Aspects of Quantum Mathematics

Hitchin Connections

and

AJ Conjectures

A PhD Dissertation by
Magnus Roed Lauridsen

Advisors
Jørgen Ellegaard Andersen
Nicolai Reshetikhin

Department of Mathematical Sciences
Aarhus University

Contents

Preface

This dissertation is the product of my PhD studies at the Center for the Topology and Quantization of Moduli Spaces (CTQM) at the Department of Mathematical Sciences, Aarhus University. The project is partially supported by the Niels Bohr Visiting Professor Initiative of the Danish National Research Foundation.

I wish to thank the CTQM and the Department of Mathematical Sciences, Aarhus University for creating a great study environment for the last eight years. I have benefitted greatly from interacting with the many visitors at the CTQM. Also, I would like to thank the Department of Mathematics at UC Berkeley for their hospitality during my visit in the fall of 2007. I thank Lars Madsen for creating some of the illustrations in this dissertation with very little notice. Also, my good friend and collaborator Niels Leth Gammelgaard for the time we spend working on our paper as well as the many formal and informal discussions. My girlfriend, who has showed great patience during the writing phase. And finally, my two advisors Jørgen Ellegaard Andersen and Nicolai Reshetikhin for all their help and patience.

Magnus Roed Lauridsen
Århus, July 28, 2010

Introduction

This Dissertation is concerned with two different areas of "quantum mathematics", namely, geometric quantization and knot theory. These have been linked via Topological Quantum Field Theory since the late 1980's. In 1988 Atiyah [Ati1] asked for a physical interpretation of the link invariant made by Jones [Jon] in 1985. Witten [Wit] in 1989 gave an outline for an answer. He argued that quantizing Chern-Simons theory with gauge group $\mathrm{SU}(n)$ would produce a $(2+1)$-dimensional Topological Quantum Field Theory and that this would be related to the Jones polynomial. On the geometric side, the two-dimensional part of the TQFT was constructed by geometric quantization of the moduli space of flat $\mathrm{SU}(n)$-connections on a surface Σ. On the other hand, the TQFT also gave a new three-manifold invariant and invariants of links related to the Jones polynomials, now called the coloured Jones polynomial. This was made mathematically rigorous by Reshetikhin and Turaev ([RT1],[RT2]) using representation theory of quantum groups. Later Blanchet, Habegger, Masbaum and Vogel ([BHMV1], [BHMV2]) gave a skein theoretical construction.

Regarding the geometric quantization, an auxilliary Kähler structure on the moduli space was needed. To remove this dependency, Hitchin [Hit] constructed a projectively flat connection in a bundle of quantum spaces over Teichmüller space. In the first part of this dissertation, we will study analogues of this connection in geometric quantization of any symplectic manifold, not necessarily moduli spaces. This is based on the construction of Andersen [And2]. As a main result, we extend his construction to a more physically sound metaplectically corrected quantization.

In the second part we investigate a conjecture in knot theory named the AJ conjecture by Garoufalidis [Gar]. This relates the coloured Jones polynomial with another knot invariant, the A-polynomial of Cooper, Culler, Gillet Long and Shalen [CCG$^+$]. The key observation here, is that the coloured Jones polynomial satisfies a non-trivial recursion relation, as proved by Garoufalidis and Lê [GL]. This relation is captured by a polynomial in two q-commuting variables. The conjecture is that in the $q = 1$ limit, this becomes the A-polynomial.

We formulate a series of related conjectures, some of which employs the ties to geometric quantization. Along the way we prove a formula for the coloured Jones polynomials for a new class of knots, which we call double twist knots. As a part of the investigation of one of these geometric AJ conjectures, we write the coloured Jones polynomial for double twist knots as a multiple contour integral using Faddeev's quantum dilogarithms. By a non-rigorous analysis of the asymptotics of these integrals, we verify the conjecture for twist knots.

The dissertation is organized as follows.

Chapter 1 is a brief introduction to the concept of geometric quantization. Also, we discuss Berezin-Toeplitz deformation quantization, primarily to introduce Toeplitz operators.

Chapter 2 reviews the construction of a Hitchin connection made by Andersen in [And2]. At the end we give a brief account of the moduli space of flat $\mathrm{SU}(n)$-connections and another example of symplectic manifolds with a *rigid* family of Kähler structures.

Chapter 3 contains original work by the author in collaboration with Andersen and Gammelgaard [AGL]. Here, we introduce a metaplectic structure in the scheme of geometric quantization. We then construct a Hitchin connection in this setting while removing several of the needed assumptions of the previous chapter. Then we discuss a setting, where the constructions from this and the previous chapter of Hitchin connections both can be carried out, and show that they agree.

Chapter 4 gives a geometric quantization of abelian varieties and writes down both a Hitchin connection and concrete formulas for Toeplitz operators as done in [And4]. This is applied to the moduli space of flat $\mathrm{SU}(2)$-connections on a genus one surface, where we discuss a good basis for the quantum spaces.

Chapter 5 introduces knots, in particular two-bridge knots and their knot groups. We also introduce a certain family of two-bridge knots called *double twist knots*. Finally, we construct the A-polynomial and discuss computations for two-bridge knots as well as a theorem of Hoste and Shanahan [HS] on the A-polynomial of twist knots.

Chapter 6 is devoted to TQFT and the coloured Jones polynomial. We discuss the construction of a TQFT in [BHMV2] and thereby introduce Skein theory. This leads to a definition of the coloured Jones polynomial of a knot or link. In the end we prove a closed formula for the coloured Jones polynomial for double twist knots.

Chapter 7 contains the AJ conjectures. We start with the original (algebraic) AJ conjecture of Garoufalidis, where we introduce the noncommutative A-polynomial and discuss the current status of the conjecture. Then, following Gukov [Guk], we introduce the Generalized Volume Conjecture, which contains an AJ conjecture. Via the link between geometric quantization of the moduli space of flat SU(2)-connections on a genus one surface and the coloured Jones polynomial of a knot, we employ the Toeplitz operators from Chapter 4 to formulate new geometric AJ conjectures. One of these are treated at the very end of the chapter for the unknot. We also make an AJ conjecture in TQFT, much more general than the conjectures in knot theory. We find this conjecture very interesting, but have no results in this direction.

Chapter 8 starts with the introduction of Faddeev's quantum dilogarithms, which we use to formulate the new result in Theorem 8.1, expressing the coloured Jones polynomial of double twist knots as a multiple countour integral. This is an extension of the work of Andersen and Hansen [AH] on the figure eight knot. We conclude by a non-rigorous asymptotic analysis of the integral, which for twist knots show the AJ conjecture in the sense of Gukov, by referring to work by Hikami [Hik2].

1

Quantization of Symplectic Manifolds

In this chapter we will discuss quantization as a mathematical concept. This, of course, has a base in the world of physics, but the physical motivation for the different approaches will only be touched upon briefly, if at all. Instead we will take a more axiomatic way of reasoning, where we will will set forth a wish-list for a quantization to fulfill. The main references are [AE], [Woo] and [Sch1]. First we discuss general axioms for quantization. As these leads to contradictions we turn to geometric quantization. This will be our preferred method of quantizing symplectic manifolds. Finally, deformation quantization is mentioned, in particular Berezin-Toeplitz quantization and Toeplitz operators.

1.1 Canonical Quantization

Quantization is the passage from a classical theory to a quantum theory. I.e., given a classical theory we seek to produce a quantum theory, which would yield back the classical theory as a certain (semi-) classical limit. For our needs, a classical system will be a symplectic manifold (M, ω). If we look at classical mechanics in $\mathbb{R}^n$, we have the phase space $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ with coordinates p_j, q_j descibing momentum and position. The symplectic form in these coordinates is $\omega = \sum dp_j \wedge dq_j$. Observables on this phase space are smooth functions $f \in C^\infty(\mathbb{R}^{2n})$. An important operator on observables is the Poisson bracket given by

$$\{f, g\} = \sum_j \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}.$$

This can be desribed by the symplectic form as

$$\{f, g\} = \omega(X_g, X_f)$$

where X_f is the Hamiltonian vector field of f defined by $df = \omega(X_f, \cdot)$.

A quantization of this system is a way of assigning to an observable f (or rather, as large a class of them as possible) a self-adjoint operator Q_f on $L^2(\mathbb{R}^n, d\mathbf{q})$. This assignment should satisfy the following 5 properties

(q1) The assignment is linear

(q2) The constant function 1 should go to the identity, $Q_1 = I$

(q3) The functional calculus for self-adjoint operators should yield $\varphi(Q_f) = Q_{\varphi\circ f}$ for $\varphi\colon \mathbb{R} \to \mathbb{R}$ where defined.

(q4) The operators corresponding to the coordinate functions p_j and q_j are

$$Q_{q_j}\psi = q_j\psi, \qquad Q_{p_j}\psi = -\frac{ih}{2\pi}\frac{\partial\psi}{\partial q_j}, \qquad \psi \in L^2(\mathbb{R}^n)$$

(q5) The canonical commutation relation $[Q_f, Q_g] = \frac{ih}{2\pi}Q_{\{f,g\}}$.

A quantization satisfying these rules is called *canonical quantization.* However, this is not possible to do without getting something trivial or a contradiction; if, for instance, the class of quantizable observables contains polynomials in p_j and q_j up to degree four, this is not possible. Indeed, one can express $p_1^2q_1^2 = (p_1q_i)^2$ in two different ways. Using (q1), (q5) and (q3) for the squaring function one obtains

$$Q_{(p_1q_1)^2} = (Q_{p_1q_1})^2 = Q_{p_1^2q_1^2} + \frac{h^2}{4\pi^2}I.$$

But this is just the tip of the iceberg in terms of contradictions for the axioms (q1)-(q5) (see [AE] for references and further discussions).

1.2 Geometric Quantization

The way we will deal with these problems, is by so-called *geometric quantization.* The idea is to drop the axiom (q3) and reduce the space of observables. We also want to quantize other symplectic manifolds, e.g. cotangent bundles. For this we impose axioms for geometric quantization.

Given a symplectic manifold (M,ω) of dimension $2m$ we assign a (separable) Hilbert space $\mathcal{H}$. Also, fix a collection of real valued functions on M as the observables $\mathcal{F}$, which are closed under $\{\cdot,\cdot\}$. The quantization assigns self-adjoint operators on $\mathcal{H}$ to functions in $\mathcal{F}$, satisfying

(Q1) The assignment is linear

(Q2) $Q_1 = I$

(Q3) $[Q_f, Q_g] = \frac{ih}{2\pi}Q_{\{f,g\}}$ for $f, g \in \mathcal{F}$

(Q4) Given two symplectic manifolds (M, ω) and $(\tilde{M}, \tilde{\omega})$ and a symplectomorphism $\varphi\colon (M, \omega) \to (\tilde{M}, \tilde{\omega})$, then for $f \in \tilde{\mathcal{F}}$ we require $Q_{f \circ \varphi}$ and $\tilde{Q}_f$ are conjugate by a unitary operator from $\mathcal{H}$ to $\tilde{\mathcal{H}}$.

(Q5) For $M = \mathbb{R}^{2m}$ with the symplectic structure as above, we recover the operators Q_{p_j} and Q_{q_j} as in (q4).

We will find a solution to the above by ignoring (Q5) and see what happens when we take the naive approach on a cotangent bundle. Namely, let $M = T^*N$ and let τ be the tautological one-form on M defined by

$$\tau(\xi) = \eta(\pi_*(\xi)), \qquad \xi \in T_{(\eta,p)}M$$

where $p \in N$ and $\pi\colon M \to N$ is the projection. In local coordinates q_j on N and (p_j, q_j) on M we get

$$\tau = \sum_{j=1}^{m} p_j dq_j$$

and thus the standard symplectic form $w = d\tau$.

Now, given a function f on M, we can write its Hamiltonian vector field in local coordinates as

$$X_f = \sum_{j=1}^{m} \frac{\partial f}{\partial q_j}\frac{\partial}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial}{\partial p_j}.$$

Since X_f acts on functions on M, one could try to set $Q_f = X_f$ and indeed $[X_f, X_g] = X_{\{f,g\}}$ so $Q_f = \frac{ih}{2\pi} X_f$ is a candidate, which satisfies (Q1), (Q3) and (Q4), but obviously not (Q2). Instead, let us try and modify it by adding multiplication by f

$$Q_f = \frac{ih}{2\pi} X_f + f.$$

One immediately sees that $X_f(g) = \{f, g\}$ and so we almost get the desired commutator

$$[Q_f, Q_g] = \frac{ih}{2\pi}\left(\frac{ih}{2\pi} X_{\{f,g\}} + 2\{f,g\}\right) = \frac{ih}{2\pi}\left(Q_{\{f,g\}} + \{f,g\}\right).$$

From observing $\tau(X_f) = -\sum p_j \frac{\partial f}{\partial p_j}$, a straight-forward calculation reveals that

$$X_f(\tau(X_g)) - X_g(\tau(X_f) = \tau(X_{\{f,g\}}) - \{f, g\}$$

and we finally arrive at the formula

$$Q_f = \frac{ih}{2\pi} X_f + \tau(X_f) + f$$

satisfying (Q1)-(Q4). This works well on a cotangent bundle, where we have a (canonical) one-form τ satisfying $d\tau = \omega$. On a general symplectic manifold we could always choose Darboux coordinates U_α with τ_α to get the desired local description, but what about on overlaps $U_\alpha \cap U_\beta$ where we have another τ_β? The difference is $\tau_\alpha - \tau\beta = du_{\alpha\beta}$ where $u_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbb{R}$. By direct computation, we see that

$$e^{-\frac{2\pi}{ih}u_{\alpha\beta}} Q_f^\alpha \left(e^{\frac{2\pi}{ih}u_{\alpha\beta}} \varphi \right) = Q_f^\beta(\varphi), \quad \varphi \in C^\infty(M).$$

This ambiguity tells us that, whereas Q_f is not defined on functions on M, it is an operator on sections of the complex line bundle $\mathcal{L}$ with transition functions $g_{\alpha\beta} = e^{\frac{2\pi i}{h}u_{\alpha\beta}}$. This, however, requires that on $U_\alpha \cap U_\beta \cap U_\gamma$

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$$

that is, if

$$\frac{u_{\alpha\beta} u_{\beta\gamma} u_{\gamma\alpha}}{h} = z_{\alpha\beta\gamma} \in \mathbb{Z}.$$

Here, with respect to the covering of $\underline{U} = \{U_\alpha\}$, $z_{\alpha\beta\gamma}$ is a cocycle representing the Chern class of $\mathcal{L}$ in $\check{H}^2(\underline{U}, \mathbb{Z})$. As we shall see, the real Chern class is represented by $\frac{\omega}{h}$, so this is really a condition on ω.

Let $\mathcal{L}$ be the line bundle descibed above. Since $|g_{\alpha\beta}| = 1$ we can choose a Hermitian structure on $\mathcal{L}$ as

$$\langle s_1, s_2 \rangle_p = s_1(p)\overline{s_2(p)} \qquad s_1, s_2 \in C^\infty(M, \mathcal{L}),\, p \in M.$$

To compute the first real Chern class of $\mathcal{L}$ we find a connection ∇, compatible with $\langle \cdot, \cdot \rangle$, and compute its curvature. The connection is descibed locally on U_α as

$$\nabla_Y(s) = Y(s) + \frac{2\pi}{ih}\tau_\alpha(Y)s$$

for a section s and a vector field Y in the complexified tangent bundle. Here τ is extended complex linearly. That ∇ is compatible with the Hermitian structure means that

$$Y\langle s_1, s_2 \rangle = \langle \nabla_Y(s_1), s_2 \rangle + \langle s_1, \nabla_{\bar{Y}}(s_2) \rangle$$

and this indeed the case. On the other hand the curvature of this connection is

$$R_\nabla(Y, Z) = \nabla_Y \nabla_Z - \nabla_Z \nabla_Y - \nabla_{[Y,Z]} = \frac{2\pi}{ih}\omega(Y, Z)$$

and we find the first real Chern class as $c_1^{\mathbb{R}}(\mathcal{L}) = \left[\frac{i}{2\pi}R_\nabla\right] = \left[\frac{\omega}{h}\right]$. The above construction can be gathered in a definition

Definition 1.1. A *prequantum line bundle* on a symplectic manifold (M, ω) is a triple $(\mathcal{L}, \nabla, \langle \cdot, \cdot, \rangle)$ of a Hermitian line bundle on M with a compatible connection satisfying the *prequantum condition*

$$F_\nabla = \frac{2\pi}{ih}\omega.$$

We say that (M, ω) is *prequantizable* if such a bundle exists, which is to say $\frac{\omega}{h}$ defines an integral cohomology class.

In the language of prequantum line bundles, the prequantum operator Q_f can now be written globally on M as

$$Q_f = \frac{ih}{2\pi}\nabla_{X_f} + f. \tag{1.1}$$

Now we have a construction satisfying (Q1)-(Q4) on any symplectic manifold. But what about the Hilbert space? We could consider the L^2-completion of $C^\infty(M, \mathcal{L})$. This is what is called *prequantization*, but we get a dimension problem for $M = \mathbb{R}^{2m}$, where we obtain $L^2(\mathbb{R}^{2m})$ instead of $L^2(\mathbb{R}^m)$. Somehow we have twice the amount of variables and so we need to choose half of them. On $\mathbb{R}^{2m}$ this is easy, but what about on a general symplectic manifold? To this end we introduce the notion of a polarization.

A polarization is a choice of a Lagrangian distribution of the tangent bundle. We will focus on complex polarization, but note that there also is a real version.

A complex polarization on (M, ω) is a complex distribution $\mathcal{P}$ of the complexified tangent bundle $TM_\mathbb{C}$, i.e. to each point $p \in M$, a (complex) subspace $\mathcal{P}_p \subset T_pM_\mathbb{C}$, satisfying the following properties:

(P1) $\mathcal{P}$ is involutive, i.e. closed under Lie bracket

(P2) It is Lagrangian, meaning $\dim_\mathbb{C} \mathcal{P}_p = m = \frac{1}{2}\dim_\mathbb{R} M$ and $\omega_{|\mathcal{P}} = 0$ for all $p \in M$

(P3) $\dim_\mathbb{C} \mathcal{P}_p \cap \bar{\mathcal{P}}_p = k$ is constant

(P4) $\mathcal{P} + \bar{\mathcal{P}}$ is involutive

A general account of quantization using a complex polarization can be found in [Woo] or [AE]. We will be using a particularly nice one, namely where $k = 0$. We will call such a polarization pseudo-kähler[1]. Let us see how one can construct such a polarization.

First, choose an almost complex structure $J \in C^\infty(M, \mathrm{End}(TM))$ on M, a smooth choice of endomorphisms of TM where each $J_p^2 = -\mathrm{id}_p$. Extending J complex linearly to $TM_\mathbb{C}$, we can split it into eigenspaces $TM_\mathbb{C} = T + \bar{T}$, where

[1] In e.g. [Woo] this is called a Kähler polarization, but we will reserve this name for a more obvious candidate.

T is the i-eigenspace and $\bar{T}$ is the $-i$-eigenspace. This is our polarization. We notice (P3) and (P4) are automatically satisfied. To satisfy (P1) exactly means that J is integrable, and so it makes M into a complex manifold.

Regarding (P2), we see that the dimension is fine and given $X, Y \in T$, we observe that

$$\omega(X,Y) = 0 \iff \omega(X,Y) = -\omega(X,Y) = \omega(iX,iY) = \omega(JX,JY)$$

so (P2) is satisfied if and only if ω is J-invariant. This amounts to (M,ω,J) being a pseudo-Kähler manifold with the pseudo-Riemannian metric $g(X,Y) = \omega(X,JY)$. If g is positive definite, we call it a Kähler polarization.

Now to define the Hilbert space H_J we simply take the completion of

$$\left\{ s \in C^\infty(M,\mathcal{L}) \mid \nabla_X s = 0, \forall X \in \bar{T} \text{ and } \int_M \langle s,s\rangle \Omega < \infty \right\}$$

where $\Omega = (-1)^{\frac{n(n-1)}{2}} \frac{\omega^m}{m!}$ is the Liouville form on M. This is the space of square integrable holomorphic sections of $\mathcal{L}$. Now, there might not be many, if any, holomorphic sections of L, but if we choose a Kähler polarization, $\mathcal{L}$ becomes an ample line bundle, so at least a power of $\mathcal{L}$ has holomorphic sections.

Now we are left with the problem of the family of obsevables $\mathcal{F}$ we can quantize. We need for $f \in \mathcal{F}$ and $s \in H_J$ that

$$Q_f(s) = Q_f = \frac{ih}{2\pi}\nabla_{X_f} s + fs \in H_J.$$

This means that for all $X \in \bar{T}$ and all $s \in H_J$ we need

$$\nabla_X(Q_f(s)) = \frac{ih}{2\pi}\nabla_{[X,X_f]} s = 0 \iff [X,X_f] \in \bar{T}.$$

As shown in [Woo] this amounts to X_f being a Killing vector field, so this means the family of quantizable functions is rather small. The dimension of the space of Killing vector fields on (M,g) is finite (bounded by $\frac{n(n-1)}{2}$)..

Also, applying this method to e.g. quantum mechanics, it can be observed that the energy levels for the harmonic oscillator is wrong ([AE]). A way to fix this is to introduce *metaplectic correction*. This technique will be explored in Chapter 3 along with the main problem addressed in this thesis, namely the dependence of the choice of polarization.

1.3 Deformation Quantization

To be able to quantize most (if not all) smooth functions we could abandon geometric quantization and turn to *deformation quantization*. As is customary, we let $\hbar = \frac{h}{2\pi}$. The idea here is to relax the canonical commutation relation (Q3) so that it only holds asymptotically in $\hbar$, that is

(Q3') $[Q_f, Q_g] = i\hbar Q_{\{f,g\}} + O(\hbar^2)$.

A way to do this is by deforming the algebra stucture on $C^\infty(M)$ by introducing an associative product $\star$ on formal power series $C^\infty[[\hbar]]$ where for $f, g \in C^\infty(M)$ we write

$$f \star g = \sum_{j=0}^{\infty} C_j(f,g)\hbar^j$$

for bilinear operators $C_j \colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying

(d1) $C_0(f,g) = fg$

(d2) $C_1(f,g) - C_1(g,f) = i\hbar\{f,g\}$

(d3) $C_j(f,1) = C_j(1,f) = 0$ for $j \leq 1$.

We do not require the power series to converge for any values of h. The hope is that the above algebra structure fits well with our quantum observables, so that

$$Q_f Q_g = Q_{f\star g}.$$

A way of securing this identity is to start from this and producing the star product accordingly. This means that we need operators on our Hilbert space with an asymptotical expansion for the product

$$Q_f Q_g \sim \sum_{j=0}^{\infty} Q_{C_j(f,g)}\hbar^j$$

where $\sim$ means

$$\|Q_f Q_g - \sum_{j=0}^{L} Q_{C_j(f,g)}\hbar^j\| \in O(\hbar^{L+1}).$$

We will now describe a specific construction of both operators and subsequently a star product which fits well together with geometric quantization with Kähler polarization. This follows Schlichenmaier [Sch1].

Let (M,ω) be a compact symplectic manifold and choose a Kähler structure J. We assume that (M,ω) is prequantizable with respect to $\hbar = \frac{h}{2\pi} = 1$ that is, $[\frac{\omega}{2\pi}]$ is an integral cohomology class. Choose a prequantum line bundle $(\mathcal{L}, \nabla, \langle\cdot,\cdot\rangle)$. This means in particular that $R_\nabla = -i\omega$. Now, for each $k \in \mathbb{N}$ we can produce another prequantizable symplectic manifold $(M, k\omega)$ with prequantum line bundle $(\mathcal{L}^k, \nabla^k, \langle\cdot,\cdot\rangle_k)$. This corresponds to setting $\hbar = \frac{1}{k}$ for the original manifold. We will drop the index k on both ∇ and $\langle \cdot, \cdot \rangle$. Carrying out the geometric quantization scheme, we produce a family of Hilbert spaces

$$H_J^{(k)} = \{s \in C^\infty(M, \mathcal{L}^k) \mid \nabla_X s = 0, \, \forall X \in \bar{T}\}.$$

The condition on s can be reformulated as follows. Using the orthogonal projections $\pi^{1,0} \colon TM_{\mathbb{C}} \to T$ and $\pi^{0,1} \colon TM_{\mathbb{C}} \to \bar{T}$ we can split $\nabla = \nabla^{1,0} + \nabla^{0,1}$ into types. Since ω is J-invariant it is of type $(1,1)$ and thus the $(0,2)$-part of the curvature of ∇ vanishes, and $\nabla^{0,1}$ defines a $\bar{\partial}$-operator on each $C^\infty(M, \mathcal{L}^k)$ giving the line bundles a holomorphic structure. In this language

$$H_J^{(k)} = \{s \in C^\infty(M, \mathcal{L}^k) \mid \nabla^{0,1} s = 0\}$$

is the space of holomorphic sections. These are the Hilbert spaces associated to the quantization of (M, ω) with $\hbar = \frac{1}{k}$ with the inner product given by the L^2-inner product induced by $\langle \cdot, \cdot \rangle$ and the Liouville form.

We now wish to produce operators on these spaces. For each k, denote by $\Pi_J^{(k)} \colon L^2(M, \mathcal{L}^k) \to H_J^{(k)}$ the orthogonal projection from the square-integrable sections to $H_J^{(k)}$. We note that, since M is compact, $H_J^{(k)}$ is finite dimensional and thus a closed subspace.

Definition 1.2. The *Toeplitz operator* $T_f^{(k)} \in \operatorname{End}(H_J^{(k)})$ associated to a function $f \in C^\infty(M)$ is given by

$$T_f^{(k)} s = \Pi_J^{(k)}(fs).$$

A few remarks on these operators are in order. First of all, the map $T \colon C^\infty(M) \to \operatorname{End}(H_J^{(k)})$ is surjective. As k goes to infinity, we also recover some faithfulness, stated in the theorem by Bordemann, Meinrenken and Schlichenmaier [BMS]

Theorem 1.3 ([BMS])**.** *Given a function $f \in C^\infty(M)$ there exists a constant $C > 0$ such that*

$$\|f\|_\infty - \frac{C}{k} \leq \|T_f^{(k)}\| \leq \|f\|_\infty$$

as $k \to \infty$.

In particular this theorem states that if $\lim_{k\to\infty} \|T_f^{(k)} - T_g^{(k)}\| = 0$ then $f = g$. Another theorem from the same paper is

Theorem 1.4 ([BMS])**.** *For functions $f, g \in C^\infty(M)$ the commutator of Toeplitz operators behaves as*

$$\|k[T_f^{(k)}, T_g^{(k)}] - iT_{\{f,g\}}^{(k)}\| \in O\left(\frac{1}{k}\right)$$

as $k \to \infty$.

The sign is different here than in the original paper, due to our choice of Poisson bracket.

From this one can show that the product of Toeplitz operators allows an asymptotical expansion in Toeplitz operators as follows

Theorem 1.5 ([Sch1]). *For functions $f, g \in C^\infty(M)$ there exists a family of functions $C_j(f,g) \in C^\infty$ satisfying (d1)-(d3) and for each $N \in \mathbb{N}$ there is a constant $K_N(f,g)$ such that*

$$\|\sum_{j=0}^{L} T^{(k)}_{C_j(f,g)} \left(\frac{1}{k}\right)^j - T^{(k)}_f T^{(k)}_g\| \leq K_N(f,g) \left(\frac{1}{k}\right)^{L+1}.$$

This means that we get a star product, which we call the Berezin-Toeplitz star product, given by

$$f \star_{\mathrm{BT}} g = \sum_{j=0}^{\infty} (-1)^j C_j(f,g) \left(\frac{1}{k}\right)^j.$$

Now, the prequantum operator Q_f from (1.1) is not a Toeplitz operator and, as we discussed, it does not allow many quantizable obervables. This was because it did not necessarily preserve the space of holomorphic sections. One way of increasing the number quantizable functions could be by considering the operator $\pi^{0,1} \circ Q_f$. This is in fact a Toeplitz operator, as proven by Tuynman.

Theorem 1.6 ([Tuy2]). *For any smooth function $f \in C^\infty(M)$, we have*

$$\pi^{0,1} \circ Q_f = T^{(k)}_{f - \frac{1}{2k}\Delta f}$$

as operators from $C^\infty(M, \mathcal{L}^k)$ to $H^{(k)}_J$, where Δ is the Laplace operator.

We will not be concerned with deformation quantization in this dissertation, but the Toeplitz operators will appear in Chapter 4 and 7.

2

The Hitchin Connection

In this chapter we address the dependence of the choice of Kähler structure for geometric quantization. Assuming that we have a family of Kähler structures parametrized by a manifold, we show that the different spaces of holomorphic sections form a vector bundle over the parametrizing manifold. To compare different choices, we thus need to descibe a connection in this bundle. This is an approach which has been used by Hitchin [Hit] and also by Axelrod, Della Pietra and Witten [APW] in the context of quantizing Chern-Simons theory to get a topological quantum field theory. The symplectic manifolds they considered were moduli spaces of flat $\mathrm{SU}(n)$-connections on a surface.

Our point of view is not that of TQFT but rather the general problem of writing down such a connection, which we will call a Hitchin connection. The following is an account of the first half of the paper [And2] in which Andersen constructs a Hitchin connection for compact symplectic manifolds.

2.1 A Hitchin Connection for Symplectic Manifolds

Let (M, ω) be a compact prequantizable symplectic manifold. Choose a prequantum line bundle $(\mathcal{L}, \nabla, \langle\cdot,\cdot\rangle)$ on M. Now, let $\mathcal{T}$ be a manifold parametrizing Kähler structures on (M, ω) by a smooth map

$$I\colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM)).$$

By smooth we mean that I gives rise to a smooth section in the pull-back bundle $\pi_M^*(\mathrm{End}(TM)) \to \mathcal{T} \times M$ under the projection on M. This means, for each $\sigma \in \mathcal{T}$, we have that $I(\sigma) = I_\sigma$ is an integrable complex structure on M, compatible with ω, such that

$$g_\sigma(X, Y) = \omega(X, I_\sigma Y)$$

is a Riemannian metric. We denote the splitting of the complexified tangent bundle by $TM_{\mathbb{C}} = T_\sigma + \bar{T}_\sigma$ and remark that the orthogonal projections are given by

$$\pi_\sigma^{1,0} = \tfrac{1}{2}(\mathrm{id} - iI_\sigma) \quad \text{and} \quad \pi_\sigma^{0,1} = \tfrac{1}{2}(\mathrm{id} + iI_\sigma)$$

respectively. This means that $\nabla^{0,1}_\sigma = \frac{1}{2}(\mathrm{id} + iI_\sigma)\nabla$. As before, we define our quantum spaces

$$H^{(k)}_\sigma = H^0(M_\sigma, \mathcal{L}^k) = \left\{ s \in C^\infty(M, \mathcal{L}^k) \mid \nabla^{0,1}_\sigma s = 0 \right\}.$$

It is not clear that these spaces form a bundle, let alone have the same dimension, but nevertheless we proceed to construct a Hitchin connection and thereby also showing that they produce a bundle over $\mathcal{T}$. The plan is to consider the infinite rank trivial bundle $\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty(M, \mathcal{L}^k)$ over $\mathcal{T}$. Clearly, $H^{(k)}_\sigma \subseteq \mathcal{H}^{(k)}_\sigma$ and so we seek a connection in $\mathcal{H}^{(k)}$ preserving these subspaces. This leads us to a definition of a Hitchin connection.

Definition 2.1. A Hitchin connection is a connection $\boldsymbol{\nabla}$ in $\mathcal{H}^{(k)}$ preserving the subspaces $H^{(k)}_\sigma$ of the form

$$\boldsymbol{\nabla}_V = \nabla^t_V + u(V), \tag{2.1}$$

where V is a vector field on $\mathcal{T}$, ∇^t is the trivial connection, and u is a one-form in $\mathcal{T}$ with values in differential operators on sections of $\mathcal{L}^k$, which we denote by $\mathcal{D}(M, \mathcal{L}^k)$.

