Representations of the Witt–Jacobson Lie algebras

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Introduction

The modern interest in modular Lie algebras, i.e. Lie algebras over fields of positive characteristic, began with Ernst Witt in the late thirties when he found a non–classical simple Lie algebra which had not been known before. The Witt algebra, as it has been named, was distinguished from the classical Lie algebras by the fact that it is not associated to a smooth algebraic group. In subsequent years, yet more non–classical Lie algebras were discovered, and a new class of restricted simple Lie algebras was established and distinguished by the name of Cartan. These Lie algebras have been classified into four categories: Contact Lie algebras K^{2n+1}, Hamiltonian Lie algebras H^{2n}, special Lie algebras S^n and Witt–Jacobson Lie algebras W^n.

The interest in modular Lie algebras was motivated by the famous Kostrikin–Shafarevich Conjecture which states that over an algebraically closed field of characteristic $p > 5$ a finite dimensional restricted simple Lie algebra is either classical or of Cartan type. The conjecture was proved for $p > 7$ by Block and Wilson and for $p = 7$ by Premet and Strade. However, in characteristic 5, there exist finite dimensional restricted simple Lie algebras which are neither classical nor of Cartan type. These Lie algebras constitute a category called the Melikyan algebras.

Unlike the situation for classical Lie algebras, the representation theory of the reduced enveloping algebra $U_\chi(\mathfrak{g})$ of a restricted Lie algebra $\mathfrak{g}$ of Cartan type is not well–known. Several efforts have been effective among a number of non–classical Lie algebras, but they have been far from successful in general. As regards the Witt–Jacobson Lie algebras, there has been slow but steady progress. In fact, the irreducible representations with characters of height at most one were computed by Holmes in [10]. Furthermore, the representation theory of $U_\chi(W^1)$ has been well understood for quite some time. A classification of the irreducible representations of $U_\chi(W^1)$ was first given by Chang in [4] and later simplified by Strade in [22].

Summary

This thesis contains two parts. The first part deals with the Witt–Jacobson Lie algebra W^1 of rank 1. The main goal here is to obtain a classification of the extensions of the simple $U_\chi(W^1)$–modules having character χ of height at most 1. The second part deals with the projective indecomposable modules of $U_\chi(W^n)$ where $n > 1$. The main goal is to determine the Cartan invariants of $U_\chi(W^n)$. The setting is kept as general as possible, but some of the results are only presented for $n = 2$.

Chapter 1. This first chapter serves as an easy start by recalling some basic concepts that are necessary for understanding the thesis.

Part I Extensions of the Witt algebra

Chapter 2. We recall several well–known facts about the Witt algebra W^1. A very brief description of the irreducible representations of W^1 will be presented.

Chapter 3. In this chapter, we give a classification of the extensions of the χ–reduced Verma modules having character χ of height at most 1. It turns out that there are two cases to consider depending on the values of the weights; each case will be examined separately. We conclude the chapter with a section summarizing our results.

Chapter 4. This chapter deals with the extensions of the simple $U_\chi(W^1)$–modules. The work done in Chapter 3 will be useful here. For characters of height 0 or 1, almost all the work has been done in Chapter 3 because the simple $U_\chi(W^1)$–modules are represented by reduced Verma modules. For characters of height -1 we still need to determine the extensions involving the trivial W^1–module and the $(p-1)$–dimensional simple W^1–module. Some of the techniques developed in Chapter 3 will be extended to the current setting.

Chapter 5. The fifth chapter applies the results obtained in Chapter 4 to give a simpler proof of the wildness of $U_\chi(W^1)$.

Part II The projective indecomposable modules

Chapter 6. This chapter lays the foundation for our study of the projective indecomposable modules. We fix a character χ of height 0 and introduce a new grading on W^n that allows us to establish a one–to–one correspondence between the irreducible representations of $U_\chi(W^n)$ and $U_\chi(W^n_{(0)})$; here $W^n_{(0)}$ is the degree zero part of the new grading.

Chapter 7. The seventh chapter is devoted to studying the projective indecomposable modules of $U_\chi(W^n_{(0)})$. The chapter consists of three main parts. The first deals with the dimension of the projective indecomposable modules. We prove that except for one case, all projective indecomposable modules have the same dimension. The second part deals with the Cartan invariants of $U_\chi(W^n_{(0)})$. We derive a formula that reduces the problem of computing the Cartan invariants to a problem of the representation theory of $\mathfrak{gl}_n$. In the third part, we prove an independence property for the Cartan invariants. We conclude the chapter by considering the case $n = 2$, which, in contrast to the general case, will be fully treated.

Chapter 8. The final chapter focuses on the Witt–Jacobson Lie algebra W^2 of rank 2. We use the general theory from Chapter 7 together with several techniques developed for $n = 2$ to compute the Cartan matrix of $U_\chi(W^2)$.

Appendix A. This appendix is needed in Chapter 6 where we choose the height 0 character χ in a specific way. In Appendix A, we prove that all height 0 characters are conjugate to this χ. Since the representation theory only depends on the conjugate classes of the characters, this reduces the height 0 case to the study of χ.

Appendix B. In this appendix, we give a classification of the irreducible representations of $U_\chi(W^2_0)$ which is needed in Chapter 8 in order to compute the Cartan invariants of $U_\chi(W^2)$. To this end, we make use of the isomorphism between W^2_0 and $\mathfrak{gl}_2$.

Appendix C. This appendix is needed in Chapter 8 in order to establish certain isomorphisms. We introduce the notion of inflation and provide some of its properties.

Appendix D. The main purpose of this appendix is to generalize some of the results obtained by Holmes in [10]. In this paper, Holmes assumes that the characteristic p is larger than 3, but it turns out that some of the results are also valid for $p = 3$.

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Dansk resumé

Hen mod slutningen af trediverne opfandt Ernst Witt en ikke-klassisk, simpel Lie-algebra, dvs. en simpel Lie-algebra, som ikke er associeret til en glat algebraisk gruppe. Dette dannede grundlag for teorien om modulære Lie-algebraer. I de forløbne år er flere ikke-klassiske, simple Lie-algebraer blevet fundet, og en ny type af simple, restringerede Lie-algebraer er blevet født og døbt: Simple, restringerede Lie-algebraer af Cartan-type. Den er inddelt i fire klasser: Witt-Jacobson Lie-algebraer W^n, specielle Lie-algebraer S^n, hamiltonske Lie-algebraer H^{2n} og kontakt-Lie-algebraer K^{2n+1}.

Vigtigheden af disse Lie-algebraer ligger i en formodning fra 1966, som siger, at en simpel, restringeret, endeligt-dimensional Lie-algebra enten er klassisk eller af Cartan-type. Formodningen er blevet bevist for $p > 5$, hvor p er karakteristikken af grundlegemet.

Denne afhandling, som er delt op i en indledning og to hoveddele, har til formål at studere Witt-Jacobson Lie-algebraer W^n. Hovedformålet med den første del er at bestemme udvidelserne for de simple moduler for den reducerede indhyldningsalgebra $U_\chi(W^n)$ relateret til W^n. Her betegner χ en p–karakter med højde mindre end eller lig med 1. Den anden hoveddel har til formål at bestemme strukturen af de projektive, indekomposable moduler for $U_\chi(W^2)$, når χ har højde lig med 0.

1 Preliminaries

We introduce here the necessary background in order to make this thesis self–contained. Since all of the results are well–known, we shall omit the proofs and instead provide references where interested readers can find the details. Unless otherwise stated, all Lie algebras, algebras and vector spaces are considered over a fixed algebraically closed field K of positive characteristic $p > 0$. The set of all natural numbers including zero is denoted by $\mathbb{N}$. Furthermore, $\mathbb{Z}$ and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ denote the ring of integers and the field having p elements, respectively. Note that we have a natural inclusion of $\mathbb{F}_p$ in K.

1.1 Restricted Lie algebras

1.1.1 Let L be a Lie algebra over K. Let $\mathfrak{gl}(L)$ and $\mathrm{ad} : L \to \mathfrak{gl}(L)$ denote the general linear algebra and the adjoint representation of L, respectively. A mapping $[p] : L \to L$, $a \mapsto a^{[p]}$, is called a *p–mapping* if

1. $\mathrm{ad}\, a^{[p]} = (\mathrm{ad}\, a)^p$ for all $a \in L$,
2. $(\alpha a)^{[p]} = \alpha^p a^{[p]}$ for all $\alpha \in K$ and $a \in L$,
3. $(a + b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a, b)$,

where $(\mathrm{ad}(a \otimes X + b \otimes 1))^{p-1}(a \otimes 1) = \sum_{i=1}^{p-1} i s_i(a, b) \otimes X^{i-1}$ in $L \otimes K[X]$ for all $a, b \in L$; here $K[X]$ is the polynomial ring over K in X. The pair $(L, [p])$ is referred to as a *restricted Lie algebra*.

1.1.2 Every associative K–algebra R gives rise to a restricted Lie algebra over K in a natural way; the Lie bracket is given by the commutator $[a, b] = ab - ba$ for all $a, b \in R$ and the p–mapping is given by the Frobenius mapping $a \to a^p$, see [23, Sec. 2.1]. Any Lie algebra $L \subset R$ satisfying $a^p \in L$ for all $a \in L$ is a restricted Lie algebra. In particular, the derivation algebra

$\mathrm{Der}_K(L) \subset \mathfrak{gl}(L)$ of a Lie algebra L is restricted. This is a consequence of Leibniz's rule which states that for all $a, b \in L$ and $n > 0$

$$D^n(ab) = \sum_{i=0}^{n} \binom{n}{i} D^i(a) D^{n-i}(b) \quad \text{for all} \quad D \in \mathrm{Der}_K(L).$$

We shall usually write $D^{[p]}$ instead of D^p.

1.1.3 Let L be a restricted Lie algebra and let $U(L)$ denote its universal enveloping algebra. If M is an L–module and $\chi \in L^*$ is a functional then we say M has *p–character* χ if and only if

$$(x^p - x^{[p]} - \chi(x)^p) \cdot M = 0 \quad \text{for all} \quad x \in L;$$

here x^p denotes the pth power of x in $U(L)$.

1.1.4 For each $\chi \in L^*$ we define *the reduced enveloping algebra* $U_\chi(L)$ of L

$$U_\chi(L) = U(L)/(x^p - x^{[p]} - \chi(x)^p \mid x \in L).$$

The bijection

$$\{L\text{–modules}\} \longleftrightarrow \{U(L)\text{–modules}\}$$

induces for each χ a bijection

$$\{L\text{–modules with } p\text{–character } \chi\} \longleftrightarrow \{U_\chi(L)\text{–modules}\}.$$

The following proposition gives a basis of $U_\chi(L)$, cf. [13, Prop. 2.8].

Proposition. *If $u_1, u_2, \ldots, u_n$ is a basis of L then $U_\chi(L)$ has basis*

$$\{u_1^{\alpha_1} u_2^{\alpha_2} \cdots u_n^{\alpha_n} \mid 0 \le \alpha_i < p \text{ for all } i\}.$$

In particular, we have $\dim U_\chi(L) = p^{\dim L}$.

1.1.5 Let M be an L–module with p–character χ. The dual vector space M^* becomes an L–module if we define for every $x \in L$, $m \in M$, $f \in M^*$

$$(x \cdot f)(m) = -f(x \cdot m).$$

One can show that M^* has p–character $-\chi$. Thus, every $U_\chi(L)$–module M induces a $U_{-\chi}(L)$–module M^* which we shall call *the dual module* of M.

1.1.6 A restricted Lie algebra L with a p–mapping $[p]$ is called *unipotent* (or *p–nilpotent*) if, for all $x \in L$, there exists an $r > 0$ such that $x^{[p^r]} = 0$, where $x^{[p^r]}$ denotes the pth power map iterated r times. An ideal of L is called unipotent if it is unipotent as a Lie algebra.

An element x in L is called *toral* if $x^{[p]} = x$.

1.2 The Witt–Jacobson Lie algebras

In this section we define and describe several properties of the Witt–Jacobson Lie algebras.

1.2.1 For any positive integer $n > 0$, let B^n be the truncated polynomial algebra over K in n indeterminates

$$B^n = K[X_1, X_2, \ldots, X_n]/(X_1^p, X_2^p, \ldots, X_n^p);$$

here $K[X_1, X_2, \ldots, X_n]$ denotes the polynomial algebra in n indeterminates $X_1, X_2, \ldots, X_n$. Denote the image of X_i in B^n by x_i. For each $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, set

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

and

$$I^n = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n \mid 0 \leq \alpha_i < p \text{ for all } i\}.$$

All x^α with $\alpha \in I^n$ form a basis for B^n. In particular, we have $\dim B^n = p^n$. The algebra B^n has a grading $B^n = \bigoplus_{i \geq 0} B_i^n$ such that x_j is homogeneous of degree 1. If we set for every $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$

$$|(\alpha_1, \alpha_2, \ldots, \alpha_n)| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$

then the x^α with $\alpha \in I^n$ and $|\alpha| = i$ is a basis for B_i^n; we have $B_i^n = 0$ for $i > n(p-1)$. We define the Witt algebra $W^n = \mathrm{Der}_K(B^n)$ as the set of all derivations of the K–algebra B^n. This is a restricted Lie subalgebra and a B^n–submodule of $\mathrm{End}_K(B^n)$; here $\mathrm{End}_K(B^n)$ denotes the algebra of K–endomorphisms of B^n.

1.2.2 The partial derivative $\partial_i = \partial/\partial X_i$ is a derivation of $K[X_1, \ldots, X_n]$ satisfying $\partial_i(X_j^p) = 0$. It therefore preserves the ideal generated by all X_i^p and induces a derivation of the factor algebra B^n. By a slight abuse of notation, we denote the induced derivation by ∂_i. Now, since $D(1) = 0$ for

$D \in W^n$, every derivation $D \in W^n$ is uniquely determined by the values $D(x_1), D(x_2), \ldots, D(x_n)$. It follows that

$$D = \sum_{i=1}^{n} D(x_i)\partial_i \tag{1.1}$$

and thus W^n is a free module over B^n with basis $\partial_1, \partial_2, \ldots, \partial_n$. Furthermore, the $x^\alpha \partial_i$ with $1 \le i \le n$ and $\alpha \in I^n$ form a basis for W^n over K; we have $\dim W^n = np^n$.

1.2.3 A simple computation shows for all $\alpha, \beta \in I^n$ and $1 \le i, j \le n$ that

$$[x^\alpha \partial_i, x^\beta \partial_j] = \beta_i x^{\alpha+\beta-\varepsilon_i}\partial_j - \alpha_j x^{\alpha+\beta-\varepsilon_j}\partial_i, \tag{1.2}$$

where $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ has a 1 in the ith position and zeros everywhere else. We interpret x^γ as 0 if $\gamma \notin \mathbb{N}^n$. The p–mapping on our basis elements is given by

$$(x^\alpha \partial_i)^{[p]} = \begin{cases} x_i\partial_i, & \text{if } \alpha = \varepsilon_i, \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

Indeed, since $(x^\alpha\partial_i)^{[p]}(x_j) = (x^\alpha\partial_i)^p(x_j) = 0$ for $i \neq j$, it follows from (1.1) that $(x^\alpha\partial_i)^{[p]} = (x^\alpha\partial_i)^p(x_i)\partial_i$. If $\alpha = \varepsilon_i$, we get $(x^\alpha\partial_i)(x_i) = x_i\partial_i(x_i) = x_i$ and thus $(x^\alpha\partial_i)^p(x_i) = x_i$. If $\alpha = 0$, the claim follows since $(x^\alpha\partial_i)^2(x_i) = \partial_i^2(x_i) = 0$. If $\alpha_i > 1$, then we have by induction that $(x^\alpha\partial_i)^r(x_i)$ is a multiple of $x^{r\alpha-(r-1)\varepsilon_i}$ which, in particular, shows that $(x^\alpha\partial_i)^p(x_i)$ is a multiple of $x_i^{p(\alpha_i-1)+1} = 0$. If $\alpha_j > 0$ for some $j \neq i$, then $(x^\alpha\partial_i)^p(x_i)$ is a multiple of $x_j^{p\alpha_j} = 0$.

1.2.4 For every integer $i \in \mathbb{Z}$, we define a subspace W_i^n of W^n by

$$W_i^n = \{D \in W^n \mid D(B_j^n) \subset B_{i+j}^n \text{ for all } j\}.$$

The sum of the W_i^n is direct and we have $[W_i^n, W_j^n] \subset W_{i+j}^n$ for all i and j. Furthermore, $D^{[p]} \in W_{pi}^n$ for all $D \in W_i^n$. We have $\partial_i \in W_{-1}^n$ because $\partial_i(x^\alpha) = \alpha_i x^{\alpha-\varepsilon_i}$ for all $\alpha \in I^n$. More generally, we have $x^\alpha\partial_i \in W_{|\alpha|-1}^n$ for all $\alpha \in I^n$. It follows that

$$W^n = \bigoplus_{i=-1}^{n(p-1)-1} W_i^n$$

is a graded restricted Lie algebra. For every $-1 \leq j \leq n(p-1)-1$, we set

$$W^n_{\geq j} = \bigoplus_{i=j}^{n(p-1)-1} W^n_i.$$

Furthermore, we set $W^n_{\geq j} = 0$ for $j > n(p-1)-1$.

1.2.5 Of central importance in the representation theory of W^n is the notion of height. The height of a character $\chi \in (W^n)^*$ is given by

$$\mathrm{ht}(\chi) = \min\{i \mid -1 \leq i \leq n(p-1) \text{ and } \chi_{|W_{\geq i}} = 0\},$$

Note that $\chi = 0$ if and only if $\mathrm{ht}(\chi) = -1$; this is usually known as the restricted case.

1.2.6 We state the following theorem without proof; a proof can be found in [23, Thm. 2.4 Ch. 4].

Theorem. *The Lie algebra W^n is simple unless $(p, n) = (2, 1)$.*

Part I

Extensions of the Witt algebra

2 Irreducible representations of the Witt algebra

2.1 Preliminaries

2.1.1 In this part, we focus on the Witt–Jacobson Lie algebra W^1 of rank 1, or the Witt algebra as it is also called. We fix here some notation and terminology which will be used later. Throughout this part, K denotes an algebraically closed field of characteristic $p > 3$. Set

$$e_i = x_1^{i+1}\partial_1 \quad \text{for all} \quad i \geq -1.$$

The e_i with $-1 \leq i \leq p-2$ form a basis for the Witt algebra. The Lie bracket and the p–mapping are given by

$$[e_i, e_j] = (j-i)e_{i+j} \quad \text{for all} \quad -1 \leq i, j \leq p-2,$$

and

$$e_i^{[p]} = \delta_{i0} e_i \quad \text{for all} \quad -1 \leq i \leq p-2.$$

2.1.2 For each $\chi \in (W^1)^*$, consider the weights

$$\Lambda(\chi) = \{\lambda \in K \mid \lambda^p - \lambda = \chi(e_0)^p\}.$$

Observe that if $\mathrm{ht}(\chi) \in \{-1, 0\}$, then $\Lambda(\chi)$ coincides with the prime field of K. For each $\lambda \in \Lambda(\chi)$, we define the χ–reduced Verma module $V_\chi(\lambda)$ by

$$V_\chi(\lambda) = U_\chi(W^1) \otimes_{U_\chi(W^1_{\geq 0})} K_\lambda,$$

where K_λ denotes K considered as a $U_\chi(W^1_{\geq 0})$–module via

$$e_i \cdot 1 = \begin{cases} \lambda \cdot 1, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The reduced Verma module $V_\chi(\lambda)$ has a basis $v_0, v_1, \ldots, v_{p-1}$ where $v_i = e_{-1}^i \otimes 1$ for all i. The action of W^1 is given by

$$e_{-1}v_i = \begin{cases} v_{i+1}, & \text{if } i < p-1, \\ \chi(e_{-1})^p v_0, & \text{if } i = p-1, \end{cases} \tag{2.1}$$

and for all $j \geq 0$

$$e_j v_i = \begin{cases} (-1)^j \frac{i!}{(i-j)!}((j+1)\lambda - i + j)v_{i-j}, & \text{if } j \leq i, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Except for the case $\mathrm{ht}(\chi) = -1$, all reduced Verma modules are simple.

2.1.3 Let M be a $U_\chi(W^1)$–module generated by a weight vector $v \in M$ that is annihilated by the action of all e_i with $i > 0$. If v has weight λ, i.e., if $e_0 v = \lambda v$, then there exists a unique surjective homomorphism of $U_\chi(W^1)$–modules $V_\chi(\lambda) \to M$ which sends v_0 to v. This is the universal property of Verma modules.

2.2 Irreducible representations

We give a very brief review of Chang's description of the simple $U_\chi(W^1)$–modules having character χ of height at most 1. Every simple $U_\chi(W^1)$–module is a homomorphic image of some reduced Verma module $V_\chi(\lambda)$. With the exception of $V_\chi(0) \simeq V_\chi(p-1)$ for $\mathrm{ht}(\chi) = 0$, the reduced Verma modules are isomorphic if and only if the corresponding weights coincide, see [4, Hilfssatz 7]. Thus, unless $\mathrm{ht}(\chi) = 0$, the isomorphism classes of simple $U_\chi(W^1)$–modules are in 1–1 correspondence with $\Lambda(\chi)$.

2.2.1 Suppose $\mathrm{ht}(\chi) = -1$ and let $v_0, v_1, \ldots, v_{p-1}$ be a basis of $V_\chi(p-1)$ as in Section 2.1.2. The socle $\mathrm{Soc}_{U_\chi(W^1)} V_\chi(p-1)$ of $V_\chi(p-1)$ is one–dimensional and spanned by v_{p-1}. This fact induces a $(p-1)$–dimensional module

$$S = V_\chi(p-1)/\,\mathrm{Soc}_{U_\chi(W^1)} V_\chi(p-1)$$

with a basis $\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_{p-2}$ such that $\bar{v}_i = v_i + \mathrm{Soc}_{U_\chi(W^1)} V_\chi(p-1)$ for all i. We have for every i

$$e_{-1}\bar{v}_i = \begin{cases} \bar{v}_{i+1}, & \text{if } i < p-2, \\ 0, & \text{if } i = p-2. \end{cases}$$

and every $j \geq 0$

$$e_j \bar{v}_i = \begin{cases} (-1)^{j+1} \frac{(i+1)!}{(i-j)!} \bar{v}_{i-j}, & \text{if } j \leq i, \\ 0, & \text{otherwise.} \end{cases}$$

Chang determined the simple $U_\chi(W^1)$–modules by a direct computation.

Theorem. *[4, Hauptsatz 2′]. If χ is of height -1, then there are p isomorphism classes of simple $U_\chi(W^1)$–modules. These modules are represented by the 1–dimensional trivial W^1–module K, the $(p-1)$–dimensional module S and the p–dimensional modules $V_\chi(\lambda)$ for $\lambda \in \{1, 2, \ldots, p-2\}$.*

2.2.2 If χ is of height 0 or 1, then the following two theorems due to Chang give a classification of the isomorphism classes of simple $U_\chi(W^1)$–modules, see [22, Part II].

Theorem (A)**.** *[4, Hauptsatz 2′] If χ is of height 0, then there are $p-1$ isomorphism classes of simple $U_\chi(W^1)$–modules each of dimension p and represented by $V_\chi(\lambda)$ for $\lambda \in \{0, 1, \ldots, p-2\}$.*

Theorem (B)**.** *[4, Hauptsatz 2′] If χ is of height 1, then there are p isomorphism classes of simple $U_\chi(W^1)$–modules each of dimension p and represented by $V_\chi(\lambda)$ for $\lambda \in \Lambda(\chi)$.*

2.2.3 If χ is of height $p-1$, then there are two cases depending on the centralizer $W^1_\chi = \{x \in W^1 \mid \chi([x, W^1]) = 0\}$ of W^1. If W^1_χ is a torus, then every simple $U_\chi(W^1)$–module is projective. If W^1_χ is unipotent, then with one exception every simple $U_\chi(W^1)$–module is projective. The remaining simple module L has a projective cover with two composition factors both isomorphic to L, see [7, Thm. 2.6].

