)). This relation is a very important part of Mourre's positive commutator method. Combined with a positive commutator estimate, one can use the virial relation to obtain local finiteness of the point spectrum. To illustrate the idea, assume that H and A are self-adjoint operators with H of class $C^1(A)$, and that there exists a bounded operator C with $\ker(C) = \{0\}$ satisfying

$$\langle \psi, [H, iA]^\circ \psi \rangle \geq \|C\psi\|^2. \tag{1.1}$$

As the virial theorem holds, the left-hand side is zero for eigenvectors. This contradicts the estimate, implying that the pure point spectrum $\sigma_{\text{pp}}(H)$ of H is empty. In fact, if H and A are bounded, H has purely absolutely continuous spectrum. This result is known as Kato-Putman's theorem, see e.g. [RS78, Theorem XIII.28] and references therein. However, as almost always the case, the situation is a bit more delicate when dealing with unbounded operators.

In models of quantum mechanics the operator H generating the dynamics – the Hamiltonian – is usually an unbounded operator, and for physical interpretation purposes one would like the spectrum of the Hamiltonian to consist of eigenvalues (bound states) and absolutely continuous spectrum (scattering states) only. For this reason, we will now go through the basics of Mourre theory, which gives conditions under which one can verify that an unbounded operator has no singular continuous spectrum.

The development of the abstract Mourre commutator method was initiated by the fundamental paper by Mourre [Mou81] and later extended and refined by many authors, see among others [PSS81], [Mou83], [JMP84], [BG92], [ABG96] and [Sah97] for developments of "regular" Mourre theory, i.e. where the commutator is comparable to the Hamiltonian and [Ski98], [MS04] and [GGM04] for developments of the "singular" Mourre theory.

The inequality in (1.1) is an example of a global commutator estimate. The essence of Mourre theory is the Mourre estimate, which is a local commutator estimate. The precise definition of a Mourre estimate is given by the following:

Definition 3.1 (Mourre estimate). Let $H \in C^1(A)$ for some self-adjoint operator A and I a bounded, open interval on $\mathbb{R}$. We say that the Mourre estimate holds true for H on I if there exists a $c > 0$ and a compact operator K such that

$$E_I(H)[H, iA]^\circ E_I(H) \geq cE_I(H) + K \tag{1.2}$$

The operator A is called a *conjugate operator*. We say that the Mourre estimate is strict, if we can choose $K = 0$.

Note that if we assume (1.2) and that $\lambda \in I$ is not an eigenvalue of H, then we can choose an $I' \ni \lambda$ and a c' such that a strict Mourre estimate holds with I and c replaced by I' and c', respectively.

To illustrate what role the compact operator plays and how the locality of the estimate comes into play, we state two simple consequences of the Mourre estimate.

Proposition 3.2. *Let H and A be self-adjoint operators with H of class $C^1(A)$ and $\lambda \in \mathbb{R}$. If there exists a neighbourhood I of λ such that a Mourre estimate holds true for H on I, then the eigenvalues of H, counted with multiplicity, cannot accumulate at λ. In particular, λ is not an eigenvalue of H of infinite multiplicity. If moreover, the Mourre estimate is strict, then there are no eigenvalues of H in I.*

So far, we have only made use of the C^k property for $k = 1$, and we have not yet discussed the nature of the essential spectrum. As we shall see, this is no coincidence. The connection goes via the limiting absorption principle, which we will now give a precise definition of (we use the somewhat standard notation $\langle x \rangle = \sqrt{x^2 + 1}$).

Definition 3.3 (Limiting absorption principle). Let H and A be self-adjoint operators, J a bounded interval on $\mathbb{R}$ and $s \geq 0$ a non-negative number. We say that the limiting absorption principle holds for H with respect to (A, J, s) if

$$\sup_{z \in J^\pm} \| \langle A \rangle^{-s} (H - z)^{-1} \langle A \rangle^{-s} \| < \infty,$$

where $J^\pm = \{ z \in \mathbb{C} \,|\, \mathrm{Re}(z) \in J, \pm \mathrm{Im}(z) > 0 \}$.

Note that the limiting absorption principle implies absolute continuity of the part of essential spectrum of the operator lying in J, see [RS78]. Clearly, if the limiting absorption principle holds for an s_0, then it holds for all $s > s_0$.

To prove absence of singular continuous spectrum, it suffices to find conditions under which the limiting absorption principle holds. It turns out that it is sufficient to prove a C^2 property and a Mourre estimate for the operator. More precisely we have:

Theorem 3.4. *Let H be of class $C^2(A)$, I an open interval on $\mathbb{R}$ and $s > \frac{1}{2}$. Assume that the strict Mourre estimate holds true for H on I. Then the limiting absorption principle with respect to (A, J, s) holds true for H, where J is any compact subinterval of I.*

For a proof, wee refer to [ABG96].

4 General Constructions in Bosonic Fock Spaces

As mentioned in the introduction, second quantisation may be seen as the process of converting quantum theory with a fixed number of particles into a quantum theory with a variable number of particles. To satisfy relativistic invariance, one is forced to take creation-annihilation processes into account in the formulation of a quantum field theory. However, for matter particles, the energy involved in creation-annihilation processes are so high that one often simplifies to a fixed matter particle number.

We will now introduce some mathematical objects and notation related to bosonic second quantisation. At the moment there seems to be no generally accepted notation for many of the objects, but we will for the most part follow the notation used in the paper by Dereziński and Gérard [DG99].

Let $\mathfrak{h}$ be a complex Hilbert space, which we call the one-particle space. It describes the quantum states for a single particle. Denote by $\mathfrak{h}^{\otimes n}$ the n-fold tensor product

$$\mathfrak{h}^{\otimes n} = \underbrace{\mathfrak{h} \otimes \cdots \otimes \mathfrak{h}}_{n \text{ times}}$$

with the usual convention that $\mathfrak{h}^{\otimes 0} = \mathbb{C}$. We define the Fock space over $\mathfrak{h}$ to be the direct sum

$$\Gamma(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} \mathfrak{h}^{\otimes n}.$$

The vector $\Omega = (1, 0, 0, \dots) \in \Gamma(\mathfrak{h})$ is called the vacuum vector.

It is not $\Gamma(\mathfrak{h})$ itself but two subspaces of $\Gamma(\mathfrak{h})$ that are used most frequently in quantum field theory, the bosonic and the fermionic subspaces. As indicated by the title of the section, we will focus on the bosonic Fock space. For any n, take the set of basis elements in $\mathfrak{h}^{\otimes n}$ invariant under permutations of the tensors, i.e.

$$\mathfrak{h}^{\otimes n} \ni \psi_1 \otimes \cdots \otimes \psi_n = \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(n)}$$

for all permutations $\sigma \in S(n)$, and denote by $\mathfrak{h}^{\otimes_s n}$ the closed linear span of this set. It is clear that

$$\Gamma_s(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} \mathfrak{h}^{\otimes_s n} \subset \Gamma(\mathfrak{h})$$

constitutes a closed subspace, which we call the *symmetric Fock space* or the *bosonic Fock space* over $\mathfrak{h}$. Let us mention that the fermionic Fock space $\Gamma_a(\mathfrak{h})$ (a for antisymmetric) is defined in a similar manner, with basis elements that changes sign according the the sign of the permutation. For later use, let S denote the projection from $\Gamma(\mathfrak{h})$ to $\Gamma_s(\mathfrak{h})$ and S_n the projection from $\mathfrak{h}^{\otimes n}$ to $\mathfrak{h}^{\otimes_s n}$. We call $\mathfrak{h}^{\otimes_s n}$ the n-particle sector of $\Gamma_s(\mathfrak{h})$.

As we exclusively work with bosonic Fock spaces, we will from now on drop the index s from the notation. An important subspace of $\Gamma(\mathfrak{h})$ is the set $\Gamma_{\text{fin}}(\mathfrak{h})$ of finite particle vectors, by which we mean a vector $\psi = (\psi^{(n)})_{n=0}^{\infty} \in \Gamma(\mathfrak{h})$ with $\psi^{(n)} = 0$ for all but finitely many n. For any vector subspace $\mathfrak{k} \subset \mathfrak{h}$, we likewise define $\Gamma_{\text{fin}}(\mathfrak{k}) \subset \Gamma_{\text{fin}}(\mathfrak{h})$ as the set of vectors $\psi = (\psi^{(n)})_{n=0}^{\infty}$ with $\psi^{(n)} \in \mathfrak{k}^{\otimes_s n}$ for all n in addition to the requirement that $\psi^{(n)} = 0$ for all but finitely many n.

Let A be an operator on $\mathfrak{h}$, essentially self-adjoint on some domain D. We define the *second quantisation* $\mathrm{d}\Gamma(A)$ of A by the closure of the operator given by linearity and

$$\mathrm{d}\Gamma(A)_{|D^{\otimes_s n}} = \sum_{j=1}^{n} \mathbb{1}^{\otimes(j-1)} \otimes A \otimes \mathbb{1}^{\otimes(n-j)}$$

for each n. The operator $\mathrm{d}\Gamma(A)$ is essentially self-adjoint on $\Gamma_{\text{fin}}(D)$. An important example is the number operator $N = \mathrm{d}\Gamma(\mathbb{1})$.

We now proceed to define the creation and annihilation operators. For $h \in \mathfrak{h}$ let $b^+(h)$ be the operator that takes each n-particle sector into the $n+1$-particle sector by the action

$$b^+(h)_{|\mathfrak{h}^{\otimes_s n}} \psi^{(n)} = S_{n+1} h \otimes \psi^{(n)} \in \mathfrak{h}^{\otimes_s (n+1)}, \quad \psi^{(n)} \in \mathfrak{h}^{\otimes_s n}.$$

Its adjoint, which we denote by $b^-(h)$, is given by

$$b^-(h)_{|\mathfrak{h}^{\otimes_s (n+1)}} \psi^{(n+1)} = \langle h, \psi_1^{(n)} \rangle \psi_2^{(n)} \otimes \cdots \otimes \psi_{n+1}^{(n)} \in \mathfrak{h}^{\otimes_s (n)},$$

where $\psi^{(n)} \in \mathfrak{h}^{\otimes_s (n+1)}$. It is easy to see that $b^+(h)$ and $b^-(h)$ are bounded operators of norm $\|h\|$. The annihilation operator $a(h)$, initially defined on $\Gamma_{\text{fin}}(\mathfrak{h})$, is then given by

$$a(h) = \sqrt{N+1} b^-(h),$$

and its adjoint $a^*(h)$ is called the creation operator and is given by

$$a^*(h) = \sqrt{N} b^+(h).$$

Both the creation and the annihilation operators are closable, and we denote their closures by the same symbol. They satisfy the cannonical commutation relations,

$$\begin{aligned} [a(h_1), a^*(h_2)] &= \langle h_1, h_2 \rangle \mathbb{1} \\ [a^*(h_1), a^*(h_2)] &= [a(h_1), a(h_2)] = 0, \end{aligned}$$

and it follows from the boundedness of $[a(h_1), a^*(h_2)]$ that they have the same domain.

It is clear from their construction that they are bounded relative to the square root of the number operator;

$$\begin{aligned} \|a^\#(h)(N+1)^{-\frac{1}{2}}\| &\leq \|h\|, \\ \|(N+1)^{-\frac{1}{2}} a^\#(h)\| &\leq \|h\|, \end{aligned}$$

where $a^\#(h)$ denotes either $a(h)$ or $a^*(h)$. On their common domain, we define the *(Segal) field operator* by

$$\Phi(h) = \frac{1}{\sqrt{2}}(a^*(h) + a(h)),$$

which is self-adjoint by Nelson's commutator theorem.

Let B be a bounded operator of norm less than or equal to 1 on $\mathfrak{h}$. We will now turn the symbol Γ into a functor, the second quantisation functor, from the category of Hilbert spaces with bounded operators of norm less than 1 to the category of (bosonic) Fock spaces with bounded operators of norm less than or equal to 1 by defining

$$\Gamma(B)_{|\mathfrak{h}^{\otimes_s n}} = B^{\otimes n}.$$

Its relation to the second quantisation of a self-adjoint operator is given by the identity

$$\Gamma(e^{itA}) = e^{it\mathrm{d}\Gamma(A)}.$$

As mentioned earlier, we will at some point need to calculate commutators of (a function of) a second quantised operator with the functor Γ of a contraction. For that purpose, the following definition is useful. Let B be a bounded operator on $\mathfrak{h}$ of norm less than or equal to 1 and B a self-adjoint operator on $\mathfrak{h}$ with domain of essential self-adjointness D. We define

$$\mathrm{d}\Gamma(B, A)_{|D^{\otimes_s n}} = \sum_{j=1}^{n} B^{\otimes(j-1)} \otimes A \otimes B^{\otimes(n-j)},$$

which clearly reduces to $\mathrm{d}\Gamma(A)$ for $B = \mathbb{1}$. The first commutator can then be computed as

$$[\mathrm{d}\Gamma(A), \Gamma(B)] = \mathrm{d}\Gamma(B, [A, B]).$$

For further constructions in bosonic Fock spaces, we refer the reader to [DG99].

We will now present some concrete examples of some of the objects introduced above. Let $\mathfrak{h} = L^2(\mathbb{R}^\nu)$. Then $\mathfrak{h}^{\otimes_s n}$ is the Hilbert subspace $L^2_{\nu\text{-sym}}(\mathbb{R}^{n\nu})$ of $L^2(\mathbb{R}^{n\nu})$ of all functions invariant under permutations of their ν-variables, i.e.

$$L^2(\mathbb{R}^{n\nu}) \ni f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}), \quad x_i \in \mathbb{R}^\nu$$

for any permutation $\sigma \in S(n)$. The Fock space $\Gamma(\mathfrak{h})$ is the set of sequences $(\psi_0, \psi_1(x_1), \psi_2(x_1, x_2), \psi_3(x_1, x_2, x_3), \dots)$ of functions ψ_n, where for each n, $\psi_n \in L^2_{\nu\text{-sym}}(\mathbb{R}^{n\nu})$. For $\mathfrak{k} = C_0^\infty(\mathbb{R}^\nu)$, $\Gamma_{\mathrm{fin}}(\mathfrak{k})$ is a dense subset of $\Gamma(\mathfrak{h})$.

If A is the self-adjoint operator of multiplication by the real function ω, then

$$(\mathrm{d}\Gamma(A)\psi)^{(n)}(x_1, \dots, x_n) = \Big(\sum_{i=1}^{n} \omega(x_i)\Big)\psi^{(n)}(x_1, \dots, x_n).$$

The creation and annihilation operators are given by

$$(a^*(h)\psi)^{(n)}(x_1, \dots, x_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(x_i)\psi^{(n-1)}(x_1, \dots, \hat{x}_i, \dots, x_n),$$
$$(a(h)\psi)^{(n)}(x_1, \dots, x_n) = \sqrt{n+1} \int_{\mathbb{R}^\nu} \bar{h}(x)\psi^{(n+1)}(x, x_1, \dots, x_n)dx,$$

where $\hat{x}_i$ denotes that x_i is omitted. Finally, for B the operator of multiplication by $j \in C^\infty(\mathbb{R}^\nu; [0, 1])$, we have

$$(\Gamma(B)\psi)^{(n)}(x_1, \dots, x_n) = \Big(\prod_{i=1}^{n} j(x_i)\Big)\psi^{(n)}(x_1, \dots, x_n).$$

5 Direct Integral Representations

Direct integrals are a generalisation of the concept of direct sums. We will now give a definition of a direct integral of Hilbert spaces. For a thorough treatment of the subject, we refer the reader to e.g. [Nai59].

Definition 5.1 (Direct integrals of Hilbert spaces). Let X denote a Borel space equipped with a countably additive measure μ. A *measurable family of Hilbert spaces* on (X, μ) is a family $\{H_x\}_{x\in X}$ which satisfies the following: There is a countable partition $\{X_i\}_{i\in\mathbb{N}\cup\{\infty\}}$ of measurable subsets of X such that

$$H_x = \mathbb{C}^n \quad \text{for } x \in X_n, n \in \mathbb{N},$$
$$H_x = \mathcal{H} \quad \text{for } x \in X_\infty,$$

where $\mathcal{H}$ is some infinite dimensional, separable Hilbert space.

A *measurable cross-section* of $\{H_x\}_{x\in X}$ is a family $\{v_x\}_{x\in X}$ such that $v_x \in H_x$ for all $x \in X$ which satisfies that the restriction of $\{v_x\}_{x\in X}$ to each partition element X_n is measurable. As usual, we identify measurable cross-sections that are equal almost everywhere.

Given a measurable family of Hilbert spaces, we define the Hilbert space direct integral

$$\int_X^\oplus H_x \mathrm{d}\mu(x)$$

as the set of measurable square integrable cross-sections of $\{H_x\}_{x\in X}$. This is a Hilbert space equipped with the inner product

$$\langle u, v\rangle = \int_X \langle u(x), v(x)\rangle_{H_x} \mathrm{d}\mu(x).$$

For counting measures, the definition reduces to a direct sum. We will now give a very simple example of a Hilbert space direct integral which is not a direct sum. Let $X = [0, 1]$ be equipped with the Lebesgue measure. For any $x \in X$, let $H_x = \mathbb{C}$. Then $\{H_x\}_{x\in X}$ is clearly a measurable family of Hilbert spaces. The set of measurable cross-sections equals the set of measurable, complex valued functions on $[0, 1]$, and

$$\int_X^\oplus H_x \mathrm{d}x = L^2([0, 1]),$$

equipped with the usual inner product. One could now proceed to construct $L^2([0, 1]^2)$ by replacing $\mathbb{C}$ with $L^2([0, 1])$ in the above example.

In the papers [MR] and [GMR], X is $\mathbb{R}^\nu$ and the measurable family $\{H_x\}_{x\in\mathbb{R}^\nu}$ is just the constant family $H_x = \mathcal{H}$ for all $x \in \mathbb{R}^\nu$.

An operator A acting on $\int_X^\oplus H_x \mathrm{d}\mu(x)$ which can be represented as $\int_X^\oplus A_x \mathrm{d}\mu(x)$ for some strongly measurable family of operators $\{A_x\}_{x\in X}$ is called *decomposable*. By a strongly measurable family $\{A_x\}_{x\in X}$ we mean a family of operators whose restriction to each X_n is strongly measurable.

The importance of direct integrals comes from the fact that if two self-adjoint operators commute, then they are simultaneously diagonalisable. Examples of such operators appear e.g. in the analysis of 3-body magnetic Hamiltonians, see [GŁ02], and of course in translation invariant models, where the Hamiltonian commutes with the operator of total momentum. This means that the Hamiltonian is diagonalisable to an operator that decomposes on the Hilbert space direct integral $\int_{\mathbb{R}^\nu}^\oplus \mathcal{H}_\xi \mathrm{d}\xi$, where ξ denotes the total momentum. The unitary operator that diagonalises the Hamiltonian in our setup was first identified in [LLP53].

6 Tools for Proving Propagation Estimates

Pseudo-differential calculus is in short the calculus of operators that are functions of the position and differential operators. The functions belong to certain symbol classes, which we will not go deeper into here. As the momentum operator is a differential operator, one may use pseudo-differential calculus to compute the commutator of functions of position and momentum, which often is needed in the proof of propagation estimates. In particular, we have the following lemma, see [FGS02].

Lemma 6.1. *Let $f \in \mathcal{S}(\mathbb{R}^\nu)$ be a Schwartz function and let $g \in C^n(\mathbb{R}^\nu)$ satisfy $\sup_{|\alpha|=n}\|\partial^\alpha g\|_\infty < \infty$. Let $p = -i\nabla$. Then*

$$\begin{aligned}[g(p), if(x)] &= i\sum_{|\alpha|=1}^{n-1}\frac{(-i)^{|\alpha|}}{\alpha!}(\partial^\alpha f)(x)(\partial^\alpha g)(p) + R_{1,n}\\ &= -i\sum_{|\alpha|=1}^{n-1}\frac{i^{|\alpha|}}{\alpha!}(\partial^\alpha g)(p)(\partial^\alpha f)(x) + R_{2,n},\end{aligned}$$

where

$$\|R_{j,n}\| \le C_n \sup_{|\alpha|=n}\|\partial^\alpha g\|_\infty \int |k|^n|\hat{f}(k)|\mathrm{d}k.$$

In particular, if $n = 2$ then

$$\begin{aligned}[g(p), if(\varepsilon x)] &= \varepsilon\nabla g(p)\cdot\nabla f(\varepsilon x) + O(\varepsilon^2)\\ &= \varepsilon\nabla f(\varepsilon x)\cdot\nabla g(p) + O(\varepsilon^2)\end{aligned}$$

in the limit $\varepsilon \to 0$.

It is the last statement one commonly needs in the computation of commutators of functions of position and momentum, respectively. The method of proof is usually based on the Fourier transform and and application of Taylor's formula. However, it is easy to see that one can use the result of [Ras] for an almost immediate proof. If one identifies $f(x) = B$ and $A = p$, then it follows from the observation that

$$\mathrm{ad}_A^{\alpha}(B) = i^{|\alpha|}\partial^{\alpha} f(x).$$

We stress that in general, pseudo-differential calculus cannot replace the results obtainable by using the result of [Ras] or vice versa.

We will now present another lemma which is used to prove propagation estimates. It is a version of the Putnam-Kato theorem mentioned in Section 3, see also [RS78, Example XIII.7.5], developed by Sigal and Soffer [SS87].

Lemma 6.2. *Let H be a self-adjoint operator and $\mathbf{D}$ the corresponding Heisenberg derivative*

$$\mathbf{D} = \frac{\mathrm{d}}{\mathrm{d}t} + [H, i\,\cdot\,].$$

Suppose that $\Phi(t)$ is a uniformly bounded family of self-adjoint operators. Suppose that there exist $C_0 > 0$ and operator valued functions $B(t)$ and $B_i(t)$, $j = 1, \ldots, n$, such that

$$\mathbf{D}\Phi(t) \geq C_0 B^*(t)B(t) - \sum_{j=1}^{n} B_j^*(t)B_j(t),$$

$$\int_1^{\infty} \|B_j(t)e^{-itH}\varphi\|^2 \mathrm{d}t \leq C\|\varphi\|^2, \quad j = 1, \ldots, n.$$

Then there exists C_1 such that

$$\int_1^{\infty} \|B(t)e^{-itH}\varphi\|^2 \mathrm{d}t \leq C_1\|\varphi\|^2. \tag{1.3}$$

The operator valued function $\Phi(t)$ is called the *propagation observable*. The main idea of the proof of the propagation estimate (1.3) is thus to find a propagation observable whose Heisenberg derivative is "essentially positive." For completeness, we will now prove the lemma.

Proof. Let $1 \leq t_1 \leq t_2$. Compute

$$\begin{aligned}
C_0 \int_{t_1}^{t_2} \|B(t)e^{-itH}\varphi\|^2 \mathrm{d}t &\leq \int_{t_1}^{t_2} \langle e^{-itH}\varphi, \mathbf{D}\Phi(t)e^{-itH}\varphi\rangle \mathrm{d}t \\
&\quad + \sum_{j=1}^{n} \int_{t_1}^{t_2} \|B_j(t)e^{-itH}\varphi\|^2 \mathrm{d}t \\
&\leq \langle e^{-it_2H}\varphi, \mathbf{D}\Phi(t_2)e^{-it_2H}\varphi\rangle - \langle e^{-it_1H}\varphi, \mathbf{D}\Phi(t_1)e^{-it_1H}\varphi\rangle \\
&\quad + \sum_{j=1}^{n} \int_{t_1}^{t_2} \|B_j(t)e^{-itH}\varphi\|^2 \mathrm{d}t \leq C_2\|\varphi\|^2,
\end{aligned}$$

from which the result follows. □

One important application of the propagation estimates is in the proof of the existence of asymptotic observables. Cook's method ([RS79, Theorem XI.4]) is based on the observation that if $f \in C^1(\mathbb{R})$ has $f' \in L^1(\mathbb{R})$, then $\lim_{t\to\infty} f(t)$ exists since

$$|f(t_2) - f(t_1)| = |\int_{t_1}^{t_2} f'(u)\mathrm{d}u| \leq \int_{t_1}^{t_2} |f'(u)|\mathrm{d}u \to 0 \tag{1.4}$$

for $T \leq t_1 \leq t_2$ in the limit $T \to \infty$. The following lemma is a variation of Cook's method due to Kato, which provides existence of asymptotic observables.

Lemma 6.3. *Let H_1 and H_2 be two self-adjoint operators. Let ${}_2\mathbf{D}_1$ be the corresponding asymmetric Heisenberg derivative:*

$${}_2\mathbf{D}_1\Phi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) + iH_2\Phi(t) - i\Phi(t)H_1.$$

Suppose that $\Phi(t)$ is a uniformly bounded function with values in self-adjoint operators. Let $\mathcal{D}_1 \subset \mathcal{H}$ be a dense subspace. Assume that for $\psi_2 \in \mathcal{H}$ and $\psi_1 \in \mathcal{D}_1$,

$$\begin{aligned}
|\langle\psi_2, {}_2\mathbf{D}_1\Phi(t)\psi_1\rangle| &\leq \sum_{j=1}^{n} \|B_{2j}(t)\psi_2\|\|B_{1j}(t)\psi_1\|, \\
\int_1^\infty \|B_{2j}(t)e^{-itH_2}\varphi\|^2 \mathrm{d}t &\leq \|\varphi\|^2, \quad \varphi \in \mathcal{H},\ j = 1, \ldots, n, \\
\int_1^\infty \|B_{1j}(t)e^{-itH_1}\varphi\|^2 \mathrm{d}t &\leq C\|\varphi\|^2, \quad \varphi \in \mathcal{D}_1,\ j = 1, \ldots, n.
\end{aligned}$$

Then the limit

$$\operatorname*{s-lim}_{t\to\infty} e^{itH_2}\Phi(t)e^{-itH_1} \tag{1.5}$$

exists.

Proof. Let $\varphi \in \mathcal{D}_1$ and $\psi \in \mathcal{H}$. Compute

$$\begin{aligned}
&|\langle \psi, e^{itH_2}\Phi(t)e^{-itH_1}\varphi\rangle - \langle \psi, e^{itH_2}\Phi(t)e^{-itH_1}\varphi\rangle| \\
&\quad \le \int_{t_1}^{t_2} |\langle \psi, e^{itH_2}({}_2\mathbf{D}_1\Phi(t))e^{-tH_1}\varphi\rangle| \mathrm{d}t \\
&\quad \le \sum_{j=1}^{n} \Big(\int_{t_1}^{t_2} \|B_{2j}e^{-itH_2}\psi\|^2 \mathrm{d}t\Big)^{\frac{1}{2}} \Big(\int_{t_1}^{t_2} \|B_{1j}e^{-itH_1}\varphi\|^2 \mathrm{d}t\Big)^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|e^{it_2H_2}\Phi(t_2)e^{-it_2H_1}\varphi - e^{it_1H_2}\Phi(t_1)e^{-it_1H_1}\varphi\| \\
&\quad = \sup_{\|\psi\|=1} |\langle \psi, e^{itH_2}\Phi(t)e^{-itH_1}\varphi\rangle - \langle \psi, e^{itH_2}\Phi(t)e^{-itH_1}\varphi\rangle| \\
&\quad \le \sum_{j=1}^{n} C\Big(\int_{t_1}^{t_2} \|B_{1j}e^{-itH_1}\varphi\|^2 \mathrm{d}t\Big)^{\frac{1}{2}}.
\end{aligned}$$

An argument similar to (1.4) now implies (1.5). □

7 The Propagation Estimates

In [GMR], we prove four propagation estimates. For the purpose of this section, their exact formulation is not too important, and hence we will not explain the notation $[\cdot]$ used below. What is important is the following: The matter particle position is denoted y and it has the dispersion relation Ω, the field particle position is denoted by x and it has the dispersion relation ω.

In each propagation estimate a function χ appears, it plays the role of an arbitrary energy cut-off. Likewise, on the right-hand side of each propagation estimate inequality, there is a constant factor C. This depends on the χ but is independent of the vector u. The left endpoint of the integral is irrelevant. For the intuitive understanding of the propagation estimates, the χ and the C is not so important, in fact one might just as well have written the propagation estimates in the form

$$\int_T^\infty \|B(t)e^{-itH}\psi\|^2 \frac{\mathrm{d}t}{t} < \infty,$$

which may be interpreted as the statement that the time-dependent observable $B(t)$ (or actually $\frac{B^*(t)B(t)}{t}$) goes to zero at an integrable rate as the arbitrary state ψ evolves according to the dynamics given by the Hamiltonian H.

We will now go through each of the propagation estimates from [GMR] in the order they are proved and state in words roughly what they say.

Theorem 7.1 (Large velocity estimate). *Let $\chi \in C_0^\infty(\mathbb{R})$. There exists a constant C_1 such that for $R' > R > C_1$, one has*

$$\int_1^\infty \big\| \big[\mathbb{1}_{[R,R']}\big(\tfrac{|x-y|}{t}\big)\big] e^{-itH}\chi(H)u\big\|^2 \frac{\mathrm{d}t}{t} \le C\|u\|^2.$$

The large velocity estimate states that the probability of finding a field particle with an average velocity relative to the matter particle larger than some critical value depending on the energy of the state goes to zero at an integrable rate.

Theorem 7.2 (Phase-space propagation estimate). *Take $\chi \in C_0^\infty(\mathbb{R})$ and let $0 < c_0 < c_1$. Write*

$$\Theta_{[c_0,c_1]}(t) = \big[\big\langle \tfrac{x-y}{t} - \nabla\omega(D_x) + \nabla\Omega(D_y), \mathbb{1}_{[c_0,c_1]}\big(\tfrac{|x-y|}{t}\big)\big(\tfrac{x-y}{t} - \nabla\omega(D_x) + \nabla\Omega(D_y)\big)\big\rangle\big].$$

Then

$$\int_1^\infty \|\Theta_{[c_0,c_1]}(t)^{\frac{1}{2}} e^{-itH}\chi(H)u\|^2 \frac{\mathrm{d}t}{t} \le C\|u\|^2.$$

The phase-space propagation estimate states that for any state where the field particle has an average velocity relative to the matter particle which is positive (i.e. larger than c_0 for an arbitrary $c_0 > 0$), the instantaneous velocity difference converges to the average velocity difference of the two particles at an integrable rate.

Theorem 7.3 (Improved phase-space propagation estimate).
Let $\chi \in C_0^\infty(\mathbb{R})$, $0 < c_0 < c_1$ and $J \in C_0^\infty(c_0 < |x| < c_1)$. Then for $1 \le i \le \nu$,

$$\int_1^\infty \big\| \big[\big|J\big(\tfrac{x-y}{t}\big)\big(\tfrac{x_i-y_i}{t} - \partial_i\omega(D_x) + \partial_i\Omega(D_y)\big) + \text{h. c.}\big|\big]^{\frac{1}{2}} e^{-itH}\chi(H)u\big\|^2 \frac{\mathrm{d}t}{t} \le C\|u\|^2.$$

As the name indicates, it states the same as the phase-space propagation estimates, only the rate of the convergence is improved.

Before stating the minimal velocity estimate, we note that due to the fibered representation of the Hamiltonian used in the formulation of the result, x no longer denotes the field particle position, but rather the relative position of the field particle with respect to the matter particle. We also note that the sets $\theta(P_0)$ and $\sigma_{\mathrm{pp}}(P_0)$ are the threshold set and the pure point spectrum, respectively, for the fiber Hamiltonian $H(P_0)$, which describes the dynamics for fixed total momentum P_0. Furthermore, we note that the union of the set of thresholds and the pure point spectrum of the fiber Hamiltonians is a closed subset of the energy-momentum spectrum.

Theorem 7.4 (Minimal velocity estimate). *Assume that* $(P_0, \lambda_0) \in \mathbb{R}^{\nu+1}$ *satisfies that* $\lambda_0 \in \mathbb{R} \setminus (\theta(P_0) \cup \sigma_{\mathrm{pp}}(P_0))$. *Then there exists an* $\varepsilon > 0$, *a neighbourhood* N *of* (P_0, λ_0) *and a function* $\chi \in C_0^\infty(\mathbb{R}^{\nu+1})$ *such that* $\chi = 1$ *on* N *and*

$$\int_1^\infty \Big\| [\mathbb{1}_{[0,\varepsilon]}]\big(\tfrac{|x|}{t}\big) \int^\oplus e^{-itH(P)} \chi(P, H(P)) \mathrm{d}P u \Big\|^2 \frac{\mathrm{d}t}{t} \leq C \|u\|^2.$$

The minimal velocity estimate states that away from thresholds and the pure point spectrum of the fixed momentum fibers, the probability of finding states with low average relative velocity goes to zero at an integrable rate.

The most important propagation estimates are the improved phase-space estimate and the minimal velocity estimate. We use the improved phase-space estimate in connection with Lemma 6.3 from the previous section to show the existence of an asymptotic observable, whose intuitive interpretation is that it is the projection onto the states where the matter and the field particles separate over time. The minimal velocity estimate is then used to prove that states that do not separate over time belong to the subspace of "bound states," for some suitable definition of this subspace.

To prove these two propagation estimates, the two other propagation estimates serve as ingredients; the large velocity estimate is used in the proof of the (first) phase-space propagation estimate, and the phase-space propagation estimate is used both in the proof of the improved phase-space estimate and in the proof of the minimal velocity estimate. The Mourre estimate again comes into play as another important ingredient of the minimal velocity estimate.

8 Scattering Theory

As mentioned in Section 3, scattering states are usually the states that "live" in the absolutely continuous part of the spectrum of a Hamiltonian. In [GMR], this is a bit more complicated due to the translation invariance that forces one to instead look at the energy-momentum spectrum of the operator. In our model, "bound states" are wave packets of eigenstates for the fixed momentum fiber Hamiltonians and hence lie in the absolutely continuous spectrum of the full Hamiltonian. We may thus define the space of bound states in the following way:

$$\mathcal{H}_{\mathrm{bd}} = \mathrm{Ran} \int_{\mathbb{R}^\nu}^{\oplus} \mathbb{1}_{\mathrm{pp}}(H(P))\mathrm{d}P,$$

where $H(P)$ is the fiber Hamiltonian at momentum P and $\mathbb{1}_{\mathrm{pp}}(H(P))$ is the projection onto the pure point spectrum of $H(P)$. After proving that there is no singular continuous spectrum, one is thus lead to define the space of scattering states as

$$\mathcal{H}_{\mathrm{scat}} = \mathrm{Ran} \int_{\mathbb{R}^\nu}^{\oplus} \mathbb{1}_{\mathrm{ac}}(H(P))\mathrm{d}P.$$

We would like to compare the dynamics given by the "interacting" Hamiltonian H to that given by a "free" Hamiltonian H_{free} on the scattering states. In quantum field theory, this is usually somewhat complicated by the fact that the "free" dynamics is not the same as the "non-interacting" dynamics; the dynamics of the "bound states" is still governed by the "interacting" Hamiltonian. One expects that scattering states emits bosons that asymptotically will evolve as free bosons until the remaining system reaches a "bound state." A way to handle this situation is by introducing *asymptotic fields* or asymptotic creation and annihilation operators, which are defined as the limits of the usual creation and annihilation operators in the so-called interaction picture:

$$a^{\#,+}(h) = \lim_{t\to\infty} e^{itH} a^{\#}(e^{-it\omega}h)e^{-itH},$$

where $a^{\#}(h)$ is either $a^*(h)$ or $a(h)$ and ω is the dispersion relation of the field particles. The operator $a^{*,+}(h)$ may thus be interpreted as the operator that adds an asymptotically free boson, and $a^+(h)$ as the operator that annihilates asymptotically free bosons. One can then define the space of bound states as the space of states annihilated by $a^+(h)$, i.e. the states with no asymptotically free bosons. However, $a^+(h)$ and $a(h)$ do not conserve momentum, and this complicates matters in connection with the fiber Hamiltonians. Moreover, for the Polaron model, the asymptotic creation and annihilation operators do not exist.

In [GMR], we avoid this problem completely. With at most one field particle, there are none left if one is removed, and hence the "free" and the "non-interacting" dynamics do in fact coincide, i.e. $H_{\mathrm{free}} = H_0$. The space $\mathcal{H}_{0,\mathrm{bd}}$ of bound states for H_0 is defined analogously to that of H and as H_0 has no singular continuous spectrum, the space of scattering states equals $\mathcal{H}_{0,\mathrm{bd}}^{\perp}$. One expects that each scattering state ψ^+ for H_0 correspond asymptotically to a scattering state ψ for H, i.e.

$$\|e^{-itH_0}\psi^+ - e^{-itH}\psi\| \to 0 \text{ for } t \to \infty,$$

or equivalently,

$$\lim_{t\to\infty} \|e^{itH}e^{-itH_0}\psi^+ - \psi\| = 0.$$

This leads to the following definition.