Our goal is to write down an explicit formula for u. But first, we need to analyze the condition that $\boldsymbol{\nabla}$ preserves the subspaces, meaning for any $\sigma \in \mathcal{T}$, any $s \in H^{(k)}_\sigma$, and any vector field V on $\mathcal{T}$ we require

$$\nabla^{0,1}_\sigma \boldsymbol{\nabla}_V s = 0.$$

As a condition on u this can be written by (2.1) as

$$\nabla^{0,1}_\sigma V[s] + \nabla^{0,1}_\sigma u(V)s = 0. \tag{2.2}$$

By taking the derivative of $\nabla^{0,1}_\sigma s = 0$ along V, we find that

$$0 = V[\nabla^{0,1}_\sigma s] = V[\tfrac{1}{2}(\mathrm{id} + iI_\sigma)\nabla s] = \tfrac{i}{2} V[I_\sigma]\nabla s + \nabla^{0,1}_\sigma V[s] \tag{2.3}$$

and so by comparing (2.2) and (2.3) and using that s is holomorphic, we get the lemma

Lemma 2.2. *The connection defined by* (2.1) *preserves the subspaces* $H^{(k)}_\sigma$ *if and only if* u *satisfies the equation*

$$\nabla^{0,1}_\sigma u(V)s = \frac{i}{2} V[I_\sigma]\nabla^{1,0}_\sigma s \tag{2.4}$$

for all $\sigma \in \mathcal{T}$ *and all vector fields* V *on* $\mathcal{T}$.

To solve (2.4) we need some extra assumptions. First, let us assume that $\mathcal{T}$ is a complex manifold. Extend I_σ, ω, and g_σ complex linearly to $TM_{\mathbb{C}}$. Given a vector field V on $\mathcal{T}$, we can differentiate our parametrization map to get

$$V[I]\colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM_{\mathbb{C}}).$$

Differentiating the identity $I^2 = -\,\mathrm{id}$ we see that $V[I]$ and I anti-commute and so $V[I]$ interchanges T and $\bar{T}$. This means that

$$V[I]_\sigma \in C^\infty(M, (T_\sigma^* \otimes \bar{T}_\sigma) \oplus (\bar{T}_\sigma^* \otimes T_\sigma))$$

and we get a splitting of $V[I]_\sigma$ in $V[I]'_\sigma \in C^\infty(M, \bar{T}_\sigma^* \otimes T_\sigma)$ and $V[I]''_\sigma \in C^\infty(M, T_\sigma^* \otimes \bar{T}_\sigma)$. Now, we make the assumption that I is *holomorphic* in the sense that

$$V'[I] = V[I]' \quad \text{and} \quad V''[I] = V[I]''$$

where V' is the $(1,0)$-part of V and V'' is the $(0,1)$-part of V. A justification for the term "holomorphic" can be seen from the following observation made in [AGL]. Namely, let J denote the (integrable almost) complex structure on $\mathcal{T}$. Induce an almost complex structure $\hat{I}$ on $\mathcal{T} \times M$ by the following

$$\hat{I}(V \oplus X) = JV \oplus I_\sigma X, \quad \text{for } V \oplus X \in T_{\sigma,p}(\mathcal{T} \times M).$$

The proposition is now

Proposition 2.3. *$\hat{I}$ is integrable if and only if the family I is holomorphic.*

Proof. We will show that the Nijenhuis tensor vanishes if and only if I is holomorphic. This suffices due to the Newlander-Nirenberg theorem. First, we remark that if two vector fields X and Y are tangent to M that is, pull-backs of vector fields on M, then the Nijenhuis tensor vanishes, since I_σ is integrable. Like-wise for vector fields tangent to $\mathcal{T}$. This means that it is enough to check on vector fields X and V, where X is tangent to M and V is tangent to $\mathcal{T}$. Also, we notice that $[V, X] = [\hat{I}V, X] = 0$ and $[V, \hat{I}X] = V[I]X$. So we calculate the Nijenhuis tensor

$$\begin{aligned}
\mathrm{Nij}(V, X) &= [\hat{I}V, \hat{I}X] - [V, X] - \hat{I}[\hat{I}V, X] - \hat{I}[V, \hat{I}X] \\
&= [JV, IX] - \hat{I}[V, IX]
\end{aligned}$$

and by splitting $V = V' + V''$ we calculate futher that

$$\begin{aligned}
&= iV'[I]X - IV'[I]X - iV''[I]X - IV''[I]X \\
&= 2i(\pi^{0,1}V'[I]X - \pi^{1,0}V''[I]X)
\end{aligned}$$

and so by a consideration of types, we see that this vanishes for all X and V if and only if we have

$$\pi^{0,1}V'[I]X = \pi^{1,0}V''[I]X = 0$$

meaning $V'[I] \in C^\infty(M, \bar{T}^* \otimes T)$ and $V''[I] \in C^\infty(M, T^* \otimes \bar{T})$ which proves the proposition. □

Similarly we can consider

$$V[g] = \omega V[I].$$

Since ω is of type (1,1) and $V[I]$ interchanges types, $V[g]$ only has components of type (2,0) and (0,2). Furthermore $V[g]$ is symmetric, since g is, and so we have $V[g]_\sigma \in C^\infty(M, S^2T^*_\sigma \oplus S^2\bar{T}^*_\sigma)$.

By (2.1) and the type of ω we can define $\tilde{G}(V)_\sigma \in C^\infty(M, (T_\sigma \otimes T_\sigma) \oplus (\bar{T}_\sigma \otimes \bar{T}_\sigma))$ by the equation

$$V[I] = \tilde{G}(V)\omega, \tag{2.5}$$

and split it in $\tilde{G}(V)_\sigma = G(V)_\sigma + \bar{G}(V)_\sigma$ with $G(V)_\sigma \in C^\infty(M, T_\sigma \otimes T_\sigma)$ for all real vector fields V on $\mathcal{T}$. Thus we see that $\tilde{G}$ and G are one-forms on $\mathcal{T}$ with values in $C^\infty(M, TM_\mathbb{C} \otimes TM_\mathbb{C})$ and $C^\infty(M, T_\sigma \otimes T_\sigma)$, respectively. By construction we have

$$V[g] = \omega\tilde{G}(V)\omega$$

and since $V[g]$ is symmetric so is $\tilde{G}(V)$ and $G(V)$. We also notice that $G(V') = G(V)$ since $V'[I] = G(V)\omega$ by the holomorphicity condition on I.

We want to construct a one-form on $\mathcal{T}$ with values in differential operators $\mathcal{D}(M, \mathcal{L}^k)$. To this end, consider $G(V)_\sigma \in C^\infty(M, S^2(T_\sigma))$. This can be viewed a linear map $C^\infty(M, T^*_\sigma M_\mathbb{C}) \to C^\infty(M, T_\sigma)$ by contraction. From this we construct the second order differential operator $\Delta_{G(V)_\sigma} \in \mathcal{D}(M_\sigma, \mathcal{L}^k)$ given by

$$\begin{tikzcd}
 & C^\infty(M_\sigma, T^*M_\mathbb{C} \otimes \mathcal{L}^k) \arrow[d, "G(V)_\sigma \otimes \mathrm{id}"] \\
\Delta_{G(V)_\sigma} \colon C^\infty(M_\sigma, \mathcal{L}^k) \arrow[ur, "\nabla"] & C^\infty(M_\sigma, T_\sigma \otimes \mathcal{L}^k) \arrow[d, "\tilde{\nabla}_\sigma \otimes \mathrm{id} + \mathrm{id} \otimes \nabla"] \\
 & C^\infty(M_\sigma, T^*M_\mathbb{C} \otimes T_\sigma \otimes \mathcal{L}^k) \arrow[ul, "\mathrm{Tr}"]
\end{tikzcd}$$

where $\tilde{\nabla}_\sigma$ is the Levi-Civita connection for g_σ. For short will write $\Delta_{G(V)_\sigma} = \mathrm{Tr}\, \nabla_\sigma G(V)_\sigma \nabla_\sigma$.

This operator will be at the heart of u, and when we test the condition (2.4) we compute $\nabla^{0,1}_\sigma \Delta_{G(V)_\sigma}$. In this computation, the trace of the curvature of M_σ appears, that is, the Ricci form ρ_σ. Since we are on a compact kähler manifold and the Ricci form is of type $(1,1)$, the Hodge decomposition and $\partial\bar{\partial}$-lemma allows us to write

$$\rho_\sigma = \rho_\sigma^H + 2i\partial_\sigma\bar{\partial}_\sigma F_\sigma,$$

where ρ_σ^H is harmonic and F_σ is a real function on M called a Ricci potential. If we choose F_σ such that $\int F_\sigma \omega^m = 0$ for all $\sigma \in \mathcal{T}$, we get a smooth family of Ricci potentials $F\colon \mathcal{T} \to C^\infty(M, \mathbb{R})$. Such families will play a big role in the next chapter.

We now define a one-form $u \in \Omega^1(\mathcal{T}, \mathcal{D}(M, \mathcal{L}^k))$ by

$$u(V) = \frac{1}{4k+2n}\big(\Delta_{G(V)} + 2\nabla_{G(V)\partial F} + 4kV'[F]\big). \tag{2.6}$$

for some $n \in \mathbb{Z}$, where $2k + n \neq 0$. This will turn out to satisfy (2.4) under some further assumptions.

Definition 2.4. We say that the family I of Kähler structures on (M, ω) is *rigid* if

$$\tilde{\nabla}_\sigma^{0,1} G(V)_\sigma = 0$$

for all vector fields V on $\mathcal{T}$ and all $\sigma \in \mathcal{T}$.

This seems like a rather restrictive condition, and attempts has been made to remove it, though without success. In Section 2.2 and 2.3 we give two examples of symplectic manifolds, satisfying this condition, starting with the moduli space of flat $\mathrm{SU}(n)$-connection as in Hitchin's original construction.

The theorem is now

Theorem 2.5 ([And2]). *Suppose that I is a holomorphic, rigid family of Kähler structures on a compact prequantizable symplectic manifold (M, ω). Furthermore assume that the first real Chern class satisfies $c_1(M, \omega) = n[\omega] \in \operatorname{Im}\big(H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})\big)$ and $H^1(M, \mathbb{R}) = 0$. Then u given by* (2.6) *satisfies* (2.4) *for all k such that $2k + n \neq 0$.*

The proof follows from the following three lemmas adapted to our conventions from [And2]. In the following we drop the σ-subscript. Also, we see that since ω is harmonic, the assumption on the Chern class implies that $\rho = n\omega + 2\pi i\partial\bar{\partial}F$.

Lemma 2.6. *For $s \in H_\sigma^{(k)}$ the equation*

$$\nabla^{0,1}\big(\Delta_G(s) + 2\nabla_{G\partial F}s\big) = i\big((2k+n)G\omega\nabla s + k\operatorname{Tr}(2G\partial F\omega + \tilde{\nabla}(G)\omega)s\big),$$

holds at any point σ in $\mathcal{T}$, where $G = G(V)$ for any tangent vector V on $\mathcal{T}$.

Proof. The proof is a straight-forward calculation

$$\begin{aligned}
\nabla^{0,1}\Delta_G s &= \operatorname{Tr} \nabla^{0,1}\nabla^{1,0} G\nabla s \\
&= \operatorname{Tr} \nabla^{1,0} G\nabla^{0,1}\nabla s - ik\omega G\nabla s - i\rho G\nabla s \\
&= ik \operatorname{Tr} \nabla^{1,0} G\omega s + ikG\omega\nabla s + iG\rho\nabla s \\
&= ik \operatorname{Tr} \tilde{\nabla}(G)\omega s + 2ik \operatorname{Tr} G\omega\nabla s + +iG\rho\nabla s \\
&= ik \operatorname{Tr} \tilde{\nabla}(G)\omega s + 2ik \operatorname{Tr} G\omega\nabla s + +inG\omega\nabla s - 2G\partial\bar{\partial}F\nabla s \\
&= ik \operatorname{Tr} \tilde{\nabla}(G)\omega s + i(2k+n) \operatorname{Tr} G\omega\nabla s + 2ik \operatorname{Tr} G\partial F\omega s \\
&\quad - 2\nabla^{0,1}\nabla_{G\partial F} s,
\end{aligned}$$

where we use that $\nabla(\omega) = 0$ and the rigidity of I. □

Lemma 2.7. *For any vector field V on $\mathcal{T}$ we get*

$$\bar{\partial}(V'[F]) = -\frac{i}{2} \operatorname{Tr} G\partial(F)\omega - \frac{i}{4}\nabla^{1,0}(G)\omega.$$

which is established from the following result

Lemma 2.8. *For any vector field V on $\mathcal{T}$ we get*

$$V'[\rho] = \frac{1}{2} d\left(\operatorname{Tr} \nabla^{1,0}(G(V))\omega\right).$$

This lemma will be proven in Section 3.2.1.

Proof of Lemma 2.7. Differentiating the decomposition $\rho = n\omega + 2id\bar{\partial}F$ along V' we find

$$\begin{aligned}
V'[\rho] &= V'[n\omega] + 2iV[d\bar{\partial}F] \\
&= -dV'[I]dF + 2id\bar{\partial}V'[F]
\end{aligned}$$

thus we can apply Lemma 2.8 and (2.5)

$$\begin{aligned}
0 &= d\bar{\partial}V'[F] + \frac{i}{2}V'[\rho] - \frac{i}{2}dV'[I]dF \\
&= d\left(\bar{\partial}V'[F] + \frac{i}{4} \operatorname{Tr} \nabla(G(V))\omega + \frac{i}{2}G(V)\omega dF\right).
\end{aligned}$$

Since $H^1(M, \mathbb{R}) = 0$

$$(\bar{\partial}V'[F] + \frac{i}{4} \operatorname{Tr} \nabla(G(V))\omega + \frac{i}{2}G(V)\omega dF$$

is an exact form and we see that it is of type $(0,1)$. But M is compact and thus it is 0 and we get the desired formula. □

Proof of Theorem 2.5. Piecing together Lemma 2.7 and Lemma 2.6 we see that

$$\nabla^{0,1} u(V)s = \frac{i}{2} G(V)\omega\nabla s = \frac{i}{2} V'[I]\nabla s = \frac{i}{2} V[I]'\nabla s = \frac{i}{2} V[I]\nabla s$$

since s is a holomophic section. □

It is worth noticing that, in the case of the moduli space of flat $\mathrm{SU}(n)$-connections, this is the same connection as the one Hitchin constructed as it is shown in [And2]. In the next section we give a short overview of the moduli space. The focus of the next chapter will be on removing some of the conditions in the above theorem, most notably the compactness and the condition on the Chern class.

2.2 The Moduli Space of Flat Connections

Regarding the TQFT mentioned in the introduction, we review the moduli space of flat $\mathrm{SU}(n)$-connections on a surface. Let Σ be a closed, compact, connected, orientable surface. The moduli space of flat $\mathrm{SU}(n)$-connections on Σ is the set of gauge equivalence classes of flat connections in a principal $\mathrm{SU}(n)$-bundle over Σ. Since Σ is a surface and $\mathrm{SU}(n)$ is simply connected, all principal $\mathrm{SU}(n)$-bundles are trivial, and thus we need not choose a specific bundle to realize the moduli space. As a set, this is in bijection with

$$\mathcal{M}_{\mathrm{SU}(n)}(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SU}(n)) \big/ \mathrm{SU}(n)$$

via the holonomy map. From now on we will make no distinction, and refer to $\mathcal{M}_{\mathrm{SU}(n)}$ as the moduli space of flat connections. This is in general not a smooth manifold, but a singular variety. However, it can be made smooth in two different ways.

First, we can restrict to the irreducible representations $\rho^{\mathrm{irr}} \colon \pi_1(\Sigma) \to \mathrm{SU}(n)$, meaning the stabilizer of ρ^{irr} is the center of $\mathrm{SU}(n)$. Then it can be shown that

$$\mathcal{M}^{\mathrm{irr}}_{\mathrm{SU}(n)}(\Sigma) = \mathrm{Hom}^{\mathrm{irr}}(\pi_1(\Sigma), \mathrm{SU}(n)) \big/ \mathrm{SU}(n) \subseteq \mathcal{M}_{\mathrm{SU}(n)}(\Sigma)$$

is an open, dense subset and a smooth manifold. Another way is considering the punctured surface $\Sigma' = \Sigma - \{p\}$ and a loop γ going once around the puncture. Then choose a $d \in \mathbb{Z}/n\mathbb{Z}$ coprime with n and let $D = e^{\frac{2\pi i d}{n}}\,\mathrm{id}$ be the corresponding central element of $\mathrm{SU}(n)$. The space

$$\mathcal{M}^{d}_{\mathrm{SU}(n)}(\Sigma') = \{\rho \in \mathrm{Hom}(\pi_1(\Sigma'), \mathrm{SU}(n)) \mid \rho(\gamma) = D\} \big/ \mathrm{SU}(n) \subseteq \mathcal{M}^{\mathrm{irr}}_{\mathrm{SU}(n)}(\Sigma')$$

is a smooth, compact manifold.

In [Gol], Goldman gave a symplectic structure for these manifolds. This is based on an identification of the tangent space with the first cohomology of Σ

with coefficients in the adjoint bundle. The symplectic form is then basically the integral of the cup product.

These moduli spaces have a prequantum line bundle (see [Qui], [BF], [Fre]). By choosing an element in the Teichmüller space $\mathcal{T}$ of Σ, we get a complex structure on Σ induced by the Hodge star operator and by Narasimhan and Seshadri [NS], the moduli space inherits a Kähler structure.

Now, we have a symplectic manifold with a prequantum line bundle and a complex manifold parametrizing Kähler structures. As to the cohomological conditions, Atiyah and Bott [AB] showed that these moduli spaces are simply connected and that the image $\operatorname{Im}(H^2(\mathcal{M}_{\mathrm{SU}(n)}, \mathbb{Z}) \to H^2(\mathcal{M}_{\mathrm{SU}(n)}, \mathbb{R}))$ is generated by $n[\omega]$. Finally, rigidity of the family of Kähler structures were part of the original construction of the Hitchin connection, made by Hitchin in [Hit].

Hitchin showed that this connection is projectively flat, allowing to identify the (projective) quantum spaces for different Kähler structures by parallel transport.

2.3 Example of Rigid Family of Kähler Structures

As promised, we now give an example, aside from the moduli spaces above, of a symplectic manifold with a rigid family of Kähler strucures. We will use the notation introduced above.

Let (M, ω) be $\mathbb{R}^2$ with the standard symplectic form $\omega = dx \wedge dy$ and let $\mathcal{T} = \mathbb{R}^l$. We want to analyze a family of complex structures

$$I_\sigma\left(\frac{\partial}{\partial x}\right) = A(\sigma, x, y)\frac{\partial}{\partial x} + B(\sigma, x, y)\frac{\partial}{\partial y}$$

given by functions $A, B \in C^\infty(\mathcal{T} \times M)$. Then the identity $I_\sigma^2 = -\operatorname{id}$ yields

$$I_\sigma\left(\frac{\partial}{\partial y}\right) = -\left(\frac{1}{B} + \frac{A^2}{B}\right)\frac{\partial}{\partial x} - A\frac{\partial}{\partial y}.$$

It is clear that ω is I_σ invariant and g_σ is positive definite when $B > 0$. From this one finds

$$\begin{aligned}\frac{\partial}{\partial z} &= \frac{1}{2}\left((1 - iA)\frac{\partial}{\partial x} - iB\frac{\partial}{\partial y}\right)\\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left((1 + iA)\frac{\partial}{\partial x} + iB\frac{\partial}{\partial y}\right)\end{aligned}$$

and thus

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\\ \frac{\partial}{\partial y} &= \frac{i - A}{B}\frac{\partial}{\partial z} - \frac{i + A}{B}\frac{\partial}{\partial \bar{z}}.\end{aligned}$$

Writing I as a tensor

$$I = \left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}\right) dx - \left(\left(\frac{1}{B} + \frac{A^2}{B}\right)\frac{\partial}{\partial x} + A\frac{\partial}{\partial y}\right) dy$$

we see that a variation along a vector field V on $\mathcal{T}$

$$\begin{aligned} V[I] = &\left(V[A]\frac{\partial}{\partial x} + V[B]\frac{\partial}{\partial y}\right) dx \\ &- \left(\frac{2ABV[A] - (1+A^2)V[B]}{B^2}\frac{\partial}{\partial x} + V[A]\frac{\partial}{\partial y}\right) dy \end{aligned}$$

and the identity $\tilde{G}(V)\omega = V[I]$ gives the formula

$$\tilde{G}(V) = -2V[A]\frac{\partial}{\partial x}\frac{\partial}{\partial y} - V[B]\frac{\partial^2}{\partial y^2} - \frac{1}{B^2}\left(2ABV[A] - (1+A^2)V[B]\right)\frac{\partial^2}{\partial x^2}.$$

By the above formulas we can write $\tilde{G}(V)$ in the basis $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$ as

$$\begin{aligned} \tilde{G}(V) = &\frac{2}{B^2}\left(-iBV[A] + V[B] + iAV[B]\right)\frac{\partial^2}{\partial z^2} \\ &+ \frac{2}{B^2}\left(iBV[A] + V[B] - iAV[B]\right)\frac{\partial^2}{\partial \bar{z}^2}. \end{aligned}$$

Let us assume that B is a function on M only. Then $\frac{\partial}{\partial \bar{z}}$ of the coefficient of $\frac{\partial^2}{\partial z^2}$ is reduced to

$$\begin{aligned} \frac{\partial}{\partial \bar{z}}\left(\frac{-2i}{B}V[A]\right) = &\frac{A}{B}\frac{\partial V[A]}{\partial x} + \frac{\partial V[A]}{\partial y} - \frac{A}{B^2}V[A]\frac{\partial B}{\partial x} - \frac{1}{B}V[A]\frac{\partial B}{\partial y} \\ &+ i\left(\frac{1}{B^2}V[A]\frac{\partial B}{\partial x} - \frac{1}{B}\frac{\partial V[A]}{\partial x}\right). \end{aligned}$$

Setting this equal to zero and splitting into real and imaginary parts, we get the equations

$$\begin{aligned} 0 &= -V[A]\frac{\partial B}{\partial y} + B\frac{\partial V[A]}{\partial y} \\ 0 &= V[A]\frac{\partial B}{\partial x} - B\frac{\partial V[A]}{\partial x}. \end{aligned}$$

These equations have solutions $B(x,y) = B_0(x,y)$ and $A(\sigma,x,y) = A_0(x,y) + \sum_{i=1}^{l}\sigma_i B_0(x,y)$ where A_0 and B_0 are arbitrary functions on (M,ω). This means that given any complex structure

$$I_0\left(\frac{\partial}{\partial x}\right) = A_0(x,y)\frac{\partial}{\partial x} + B_0(x,y)\frac{\partial}{\partial y}$$

we have obtained a rigid family of deformations parametrized by $\mathbb{R}^l$. This means, that for any symplectic two-manifold with a Kähler structure, we can find a rigid family of Kähler structures locally.

3

The Hitchin Connection and Metaplectic Correction

This chapter contains one of the main results of this thesis, namely an extension of the Hitchin connection from the previous chapter to geometric quantization with *metaplectic correction*. This serves two purposes. First and foremost, this is the "right" theory in terms of physics ([AE], [Woo]). Second, it removes many of the assumptions of the previous chapter. Most notably, there is no requirement on the first real Chern class of the manifold, other than it should be divisible by two. But also, we need not assume M compact, $\mathcal{T}$ complex, or I holomorphic to descibe the connection. The price, however, is the loss of the trivial connection, which served as reference point for Definition 2.1. This results in the construction of a *reference connection* in Section 3.2. A new Hitchin connection is defined and constructed. At the end of the chapter, the setting from Chapter 2 is introduced and a comparison between the Hitchin connections is carried out, culminating in Theorem 3.18. But before all this, we need to introduce the metaplectic structure on (M, ω).

3.1 Metaplectic Structure

As mentioned in Chapter 1, geometric quantization does not give the right theory in terms of physics. As it turns out, the prequantum line bundle must be corrected by a square root of the canonical line bundle. However, this bundle depends on the choice of Kähler structure, and so we need to take the square root of the canonical line bundles associated to all Kähler structures simultaneously. This is the notion of a metaplectic structure. It can be formulated in terms of structure groups, as the metaplectic group is the double cover of the symplectic group (a symplectic analogue of a spin structure, if you will). This in not the approach we will take. We will construct the bundle itself following [Woo] (in the spirit of Rawnsley [Raw]).

Let (M, ω) be a symplectic manifold of dimension $2m$, not necessarily compact. Given a compatible almost complex structure J on (M, ω), we seek to find a line bundle δ_J which satisfies that $\delta_J \otimes \delta_J = K_J = \bigwedge T_J^*$, i.e. a square root of the canonical line bundle. Clearly, there is more than one choice of a square root of K_J (when it exists), and we would like to choose δ_J in a unified way for different J.

Consider the *positive Lagrangian Grassmannian* L^+M consisting of pairs (p, J_p), where $p \in M$ and J_p is a compatible almost complex structure on the tangent space T_pM. This space has the structure of a smooth bundle over M, with the obvious projection, and with sections corresponding precisely to compatible almost complex structures on (M, ω).

At each point $(p, J_p) \in L^+M$, we can consider the one dimensional space $K_{J_p} = \bigwedge^m T^*_{J_p}$. These form a smooth bundle K over L^+M, and the pullback by a section of L^+M yields the canonical line bundle associated to the almost complex structure on M given by the section.

We want to find a square root $\delta \to L^+M$ of the bundle $K \to L^+M$. Such a square root is called a metaplectic structure on M. Since the space of compatible almost complex structures is contractible, L^+M has contractible fibers, and so we can find local trivializations of K with constant transition functions along the fibers. The construction of a metaplectic structure on M amounts to choosing square roots of these transition functions in such a way that they still satisfy the cocycle conditions. But since the transition functions are constant along the fibers, we only have to choose a square root at a single point in each fiber. In other words, a square root δ_J of K_J, for a single almost complex structure J on M, determines a metaplectic structure. We summarize this in a proposition.

Proposition 3.1. *Let M be a manifold with vanishing second Stiefel-Whitney class, and let ω be any symplectic structure on M. Then (M, ω) admits a metaplectic structure $\delta \to L^+M$.*

For the rest of this chapter, we shall assume that M satisifies the conditions of this proposition, and fix a metaplectic structure δ. In this way, for every almost complex structure J on M, viewed as a section of L^+M, we have a canonical choice of square root of the canonical line bundle, given as the pullback of δ by J.

3.2 The Reference Connection

As before let $I \colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$ be a smooth family of Kähler structures on (M, ω) parametrized by a manifold $\mathcal{T}$. We can also view I as a map $I \colon \mathcal{T} \times M \to L^+M$. This allows us to pull back δ to get a line bundle $\delta \to \mathcal{T} \times M$ which we will call the metaplectic line bundle on $\mathcal{T} \times M$. We note that the restriction $\delta_\sigma \to \{\sigma\} \times M = M$ is a square root of the canonical line bundle K_σ on M_σ. The Riemannian metric g_σ on M_σ induces a Hermitian metric h^T_σ in $T_\sigma \to M_\sigma$ by

$$h^T_\sigma(X, Y) = g_\sigma(X, \bar{Y}).$$

Likewise, the Levi-Civita connection $\tilde{\nabla}_\sigma$ induces a compatible connection ∇^T_σ in T_σ. We further let $(h^T_\sigma, \nabla^T_\sigma)$ induce compatible Hermitian metric and connection $(h^\delta_\sigma, \nabla^\delta_\sigma)$ in $\delta_\sigma \to M_\sigma$.

Pick a prequantum line bundle $(\mathcal{L}, \nabla^{\mathcal{L}}, h^{\mathcal{L}})$ on (M, ω). Instead of considering holomorphic sections of $\mathcal{L}^k$ we look at the bundle $\mathcal{L}^k \otimes \delta_\sigma \to M_\sigma$. In this bundle we have a Hermitian structure induced from $h^{\mathcal{L}}$ and h^δ_σ and a compatible connection

$$\nabla_\sigma = \left(\nabla^{\mathcal{L}}\right)^k \otimes \mathrm{id} + \mathrm{id} \otimes \nabla^\delta_\sigma. \tag{3.1}$$

This connection also splits $\nabla_\sigma = \nabla^{1,0}_\sigma + \nabla^{0,1}_\sigma$ by the projections $\pi^{1,0}_\sigma$ and $\pi^{0,1}_\sigma$. Letting $\mathcal{H}^{(k)}_\sigma = C^\infty(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma)$ we again look at the subspace of holomorphic sections

$$H^{(k)}_\sigma = \{s \in \mathcal{H}^{(k)}_\sigma \mid \nabla^{0,1}_\sigma s = 0\}.$$

Our goal is, as before, to write down a connection in $\mathcal{H}^{(k)} \to \mathcal{T}$ which preserves the subspaces $H^{(k)}_\sigma$ and thereby proving that these form a subbundle $H^{(k)} \subseteq \mathcal{H}^{(k)}$ and at the same time providing it with a connection. But this time the bundle $\mathcal{H}^{(k)} \to \mathcal{T}$ is not a trivial bundle, as the fibers vary along $\mathcal{T}$, and so we do not have the trivial connection we used in Definition 2.1 to define a Hitchin connection. Therefore, we wish to construct some kind of reference connection.

It has proven useful to work on the product space $\mathcal{T} \times M$ and so we pull back our prequantum line bundle $\mathcal{L}$ to $\hat{\mathcal{L}} \to \mathcal{T} \times M$ by the projection π_M. We will in general but a hat on object which are extended or pulled back to $\mathcal{T} \times M$. In that fashion, we let $\hat{h}^{\mathcal{L}}$ be the Hermitian metric and so $\hat{\mathcal{L}} \otimes \delta$ becomes a Hermitian line bundle over $\mathcal{T} \times M$ with the metric $\hat{h}$ induced by $\hat{h}^{\mathcal{L}}$ and h^δ, coming from h^δ_σ. Also, we let $\hat{\nabla}^{\mathcal{L}}$ denote the pullback connection which is explicitly given by the following:

Let s be a section of $\hat{\mathcal{L}}$ and $(\sigma, p) \in \mathcal{T} \times M$. Then for a vector field X on $\mathcal{T} \times M$, which is the pullback of a vector field on M, we have

$$(\hat{\nabla}^{\mathcal{L}}_X s)_{(\sigma,p)} = (\nabla_X s_\sigma)_p \,.$$

For the vector field V, a pullback of a vector field on $\mathcal{T}$, we get

$$(\hat{\nabla}^{\mathcal{L}}_V s)_{(\sigma,p)} = V[s_p]_\sigma,$$

which is the derivative along V at $\sigma \in \mathcal{T}$ of the section s_p of the trivial bundle $\mathcal{T} \times \mathcal{L}_p \to \mathcal{T} \times \{p\} = \mathcal{T}$. This connection is compatible with $\hat{h}^{\mathcal{L}}$ and has curvature

$$R_{\hat{\nabla}^{\mathcal{L}}} = \pi^*_M R_{\nabla^{\mathcal{L}}} = -i\pi^*_M \omega. \tag{3.2}$$

We now wish to describe a connection in $\delta \to \mathcal{T} \times M$. First, we consider the bundle $T \to \mathcal{T} \times M$ with the obvious fibers $T_{\sigma,p} = (T_\sigma)_p$. In this bundle we have a natural connection $\hat{\nabla}^T$.

Let Y be a section of T. In the directions of M, that is, for a vector field X that is a pullback of a vector field on M, we use the connection ∇^T_σ from above

$$(\hat{\nabla}^T_X Y)_{(\sigma,p)} = ((\nabla^T_\sigma)_X Y)_p.$$

In the directions of $\mathcal{T}$, we use the trivial connection in $\mathcal{T} \times TM_{\mathbb{C}} \to \mathcal{T}$ and the projections $\pi^{1,0}_\sigma$ to define

$$(\hat{\nabla}^T_V Y)_{(\sigma,p)} = \pi^{1,0}_\sigma V[Y_p]_\sigma.$$

Here, as before, we view Y_p as a section of $\mathcal{T} \times T_pM_{\mathbb{C}} \to \mathcal{T}$.

This connection induces a connection $\hat{\nabla}^K$ in $K \to \mathcal{T} \times M$ and thus a connection $\hat{\nabla}^\delta$ in $\delta \to \mathcal{T} \times M$. This leads us to the following definition.