Remark. If $1 < \mathrm{ht}(\chi) < p-1$, then there exists a unique simple $U_\chi(W^1)$–module up to isomorphism [4, Hauptsatz 1]. This module has a non–trivial self–extension.

2.3 The restricted case

In this section, we derive some basic results which are necessary for the proofs of the main results of Section 4.1. We assume that $\mathrm{ht}(\chi) = -1$, or equivalently, that $\chi = 0$.

2.3.1 The kernel of the canonical surjection $V_\chi(p-1) \to S$ is one–dimensional and thus isomorphic to the trivial module K. This gives rise to the following short exact sequence

$$0 \to K \to V_\chi(p-1) \to S \to 0. \tag{2.3}$$

By the universal property of Verma modules one has the following short exact sequence

$$0 \to S \to V_\chi(0) \to K \to 0. \tag{2.4}$$

2.3.2 The dual Verma modules are given by the following proposition

Proposition (A)**.** *We have for every* $\lambda \in \{0, 1, \ldots, p-1\}$

$$V_\chi(\lambda)^* \simeq V_{-\chi}(p-1-\lambda).$$

Proof. We saw in Section 1.1.5 that $V_\chi(\lambda)^*$ has a natural module structure over $U_{-\chi}(W^1)$. Let $v_0, v_1, \ldots, v_{p-1}$ be a basis of $V_\chi(\lambda)$ as in Section 2.1.2. Define $f_0 \in V_\chi(\lambda)^*$ such that $f_0(v_i) = \delta_{p-1,i}$ and set $f_i = e_{-1}^i f_0$ for all $i = 1, 2, \ldots, p-1$. Since for all i and j we have

$$f_i(v_j) = (-1)^i f_0(e_{-1}^i v_j) = (-1)^i \delta_{p-1,i+j},$$

it follows that $f_0, f_1, \ldots, f_{p-1}$ are linearly independent and hence form a basis of the dual module $V_\chi(\lambda)^*$. We have for every j

$$(e_0 f_0)(v_j) = -f_0(e_0 v_j) = -(\lambda - j) f_0(v_j) = (p-1-\lambda)\delta_{p-1,j}.$$

Thus

$$e_0 f_0 = (p-1-\lambda) f_0.$$

Furthermore, we have for every $i > 0$

$$(e_i f_0)(v_j) = -f_0(e_i v_j) = 0,$$

so

$$e_i f_0 = 0 \quad \text{for all} \quad i > 0.$$

More generally, one can show that for every $j \geq 0$ and i

$$e_j f_i = \begin{cases} (-1)^j \frac{i!}{(i-j)!}((j+1)(p-1-\lambda) - i + j) f_{i-j}, & \text{if } j \leq i, \\ 0, & \text{otherwise.} \end{cases}$$

Now, let $w_0, w_1, \ldots, w_{p-1}$ be a basis of $V_{-\chi}(p-1-\lambda)$ as in Section 2.1.2; here w_0 is the analogue to v_0 and $w_i = e_{-1}^i w_0$ for all i. A very simple computation shows that the mapping $V_\chi(\lambda)^* \to V_{-\chi}(p-1-\lambda)$ which sends f_i to w_i is an isomorphism of W^1–modules. □

Evidently, K is a self–dual module in the sense that it is isomorphic to its dual. Furthermore, we have

Proposition (B). *The module S is self–dual.*

Proof. We consider the basis $\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_{p-2}$ constructed in Section 2.2.1. Define $g_0 \in S^*$ such that $g_0(\bar{v}_i) = \delta_{p-2,i}$ and set $g_i = e_{-1}^i g_0$ for all $i = 1, 2, \ldots, p-2$. Since for all i and j, we have

$$g_i(\bar{v}_j) = (-1)^i g_0(e_{-1}^i \bar{v}_j) = (-1)^i \delta_{p-2,i+j},$$

it follows that $g_0, g_1, \ldots, g_{p-2}$ are linearly independent and hence form a basis of S^*. One can show that for every $j \geq 0$ and i

$$e_j g_i = \begin{cases} (-1)^{j+1} \frac{(i+1)!}{(i-j)!} g_{i-j}, & \text{if } j \leq i, \\ 0, & \text{otherwise.} \end{cases}$$

A very simple computation shows that the mapping $S^* \to S$ that sends g_i to $\bar{v}_i$ is an isomorphism of W^1–modules. □

2.3.3 We denote the space of $U_\chi(W^1)$–homomorphisms between two modules M and N by $\mathrm{Hom}_{U_\chi(W^1)}(M, N)$. The following two lemmas are needed later.

Lemma (A). *If* $\mathrm{ht}(\chi) = -1$, *then*

$$\mathrm{Hom}_{U_\chi(W^1)}(V_\chi(0), K) \simeq K.$$

Proof. It follows from (2.4) that there exists a surjective homomorphism $V_\chi(0) \to K$ and then that $\mathrm{Hom}_{U_\chi(W^1)}(V_\chi(0), K) \neq 0$. Let $v_0, v_1, \ldots, v_{p-1}$ be a basis of $V_\chi(\lambda)$ as in Section 2.1.2. Every nonzero homomorphism $\varphi \in \mathrm{Hom}_{U_\chi(W^1)}(V_\chi(0), K)$ is uniquely determined by the value of $\varphi(v_0)$ because

$$\varphi(v_i) = e_{-1}\varphi(v_{i-1}) = 0 \quad \text{for all} \quad i = 1, 2 \ldots, p-1.$$

This proves the lemma. □

Lemma (B). *If* $\mathrm{ht}(\chi) = -1$, *then*

$$\mathrm{Hom}_{U_\chi(W^1)}(V_\chi(0), S) = 0.$$

Proof. Every nonzero homomorphism $\varphi : V_\chi(0) \to S$ is surjective and thus has a kernel of dimension 1; say $\ker \varphi = Kx$ for some $x \in V_\chi(0)$. Evidently, $\ker \varphi$ is simple and hence isomorphic to the trivial module K. This implies $e_i x = 0$ for all i and in particular $e_0 x = 0$. Since the 0 weight space in $V_\chi(0)$ is one–dimensional, we may choose x such that $x = v_0$; here we use the notation introduced in Section 2.1.2. But this contradicts the fact that $e_{-1} v_0 \neq 0$. □

3 Extensions of the Verma modules

In this chapter, we shall determine the extensions of the reduced Verma modules having character χ of height at most one. As noted in Chapter 2, $\Lambda(\chi)$ coincides with the prime field of K for $\mathrm{ht}(\chi) \in \{-1, 0\}$. This is not the case if the height is 1 because $\mathrm{ht}(\chi) = 1$ implies $\lambda^p \neq \lambda$ for all $\lambda \in \Lambda(\chi)$. Luckily, this is not as bad as it might seem at first because we always have $\lambda_1 - \lambda_2 \in \mathbb{F}_p$ for all $\lambda_1, \lambda_2 \in \Lambda(\chi)$. This fact turns out to be crucial to make our arguments work in general. The proof is simple and follows immediately from Freshman's dream. Suppose, indeed, that $\lambda_1, \lambda_2 \in \Lambda(\chi)$. Then, by definition, we have $\lambda_1^p - \lambda_1 = \lambda_2^p - \lambda_2$ which implies $(\lambda_1 - \lambda_2)^p = \lambda_1 - \lambda_2$, thereby the claim.

3.1 Preliminaries

3.1.1 It is sometimes useful to abuse the notation and view integers as elements in $\mathbb{F}_p$; this will be clear from the context and will cause no confusion. For each $\mu \in \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, we let $[\mu] \in \{0, 1, \ldots, p-1\}$ denote the unique representative of μ. We shall usually denote the inverse of $\mu \neq 0$ by $1/\mu$. The following lemma will be used several times, often without any reference.

Lemma. *If $i \in \{1, 2, \ldots, p\}$, then*

$$(p-i)! = \frac{(-1)^i}{(i-1)!} \quad \textit{in} \quad \mathbb{F}_p.$$

Proof. The case $i = 1$ is an immediate consequence of Wilson's theorem. We proceed by induction on i; we assume that the assertion is true for $1 < i < p$. Since

$$(p-i)! = (p-i)(p-(i+1))!,$$

it follows that

$$(p-(i+1))! = \frac{(-1)^i}{(p-i)(i-1)!} = \frac{(-1)^{i+1}}{i!},$$

proving the lemma. □

3.1.2 Let L be a Lie algebra over K and let $\chi \in L^*$. Suppose that M is a $U_\chi(L)$–module. For each $i \geq 0$, we define the ith right derived functor

$$\operatorname{Ext}^i_{U_\chi(L)}(M, -) = R^i \operatorname{Hom}_{U_\chi(L)}(M, -).$$

If M and N are $U_\chi(L)$–modules, then $\operatorname{Ext}^1_{U_\chi(L)}(M,N)$ has an interpretation as the group of extensions $\operatorname{Ext}_{U_\chi(L)}(M,N)$ with elements

$$0 \to N \to E \to M \to 0,$$

see [20, Thm. 2.4]. We shall usually use the notation $\operatorname{Ext}_{U_\chi(L)}(M,N)$ instead of $\operatorname{Ext}^1_{U_\chi(L)}(M,N)$. There are similar "higher" extensions corresponding to $\operatorname{Ext}^i_{U_\chi(L)}(M,N)$ for $i > 1$. For every short exact sequence of $U_\chi(L)$–modules

$$0 \to N' \to N \to N'' \to 0,$$

we get a long exact sequence of K–vector spaces

$$\begin{aligned} 0 &\to \operatorname{Hom}_{U_\chi(L)}(M,N') \to \operatorname{Hom}_{U_\chi(L)}(M,N) \to \operatorname{Hom}_{U_\chi(L)}(M,N'') \\ &\to \operatorname{Ext}_{U_\chi(L)}(M,N') \to \operatorname{Ext}_{U_\chi(L)}(M,N) \to \operatorname{Ext}_{U_\chi(L)}(M,N'') \\ &\to \operatorname{Ext}^2_{U_\chi(L)}(M,N') \to \cdots . \end{aligned}$$

Similarly, if we fix N then a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of $U_\chi(L)$–modules leads to the long exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}_{U_\chi(L)}(M'',N) \to \operatorname{Hom}_{U_\chi(L)}(M,N) \to \operatorname{Hom}_{U_\chi(L)}(M',N) \\ &\to \operatorname{Ext}_{U_\chi(L)}(M'',N) \to \operatorname{Ext}_{U_\chi(L)}(M,N) \to \operatorname{Ext}_{U_\chi(L)}(M',N) \\ &\to \operatorname{Ext}^2_{U_\chi(L)}(M'',N) \to \cdots . \end{aligned}$$

We have for all finite dimensional $U_\chi(L)$–modules M and N a natural isomorphism

$$\operatorname{Hom}_{U_\chi(L)}(M,N) \simeq \operatorname{Hom}_{U_{-\chi}(L)}(N^*, M^*),$$

which gives rise to the isomorphism

$$\operatorname{Ext}^n_{U_\chi(L)}(M,N) \simeq \operatorname{Ext}^n_{U_{-\chi(L)}}(N^*, M^*).$$

for all $n \geq 0$.

3.2 Setting

3.2.1 Except for Section 4.3.1, we will always assume that χ is a character of height at most 1. Now, classifying the extensions of the reduced Verma modules can be quite complicated. The following lemma will, however, make it much easier to achieve our goal.

Lemma. *Let $\chi \in (W^1)^*$ and $\lambda \in \Lambda(\chi)$. If M is a $U_\chi(W^1)$–module, then*

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda), M) \simeq \mathrm{Ext}_{U_\chi(W^1_{\geq 0})}(K_\lambda, M).$$

Proof. We regard $U_\chi(W^1)$ as a free $U_\chi(W^1_{\geq 0})$–module in the natural way. This implies [19, Prop. 7.2.1]

$$\mathrm{Tor}_n^{U_\chi(W^1_{\geq 0})}(U_\chi(W^1), K_\lambda) = 0 \quad \text{for all} \quad n > 0,$$

where $\mathrm{Tor}_n^{U_\chi(L)}(-, M)$ is the nth left derived functor of $- \otimes_{U_\chi(L)} M$. The claim follows from [9, Prop. VI. 4.1.3]. □

As an immediate consequence, we obtain an isomorphism

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda'), V_\chi(\lambda)) \simeq \mathrm{Ext}_{U_\chi(W^1_{\geq 0})}(K_{\lambda'}, V_\chi(\lambda)),$$

that reduces the problem of classifying the extensions of Verma modules to that of classifying the extensions of $K_{\lambda'}$ by $V_\chi(\lambda)$.

3.2.2 Consider a short exact sequence of $U_\chi(W^1_{\geq 0})$–modules

$$0 \to V_\chi(\lambda) \xrightarrow{f} M \xrightarrow{g} K_{\lambda'} \to 0. \tag{3.1}$$

Let $v_0, v_1, \ldots, v_{p-1}$ be a basis for $V_\chi(\lambda)$ as in Section 2.1.2 and set $w_i = f(v_i)$ for all i. Since $U_\chi(W_0)$ is a semisimple algebra, there exists $w' \in M$ such that $e_0 w' = \lambda' w'$ and $g(w') = 1$. Thus, we obtain a basis for M

$$w_0, w_1, \ldots, w_{p-1}, w', \tag{3.2}$$

such that for all $0 \leq j \leq p-2$

$$e_j w_i = \begin{cases} (-1)^j \frac{i!}{(i-j)!}((j+1)\lambda - i + j) w_{i-j}, & \text{if } j \leq i, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Furthermore, if $1 \leq j \leq p-2$ then

$$e_0 e_j w' = (e_j e_0 + j e_j) w' = (\lambda' + j) e_j w',$$

hence $e_j w'$ belongs to the 1–dimensional weight space $M_{\lambda'+j}$ spanned by $w_{[\lambda-\lambda'-j]}$. It follows that

$$e_j w' = \mathfrak{a}_j w_{[\lambda-\lambda'-j]} \quad \text{for some} \quad \mathfrak{a}_j \in K. \tag{3.4}$$

3.2.3 The w' from the previous section is not unique. Since the λ' weight space in M is spanned by w' and $w_{[\lambda-\lambda']}$ any different choice for w' has the form $w' + bw_{[\lambda-\lambda']}$ for some $b \in K$. Obviously, this leads to the same $e_j w'$ if $e_j w_{[\lambda-\lambda']} = 0$. Thus, if $\lambda = \lambda'$ then $(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_{p-2})$ is determined by the extension (3.1). Furthermore, we have

Lemma (A)**.** *Let* $\lambda, \lambda' \in \Lambda(\chi)$*. If* $(\lambda, \lambda') = (0, p-1)$ *or* $(\lambda, \lambda') = (p-1, 0)$*, then* $e_j w_{[\lambda-\lambda']} = 0$ *for all* $j \geq 1$*.*

Proof. By use of (3.3), we obtain

$$e_1 w_{[\lambda-\lambda']} = -(\lambda - \lambda')(\lambda + \lambda' + 1) w_{[\lambda-\lambda'-1]},$$

which equals 0 for $\lambda + \lambda' = p - 1$. If $(\lambda, \lambda') = (0, p-1)$ then $[\lambda - \lambda'] < 2$ and thus $e_j w_{[\lambda-\lambda']} = 0$ for all $j \geq 2$. If $(\lambda, \lambda') = (p-1, 0)$, we have for every $2 \leq j \leq p-2$

$$e_j w_{[\lambda-\lambda']} = (-1)^j \frac{[\lambda - \lambda']!}{([\lambda - \lambda'] - j)!} (j(\lambda + 1) + \lambda') w_{[\lambda-\lambda'-j]} = 0,$$

proving the lemma. □

Note that the assumptions of the lemma imply $\mathrm{ht}(\chi) < 1$ as $\lambda, \lambda' \notin \mathbb{F}_p$ for $\mathrm{ht}(\chi) = 1$. We let $\Theta(\chi) \subset \Lambda(\chi) \times \Lambda(\chi)$ denote the subset given by

$$(\{(0, p-1), (p-1, 0)\} \cap \Lambda(\chi) \times \Lambda(\chi)) \cup \{(\mu, \mu) \mid \mu \in \Lambda(\chi)\}.$$

Lemma A and the discussion at the beginning of this section show that all the $\mathfrak{a}_j$ are determined by M if $(\lambda, \lambda') \in \Theta(\chi)$. This is not the case for $(\lambda, \lambda') \notin \Theta(\chi)$. However, we have

Lemma (B)**.** *Let* $\lambda, \lambda' \in \Lambda(\chi)$ *such that* $(\lambda, \lambda') \notin \Theta(\chi)$*. If furthermore* $\lambda + \lambda' \neq p - 1$ *(resp.* $\lambda + \lambda' = p - 1$*), then there is a unique choice for* w' *such that* $e_1 w' = 0$ *(resp.* $e_2 w' = 0$*).*

Proof. Suppose that $(\lambda, \lambda') \notin \Theta(\chi)$. Since

$$e_1 w_{[\lambda-\lambda']} = -(\lambda - \lambda')(\lambda + \lambda' + 1) w_{[\lambda-\lambda'-1]},$$

it follows that $e_1 w_{[\lambda-\lambda']} \neq 0$ for $\lambda + \lambda' \neq p - 1$. The discussion at the beginning of this section shows that in this case we can choose w' uniquely such that $e_1 w' = 0$. Assume next that $\lambda + \lambda' = p - 1$. We have $\lambda - \lambda' \neq 1$ and $2\lambda + \lambda' + 2 \neq 0$ since otherwise this would imply $(\lambda, \lambda') = (0, p-1)$ or $(\lambda, \lambda') = (p-1, 0)$, respectively. It follows that

$$e_2 w_{[\lambda-\lambda']} = (\lambda - \lambda')(\lambda - \lambda' - 1)(2\lambda + \lambda' + 2) w_{[\lambda-\lambda'-2]} \neq 0,$$

so there is a unique choice for w' such that $e_2 w' = 0$. □

Remark (A)**.** For each pair $(\lambda, \lambda') \notin \Theta(\chi)$, we will always assume that w' is the unique choice from the lemma. It then follows that the tuple $(\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_{p-2})$ determines the extension (3.1). We call this tuple *the* $\mathfrak{a}$–*datum of* (3.1).

Remark (B)**.** If χ is of height 1, then $\Theta(\chi) = \{(\mu, \mu) \mid \mu \in \Lambda(\chi)\}$. Moreover, the fact that $\lambda - \lambda' \in \mathbb{F}_p$ implies $\lambda + \lambda' \notin \mathbb{F}_p$ because $\lambda, \lambda' \notin \mathbb{F}_p$ for $\mathrm{ht}(\chi) = 1$. Therefore, Lemma B becomes: *If* $\lambda \neq \lambda'$*, then there is a unique choice for* w' *such that* $e_1 w' = 0$.

3.2.4

Consider now a second extension

$$0 \to V_\chi(\lambda) \xrightarrow{f'} M' \xrightarrow{g'} K_{\lambda'} \to 0 \tag{3.5}$$

of $U_\chi(W^1_{\geq 0})$–modules. If (3.1) and (3.5) are equivalent extensions then there is an isomorphism $h : M \xrightarrow{\sim} M'$ compatible with the identities in both $V_\chi(\lambda)$ and $K_{\lambda'}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_\chi(\lambda) & \xrightarrow{f} & M & \xrightarrow{g} & K_{\lambda'} & \longrightarrow & 0 \\ & & \| & & \downarrow h & & \| & & \\ 0 & \longrightarrow & V_\chi(\lambda) & \xrightarrow{f'} & M' & \xrightarrow{g'} & K_{\lambda'} & \longrightarrow & 0. \end{array}$$

We can choose $w'' = h(w')$ to be the analogue to w'. Then $e_i w'' = h(e_i w') = h(f(\mathfrak{a}_i v_{[\lambda-\lambda'-i]})) = f'(\mathfrak{a}_i v_{[\lambda-\lambda'-i]})$ for all $1 \leq i \leq p-2$. Hence $f^{-1}(e_i w')$ depends only on the class of the extension (3.1) and we obtain a well–defined map

$$\Phi^i_{\lambda,\lambda'} \colon \operatorname{Ext}_{U_\chi(W^1_{\geq 0})}(K_{\lambda'}, V_\chi(\lambda)) \to K, \tag{3.6}$$

that sends the class of (3.1) to $\mathfrak{a}_i$. In Section 3.2.5 we prove that $\Phi^i_{\lambda,\lambda'}$ is linear and in Section 3.2.6 we show that every $\mathfrak{a}_i$ with $i \geq 3$ can be expressed as a linear combination of $\mathfrak{a}_1$ and $\mathfrak{a}_2$. It follows that $\Phi_{\lambda,\lambda'} = (\Phi^1_{\lambda,\lambda'}, \Phi^2_{\lambda,\lambda'})$ maps $\operatorname{Ext}_{U_\chi(W^1_{\geq 0})}(K_{\lambda'}, V_\chi(\lambda))$ injectively into K^2.

Remark. If $(\lambda, \lambda') \notin \Theta(\chi)$ then $\dim \operatorname{Ext}_{U_\chi(W^1_{\geq 0})}(K_{\lambda'}, V_\chi(\lambda)) \leq 1$. This is a consequence of Lemma 3.2.3 B.

3.2.5

We claim that $\Phi^i_{\lambda,\lambda'}$ is a linear map. If we have extensions as in (3.1) and (3.5), then the Baer sum is represented by

$$0 \to V_\chi(\lambda) \xrightarrow{\tilde{f}} N/N' \xrightarrow{\tilde{g}} K_{\lambda'} \to 0,$$

where $N \subset M \oplus M'$ is the submodule of all (m, m') with $g(m) = g'(m')$ and $N' \subset N$ is the submodule of all $(f(x), -f'(x))$ with $x \in V_\chi(\lambda)$. The homomorphisms $\widetilde{f}$ and $\widetilde{g}$ are given by $\widetilde{f}(x) = (f(x), 0)+N' = (0, f'(x))+N'$ and $\widetilde{g}((m, m')+N') = g(m) = g'(m')$. We can choose $(w', w'')+N'$ to be the analogue to w'. Let $f(x) = e_i w'$ and $f'(x') = e_i w''$ for suitable $x, x' \in V_\chi(\lambda)$. Then

$$e_i((w', w'') + N') = (f(x), f'(x')) + N' = \widetilde{f}(x) + \widetilde{f}(x') = \widetilde{f}(x + x'),$$

hence ϕ_i is a group homomorphism. To prove that it is also closed under scalar multiplication, let $b \in K \setminus \{0\}$ and assume that the class of (3.5) is b times the class of (3.1). Then there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_\chi(\lambda) & \xrightarrow{f} & M & \xrightarrow{g} & K_{\lambda'} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle b\cdot\mathrm{id}} & & \downarrow{\scriptstyle h} & & \| & & \\ 0 & \longrightarrow & V_\chi(\lambda) & \xrightarrow{f'} & M' & \xrightarrow{g'} & K_{\lambda'} & \longrightarrow & 0. \end{array}$$

If we again choose $w'' = h(w')$ we get

$$(f')^{-1}(e_i w'') = (f')^{-1}(h(e_i w')) = b f^{-1}(e_i w'),$$

thereby the claim.

3.2.6 In this section, we prove that every $\mathfrak{a}_i$ with $i \geq 3$ can be expressed as a linear combination of $\mathfrak{a}_1$ and $\mathfrak{a}_2$. Since M is a $U_\chi(W^1_{\geq 0})$–module, we have

$$[e_i, e_j]w' = (e_i e_j - e_j e_i)w' \quad \text{for all } i \text{ and } j.$$

We will need the full strength of this formula later but for now we are content to remark that when $i = 1$ and $2 \leq j \leq p-3$, it yields

$$\begin{aligned}(j-1)\mathfrak{a}_{j+1} = {} & -(-1)^j \left(\prod_{k=1}^{j}(\lambda - \lambda' - k)\right)(j(\lambda+1) + \lambda' + 1)\mathfrak{a}_1 \\ & -(\lambda - \lambda' - j)(\lambda + \lambda' + j + 1)\mathfrak{a}_j,\end{aligned}$$

or, equivalently, by induction

$$\mathfrak{a}_j = A_j \mathfrak{a}_1 + B_j \mathfrak{a}_2 \quad \text{for all} \quad 3 \leq j \leq p-2, \tag{3.7}$$

where

$$\begin{aligned}A_j = \frac{(-1)^j}{j-2}\left(\prod_{k=1}^{j-1}(\lambda - \lambda' - k)\right)\Bigg((j-1)\lambda + \lambda' + j + \sum_{k=4}^{j}\frac{(j-k)!}{(j-3)!} \\ \cdot ((j+2-k)\lambda + \lambda' + (j+3-k))\prod_{l=0}^{k-4}(\lambda + \lambda' + j - l)\Bigg),\end{aligned}$$

(the summation $\sum_{k=4}^{j}$ is understood to be 0 when $j = 3$) and

$$B_j = \frac{(-1)^j}{(j-2)!} \prod_{k=2}^{j-1} (\lambda - \lambda' - k)(\lambda + \lambda' + k + 1).$$

Consequently, all the $\mathfrak{a}_i$ with $i \geq 3$ are determined by $\mathfrak{a}_1$ and $\mathfrak{a}_2$.