Definition 8.1 (Wave operators). Let H and H_0 be self-adjoint operators such that H_0 has no singular continuous spectrum, and let $P_{\mathrm{bd}}^{\perp}(H_0)$ denote the projection onto a subspace of the absolutely continuous spectrum of H_0 which we call the *space of scattering states* for H_0. The *wave operators*, if they exist, are the operators given by

$$W^{\pm} = \mathrm{s}\text{-}\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}P_{\mathrm{bd}}^{\perp}(H_0).$$

Asymptotic completeness in quantum mechanics is the statement that the space of states splits into a direct sum of bound and scattering states, that the wave operator exists and that all scattering states for H for large times evolve as a scattering state for the "free" dynamics, i.e.

$$\operatorname{Ran} W^{\pm} = \operatorname{Ran} P_{\mathrm{bd}}^{\perp}(H),$$

where $P_{\mathrm{bd}}^{\perp}(H)$ denotes the projection onto the space of scattering states of H.

For so-called short-range interactions, there is a connection between large times and large distances. This leads to the concept of "geometric asymptotic completeness," which roughly speaking is the statement that the states that are asymptotically comparable to "free" states are the states where the particles are far apart for large times.

To prove such a statement, one may introduce an *asymptotic observable* $P_0^{\pm}$, which projects onto the states where the particles in the distant future (or distant past, according to the sign) are far apart. Hence, geometric asymptotic completeness may be stated as

$$\operatorname{Ran} P_0^{\pm} = \operatorname{Ran} W^{\pm}.$$

Hence, if geometric asymptotic completeness is obtained, the proof of asymptotic completeness is reduced to the proof of

$$\operatorname{Ran} P_0^{\pm} = \operatorname{Ran} P_{\mathrm{bd}}^{\perp}(H).$$

A minimal velocity estimate is usually an important ingredient in the proof of this statement.

9 Bibliography

[ABG96] W. O. Amrein, A. Boutet de Monvel, and V. Georgescu, *C_0-groups, commutator methods and spectral theory of N-body hamiltonians*, Birkhäuser, 1996.

[BG92] A. Boutet de Monvel and V. Georgescu, *Spectral and scattering theory by the conjugate operator method*, Alg. and Anal. **4** (1992), 73–116.

[Dav95] E. B. Davies, *Spectral theory and differential operators*, Cambridge Studies in Advanced Mathematics, vol. 42, Cambridge University Press, Cambridge, 1995.

[DG99] J. Dereziński and C. Gérard, *Asymptotic completeness in quantum field theory. massive Pauli-Fierz Hamiltonians*, Rev. Math. Phys. **11** (1999), 383–450.

[DS99] M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, London Mathematical Society Lecture Note Series, vol. 268, Cambridge University Press, 1999.

[FGS02] J. Fröhlich, M. Griesemer, and B. Schlein, *Asymptotic completeness for Rayleigh scattering*, Ann. Henri Poincaré **3** (2002), 107–170.

[FGS08] J. Fröhlich, M. Griesemer, and I. M. Sigal, *Spectral theory for the standard model of non-relativistic QED*, Commun. Math. Phys. **283** (2008), 613–646.

[GG99] V. Georgescu and C. Gérard, *On the virial theorem in quantum mechanics*, Communications in Mathematical Physics **208** (1999), no. 2, 275–281.

[GGM04] V. Georgescu, C. Gérard, and J. S. Møller, *Commutators, C_0-semigroups and resolvent estimates*, J. Funct. Anal. **216** (2004), 303–361.

[GŁ02] C. Gérard and I. Łaba, *Multiple quantum scattering in constant magnetic fields*, Mathematical Surveys and Monographs, vol. 90, American Mathematical Society, Rhode Island, 2002.

[GMR] Christian Gérard, J. S. Møller, and M. G. Rasmussen, *Asymptotic completeness in quantum field theory: Translation invariant Nelson type models restricted to the vacuum and one-particle sectors*, Submitted.

[JMP84] A. Jensen, E. Mourre, and P. Perry, *Multiple commutator estimates and resolvent smoothness in quantum scattering theory*, Ann. Inst. Henri Poincaré **41** (1984), 207–225.

[LLP53] T. D. Lee, F. E. Low, and D. Pines, *The motion of slow electrons in a polar crystal*, Phys. Rev. **90** (1953), 297–302.

[Mou81] E. Mourre, *Absence of Singular Continuous Spectrum for Certain Self-Adjoint Operators*, Commun. Math. Phys. **78** (1981), 391–408.

[Mou83] ______, *Opérateurs conjugués et propriété de propagation*, Commun. Math. Phys. **91** (1983), 279–300.

[MR] J. S. Møller and M. G. Rasmussen, *The translation invariant massive Nelson model: II. The lower part of the essential spectrum*, In prep.

[MS04] J. S. Møller and E. Skibsted, *Spectral theory of time-periodic many-body systems*, Advances in Math. **188** (2004), 137–221.

[Møl05] J. S. Møller, *The translation invariant massive Nelson model: I. The bottom of the spectrum*, Ann. Henri Poincaré **6** (2005), 1091–1135.

[Møl06] ______, *The polaron revisited*, Rev. Math. Phys. **18** (2006), no. 5, 485–517.

[Nai59] M. A. Naimark, *Normed rings*, P. Noordhoff, Groningen, 1959.

[PSS81] P. Perry, I. M. Sigal, and B Simon, *Spectral analysis of N-body Schrödinger operators*, Ann. Math. **114** (1981), 519–567.

[Ras] M. G. Rasmussen, *A Taylor-like expansion of a commutator with a function of self-adjoint, pairwise commuting operators*, Submitted.

[RS78] M. Reed and B. Simon, *Methods of modern mathematical physics IV, Analysis of operators*, Academic Press, 1978.

[RS79] ______, *Methods of modern mathematical physics III, Scattering theory*, Academic Press, 1979.

[Sah97] J. Sahbani, *The conjugate operator method for locally regular Hamiltonians*, J. Operator Theory **38** (1997), 297–322.

[Ski98] E. Skibsted, *Spectral analysis of N-body systems coupled to a bosonic field*, Rev. Math. Phys. **10** (1998), 989–1026.

[SS87] I. M. Sigal and A. Soffer, *The N-particle scattering problem: Asymptotic completeness for short-range quantum systems*, Ann. Math. **125** (1987), 35–108.

CHAPTER 2

The Translation Invariant Massive Nelson Model: II. The Lower Part of the Essential Spectrum

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1 The Model

In this paper we study the structure of the lowest branch of continuous energy-momentum spectrum of a class of massive translation invariant models describing one quantum particle linearly coupled to a second quantized radiation field. Included in the class of models we study are the translation invariant massive Nelson model, [Nel64, Can71, Frö74, Møl05], and Fröhlich's polaron model, [Frö54, Spo04, Møl06b, AD10].

This paper is a natural continuation of [Møl05, Møl06b], where the structure of the groundstate mass shell and the bottom of the continuous energy-momentum spectrum was studied for the class of models considered here.

Before describing our results in detail, we pause to introduce the class of models we consider.

1.1 A class of translation invariant massive scalar field models

We consider a particle (from now on refered to as "the electron") moving in $\mathbb{R}^\nu$ linearly coupled to a scalar field of massive field particles ("photons"). Note that the terms electron and photon are somewhat arbitrary, replaceable with e.g. the terms "particle" and "phonon". The electron Hilbert space is

$$\mathcal{K} := L^2(\mathbb{R}^\nu_x)$$

where x is the electron position. The free electron Hamiltonian is $\Omega(p)$, where $p := -i\nabla$. We will later impose some conditions on the electron dispersion relation Ω, see Condition 1.1.

The photon Hilbert space is

$$\mathfrak{h}_{\mathrm{ph}} := L^2(\mathbb{R}^\nu_k)$$

where k is the photon momentum, and the one-photon dispersion relation is $\omega(k)$. See Condition 1.2 for the conditions imposed on ω.

The Hilbert space for the field is the bosonic Fock space

$$\begin{aligned} \mathcal{F} = \Gamma(\mathfrak{h}_{\mathrm{ph}}) &:= \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}, \quad \text{where} \\ \mathcal{F}^{(n)} = \Gamma^{(n)}(\mathfrak{h}_{\mathrm{ph}}) &:= \mathfrak{h}_{\mathrm{ph}}{}^{\otimes_s n}. \end{aligned}$$

Here $\mathfrak{h}_{\mathrm{ph}}{}^{\otimes_s n}$ is the symmetric tensor product of n copies of $\mathfrak{h}_{\mathrm{ph}}$. We denote the vacuum by $\Omega = (1, 0, 0, \dots)$. The creation and annihilation operators $a^*(k)$ and $a(k)$ satisfy the following distributional form identities, known as the canonical commutation relations.

$$\begin{aligned} [a^*(k), a^*(k')] &= [a(k), a(k')] = 0, \\ [a(k), a^*(k')] &= \delta(k - k') \quad \text{and} \\ a(k)\Omega &= 0. \end{aligned}$$

The free photon energy is the second quantization of the one-photon dispersion relation,

$$d\Gamma(\omega) = \int_{\mathbb{R}^\nu} \omega(k) a^*(k) a(k)\, dk.$$

The Hilbert space of the combined system is

$$\mathcal{H} := \mathcal{K} \otimes \mathcal{F},$$

on which we make the following identification.

$$\mathcal{H} := L^2(\mathbb{R}^\nu; \mathcal{F}).$$

The free and coupled Hamiltonians for the combined system are

$$\begin{aligned} H_0 &:= \Omega(p) \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{K}} \otimes d\Gamma(\omega) \quad \text{and} \\ H &:= H_0 + V \end{aligned}$$

where the interaction V is given by

$$V := \int_{\mathbb{R}^\nu} \big(e^{-ik\cdot x}\, v(k)\, \mathbb{1}_{\mathcal{K}} \otimes a^*(k) + e^{ik\cdot x}\, \overline{v(k)}\, \mathbb{1}_{\mathcal{K}} \otimes a(k)\big)\, dk$$

where $v \in \mathfrak{h}_{\text{ph}} = L^2(\mathbb{R}^\nu)$ is a real-valued coupling function. A natural choice for v would be $v(k) = \chi(k)/\sqrt{\omega(k)}$, where χ is an ultra-violet cutoff function which insures the $v \in L^2(\mathbb{R}^\nu)$ requirement. We hope to be able to remove this cutoff in future work.

The total momentum of the combined system is given by

$$P = -i\nabla \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{K}} \otimes d\Gamma(k).$$

The operators H_0 and H commutes with P, i.e. H_0 and H are translation invariant. This implies that H_0 and H are fibered operators. Using the unitary transform I_{LLP} first introduced in [LLP53] and given by

$$I_{\text{LLP}} := (F \otimes \mathbb{1}_{\mathcal{F}}) \circ \Gamma(e^{-ik\cdot x})$$

we can identify the fibers of H_0 respectively H. Here F is the Fourier transform. We get

$$\begin{aligned} I_{\text{LLP}}\, H_0\, I_{\text{LLP}}^* &= \int_{\mathbb{R}^\nu}^{\oplus} H_0(\xi)\, d\xi \quad \text{and} \\ I_{\text{LLP}}\, H\, I_{\text{LLP}}^* &= \int_{\mathbb{R}^\nu}^{\oplus} H(\xi)\, d\xi, \end{aligned}$$

where $H_0(\xi)$ and $H(\xi)$ are operators on $\mathcal{F}$ and given by

$$\begin{aligned} H_0(\xi) &= d\Gamma(\omega) + \Omega(\xi - d\Gamma(k)) \quad \text{and} \\ H(\xi) &= H_0(\xi) + \Phi(v). \end{aligned}$$

Here $\Phi(v)$ is the field operator given by

$$\Phi(v) = \int_{\mathbb{R}^\nu} \big(v(k)\, a^*(k) + \overline{v(k)}\, a(k)\big)\, dk.$$

See also [RS75] and [DG99] for general constructions related to bosonic Fock space.

The set $\{(\xi,\lambda)|\lambda \in \sigma(H(\xi))\}$ is called the energy-momentum spectrum of the Hamiltonian H. In Figure 2.1 such a spectrum is depicted. The grey region is the continuous part of the energy-momentum spectrum, and the black solid curve is the ground state mass shell. This part of the depicted spectrum is in fact that of the free Hamiltonian H_0, with $\Omega(\eta) = \eta^2/2$ and $\omega(k) = \sqrt{k^2+1}$. The black dotted curve that extends the ground state mass shell into the continuous spectrum is an embedded eigenvalue for the uncoupled model. It is expected to disappear when the coupling is turned on, which has been established if $\nu \geq 3$ for a class of models including the polaron model in [AMZ05], at least in the region between the two lowest solid red curves representing the lowest 1- and 2-body thresholds (see subsection 1.3).

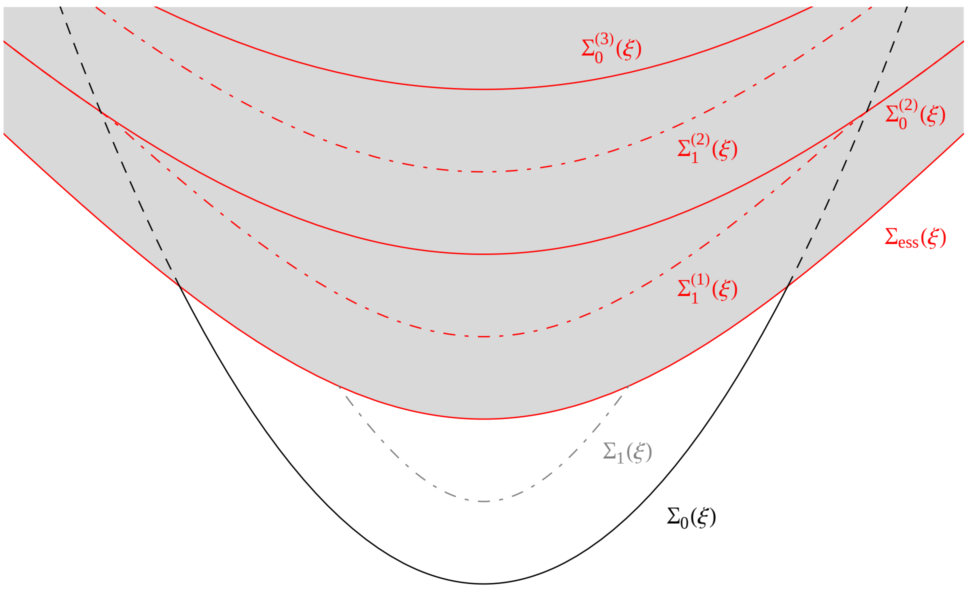

Figure 2.1: The lower branches of the energy-momentum spectrum.

It is a curious fact that in dimension 1 and 2, the coupled system will have an isolated ground state mass shell for all k (if v is nowhere vanishing). This is orginally due to Spohn, see [Spo04, Møl06b].

If the coupling constant is small and $\nu \geq 3$, one can show, again for a class of models including the polaron model, that there are no excited mass shells below the lowest 2-body threshold. See [AMZ05]. At large coupling one cannot rule out the existence of excited isolated mass shells (or even embedded ones). Such exited mass shells will also give rise to thresholds. An exited mass shell and associated 1- and 2-body thresholds are depicted in Figure 2.1 as the dash-dotted curves. Based on the work of [AMZ05] one is lead to conjecture that in general there should at most be finitely many excited isolated mass shells.

In the present paper we study the structure of the lowest part of the continuous energy-momentum spectrum, that is the region lying between the bottom of the continuous spectrum and the first of the drawn red curves. I.e. the first 2-body threshold, if there no excited isolated mass shells, or the 1-body threshold pertaining to the first isolated excited mass shell if it exists.

In order to explain the significance of this choice of region, we discuss briefly in scattering terms what the structure of the associated spectral subspace should be. Suppose we start out at total momentum ξ and energy λ, with (ξ, λ) in the region just discussed in the previous paragraph. Let ψ be a state localized in momentum and energy at (ξ, λ). Then ψ can be decomposed into a linear combination of possible bound states, corresponding to embedded eigenvalues at total momentum ξ, and a scattering state that should emit a field particle at momentum k, leaving behind a bound interacting state at momentum $\xi - k$, with energy $\Sigma_0(\xi - k)$ – the ground state energy at momentum $\xi - k$. Due to energy conservation we must have $\lambda = \Sigma_0(\xi - k) + \omega(k)$. conversely, any such compound state, emitted field particle and interacting bound state, satisfying energy and momentum conservation, should be attained by some scattering state. This description is in fact that of *asymptotic completeness*.

If we had started between the red dash-dotted curve and the 2-body threshold coming from the ground state mass shell, the second solid red curve, the scattering process is more complicated. Here there are two available channels. In either case the state emits 1 field-particle, but it now has two bound states available to the remaining interacting system. Either the ground state, or the first excited mass shell. The further one moves up into the continuous spectrum, the more scattering channels become available. Including the emission of multiple field particles if one starts above the first 2-body threshold.

The significance of the thresholds is also best explained in a dynamical picture. If we had started on a threshold, then there would be momenta such that if field particles were emitted at these momenta, then the remaining interacting state, which travels with an effective dispersion relation equal to the mass shell it ends up at, would not separate from the emitted field particles. I.e., the emitted particles cannot be treated as uncoupled from the remaining bound state.

In Figure 2.1 we have chosen to depict a situation where the mass shells are convex. Unfortunately it is a very hard question, outside of the weak coupling regime, to determine if a mass shell is convex or even monotonically increasing away from $\xi = 0$. If a mass shell is not convex, then this will potentially give rise to extra thresholds falling in between the thresholds depicted in Figure 2.1. Our methods however are capable of dealing with such additional thresholds, so we are not required to make implicit assumptions on the structure of mass shells. In addition, our method is not sensitive to the possible existence of embedded mass shells in the region considered. In fact, we can prove that at fixed momentum, embedded eigenvalues are locally finite away from thresholds, with only possible accumulation points at thresholds.

We make use of the fact that in the region considered there is only one available scattering channel to construct an operator A_ξ conjugate to $H(\xi)$ in the sense of Mourre. This enables us to deduce information about the structure of the continuous spectrum, such as absence of singular continuous spectrum. If $\Omega(\eta) = \eta^2/2$ (or a multiple thereof), we can combine with recent results of [FMSa, FMSb, MW] to conclude that the embedded eigenvalues together with the threshold set, in the region considered, form a closed subset of energy-momentum space, with the property that at fixed total momentum this set becomes at most countable. The precise formulation of the main results are contained in Theorem 3.6 and Corollary 3.7, see pages 60–61.

We remark that this model has an interesting technical feature. If $\Omega(\eta) = \eta^2/2$, the fiber Hamiltonians are of class $C^2(A_\xi)$, but not of class $C^3(A_\xi)$, see [ABG96] or subsection 1.6. In fact, neither the domain nor the form domain of the Hamiltonian is invariant under the unitary group generated by A_ξ. See Remark 1.23. That is, we need the full force of the Amrein-Boutet de Monvel-Georgescu extension of Mourre's commutator method [ABG96, Gér08, GJ06].

1.2 Conditions on Ω, ω and v

We will need a combination of the following conditions.

Condition 1.1 (Electron dispersion relation). Let $\Omega \in C^\infty(\mathbb{R}^\nu)$ be a non-negative function. There exists an $s_\Omega \in [0,2]$ such that Ω satisfies:

(i) There exists $c > 0$ such that $\Omega(\eta) \geq c^{-1}\langle\eta\rangle^{s_\Omega} - c$.

(ii) For any multi-index α there exists a positive constant $c_\alpha > 0$ such that $|\partial^\alpha \Omega(\eta)| \leq c_\alpha \langle\eta\rangle^{s_\Omega - |\alpha|}$.

(iii) Rotation invariance of Ω, i.e. $\Omega(O\xi) = \Omega(\xi)$ for all $\xi \in \mathbb{R}^\nu$ and $O \in O(\nu)$.

(iv) Analyticity of Ω, i.e. Ω is real analytic.

Note that the standard non-relativistic and relativistic choices $\Omega(\eta) = \frac{\eta^2}{2M}$ and $\Omega(\eta) = \sqrt{\eta^2 + M^2}$ satisfies Condition 1.1 with $s_\Omega = 2$ and $s_\Omega = 1$, respectively.

Condition 1.2 (Photon dispersion relation). Let $\omega \in C^\infty(\mathbb{R}^\nu)$ satisfy

(i) There exists a positive constant $m > 0$, which we call the photon mass, such that $\inf_{k\in\mathbb{R}^\nu} \omega(k) = m$.

(ii) ω is strictly subadditive, $\omega(k_1 + k_2) < \omega(k_1) + \omega(k_2)$.

(iii) Rotation invariance of ω, i.e. $\omega(O\xi) = \omega(\xi)$ for all $\xi \in \mathbb{R}^\nu$ and $O \in O(\nu)$.

(iv) Analyticity of ω, i.e. ω is real analytic.

(v) For any multi-index α with $|\alpha| \geq 1$, we have $\sup_{k\in\mathbb{R}^\nu} |\partial^\alpha \omega(k)| < \infty$.

(vi) There exists $c > 0$ such that $|k||\nabla\omega(k)| \leq c\omega(k)$.

(vii) There exists $c > 0$ such that $|k|^2 \|\nabla^2\omega(k)\| \leq c\omega(k)$.

Condition 1.2 is e.g. satisfied for $\omega(k) = \sqrt{k^2 + m^2}$ and $\omega(k) = \omega_0 > 0$.

Condition 1.3 (Coupling function). Let v have 2 distributional derivatives and satisfy

(i) We have that $v \in L^2(\mathbb{R}^\nu)$.

(ii) We have that $\langle\cdot\rangle|\nabla v|, \partial_j v \in L^2(\mathbb{R}^\nu)$, for $1 \leq j \leq \nu$.

(iii) Rotation invariance of v, i.e. $v(O\xi) = v(\xi)$ for a.e. $\xi \in \mathbb{R}^\nu$ and $O \in O(\nu)$.

(iv) We have $\langle\cdot\rangle\|\nabla^2 v\| \in L^2(\mathbb{R}^\nu)$.

Condition 1.4 (Dispersion relation behavior at infinity). The dispersion relations Ω and ω satisfy one of the following conditions.

(i) The photon dispersion relation satisfies $\lim_{|k|\to\infty} \omega(k) = \infty$.

(ii) The dispersion relations satisfy $\sup_k \omega(k) < \infty$ and $\lim_{|\eta|\to\infty} \Omega(\eta) = \infty$.

We note that any combination of Ω and ω as in one the examples above satisfies Condition 1.4, i.e. we are able to cover the Fröhlich Hamiltonian, $\Omega(\eta) = \eta^2/(2M_{\mathrm{eff}})$ and $\omega(k) = h\omega_0 > 0$, with a (sufficiently smooth) ultraviolet cutoff in the coupling function v.

1.3 Some preliminaries

In this subsection we recall some known results and establishes some notation used throughout the paper. Apart from a lemma about the structure of the thresholds, the results are all from [Møl05] or [Møl06b]. We also need a corollary to a result from [Møl06b] and lemma, which is an easy consequence of this corollary.

Proposition 1.5. *Assume Conditions 1.1*(i), (ii)*, 1.2*(i) *and 1.3*(i)*. Then*

(i) The operator $H_0(\xi)$ is essentially self-adjoint on $\mathcal{C}_0^\infty := \Gamma_{\mathrm{fin}}(C_0^\infty(\mathbb{R}^\nu))$.

(ii) The domain $\mathcal{D} := \mathcal{D}(H_0(\xi))$ is independent of ξ.

(iii) The field operator $\Phi(v)$ is $H_0(\xi)$-bounded with relative bound 0*. In particular $H(\xi)$ is bounded from below, self-adjoint on $\mathcal{D}$ and essentially self-adjoint on $\mathcal{C}_0^\infty$.*

(iv) The bottom of the spectrum of the fiber Hamiltonians,

$$\xi \mapsto \inf \sigma(H(\xi)),$$

is Lipschitz continuous.

The proof, which uses the identity $H_0(\xi) - H_0(0) = \xi \cdot \int_0^1 \nabla\Omega(t\xi - d\Gamma(k))\, dt$, the $H_0(\xi)^{\frac{1}{2}}$-boundedness of $N^{\frac{1}{2}}$, where $N = d\Gamma(\mathbb{1}_{\mathcal{F}})$ is the number operator, a standard estimate on creation and annihilation operators and the Kato-Rellich theorem twice, can be found in [Møl05, Chapter 3].

Proposition 1.5 also holds with the pair $(H_0(\xi), H(\xi))$ replaced by either of the pairs $(H_0^{\mathrm{ext}}(\xi), H^{\mathrm{ext}}(\xi))$ or $(H_0^{(\ell)}(\xi), H^{(\ell)}(\xi))$, where $H_0^{\mathrm{ext}}(\xi)$, $H^{\mathrm{ext}}(\xi)$, $H_0^{(\ell)}(\xi)$ and $H^{(\ell)}(\xi))$ are the operators defined i subsection 1.4 respectively subsection 1.5.

We now introduce some notation. We denote the bottom of the spectrum of the fiber Hamiltonians

$$\Sigma_0(\xi) := \inf \sigma(H(\xi)).$$

The bottom of the spectrum of the full operator:

$$\Sigma_0 := \inf_{\xi\in\mathbb{R}^\nu} \Sigma_0(\xi) > -\infty$$

where the inequality follows from Proposition 1.5. Let $n \in \mathbb{N}$ be some positive integer and $k = (k_1, \ldots, k_n) \in \mathbb{R}^{n\nu}$. We introduce the bottom of the spectrum of a composite system consisting of a copy of an interacting system at momentum $\xi - \sum_{j=1}^n k_j$ and n non-interacting photons with momenta k.

$$\Sigma_0^{(n)}(\xi; k) := \Sigma_0(\xi - \textstyle\sum_{j=1}^n k_j) + \sum_{j=1}^n \omega(k_j).$$

The following functions are so-called thresholds. We need them to outline the region in which our Mourre estimate is valid.

$$\Sigma_0^{(n)}(\xi) := \inf_{k\in\mathbb{R}^{n\nu}} \Sigma_0^{(n)}(\xi; k).$$

$\Sigma_0^{(n)}(\xi)$ is the first n-particle threshold. If ω and $\Sigma_0(\cdot)$ are convex, this is in fact the only n-particle threshold pertaining to the ground state mass shell. It turns out that the bottom of the essential spectrum can be expressed in terms of these threshold functions. More precisely we have

$$\Sigma_{\mathrm{ess}}(\xi) := \inf_{n\geq 1} \Sigma_0^{(n)}(\xi), \tag{2.1}$$

see Theorem 1.7 below. If ω satisfies Condition 1.2(ii), then $\Sigma_0^{(n)}(\xi) \geq \Sigma_0^{(n')}(\xi)$ when $n > n'$, see also Proposition 1.11. Hence (2.1) reduces to $\Sigma_{\mathrm{ess}}(\xi) := \Sigma_0^{(1)}(\xi)$.

Let $\mathcal{I}_0 := \{\eta \in \mathbb{R}^\nu | \Sigma_0(\eta) < \Sigma_{\mathrm{ess}}(\eta)\}$, i.e. $\mathcal{I}_0$ is the region of momenta of the interacting system where the bottom of the spectrum of the fiber Hamiltonians are isolated eigenvalues. For $\xi \in \mathbb{R}^\nu$ and $n \in \mathbb{N}$ we define

$$\mathcal{I}_0^{(n)}(\xi) := \{k \in \mathbb{R}^{n\nu} \,|\, \xi - \textstyle\sum_{j=1}^n k_j \in \mathcal{I}_0\}. \tag{2.2}$$

For $0 \leq p < \infty$, we let $\Sigma_p(\xi) \leq \Sigma_{\mathrm{ess}}(\xi)$ denote the p'th isolated eigenvalue of $H(\xi)$ below the essential spectrum, counted from 0 and without multiplicity and with the convention that $\Sigma_p(\xi) = \Sigma_{\mathrm{ess}}(\xi)$ if there are less than $p+1$ isolated eigenvalues at total momentum ξ. Note that if $\Sigma_p(\xi) = \Sigma_{\mathrm{ess}}(\xi)$, it is not necessarily an eigenvalue. In fact, if $4 \geq \nu \geq 3$ and $\xi \notin \mathcal{I}_0$, then $\Sigma_0(\xi)$ is not an eigenvalue, see [Møl06b]. Let

$$p_{\max} := \sup\{p + 1 \in \mathbb{N} \,|\, \exists \xi \in \mathbb{R}^\nu \colon \Sigma_p(\xi) < \Sigma_{\mathrm{ess}}(\xi)\},$$

then for $p = 0$ we have the bottom of the spectrum, for $p \geq 1$ we have excited states and $p_{\max}$ counts the number of mass shells.

Let $n \in \mathbb{N}$, $k = (k_1, \ldots, k_n) \in \mathbb{R}^{n\nu}$ and $p < p_{\max}$. As for the ground state, we introduce excited states of a composite system consisting of a copy of an interacting system at momentum $\xi - \sum_{j=1}^n k_j$ and n non-interacting photons with momenta k.

$$\Sigma_p^{(n)}(\xi; k) := \Sigma_p(\xi - \textstyle\sum_{j=1}^n k_j) + \sum_{j=1}^n \omega(k_j).$$

We also define the corresponding

$$\Sigma_p^{(n)}(\xi) := \inf_{k \in \mathbb{R}^{n\nu}} \Sigma_p^{(n)}(\xi; k),$$
$$\mathcal{I}_p := \{\eta \in \mathbb{R}^\nu | \Sigma_p(\eta) < \Sigma_{\mathrm{ess}}(\eta)\}$$

and

$$\mathcal{I}_p^{(n)}(\xi) := \{k \in \mathbb{R}^{n\nu} \mid \xi - \textstyle\sum_{j=1}^n k_j \in \mathcal{I}_p\}$$

for $0 < p < p_{\max}$. The $\Sigma_p^{(n)}(\xi)$ are the first n-particle thresholds for the mass shell Σ_p. If ω and $\Sigma_p(\cdot)$ are convex, they are the only ones.

We need the following lemma about the structure of the thresholds.

Lemma 1.6. *Assume Conditions 1.1*(i), (ii)*, 1.2*(i), (ii) *and 1.3*(i)*, $n \geq 1$, $\xi \in \mathbb{R}^\nu$, $0 \leq p < p_{\max}$ and $k \in \mathbb{R}^{n\nu}$. If $\Sigma_p^{(n)}(\xi; k) < \Sigma_0^{(n+1)}(\xi)$, then $k \in \mathcal{I}_p^{(n)}(\xi)$.*

Proof. Assume $k \notin \mathcal{I}_p^{(n)}(\xi)$. Then $\Sigma_p(\xi - \sum_{i=1}^n k_i) = \Sigma_{\mathrm{ess}}(\xi - \sum_{i=1}^n k_i)$. But $\Sigma_{\mathrm{ess}}(\eta) = \inf_{k \in \mathbb{R}^\nu} \Sigma_0(\eta; k)$. Let $2\varepsilon = \Sigma_0^{(n+1)}(\xi) - \Sigma_p^{(n)}(\xi; k) > 0$. Choose k_ε such that $\Sigma_{\mathrm{ess}}(\xi - \sum_{i=1}^n k_i) + \varepsilon > \Sigma_0(\xi - \sum_{i=1}^n k_i; k_\varepsilon)$. Then

$$\Sigma_p^{(n)}(\xi; k) + \varepsilon > \Sigma_0(\xi - \sum_{i=1}^n k_i - k_\varepsilon) + \sum_{i=1}^n \omega(k_i) + \omega(k_\varepsilon) \geq \Sigma_0^{(n+1)}(\xi),$$

which is a contradiction. □

The following HVZ-type theorem on the structure of the spectrum of $H(\xi)$ is crucial for our arguments in the proof of the virial-like theorem and the Mourre estimate.

Theorem 1.7. *Assume Conditions 1.1*(i), (ii)*, 1.2*(i) *and 1.3*(i)*. Then*

(i) The spectrum of $H(\xi)$ below $\Sigma_{\mathrm{ess}}(\xi)$ consists at most of eigenvalues of finite multiplicity, with $\Sigma_{\mathrm{ess}}(\xi)$ as the only possible accumulation point.

(ii) If Condition 1.4 is also satisfied, then $\sigma_{\mathrm{ess}}(H(\xi)) = [\Sigma_{\mathrm{ess}}(\xi), \infty)$.

For a proof, we refer the reader to [Møl06b].

The following theorem, [Møl06b, Theorem 2.3], has two important consequences, Theorem 1.9 and Proposition 1.12.

Theorem 1.8. *Let $\xi \in \mathbb{R}^\nu$. Assume Conditions 1.1*(i), (ii), *1.2*(i) *and 1.3*(i). *If either $\xi \in \mathcal{I}_0$ or $\Sigma_0(\xi)$ is an eigenvalue of $H(\xi)$ and $v \neq 0$ a.e. then $\Sigma_0(\xi)$ is non-degenerate.*

Theorem 1.9. *Assume Conditions 1.1*(i), (ii), *1.2*(i), *1.3*(i) *and 1.4*(ii). *We have the limit*

$$\lim_{|\xi|\to\infty} \Sigma_{\text{ess}}(\xi) - \Sigma_0(\xi) = 0.$$

This theorem is a slightly simplified version of [Møl06b, Theorem 2.4]. The simplification comes from the fact that

$$\Sigma_{\text{ess}}(\xi) = \inf_{k\in\mathbb{R}^\nu} (\Sigma_0(\xi - k) + \omega(k)) \leq \inf_{k\in\mathbb{R}^\nu} \Sigma_0(\xi - k) + \sup_{k\in\mathbb{R}^\nu} \omega(k)$$

which is independent of ξ and bounded under Condition 1.4(ii), and hence $\Sigma_0(\cdot)$ is bounded.

Corollary 1.10. *Assume Conditions 1.1*(i), (ii), *1.2*(i), (ii), *1.3*(i) *and 1.4. Then*

$$\limsup_{|k|\to\infty} \Sigma_0^{(2)}(\xi) - \Sigma_0^{(1)}(\xi; k) \leq 0.$$

Proof. Condition 1.4 implies that either $\lim_{|k|\to\infty} \omega(k) = \infty$, in which case the result is trivial, or, Condition 1.4(ii) is satisfied, in which case Theorem 1.9 applies. We see that

$$\begin{aligned}
&\Sigma_0^{(2)}(\xi) - \Sigma_0^{(1)}(\xi; k) \\
&\quad \leq \inf_{k'\in\mathbb{R}^\nu} \Sigma_0(\xi - k' - k) + \omega(k') + \omega(k) - (\Sigma_0(\xi - k) + \omega(k)) \\
&\quad = \Sigma_0^{(1)}(\xi - k) - \Sigma_0(\xi - k) \\
&\quad = \Sigma_{\text{ess}}(\xi - k) - \Sigma_0(\xi - k) \to 0,
\end{aligned}$$

which proves the corollary. □

The following proposition ensures that our main result is not an empty statement.

Proposition 1.11. *Assume Conditions 1.1*(i), (ii), *1.2*(i), (ii), *1.3*(i), *1.4 with the addition that if* $\sup_k \omega(k) < \infty$ *then*

$$2 \liminf_{|k|\to\infty} \omega(k) > \sup_k \omega(k).$$

Then, for any $\xi \in \mathbb{R}^\nu$ *and* $n \geq 1$, *we have* $\Sigma_0^{(n)}(\xi) < \Sigma_0^{(n+1)}(\xi)$.

Again, the proof is found in [Møl06b].

Assume Conditions 1.1(iii), 1.2(iii) and 1.3(iii), i.e. Ω, ω and v are rotation invariant. Then clearly $\Sigma_0(\xi)$ is rotation invariant and all information can be obtained by the function $\sigma\colon \mathbb{R} \to \mathbb{R}$ defined by $\sigma(t) := \Sigma_0(tu)$, where u is an arbitrary unit vector in $\mathbb{R}^\nu$. We have the following result on σ, which follows from Theorem 1.8.

Proposition 1.12. *Assume Conditions 1.1, 1.2*(i), (iii) *1.3*(i) *and* (iii). *Then the map* $t \mapsto \sigma(t)$ *is real analytic on* $\{t \in \mathbb{R} \,|, tu \in \mathcal{I}_0\}$.

The proof goes back to [Frö73].