Definition 3.2. The *reference connection* $\hat{\nabla}^r$ in $\hat{\mathcal{L}}^k \otimes \delta \to \mathcal{T} \times M$ is

$$\hat{\nabla}^r = (\hat{\nabla}^{\mathcal{L}})^k \otimes \mathrm{id} + \mathrm{id} \otimes \hat{\nabla}^\delta.$$

A few remarks on this connection are in order. First of all, let s be a section of $\mathcal{H}^{(k)} \to \mathcal{T}$, i.e. a map $\sigma \mapsto s_\sigma \in C^\infty(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma)$, we see that this is also a section of $\hat{\mathcal{L}}^k \otimes \delta$ over $\mathcal{T} \times M$. Choosing a vector field V on $\mathcal{T}$ we therefore get a connection in $\mathcal{H}^{(k)} \to \mathcal{T}$ by $\hat{\nabla}^r_V s$. Furthermore, if we fix a point $\sigma \in \mathcal{T}$ and restrict $\hat{\nabla}^r$ to M we get the connection ∇_σ in $\mathcal{L}^k \otimes \delta_\sigma \to M_\sigma$ defined in (3.1). In this way, the reference connection is both a connection in the bundle $\mathcal{H}^{(k)} \to \mathcal{T}$ and our prequantum connection ∇_σ in $\mathcal{L}^k \otimes \delta_\sigma \to M_\sigma$.

3.2.1 Curvature

We will now calculate the curvature of the reference connection. This is split into three different formulas, namely the curvature in the pure directions of $\mathcal{T}$ and M, and in the mixed direction. First, we address the curvature in the direction of M.

Proposition 3.3. *Let X, Y be pullbacks of tangent vectors on M to $\mathcal{T} \times M$. Then the curvature of $\hat{\nabla}^r$ in $\mathcal{L}^k \otimes \delta$ is*

$$R_{\hat{\nabla}^r}(X,Y) = -ik\omega(X,Y) + \frac{i}{2}\rho_\sigma(X,Y)$$

at $\sigma \in \mathcal{T}$.

Proof. The first term is from (3.2) and the second term is by the formula

$$R_{\hat{\nabla}^\delta} = \frac{1}{2} R_{\hat{\nabla}^K} \tag{3.3}$$

and the fact that the the curvature of the canonical line bundle K_σ is $i\rho_\sigma$. □

We also note for later use that

$$R_{\hat{\nabla}^K} = -\operatorname{Tr} R_{\hat{\nabla}^T}, \tag{3.4}$$

which is part of the calculation of the curvature of K_σ.

As in the previous chapter, we consider the variation of I along a vector field V on $\mathcal{T}$. From this we can construct a two-form θ on $\mathcal{T}$ with values in $C^\infty(M)$. Namely, let V, W be vector fields on $\mathcal{T}$. Then the commutator

$$[V[I], W[I]]_\sigma \in C^\infty(M_\sigma, (T_\sigma^* \otimes T_\sigma) \oplus (\bar{T}_\sigma^* \otimes \bar{T}_\sigma))$$

since both $V[I]$ and $W[I]$ interchange types. If we precompose with $\pi_\sigma^{1,0}$ we get the restriction to a section in the first summand

$$[V[I], W[I]]_\sigma \pi_\sigma^{1,0} \in C^\infty(M_\sigma, T_\sigma^* \otimes T_\sigma).$$

From this we define

$$\theta(V, W) = -\frac{i}{4} \operatorname{Tr}([V[I], W[I]]\pi^{1,0})$$

which is a real two-form on $\mathcal{T}$. This will turn out to be the curvature in the direction on $\mathcal{T}$. Another useful observation is, given V tangent to $\mathcal{T}$ and Y a section of T,

$$V[Y] = V[\pi^{1,0}Y] = V[\pi^{1,0}]Y + \pi^{1,0}V[Y] = -\frac{i}{2}V[I]Y + \hat{\nabla}_V^r Y. \tag{3.5}$$

We are now ready to formulate and prove the following proposition.

Proposition 3.4. *For vector fields V and W tangent to $\mathcal{T}$ we have*

$$R_{\hat{\nabla}^r}(V, W) = \frac{i}{2}\theta(V, W).$$

Proof. Let V and W be the pullback of commuting vector fields on $\mathcal{T}$. Applying (3.5) we find, for a section Y of T

$$\begin{aligned}
\hat{\nabla}_V^T \hat{\nabla}_W^T Y &= \hat{\nabla}_V^T (W[Y] + \frac{i}{2}W[I]Y) \\
&= VW[Y] + \frac{i}{2}V[I]W[Y] + \frac{i}{2}(V[W[I]Y] + \frac{i}{2}V[I]W[I]Y) \\
&= VW[Y] + \frac{i}{2}V[I]W[Y] + \frac{i}{2}VW[I]Y + \frac{i}{2}W[I]V[Y] - \frac{1}{4}V[I]W[I]Y.
\end{aligned}$$

Using that $[V, W] = 0$ we get the curvature

$$\begin{aligned}
R_{\hat{\nabla}^T}(V, W)Y &= -\frac{1}{4}(V[I]W[I]Y - W[I]V[I]Y) \\
&= \frac{1}{4}[V[I], W[I]]Y
\end{aligned}$$

and we see that

$$R_{\hat{\nabla}^r}(V,W) = kR_{\hat{\nabla}^{\mathcal{L}}}(V,W) - \frac{1}{2}\operatorname{Tr} R_{\hat{\nabla}^T}(V,W) = \frac{i}{2}\theta(V,W)$$

since $R_{\hat{\nabla}^{\mathcal{L}}}(V,W) = 0$. $\square$

For the calculation of the curvature in the mixed direction, we recall the tensors $\tilde{G}(V)$ and $G(V)$ from Chapter 2.

Proposition 3.5. *For vector fields V and X, tangent to $\mathcal{T}$ and M respectively, we have*

$$R_{\hat{\nabla}^r}(V,X) = \frac{i}{4}\operatorname{Tr}\tilde{\nabla}(\tilde{G}(V))\omega X$$

Proof. First we calculate the curvature of $\hat{\nabla}^T$. Let X and V be pullbacks of real vector fields on M and $\mathcal{T}$ respectively, and let Y be any section of T. Then we get

$$\begin{aligned}
R_{\hat{\nabla}^T}(V,X)Y &= \hat{\nabla}^T_V\hat{\nabla}^T_X Y - \hat{\nabla}^T_X\hat{\nabla}^T_V Y \\
&= \pi^{1,0}V[\tilde{\nabla}_X Y] - \tilde{\nabla}_X\pi^{1,0}V[Y] \\
&= \pi^{1,0}V[\tilde{\nabla}_X Y] - \pi^{1,0}\tilde{\nabla}_X V[Y] \\
&= \pi^{1,0}V[\tilde{\nabla}]_X Y
\end{aligned}$$

By Theorem 1.174 in [Bes], we get that the variation of the Levi-Civita connection in the tangent bundle is a symmetric (2,1)-tensor given by

$$\begin{aligned}
g(V[\tilde{\nabla}]_X Y, Z) = \tfrac{1}{2}(&\tilde{\nabla}_X(V[g])(Y,Z) \\
&+\tilde{\nabla}_Y(V[g])(X,Z) \\
&-\tilde{\nabla}_Z(V[g])(X,Y))
\end{aligned}$$

for vector fields X, Y and Z on M and V on $\mathcal{T}$. We focus our attention on a point $p \in M$, and let $e_1, \dots, e_m$ be a basis of T_pM satisfying the orthogonality condition that $g(e'_j, e''_l) = \delta_{jl}$. Then

$$\operatorname{Tr} R_{\hat{\nabla}^T}(V,X) = \operatorname{Tr}\pi^{1,0}V[\tilde{\nabla}]_X\pi^{1,0} = \sum_\nu g(V[\tilde{\nabla}]_X e'_\nu, e''_\nu).$$

But taking into account the type of $V[g]$, and the fact that $\tilde{\nabla}$ preserves types, we get

$$\begin{aligned}
g(V[\tilde{\nabla}]_X e'_\nu, e''_\nu) &= \frac{1}{2}\tilde{\nabla}_{e'_\nu}(V[g])(X, e''_\nu) - \frac{1}{2}\tilde{\nabla}_{e''_\nu}(V[g])(X, e'_\nu) \\
&= \frac{1}{2}X\omega\tilde{\nabla}_{e'_\nu}(\tilde{G}(V))\omega e''_\nu - \frac{1}{2}X\omega\tilde{\nabla}_{e''_\nu}(\tilde{G}(V))\omega e'_\nu \\
&= \frac{i}{2}X\omega\tilde{\nabla}_{e'_\nu}(G(V))g e''_\nu + \frac{i}{2}X\omega\tilde{\nabla}_{e''_\nu}(\bar{G}(V))g e'_\nu \\
&= -\frac{i}{2}g(\tilde{\nabla}_{e'_\nu}(G(V))\omega X, e''_\nu) - \frac{i}{2}g(\tilde{\nabla}_{e''_\nu}(\bar{G}(V))\omega X, e'_\nu).
\end{aligned}$$

Summing over ν, we conclude that

$$\begin{aligned}\operatorname{Tr} R_{\hat{\nabla}^T}(V,X) &= -\frac{i}{2}\operatorname{Tr}\tilde{\nabla}(G(V))\omega X - \frac{i}{2}\operatorname{Tr}\tilde{\nabla}(\bar{G}(V))\omega X \\ &= -\frac{i}{2}\operatorname{Tr}\tilde{\nabla}(\tilde{G}(V))\omega X,\end{aligned}$$

at the point p which was arbitrary. Finally we get by (3.2) and (3.4) that

$$\begin{aligned}R_{\hat{\nabla}^r}(V,X) &= R^{(k)}_{\hat{\nabla}^{\mathcal{L}}}(V,X) - \frac{1}{2}\operatorname{Tr} R_{\hat{\nabla}^T}(V,X) \\ &= \frac{i}{4}\operatorname{Tr}\tilde{\nabla}(\tilde{G}(V))\omega X,\end{aligned}$$

which was the claim. □

We have now calculated the curvature for the reference connection. This allows us to prove Lemma 2.8.

Proof of Lemma 2.8. From the Bianchi identity for $R_{\hat{\nabla}^K}$ with vector fields X and Y along M and V along $\mathcal{T}$ we find

$$\begin{aligned}V[\rho](X,Y) &= -i\hat{\nabla}^K_V R_{\hat{\nabla}^K}(X,Y) \\ &= i\left(\hat{\nabla}^K_X R_{\hat{\nabla}^K}(Y,V) + \hat{\nabla}^K_Y R_{\hat{\nabla}^K}(V,X)\right) \\ &= -\frac{1}{2}\left(\hat{\nabla}^K_X(\operatorname{Tr}\tilde{\nabla}(\tilde{G}(V))\omega Y) - \hat{\nabla}^K_X(\operatorname{Tr}\tilde{\nabla}(\tilde{G}(V))\omega X)\right) \\ &= \frac{1}{2}d(\operatorname{Tr}\tilde{\nabla}(\tilde{G}(V))\omega)(X,Y)\end{aligned}$$

by Proposition 3.5. Letting $V = V'$ be of type $(1,0)$ we get $\tilde{G}(V) = G(V)$ and thus by rigidity of our family I (Definition 2.4), we get

$$V'[\rho] = \frac{1}{2}d(\operatorname{Tr}\nabla^{1,0}(G(V))\omega)$$

which the statement of the lemma. □

3.3 The Hitchin Connection with Metaplectic Correction

As in Chapter 2 we wish to define and construct a connection in $\mathcal{H}^{(k)} \to \mathcal{T}$ which preserves the subspaces $H^{(k)}_\sigma$. Now, let $\mathcal{D}(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma)$ denote the space of differential operators on $\mathcal{H}^{(k)}_\sigma$. These are collected as the fibers of a bundle $\mathcal{D}(M, \hat{\mathcal{L}}^k \otimes \delta)$ over $\mathcal{T}$. Again, we adopt the viewpoint of bundles over $\mathcal{T} \times M$ where we see $\mathcal{D}(M, \hat{\mathcal{L}}^k \otimes \delta)$ as differential operators on sections of $\hat{\mathcal{L}}^k \otimes \delta$, which are of order zero in the $\mathcal{T}$ direction. We can now define a Hitchin connection on $\mathcal{H}^{(k)} \to \mathcal{T}$.

Definition 3.6. By a Hitchin connection $\boldsymbol{\nabla}$ in $\mathcal{H}^{(k)} \to \mathcal{T}$ we mean a connection preserving the subspaces $H^{(k)}_\sigma$ of the form

$$\boldsymbol{\nabla}_V = \hat{\nabla}^r_V + u(V)$$

for any vector field V on $\mathcal{T}$, where u is a one-form on $\mathcal{T}$ with values in $\mathcal{D}(M, \hat{\mathcal{L}}^k \otimes \delta)$.

The aim of this section is of course to find such a connection. The first step is to find a parallel to Lemma 2.2.

Lemma 3.7. *A Hitchin connection $\boldsymbol{\nabla}$ preseves the subspaces $H^{(k)}_\sigma$ if and only if*

$$\nabla^{0,1} u(V)s = \frac{i}{2} V[I]\nabla s + \frac{i}{4} \operatorname{Tr} \tilde{\nabla}(G(V))\omega s$$

for all vector fields V on $\mathcal{T}$ and all $s \in H^{(k)}$.

Proof. The proof is again a straight forward calculation. Let X and V be pullbacks of vector field on M and $\mathcal{T}$, respectively, to $\mathcal{T} \times M$. Let $s \in C^\infty(\mathcal{T}, \mathcal{H}^{(k)})$ such that $s_\sigma \in H^{(k)}_\sigma$. Then

$$\begin{aligned}
\nabla_{X''} \boldsymbol{\nabla}_V s &= \hat{\nabla}^r_{X''} \hat{\nabla}^r_V s + \nabla_{X''} u(V)s \\
&= \hat{\nabla}^r_V \hat{\nabla}^r_{X''} s - R_{\hat{\nabla}^r}(V, X'')s - \hat{\nabla}^r_{[V,X'']} s + \nabla_{X''} u(V)s \\
&= -\frac{i}{4} \operatorname{Tr}(\tilde{\nabla}(G(V))\omega X)s - \frac{i}{2} \nabla_{V[I]X} s + \nabla_{X''} u(V)s
\end{aligned}$$

at $\sigma \in \mathcal{T}$. Then simply apply Proposition 3.5 to get the desired. □

To solve this we turn to the same second order differential operator Δ_G. At a point $\sigma \in \mathcal{T}$, $\Delta_{G(V)\sigma}$ is the operator in $\mathcal{D}(M, \mathcal{L}^k, \otimes \delta_\sigma)$ given by the diagram

$$\begin{tikzcd}
 & C^\infty(M_\sigma, T^*M_{\mathbb{C}} \otimes \mathcal{L}^k \otimes \delta_\sigma) \arrow[d, "G(V)_\sigma \otimes \mathrm{id} \otimes \mathrm{id}"] \\
C^\infty(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma) \arrow[ur, "\nabla_\sigma"] & C^\infty(M_\sigma, T_\sigma \otimes \mathcal{L}^k \otimes \delta_\sigma) \arrow[d, "\tilde{\nabla}_\sigma \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_\sigma"] \\
 & C^\infty(M_\sigma, T^*M_{\mathbb{C}} \otimes T_\sigma \otimes \mathcal{L}^k \otimes \delta_\sigma) \arrow[ul, "\mathrm{Tr}"]
\end{tikzcd}$$

which again can be written $\Delta_{G(V)\sigma} = \operatorname{Tr} \nabla_\sigma G(V)_\sigma \nabla_\delta$ for short. Now, assume that the family I of Kähler structures is rigid. The following lemma calculates $\nabla^{0,1}\Delta_{G(V)}$.

Lemma 3.8. *At every point $\sigma \in \mathcal{T}$, the operator $\Delta_{G(V)}$ satisfies*

$$\nabla^{0,1}\Delta_{G(V)} s = 2ikV[I]'\nabla s + ik \operatorname{Tr} \tilde{\nabla}(G(V))\omega s - \tfrac{i}{2} \operatorname{Tr} \tilde{\nabla}(G(V)\rho)s$$

for all vector fields V on $\mathcal{T}$ and all (local) holomorphic sections s of the line bundle $\mathcal{L}^k \otimes \delta \to M$.

Proof. The proof is, as in the previous chapter, by direct calculation. For clarity, we comment on the steps, although the arguments are similar. Letting G denote $G(V)$ we have

$$\nabla^{0,1}\Delta_G s = \nabla^{0,1} \operatorname{Tr} \nabla G \nabla s = \operatorname{Tr} \nabla^{0,1} \nabla G \nabla s.$$

Working further on the right side we commute the two connections, giving as ekstra terms the curvature of M_σ and of the line bundle $\mathcal{L}^k \otimes \delta_\sigma$,

$$\nabla^{0,1}\Delta_G s = \operatorname{Tr} \nabla \nabla^{0,1} G \nabla s - ik\omega G \nabla s + \tfrac{i}{2}\rho G \nabla s - i\rho G \nabla s.$$

Collecting the last two terms, and using the fact that J is rigid on the first, we obtain

$$\nabla^{0,1}\Delta_G s = \operatorname{Tr} \nabla G \nabla^{0,1} \nabla s - ik\omega G \nabla s - \tfrac{i}{2}\rho G \nabla s.$$

Commuting the two connections, and using that s is holomorphic, we get

$$\nabla^{0,1}\Delta_G s = ik \operatorname{Tr} \nabla G \omega s - \tfrac{i}{2} \operatorname{Tr} \nabla G \rho s - ik\omega G \nabla s - \tfrac{i}{2}\rho G \nabla s.$$

Expanding the covariant derivatives in the first two terms by the Leibniz rule, and using the fact that ω is parallel, we get the following, after collecting and cancelling terms,

$$\nabla^{0,1}\Delta_G s = ik \operatorname{Tr} \tilde{\nabla}(G)\omega s - 2ik\omega G \nabla s - \tfrac{i}{2} \operatorname{Tr} \tilde{\nabla}(G\rho) s$$

Finally applying $V[I]' = \omega G(V)$ was the desired expression. Moreover we notice, that the above is a local computation, so that the identity is valid for local holomorphic sections of $\mathcal{L}^k \otimes \delta$ as well. □

We notice that for $s \in H^{(k)}_\sigma$ we have

$$V[I]\nabla_X s = \nabla_{V[I]X} s = \nabla_{V[I]'X} s$$

since $\nabla s = \nabla^{1,0} s$ and $\pi^{1,0}_\sigma(V[I]X) = V[I]'X$ by definiton. Thus we see that $\frac{1}{4k}\Delta_{G(V)}$ solves Lemma 3.7 if not for an error term. Fortunately, this can be dealt with under an additional assumption.

Lemma 3.9. *Provided that $H^{0,1}(M_\sigma) = 0$ for all $\sigma \in \mathcal{T}$, we have that $\operatorname{Tr} \tilde{\nabla}(G(V)\rho)$ is exact with respect to the $\bar{\partial}$-operator on M_σ.*

Proof. By appealing to Lemma 3.8 in the case where $k = 0$, we get for any local holomorphic section s of $\mathcal{L}^k \otimes \delta_\sigma \to M_\sigma$ that

$$0 = \frac{i}{2}\nabla^{0,1}_\sigma \operatorname{Tr} \tilde{\nabla}_\sigma(G(V)_\sigma \rho_\sigma) s = \frac{i}{2}\bar{\partial}_\sigma(\operatorname{Tr} \tilde{\nabla}_\sigma(G(V)_\sigma \rho_\sigma)) s.$$

This immediately implies that

$$0 = \bar{\partial}_\sigma(\operatorname{Tr} \tilde{\nabla}_\sigma(G(V)_\sigma \rho_\sigma)),$$

and since $H^{0,1}(M) = 0$, the corollary follows. □

This extra assumption is reminescent of the assumption $H^1(M) = 0$ from Chapter 2, and indeed for a compact Kähler manifold the Hodge decomposition theorem implies $H^{0,1}(M) = 0$ if $H^1(M) = 0$.

So, assuming $H^{0,1}(M_\sigma) = 0$ for all $\sigma \in \mathcal{T}$, we can choose a one-form $\beta \in \Omega^1(M, C^\infty(M))$ satisfying

$$\bar{\partial}\beta(V) = \frac{i}{2} \operatorname{Tr} \tilde{\nabla}(G(V)\rho),$$

for all vector fields V on $\mathcal{T}$. This implies the theorem.

Theorem 3.10. *Let (M, ω) be a prequantizable symplectic manifold with vanishing second Stiefel-Whitney class. Further, let I be a rigid family of Kähler structures on M, all satifying $H^{0,1}(M) = 0$. Then, there exists a one-form $\beta \in \Omega^1(\mathcal{T}, C^\infty(M))$ such that the connection $\boldsymbol{\nabla}$, in the bundle $\mathcal{H}^{(k)}$, given by*

$$\boldsymbol{\nabla}_V = \hat{\nabla}^r_V + \frac{1}{4k}(\Delta_{G(V)} + \beta(V)),$$

is a Hitchin connection. The connection is unique up to addition of the pullback of an ordinary one-form on $\mathcal{T}$.

3.4 Comparing the Connections

In this section we compare the constructions of Hitchin connections in Section 2.1 and Section 3.3. While this means that we need to adopt all of the assumptions from both constructions, the upshot is a formula for the one-form β from Theorem 3.10.

The setup is now the following. We let (M, ω) be a compact prequantizable symplectic manifold with $H^1(M) = 0$ and choose a prequantum line bundle $(\mathcal{L}, h^{\mathcal{L}}, \nabla^{\mathcal{L}})$. Let I be a rigid, holomorphic family of Kähler structures on (M, ω) parametrized by a complex manifold $\mathcal{T}$. Also, we assume that the first real Chern class of (M, ω) satisfies $c_1(M, \omega) = n[\frac{\omega}{2\pi}]$ for some $n \in 2\mathbb{Z}$. This ensures that $w_2(M) = 0$ which allows us to choose a metaplectic structure δ.

In order to distiguish the exterior differentials we let $\hat{d}$ denote the differential on $\mathcal{T} \times M$. Similarly, by Proposition 2.3 the holomorphicity of I gives rise to $\hat{\partial}$ and $\hat{\bar{\partial}}$. When necessary, we will adopt the notation

$$\hat{d} = d_{\mathcal{T}} + d_M$$

for the splitting into differentials on $\mathcal{T}$ and M. Likewise for ∂ and $\bar{\partial}$.

In Section 2.1 we chose a particular family of Ricci potentials $F \colon \mathcal{T} \to C^\infty(M)$ satisfying

$$\rho - n\omega = 2i\partial_\sigma\bar{\partial}_\sigma F_\sigma \quad \text{and} \quad \int_M F_\sigma \omega^m = 0.$$

The latter condition was only to ensure smoothness (and uniqueness). But now we will have use for the more general concept.

Definition 3.11. A *smooth family of Ricci potentials* on (M,ω) is a function $F \in C^\infty(\mathcal{T} \times M)$ satisfying

$$\rho = n\omega + 2i\partial_M \bar{\partial}_M F. \tag{3.6}$$

The following proposition gives a formula for β.

Proposition 3.12. *Let F be any smooth family of Ricci potentials. Then the one-form $\beta \in \Omega^1(\mathcal{T}, C^\infty(M))$ given by*

$$\beta(V) = -2nV'[F] - \partial_M F G(V) \partial_M F - \operatorname{Tr} \tilde{\nabla}(G(V)\partial_M F)$$

satisfies $\bar{\partial}_M \beta(V) = \frac{i}{2}\tilde{\nabla}(G(V)\rho)$.

Proof. Since we only consider ∂ and $\bar{\partial}$ in directions of M, we will drop the subscript in the proof.

Using (3.6) we calculate

$$\begin{aligned}\operatorname{Tr} \tilde{\nabla}(G(V)\rho) &= \operatorname{Tr} \tilde{\nabla}(G(V)(n\omega + 2i\partial\bar{\partial}F)) \\ &= n \operatorname{Tr} \tilde{\nabla}(G(V))\omega + 2i \operatorname{Tr} \tilde{\nabla}(G(V)\partial\bar{\partial}F)\end{aligned}$$

where the last equality comes from the fact that $\tilde{\nabla}(\omega) = 0$. Examining the last term we see that it can be found in the calculation

$$\bar{\partial} \operatorname{Tr} \tilde{\nabla}(G(V)\partial F) = \operatorname{Tr} \tilde{\nabla}(G(V)\partial\bar{\partial}F) - i\rho G(V)\partial F$$

where we use that I is rigid. Furthermore, by using (3.6), rigidity of I, and the symmetry of $G(V)$ we see that

$$\begin{aligned}-i\rho G(V)\partial F &= i\partial F G(V)\rho \\ &= in\partial F G(V)\omega - 2\partial F G(V)\partial\bar{\partial}F \\ &= in\partial F G(V)\omega - \bar{\partial}\left(\partial F G(V)\partial F\right).\end{aligned}$$

Piecing these calculations together yields the expression

$$\begin{aligned}\frac{i}{2}\operatorname{Tr} \tilde{\nabla}(G(V)\rho) &= \frac{i}{2} n \operatorname{Tr} \tilde{\nabla}(G(V))\omega + in\partial F G(V)\omega \\ &\quad + \bar{\partial}\left(\operatorname{Tr} \tilde{\nabla}(G(V)\partial F) + \partial F G(V)\partial F\right)\end{aligned}$$

an so by Lemma 2.7 we have the desired formula. $\square$

Having calculated the curvature of the reference connection in all directions, we see that it is of type (1,1) over $\mathcal{T} \times M$ and thus the (0,2)-part of the curvature vanishes. This means that the reference connection defines a holomophic structure on the line bundle $\hat{\mathcal{L}}^k \otimes \delta$, over the complex manifold $\mathcal{T} \times M$. Moreover, we observe that $(\hat{\nabla}^r)^{0,1}$ preserves the bundle $H^{(k)} \to \mathcal{T}$, since $u(V'') = 0$ solves (3.3). Thus the reference connection defines a holomorphic structure on the bundle $H^{(k)} \to \mathcal{T}$.

These families of Ricci potentials carry a lot of local information about the curvature of the reference connection as will be stated in Theorem 3.16. This is established through three lemmas.

Lemma 3.13. *For any smooth family of Ricci potentials $F \in C^\infty(\mathcal{T} \times M)$ and vector fields V along $\mathcal{T}$ and X along M the curvature of the reference connection satisfies*

$$R_{\hat{\nabla}^r}(V, X) = -\hat{\partial}\hat{\bar{\partial}} F(V, X).$$

Proof. Since $R_{\hat{\nabla}^r}$ is of type (1,1), we split V and X and calculate from the right-hand side

$$\begin{aligned}
\hat{\bar{\partial}}\hat{\partial} F(X'', V') &= \hat{d}\hat{\partial} F(X'', V') \\
&= X''(\hat{\partial} F(V')) - V'(\hat{\partial} F(X'')) - \hat{\partial} F([X'', V']) \\
&= X'' V'[F] + \frac{i}{2}\hat{\partial} F(V'[I]X) \\
&= X'' V'[F] + \frac{i}{2}\partial_M F G(V)\omega X''.
\end{aligned}$$

Now we can apply Lemma 2.7 and conclude

$$\begin{aligned}
\hat{\bar{\partial}}\hat{\partial} F(X'', V') &= -\frac{i}{4}\operatorname{Tr}\tilde{\nabla}(G(V))\omega X'' \\
&= -R_{\hat{\nabla}^r}(V', X'').
\end{aligned}$$

The case of X' and V'' follows from

$$\partial_M V''[F] = \frac{i}{4}\operatorname{Tr}\tilde{\nabla}(\bar{G}(V)\omega) + \frac{i}{2}\bar{\partial}_M F \bar{G}(V)\omega$$

which is just the conjugate of 2.7. □

Lemma 3.14. *For any smooth family F of Ricci potentials, the expression*

$$\theta - 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}} F \tag{3.7}$$

defines an ordinary two-form on $\mathcal{T}$.

Proof. Take V, W and X to be commuting vector fields so that V and W are tangent to $\mathcal{T}$ and X is tangent to M. We must prove that (3.7) takes values in constant functions on M, i.e. that

$$0 = X\big[\theta(V,W) - 2i\hat{\partial}\hat{\bar{\partial}}F(V,W)\big].$$

Now, by the Bianchi identity and Proposition 3.4 we have

$$\begin{aligned}0 &= \hat{\nabla}^r_X R_{\hat{\nabla}^r}(V,W) + \hat{\nabla}^r_V R_{\hat{\nabla}^r}(W,X) + \hat{\nabla}^r_W R_{\hat{\nabla}^r}(X,V)\\ &= \frac{i}{2}X[\theta(V,W)] - \hat{\nabla}^r_V R_{\hat{\nabla}^r}(X,W) + \hat{\nabla}^r_W R_{\hat{\nabla}^r}(X,V).\end{aligned}$$

Then Lemma 3.13 yields

$$\begin{aligned}\frac{i}{2}X[\theta(V,W)] &= W[\hat{\partial}\hat{\bar{\partial}}F(X,V)] - V[\hat{\partial}\hat{\bar{\partial}}F(X,W)]\\ &= WXV''[F] - WVX''[F] - VXW''[F] + VWX''[F]\\ &= XWV''[F] - XVW''[F]\\ &= -X[\hat{\partial}\hat{\bar{\partial}}F(V,W)]\end{aligned}$$

as desired. □

Lemma 3.15. *Over any open subset U of $\mathcal{T}$ with $H^1(U,\mathbb{R}) = 0$, we can find a family $\tilde{F}$ of Ricci potentials satisfying*

$$\theta = 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}\tilde{F}. \tag{3.8}$$

Proof. Let $\sigma \in \mathcal{T}$ and fix a smooth family F of Ricci potentials. Let V and W be vectorfields tangent to $\mathcal{T}$. Then, by Lemma 3.14, we can define a two-form $\alpha \in \Omega^{1,1}(\mathcal{T})$ by

$$\alpha = \theta - 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}F.$$

By applying the Bianchi identity to the reference connection it follows that θ is closed on $\mathcal{T}$. Thus, we see that α is a closed two-form on $\mathcal{T}$. Since θ is real, so is α, and therefore we can find a real function A on U such that

$$\alpha|_U = 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}A.$$

But then $\tilde{F} = F|_U + A$ defines a new smooth family of Ricci potentials with the desired property. □

We are now ready to establish a theorem relating the curvature of the reference connection and the curvature of $\hat{\nabla}^{\mathcal{L}}$.

Theorem 3.16. *Let (M,ω) be a compact prequantizable symplectic manifold with the real first Chern class satisfying $c_1(M,\omega) = n[\frac{\omega}{2\pi}]$, $n \in 2\mathbb{Z}$, and $H^1(M,\mathbb{R}) = 0$. Let I be a rigid, holomorphic family of Kähler structures on M, parametrized by a complex manifold $\mathcal{T}$. Then, for any open subset U of $\mathcal{T}$ with $H^1(U,\mathbb{R}) = 0$ there exists a family of Ricci potentials $\tilde{F}$ over U such that*

$$R^{(k)}_{\hat{\nabla}^r} = R^{(k-n/2)}_{\hat{\nabla}^{\mathcal{L}}} - \hat{\partial}\hat{\bar{\partial}}\tilde{F}, \tag{3.9}$$

where $R^{(k)}_{\hat{\nabla}^r}$ denotes curvature of the reference connection in $\hat{\mathcal{L}}^k \otimes \delta$ and $R^{(k-n/2)}_{\hat{\nabla}^{\mathcal{L}}}$ denotes the curvature of $\hat{\nabla}^{\mathcal{L}}$ in $\hat{\mathcal{L}}^{k-n/2}$.