3.3 The possible $\mathfrak{a}$–data

3.3.1 We have proved that $\Phi_{\lambda,\lambda'}$ maps $\mathrm{Ext}_{U_\chi(W^1_{\geq 0})}(K_{\lambda'}, V_\chi(\lambda))$ injectively into K^2. In the following, we shall be concerned with the image of $\Phi_{\lambda,\lambda'}$. We want to describe all possible $\mathfrak{a}$–data of extensions as in (3.1). To this end, consider an arbitrary pair $(\mathfrak{a}_1, \mathfrak{a}_2) \in K^2$ and use (3.7) to extend it to a tuple $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_{p-2})$ in K^{p-2}. In order to simplify notation set $\mathfrak{a}_i = 0$ for all $i > p-2$. Consider a vector space $M_\mathfrak{a}$ with a basis $w_0, w_1, \ldots, w_{p-1}, w'$ and define endomorphisms E_i of $M_\mathfrak{a}$ acting as e_i on the basis vectors. We have on $M_\mathfrak{a}$ a $W^1_{\geq 0}$–module structure if and only if

$$[E_i, E_j]w = (j-i)E_{i+j}w \quad \text{for all } i, j \geq 0 \text{ and } w \in M_\mathfrak{a}. \tag{3.8}$$

It suffices to check this for all w in our basis. The linear map $f\colon V_\chi(\lambda) \to M_\mathfrak{a}$ with $f(v_i) = w_i$ for all i satisfies $E_j f(v) = f(E_j v)$ for all $v \in V_\chi(\lambda)$ and all j. Consequently, (3.8) holds for all w_i and $M_\mathfrak{a}$ has the desired structure if and only if

$$[E_i, E_j]w' = (j-i)E_{i+j}w' \quad \text{for all } i, j \geq 0. \tag{3.9}$$

3.3.2 One verifies easily that (3.9) holds for $i = 0$. It also holds for $i = j$. For $i = 1$ and $2 \leq j \leq p-3$ it is equivalent to (3.7) which holds by the definition of the $\mathfrak{a}_j$ ($j \geq 3$). Furthermore, for every (i, j) in

$$\{(1, p-2), (p-2, 1)\} \cup \{(a, b) \in \mathbb{Z}^2 \mid a \neq b \text{ and } 2 \leq a, b \leq p-2\},$$

formula (3.9) is equivalent to the following conditions

If $i > [\lambda - \lambda' - j]$ and $j > [\lambda - \lambda' - i]$, then

$$\mathfrak{a}_{i+j} = 0. \tag{3.10}$$

If $i > [\lambda - \lambda' - j]$ and $j \leq [\lambda - \lambda' - i]$, then

$$(j-i)\mathfrak{a}_{i+j} = (-1)^{j+1} \frac{[\lambda - \lambda' - i]!}{([\lambda - \lambda' - i] - j)!} (j(\lambda+1) + \lambda' + i)\mathfrak{a}_i. \tag{3.11}$$

If $i \leq [\lambda - \lambda' - j]$ and $j \leq [\lambda - \lambda' - i]$, then

$$\begin{aligned}(j-i)\mathfrak{a}_{i+j} = (-1)^i \frac{[\lambda - \lambda' - j]!}{([\lambda - \lambda' - j] - i)!}(i(\lambda+1) + \lambda' + j)\mathfrak{a}_j \\ - (-1)^j \frac{[\lambda - \lambda' - i]!}{([\lambda - \lambda' - i] - j)!}(j(\lambda+1) + \lambda' + i)\mathfrak{a}_i.\end{aligned} \tag{3.12}$$

It should be noted that there is no deep mathematics involved here; only straightforward but tedious computations.

3.3.3 Suppose now that $\mathfrak{a}$ satisfies (3.10)–(3.12) and hence that $M_{\mathfrak{a}}$ is a $W^1_{\geq 0}$–module with each e_i acting as E_i. We have then a short exact sequence

$$0 \to V_\chi(\lambda) \xrightarrow{f} M_{\mathfrak{a}} \xrightarrow{g} K_{\lambda'} \to 0. \tag{3.13}$$

Evidently, $M_{\mathfrak{a}}$ has p–character χ if and only if $(e_i^p - e_i^{[p]} - \chi(e_i)^p)w = 0$ for all w in our basis. This is clearly true for all w_i. Moreover, since $\Lambda(\chi) = \{\mu \in K \mid \mu^p - \mu = \chi(e_0)^p\}$, we have $(e_0^p - e_0^{[p]} - \chi(e_0)^p)w' = 0$. Thus, to prove that χ is the p–character of $M_{\mathfrak{a}}$, it suffices to check that $e_i^p w' = 0$ for all $i \geq 1$, or, equivalently, that $\mathfrak{a}_i e_i^{p-1} w_{[\lambda-\lambda'-i]} = 0$ for all $i \geq 1$. Since this is trivial for $i > 1$, we conclude that $M_{\mathfrak{a}}$ has p–character χ if and only if e_1^p annihilates w'. Now, $\mathfrak{a}_1 e_1^{p-1} w_{[\lambda-\lambda'-1]} = 0$ if $\lambda \neq \lambda'$, because the subfix of $w_{[\lambda-\lambda'-1]}$ drops by 1 each time we apply e_1. For $\lambda = \lambda'$, we have

$$\mathfrak{a}_1 e_1^{p-1} w_{p-1} = -\mathfrak{a}_1 \left(\prod_{j=0}^{p-2} (2\lambda - j) \right) w_0.$$

The term inside the bracket does not vanish if the height of χ is 1 because $\lambda \notin \mathbb{F}_p$ in this case. We obtain the following lemma.

Lemma. *$M_{\mathfrak{a}}$ has p–character χ if and only if one of the following conditions holds*

1. *If* $\mathrm{ht}(\chi) \in \{-1, 0\}$ *and* $\lambda = \lambda'$ *and* $2\lambda = p - 1$, *then* $\mathfrak{a}_1 = 0$.
2. *If* $\mathrm{ht}(\chi) = 1$ *and* $\lambda = \lambda'$, *then* $\mathfrak{a}_1 = 0$.

3.4 Case $(\lambda, \lambda') \in \Theta(\chi)$

3.4.1 We first look at the case where $\lambda = \lambda'$.

Proposition. *We have*

$$\operatorname{Ext}_{U_\chi(W^1_{\geq 0})}(K_\lambda, V_\chi(\lambda)) \simeq \begin{cases} K, & \text{if } \lambda \in \{0, p-1\}, \\ 0, & \text{otherwise.} \end{cases}$$

It is convenient to break up the proof of the proposition into two lemmas.

Lemma (A). *If* $\operatorname{ht}(\chi) = 1$*, then*

$$\operatorname{Ext}_{U_\chi(W^1_{\geq 0})}(K_\lambda, V_\chi(\lambda)) = 0.$$

Proof. Every extension of K_λ by $V_\chi(\lambda)$ can be represented by a short exact sequence of $U_\chi(W^1_{\geq 0})$–modules

$$0 \to V_\chi(\lambda) \to M_{\mathfrak{a}} \to K_\lambda \to 0.$$

The claim to be proved amounts to saying that the above sequence splits, or, equivalently, that $\mathfrak{a} = 0$. Since all the $\mathfrak{a}_j$ can be expressed as a linear combination of $\mathfrak{a}_1$ and $\mathfrak{a}_2$, it suffices to prove that both $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are 0. Note that $\mathfrak{a}_1 = 0$ follows from Lemma 3.3.3. Thus, by definition, we have

$$\mathfrak{a}_j = (j-1)\left(\prod_{k=3}^{j}(2\lambda + k)\right)\mathfrak{a}_2 \quad \text{for} \quad 3 \leq j \leq p-2.$$

Since the height of χ is 1, it follows that $2\lambda + k \neq 0$ for all k. Therefore, in order to prove $\mathfrak{a}_2 = 0$ it suffices to show $\mathfrak{a}_j = 0$ for any $2 \leq j \leq p-2$. Now, if we insert $(i, j) = (1, p-2)$ into (3.12) and use the fact that $(p-1)! = -1$ we get

$$-2(2\lambda - 1)\mathfrak{a}_{p-2} = 0,$$

proving the claim. □

Lemma (B). *If* $\operatorname{ht}(\chi) \in \{-1, 0\}$*, then*

$$\operatorname{Ext}_{U_\chi(W^1_{\geq 0})}(K_\lambda, V_\chi(\lambda)) \simeq \begin{cases} K, & \text{if } \lambda \in \{0, p-1\}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $(\mathfrak{a}_1, \mathfrak{a}_2)$ be an arbitrary pair in K^2 and consider the vector space $M_{\mathfrak{a}}$ constructed in Section 3.3.1. By definition, we have

$$\mathfrak{a}_j = A_j \mathfrak{a}_1 + B_j \mathfrak{a}_2 \quad \text{for all} \quad 3 \leq j \leq p-2, \tag{3.14}$$

where

$$A_j = -\frac{\lambda+1}{j-2}\left(j! + \sum_{k=4}^{j}(k-1)!\prod_{l=k}^{j}\frac{l-1}{l-3}(2\lambda+l)\right),$$

(the summation $\sum_{k=4}^{j}$ is understood to be 0 when $j = 3$) and

$$B_j = (j-1)\prod_{k=3}^{j}(2\lambda+k).$$

We rewrite (3.10)–(3.12) for the present case. First, note that (3.10) and (3.11) hold trivially since $[\lambda-\lambda-i] = p-i$ and $[\lambda-\lambda-j] = p-j$ for all $1 \le i,j \le p-2$. If we insert $(i,j) = (1,p-2)$ into (3.12) and use $(p-1)! = -1$, we see that

$$2(2\lambda-1)\mathfrak{a}_{p-2} = (\lambda+1)\mathfrak{a}_1. \tag{3.15}$$

If $2 \le i,j \le p-2$ and $i+j \le p$, the same formula yields

$$(j-i)\mathfrak{a}_{i+j} = \frac{(i+j-1)!}{(j-1)!}((i+1)\lambda+i+j)\mathfrak{a}_j - \frac{(i+j-1)!}{(i-1)!}((j+1)\lambda+i+j)\mathfrak{a}_i. \tag{3.16}$$

We now proceed by showing how these formulas are related. Let us begin by assuming that $\lambda \notin \{0, p-1\}$. The claim to be proved amounts to saying that $\mathfrak{a} = 0$, or, equivalently, that both $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are 0. If $2\lambda = p-1$ then Lemma 3.3.3 implies $\mathfrak{a}_1 = 0$, which together with (3.14) implies

$$\mathfrak{a}_j = (j-1)(j-1)!\mathfrak{a}_2 \quad \text{for all} \quad 3 \le j \le p-2. \tag{3.17}$$

It follows that $\mathfrak{a}_2 = 0$ because otherwise this would mean that $\mathfrak{a}_{p-2} \neq 0$ and then by (3.15) that $2\lambda = 1$ in contradiction with our assumption.

Suppose next that $2\lambda \neq p-1$. If we insert $(i,j) = (2,p-2)$ into (3.16), we obtain

$$6\mathfrak{a}_{p-2} = \mathfrak{a}_2. \tag{3.18}$$

Assume first that $2\lambda = 1$. Since $\lambda \neq p-1$, eq. (3.15) implies $\mathfrak{a}_1 = 0$ hence (3.14) reduces to

$$\mathfrak{a}_j = \frac{1}{6}(j-1)(j+1)!\mathfrak{a}_2 \quad \text{for all} \quad 3 \le j \le p-2.$$

In particular, we have $2\mathfrak{a}_{p-2} = \mathfrak{a}_2$, which together with (3.18) implies $\mathfrak{a}_2 = 0$, as desired.

Assume next that $2\lambda \neq 1$. Eq. (3.15) and (3.18) give the following relation

$$3(\lambda+1)\mathfrak{a}_1 = (2\lambda - 1)\mathfrak{a}_2, \tag{3.19}$$

which means that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are either both 0 or both nonzero. We assume the latter and derive a contradiction. By definition, we have $\mathfrak{a}_3 = -6(\lambda + 1)\mathfrak{a}_1 + 2(2\lambda+3)\mathfrak{a}_2$. Hence it follows from (3.19) that $\mathfrak{a}_3 = 8\mathfrak{a}_2$. Thus, if we insert $(i,j) = (3, p-3)$ into (3.16), we obtain

$$3\mathfrak{a}_{p-3} = -\mathfrak{a}_2.$$

Likewise, if we insert $(i,j) = (2, p-3)$ into (3.16), we obtain

$$6(3\lambda - 1)\mathfrak{a}_{p-3} = -(2\lambda+1)\mathfrak{a}_2.$$

Since $2\lambda + 1 \neq 0$, we can determine λ by eliminating $\mathfrak{a}_2$ and $\mathfrak{a}_{p-3}$. This yields

$$4\lambda = 3.$$

Note that $3\mathfrak{a}_{p-3} = -\mathfrak{a}_2$ implies $p > 5$. If we insert $(i,j) = (2,3)$ into (3.16), we see

$$\mathfrak{a}_5 = (192\lambda + 360)\mathfrak{a}_2 = 504\mathfrak{a}_2.$$

However, by definition we have

$$\mathfrak{a}_5 = \frac{1}{3}(128\lambda^2 + 656\lambda + 1080)\mathfrak{a}_2 = 548\mathfrak{a}_2,$$

which implies first that $p = 11$ and then that $\lambda = 9$. If we put all this together and use (3.14), we see $\mathfrak{a}_1 = \mathfrak{a}_4$. But inserting $(i,j) = (4,6)$ into (3.16) implies $\mathfrak{a}_4 = 0$ and thus $\mathfrak{a}_1 = 0$. Contradiction!

We now move to the case where $\lambda = 0$. A necessary condition for $M_{\mathfrak{a}}$ to be a $U_\chi(W^1_{\geq 0})$–module is that $2\mathfrak{a}_{p-2} = -\mathfrak{a}_1$, cf. (3.15). Since by definition

$$\mathfrak{a}_{p-2} = 4\mathfrak{a}_1 - \frac{3}{2}\mathfrak{a}_2,$$

this implies $\mathfrak{a}_2 = 3\mathfrak{a}_1$. Now, the claim to be proved amounts to saying that each pair $(\mathfrak{a}_1, \mathfrak{a}_2)$ with $\mathfrak{a}_2 = 3\mathfrak{a}_1$ induces a $U_\chi(W^1_{\geq 0})$–module $M_{\mathfrak{a}}$ in the way described in Section 3.3.1. Using our previous notation, we have

$$A_j = -(j-2)j! \quad \text{and} \quad B_j = \frac{1}{2}(j-1)j!.$$

Hence $\mathfrak{a}_2 = 3\mathfrak{a}_1$ implies

$$\mathfrak{a}_j = \frac{1}{2}(j+1)!\mathfrak{a}_1 \quad \text{for all} \quad 1 \leq j \leq p-2.$$

In particular, we have $\mathfrak{a}_{p-2} = -1/2\mathfrak{a}_1$ in consistence with (3.15). A straightforward computation shows that

$$(j-i)A_{i+j} = (i+j)!\left(\frac{A_j}{(j-1)!} - \frac{A_i}{(i-1)!}\right),$$
$$(j-i)B_{i+j} = (i+j)!\left(\frac{B_j}{(j-1)!} - \frac{B_i}{(i-1)!}\right).$$

proving (3.16) for $i+j \leq p-2$. Likewise, a simple computation shows

$$\mathfrak{a}_j(p-j-2)! = \mathfrak{a}_{p-j-1}(j-1)!,$$

proving (3.16) for $i+j = p-1$. Since $i+j = p$ gives 0 on both sides of (3.16), the claim follows from Lemma 3.3.3.

Finally, we consider the case where $\lambda = p-1$. Here we have $A_j = 0$ for all $j \geq 3$ and it follows from (3.15) that $\mathfrak{a}_{p-2} = 0$. Therefore, since

$$\mathfrak{a}_j = (j-1)!\mathfrak{a}_2 \quad \text{for all} \quad 2 \leq j \leq p-2,$$

this implies first $\mathfrak{a}_2 = 0$ and then $\mathfrak{a}_j = 0$ for all $j \geq 2$. The rest of the proof is straightforward. □

3.4.2 Next, we address the cases where $(\lambda, \lambda') = (0, p-1)$ and $(\lambda, \lambda') = (p-1, 0)$. This can only occur if the height of χ is less than 1 because $\lambda, \lambda' \notin \mathbb{F}_p$ for $\mathrm{ht}(\chi) = 1$. The arguments presented previously would apply equally well to these cases, but we will give another, simpler, proof which relies on the fact that $\mathrm{Ext}_{U_\chi(W^1_{\geq 0})}(K_0, V_\chi(0))$ is 1–dimensional, cf. Proposition 3.4.1.

Proposition. *If* $(\lambda, \lambda') = (0, p-1)$ *or* $(\lambda, \lambda') = (p-1, 0)$*, then*

$$\mathrm{Ext}_{U_\chi(W^1_{\geq 0})}(K_{\lambda'}, V_\chi(\lambda)) \simeq K.$$

Proof. Let $\eta \in (W^1)^*$ be a character of height 0. We have

$$\mathrm{Ext}_{U_\eta(W^1)}(V_\eta(0), V_\eta(0)) \simeq \mathrm{Ext}_{U_\chi(W^1_{\geq 0})}(K_0, V_\chi(0)) \simeq K.$$

Since the reduced Verma modules $V_\eta(0)$ and $V_\eta(p-1)$ are isomorphic [4, Hilfssatz 7], we obtain

$$\mathrm{Ext}_{U_\chi(W^1_{\geq 0})}(K_0, V_\chi(p-1)) \simeq \mathrm{Ext}_{U_\eta(W^1)}(V_\eta(0), V_\eta(p-1)) \simeq K,$$

as claimed. The other case can be handled similarly. □

3.4.3 The results of the previous sections can be summarized into the following proposition.

Proposition. *If* $(\lambda, \lambda') \in \Theta(\chi)$, *then*

$$\mathrm{Ext}_{U_\chi(W^1_{\geq 0})}(K_{\lambda'}, V_\chi(\lambda)) \simeq \begin{cases} K, & \text{if } \lambda = \lambda' \text{ and } \lambda \in \{0, p-1\}, \\ K, & \text{if } (\lambda, \lambda') \in \{(0, p-1), (p-1, 0)\}, \\ 0, & \text{otherwise.} \end{cases}$$

3.5 Case $(\lambda, \lambda') \notin \Theta(\chi)$

3.5.1 It is convenient to divide the case $(\lambda, \lambda') \notin \Theta(\chi)$ into two subcases depending on whether or not $\lambda + \lambda' = p - 1$. Recall that $\lambda + \lambda' = p - 1$ can only occur if $\mathrm{ht}(\chi) < 1$. The following lemma will be needed in the sequel.

Lemma. *Let i and j be two integers with $2 \leq i, j \leq p - 2$.*

1. *If $i > [\lambda - \lambda' - j]$ and $i + j \leq p$, then $[\lambda - \lambda' - j] = [\lambda - \lambda'] - j$ and $i + j > [\lambda - \lambda']$.*

2. *If $i > [\lambda - \lambda' - j]$, then $i + j > [\lambda - \lambda']$.*

3. *If $j \leq [\lambda - \lambda' - i]$ and $i + j > [\lambda - \lambda']$ then $i > [\lambda - \lambda']$.*

Proof. (1) We have clearly $[\lambda - \lambda' - j] = [[\lambda - \lambda'] - j]$ and

$$-p < [\lambda - \lambda'] - j < p,$$

so if $[\lambda - \lambda' - j] \neq [\lambda - \lambda'] - j$, this implies $[\lambda - \lambda' - j] = [\lambda - \lambda'] - j + p$. Since $i > [\lambda - \lambda' - j]$, it follows that $i + j > [\lambda - \lambda'] + p$, in contradiction to our assumption that $i + j \leq p$. (2) We may assume that $i + j \leq p$ since $[\lambda - \lambda'] < p$. But then the assertion follows from the first claim. (3) If $i \leq [\lambda - \lambda']$, then $[\lambda - \lambda' - i] = [\lambda - \lambda'] - i$ hence the assumption $j \leq [\lambda - \lambda' - i]$ implies $i + j \leq [\lambda - \lambda']$. □

3.5.2 Since K is algebraically closed every polynomial in $K[x]$ of degree ≥ 1 has a root in K. We denote the roots of the polynomial $x^2 - a \in K[x]$ by $\pm\sqrt{a}$.

Proposition. *If* $(\lambda, \lambda') \notin \Theta(\chi)$ *and* $\lambda + \lambda' \neq p-1$*, then*

$$\mathrm{Ext}_{U_\chi(W^1_{\geq 0})}(K_{\lambda'}, V_\chi(\lambda)) \simeq \begin{cases} K, & \textit{if } [\lambda-\lambda'] = 1 \textit{ and } \lambda \in \{1, p-1\}, \\ K, & \textit{if } [\lambda-\lambda'] \in \{2,3\}, \\ K, & \textit{if } [\lambda-\lambda'] = 4 \textit{ and } p > 5, \\ K, & \textit{if } [\lambda-\lambda'] = 4 \textit{ and } \lambda \in \{0,3\}, \\ K, & \textit{if } [\lambda-\lambda'] = 5 \textit{ and } \lambda \in \{0,4\}, \\ K, & \textit{if } [\lambda-\lambda'] = 6 \textit{ and } 2\lambda = 5 \pm \sqrt{19} \textit{ and } p > 7, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Every extension of $K_{\lambda'}$ by $V_\chi(\lambda)$ can be represented by a short exact sequence of $U_\chi(W^1_{\geq 0})$–modules

$$0 \to V_\chi(\lambda) \to M_{\mathfrak{a}} \to K_{\lambda'} \to 0, \tag{3.20}$$

where $\mathfrak{a}_1 = 0$ (see Lemma 3.2.3 B and Remark 3.2.3 A) and

$$\mathfrak{a}_j = \frac{(-1)^j}{(j-2)!}\left(\prod_{k=2}^{j-1}(\lambda-\lambda'-k)(\lambda+\lambda'+k+1)\right)\mathfrak{a}_2, \quad 3 \leq j \leq p-2. \tag{3.21}$$

Note that $\mathfrak{a}_j = 0$ for all $j > [\lambda-\lambda']$ if $[\lambda-\lambda'] \geq 2$. This is a very useful observation which we shall use several times, often without any reference. We divide the proof into several steps depending on $[\lambda-\lambda']$.

Case 1. Suppose that $[\lambda-\lambda'] = 1$.

For the sake of simplicity, we assume that $p > 5$. (The case $p = 5$ is left to the reader.) Eq. (3.21) becomes

$$\mathfrak{a}_j = \left(\prod_{k=2}^{j-1}(2\lambda+k)\right)\mathfrak{a}_2 \quad \text{for all} \quad 3 \leq j \leq p-2.$$

If we insert $(i,j) = (2,3)$ into (3.12) and then use the above formula to express $\mathfrak{a}_3$ as a scalar multiple $\mathfrak{a}_2$ we obtain

$$\mathfrak{a}_5 = (2\lambda+2)(18\lambda+12)\mathfrak{a}_2.$$

A necessary condition for (3.20) to be non-split is that $\mathfrak{a} \neq 0$, or equivalently, that $\mathfrak{a}_2 \neq 0$. If this is the case, then

$$\prod_{k=2}^{4}(2\lambda+k) = (2\lambda+2)(18\lambda+12),$$

which implies $\lambda \in \{1, p-1\}$. (It should be noted that $\lambda = 0$ is a solution to the above equation, but it has been deliberately omitted because otherwise we would have $\lambda' = p-1$ and thus $\lambda + \lambda' = p-1$.)