The following threshold set is needed in our argument to make sure that in the proof of the Mourre estimate, Theorem 3.5, we get something positive from the virial-like theorem, Theorem 3.1.

$$\mathcal{T}_0^{(1)}(\xi) := \{\lambda \in \mathbb{R} \,|\, \exists k \in \mathcal{I}_0^{(1)}(\xi)\colon \Sigma_0^{(1)}(\xi;k) = \lambda \text{ and } \nabla_k \Sigma_0^{(1)}(\xi;k) = 0\}.$$

Note that $\Sigma_0^{(1)}(\xi) \in \mathcal{T}_0^{(1)}(\xi)$ is a lower bound. We are now ready to prove the following lemma.

Lemma 1.13. *Assume Conditions 1.1, 1.2*(i), (ii), (iii), (iv), *1.3*(i), (iii) *and 1.4. Then* $\mathcal{T}_0^{(1)}(\xi) \cap [\Sigma_0^{(1)}(\xi), \Sigma_0^{(2)}(\xi))$ *is at most countable with* $\Sigma_0^{(2)}(\xi)$ *the only possible accumulation point.*

Proof. Let $\xi \in \mathbb{R}^\nu$. Assume first that $\xi \neq 0$. If $\nabla_k \Sigma_0^{(1)}(\xi;k) = 0$ and $\nabla\omega(k) \neq 0$ for some k, then it follows from the rotation invariance that $k = \theta\xi$ for some $\theta \in \mathbb{R}$, see [Møl06a, Lemma 3.2]. By analyticity $-\nabla\Sigma_0^{(1)}(\xi - \theta\xi) + \nabla\omega(\theta\xi) = 0$ can only be true for countably many θ and hence countably many k, with the possible accumulation points at the boundary of $\mathcal{I}_0^{(1)}(\xi)$. If $k_n \to k \in \partial\mathcal{I}_0^{(1)}(\xi)$, then it follows that $\Sigma_0^{(1)}(\xi;k_n) = \Sigma_0(\xi - k_n) + \omega(k_n) \to \Sigma_0^{(1)}(\xi - k) + \omega(k) \geq \Sigma_0^{(2)}(\xi)$. Hence the set

$$\{\lambda \in \mathbb{R} \,|\, \exists k \in \mathcal{I}_0^{(1)}(\xi)\colon \nabla_k \Sigma_0^{(1)}(\xi;k) = 0,\ \nabla\omega(k) \neq 0 \text{ and } \Sigma_0^{(1)}(\xi;k) = \lambda\}$$

is countable with the only possible accumulation points being greater than or equal to $\Sigma_0^{(2)}(\xi)$. If $\nabla\omega(k) = 0$, then by rotation invariance, $\nabla\omega(tu) = 0$, $t = |k|$, $u \in \mathbb{R}^\nu$ any unit vector. By analyticity this can only be true for locally finitely many $\{t_n\}$. Clearly, if there are infinitely many t_n's, then $t_n \to \infty$, so Corollary 1.10 implies that the set

$$\{\lambda \in \mathbb{R} \mid \exists k \in \mathcal{I}_0^{(1)}(\xi)\colon \nabla\omega(k) = 0 \text{ and } \Sigma_0^{(1)}(\xi; k) = \lambda\}$$

is countable with all possible accumulation points greater than or equal to $\Sigma_0^{(2)}$. Since $\mathcal{T}_0^{(1)}$ is contained in the union of these two sets we are done. The case $\xi = 0$ can be handled by similar but easier arguments. □

1.4 The extended space and a partition of unity

We introduce introduce operators $\check{\Gamma}(b)\colon \mathcal{F} \to \mathcal{F}^{\text{ext}}$, where the extended space $\mathcal{F}^{\text{ext}}$ is the Hilbert space defined by

$$\mathcal{F}^{\text{ext}} := \mathcal{F} \otimes \mathcal{F}$$

and $b = (b_0, b_\infty)$ with $b_0, b_\infty \in \mathcal{B}(\mathfrak{h}_{\text{ph}})$ and

$$b_0^* b_0 + b_\infty^* b_\infty = \mathbb{1}_{\mathfrak{h}_{\text{ph}}}. \tag{2.3}$$

We identify b with the bounded operator

$$\begin{aligned} b\colon \mathfrak{h}_{\text{ph}} &\to \mathfrak{h}_{\text{ph}} \oplus \mathfrak{h}_{\text{ph}}, \\ b\psi &= (b_0\psi, b_\infty\psi). \end{aligned}$$

It is easy to see that $b^*\colon \mathfrak{h}_{\text{ph}} \oplus \mathfrak{h}_{\text{ph}} \to \mathfrak{h}_{\text{ph}}$ is given by $b^*(\psi, \varphi) = b_0^*\psi + b_\infty^*\varphi$. Hence $b^*b = b_0^*b_0 + b_\infty^*b_\infty$ and (2.3) implies that $\|b\| = 1$.

Define $U\colon \Gamma(\mathfrak{h}_{\text{ph}} \oplus \mathfrak{h}_{\text{ph}}) \to \Gamma(\mathfrak{h}_{\text{ph}}) \otimes \Gamma(\mathfrak{h}_{\text{ph}}) = \mathcal{F}^{\text{ext}}$ by

$$\begin{aligned} U\Omega &= \Omega \otimes \Omega, \\ U(a^*(f,g)) &= (a^*(f) \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{F}} \otimes a^*(g))U \end{aligned}$$

and linearity. Since vectors of the form $a^*(f_1)\cdots a^*(f_n)\Omega$ form a total set in $\mathcal{F}$ and since U preserves the canonical commutation relations, we see that U extends uniquely to a unitary operator, which we also call U. Let b be as before. Then it is easy to check that

$$Ud\Gamma(b) = (d\Gamma(b_0) \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{F}} \otimes d\Gamma(b_\infty))U$$

as an identity on $\Gamma_{\text{fin}}(\mathfrak{h}_{\text{ph}} \oplus \mathfrak{h}_{\text{ph}})$. Define $\check{\Gamma}(b)$ by

$$\check{\Gamma}\colon \mathcal{F} \to \mathcal{F}^{\text{ext}},$$
$$\check{\Gamma}(b) = U\Gamma(b).$$

Note that (2.3) implies that $\check{\Gamma}(b)$ is an isometry:

$$\check{\Gamma}(b)^*\check{\Gamma}(b) = \mathbb{1}_{\mathcal{F}}.$$

We will interpret $\check{\Gamma}(b)$ as a partition of unity. We note that our $\check{\Gamma}$ is a special case of a more general construction, see e.g. [Møl05].

We will use two different choices for b. One will be the family $j^R, R > 1$ given by

$$j^R = (j_0^R, j_\infty^R) := (j_0(x/R), j_\infty(x/R)),$$

where $x = i\nabla_k$ and $j_0, j_\infty \in C^\infty(\mathbb{R}^\nu)$ are real and non-negative and satisfies that $j_0(k) = 1$ for $|k| \le 1$, $j_0(k) = 0$ for $|k| > 2$ and $j_0^2 + j_\infty^2 = 1$. By the last condition, (2.3) is satisfied for $b = j^R$. The other choice will be the family $J^r = (\chi_{\{|k|<r\}}, \chi_{\{|k|\ge r\}})$, $0 \le r \le \infty$, where χ_A is the characteristic function of the set A. One should think of $\check{\Gamma}(j^R)$ as a decomposition of a state in $\mathcal{F}$ into two parts, one containing the photons near the electron, and one containing photons near infinity. Intuitively, photons near infinity should be more or less non-interacting. Under certain conditions, this is true in a very precise sense, see Corollary 3.3.

If we let

$$\Phi_r(v) = \int_{|k|<r} \big(v(k)a^*(k) + \overline{v(k)}a(k)\big)dk$$

and define $H_r(\xi) = H_0(\xi) + \Phi_r(v)$ for $0 \le r \le \infty$, then clearly $H_r(\xi)$ is well-defined for $r = 0$ and $H_\infty(\xi) = H(\xi)$.

We now introduce some operators on $\mathcal{F}^{\text{ext}}$. If a is an essentially self-adjoint operator on $\mathfrak{h}_{\text{ph}}$ with domain $\mathcal{D}(a)$, then

$$d\Gamma^{\text{ext}}(a) = d\Gamma(a) \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{F}} \otimes d\Gamma(a)$$

defines an operator on $\mathcal{F}^{\text{ext}}$ with domain $\mathcal{D}(a) \otimes \mathcal{D}(a)$. If a is essentially self-adjoint, so is $d\Gamma^{\text{ext}}(a)$, and the self-adjoint extension will also be denoted by $d\Gamma^{\text{ext}}(a)$. In particular, we have for $a = \mathbb{1}_{\mathfrak{h}_{\text{ph}}}$

$$N^{\text{ext}} := d\Gamma^{\text{ext}}(\mathbb{1}_{\mathfrak{h}_{\text{ph}}}) = d\Gamma(\mathbb{1}_{\mathfrak{h}_{\text{ph}}}) \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{F}} \otimes d\Gamma(\mathbb{1}_{\mathfrak{h}_{\text{ph}}}).$$

The extended free Hamiltonian is given by

$$H_0^{\text{ext}}(\xi) := d\Gamma^{\text{ext}}(\omega) + \Omega(\xi - d\Gamma^{\text{ext}}(k))$$

and is essentially self-adjoint on $\mathcal{C}_0^\infty \otimes \mathcal{C}_0^\infty$. As for $\mathcal{D}$, $\mathcal{D}^{\text{ext}} := \mathcal{D}(H_0^{\text{ext}}(\xi))$ is independent of ξ by Proposition 1.5.

The extended Hamiltonian, which in the spirit of the previous discussion treats photons in the first part of $\mathcal{F}^{\text{ext}}$ as interacting and photons in the second part as non-interacting, is defined as

$$H^{\text{ext}}(\xi) := H_0^{\text{ext}}(\xi) + \Phi(v) \otimes \mathbb{1}_{\mathcal{F}}.$$

Again by Proposition 1.5, $\Phi(v) \otimes \mathbb{1}_{\mathcal{F}}$ is $H_0^{\text{ext}}(\xi)$-bounded with relative bound 0, so $H^{\text{ext}}(\xi)$ is self-adjoint on $\mathcal{D}^{\text{ext}}$ and essentially self-adjoint on $\mathcal{C}_0^\infty \otimes \mathcal{C}_0^\infty$. Likewise, we define $H_r^{\text{ext}}(\xi) = H_0^{\text{ext}}(\xi) + \Phi_r(v) \otimes \mathbb{1}_{\mathcal{F}}$.

1.5 Auxiliary spaces and auxiliary operators

We introduce auxiliary Hilbert spaces for a system consisting of a copy of the original system and a fixed number ℓ of non-interacting photons. More precisely we define

$$\mathcal{H}^{(\ell)} := \mathcal{F} \otimes \mathcal{F}^{(\ell)}.$$

As before, we can identify $\mathcal{H}^{(\ell)} = L^2_{\text{sym}}(\mathbb{R}^{\ell\nu}; \mathcal{F})$, where $_{\text{sym}}$ indicates that the functions are symmetric under permutations from $S(\ell)$. We extend the notation of second quantization by setting

$$d\Gamma^{(\ell)}(a) = d\Gamma(a) \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}} + \mathbb{1}_{\mathcal{F}} \otimes d\Gamma(a)_{|\mathcal{F}^{(\ell)}}$$

for a self-adjoint operator a. The operator $d\Gamma^{(\ell)}(a)$ is essentially self-adjoint. The auxiliary Hamiltonian is given as

$$\begin{aligned} H_r^{(\ell)}(\xi) &:= H_0^{(\ell)}(\xi) + \Phi_r(v) \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}} \text{ where} \\ H_0^{(\ell)}(\xi) &:= d\Gamma^{(\ell)}(\omega) + \Omega(\xi - d\Gamma^{(\ell)}(k)). \end{aligned}$$

Proposition 1.5 tells us that $\mathcal{D}^{(\ell)} := \mathcal{D}(H_0^{(\ell)}(\xi))$ is independent of ξ and that $\Phi_r(v) \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}}$ is $H_0^{(\ell)}(\xi)$-bounded with relative bound 0, so $H_r^{(\ell)}(\xi)$ is essentially self-adjoint on

$$\mathcal{C}_0^{\infty\,(\ell)} := \mathcal{C}_0^\infty \otimes \Gamma^{(\ell)}(C_0^\infty(\mathbb{R}^\nu))$$

and self-adjoint on $\mathcal{D}^{(\ell)}$. We abbreviate $H^{(\ell)}(\xi) = H_\infty^{(\ell)}(\xi)$.

Define

$$H_r^{(\ell)}(\xi; k) := H_r(\xi - \textstyle\sum_{j=1}^{\ell} k_j) + (\sum_{j=1}^{\ell} \omega(k_j))\mathbb{1}_{\mathcal{F}},$$

and again we write $H^{(\ell)}(\xi;k) = H^{(\ell)}_\infty(\xi;k)$ for short. The auxiliary Hamiltonian can then be written as a direct integral representation as

$$H^{(\ell)}_r(\xi) = \int^{\oplus}_{\mathbb{R}^{\ell\nu}} H^{(\ell)}_r(\xi;k)dk. \tag{2.4}$$

These fiber operators are clearly self-adjoint on $\mathcal{D}$ and essentially self-adjoint on $\mathcal{C}_0^\infty$. Note that we have

$$\begin{aligned}\Sigma^{(\ell)}_0(\xi,k) &= \inf\{\sigma(H^{(\ell)}(\xi;k))\} \quad \text{and}\\ \Sigma^{(\ell)}_0(\xi) &= \inf\{\sigma(H^{(\ell)}(\xi))\}.\end{aligned} \tag{2.5}$$

Using this notation, the extended space and the extended Hamiltonian defined in the previous subsection can be written as

$$\mathcal{F}^{\mathrm{ext}} = \bigoplus_{\ell=0}^{\infty} \mathcal{F}\otimes\mathcal{F}^{(\ell)} = \mathcal{F}\oplus\bigoplus_{\ell=1}^{\infty}\mathcal{F}\otimes\mathcal{F}^{(\ell)} = \mathcal{F}\oplus\Big(\bigoplus_{\ell=1}^{\infty}\mathcal{H}^{(\ell)}\Big)$$

and

$$H^{\mathrm{ext}}_r(\xi) = H_r(\xi)\oplus\Big(\bigoplus_{\ell=1}^{\infty}H^{(\ell)}_r(\xi)\Big). \tag{2.6}$$

1.6 Limiting absorption principle

We briefly recall the definition of the regularity property $C^k(A)$ of operators on a Hilbert space $\mathcal{H}$ for a self-adjoint operator A on $\mathcal{H}$ and establish some results regarding this property. Throughout this subsection, A will denote a self-adjoint operator on a separable Hilbert space $\mathcal{H}$.

Definition 1.14. Let $B \in \mathcal{B}(\mathcal{H})$ be a bounded operator and $k \in \mathbb{N}$. We say that $B \in C^k(A)$ if, for all $\varphi \in \mathcal{H}$, the map $\mathbb{R} \ni s \mapsto e^{-isA}Be^{isA}\varphi \in \mathcal{H}$ is k times continuously differentiable. If $B \in C^k(A)$, B is said to be *of class* $C^k(A)$. Let H be a self-adjoint operator on $\mathcal{H}$. If for some (and hence all) $z \in \rho(H)$, $(H-z)^{-1} \in C^1(A)$, we say that H is of class $C^1(A)$.

The following equivalences are well-known, see e.g. [ABG96].

Proposition 1.15. *Let $B \in \mathcal{B}(\mathcal{H})$. The following are equivalent.*

(i) $B \in C^1(A)$.

(ii) There is a constant C such that for all $\psi,\varphi \in \mathcal{D}(A)$,

$$|\langle A\psi, B\varphi\rangle - \langle\psi, BA\varphi\rangle| \le C\|\varphi\|\|\psi\|.$$

(iii) *B maps $\mathcal{D}(A)$ into itself and $AB - BA\colon \mathcal{D}(A) \to \mathcal{H}$ extends to a bounded operator on $\mathcal{H}$.*

(iv) *There exists a core $\mathcal{C}$ for A such that $B\mathcal{C} \subset \mathcal{D}(A)$ and $AB - BA$ extends from $\mathcal{C}$ to a bounded operator on $\mathcal{H}$.*

We denote the extension of $AB - BA$ from $\mathcal{D}(A)$ to $\mathcal{H}$ by $[A, B]^\circ$.

Proposition 1.16. *If H is a self-adjoint operator of class $C^1(A)$ and $W_t = e^{itA}$ is the unitary group associated to the self-adjoint operator A and $\psi, \varphi \in \mathcal{D}(H)$, then we have*

$$\langle \psi, [H, iA]^\circ \varphi \rangle = \lim_{s \to 0} \frac{1}{s} \big(\langle H\psi, W_s \varphi \rangle - \langle \psi, W_s H \varphi \rangle \big).$$

The proof is left to the reader.

Definition 1.17. Let H be a self-adjoint operator and $k \in \mathbb{N}$. We say that H is *of class* $C^k(A)$ if $(H - z)^{-1} \in C^k(A)$ for some (and hence all) $z \in \rho(H)$.

If H is of class $C^k(A)$ it follows that the form $[H, A]$ extends from $\mathcal{D}(A) \cap \mathcal{D}(H)$ to $\mathcal{D}(H)$. This extension is also denoted $[H, A]^\circ$.

As mentioned earlier, we will obtain a Mourre estimate and a C^2 property of our Hamiltonians to prove a version of the limiting absorption principle. We begin by recalling the definition of a Mourre estimate.

Definition 1.18 (Mourre estimate). Let $H \in C^1(A)$ for some self-adjoint operator A on a Hilbert space $\mathcal{H}$ and I a bounded, open interval on $\mathbb{R}$. We say that the Mourre estimate holds true for H on I if there exists a $c > 0$ and a compact operator K such that

$$E_I(H)[H, iA]E_I(H) \geq cE_I(H) + K \tag{2.7}$$

as a form on $\mathcal{H}$. We say that the Mourre estimate is strict, if we can choose $K = 0$.

Remark 1.19. Assume (2.7) and that $\lambda \in I$ is not an eigenvalue of H. Then we can choose an $I' \ni \lambda$ and a c' such that a strict Mourre estimate holds with I and c replaced by I' and c', respectively. To see this, pick $I_n \subset I, n \in \mathbb{N}$ such that $\lambda \in I_n$ and $|I_n| \to 0$ for $n \to \infty$. As λ is not an eigenvalue of H, s-lim$(E_{I_n}(H)) = 0$ and hence $\|KE_{I_n}(H)\| \to 0$. Choose $I' = I_N$ for N so large that $\|KE_{I_N}(H)\| < \frac{c}{2}$ and $c' = \frac{c}{2}$. If we now sandwich both sides of the inequality (2.7) with $E_{I'}$, we easily arrive at the desired inequality.

Remark 1.20. Assume (2.7) and that H is of class $C^1(A)$. Then the so-called Virial Theorem, $E_\lambda(H)[H, iA]^\circ E_\lambda(H) = 0$, holds by [ABG96, Proposition 7.2.10]. This in turn implies by [ABG96, Corollary 7.2.11] that the total multiplicity of eigenvalues in I is finite.

By the limiting absorption principle, we mean the following.

Definition 1.21 (Limiting absorption principle). Let H and A be self-adjoint operators on the Hilbert space $\mathcal{H}$, A self-adjoint, J a bounded interval on $\mathbb{R}$ and $s \geq 0$ a non-negative number. We say that the limiting absorption principle holds for H with respect to (A, J, s) if

$$\sup_{z \in J^\pm} \| \langle A \rangle^{-s} (H - z)^{-1} \langle A \rangle^{-s} \| < \infty,$$

where $J^\pm = \{z \in \mathbb{C} | \operatorname{Re}(z) \in J, \pm \operatorname{Im}(z) > 0\}$.

Note that if the limiting absorption principle holds for an s_0, then it holds for all $s > s_0$. Note that the limiting absorption principle implies absolute continuity of the part of essential spectrum of the operator lying in J.

To obtain a version of the limiting absorption principle, it is sufficient to prove a Mourre estimate and a C^2 property of the Hamiltonian. More precisely the following theorem holds.

Theorem 1.22. *Let H be of class $C^2(A)$, I an open interval on $\mathbb{R}$ and $s > \frac{1}{2}$. Assume that the strict Mourre estimate holds true for H on I. Then the limiting absorption principle with respect to (A, J, s) holds true for H, where J is any compact subinterval of I.*

For a proof, see e.g. [GJ06] or [Gér08]. We note that this is a generalization of Mourre's original result, see [Mou81].

Remark 1.23. Mourre assumed a list of technical conditions, among these the condition that $e^{itA}\mathcal{D}(H) \subset \mathcal{D}(H)$, a condition which is not true in all the cases covered in this work. An example where this condition fails is $\nu = 1$, $v_\xi(k) = Ck + b(k)$ and $\Omega(\eta) = \eta^2$, where $C = 0$ and $b = 1$. Calculate on $\mathcal{C}_0^\infty$

$$\begin{aligned} e^{iA_\xi}(\xi - d\Gamma(k))^2 f &= (\xi - d\Gamma(k-1))^2 e^{iA_\xi} f \\ &= (\xi - d\Gamma(k) + N)^2 e^{iA_\xi} f. \end{aligned}$$

2 Regularity Properties of the Hamiltonian with Respect to a Conjugate Operator

In this section we will define the conjugate operator A_ξ and prove that $H_0(\xi)$ and $H(\xi)$ are of class $C^2(A_\xi)$.

2.1 A tool for proving the $C^2(A)$ property

The following proposition will be used to prove the regularity property of the fiber Hamiltonians.

Proposition 2.1. *Let H_0 be a self-adjoint operator, V a symmetric operator and C_0 a form on $\mathcal{D}(H_0)$. Write $R_0(z) = (H_0 - z)^{-1}$ for $z \in \rho(H_0)$ and $\mathcal{H}_s$, $-1 \le s \le 1$, for the scale of spaces associated to H_0. Assume that*

(i) $\mathcal{C} \subset \mathcal{D}(A) \cap \mathcal{D}(H_0)$ *is a core for H_0 and A,*

(ii) $[H_0, iA] = C_0$ *as a form identity on $\mathcal{C}$,*

(iii) there exists $z_0 \in \rho(H_0)$ such that $(H_0 - z_0)\mathcal{C}$ and $(H_0 - \bar{z}_0)\mathcal{C}$ are cores for A,

(iv) $|\langle \psi, C_0\varphi\rangle| \le c(\|H_0\psi\|^2 + \||H_0|^{\frac{1}{2}}\varphi\|^2 + \|\psi\|^2 + \|\varphi\|^2)$ *for some $c > 0$ and all $\psi, \varphi \in \mathcal{C}$.*

(v) $VR_0(z)^{\frac{1}{2}}$ *is bounded,*

(vi) $|\langle V\psi, iA\varphi\rangle - \langle A\psi, iV\varphi\rangle| \le c(\|H_0\psi\|^2 + \|\psi\|^2 + \|\varphi\|^2)$ *for some $c > 0$ and all $\psi, \varphi \in \mathcal{C}$ and*

(vii) $V\mathcal{C} \subset \mathcal{D}(A)$.

Then the self-adjoint operator $H = H_0 + V$ with domain $\mathcal{D}(H_0)$ is of class $C^1(A)$ with $H' = [H, iA]^\circ \in \mathcal{B}(\mathcal{H}_{1-t}, \mathcal{H}_{-\frac{1}{2}-t})$ for $0 \le t \le \frac{1}{2}$.

Write C_V for the H_0-bounded operator associated with the form $[V, iA]$, cf. (vi). *Assume furthermore that D_0 is a form on $\mathcal{D}(H_0)$ and*

(viii) $A\mathcal{C} \subset \mathcal{C}$ *and* $[C_0, iA] = D_0$ *as a form on $\mathcal{C}$,*

(ix) $C_V\mathcal{C} \subset \mathcal{D}(A)$ *and* $|\langle C_V\psi, iA\psi\rangle - \langle A\psi, iC_V\psi\rangle| \le c(\|H\psi\|^2 + \|\psi\|^2)$ *for some $c > 0$ and all $\psi \in \mathcal{C}$ and*

(x) $|\langle \psi, D_0\psi\rangle| \le c(\|H_0\psi\|^2 + \|\psi\|^2)$ *for some $c > 0$ and all $\psi \in \mathcal{C}$.*

Then H is of class $C^2(A)$.

Proof. For any $\psi \in (H_0 - \bar{z}_0)\mathcal{C}$, $\varphi \in (H_0 - z_0)\mathcal{C}$,

$$\langle \psi, [(H_0 - z_0)^{-1}, iA]\varphi\rangle = -\langle (H_0 - \bar{z}_0)^{-1}\psi, [H_0, iA](H_0 - z_0)^{-1}\varphi\rangle.$$

Then by (i), (ii) and (iv),

$$\begin{aligned} &|\langle \psi, [(H_0 - z_0)^{-1}, iA]\varphi\rangle| \\ &\quad \le K(\|\psi\|^2 + \|\varphi\|^2 + \|(H_0 - \bar{z}_0)^{-1}\psi\|^2 + \|(H_0 - z_0)^{-1}\varphi\|^2) \\ &\quad \le c(\|\psi\|^2 + \|\varphi\|^2) \end{aligned}$$

for some constant $c > 0$. By (iii), this proves that $(H_0 - z_0)^{-1} \in C^1(A)$. Hence H_0 is of class $C^1(A)$ and $[H_0, iA]$ has a unique extension from $\mathcal{D}(H_0) \cap \mathcal{C}(A)$ to a continuous form $[H_0, iA]^\circ$ on $\mathcal{D}(H_0)$. By noting that (iv) extends to $\mathcal{D}(H_0)$ and using symmetry and an approximation argument, one sees that $[H_0, iA]^\circ = C_0$ as an operator in $\mathcal{B}(\mathcal{H}_{1-t}, \mathcal{H}_{-\frac{1}{2}-t})$ for $0 \le t \le \frac{1}{2}$.

It is clear that (v) implies that $H = H_0 + V$ is self-adjoint on $\mathcal{D}(H_0)$. Furthermore, it follows that we can choose z_1 such that $\|VR_0(z_1)\| < 1$. We can now write

$$R(z_1) := (H - z_1)^{-1} = R_0(z_1)(I + VR_0(z_1))^{-1}.$$

As $C^1(A)$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ and as $S \in C^1(A)$ and z in the connected component of infinity of $\rho(S)$ implies $(S - z)^{-1} \in C^1(A)$ (see [GGM04, Proposition 2.6]), it suffices to show that $VR_0(z_1) \in C^1(A)$ in order to prove that H is of class $C^1(A)$. Calculate for $\psi \in \mathcal{C}$, $\varphi \in (H_0 - z_1)\mathcal{C}$

$$\begin{aligned} &\langle \psi, VR_0(z_1)iA\varphi\rangle - \langle A\psi, iVR_0(z_1)\varphi\rangle \qquad (2.8)\\ &\quad = \langle V\psi, iAR_0(z_1)\varphi\rangle - \langle A\varphi, iVR_0(z_1)\varphi\rangle - \langle \psi, VR_0(z_1)[H_0, iA]R_0(z_1)\varphi\rangle. \end{aligned}$$

By using that $[H_0, iA]^\circ \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-\frac{1}{2}})$, (v) and (vi), it follows that (2.8) can be bounded by $c(\|\psi\|^2 + \|\psi\|^2)$ for some $c > 0$. Then by (i) and (vii) we may apply Proposition 1.15. It is now easy to see that $H' \in \mathcal{B}(\mathcal{H}_{1-t}, \mathcal{H}_{-\frac{1}{2}-t})$ for $0 \le t \le \frac{1}{2}$.

Write C_V for the unique H_0-bounded operator from $\mathcal{D}(H_0)$ to $\mathcal{H}$ associated with the form $[V, iA]$ on $\mathcal{C}$, cf. (vi). Note that (v) implies that $R(z)V$ can be extended to a bounded operator and that one by an argument similar to the one above can prove that $R(z)V \in C^1(A)$ using (viii). We have the identity

$$\begin{aligned} [R(z_1), iA]^\circ = &-R(z_1)V[R_0(z_1), iA]^\circ(I + VR_0(z_1))^{-1} \\ &- R(z_1)C_V R(z_1) \\ &- [R_0(z_1), iA]^\circ(I + VR_0(z_1))^{-1}. \end{aligned}$$

Note that $R_0(z_1) = R_0(z_0)((z_0 - z_1)R_0(z_0) - I)$. Thus, to show that H is of class $C^2(A)$, it suffices to show that $R_0(z_0)C_V R_0(z_0)$ and $[R_0(z_0), iA]$ are in $C^1(A)$.

We begin with $[R_0(z_0), iA]^\circ \in C^1(A)$. Let $\psi \in (H_0 - \bar{z}_0)\mathcal{C}$ and $\varphi \in (H_0 - z_0)\mathcal{C}$. Then by the assumptions

$$\begin{aligned}&\langle \psi, [R_0(z_0), iA]^\circ iA\varphi\rangle + \langle A\psi, i[R_0(z_0), iA]^\circ \varphi\rangle \\ &\quad = \langle \psi, R_0(z_0)D_0R_0(z_0)\varphi\rangle \\ &\qquad - \langle \psi, R_0(z_0)C_0R_0(z_0)C_0R_0(z_0)\varphi\rangle \\ &\qquad - \langle R_0(\bar{z}_0)^{\frac{1}{2}}C_0R_0(\bar{z}_0)\psi, R_0(z_0)^{\frac{1}{2}}C_0R_0(z_0)\varphi\rangle,\end{aligned}$$

can be bounded by $c(\|\psi\|^2 + \|\varphi\|^2)$ for some $c > 0$. Hence Proposition 1.15 can be applied and $[R_0(z_0), iA]^\circ \in C^1(A)$.

By the assumptions, the following form identity on $(H_0 - \bar{z}_0)\mathcal{C} \times (H_0 - z_0)\mathcal{C}$ is true,

$$\begin{aligned}[R_0(z_0)C_VR_0(z_0), iA] = {}& R_0(z_0)C_V[R_0(z_0), iA]^\circ \\ &+ R_0(z_0)[C_V, iA]R_0(z_0) \\ &+ [R_0(z_0), iA]^\circ C_VR_0(z_0)\end{aligned}$$

and again one finds that Proposition 1.15 can be applied. □

2.2 Definition and self-adjointness of the conjugate operator

We choose the conjugate operator as an operator of the usual form $d\Gamma(a_\xi)$ with $a_\xi = \frac{1}{2}(v_\xi \cdot x + x \cdot v_\xi)$, where $x := i\nabla_k$ and v_ξ is a sufficiently nice vector field. More precisely, we assume that v_ξ satisfies the following condition.

Condition 2.2. Let $v_\xi \in C^\infty(\mathbb{R}^\nu; \mathbb{R}^\nu)$. For any multi-index α, $|\alpha| \in \{0, 1, 2\}$, there exists a constant c_α such that $|\partial^\alpha v_\xi(\eta)| \le c_\alpha \langle \eta \rangle^{1-|\alpha|}$.

In order to define A_ξ as $d\Gamma(a_\xi)$, we need to make sure that $\frac{1}{2}(v_\xi \cdot x + x \cdot v_\xi)$ represents a well-defined self-adjoint operator. The following proposition takes care of this and implies the essential self-adjointness of the operator $A_\xi := d\Gamma(a_\xi)$ on $\mathcal{C}_0^\infty$.

Proposition 2.3. *Let v_ξ satisfy Condition 2.2. Let $x := i\nabla$. Then the operator given by $a_\xi = \frac{1}{2}(v_\xi \cdot x + x \cdot v_\xi)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$.*

Proof. Let v_ξ satisfy the assumptions. Then v_ξ is globally Lipschitz and hence we can define the flow $\gamma_s \colon \mathbb{R}^n \to \mathbb{R}^n$ generated by v_ξ as the unique solution to the ODE

$$\frac{d}{ds}\gamma_s(k) = v_\xi(\gamma_s(k)), \quad \gamma_0(k) = k.$$

Then $\gamma(s,k) := \gamma_s(k)$ is smooth in (s,k). By differentiating the Jacobian of γ_s with respect to s we get

$$\frac{d}{ds}D\gamma_s(k) = Dv_\xi(\gamma_s(k))D\gamma_s(k). \tag{2.9}$$

Note also that $D\gamma_0(k) = I$ and that $\operatorname{Tr} Dv_\xi = \nabla \cdot v_\xi$. Now differentiating the identity $\det A(s) = \exp \operatorname{Tr} \ln A(s)$, which holds for differentiable, quadratic matrix functions A with $A(0) = 1$ when s is sufficiently small, we get

$$\frac{d}{ds}\det A(s) = \operatorname{Tr}\big(\tfrac{d}{ds}A(s)A^{-1}(s)\big)\det A(s). \tag{2.10}$$

Hence we see by combining (2.9) and (2.10) that for small s

$$\frac{d}{ds}\det D\gamma_s(k) = \nabla \cdot v_\xi(\gamma_s(k))\det D\gamma_s(k), \quad \det D\gamma_0(k) = 1.$$

This implies that the function $J(s,k) = \det D\gamma_s(k)$ is given by

$$J(s,k) = e^{\int_0^s \nabla\cdot v_\xi(\gamma_t(k))dt}, \quad s \text{ small}.$$

We will now define a one-parameter group of unitary operators. We begin by setting

$$\psi_s(k) = \sqrt{J(s,k)}\psi(\gamma_s(k)), \quad \psi \in C_0^\infty(\mathbb{R}^n),$$

for s sufficiently small. Clearly, ψ_s is again in $C_0^\infty(\mathbb{R}^n)$. Straightforward computations (using the definition of J) show that $(\psi_s)_t = \psi_{s+t}$ and that $\|\psi_s\| = \|\psi\|$. By repeated use of the group property, we extend the definition of ψ_s to arbitrary s, so the maps $\psi \mapsto \psi_s$ extend to a strongly continuous one-parameter group of unitary operators U_t on $\mathcal{H}$, essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$.

The calculation

$$\begin{aligned} i\frac{d}{ds}\psi_s(k)|_{s=0} &= \psi(k)i\frac{d}{ds}\sqrt{J(s,k)}|_{s=0} + v_\xi(k)\cdot x\psi(k) \\ &= (\tfrac{i}{2}\operatorname{div}(v_\xi) + v_\xi \cdot x)\psi(k) = \tfrac{1}{2}(v_\xi\cdot x + x\cdot v_\xi)\psi(k). \end{aligned}$$

show that this group is in fact generated by $\frac{1}{2}(v_\xi \cdot x + x \cdot v_\xi)$. ☐

Lemma 2.4. *Let a_ξ be as in Proposition 2.3, and assume Conditions 1.3*(i),(ii),(iv) *and 2.2. Then $a_\xi v, a_\xi^2 v \in L^2(\mathbb{R}^\nu)$.*

Proof. We calculate $-ia_\xi v = \frac{1}{2}\operatorname{div}(v_\xi)v + v_\xi \cdot \nabla v$, which clearly is in $L^2(\mathbb{R}^\nu)$ by the assumptions. Similarly

$$\begin{aligned}
-a_\xi^2 v &= -ia_\xi(\tfrac{1}{2}\operatorname{div}(v_\xi)v + v_\xi \cdot \nabla v) \\
&= \tfrac{1}{4}(\operatorname{div} v_\xi)^2 \cdot v + \tfrac{1}{2}(\operatorname{div} v_\xi)v_\xi \cdot \nabla v \\
&\quad + \tfrac{1}{2}v_\xi \cdot ((\nabla \operatorname{div}(v_\xi))v + \operatorname{div}(v_\xi)\nabla v) \\
&\quad + \langle v_\xi, (\nabla^2 v)v_\xi \rangle + \langle (\nabla v_\xi)v_\xi, \nabla v \rangle,
\end{aligned}$$

where $\nabla^2 v$ is the Hessian of v and ∇v_ξ is the Jacobian of v_ξ. It follows from the assumptions that each term is in $L^2(\mathbb{R}^\nu)$. □

2.3 The $C^2(A_\xi)$ property of the Hamiltonian

In this subsection we prove that $H(\xi)$ is of class $C^2(A_\xi)$. In fact, we will prove a little more than that. In the following $\mathcal{D}_s$ will be used to denote the scale of spaces associated to $H_0(\xi)$. Note that by Proposition 1.5 $\mathcal{D}_s$ is independent of ξ, that replacing $H_0(\xi)$ with $H(\xi)$ leaves $\mathcal{D}_s$ unchanged for $|s| \leq 1$ and that $\mathcal{D}_1 = \mathcal{D}$.