Proof. Let X and Y be vector fields tangent to M, and let V and W be vector-fields tangent to $\mathcal{T}$. Use Lemma 3.15 to find a family of Ricci potentials over U satisfying (3.8). Then, by Proposition 3.3, Proposition 3.4 and Lemma 3.13 we have that

$$\begin{aligned}
R_{\hat{\nabla}^r}(X+V,Y+W) &= -ik\omega(X,Y) + \frac{i}{2}\rho(X,Y) \\
&\quad + \frac{i}{2}\theta(V,W) + R_{\hat{\nabla}^r}(V,Y) + R_{\hat{\nabla}^r}(X,W) \\
&= -\, i(k - \tfrac{n}{2})\omega(X,Y) - \partial_M\bar{\partial}_M\tilde{F}(X,Y) \\
&\quad - \partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}\tilde{F}(V,W) - \hat{\partial}\hat{\bar{\partial}}\tilde{F}(V,Y) - \hat{\partial}\hat{\bar{\partial}}\tilde{F}(X,W) \\
&= R^{(k-n/2)}_{\hat{\nabla}^{\mathcal{L}}}(X,Y) - \hat{\partial}\hat{\bar{\partial}}\tilde{F}(X+V,Y+W) \\
&= R^{(k-n/2)}_{\hat{\nabla}^{\mathcal{L}}}(X+V,Y+W) - \hat{\partial}\hat{\bar{\partial}}\tilde{F}(X+V,Y+W)
\end{aligned}$$

since the curvature $R^{(k-n/2)}_{\hat{\nabla}^{\mathcal{L}}}$ vanishes in all other directions than M (see (3.2)). ☐

We are now in good shape to compare the formulas in Section 2.1 and Proposition 3.12 for Hitchin connections. However, this requires a comparable settting. As it turns out, we must be more careful in choosing the prequantum line bundle. Because even though δ and $\mathcal{L}^{-n/2}$ have the same first real Chern class, they may not be isomorphic, as there could be some torsion lost in the passage from integral cohomology to real cohomology. The following lemma ensures us that there exists a prequantum line bundle, which will satisfy our needs.

Lemma 3.17. *If $c_1(M,\omega)$ is divisible by n in $H^2(M,\mathbb{Z})$, there exists a prequantum line bundle $\mathcal{L}$ over M such that*

$$\tfrac{n}{2}c_1(\mathcal{L}) = -c_1(\delta),$$

where c_1 is the first Chern class in $H^2(M,\mathbb{Z})$.

Proof. Let $\mathcal{L}_0$ be any prequantum line bundle on M and pick an auxiliary Kähler structure J on M. Let F_J be a Ricci potential on M_J and consider the line bundles $(\mathcal{L}_0^{-n/2}, e^{F_J} h^{\mathcal{L}_0})$ and (δ_J, h_J^δ) over M. Then a small calculation

$$\begin{aligned}\bar\partial\partial \log(e^{F_J} h^{\mathcal{L}_0}) &= \bar\partial\partial F_J + i\tfrac{n}{2}\omega \\ &= \tfrac{i}{2}(n\omega + 2i\partial\bar\partial F_J) \\ &= \tfrac{i}{2}\rho_J\end{aligned}$$

shows that the line bundles have the same curvature. Thus, the tensor product of the former with the dual of the latter yields a flat Hermitian line bundle L_1. Since $c_1(\delta)$ is divisible by $\frac{n}{2}$, there exists a flat Hermitian line bundle L_2 such that $L_2^{n/2} \cong L_1$. Finally, the line bundle $\mathcal{L} = \mathcal{L}_0 \otimes L_2$ has the structure of a prequantum line bundle, and $\frac{n}{2}c_1(\mathcal{L}) = c_1(\mathcal{L}^{n/2}) = -c_1(\delta)$. Thus $\mathcal{L}$ is the desired prequantum line bundle. □

From now on assume we have chosen a prequantum line bundle $\mathcal{L}$ as in Lemma 3.17. Note that, only when $H^2(M, \mathbb{Z})$ has torsion is this a further restriction.

Next, let $\tilde{F}$ be a family of Ricci potentials over U, with $H^1(U, \mathbb{R}) = 0$, such that (3.9) in Theorem 3.16 is satisfied. We wish to construct an isomorphism $\hat\varphi$ of holomorphic Hermitian line bundles over $U \times M$

$$\hat\varphi \colon (\hat{\mathcal{L}}^{k-n/2}, e^{\tilde F}\hat h^{\mathcal{L}}) \to (\hat{\mathcal{L}}^k \otimes \delta, \hat h).$$

Since $\frac{n}{2}c_1(\mathcal{L}) = -c_1(\delta)$, the line bundles are isomorphic as complex line bundles, and with the given Hermitian structures, a simple calculation and application of (3.9) reveals that they have the same curvature. Thus, the obstruction to finding the structure preserving isomorphism $\hat\varphi$ lies in the first cohomology of $U \times M$. But this is trivial by the Künneth formula, since $H^1(U, \mathbb{R}) = 0$ and $H^1(M, \mathbb{R}) = 0$ by assumption.

Moreover, it is easily seen that the pullback under $\hat\varphi$ of the reference connection is given by

$$\hat\varphi^* \hat\nabla^r = \hat\nabla^{\mathcal{L}} + \hat\partial\tilde F, \tag{3.10}$$

since the right hand side is the unique Hermitian connection compatible with the holomorphic structure of $\hat{\mathcal{L}}^{k-n/2}$.

Now the final theorem compares the two constructed Hitchin connections.

Theorem 3.18. *Let (M, ω) be a compact prequantizable symplectic manifold with $H^1(M, \mathbb{R}) = 0$. Further, let I be a rigid, holomorphic family of Kähler structures on M parametrized by a complex manifold $\mathcal{T}$. Assume that the first Chern class of (M, ω) is divisible by an even integer n and that its image in $H^2(M, \mathbb{R})$ satisfies*

$$c_1(M, \omega) = n[\tfrac{\omega}{2\pi}].$$

Then, around every point $\sigma \in \mathcal{T}$, there exists an open neighborhood U, a local smooth family $\tilde{F}$ of Ricci potentials on M over U and an isomorphism of vector bundles over U

$$\varphi\colon \tilde{H}^{(k-n/2)}|_U \to H^{(k)}|_U,$$

such that

$$\varphi^*\boldsymbol{\nabla} = \tilde{\boldsymbol{\nabla}},$$

where $\varphi^*\boldsymbol{\nabla}$ is the pullback of the Hitchin connection given by Theorem 3.10, and $\tilde{\boldsymbol{\nabla}}$ is the Hitchin connection in $H^0(M, \mathcal{L}^{(k-n/2)})$ constructed in Section 2.1, both of which are expressed in terms of $\tilde{F}$.

Proof. First, let us set the notation. The connection from Section 2.1 will be denoted by

$$\tilde{\boldsymbol{\nabla}}_V = \hat{\nabla}^{\mathcal{L}}_V + \tilde{u}(V)$$

where

$$\tilde{u}(V) = \frac{1}{4k+2n}\left(\Delta^{\mathcal{L}}_{G}(V) + 2\nabla^{\mathcal{L}}_{G(V)\partial_M \tilde{F}} + 4kV'[\tilde{F}]\right). \tag{3.11}$$

The operator $\Delta_{G(V)}$ from Section 3.3 was given by $\operatorname{Tr} \nabla_\sigma G(V) \nabla_\sigma$, but by the remark after Definition 3.2 suggests the viewpoint of the reference connection, from which (3.10) gives us

$$\begin{aligned}
\hat{\varphi}^*(\Delta_{G(V)}) &= \hat{\varphi}^*\left(\operatorname{Tr}(\tilde{\nabla} \otimes \mathrm{id} + \mathrm{id} \otimes \hat{\nabla}^r) G(V) \hat{\nabla}^r\right) \\
&= \operatorname{Tr}\left(\tilde{\nabla} \otimes \mathrm{id} + \mathrm{id} \otimes (\hat{\nabla}^{\mathcal{L}} + \partial_M \tilde{F})\right) G(V)(\hat{\nabla}^{\mathcal{L}} + \partial_M \tilde{F}) \\
&= \Delta^{\mathcal{L}}_{G(V)} + 2\hat{\nabla}^{\mathcal{L}}_{G(V)\partial_M \tilde{F}} + \operatorname{Tr} \tilde{\nabla}(G(V)\partial_M \tilde{F}) + \partial_M \tilde{F} G(V) \partial_M \tilde{F} \\
&= \Delta^{\mathcal{L}}_{G(V)} + 2\hat{\nabla}^{\mathcal{L}}_{\tilde{G}(V)\partial_M \tilde{F}} - \beta(V) - 2nV'[\tilde{F}]
\end{aligned}$$

where the last equality is by Proposition 3.12. But in $\mathcal{L}^{k-n/2}$ the formula (3.11) becomes

$$\begin{aligned}
\tilde{u}(V) &= \frac{1}{4k}\left(\Delta^{\mathcal{L}}_{G(V)} + 2\hat{\nabla}^{\mathcal{L}}_{\tilde{G}(V)\partial_M \tilde{F}} + (4k-2n)V'[\tilde{F}]\right) \\
&= \frac{1}{4k}\left(\Delta^{\mathcal{L}}_{G(V)} + 2\hat{\nabla}^{\mathcal{L}}_{\tilde{G}(V)\partial_M \tilde{F}} - 2nV'[\tilde{F}]\right) + V'[\tilde{F}] \\
&= \frac{1}{4}\left(\hat{\varphi}^*\Delta_{G(V)} + \beta(V)\right) V'[\tilde{F}] \\
&= \hat{\varphi}^* u(V) + V'[\tilde{F}]
\end{aligned}$$

where u is the one-form from Theorem 3.10. Thus the pullback of the Hitchin connection is

$$\begin{aligned}\hat{\varphi}^*\boldsymbol{\nabla}_V &= \hat{\varphi}^*\hat{\nabla}^r + \hat{\varphi}^*u(V)\\ &= \hat{\nabla}^{\mathcal{L}} + V'[\tilde{F}] + \hat{\varphi}^*u(V)\\ &= \hat{\nabla}^{\mathcal{L}} + \tilde{u}(V)\\ &= \tilde{\boldsymbol{\nabla}}_V\end{aligned}$$

and so the two connections agree in the bundle $\tilde{H}^{(k)}$. □

In the case of the moduli space of flat connection, a Hitchin connection was constructed, using Hitchin's techniques, by Scheinost and Schottenloher [SS]. However, we believe that the scope of our construction is much larger, since geometric quantization of symplectic manifolds is a general quantization scheme. As for the curvature of this connection, Gammelgaard has been addressing this and has shown some partial results in that direction.

4

Abelian Varieties and Toeplitz Operators

In the previous chapters we have constructed Hitchin connections under various assumptions. In this chapter we look at a case where the conditions on the first cohomology ($H^1(M) = 0$ in Chapter 2 and $H^{0,1}(M) = 0$ in Chapter 3) is not satisfied. Indeed, we will discuss geometric quantization of abelian varieties, including a Hitchin connection and Toeplitz operators. We start by following the paper [And4]. Regarding moduli spaces, the tori we will consider can also be viewed as moduli spaces of flat U(1)-connections on surfaces. Later we discuss the quantization of SU(2)-moduli space of a genus one surface (see [Jef]) and consider Toeplitz operators in this setting.

4.1 Geometric Quantization of Abelian Varieties

Let V be a real vector space of dimension $2m$ with a symplectic structure ω. Let Λ be a maximal lattice in V such that ω is integral and unimodular on Λ. Then there is a lemma in [GH] saying that Λ admits a basis $\lambda_1, \dots, \lambda_{2m}$ with dual coordinates $x_1, \dots, x_m, y_1 \dots, y_m$ such that

$$\omega = \sum_{j=1}^{m} dx_j \wedge dy_j.$$

Let

$$\mathbb{H} = \{Z \in \mathrm{Mat}_m(\mathbb{C}) \mid Z^T = Z, \, \mathrm{Im}(Z) \text{ is positive definite}\}$$

denote the *Siegel generalized upper half space.* Since ω is positive, by the third Riemann condition in [GH], any $Z \in \mathbb{H}$ determines a complex structure on $M = V/\Lambda$ compatible with ω, with complex coordinates

$$z = x + Zy.$$

If we denote by $I(Z)$ the corresponding integrable almost complex structure on M, we find that $(M, \omega, I(Z))$ is Kähler. We now wish to proceed with geometric quantization of (M, ω). But rather than choosing a fixed prequantum line bundle on (M, ω), we will instead choose a family of holomorphic line bundles parametrized by $\mathbb{H}$. First, let us review the theory of line bundles on abelian varieties.

Suppose $\mathcal{L} \to M$ is a line bundle. Then we can pull back this bundle by the quotient map $\pi \colon V \to V/\Lambda$ to a bundle $\pi^*\mathcal{L} \to V$. Now, every bundle over V is trivializable since V is contractible, and so we let $\varphi \colon \pi^*\mathcal{L} \to V \times \mathbb{C}$ be a global trivialization. If we let $z \in V$ and $\lambda \in \Lambda$ we know that the fibers $\pi^*\mathcal{L}_z$ and $\pi^*\mathcal{L}_{z+\lambda}$ coincide and so there is a (nonzero) complex number $e_\lambda(z)$ making the diagram of linear maps

$$\begin{tikzcd}
\pi^*\mathcal{L}_z \arrow[r, "\varphi_z"] \arrow[d, equal] & \mathbb{C} \arrow[d, "e_\lambda(z)"] \\
\pi^*\mathcal{L}_{z+\lambda} \arrow[r, "\varphi_{z+\lambda}"] & \mathbb{C}
\end{tikzcd}$$

commute. This way we get a family of nonzero complex-valued functions e_λ, $\lambda \in \Lambda$. These are called *multipliers*. We note that if $\mathcal{L}$ is a holomorphic line bundle, the multipliers become holomorphic. Considering the slightly more complicated diagram

$$\begin{tikzcd}
\pi^*\mathcal{L}_z \arrow[r, "\varphi_z"] \arrow[d, equal] & \mathbb{C} \arrow[d, "e_\lambda(z)"'] \arrow[dd, bend left, "e_{\lambda'}(z)"] \arrow[ddd, bend left=60, "e_{\lambda+\lambda'}(z)"] \\
\pi^*\mathcal{L}_{z+\lambda} \arrow[r, "\varphi_{z+\lambda}"] \arrow[d, equal] & \mathbb{C} \arrow[d, "e_{\lambda'}(z+\lambda)"'] \arrow[dd, bend left, "e_{\lambda'}(z+\lambda)"] \\
\pi^*\mathcal{L}_{z+\lambda'} \arrow[r, "\varphi_{z+\lambda'}"] \arrow[d, equal] & \mathbb{C} \arrow[d, "e_\lambda(z+\lambda')"'] \\
\pi^*\mathcal{L}_{z+\lambda+\lambda'} \arrow[r, "\varphi_{z+\lambda+\lambda'}"] & \mathbb{C}
\end{tikzcd}$$

we get the equations

$$e_{\lambda'}(z + \lambda)e_\lambda(z) = e_\lambda(z + \lambda')e_{\lambda'}(z) = e_{\lambda+\lambda'}(z) \tag{4.1}$$

for all $\lambda, \lambda' \in \Lambda$.

Conversely, given a collection of nonzero holomorphic functions $\{e_\lambda \in \mathcal{O}^*(V)\}_{\lambda\in\Lambda}$ satifying the above relation (4.1), we can construct a line bundle as the quotient

$$V \times \mathbb{C}/\!\sim$$

where $(z, v) \sim (z + \lambda, e_\lambda(z)v)$.

Now suppose a line bundle $\mathcal{L}$ is given by multipliers $\{e_\lambda \in \mathcal{O}^*(V)\}_{\lambda\in\Lambda}$. Given a function $f\colon V \to \mathbb{C}$ it defines a section of $\mathcal{L}$ if and only if it satisfies the equation

$$f(z + \lambda) = e_\lambda(z)f(z)$$

for all $\lambda \in \Lambda$. Let $\tilde{f}$ denote the corresponding section. Furthermore, given a metric $\|\cdot\|$ on $\mathcal{L}$ we can compare $\|\tilde{f}(z)\|^2$ and $|f(z)|^2$, where $|\cdot|$ is the standard metric on $\mathbb{C}$. Then there must exist a positive smooth function $h \in C^\infty(M, \mathbb{R}_+)$ such that

$$\|\tilde{f}(z)\|^2 = h(z)|f(z)|^2$$

which clearly must satisfy a similar equation

$$h(z + \lambda) = |e_\lambda(z)|^2 h(z) \tag{4.2}$$

for all $\lambda \in \Lambda$. Again, any metric can be given by such a function h.

Since $H^2(M, \mathbb{Z})$ has no torsion, we can compute the Chern class of this line bundle by computing the curvature of the canonical connection corresponding to h. Indeed, setting $\Theta = \bar{\partial}\partial \log h$, we get that

$$c_1(\mathcal{L}) = [\tfrac{i\Theta}{2\pi}].$$

Now, fix $Z \in \mathbb{H}$. We define multipliers on our basis by

$$e_{\lambda_j}(z) = 1 \qquad e_{\lambda_{m+j}}(z) = e^{-2\pi i z_j - \pi i Z_{jj}}, \qquad j = 1, \ldots, m$$

and extend them uniquely to all of Λ. This defines a line bundle $\mathcal{L} = \mathcal{L}_Z$ on M. We choose a Hermitian metric on $\mathcal{L}$ given by the function

$$h(z) = e^{-2\pi y \cdot Y y},$$

where $Z = X + iY$. This can also be written in terms of z and $\bar{z}$ as

$$h(z) = e^{\frac{\pi}{2}\sum_{jk} W_{jk}(z_j - \bar{z}_j)(z_k - \bar{z}_k)},$$

where $W = Y^{-1}$. For this to be a Hermitian structure, we need to check (4.2). Let $1 \leq j \leq n$. For λ_j it is immediate, since it only affects the real part of z.

now,

$$\begin{aligned}\log h(z+\lambda_{m+j}) &= \frac{\pi}{2}\sum_{kl}(z_k-\bar{z}_k+Z_{kj}-\bar{Z}_{kj})(z_l-\bar{z}_l+Z_{lj}-\bar{Z}_{lj})\\ &= \frac{\pi}{2}\sum_{kl}W_{kl}(z_k-\bar{z}_k)(z_l-\bar{z}_l)\\ &\quad+\pi i\sum_{kl}W_{kl}\left(Y_{kj}(z_l-\bar{z}_l)+Y_{lj}(z_k-\bar{z}_k)+2iY_{kj}Y_{lj}\right)\\ &= \log h(z)\\ &\quad+\pi i\sum_{kl}\left(\delta_{lj}(z_l-\bar{z}_l)+\delta_{kj}(z_k-\bar{z}_k)+2i\delta_k jY_{kj}\right)\\ &= \log h(z)+2\pi i(z_j-\bar{z}_j)-2\pi Y_{jj}\end{aligned}$$

which shows (4.2). Since we have a Hermitian structure on $\mathcal{L}$ we can choose the canonical compatible connection (Chern connection). The curvature of this connection is given by $\Theta=\bar{\partial}\partial\log h$. A direct computation of this reveals that $\Theta=-2\pi i\omega$ and so $c_1(\mathcal{L})=[\omega]$. We note that this is a different normalization than in the previous chapters.

The space of holomorphic sections $H_Z^{(k)}=H^0(M_Z,\mathcal{L}^k)$ has dimension k^m and a basis given by Theta-functions

$$\theta_{\gamma,k}(Z,z)=\sum_{l\in\mathbb{Z}^m}e^{i\pi k(l+\frac{\gamma}{k})\cdot Z(l+\frac{\gamma}{k})+2\pi ik(l+\frac{\gamma}{k})\cdot z},\qquad \gamma\in\{0,\dots,k-1\}^m.$$

We also induce the L^2-inner product from h on $H^0(M_Z,\mathcal{L}^k)$ given by

$$(s_1,s_2)=\int_M s_1(z)\overline{s_2(z)}h(z)dxdy.$$

We wish to produce a Hitchin connection in the bundle $H^{(k)}\to\mathbb{H}$. To this end, consider the trivial bundle $\mathbb{H}\times C^\infty(\mathbb{C}^m)\to\mathbb{H}$. For each $Z\in\mathbb{H}$ we can view $H_Z^{(k)}$ as a subspace of $C^\infty(\mathbb{C}^m)$.

Writing out the Theta-functions

$$\theta_{\gamma,k}(Z,z)=\sum_{l\in\mathbb{Z}^m}e^{i\pi k\sum_{ab}Z_{ab}(l_a+\frac{\gamma_a}{k})(l_b+\frac{\gamma_b}{k})+2\pi ik\sum_a(l_a+\frac{\gamma_a}{k})z_a}$$

we see that

$$\frac{\partial^2\theta_{\gamma,k}}{\partial z_a\partial z_b}=4\pi ik\frac{\partial\theta_{\gamma,k}}{\partial Z_{ab}}.$$

And so this heat equation yields a Hitchin connection given by

$$\boldsymbol{\nabla}_{\frac{\partial}{\partial Z_{ab}}}=\frac{\partial}{\partial Z_{ab}}-\frac{1}{4\pi ik}\frac{\partial^2}{\partial z_a\partial z_b}$$

preserving $H^{(k)}$. This is obviously a flat connection in $H^{(k)} \to \mathbb{H}$ and so we can identify the different quantum spaces by parallel transport with respect to $\boldsymbol{\nabla}$. This means we get a canonical quantum space by considering covarant constant holomorphic sections, which also have a basis of Theta-functions.

We wish to compute Toeplitz operators for certain phase functions on M, so to do this we investigate the inner products of Theta-functions. Writing out the integral for $(\theta_{\alpha,k}, \theta_{\beta,k})$ one interchanges the sum and the integration and consider the x-integrals for each factor. These are easily shown to be

$$\delta_{(l+\frac{\alpha}{k}),(j+\frac{\beta}{k})}$$

where l is the summing variable for $\theta_{\alpha,k}$ and j is the summing variable for $\theta_{\beta,k}$. Thus the Theta-functions are orthogonal and their inner product is a single sum over $\mathbb{Z}^m$ of integrals. Now, computing the norm, one finds

$$\begin{aligned}
\|\theta_{\gamma,k}\|^2 &= \sum_{l\in\mathbb{Z}^m} \int_{[0,1]^m} e^{-2\pi k\left((l+\frac{\gamma}{k})\cdot Y(l+\frac{\gamma}{k})+2(l+\frac{\gamma}{k})\cdot Yy+y\cdot Yy\right)} dy \\
&= \sum_{l\in\mathbb{Z}^m} \int_{[0,1]^m} e^{-2\pi k(l+\frac{\gamma}{k}+y)\cdot Y(l+\frac{\gamma}{k}+y)} dy \\
&= \int_{[0,1]^m} e^{-2\pi k(l+\frac{\gamma}{k}+y)\cdot Y(l+\frac{\gamma}{k}+y)} dy \\
&= \int_{\mathbb{R}^m} e^{-2\pi k(\frac{\gamma}{k}+y)\cdot Y(\frac{\gamma}{k}+y)} dy \\
&= \sqrt{\left(\frac{(2\pi)^m}{(4\pi k)^m \det(Y)}\right)}.
\end{aligned}$$

This leads us to normalizing the inner product to

$$(\cdot,\cdot)_Y = \sqrt{2^m k^m \det(Y)}(\cdot,\cdot).$$

This normalized Hermitian structure is compatible with $\boldsymbol{\nabla}$ on $H^{(k)}$.

The functions on M we consider are phase functions of the type

$$F_{r,s}(x,y) = e^{2\pi i(r\cdot x+s\cdot y)},$$

where $r, s \in \mathbb{Z}^m$. Now, fix r, s. We want a formula for the matrix coefficients $(F_{r,s}\theta_{\alpha,k}, \theta_{\beta,k})_Y$. As before, we consider the x-integral first. Writing $r_j = p_j k + [r]_j$, $p \in \mathbb{Z}$, $0 \le [r]_j \le k-1$, the formula in [And4] states that

$$(F_{r,s}\theta_{\alpha,k}, \theta_{\beta,k})_Y = \delta_{\alpha-\beta,-[r]} e^{-\frac{\pi i}{k} r\cdot\bar{Z}r - 2\pi i s\cdot\frac{\alpha}{k} - \pi^2(s-\bar{Z}r)\cdot(2\pi k Y)^{-1}(s-\bar{Z}r)}.$$

This is the coefficient $(T^{(k)}_{F_{r,s}})_{\alpha,\beta}$ of the Toeplitz operator written in the basis of the Theta-functions. Evidently, these depend on Z and are thus not covariantly constant with respect to the Hitchin connection induced in the endomorphism bundle of $H^{(k)}$ by $\boldsymbol{\nabla}$. This fixed by a function depending on k

$$f(r,s,Z)(k) = e^{\frac{\pi}{2k}(s-Xr)\cdot Y^{-1}(s-Xr)} e^{\frac{\pi}{2k} r\cdot Yr} e^{-\frac{\pi i}{k} s\cdot r}$$

which lets us define

$$\left(\hat{T}^{(k)}_{F_{r,s}}\right)_{\alpha,\beta} = \left(T^{(k)}_{f(r,s,Z)(k)F_{r,s}}\right)_{\alpha,\beta} = \delta_{\alpha-\beta,-[r]} e^{-\frac{2\pi i}{k}(s\cdot(r+\alpha))},$$

covariant constant operators on $H^{(k)}$. These are slightly different than in [And4], but they fit better with the use we have in mind in Section 7.2

Example 4.1. A good example of such abelian varieties are the moduli spaces of flat U(1)-connections on a closed, compact surface Σ_g of genus g.

$$M_g = \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{U}(1)).$$

since U(1) is abelian the Universal Coefficient Theorem and the long exact cohomology sequence tells us that

$$M_g = H^1(\Sigma_g, \mathrm{U}(1)) = H^1(\Sigma_g, \mathbb{R})/H^1(\Sigma_g, \mathbb{Z}).$$

The symplectic structure on $H^1(\Sigma_g, \mathbb{R})$ given by the cup-product makes M_g into an abelian variety as above (see e.g. [GH] p. 306 – 307).

4.2 The SU(2) Moduli Space of a Genus One Surface

Let us consider geometric quantization of the moduli space of flat SU(2)-connections on a genus one surface T. As in Section 2.2 we look at

$$\mathcal{M}_{\mathrm{SU}(2)}(T) = \mathrm{Hom}(\pi_1(T), \mathrm{SU}(2)) / \mathrm{SU}(2)$$

Since $\pi_1(T) \cong \mathbb{Z}\times\mathbb{Z}$ is abelian, it maps into the maximal torus of SU(2) which is $U(1)$. Thus the moduli space reduces to

$$\mathcal{M}_{\mathrm{SU}(2)}(T) = \mathrm{U}(1) \times \mathrm{U}(1)/W$$

where W is the Weyl group of SU(2), so $W = \mathbb{Z}_2$, and we see that $\mathcal{M}_{\mathrm{SU}(2)}(T)$ is topologically a sphere with a torus as a double cover. It has singularities, however, so instead of choosing a prequantum line bundle over $\mathcal{M}_{\mathrm{SU}(2)}(T)$, we pick a prequantum line bundle over the torus $T = \mathrm{U}(1) \times \mathrm{U}(1)$ with a $\mathbb{Z}_2$ action.

Since T is a double cover, the line bundle $\tilde{\mathcal{L}}$ corresponding to the fundamental class of $\mathcal{M}_{\mathrm{SU}(2)}(T)$ pulls back to a line bundle $\tilde{\mathcal{L}}$ corresponding to twice the fundamental class on T. Thus $\tilde{\mathcal{L}} \cong \mathcal{L}^2$ over T, for the prequantum line bundle $\mathcal{L}$ on T with Chern class 1.

4.2.1 Theta-functions

Choose a Kähler structure $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. Above we saw that $\dim_{\mathbb{C}} H^0(T, \mathcal{L}^l) = l$, with a basis given by

$$\theta_{\gamma,l}(\tau, z) = \sum_{m\in\mathbb{Z}} e^{i\pi l\tau(m+\frac{\gamma}{l})^2+2\pi ilz(m+\frac{\gamma}{l})}, \qquad \gamma = 0, \dots, l-1.$$

We construct the Weyl invariant Theta-functions $\theta^+_{\theta,l}$ by

$$\begin{aligned}\theta^+_{\gamma,l}(\tau, z) &= \theta_{\gamma,l}(\tau, z) + \theta_{\gamma,l}(\tau, -z)\\ &= \theta_{\gamma,l}(\tau, z) + \theta_{l-\gamma,l}(\tau, z).\end{aligned}$$

Let $H^0(T, \mathcal{L}^l)^W$ denote the Weyl invariant subspace of holomorphic section. Let us calculate the dimension.

First we notice that if $\gamma \in \{0, \dots, l-1\}$ then so is $l-\gamma$ and $\theta^+_{\gamma,l} = \theta^+_{l-\gamma,l}$. Assume l even. The Weyl action has two fixed points, namely 0 and $\frac{l}{2}$. Away from these points, W maps the subset $\{1, \dots, \frac{l}{2}-1\}$ bijectively to $\{\frac{l}{2}+1, \dots, l-1\}$. from this we conclude that we get the basis $\{\theta^+_{\gamma,l} \mid \gamma = 0, \dots, \frac{l}{2}\}$ and the dimension becomes $\frac{l}{2}+1$.

Now, our level k quantum space is $H^0(T, \tilde{\mathcal{L}}^k)^W = H^0(T, \mathcal{L}^{2k})^W$ and therefore has dimension $k+1$.

However, this is not the basis we are interested in using. In [APW] they consider the following corrected Theta-functions, as they are parallel with respect to the Hitchin connection. Consider the space $H^0(T, \mathcal{L}^l)^{W_-}$ of Weyl anti-invariant holomorphic sections. Clearly this is spanned by the Weyl anti-invariant theta functions

$$\theta^-_{\gamma,l} = \theta_{\gamma,l} - \theta_{l-\gamma,l}, \qquad \gamma = 0, \dots, l-1.$$

As above we find seek to find a basis from this spanning set.

Clearly $\theta^-_{\gamma,l} = -\theta^-_{l-\gamma,l}$ and if $[\gamma]_l = [-\gamma]_l$ we find that $\theta^-_{\gamma,l} = 0$. So for l even, we see that we have a basis

$$\{\theta^-_{\gamma,l} \mid \gamma = 1, \dots, \tfrac{l}{2}-1\}$$

and for l odd

$$\{\theta^-_{\gamma,l} \mid \gamma = 1, \dots, \tfrac{l-1}{2}\}.$$

From this we see that $\dim H^0(T, \mathcal{L}^{2(k+2)})^{W_-} = k+1 = \dim H^0(T, \mathcal{L}^{2k})^W$. We want to construct an isomorphism between these spaces.

Let $\varphi\colon H^0(T, \mathcal{L}^{2k}) \otimes H^0(T, \mathcal{L}^4) \to H^0(T, \mathcal{L}^{2k+4})$ be the natural map induced from the tensor product. If we restrict φ to

$$\varphi'\colon H^0(T, \mathcal{L}^{2k})^W \otimes H^0(T, \mathcal{L}^4)^{W_-} \to H^0(T, \mathcal{L}^{2k+4})$$

we see that $\operatorname{Im}(\varphi') \subseteq H^0(T, \mathcal{L}^{2k+4})^{W_-}$. We notice that $\dim H^0(T, \mathcal{L}^4)^{W_-} = 1$ and thus we have the isomorphism

$$\varphi'' \colon H^0(T, \mathcal{L}^{2k})^W \to H^0(T, \mathcal{L}^{2k})^W \otimes H^0(T, \mathcal{L}^4)^{W_-}$$

given by $\varphi''(s) = s \otimes \theta^-_{1,4}$. Now we can define the map

$$\Phi = \varphi' \circ \varphi'' \colon H^0(T, \mathcal{L}^{2k})^W \to H^0(T, \mathcal{L}^{2k+4})^{W_-}.$$

We want to show that this map is injective and thus an isomorphism.

Suppose $\Phi(s) = 0$. This means that

$$0 = \varphi''(s \otimes \theta^-_{1,4}) = s \otimes \theta^-_{1,4} \in C^\infty(T, \mathcal{L}^{2k}) \otimes C^\infty(T, \mathcal{L}^4)$$

which implies that s is the zero section.

Now we can pull back the basis on $H^0(T, \mathcal{L}^{2k+4})^{W_-}$ by Φ to get the basis

$$\psi_{\gamma,k} = \Phi^{-1}(\theta^-_{\gamma,2k+4}), \qquad \gamma = 1, \ldots, k+1$$

for $H^0(T, \mathcal{L}^{2k})^W$.