Conversely, suppose that $\lambda = 1$ or $\lambda = p-1$. Since $\lambda - \lambda' = 1$, this implies $\lambda' = 0$ or $\lambda' = p-2$, respectively. Let $(\mathfrak{a}_1, \mathfrak{a}_2)$ be a pair in K^2 such that $\mathfrak{a}_1 = 0$ and consider the corresponding vector space $M_{\mathfrak{a}}$. If $\lambda = p-1$, then $\lambda + \lambda' = p-3$ hence by definition $\mathfrak{a}_j = 0$ for all $j \neq 2$. It follows that (3.10)−(3.12) hold trivially for all $i, j \neq 2$. That they hold for $i = 2$ follows from

$$2 < [\lambda - \lambda' - j] \quad \text{and} \quad j < [\lambda - \lambda' - 2],$$

together with the fact that $j(\lambda+1) + \lambda' + 2 = 0$, cf. (3.12). The case $j = 2$ is treated similarly. Suppose next that $\lambda = 1$. We have then

$$p-2 > [\lambda - \lambda' - 1] \quad \text{and} \quad 1 < [\lambda - \lambda' - (p-2)].$$

If we insert $(i, j) = (p-2, 1)$ into (3.11), we obtain zero on both sides because $(\lambda+1) + \lambda' - 2 = 0$ and $\mathfrak{a}_{p-1} = 0$. Furthermore, if $2 \le i, j \le p-2$ then

$$[\lambda - \lambda' - i] = p + 1 - i \quad \text{and} \quad [\lambda - \lambda' - j] = p + 1 - j.$$

Thus, we may assume that

$$i \le [\lambda - \lambda' - j] \quad \text{and} \quad j \le [\lambda - \lambda' - i],$$

or, equivalently, that $i + j \le p + 1$. We have

$$\mathfrak{a}_k = \frac{1}{6}(k+1)!\mathfrak{a}_2 \quad \text{for all} \quad 2 \le k \le p-2,$$

which, when inserted into (3.12), yields

$$\begin{aligned}(j-i)\mathfrak{a}_{i+j} = {} & \frac{1}{6}(i+j-2)!(2i+j)(j-1)j(j+1)\mathfrak{a}_2 \\ & - \frac{1}{6}(i+j-2)!(2j+i)(i-1)i(i+1)\mathfrak{a}_2.\end{aligned}$$

(Here we have used Lemma 3.1.1.) Since

$$(j-i)\mathfrak{a}_{i+j} = \begin{cases} \frac{1}{6}(j-i)(i+j+1)!\mathfrak{a}_2, & \text{if } i+j \le p-2, \\ 0, & \text{if } i+j \in \{p, p \pm 1\}, \end{cases}$$

the claim follows from a straightforward computation.

Case 2. Suppose that $[\lambda - \lambda'] \in \{2, 3\}$.

Let $(\mathfrak{a}_1, \mathfrak{a}_2)$ be a pair in K^2 such that $\mathfrak{a}_1 = 0$ and consider the corresponding vector space $M_{\mathfrak{a}}$. Together with the remark following eq. (3.21), Lemma 3.5.1(2) yields (3.10) for all distinct integers $2 \leq i, j \leq p-2$. Meanwhile, (3.11) and (3.12) follow from Lemma 3.5.1(3) since $i + j > [\lambda - \lambda']$ for $2 \leq i, j \leq p-2$. We have $p - 2 > [\lambda - \lambda']$ for $p > 5$ which, together with the remark following (3.21), implies $\mathfrak{a}_{p-2} = 0$. Since $\mathfrak{a}_1 = 0$, this proves (3.10)–(3.12) for $\{i, j\} = \{1, p-2\}$. (Note that all of the above reasoning works just as well in the case where $\lambda - \lambda' = 4$.) Now, the same argument can be repeated to prove (3.10) for $p = 5$ and $\lambda - \lambda' = 2$. For $p = 5$ and $\lambda - \lambda' = 3$ we have

$$3 > [\lambda - \lambda' - 1] \quad \text{and} \quad 1 > [\lambda - \lambda' - 3],$$

which, since $\mathfrak{a}_4 = 0$, implies (3.10).

Case 3. Suppose that $[\lambda - \lambda'] = 4$.

As noted above, we may assume $p = 5$ and $(i, j) \in \{(1, 3), (3, 1)\}$. We have

$$3 = [\lambda - \lambda' - 1] \quad \text{and} \quad 1 = [\lambda - \lambda' - 3].$$

Thus, if we insert $(i, j) = (1, 3)$ into (3.12) and then use the definition of $\mathfrak{a}_3$ to express it as a scalar multiple of $\mathfrak{a}_2$, we obtain

$$2(\lambda + \lambda' + 3)(\lambda + \lambda' + 4)\mathfrak{a}_2 = 0.$$

Now, every extension of $K_{\lambda'}$ by $V_\chi(\lambda)$ can be represented by a short exact sequence as in (3.20) and such that $\mathfrak{a}_1 = 0$. The above computation shows that a necessary (and sufficient) condition for (3.20) to be non-split is that $\lambda + \lambda' \in \{1, 2\}$, or equivalently, that $\lambda \in \{0, 3\}$.

Case 4. Suppose that $[\lambda - \lambda'] = 5$.

First, note that the assumption implies $p > 5$ since $[\lambda - \lambda'] < p$. We consider a short exact sequence of $U_\chi(W^1_{\geq 0})$–modules as in (3.20). If we insert $(i, j) = (2, 3)$ into (3.12) and then use the definition of $\mathfrak{a}_5$ to express it as a scalar multiple $\mathfrak{a}_2$, we get

$$\frac{1}{6}\left(\prod_{k=3}^{5}(\lambda + \lambda' + k)\right)\mathfrak{a}_2 = (2\lambda + \lambda' + 5)(\lambda + \lambda' + 3)\mathfrak{a}_2 - (3\lambda + \lambda' + 5)\mathfrak{a}_2, \tag{3.22}$$

Setting $\mu = \lambda + \lambda'$ in the above equation gives

$$\frac{1}{6}\left(\prod_{k=3}^{5}(\mu + k)\right)\mathfrak{a}_2 = ((\mu + 1)\lambda + (\mu + 2)(\mu + 5))\mathfrak{a}_2,$$

which can be rewritten as

$$\frac{1}{6}\mu(\mu+1)(\mu+5)\mathfrak{a}_2 = (\mu+1)\lambda\mathfrak{a}_2. \tag{3.23}$$

Since by assumption $\lambda + \lambda' \neq p - 1$, we can reduce the above equation by dividing both sides by $\mu + 1$

$$((\lambda+\lambda')(\lambda+\lambda'+5) - 6\lambda)\mathfrak{a}_2 = 0.$$

Thus, the term inside the parenthesis must equal 0 in order for (3.20) to be non-split. Note that the above reasoning works just as well in the case $[\lambda - \lambda'] \geq 5$. We shall make use of this later, but for now we are content to remark that in our present case this implies $\lambda \in \{0, 4\}$.

Conversely, suppose that $\lambda \in \{0, 4\}$ and let $(\mathfrak{a}_1, \mathfrak{a}_2)$ be a pair in K^2 such that $\mathfrak{a}_1 = 0$. We extend $(\mathfrak{a}_1, \mathfrak{a}_2)$ as usual to a tuple $\mathfrak{a} \in K^{p-2}$. Suppose that $2 \leq i, j \leq p - 2$ are distinct. Eq. (3.10) follows immediately from Lemma 3.5.1(2). To prove (3.11) and (3.12), we may assume that $i + j \leq [\lambda - \lambda']$, see Lemma 3.5.1(3). It follows then that $\{i, j\} = \{2, 3\}$ and since we have chosen λ in such a way that (3.12) holds for $(i, j) = (2, 3)$, the claim becomes trivial. We move to the case where $\{i, j\} = \{1, p - 2\}$. If $p = 7$ then

$$p - 2 > [\lambda - \lambda' - 1] \quad \text{and} \quad 1 > [\lambda - \lambda' - (p-2)].$$

Since by definition $\mathfrak{a}_{p-1} = 0$, this proves (3.10). If $p > 7$, then

$$p - 2 > [\lambda - \lambda' - 1] \quad \text{and} \quad 1 < [\lambda - \lambda' - (p-2)].$$

The remark following eq. (3.21) implies $\mathfrak{a}_{p-2} = 0$ and thus (3.11), thereby proving the claim.

Case 5. Suppose that $[\lambda - \lambda'] = 6$.

As previously noted, every nontrivial extension of $K_{\lambda'}$ by $V_\chi(\lambda)$ can be represented by a sequence of $U_\chi(W^1_{\geq 0})$–modules as in (3.20) and such that $\mathfrak{a}_1 = 0$ and $(\lambda+\lambda')(\lambda+\lambda'+5) = 6\lambda$. In our present case, this amounts to

$$\lambda = \frac{1}{2}(5 \pm \sqrt{19}).$$

We claim that this implies $p > 7$. Indeed, for $\mathrm{ht}(\chi) < 1$ we have $\lambda \in \mathbb{F}_p$ and since $x^2 - 19$ does not split in $\mathbb{F}_7[x]$, this shows that $p > 7$. For $\mathrm{ht}(\chi) = 1$ we have $\lambda + \lambda' \notin \mathbb{F}_p$. If $p = 7$ then

$$p - 2 = [\lambda - \lambda' - 1] \quad \text{and} \quad 1 = [\lambda - \lambda' - (p-2)].$$

But if we insert $(i,j) = (1, p-2)$ into (3.12) and then express $\mathfrak{a}_{p-2}$ as a multiple of $\mathfrak{a}_2$ we obtain

$$\frac{1}{6}(\lambda + \lambda' - 1)\left(\prod_{k=2}^{4}(\lambda - \lambda' - k)(\lambda + \lambda' + k + 1)\right)\mathfrak{a}_2 = 0.$$

This is only possible if $\mathfrak{a}_2 = 0$, or, equivalently, if $\mathfrak{a} = 0$ contradicting the fact that the sequence (3.20) does not split.

Conversely, assume that $p > 7$ and let λ be as above. Let $(\mathfrak{a}_1, \mathfrak{a}_2) \in K^2$ be a pair such that $\mathfrak{a}_1 = 0$ and consider the corresponding vector space $M_{\mathfrak{a}}$. We have

$$p - 2 > [\lambda - \lambda' - 1] \quad \text{and} \quad 1 < [\lambda - \lambda' - (p-2)].$$

The remark following eq. (3.21) gives $\mathfrak{a}_{p-2} = 0$, which, since $\mathfrak{a}_{p-1} = 0$, implies (3.11) for $\{i,j\} = \{1, p-2\}$. Suppose that $2 \le i, j \le p-2$ are distinct. Eq. (3.10) follows from Lemma 3.5.1(2). To prove (3.11) and (3.12), we may assume that $i + j \le [\lambda - \lambda']$. It then follows that $\{i,j\} = \{2,3\}$ or $\{i,j\} = \{2,4\}$ and since we have chosen λ in such a way that (3.12) holds for $(i,j) = (2,3)$, it suffices to consider the case $\{i,j\} = \{2,4\}$. We have

$$4 = [\lambda - \lambda' - 2] \quad \text{and} \quad 2 = [\lambda - \lambda' - 4].$$

If we insert $(i,j) = (2,4)$ into (3.12) and then use the definition of $\mathfrak{a}_4$ to express it as a scalar multiple of $\mathfrak{a}_2$, we obtain

$$\begin{aligned}\frac{1}{12}\left(\prod_{k=3}^{6}(\lambda + \lambda' + k)\right)\mathfrak{a}_2 &= \frac{1}{2}(2\lambda + \lambda' + 6)\left(\prod_{k=3}^{4}(\lambda + \lambda' + k)\right)\mathfrak{a}_2 \\ &\quad - (4\lambda + \lambda' + 6)\mathfrak{a}_2.\end{aligned}$$

We set $\mu = \lambda + \lambda'$ so the above equation becomes

$$\begin{aligned}\frac{1}{12}\left(\prod_{k=3}^{6}(\mu + k)\right)\mathfrak{a}_2 &= \frac{1}{2}(\mu + 2)(\mu + 5)(\mu + 6)\mathfrak{a}_2 \\ &\quad + \frac{1}{2}(\mu + 1)(\mu + 6)\lambda\mathfrak{a}_2,\end{aligned}$$

which gives

$$\frac{1}{6}\mu(\mu + 1)(\mu + 5)(\mu + 6)\mathfrak{a}_2 = (\mu + 1)(\mu + 6)\lambda\mathfrak{a}_2.$$

Now, since $\lambda - \lambda' = 6$, we have $\lambda + \lambda' \neq p - 6$ since otherwise $\lambda = 0$ in contradiction with our assumption that $\lambda = (5 \pm \sqrt{19})/2$. If we divide both sides of the above equation by $\mu + 6$, we get an equation similar to (3.23) which holds by the definition of λ.

Case 6. Suppose that $[\lambda - \lambda'] \geq 7$.

Every extension of $K_{\lambda'}$ by $V_\chi(\lambda)$ can be represented by a short exact sequence of $U_\chi(W^1_{\geq 0})$–modules as in (3.20) and such that $\mathfrak{a}_1 = 0$. A necessary condition for (3.20) to be non-split is that $(\lambda + \lambda')(\lambda + \lambda' + 5) = 6\lambda$ which, in our present case, implies

$$\lambda' = \frac{1}{2}(-5 \pm \sqrt{24\lambda + 25}) - \lambda. \tag{3.24}$$

The claim to be proved amounts to saying that $\mathfrak{a} = 0$, or, equivalently, that $\mathfrak{a}_2 = 0$. Assume towards contradiction that this is not the case. We have

$$5 \leq [\lambda - \lambda' - 2] \quad \text{and} \quad 2 \leq [\lambda - \lambda' - 5].$$

Thus, if we insert $(i, j) = (2, 5)$ into (3.12) and then use the definition of $\mathfrak{a}_5$ to write it as a scalar multiple of $\mathfrak{a}_2$, we get

$$\begin{aligned}\frac{1}{40}\left(\prod_{k=3}^{7}(\lambda + \lambda' + k)\right)\mathfrak{a}_2 = \frac{1}{6}(2\lambda + \lambda' + 7)&\left(\prod_{k=3}^{5}(\lambda + \lambda' + k)\right)\mathfrak{a}_2 \\ &- (5\lambda + \lambda' + 7)\mathfrak{a}_2.\end{aligned}$$

which, together with (3.24), yields

$$\pm(\lambda^2 + 5\lambda + 4)\sqrt{24\lambda + 25} = 15\lambda^2 + 35\lambda + 20.$$

(Here we have used the assumption that $\mathfrak{a}_2 \neq 0$.) Squaring both sides and subtracting one from the other yields the equation

$$\frac{9}{50}\lambda(\lambda - 1)(\lambda + 1)^2(3\lambda + 2) = 0,$$

which in turn yields $\lambda \in \{0, \pm 1, -2/3\}$. Now we can insert this back into (3.24) to determine λ'. But first note that if

$$\lambda' = \frac{1}{2}(-5 - \sqrt{24\lambda + 25}) - \lambda,$$

then $\lambda = p - 1$ because $0, 1, -2/3$ are not solutions to

$$-(\lambda^2 + 5\lambda + 4)\sqrt{24\lambda + 25} = 15\lambda^2 + 35\lambda + 20.$$

Putting all of this together, we obtain

$$(\lambda, \lambda') \in \{(p-1, p-1), (p-1, p-2), (1, 0), (0, 0), (-2/3, -1/3)\}.$$

But none of these cases are possible since by assumption $[\lambda - \lambda'] \geq 7$ and $\lambda + \lambda' \neq p - 1$; a contradiction which can only be avoided if $\mathfrak{a}_2 = 0$. This completes the proof of the proposition. □

3.5.3 We are left with the case where $(\lambda, \lambda') \notin \Theta(\chi)$ and $\lambda + \lambda' = p - 1$. Note that this implies $\mathrm{ht}(\chi) < 1$.

Proposition. *If* $(\lambda, \lambda') \notin \Theta(\chi)$ *and* $\lambda + \lambda' = p - 1$, *then*

$$\mathrm{Ext}_{U_\chi(W^1_{\geq 0})}(K_{\lambda'}, V_\chi(\lambda)) \simeq \begin{cases} K, & \text{if } [\lambda - \lambda'] \in \{2, 3, 4\}, \\ K, & \text{if } [\lambda - \lambda'] = 6 \text{ and } p = 19, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $(\lambda, \lambda') \notin \Theta(\chi)$ such that $\lambda + \lambda' = p - 1$. Every extension of $K_{\lambda'}$ by $V_\chi(\lambda)$ can be represented by a short exact sequence of $U_\chi(W^1_{\geq 0})$–modules as (3.20). The assumption $\lambda + \lambda' = p - 1$ implies $\mathfrak{a}_2 = 0$, see Lemma 3.2.3 B and Remark 3.2.3 A. Using the foregoing notation, we have

$$A_j = \frac{(-1)^j}{j-2} \left(\prod_{k=0}^{j-2} (2\lambda - k) \right) \Bigg((j-2)\lambda - 1 + j + \sum_{k=4}^{j} \frac{(j-k)!}{(j-3)!} \cdot ((j+1-k)\lambda + (j+2-k)) \prod_{l=1}^{k-3} (j - l) \Bigg).$$

(The summation $\sum_{k=4}^{j}$ is understood to be 0 when $j = 3$.) For our present purpose it is not necessary to write down the explicit formula of B_j because $\mathfrak{a}_2 = 0$. Note that $\mathfrak{a}_j = 0$ for all $j > [\lambda - \lambda']$. This is a consequence of the fact that $\lambda + \lambda' = p - 1$ which implies

$$\prod_{k=0}^{j-2} (2\lambda - k) = \prod_{k=1}^{j-1} (\lambda - \lambda' - k).$$

Furthermore, it should also be noted that $\lambda - \lambda' \neq 1$ since otherwise we would have $(\lambda, \lambda') = (0, p - 1)$ in contradiction to the assumption that $(\lambda, \lambda') \notin \Theta(\chi)$.

Case 1. Suppose that $[\lambda - \lambda'] \in \{2, 3, 4\}$.

Conversely, every pair $(\mathfrak{a}_1, \mathfrak{a}_2) \in K^2$ with $\mathfrak{a}_2 = 0$ gives rise to a module $M_\mathfrak{a}$ as described in Section 3.3.1. Lemma 3.5.1(2) yields (3.10) for all distinct $2 \leq i, j \leq p-2$. Meanwhile, (3.11) and (3.12) follow from Lemma 3.5.1(3) since $i + j > [\lambda - \lambda']$ for all distinct integers $2 \leq i, j \leq p-2$. Let $\{i, j\} = \{1, p-2\}$ and assume that $p > 5$. We have then $p - 2 > [\lambda - \lambda']$, which implies $\mathfrak{a}_{p-2} = 0$. Since

$$p - 2 > [\lambda - \lambda' - 1] \quad \text{and} \quad 1 < [\lambda - \lambda' - (p-2)],$$

this implies (3.11) for $p > 5$. Note that this completes the proof of the claim for $[\lambda - \lambda'] = 4$ since in this case $p > 5$; otherwise we would have $(\lambda, \lambda') = (4, 0)$ in contradiction with the assumption that $(\lambda, \lambda') \notin \Theta(\chi)$. Now, the same argument can be repeated to prove (3.10) for $p = 5$ and $[\lambda - \lambda'] = 2$. For $p = 5$ and $[\lambda - \lambda'] = 3$ we have

$$3 > [\lambda - \lambda' - 1] \quad \text{and} \quad 1 > [\lambda - \lambda' - 3],$$

which, since $\mathfrak{a}_4 = 0$, implies the desired result.

Case 2. Suppose that $[\lambda - \lambda'] = 5$.

Consider a short exact sequence of $U_\chi(W^1_{\geq 0})$–modules as in (3.20). If we insert $(i, j) = (2, 3)$ into (3.12) and then insert the expressions $\mathfrak{a}_3 = A_3\mathfrak{a}_1$ and $\mathfrak{a}_5 = A_5\mathfrak{a}_1$ into the result we get

$$\lambda(\lambda+2)(\lambda+4)\left(\prod_{k=1}^{3}(2\lambda - k)\right)\mathfrak{a}_1 = \frac{1}{3}\lambda(13\lambda + 22)\left(\prod_{k=1}^{3}(2\lambda - k)\right)\mathfrak{a}_1.$$

Now, the assumptions $\lambda + \lambda' = p - 1$ and $(\lambda, \lambda') \notin \Theta(\chi)$ imply $\lambda \neq 0$. Since, in addition, $[\lambda - \lambda'] = 5$, the above equation reduces to

$$3(\lambda+2)(\lambda+4)\mathfrak{a}_1 = (13\lambda + 22)\mathfrak{a}_1,$$

which, by subtracting the right-hand side from the left-hand side, gives

$$(3\lambda^2 + 5\lambda + 2)\mathfrak{a}_1 = 0.$$

We get a quadratic equation which can be solved easily; the roots are $p - 1$ and $-2/3$. But, we can immediately exclude $\lambda = p - 1$ since this would imply $(\lambda, \lambda') = (p-1, 0)$. All the above reasoning works just as well in the case where $[\lambda - \lambda'] \geq 5$. We shall make use of this later, but for now we are content to remark that in our present case $\lambda + \lambda' = p - 1$ implies $(\lambda, \lambda') = (-2/3, -1/3)$ contradicting the fact that $\lambda - \lambda' = 5$.

Case 3. Suppose that $[\lambda - \lambda'] = 6$.

As noted previously, a necessary condition for (3.20) to be non-split is that $\lambda = -2/3$. The assumption $\lambda + \lambda' = p - 1$ implies $\lambda - \lambda' = 2\lambda + 1$ which, since $\lambda - \lambda' = 6$, means $\lambda = 5/2$. But $-2/3 = 5/2$ if and only if $p = 19$. Conversely, let $p = 19$ and $(\lambda, \lambda') = (12, 6)$. (Note that $12 = 5/2$ in characteristic $p = 19$.) Let $(\mathfrak{a}_1, \mathfrak{a}_2)$ be a pair in K^2 such that $\mathfrak{a}_2 = 0$ and consider the corresponding vector space $M_{\mathfrak{a}}$. Eq. (3.10)–(3.12) follow immediately for $\{i, j\} = \{1, p-2\}$ since $\mathfrak{a}_{p-2} = 0$. For all distinct integers $2 \le i, j \le p - 2$, eq. (3.10) follows from Lemma 3.5.1(2). Furthermore, in order to prove (3.11) and (3.12), we may assume that $i + j \le [\lambda - \lambda']$, see Lemma 3.5.1(3). But since we have chosen λ in such a way that (3.12) holds for $(i, j) = (2, 3)$, we shall only be concerned with $\{i, j\} = \{2, 4\}$. We have

$$2 = [\lambda - \lambda' - 4] \quad \text{and} \quad 4 = [\lambda - \lambda' - 2].$$

If we insert $(i, j) = (2, 4)$ into (3.12) and then insert the expressions $\mathfrak{a}_4 = A_4\mathfrak{a}_1$ and $\mathfrak{a}_6 = A_6\mathfrak{a}_1$ into the result, we get

$$\frac{1}{3}\lambda(77\lambda + 125)\left(\prod_{k=1}^{4}(2\lambda - k)\right)\mathfrak{a}_1 = \lambda(\lambda + 5)(5\lambda + 9)\left(\prod_{k=1}^{4}(2\lambda - k)\right)\mathfrak{a}_1.$$

The assumptions $\lambda + \lambda' = p - 1$ and $(\lambda, \lambda') \notin \Theta(\chi)$ imply $\lambda \neq 0$. Since, in addition, $[\lambda - \lambda'] = 6$, the above equation reduces to

$$(77\lambda + 125)\mathfrak{a}_1 = 3(\lambda + 5)(5\lambda + 9)\mathfrak{a}_1,$$

which, by subtracting the left-hand side from the right-hand side, gives

$$(15\lambda^2 + 25\lambda + 10)\mathfrak{a}_1 = 0.$$

Keeping in mind that the characteristic is 19, a very simple computation shows that the above equation holds for all $\mathfrak{a}_1$.