Proposition 2.5. *Assume Conditions 1.1*(i), (ii), *1.2*(i), (v), (vi), *1.3*(i), (ii), (iv) *and 2.2 and that there exists a constants c_1, c_2 and a function $b\colon \mathbb{R}^\nu \to \mathbb{R}^\nu$ such that $\|b(k)\| \leq c_1\omega(k)$ and $v_\xi(k) = c_2 k + b(k)$. Then the fiber Hamiltonians $H(\xi)$ are of class $C^2(A_\xi)$ and*

$$[H(\xi), iA_\xi]^\circ = d\Gamma(v_\xi \cdot \nabla\omega) - d\Gamma(v_\xi) \cdot \nabla\Omega(\xi - d\Gamma(k)) - \Phi(ia_\xi v)$$

is contained in $\mathcal{B}(\mathcal{D}, \mathcal{D}_{-\frac{1}{2}}) \cap \mathcal{B}(\mathcal{D}_{-\frac{1}{2}}, \mathcal{D})$.

Note that the proposition also holds true with $H(\xi)$ and A_ξ replaced by $H^{\text{ext}}(\xi)$ and A_ξ^{ext} or $H^{(\ell)}(\xi)$ and $A^{(\ell)}(\xi)$, respectively, by the same arguments. We remark that we do not make use of the result for $c_2 \neq 0$. However, we have another application in mind that requires $c_2 \neq 0$.

Proof. We will show this by applying Proposition 2.1 with $H_0 = H_0(\xi)$, $V = \Phi(v)$, $A = A_\xi$ and $C_0 = d\Gamma(v_\xi \cdot \nabla\omega) - d\Gamma(v_\xi) \cdot \nabla\Omega(\xi - d\Gamma(k))$. Clearly $\Phi(v)$ is symmetric, C_0 is a form on $\mathcal{D}$ and $H_0(\xi)$ and A_ξ are self-adjoint by Proposition 1.5 and Proposition 2.3, respectively. We choose

$\mathcal{C}_0^\infty$ as our common core. Note that $(H_0(\xi) - z)^{-1}\mathcal{C}_0^\infty = \mathcal{C}_0^\infty$ for any $z \in \rho(H_0(\xi))$. On $\mathcal{C}_0^\infty$, the identity

$$[H_0(\xi), iA_\xi] = C_0 \tag{2.11}$$

holds. Indeed, by noting that $H_0(\xi)$ leaves particle sectors invariant and restricting to the n'th particle sector $\mathcal{F}^{(n)}$, (2.11) is easily seen by direct computation.

First we show that the following holds:

$$\exists c > 0\colon |v_\xi(k) \cdot \nabla\omega(k)| \leq c\omega(k). \tag{2.12}$$

Condition 1.2(v) imply that $\|\nabla\omega\|_\infty$ is finite and Conditions 1.2(i) and (vi) imply that

$$\begin{aligned} |v_\xi(k) \cdot \nabla\omega(k)| &\leq |ck \cdot \nabla\omega(k)| + |b(k) \cdot \nabla\omega(k)| \\ &\leq (c' + \|\nabla\omega\|_\infty)\omega(k), \end{aligned}$$

for a suitable constant c'.

We now show that:

$$\begin{aligned} &\exists c > 0\colon |\langle\psi, d\Gamma(v_\xi) \cdot \nabla\Omega(\xi - d\Gamma(k))\varphi\rangle| \\ &\qquad \leq c(\|H_0(\xi)^{\frac{1}{2}}\psi\|^2 + \|H_0(\xi)\varphi\|^2 + \|\psi\|^2 + \|\varphi\|^2) \end{aligned} \tag{2.13}$$

First observe that by Condition 1.1(i) and (ii), for any $\eta, \xi \in \mathbb{R}^\nu$ we have

$$\begin{aligned} |-\eta \cdot \nabla\Omega(\eta) + \xi \cdot \nabla\Omega(\eta)| &\leq \sum_{j=1}^{\nu}(|\eta_j\partial_j\Omega(\eta)| + |\xi_j\partial_j\Omega(\eta)|) \\ &\leq \sum_{j=1}^{\nu}(|\eta_j| + |\xi_j|)c\langle\eta\rangle^{s_\Omega - 1} \\ &\leq c'\langle\eta\rangle^{s_\Omega} \leq c''\Omega(\eta) + c''', \end{aligned} \tag{2.14}$$

where c, c', c'' and c''' are suitable constants. Calculate

$$\begin{aligned} d\Gamma(v_\xi) \cdot \nabla\Omega(\xi - d\Gamma(k)) &= d\Gamma(Ck + b(k)) \cdot \nabla\Omega(\xi - d\Gamma(k)) \\ &= c(d\Gamma(k) - \xi) \cdot \nabla\Omega(\xi - d\Gamma(k)) \\ &\quad + c^{-1}\xi \cdot \nabla\Omega(\xi - d\Gamma(k)) \\ &\quad + d\Gamma(b(k)) \cdot \nabla\Omega(\xi - d\Gamma(k)). \end{aligned}$$

To treat the first two terms, note that (2.14) implies that

$$\begin{aligned} &|\langle\psi, c(d\Gamma(k) - \xi) \cdot \nabla\Omega(\xi - d\Gamma(k)) + c^{-1}\xi \cdot \nabla\Omega(\xi - d\Gamma(k))\varphi\rangle| \\ &\quad \leq c'(\|\Omega(\xi - d\Gamma(k))\varphi\|^2 + \|\psi\|^2 + \|\varphi\|^2) \end{aligned} \tag{2.15}$$

for a suitable c'. To treat the last term, note that

$$d\Gamma(b(k)) \cdot \nabla\Omega(\xi - d\Gamma(k))|_{\mathcal{F}^{(n)}} = \sum_{j=1}^{\nu}\sum_{i=1}^{n} b(k_i)_j \partial_j \Omega(\xi - \textstyle\sum_l^n k_l).$$

Condition 1.1(i) and (ii) together implies that there exist constants $c > 0$ and $c' > 0$ such that $|\partial_j \Omega(\eta)|^2 \leq C\langle\eta\rangle^{2s_\Omega - 2} \leq c\langle\eta\rangle^{s_\Omega} \leq c'(|\Omega(\eta)| + c')$. Let $\psi, \varphi \in \mathcal{F}^{(n)} \cap \mathcal{C}_0^\infty$. Then

$$\begin{aligned} &|\langle\psi, d\Gamma(b(k)) \cdot \nabla\Omega(\xi - d\Gamma(k))\varphi\rangle| \\ &\quad\leq \sum_{j=1}^{\nu} |\langle\partial_j\Omega(\xi - \textstyle\sum_l^n k_l)\psi, \textstyle\sum_i^n b(k_i)_j\varphi\rangle| \\ &\quad\leq c\|(\Omega(\xi - \textstyle\sum_l^n k_l) + c)^{\frac{1}{2}}\psi\| \|\textstyle\sum_i^n \omega(k_i)_j\varphi\|. \end{aligned} \tag{2.16}$$

As $d\Gamma(\omega)$ and $\Omega(\xi - d\Gamma(k))$ are both bounded from below, this implies that for any $\psi, \varphi \in \mathcal{F}$,

$$\begin{aligned} &|\langle\psi, d\Gamma(b(k)) \cdot \nabla\Omega(\xi - d\Gamma(k))\varphi\rangle| \\ &\quad\leq c(\|H_0(\xi)^{\frac{1}{2}}\psi\|^2 + \|H_0(\xi)\varphi\|^2 + \|\psi\|^2 + \|\varphi\|^2), \end{aligned}$$

so (2.15) and (2.16) implies (2.13).

By combining (2.12) and (2.13) and using the semiboundedness we see that

$$|\langle\psi, [H_0(\xi), iA_\xi]\varphi\rangle| \leq c(\|H_0(\xi)^{\frac{1}{2}}\psi\|^2 + \|H_0(\xi)\varphi\|^2 + \|\psi\|^2 + \|\varphi\|^2).$$

That $\Phi(v)(H_0(\xi) - z)^{-\frac{1}{2}}$ is bounded follows from the positive mass assumption and standard arguments.

By Lemma 2.4 $ia_\xi v \in L^2(\mathbb{R}^\nu)$. This implies that $\Phi(v)\mathcal{C}_0^\infty \subset \mathcal{D}(A_\xi)$. It also implies that

$$\begin{aligned} |\langle\Phi(v)\psi, iA_\xi\varphi\rangle - \langle A_\xi\psi, i\Phi(v)\varphi\rangle| &= |\langle\psi, \Phi(ia_\xi v)\varphi\rangle| \\ &\leq c(\|H_0(\xi)\psi\|^2 + \|\psi\|^2 + \|\varphi\|^2) \end{aligned}$$

for all $\psi, \varphi \in \mathcal{C}_0^\infty$, which shows that the first part of Proposition 2.1 is satisfied.

To get the $C^2(A_\xi)$ property, let $C_V = \Phi(v)$ and

$$\begin{aligned} D_0 = d\Gamma(\langle v_\xi, (\nabla^2\omega)v_\xi\rangle) &+ d\Gamma(\langle(\nabla v_\xi)v_\xi, \nabla\omega\rangle) \\ &+ \langle d\Gamma(v_\xi), \nabla^2\Omega(\xi - d\Gamma(k))d\Gamma(v_\xi)\rangle \\ &- d\Gamma((\nabla v_\xi)v_\xi) \cdot \nabla\Omega(\xi - d\Gamma(k)). \end{aligned}$$

We clearly have that $A_\xi \mathcal{C}_0^\infty \subset \mathcal{C}_0^\infty$. One may check by direct calculations on each particle sector that $[C_0, iA] = D_0$ as a form on $\mathcal{C}_0^\infty$. That $C_V \mathcal{C}_0^\infty \subset \mathcal{D}(A)$ and $|\langle C_V\psi, iA_\xi\psi\rangle - \langle A\psi, iC_V\psi\rangle| \leq c(\|H(\xi)\psi\|^2 + \|\psi\|^2)$ for all $\psi \in \mathcal{C}_0^\infty$ follows from Lemma 2.4.

The rest of the proof deals with showing the inequality

$$|\langle\psi, D_0\psi\rangle| \leq c(\|H_0\psi\|^2 + \|\psi\|^2).$$

By Condition 1.2(vii) and the assumption on v_ξ, we have

$$\langle v_\xi, (\nabla^2\omega)v_\xi\rangle \leq c\omega \tag{2.17}$$

for some constant C. That

$$\langle(\nabla v_\xi)v_\xi, \nabla\omega\rangle \leq c\omega \tag{2.18}$$

for a constant c follows by the boundedness of ∇v_ξ and the same arguments as in the proof of (2.12). Likewise, the inequality

$$\begin{aligned}|\langle\psi, d\Gamma((\nabla v_\xi)v_\xi) \cdot \nabla\Omega(\xi - d\Gamma(k))\varphi\rangle| \\ \leq c(\|H_0(\xi)^{\frac{1}{2}}\psi\|^2 + \|H_0(\xi)\varphi\|^2 + \|\psi\|^2 + \|\varphi\|^2)\end{aligned} \tag{2.19}$$

can be proved by the same arguments as in the proof of (2.13) and using the boundedness of ∇v_ξ.

Calculate

$$\begin{aligned}
&\langle d\Gamma(v_\xi), \nabla^2\Omega(\xi - d\Gamma(k))d\Gamma(v_\xi)\rangle \\
&\quad = c^2\langle d\Gamma(k) - \xi, \nabla^2\Omega(\xi - d\Gamma(k))(d\Gamma(k) - \xi)\rangle && (2.20a)\\
&\qquad + c^2\langle\xi, \nabla^2\Omega(\xi - d\Gamma(k))\xi\rangle && (2.20b)\\
&\qquad + 2c^2\,\mathrm{Re}\langle\xi, \nabla^2\Omega(\xi - d\Gamma(k))(d\Gamma(k) - \xi)\rangle && (2.20c)\\
&\qquad + 2c\,\mathrm{Re}\langle d\Gamma(b(k)), \nabla^2\Omega(\xi - d\Gamma(k))(d\Gamma(k) - \xi)\rangle \\
&\qquad + 2c\,\mathrm{Re}\langle\xi, \nabla^2\Omega(\xi - d\Gamma(k))d\Gamma(b(k))\rangle \\
&\qquad + \langle d\Gamma(b(k)), \nabla^2\Omega(\xi - d\Gamma(k))d\Gamma(b(k))\rangle.
\end{aligned}$$

The inequalities $\langle\eta\rangle^\beta|\partial^\alpha\Omega(\eta)| \leq c\Omega(\eta) + c'$, for $\beta = 0, 1, 2$ and α a multi-index with $|\alpha| = 2$, which follows from Condition 1.1(i) and (ii), shows that $\langle\eta, \nabla^2\Omega(\eta)\eta\rangle$, $\mathrm{Re}\langle\eta, \nabla^2\Omega(\eta)\xi\rangle$ and $\langle\xi, \nabla^2\Omega(\eta)\xi\rangle$ are dominated by $c\Omega(\eta) + c'$. This implies that $|\langle\psi, T\varphi\rangle|$ where T is any of the operators (2.20a), (2.20b) or (2.20c) is bounded by $c''(\|\Omega(\xi - d\Gamma(k))\varphi\|^2 + \|\psi\|^2 + \|\varphi\|^2)$.

The inequalities $\|b(k)\| \leq c\omega(k)$ and $\|\nabla^2\Omega(\eta)\eta\| \leq c\Omega(\eta) + c'$, give that

$$\begin{aligned}&|\langle \psi, \mathrm{Re}\langle d\Gamma(b), \nabla^2\Omega(\xi - d\Gamma(k))(\xi - d\Gamma(k))\rangle \varphi\rangle| \\ &\quad \leq c(\|d\Gamma(\omega)\psi\|^2 + \|\Omega(\xi - d\Gamma(k))\varphi\|^2 + \|\psi\|^2 + \|\varphi\|^2).\end{aligned}$$

Note that $\langle d\Gamma(v_\xi), \nabla^2\Omega(\xi - d\Gamma(k))d\Gamma(v_\xi)\rangle$ leaves the particle sectors invariant. Let $\psi, \varphi \in \mathcal{F}^{(n)} \cap \mathcal{C}_0^\infty$. Then as $s_\Omega \leq 2$

$$\begin{aligned}&|\langle \psi, \langle d\Gamma(b(k)), \nabla^2\Omega(\xi - d\Gamma(k))d\Gamma(b(k))\rangle \varphi\rangle| \\ &\quad \leq c|\langle \psi, \langle \textstyle\sum_i^n b(k_i), \sum_j^n b(k_j)\rangle \varphi\rangle| \\ &\quad \leq c'\Big(\sum_{i=1}^n \|\omega(k_i)\psi\|^2 + \sum_{j=1}^n \|b(k_j)\varphi\|^2\Big),\end{aligned}$$

which proves that

$$\begin{aligned}&|\langle \psi, \langle d\Gamma(b(k)), \nabla^2\Omega(\xi - d\Gamma(k))d\Gamma(v_\xi)\rangle \varphi\rangle| \\ &\quad \leq c(\|d\Gamma(\omega)\psi\|^2 + \|d\Gamma(\omega)\varphi\|^2 + \|\psi\|^2 + \|\varphi\|^2).\end{aligned}$$

Finally, we se that $\mathrm{Re}\langle \xi, \nabla^2\Omega(\eta)d\Gamma(b)\rangle$ is bounded by $c\omega$. All in all, we have proved that

$$\begin{aligned}&|\langle \psi, \langle d\Gamma(v_\xi), \nabla^2\Omega(\xi - d\Gamma(k))d\Gamma(v_\xi)\rangle \varphi\rangle| \\ &\quad \leq c(\|d\Gamma(\omega)\psi\|^2 + \|\Omega(\xi - d\Gamma(k))\varphi\|^2 + \|\psi\|^2 + \|\varphi\|^2).\end{aligned} \tag{2.21}$$

Combining (2.17), (2.18), (2.19) with (2.21), we get the inequality

$$|\langle \psi, [[H_0(\xi), iA_\xi]^\circ, iA_\xi]\varphi\rangle| \leq c(\|H_0(\xi)\psi\|^2 + \|H_0(\xi)\varphi\|^2 + \|\psi\|^2 + \|\varphi\|^2),$$

so $H(\xi)$ is of class $C^2(A_\xi)$ by Proposition 2.1. □

3 Mourre Theory and a Limiting Absorption Principle

In this section, we prove a Mourre estimate. The Mourre estimate holds in the energy interval between $\Sigma_0^{(1)}(\xi)$ and $\Sigma_1^{(1)}(\xi)$ away from the threshold set $\mathcal{T}_0^{(1)}(\xi)$ at the bottom of the essential energy-momentum spectrum.

We hope to extend the result to cases where ω is bounded, so we are able to cover the polaron model. We also wish to extend this result to cover a larger part of the essential spectrum in a future work.

We conclude the section by obtaining a limiting absorption principle, implying absolute continuity of the essential spectrum.

3.1 A virial-like theorem

In the following $E_{\lambda,\kappa}$ denotes the characteristic function of the set $[\lambda - \kappa, \lambda + \kappa]$ and E_λ denotes the characteristic function of $\{\lambda\}$.

Theorem 3.1. *Assume Conditions 1.1*(i), (ii), *1.2*(i), (v), (vi), *1.3*(i), (ii), (iv) *and 2.2. Let $\mathcal{O} \subset \mathcal{I}_0^{(1)}(\xi)$ be open and $\lambda \in \mathbb{R}$ and $\kappa > 0$ be such that*

(i) For all $k \in \mathcal{O}$ we have $\Sigma_1^{(1)}(\xi;k) > \lambda + \kappa$

(ii) There exists $k \in \mathcal{O}$ such that $\lambda - \kappa < \Sigma_0^{(1)}(\xi;k) < \lambda + \kappa$

Then

$$\begin{aligned} &\mathbb{1}_{\mathcal{O}} E_{\lambda,\kappa}(H^{(1)}(\xi))[H^{(1)}(\xi), iA_\xi^{(1)}]^\circ E_{\lambda,\kappa}(H^{(1)}(\xi))\mathbb{1}_{\mathcal{O}} \\ &= \int_{\mathcal{O}}^{\oplus} v_\xi(k) \cdot \nabla \Sigma_0^{(1)}(\xi;k) \mathbb{1}_{\mathcal{F}}\, dk\, E_{\lambda,\kappa}(H^{(1)}(\xi)), \end{aligned} \tag{2.22}$$

as an identity on $L^2(\mathbb{R}^\nu; \mathcal{F}) = \mathcal{H}^{(1)}$, where $\mathbb{1}_{\mathcal{O}} = \int_{\mathcal{O}}^{\oplus} \mathbb{1}_{\mathcal{F}} dk$.

Proof. Since $\sigma(H^{(1)}(\xi;k)) = \sigma(H(\xi - k)) + \omega(k)$ the assumptions on $\mathcal{O}$, λ and κ imply that

$$E_{\lambda,\kappa}(H^{(1)}(\xi;k)) = E_{\Sigma_0(\xi-k)}(H(\xi - k)) = E_{\Sigma_0^{(1)}(\xi;k)}(H^{(1)}(\xi;k)).$$

Since $H^{(1)}(\xi)$ is fibered, $\mathbb{1}_{\mathcal{O}}$ and $E_{\lambda,\kappa}(H^{(1)}(\xi))$ commute. In fact,

$$\begin{aligned} \mathbb{1}_{\mathcal{O}} E_{\lambda,\kappa}(H^{(1)}(\xi)) &= \int_{\mathcal{O}}^{\oplus} E_{\lambda,\kappa}(H^{(1)}(\xi;k))dk = E_{\lambda,\kappa}(H^{(1)}(\xi))\mathbb{1}_{\mathcal{O}} \\ &= \int_{\mathcal{O}}^{\oplus} E_{\Sigma_0^{(1)}(\xi;k)}(H^{(1)}(\xi;k))dk. \end{aligned} \tag{2.23}$$

Write $W_t^{(1)} = e^{itA_\xi^{(1)}}$ for the unitary group associated to the self-adjoint operator $A_\xi^{(1)}$. Then by Proposition 1.16, we have for $\psi^{(1)}, \varphi^{(1)} \in \mathcal{D}^{(1)}$ that

$$\begin{aligned} &\langle \psi^{(1)}, [H^{(1)}(\xi), iA_\xi^{(1)}]^\circ \varphi^{(1)} \rangle \\ &= \lim_{s \to 0} \frac{1}{s} \Big(\langle H^{(1)}(\xi)\psi^{(1)}, W_s^{(1)} \varphi^{(1)} \rangle - \langle \psi^{(1)}, W_s^{(1)} H^{(1)}(\xi) \varphi^{(1)} \rangle \Big). \end{aligned} \tag{2.24}$$

Let $\psi^{(1)}, \varphi^{(1)} \in \mathrm{Ran}(E_{\lambda,\kappa}(H^{(1)}(\xi))\mathbb{1}_{\mathcal{O}})$. We now calculate using (2.23) and (2.4)

$$\langle H^{(1)}(\xi)\psi^{(1)}, W_s^{(1)} \varphi^{(1)} \rangle$$

$$
\begin{aligned}
&= \langle H^{(1)}(\xi)\, E_{\lambda,\kappa}(H^{(1)}(\xi))\mathbb{1}_{\mathcal{O}}\psi^{(1)}, W_s^{(1)}\varphi^{(1)}\rangle \\
&= \Big\langle \int_{\mathbb{R}^\nu}^{\oplus} H^{(1)}(\xi;k)dk \int_{\mathcal{O}}^{\oplus} E_{\Sigma_0^{(1)}(\xi;k)}(H^{(1)}(\xi;k))\psi^{(1)}(k)dk, W_s^{(1)}\varphi^{(1)}\Big\rangle \\
&= \Big\langle \int_{\mathcal{O}}^{\oplus} H^{(1)}(\xi;k)E_{\Sigma_0^{(1)}(\xi;k)}(H^{(1)}(\xi;k))\psi^{(1)}(k)dk, W_s^{(1)}\varphi^{(1)}\Big\rangle \\
&= \Big\langle \int_{\mathcal{O}}^{\oplus} \Sigma_0^{(1)}(\xi;k)E_{\Sigma_0^{(1)}(\xi;k)}(H^{(1)}(\xi;k))\psi^{(1)}(k)dk, W_s^{(1)}\varphi^{(1)}\Big\rangle \\
&= \Big\langle \int_{\mathcal{O}}^{\oplus} \Sigma_0^{(1)}(\xi;k)\mathbb{1}_{\mathcal{F}}dk\, E_{\lambda,\kappa}(H^{(1)}(\xi))\mathbb{1}_{\mathcal{O}}\psi^{(1)}, W_s^{(1)}\varphi^{(1)}\Big\rangle \\
&= \langle \mathbb{1}_{\mathcal{F}} \otimes \Sigma_0^{(1)}(\xi;\cdot)\psi^{(1)}, W_s^{(1)}\varphi^{(1)}\rangle. \qquad (2.25)
\end{aligned}
$$

Similarly,

$$
\langle \psi^{(1)}, W_s^{(1)}H^{(1)}(\xi)\varphi^{(1)}\rangle = \langle \psi^{(1)}, W_s^{(1)}\mathbb{1}_{\mathcal{F}} \otimes \Sigma_0^{(1)}(\xi;\cdot)\varphi^{(1)}\rangle. \qquad (2.26)
$$

Let $\chi \in C_0^\infty(\mathcal{I}_0^{(1)}(\xi))$ be such that $\chi \equiv 1$ on $\mathcal{O}$. Since $\psi^{(1)}, \varphi^{(1)} \in \mathrm{Ran}(\mathbb{1}_{\mathcal{O}})$, multiplication of $\Sigma_0^{(1)}(\xi;\cdot)$ with χ in (2.25) and (2.26) leaves the expressions invariant. This means that (2.24) equals

$$
\begin{aligned}
&\lim_{s\to 0}\frac{1}{s}\big(\langle \mathbb{1}_{\mathcal{F}} \otimes \chi\Sigma_0^{(1)}(\xi;\cdot)\psi^{(1)}, W_s^{(1)}\varphi^{(1)}\rangle - \langle \psi^{(1)}, W_s^{(1)}\mathbb{1}_{\mathcal{F}} \otimes \chi\Sigma_0^{(1)}(\xi;\cdot)\varphi^{(1)}\rangle\big) \\
&\quad = \langle \psi^{(1)}, [\mathbb{1}_{\mathcal{F}} \otimes \chi\Sigma_0^{(1)}(\xi;\cdot), iA_\xi^{(1)}]^\circ\varphi^{(1)}\rangle \qquad (2.27)
\end{aligned}
$$

if $\mathbb{1}_{\mathcal{F}} \otimes \chi\Sigma_0^{(1)}(\xi;\cdot) \in C^1(A_\xi^{(1)})$. But a simple computation on a core shows that

$$
\begin{aligned}
&[\mathbb{1}_{\mathcal{F}} \otimes \chi\Sigma_0^{(1)}(\xi;\cdot), iA_\xi^{(1)}] \\
&\quad = \mathbb{1}_{\mathcal{F}} \otimes \chi v_\xi \cdot (\nabla_k\Sigma_0^{(1)})(\xi;\cdot) + \mathbb{1}_{\mathcal{F}} \otimes \Sigma_0^{(1)}(\xi;\cdot)v_\xi \cdot (\nabla\chi), \qquad (2.28)
\end{aligned}
$$

which clearly extends to a bounded operator under the assumed conditions, so Proposition 1.15(v) shows that $\mathbb{1}_{\mathcal{F}} \otimes \chi\Sigma_0^{(1)}(\xi;\cdot) \in C^1(A_\xi^{(1)})$. Since $\nabla\chi \equiv 0$ on $\mathcal{O}$, (2.27) reduces to (2.22) when inserting (2.28). □

3.2 Localization errors

Let $P\colon \mathcal{F}^{\mathrm{ext}} \to \mathcal{F}$ be the projection $\mathcal{F}^{\mathrm{ext}} = \mathcal{F} \oplus \left(\bigoplus_{\ell=1}^{\infty} \mathcal{H}^{(\ell)}\right) \ni (u,v) \mapsto u \in \mathcal{F}$ and $I\colon \mathcal{F} \to \mathcal{F}^{\mathrm{ext}}$ the injection $\mathcal{F} \ni u \mapsto (u,0) \in \mathcal{F} \oplus \left(\bigoplus_{\ell=1}^{\infty} \mathcal{H}^{(\ell)}\right)$. Define $\check{\Gamma}^{\mathrm{ext}}(j^R)\colon \mathcal{F}^{\mathrm{ext}} \to \mathcal{F}^{\mathrm{ext}}$ by $\check{\Gamma}^{\mathrm{ext}}(j^R) = \check{\Gamma}(j^R)P$. Note that PI is the identity and that $\check{\Gamma}^{\mathrm{ext}}(j^R)I = \check{\Gamma}(j^R)$, $HI = IH(\xi)$, $AI = IA_\xi$ and $\Phi(v) \otimes \mathbb{1}_{\mathcal{F}}I = I\Phi(v)$.

Lemma 3.2. *Assume Conditions 1.1*(i), (ii), *1.2*(i), (v), (vi), *1.3*(i), (ii), (iv) *and 2.2. Let* $f \in C_0^\infty(\mathbb{R})$. *The following is then true.*

(i) $[\check{\Gamma}^{\mathrm{ext}}(j^R), f(H^{\mathrm{ext}}(\xi))] = o_R(1)$.

(ii) $[f(H^{\mathrm{ext}}(\xi))[H^{\mathrm{ext}}(\xi), A_\xi^{\mathrm{ext}}]f(H^{\mathrm{ext}}(\xi)), \check{\Gamma}^{\mathrm{ext}}(j^R)] = o_R(1)$.

Proof. We will start by proving the following statements:

(a) $\check{\Gamma}^{\mathrm{ext}}(j^R)f(H^{\mathrm{ext}}(\xi))\colon \mathcal{H}^{\mathrm{ext}} \to \mathcal{D}_{\frac{1}{2}}^{\mathrm{ext}}$ and $f(H^{\mathrm{ext}}(\xi))\check{\Gamma}^{\mathrm{ext}}(j^R)\colon \mathcal{D}_{\frac{1}{2}}^{\mathrm{ext}*} \to \mathcal{H}^{\mathrm{ext}}$ for any $R > 1$ and,

$$(H^{\mathrm{ext}}(\xi) - i)^{-\frac{1}{2}}[\check{\Gamma}^{\mathrm{ext}}(j^R), H^{\mathrm{ext}}(\xi)]f(H^{\mathrm{ext}}(\xi)) = o_R(1) \text{ and}$$
$$f(H^{\mathrm{ext}}(\xi))[\check{\Gamma}^{\mathrm{ext}}(j^R), H^{\mathrm{ext}}(\xi)](H^{\mathrm{ext}}(\xi) - i)^{-\frac{1}{2}} = o_R(1).$$

(b) $[\check{\Gamma}^{\mathrm{ext}}(j^R), \partial_\ell \Omega(\xi - d\Gamma^{\mathrm{ext}}(k))]f(H^{\mathrm{ext}}(\xi)) = o_R(1)$.

(c) $f(H^{\mathrm{ext}}(\xi))[d\Gamma^{\mathrm{ext}}(v_\xi)_\ell, \check{\Gamma}^{\mathrm{ext}}(j^R)](H_0^{\mathrm{ext}}(\xi) - i)^{-\frac{1}{2}} = o_R(1)$.

(d) $f(H^{\mathrm{ext}}(\xi))[\Phi(ia_\xi v) \otimes \mathbb{1}_{\mathcal{F}}, \check{\Gamma}^{\mathrm{ext}}(j^R)]f(H^{\mathrm{ext}}(\xi)) = o_R(1)$.

We will use the following abbrevations:

$$\Gamma = \check{\Gamma}^{\mathrm{ext}}(j^R), \qquad H = H^{\mathrm{ext}}(\xi),$$
$$A = A_\xi^{\mathrm{ext}} \qquad \text{and} \qquad \Phi^{\mathrm{ext}}(v) = \Phi(v) \otimes \mathbb{1}_{\mathcal{F}}.$$

Also, for notational convenience, we write $M \overset{o}{=} N$ if $M = N + o_R(1)$.

(a) We only prove half of the statement as the other half follows by a symmetric argument. Note that $(H - i)^{-\frac{1}{2}}[\Gamma, H - \Omega(\xi - d\Gamma^{\mathrm{ext}}(k))]f(H) = o_R(1)$ by (the proof of) [Møl05, Lemma 3.2]. Hence, to prove the statement, we need only show that $(H - i)^{-\frac{1}{2}}[\Gamma, \Omega(\xi - d\Gamma^{\mathrm{ext}}(k))]f(H) = o_R(1)$. We write, using [Møl05, Lemma 3.6],

$$\begin{aligned}&(H - i)^{-\frac{1}{2}}[\Gamma, \Omega(\xi - d\Gamma^{\mathrm{ext}}(k))]f(H)\\ &\quad = (H - i)^{-\frac{1}{2}}[\Gamma(N^{\mathrm{ext}} + 1)^{-3}, \Omega(\xi - d\Gamma^{\mathrm{ext}}(k))](N^{\mathrm{ext}} + 1)^3 f(H)\end{aligned}$$

The commutator $[\Gamma(N^{\mathrm{ext}} + 1)^{-3}, \Omega(\xi - d\Gamma^{\mathrm{ext}}(k))]$ satisfies the assumptions of Theorem A.3 with $B = \Gamma(N^{\mathrm{ext}} + 1)^{-3}$, $A = \xi - d\Gamma^{\mathrm{ext}}(k)$, $f_\lambda = \Omega$, $s = s_\Omega$, $n_0 = 3$ and $n = 2$ so

$$\begin{aligned}&[\Gamma(N^{\mathrm{ext}} + 1)^{-3}, \Omega(\xi - d\Gamma^{\mathrm{ext}}(k))]\\ &\quad = \sum_{|\alpha|=1}^{2} \frac{1}{\alpha!}\partial^\alpha \Omega(\xi - d\Gamma^{\mathrm{ext}}(k))\, \mathrm{ad}^\alpha_{\xi - d\Gamma^{\mathrm{ext}}(k)}(\Gamma(N^{\mathrm{ext}} + 1)^{-3})\\ &\quad\quad + R_2(\xi - d\Gamma^{\mathrm{ext}}(k), \Gamma(N^{\mathrm{ext}} + 1)^{-3}).\end{aligned}$$

Now one can readily verify that

$$\mathrm{ad}^{\alpha}_{d\Gamma^{\mathrm{ext}}(k)}(\Gamma)_{|\mathcal{F}^{(n)}\otimes\mathcal{F}} = U \sum_{\sum \alpha_i = \alpha} \frac{\alpha!}{\prod_{i=1}^{n} \alpha_i!} \bigotimes_{i=1}^{n} (\mathrm{ad}_k^{\alpha_i}(j_0^R), \mathrm{ad}_k^{\alpha_i}(j_\infty^R)) P_{|\mathcal{F}^{(n)}\otimes\mathcal{F}},$$

that $\sum_{\sum \alpha_i = \alpha} \frac{\alpha!}{\prod_i \alpha_i!} = n^{|\alpha|}$, that $\mathrm{ad}_k^{\alpha_i}(j_\#^R) = O(R^{-|\alpha_i|})$, where the sums are over all ordered sets of multi-indices $\{\alpha_i\}_{i=1}^n$ such that $\sum_{i=1}^n \alpha_i = \alpha$. It follows that $\mathrm{ad}^{\alpha}_{d\Gamma^{\mathrm{ext}}(k)}(\Gamma)(N^{\mathrm{ext}}+1)^{-|\alpha|} = O(R^{-|\alpha|})$ and hence that

$$\sum_{|\alpha|=1}^{2} \|\mathrm{ad}^{\alpha}_{\xi - d\Gamma^{\mathrm{ext}}(k)}(\Gamma(N^{\mathrm{ext}}+1)^{-3})\| \\ + \|R_2(\xi - d\Gamma^{\mathrm{ext}}(k), \Gamma(N^{\mathrm{ext}}+1)^{-3})\| = O(R^{-1}).$$

As $s_\Omega \leq 2$, $(H-i)^{-\frac{1}{2}}\partial^\alpha \Omega(\xi - d\Gamma^{\mathrm{ext}}(k))$ is bounded. Hence (a) follows.

By an analogous argument we get (b). The proof of (c) and (d) can be found in the proof of [Møl05, Lemma 3.2].

(i) By symmetry it suffices to show that $f(H)\chi(H)\Gamma \overset{o}{=} f(H)\Gamma\chi(H)$ for any $\chi \in C_0^\infty(\mathbb{R})$, which follows by the identity

$$f(H)\chi(H)\Gamma = f(H)\Gamma\chi(H) + \int \bar{\partial}\tilde{\chi}(z)(H-z)^{-1} f(H)[\Gamma, H](H-z)^{-1} dz$$

and (a).

(ii) Choose $\chi \in C_0^\infty(\mathbb{R})$ such that $f = f\chi$. By (i) and (a) we see that

$$\begin{aligned} & f(H)[H,A]f(H)\Gamma \\ & \quad = f(H)[H,A]f(H)\Gamma\chi(H) + f(H)[H,A]f(H)o_R(1) \\ & \quad \overset{o}{=} f(H)[H,A]\Gamma f(H) \\ & \qquad + \int f(H)[H,A]\bar{\partial}\tilde{f}(z)(H-z)^{-1}[\Gamma, H]\chi(H)(H-z)^{-1} dz \\ & \quad \overset{o}{=} f(H)[H,A]\Gamma f(H), \end{aligned}$$

which this splits into

$$f(H) d\Gamma^{\mathrm{ext}}(v_\xi \cdot \nabla\omega)\Gamma f(H) \tag{2.29a}$$
$$- f(H) d\Gamma^{\mathrm{ext}}(v_\xi) \cdot \nabla\Omega(\xi - d\Gamma(k))\Gamma f(H) \tag{2.29b}$$
$$- f(H)\Phi^{\mathrm{ext}}(i a_\xi v)\Gamma f(H) \tag{2.29c}$$

Now by (c)

$$\text{(2.29a)} \overset{o}{=} f(H)\Gamma d\Gamma^{\mathrm{ext}}(v_\xi \cdot \nabla\omega) f(H),$$

by (b), (c) and (a)

$$
\begin{aligned}
(2.29b) &\overset{o}{=} -f(H)\sum_{\ell=1}^{\nu} d\Gamma^{\mathrm{ext}}(v_\xi)_\ell \Gamma \partial_\ell \Omega(\xi - d\Gamma(k))f(H)\\
&= -f(H)\Gamma d\Gamma^{\mathrm{ext}}(v_\xi)\cdot\nabla\Omega(\xi - d\Gamma(k))f(H)\\
&\quad - f(H)\sum_{\ell=1}^{\nu}[d\Gamma^{\mathrm{ext}}(v_\xi)_\ell,\Gamma](H-i)^{-\frac{1}{2}}\partial_\ell\Omega(\xi - d\Gamma(k))(H-i)^{\frac{1}{2}}f(H)
\end{aligned}
$$

and by (d)

$$(2.29c) \overset{o}{=} -f(H)\Gamma\Phi^{\mathrm{ext}}(ia_\xi v)f(H).$$

Putting this together – and again using (a) and (i) – we see that

$$
\begin{aligned}
(2.29) &\overset{o}{=} f(H)\Gamma[H,A]f(H)\\
&= \chi(H)\Gamma f(H)[H,A]f(H)\\
&\quad + \int \bar{\partial}\tilde{f}(z)(H-z)^{-1}\chi(H)[\Gamma,H](H-z)^{-1}[H,A]f(H)dz\\
&\overset{o}{=} \Gamma f(H)[H,A]f(H) + o_R(1)f(H)[H,A]f(H),
\end{aligned}
$$

as wanted. □

We get the following important corollary to (i) and (ii) of Lemma 3.2.