The Toeplitz operators on $H^0(T, \mathcal{L}^l)$ from the previous section can now be written as

$$(\hat{T}^{(l)}_{F_{r,s}})_{\alpha,\beta} = \delta_{\alpha-\beta,-[r]} e^{-\frac{2\pi i s}{l}(r+\alpha)}.$$

The inner product $(\cdot,\cdot)_Y$ on $H^0(T, \mathcal{L}^l)$ induce an inner product $(\cdot,\cdot)_{W_-}$ given by

$$(\theta^-_{\alpha,l}, \theta^-_{\beta,l})_{W_-} = \frac{1}{2}(\theta^-_{\alpha,l}, \theta^-_{\beta,l})_Y$$

in which the basis is orthonormal.

Let us investigate which combinations of phase functions have Toeplitz operators that preserve the Weyl anti-invariant subspaces.

Let $\mathbf{r} = (r_1, \ldots, r_d)$ and $\mathbf{s} = (s_1, \ldots, s_d)$ be vectors in $\mathbb{Z}^d$ and consider the function

$$G_{\mathbf{r},\mathbf{s}}(x, y) = \sum_{j=1}^{d} c_j F_{r_j,s_j}(x, y)$$

for some constants $c_j \in \mathbb{C}\backslash\{0\}$. We now apply the corrected Toeplitz operators for such a function to a Weyl anti-invariant Theta-function

$$\begin{aligned}
\hat{T}^{(l)}_{G_{\mathbf{r},\mathbf{s}}}\theta^-_{\alpha,l} &= \hat{T}^{(l)}_{G_{\mathbf{r},\mathbf{s}}}\theta_{\alpha,l} - \hat{T}^{(l)}_{G_{\mathbf{r},\mathbf{s}}}\theta_{l-\alpha,l} \\
&= \sum_{\beta=0}^{l}(\hat{T}^{(l)}_{G_{\mathbf{r},\mathbf{s}}})_{\alpha,\beta}\theta_{\beta,l} - (\hat{T}^{(l)}_{G_{\mathbf{r},\mathbf{s}}})_{l-\alpha,\beta}\theta_{\beta,l} \\
&= \sum_{\beta=0}^{l}(\hat{T}^{(l)}_{G_{\mathbf{r},\mathbf{s}}})_{\alpha,\beta}\theta_{\beta,l} - (\hat{T}^{(l)}_{G_{\mathbf{r},\mathbf{s}}})_{l-\alpha,l-\beta}\theta_{l-\beta,l} \\
&= \sum_{\beta=0}^{l}\sum_{j=1}^{d} c_j(\hat{T}^{(l)}_{F_{r_j,s_j}})_{\alpha,\beta}\theta_{\beta,l} - c_j(\hat{T}^{(l)}_{F_{r_j,s_j}})_{l-\alpha,l-\beta}\theta_{l-\beta,l}.
\end{aligned}$$

To preserve the Weyl anti-invariant subspace, we need the coefficient for $\theta_{\alpha,\beta}$ to agree with the coefficient for $\theta_{l-\alpha,l-\beta}$. I.e. we need a bijection permutation $\sigma \in \mathcal{S}_d$ satisfying

$$c_j(\hat{T}^{(l)}_{F_{r_j,s_j}})_{\alpha,\beta} = c_p(\hat{T}^{(l)}_{F_{r_p,s_p}})_{l-\alpha,l-\beta}, \qquad \sigma(j) = p$$

for all α, β. Writing out the condition

$$\begin{aligned}
c_j\delta_{\alpha-\beta,-[r_j]}e^{-\frac{2\pi i s_j}{l}(r_j+\alpha)} &= c_p\delta_{\beta-\alpha,-[r_p]}e^{-\frac{2\pi i s_p}{l}(r_j+l-\alpha)} \\
&= c_p\delta_{\alpha-\beta,[r_p]}e^{-\frac{2\pi i s_p}{l}(r_p-\alpha)}.
\end{aligned}$$

This is satisfied if

$$c_j = c_p, \quad r_j = -r_p \quad \text{and} \quad s_j = -s_p.$$

In particular we see if $r_j \neq 0$ for all j this yields that d must be even and we get an l-independent family of functions

$$G_{\mathbf{r},\mathbf{s}}(x,y) = \sum_{j=1}^{d/2} c_j\left(e^{2\pi i(r_j x+s_j y)} + e^{-2\pi i(r_j x+s_j y)}\right)$$

preserving the Weyl anti-invariant subspace. We shall use this type of functions later in Section 7.3.

5

Knots

The goal of this chapter is to introduce the A-polynomial of a knot. We start by defining the knot group and introduce two-bridge knots, a particularly nice class of knots. We discuss the knot group of two-bridge knots and calculate it for twist knots and double twist knots. Then we construct the A-polynomial and show how it can be calculated for two-bridge knots. We end with a theorem by Hoste adn Shanahan [HS] giving a recursive formula for the A-polynomial of twist knots.

5.1 Knots and Knot Groups

In this dissertation, a knot K is an embedding of S^1 in either S^3 or $\mathbb{R}^3$ and the equivalence of knots is ambient isotopy. We will not distinguish the knot itself and its equivalence class and whenever the existence of a certain projection for a knot is needed, it will be implied that we refer to the equivalence class of the knot.

We start by discussing the *knot group* of a knot, namely the fundamental group of the knot complement, $\pi_1(S^3 - K)$. We will give the Wirtinger presentation of the knot group, which can be describe directly from a diagram.

Namely, place the knot "close" to its projection and label the arcs in the projection from undercrossing to undercrossing by $v_1, \ldots, v_n$. Orient the knot. Now, pick a basepoint x_0 above the knot and for each v_i choose a loop s_i based at x_0 circling v_i according to the right-hand rule (see Figure 5.1). It is clear that $s_1, \ldots, s_n$ generate $\pi_1(\mathbb{R}^3 - K) = \pi_1(S^3 - K)$. Now, let us find the generators. Make a small circle c below the crossing around the double point of the projection and connect it to x_0 by a path l (Figure 5.2). Then lcl^{-1} is a loop based at x_0 and it is contractible. This corresponds to a word in the generators involved in the crossing and so we can read off the relations corresponding to positive and negative crossings as in the figure. This gives rise to n relations $r_1, \ldots, r_n$. These can be shown, e.g. by a Van Kampen argument, to be a complete set of relations and we have

$$\pi_1(\mathbb{R}^3 - K) = \langle s_1, \ldots, s_n \mid r_1, \ldots, r_n \rangle.$$

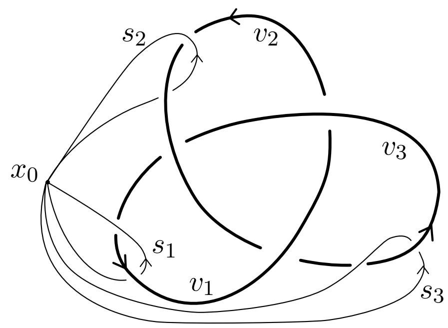
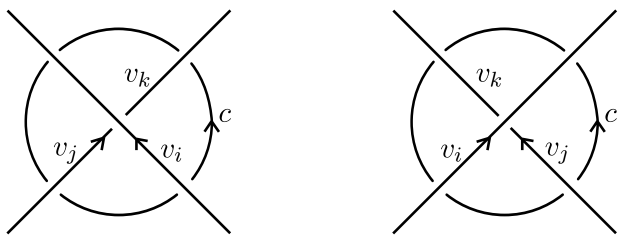


Figure 5.1: Wirtinger generators for the trefoil.



$s_j = s_i^{-1} s_k s_i$ $\qquad$ $s_j = s_i s_k s_i^{-1}$

Figure 5.2: Wirtinger relation at a positive and negative crossing.

One thing to observe, however, is that a loop encircling the whole projection is the product of (conjugates) of all of the above relations, and so we find that any single relation is a consequence of all of the others.

The knot group is in itself a powerful knot invariant, but will not be of direct interest to us, as it will only be used to build the invariant called the A-polynomial.

5.2 Two-bridge Knots

In this section we will focus our attention to a certain well-understood class of knots, called the two-bridge knots. First, we will review the concept of braids.

Definition 5.1. Choose a rectangle in $\mathbb{R}^3$ and place n equidistant points directly across from each other on two opposite sides. An *n-braid* is a collection of simple disjoint arcs connecting points on opposite sides such that for all

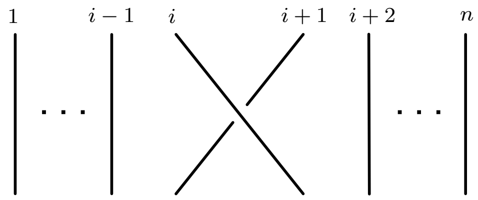


Figure 5.3: The generator σ_i for the braid group on n strands.

planes orthogonal to the rectangle and parallel to the distinguished sides intersects each arc only once.

We consider braids equivalent if they are isotopic by an ambient isotopy relative to the endpoints. A braid gives rise to a knot or a link by connecting the endpoints by simple arcs. In fact all links and knots can be presented by a closed braid. This can, of course, be done in different ways but for $2m$-braids the closure called the *plat*-closure will be of special interest. This consists of connecting the endpoints with a neighbouring endpoint on both ends of the braid

Braids on n strands can be composed in the obvious way and this gives rise to the *braid group* $\mathcal{B}_n$. This has $n-1$ generators σ_i (Figure 5.3) which is a positive crossing of the i'th and the $(i+1)$'th strand. It is immediate that a complete set of relations is

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{and} \quad \sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| \geq 2.$$

Definition 5.2. A knot $K \subseteq \mathbb{R}^3$ is said to have an *m-bridge presentation* if a plane V can be placed in $\mathbb{R}^3$ in such a way that it intersects K in $2m$ distinct points (bridge points) and for each of the resulting half-spaces, the part of K it contains projects orthogonally onto V as m simple and disjoint arcs (bridges). The minimal number of bridges is called the *bridge number*.

We notice that all knots (and links) have an m-bridge presentation. Indeed, project the knot to a diagram and choose points between over- and undercrossings. Then lift overcrossing strands up from the plane and push the undercrossing strands to the other side. This gives a bridge presentation (and shows that the bridge number is bounded by the number of crossings in any diagram).

Given this m-bridge presentation, one can show that all knots (and links) can be obtained as the closure of a braid. In fact, we will show that m-bridge knots and $2m$-plats are the same.

Lemma 5.3. *An m-bridge knot allows a presentation as a $2m$-plat.*

Proof. By ambient isotopy, it suffices to show that a given m-bridge knot can be deformed into having m maxima and m minima in the z-direction. First, arrange the knot so that the plane containing the bridge points is the xy-plane. We can arrange the m arcs in the upper-half plane such that their vertical projections onto the xy-plane does not intersect. This means that their projection cylinders in the upper-half plane are disjoint. Now we can deform the strands within their projection cylinders to have only one maximum. A similar thing can be done in the lower-half plane and we have proved the lemma. □

The converse can be shown through the following lemma (Lemma 10.4 in [BZ2])

Lemma 5.4. *An n-braid has a projection with no double points.*

Proof. Place the rectangle of the braid at an 45° angle to the xy-plane so the projections onto the xy-plane and the yz-plane are regular. Choose the lowest double point (in the z- and hence x-direction) of the xy-projection and pull the overcrossing strand in the x-direction until the double point vanishes. Continue this process until the projection onto the xy-plane has no double points. □

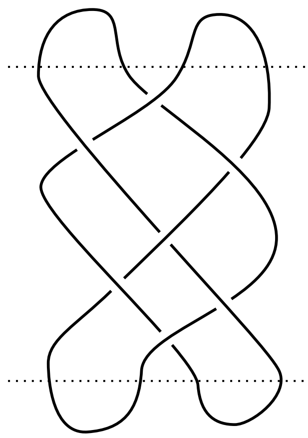


Figure 5.4: A two-bridge knot.

Proposition 5.5. *The m-bridge knots are the $2m$-plats.*

Proof. From the above lemmas, we only need to construct an m-bridge presentation from the projection of a $2m$-braid with no double points. From the above construction, we see that the $2m$ endpoints of the projection furthest in the x-direction can be connected pairwise by m simple closed arcs in the plane. Now, connect the remaining $2m$ endpoints by m simple closed arcs in the lower-half space. This gives an m-bridge presentation. □

In particular, this shows that two-bridge knots are the 4-plats (see Figure 5.4).

These are well-understood and are for instance prime and invertible, as well as alternating. These knots have also been classified by Schubert ([Sch2]) by associating a pair of coprime integers to the knot. We will now describe how to obtain these integers. First, we introduce the *reduced* diagram of a two-bridge knot (Figure 5.5).

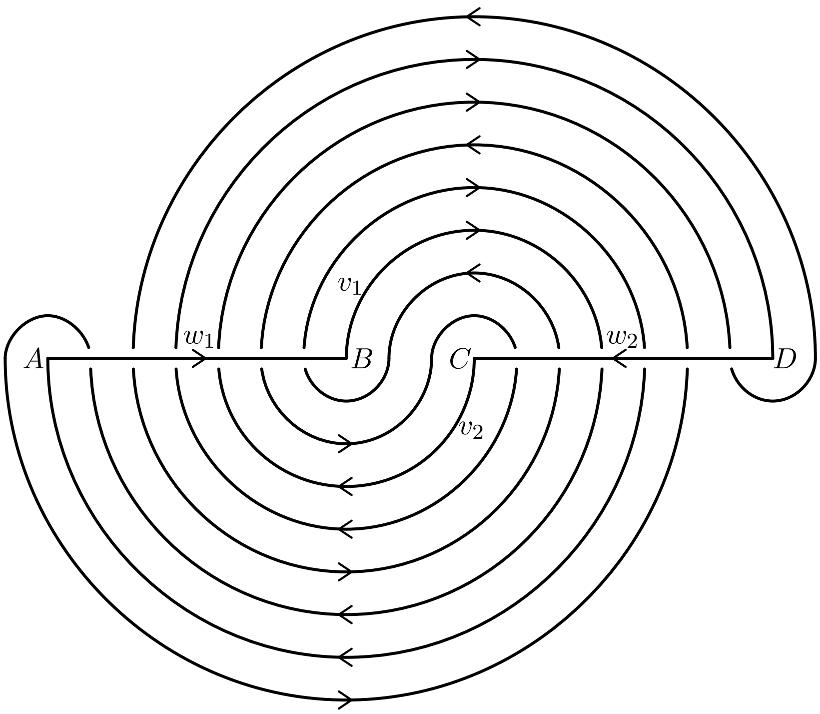


Figure 5.5: Reduced diagram for the two-bridge knot $K(7,3)$.

We can arrange the knot in its two-bridge presentation such that the two upper bridges projects to two intervals of the same straight line and these are directed towards each other. We refer to the points on the plane as A, B, C, D and $w_1 = [A, B]$, $w_2 = [D, C]$. Let v_1 be the curve under the plane from B to D, and v_2 the curve connecting C from A.

Starting by following v_1 we can arrange that it crosses w_2 first. Indeed, if it crosses w_1 first, the bi-gon this produces can be emptied for v_i's and subsequently this crossing can be eliminated. Similarly, it can be arranged to meet the w_i's alternately, ending at D on w_2. The same can be done for v_2, crossing w_1 first. This also means that the w_i's meet the v_i's alternately. Label the double points on along w_1 from B to A by numbers 0 to α. We remark that α is odd. Then follow v_2 from C to the first intersection with w_1. We denote the label $\hat{\beta}$. Notice that this number is also odd. If v_2 crosses w_1 from below, we let $\beta = \hat{\beta}$. If it crosses from above, then $\beta = -\hat{\beta}$. These numbers (α, β) classify the two-bridge knots by the following theorem

Theorem 5.6 ([Sch2]). *Let $K(\alpha,\beta)$ and $K(\alpha',\beta')$ are equvalent if and only if*

$$\alpha = \alpha' \quad \textit{and} \quad \beta^{\pm 1} \equiv \beta' \mod \alpha.$$

Furthermore $K(\alpha, -\beta)$ is the mirror image of $K(\alpha,\beta)$.

The numbers α and β have the following geometric meaning. Project the reduced diagram to the plane and consider the one-point campactification of $\mathbb{R}^2$ so it becomes a diagram on S^2. The sphere has a torus as a two-fold branched covering, branched over A, B, C and D. Choose a preferred sheet. The lift of w_1 to the preferred sheet concatenated with the inverse of the other lift yields a meridian m. By lifting a small circle around B and C we get a longitude l, which together with m generate $H_1(T)$. Similarly, v_1 lifts to a simple curve from B to C on the preferred sheet, and by concatenating with the inverse of the other lift, we get an element $\hat{v} \in H_1(T)$ which has exactly the coefficients

$$\hat{v} = \beta m + \alpha l.$$

A proposition, a proof of which can be found in [BZ2], gives an algorithmic way of constructing $K(\alpha,\beta)$ as a 4-plat

Proposition 5.7. *Let $0 < \beta < \alpha$, α odd and $\gcd(\alpha,\beta) = 1$. Then $K(\alpha,\beta)$ is the plat closure of $\xi = \sigma_2^{a_1}\sigma_1^{-a_2}\sigma_2^{a_3}\cdots\sigma_2^{\alpha_m}$ where m is odd and*

$$\frac{\beta}{\alpha} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots \cfrac{1}{a_{m-1} + \frac{1}{a_m}}}}}.$$

We remark in the above that if the euclidean algorithm producing the a_i's yields m even, we can just replace a_m by $a_m \pm 1$ and let $a_{m+1} = \mp 1$. In particular the above proposition shows that 4-plats can all be obtained as the plat closure of a 3-braid with a trivial fourth strand. Also, given a knot with a 4-plat diagram in only the two first generators, we can easily find α and β.

Now we want to compute the knot group of a two-bridge knot. This turns out to be done the easiest from a reduced diagram. Fortunately, there is an algorithm for drawing such a diagram, knowing α and β. The proof of this algorithm comes from the geometrical picture on the torus, where the curve $\hat{v}$ is lifted to $\mathbb{R}^2$.

Let $0 < \beta < \alpha$ α, β odd and $\gcd \alpha, \beta = 1$. For $i = 1, \ldots, \alpha - 1$, Let $0 < \hat{\gamma}_i < \alpha$ such that $i\beta \equiv \hat{\gamma}_i \mod \alpha$. If $\hat{\gamma}_i \equiv i \mod 2$, define $\gamma_i = \hat{\gamma}_i$.

Otherwise, let $\gamma_i = \hat{\gamma}_i - \alpha$. Now draw w_1 and w_2 as above and draw v_1 from point B by the following prescription: Let v_1 cross w_2 at $|\gamma_1|$ from above, if $\gamma_1 > 0$, and from below otherwise. From there, let v_1 cross w_1 at $|\gamma_2|$ from *below*, if $\gamma_2 > 0$, and from above otherwise. Continue in this fashion until $|\gamma_{\alpha_1}|$ on ω_1, from where v_1 is continued to D. Then rotate v_1 by $180°$ to get v_2.

The knot group of a two-bridge knot has a presentation with the two bridges as generators. Let x be the Wirtinger generator corresponding to w_1 and y corresponding to w_2. We wish to read off the relations from the reduced diagram, following v_1. We notice that at the starting point v_1 is just w_1. The first time v_1 meets w_2, x is conjugated by y to get $y^{-1}xy$ if v_1 crosses from above and yxy^{-1} if from below, or $y^{-\varepsilon_1}xy^{\varepsilon_1}$ where $\varepsilon_i = \frac{\gamma_i}{|\gamma_i|}$. This number can also be computed as $\varepsilon_i = (-1)^{k_i}$, where $i\beta = k_i\alpha_i + \hat{\gamma}_i$. Continuing along v_1 we conjugate by x when meeting w_1 to $x^{-\varepsilon_2}y^{-\varepsilon_1}xy^{\varepsilon_1}x^{\varepsilon_2}$. This procedure stops when v_1 meets D and we get the equation

$$y = W^{-1}xW, \qquad W = y^{\varepsilon_1}x^{\varepsilon_2}\cdots x^{\varepsilon_{\alpha-1}}.$$

Now, doing the same for v_2 we get the relation

$$x = \bar{W}^{-1}y\bar{W},$$

where $\bar{W}$ is the inverse word of W. However, the one relation implies the other as remarked in Section 5.1. So we are left with the single relation and the following proposition.

Proposition 5.8. *Let* $0 < \beta < \alpha$, *where* α, β *odd and* $\gcd(\alpha, \beta) = 1$. *Then the knot group is*

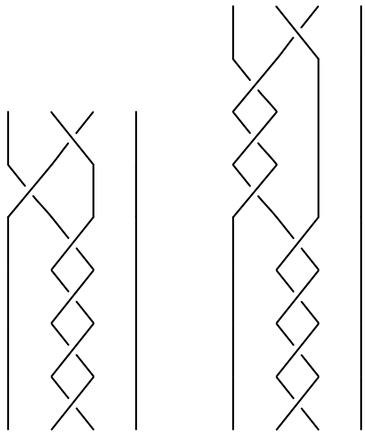
$$\pi_1(S^3 - K(\alpha,\beta)) = \langle x, y \mid xW = Wy\rangle, \qquad W = y^{\varepsilon_1}x^{\varepsilon_2}\cdots x^{\varepsilon_{\alpha-1}}$$

where $\varepsilon_i = (-1)^{\lfloor \frac{i\beta}{\alpha} \rfloor}$.

Note that since β is odd, we have $\varepsilon_i = \varepsilon_{\alpha-i}$.

From the reduced diagram we can also give a presentation of the *longitude* of the knot, i.e. the longitude of the boundary torus of a tubular neighbourhood. This is a parallel to the knot inself, which does not link with the knot. Starting from point B again, we follow the knot along v_1 and every time we go under a strand we pick up the corresponding generator with the appropriate sign. When we get to point D we have exactly the word W and continuing along w_2, v_2 and finally w_1, we pick up $\bar{W}$. To unlink the longitude from the knot, we then run around w_1, i.e. add x, enough times to make the total exponent zero. This yields the longitude

$$l = W\bar{W}x^{-2\sum \varepsilon_i}. \tag{5.1}$$

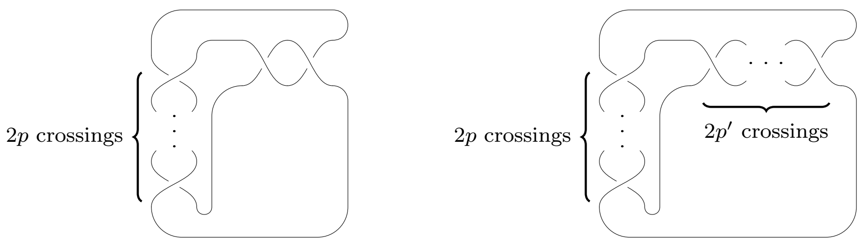


(a) K_2 (b) $K_{2,2}$

Figure 5.6: Braid presentation for the knots K_2 and $K_{2,2}$.

We now turn our attention to two special classes of two-bridge knots called twist and double twist knots.

Definition 5.9. given $p \in \mathbb{Z}$, the p-twist knot K_p is the plat closure of the braid $\sigma_2 \sigma_1^{-1} \sigma_2^{-2p}$. In general, given $p, p' \in \mathbb{Z}$, the *double twist knot* $K_{p,p'}$ (In [HS], denoted $J(2p, 2p')$) is the plat closure of the braid $\sigma_2 \sigma_1^{-(2p'-1)} \sigma_2^{-2p}$. See Figure 5.6 and 5.7.



(a) K_p with $2p$ vertical crossings.

(b) $K_{p,p'}$ with $2p$ vertical crossings on the left and $2p'$ horizontal crossings on the right.

Figure 5.7: Twist and double twist knots.

The first few (double) twist knots can be found in the Rolfsen table of knots at the Knot Atlas [BNM]. We have listed them with a note on whether these are the actual knots in the table or the mirror image. The knot $K_{-p,-p'}$ is the mirror of $K_{p,p'}$, and so we will let $p' \geq 1$ and $p \in \mathbb{Z}$.

p	p'	Name	Mirror
1	1	3_1	No
-1	1	4_1	achiral
2	1	5_2	No
-2	1	6_1	Yes
3	1	7_2	No
2	2	7_4	Yes
-3	1	8_1	Yes
-2	2	8_3	achiral
4	1	9_2	No
3	2	9_5	Yes
-4	1	10_1	Yes
-3	2	10_3	No

From the definition, we can use Proposition 5.7 to calculate α and β for the double twist knots, where we let $p' \geq 1$.

$$\frac{\beta}{\alpha} = \frac{1}{1 + \frac{1}{2p'-1+\frac{1}{-2p}}} = \frac{4pp' - 2p - 1}{4pp' - 1}.$$

So for $p > 0$, we then get $K_{p,p'} = K(4pp' - 1, 4pp' - 2p - 1)$ and, by imposing $\alpha > 0$, $K_{-p,p'} = K(4pp' + 1, 4pp' - 2p + 1)$. As an example, we can calculate the knot groups for the twist knots. For K_{-p} we see that

$$i\beta = i(2p + 1) = \frac{i}{2}(4p + 1) + \frac{i}{2}$$

and thus $\varepsilon_i = (-1)^{\lfloor \frac{i}{2} \rfloor}$ and $W = (yx^{-1}y^{-1}x)^p$. Similarly, for K_p

$$i\beta = i(2p - 1) = \frac{i-1}{2}(4p - 1) + \frac{4p - 1 - i}{2}$$

and we get $W = (yxy^{-1}x^{-1})^{p-1}yx$.

For the more general double twist knots we find by similar methods that for $K_{p,p'}$ we get $\varepsilon_i = (-1)^{i-1-\lfloor \frac{i}{2p'} \rfloor}$ and thus

$$W = \left(y(x^{-1}y)^{p'-1}(xy^{-1})^{p'}x^{-1}\right)^{p-1} y(x^{-1}y)^{p'-1}(xy^{-1})^{p'-1}x,$$

and for $K_{-p,p'}$ we find $\varepsilon_i = (-1)^{i-1-\lfloor \frac{i-1}{2p'} \rfloor}$ giving

$$W = \left((yx^{-1})^{p'}(y^{-1}x)^{p'}\right)^{p}.$$

5.3 The A-Polynomial

We now introduce a very strong knot invariant, the A-polynomial, as defined in [CCG+]. First, for a space M we define the representation variety $\mathrm{Rep}(M) = \mathrm{Hom}(\pi_1(M), \mathrm{SL}(2,\mathbb{C}))$. Also, we denote by $\chi(M)$ the set af characters of the representations, which by [CS] is a closed algebraic set, so we call it the *character variety.* We remark that the map $t\colon \mathrm{Rep}(M) \to \chi(M)$ is surjective and that t factors through the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$-connections. In [CS] it is shown that this map is injective on the irreducible representations.

Now, let $K \subset S^3$ be a knot and let $N(K)$ be a tubular neighbourhood of K. Also, we write the knot complement as $X_K = S^3 - N(K)$. Fix and isomorphism $\partial N(K) = \partial X_K \cong T$ to a standard torus. Choosing standard generators $\pi_1(T) = \langle l, m\rangle$ we see these are mapped to a peripheral system on $\partial N(K)$. Using the inclusion of the boundary, we get the induced maps $r\colon \mathrm{Rep}(X_K) \to \mathrm{Rep}(T)$ and $r\colon \chi(X_K) \to \chi(T)$.

The aim is to describe the image of r in $\chi(T)$ in a nice way. To this end, consider the subset

$$\Delta = \left\{ \rho \in \mathrm{Rep}(T) \,\middle|\, \rho(l) = \begin{pmatrix} L & 0 \\ 0 & L^{-1} \end{pmatrix}, \rho(m) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} \; L, M \in \mathbb{C}^* \right\}$$

of diagonal representations in $\rho(T)$. Clearly, the restriction $t_{|\Delta}\colon \Delta \to \chi(T)$ is still surjective, and generically $2:1$. By choosing the upper-left entry, we can identify Δ with $\mathbb{C}^* \times \mathbb{C}^*$. Call this isomorphism p. Also, we notice that

$$\chi(T) \cong \mathbb{C}^* \times \mathbb{C}^* / (L, M) \sim (L^{-1}, M^{-1}).$$

The above is summarized in the diagram below.

$$\begin{tikzcd}
\mathrm{Rep}(X_K) \arrow[r, "r"] \arrow[d, "t"'] & \mathrm{Rep}(T) \arrow[d, "t"'] & \supseteq \; \Delta \arrow[r, "p"] \arrow[dl, "t_\Delta"] & \mathbb{C}^* \times \mathbb{C}^* \\
\chi(X_K) \arrow[r, "r"] & \chi(T) & &
\end{tikzcd}$$

Now, we define the *deformation variety* in $\mathbb{C}^* \times \mathbb{C}^*$. Let Y be the collection of components in $\chi(X_K)$ such that for all $V \in Y$, $r(V)$ has one-dimensional closure in $\chi(T)$. The deformation variety D_K is

$$D_K = p\left(\bigcup_{V \in Y} t_{|\Delta}^{-1}(\overline{r(V)}) \right) \subseteq \mathbb{C}^* \times \mathbb{C}^*$$

Definition 5.10. The *A-polynomial* of a knot K is the defining polynomial $A_K(L, M)$ of the closure of D_K in $\mathbb{C}^2$. This is unique up to a scalar.

It can be shown that this can be chosen with integer coeficients ([CCG$^+$]) and is a knot invariant.

Let $V \in Y$ be a component of characters of reducible representations of $\pi_1(X_K)$. Let $\rho \in \Delta$ be a diagonal reducible representation, such that $t_{|\Delta}(\rho) \in r(V)$. In [CS] it is shown that for any $c \in [\pi_1(X_K), \pi_1(X_K)]$, $\mathrm{Tr}(\rho(c)) = 2$. Since the longitude is in the commutator subgroup, we see that $\rho(L) = I$ and so V in D_K is just the component given by $L = 1$. This means that $L - 1$ is always a factor of $A_K(L, M)$. On the other hand, if K is the unknot all representations are reducible and so $A_U(L, M) = L - 1$.

In [CCG$^+$] it was shown, that the A-polynomial is non-trivial (i.e. $A_K \neq L - 1$) for torus knots and hyperbolic knots. Dunfield and Garoufalidis [DG] and Boyer and Zhang [BZ1] independently showed that the A-polynomial is non-trivial for all non-trivial knots in S^3. Both of their proofs are based on the following theorem by Kronheimer and Mrowka.

Theorem 5.11 ([KM2])**.** *Let K be a non-trivial knot in S^3, and let Y_r be the three-manifold obtained by Dehn surgery on K with surgery-coefficient $r \in \mathbb{Q}$. If $|r| \leq 2$, then there is a homomorphism $\rho\colon \pi_1(Y_r) \to \mathrm{SU}(2)$ with non-cyclic image.*

Remark 5.12. The map $r\colon \chi(X_K) \to \chi(T)$, whose image basically defines D_K can also be viewed as a map $\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}(X_K) \to \mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}(T)$. In this language, D_K is the variety of flat $\mathrm{SL}(2, \mathbb{C})$-connections on ∂X_K which extend to flat connections on all of X_K. We will use this geometric interpretation in Section 7.1.

5.3.1 Computing the A-polynomial

Given a presentation of the knot group, in principle, the A-polynomial is computable through elimination theory. In case of two-bridge knots, the simple presentation of the knot group allows for a direct calculation, which we will describe below.

We are looking for representations $\rho\colon \pi_1(X_K) \to \mathrm{SL}_2(\mathbb{C})$ that restricts to a diagonal representation on the peripheral subgroup, that is, the subgroup $\pi_1(\partial X_K)$ generated by m and l. Now, any representation can be conjugated to a representation, which is uppertriangular on l and thus on $\pi_1(\partial X_K)$, since this is abelian. But given such a representation ρ, we can construct another ρ'

$$\rho(\gamma) = \begin{pmatrix} a(\gamma) & b(\gamma) \\ 0 & d(\gamma) \end{pmatrix} \qquad \rho'(\gamma) = \begin{pmatrix} a(\gamma) & 0 \\ 0 & d(\gamma) \end{pmatrix}$$

which is an element of Δ. Thus we get a point $(a(l), a(m))$ on D_K. This procedure produces a dense subset of D_K.