Case 4. Suppose that $[\lambda - \lambda'] \ge 7$.

Suppose we are given a short exact sequence of $U_\chi(W^1_{\ge 0})$–modules as in (3.20) and such that $\mathfrak{a}_2 = 0$ and $\lambda = -2/3$. We have

$$2 \le [\lambda - \lambda' - 5] \quad \text{and} \quad 5 \le [\lambda - \lambda' - 2].$$

If we insert $(i, j) = (2, 5)$ into (3.12), we obtain

$$\frac{6}{10}\lambda(87\lambda + 137)\left(\prod_{k=1}^{5}(2\lambda - k)\right)\mathfrak{a}_1 = \frac{2}{3}\lambda(\lambda + 6)(13\lambda + 22) \cdot \left(\prod_{k=1}^{5}(2\lambda - k)\right)\mathfrak{a}_1.$$

Since $\lambda \neq 0$ and $[\lambda - \lambda'] \geq 7$, the above equation reduces to

$$9\,(87\lambda + 137)\,\mathfrak{a}_1 = 10(\lambda + 6)(13\lambda + 22)\mathfrak{a}_1,$$

which, by subtracting the left-hand side from the right-hand side, gives

$$(130\lambda^2 + 217\lambda + 87)\mathfrak{a}_1 = 0.$$

The claim follows since $-2/3$ is not a root of the polynomial inside the brackets. □

3.6 Summary

3.6.1 We summarize the preceding results as a theorem as follows.

Theorem. *We have the following three cases*

1. *If* $\lambda' \in \{0, p-1\}$, *then*

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda'), V_\chi(\lambda)) \simeq \begin{cases} K, & \text{if } \lambda \in \{0,1,2,3,4,p-1\}, \\ 0, & \text{otherwise.} \end{cases}$$

2. *If* $\lambda \in \{0, p-1\}$, *then*

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda'), V_\chi(\lambda)) \simeq \begin{cases} K, & \text{if } \lambda' \in \{0\} \cup \{p-i \mid 1 \leq i \leq 5\}, \\ 0, & \text{otherwise.} \end{cases}$$

3. *If* $\lambda, \lambda' \notin \{0, p-1\}$, *then*

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda'), V_\chi(\lambda)) \simeq \begin{cases} K, & \text{if } [\lambda - \lambda'] \in \{2,3\}, \\ K, & \text{if } [\lambda - \lambda'] = 4 \text{ and } p \neq 5, \\ K, & \text{if } [\lambda - \lambda'] = 6 \text{ and } 2\lambda = 5 \pm \sqrt{19} \text{ and } p > 7, \\ 0, & \text{otherwise.} \end{cases}$$

Remark. The above results are consistent with those in [3] when χ has height equal to or less than 0 or χ has height 1 and $\chi(e_0) = 1$. If $\mathrm{ht}(\chi) \in \{-1, 0\}$, then we need $\lambda \in \mathbb{F}_p$; since $5 \pm \sqrt{19} \notin \mathbb{F}_7$, we could replace $p > 7$ by $p \geq 7$ in the case where $[\lambda - \lambda'] = 6$, as in [3, Thm. 2.2] and [3, Thm. 4.1]. Furthermore, if $2\lambda = 5 \pm \sqrt{19}$ occurs for height 1 then one checks

$\chi(e_0)^p = \lambda^p - \lambda \neq 1$. This case does not appear in [3, Thm. 4.2] where it is assumed that $\chi(e_0) = 1$. However, [3, Thm. 4.2] does not cover all χ of height 1. At the beginning of Section 4.2 in [3], the authors claim that by conjugation one may assume $\chi(e_0) = 1$. This is not true. The action of the automorphism group $\operatorname{Aut} W^1$ of W^1 on a character θ of height 1 does not change the value of $\theta(e_0)$ because every automorphism of W^1 maps e_0 to $e_0 + f$ for some $f \in W^1_1$. The orbits of characters under the action of $\operatorname{Aut} W^1$ were computed in [7] by Feldvoss and Nakano. However, [7, Thm. 3.1(a)] does not hold for height $r = 1$. In the proof, the authors claim that every character θ of height r is conjugate under the action of a torus T to a character ξ with $\xi(e_{r-1}) = 1$. This is false for $r = 1$.

4 Extensions of the simple modules

4.1 Height -1

Throughout this section we will assume that $\mathrm{ht}(\chi) = -1$, or, equivalently, that $\chi = 0$. Furthermore, we let as usual λ and λ' be elements in $\Lambda(0) \simeq \mathbb{F}_p$.

4.1.1 $V_\chi(\lambda)$ and $V_\chi(\lambda')$ This is fully described in Theorem 3.6.1.

4.1.2 $V_\chi(\lambda)$ and K Our approach will follow the one taken earlier in Chapter 3. We have an isomorphism of vector spaces

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda), K) \simeq \mathrm{Ext}_{U_\chi(W^1_{\geq 0})}(K_\lambda, K),$$

where K denotes the trivial W^1–module. Suppose that we have a short exact sequence of $U_\chi(W^1_{\geq 0})$–modules

$$0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} K_\lambda \longrightarrow 0, \tag{4.1}$$

and fix bases v and v' for K and K_λ, respectively. Let $\{w, w'\}$ be a basis for M such that $f(v) = w$ and $g(w') = v'$. Since $U_\chi(W^1_0)$ is semisimple, we may choose w' such that $e_0 w' = \lambda w'$. Note that the λ weight space in M is one dimensional so w' is unique with these properties. Now, we have clearly $e_i w = 0$ for all i. Furthermore, if $e_1 w'$ and $e_2 w'$ are both 0, then $e_i w' = 0$ for all $i > 0$. This can be seen by a simple induction argument since

$$(i-1)e_{i+1} w' = (e_1 e_i - e_i e_1) w'.$$

Therefore, a necessary condition for (4.1) to be non-split is $e_1 w' \neq 0$ or $e_2 w' \neq 0$. We can interpret this condition in terms of λ. Indeed, since $e_i w'$ belongs to the weight space $M_{\lambda+i}$, we have $e_i w' = 0$ if $\lambda + i \neq 0$ and $i > 0$. Thus $e_1 w' = 0$ and $e_2 w' = 0$ for $\lambda \notin \{p-1, p-2\}$. This leads to the following lemma

Lemma (A). *If* $\lambda \notin \{p-1, p-2\}$ *then*

$$\mathrm{Ext}_{U_\chi(W^1_{\geq 0})}(K_\lambda, K) = 0.$$

We consider the cases $\lambda \in \{p-1, p-2\}$ separately. Note that $\lambda = p-1$ is not interesting for our purpose because $V_\chi(p-1)$ is not simple; we shall, nevertheless, include it in our study for the sake of completeness. Fix $\lambda = p-k$ for $k \in \{1, 2\}$. There exists $\mathfrak{a} \in K$ such that for every $i > 0$

$$e_i w' = \begin{cases} \mathfrak{a}w, & \text{if } i = k, \\ 0, & \text{otherwise.} \end{cases}$$

As in Chapter 3, we construct a well–defined injective homomorphism

$$\mathrm{Ext}_{U_\chi(W^1_{\geq 0})}(K_\lambda, K) \to K,$$

which sends the class of (4.1) to $\mathfrak{a}$. Conversely, for each $\mathfrak{a} \in K$, consider a short exact sequence of vector spaces

$$0 \longrightarrow K \xrightarrow{f} M_\mathfrak{a} \xrightarrow{g} K_\lambda \longrightarrow 0,$$

and choose a basis $\{w, w'\}$ for $M_\mathfrak{a}$ such that $f(v) = w$ and $g(w') = v'$ where, as before, v and v' are bases for K and K_λ, respectively. For each $0 \leq i \leq p-2$ define an endomorphism $E_i \in \mathrm{End}_K(M_\mathfrak{a})$ of $M_\mathfrak{a}$ such that $E_i w = 0$ for all i and

$$E_i w' = \begin{cases} \lambda w', & \text{if } i = 0, \\ \mathfrak{a}w, & \text{if } i = k, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, set $E_i^{[p]} = \delta_{i0} E_i$ for all $i = 0, 1, \ldots, p-2$ and $E_i = 0$ for $i \neq 0, 1, \ldots, p-2$. We claim that

$$(E_i^p - E_i^{[p]})w' = 0 \quad \text{for all} \quad 0 \leq i \leq p-2.$$

Indeed, that $E_0^p - E_0$ annihilates w' follows from the fact that $\lambda^p = \lambda$. If $i = k$, we have $E_k^p w' = \mathfrak{a} E_k^{p-1} w$ which is 0 since $E_i w = 0$ for all i. If $i \notin \{0, k\}$ then already $E_i w' = 0$ and hence also $E_i^p w' = 0$.

Lemma (B). *If* $\lambda = p-1$ *or* $\lambda = p-2$, *then*

$$\mathrm{Ext}_{U_\chi(W^1_{\geq 0})}(K_\lambda, K) \simeq K.$$

Proof. Keeping the notation introduced above, we only have to show

$$[E_i, E_j]w' = (j-i)E_{i+j}w' \quad \text{for all } i \text{ and } j,$$

We may assume that $i \in \{0, k\}$ and that $i \neq j$ since otherwise we would obtain 0 on both sides of the equation. If $i = 0$, the left-hand side becomes $(E_0E_j - E_jE_0)w'$ which equals 0 if $j \neq k$ and $-\lambda\mathfrak{a}w = k\mathfrak{a}w$ otherwise. The equality holds since the right-hand side is jE_jw'. Using the same argument as above we may assume that $j \neq 0$. But then $i = k$ implies $E_{k+j}w' = 0$ and $(E_kE_j - E_jE_k)w' = 0$, proving the lemma. □

We summarize the results in this section into the following proposition

Proposition. *We have*

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda), K) \simeq \begin{cases} K, & \text{if } \lambda \in \{p-1, p-2\}, \\ 0, & \text{otherwise.} \end{cases}$$

Evidently, K is self–dual in the sense that it is isomorphic to its dual. Since $V_\chi(\lambda)^*$ and $V_\chi(p-1-\lambda)$ are isomorphic we obtain

Corollary. *We have*

$$\mathrm{Ext}_{U_\chi(W^1)}(K, V_\chi(\lambda)) \simeq \begin{cases} K, & \text{if } \lambda = \{0, 1\}, \\ 0, & \text{otherwise.} \end{cases}$$

4.1.3 $V_\chi(\lambda)$ and S

We proceed to describe $\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda), S)$ where S denotes the $(p-1)$–dimensional simple $U_\chi(W^1)$–module. We have an isomorphism

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda), S) \simeq \mathrm{Ext}_{U_\chi(W^1_{\geq 0})}(K_\lambda, S).$$

Thus, classifying the extensions of $V_\chi(\lambda)$ by S reduces to classifying the extensions of K_λ by S. Suppose that we have a short exact sequence of $U_\chi(W^1_{\geq 0})$–modules

$$0 \longrightarrow S \xrightarrow{f} M \xrightarrow{g} K_\lambda \longrightarrow 0, \tag{4.2}$$

and fix bases $v_0, v_1, \ldots, v_{p-2}$ and v' of S and K_λ, respectively. Let

$$w_0, \ldots, w_{p-2}, w' \tag{4.3}$$

be a basis of M such that $f(v_i) = w_i$ for all i and $g(w') = v'$; we may choose w' such that $e_0 w' = \lambda w'$. We have

$$e_j w_i = \begin{cases} (-1)^{j+1} \frac{(i+1)!}{(i-j)!} w_{i-j}, & \text{if } j \le i, \\ 0, & \text{otherwise.} \end{cases}$$

By weight considerations there exists $\mathfrak{a}_j \in K$ $(j > 0)$ such that

$$e_j w' = \begin{cases} \mathfrak{a}_j w_{[-\lambda-j-1]}, & \text{if } \lambda + j \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, we set

$$\mathfrak{a}_j = 0 \quad \text{if} \quad \lambda + j = 0 \;\text{ or }\; j \notin \{1, 2, \ldots, p-2\}. \tag{4.4}$$

For $\lambda \neq 0$ the λ weight space in M is two–dimensional; it is generated by w' and $w_{[-\lambda-1]}$ so any different choice for w' has the form $w' + b w_{[-\lambda-1]}$ for some $b \in K$. Obviously, this leads to the same $e_j w'$ if $e_j w_{[-\lambda-1]} = 0$. In particular, we see that all the $\mathfrak{a}_j$ are determined by M if $\lambda = p-1$. The same holds for $\lambda = 0$ since w' is unique in this case; we have $M_0 = Kw'$.

Lemma (A)**.** *If $\lambda \in \{0, p-1\}$ then all the $\mathfrak{a}_j$ are determined by M.*

For $\lambda \notin \{0, p-1\}$ we have $e_1 w_{[-\lambda-1]} \neq 0$. Thus, the discussion preceding Lemma A shows that we can choose w' uniquely such that $e_1 w' = 0$.

Lemma (B)**.** *If $\lambda \notin \{0, p-1\}$ then there is a unique choice for w' such that $e_1 w' = 0$.*

We will henceforth always assume that w' is the unique choice from Lemma B if $\lambda \notin \{0, p-1\}$. As in Chapter 3 we obtain an injective homomorphism $\operatorname{Ext}_{U_\chi(W^1_{\ge 0})}(K_\lambda, S) \to K^2$ which sends the class of (4.2) to $(\mathfrak{a}_1, \mathfrak{a}_2) \in K^2$ and such that $\mathfrak{a}_1 = 0$ for $\lambda \notin \{0, p-1\}$. A simple induction shows that $e_1 w' = 0$ and $e_2 w' = 0$ imply $e_i w' = 0$ for all $i > 0$. More precisely, we have

$$(j-i) e_{i+j} w' = (e_i e_j - e_i e_i) w' \;\text{ for all } i \text{ and } j,$$

which for $i = 1$ and $2 \le j \le p-3$ yields

$$(j-1)\mathfrak{a}_{j+1} = -\Bigl(\prod_{k=1}^{j+1} (\lambda + k)\Bigr)\mathfrak{a}_1 + (\lambda + j)(\lambda + j + 1)\mathfrak{a}_j,$$

or, equivalently, by induction

$$\mathfrak{a}_j = A_j\mathfrak{a}_1 + B_j\mathfrak{a}_2 \quad \text{for all} \quad 3 \le j \le p-2, \tag{4.5}$$

where

$$A_j = -\frac{1}{j-2}\Big(\prod_{k=1}^{j}(\lambda+k)\Big)\Big(1+\sum_{k=4}^{j}\frac{(j-k)!}{(j-3)!}\prod_{l=1}^{k-3}(\lambda+j-l)\Big),$$

(the summation $\sum_{k=4}^{j}$ is understood to be 0 when $j=3$) and

$$B_j = \frac{(-1)^j}{(j-2)!}\prod_{k=2}^{j-1}(-\lambda-1-k)(\lambda+k).$$

Conversely, let $(\mathfrak{a}_1, \mathfrak{a}_2)$ be a pair in K^2 such that (see (4.4))

$$\mathfrak{a}_1 = 0 \quad \text{if} \quad \lambda = p-1,$$

$$\mathfrak{a}_2 = 0 \quad \text{if} \quad \lambda = p-2.$$

We extend $(\mathfrak{a}_1, \mathfrak{a}_2)$ to a tuple $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_{p-2}) \in K^{p-2}$ by using (4.5) and consider a short exact sequence

$$0 \longrightarrow S \xrightarrow{f} M_{\mathfrak{a}} \xrightarrow{g} K_\lambda \longrightarrow 0, \tag{4.6}$$

where $M_{\mathfrak{a}}$ is a vector space with a basis as in (4.3). For each $0 \le j \le p-2$ we define an endomorphism $E_j \in \operatorname{End}_K(M_{\mathfrak{a}})$ such that

$$E_j w_i = \begin{cases} (-1)^{j+1}\frac{(i+1)!}{(i-j)!}w_{i-j}, & \text{if } j \le i, \\ 0, & \text{otherwise,} \end{cases}$$

and $E_0 w' = \lambda w'$ and for every $1 \le j \le p-2$

$$E_j w' = \begin{cases} \mathfrak{a}_j w_{[-\lambda-j-1]}, & \text{if } \lambda + j \ne 0, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, we set $E_j^{[p]} = \delta_{j0}E_j$ for all j and $E_j = 0$ for $j \notin \{0, 1, \dots, p-2\}$. We claim that

$$(E_j^p - E_j^{[p]})w' = 0 \quad \text{for all } j.$$

Indeed, that $E_0^p - E_0$ annihilates w' follows from the fact that $\lambda^p = \lambda$. For $j > 0$ we have $E_j^p w' = \mathfrak{a}_j E_j^{p-1} w_{[-\lambda-j-1]}$ if $\lambda + j \ne 0$ and 0 otherwise. But since $[-\lambda-j-1] < p-1$ for $\lambda+j \neq 0$ then $E_j^p w' = 0$ for all $j > 0$, thereby proving the claim.

Except for the change in notation, the formula $[E_i, E_j]w' = (j-i)E_{i+j}w'$ leads to the same equations as in Section 3.3.2; we include them here for completeness:

If $i > [-1-\lambda-j]$ and $j > [-1-\lambda-i]$, then

$$\mathfrak{a}_{i+j} = 0. \tag{4.7}$$

If $i > [-1-\lambda-j]$ and $j \leq [-1-\lambda-i]$, then

$$(j-i)\mathfrak{a}_{i+j} = (-1)^{j+1}\frac{[-1-\lambda-i]!}{([-1-\lambda-i]-j)!}(\lambda+i)\mathfrak{a}_i. \tag{4.8}$$

If $i \leq [-1-\lambda-j]$ and $j \leq [-1-\lambda-i]$, then

$$\begin{aligned}(j-i)\mathfrak{a}_{i+j} = (-1)^i &\frac{[-1-\lambda-j]!}{([-1-\lambda-j]-i)!}(\lambda+j)\mathfrak{a}_j \\ -(-1)^j &\frac{[-1-\lambda-i]!}{([-1-\lambda-i]-j)!}(\lambda+i)\mathfrak{a}_i.\end{aligned} \tag{4.9}$$

Proposition. *We have*

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda), S) \simeq \begin{cases} K, & \text{if } \lambda \in \{0\} \cup \{p-i \mid 3 \leq i \leq 5\}, \\ 0, & \text{otherwise.}\end{cases}$$

Proof. The short exact sequence

$$0 \longrightarrow S \longrightarrow V_\chi(0) \xrightarrow{\pi} K \longrightarrow 0 \tag{4.10}$$

induces the long exact sequence of vector spaces

$$\begin{aligned}0 &\to \mathrm{Hom}_{U_\chi(W^1)}(V_\chi(\lambda), S) \to \mathrm{Hom}_{U_\chi(W^1)}(V_\chi(\lambda), V_\chi(0)) \\ &\to \mathrm{Hom}_{U_\chi(W^1)}(V_\chi(\lambda), K) \to \mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda), S) \\ &\to \mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda), V_\chi(0)) \to \mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda), K) \to \cdots .\end{aligned} \tag{4.11}$$

Due to Schur's Lemma, we have $\mathrm{Hom}_{U_\chi(W^1)}(V_\chi(\lambda), K) = 0$ for $\lambda \notin \{0, p-1\}$. We obtain, in these cases, the exact sequence

$$\begin{aligned}0 &\to \mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda), S) \to \mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda), V_\chi(0)) \\ &\to \mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda), K) \to \cdots .\end{aligned} \tag{4.12}$$

Lemma 2.3.3 A shows that $\mathrm{Hom}_{U_\chi(W)}(V_\chi(0), K)$ is 1–dimensional; it is generated by the surjection π in (4.10). The map $\mathrm{Hom}_{U_\chi(W^1)}(V_\chi(0), V_\chi(0)) \to \mathrm{Hom}_{U_\chi(W^1)}(V_\chi(0), K)$ appearing in (4.11) is surjective because it maps the identity $\mathrm{id}_{U_\chi(W^1)}$ on $U_\chi(W^1)$ to π. Thus, we obtain an exact sequence as in (4.12)

$$\begin{aligned} 0 &\to \mathrm{Ext}_{U_\chi(W^1)}(V_\chi(0), S) \to \mathrm{Ext}_{U_\chi(W^1)}(V_\chi(0), V_\chi(0)) \\ &\to \mathrm{Ext}_{U_\chi(W^1)}(V_\chi(0), K) \to \cdots . \end{aligned}$$

Consequently, we have for every $\lambda \neq p-1, p-2$

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda), S) \simeq \mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda), V_\chi(0)),$$

which, by Proposition 3.6.1, implies

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda), S) \simeq \begin{cases} K, & \text{if } \lambda \in \{0\} \cup \{p-i \mid 3 \leq i \leq 5\}, \\ 0, & \text{if } \lambda \notin \{0\} \cup \{p-i \mid 1 \leq i \leq 5\}. \end{cases}$$

Next, suppose that $\lambda \in \{p-1, p-2\}$. We prove that every short exact sequence of $U_\chi(W^1)$–modules as in (4.6) splits, or, equivalently, that $\mathfrak{a}_1 = 0$ and $\mathfrak{a}_2 = 0$. The case $\lambda = p-2$ comes almost at once from the observations preceding the proposition; indeed $\mathfrak{a}_1 = 0$ because of Lemma B and $\mathfrak{a}_2 = 0$ by definition, see (4.4). For $\lambda = p-1$ it follows from (4.4) that $\mathfrak{a}_1 = 0$. To prove $\mathfrak{a}_2 = 0$, note first that (4.5) implies $\mathfrak{a}_2 = -2\mathfrak{a}_{p-2}$. However, since

$$1 \leq [-1-\lambda-(p-2)] \quad \text{and} \quad p-2 \leq [-1-\lambda-1],$$

we see by inserting $(i,j) = (1, p-2)$ into (4.9) that $\mathfrak{a}_{p-2} = 0$. This is only possible if $\mathfrak{a}_2 = 0$ so the proposition is proved. □

The module S is self–dual. Since $V_\chi(\lambda)^* \simeq V_\chi(p-1-\lambda)$, we obtain

Corollary. *We have*

$$\mathrm{Ext}_{U_\chi(W^1)}(S, V_\chi(\lambda)) \simeq \begin{cases} K, & \text{if } \lambda \in \{2,3,4,p-1\}, \\ 0, & \text{otherwise.} \end{cases}$$

4.1.4 Self–extensions of K

There are no nontrivial self–extensions of the trivial module K.

Proposition. *We have*

$$\mathrm{Ext}_{U_\chi(W^1)}(K, K) = 0$$

Proof. Suppose we have a short exact sequence of $U_\chi(W^1)$–modules

$$0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} K \longrightarrow 0, \tag{4.13}$$

and let $\{w, w'\}$ be a basis of M such that $Kw = \operatorname{im} f$. We clearly have $W^1 w = 0$. Furthermore, $W^1 w' \subset Kw$ since $W^1 w' \subset \ker g$. Thus, every $x \in W^1$ acting on M can be represented by a matrix

$$\begin{bmatrix} 0 & \phi(x) \\ 0 & 0 \end{bmatrix},$$

where $\phi : W^1 \to K$ is a homomorphism of Lie algebras. That ϕ preserves the Lie algebra structure follows from the fact that M is a W^1–module and hence that $W^1 \to \mathfrak{gl}(M) \simeq \mathfrak{gl}(2, K)$ is a homomorphism of Lie algebras. Now, if (4.13) is non-split, then $\phi(x) \neq 0$ for some $x \in W^1$. The kernel of ϕ is then an ideal of codimension 1, in apparent contradiction with the fact that W^1 is simple. □

4.1.5 Self–extensions of S

Proposition. *We have*

$$\operatorname{Ext}_{U_\chi(W^1)}(S, S) = 0.$$

Proof. The short exact sequence

$$0 \longrightarrow S \longrightarrow V_\chi(0) \longrightarrow K \longrightarrow 0$$

induces the long exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}_{U_\chi(W^1)}(S, S) \to \operatorname{Hom}_{U_\chi(W^1)}(S, V_\chi(0)) \\ &\to \operatorname{Hom}_{U_\chi(W^1)}(S, K) \to \operatorname{Ext}_{U_\chi(W^1)}(S, S) \\ &\to \operatorname{Ext}_{U_\chi(W^1)}(S, V_\chi(0)) \to \cdots , \end{aligned}$$

which, by Corollary 4.1.3 and Schur's lemma, implies the claim. □

4.1.6 S and K The proof of the next proposition makes use of the fact that $\operatorname{Hom}_{U_\chi(W^1)}(V_\chi(0), S) = 0$, see Lemma 2.3.3 B.