Corollary 3.3. *Assume Conditions 1.1*(i), (ii), *1.2*(i), (v), (vi), *1.3*(i), (ii), (iv) *and 2.2. Let $f \in C_0^\infty(\mathbb{R})$. Then*

(i) $\check{\Gamma}(j^R)f(H(\xi)) = f(H^{\mathrm{ext}}(\xi))\check{\Gamma}(j^R) + o_R(1)$

(ii) $\check{\Gamma}(j^R)f(H(\xi))[H(\xi), iA_\xi]^\circ f(H(\xi))$
$= f(H^{\mathrm{ext}}(\xi))[H^{\mathrm{ext}}(\xi), iA_\xi^{\mathrm{ext}}]^\circ f(H^{\mathrm{ext}}(\xi))\check{\Gamma}(j^R) + o_R(1)$

We note that the first part of this corollary was already proved in [Møl05] in the case $s_\Omega \in \{0,1,2\}$. As the assumption of s_Ω being integer is only used in the proof of this result in [Møl05], this new proof implies the validity of the results in [Møl05] for non-integer values of s_Ω.

Lemma 3.4. *Assume Conditions 1.1*(i), (ii), *1.2*(i) *and 1.3*(i). *Then*

$$\check{\Gamma}(J^r)f(H(\xi)) = f(H_r^{\mathrm{ext}}(\xi))\check{\Gamma}(J^r) + o_r(1).$$

Proof. Note that

$$H(\xi) = \check{\Gamma}(J^r)^* H_r^{\mathrm{ext}}(\xi)\check{\Gamma}(J^r) + \int_{|k|\geq r} \big(v(k)a^*(k) + \overline{v(k)}a(k)\big)dk.$$

The operator $\check{\Gamma}(J^r)\check{\Gamma}(J^r)^*$ projects $\mathcal{F}^{\mathrm{ext}}$ onto $\Gamma(L^2(\Lambda_r)) \otimes \Gamma(L^2(\Lambda_r^{\mathrm{C}}))$ where $\Lambda_r = \{k \in \mathbb{R}^\nu \mid |k| < r\}$ and it commutes with $H_r^{\mathrm{ext}}(\xi)$, hence

$$\check{\Gamma}(J^r)H(\xi) = H_r^{\mathrm{ext}}(\xi)\check{\Gamma}(J^r) + \check{\Gamma}(J^r) \int_{|k|\geq r} \big(v(k)a^*(k) + \overline{v(k)}a(k)\big)dk.$$

Subtracting $z\check{\Gamma}(J^r)$ on both sides and multiplying with $(H_r^{\mathrm{ext}}(\xi) - z)^{-1}$ and $(H(\xi) - z)^{-1}$ from left respectively right, we get

$$\begin{aligned}(H_r^{\mathrm{ext}}(\xi) - z)^{-1}\check{\Gamma}(J^r) &= \check{\Gamma}(J^r)(H(\xi) - z)^{-1}\\ &+ (H_r^{\mathrm{ext}}(\xi) - z)^{-1}\check{\Gamma}(J^r) \int_{|k|\geq r} \big(v(k)a^*(k) + \overline{v(k)}a(k)\big)dk(H(\xi) - z)^{-1},\end{aligned}$$

where the expression on the last line of the equation is $|\mathrm{im}\, z|^{-2} o_r(1)$. The result is now obtained using calculus of almost analytic extensions. □

3.3 The Mourre estimate

Theorem 3.5 (Mourre Estimate). *Assume Conditions 1.1, 1.2*(i), (ii), (iii), (iv), (v), (vi), *1.3 and 1.4. Let $\xi \in \mathbb{R}^\nu$, $\lambda \notin \mathcal{T}_0^{(1)}(\xi)$ and suppose that $\Sigma_0^{(1)}(\xi) < \lambda < \Sigma_1^{(1)}(\xi)$. Then there exist $\kappa > 0$ and $c > 0$ such that*

$$E_{\lambda,\kappa}(H(\xi))[H(\xi), iA_\xi]^\circ E_{\lambda,\kappa}(H(\xi)) \geq cE_{\lambda,\kappa}(H(\xi)) + K. \tag{2.30}$$

Proof. Let $v_\xi(k) = \chi(k)\nabla_k\Sigma_0^{(1)}(\xi;k)$ with $\chi \in C_0^\infty(\mathcal{I}_0^{(1)}(\xi))$, see (2.2). Note that the function $k \mapsto \Sigma_0^{(1)}(\xi;k)$ is differentiable in $\mathcal{I}_0^{(1)}(\xi)$ by Proposition 1.12 and the assumptions. Clearly the conditions of Proposition 2.5 are satisfied for this choice of v_ξ, and we get that $H(\xi)$, $H^{\mathrm{ext}}(\xi)$ and $H^{(1)}(\xi)$ are of classes $C^1(A_\xi)$, $C^1(A_\xi^{\mathrm{ext}})$ and $C^1(A_\xi^{(1)})$, respectively.

Let $f \in C_0^\infty\big((\Sigma_0^{(1)}(\xi), \Sigma_1^{(1)}(\xi))\big)$. Calculate using Corollary 3.3

$$\begin{aligned}&f(H(\xi))\,[H(\xi), iA_\xi]^\circ\, f(H(\xi))\\ &\quad = \check{\Gamma}(j^R)^*\,\check{\Gamma}(j^R)\, f(H(\xi))\,[H(\xi), iA_\xi]^\circ\, f(H(\xi))\\ &\quad = \check{\Gamma}(j^R)^*\, f(H^{\mathrm{ext}}(\xi))\,[H^{\mathrm{ext}}(\xi), iA_\xi^{\mathrm{ext}}]^\circ\, f(H^{\mathrm{ext}}(\xi))\,\check{\Gamma}(j^R) + o_R(1)\end{aligned} \tag{2.31}$$

In analogy with (2.6), one sees that

$$f(H^{\text{ext}}(\xi))\,[H^{\text{ext}}(\xi), iA^{\text{ext}}_{\xi}]^{\circ}\,f(H^{\text{ext}}(\xi)) = \bigoplus_{\ell=0}^{\infty} f(H^{(\ell)}(\xi))\,[H^{(\ell)}(\xi), iA^{(\ell)}_{\xi}]^{\circ}\,f(H^{(\ell)}(\xi)), \tag{2.32}$$

where $H^{(0)}(\xi) := H(\xi)$ and $A^{(0)}_{\xi} := A_{\xi}$. If we insert (2.32) into (2.31) and look at the $\ell = 0$ contribution, we get

$$\begin{aligned}\Gamma(j_0^R)^*\,f(H(\xi))\,[H(\xi), iA_{\xi}]^{\circ}\,f(H(\xi))\,\Gamma(j_0^R)\\ = \Gamma(j_0^R)^*\,f(H(\xi))\,[H(\xi), iA_{\xi}]^{\circ}\,g(H(\xi))\,f(H(\xi))\,\Gamma(j_0^R) = BK\end{aligned} \tag{2.33}$$

for

$$\begin{aligned}&B = \Gamma(j_0^R)^*\,f(H(\xi))\,[H(\xi), iA_{\xi}]^{\circ}\,g(H(\xi)) \quad \text{and}\\ &K = f(H(\xi))\,\Gamma(j_0^R)\end{aligned}$$

where $g \in C_0^{\infty}(\mathbb{R})$ equals 1 on the support of f. Note that B is bounded, so to see that BK is compact, it is enough to prove that K is compact. Now by Lemma 3.4

$$K = \check{\Gamma}(J^r)^* f(H_r^{\text{ext}}(\xi))\check{\Gamma}(J^r)\Gamma(j_0^R) + o_r(1).$$

Like before, we split

$$\begin{aligned}&\check{\Gamma}(J^r)^* f(H_r^{\text{ext}}(\xi))\check{\Gamma}(J^r)\Gamma(j_0^R)\\ &\quad = \check{\Gamma}(J^r)^* f(H_r(\xi)) \oplus \Big(\bigoplus_{\ell=1}^{\infty}\int^{\oplus} f(H_r^{(\ell)}(\xi;k))dk\Big)\check{\Gamma}(J^r)\Gamma(j_0^R).\end{aligned} \tag{2.34}$$

The operator $\check{\Gamma}(J^r)$ maps $\mathcal{F}$ onto the subset $\Gamma(L^2(\Lambda_r)) \otimes \Gamma(L^2(\Lambda_r^{\mathsf{C}})) \subset \mathcal{F}^{\text{ext}}$, where again $\Lambda_r = \{k \in \mathbb{R}^{\nu} \mid |k| < r\}$. We split the subset

$$\Gamma\big(L^2(\Lambda_r)\big) \otimes \Gamma\big(L^2(\Lambda_r^{\mathsf{C}})\big) = \Gamma\big(L^2(\Lambda_r)\big) \oplus \Big(\bigoplus_{\ell=1}^{\infty}\Gamma\big(L^2(\Lambda_r)\big) \otimes \Gamma^{(\ell)}\big(L^2(\Lambda_r^{\mathsf{C}})\big)\Big),$$

and $\Gamma(L^2(\Lambda_r)) \otimes \Gamma^{(\ell)}(L^2(\Lambda_r^{\mathsf{C}}))$ we identify with $L^2_{\text{sym}}((\Lambda_r^{\mathsf{C}})^{\ell}; \Gamma(L^2(\Lambda_r)))$. Now $H_r(\xi) = H(\xi)$ and $H_r(\xi;k) = H(\xi;k)$ on $\Gamma(L^2(\Lambda_r))$ and the integrand in (2.34) is killed by $\check{\Gamma}(J^r)$ if $k < r$, so

$$(2.34) = \check{\Gamma}(J^r)^* f(H_r(\xi)) \oplus \Big(\bigoplus_{\ell=1}^{\infty}\int^{\oplus}_{(\Lambda_r^{\mathsf{C}})^{\ell}} f(H^{(\ell)}(\xi;k))dk\Big)\check{\Gamma}(J^r)\Gamma(j_0^R), \tag{2.35}$$

Note that

$$\Sigma_1^{(1)}(\xi) \le \Sigma_0^{(2)}(\xi), \tag{2.36}$$

hence we see that $H^{(\ell)}(\xi;k) \ge H^{(\ell)}(\xi) \ge \Sigma_0^{(\ell)}(\xi)\mathbb{1}_{\mathcal{H}^{(\ell)}} \ge \Sigma_1^{(1)}\mathbb{1}_{\mathcal{H}^{(\ell)}}$ for $\ell \ge 2$, cf. (2.5). It follows that $f(H^{(\ell)}(\xi;k)) = 0$ for $\ell \ge 2$ and by Corollary 1.10 r can be chosen so large that $f(H^{(\ell)}(\xi;k)) = 0$ for $\ell = 1$ and $|k| \ge r$. The remaining part of (2.35) equals $f(H_r(\xi))\Gamma(\chi_{\Lambda_r})\Gamma(j_0^R)$, which clearly is compact, hence we see by letting $r \to \infty$ that K is compact.

By the same argument as above, we only get one more contribution from (2.32), namely

$$f(H^{(1)}(\xi))\,[H^{(1)}(\xi), iA_\xi^{(1)}]^\circ f(H^{(1)}(\xi)).$$

Since we have $\Sigma_0^{(1)}(\xi) \le \Sigma_0^{(1)}(\xi;k)$ and $\Sigma_1^{(1)}(\xi) \le \Sigma_n^{(\ell)}(\xi;k)$ for any $n,\ell \ge 1$ while $\Sigma_0^{(1)}(\xi) < \lambda < \Sigma_1^{(1)}(\xi)$, the only possible solution for $\lambda = \Sigma_n^{(\ell)}(\xi;k)$ is with $n = 0$ and $\ell = 1$.

Note that (2.36) and Lemma 1.13 implies that locally there are only finitely many points in $\mathcal{T}_0^{(1)}(\xi)$ between $\Sigma_0^{(1)}(\xi)$ and $\Sigma_1^{(1)}(\xi)$.

Let $k_0 \in \mathbb{R}^\nu$. Assume $\lambda = \Sigma_0^{(1)}(\xi;k_0)$. Then by Lemma 1.13, we can choose a number κ_{k_0} such that $\mathrm{dist}(\lambda, \mathcal{T}_0^{(1)} \cup \{\Sigma_0^{(1)}\} \cup \{\Sigma_0^{(1)}\}) > \kappa_{k_0} > 0$ and a neighbourhood $\mathcal{O}_{k_0}$ of k_0 such that the conditions of Theorem 3.1 are satisfied. This implies that

$$\begin{aligned} &\mathbb{1}_{\mathcal{O}_{k_0}} E_{\lambda,\kappa_0}(H^{(1)}(\xi))[H^{(1)}(\xi), iA_\xi^{(1)}]^\circ E_{\lambda,\kappa_0}(H^{(1)}(\xi))\mathbb{1}_{\mathcal{O}_{k_0}} \\ &\quad = \int_{\mathcal{O}_{k_0}}^\oplus \chi(k)|\nabla\Sigma_0^{(1)}(\xi;k)|^2 \mathbb{1}_{\mathcal{F}}\,dk\, E_{\lambda,\kappa_0}(H^{(1)}(\xi)). \end{aligned} \tag{2.37}$$

Now assume that $\Sigma_0^{(1)}(\xi;k_0) \ne \lambda$. Then we can choose a number κ_{k_0} such that $\mathrm{dist}(\lambda, \mathcal{T}_0^{(1)} \cup \{\Sigma_0^{(1)}\} \cup \{\Sigma_0^{(1)}\}) > \kappa_{k_0} > 0$ and a neighbourhood $\mathcal{O}_{k_0}$ of k_0 such that $\mathbb{1}_{\mathcal{O}_{k_0}} E_{\lambda,\kappa_0}(H^{(1)}(\xi)) = 0$. Then the following trivially holds, e.g. with $c_{k_0} = 1$.

$$\begin{aligned} &\mathbb{1}_{\mathcal{O}_{k_0}} E_{\lambda,\kappa_{k_0}}(H^{(1)}(\xi))[H^{(1)}(\xi), iA_\xi^{(1)}]^\circ E_{\lambda,\kappa_{k_0}}(H^{(1)}(\xi))\mathbb{1}_{\mathcal{O}_{k_0}} \\ &\quad \ge c_{k_0}\mathbb{1}_{\mathcal{O}_{k_0}} E_{\lambda,\kappa_{k_0}}(H^{(1)}(\xi)). \end{aligned} \tag{2.38}$$

In the Corollary 1.10 argument given above, we found an r such that if $|k| \ge r$ then $f(H^{(1)}(\xi;k)) = 0$. As Λ_r is compact, there exists a finite cover $\cup_{k\in F}\mathcal{O}_k$ of Λ_r.

Let $\kappa = \frac{1}{2}\min_{k\in F}\{\kappa_k\}$ and $X = \{k \in \mathbb{R}^\nu \,|\, \Sigma_0^{(1)}(\xi;k) \in [\lambda-\kappa, \lambda+\kappa]\}$. Then we have $X \subset \mathcal{I}_0^{(1)}(\xi)$ and again Corollary 1.10 gives compactness of X. Choose $\chi \in C_0^\infty(\mathcal{I}_0^{(1)}(\xi))$ such that $\chi = 1$ on X. For this χ, (2.37) implies that for all $k_0 \in F$ for which $\Sigma_0^{(1)}(\xi;k_0) = \lambda$ there exists a $c_{k_0} > 0$ such that (2.38) holds.

If we now take $c = \frac{1}{2}\min_{k\in F}\{c_k\}$, multiply both sides of (2.38) from left and right with $E_{\lambda,2\kappa}(H^{(1)}(\xi))$, then it follows that

$$\begin{aligned} &E_{\lambda,2\kappa}(H^{(1)}(\xi))[H^{(1)}(\xi), iA_\xi^{(1)}]^\circ E_{\lambda,2\kappa}(H^{(1)}(\xi)) \\ &\quad \geq 2cE_{\lambda,2\kappa}(H^{(1)}(\xi)). \end{aligned} \tag{2.39}$$

Choose now an $f \in C_0^\infty((\lambda-2\kappa,\lambda+2\kappa);[0,1])$ such that $E_{\lambda,\kappa} \leq f \leq E_{\lambda,2\kappa}$. Then we get by inserting (2.33) and (2.39) into (2.31) via (2.32) that

$$\begin{aligned} &f(H(\xi))\,[H(\xi), iA_\xi]^\circ f(H(\xi)) \\ &\quad \geq 2c\check{\Gamma}(j^R) f(H^{\text{ext}}(\xi))^2 \check{\Gamma}(j^R) + o_R(1) + K \\ &\quad \geq 2cf(H(\xi))^2 + o_R(1) + K. \end{aligned} \tag{2.40}$$

Now choose R so large that $\|o_R(1)\| \leq c$ and sandwich both ends of (2.40) with $E_{\lambda,\kappa}(H(\xi))$, then we get

$$\begin{aligned} &E_{\lambda,\kappa}(H(\xi))\,[H(\xi), iA_\xi]^\circ E_{\lambda,\kappa}(H(\xi)) \\ &\quad = E_{\lambda,\kappa}(H(\xi))\, f(H(\xi))\,[H(\xi), iA_\xi]^\circ f(H(\xi))\, E_{\lambda,\kappa}(H(\xi)) \\ &\quad \geq 2cE_{\lambda,\kappa}(H(\xi))\big(f(H(\xi))^2 + o_R(1) + K\big)E_{\lambda,\kappa}(H(\xi)) \\ &\quad \geq cE_{\lambda,\kappa}(H(\xi)) + E_{\lambda,\kappa}(H(\xi))\, K\, E_{\lambda,\kappa}(H(\xi)), \end{aligned}$$

which is of the form (2.30). □

3.4 The limiting absorption principle

Theorem 3.6. *Assume Conditions 1.1, 1.2, 1.3 and 1.4. Let $\xi \in \mathbb{R}^\nu$ and $J \subset \big(\inf \sigma_{\text{ess}}(H(\xi)), \Sigma_1^{(1)}(\xi)\big) \setminus \big(\mathcal{T}_0^{(1)}(\xi) \cup \mathcal{E}(\xi)\big)$ be closed, where $\mathcal{E}(\xi)$ denotes the set of eigenvalues of $H(\xi)$, and $s > \frac{1}{2}$. Then*

$$\sup_{z\in J^\pm} \|\langle A_\xi\rangle^{-s}(H(\xi)-z)^{-1}\langle A_\xi\rangle^{-s}\| < \infty,$$

where $J^\pm = \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \in J, \pm\,\mathrm{Im}(z) > 0\}$.

Note that $\mathcal{E}(\xi)$ is also locally finite in $\big(\inf \sigma_{\text{ess}}(H(\xi)), \Sigma_1^{(1)}(\xi)\big)$.

Proof. By Remark 1.19, Theorem 3.5 and Proposition 2.5 with v_ξ chosen as in the proof of Theorem 3.5, Theorem 1.22 is applicable. □

Combined with the results of [FMSa], [FMSb] and [MW], Theorem 3.6 implies the following.

Corollary 3.7. *Assume Conditions 1.2, 1.3 and 1.4 and that* $\Omega(\eta) = \frac{\eta^2}{2M}$. *Then*

$$\{(\xi, \lambda) \in \mathbb{R}^{\nu+1} \,|\, \Sigma_{\text{ess}}(\xi) \leq \lambda \leq \Sigma_0^{(1)}(\xi), \lambda \in \mathcal{E}(\xi) \cup \Theta(\xi)\}$$

is closed and $\mathcal{E}(\xi) \cup \Theta(\xi) \cap [\Sigma_{\text{ess}}(\xi), \Sigma_0^{(1)}(\xi)]$ *is at most countable.*

A A Taylor-like Expansion of $[B, f(A_1, \dots, A_\nu)]$

We now recall a result from [Ras].

In the following, $A = (A_1, \dots, A_\nu)$ is a vector of self-adjoint, pairwise commuting operators acting on a Hilbert space $\mathcal{H}$, and $B \in \mathcal{B}(\mathcal{H})$ is a bounded operator on $\mathcal{H}$. We shall use the notion of B being of class $C^{n_0}(A)$ introduced in [ABG96]. For notational convenience, we adobt the following convention: If $0 \leq j \leq \nu$, then δ_j denotes the multi-index $(0, \dots, 0, 1, 0, \dots, 0)$, where the 1 is in the j'th entry.

Definition A.1. Let $n_0 \in \mathbb{N} \cup \{\infty\}$. Assume that the multi-commutator form defined iteratively by $\text{ad}_A^0(B) = B$ and $\text{ad}_A^\alpha(B) = [\text{ad}_A^{\alpha-\delta_j}(B), A_j]$ as a form on $\mathcal{D}(A_j)$, where $\alpha \geq \delta_j$ is a multi-index and $1 \leq j \leq \nu$, can be represented by a bounded operator also denoted by $\text{ad}_A^\alpha(B)$, for all multi-indices α, $|\alpha| < n_0 + 1$. Then B is said to be of class $C^{n_0}(A)$ and we write $B \in C^{n_0}(A)$.

Remark A.2. The definition of $\text{ad}_A^\alpha(B)$ does not depend on the order of the iteration since the A_j are pairwise commuting. We call $|\alpha|$ the *degree* of $\text{ad}_A^\alpha(B)$.

In the following, $\mathcal{H}_A^s := D(|H|^s)$ for $s \geq 0$ will be used to denote the scale of spaces associated to A. For negative s, we define $\mathcal{H}_A^s := {\mathcal{H}_A^s}^*$.

Theorem A.3. *Assume that* $B \in C^{n_0}(A)$ *for some* $n_0 \geq n + 1 \geq 1$, $0 \leq t_1 \leq n + 1$, $0 \leq t_2 \leq 1$ *and that* $\{f_\lambda\}_{\lambda \in I}$ *satisfies*

$$\forall \alpha \, \exists C_\alpha \colon |\partial^\alpha f_\lambda(x)| \leq C_\alpha \langle x \rangle^{s-|\alpha|}$$

uniformly in λ for some $s \in \mathbb{R}$ such that $t_1 + t_2 + s < n + 1$. Then

$$[B, f_\lambda(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!} \partial^\alpha f_\lambda(A)\ \mathrm{ad}_A^\alpha(B) + R_{\lambda,n}(A, B)$$

as an identity on $\mathcal{D}(\langle A\rangle^s)$, where $R_{\lambda,n}(A, B) \in \mathcal{B}(\mathcal{H}_A^{-t_2}, \mathcal{H}_A^{t_1})$ and there exist a constant C independent of A, B and λ such that

$$\|R_{\lambda,n}(A, B)\|_{\mathcal{B}(\mathcal{H}_A^{-t_2}, \mathcal{H}_A^{t_1})} \leq C \sum_{|\alpha|=n+1} \|\mathrm{ad}_A^\alpha(B)\|.$$

B Bibliography

[ABG96] W. O. Amrein, A. Boutet de Monvel, and V. Georgescu, *C_0-groups, commutator methods and spectral theory of N-body hamiltonians*, Birkhäuser, 1996.

[AD10] A. S. Alexandrov and J. T. Devreese, *Advances in polaron physics*, Springer Series in Solid-State Sciences, Springer, Heidelberg, 2010.

[AMZ05] N. Angelescu, R. Minlos, and V. Zagrebnov, *Lower spectral branches of a particle coupled to a Bose field*, Rev. Math. Phys. **17** (2005), 1111–1142.

[Can71] J. T. Cannon, *Quantum field theoretic properties of a model of Nelson: Domain and eigenvector stability for perturbed linear operators*, J. Funct. Anal. **8** (1971), no. 1, 101–152.

[DG99] J. Dereziński and C. Gérard, *Asymptotic completeness in quantum field theory. massive Pauli-Fierz Hamiltonians*, Rev. Math. Phys. **11** (1999), 383–450.

[FMSa] J. Faupin, J. S. Møller, and E. Skibsted, *Regularity of bound states*, In prep.

[FMSb] ———, *Second order perturbation theory for embedded eigenvalues*, In prep.

[Frö54] H. Fröhlich, *Electrons in lattice fields*, Adv. in Phys. **3** (1954), 325–362.

[Frö73] J. Fröhlich, *On the infrared problem in a model of scalar electrons and massless scalar bosons*, Ann. Inst. Henri Poincaré **19** (1973), 1–103.

[Frö74] ______, *Existence of dressed one-electron states in a class of persistent models*, Fortschr. Phys. **22** (1974), 159–198.

[Gér08] C. Gérard, *A proof of the abstract limiting absorption principle by energy estimates*, J. Funct. Anal. **254** (2008), no. 11, 2707–2724.

[GGM04] V. Georgescu, C. Gérard, and J. S. Møller, *Commutators, C_0-semigroups and resolvent estimates*, J. Funct. Anal. **216** (2004), 303–361.

[GJ06] S. Golenia and T. Jecko, *A new look at Mourre's commutator theory*, 2006.

[LLP53] T. D. Lee, F. E. Low, and D. Pines, *The motion of slow electrons in a polar crystal*, Phys. Rev. **90** (1953), 297–302.

[Mou81] E. Mourre, *Absence of Singular Continuous Spectrum for Certain Self-Adjoint Operators*, Commun. Math. Phys. **78** (1981), 391–408.

[MW] J. S. Møller and M. Westrich, *Regularity of eigenstates in regular mourre theory*, In prep.

[Møl05] J. S. Møller, *The translation invariant massive Nelson model: I. The bottom of the spectrum*, Ann. Henri Poincaré **6** (2005), 1091–1135.

[Møl06a] J. S. Møller, *On the essential spectrum of the translation invariant Nelson model*, Mathematical Physics of Quantum Mechanics (J. Asch and A. Joye, eds.), Lecture Notes in Physics, vol. 690, Springer, 2006, pp. 179–195.

[Møl06b] J. S. Møller, *The polaron revisited*, Rev. Math. Phys. **18** (2006), no. 5, 485–517.

[Nel64] E. Nelson, *Interaction of non-relativistic particles with a quantized scalar field*, J. Math. Phys. **5** (1964), 1190–1197.

[Ras] M. G. Rasmussen, *A Taylor-like expansion of a commutator with a function of self-adjoint, pairwise commuting operators*, Submitted.

[RS75] M. Reed and B. Simon, *Methods of modern mathematical physics II, fourier analysis, self-adjointness*, Academic Press, 1975.

[Spo04] H. Spohn, *Dynamics of charged particles and their radiation field*, Cambridge University Press, Cambridge, 2004.

Chapter 3

Asymptotic Completeness in Quantum Field Theory: Translation Invariant Nelson Type Models Restricted to the Vacuum and One-Particle Sectors

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Abstract

Time-dependent scattering theory for a large class of translation invariant models, including the Nelson and Polaron models, restricted to the vacuum and one-particle sectors is studied. Asymptotic completeness of these Hamiltonians is shown. The translation invariance imply that the Hamiltonian is fibered with respect to the total momentum. On the way to asymptotic completeness we determine the spectral structure of the fiber Hamiltonians, establish a Mourre estimate and derive a geometric asymptotic completeness statement as an intermediate step.

Keywords: quantum field theory, time-dependent scattering theory, asymptotic completeness, translation invariance
Mathematics Subject Classification (2010): 81Q10, 47A40, 81T10, 81U30

1 Introduction and motivation

In this paper, we study the spectral and scattering theory of a class of Hamiltonians that arise when one restricts e.g. the Nelson or Polaron model to the subspace of at most one field particle. As our results are valid for both models, we will use the term "field particles" rather than photons or phonons, and in the same spirit, we will use the term "matter particle" rather than electron or positron.

In [MR], two of the authors prove a Mourre estimate and C^2 regularity for the full model, with respect to a suitably chosen conjugate operator. The estimate holds in the part of the energy-momentum spectrum lying between the bottom of the essential energy-momentum spectrum and either the two-body threshold, if there are no exited isolated mass shells, or the one-body threshold pertaining to the first exited isolated mass shell, if it exists. This is a natural first step for scattering theory. As the full model in that energy-momentum regime is expected to resemble the model with at most one field particle in many aspects, the scattering theory of the cut-off model is of obvious interest. We note that in [GJY03], the spectral and scattering theory of the massless Nelson model is studied, and that the stationary methods used there would to some extend also work on the class of models considered here. However, the scattering theory in [GJY03] is obtained via a Kato-Birman argument, a method one cannot hope to work on the full model.

In recent years a lot of effort was put into investigating the spectral and scattering theory of various models of quantum field theory (see among many other papers [Amm00], [AMZ05], [DG99], [FGS04], [FGS08], [Gér96], [Piz03], [Spo04] and references therein). Substantial progress was made by applying methods originally developed in the study of N-particle Schrödinger operators namely the Mourre positive commutator method and the method of propagation observables to study the behavior of the unitary group e^{-itH} for large times. Up to now, the most complete results on the scattering theory for these models have only been available for models where the translation invariance is broken [Amm00], [DG99], [Gér96], [Piz03], [Spo04], or for small coupling constants [FGS04]. In fact the only asymptotic completeness result valid for arbitrary coupling strength, in time-dependent scattering theory of translation invariant models known to us are variations of the N-body problem, where the dispersion relations are of the non-relativistic form $\frac{p^2}{M}$. Our results hold for a large class of dispersion relations, including a combination of the relativistic and non-relativistic choices.

In order to appreciate the difficulties associated with proving asymptotic completeness for translation invariant models of QFT, we explain the structure of scattering channels. If a system starts in a scattering state at total momentum ξ and energy E, it will emit field particles with momenta $k_1, \dots, k_n$ until the remaining interacting system reaches a a total momentum ξ' and an eigenvalue $E'(\xi')$ for the Hamiltonian at total momentum ξ'. In order to conserve energy and momentum we must have $\xi = \xi' + k_1 + \cdots + k_n$ and $E = E'(\xi') + \omega(k_1) + \cdots + \omega(k_n)$, where ω is the dispersion relation for the field.

That is, the scattering channels are labeled by bound states at momenta ξ' and the number of emitted field particles n, under the constraint of conservation of energy and total momentum. The resulting bound particle will not be at rest but rather move according to a dispersion relation which is in fact the eigenvalue band, or mass shell, to which it belongs. This band may a priori be an isolated mass shell or an embedded one. If one wants to capture the behaviour of scattering states through a Mourre estimate, then one needs to build into a conjugate operator the dynamics of all the mass shells that appear in the available channels. This is a difficult task. The thresholds at total momentum ξ are energies E that has a scattering channel with the property that the bound state and the emitted field particles do not separate over time.

When introducing a number cutoff in the model, one simplifies the situation in that the scattering channels are now labeled by bound states of Hamiltonians with strictly fewer field particles. In particular in our case, we can label the scattering channels by mass shells of the Hamiltonian on the vacuum sector, which are easily understood. Indeed, there is in fact only one mass shell and it is identical to the matter dispersion relation Ω.

Finally, we will briefly outline the contents of this paper. In Section 2 we introduce the model in details and state our main result, the asymptotic completeness. In Section 3 we briefly go through the spectral theory for the fiber Hamiltonians, in particular we prove an HVZ theorem, a Mourre estimate, absence of singular continuous spectrum and a semicontinuity statement about the Mourre estimate. In Section 4 we prove the following propagation estimates: A large velocity estimate, a phase-space propagation estimate, an improved phase-space propagation estimate and a minimal velocity estimate. These form the technical foundation for Section 5, where we introduce the asymptotic observable, the spaces of asymptotically bound resp. free particles, the wave operators and prove asymptotic completeness via so-called geometric asymptotic completeness.

2 The model and the result

The Hilbert space for the Hamiltonian is

$$\mathcal{H} = L^2(\mathbb{R}^\nu, \mathrm{d}y) \otimes (\mathbb{C} \oplus L^2(\mathbb{R}^\nu, \mathrm{d}x)) = L^2(\mathbb{R}^\nu, \mathrm{d}y) \oplus L^2(\mathbb{R}^{2\nu}, \mathrm{d}x\mathrm{d}y),$$

where $\nu \in \mathbb{N}$. We write $D_x = -\mathrm{i}\nabla_x$, $D_y = -\mathrm{i}\nabla_y$ for the respective momentum operators. The Hamiltonian we wish to study the spectral and scattering theory of is given by

$$H = H_0 + V = \begin{pmatrix} \Omega(D_y) & 0 \\ 0 & \Omega(D_y) + \omega(D_x) \end{pmatrix} + \begin{pmatrix} 0 & v^* \\ v & 0 \end{pmatrix},$$

where

$$(vu_0)(x,y) = \rho(x-y)u_0(y) \quad \text{and} \quad (v^*u_1)(x) = \int \rho(x-y)u_1(x,y)dy$$

for some $\rho \in L^2(\mathbb{R}^\nu)$. Here Ω is the dispersion relation for the matter particle, ω the dispersion relation for the field particles and ρ a coupling function. One may view it as the translation invariant Nelson or Polaron model restricted to the subspace with at most one field particle, depending on the choice of dispersion relations.

The coupling function will be assumed to satisfy a short-range condition which implies a UV-cutoff (see Condition 2.3). We work with more general dispersion relations ω and Ω than $\omega(k) = \sqrt{k^2 + m^2}$ or $\omega(k) = \omega_0 > 0$ and $\Omega(\eta) = \eta^2/2M$ respectively (see Conditions 2.1 and 2.2 for details). As the infrared problem is not present in this model due to the finite number of field particles, the mass of the field particle is not important. However, the singular behavior of the dispersion relation $\omega(k) = |k|$ at $k = 0$ makes this choice fall outside of what can be handled in this treatment, although it seems likely that one with minor adjustments may include this case in the same framework. For a treatment of the case where $\Omega(\eta) = \frac{1}{2}\eta^2$ and $\omega(k) = |k|$, see [GJY03].

The operator H commutes with the operator of total momentum, $\mathrm{P}=\begin{pmatrix} D_y & 0 \\ 0 & D_x+D_y \end{pmatrix}$, and hence H is fibered, $H = U^{-1}\int_{\mathbb{R}^\nu}^{\oplus} H(P)\mathrm{d}PU$, where

$$U(u_0, u_1)(x, y) = (u_0(y), u_1(y, x + y))$$

and

$$H(P) = H_0(P) + \tilde{V} = \begin{pmatrix} \Omega(P) & 0 \\ 0 & \Omega(P - D_x) + \omega(D_x) \end{pmatrix} + \begin{pmatrix} 0 & \langle\rho| \\ |\rho\rangle & 0 \end{pmatrix},$$

where $\langle\cdot|$ and $|\cdot\rangle$ denote the Dirac brackets. The fiber Hamiltonians are operators on the Hilbert space $\mathcal{K} = \mathbb{C} \oplus L^2(\mathbb{R}^\nu)$.

The precise assumptions on Ω, ω and ρ are given below.

Condition 2.1 (Matter particle dispersion relation). Let $\Omega \in C^\infty(\mathbb{R}^\nu)$ be a non-negative, real-analytic and rotation invariant[1] function. There exists $s_\Omega \in [0, 2]$ such that Ω satisfies:

(i) There is a $C > 0$ such that $\Omega(\eta) \geq C^{-1}\langle\eta\rangle^{s_\Omega} - C$.

(ii) For any multi-index α there is a $C_\alpha > 0$ such that $|\partial^\alpha\Omega(\eta)| \leq C_\alpha\langle\eta\rangle^{s_\Omega - |\alpha|}$.

Note that this assumption is satisfied by the standard non-relativistic and relativistic choices, $\Omega(\eta) = \frac{\eta^2}{2M}$ and $\Omega(\eta) = \sqrt{\eta^2 + M^2}$.