Let $K \subset S^3$ be a two-bridge knot and $\pi_1(X_K) = \langle x, y \mid xW = Wy \rangle$ a presentation of the knot group, where $W = y^{\varepsilon_1} x^{\varepsilon_2} \dots x^{\varepsilon_{\alpha-1}}$ as in Proposition 5.8.

We choose $m = x$ and get a longitude as $l = W\overline{W}x^{-2\sum \varepsilon_i}$ by (5.1). Now, let ρ be an irreducible representation of $\pi_1(X_K)$. Since ρ is irreducible, we can only get a point of D_K away from the component $L = 1$. Also, $\rho(x)$ and $\rho(y)$ have the same trace and does not commute, Lemma 7 in [Ril2] allows us to conjugate ρ to the form

$$\rho(x) = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix} \qquad \rho(y) = \begin{pmatrix} M & 0 \\ t & M^{-1} \end{pmatrix}.$$

For this to be a representation, the equation $\rho(xW) - \rho(Wy) = 0$ must hold. From now on, we drop ρ from the notation, where it causes no confusion. To analyze this, we need a lemma.

Lemma 5.13 ([Ril1])**.** *Let*

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$$

written in x and y as above. Then $W_{21} = tW_{12}$.

Proof. Let

$$V = \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \sqrt{t}^{-1} \end{pmatrix}.$$

Conjugating (a_{ij}) by this matrix multiplies a_{21} by t^{-1} and a_{12} by t. Thus we need to show that $VWV^{-1} = W^T$. Evidently, $VxV^{-1} = y^T$ and $VyV^{-1} = x^T$. Using that $\varepsilon_i = \varepsilon_{\alpha - i}$ we then see that

$$\begin{aligned} VWV^{-1} &= (x^T)^{\varepsilon_1}(y^T)^{\varepsilon_2} \dots (x^T)^{\varepsilon_{\alpha-2}}(y^T)^{\varepsilon_{\alpha-1}} \\ &= (x^T)^{\varepsilon_{\alpha-1}}(y^T)^{\varepsilon_{\alpha-2}} \dots (x^T)^{\varepsilon_2}(y^T)^{\varepsilon_1} \\ &= W^T \end{aligned}$$

and we have the desired. □

Computing

$$\rho(xW) - \rho(Wy) = \begin{pmatrix} W_{21} - tW_{12} & (M - M^{-1})W_{12} + W_{22} \\ -(M - M^{-1})W_{21} - tW_{22} & 0 \end{pmatrix}$$

we see that by Lemma 5.13 this vanishes if

$$p(M, t) = (M - M^{-1})W_{12} + W_{22} = 0.$$

To get the L-coordinate in D_K, we need the upperleft entry of the matrix $\rho(W\overline{W}x^{-2\sum \varepsilon_i})$. Call this polynomial $q(M, L)$.

From this we can produce $A'_K(L, M) = (L - 1)^{-1}A_K(L, M)$ as the resultant of $M^r p(M, t)$ and $M^s(q(M, t) - L)$ with respect to t, where r and s are chosen to clear negative powers of M. One should clear any monomial and repeated factors from A'_K to get the unique expression.

For twist knots, Hoste and Shanahan proved a recursive formula for the A-polynomial in [HS]. Their theorem is

Theorem 5.14 ([HS]). *For $p \neq -1, 0, 1, 2$, the A-polynomial for the K_p twist knot is given recursively by*

$$A'_{K_p}(L,M) = xA'_{K_{p-p/|p|}}(L,M) - yA'_{K_{p-2p/|p|}}(L,M),$$

where

$$\begin{aligned} x &= L^2(1+M^4) + L(-1+2M^2+2M^4+2M^6-M^8) + M^4 + M^8 \\ y &= M^4(L+M^2)^4, \end{aligned}$$

with initial conditions

$$\begin{aligned} A'_{K_2}(L,M) &= L^3 + L^2(-1+2M^2+2M^4-M^8+M^{10}) \\ &\quad + L(M^4-M^6+2M^{10}+2M^{12}-M^{14}) + M^{14} \\ A'_{K_1}(L,M) &= L + M^6 \\ A'_{K_0}(L,M) &= 1 \\ A'_{K_{-1}}(L,M) &= L^2M^4 + L(-1+M^2+2M^4+M^6-M^8) + M^4. \end{aligned}$$

6 TQFT and the Coloured Jones Polynomial

In this chapter we introduce Topological Quantum Field Theory in the spirit or Blanchet, Habegger, Masbaum and Vogel [BHMV2]. This is done by producing a quantum invariant of three-manifolds from Skein theory, which has been shown to give rise to a TQFT via the *universal construction*. We will describe just enough Skein theory to formulate the invariant. At the core of this invariant lies an invariant of links which generalizes the Jones polynomial, called the coloured Jones polynomial. We conclude the chapter by proving a new formula for the coloured Jones polynomial of double twist knots.

6.1 Topological Quantum Field Theory

We will now descibe a Topological Quantum Field Theory, as is defined in [BHMV2]. This is very closely related to the Atiyah-Segal axioms for TQFT (see [Ati2]). It is build from a *quantization functor*. Choose a cobordism category $\mathcal{C}$, with oriented $d-1$-dimensional manifolds (maybe with some extra structure) as objects and oriented d-manifolds bounded by $d-1$-manifolds as morphisms. Also, we let disjoint union be a monoidal structure on objects with unit object $\emptyset$. Let $\mathcal{Z}\colon \mathcal{C} \to \mathrm{Vect}_{\mathbb{C}}$ be a functor to the category of complex vector spaces. The notation is the following:

If M is a cobordism from Σ_1 to Σ_2, then $\mathcal{Z}_M\colon \mathcal{Z}(\Sigma_1) \to \mathcal{Z}(\Sigma_2)$. Let $\mathcal{Z}$ satisfy

$$\mathcal{Z}(\emptyset) = \mathbb{C}.$$

Then, if M is a cobordism from $\emptyset$ to ∂M, then $\mathcal{Z}(M) = \mathcal{Z}_M(1) \in \mathcal{Z}(\partial M)$. If M is closed then $\mathcal{Z}(M) \in \mathbb{C}$ and we denote the number $\tau(M)$, called the quantum invariant of M.

Definition 6.1. A *quantization functor* is a functor $\mathcal{Z}$ as above from a cobordism category $\mathcal{C}$ to $\mathrm{Vect}_{\mathbb{C}}$ with a sesquilinear form $\langle \cdot, \cdot \rangle_\Sigma$ on $\mathcal{Z}(\Sigma)$ for every $\Sigma \in \mathcal{C}$ such that for corbordisms M_1, M_2 with $\partial M_1 = \partial M_2 = \Sigma$ we get

$$\langle \mathcal{Z}_{M_1}, \mathcal{Z}_{M_2} \rangle_\Sigma = \tau(M_1 \cup_\Sigma -M_2) \in \mathbb{C}.$$

We say that a quantization functor is *cobordism generated* if the elements $\mathcal{Z}(M)$, with $\partial M = \Sigma$, generate $\mathcal{Z}(\Sigma)$. The quantum invariant τ is called *multiplicative* if

$$\tau(M_1 \sqcup M_2) = \tau(M_1)\tau(M_2)$$

and *involutive* if

$$\tau(-M) = \overline{\tau(M)}.$$

Now, given a quantization functor, the corresponding quantum invariant is multiplicative and involutive. The converse is also true.

Proposition 6.2. *If τ is a multiplicative and involutive invariant of closed cobordisms of $\mathcal{C}$, the there is a unique cobordism generated quantization functor extending it.*

The universal construction of the functor from the quantum invariant τ is the following:

Let $V(\Sigma)$ denote the $\mathbb{C}$-vector space generated by all cobordisms M with $\partial M = \Sigma$ (as a cobordism from $\emptyset$ to Σ). Then we define our sesquilinear form on $V(\Sigma)$

$$\langle M_1, M_2 \rangle = \tau(M_1 \cup_\Sigma M_2).$$

Let

$$N(\Sigma) = \{M \in V(\Sigma) \mid \langle M, M' \rangle = 0, \forall M' \in V(\Sigma)\}$$

and define

$$\mathcal{Z}(\Sigma) = V(\Sigma)/N(\Sigma).$$

The morphisms are defined by gluing, i.e. given M a cobordism from Σ_1 to Σ_2, then for $M' \in \mathcal{Z}(\Sigma_1)$

$$\mathcal{Z}_M(M') = M' \cup_{\Sigma_1} M \in \mathcal{Z}(\Sigma_2).$$

There are natural maps $\mathcal{Z}(-\Sigma) \to \mathcal{Z}(\Sigma)^*$ and $\mathcal{Z}(\Sigma_1) \otimes \mathcal{Z}(\Sigma_2) \to \mathcal{Z}(\Sigma_1 \sqcup \Sigma_2)$. We say that the quantization functor is involutive if the former is an isomorphism and multiplicative if the latter in an isomorphism. Also, we can impose a finiteness condition (F) saying that $\mathcal{Z}(\Sigma)$ is of finite rank and $\langle \cdot, \cdot \rangle_\Sigma$ induces an isomorphism $\mathcal{Z}(\Sigma) \to \mathcal{Z}(\Sigma)^*$.

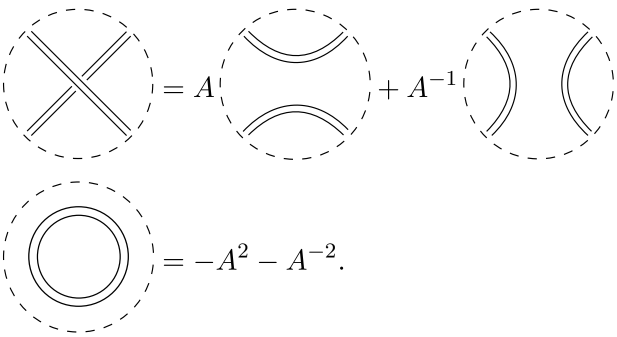
Definition 6.3. A Topological Quantum Field Theory on a cobordism category $\mathcal{C}$ is an involutive, multiplicative cobordism generated quantization functor satisfying the property (F).

We are interested in a $(2+1)$-dimensional TQFT, that is, the objects are surfaces and the cobordisms are three-manifolds. By the above proposition, we can construct such a functor by finding an involutive and multiplicative three-manifold invariant.

Such quantum invariants was introduced by Witten in [Wit] in the context of quantum Chern-Simons theory and its relation to the Jones polynomial of links. Witten's invariants were constructed by Reshetikhin and Turaev ([RT1], [RT2]) by the use of the representation theory of $\mathfrak{sl}_2$-quantum groups at a root of unity. Blanchet, Habegger, Masbaum and Vogel gave a skein theoretical construction of the quantum invariant in [BHMV1] based on work by Lickorish ([Lic]). These were shown to produce a TQFT in ([BHMV2]). Also, Kirby and Melvin gave a very nice presentation of invariants by Reshetikhin and Turaev in [KM1].

6.2 Quantum Invariants from Skein Theory

We will present the quantum invariant from [BHMV1]. Let M be a compact, oriented three-manifold. By a banded link in M we mean a link in M provided with a framing on each component, that is, an isotopy class of disjoint embedded annuli in M. Let A be an indeterminate and define the *Kauffman bracket skein module of* M $K(M)$ as the $\mathbb{Z}[A, A^{-1}]$-module generated by banded links in M, subject to the so-called skein relations



Given a banded link $L \subset M$ we denote the image of L in $K(M)$ by $\langle L \rangle$. This is called the Kauffman bracket. It can be shown that

(i) $K(S^3) \cong \mathbb{Z}[A, A^{-1}]$ by letting the empty link have value 1.

(ii) The skein module for the solid torus is $K(S^1 \times I \times I) = \mathbb{Z}[A, A^{-1}]z$, where z is the banded link given by $S^1 \times [\frac{1}{4}, \frac{3}{4}] \times \{\text{pt}\}$.

We denote the latter by $\mathcal{B} = K(S^1 \times I \times I)$. Now, given a banded link $L = L1 \sqcup \cdots \sqcup L_n \subset S^3$, we can supply each component by a tubular neighborhood to get n disjoint copies of a (knotted) solid torus, where we indentify L_i with the element z. The skein module of the disjoint union of the tori is simply $\mathcal{B}^{\otimes n}$ and so L induces a multilinear map

$$\langle \cdot, \dots, \cdot \rangle_L \colon \mathcal{B}^n \to \mathbb{Z}[A, A^{-1}]$$

which we will refer to as the *meta-bracket.* This meta-bracket acts as follows. Given a collection of monomials $z^{a_1}, \dots, z^{a_n}$, the meta-bracket $\langle z^{a_1}, \dots, z^{a_n} \rangle_L$ is the Kauffman bracket of the banded link obtained from L by replacing L_i by a_i parallel copies. We say that the component L_i is coloured by z^{a_i}.

We will now describe the necessary ingredients to create a three-manifold invariant from the Kauffman bracket. Let t be the map on $\mathcal{B}$ induced from a full positive twist of the torus. Also, let c be the map that adds an annulus coloured by z linking once with the element of $\mathcal{B}$ it acts on.

Lemma 6.4 ([BHMV1])**.** *There is a family e_i of eigenvectors for t and c given by $e_0 = 1$, $e_1 = z$ and $e_n = ze_{n-1} - e_{n-2}$, for $n \geq 2$ with*

$$t(e_n) = \mu_n e_n \quad \textit{and} \quad c(e_n) = \lambda_n e_n,$$

where

$$\mu_n = (-1)^n A^{n^2-2n} \quad \textit{and} \quad \lambda_n = -(A^{2n+2} + A^{-2n-2}).$$

Now, let Λ be an integral domain containing a homomorphic image of $\mathbb{Z}[A, A^{-1}]$ and let φ_d be the d-th cyclotomic polynomial. We define the quotient ring $\Lambda_p = \mathbb{Z}[A, A^{-1}]/\varphi_{2p}(A)$ and change coefficients in $\mathcal{B}$ to get

$$\mathcal{B}_p = \mathcal{B} \otimes \Lambda_p.$$

We fix the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{B}$ induced by the zero-framed Hopf link. As in the universal construction above we let

$$N_p = \{u \in \mathcal{B}_p \mid \langle u, v \rangle = 0 \, \forall v \in \mathcal{B}_p\}.$$

This allows us to define the Λ_p-algebra $V_p = \mathcal{B}_p/N_p$. It is also a free Λ_p-module of rank $n(p) = \lfloor \frac{p-1}{2} \rfloor$. From now on assume that p is even and let $p = 2r$, $r \geq 2$. Then $n(p) = r - 1$ and V_p has basis $e_0, \dots, e_{r-2}$. As is shown in [BHMV1], the twist map t preserves N_p and thus it induces a map on V_p. In V_p we define the element

$$\Omega_p = \sum_{i=0}^{r-2} \langle e_i \rangle e_i.$$

Now we can define the quantum invariant.

Theorem 6.5 ([BHMV1]). *Let M be a three-manifold and let $L \subset S^3$ be a banded link such that surgery on L yields M. Then*

$$\tau_r(M) = \frac{\langle \Omega_{2r}, \dots, \Omega_{2r} \rangle_L}{\langle t(\Omega_{2r}) \rangle^{b_+(L)} \langle t^{-1}(\Omega_{2r}) \rangle^{b_-(L)}} \in \Lambda_{2r}[\tfrac{1}{2r}]$$

is an invariant of M, where $b_+(L)$ ($b_-(L)$) is the number of positive (resp. negative) eigenvalues of the linking matrix of L.

Also, they prove

Theorem 6.6 ([BHMV2]). *Let $\Lambda = \mathbb{C}$ and $A = e^{\frac{\pi i}{2r}}$. The quantization functor induced by the quantum invariant τ_r is a TQFT.*

For completeness sake, we remark that the cobordism category for which this gives a TQFT is extended by so-called p_1-structures to get rid of anomalies.

Remark 6.7. Andersen and Ueno has made a correspondence in a series of papers ([AU1],[AU2],[AU3],[AU4] building on [TUY1]) between this TQFT and the one proposed by Witten. Concretely, letting $r = k + 2$, the two-dimensional part of the BHMV-construction agrees with the quantum spaces from Chapter 2, when geometrically quantizing the moduli space of flat SU(2)-connections. We call k the level of the quantization and denote the corresponding TQFT functor by $\mathcal{Z}^k$.

The Kauffman bracket of the e_n can be calculated to be

$$\langle e_n \rangle = (-1)^n [n+1]$$

where $[n+1]$ is the quantum integer defined by

$$[k] = \frac{A^{2k} - A^{-2k}}{A^2 - A^{-2}}.$$

So, by this we can express the quantum invariant as

$$\tau_r(M) = \alpha(L) \sum_{k_1=1}^{r-1} \cdots \sum_{k_n=1}^{r-1} \prod_{i=1}^{n} (-1)^{k_i-1} [k_i] \langle e_{k_1-1}, \dots, e_{k_n-1} \rangle_L$$

where $C_{r,n} = \{1, \dots, r-1\}^n$ and $\alpha(L)$ is the normalization factor of τ_r coming from the denominator. At the heart of the quantum invariant is an invariant of the banded link.

Definition 6.8. The *coloured Jones polynomial* of a banded link $L = L_1 \sqcup \cdots \sqcup L_n$ with colour $(k_1, \dots, k_n)$ is

$$J_L(k_1, \dots, k_n) = (-1)^{\sum_i (k_i - 1)} \langle e_{k_1-1}, \dots, e_{k_n-1} \rangle_L \in \mathbb{Z}[A, A^{-1}].$$

This invariant generalizes the Jones polynomial, in the sense that when all colours are 2, it gives the original Jones polynomial. This invariant was constructed by Reshetikhin and Turaev in [RT1] and a colour k_i can be interpreted as the unique irreducible k_i-dimensional representation of $\mathfrak{sl}_2$. Furthermore, Andersen notes that it detects the unknot [And3].

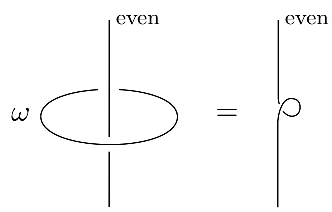
Remark 6.9. In the above TQFT, the vector space associated to a genus one surface is exactly V_p. As mentioned, this has the basis $e_0, \ldots, e_{r-2}$. Now, considering the complement of a knot $X_K = S^3 \setminus N(K)$ as in Section 5.3, we see that the TQFT yields a vector in V_p. The coefficient (up to an overall normalization) of e_{n-1} is the coloured Jones polynomial of colour n, and so we can write

$$\mathcal{Z}^k(X_K) = \sum_{n=1}^{k+1} J_K(n; A = e^{\frac{\pi i}{2(k+2)}}) e_{n-1}. \tag{6.1}$$

6.3 A Formula for the Coloured Jones Polynomial of Double Twist Knots

In the following, let $q = A^4$. In [Mas], Masbaum produces a closed formula for the coloured Jones polynomial of twist knots. We observe, that this construction can be modified to include all double twist knots. We follow the notation of [Mas] closely, and refer the reader to this paper for details on many of the formulas, as we only state what is needed here.

We are searching for an element ω such that when encircling an even number of strands it adds a full twist.



This can also be expressed as

$$\langle \omega, x \rangle = \langle t(x) \rangle,$$

for any even element $x \in \mathcal{B}$. We can equip $\mathcal{B}$ with the basis

$$R_n = \prod_{i=0}^{n-1} (z - \lambda_{2i}).$$

By definition of c we observe

$$\langle z^k, e_i\rangle = \langle 1, c^k(e_i)\rangle = \lambda_i^k \langle e_i\rangle.$$

We see directly that $\langle R_n, e_{2k}\rangle = 0$ if $k < n$. Since e_{2k} consist only of even powers of z, we also get

$$\langle R_n, z^{2k}\rangle = 0 \tag{6.2}$$

when $k < n$. For later use, we record the fact that

$$\langle R_n, e_{2n}\rangle = (-1)^n \frac{\{2n+1\}!}{\{1\}} \tag{6.3}$$

where $\{n\} = A^{2n} - A^{-2n}$. And so in this basis we are seaching for coefficients such that

$$\omega = \sum_{n=0}^{\infty} c_n R_n.$$

We also want to consider powers of ω, to get more twists, i.e.

$$\langle \omega^p, x\rangle = \langle t^p(x)\rangle, \quad p \in \mathbb{Z}$$

and find coefficients such that

$$\omega^p = \sum_{n=0}^{\infty} c_{n,p} R_n.$$

Following [Mas] we let $R'_n = (\{n\}!)^{-1} R_n$ and thus $c'_{n,p} = \{n\}! c_{n,p}$ to let $\omega^p = \sum_{n=0}^{\infty} c'_{n,p} R'_n$. These coefficients are computed in [Mas].

Assume $p \geq 1$. First, we consider a vector $\underline{k} = (k_1, \dots, k_p)$ such that $k_i \geq 0$ and $\sum k_i = n$. From this we can define

$$\varphi(\underline{k}) = \frac{1}{2} \sum_{i=1}^{p-1} (n - s_i)(2n - s_i - s_{i-1} + 2) \tag{6.4}$$

where $s_i = \sum_{l=1}^{i} k_l$. Second, we define

$$\begin{bmatrix} n \\ \underline{k} \end{bmatrix} = \frac{[n]!}{[k_1]! \cdots [k_p]!}$$

to write the expression

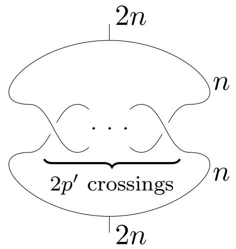
$$c'_{n,p} = (-1)^n q^{\frac{n(n+3)}{4}} \sum_{\underline{k}} q^{\varphi(\underline{k})} \begin{bmatrix} n \\ \underline{k} \end{bmatrix} \tag{6.5}$$

for $p \geq 1$. For negative p the conversion goes as

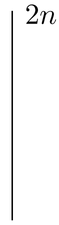
$$c'_{n,-p} = (-1)^n \overline{c'_{n,p}}.$$

We now wish to extend a formula for the coloured Jones polynomial for twist knots given in [Mas] to the double twist knots $K_{p,p'}$ as in Definition 5.9. This will make use of the concept of *admissibly coloured trivalent graphs* as in [MV], but since all the formulas needed are stated in [Mas], we will use this as a 'black box'. An essential formula we will need is

$\mu_n^{2p'}$



$= (-1)^n\{n\}c'_{n,p'}$



Recall the coloured Jones polynomial of a knot $K \subset S^3$

$$J_K(k) = (-1)^{k-1}\langle e_{k-1}\rangle.$$

By using the recursive definition of e_n, these can be written in the basis R_n as

$$e_{k-1} = \sum_{n=0}^{k-1}(-1)^{k-1-n}\begin{bmatrix} k+n \\ k-1-n \end{bmatrix} R_n. \tag{6.6}$$

We are now ready to state and prove the theorem, a corollary to the theorem of Masbaum.

Theorem 6.10. *The coloured Jones polynomial of the double twist knot $K_{p,p'}$, $p, p' \in \mathbb{Z}$ is given by*

$$J_{K_{p,p'}}(k) = \sum_{n=0}^{k-1}(-1)^n \frac{1-q^k}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} q^{-nk-\frac{k}{2}} c'_{n,p} c'_{n,p'} \prod_{l=1}^{n}(1-q^{k-l})(1-q^{k+l}). \tag{6.7}$$

Proof. We use the surgery description of $K_{p,p'}$ in Figure 6.1.

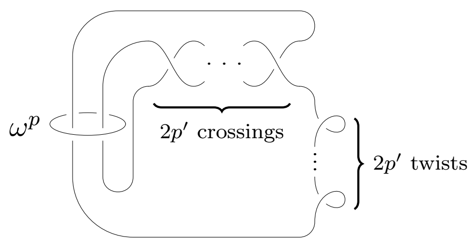
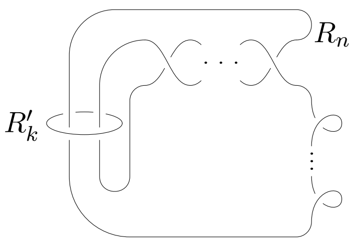


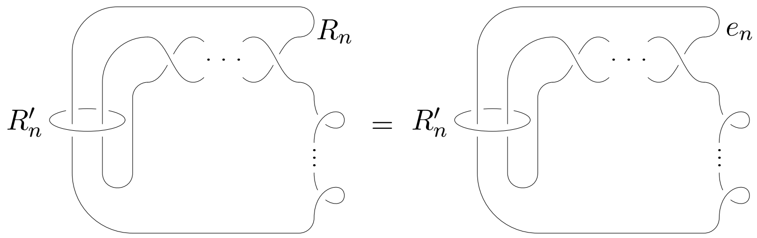
Figure 6.1: Surgery description for unframed $K_{p,p'}$.

The crucial observation made in [Mas] (referring to Lê) is that



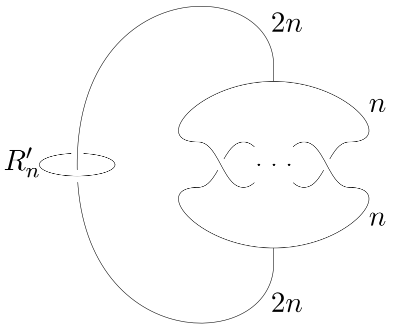
$$= 0 \tag{6.8}$$

when $k \neq n$. This is by (6.2), since circling with R_n annihilates all even polynomials z^{2k} when $k < n$. Both components are unframed unknots having spanning disks pierced twice by the other component. Also, since both R_n and e_n are monic polynomials of degree n, we get that



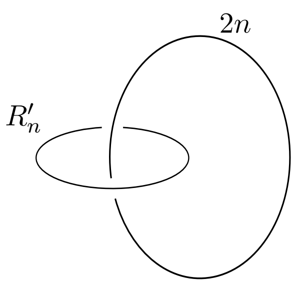
Each twist gives rise to a μ_n and so, in the notation of [MV] we get

$\mu_n^{2p'}\cdot$



and so by (6.8) we can write it as

$(-1)^n\{n\}!c'_{n,p'}\cdot$



$$= (-1)^n\{n\}!c'_{n,p'}\langle R'_n, e_{2n}\rangle$$

$$= c'_{n,p'}\frac{\{2n+1\}!}{\{1\}}$$

by (6.3). Combining this with (6.6) we get the equation

$$
\begin{aligned}
J_{K_{p,o}}(k) &= \sum_{n=0}^{k-1} (-1)^n \begin{bmatrix} k+n \\ k-1-n \end{bmatrix} c'_{n,p} c'_{n,o} \frac{\{2n+1\}!}{\{1\}} \\
&= \sum_{n=0}^{k-1} (-1)^n c'_{n,p} c'_{n,p'} \frac{\{k+n\} \cdots \{k-n\}}{\{1\}}.
\end{aligned}
$$

Using that $\{l\} = -q^{-\frac{l}{2}}(1-q^l)$ we get the desired formula. $\square$

7

The AJ Conjecture

In this chapter we will discuss various conjecures related to the AJ conjecture of Garoufalidis [Gar] . The AJ conjectures relate the coloured Jones polynomial and the A-polynomial of the two previous chapters. We will formulate different types of such conjectures, the original algebraic AJ conjecture, geometric versions, and a more general conjecture in TQFT. One of the geometric conjectures will arise from Gukov's Generalized Volume Conjecture [Guk]. Then we will observe that the Toeplitz operators of Chapter 4 corresponding to the A-polynomial are well-suited to formulate another geometric version of the AJ conjecture. Finally, we go out on a limb and give a very general conjecture in TQFT. At the end of the chapter we prove the Toeplitz operator-version for the unknot.

7.1 The Algebraic AJ Conjecture

The starting point of the algebraic AJ conjecture is an action of a *non-commutative torus* on the coloured Jones polynomial. From this we construct the *non-commutative A-polynomial* $\hat{A}_K$. For this, it is convenient to write the coloured Jones polynomial as a function

$$J_K\colon \mathbb{N} \to \mathbb{Z}[q^{\pm\frac{1}{4}}]$$

or even as a formal power series

$$\mathcal{J}_K(h) = \sum_{n=1}^{\infty} J_K(n) h^n. \tag{7.1}$$

Let $\mathcal{A} = \mathbb{Z}[q^{\pm}]\langle E, Q\rangle \big/ EQ = qQE$ be what we will call the non-commutative torus (also known as the q-Weyl algebra). We let E and Q act on J_K by

$$E(J_K)(n) = J_K(n+1) \qquad Q(J_K)(n) = q^n J_K \tag{7.2}$$

and see that this gives an action of $\mathcal{A}$. Now we can ask if any elements of $\mathcal{A}$ annihilates J_K and thus form the annihilation ideal

$$\mathcal{I}_K = \{P \in \mathcal{A} \mid PJ_K = 0\} \subseteq \mathcal{A}.$$

It was shown in [GL] that this ideal is non-trivial by showing that the coloured Jones function is *q-holonomic*, which in our case precisely states that $\mathcal{I}_K \neq 0$. The $\hat{A}_K$ is should now be defined as a generator for $\mathcal{I}_K$. But this would require that $\mathcal{I}_K$ is a principal left ideal, which in general it is not. A way to fix this is by inverting polynomials in Q by going to the *Ore algebra* $\mathcal{A}_{\text{loc}}$. This is an algebra over the field of fractions $\mathbb{Q}(q, Q)$, which additively is

$$\mathcal{A}_{\text{loc}} = \left\{ \sum a_k E^k \mid a_k \in \mathbb{Q}(q, Q), \quad a_k = 0 \text{ for } k >> 0 \right\}$$

and has multiplication given by

$$aE^k \cdot bE^l = a\sigma^k(b)E^{k+l}, \qquad \sigma(f)(q, Q) = f(q, qQ)$$

on monomials. It turns out that every left ideal over $\mathcal{A}_{\text{loc}}$ is principal (see reference in [Gar]). To define the annihilation ideal over this algebra, consider the ring

$$\mathcal{F} = \{f \colon \mathbb{N} \to \mathbb{Q}(q)\} \,/ \sim$$

where $f \sim g$ if f and g agree everywhere but a finite set. $\mathcal{A}_{\text{loc}}$ acts on this ring and thus we can define the annihilation ideal $\tilde{\mathcal{I}}_K$ of a knot in this setting. It turns out that $\mathcal{I}_K \neq 0$ if and only if $\tilde{\mathcal{I}}_K \neq 0$. As these are proven not to vanish, we can find a generator for $\tilde{\mathcal{I}}_K$, denoted $\hat{A}_K$, satisfying

- $\hat{A}_K \in \mathcal{A}$
- $\hat{A}_K$ has minimal E-degree
- We can write $\hat{A}_K = \sum a_k E^k$ where $a_k \in \mathbb{Z}[q, Q]$ are coprime.

This is called the *non-commutative A-polynomial.*

We now wish to relate this polynomial to the A-polynomial of Section 5.3 as limit when $q \to 1$. For polynomials $f, g \in \mathbb{C}[L, M]$, we say that polynomials f and g are *M-essentially equal*, denoted by $f \overset{M}{=} g$, if f/g does not depend on L. Similarly, two algebraic sets in $\mathbb{C}^2$ with coordinates L and M are M-essentially equal if they are the same up to adding lines parallel to the L-axis. Clearly, if $f \overset{M}{=} g$ then $\{f = 0\} \overset{M}{=} \{g = 0\}$.

We can now formulate the AJ conjecture of Garoufalidis ([Gar]), which he proved in the same paper for the trefoil and the figure eight knot.

Conjecture 7.1 (AJ conjecture)**.** *For every knot $K \subset S^3$, $A_K(L, M) \overset{M}{=} \hat{A}_K(L, M^2)_{|q=1}$.*

This have also been proven true for torus knots by Hikami in [Hik1] and for twist knots (and some other 2-bridge knots) by T. T. Q. Lê in [Lê], as the following theorem states.

Theorem 7.2 ([Lê])**.** *For the two-bridge knot $K = K(\alpha, \beta)$ the algebraic set $\{(\hat{A}_K)_{|q=1} = 0\}$ is M-essentially equal to an algebraic subset of the closure of the deformation variety $\overline{D_K}$.*

Furthermore, if $A_K/(L-1)$ is irreducible over $\mathbb{Z}$ and the L-degree is $\frac{\alpha-1}{2}$ then Conjecture 7.1 holds.