Proposition. *We have*

$$\operatorname{Ext}_{U_\chi(W^1)}(S, K) \simeq \operatorname{Ext}_{U_\chi(W^1)}(K, S) \simeq K^2.$$

Proof. The first isomorphism follows from the fact that S and K are self–dual. The short exact sequence

$$0 \longrightarrow S \longrightarrow V_\chi(0) \longrightarrow K \longrightarrow 0$$

induces the long exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{U_\chi(W^1)}(K, S) \to \mathrm{Hom}_{U_\chi(W^1)}(V_\chi(0), S) \\ &\to \mathrm{Hom}_{U_\chi(W^1)}(S, S) \to \mathrm{Ext}_{U_\chi(W^1)}(K, S) \\ &\to \mathrm{Ext}_{U_\chi(W^1)}(V_\chi(0), S) \to \mathrm{Ext}_{U_\chi(W^1)}(S, S) \to \cdots, \end{aligned}$$

which in turn induces the exact sequence

$$0 \to \mathrm{Hom}_{U_\chi(W^1)}(S, S) \to \mathrm{Ext}_{U_\chi(W^1)}(K, S) \to \mathrm{Ext}_{U_\chi(W^1)}(V_\chi(0), S) \to 0.$$

The claim follows from Schur's lemma and Proposition 4.1.3. □

4.1.7 We summarize the results on the extensions between all restricted simple modules as follows.

Theorem. *Let $\lambda, \lambda' \in \{1, 2, \ldots, p-2\}$. Then*

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda'), V_\chi(\lambda))$$

$$\simeq \begin{cases} K, & \text{if } [\lambda - \lambda'] \in \{2, 3\}, \\ K, & \text{if } [\lambda - \lambda'] = 4 \text{ and } p \neq 5, \\ K, & \text{if } [\lambda - \lambda'] = 6 \text{ and } 2\lambda = 5 \pm \sqrt{19} \text{ and } p > 7, \\ 0, & \text{otherwise.} \end{cases}$$

The only non–trivial extensions including K and $V_\chi(\lambda)$ are

$$\mathrm{Ext}_{U_\chi(W^1)}(K, V_\chi(1)) \simeq \mathrm{Ext}_{U_\chi(W^1)}(V_\chi(p-2), K) \simeq K.$$

We have

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda), S) \simeq \begin{cases} K, & \text{if } \lambda \in \{p - i \mid 3 \leq i \leq 5\}, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathrm{Ext}_{U_\chi(W^1)}(S, V_\chi(\lambda)) \simeq \begin{cases} K, & \text{if } \lambda \in \{2, 3, 4\}, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, we have

$$\mathrm{Ext}_{U_\chi(W^1)}(K, S) \simeq \mathrm{Ext}_{U_\chi(W^1)}(S, K) \simeq K^2,$$

$$\mathrm{Ext}_{U_\chi(W^1)}(K, K) \simeq \mathrm{Ext}_{U_\chi(W^1)}(S, S) \simeq 0.$$

Remark. There are no non–trivial self–extensions between simple modules over $U_\chi(W^1)$. This is proved more generally for every Lie algebra of Cartan type W or CS by Lin and Nakano in [16].

4.2 Height 0 and 1

All the work has been done in Chapter 3; Theorem 3.6.1 gives a complete classification of the χ–reduced Verma modules having character χ at most 1. Nevertheless, we state the theorem here again for the sake of completeness.

4.2.1 Height 0 The χ–reduced Verma modules $V_\chi(0)$ and $V_\chi(p-1)$ are isomorphic for $\mathrm{ht}(\chi) = 0$. Therefore

Proposition. *If* $\mathrm{ht}(\chi) = 0$ *and* $\lambda, \lambda' \in \{0, 1, \dots, p-2\}$, *then*

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(0), V_\chi(\lambda)) \simeq \begin{cases} K, & \text{if } \lambda \in \{0,1,2,3,4\}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda'), V_\chi(0)) \simeq \begin{cases} K, & \text{if } \lambda' \in \{0\} \cup \{p-i \mid 2 \le i \le 5\}, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if $\lambda, \lambda' \neq 0$ *then*

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda'), V_\chi(\lambda)) \simeq \begin{cases} K, & \text{if } [\lambda - \lambda'] \in \{2,3\}, \\ K, & \text{if } [\lambda - \lambda'] = 4 \text{ and } p \neq 5, \\ K, & \text{if } [\lambda - \lambda'] = 6 \text{ and } 2\lambda = 5 \pm \sqrt{19}, \\ 0, & \text{otherwise.} \end{cases}$$

Remark. We have removed the condition $p > 7$ for $[\lambda - \lambda'] = 6$ because $x^2 - 19$ does not split in $\mathbb{F}_7[x]$.

4.2.2 Height 1 Since $\Lambda(\chi) \cap \mathbb{F}_p = \emptyset$ for $\mathrm{ht}(\chi) = 1$, Theorem 3.6.1 becomes

Proposition. *If* $\mathrm{ht}(\chi) = 1$, *then*

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda'), V_\chi(\lambda)) \simeq \begin{cases} K, & \text{if } [\lambda - \lambda'] \in \{2,3\}, \\ K, & \text{if } [\lambda - \lambda'] = 4 \text{ and } p \neq 5, \\ K, & \text{if } [\lambda - \lambda'] = 6 \text{ and } 2\lambda = 5 \pm \sqrt{19} \text{ and } p > 7, \\ 0, & \text{otherwise.} \end{cases}$$

Remark. There are no non–trivial self–extensions between simple modules for all small heights with exactly one exception occurring in height 0, namely $V_\chi(0)$ which seems to come from the anomaly that it is isomorphic to $V_\chi(p-1)$.

4.3 Height $p-1$

4.3.1 Recall that if the centralizer $(W^1)^\chi$ of W^1 is a torus, then $U_\chi(W^1)$ is semisimple. If $(W^1)^\chi$ is unipotent then every simple $U_\chi(W^1)$–module with one exception is projective. The remaining simple module L has a projective cover with two composition factors both isomorphic to L. We have a short exact sequence

$$0 \longrightarrow L \longrightarrow P \longrightarrow L \longrightarrow 0,$$

where P is a projective module. This induces the long exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{U_\chi(W^1)}(L,L) \to \mathrm{Hom}_{U_\chi(W^1)}(P,L) \\ &\to \mathrm{Hom}_{U_\chi(W^1)}(L,L) \to \mathrm{Ext}_{U_\chi(W^1)}(L,L) \\ &\to \mathrm{Ext}_{U_\chi(W^1)}(P,L) \to \cdots, \end{aligned}$$

which in turn implies $\mathrm{Ext}_{U_\chi(W^1)}(L,L) \simeq K$. We have therefore

Proposition. *Let M and N be two simple $U_\chi(W^1)$–modules. If $(W^1)^\chi$ is a torus then*

$$\mathrm{Ext}_{U_\chi(W^1)}(M,N) \simeq 0.$$

If $(W^1)^\chi$ is unipotent, then

$$\mathrm{Ext}_{U_\chi(W^1)}(M,N) \simeq \begin{cases} K, & \text{if } N = M = L, \\ 0, & \text{otherwise.} \end{cases}$$

5 Wildness of the Witt algebra

5.1 Representation types of algebras

This section is based on Section 4.4 in [2]. We shall introduce the notion of the representation type of a finite dimensional algebra and present a fundamental result due to Drozd.

5.1.1 Let R be a finite dimensional algebra over a field k. By the Krull-Schmidt Theorem, every finite dimensional R–module decomposes uniquely as a direct sum of indecomposable R–modules. Thus, the representation theory of R reduces naturally to the study of the representations of the indecomposables. We say that R is of *finite representation type* if there are only finitely many indecomposables; otherwise it is of *infinite representation type*. In infinite type we distinguish between algebras of *tame representation type* (or, equivalently, *tame* algebras) and algebras of *wild representation type* (or, equivalently, *wild* algebras). We say that R is tame if the indecomposables in each dimension come in finitely many one–parameter families with finitely many exceptions. Tameness, in other words, suggests that we can classify all the isomorphism classes of the indecomposable R–modules of each dimension. On the other hand, we say R is wild if the category mod_R of finite dimensional R–modules contains a copy of the category $\text{mod}_{k\langle X,Y\rangle}$ of finite dimensional modules over the free algebra $k\langle X,Y\rangle$ in two variables. The latter "includes" the representation theory of an arbitrary finite dimensional algebra and the consensus feeling is that wild algebras are not "well–behaved" because their representations are not classifiable.

5.1.2 We gather our discussion into a formal definition

Definition. Let R be a finite dimensional associative algebra.

- R is of finite representation type if there are only finitely many indecomposable R–modules; otherwise it is of infinite representation type.

- R is of tame representation type if it is of infinite type and if for every dimension d there exists a finite set of R–$k[X]$–bimodules M_i such that

  * the M_i are free as right $k[X]$–modules,
  * all but finitely many indecomposable R–modules of dimension d can be expressed as $M_i \otimes_{k[X]} M$ for some i and some indecomposable $k[X]$–module M.

- R is of wild representation type if it is of infinite type and if there is a finitely generated R–$k\langle X, Y\rangle$–bimodule M such that

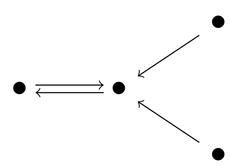
  * M is free as a right $k\langle X, Y\rangle$–module,
  * the functor $M \otimes_{k\langle X,Y\rangle} -$ from finite dimensional $k\langle X, Y\rangle$–modules to finite dimensional R–modules preserves indecomposability and isomorphism classes.

The following theorem due to Drozd is a fundamental result on the representation type.

Theorem. *Over an algebraically closed field, every finite dimensional algebra is either of finite, tame or wild representation type.*

5.2 Quivers

5.2.1 A quiver is an oriented graph, possibly with multiple arrows and loops. Quivers play an important role in the representation theory of finite dimensional algebras over an algebraically closed fields.



Suppose R is a finite dimensional algebra over (the algebraically closed field) K. Let $E_1, E_2, \ldots, E_n$ denote the isomorphism classes of simple R–modules. *The* Ext*–quiver* Q_R of R is a quiver with n vertices $v_1, v_2, \ldots, v_n$ such that the number of arrows from v_i to v_j is the dimension of $\mathrm{Ext}_R(E_i, E_j)$. (In some literature this is called *the Gabriel quiver* of R.)

Definition. Suppose Q is a quiver with n vertices $v_1, v_2, \dots, v_n$. We attach to Q a quiver Q_s with $2n$ vertices $v_1, , \dots, v_n, v'_1, \dots, v'_n$ and an arrow $v_i \to v'_j$ for every arrow $v_i \to v_j$ in Q. We call Q_s *the separated quiver* of Q.

5.2.2 Let J denote the Jacobson radical of R. If the factor algebra $R' = R/J^2$ is of wild representation type, then R is of wild representation type. The algebra R' has Jacobson radical $J' = J/J^2$ and Ext–quiver $Q_{R'} = Q_R$. Furthermore, the triangular matrix algebra

$$T = \begin{pmatrix} R'/J & 0 \\ J & R'/J \end{pmatrix}$$

is hereditary with quiver $Q_T = (Q_{R'})_s$, see [1, Thm. 2.4 X.2]. There is a well–known functor $\mathrm{mod}_{R'} \to \mathrm{mod}_T$ which reflects isomorphisms and indecomposability. This functor reaches all but finitely many indecomposables in mod_T. Hence if R' is of finite or tame representation type, then T is of finite or tame representation type. Gabriel determined the hereditary algebras of finite representation type in [8] whereas the tame hereditary algebras were classified independently by Donovan–Freislich in [5] and Nazarova in [18]. Putting all this together, we get the following theorem

Theorem. *Let R be a finite dimensional associative algebra. If the factor algebra R/J^2 is of finite or tame representation type, then the separated quiver of the* Ext*–quiver of R/J^2 is (when the directions of the arrows are ignored) a union of Dynkin diagrams of types A, D, E or Euclidean diagrams of types $\tilde{A}$, $\tilde{D}$, $\tilde{E}$.*

5.3 Wildness of the Witt algebra

5.3.1 The determination of the representation type of the reduced enveloping algebra $U_\chi(W^1)$ of W^1 has been (partially) determined for quite some time. It is now well–known that in the case where the characteristic of the ground field is larger than 7 then

$$U_\chi(W^1) \text{ is wild if and only if } \mathrm{ht}(\chi) \leq p - 4.$$

This can be proved, for example, by using support varieties, see [6, Thm. 5.2]. Our goal is to improve this statement in some special cases and to present a more elementary approach than the one referred to above. To this end, let J_χ denote the Jacobson radical of $U_\chi(W^1)$. From now on, we will restrict ourselves to the case where $\mathrm{ht}(\chi) \leq 1$. We claim that

$$U_\chi(W^1)/J_\chi^2 \text{ is wild if } (\mathrm{ht}(\chi), p) \neq (1, 5).$$

Observe that this implies the wildness of $U_\chi(W^1)$ since the category of finite dimensional $U_\chi(W^1)/J_\chi^2$–modules is included in the category of finite dimensional $U_\chi(W^1)$–modules.

Proposition. *We have* $J_\chi^2 \neq 0$ *for all* $\chi \in (W^1)^*$.

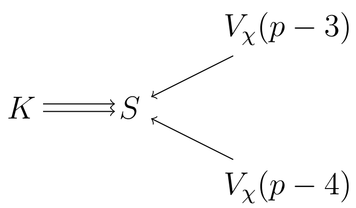
Proof. Let P be a projective and indecomposable $U_\chi(W^1)$–module which is not simple. If $J_\chi^2 = 0$ then $J_\chi^2 P = 0$ and hence $J_\chi P \subset P$ is a nonzero semisimple submodule. This shows $J_\chi P \subset \mathrm{Soc}_{U_\chi(W^1)} P$. But the socle of P is simple since $U_\chi(W^1)$ is Frobenius so P has length 2 in contradiction with [7, Sec. 2]. □

Theorem. *Suppose that* $\chi \in (W^1)^*$ *is of height at most 1. Then, the algebra* $U_\chi(W^1)/J_\chi^2$ *is of wild representation type if* $(\mathrm{ht}(\chi), p) \neq (1, 5)$.

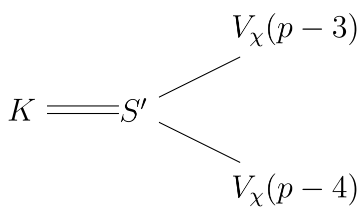
Proof. Suppose $(\mathrm{ht}(\chi), p) \neq (1, 5)$. We shall show that the separated quiver $(Q_{W^1})_s$ of the Ext–quiver of $U_\chi(W^1)$ is not a union of diagrams of types A, D, E, $\tilde{A}$, $\tilde{D}$, $\tilde{E}$. The wildness of $U_\chi(W^1)$ will then follow from Theorem 5.2.2. We begin by considering the case $\mathrm{ht}(\chi) = -1$ where we have isomorphisms

$$\begin{aligned} &\mathrm{Ext}_{U_\chi(W^1)}(K, S) \simeq K^2, \\ &\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(p-3), S) \simeq K, \\ &\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(p-4), S) \simeq K. \end{aligned}$$

This yields a subquiver of the Ext–quiver of the form



To construct the corresponding separated quiver, we merely need to switch S to S'. (We ignore the vertices K', $V_\chi(p-3)'$ and $V_\chi(p-4)'$.) If, furthermore, we ignore the directions of the arrows, we obtain



which is clearly not a union of diagrams of types A, D, E, $\tilde{A}$, $\tilde{D}$, $\tilde{E}$. We now move to the case $\mathrm{ht}(\chi) = 0$ where we have the following isomorphisms

$$\begin{aligned}
\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(0), V_\chi(0)) &\simeq K,\\
\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(p-2), V_\chi(0)) &\simeq K,\\
\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(p-3), V_\chi(0)) &\simeq K,\\
\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(p-4), V_\chi(0)) &\simeq K,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(0), V_\chi(1)) &\simeq K,\\
\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(0), V_\chi(2)) &\simeq K,\\
\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(0), V_\chi(3)) &\simeq K.
\end{aligned}$$

This yields a subquiver of the separated quiver of the form (the directions of the arrows are ignored)

$$\begin{array}{ccccccc}
V_\chi(p-2) & & & & & & V_\chi(1)' \\
 & \diagdown & & & & \diagup & \\
V_\chi(p-3) & \text{——} & V_\chi(0)' & \text{——} & V_\chi(0) & \text{——} & V_\chi(2)' \\
 & \diagup & & & & \diagdown & \\
V_\chi(p-4) & & & & & & V_\chi(3)'.
\end{array}$$

Again, this is not a union of diagrams of types A, D, E, $\tilde{A}$, $\tilde{D}$, $\tilde{E}$. Assume, next, that $\mathrm{ht}(\chi) = 1$. The isomorphism

$$\mathrm{Ext}_{U_\chi(W^1)}(V_\chi(\lambda'), V_\chi(\lambda)) \simeq K \quad \text{if} \quad [\lambda - \lambda'] = 2, 3$$

yields a subquiver of the form $\tilde{A}_{2p}$. If $p > 5$, we have additional extensions between Verma modules with $[\lambda - \lambda'] = 4$. This proves the claim. $\square$

Part II

The projective indecomposable modules

6 Representations of the Witt–Jacobson Lie algebras

6.1 Preliminaries

From now on we will focus more on the restricted Witt–Jacobson Lie algebras W^n of rank $n > 1$. We will keep the setting as general as possible, but some of the results are presented only for $n = 2$. Again, the ground field K is taken to be algebraically closed and of positive characteristic. However, unlike before where the characteristic p was assumed to be strictly greater than 3 (the Witt algebra W^1 is isomorphic to $\mathfrak{sl}_2$ when $p = 3$), we will, as far as possible, cover all cases $p > 0$.

6.1.1 We introduce the notion of the character of a module. Let T be a torus in a restricted Lie algebra L. Let M be a finite dimensional module over T and let $\mathbb{Z}[T^*]$ be the group algebra of T^* with basis elements $e(\nu)$ for $\nu \in T^*$. We set

$$\operatorname{ch} M = \sum_{\nu \in T^*} \dim M_\nu e(\nu) \in \mathbb{Z}[T^*],$$

where $M_\nu = \{m \in M \mid hm = \nu(h)m \text{ for all } h \in T\}$. This is called *the formal character* of M or, more commonly, *the character* of M. Note that the $e(\nu)$ are multiplied according to the rule

$$e(\nu_1)e(\nu_2) = e(\nu_1 + \nu_2). \tag{6.1}$$

In particular, we have $e(\nu)^p = 1$ for all $\nu \in T^*$. One can prove that given a short exact sequence

$$0 \to M' \to M \to M'' \to 0,$$

then $\operatorname{ch} M = \operatorname{ch} M' + \operatorname{ch} M''$. Thus, any $\operatorname{ch} M$ is determined uniquely by the formal characters of the composition factors of M, taken with multiplicity.

Furthermore, if $M = M' \otimes M''$, then one can prove that

$$\operatorname{ch} M = \operatorname{ch} M' \operatorname{ch} M''.$$

We say that the character of a tensor product is the product of the characters.

6.1.2 Let $\mathfrak{h}$ be a restricted Lie subalgebra in a restricted Lie algebra L. Suppose $\chi \in L^*$. Then $\chi \in \mathfrak{h}^*$ by restriction. The induction functor

$$U_\chi(L) \otimes_{U_\chi(\mathfrak{h})} -\colon \operatorname{mod} U_\chi(\mathfrak{h}) \to \operatorname{mod} U_\chi(L)$$

which takes a $U_\chi(\mathfrak{h})$–module M to the induced $U_\chi(L)$–module $U_\chi(L) \otimes_{U_\chi(\mathfrak{h})} M$ will be denoted by $\operatorname{ind}_{U_\chi(\mathfrak{h})}^{U_\chi(L)} -$. This is an exact functor.

If also $\chi' \in L^*$ and M and M' are modules over $U_\chi(L)$ and $U_{\chi'}(L)$, respectively, then $M \otimes M'$ is a module over $U_{\chi+\chi'}(L)$. This also applies for $\mathfrak{h}$ instead of L. Suppose that N is a module over $U_{\chi'}(\mathfrak{h})$. Then there is an isomorphism of $U_{\chi+\chi'}(L)$–modules

$$M \otimes \operatorname{ind}_{U_{\chi'}(\mathfrak{h})}^{U_{\chi'}(L)} N \simeq \operatorname{ind}_{U_{\chi+\chi'}(\mathfrak{h})}^{U_{\chi+\chi'}(L)} (M \otimes N).$$

This is called *the tensor identity*, see e.g. [14, Sec. 1.12].

6.1.3 The following lemma is a well–known result that will be used several times, often without any reference. See e.g. [21, Lem. 6.3.1].

Lemma. *Let L be a restricted Lie algebra with a p–mapping $[p]$ and let $I \subset L$ be a unipotent ideal such that $x^{[p]} \in I$ for all $x \in I$. If V is a simple module over L with p–character χ such that $\chi(I) = 0$, then $IV = 0$.*

6.1.4 Let L be a graded restricted Lie algebra

$$L = L_{-s} \oplus \cdots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_t.$$

Suppose that

$$N^+ = \bigoplus_{i>0} L_i \quad \text{and} \quad N^- = \bigoplus_{i<0} L_i$$

are unipotent subalgebras of L and set

$$B^+ = L_0 \oplus N^+ \quad \text{and} \quad B^- = L_0 \oplus N^-.$$

Every L_0–module can be extended to $B^\pm$ by letting $N^\pm$ act trivially. On the other hand, the simple modules over $B^\pm$ are just the simple modules over L_0 with $N^\pm$ acting trivially. Now, suppose that $\chi \in L^*$ with $\chi(N^\pm) = 0$. Then again, we can extend any $U_\chi(L_0)$–module to $U_\chi(B^\pm)$ by letting $U_\chi(N^\pm)$ act trivially. Furthermore, the simple $U_\chi(B^\pm)$–modules correspond to the simple modules over $U_\chi(L_0)$ with $U_\chi(N^\pm)$ acting trivially.

6.2 New grading

6.2.1 From now on, we fix a character $\chi \in (W^n)^*$ of height 0. According to Appendix A, we may assume that $\chi(\partial_1) \neq 0$ and $\chi(\partial_i) = 0$ for all $i > 1$. In light of the observations made in Section 6.1.4, we introduce a new grading on W^n

$$W^n = \bigoplus_{i=-1}^{(n-1)(p-1)} W^n_{(i)},$$

such that $\chi(W^n_{(>0)}) = 0$ and $\chi(W^n_{(<0)}) = 0$. Indeed, the truncated polynomial algebra $B^n = K[X_1, X_2, \ldots, X_n]/(X_1^p, X_2^p, \ldots, X_n^p)$ has a grading $B^n = \bigoplus_{i\in\mathbb{Z}} B^n_{(i)}$ which assigns degree 0 to X_1 and degree 1 to X_i for $i > 1$. This induces a grading on $W^n = \operatorname{Der}_K B^n$ in the following way: For all $i \in \mathbb{Z}$, set

$$W^n_{(i)} = \{D \in W^n \mid D(B^n_{(m)}) \subset B^n_{(m+i)} \text{ for all } m\}.$$

Then $W^n_{(i)}$ is a subspace of W^n and the sum of the $W^n_{(i)}$ is direct. We also have $[W^n_{(i)}, W^n_{(j)}] \subset W^n_{(i+j)}$ for all i, j and the grading is restricted: If $D \in W^n_{(i)}$ then $D^{[p]} \in W^n_{(pi)}$. Using the notation introduced in Section 1.2.1, we have

$$\deg(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \partial_k) = \begin{cases} i_2 + i_3 + \cdots + i_n, & \text{if } k = 1, \\ i_2 + i_3 + \cdots + i_n - 1, & \text{if } k \neq 1. \end{cases}$$

For all $i > (n-1)(p-1)$ and all $i < -1$, we have $W^n_{(i)} = 0$. Furthermore,

$$W^n_{(-1)} = \bigoplus_{\substack{0\le r<p \\ 2\le i\le n}} K x_1^r \partial_i,$$

and

$$W^n_{(0)} = \bigoplus_{0\le r<p} K x_1^r \partial_1 \oplus \bigoplus_{\substack{0\le r<p \\ 2\le i,j\le n}} K x_1^r x_i \partial_j.$$

From now on, set

$$N^+ = W^n_{(>0)} \quad \text{and} \quad N^- = W^n_{(<0)},$$
$$B^+ = W^n_{(\geq 0)} \quad \text{and} \quad B^- = W^n_{(\leq 0)}.$$

The subalgebras $N^\pm$ are unipotent because $D \in W^n_{(i)}$ implies $D^{[p]} \in W^n_{(pi)}$ and thus $D^{[p^r]} \in W^n_{(p^r i)}$ which is zero for r large enough. Furthermore, observe that $\chi(N^\pm) = 0$. In fact, we have $\chi(W^n_{(i)}) = 0$ for all $i \neq 0$.