Condition 2.2 (Field particle dispersion relation). Let $\omega \in C^\infty(\mathbb{R}^\nu)$ be non-negative, real-analytic, rotation invariant and satisfy:

(i) For any multi-index α with $|\alpha| \geq 1$, we have $\sup_{k\in\mathbb{R}^\nu}|\partial^\alpha\omega(k)| < \infty$.

(ii) If $s_\Omega = 0$, then $\omega(k) \to \infty$ as $|k| \to \infty$.

This is satisfied e.g. for $\omega(k) = \sqrt{k^2 + m^2}$, $m \neq 0$, and if $s_\Omega \neq 0$, also for the Polaron[2], $\omega(k) = \omega_0$.

Condition 2.3 (Coupling function). Let $\rho \in L^2(\mathbb{R}^\nu)$ be rotation invariant and satisfy that

(i) $\hat{\rho} \in C^2(\mathbb{R}^\nu)$.

(ii) $\langle\cdot\rangle|\nabla\hat{\rho}|, \partial_j\hat{\rho}, \langle\cdot\rangle\|\nabla^2\hat{\rho}\| \in L^2(\mathbb{R}^\nu)$.

[1]By rotation invariance of a function f we mean that $f(\eta) = f(O\eta)$ a.e. for any $O \in O(\nu)$ where $O(\nu)$ denotes the ν-dimensional orthogonal group.

[2]In fact the Fröhlich Polaron has $\Omega(\eta) = \frac{\eta^2}{2M_{\mathrm{eff}}}$, so $s_\Omega = 2 \neq 0$.

(iii) There exist constants $C, \mu > 0$ such that $|\rho(x)| \leq C\langle x\rangle^{-1-\frac{\nu}{2}-\mu}$.

Condition 2.3 (iii) is the so-called short-range condition. Note that it implies that for $J \in C^\infty(\mathbb{R}^\nu)$ with support away from 0, we have

$$|\rho(x)J(\tfrac{x}{t})| = O(t^{-1-\mu}). \tag{3.1}$$

For the rest of this paper, Conditions 2.1, 2.2 and 2.3 will tacitly be assumed to be fulfilled, and under this assumption, our main result will be the following

Theorem 2.4 (Asymptotic completeness). *The wave operator*

$$W^+ = \text{s-}\lim_{t\to\infty} e^{\mathrm{i}tH}e^{-\mathrm{i}tH_0}P^+(H_0)$$

exists, where $P^+(H_0)$ is the projection onto $\{0\} \oplus L^2(\mathbb{R}^{2\nu})$, and the system is asymptotically complete:

$$\operatorname{Ran} W^+ = \mathcal{H}_{\mathrm{bd}}^\perp,$$

where $\mathcal{H}_{\mathrm{bd}} = U^{-1}\int_{\mathbb{R}^\nu}^\oplus \mathbb{1}_{\mathrm{pp}}(H(P))\mathrm{d}PU\mathcal{H}$.

Remark 2.5. That $P \mapsto \mathbb{1}_{\mathrm{pp}}(H(P))$ is weakly – and hence strongly – measurable follows from an application of the RAGE theorem, [CFKS87, Theorem 5.8], see the proof of [CFKS87, Theorem 9.4] for details.

3 Spectral analysis

We begin by recalling the following well-known properties of the fibered Hamiltonian. The Hamiltonian $H_0(P)$ is essentially self-adjoint on $\mathbb{C} \oplus C_0^\infty(\mathbb{R}^\nu)$ and the domain $\mathcal{D} = \mathcal{D}(H_0(P))$ is independent of P. As $\tilde{V}$ is bounded, the Kato-Rellich theorem implies that the same is true for $H(P)$ and that $\mathcal{D}(H(P)) = \mathcal{D}$.

The following threshold set will play an important role in our analysis:

$$\theta(P) = \{\lambda \in \mathbb{R} \,|\, \exists k \in \mathbb{R}^\nu : \lambda = \Sigma(P-k) + \omega(k), \nabla\Omega(P-k) = \nabla\omega(k)\}.$$

By rotation invariance and analyticity it is easy to see that $\theta(P)$ is locally finite and closed.

The following results, Theorems 3.1 to 3.4, correspond to completely analogous statements for the full model, see [MR].

Theorem 3.1. *Assume that the vector field $v_P \in C^\infty(\mathbb{R}^\nu;\mathbb{R}^\nu)$ satisfies that for any multi-index α, $|\alpha| \in \{0,1,2\}$, there is a constant $C_\alpha > 0$ such that*

$|\partial^\alpha v_P(\eta)| \leq C_\alpha \langle \eta \rangle^{1-|\alpha|}$. *Then the operator* $a_P = \frac{1}{2}(v_P(D_x) \cdot x + x \cdot v_P(D_x))$ *is essentially self-adjoint on the Schwarz space* $\mathcal{S}$ *and* $H(P)$ *is of class* $C^2(A_P)$, *where* $A_P = \left(\begin{smallmatrix} 0 & 0 \\ 0 & a_P \end{smallmatrix}\right)$ *is self-adjoint on* $\mathcal{D}(A_P)$. *The first commutator is given by*

$$[H(P), \mathrm{i}A_P]^\circ = \begin{pmatrix} 0 & \langle \mathrm{i}a_P\rho| \\ |\mathrm{i}a_P\rho\rangle & v_P(D_x) \cdot \nabla(\omega(D_x) + \Omega(P - D_x)) \end{pmatrix}$$

as a form on $\mathcal{D}$.

This can be seen either by direct computations or by following [MR].

We now introduce the extended space $\mathcal{K}^{\mathrm{ext}} = \mathcal{K} \oplus L^2(\mathbb{R}^\nu)$ to be able to make a geometric partition of unity in configuration space. The partition of unity is similar to what is done in the analysis of the N-body Schrödinger operator (see e.g. [DG97]) and in complete analogy with what is done in e.g. [DG99] and [Møl05]. The partition of unity used here may actually be seen as the partition of unity introduced in [DG99] restricted to the subspace with at most 1 field particle.

Let $j_0, j_\infty \in C^\infty(\mathbb{R}^\nu)$ be real, non-negative functions satisfying $j_0 = 1$ on $\{x \,|\, |x| \leq 1\}$, $j_0 = 0$ on $\{x \,|\, |x| > 2\}$ and $j_0^2 + j_\infty^2 = 1$. We now define

$$\begin{aligned} j^R &: \mathcal{K} \to \mathcal{K}^{\mathrm{ext}} \\ j^R(v_0, v_1) &= (v_0, j_0(\tfrac{\cdot}{R})v_1) \oplus (j_\infty(\tfrac{\cdot}{R})v_1). \end{aligned}$$

Clearly, j^R is isometric.

We introduce two self-adjoint operators, the extended Hamiltonian, $H^{\mathrm{ext}}(P)$, and the extended conjugate operator, A_P^{ext}, acting in $\mathcal{K}^{\mathrm{ext}}$,

$$\begin{aligned} H^{\mathrm{ext}}(P) &= H(P) \oplus F_P(D_x) \text{ and} \\ A_P^{\mathrm{ext}} &= A_P \oplus a_P, \end{aligned}$$

where $F_P(D_x) = \omega(D_x) + \Omega(P - D_x)$, with the obvious domains denoted by $\mathcal{D}^{\mathrm{ext}}$ and $\mathcal{D}(A_p^{\mathrm{ext}})$. The extended Hamiltonian describes an interacting system and a system with a free field particle. It is easy to see that Theorem 3.1 holds true with $H(P)$ and A_P replaced by $H^{\mathrm{ext}}(P)$ and A_P^{ext}, respectively, and the commutator equal to

$$[H^{\mathrm{ext}}(P), \mathrm{i}A_p^{\mathrm{ext}}]^\circ = [H(P), \mathrm{i}A_P]^\circ \oplus \big(v_P(D_x) \cdot (\nabla\omega(D_x) - \nabla\Omega(P - D_x))\big).$$

We have the following localisation error when applying j^R.

Lemma 3.2. *Let* $f \in C_0^\infty(\mathbb{R})$. *Then*

$$\begin{aligned} &j^R f(H(P)) = f(H^{\mathrm{ext}}(P))j^R + o_R(1) \quad \textit{and} \\ &j^R f(H(P))[H(P), \mathrm{i}A_P]^\circ f(H(P)) \\ &\qquad = f(H^{\mathrm{ext}}(P))[H^{\mathrm{ext}}(P), \mathrm{i}A_P^{\mathrm{ext}}]^\circ f(H^{\mathrm{ext}}(P))j^R + o_R(1), \end{aligned}$$

for $R \to \infty$.

This can be seen either by a direct computation or by applying [MR, Corollary 5.3]. The following two results, an HVZ theorem and a Mourre estimate, are now almost immediate.

Theorem 3.3. *The spectrum of* $H(P)$ *below* $\Sigma_{\mathrm{ess}}(P) = \inf_{k\in\mathbb{R}^\nu}\{\Omega(P-k)+\omega(k)\}$ *consists at most of eigenvalues of finite multiplicity and can only accumulate at* $\Sigma_{\mathrm{ess}}(P)$. *The essential spectrum is given by* $\sigma_{\mathrm{ess}}(H(P)) = [\Sigma_{\mathrm{ess}}(P), \infty)$.

Proof. Using Lemma 3.2 for an $f \in C_0^\infty(\mathbb{R})$ supported in $(-\infty, \Sigma_{\mathrm{ess}}(P))$ and letting R tend to infinity shows that $f(H(P))$ is compact. This proves the first part.

To prove the last part, let $\lambda \in [\Sigma_{\mathrm{ess}}(P), \infty)$ and note that there exists a $k_0 \in \mathbb{R}^\nu$ such that $\lambda = \Omega(P-k_0) + \omega(k_0)$. Now choose $u_n = (0, u_{1n}) \in \mathbb{C} \oplus L^2(\mathbb{R}^\nu)$ with $\hat{u}_{1n}(\cdot) = n^{\frac{\nu}{2}} f(n(\cdot - k_0))$ for some $f \in C_0^\infty(\mathbb{R}^\nu)$ with $f \geq 0$ and $f(0) = 1$. One may now check that u_n is a Weyl sequence for the energy λ. □

Theorem 3.4. *Assume that* $\lambda \notin \theta(P)$. *Let* A_P *be given as in Theorem 3.1 with* $v_P(D_x) = \nabla\omega(D_x) - \nabla\Omega(P - D_x))$. *Then there exist constants* $\kappa, c > 0$ *and a compact operator* K *such that*

$$E_{\lambda,\kappa}(H(P))[H(P), \mathrm{i}A_P]^\circ E_{\lambda,\kappa}(H(P)) \geq cE_{\lambda,\kappa}(H(P)) + K,$$

where $E_{\lambda,\kappa}$ *denotes the characteristic function of the interval* $[\lambda-\kappa, \lambda+\kappa]$.

Proof. We may find a κ such that $[\lambda - 2\kappa, \lambda + 2\kappa] \cap \theta(P) = \emptyset$. Choose $f \in C_0^\infty(\mathbb{R})$ with support in $[\lambda-2\kappa, \lambda+2\kappa]$ and equal to 1 on $[\lambda-\kappa, \lambda+\kappa]$. Note that

$$\begin{aligned}
&f(H(P))[H(P), \mathrm{i}A_P]^\circ f(H(P)) \\
&\quad = j^{R*} j^R f(H(P))[H(P), \mathrm{i}A_P]^\circ f(H(P)) \\
&\quad = j^{R*} f(H^{\mathrm{ext}}(P))[H^{\mathrm{ext}}(P), \mathrm{i}A_P^{\mathrm{ext}}]^\circ f(H^{\mathrm{ext}}(P)) j^R + o_R(1),
\end{aligned}$$

by Lemma 3.2. Note that

$$\begin{aligned}
&f(H^{\mathrm{ext}}(P))[H^{\mathrm{ext}}(P), \mathrm{i}A_P^{\mathrm{ext}}]^\circ f(H^{\mathrm{ext}}(P)) j^R \\
&\quad = f(H(P))[H(P), \mathrm{i}A_P]^\circ f(H(P)) \begin{pmatrix} 1 \\ j_0(\frac{\cdot}{R}) \end{pmatrix} \\
&\qquad \oplus f(F_P(D_x))|\nabla\omega(D_x) - \nabla\Omega(P-D_x)|^2 f(F_P(D_x)) j_\infty(\tfrac{\cdot}{R}).
\end{aligned} \tag{3.2}$$

Taking the support of f into account, one finds that

$$f(F_P(D_x))|\nabla\omega(D_x) - \nabla\Omega(P - D_x)|^2 f(F_P(D_x)) \geq 2cf^2(F_P(D_x))$$

for some positive constant $c > 0$. The operator $K(R) = f(H(P))\left(\begin{smallmatrix}1\\ j_0(\frac{\cdot}{R})\end{smallmatrix}\right)$ is easily seen to be compact. Let $g \in C_0^\infty(\mathbb{R})$ equal 1 on the support of f. Then

$$B = f(H(P))[H(P), \mathrm{i}A_P]^\circ g(H(P))$$

is bounded and (3.2) equals $BK(R)$. Hence by Lemma 3.2

$$\begin{aligned}
&f(H(P))[H(P), \mathrm{i}A_P]^\circ f(H(P)) \\
&\quad \geq j^{R*} 2cf^2(H(P)) \begin{pmatrix} 1 \\ j_0(\frac{\cdot}{R}) \end{pmatrix} \oplus 2cf^2(F_P(D_x)) j_\infty(\tfrac{\cdot}{R}) \\
&\qquad + j^{R*}(B - 2cf(H(P)))K(R) \oplus 0 + o_R(1) \\
&\quad = 2cf^2(H(P)) + K_R + o_R(1),
\end{aligned}$$

for some compact operator K_R depending on R. One may now choose R so large that $\|o_R(1)\| \leq c$ and sandwich the inequality with $E_{\lambda,\kappa}(H(P))$ on both sides to arrive at the desired result. □

We infer the following corollary of Theorems 3.1 and 3.4 by standard arguments of regular Mourre theory.

Corollary 3.5. *The essential spectrum of the fiber Hamiltonians is non-singular:*

$$\sigma_{\mathrm{sing}}(H(P)) = \emptyset.$$

Theorem 3.6. *Let $(P_0, \lambda_0) \in \mathbb{R}^{\nu+1}$. Assume that $\lambda_0 \notin \theta(P_0) \cup \sigma_{\mathrm{pp}}(P_0)$. Then there exists a constant $C > 0$, a neighbourhood $\mathcal{O}$ of P_0 and a function $f \in C_0^\infty(\mathbb{R})$ with $f = 1$ in a neighbourhood of λ_0 such that for all $P \in \mathcal{O}$,*

$$f(H(P))[H(P), \mathrm{i}A_{P_0}]^\circ f(H(P)) \geq Cf^2(H(P))$$

where A_{P_0} is given as in Theorem 3.4.

Proof. We begin by noting that the object $[H(P), \mathrm{i}A_{P_0}]^\circ$ is well-defined by Theorem 3.1. By standard arguments using the fact that $\lambda_0 \notin \sigma_{\mathrm{pp}}(P_0)$ and Theorem 3.4, there exist a function $\tilde{f} \in C_0^\infty(\mathbb{R})$ and a constant $\tilde{C}$ such that

$$\tilde{f}(H(P_0))[H(P_0), \mathrm{i}A_{P_0}]^\circ \tilde{f}(H(P_0)) \geq \tilde{C}\tilde{f}^2(H(P_0)),$$

with $\tilde{f} = 1$ on a neighbourhood of λ_0. It is easy to see that the operators $(H(P) - z)^{-1}(H_0(0) - \mathrm{i})$ and $(H_0(0) - \mathrm{i})^{-1}[H(P), \mathrm{i}A_{P_0}]^\circ(H_0(0) - \mathrm{i})^{-1}$ are norm continuous as functions of P, and hence it follows by an application of the functional calculus of almost analytic extensions that $\tilde{f}^2(H(P))$ and $\tilde{f}(H(P))[H(P), \mathrm{i}A_{P_0}]^\circ\tilde{f}(H(P))$ are norm continuous as functions of P.

Let $\mathcal{O} \ni P_0$ be a neighbourhood such that

$$\|\tilde{f}^2(H(P)) - \tilde{f}^2(H(P_0))\| \le \frac{\tilde{C}}{3} \quad \text{and}$$

$$\|\tilde{f}(H(P))[H(P), \mathrm{i}A_{P_0}]^\circ\tilde{f}(H(P)) - \tilde{f}(H(P_0))[H(P_0), \mathrm{i}A_{P_0}]^\circ\tilde{f}(H(P_0))\| \le \frac{\tilde{C}}{3}$$

for all $P \in \mathcal{O}$. Then

$$\tilde{f}(H(P))[H(P), \mathrm{i}A_{P_0}]^\circ\tilde{f}(H(P)) \ge -\frac{2\tilde{C}}{3}I + \tilde{C}\tilde{f}^2(H(P)). \tag{3.3}$$

Choose now $C = \frac{\tilde{C}}{3}$ and $f \in C_0^\infty(\mathbb{R})$ such that $f = 1$ on a neighbourhood of λ_0 and $f = f\tilde{f}$. The result is then obtained by multiplying (3.3) from both sides with $f(H(P))$. □

4 Propagation estimates

We will write $\mathbf{D} = [H, \mathrm{i}\,\cdot\,]$ and $\mathbf{d}_0 = [\Omega(D_x + D_y) + \omega(D_x), \mathrm{i}\,\cdot\,]$ for the Heisenberg derivatives. The following abbreviation will be used to ease the notation:

$$[B] := \left(\begin{smallmatrix} 0 & 0 \\ 0 & B \end{smallmatrix}\right).$$

Theorem 4.1 (Large velocity estimate). *Let $\chi \in C_0^\infty(\mathbb{R})$. There exists a constant C_1 such that for $R' > R > C_1$, one has*

$$\int_1^\infty \left\| \left[\mathbb{1}_{[R,R']}\left(\tfrac{|x-y|}{t}\right)\right] e^{-\mathrm{i}tH}\chi(H)u \right\|^2 \frac{\mathrm{d}t}{t} \le C\|u\|^2$$

Proof. Let C_1 be a constant to be specified later and $R' > R > C_1$. Let $F \in C^\infty(\mathbb{R})$ equal 0 near the origin and 1 near infinity such that $F'(s) \ge c\mathbb{1}_{[R,R']}(s)$ for some positive constant $c > 0$. Let

$$\Phi(t) = -\chi(H)[F(\tfrac{|x-y|}{t})]\chi(H),$$
$$b(t) = -\mathbf{d}_0 F(\tfrac{|x-y|}{t}).$$

By using e.g. Theorem B.3 or pseudo-differential calculus one sees that

$$b(t) = \frac{1}{t}\big(\tfrac{|x-y|}{t} - (\nabla\Omega(D_y) - \nabla\omega(D_x))\tfrac{x-y}{|x-y|}\big)F'(\tfrac{|x-y|}{t}) + O(t^{-2}).$$

Hence for any $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ such that $\chi = \chi\tilde{\chi}$ one finds that

$$\begin{aligned}
&-\chi(H)[b(t)]\chi(H)\\
&= \frac{1}{t}\chi(H)\big(\tfrac{|x-y|}{t} - (\nabla\Omega(D_y) - \nabla\omega(D_x))\tfrac{x-y}{|x-y|}\big)F'(\tfrac{|x-y|}{t})\chi(H) + O(t^{-2})\\
&= \frac{1}{t}\chi(H)\big(\tfrac{|x-y|}{t} - \tilde{\chi}(H)(\nabla\Omega(D_y) - \nabla\omega(D_x))\tfrac{x-y}{|x-y|}\big)\mathbb{1}_{[C_1,\infty)}(\tfrac{|x-y|}{t})\\
&\qquad \times F'(\tfrac{|x-y|}{t})\chi(H) + O(t^{-2})\\
&\geq \frac{C_0}{t}\chi(H)F'(\tfrac{|x-y|}{t})\chi + O(t^{-2})
\end{aligned}$$

for some $C_0 > 0$ if one chooses $C_1 > \|\tilde{\chi}(H)(\nabla\Omega(D_y) - \nabla\omega(D_x))\frac{x-y}{|x-y|}\|$.

It follows from Condition 2.3 (iii) that

$$[V, \mathrm{i}[F(\tfrac{|x-y|}{t})]] = O(t^{-1-\mu}),$$

cf. (3.1). Putting this together, we get

$$\mathbf{D}\Phi(t) \geq \tfrac{C_0}{t}\chi(H)[F'(\tfrac{|x-y|}{t})]\chi(H) + O(t^{-1-\mu}),$$

which combined with Lemma A.1 implies the result. □

Theorem 4.2 (Phase-space propagation estimate). *Take $\chi \in C_0^\infty(\mathbb{R})$ and let $0 < c_0 < c_1$. Write*

$$\begin{aligned}
&\Theta_{[c_0,c_1]}(t) =\\
&\big[\big\langle \tfrac{x-y}{t} - \nabla\omega(D_x) + \nabla\Omega(D_y), \mathbb{1}_{[c_0,c_1]}(\tfrac{|x-y|}{t})\big(\tfrac{x-y}{t} - \nabla\omega(D_x) + \nabla\Omega(D_y)\big)\big\rangle\big].
\end{aligned}$$

Then

$$\int_1^\infty \|\Theta_{[c_0,c_1]}(t)^{\frac{1}{2}} e^{-\mathrm{i}tH}\chi(H)u\|^2 \frac{\mathrm{d}t}{t} \leq C\|u\|^2.$$

Proof. The following construction is taken from [DG99] but ultimately goes back to a construction of Graf, see e.g. [Gra90]. There exists a function $R_0 \in C^\infty(\mathbb{R}^\nu)$ such that

$$\begin{aligned}
R_0(x) &= 0 && \text{for } |x| \leq \tfrac{c_0}{2},\\
R_0(x) &= \tfrac{1}{2}x^2 + c && \text{for } |x| \geq 2c_1,\\
\nabla^2 R_0(x) &\geq \mathbb{1}_{[c_0,c_1]}(|x|).
\end{aligned}$$

Without loss of generality, we may assume that $c_1 > C_1 + 1$, where C_1 is the constant whose existence is ensured by Theorem 4.1. Choose a constant $c_2 > c_1 + 1$ and a smooth function F such that $F(s) = 1$ for $s < c_1$ and $F(s) = 0$ for $s \geq c_2$. Let

$$R(x) = F(|x|)R_0(x).$$

Then R satisfies

$$\begin{aligned} &\nabla^2 R(x) \geq \mathbb{1}_{[c_0,c_1]}(|x|) - C\mathbb{1}_{[C_1+1,c_2]}(|x|), \\ &|\partial^\alpha R(x)| \leq C_\alpha. \end{aligned} \tag{3.4}$$

Write $X = \frac{x-y}{t} - \nabla\omega(D_x) + \nabla\Omega(D_y)$ and let

$$\Phi(t) = \chi(H)[b(t)]\chi(H),$$

where

$$b(t) = R(\tfrac{x-y}{t}) - \tfrac{1}{2}\big(\langle \nabla R(\tfrac{x-y}{t}), X\rangle + \mathrm{h.c.}\big).$$

By using Condition 2.3 (iii) and pseudo-differential calculus, one sees that

$$\left\| \chi(H) \begin{pmatrix} 0 & 0 \\ -\mathrm{i}b(t)\rho(x-\cdot) & 0 \end{pmatrix} \chi(H) \right\| \in O(t^{-1-\mu})$$

and hence

$$\chi(H)[V, \mathrm{i}[b(t)]]\chi(H) \in O(t^{-1-\mu}).$$

Compute

$$\begin{aligned} \tfrac{d}{dt} b(t) = &-\tfrac{1}{t}\langle \tfrac{x-y}{t}, \nabla R(\tfrac{x-y}{t})\rangle \\ &+ \tfrac{1}{2}\tfrac{1}{t}\big(\langle \tfrac{x-y}{t}, \nabla^2 R(\tfrac{x-y}{t})X\rangle + \mathrm{h.c.}\big) \\ &+ \tfrac{1}{t}\langle \nabla R(\tfrac{x-y}{t}), \tfrac{x-y}{t}\rangle \\ = &\tfrac{1}{2}\tfrac{1}{t}\big(\langle \tfrac{x-y}{t}, \nabla^2 R(\tfrac{x-y}{t})X\rangle + \mathrm{h.c.}\big), \end{aligned}$$

and by pseudo-differential calculus one sees that

$$\begin{aligned} [\omega(D_x)+\Omega(D_y), \mathrm{i}b(t)] = &\tfrac{1}{2}\tfrac{1}{t}\big(\langle \nabla\omega(D_x) - \nabla\Omega(D_y), \nabla R(\tfrac{x-y}{t})\rangle + \mathrm{h.c.}\big) \\ &- \tfrac{1}{2}\tfrac{1}{t}\big(\langle \nabla\omega(D_x) - \nabla\Omega(D_y), \nabla^2 R(\tfrac{x-y}{t})X\rangle + \mathrm{h.c.}\big) \\ &- \tfrac{1}{2}\tfrac{1}{t}\big(\langle \nabla R(\tfrac{x-y}{t}), \nabla\omega(D_x) - \nabla\Omega(D_y)\rangle + \mathrm{h.c.}\big) \\ &+ O(t^{-2}) \\ = &-\tfrac{1}{2}\tfrac{1}{t}\big(\langle \nabla\omega(D_x) - \nabla\Omega(D_y), \nabla^2 R(\tfrac{x-y}{t})X\rangle + \mathrm{h.c.}\big) \\ &+ O(t^{-2}), \end{aligned}$$

hence by using (3.4), it follows that

$$\begin{aligned}
&\chi(H)[\mathbf{d}_0 b(t)]\chi(H)\\
&\quad= \tfrac{1}{t}\chi(H)[\langle X, \nabla^2 R(\tfrac{x-y}{t})X\rangle]\chi(H) + O(t^{-2})\\
&\quad\geq \tfrac{1}{t}\chi(H)\big[\big\langle X, \mathbb{1}_{[c_0,c_1]}\big(\tfrac{|x-y|}{t}\big)X\big\rangle\big]\chi(H)\\
&\qquad - \tfrac{C}{t}\chi(H)\big[\big\langle X, \mathbb{1}_{[C_1+1,c_2]}\big(\tfrac{|x-y|}{t}\big)X\big\rangle\big]\chi(H) + O(t^{-2})
\end{aligned}$$

By introducing $J \in C_0^\infty(\mathbb{R};[0,1])$ supported above C_1 with $J\mathbb{1}_{[C_1+1,c_2]} = \mathbb{1}_{[C_1+1,c_2]}$ and $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ with $\tilde{\chi}\chi = \chi$ and using pseudo-differential calculus, the functional calculus of almost analytic extensions and Condition 2.3 (iii) again, one gets that

$$\begin{aligned}
&\tfrac{C}{t}\chi(H)\big[X_i\mathbb{1}_{[C_1+1,c_2]}\big(\tfrac{|x-y|}{t}\big)X_i\big]\chi(H)\\
&\quad\leq \tfrac{C}{t}\chi\tilde{\chi}(H)\big[X_i J^3\big(\tfrac{|x-y|}{t}\big)X_i\big]\tilde{\chi}\chi(H)\\
&\quad= \tfrac{C}{t}\chi(H)\big[J\big(\tfrac{|x-y|}{t}\big)\big]\tilde{\chi}(H)\big[X_i J\big(\tfrac{|x-y|}{t}\big)X_i\big]\tilde{\chi}(H)\big[J\big(\tfrac{|x-y|}{t}\big)\big]\chi(H)+O(t^{-2})\\
&\quad\leq \tfrac{C'}{t}\chi(H)\big[J^2\big(\tfrac{|x-y|}{t}\big)\big]\chi(H) + Ct^{-2},
\end{aligned}$$

where we estimated $\tilde{\chi}(H)\big[X_i J\big(\tfrac{|x-y|}{t}\big)X_i\big]\tilde{\chi}(H)$ by a constant. Putting it all together yields

$$\mathbf{D}\Phi(t) \geq \tfrac{1}{t}\chi(H)\Theta_{[c_0,c_1]}(t)\chi(H) - \tfrac{C}{t}\chi(H)[J^2(\tfrac{|x-y|}{t})]\chi(H) + O(t^{-1-\mu}),$$

where the second term is integrable along the evolution by Theorem 4.1, so the result now follows from Lemma A.1. □

Theorem 4.3 (Improved phase-space propagation estimate).
Let $0 < c_0 < c_1$, $J \in C_0^\infty(c_0 < |x| < c_1)$, $\chi \in C_0^\infty(\mathbb{R})$. *Then for* $1 \leq i \leq \nu$

$$\int_1^\infty \Big\| \big[\big|J(\tfrac{x-y}{t})\big(\tfrac{x_i-y_i}{t} - \partial_i\omega(D_x) + \partial_i\Omega(D_y)\big) + \text{h. c.}\big|\big]^{\frac{1}{2}} e^{-\mathrm{i}tH}\chi(H)u\Big\|^2 \tfrac{\mathrm{d}t}{t} \leq C\|u\|^2$$

Proof. For brevity, we write $X = \frac{x-y}{t} - \nabla\omega(D_x) + \nabla\Omega(D_y)$ and $R_0 = (H_0-\lambda)^{-1}$ for some real $\lambda \in \rho(H_0)$. Let

$$A = X^2 + t^{-\delta},$$

$\delta > 0$. Note that $[J(\frac{x-y}{t})A^{\frac{1}{2}}]R_0$ is uniformly bounded in $t \geq 1$.

The following identities hold as forms on $C_0^\infty(\mathbb{R}^\nu)$.

$$e^{\mathrm{i}t(\omega(D_x)+\Omega(D_y))}Xe^{-\mathrm{i}t(\omega(D_x)+\Omega(D_y))} = \tfrac{x-y}{t},$$

$$e^{\mathrm{i}t(\omega(D_x)+\Omega(D_y))}A^{\frac{1}{2}}e^{-\mathrm{i}t(\omega(D_x)+\Omega(D_y))} = ((\tfrac{x-y}{t})^2 + t^{-\delta})^{\frac{1}{2}} := A_0^{\frac{1}{2}} \tag{3.5}$$

and

$$e^{\mathrm{i}t(\omega(D_x)+\Omega(D_y))}J(X)e^{-\mathrm{i}t(\omega(D_x)+\Omega(D_y))} = J(\tfrac{x-y}{t}). \tag{3.6}$$

That the following commutator, viewed as a form on $C_0^\infty(\mathbb{R}^\nu)$, extends by continuity to a bounded form on $L^2(\mathbb{R}^\nu)$ can be seen using pseudo-differential calculus:

$$[X, A_0^{\frac{1}{2}}] = [\nabla\omega(D_x), A_0^{\frac{1}{2}}] - [\nabla\Omega(D_y), A_0^{\frac{1}{2}}] = O(t^{-\min\{1,2-\frac{\delta}{2}\}}).$$

Together with the functional calculus of almost analytic extensions this implies that

$$[J(X), A_0^{\frac{1}{2}}] = O(t^{-\min\{1,2-\frac{\delta}{2}\}}),$$

and hence using (3.5) and (3.6) that

$$[J(\tfrac{x-y}{t}), A^{\frac{1}{2}}] = O(t^{-\varepsilon}), \tag{3.7}$$

where $\varepsilon = \min\{1, 2-\frac{\delta}{2}\}$. Write $h = \Omega(D_y) + \omega(D_x)$. Note that

$$\begin{aligned} e^{\mathrm{i}th}\mathbf{d}_0 A^{\frac{1}{2}}e^{-\mathrm{i}th} &= e^{\mathrm{i}th}[h, \mathrm{i}A^{\frac{1}{2}}]e^{-\mathrm{i}th} + e^{\mathrm{i}th}(\tfrac{d}{dt}A^{\frac{1}{2}})e^{-\mathrm{i}th} \\ &= \tfrac{d}{dt}\left(e^{\mathrm{i}th}A^{\frac{1}{2}}e^{-\mathrm{i}th}\right) = \tfrac{d}{dt}A_0^{\frac{1}{2}} \\ &= -\tfrac{1}{t}A_0^{\frac{1}{2}} - \frac{(2-\delta)t^{-\delta-1}}{2((\frac{x-y}{t})^2+t^{-\delta})^{\frac{1}{2}}}, \end{aligned}$$

so

$$\mathbf{d}_0 A^{\frac{1}{2}} = -\tfrac{1}{t}A^{\frac{1}{2}} + O(t^{-1-\frac{\delta}{2}}). \tag{3.8}$$

In addition

$$[R_0, [X_i]] = R_0^{\frac{1}{2}+\rho_1}O(t^{-1})R_0^{1-\rho_1} \tag{3.9}$$

for any ρ_1, $0 < \rho_1 < \frac{1}{2}$ and that

$$[R_0, [A^{\frac{1}{2}}]] = R_0^{\rho_2}O(t^{\frac{\delta}{2}-1})R_0^{1-\rho_2} \tag{3.10}$$

for any ρ_2, $0 < \rho_2 < 1$. The identity (3.10) can be seen e.g. by using (3.9) and the representation formula

$$s^{-\frac{1}{2}} = \frac{1}{\pi}\int_0^\infty (s+y)^{-1}y^{-\frac{1}{2}}\mathrm{d}y,$$

which can be verified for $t > 0$ by direct computations.

Let $J_1, J_2 \in C_0^\infty(c_0 < |x| < c_1)$ such that $JJ_1 = J$ and $J_1 J_2 = J_1$ and write for $i = 1, \dots, \nu$:

$$B_{0,i} = R_0[J(\tfrac{x-y}{t})X_i]R_0 + \text{h.c.}$$

and

$$B_1 = R_0[J_1(\tfrac{x-y}{t})A^{\frac{1}{2}}J_1(\tfrac{x-y}{t})]R_0. \tag{3.11}$$

We compute using (3.7), (3.9) and (3.10):

$$\begin{aligned}
B_{0,i}^2 &= 4R_0[X_i J(\tfrac{x-y}{t})]R_0^2[J(\tfrac{x-y}{t})X_i]R_0 + O(t^{-1}) \\
&= 4R_0^2[X_i J^2(\tfrac{x-y}{t})X_i]R_0^2 + O(t^{-1}) \\
&\le CR_0^2[X_i J_1^4(\tfrac{x-y}{t})X_i]R_0^2 + Ct^{-1} \\
&= CR_0^2[J_1^2(\tfrac{x-y}{t})X_i^2 J_1^2(\tfrac{x-y}{t})]R_0^2 + O(t^{-1}) \\
&\le CR_0^2[J_1^2(\tfrac{x-y}{t})AJ_1^2(\tfrac{x-y}{t})]R_0^2 + O(t^{-\delta}) \\
&= CR_0[J_1^2(\tfrac{x-y}{t})A^{\frac{1}{2}}]R_0^2[A^{\frac{1}{2}}J_1^2(\tfrac{x-y}{t})]R_0 + O(t^{-\min\{1-\frac{\delta}{2},\delta\}}) \\
&= CR_0[J_1(\tfrac{x-y}{t})A^{\frac{1}{2}}J_1(\tfrac{x-y}{t})]R_0^2[J_1(\tfrac{x-y}{t})A^{\frac{1}{2}}J_1(\tfrac{x-y}{t})]R_0 + O(t^{-\min\{1-\frac{\delta}{2},\delta\}}) \\
&= CB_1^2 + O(t^{-\kappa}),
\end{aligned}$$

where $\kappa = \min\{1 - \frac{\delta}{2}, \delta\}$. By the matrix monotonicity of $\lambda \mapsto \lambda^{\frac{1}{2}}$ [BR81, Sec. 2.2.2], we deduce that

$$|B_{0,i}| \le CB_1 + Ct^{-\frac{\kappa}{2}}. \tag{3.12}$$

Now let

$$\Phi(t) = -\chi(H)[J(\tfrac{x-y}{t})A^{\frac{1}{2}}J(\tfrac{x-y}{t})]\chi(H)$$

It follows from (3.7) that

$$\Phi(t) = -\chi(H)[J(\tfrac{x-y}{t})^2 A^{\frac{1}{2}}]\chi(H) + O(t^{-\varepsilon})$$

is uniformly bounded for $t > 1$.