By Theorem 5.14 the L-degree of positive twist knots $K_p = K(4p-1, 2p-1)$ is $2p-1$ and for negative twist knots $K_{-p} = K(4p+1, 2p+1)$ is $2p$. Hoste and Shanahan also show irreducibility of A'_{K_p} in [HS], and so the above theorem proves the AJ conjecture for twist knots. It should also be noted that the proof does not rely on explicit formulas for the coloured Jones polynomial as opposed to previous results.

7.2 Geometric AJ Conjectures

The AJ conjecture above relates the annihilation of the coloured Jones polynomial with the zero-locus of the A-polynomial. We will in this section formulate related conjectures in a more geometric setting. Around the same time as Garoufalidis formulated his AJ conjecture, Gukov [Guk] made his Generalized Volume Conjecture, which contains a more geometric picture.

It studies the asymptotic behaviour of the coloured Jones polynomial $J_K(k)$ at $q = ^{\frac{2\pi i}{r}}$ as both k and r goes to infinity. This is, however, done in a controlled manor, as the ratio $a = \frac{k}{r}$ is fixed in the limit. Allowing colours from $\mathbb{R}$, we can view this as

$$\lim_{r\to\infty} J_K(ra; q = e^{\frac{2\pi i}{r}}), \qquad a \in]0,1[.$$

The conjecture made was for hyperbolic knots and carried information of the value of the limit. We introduce the generalizations of the hyperbolic volume and the Chern-Simons invariant of the knot complement to the deformation variety

$$\begin{aligned} \mathrm{Vol}(L,M) &= \mathrm{Vol}(X_K) + 2\int (-\log|L| d(\arg M) + \log|M| d(\arg L)) \\ \mathrm{Vol}(L,M) &= \mathrm{CS}(X_K) - \frac{1}{\pi^2}\int (\arg|L| d(\arg M) + \arg|M| d(\arg L)), \end{aligned}$$

where $A_K(L,M) = 0$.

Conjecture 7.3 (Generalized Volume Conjecture)**.** *For a (hyperbolic) knot $K \in S^3$ the limit of the coloured Jones polynomial is*

$$\lim_{r\to infty} \frac{\log J_K(ra; q = e^{\frac{2\pi i}{r}})}{r} = \frac{1}{2\pi}(\mathrm{Vol}(L,M) + i2\pi^2\,\mathrm{CS}(L,M)),$$

where $A_K(L,M) =$ and $M = e^{\pi i a}$, $L = \frac{\partial}{\partial a} \lim_{r\to\infty} \frac{\log J_K(ra; q=e^{\frac{2\pi i}{r}})}{r}$.

The actual value of the limit will not be studied here, but we will try to recover the A-polynomial though an analysis of the left-hand side for not necessarily hyperbolic knots in Chapter 8.

Now we turn to geometric quantization and TQFT. We will start by discussing a unified setting for the coloured Jones polynomial and the A-polynomial. As mentioned in Remark 5.12, we can view the deformation variety D_K as a subvariety of the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$-connections on a torus. Inside this moduli space lies the SU(2)-moduli space of a torus, which we discussed a bit in Section 4.2. This was the $\mathbb{Z}_2$-quotient of the torus $\mathrm{U}(1)\times\mathrm{U}(1)$. The qoutient map is the map t as in Section 5.3.

$$\begin{tikzcd} \mathrm{U}(1)\times\mathrm{U}(1) \arrow[r, hook] \arrow[d, "t"'] & \mathbb{C}^*\times\mathbb{C}^* \arrow[d, "t"] \\ \mathcal{M}_{\mathrm{SU}(2)}(T) \arrow[r, hook, "i"] & \mathcal{M}_{\mathrm{SL}(2,\mathbb{C})}(T) \end{tikzcd}$$

From this diagram we can form the Weyl-invariant subset

$$\hat{D}_K = t^{-1}(i^{-1}t(D_K)) \in \mathrm{U}(1)\times\mathrm{U}(1).$$

This is a restriction from an $\mathrm{SL}(2,\mathbb{C})$ theory to an SU(2) theory.

Question 7.4. Does $\hat{D}_K$ determine the A-polynomial?

As mentioned in Chapter 6, the TQFT from the BHMV-construction is the same as the one arising from geometric quantization. In particular, the vector space $\mathcal{Z}^k(T)$ is the same as $H^0(T,\mathcal{L}^{2k})^W$. This means that the TQFT boundary vector of the complement of a knot $\mathcal{Z}^k(X_K)$ can be viewed as a holomorphic section of $\mathcal{L}^{2k}$ over T. If we make the correspondence $e_{n-1}\mapsto\psi_{n,k}$, we can use the description (6.1) to write the section

$$\mathcal{Z}^k(X_K) = \sum_{n=1}^{k+1} J_K(n; q = e^{\frac{2\pi i}{k+2}})\psi_{n,k} \in H^0(T,\mathcal{L}^{2k})^W,$$

resembling (7.1). This correspondence is consistent with the original interpretation of Witten (see Appendix A.3 in [Jef]). The conjecture is now

Conjecture 7.5. *The family of functions P^k on $T=\mathrm{U}(1)\times\mathrm{U}(1)$ given by*

$$P^k(z) = \frac{\|\mathcal{Z}^k(X_K)(z)\|}{\max_{x\in T}\|\mathcal{Z}(X_K)(x)\|}$$

satisfies $\lim_{k\to\infty} P^k(z) = 0$ if and only if $z\in\hat{D}_K$.

One could take this a step further, and consider any three-manifold M with boundary $\partial M = \Sigma$. The TQFT vector space associated to Σ is by Chapter 2

$\mathcal{Z}^k(\Sigma) = H^0(\mathcal{M}_{\mathrm{SU}(2)}(\Sigma), \mathcal{L}^k)$ (or at least covariant constant holomorphic sections, with respect to the Hitchin connection) and we can make an even more general conjecture.

Conjecture 7.6. *The family of functions P_M^k on $\mathcal{M}_{\mathrm{SU}(2)}(\Sigma)$ given by*

$$P_M^k(z) = \frac{\|\mathcal{Z}^k(M)(z)\|}{\max_{x\in\mathcal{M}_{\mathrm{SU}(2)}(\Sigma)}\|\mathcal{Z}^k(M)(x)\|}$$

satisfies $\lim_{k\to\infty} P_M^k(z) = 0$ if and only if z corresponds to a flat $\mathrm{SU}(2)$-connection on Σ that extends to a flat $\mathrm{SU}(2)$-connection on M.

Let us investigate Conjecture 7.5 in connection with the original AJ conjecture. Since the polynomial $\hat{A}_K$ is conjectures to have A_K as the limit $q \to 1$, we could see it as a quantization of the A-polynomial. This inspires us to try and write down a Toeplitz operator as a quantization of the A-polynomial. First, we restrict the A-polynomial to a function on $\mathrm{U}(1)\times\mathrm{U}(1)$ and since $\hat{D}_K$ is Weyl invariant, we make the Weyl invariant A-polynomial

$$A_K^+(L, M) = A_K(L, M) + A_K(L^{-1}, M^{-1}).$$

Given a monomial term L^tM^s in A_K, we thus get it replaced by $L^tM^s + L^{-t}M^{-s}$. Since L and M are the two coordinates on the torus, we can represent them by the phase functions $F_{1,0}$ and $F_{0,1}$ from Chapter 4. This means that we make the correspondence

$$L^tM^s + L^{-t}M^{-s} \mapsto F_{t,s} + F_{-t,-s} =: F_{t,s}^+.$$

By Section 4.2.1, we see that the Toeplitz operators for $F_{t,s}^+$ preserves the Weyl anti-invariant space $H^0(T, \mathcal{L}^{2k+4})^{W_-}$, as is needed. For brevity, let $l = 2k+4 = 2r$. We compute the action of the Toeplitz operators for these functions on our basis.

$$\begin{aligned}
\hat{T}^{(l)}_{F^-_{t,s}}\theta^-_{n,l} &= \hat{T}^{(l)}_{F_{t,s}}\theta_{n,l} + \hat{T}^{(l)}_{F_{-t,-s}}\theta_{n,l}\\
&\quad - \hat{T}^{(l)}_{F_{t,s}}\theta_{l-n,l} - \hat{T}^{(l)}_{F_{-t,-s}}\theta_{l-n,l}\\
&= e^{-\frac{2\pi is}{l}(t+n)}\theta_{n+t,l} + e^{\frac{2\pi is}{l}(-t+n)}\theta_{n-t,l}\\
&\quad - e^{-\frac{2\pi is}{l}(t+l-n)}\theta_{l-n+t,l} - e^{\frac{2\pi is}{l}(-t+l-n)}\theta_{l-n-t,l}\\
&= e^{-\frac{2\pi is}{l}(t+n)}\theta^-_{n+t,l} + e^{-\frac{2\pi is}{l}(t-n)}\theta^-_{n-t,l}.
\end{aligned}$$

The action of E and Q on the coloured Jones polynomial in (7.2) induces the action on the Theta-functions given by

$$E\theta^-_{n,l} = \begin{cases}\theta^-_{n-1,l}, & n \geq 2\\ 0, & \gamma = 1\end{cases} \qquad Q\theta^-_{n,l} = q^n\theta^-_{n,l}.$$

By this we see that

$$\hat{T}^{(l)}_{F^+_{t,2s}}\theta^-_{n,l} = (Q^s E^t + Q^{-s}E^{-t})\theta^-_{n,l},$$

which fits perfectly with the correspondence $L \mapsto E$, $M^2 \mapsto Q$ as in Conjecture 7.1.

In this setting, we could the make the following conjecture, where $T^{(k)}_{A_K}$ is short-hand for the operator associated to the A-polynomials, described on monomials above.

Conjecture 7.7. *For any knot $K \subset S^3$ A_K there is an element $\hat{A}_K \in \mathcal{I}_K$ such that*

$$\|T^{(k)}_{A_K} - \hat{A}_K\| \in O(\tfrac{1}{k}).$$

In particular, this will imply this statement, which also shows the ties to TQFT.

Conjecture 7.8. *For any knot $K \subset S^3$ the A-polynomial A_K and the TQFT boundary vector $\mathcal{Z}^k(X_K)$ satisfy*

$$\|T^{(k)}_{A_K}\mathcal{Z}^k(X_K)\| \leq c(k)\|\mathcal{Z}^k(X_K)\|$$

where $c(k) \in O(\frac{1}{k})$.

We find these two conjectures very interesting, since a whole new set of tools from the world of Toeplitz operators, such as the techniques used by Andersen in e.g. [And1] could be applied here. In the following section, we will do a small example, where Conjecture 7.8 will be proved for the unknot.

7.3 Unknot and Toeplitz Operators

Recall the A-polynomial and coloured Jones polynomial for the unknot $U \subset S^3$

$$A_U(L, M) = L - 1 \qquad J_U(n) = [n].$$

We fix the level k and let $q = e^{\frac{2\pi i}{k+2}}$. It is not hard to see that if $m + n = k + 2$, then $[m] = [n]$. The norm of the TQFT boundary vector is

$$\|v^{(k)}\|^2 = \sum_{n=1}^{k+1}[n]^2,$$

where $\mathcal{Z}^{(k)}(X_U) = v^{(k)}$ for short. The Toeplitz operator for A_U has the matrix form

$$T_{A_U}^{(k)} = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}$$

and so

$$T_{A_U}^{(k)} v^{(k)} = \begin{pmatrix} -2[1]+[2] \\ [1]-2[2]+[3] \\ [2]-2[3]+[4] \\ \vdots \\ [k-1]-2[k]+[k+1] \\ [k]-2[k+1] \end{pmatrix}.$$

Computing the norm of this amounts to

$$\begin{aligned}
\|T_{A_U}^{(k)} v^{(k)}\|^2 &= \big(-2[1]+[2]\big)^2 \\
&\quad + \sum_{n=1}^{k-1} \big([n]-2[n+1]+[n+2]\big)^2 + \big([k]-2[k+1]\big)^2 \\
&= 4[1]^2 + [2]^2 - 4[1][2] \\
&\quad + [1]^2 + 4[2]^2 + [3]^2 - 4[1][2] + 2[1][3] - 4[2][3] \\
&\quad + [2]^2 + 4[3]^2 + [4]^2 - 4[2][3] + 2[2][4] - 4[3][4] \\
&\quad + [3]^2 + 4[4]^2 + [5]^2 - 4[3][4] + 2[3][5] - 4[4][5] \\
&\qquad \vdots \\
&\quad + [k-1]^2 + 4[k]^2 + [k+1]^2 - 4[k-1][k] \\
&\quad + 2[k-1][k+1] - 4[k][k+1] \\
&\quad + [k]^2 + 4[k+1]^2 - 4[k][k+1] \\
&= -[1]^2 + \sum_{n=1}^{k-1} [n]\big(6[n] - 8[n+1] + 2[n+2]\big) \\
&\quad + 6[k]^2 + 5[k+1]^2 - 8[k][k+1] \\
&= \sum_{n=1}^{k+1} [n]\big(6[n] - 8[n+1] + 2[n+2]\big)
\end{aligned}$$

where the last equation use the facts $[1] = [k+1] = -[k+3] = 1$ and $[k+2] = 0$.

Recall the Euler-Maclaurin formula

$$\sum_{j=a}^{n} f(j) = \int_a^n f(x)dx + \tfrac{1}{2}(f(n) + f(a)) \\ + \sum_{s=1}^{m-1} \frac{B_{2s}}{(2s)!}(f^{(2s-1)}(n) - f^{(2s-1)}(a)) + R_m(n)$$

where B_i is the i'th Bernoulli number and

$$|R_m(n)| \leq \frac{2}{(2\pi)^{m-1}} \int_a^n |f^{(m-1)}(x)|dx.$$

Let $g_{r,l}(x) = \sin \frac{\pi x}{r} \sin \frac{\pi(x+l)}{r}$. Then we see that

$$\sum_{\alpha=1}^{k+1} [\alpha][\alpha + l] = \frac{1}{\sin^2 \frac{\pi}{r}} \sum_{j=1}^{r-1} g_{r,l}(j).$$

Calculating derivatives of $g_{r,l}$ we find that

$$g_{r,l}^{(2s-1)}(x) = (-1)^{s-1} \frac{(2\pi)^{2s-1}}{2r^{2s-1}} \sin \tfrac{\pi(2x+l)}{r}.$$

So we can reduce the sum to

$$\sum_{j=1}^{r-1} g_{r,l}(j) = I_{r,l} + \sin \tfrac{\pi}{r} \sin \tfrac{\pi(l+1)}{r} \\ + \sum_{s=1}^{m-1} \frac{B_{2s}}{(2s)!}(-1)^{s-1} \frac{(2\pi)^{2s-1}}{2r^{2s-1}} \left(\sin \tfrac{\pi(l-2)}{r} - \sin \tfrac{\pi(l+2)}{r}\right) + R_m(r-1)$$

$$= I_{r,l} + \sin \tfrac{\pi}{r} \sin \tfrac{\pi(l+1)}{r} \\ + \sum_{s=1}^{m-1} \frac{B_{2s}}{(2s)!}(-1)^{s} \frac{(2\pi)^{2s-1}}{r^{2s-1}} \cos \tfrac{\pi l}{2} \sin \tfrac{2\pi}{r} + R_m(r-1)$$

where

$$I_{r,l} = \int_1^{r-1} \sin \tfrac{\pi x}{r} \sin \tfrac{\pi(x+l)}{r} dx.$$

By substitution and the addition formula for sine, we see that

$$I_{r,l} = \cos \tfrac{\pi l}{r} I_{r,0}.$$

From the above bound on the remainder term, we see that $|R_m(r-1)| \leq \frac{1}{r^{m-2}}$ so we find the asymptotic behaviour as

$$\frac{1}{\sin^2 \frac{\pi}{r}} \sum_{j=1}^{r-1} g_{r,l}(j) = \frac{\cos \frac{\pi l}{r}}{\sin^2 \frac{\pi}{r}} I_{r,0} + O(1).$$

The integral $I_{r,0}$ can be calculated to be

$$I_{r,0} = \frac{r}{2} - 1 + \frac{r}{\pi} \sin \tfrac{\pi}{r} \cos \tfrac{\pi}{r} \in \Theta(r)$$

and so

$$\frac{1}{\sin^2 \frac{\pi}{r}} \sum_{j=1}^{r-1} g_{r,l}(j) \in \Theta(r^3).$$

In particular, this shows that $\|v^{(k)}\|^2 \in \Theta(r^3)$. On the other hand

$$\begin{aligned}
\|T^{(k)}_{A_U} v^{(k)}\|^2 &= \sum_{\alpha=1}^{k+1} [\alpha]\big(6[\alpha] - 8[\alpha+1] + 2[\alpha+2]\big) \\
&= \frac{2}{\sin^2 \frac{\pi}{r}} \left(3 \sum_{j=1}^{r-1} g_{r,0}(j) - 4 \sum_{j=1}^{r-1} g_{r,1}(j) + \sum_{j=1}^{r-1} g_{r,2}(j) \right) \\
&= \frac{2}{\sin^2 \frac{\pi}{r}} \left(3 - 4 \cos \tfrac{\pi}{r} + \cos \tfrac{2\pi}{r}\right) I_{r,0} + O(1) \\
&= \left(\frac{8(1 - \cos \frac{\pi}{r})}{\sin^2 \frac{\pi}{r}} - 4 \right) I_{r,0} + O(1) \in O(r)
\end{aligned}$$

and we have the desired estimate.

8

Asymptotics of the Coloured Jones Polynomial for Double Twist Knots

In is chapter we address Conjecture 7.3, or at least the conjectured connection to the A-polynomial. We approach the formula in Theorem 6.10 by Faddeev's quantum dilogarithm and write the coloured Jones polynomial of double twist knots as a multiple contour integral. The hope is that this can be used to give a full asymptotic expansion of the coloured Jones polynomial. As a part of this, we investigate the leading order asymptotic behaviour of the integral and produce equations that, for twist knots, have been shown to give the A-polynomial.

8.1 The Coloured Jones Polynomial Using Quantum Dilogarithms

We start by introducing Faddeev's quantum dilogarithms

$$S_\gamma(\zeta) = \exp\left(\frac{1}{4}\int_{C_R} \frac{e^{\zeta z}}{\sinh(\pi z)\sinh(\gamma z) z} dz\right)$$

for $|\mathrm{Re}(\zeta)| < \pi + \gamma$, where $0 < \gamma < 1$ and C_R is the contour from $-\infty$ to ∞ going clock-wise around the half-circle of radius $R < 1$. S_γ was constructed to solve the functional equation

$$S_\gamma(\zeta - \gamma) = (1 + e^{i\zeta}) S_\gamma(\zeta + \gamma) \tag{8.1}$$

(see [Fad] and [AH] for a proof). This allows us to extend the function toa meromorphic function on the complex plane. Letting $\gamma = \frac{\pi}{r}$ for an integer $r > 3$, (8.1) implies that

$$S_\gamma(-\pi + 2\pi x) = (1 + e^{2\pi i x r}) S_\gamma(-\pi + 2\pi(x + 1)). \tag{8.2}$$

The function

$$x \mapsto S_\gamma(-\pi + 2\gamma x + \gamma)$$

is analytic on $\mathbb{C} \setminus \{r, r+1, \ldots\}$. It has poles of order m on the set

$$P_m = \{mr, mr+1, \ldots, mr+r-1\}$$

and zeroes of order m on

$$N_m = \{-mr, -mr+1, \ldots, -mr+r-1\}. \tag{8.3}$$

The functional equation (8.2) implies the formulas

$$\begin{aligned}
\prod_{l=0}^{n}(1-e^{\frac{2\pi i}{r}(k-l)}) &= \frac{S_\gamma(-\pi+2\gamma(k-n)-\gamma)}{S_\gamma(-\pi+2\gamma k+\gamma)} \\
\prod_{l=0}^{n}(1-e^{\frac{2\pi i}{r}(k+l)}) &= \frac{S_\gamma(-\pi+2\gamma k-\gamma)}{S_\gamma(-\pi+2\gamma(k+n)+\gamma)}.
\end{aligned}$$

We notice here that, if $k+n \geq r$, there is a pole in the denominator reflecting the fact that the left-hand side is zero in that case.

Introducing the notation $\tilde{S}_\gamma(z) = S_\gamma(-\pi+2\gamma z+\gamma)$ and using the above lets us write

$$\prod_{l=0}^{n}(1-e^{\frac{2\pi i}{r}(k-l)})(1-e^{\frac{2\pi i}{r}(k+l)}) = (1-e^{\frac{2\pi i}{r}k})\frac{\tilde{S}_\gamma(k-n-1)}{\tilde{S}_\gamma(k+n)}.$$

If we fix $q = e^{\frac{2\pi i}{r}}$, we can rewrite (6.7) from Theorem 7.2 as

$$\begin{aligned}
J_{K_{p,p'}}(k) &= \sum_{n=0}^{k-1}(-1)^n \frac{q^{-nk-\frac{k}{2}}}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})(1-q^k)} c'_{n,p}c'_{n,p'} \prod_{l=0}^{n}(1-q^{k-l})(1-q^{k+l}) \\
&= \frac{i}{2\sin(\frac{\pi}{r})} \sum_{n=0}^{k-1}(-1)^n q^{-nk-\frac{k}{2}} c'_{n,p}c'_{n,p'} \frac{\tilde{S}_\gamma(k-n-1)}{\tilde{S}_\gamma(k+n)},
\end{aligned} \tag{8.4}$$

where the coefficients $c'_{n,p}$ were defined in (6.5). The formula for $c'_{n,p}$ contained the quantum binomials $\left[\begin{smallmatrix} n \\ \underline{k} \end{smallmatrix}\right]$. These can be written in terms of the sums $s_i = k_1 + \cdots + k_i$ (writing $n = s_p$ for brevity) as

$$\begin{bmatrix} n \\ \underline{k} \end{bmatrix} = \prod_{i=1}^{p} \begin{bmatrix} s_i \\ s_{i-1} \end{bmatrix}.$$

Using the functional equation (8.2) one can calculate that

$$\begin{bmatrix} n \\ j \end{bmatrix} = q^{\frac{j(j-n)}{2}} \frac{\tilde{S}_\gamma(n-j)\tilde{S}_\gamma(j)}{\tilde{S}_\gamma(n)\tilde{S}_\gamma(0)}$$

and so we get

$$\begin{aligned}\begin{bmatrix} n \\ \underline{k} \end{bmatrix} &= q^{\tilde{\varphi}(\underline{k})} \prod_{i=1}^{p} \frac{\tilde{S}_\gamma(s_i - s_{i-1})\tilde{S}_\gamma(s_{i-1})}{\tilde{S}_\gamma(s_i)\tilde{S}_\gamma(0)} \\ &= q^{\tilde{\varphi}(\underline{k})} \frac{\tilde{S}_\gamma(n - s_{p-1}) \prod_{i=1}^{p-1} \tilde{S}_\gamma(s_i - s_{i-1})}{\tilde{S}_\gamma(n) \left(\tilde{S}_\gamma(0)\right)^{p-1}}\end{aligned}$$

where

$$\tilde{\varphi}(\underline{k}) = \frac{1}{2} \sum_{i=1}^{p} s_{i-1}(s_{i-1} - s_i).$$

Adding φ from (6.4) and $\tilde{\varphi}$ yields

$$\varphi(\underline{k}) + \tilde{\varphi}(\underline{k}) = (p-1)n(n+1) + \sum_{i=1}^{p-1} s_i(s_i - 2n - 1)$$

and so we get the formula

$$\begin{aligned} c'_{n,p} = (-1)^n q^{\frac{n(n+3)}{4}} &\sum_{s_{p-1}=0}^{n} \cdots \sum_{s_i=0}^{s_{i+1}} \cdots \sum_{s_1=0}^{s_2} q^{(p-1)n(n+1)+\sum_{i=1}^{p-1} s_i(s_i-2n-1)} \\ &\times \frac{\tilde{S}_\gamma(n - s_{p-1}) \prod_{i=1}^{p-1} \tilde{S}_\gamma(s_i - s_{i-1})}{\tilde{S}_\gamma(n) \left(\tilde{S}_\gamma(0)\right)^{p-1}}. \end{aligned}$$

We now wish to extend the sums in the above. Namely, consider the case where $s_i < s_{i-1}$, $i = 2, \ldots, p$. Then by (8.3) we see that the factor $\tilde{S}_\gamma(s_i - s_{i-1})$ vanishes, and thus we can extend all the sums to

$$\begin{aligned} c'_{n,p} = (-1)^n q^{\frac{n(n+3)}{4}} &\sum_{s_{p-1}=0}^{k-1} \cdots \sum_{s_1=0}^{k-1} q^{(p-1)n(n+1)+\sum_{i=1}^{p-1} s_i(s_i-2n-1)} \\ &\times \frac{\tilde{S}_\gamma(n - s_{p-1}) \prod_{i=1}^{p-1} \tilde{S}_\gamma(s_i - s_{i-1})}{\tilde{S}_\gamma(n) \left(\tilde{S}_\gamma(0)\right)^{p-1}}. \end{aligned} \tag{8.5}$$

Recall that $c'_{n,-p} = (-1)^n \overline{c'_{n,p}}$. To this end, notice that $\left[\begin{smallmatrix} n \\ \underline{k} \end{smallmatrix}\right]$ is real since the quantum integers are real. Thus

$$c'_{n,-p} = q^{-\frac{n(n+3)}{4}} \sum_{s_{p-1}=0}^{k-1} \cdots \sum_{s_1=0}^{k-1} q^{\tilde{\varphi}(\underline{k}) - \varphi(\underline{k})} \frac{\tilde{S}_\gamma(n - s_{p-1}) \prod_{i=1}^{p-1} \tilde{S}_\gamma(s_i - s_{i-1})}{\tilde{S}_\gamma(n) \left(\tilde{S}_\gamma(0)\right)^{p-1}}.$$

As before we find

$$\tilde{\varphi}(\underline{k}) - \varphi(\underline{k}) = -(p-1)n(n+1) - ns_{p-1} + \sum_{i=1}^{p-1} s_i(2n - s_{i-1} - 1)$$

and so

$$\begin{aligned} c'_{n,-p} = q^{-\frac{n(n+3)}{4}} \sum_{s_{p-1}=0}^{k-1} \cdots \sum_{s_1=0}^{k-1} q^{-(p-1)n(n+1)-ns_{p-1}+\sum_{i=1}^{p-1} s_i(2n-s_{i-1}-1)} \\ \times \frac{\tilde{S}_\gamma(n-s_{p-1}) \prod_{i=1}^{p-1} \tilde{S}_\gamma(s_i - s_{i-1})}{\tilde{S}_\gamma(n) \left(\tilde{S}_\gamma(0)\right)^{p-1}}. \end{aligned} \tag{8.6}$$

The idea is to write the formula for the coloured Jones polynomial as the sum of evaluations of analytic functions. Thus, we define the following functions, where $\underline{z} = (z_1, \ldots, z_{p-1})$

$$\begin{aligned} g(a,y) &= e^{\pi i r y} e^{\frac{2\pi i}{r}\left(-r^2 y a - \frac{ra}{2}\right)} \frac{\tilde{S}_\gamma(r(a-y)-1)}{\tilde{S}_\gamma(r(a+y))} \\ f_p(a,y,\underline{z}) &= e^{\pi i r y + \frac{2\pi i}{r}\left(\frac{ry(ry+3)}{4} + (p-1)ry(ry+1) + \sum_{i=1}^{p-1} r z_i(r z_i - 2ry - 1)\right)} \\ &\quad \times \frac{\tilde{S}_\gamma(r(y - z_{p-1})) \prod_{i=1}^{p-1} \tilde{S}_\gamma(r(z_i - z_{i-1}))}{\tilde{S}_\gamma(ry)} \\ f_{-p}(a,y,\underline{z}) &= e^{\frac{2\pi i}{r}\left(\frac{-ry(ry+3)}{4} - (p-1)ry(ry+1) - r^2 y z_{p-1} - \sum_{i=1}^{p-1} r z_i(2ry + r z_{i-1} - 1)\right)} \\ &\quad \times \frac{\tilde{S}_\gamma(r(y - z_{p-1})) \prod_{i=1}^{p-1} \tilde{S}_\gamma(r(z_i - z_{i-1}))}{\tilde{S}_\gamma(ry)} \end{aligned}$$

Let $\varepsilon, \varepsilon' \in \{\pm 1\}$. Then by (8.4), (8.5) and (8.6) we can write the coloured Jones polynomial of the double twist knot $K_{\varepsilon p, \varepsilon' p'}$ as

$$J_{K_{\varepsilon p,\varepsilon' p'}}(k) = \beta_r(p,p') \sum_{n,s_i,t_i=0}^{k-1} f_{\varepsilon p}\left(\tfrac{k}{r}, \tfrac{n}{r}, \tfrac{1}{r}\underline{s}\right) f_{\varepsilon' p'}\left(\tfrac{k}{r}, \tfrac{n}{r}, \tfrac{1}{r}\underline{t}\right) g\left(\tfrac{k}{r}, \tfrac{n}{r}\right)$$

Where $\frac{1}{r}\underline{s} = \left(\frac{s_1}{r}, \ldots, \frac{s_{p-1}}{r}\right)$, $\frac{1}{r}\underline{t} = \left(\frac{t_1}{r}, \ldots, \frac{t_{p'-1}}{r}\right)$ and

$$\beta_r(p,p') = \frac{i}{2\sin(\frac{\pi}{r}) \left(\tilde{S}_\gamma(0)\right)^{p+p'-2}}.$$

We are now in good shape to write the coloured jones polynomial as an integral. Namely, let $C_{k,r}$ be the curve in $\mathbb{C}$ parametrized as

$$C_{k,r}(t) = \frac{k-1}{2r} + \frac{k-\frac{1}{2}}{2r} e^{2\pi i t}, \qquad t \in [0,1],$$

that is, a circle with center in $\frac{k-1}{2r}$, encircling the points $0, \frac{1}{r}, \dots, \frac{k-1}{r}$ with a $\frac{1}{4r}$ margin. Then we can apply the Residue Theorem $p+p'-1$ times to obtain the following theorem.

Theorem 8.1. *For $p, p' \in \mathbb{N}$ and $\varepsilon, \varepsilon' \in \{\pm 1\}$, the coloured Jones polynomial of the $(\varepsilon p, \varepsilon' p')$ double twist knot $K_{\varepsilon p, \varepsilon' p'}$ at $q = e^{\frac{2\pi i}{r}}$ is*

$$J_{K_{\varepsilon p,\varepsilon' p'}}(k) = \alpha_{p,p'}(r) \int_{T_{k,r}^{p+p'-1}} f_{\varepsilon p}(\tfrac{k}{r}, y, \underline{z}) f_{\varepsilon' p'}(\tfrac{k}{r}, y, \underline{w}) g(\tfrac{k}{r}, y) \times \cot(\pi r y) \prod_{l=1}^{p-1} \cot(\pi r z_l) \prod_{j=1}^{p'-1} \cot(\pi r w_j)\, dy d\underline{z} d\underline{w}, \tag{8.7}$$

where $T_{k,r}^{p+p'-1} = (C_{k,r})^{p+p'-1}$, $\underline{z} = (z_1, \dots, z_{p-1})$, $\underline{w} = (w_1, \dots, w_{p'-1})$, and

$$\alpha_{p,p'}(r) = \frac{-r^{p+p'-1}}{(2i)^{p+p'} \sin(\frac{\pi}{r}) \left(\tilde{S}_\gamma(0)\right)^{p+p'-1}}.$$

8.2 Asymptotic Behaviour

We now wish to investigate the asymptotic behaviour of the integral above, where we let the ratio $\frac{k}{r} \to a$ as $r \to \infty$, for some fixed $a \in]0,1[$. This is to address the version of the AJ conjecture coming from Gukov's Generalized Volume Conjecture (Conjecture 7.3). As we are only interested in the large r asymptotics, we replace the ratio $\frac{k}{r}$ by a. We can not do the analysis completely rigorously yet, but we will sketch how we believe one could get to a point where the method of steepest descend could be applied. This done through a number of leading order estimates on the factors of the integral.