Remark. There is a natural inclusion $W^1 \subset W^n_{(0)}$ which maps $x_1^r \partial_1$ in W^1 to $x_1^r \partial_1$ in $W^n_{(0)}$.

6.3 Irreducible representations

6.3.1 This section is devoted to classifying the irreducible representations of $U_\chi(W^n)$. Thanks to Proposition 1.2.4 in Nakano's paper [17], this query amounts to classifying the irreducible representations of the subalgebra $U_\chi(W^n_{(0)})$.

Theorem. *There is a one–to–one correspondence between the irreducible representations of $U_\chi(W^n)$ and $U_\chi(W^n_{(0)})$.*

If, indeed, L is a simple module over $U_\chi(W^n_{(0)})$, then the induced module $U_\chi(W^n) \otimes_{U_\chi(B^+)} L$ has a unique maximal submodule and the set of heads of all $U_\chi(W^n) \otimes_{U_\chi(B^+)} L$ is a complete set of pairwise non–isomorphic simple modules over $U_\chi(W^n)$.

6.3.2 To classify the irreducible representations of $U_\chi(W^n_{(0)})$ we consider the restricted Lie subalgebras $A, B \subset W^n_{(0)}$ defined by

$$A = \bigoplus_{2 \leq i,j \leq n} K x_i \partial_j \quad \text{and} \quad B = \bigoplus_{0 \leq r < p} K x_1^r \partial_1.$$

Observe that B is isomorphic to W^1. Furthermore, A is isomorphic to $\mathfrak{gl}_{n-1}$ via the map that sends $x_i \partial_j$ to the $(n-1) \times (n-1)$ matrix E_{ij} with 1 in the (i,j)th position and 0 elsewhere. This isomorphism defines a triangular decomposition of $A = A^- \oplus A^0 \oplus A^+$ where

$$A^- = \bigoplus_{2 \le j < i \le n} K x_i \partial_j,$$
$$A^0 = \bigoplus_{2 \le i \le n} K x_i \partial_i,$$
$$A^+ = \bigoplus_{2 \le i < j \le n} K x_i \partial_j.$$

For every $2 \le j \le n$, we define a linear map $\varepsilon_j \colon A^0 \to K$ such that $\varepsilon_j(\sum_{i=2}^n \alpha_i x_i \partial_i) = \alpha_j$. Consider the weights

$$\Lambda = \{\sum_{i=2}^{n} \alpha_i \varepsilon_i \mid \alpha_i \in \mathbb{F}_p\} \subset (A^0)^*.$$

Every $\lambda \in \Lambda$ defines a 1–dimensional A^0–module K_λ such that every $x \in A^0$ acts as multiplication by $\lambda(x)$. This module can be extended to a module over $A^0 \oplus A^+$ by letting A^+ act trivially. The induced module

$$Z(\lambda) = U_\chi(A) \otimes_{U_\chi(A^0 \oplus A^+)} K_\lambda,$$

has a unique maximal submodule. See e.g. [11, Prop. 1.2], but as mentioned in [11], this result goes back to Braden. (Note that [11] deals with Lie algebras of semisimple algebraic groups, but the arguments work just as well for A.) If $E(\lambda)$ denotes the simple module that corresponds to $Z(\lambda)$, then the set of all $E(\lambda)$ forms a complete set of pairwise non–isomorphic simple modules of $U_\chi(A)$, see e.g. [17, Prop. 1.2.3 & Prop. 1.2.4]. In particular, $U_\chi(A)$ has p^{n-1} isomorphism classes of simple modules. We will denote the trivial module by K.

Remark. If $n = 2$, then $A = A^0$ and $A^+ = A^- = 0$. Moreover, we have $Z(\lambda) = E(\lambda) = K_\lambda$.

6.3.3 We consider the subsets $J \subset H$ in $W^n_{(0)}$ defined by

$$H = \bigoplus_{1 \le r < p} K x_1^r \partial_1 \oplus \bigoplus_{\substack{0 \le r < p \\ 2 \le i,j \le n}} K x_1^r x_i \partial_j,$$
$$J = \bigoplus_{2 \le r < p} K x_1^r \partial_1 \oplus \bigoplus_{\substack{1 \le r < p \\ 2 \le i,j \le n}} K x_1^r x_i \partial_j.$$

We have the following commutator formulas

$$[x_1^r\partial_1, x_1^s\partial_1] = (s-r)x_1^{r+s-1}\partial_1, \tag{6.2}$$
$$[x_1^r\partial_1, x_1^s x_i\partial_j] = s x_1^{r+s-1}x_i\partial_j, \tag{6.3}$$
$$[x_1^r x_i\partial_j, x_1^s x_k\partial_l] = \delta_{jk}x_1^{r+s}x_i\partial_l - \delta_{il}x_1^{r+s}x_k\partial_j, \tag{6.4}$$

which first imply that H and J are (restricted) Lie subalgebras in $W^n_{(0)}$ and then that J is an ideal in H. Furthermore, J is unipotent as it is contained in $W^n_{>0}$. We have $\chi(J) = 0$ since $\partial_1 \notin J$. It follows from Lemma 6.1.3 that every $U_\chi(H)$–module is a $U_\chi(H/J)$–module. In particular, the simple modules over $U_\chi(H)$ are exactly the simple modules over $U_\chi(H/J)$. Now, since A and $Kx_1\partial_1$ commute, the Lie algebra $H/J \simeq Kx_1\partial_1 \oplus A$ is the direct product of $Kx_1\partial_1$ and A. Therefore, the set of tensor products $E_1 \otimes E_2$ where E_1 and E_2 are simple over $U_\chi(Kx_1\partial_1)$ and $U_\chi(A)$, respectively, is a complete set of simple $U_\chi(H/J)$–modules.

Lemma. *The induced module* $\operatorname{ind}_{U_\chi(H)}^{U_\chi(W^n_{(0)})}(E_1 \otimes E_2)$ *is simple for all simple modules* E_1 *and* E_2 *over* $U_\chi(Kx_1\partial_1)$ *and* $U_\chi(A)$, *respectively. In fact,* $\operatorname{ind}_{U_\chi(H)}^{U_\chi(W^n_{(0)})}(E_1 \otimes E_2)$ *is simple over the subalgebra* $U_\chi(B \times A)$.

Proof. It suffices to prove that $\operatorname{ind}_{U_\chi(H)}^{U_\chi(W^n_{(0)})}(E_1 \otimes E_2)$ is simple over $B \oplus A$. The assertion will then follow because $B \oplus A$ is a subalgebra of $W^n_{(0)}$. Note that B and A commute so $B \oplus A$ is the direct product of B and A. Thus, if we can prove

$$\operatorname{ind}_{U_\chi(H)}^{U_\chi(W^n_{(0)})}(E_1 \otimes E_2) \simeq (\operatorname{ind}_{U_\chi(B\cap H)}^{U_\chi(B)} E_1) \otimes E_2,$$

then the assertion follows since $\operatorname{ind}_{U_\chi(B\cap H)}^{U_\chi(B)} E_1 \simeq \operatorname{ind}_{U_\chi(W^1_{\geq 0})}^{U_\chi(W^1)} E_1$ is simple over $U_\chi(B) \simeq U_\chi(W^1)$. The desired isomorphism can be obtained from the map that sends every $\partial_1^r \otimes (e_1 \otimes e_2)$ into $(\partial_1^r \otimes e_1) \otimes e_2$ where e_1 and e_2 are basis elements in E_1 and E_2, respectively. Indeed, this is clearly an isomorphism of vector spaces. It is invariant under the action of A since A commutes with ∂_1^r. If $x \in B$, then the product $x\partial_1^r$ can be written as $x\partial_1^r = \sum_{i\in I}\partial_1^i u_i$ for some $u_i \in U_\chi(\bigoplus_{i=1}^{p-1} Kx_1^i\partial_1)$. (This is a consequence of the PBW theorem and the fact that B has a basis consisting of the elements $x_1^i\partial_1$ where $i = 0, 1, \ldots, p-1$.) Thus,

$$\begin{aligned} x\partial_1^r \otimes (e_1 \otimes e_2) &= \sum_{i\in I}\partial_1^i u_i \otimes (e_1 \otimes e_2) \\ &= \sum_{i\in I}\partial_1^i \otimes (u_i e_1 \otimes e_2). \end{aligned}$$

This proves that the map is invariant under the action of B and hence the lemma. □

Remark. Let V be a simple $U_\chi(Kx_1\partial_1)$–module. Since K is algebraically closed, $x_1\partial_1$ has an eigenvalue $\mu \in K$ for some nonzero $v \in V$. It follows that V is 1–dimensional. Since, furthermore, $x_1\partial_1$ is toral and $\chi(x_1\partial_1) = 0$, we deduce that $\mu \in \mathbb{F}_p$. Conversely, every $\mu \in \mathbb{F}_p$ gives rise to a simple 1–dimensional module K_μ via $x_1\partial_1 \cdot 1 = \mu \cdot 1$. Therefore, there are exactly p isomorphism classes of simple $U_\chi(Kx_1\partial_1)$–modules.

For every simple $U_\chi(Kx_1\partial_1)$–module K_μ and every simple $U_\chi(A)$–module E, we set

$$L(\mu, E) = \operatorname{ind}_{U_\chi(H)}^{U_\chi(W^n_{(0)})}(K_\mu \otimes E).$$

For convenience, we change the notation from Part I and let $L(\mu) = V_\chi(\mu)$ denote the simple modules of W^1. Note that in the proof of the lemma we have established the following isomorphism $L(\mu, E) \simeq L(\mu) \otimes E$.

In the subsequent theorem, the notation $(\mu, E) \neq (p-1, K)$ means that $\mu \neq p-1$ and $E \not\simeq K$.

Theorem. *The set of all $L(\mu, E)$ with $(\mu, E) \neq (p-1, K)$ forms a complete set of pairwise non–isomorphic simple modules over $U_\chi(W^n_{(0)})$. In particular, there exist exactly $p^n - 1$ isomorphism classes of simple $U_\chi(W^n_{(0)})$–modules.*

Proof. Let V be a simple module over $U_\chi(W^n_{(0)})$. For some K_μ and E there exists a monomorphism of $U_\chi(H)$–modules

$$\kappa\colon K_\mu \otimes E \to V.$$

It follows that κ extends to a homomorphism of $U_\chi(W^n_{(0)})$–modules from $L(\mu, E)$ to V and therefore $L(\mu, E) \simeq V$.

Next, we prove that with the exception of $L(0, K) \simeq L(p-1, K)$, two modules $L(\mu, E)$ and $L(\mu', E')$ are isomorphic if and only if $\mu = \mu'$ and $E \simeq E'$. Suppose we have an isomorphism $\phi\colon L(\mu, E) \to L(\mu', E')$. Then, by restriction, we get an isomorphism $L(\mu) \otimes E \simeq L(\mu') \otimes E'$ of $U_\chi(B \times A)$–modules which, obviously, implies $L(\mu) \simeq L(\mu')$ and $E \simeq E'$. Thus, $\mu = \mu'$ or $\{\mu, \mu'\} = \{0, p-1\}$ so without loss of generality we may assume that $\mu = p-1$ and $\mu' = 0$ and $E = E'$. The isomorphism $L(p-1) \to L(0)$ which maps $\partial_1^r \otimes 1$ into $\partial_1^{[r+1]} \otimes 1$ can be extended to an isomorphism $\varphi\colon L(p-1) \otimes E \to L(0) \otimes E$. (Here we use the notation introduced in

Section 3.1.1.) It follows from Schur's Lemma that we may assume that $\phi = \varphi$ hence

$$\phi(\partial_1^r \otimes (1 \otimes e)) = \partial_1^{[r+1]} \otimes (1 \otimes e).$$

For $i, j \geq 2$, we have

$$\phi(x_1 x_i \partial_j (1 \otimes (1 \otimes e))) = x_1 x_i \partial_j (\partial_1 \otimes (1 \otimes e).$$

The left-hand side is equal to 0 as $x_1 x_i \partial_j \in J$ annihilates $1 \otimes e$. The right-hand side is equal to $\partial_1 \otimes (x_1 x_i \partial_j (1 \otimes e)) - 1 \otimes (1 \otimes x_i \partial_j e) = -1 \otimes (1 \otimes x_i \partial_j e)$. It follows that $x_i \partial_j e = 0$ and since i and j and e are arbitrarily chosen, we deduce $E = K$. What remains to prove is that $L(0, K)$ and $L(p-1, K)$ are, in fact, isomorphic or, equivalently, that the map constructed above is invariant under the action of $x_1^r x_i \partial_j$ for $r > 0$ and $i, j \geq 2$. To this end, note that $x_1^r x_i \partial_j (1 \otimes (1 \otimes e)) = 0$ for all $r > 0$. If $s > 0$, then we have

$$\begin{aligned} x_1^r x_i \partial_j (\partial_1^s \otimes (1 \otimes e)) &= \partial_1 x_1^r x_i \partial_j (\partial_1^{s-1} \otimes (1 \otimes e)) \\ &\quad - r x_1^{r-1} x_i \partial_j (\partial_1^{s-1} \otimes (1 \otimes e)). \end{aligned}$$

Thus, by induction, we conclude that $x_1^r x_i \partial_j$ annihilates all $\partial_1^s \otimes (1 \otimes e)$ hence the theorem. □

7 Projective indecomposable modules of $W^n_{(0)}$

7.1 Preliminaries

This section is devoted to studying the projective indecomposable modules of the algebra $U_\chi(W^n_{(0)})$. We will keep the setting as general as possible, but the general case is very difficult to describe completely. The main results are Theorem 7.3.1 together with Proposition 7.2.2 and 7.4.3. For $n = 2$, we will give a complete classification of all projective indecomposable modules and we will compute the Cartan invariants of $U_\chi(W^2_{(0)})$, see Corollary 7.4.3.

7.1.1 Let R be a finite dimensional K–algebra. If P is a finite dimensional, projective indecomposable R–module, then the radical $\operatorname{rad} P$ of P is a maximal submodule of P. It follows that $P/\operatorname{rad} P$ is a simple R–module. The map that sends the isomorphism class of P to that of $P/\operatorname{rad} P$ is a bijection from the set of isomorphism classes of finite dimensional, indecomposable projective R–modules to the set of isomorphism classes of simple R–modules. Suppose E is a simple module isomorphic to $P/\operatorname{rad} P$, then one calls P the projective cover of E. Furthermore, if M is a finite dimensional R–module then $\dim \operatorname{Hom}_R(P, M) = [M\colon E]$ where $[M\colon E]$ denotes the multiplicity of E as a composition factor of M. We let $[M]$ denote the class of M in the Grothendieck group. Furthermore, if $E_1, E_2, \ldots, E_r$ is a system of representatives for the isomorphism classes of simple modules and $P_{E_1}, P_{E_2}, \ldots, P_{E_r}$ are the corresponding projective modules then there is an isomorphism

$$R \simeq P_{E_1}^{\dim E_1} \oplus \cdots \oplus P_{E_r}^{\dim E_r}$$

of R–modules. The elements $c_{ij} = [P_{E_i} : E_j]$ are called *Cartan invariants* and the matrix (c_{ij}) is called *the Cartan martix* of R.

7.1.2 Let L be a finite dimensional restricted Lie algebra and let $\delta_L\colon L \to K$ be the map that sends every x in L into $\operatorname{tr}(\operatorname{ad} x)$ in K. We have for $x, y \in L$

$$\delta_L([x,y]) = \operatorname{tr}(\operatorname{ad}[x,y]) = \operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y - \operatorname{ad} y \circ \operatorname{ad} x) = 0.$$

Thus, δ_L defines a 1–dimensional L–module K_{δ_L} where every $x \in L$ acts as multiplication by $\delta_L(x)$. Furthermore, we have

$$\delta_L(x^{[p]}) = \operatorname{tr}((\operatorname{ad} x)^p) = (\operatorname{tr} \operatorname{ad} x)^p = \delta_L(x)^p.$$

Hence, the module structure of K_{δ_L} can be extended to give K_{δ_L} the structure of a $U_0(L)$–module. These observations carry over to the map $-\delta_L$ and in a similar way we obtain a $U_0(L)$–module $K_{-\delta_L}$. Now, let $\varphi \in L^*$ and let E be a simple module over $U_\varphi(L)$. It is well–known that if P_E is the projective cover of E, then

$$\operatorname{Soc}_{U_\varphi(L)} P_E \simeq E \otimes K_{-\delta_L}.$$

See e.g. [14, Prop. 1.9 & Formula 1.9(4)]. Furthermore, if E and E' are simple modules over $U_\varphi(L)$ with corresponding projective covers P_E and $P_{E'}$, then

$$\operatorname{Hom}_{U_\varphi(L)}(E, P_{E'}) \simeq \begin{cases} K, & \text{if } E \simeq E' \otimes K_{-\delta_L}, \\ 0, & \text{otherwise.} \end{cases} \tag{7.1}$$

Furthermore, every finite dimensional projective module P decomposes into a direct sum of projective indecomposable modules $P_{E_1}, P_{E_2}, \dots, P_{E_r}$

$$P \simeq P_{E_1}^{m(E_1)} \oplus P_{E_2}^{m(E_2)} \oplus \cdots \oplus P_{E_r}^{m(E_r)},$$

where $m(E_i) = \dim \operatorname{Hom}_{U_\varphi(L)}(E_i \otimes K_{-\delta_L}, P)$.

7.1.3 The dimensions of the projective indecomposable modules and the Cartan invariants of $U_\chi(W^1)$ were computed for $p > 3$ by Nakano and Feldvoss in [7, Thm. 2.3]. The case $p = 3$ is easy to handle since W^1 is isomorphic to $\mathfrak{sl}_2$, cf. [13, Prop. 10.10]. In the following, we let $P(\mu)$ denote the projective cover of the simple $U_\chi(W^1)$–module $L(\mu)$.

Theorem. *If $p > 2$, then $\dim P(0) = 2p^{p-2}$ and $\dim P(\mu) = p^{p-2}$ for $\mu \neq 0$. Furthermore, we have for $p > 3$*

1. $[P(0)] = 4p^{p-4}[L(0)] + \sum_{\mu'=1}^{p-2} 2p^{p-4}[L(\mu')]$,
2. $[P(\mu)] = 2p^{p-4}[L(0)] + \sum_{\mu'=1}^{p-2} p^{p-4}[L(\mu')]$ *for* $\mu \neq 0$.

For $p = 3$, we have $[P(0)] = 2[L(0)]$ and $[P(1)] = [L(1)]$.

7.1.4 It is well–known that every projective indecomposable module P_E of $U_\chi(A)$ has a filtration with factors $Z(\lambda)$ each occuring with multiplicity $[Z(\lambda) : E]$

$$[P_E] = \sum_{\lambda\in\Lambda}[Z(\lambda) : E][Z(\lambda)].$$

(Here E is the simple $U_\chi(A)$–module corresponding to P_E.) This is proved for reductive groups in [15, Prop. II. 11.4] (see also Formula II. 11.3(3), Prop. II. 9.5(e) and Formula II. 11.5(4)), but for Lie algebras it goes back to [12, Satz 4.3] and [11, Thm. 4.5].

7.2 Dimension

7.2.1 The projective indecomposable modules of $U_\chi(A)$ will be denoted by P_E, where E is the corresponding simple module. The set of all $P(\mu)\otimes P_E$ forms a complete set of projective indecomposable modules of $U_\chi(B \times A)$. We consider the module

$$Q(\mu, E) = U_\chi(W^n_{(0)}) \otimes_{U_\chi(B\times A)} (P(\mu) \otimes P_E).$$

Since $P(\mu) \otimes P_E$ is projective over $U_\chi(B \times A)$, it follows that $Q(\mu, E)$ is projective over $W^n_{(0)}$. Furthermore, we have

$$\begin{aligned}\operatorname{Hom}_{U_\chi(W^n_{(0)})}(Q(\mu, E), L(\mu', E') &\simeq \operatorname{Hom}_{U_\chi(B\times A)}(P(\mu) \otimes P_E, L(\mu') \otimes E')\\ &\simeq \begin{cases} K, & \text{if } \mu = \mu' \text{ and } E \simeq E',\\ K, & \text{if } \{\mu, \mu'\} = \{0, p-1\} \text{ and } E \simeq E',\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

We let $P(\mu, E)$ denote the projective cover of $L(\mu, E)$ as a $U_\chi(W^n_0)$–module. The above discussion yields the following Lemma.

Lemma. *We have*

1. $Q(0, K) \simeq P(0, K)$,
2. $Q(\mu, E) \simeq P(\mu, E)$ *if* $\mu \neq 0, p - 1$,
3. $Q(0, E) \simeq P(0, E) \oplus P(p - 1, E)$ *if* $E \not\simeq K$.

7.2.2 The main goal of this section is to prove the following proposition.

Proposition. *We have* $\dim P(\mu, E) = \dim P(0, E)$ *if* $E \not\simeq K$. *Furthermore,* $\dim P(0, K) = 2 \dim P(\mu, K)$ *for all* $\mu \neq 0, p-1$.

We set $a = \delta_{W^n_{(0)}}(x_1\partial_1)$, where $\delta_{W^n_{(0)}}$ is the map introduced in Section 7.1.2. A very simple calculation shows that $a = 0$ for $p > 2$ and $a = n^2$ for $p = 2$. Furthermore, we have $\delta_H(x_1\delta_1) = a + 1$. For the remaining basis elements $x^\alpha\partial_i$, we have $\delta_{W^n_{(0)}}(x^\alpha\partial_i) = 0$ and $\delta_H(x^\alpha\partial_i) = 0$. (Here we are using the notation introduced in Section 1.2.1.)

We let $P_H(\mu, E)$ denote the projective cover of the simple $U_\chi(H)$–module $K_\mu \otimes E$. Together with the discussion in Section 7.1.2, the isomorphism $(K_\mu \otimes E) \otimes K_{-\delta_H} \simeq K_{\mu-(a+1)} \otimes E$ implies

$$\mathrm{Soc}_{U_\chi(H)}\, P_H(\mu, E) \simeq K_{\mu-(a+1)} \otimes E.$$

Similarly, together with the tensor identity, the isomorphism $(K_\mu \otimes E) \otimes K_{-\delta_{W^n_0}} \simeq K_{\mu-a} \otimes E$ implies $L(\mu, E) \otimes K_{-\delta_{W^n_{(0)}}} \simeq L(\mu - a, E)$ which, in turn, yields

$$\mathrm{Soc}_{U_\chi(W^n_{(0)})}\, P(\mu, E) \simeq L(\mu - a, E).$$

The module $P(\mu, E)$ is projective over $U_\chi(H)$ because it is projective over $U_\chi(W^n_{(0)})$ and $U_\chi(W^n_{(0)})$ is free over $U_\chi(H)$. Thus, $P(\mu, E)$ decomposes into a direct sum of $P_H(\mu', E')$ each with multiplicity

$$\begin{aligned} m(K_{\mu'} \otimes E') &= \dim \mathrm{Hom}_{U_\chi(H)}(K_{\mu'-(a+1)} \otimes E', P(\mu, E)) \\ &= \dim \mathrm{Hom}_{U_\chi(W^n_{(0)})}(L(\mu' - (a+1), E'), P(\mu, E)) \\ &= \begin{cases} 1, & \text{if } L(\mu' - (a+1), E') \simeq L(\mu - a, E), \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where the last equality follows from (7.1). We will write $P(\mu, E)_{|H}$ instead of $P(\mu, E)$ when we consider $P(\mu, E)$ as a module over $U_\chi(H)$. Now, the proposition follows from the following two lemmas.