We compute

$$\begin{aligned}
&-\mathbf{D}\Phi(t) = \\
&\chi(H)[V, \mathrm{i}[J(\tfrac{x-y}{t})A^{\frac{1}{2}}J(\tfrac{x-y}{t})]]\chi(H) + \chi(H)\big[\mathbf{d}_0\big(J(\tfrac{x-y}{t})A^{\frac{1}{2}}J(\tfrac{x-y}{t})\big)\big]\chi(H)
\end{aligned}$$

Using Condition 2.3 (iii) we see that

$$\chi(H)[V, \mathrm{i}[J(\tfrac{x-y}{t})A^{\frac{1}{2}}J(\tfrac{x-y}{t})]]\chi(H) = O(t^{-1-\mu}).$$

Indeed,

$$
\begin{aligned}
&\chi(H)[V,\mathrm{i}[J(\tfrac{x-y}{t})A^{\frac12}J(\tfrac{x-y}{t})]]\chi(H)\\
&\quad=\chi(H)\begin{pmatrix}0 & 0\\ -\mathrm{i}J(\frac{x-y}{t})A^{\frac12}J(\frac{x-y}{t})v & 0\end{pmatrix}\chi(H)+\text{h. c.}\\
&\quad=\chi(H)(H_0-\lambda)R_0\begin{pmatrix}0 & 0\\ -\mathrm{i}\big(A^{\frac12}J(\frac{x-y}{t})+O(t^{-\varepsilon})\big)J(\frac{x-y}{t})v & 0\end{pmatrix}\chi(H)+\text{h. c.}
\end{aligned}
$$

Now by Condition 2.3 (iii) we have that $\|J(\frac{x-y}{t})v\|=O(t^{-1-\mu})$ and hence we also have that $R_0\begin{pmatrix}0 & 0\\ -\mathrm{i}\big(A^{\frac12}J(\frac{x-y}{t})+O(t^{-\varepsilon})\big)J(\frac{x-y}{t})v & 0\end{pmatrix}=O(t^{-1-\mu})$.

Note that

$$
\mathbf{d}_0J(\tfrac{x-y}{t})=-\tfrac{1}{t}\nabla J(\tfrac{x-y}{t})\cdot v+O(t^{-2}) \tag{3.13}
$$

and using (3.8) and (3.12) (cf. (3.11)),

$$
\begin{aligned}
&-\chi(H)[J(\tfrac{x-y}{t})(\mathbf{d}_0A^{\frac12})J(\tfrac{x-y}{t})]\chi(H)\\
&\qquad\geq\tfrac{C_0}{t}\chi(H)[|J(\tfrac{x-y}{t})X_i+\text{h. c.}|]\chi(H)-Ct^{-1-\frac{\kappa}{2}}.
\end{aligned}
$$

Again we compute using (3.7):

$$
\begin{aligned}
&R_0[\nabla J(\tfrac{x-y}{t})\cdot XA^{\frac12}J(\tfrac{x-y}{t})]R_0+\text{h. c.}\\
&\quad=R_0[J_2(\tfrac{x-y}{t})X\cdot\nabla J(\tfrac{x-y}{t})J(\tfrac{x-y}{t})A^{\frac12}J_2(\tfrac{x-y}{t})]R_0+\text{h. c.}+O(t^{-1})\\
&\quad=\sum_{i=1}^{\nu}R_0[J_2(\tfrac{x-y}{t})A^{\frac12}X_iA^{-\frac12}\partial_iJ(\tfrac{x-y}{t})J(\tfrac{x-y}{t})A^{\frac12}J_2(\tfrac{x-y}{t})]R_0+\text{h. c.}+O(t^{-1})\\
&\quad\leq CR_0[J_2(\tfrac{x-y}{t})AJ_2(\tfrac{x-y}{t})]R_0+Ct^{-1}\\
&\quad\leq CR_0[J_2(\tfrac{x-y}{t})X^2J_2(\tfrac{x-y}{t})]R_0+O(t^{-\min\{1,\delta\}})\\
&\quad\leq CR_0[\langle X,J_2^2(\tfrac{x-y}{t})X\rangle]R_0+Ct^{-\min\{1,\varepsilon\}}.
\end{aligned}
$$

Hence (cf. (3.13))

$$
\begin{aligned}
&-\chi(H)\big[\mathbf{d}_0\big(J(\tfrac{x-y}{t})A^{\frac12}J(\tfrac{x-y}{t})\big)\big]\chi(H)\\
&\qquad=\chi(H)[(\mathbf{d}_0J(\tfrac{x-y}{t}))A^{\frac12}J(\tfrac{x-y}{t})]\chi(H)+\text{h. c.}\\
&\qquad\quad+\chi(H)[J(\tfrac{x-y}{t})(\mathbf{d}_0A^{\frac12})J(\tfrac{x-y}{t})]\chi(H)\\
&\qquad\geq\tfrac{C_0}{t}\chi(H)[|J(\tfrac{x-y}{t})X_i+\text{h. c.}|]\chi(H)\\
&\qquad\quad-\tfrac{C}{t}\chi(H)[\langle X,J_2^2(\tfrac{x-y}{t})X\rangle]\chi(H)+O(t^{-1-\gamma})
\end{aligned} \tag{3.14}
$$

for some $\gamma>0$. Since by Theorem 4.2 the second term in the r.h.s. of (3.14) is integrable along the evolution, the theorem follows from Lemma A.1.□

Theorem 4.4 (Minimal velocity estimate). *Assume that* $(P_0, \lambda_0) \in \mathbb{R}^{\nu+1}$ *satisfies that* $\lambda_0 \in \mathbb{R} \setminus (\theta(P_0) \cup \sigma_{\mathrm{pp}}(P_0))$. *Then there exists an* $\varepsilon > 0$, *a neighbourhood* N *of* (P_0, λ_0) *and a function* $\chi \in C_0^\infty(\mathbb{R}^{\nu+1})$ *such that* $\chi = 1$ *on* N *and*

$$\int_1^\infty \Big\| [\mathbb{1}_{[0,\varepsilon]}](\tfrac{|x|}{t}) \int^\oplus e^{-\mathrm{i}tH(P)} \chi(P, H(P)) \mathrm{d}P u \Big\|^2 \frac{\mathrm{d}t}{t} \le C\|u\|^2$$

Proof. By Theorem 3.6, it follows that there exists a neighbourhood $\mathcal{O}$ of P_0 and a function f with $f = 1$ in a neighbourhood of λ_0 such that

$$f(H(P))[H(P), \mathrm{i}A_{P_0}]f(H(P)) \ge Cf^2(H(P))$$

for all P in $\mathcal{O}$. Let $\chi \in C_0^\infty(\mathbb{R}^{\nu+1}; [0,1])$ be supported in the set $\mathcal{O} \times \{\lambda \,|\, f(\lambda) = 1\}$ and $\chi = 1$ in a neighbourhood N of (P_0, λ_0). It follows that

$$\chi(P, H(P))[H(P), \mathrm{i}A_{P_0}]\chi(P, H(P)) \ge \tfrac{C}{2}\chi^2(P, H(P)). \tag{3.15}$$

Let $q \in C_0^\infty(\{|x| \le 2\varepsilon\})$ satisfy $0 \le q \le 1$, $q = 1$ in a neighbourhood of $\{|x| \le \varepsilon\}$ for some $\varepsilon > 0$ to be specified later on. Write

$$Q(t) = \begin{pmatrix} 1 & 0 \\ 0 & q(\frac{x}{t}). \end{pmatrix}$$

Let

$$\Phi(t) = \int^\oplus \chi(P, H(P)) Q(t) \frac{A_{P_0}}{t} Q(t) \chi(P, H(P)) \mathrm{d}P.$$

Taking into account the support of q and that v_{P_0} is ω-bounded, and using pseudo-differential calculus, it is easy to see that $\Phi(t)$ is uniformly bounded.

We compute the Heisenberg derivative:

$$\begin{aligned}
\mathbf{D}\Phi(t) &= \int^\oplus \chi(P, H(P))[\mathbf{d}_0 q(\tfrac{x}{t})] \frac{A_{P_0}}{t} Q(t) \chi(P, H(P)) \mathrm{d}P + \mathrm{h.\,c.} \\
&\quad + \int^\oplus \chi(P, H(P))[V, \mathrm{i}Q(t)] \frac{A_{P_0}}{t} Q(t) \chi(P, H(P)) \mathrm{d}P + \mathrm{h.\,c.} \\
&\quad + \frac{1}{t} \int^\oplus \chi(P, H(P)) Q(t) [H(P), \mathrm{i}A_{P_0}] Q(t) \chi(P, H(P)) \mathrm{d}P \\
&\quad - \frac{1}{t} \int^\oplus \chi(P, H(P)) Q(t) \frac{A_{P_0}}{t} Q(t) \chi(P, H(P)) \mathrm{d}P \\
&= R_1 + R_2 + R_3 + R_4.
\end{aligned}$$

By the same arguments as before it follows that $\frac{A_{P_0}}{t}Q(t)\chi(P,H(P))$ is uniformly bounded. Using pseudo-differential calculus gives

$$R_1 = \frac{1}{t}\int^{\oplus}\chi(P,H(P))[\langle\tfrac{x}{t}-\nabla\omega(D_x)+\nabla\Omega(D_y),\nabla q(\tfrac{x}{t})\rangle]\tfrac{A_{P_0}}{t}Q(t)\chi(P,H(P))\mathrm{d}P + \text{h.c.} + O(t^{-2}).$$

Let

$$B_1 = \int^{\oplus}\chi(P,H(P))[\langle\tfrac{x}{t}-\nabla\omega(D_x)+\nabla\Omega(D_y),\nabla q(\tfrac{x}{t})\rangle]\mathrm{d}P$$

and

$$B_2 = \int^{\oplus}\chi(P,H(P))Q(t)\frac{A_{P_0}}{t}\mathrm{d}P.$$

Then

$$R_1 = \tfrac{1}{t}B_1B_2^* + \tfrac{1}{t}B_2B_1^* \geq -\varepsilon_0^{-1}\tfrac{1}{t}B_1B_1^* - \varepsilon_0\tfrac{1}{t}B_2B_2^*.$$

Now by Theorem 4.2, we get that $\frac{1}{t}B_1B_1^*$ is integrable along the evolution. Using pseudo-differential calculus and functional calculus of almost analytic extensions one can verify that

$$[\chi(P,H(P)),Q(t)] = (H_0(P)-R)^{-1+\rho}O(t^{-1})(H_0(P)-R)^{-\frac{1}{2}-\rho} \tag{3.16}$$

for any $R \in \mathbb{R}\setminus\sigma(H_0(P))$ and any ρ, $0 \leq \rho \leq \frac{1}{2}$. Hence it follows by introducing cutoff functions $\tilde{\chi} \in C_0^\infty(\mathbb{R}^{\nu+1})$ and $\tilde{q} \in C_0^\infty(\mathbb{R}^{\nu})$ with $\tilde{\chi}\chi = \chi$ and $\tilde{q}q = q$ that

$$\begin{aligned}
-\tfrac{1}{t}B_2B_2^* &= -\frac{1}{t}\int^{\oplus}Q(t)\chi\tilde{\chi}(P,H(P))[\tilde{q}(\tfrac{x}{t})]\frac{A_{P_0}^2}{t^2}[\tilde{q}(\tfrac{x}{t})]\tilde{\chi}\chi(P,H(P))Q(t)\mathrm{d}P \\
&\quad + O(t^{-2}) \\
&\geq -\frac{C_1}{t}\int^{\oplus}Q(t)\chi^2(P,H(P))Q(t)\mathrm{d}P + O(t^{-2}) \\
&= -\frac{C_1}{t}\int^{\oplus}\chi(P,H(P))Q^2(t)\chi(P,H(P))\mathrm{d}P + O(t^{-2})
\end{aligned} \tag{3.17}$$

By Condition 2.3 (iii) it follows that $\begin{pmatrix} 0 & 0 \\ \mathrm{i}(1-q(\frac{x}{t}))|\rho\rangle & 0\end{pmatrix} \in O(t^{-1-\mu})$ and hence

$$R_2 \in O(t^{-1-\mu}) \tag{3.18}$$

Using (3.15) and (3.16) twice, we see that

$$
\begin{aligned}
R_3 &= \frac{1}{t}\int^{\oplus} Q(t)\chi(P,H(P))[H(P),\mathrm{i}A_{P_0}]\chi(P,H(P))Q(t)\mathrm{d}P + O(t^{-2}) \\
&\geq \frac{C_2}{t}\int^{\oplus} Q(t)\chi^2(P,H(P))Q(t)\mathrm{d}P + O(t^{-2}) \\
&\geq \frac{C_2}{t}\int^{\oplus} \chi(P,H(P))Q(t)^2\chi(P,H(P))\mathrm{d}P + O(t^{-2}). \qquad (3.19)
\end{aligned}
$$

Again using the cutoff functions and pseudo-differential calculus and taking into account the support of q, we see that

$$
\begin{aligned}
&\pm\chi(P,H(P))Q(t)\frac{A_{P_0}}{t}Q(t)\chi(P,H(P)) \\
&\quad = \pm Q(t)\chi\tilde{\chi}(P,H(P))[\tilde{q}(\tfrac{x}{t})]\frac{A_{P_0}}{t}[\tilde{q}(\tfrac{x}{t})]\tilde{\chi}\chi(P,H(P))Q(t) \pm O(t^{-1}) \\
&\quad \leq \varepsilon C_3 Q(t)\chi^2(P,H(P))Q(t) + O(t^{-1}) \\
&\quad = \varepsilon C_3\chi(P,H(P))Q(t)^2\chi(P,H(P)) + O(t^{-1})
\end{aligned}
$$

so

$$
R_4 \geq -\frac{C_3\varepsilon}{t}\int^{\oplus}\chi(P,H(P))Q(t)^2\chi(P,H(P))\mathrm{d}P + O(t^{-2}). \qquad (3.20)
$$

Putting (3.17), (3.18), (3.19) and (3.20) together, we see that

$$
\begin{aligned}
\mathbf{D}\Phi(t) \geq{}& \frac{-\varepsilon_0 C_1 + C_2 - \varepsilon C_3}{t}\int^{\oplus}\chi(P,H(P))Q(t)^2\chi(P,H(P))\mathrm{d}P \\
&- \frac{1}{\varepsilon t}B_1B_1^* + O(t^{-1-\mu}).
\end{aligned}
$$

Now choosing ε and ε_0 so small that $-\varepsilon_0 C_1 + C_2 - \varepsilon C_3 > 0$ together with Lemma A.1 yields the result. □

5 The asymptotic observable and asymptotic completeness

Theorem 5.1 (Asymptotic observable). *Let $p \in C^\infty(\mathbb{R}^\nu)$ be a smooth function satisfying that $p(x) \leq p(y)$ for $|x| \leq |y|$, $p(x) = 0$ for $|x| \leq \frac{1}{2}$ and*

$p(x) = 1$ *for* $|x| \geq 1$. *Define* $p_\delta(x) = p(\frac{x}{\delta})$. *Then the limits*

$$\begin{aligned} P_\delta^+(H) &= \text{s-}\lim_{t\to\infty} e^{itH}[p_\delta(\tfrac{x-y}{t})]e^{-itH}, \qquad (3.21)\\ P_0^+(H) &= \text{s-}\lim_{\delta\to 0} P_\delta^+(H),\\ P_0^+(H_0,H) &= \text{s-}\lim_{\delta\to 0}\text{s-}\lim_{t\to\infty} e^{itH}[p_\delta(\tfrac{x-y}{t})]e^{-itH_0},\\ P_0^+(H,H_0) &= \text{s-}\lim_{\delta\to 0}\text{s-}\lim_{t\to\infty} e^{itH_0}[p_\delta(\tfrac{x-y}{t})]e^{-itH} \end{aligned}$$

exist and $P_0^+(H)$ *is a projection.*

Remark 5.2. Note that $\delta \mapsto P_\delta^+(H)$ is increasing, i.e. $P_\delta^+(H) \leq P_{\delta'}^+(H)$ for $0 < \delta' < \delta$. We leave it to the reader to verify that the definition of $P_0^+(H)$ is independent of the choice of p, and that one in fact could have chosen any family of functions $\{p_\delta\}$ satisfying $p_\delta(x) \leq p_\delta(y)$ for $|x| \leq |y|$, $p_\delta(x) = 0$ for $|x| \leq \frac{\delta}{2}$ and $p_\delta(x) = 1$ for $|x| \geq \delta$.

Proof. We will prove the statements about $P_\delta^+(H)$ and $P_0^+(H)$. The statements about $P_0^+(H_0,H)$ and $P_0^+(H,H_0)$ are proved completely analogously.

Let

$$\Phi(t) = -\chi(H)[p_\delta(\tfrac{x-y}{t})]\chi(H),$$

and calculate using pseudo-differential calculus

$$\mathbf{d}_0 p_\delta(\tfrac{x-y}{t}) = -\frac{1}{2}\frac{1}{t}\Big(\big(\tfrac{x-y}{t} - \nabla\omega(D_x) + \nabla\Omega(D_y)\big)\cdot\nabla p_\delta(\tfrac{x-y}{t}) + \text{h. c.}\Big) + O(t^{-2}).$$

This in combination with Condition 2.3 (iii) gives

$$\mathbf{D}\Phi(t) = \frac{1}{t}\chi(H)[\tfrac{1}{2}X\cdot\nabla p_\delta(\tfrac{x-y}{t}) + \text{h. c.}]\chi(H) + O(t^{-\min\{1+\mu,2\}}),$$

where $X = \frac{x-y}{t} - \nabla\omega(D_x) + \nabla\Omega(D_y)$, so Theorem 4.3 in combination with Lemma A.2 gives the existence of the limit (3.21).

The existence of the weak limit w-$P_0^+(H) =$ w-$\lim_{\delta\to 0} P_\delta^+(H)$ is obvious. Moreover, for every $\delta > 0$, it is clear from Lemma A.3 that the strong limit $\text{s-}\lim_{n\to\infty} P_{\frac{\delta}{2^n}}^+(H)$ exists, is a projection and equals w-$P_0^+(H)$. The inequality $P_\delta^+(H)^2 \leq P_\delta^+(H)$ implies

$$\begin{aligned} \lim_{\delta\to 0}\|(\text{w-}P_0^+(H) - P_\delta^+(H))u\|^2 &= \lim_{\delta\to 0}\langle(\text{w-}P_0^+(H) + P_\delta^+(H)^2 - 2P_\delta^+(H))u,u\rangle\\ &\leq \lim_{\delta\to 0}\langle(\text{w-}P_0^+(H) - P_\delta^+(H))u,u\rangle = 0. \end{aligned}$$

This finishes the argument. □

Proposition 5.3. *Let* $\Sigma = \{(P,\lambda) \in \mathbb{R}^{\nu+1} \,|\, \lambda \in \sigma_{\mathrm{pp}}(H(P))\}$ *denote the set in energy-momentum space consisting of eigenvalues for the fibered Hamiltonian and* $\Theta = \{(P,\lambda) \in \mathbb{R}^{\nu+1} \,|\, \lambda \in \theta(P)\}$ *the corresponding set of thresholds. Then* $\Sigma \cup \Theta$ *is a closed set of Lebesgue measure* 0*. Moreover,* $(\Sigma \cup \Theta)(P) = \sigma_{\mathrm{pp}}(P) \cup \theta(P)$ *is at most countable.*

Proof. By the usual arguments, Theorems 3.1 and 3.4 imply that eigenvalues of $H(P)$ can only accumulate at thresholds (see e.g. [ABG96] for details), and by analyticity, the threshold set $\theta(P)$ is at most countable. Hence, if $\Sigma \cup \Theta$ is closed, it is in particular of measure 0.

Let $(P_0, \lambda_0) \notin \Sigma \cup \Theta$. Then by Theorem 3.6, there are neighbourhoods $\mathcal{O}$ of P_0 and I of λ_0 such that for all $P \in \mathcal{O}$, a strict Mourre estimate holds for $H(P)$ on the energy interval I with conjugate operator A_{P_0} given as in Theorem 3.4 and $H(P)$ is of class $C^2(A_{P_0})$ by Theorem 3.1, which by the Virial Theorem implies that there are no eigenvalues for $H(P)$ in I for any $P \in \mathcal{O}$. Clearly,

$$\Theta = \{(P,\lambda) \in \mathbb{R}^{\nu+1} | \exists k \in \mathbb{R}^{\nu} \colon \lambda = \Omega(P-k) + \omega(k), \nabla\omega(k) - \nabla\Omega(P-k) = 0\}$$

is a closed set. Hence, possibly after chosing smaller $\mathcal{O}$ and I, $\mathcal{O} \times I$ is a neighbourhood of (P_0, λ_0) which does not intersect $\Sigma \cup \Theta$. □

Let $\mathcal{H}_{\mathrm{bd}} = E_{\Sigma\cup\Theta}((\mathrm{P}, H))\mathcal{H}$ and similarly $\mathcal{H}_{0,\mathrm{bd}} = E_{\Sigma_0\cup\Theta}((\mathrm{P}, H_0))\mathcal{H}$. We remark that if we for a fixed P take the fiber $(\Sigma \cup \Theta)(P) = \{\lambda \,|\, (\lambda, P) \in \Sigma \cup \Theta\}$, then we have $E_{(\Sigma\cup\Theta)(P)}(H(P)) = \mathbb{1}_{\mathrm{pp}}(H(P))$.

Theorem 5.4. *With* $\mathcal{H}_{\mathrm{bd}}$ *and* $P_0^+(H)$ *given as above, we have* $\mathcal{H}_{\mathrm{bd}} = (1 - P_0^+(H))\mathcal{H}$*.*

Proof. Let $(\lambda_0, P_0) \in \mathbb{R}^{\nu+1} \setminus (\Sigma \cup \Theta)$. Let the neighbourhood N and $\varepsilon > 0$ be those of Theorem 4.4 corresponding to the point (λ_0, P_0). Let $\psi \in E_N(\mathrm{P}, H)\mathcal{H}$. Then by Theorem 4.4, there exists a sequence $t_n \to \infty$ such that

$$\psi = e^{\mathrm{i}t_n H} p_\varepsilon(\tfrac{x-y}{t_n}) e^{-\mathrm{i}t_n H}\psi + e^{\mathrm{i}t_n H}(1 - p_\varepsilon(\tfrac{x-y}{t_n}))e^{-\mathrm{i}t_n H}\psi \to P_\varepsilon^+(H)\psi + 0,$$

which implies that $\psi \in P_0^+(H)\mathcal{H}$. As the span of such ψ is dense in $\mathcal{H}_{\mathrm{bd}}^\perp$ and $P_0^+(H)\mathcal{H}$ is closed, this implies that $\mathcal{H}_{\mathrm{bd}} \supset (1 - P_0^+(H))\mathcal{H}$.

By Proposition 5.3, $\Sigma \cup \Theta$ may be written as an at most countable union of graphs Σ_i of Borel functions from (subsets of) $\mathbb{R}^\nu$ to $\mathbb{R}$ (see [Ra80, Théorème 21, p. 226]). Let $\varphi = U\int^{\oplus} \varphi_P \mathrm{d}P \in \mathcal{H}$. Then $\psi = E_{\Sigma_j}(\mathrm{P}, H)\varphi = U\int^{\oplus} E_{\Sigma_j(P)}(H)\varphi_P \mathrm{d}P$. This implies that ψ can be written as

$$\psi = U\int^{\oplus} \psi_P \mathrm{d}P,$$

where ψ_P is an eigenvector for $H(P)$ with eigenvalue $\Sigma_j(P)$. Note that this ensures that ψ_P is Borel as a function of P. Now

$$\begin{aligned} P_\delta^+(H)\psi &= \mathrm{s-}\lim_{t\to\infty} e^{\mathrm{i}tH}[p_\delta(\tfrac{x-y}{t})]e^{-\mathrm{i}tH}\psi \\ &= \mathrm{s-}\lim_{t\to\infty} U\int^{\oplus} e^{\mathrm{i}tH(P)}[p_\delta(\tfrac{x}{t})]e^{-\mathrm{i}tH(P)}\psi_P \mathrm{d}P \\ &= \mathrm{s-}\lim_{t\to\infty} e^{\mathrm{i}tH}U\int^{\oplus} [p_\delta(\tfrac{x}{t})]e^{-\mathrm{i}t\Sigma_j(P)}\psi_P \mathrm{d}P, \end{aligned}$$

where the last integrand goes pointwise to 0 and hence by the dominated convergence theorem, the limit is 0. As δ was arbitrary, this shows that $P_0^+(H)\psi = 0$.

Since the span of the set of ψ we have covered is dense in $\mathcal{H}_{\mathrm{bd}}$ and $P_0^+(H)$ is closed, we conclude that $\mathcal{H}_{\mathrm{bd}} \subset (1 - P_0^+(H))\mathcal{H}$. □

Theorem 5.5 (Existence of wave operators). *The wave operator $W^+ \colon \mathcal{H} \mapsto \mathcal{H}$ given by*

$$W^+u = \mathrm{s-}\lim_{t\to\infty} e^{\mathrm{i}tH}e^{-\mathrm{i}tH_0}P_0^+(H_0)u,$$

where $P_0^+(H_0)$ is the projection onto $\{0\} \oplus L^2(\mathbb{R}^{2\nu}) = \mathcal{H}_{0,\mathrm{bd}}^{\perp}$, exists.

Proof. From Theorem 5.1 and Theorem 5.4 with $H = H_0$ it follows that $P_0^+(H_0)$ can be given as in Theorem 5.1, and by passing to the fibered representation, it is easy to see that the assumptions on Ω and ω imply that $\mathcal{H}_{0,\mathrm{bd}} = L^2(\mathbb{R}^\nu) \oplus \{0\}$.

By Theorem 5.1,

$$e^{\mathrm{i}tH}[p_\delta(\tfrac{x-y}{t})]e^{-\mathrm{i}tH_0} = e^{\mathrm{i}tH}e^{-\mathrm{i}tH_0}e^{\mathrm{i}tH_0}[p_\delta(\tfrac{x-y}{t})]e^{-\mathrm{i}tH_0}$$

tends strongly to $P_0^+(H_0, H)$ when $t \to \infty$ and $\delta \to 0$ (in that order). On the other hand,

$$e^{\mathrm{i}tH_0}[p_\delta(\tfrac{x-y}{t})]e^{-\mathrm{i}tH_0}$$

tends strongly to $P_0^+(H_0)$ in the same limit. This implies that

$$P_0^+(H_0, H) = \mathrm{s-}\lim_{t\to\infty}(e^{\mathrm{i}tH}e^{-\mathrm{i}tH_0})P_0^+(H_0)$$

exists. □

Theorem 5.6 (Geometric asymptotic completeness). *With W^+ as in Theorem 5.5,* $\operatorname{Ran} W^+ = P_0^+(H)\mathcal{H}$*.*

Proof. Consider

$$\begin{aligned}
W^+ &= \text{s–}\lim_{t\to\infty} e^{\mathrm{i}tH} e^{-\mathrm{i}tH_0} P_0^+(H_0)^2 \\
&= P_0^+(H_0, H) P_0^+(H_0) \\
&= \text{s–}\lim_{\delta\to 0} \text{s–}\lim_{t\to\infty} e^{\mathrm{i}tH} [p_\delta(\tfrac{x-y}{t})] e^{-\mathrm{i}tH_0} P^+(H_0) \\
&= \text{s–}\lim_{\delta\to 0} \text{s–}\lim_{t\to\infty} \big(e^{\mathrm{i}tH}[p_\delta(\tfrac{x-y}{t})] e^{-\mathrm{i}tH}\big) \text{s–}\lim_{\delta\to 0} \text{s–}\lim_{t\to\infty} \big(e^{-\mathrm{i}tH} e^{-\mathrm{i}tH_0}\big) P_0^+(H_0) \\
&= P_0^+(H) W^+,
\end{aligned}$$

which proves that $\operatorname{Ran} W^+ \subset P_0^+(H)\mathcal{H}$. For the other inclusion, we similarly calculate

$$\begin{aligned}
P_0^+(H) &= \text{s–}\lim_{\delta\to 0} \text{s–}\lim_{t\to\infty} e^{\mathrm{i}tH} [p_\delta(\tfrac{x-y}{t})] e^{-\mathrm{i}tH} P_0^+(H) \\
&= \text{s–}\lim_{\delta\to 0} \text{s–}\lim_{t\to\infty} e^{\mathrm{i}tH} e^{-\mathrm{i}tH_0} e^{\mathrm{i}tH_0} [p_\delta(\tfrac{x-y}{t})] e^{-\mathrm{i}tH} P_0^+(H) \\
&= \text{s–}\lim_{\delta\to 0} \text{s–}\lim_{t\to\infty} e^{\mathrm{i}tH} e^{-\mathrm{i}tH_0} P_0^+(H, H_0) P_0^+(H) \\
&= \text{s–}\lim_{\delta\to 0} \text{s–}\lim_{t\to\infty} e^{\mathrm{i}tH} e^{-\mathrm{i}tH_0} P_0^+(H_0) P_0^+(H, H_0) \\
&= W^+ P_0^+(H, H_0),
\end{aligned}$$

which proves $\operatorname{Ran} P_0^+(H) \subset \operatorname{Ran} W^+$. □

Theorem 2.4 now follows from Proposition 5.3, Theorem 5.4 and Theorem 5.6.

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A Appendix A

For easy reference, we list the following lemmata, which are taken from the appendix of [DG99]. The first lemma which is used to prove the propagation estimates, is a version of the Putnam-Kato theorem developed by Sigal–Soffer [SS87].

Lemma A.1. *Let H be a self-adjoint operator and $\mathbf{D}$ the corresponding Heisenberg derivative*

$$\mathbf{D} = \frac{\mathrm{d}}{\mathrm{d}t} + [H, \mathrm{i}\,\cdot\,].$$

Suppose that $\Phi(t)$ is a uniformly bounded family of self-adjoint operators. Suppose that there exist $C_0 > 0$ and operator valued functions $B(t)$ and $B_i(t)$, $i = 1, \ldots, n$, such that

$$\mathbf{D}\Phi(t) \geq C_0 B^*(t)B(t) - \sum_{i=1}^{n} B_i^*(t)B_i(t),$$

$$\int_1^\infty \|B_i(t)e^{-\mathrm{i}tH}\varphi\|^2 \mathrm{d}t \leq C\|\varphi\|^2, \quad i = 1, \ldots, n.$$

Then there exists C_1 such that

$$\int_1^\infty \|B(t)e^{-\mathrm{i}tH}\varphi\|^2 \mathrm{d}t \leq C_1\|\varphi\|^2.$$

The next lemma shows how to use propagation estimates to prove the existence of asymptotic observables and is a version of Cook's method due to Kato.

Lemma A.2. *Let H_1 and H_2 be two self-adjoint operators. Let ${}_2\mathbf{D}_1$ be the corresponding asymmetric Heisenberg derivative:*

$${}_2\mathbf{D}_1\Phi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) + \mathrm{i}H_2\Phi(t) - \mathrm{i}\Phi(t)H_1.$$

Suppose that $\Phi(t)$ is a uniformly bounded function with values in self-adjoint operators. Let $\mathcal{D}_1 \subset \mathcal{H}$ be a dense subspace. Assume that

$$|\langle\psi_2, {}_2\mathbf{D}_1\Phi(t)\psi_1\rangle| \leq \sum_{i=1}^{n} \|B_{2i}(t)\psi_2\| \|B_{1i}(t)\psi_1\|,$$

$$\int_1^\infty \|B_{2i}(t)e^{-\mathrm{i}tH_2}\varphi\|^2 \mathrm{d}t \leq \|\varphi\|^2, \quad \varphi \in \mathcal{H},\ i = 1, \ldots, n,$$

$$\int_1^\infty \|B_{1i}(t)e^{-\mathrm{i}tH_1}\varphi\|^2 \mathrm{d}t \leq C\|\varphi\|^2, \quad \varphi \in \mathcal{D}_1,\ i = 1, \ldots, n.$$

Then the limit

$$\text{s-}\lim_{t\to\infty} e^{\mathrm{i}tH_2}\Phi(t)e^{-\mathrm{i}tH_1}$$

exists.

The final lemma gives us the actual asymptotic observable.

Lemma A.3. *Let Q_n be a commuting sequence of self-adjoint operators such that:*

$$0 \le Q_n \le 1, \quad Q_n \le Q_{n+1}, \quad Q_{n+1}Q_n = Q_n.$$

Then the limit

$$Q = \text{s-}\lim_{n\to\infty} Q_n$$

exists and is a projection.

B Appendix B

In this section, we recall a result from [Ras].

In the following, $A = (A_1, \ldots, A_\nu)$ is a vector of self-adjoint, pairwise commuting operators acting on a Hilbert space $\mathcal{H}$, and $B \in \mathcal{B}(\mathcal{H})$ is a bounded operator on $\mathcal{H}$. We shall use the notion of B being of class $C^{n_0}(A)$ introduced in [ABG96]. For notational convenience, we adopt the following convention: If $0 \le j \le \nu$, then δ_j denotes the multi-index $(0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the j'th entry.

Definition B.1. Let $n_0 \in \mathbb{N} \cup \{\infty\}$. Assume that the multi-commutator form defined iteratively by $\mathrm{ad}_A^0(B) = B$ and $\mathrm{ad}_A^\alpha(B) = [\mathrm{ad}_A^{\alpha-\delta_j}(B), A_j]$ as a form on $\mathcal{D}(A_j)$, where $\alpha \ge \delta_j$ is a multi-index and $1 \le j \le \nu$, can be represented by a bounded operator also denoted by $\mathrm{ad}_A^\alpha(B)$, for all multi-indices α, $|\alpha| < n_0 + 1$. Then B is said to be of class $C^{n_0}(A)$ and we write $B \in C^{n_0}(A)$.

Remark B.2. The definition of $\mathrm{ad}_A^\alpha(B)$ does not depend on the order of the iteration since the A_j are pairwise commuting. We call $|\alpha|$ the *degree* of $\mathrm{ad}_A^\alpha(B)$.

In the following, $\mathcal{H}_A^s := D(|H|^s)$ for $s \ge 0$ will be used to denote the scale of spaces associated to A. For negative s, we define $\mathcal{H}_A^s := {\mathcal{H}_A^s}^*$.

Theorem B.3. *Assume that* $B \in C^{n_0}(A)$ *for some* $n_0 \geq n+1 \geq 1$, $0 \leq t_1, t_2$, $t_1 + t_2 \leq n+2$ *and that* $\{f_\lambda\}_{\lambda \in I}$ *satisfies*

$$\forall \alpha \, \exists C_\alpha \colon |\partial^\alpha f_\lambda(x)| \leq C_\alpha \langle x \rangle^{s-|\alpha|}$$

uniformly in λ *for some* $s \in \mathbb{R}$ *such that* $t_1 + t_2 + s < n+1$. *Then*

$$[B, f_\lambda(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!} \partial^\alpha f_\lambda(A) \, \mathrm{ad}_A^\alpha(B) + R_{\lambda,n}(A,B)$$

as an identity on $\mathcal{D}(\langle A \rangle^s)$, *where* $R_{\lambda,n}(A,B) \in \mathcal{B}(\mathcal{H}_A^{-t_2}, \mathcal{H}_A^{t_1})$ *and there exist a constant* C *independent of* A, B *and* λ *such that*

$$\|R_{\lambda,n}(A,B)\|_{\mathcal{B}(\mathcal{H}_A^{-t_2}, \mathcal{H}_A^{t_1})} \leq C \sum_{|\alpha|=n+1} \|\mathrm{ad}_A^\alpha(B)\|.$$

C Bibliography

[ABG96] W. O. Amrein, A. Boutet de Monvel, and V. Georgescu, *C_0-groups, commutator methods and spectral theory of N-body hamiltonians*, Birkhäuser, 1996.

[Amm00] Z. Ammari, *Asymptotic completeness for a renormalized nonrelativistic Hamiltonian in quantum field theory: The Nelson model*, Math. Phys. Anal. Geom. **3** (2000), 217–285.

[AMZ05] N. Angelescu, R. Minlos, and V. Zagrebnov, *Lower spectral branches of a particle coupled to a Bose field*, Rev. Math. Phys. **17** (2005), 1111–1142.

[BR81] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics*, Springer, Berlin, 1981.

[CFKS87] H. Cycon, R. Froese, W. Kirsch, and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*, Springer, Berlin, 1987.

[DG97] J. Dereziński and C. Gérard, *Scattering theory of classical and quantum N-particle systems*, Texts and Monographs in Physics, Springer, Berlin, 1997.

[DG99] ______, *Asymptotic completeness in quantum field theory. massive Pauli-Fierz Hamiltonians*, Rev. Math. Phys. **11** (1999), 383–450.

[FGS04] J. Fröhlich, M. Griesemer, and B. Schlein, *Asymptotic completeness for Compton scattering*, Commun. Math. Phys. **252** (2004), 415–476.

[FGS08] J. Fröhlich, M. Griesemer, and I. M. Sigal, *Spectral theory for the standard model of non-relativistic QED*, Commun. Math. Phys. **283** (2008), 613–646.