8.2.1 Asymptotics of Quantum Dilogarithms

Recall our notation $\tilde{S}_\gamma(x) = S_\gamma(-\pi + 2\gamma x + \gamma)$. In [AH] the $\gamma \to 0$ behaviour for S_γ is investigated. This is based on Euler's dilogarithm

$$\mathrm{Li}_2(w) = -\int_0^w \frac{\mathrm{Log}(1-z)}{z} dz$$

defined on $\mathbb{C} \setminus]1, \infty[$.

Assume

$$-\tfrac{1}{2r} < \mathrm{Re}(x) < 1 - \tfrac{1}{2r}, \tag{8.8}$$

then

$$S_\gamma(rx) = \exp\left(\frac{r}{2\pi i} \mathrm{Li}_2(e^{2\pi i x + \frac{\pi i}{r}}) + R_\gamma(x)\right) \tag{8.9}$$

where

$$|R_\gamma(x)| \le \left(A_R\left(\frac{1}{1-\frac{1}{2r}-\mathrm{Re}(x)}+\frac{1}{\frac{1}{2r}+\mathrm{Re}(x)}\right)+B_R\left(1+e^{-2\pi R\,\mathrm{Im}(x)}\right)\right)\frac{1}{r}$$

for some constants A_R and B_R only depending on R. This means that in a leading order asymptotical analysis of the integral in Theorem 8.1, we should be able to replace the quantum dilogarithms by Euler's dilogarithms. This, however, is only where the condition (8.8) is satisfied. Let us take a look at how the quantum dilogarithms enter in Theorem 8.1. There are four ways:

(1) $\tilde{S}_\gamma(ry)$

(2) $\tilde{S}_\gamma(r(a-y)-1)$

(3) $\tilde{S}_\gamma(r(a+y))$

(4) $\tilde{S}_\gamma(r(z_i-z_{i-1}))$.

First, let us rewrite the curve $C_{k,r}$ as $C_{a,r}$ with

$$C_{a,r}(t)=\frac{a}{2}-\frac{1}{2r}+\left(\frac{a}{2}-\frac{1}{4r}\right)e^{2\pi it}, \qquad t\in[0,1].$$

We then see that for $x\in C_{a,r}$

$$\mathrm{Re}(x)\in[-\tfrac{1}{4r},a-\tfrac{3}{4r}].$$

Let us consider the different quantum dilogarithms one by one. In (1) we see that (8.8) is satisfied immediately.

In (2) we can rewrite it as $\tilde{S}_\gamma(r(a-y-\frac{1}{r}))$ and we find that

$$-\frac{1}{4r}\le a-\mathrm{Re}(y)-\frac{1}{r}\le 1-\frac{3}{4r}$$

and we have the estimate (8.8).

(3) is a bit more tricky, as it depends on what a is. We immediately get the lower bound satisfied, as $a+y\ge a-\frac{1}{4r}$ and if $a\le\frac{1}{2}$, we see that

$$a+\mathrm{Re}(y)\le 1-\frac{3}{4r}.$$

But if $a\ge\frac{1}{2}+\frac{1}{8r}$, the upper bound may not hold. Indeed, then there is $y\in C_{a,r}$ such that $\mathrm{Re}(y)\ge 1-a-\frac{1}{2r}$, which implies that $a+\mathrm{Re}(y)\ge 1-\frac{1}{2r}$. But then $a+y-1$ will satisfy the bounds in (8.8). Using the functional equation (8.2) we see that

$$\begin{aligned}\tilde{S}_\gamma(r(a+y)) &= (1+e^{2\pi ir(a+y-1+\frac{1}{2r})})^{-1}\tilde{S}_\gamma(r(a+y-1))\\ &= (1-e^{2\pi ir(a+y-1)})^{-1}\tilde{S}_\gamma(r(a+y-1))\end{aligned} \tag{8.10}$$

And we can use the estimates on this instead. This requires a change of contour. First assume that r satisfies that $r(1-a)-\frac{1}{2}\in\mathbb{R}\setminus\mathbb{Z}$, and let $\varepsilon_a(r)>0$ such that

$$[r(1-a)-\frac{1}{2}-\varepsilon_a(r), r(1-a)-\frac{1}{2}+\varepsilon_a(r)]\cap\mathbb{Z}=\emptyset$$

Define the contours

$$\begin{aligned}
C^1_{a,r}(t)&=\frac{1-a}{2}-\frac{3}{8r}-\frac{\varepsilon_a(r)}{2}+\left(\frac{1-a}{2}-\frac{1}{8r}-\frac{\varepsilon_a(r)}{2}\right)e^{2\pi it},\quad t\in[0,1]\\
C^2_{a,r}(t)&=\frac{1}{2}-\frac{5}{8r}+\frac{\varepsilon_a(r)}{2}+\left(a-\frac{1}{2}-\frac{1}{8r}-\frac{\varepsilon_a(r)}{2}\right)e^{2\pi it},\quad t\in[-\tfrac{1}{2},\tfrac{1}{2}]\\
C^3_{a,r}&=\left[(1-a)-\frac{1}{2r}-\varepsilon_a(r),(1-a)-\frac{1}{2r}+\varepsilon_a(r)\right].
\end{aligned}$$

Here, $C^1_{a,r}$ intersects the real axis in $-\frac{1}{4r}$ and $1-a-\frac{1}{2r}-\varepsilon_a(r)$, and $C^2_{a,r}$ intersect in $(1-a)-\frac{1}{2r}+\varepsilon_a(r)$ and $a-\frac{3}{4r}$. The contour $C^1_{a,r}+C^3_{a,r}+C^2_{a,r}-C^3_{a,r}$ is a deformation of $C_{a,r}$ encircling the same part of the real axis, except that it meets the real axis twice in $C^3_{a,r}$. By the choice of r and $\varepsilon_a(r)$, all the poles for $\cot(\pi ry)$ are on the inside of either $C^1_{a,r}$ or $C^2_{a,r}$. Then we can use the estimate (8.9) directly on the integral over $C^1_{a,r}$ and use the transformation (8.10) on $C^2_{a,r}$ to again apply (8.9).

An analysis of case (4) poses an even greater challenge. When $\mathrm{Re}(z_i)-\mathrm{Re}(z_{i-1})>-\frac{1}{2r}$, the other inequality also holds, as

$$\mathrm{Re}(z_i)-\mathrm{Re}(z_{i-1})\le a-\frac{1}{2r}<1-\frac{1}{2r}.$$

And if $\mathrm{Re}(z_i)-\mathrm{Re}(z_{i-1})<-\frac{1}{2r}$, then

$$1-\frac{1}{2r}>1+\mathrm{Re}(z_i)-\mathrm{Re}(z_{i-1})>1-\frac{1}{4r}-a+\frac{3}{4r}>\frac{1}{2r},$$

and so we can use the transformation

$$\tilde{S}_\gamma(r(z_i-z_{i-1}))=(1-e^{2\pi ir(z_i-z_{i-1}+1)})^{-1}\tilde{S}_\gamma(r(z_i-z_{i-1}+1)).\quad(8.11)$$

But when $\mathrm{Re}(z_i-z_{i-1})=-\frac{1}{2r}$ we have no estimate. In [AH], they have another estimate, when $\mathrm{Im}(z_i-z_{i-1})\ge 0$, but this is obviously not always satisfied. This is indeed a place where the analysis is lacking and should be addressed more in depth.

This concludes the discussion of the asymptotical behaviour of the quantum dilogarithms.

8.2.2 Approximating Cotangents

In [AH], $\tan(\pi r y)$ is approximated by $\pm i$ away from the real axis as

$$|\tan(\pi r y) - i| \leq \begin{cases} 4e^{-2\pi i \operatorname{Im}(y)}, & \operatorname{Im}(y) \geq \frac{1}{\pi r} \\ 2e^{-2\pi i \operatorname{Im}(y)}, & \operatorname{Im}(y) \geq 0,\, r\operatorname{Re}(y) \in \mathbb{Z} \end{cases}$$

$$|\tan(\pi r y) + i| \leq \begin{cases} 4e^{2\pi i \operatorname{Im}(y)}, & \operatorname{Im}(y) \leq -\frac{1}{\pi r} \\ 2e^{2\pi i \operatorname{Im}(y)}, & \operatorname{Im}(y) \leq 0,\, r\operatorname{Re}(y) \in \mathbb{Z}. \end{cases}$$

Using that $\cot(\pi r y) = \tan(\pi r(\frac{1}{2r} - y))$ we see that it switches

$$|\cot(\pi r y) + i| \leq \begin{cases} 4e^{-2\pi i \operatorname{Im}(y)}, & \operatorname{Im}(y) \geq \frac{1}{\pi r} \\ 2e^{-2\pi i \operatorname{Im}(y)}, & \operatorname{Im}(y) \geq 0,\, \frac{1}{2} - r\operatorname{Re}(y) \in \mathbb{Z} \end{cases}$$

$$|\cot(\pi r y) - i| \leq \begin{cases} 4e^{2\pi i \operatorname{Im}(y)}, & \operatorname{Im}(y) \leq -\frac{1}{\pi r} \\ 2e^{2\pi i \operatorname{Im}(y)}, & \operatorname{Im}(y) \leq 0,\, \frac{1}{2} - r\operatorname{Re}(y) \in \mathbb{Z}. \end{cases}$$

With these estimates, they replace tan by $\pm i$ in a setting similar to ours, albeit not as complicated. Their proof is somewhat technical, but with a careful analysis, it should be possible to show a similar result in our case. This will be addressed in future work.

8.2.3 Leading Order Asymptotics

We now assume that our integral has the same large r asymptotic behaviour as an integral of the form

$$\int_D h(y, \underline{z}, \underline{w}) e^{2\pi i r \Phi(y,\underline{z},\underline{w})} dy d\underline{z}\underline{w},$$

where D is a deformation of $T_{a,r}^{p+p'-1}$ such that the method of steepest descend can be applied. We search for critical values of Φ, as these stationary point determine the leading order asymptotics. The leading order term for such an integral near a stationary point x for Φ is

$$\left(\frac{2\pi}{r}\right)^{\frac{n}{2}} \frac{e^{\frac{i\pi}{4}\operatorname{sign}(A)}}{|\det A|^{\frac{1}{2}}} h(x) e^{2\pi i r \Phi(x)} \tag{8.12}$$

where n is the real dimension of D and A is the Hessian of Φ (for precise condition where this holds, see [Won]).

For this, we rewrite the functions f_p and g to determine a possible Φ.

Letting

$$\Phi_p^1(a, y, \underline{z}) = \frac{1}{2}y + \frac{1}{4}y^2 + (p-1)y^2 + \sum_{i=1}^{p-1}(z_i^2 - 2z_i y)$$

and

$$\Phi^2_p(a,y,\underline{z}) = \frac{3}{4}y + (p-1)y - \sum_{i=1}^{p-1} z_i$$

we see that

$$f_p(a,y,\underline{z}) = e^{2\pi i r\Phi^1_p(a,y,\underline{z})} e^{2\pi i \Phi^2_p(a,y,\underline{z})} \frac{\tilde{S}_\gamma(r(y-z_{p-1}))\prod_{i=1}^{p-1}\tilde{S}_\gamma(r(z_i-z_{i-1}))}{\tilde{S}_\gamma(ry)}.$$

Similarly,

$$\Phi^1_{-p}(a,y,\underline{z}) = -\frac{1}{4}y^2 - (p-1)y^2 - yz_{p-1} + \sum_{i=1}^{p-1}(2z_iy - z_iz_{i-1})$$

and

$$\Phi^2_{-p}(a,y,\underline{z}) = -\frac{3}{4}y - (p-1)y + \sum_{i=1}^{p-1} z_i = -\Phi^2_p(a,y,\underline{z})$$

gives

$$f_{-p}(a,y,\underline{z}) = e^{2\pi i r\Phi^1_{-p}(a,y,\underline{z})} e^{2\pi i \Phi^2_{-p}(a,y,\underline{z})} \frac{\tilde{S}_\gamma(r(y-z_{p-1}))\prod_{i=1}^{p-1}\tilde{S}_\gamma(r(z_i-z_{i-1}))}{\tilde{S}_\gamma(ry)}.$$

Finally,

$$\Psi(a,y) = \frac{1}{2}y - ay$$

lets us write

$$g(a,y) = e^{2\pi i r\Psi_p(a,y)} e^{-\pi i a} \frac{\tilde{S}_\gamma(r(a-y)-1)}{\tilde{S}_\gamma(r(a+y))}$$

Assuming we can approximate the quantum dilogarithms by Euler's dilogarithm, we need the following functions

$$\Lambda_p(a,y,\underline{z}) = -\frac{1}{4\pi^2}\left(\mathrm{Li}_2(e^{2\pi i(y-z_{p-1})}) + \sum_{i=1}^{p-1}\mathrm{Li}_2(e^{2\pi i(z_i-z_{i-1})}) - \mathrm{Li}_2(e^{2\pi i y})\right)$$

$$\Gamma(a,y) = \Psi(a,y) - \frac{1}{4\pi^2}\left(\mathrm{Li}_2(e^{2\pi i(a-y)}) - \mathrm{Li}_2(e^{2\pi i(a+y)})\right)$$

Note that Λ_1 is constant. However, there is the delicate matter of the shifts in (8.10) and (8.11), but let us ignore this for a moment.

If we can approximate cot by constants, we see that Φ is composed by $\Phi^1_{\pm p}$'s, ψ, Λ_p and Γ. More precisely, for $K_{\varepsilon p,\varepsilon' p'}$ we get

$$\Phi = \Phi^1_{\varepsilon p} + \Lambda_{\varepsilon p} + \Phi^1_{\varepsilon' p'} + \Lambda_{\varepsilon' p'} + \psi + \Gamma.$$

We need the derivatives of Φ. As we are aiming for the A-polynomial with the correspondence

$$M = e^{\pi i a}, \quad L = \frac{\partial}{\partial a} \lim_{r\to\infty} \frac{\log J_K(ra; q = e^{\frac{2\pi i}{r}})}{r} \tag{8.13}$$

we will write write the expressions for the derivatives in the coordinates

$$M = e^{\pi i a}, \quad x_i = e^{2\pi i z_{p-i}}, \quad x_0 = e^{2\pi i y}.$$

The possible shift factors for the quantum dilogarithm will be integers, and so they disappear in these coordinates. We calculate

$$\begin{aligned}
e^{2\pi i \frac{\partial \Phi^1_p}{\partial y}} &= -x_0^{\frac{1}{2}} x_0^{2(p-1)} \prod_{i=1}^{p-1} x_i^{-2} \\
e^{2\pi i \frac{\partial \Phi^1_p}{\partial z_{p-i}}} &= x_i^2 x_0^{-2}, \qquad \text{for } p > 1 \\
e^{2\pi i \frac{\partial \Phi^1_{-p}}{\partial y}} &= x_0^{-\frac{1}{2}} x_0^{-2(p-1)} x_1^{-1} \prod_{i=1}^{p-1} x_i^2 \\
e^{2\pi i \frac{\partial \Phi^1_{-p}}{\partial z_{p-i}}} &= x_0^2 (x_{i+1} x_{i-1})^{-1}, \qquad \text{for } p > 1 \\
e^{2\pi i \frac{\partial \Lambda_p}{\partial y}} &= (1 - x_0)(1 - x_0 x_1^{-1})^{-1}, \qquad \text{for } p > 1 \\
e^{2\pi i \frac{\partial \Lambda_p}{\partial z_{p-i}}} &= (1 - x_{i-1} x_i^{-1})(1 - x_i x_{i+1}^{-1})^{-1}, \qquad \text{for } p > 1 \\
e^{2\pi i \frac{\partial \Gamma}{\partial y}} &= -M^{-2}(1 - M^2 x_0^{-1})(1 - M^2 x_0).
\end{aligned}$$

From these, we can piece together equations for the stationary points for the integral (8.7) for $K_{\varepsilon p, \varepsilon' p'}$ as the equations

$$1 = e^{2\pi i \frac{\partial \Phi^1_{\varepsilon p}}{\partial y}} e^{2\pi i \frac{\partial \Phi^1_{\varepsilon' p'}}{\partial y}} e^{2\pi i \frac{\partial \Lambda_p}{\partial y}} e^{2\pi i \frac{\partial \Lambda_{p'}}{\partial y}} e^{2\pi i \frac{\partial \Gamma}{\partial y}} \tag{8.14}$$

$$1 = e^{2\pi i \frac{\partial \Phi^1_{\varepsilon p}}{\partial z_{p-i}}} e^{2\pi i \frac{\partial \Lambda_p}{\partial z_{p-i}}}, \quad i = 1, \dots, p-1. \tag{8.15}$$

$$1 = e^{2\pi i \frac{\partial \Phi^1_{\varepsilon' p'}}{\partial z_{p-i}}} e^{2\pi i \frac{\partial \Lambda_{p'}}{\partial z_{p'-i}}}, \quad i = 1, \dots, p'-1. \tag{8.16}$$

By a small calculation (8.15) (and similarly (8.16)) becomes

$$1 = \left(\frac{x_i^2}{x_0^2}\right)^{\varepsilon} \frac{1 - \left(\frac{x_{i-1}}{x_i}\right)^{\varepsilon}}{1 - \left(\frac{x_i}{x_{i+1}}\right)^{\varepsilon}}$$

which implies

$$\prod_{i=1}^{p-1} x_i^{2\varepsilon} = x_0^{2\varepsilon(p-1)} \frac{1 - x_{p-1}^{\varepsilon}}{1 - \left(\frac{x_0}{x_1}\right)^{\varepsilon}}.$$

This lets us rewrite

$$x_0^{2(p-1)} \prod_{i=1}^{p-1} x_i^{-2} = \frac{1-\left(\frac{x_0}{x_1}\right)}{1-x_{p-1}}$$

and

$$x_0^{-2(p-1)} x_1^{-1} \prod_{i=1}^{p-1} x_i^2 = -x_0^{-1} \frac{1-\left(\frac{x_0}{x_1}\right)}{1-x_{p-1}^{-1}}.$$

Writing out (8.14) for $\varepsilon = \varepsilon' = 1$ we get

$$1 = \frac{(1-x_0)^2}{(1-x_{p-1})\left(1-v_{p'-1}\right)} \left(1 - \frac{x_0}{m_2}\right)(1-m^2 x_0),$$

for $\varepsilon = -1$, $\varepsilon' = 1$

$$x_0^2(1-x_{p-1}^{-1})(1-v_{p-1}) = -(1-x_0)^2 \left(1 - \frac{x_0}{m^2}\right)(1-m^2 x_0)$$

and finally, for $\varepsilon = \varepsilon' = -1$

$$x_0^4(1-x_{p-1}^{-1})(1-v_{p-1}^{-1}) = -(1-x_0)^2 \left(1 - \frac{x_0}{M^2}\right)(1-M^2 x_0).$$

As for L, we use (8.13) and (8.12) to compute, for a critical point $(y_0, \underline{z}_0, \underline{w}_0)$,

$$L = e^{2\pi i \frac{\partial \Phi}{\partial a}(a, y_0, \underline{z}_0, \underline{w}_0)} = e^{2\pi i \frac{\partial (\Gamma)}{\partial a}(a, y_0)} = \frac{1 - x_0 M^2}{x_0 - M^2}.$$

Isolating x_0, one finds

$$x_0 = \frac{1 + LM^2}{L + M^2}. \tag{8.17}$$

This means, that if we can eliminate $x_1, \ldots, x_{p-1}$ and $v_1, \ldots, v_{p'-1}$ in the above system of equations, we get a polynomial in x_0 and M, in which we insert (8.17) to obtain what should be the A-polynomial.

This has so far only been succesful for the twist knots $K_{\varepsilon p,1}$, where the equations to solve for $\varepsilon = 1$ take the form

$$1 - x_{p-1} = (1-x_0)\left(1 - \frac{x_0}{m_2}\right)(1 - m^2 x_0) \tag{8.18}$$

$$1 = \frac{x_i^2}{x_0^2} \frac{1 - \frac{x_{i-1}}{x_i}}{1 - \frac{x_i}{x_{i+1}}}, \qquad i = 1, \ldots, p-1, \tag{8.19}$$

and for $\varepsilon = -1$

$$x_0^2(1 - x_{p-1}^{-1}) = -(1 - x_0)\left(1 - \frac{x_0}{m^2}\right)(1 - m^2 x_0) \tag{8.20}$$

$$1 = \frac{x_0^2}{x_i^2}\frac{1 - \frac{x_i}{x_{i-1}}}{1 - \frac{x_{i+1}}{x_i}}, \qquad i = 1, \ldots, p-1. \tag{8.21}$$

These are exactly the formulas obtained in [Hik2]. We solved these for $p = -5, \ldots, 7$ by use of computers, but since Hikami has solved these equations for all p, we present his solution.

8.2.4 Recursively defined solution

We follow [Hik2] in recursively defining a family of rational functions C_k giving rise the A-polynomial.

We let $x_{p-k} = x_0 C_k(x_0)$. Rewriting (8.19) for $i = p - k$ and inserting the corresponding C_k's, we find that

$$\begin{aligned} 1 &= \frac{x_{p-k}}{x_0}\left(\frac{x_{p-k}}{x_0} - \frac{x_{p-k-1}}{x_0}\right)\left(1 - \frac{x_{p-k}}{x_{p-k+1}}\right)^{-1} \\ &= C_k(x_0)(C_k(x_0) - C_{k+1}(x_0))\left(1 - \frac{C_k(x_0)}{C_{k-1}(x_0)}\right)^{-1} \end{aligned}$$

Which implies the relation

$$C_{k+1}(x) = C_k(x) - \frac{1}{C_k(x)} + \frac{1}{C_{k-1}(x)}.$$

As for the base cases (8.18) gives rise to the equation

$$C_1(x) = \frac{1 - (1-x)(1-M^2x)(1-M^{-2}x)}{x}$$

Finally, we have the obvious condition $C_0(x) = \frac{1}{x}$. The A-polynomial is now determined by $C_p(x_0) = 1$ and the equation

$$x_0 = \frac{1 + LM^2}{L + M^2}.$$

A similar recursive formula was also established for $\varepsilon = -1$ from the equations (8.20) and (8.21).

The way Hikami showed this gave the Apolynomial, was by showing that the equation $C_p(x_0) - 1$ satisfied the recursion relation of 5.14.

Unfortunately, a similar approach to $p' \geq 2$ does not seem to work, as of this moment. Moreover, if we eliminate the variables x_1 and v_1 for $p = p' = 2$ using Gröbner bases, we do not get anything related to the A-polynomial. This is a particular interesting case, since the knot $K_{2,2} = 7_4$ (mirror) does not satisfy the condition in Theorem 7.2 for the full AJ conjecture to hold. Therefore, it would be of interest to consider this knot in greater detail, as we plan to do in the future.

Bibliography

[AB] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.

[AE] S. T. Ali and M. Engliš. Quantization methods: a guide for physicists and analysts. *Rev. Math. Phys.*, 17(4):391–490, 2005.

[AGL] J. E. Andersen, N. L. Gammelgaard, and M. R. Lauridsen. Hitchin's connection in half-form quantization. arXiv:math.QA/0711.3995.

[AH] J. E. Andersen and S. K. Hansen. Asymptotics of the quantum invariants for surgeries on the figure 8 knot. *J. Knot Theory Ramifications*, 15(4):479–548, 2006.

[And1] J. Andersen. Asymptotic faithfulness of the quantum SU(n) representations of the mapping class groups. *Ann. of Math. (2)*, 163(1):347–368, 2006.

[And2] J. E. Andersen. Hitchin's connection, Toeplitz operators and symmetry invariant deformation quantization. arXiv:math.DG/0611126.

[And3] J. E. Andersen. The Homepage of Jørgen Ellegaard Andersen. `http://home.imf.au.dk/andersen/`.

[And4] J. E. Andersen. Deformation quantization and geometric quantization of abelian moduli spaces. *Comm. Math. Phys.*, 255(3):727–745, 2005.

[APW] S. Axelrod, S. D. Pietra, and E. Witten. Geometric quantization of Chern-Simons gauge theory. *J. Differential Geom.*, 33(3):787–902, 1991.

[Ati1] M. Atiyah. New invariants of 3- and 4-dimensional manifolds. In *The mathematical heritage of Hermann Weyl (Durham, NC, 1987)*, volume 48 of *Proc. Sympos. Pure Math.*, pages 285–299. Amer. Math. Soc., Providence, RI, 1988.

[Ati2] M. Atiyah. *The geometry and physics of knots.* Lezioni Lincee. [Lincei Lectures]. Cambridge University Press, Cambridge, 1990.

[AU1] J. E. Andersen and K. Ueno. Construction of the Reshetikhin-Turaev TQFT from conformal field theory. In preparation.

[AU2] J. E. Andersen and K. Ueno. Modular functors are determined by their genus zero data. arXiv:math.QA/0611087.

[AU3] J. E. Andersen and K. Ueno. Abelian conformal field theory and determinant bundles. *Internat. J. Math.*, 18(8):919–993, 2007.

[AU4] J. E. Andersen and K. Ueno. Geometric construction of modular functors from conformal field theory. *J. Knot Theory Ramifications*, 16(2):127–202, 2007.

[Bes] A. L. Besse. *Einstein manifolds*, volume 10 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1987.

[BF] J.-M. Bismut and D. S. Freed. The analysis of elliptic families. I. Metrics and connections on determinant bundles. *Comm. Math. Phys.*, 106(1):159–176, 1986.

[BHMV1] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel. Three-manifold invariants derived from the Kauffman bracket. *Topology*, 31(4):685–699, 1992.

[BHMV2] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel. Topological quantum field theories derived from the Kauffman bracket. *Topology*, 34(4):883–927, 1995.

[BMS] M. Bordemann, E. Meinrenken, and M. Schlichenmaier. Toeplitz quantization of Kähler manifolds and gl(N), $N \to \infty$ limits. *Comm. Math. Phys.*, 165(2):281–296, 1994.

[BNM] D. Bar-Natan and S. Morrison. The knot atlas. `http://katlas.math.toronto.edu/wiki/Main_Page`.

[BZ1] S. Boyer and X. Zhang. Every nontrivial knot in S^3 has nontrivial A-polynomial. *Proc. Amer. Math. Soc.*, 133(9):2813–2815 (electronic), 2005.

[BZ2] G. Burde and H. Zieschang. *Knots*, volume 5 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 2003.

[CCG+] D. Cooper, M. Culler, H. Gillet, D. D. Long, and P. B. Shalen. Plane curves associated to character varieties of 3-manifolds. *Invent. Math.*, 118(1):47–84, 1994.

[CS] M. Culler and P. B. Shalen. Varieties of group representations and splittings of 3-manifolds. *Ann. of Math. (2)*, 117(1):109–146, 1983.

[DG] N. M. Dunfield and S. Garoufalidis. Non-triviality of the A-polynomial for knots in S^3. *Algebr. Geom. Topol.*, 4:1145–1153 (electronic), 2004.

[Fad] L. D. Faddeev. Discrete Heisenberg-Weyl group and modular group. *Lett. Math. Phys.*, 34(3):249–254, 1995.

[Fre] D. S. Freed. Classical Chern-Simons theory. I. *Adv. Math.*, 113(2):237–303, 1995.

[Gar] S. Garoufalidis. On the characteristic and deformation varieties of a knot. In *Proceedings of the Casson Fest*, volume 7 of *Geom. Topol. Monogr.*, pages 291–309 (electronic). Geom. Topol. Publ., Coventry, 2004.

[GH] P. Griffiths and J. Harris. *Principles of algebraic geometry.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[GL] S. Garoufalidis and T. T. Q. Lê. The colored Jones function is q-holonomic. *Geom. Topol.*, 9:1253–1293 (electronic), 2005.

[Gol] W. M. Goldman. The symplectic nature of fundamental groups of surfaces. *Adv. in Math.*, 54(2):200–225, 1984.

[Guk] S. Gukov. Three-dimensional quantum gravity, Chern-Simons theory, and the A-polynomial. *Comm. Math. Phys.*, 255(3):577–627, 2005.

[Hik1] K. Hikami. Difference equation of the colored Jones polynomial for torus knot. *Internat. J. Math.*, 15(9):959–965, 2004.

[Hik2] K. Hikami. Asymptotics of the colored Jones polynomial and the A-polynomial. *Nuclear Phys. B*, 773(3):184–202, 2007.

[Hit] N. J. Hitchin. Flat connections and geometric quantization. *Comm. Math. Phys.*, 131(2):347–380, 1990.

[HS] J. Hoste and P. D. Shanahan. A formula for the A-polynomial of twist knots. *J. Knot Theory Ramifications*, 13(2):193–209, 2004.

[Jef] L. C. Jeffrey. Chern-Simons-Witten invariants of lens spaces and torus bundles, and the semiclassical approximation. *Comm. Math. Phys.*, 147(3):563–604, 1992.

[Jon] V. F. R. Jones. A polynomial invariant for knots via von Neumann algebras. *Bull. Amer. Math. Soc. (N.S.)*, 12(1):103–111, 1985.

[KM1] R. Kirby and P. Melvin. The 3-manifold invariants of Witten and Reshetikhin-Turaev for $\mathfrak{sl}(2, \mathbb{C})$. *Invent. Math.*, 105(3):473–545, 1991.

[KM2] P. B. Kronheimer and T. S. Mrowka. Dehn surgery, the fundamental group and SU(2). *Math. Res. Lett.*, 11(5-6):741–754, 2004.

[Lê] T. T. Q. Lê. The colored Jones polynomial and the A-polynomial of knots. *Adv. Math.*, 207(2):782–804, 2006.

[Lic] W. B. R. Lickorish. Three-manifolds and the Temperley-Lieb algebra. *Math. Ann.*, 290(4):657–670, 1991.

[Mas] G. Masbaum. Skein-theoretical derivation of some formulas of Habiro. *Algebr. Geom. Topol.*, 3:537–556 (electronic), 2003.

[MV] G. Masbaum and P. Vogel. 3-valent graphs and the Kauffman bracket. *Pacific J. Math.*, 164(2):361–381, 1994.

[NS] M. S. Narasimhan and C. S. Seshadri. Stable and unitary vector bundles on a compact Riemann surface. *Ann. of Math. (2)*, 82:540–567, 1965.

[Qui] D. Quillen. Determinants of Cauchy-Riemann operators on Riemann surfaces. *Functional Anal. Appl. 19*, 1(1):31–34, 1985.

[Raw] J. H. Rawnsley. Some properties of half-forms. In *Differential geometrical methods in mathematical physics, II (Proc. Conf., Univ. Bonn, Bonn, 1977)*, volume 676 of *Lecture Notes in Math.*, pages 311–314. Springer, Berlin, 1978.

[Ril1] R. Riley. Nonabelian representations of 2-bridge knot groups. *Quart. J. Math. Oxford Ser. (2)*, 35(138):191–208, 1984.

[Ril2] R. Riley. Holomorphically parameterized families of subgroups of $\mathrm{SL}(2, \mathbb{C})$. *Mathematika*, 32(2):248–264 (1986), 1985.

[RT1] N. Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.*, 127(1):1–26, 1990.

[RT2] N. Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.*, 103(3):547–597, 1991.

[Sch1] M. Schlichenmaier. Zwei anwendungen algebraisch-geometrischer methoden in der theoretischen physik: Berezin-toeplitz-quantisierung und globale algebren der zweidimensionalen konformen feldtheorie. 1996. Habiliationsschrift Universität Mannheim.

[Sch2] H. Schubert. Knoten mit zwei Brücken. *Math. Z.*, 65:133–170, 1956.

[SS] P. Scheinost and M. Schottenloher. Metaplectic quantization of the moduli spaces of flat and parabolic bundles. *J. Reine Angew. Math.*, 466:145–219, 1995.

[TUY1] A. Tsuchiya, K. Ueno, and Y. Yamada. Conformal field theory on universal family of stable curves with gauge symmetries. In *Integrable systems in quantum field theory and statistical mechanics*, volume 19 of *Adv. Stud. Pure Math.*, pages 459–566. Academic Press, Boston, MA, 1989.

[Tuy2] G. M. Tuynman. Quantization: towards a comparison between methods. *J. Math. Phys.*, 28(12):2829–2840, 1987.

[Wit] E. Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121(3):351–399, 1989.

[Won] R. Wong. *Asymptotic approximations of integrals.* Computer Science and Scientific Computing. Academic Press Inc., Boston, MA, 1989.

[Woo] N. M. J. Woodhouse. *Geometric quantization.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1992. Oxford Science Publications.