[Gér96] C. Gérard, *Asymptotic completeness for the spin-boson model with a particle number cutoff*, Rev. Math. Phys **8** (1996), 549–589.

[GJY03] A. Galtbayar, A. Jensen, and K. Yajima, *The Nelson model with less than two photons*, Ann. Henri Poincaré **4** (2003), 239–273.

[Gra90] G. M. Graf, *Asymptotic completeness for N-body short range quantum systems: A new proof*, Commun. Math. Phys. **132** (1990), 73–101.

[MR] J. S. Møller and M. G. Rasmussen, *The translation invariant massive Nelson model: II. The lower part of the essential spectrum*, In prep.

[Møl05] J. S. Møller, *The translation invariant massive Nelson model: I. The bottom of the spectrum*, Ann. Henri Poincaré **6** (2005), 1091–1135.

[Piz03] A. Pizzo, *One-particle (improper) states and scattering states in Nelson's massless model*, Ann. Henri Poincaré **4** (2003), 439–483.

[Ra80] C. A. Rogers and al., *Analytic sets*, Academic Press, 1980.

[Ras] M. G. Rasmussen, *A Taylor-like expansion of a commutator with a function of self-adjoint, pairwise commuting operators*, Submitted.

[Spo04] H. Spohn, *Dynamics of charged particles and their radiation field*, Cambridge University Press, Cambridge, 2004.

[SS87] I. M. Sigal and A. Soffer, *The N-particle scattering problem: Asymptotic completeness for short-range quantum systems*, Ann. Math. **125** (1987), 35–108.

Chapter 4

A Taylor-like Expansion of a Commutator with a Function of Self-adjoint, Pairwise Commuting Operators

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Abstract

Let A be a ν-vector of self-adjoint, pairwise commuting operators and B a bounded operator of class $C^{n_0}(A)$. We prove a Taylor-like expansion of the commutator $[B, f(A)]$ for a large class of functions $f\colon \mathbb{R}^\nu \to \mathbb{R}$, generalising the one-dimensional result where A is just a self-adjoint operator. This is done using almost analytic extensions and the higher-dimensional Helffer-Sjöstrand formula.

Keywords: commutator expansions, functional calculus, almost analytic extensions, Helffer-Sjöstrand formula
Mathematics Subject Classification (2010): 47B47

1 Introduction

It is well-known that if A is a self-adjoint operator, B is a bounded operator of class $C^{n_0}(A)$ in the sense of [ABG96] and f satisfies that for all n, $|f^{(n)}(x)| \leq C_n \langle x \rangle^{s-n}$, then for $0 \leq t_1 \leq n_0$, $0 \leq t_2 \leq 1$ with $s + t_1 + t_2 < n_0$,

$$[B, f(A)] = \sum_{k=1}^{n_0-1} \frac{1}{k!} f^{(k)}(A) \operatorname{ad}_A^k(B) + R_{n_0}(A, B)$$

where $\operatorname{ad}_A^k(B)$ is the k'th iterated commutator, $R_{n_0}(A, B) \in \mathcal{B}(\mathcal{H}_A^{-t_2}; \mathcal{H}_A^{t_1})$ and $\mathcal{H}_A^t$ is defined as $\mathcal{D}(\langle A \rangle^t)$ equipped with the graph-norm $\|v\|_t = \|\langle A \rangle^t v\|$ for $t \geq 0$ and $\mathcal{H}_A^{-t}$ is the dual space of $\mathcal{H}_A^t$. This follows relatively easily from using the (one-dimensional) Helffer-Sjöstrand formula

$$f(A) = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z)(A - z)^{-1} dz, \tag{4.1}$$

where $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ and $\tilde{f}$ is an almost analytic extension of f, and the identity

$$\begin{aligned}[B, f(A)] = &\sum_{k=1}^{n_0-1} \frac{1}{k!} \frac{k!}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z)(-1)^k (A - z)^{-k-1} dz \\ &+ \frac{(-1)^{n_0}}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z)(A - z)^{-n_0} \operatorname{ad}_A^{n_0}(B)(A - z)^{-1} dz\end{aligned}$$

when $\frac{k!}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z)(-1)^k (A - z)^{-k-1} dz$ is recognised as $f^{(k)}(A)$ using (4.1). See e.g. [Møl00] for details. Due to the higher complexity of the general Helffer-Sjöstrand formula, these calculations do not lead directly to the generalised result where A is a vector of self-adjoint, pairwise commuting operators. However, we will follow the same idea.

The theorem may be viewed as an abstract analogue of pseudo-differential calculus. The one-dimensional version is an often used result, see e.g. [DG97] and [Møl00]. Apart from the obvious interest in generalising the result to higher dimensions, our improvement has proven useful in the treatment of models in quantum field theory, see [MR]. In particular, a lemma in [MR] whose proof depends on our result, extends the results of [Møl05] to a larger class of models.

2 The setting and result

In the following, $A = (A_1, \ldots, A_\nu)$ is a vector of self-adjoint, pairwise commuting operators acting on a Hilbert space $\mathcal{H}$, and $B \in \mathcal{B}(\mathcal{H})$ is a bounded operator on $\mathcal{H}$. We shall use the notion of B being of class $C^{n_0}(A)$ introduced in [ABG96]. For notational convenience, we adobt the following convention: If $0 \leq j \leq \nu$, then δ_j denotes the multi-index $(0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the j'th entry.

Definition 2.1. Let $n_0 \in \mathbb{N} \cup \{\infty\}$. Assume that the multi-commutator form defined iteratively by $\mathrm{ad}_A^0(B) = B$ and $\mathrm{ad}_A^\alpha(B) = [\mathrm{ad}_A^{\alpha-\delta_j}(B), A_j]$ as a form on $\mathcal{D}(A_j)$, where $\alpha \geq \delta_j$ is a multi-index and $1 \leq j \leq \nu$, can be represented by a bounded operator also denoted by $\mathrm{ad}_A^\alpha(B)$, for all multi-indices α, $|\alpha| < n_0 + 1$. Then B is said to be of class $C^{n_0}(A)$ and we write $B \in C^{n_0}(A)$.

Remark 2.2. The definition of $\mathrm{ad}_A^\alpha(B)$ does not depend on the order of the iteration since the A_j are pairwise commuting. We call $|\alpha|$ the *degree* of $\mathrm{ad}_A^\alpha(B)$.

In the following, $\mathcal{H}_A^s := D(|A|^s)$ for $s \geq 0$ will be used to denote the scale of spaces associated to A. For negative s, we define $\mathcal{H}_A^s := (\mathcal{H}_A^{-s})^*$.

Theorem 2.3. *Assume that $B \in C^{n_0}(A)$ for some $n_0 \geq n+1 \geq 1$, $0 \leq t_1 \leq n+1$, $0 \leq t_2 \leq 1$ and that $\{f_\lambda\}_{\lambda \in I}$ satisfies*

$$\forall \alpha\, \exists C_\alpha : |\partial^\alpha f_\lambda(x)| \leq C_\alpha \langle x \rangle^{s-|\alpha|}$$

uniformly in λ for some $s \in \mathbb{R}$ such that $t_1 + t_2 + s < n+1$. Then

$$[B, f_\lambda(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!} \partial^\alpha f_\lambda(A)\, \mathrm{ad}_A^\alpha(B) + R_{\lambda,n}(A, B)$$

as an identity on $\mathcal{D}(\langle A \rangle^s)$, where $R_{\lambda,n}(A, B) \in \mathcal{B}(\mathcal{H}_A^{-t_2}, \mathcal{H}_A^{t_1})$ and there exist a constant C independent of A, B and λ such that

$$\|R_{\lambda,n}(A, B)\|_{\mathcal{B}(\mathcal{H}_A^{-t_2}, \mathcal{H}_A^{t_1})} \leq C \sum_{|\alpha|=n+1} \|\mathrm{ad}_A^\alpha(B)\|.$$

Remark 2.4. A similar statement holds with the $\mathrm{ad}_A^\alpha(B)$ and $\partial^\alpha f_\lambda(A)$ interchanged at the cost of a sign correction given by $(-1)^{|\alpha|-1}$, and the corresponding remainder term $R'_{\lambda,n}(A, B) \in \mathcal{B}(\mathcal{H}_A^{-t_1}, \mathcal{H}_A^{t_2})$. This can be seen either by proving it analogously or by taking the adjoint equation and replacing B by $-B$.

Remark 2.5. If $k \leq t_1$ and $n_0 \geq n + 1 + k$, then $R_{\lambda,n}(A, B)$ can be replaced by $R_{\lambda,n}^k(A, B) \in \mathcal{B}(\mathcal{H}_A^{-t_2+k}, \mathcal{H}_A^{t_1-k})$. This can be seen by commuting $\mathrm{ad}_A^\alpha(B)$ and $|A - z|^{-2}$ in the terms of the remainder, see page 101.

3 The Proof

Let $z \in \mathbb{C}^\nu$, $\operatorname{Im} z \neq 0$, $1 \leq \ell \leq \nu$ and $g, g_\ell \colon \mathbb{R}^\nu \to \mathbb{C}$ be given as $g(t) = |t-z|^{-2}$ and $g_\ell(t) = t_\ell - \bar{z}_\ell$. Write for $2\beta \leq \alpha$

$$T_\alpha^\beta(t,z) := \tfrac{(-2)^{|\alpha-\beta|}|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(t-\operatorname{Re} z)^{\alpha-2\beta}|t-z|^{-2|\alpha-\beta|}.$$

Lemma 3.1. *Let g be as above and α be any multi-index. Then*

$$\partial^\alpha g(t) = \sum_{2\beta\leq\alpha} \alpha! T_\alpha^\beta(t,z)|t-z|^{-2}.$$

Proof. For brevity, we will write α^i or β^i for $\alpha+\delta_i$ or $\beta+\delta_i$, respectively. The formula is obviously true for $|\alpha| \leq 1$. Now assume that we have proven the formula for $|\alpha| \leq k$. Let $|\alpha| = k$ and $0 \leq i \leq \nu$ be arbitrary. It suffices to prove the formula for α^i. One easily verifies using the chain rule that

$$(\partial^{\delta_i} g^n)(t) = -2n(t_i - \operatorname{Re} z_i)|t-z|^{-2n-2}. \tag{4.2}$$

Now by the induction hypothesis, we see that

$$\begin{aligned}
\partial^{\alpha+\delta_i} g(t) &= \partial_t^{\delta_i} \sum_{2\beta\leq\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(t-\operatorname{Re} z)^{\alpha-2\beta}|t-z|^{-2|\alpha-\beta|-2} \\
&= \sum_{2\beta\leq\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(\partial_t^{\delta_i}(t-\operatorname{Re} z)^{\alpha-2\beta})|t-z|^{-2|\alpha-\beta|-2} \qquad (4.3)\\
&\quad + \sum_{2\beta\leq\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(t-\operatorname{Re} z)^{\alpha-2\beta}(\partial_t^{\delta_i}|t-z|^{-2|\alpha-\beta|-2}). \qquad (4.4)
\end{aligned}$$

For the sake of clarity, we will now consider each sum independently.

$$\begin{aligned}
(4.3) &= \sum_{2\beta\leq\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(\alpha_i - 2\beta_i)(t-\operatorname{Re} z)^{\alpha-2\beta-\delta_i}|t-z|^{-2|\alpha-\beta|-2} \\
&= \sum_{\substack{2\beta\leq\alpha \\ 2\beta_i<\alpha_i}} 2(\beta_i+1)\tfrac{(-2)^{|\alpha^i-\beta^i|}\alpha!|\alpha^i-\beta^i|!}{2^{|\beta^i|}\beta^i!(\alpha^i-2\beta^i)!}(t-\operatorname{Re} z)^{\alpha^i-2\beta^i}|t-z|^{-2|\alpha^i-\beta^i|-2} \\
&= \sum_{2\beta\leq\alpha+\delta_i} 2\beta_i \tfrac{(-2)^{|\alpha^i-\beta|}\alpha!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t-\operatorname{Re} z)^{\alpha^i-2\beta}|t-z|^{-2|\alpha^i-\beta|-2}. \qquad (4.5)
\end{aligned}$$

Using (4.2), we see that (4.4) equals

$$\sum_{2\beta\leq\alpha}\frac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(t-\operatorname{Re}z)^{\alpha-2\beta}(-2)(|\alpha-\beta|+1)(t_i-\operatorname{Re}z_i)|t-z|^{-2|\alpha-\beta|-4}$$

$$=\sum_{2\beta\leq\alpha}(\alpha_i+1-2\beta_i)\frac{(-2)^{|\alpha^i-\beta|}\alpha!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t-\operatorname{Re}z)^{\alpha^i-2\beta}|t-z|^{-2|\alpha^i-\beta|-2}$$

$$=\sum_{2\beta\leq\alpha}\frac{(-2)^{|\alpha^i-\beta|}\alpha^i!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t-\operatorname{Re}z)^{\alpha^i-2\beta}|t-z|^{-2|\alpha^i-\beta|-2} \tag{4.6}$$

$$-\sum_{2\beta\leq\alpha}2\beta_i\frac{(-2)^{|\alpha^i-\beta|}\alpha!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t-\operatorname{Re}z)^{\alpha^i-2\beta}|t-z|^{-2|\alpha^i-\beta|-2}. \tag{4.7}$$

Now (4.7) cancels (4.5) except for possible terms with $2\beta=\alpha+\delta_i$:

$$(4.5)+(4.7)=\sum_{2\beta=\alpha+\delta_i}\frac{(-2)^{|\alpha^i-\beta|}\alpha^i!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t-\operatorname{Re}z)^{\alpha^i-2\beta}|t-z|^{-2|\alpha^i-\beta|-2}. \tag{4.8}$$

Adding (4.6) and (4.8) finishes the induction. □

Lemma 3.2. *Let $B\in C^{n_0}(A)$ for some $n_0\geq 1$ and let $n\in\mathbb{N}_0$ and α_0 be a multi-index satisfying $|\alpha_0|+n+1\leq n_0$. Then*

$$[\operatorname{ad}_A^{\alpha_0}(B),g(A)]=\sum_{|\alpha|=1}^{n}\frac{1}{\alpha!}\partial^\alpha g(A)\operatorname{ad}_A^{\alpha_0+\alpha}(B)+R_n^g(A,\operatorname{ad}_A^{\alpha_0}(B)), \tag{4.9}$$

where

$$R_n^g(A,\operatorname{ad}_A^{\alpha_0}(B))$$

$$=\sum_{\substack{|\alpha|=n-1\\2\beta\leq\alpha}}\sum_{i=1}^{\nu}\frac{\beta_i+1}{|\alpha+\delta_i-\beta|}T_{\alpha+2\delta_i}^{\beta+\delta_i}(A,z)\operatorname{ad}_A^{\alpha_0+\alpha+2\delta_i}(B)|A-z|^{-2} \tag{4.10}$$

$$+\sum_{\substack{|\alpha|=n\\2\beta\leq\alpha}}\sum_{i=1}^{\nu}\frac{\beta_i+1}{|\alpha+\delta_i-\beta|}T_{\alpha+2\delta_i}^{\beta+\delta_i}(A,z)(A_i-\bar{z}_i)\operatorname{ad}_A^{\alpha_0+\alpha+\delta_i}(B)|A-z|^{-2} \tag{4.11}$$

$$+\sum_{\substack{|\alpha|=n\\2\beta\leq\alpha}}\sum_{i=1}^{\nu}\frac{\beta_i+1}{|\alpha+\delta_i-\beta|}T_{\alpha+2\delta_i}^{\beta+\delta_i}(A,z)\operatorname{ad}_A^{\alpha_0+\alpha+\delta_i}(B)(A_i-z_i)|A-z|^{-2}. \tag{4.12}$$

Proof. The proof goes by induction. One may check by inspection of the following identity that the statement is true for $n = 0$.

$$\begin{aligned}[\mathrm{ad}_A^{\alpha_0}(B), |A-z|^{-2}] = &-\sum_{i=1}^{\nu} |A-z|^{-2}(A_i-\bar{z}_i)\,\mathrm{ad}_A^{\alpha_0+\delta_i}(B)|A-z|^{-2} \\ &-\sum_{i=1}^{\nu} |A-z|^{-2}\,\mathrm{ad}_A^{\alpha_0+\delta_i}(B)(A_i-z_i)|A-z|^{-2}.\end{aligned} \tag{4.13}$$

Now assume that we have proven the formula for $k \leq n$, $|\alpha_0| + n + 2 \leq n_0$. We will now show that this implies that the formula holds for $k = n + 1$. We begin by noting two useful identities.

$$T_\alpha^\beta(t,z)|t-z|^{-2} = -\tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(t,z). \tag{4.14}$$

$$(\beta_i+1)T_{\alpha+2\delta_i}^{\beta+\delta_i}(t,z)2(t_i - \operatorname{Re} z_i) = (\alpha_i + 1 - 2\beta_i)T_{\alpha+\delta_i}^{\beta}(t,z). \tag{4.15}$$

Now using (4.13) and (4.14) we see that

$$(4.10) = \sum_{|\alpha|=n-1}\sum_{2\beta\leq\alpha}\sum_{i=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|} T_{\alpha+2\delta_i}^{\beta+\delta_i}(A,z)|A-z|^{-2}\mathrm{ad}_A^{\alpha_0+\alpha+2\delta_i}(B) \tag{4.16}$$

$$\begin{aligned}&+\sum_{|\alpha|=n-1}\sum_{2\beta\leq\alpha}\sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|}\tfrac{\beta_j+\delta_{ij}+1}{|\alpha+\delta_i+\delta_j-\beta|} T_{\alpha+2\delta_i+2\delta_j}^{\beta+\delta_i+\delta_j}(A,z) \\ &\quad\times (A_j-\bar{z}_j)\,\mathrm{ad}_A^{\alpha_0+\alpha+2\delta_i+\delta_j}(B)|A-z|^{-2}\end{aligned} \tag{4.17}$$

$$\begin{aligned}&+\sum_{|\alpha|=n-1}\sum_{2\beta\leq\alpha}\sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|}\tfrac{\beta_j+\delta_{ij}+1}{|\alpha+\delta_i+\delta_j-\beta|} T_{\alpha+2\delta_i+2\delta_j}^{\beta+\delta_i+\delta_j}(A,z) \\ &\quad\times \mathrm{ad}_A^{\alpha_0+\alpha+2\delta_i+\delta_j}(B)(A_j-z_j)|A-z|^{-2},\end{aligned} \tag{4.18}$$

and by reordering and reindexing the sum in (4.16), (4.17) and (4.18), we get

$$(4.16) = \sum_{i=1}^{\nu}\sum_{\substack{|\alpha|=n+1\\ \alpha_i\geq 2}}\sum_{\substack{2\beta\leq\alpha\\ \beta_i\geq 1}} \tfrac{\beta_i}{|\alpha-\beta|} T_\alpha^\beta(A,z)|A-z|^{-2}\,\mathrm{ad}_A^{\alpha_0+\alpha}(B), \tag{4.19}$$

and (4.17) equals

$$\begin{aligned}&\sum_{i=1}^{\nu}\sum_{\substack{|\alpha|=n+1\\ \alpha_i\geq 2}}\sum_{\substack{2\beta\leq\alpha\\ \beta_i\geq 1}}\sum_{j=1}^{\nu} \tfrac{\beta_i}{|\alpha-\beta|}\tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(A,z) \\ &\quad\times (A_j-\bar{z}_j)\,\mathrm{ad}_A^{\alpha_0+\alpha+\delta_j}(B)|A-z|^{-2}\end{aligned} \tag{4.20}$$

and similarly for (4.18) with the factor $(A_j - \bar{z}_j)\,\mathrm{ad}_A^{\alpha_0+\alpha+\delta_j}(B)$ replaced by $\mathrm{ad}_A^{\alpha_0+\alpha+\delta_j}(B)(A_j - z_j)$. Note that we may relax the extra conditions on α and β in the above statements, as a term with $\beta_i = 0$ contributes nothing.

Instead of continuing in the same fashion with (4.11) and (4.12), we note using (4.15) that

$$
\begin{aligned}
&(4.11)+(4.12) = \\
&\quad \sum_{|\alpha|=n} \sum_{2\beta\le\alpha} \sum_{i=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|} T_{\alpha+2\delta_i}^{\beta+\delta_i}(A,z)\,\mathrm{ad}_A^{\alpha_0+\alpha+2\delta_i}(B)|A-z|^{-2} &(4.21)\\
&\quad + \sum_{|\alpha|=n} \sum_{2\beta\le\alpha} \sum_{i=1}^{\nu} \tfrac{\alpha_i+1-2\beta_i}{|\alpha+\delta_i-\beta|} T_{\alpha+\delta_i}^{\beta}(A,z)\,\mathrm{ad}_A^{\alpha_0+\alpha+\delta_i}(B)|A-z|^{-2}, &(4.22)
\end{aligned}
$$

so we may focus our attention on (4.22):

$$
\begin{aligned}
(4.22) = &\sum_{i=1}^{\nu} \sum_{\substack{|\alpha|=n+1\\ \alpha_i\ge1}} \sum_{\substack{2\beta\le\alpha\\ 2\beta_i<\alpha_i}} \tfrac{\alpha_i-2\beta_i}{|\alpha-\beta|} T_{\alpha}^{\beta}(A,z)|A-z|^{-2}\,\mathrm{ad}_A^{\alpha_0+\alpha}(B) &(4.23)\\
&+\sum_{i=1}^{\nu} \sum_{\substack{|\alpha|=n+1\\ \alpha_i\ge1}} \sum_{\substack{2\beta\le\alpha\\ 2\beta_i<\alpha_i}} \sum_{j=1}^{\nu} \tfrac{\alpha_i-2\beta_i}{|\alpha-\beta|} \tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(A,z) \\
&\qquad \times (A_j-\bar{z}_j)\,\mathrm{ad}_A^{\alpha_0+\alpha+\delta_j}(B)|A-z|^{-2}. &(4.24)\\
&+\sum_{i=1}^{\nu} \sum_{\substack{|\alpha|=n+1\\ \alpha_i\ge1}} \sum_{\substack{2\beta\le\alpha\\ 2\beta_i<\alpha_i}} \sum_{j=1}^{\nu} \tfrac{\alpha_i-2\beta_i}{|\alpha-\beta|} \tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(A,z) \\
&\qquad \times \mathrm{ad}_A^{\alpha_0+\alpha+\delta_j}(B)(A_j-z_j)|A-z|^{-2} &(4.25)
\end{aligned}
$$

We note again that the additional conditions on α and β are superfluous.

We may now recollect the terms. First we see using Lemma 3.1:

$$
\begin{aligned}
&\sum_{|\alpha|=1}^{n} \frac{1}{\alpha!}\partial^\alpha g(A)\,\mathrm{ad}_A^{\alpha_0+\alpha}(B) + (4.19) + (4.23) = \\
&\qquad\qquad \sum_{|\alpha|=1}^{n+1} \frac{1}{\alpha!}\partial^\alpha g(A)\,\mathrm{ad}_A^{\alpha_0+\alpha}(B), &(4.26)
\end{aligned}
$$

then

$$
\begin{aligned}
&(4.20) + (4.24) = \\
&\sum_{\substack{|\alpha|=n+1\\ 2\beta\le\alpha}} \sum_{j=1}^{\nu} \tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(A,z)(A_j-\bar{z}_j)\,\mathrm{ad}_A^{\alpha_0+\alpha+\delta_j}(B)|A-z|^{-2}, &(4.27)
\end{aligned}
$$

and

$$(4.18)+(4.25)=$$
$$\sum_{\substack{|\alpha|=n+1\\ 2\beta\leq\alpha}}\sum_{j=1}^{\nu}\tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|}T^{\beta+\delta_j}_{\alpha+2\delta_j}(A,z)\,\mathrm{ad}_A^{\alpha_0+\alpha+\delta_j}(B)(A_j-z_j)|A-z|^{-2}, \tag{4.28}$$

so adding up, we have proved that (4.9) equals the sum of (4.26), (4.21), (4.27) and (4.28) as stated. □

The following lemma plays the same role for g_ℓ as Lemma 3.2 plays for g, but contrary to Lemma 3.2, the proof is trivial.

Lemma 3.3. *Let $B\in C^{n_0}(A)$ for some $n_0\geq 1$ and let $n\in\mathbb{N}_0$ and α_0 be a multi-index satisfying $|\alpha_0|+n+1\leq n_0$. Then*

$$[\mathrm{ad}_A^{\alpha_0}(B),g_\ell(A)]=\sum_{|\alpha|=1}^{n}\frac{1}{\alpha!}\partial^\alpha g_\ell(A)\,\mathrm{ad}_A^{\alpha_0+\alpha}(B)+R_n^{g_\ell}(A,\mathrm{ad}_A^{\alpha_0}(B)),$$

where $R_n^{g_\ell}(A,\mathrm{ad}_A^{\alpha_0}(B))=0$ for $n\geq 1$, $R_0^{g_\ell}(A,\mathrm{ad}_A^{\alpha_0}(B))=\mathrm{ad}_A^{\alpha_0+\delta_\ell}(B)$.

The following lemma also follows by induction.

Lemma 3.4. *Let $B\in C^{n_0}(A)$ for some $n_0\geq 1$. Assume that $h_i\in C^\infty(\mathbb{R}^\nu)$, $1\leq i\leq k$, satisfies*

$$[\mathrm{ad}_A^{\alpha_0}(B),h_i(A)]=\sum_{|\alpha|=1}^{n}\frac{1}{\alpha!}\partial^\alpha h_i(A)\,\mathrm{ad}_A^{\alpha_0+\alpha}(B)+R_n^{h_i}(A,\mathrm{ad}_A^{\alpha_0}(B)),$$

where $R_n^{h_i}(A,\mathrm{ad}_A^{\alpha_0}(B))$ is bounded for all $n\in\mathbb{N}_0$ and multi-indices α_0 satisfying $|\alpha_0|+n+1\leq n_0$ and $\partial^\alpha h_i(A)$ is bounded for all $1\leq|\alpha|\leq n_0-1$. Then

$$\Big[B,\prod_{i=1}^{k}h_i(A)\Big]=\sum_{|\alpha|=1}^{n}\frac{1}{\alpha!}\partial^\alpha\Big(\prod_{i=1}^{k}h_i\Big)(A)\,\mathrm{ad}_A^{\alpha}(B)$$
$$+\sum_{j=1}^{k}\sum_{|\alpha|=0}^{n}\frac{1}{\alpha!}\partial^\alpha\Big(\prod_{i=1}^{j-1}h_i\Big)(A)R^{h_j}_{n-|\alpha|}(A,\mathrm{ad}_A^{\alpha}(B))\prod_{i=j+1}^{k}h_i(A).$$

Let $n+1\leq n_0$. If we put $k=\nu+1$, $h_i=g$ for $i\neq\nu$, $h_\nu=g_\ell$ and apply Lemma 3.2, 3.3 and 3.4 we see that

$$[B,|A-z|^{-2\nu}(A_\ell-\bar z_\ell)]=$$
$$\sum_{|\alpha|=1}^{n}\frac{1}{\alpha!}\partial^\alpha\big(|\cdot-z|^{-2\nu}(\cdot_\ell-\bar z_\ell)\big)(A)\,\mathrm{ad}_A^{\alpha}(B)+R_{\ell,n}(A,B), \tag{4.29}$$

where

$$R_{\ell,n}(A,B) =$$
$$\sum_{j=1}^{\nu-1}\sum_{|\alpha|=0}^{n}\frac{1}{\alpha!}\partial^\alpha(g^{j-1})(A)R^g_{n-|\alpha|}(A,\mathrm{ad}^\alpha_A(B))|A-z|^{-2(\nu-j)}(A_\ell-\bar{z}_\ell) \quad (4.30)$$
$$+\sum_{|\alpha|=n}\frac{1}{\alpha!}\partial^\alpha(g^{\nu-1})(A)\,\mathrm{ad}_A^{\alpha+\delta_\ell}(B)|A-z|^{-2} \quad (4.31)$$
$$+\sum_{|\alpha|=0}^{n}\frac{1}{\alpha!}\partial^\alpha(g^{\nu-1}g_\ell)(A)R^g_{n-|\alpha|}(A,\mathrm{ad}^\alpha_A(B)) \quad (4.32)$$

In the following, we will refer to the terms of $R_{\ell,n}(A,B)$ as the remainder terms. Let $0\le t_1\le n+1$ and $0\le t_2\le 1$. By Hadamard's three-line lemma and using (4.10–4.12), (4.30–4.32), Lemma 3.1 and the identity

$$\partial^\alpha\Big(\prod_{i=1}^j f_i\Big)=\sum_{\sum\alpha_i=\alpha}\frac{\alpha!}{\prod_{i=1}^j\alpha_i!}\prod_{i=1}^j\partial^{\alpha_i}f_i,$$

we may inspect that each remainder term (with $R_{\ell,n}(A,B)$ replaced by the remainder term) and hence $R_{\ell,n}(A,B)$ satisfies the inequality

$$\|\langle A\rangle^{t_1}R_{\ell,n}(A,B)\langle A\rangle^{t_2}\|\le C\langle z\rangle^{t_1+t_2}|\mathrm{Im}\,z|^{-n-2\nu}. \quad (4.33)$$

We will now use the functional calculus of almost analytic extensions. See e.g. [DS99] for details. In the following, we write $\bar\partial=(\bar\partial_1,\dots,\bar\partial_\nu)$ where $\bar\partial_j=\frac{1}{2}(\partial_{u_j}+i\partial_{v_j})$ and $u_j+v_j=z_j\in\mathbb{C}$, $z=(z_1,\dots,z_n)\in\mathbb{C}^\nu$. The following proposition is inspired by [Tre80, Chap. X.2] and [Møl00].

Proposition 3.5. *Let $s\in\mathbb{R}$ and $\{f_\lambda\}_{\lambda\in I}\subset C^\infty(\mathbb{R}^\nu)$ satisfy*

$$\forall\alpha\ \exists C_\alpha\colon|\partial^\alpha f_\lambda(x)|\le C_\alpha\langle x\rangle^{s-|\alpha|}.$$

There exists a family of almost analytic extensions $\{\tilde f_\lambda\}_{\lambda\in I}\subset C^\infty(\mathbb{C}^\nu)$ satisfying

(i) $\mathrm{supp}(\tilde f_\lambda)\subset\{u+iv\mid u\in\mathrm{supp}(f_\lambda),|v|\le C\langle u\rangle\}$.

(ii) $\forall\ell\ge 0\ \exists C_\ell\colon|\bar\partial\tilde f_\lambda(z)|\le C_\ell\langle z\rangle^{s-\ell-1}|\mathrm{Im}\,z|^\ell$.

Proof. We define a mapping $C^\infty(\mathbb{R}^\nu)\ni f\mapsto\tilde f\in C^\infty(\mathbb{C}^\nu)$ in the following way. Choose a function $\kappa\in C_0^\infty(\mathbb{R})$ which equals 1 in a neighbourhood of 0 and put $\lambda_0=C_0$, $\lambda_k=\max\{\max_{|\alpha|=k}C_\alpha,\lambda_{k-1}+1\}$ for $k\ge1$. Writing $z=u+iv\in\mathbb{R}^\nu\oplus i\mathbb{R}^\nu$, we now define

$$\tilde f(z)=\sum_\alpha\frac{\partial^\alpha f(u)}{\alpha!}(iu)^\alpha\prod_{j=1}^\nu\kappa\Big(\frac{\lambda_{|\alpha|}v_j}{\langle u\rangle}\Big).$$

One can now check that the properties hold. □

Remark 3.6. Note that if we for a $\chi \in C_0^\infty(\mathbb{R}^\nu; [0,1])$ with $\chi(0) = 1$ define a sequence of functions by $f_{k,\lambda}(x) = \chi(\frac{x}{k}) f_\lambda(x)$, then

$$[B, f_\lambda(A)] = \lim_{k\to\infty} [B, f_{k,\lambda}(A)]$$

as a form identity on $\mathcal{D}(\langle A\rangle^s)$ and we have the dominated pointwise convergence

$$\bar\partial \tilde f_{k,\lambda}(x) \to \bar\partial \tilde f_\lambda(x) \text{ for } k \to \infty.$$

Let $\{f_\lambda\}_{\lambda\in I}$ satisfy the assumption of Proposition 3.5 with $s < 0$. Then the almost analytic extensions provide a functional calculus via the formula

$$f_\lambda(A) = C_\nu \sum_{\ell=1}^{\nu} \int_{\mathbb{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z)(A_\ell - \bar z_\ell)|A - z|^{-2\nu} dz, \tag{4.34}$$

where C_ν is a positive constant (again we refer to [DS99] for details). Note that the integrals are absolutely convergent by Proposition 3.5(ii).

Multiplying $\langle A\rangle^{t_1} R_{\ell,n}(A,B)\langle A\rangle^{t_2}$ with $\bar\partial \tilde f_\lambda(z)$, we get from (4.33) and Proposition 3.5 (ii) that

$$\|\langle A\rangle^{t_1} \bar\partial \tilde f_\lambda(z) R_{\ell,n}(A,B)\langle A\rangle^{t_2}\| \le C\langle z\rangle^{t_1+t_2+s-n-1-2\nu}. \tag{4.35}$$

Hence, if $t_1 + t_2 + s < n + 1$, $\langle A\rangle^{t_1} \bar\partial \tilde f_\lambda(z) R_{\ell,n}(A,B)\langle A\rangle^{t_2}$ is integrable over $\mathbb{C}^\nu$. Using (4.29), (4.34) and (4.35), we see that

$$\begin{aligned}
[B, f_\lambda(A)] &= C_\nu \sum_{\ell=1}^{\nu} \int_{\mathbb{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z)[B, (A_\ell - \bar z_\ell)|A - z|^{-2\nu}]\, dz \\
&= C_\nu \sum_{\ell=1}^{\nu} \int_{\mathbb{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z) \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!} \partial^\alpha \big(|\cdot - z|^{-2\nu}(\cdot_\ell - \bar z_\ell)\big)(A)\, dz\, \mathrm{ad}_A^\alpha(B) \\
&\quad + C_\nu \sum_{\ell=1}^{\nu} \int_{\mathbb{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z) R_{\ell,n}(A,B)\, dz.
\end{aligned} \tag{4.36}$$

We denote (4.36) by $R_{\lambda,n}(A,B)$. Note that

$$\begin{aligned}
&\sum_{\ell=1}^{\nu} \int_{\mathbb{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z) \frac{1}{\alpha!} \partial_t^\alpha \big(|t - z|^{-2\nu}(t_\ell - \bar z_\ell)\big)\, dz \\
&\quad = \frac{1}{\alpha!} \partial_t^\alpha \sum_{\ell=1}^{\nu} \int_{\mathbb{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z)|t - z|^{-2\nu}(t_\ell - \bar z_\ell)\, dz = \frac{1}{\alpha!} \partial^\alpha f_\lambda(t),
\end{aligned}$$

which implies

$$[B, f_\lambda(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!} \partial^\alpha f_\lambda(A) \ \mathrm{ad}_A^\alpha(B) + R_{\lambda,n}(A,B).$$

We have now proved Theorem 2.3 in the case $s < 0$. For the general case, we use Remark 3.6 to see that $[B, f_\lambda(A)] = \lim_{k\to\infty}[B, f_{k,\lambda}(A)]$ and clearly, $f_{k,\lambda}$ satisfies the assumption of Proposition 3.5 with the same s, so the estimate corresponding to (4.35) is now uniform in k and λ. The pointwise convergence and Lebesgue's theorem on dominated convergence now finishes the argument.

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Bibliography

[ABG96] W. O. Amrein, A. Boutet de Monvel, and V. Georgescu, *C_0-groups, commutator methods and spectral theory of N-body hamiltonians*, Birkhäuser, 1996.

[DG97] J. Dereziński and C. Gérard, *Scattering theory of classical and quantum N-particle systems*, Texts and Monographs in Physics, Springer, Berlin, 1997.

[DS99] M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, London Mathematical Society Lecture Note Series, vol. 268, Cambridge University Press, 1999.

[MR] J. S. Møller and M. G. Rasmussen, *The translation invariant massive Nelson model: II. The lower part of the essential spectrum*, In prep.

[Møl00] J. S. Møller, *An abstract radiation condition and application to N-body systems*, Rev. Math. Phys. **12** (2000), no. 5, 767–803.

[Møl05] ———, *The translation invariant massive Nelson model: I. The bottom of the spectrum*, Ann. Henri Poincaré **6** (2005), 1091–1135.

[Tre80] F. Treves, *Introduction to pseudodifferential operators and fourier integral operators*, vol. 2, Plenum Press, 1980.