Lemma (A)**.** *We have*

1. $P(\mu, E)_{|H} \simeq P_H(\mu + 1, E)$ *for* $E \not\simeq K$.
2. $P(\mu, K)_{|H} \simeq P_H(\mu + 1, K)$ *for* $\mu \neq 0, p-1$.
3. $P(0, K)_{|H} \simeq P_H(0, K) \oplus P_H(1, K)$.

Proof. In light of Theorem 6.3.3, we see that if $E \not\simeq K$ then the isomorphism $L(\mu' - (a+1), E') \simeq L(\mu - a, E)$ amounts to $\mu' = \mu + 1$ and $E \simeq E'$. Therefore, the discussion preceding the lemma implies (1). If $E \simeq K$, we may assume that $\mu \neq p - 1$. If, furthermore, $\mu \neq 0$ then $p > 2$ which in turn implies $a = 0$. Thus, $L(\mu' - 1, K) \simeq L(\mu, K)$ if and only if $\mu' = \mu + 1$. This proves (2). The last assertion is clear for $p = 2$ because all $L(\mu, K)$ are isomorphic in this case. If $p > 2$, then $a = 0$ and $L(\mu' - 1, K) \simeq L(0, K)$ is equivalent to $\mu' = 1$ or $\mu' = 0$. □

Lemma (B). *We have $P_H(\mu, E) \simeq (K_\mu \otimes K) \otimes P_H(0, E)$ for any $\mu \in \mathbb{F}_p$ and any simple $U_\chi(A)$–module E. In particular,* $\dim P_H(\mu, E) = \dim P_H(0, E)$.

Proof. The module $(K_\mu \otimes K) \otimes P_H(0, E)$ is projective because the tensor product of a projective module with a finite dimensional module is projective. The module $(K_\mu \otimes K) \otimes P_H(0, E)$ is indecomposable because tensoring with a 1–dimensional module sends indecomposables to indecomposables. The claim follows as $(K_\mu \otimes K) \otimes P_H(0, E)$ has a submodule isomorphic to $(K_\mu \otimes K) \otimes (K_{-(a+1)} \otimes E) \simeq K_{\mu-(a+1)} \otimes E$. □

7.3 Cartan invariants

Our goal is to describe the projective indecomposable modules $P(\mu, E)$ of $U_\chi(W^n_{(0)})$. With the exception of the cases where $\mu \in \{0, p-1\}$ and $E \simeq K$, Lemma 7.2.1 reduces this problem to that of describing the projective modules $Q(\mu, E)$. Later, we will prove that the formal sum of composition factors of $P(\mu, E)$ in the Grothendieck group does not depend on μ, see Proposition 7.4.3. Thus, together with Theorem 7.3.1, this will allow us, at least for $n = 2$, to determine the Cartan invariants of $U_\chi(W^n_{(0)})$. For general n it is not known how to compute the multiplicities $[Z(\lambda') : E]$ in Theorem 7.3.1. The case $n = 2$ will be treated fully in Corollary 7.3.2 and 7.4.3.

7.3.1 We consider the subset

$$g = \bigoplus_{\substack{1 \le r < p \\ 2 \le i,j \le n}} K x_1^r x_i \partial_j.$$

The commutator formulas (6.3)–(6.4) show that g is an ideal in H. We can therefore consider $U_\chi(g)$ as a $U_\chi(K x_1 \partial_1 \times A)$–module via the adjoint representation. As a module over $U_\chi(K x_1 \partial_1 \times A)$, we have

$$[U_\chi(g)] = \sum_{i \in X} [K_{\mu_i} \otimes E_i], \tag{7.2}$$

for some index set $X \subset \mathbb{N}$. (Here $K_{\mu_i} \otimes E_i$ is simple over $U_\chi(Kx_1\partial_1 \times A)$.) Now, E_i and $U_\chi(g)$ admit weight space decompositions with respect to A^0 and $Kx_1\partial_1 \times A^0$, respectively

$$E_i = \bigoplus_{\lambda \in \Lambda} (E_i)_\lambda \quad \text{and} \quad U_\chi(g) = \bigoplus_{\mu \in \mathbb{F}_p} \bigoplus_{\lambda \in \Lambda} U_\chi(g)_{\mu,\lambda}. \tag{7.3}$$

Lemma (A). *We have*

$$\dim U_\chi(g)_{\mu,\lambda} = \begin{cases} \dim(E_i)_\lambda, & \textit{if there exists } i \in X \textit{ such that } \mu = \mu_i, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. We have

$$\operatorname{ch} U_\chi(g) = \sum_{\mu \in \mathbb{F}_p} \sum_{\lambda \in \Lambda} \dim U_\chi(g)_{\mu,\lambda} e(\mu, \lambda).$$

On the other hand, the character of $U_\chi(g)$ is determined by the characters of the composition factors of $U_\chi(g)$

$$\begin{aligned} \operatorname{ch} U_\chi(g) &= \sum_{i \in X} \operatorname{ch}(K_{\mu_i} \otimes E_i) \\ &= \sum_{i \in X} \sum_{\lambda \in \Lambda} \dim(E_i)_\lambda e(\mu_i, \lambda). \end{aligned}$$

This proves the lemma. □

We let $\tilde{\Lambda} \subset \Lambda$ be the subspace generated by all $\varepsilon_i - \varepsilon_j$. Then

$$\tilde{\Lambda} = \left\{ \sum_{i=2}^n a_i \varepsilon_i \mid a_i \in \mathbb{F}_p \text{ and } \sum_{i=2}^n a_i = 0 \right\}.$$

Lemma (B). *We have*

$$\operatorname{ch} U_\chi(g) = p^{(p-1)(n-1)^2-(n-1)} \sum_{\mu \in \mathbb{F}_p} \sum_{\lambda \in \tilde{\Lambda}} e(\mu, \lambda).$$

Proof. The Lie algebra g has a basis consisting of elements $x_1^r x_i \partial_j$ of weights $(r, \varepsilon_i - \varepsilon_j)$. Thus, every $(x_1^r x_i \partial_j)^s$ is of weight $(sr, s(\varepsilon_i - \varepsilon_j))$. Since the character of a tensor product is the product of the characters and since $U_\chi(g)$ is the tensor product of all $\oplus_{s=0}^{p-1} K(x_1^r x_i \partial_j)^s$ with $1 \leq r < p$ and $2 \leq i, j \leq n$, it follows that

$$\operatorname{ch} U_\chi(g) = \prod_{r=1}^{p-1} \prod_{i=2}^n \prod_{j=2}^n \left(\sum_{s=0}^{p-1} e(r, \varepsilon_i - \varepsilon_j)^s \right).$$

Note that all the weights of $U_\chi(g)$ belong to $\mathbb{F}_p \times \tilde{\Lambda}$. Since $e(\mu,\lambda)^p = 1$ for all $\mu \in \mathbb{F}_p$ and $\lambda \in \tilde{\Lambda}$ the formula $0 = 1 - e(\mu,\lambda)^p = (1 - e(\mu,\lambda))(1 + e(\mu,\lambda) + \cdots + e(\mu,\lambda)^{p-1})$ implies $(1 - e(r, \varepsilon_i - \varepsilon_j)) \operatorname{ch} U_\chi(g) = 0$. Now, suppose that

$$\operatorname{ch} U_\chi(g) = \sum_{\mu \in \mathbb{F}_p} \sum_{\lambda \in \tilde{\Lambda}} c_{\mu,\lambda} e(\mu,\lambda).$$

By multiplying both sides by $1 - e(r, \varepsilon_i - \varepsilon_j)$ we obtain

$$\sum_{\mu \in \mathbb{F}_p} \sum_{\lambda \in \tilde{\Lambda}} (c_{\mu,\lambda} - c_{\mu-r,\lambda-(\varepsilon_i-\varepsilon_j)}) e(\mu,\lambda) = 0.$$

which implies $c_{\mu,\lambda} = c_{\mu-r,\lambda-(\varepsilon_i-\varepsilon_j)}$ for all $1 \le r < p$ and $2 \le i,j \le n$. In particular, we have $c_{\mu,\lambda} = c_{\mu-r,\lambda}$ which means that the coefficients having the same second index are equal. We have $c_{\mu,\lambda} = c_{\mu-r,\lambda-(\varepsilon_i-\varepsilon_j)} = c_{\mu,\lambda-(\varepsilon_i-\varepsilon_j)}$ hence by using $c_{\mu,\lambda} = c_{\mu,\lambda-(\varepsilon_i-\varepsilon_j)}$ sufficiently many times we conclude that all the coefficients are equal. Now, a dimension argument yields the claim. □

Theorem. *We have for every $\mu \in \mathbb{F}_p$ and every simple $U_\chi(A)$–module E*

$$[Q(\mu,E)] = p^{(p-1)(n-1)^2-(n-1)} \sum_{\substack{\mu' \in \mathbb{F}_p \\ \mu' \neq p-1}} \sum_{\lambda' \in \Lambda} \sum_{\mu'' \in \mathbb{F}_p} \sum_{\lambda'' \in \lambda' + \tilde{\Lambda}} [P(\mu) : L(\mu')] \cdot [Z(\lambda') : E][U_\chi(W^n_{(0)}) \otimes_{U_\chi(H)} (K_{\mu''} \otimes Z(\lambda''))].$$

Proof. It follows from Section 7.1.4 that the projective cover P_E of E has a filtration with factors $Z(\lambda')$ each occuring with multiplicity $[Z(\lambda') : E]$. Since, furthermore, the composition factor multiplicity of $L(\mu')$ in $P(\mu)$ is $[P(\mu) : L(\mu')]$, we have

$$[P(\mu) \otimes P_E] = \sum_{\substack{\mu' \in \mathbb{F}_p \\ \mu' \neq p-1}} \sum_{\lambda' \in \Lambda} [P(\mu) : L(\mu')][Z(\lambda') : E][L(\mu') \otimes Z(\lambda')].$$

Inducing to $U_\chi(W^n_{(0)})$–modules, we get

$$[Q(\mu,E)] = \sum_{\substack{\mu' \in \mathbb{F}_p \\ \mu' \neq p-1}} \sum_{\lambda' \in \Lambda} [P(\mu) : L(\mu')][Z(\lambda') : E] \cdot [U_\chi(W^n_{(0)}) \otimes_{U_\chi(B \times A)} (L(\mu') \otimes Z(\lambda'))].$$

For every $\mu' \in \mathbb{F}_p$ and $\lambda' \in \Lambda$, set

$$\tilde{V}(\mu',\lambda') = U_\chi(W^n_{(0)}) \otimes_{U_\chi(B \times A)} (L(\mu') \otimes Z(\lambda')).$$

Since $L(\mu') = U_\chi(B) \otimes_{U_\chi(B\cap H)} K_{\mu'}$, we obtain an isomorphism

$$L(\mu') \otimes Z(\lambda') \simeq U_\chi(B \times A) \otimes_{U_\chi((B\cap H)\times A)} (K_{\mu'} \otimes Z(\lambda')),$$

which implies

$$\tilde{V}(\mu', \lambda') \simeq U_\chi(W_{(0)}^n) \otimes_{U_\chi((B\cap H)\times A)} (K_{\mu'} \otimes Z(\lambda')).$$

If we set $V(\mu', \lambda') = U_\chi(H) \otimes_{U_\chi((B\cap H)\times A)} (K_{\mu'} \otimes Z(\lambda'))$, then

$$\tilde{V}(\mu', \lambda') \simeq U_\chi(W_{(0)}^n) \otimes_{U_\chi(H)} V(\mu', \lambda').$$

By Lemma 6.3.3, we see that induction takes simple $U_\chi(H)$–modules to simple $U_\chi(W_{(0)}^n)$–modules. Thus, it suffices to determine the composition factors of $V(\mu', \lambda')$ as a $U_\chi(H)$–module. In fact, since the simple $U_\chi(H)$–modules are exactly the simple modules over $U_\chi(H/J) \simeq U_\chi(Kx_1\partial_1 \times A)$, it is enough to determine the composition factors of $V(\mu', \lambda')$ as a module over $U_\chi(Kx_1\partial_1 \times A)$. Now, we have an isomorphism of $U_\chi(Kx_1\partial_1 \times A)$–modules

$$V(\mu', \lambda') \simeq U_\chi(g) \otimes (K_{\mu'} \otimes Z(\lambda')).$$

It follows from (7.2) that $V(\mu', \lambda')$ has a filtration with factors $(K_{\mu_i} \otimes E_i) \otimes (K_{\mu'} \otimes Z(\lambda')) \simeq K_{\mu'+\mu_i} \otimes (E_i \otimes Z(\lambda'))$

$$[V(\mu', \lambda')] = \sum_{i\in X} [K_{\mu'+\mu_i} \otimes (E_i \otimes Z(\lambda'))].$$

Recall that E_i can be expressed as a direct sum of weight spaces $(E_i)_{\lambda''}$ with respect to A^0. Together with the tensor identity

$$E_i \otimes (U_\chi(A) \otimes_{U_\chi(A^0\oplus A^+)} K_{\lambda'}) \simeq U_\chi(A) \otimes_{U_\chi(A^0\oplus A^+)} (E_i \otimes K_{\lambda'}),$$

this implies that $E_i \otimes Z(\lambda')$ has a filtration with factors $Z(\lambda' + \lambda'')$ each occuring with multiplicity $\dim(E_i)_{\lambda''}$

$$[E_i \otimes Z(\lambda')] = \sum_{\lambda''\in\Lambda} \dim(E_i)_{\lambda''} [Z(\lambda' + \lambda'')].$$

Going back to $V(\mu', \lambda')$, all this means

$$\begin{aligned}[V(\mu', \lambda')] &= \sum_{i\in X} \sum_{\lambda''\in\Lambda} \dim(E_i)_{\lambda''} [K_{\mu'+\mu_i} \otimes Z(\lambda' + \lambda'')] \\ &= \sum_{\mu''\in\mathbb{F}_p} \sum_{\lambda''\in\Lambda} \dim U_\chi(g)_{\mu'',\lambda''} [K_{\mu'+\mu''} \otimes Z(\lambda' + \lambda'')],\end{aligned}$$

where the last equality follows from Lemma A. Together with Lemma B, this implies

$$\begin{aligned}[V(\mu',\lambda')] &= p^{(p-1)(n-1)^2-(n-1)} \sum_{\mu''\in\mathbb{F}_p} \sum_{\lambda''\in\tilde{\Lambda}} [K_{\mu'+\mu''} \otimes Z(\lambda'+\lambda'')] \\ &= p^{(p-1)(n-1)^2-(n-1)} \sum_{\mu''\in\mathbb{F}_p} \sum_{\lambda''\in\lambda'+\tilde{\Lambda}} [K_{\mu''} \otimes Z(\lambda'')],\end{aligned}$$

proving the theorem. □

7.3.2 As an example of how Theorem 7.3.1 may be applied, we consider the case $n = 2$. According to Theorem 6.3.3, the simple $W^2_{(0)}$–modules are given by the set of all $L(\mu, E)$, where E is simple over $U_\chi(A)$. Obviously, in the present setting, we have $A = Kx_2\partial_2$, cf. Remark 6.3.2. This Lie algebra has p isomorphism classes of simple modules each represented by a 1–dimensional module K_λ where $\lambda \in \Lambda \simeq \mathbb{F}_p$. Note that K_0 is the trivial module over $U_\chi(A)$. The projective indecomposable modules of $U_\chi(A)$ are all 1–dimensional $P_{K_\lambda} \simeq K_\lambda$. When $n = 2$, we let

$$L(\mu,\lambda) = U_\chi(W^2_{(0)}) \otimes_{U_\chi(H)} K_{\mu,\lambda}$$

denote the simple $U_\chi(W^2_{(0)})$–modules. Note that $L(\mu,\lambda) \simeq L(\mu',\lambda')$ if and only if $(\mu,\lambda) = (\mu',\lambda')$ or $\{\mu,\mu'\} = \{0, p-1\}$ and $\lambda = \lambda'$. Furthermore, we let $P(\mu,\lambda) = P(\mu, K_\lambda)$ denote the projective cover of $L(\mu,\lambda)$ and we let $Q(\mu,\lambda) = Q(\mu, K_\lambda)$.

Corollary. *If $n = 2$ and $p > 2$, then*

1. $[P(0,0)] = 4p^{2p-5}[L(0,0)] + 2p^{2p-5}\sum_{\mu'=1}^{p-2}[L(\mu',0)]$,
2. $[P(\mu,0)] = 2p^{2p-5}[L(0,0)] + p^{2p-5}\sum_{\mu'=1}^{p-2}[L(\mu',0)]$ *for* $\mu \neq 0$,
3. $[P(\mu,\lambda)] = p^{2p-5}\sum_{\mu'=0}^{p-1}[L(\mu',\lambda)]$ *for* $\mu \neq 0, p-1$ *and* $\lambda \neq 0$.

Proof. If $n = 2$, then $\tilde{\Lambda} = 0$ and Theorem 7.3.1 becomes

$$[Q(\mu,\lambda)] = p^{p-2} \sum_{\substack{\mu'\in\mathbb{F}_p \\ \mu'\neq p-1}} \sum_{\mu''\in\mathbb{F}_p} [P(\mu) : L(\mu')][L(\mu'',\lambda)].$$

The multiplicities of $L(\mu')$ in $P(\mu)$ are computed in Theorem 7.1.3. If $\mu \neq 0, p-1$, we have

$$[Q(\mu,\lambda)] = p^{2p-5} \sum_{\mu''=0}^{p-1} [L(\mu'',\lambda)].$$

If, furthermore, $\lambda \neq 0$, then all $L(\mu'', \lambda)$ are pairwise non–isomorphic so (3) follows from Lemma 7.2.1. If, on the other hand, $\lambda = 0$, we have an isomorphism $L(0,0) \simeq L(p-1,0)$ so

$$[Q(\mu,0)] = 2p^{2p-5}[L(0,0)] + p^{2p-5} \sum_{\mu''=1}^{p-2} [L(\mu'',0)],$$

which together with Lemma 7.2.1 yields (2). The proof of (1) is omitted as it uses exactly the same argument as in the proof of (2). □

Remark. The results of the corollary will be generalized later to include the cases $\mu = 0, p-1$ for $\lambda \neq 0$, see Corollary 7.4.3.

7.4 Independence property

7.4.1 The main goal of this section is Proposition 7.4.3 which proves that the formal sum of composition factors of $P(\mu, E)$ in the Grothendieck group does not depend on μ. Together with Theorem 7.3.1 and Lemma 7.2.1, this is needed in order to give a complete description of the projective indecomposable modules of $U_\chi(W_{(0)}^n)$.

We consider the restricted Lie subalgebra $C \subset B$ given by

$$C = K\partial_1 + Kx_1\partial_1.$$

Lemma. *The simple $U_\chi(B)$–modules are simple over $U_\chi(C)$. Furthermore, all these modules are isomorphic over $U_\chi(C)$.*

Proof. Every $L(\mu)$ has a basis consisting of elements $\partial_1^r \otimes 1$ with $0 \leq r < p$. We have

$$x_1\partial_1(\partial_1^r \otimes 1) = (\mu - r)(\partial_1^r \otimes 1).$$

Thus, the set of all $\partial_1^r \otimes 1$ is a basis consisting of eigenvectors of $x_1\partial_1$. It follows that $x_1\partial_1$ acts diagonally on $L(\mu)$ and on every $U_\chi(C)$–submodule M. If $M \neq 0$, then it must contain some eigenvector of $x_1\partial_1$. Thus, there exists an s such that $\partial_1^s \otimes 1 \in M$. By applying ∂_1 sufficiently many times, we deduce that $\partial_1^r \otimes 1 \in M$ for all r and thus $M = L(\mu)$. The second assertion follows from the isomorphism $L(\mu) \to L(\mu + i)$ that maps $\partial_1^r \otimes 1$ into $\partial_1^{[r+i]} \otimes 1$. This proves that $L(\mu) \simeq L(\mu')$ for all μ and μ'. □

We consider the elements

$$h = \sum_{i=2}^{n} x_i\partial_i \quad \text{and} \quad y = \sum_{i=2}^{n} x_1x_i\partial_i,$$

and the commutators

$$[\partial_1, y] = h \quad \text{and} \quad [x_1\partial_1, y] = y \quad \text{and} \quad [x_i\partial_j, y] = 0,$$

where $i, j \geq 2$. Since $y^{[p]} = 0$, this induces the following restricted Lie subalgebras $F_0 \subset F_1$,

$$F_0 = Kx_1\partial_1 \oplus A \oplus Ky \quad \text{and} \quad F_1 = C \oplus A \oplus Ky.$$

Now, every simple $U_\chi(Kx_1\partial_1 \times A)$–module $K_\mu \otimes E$ can be extended to a module over $U_\chi(F_0)$ by letting y act trivially. In fact, the set of all $K_\mu \otimes E$ forms a complete set of isomorphism classes of simple $U_\chi(F_0)$–modules. The module $U_\chi(F_1) \otimes_{U_\chi(F_0)} (K_\mu \otimes E)$ has a basis consisting of all $\partial_1^r \otimes (1 \otimes e)$ where the e are basis elements in E. We consider $U_\chi(F_1) \otimes_{U_\chi(F_0)} (K_\mu \otimes E)$ as the restriction of $L(\mu, E) = U_\chi(W^n_{(0)}) \otimes_{U_\chi(H)} (K_\mu \otimes E)$ to $U_\chi(F_1)$. It follows from the lemma that $L(\mu, E)$ is simple over $U_\chi(C \times A)$ and hence over $U_\chi(F_1)$. On the other hand, it is clear that any simple $U_\chi(F_1)$–module is isomorphic to some $L(\mu, E)$. We now proceed to determine when two modules $L(\mu, E)$ and $L(\mu', E')$ are isomorphic over $U_\chi(F_1)$. Clearly, such an isomorphism gives rise to an isomorphism $E \simeq E'$ of $U_\chi(A)$–modules. Now, being central in A, the element h acts as multiplication with a scalar a_E on every simple $U_\chi(A)$–module E.

Proposition. *The set of all $L(\mu, E)$ where $\mu \in \mathbb{F}_p$ and E is a simple module over $U_\chi(A)$ forms a complete set of isomorphism classes of simple $U_\chi(F_1)$–modules. If $L(\mu, E) \simeq L(\mu', E')$, then $E \simeq E'$. Furthermore,*

1. *If $a_E = 0$, then $L(\mu, E) \simeq L(\mu', E)$ for all μ and μ'.*
2. *If $a_E \neq 0$, then $L(\mu, E) \simeq L(\mu', E)$ if and only if $\mu = \mu'$.*

Proof. We have an isomorphism $L(\mu, E) \simeq L(\mu', E)$ of $U_\chi(C \times A)$–modules. This isomorphism extends to an isomorphism of $U_\chi(F_1)$–modules if $a_E = 0$. Indeed, the commutators $[\partial_1, y] = h$ and $[\partial_1, h] = 0$ imply for every $r > 0$

$$y\partial_1^r = \partial_1^r y - r\partial_1^{r-1}h.$$

Thus, for every $e \in E$, we have

$$y(\partial_1^r \otimes (1 \otimes e)) = \begin{cases} -r\partial_1^{r-1} \otimes (1 \otimes he), & \text{if } r > 0, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that if $a_E = 0$, then $he = 0$ and therefore $y(L(\mu) \otimes E) = 0$ proving the claim. If, on the other hand, $a_E \neq 0$, then $y(\partial_1^r \otimes (1 \otimes e)) \neq 0$ for $r > 0$.

The isomorphism $L(\mu) \to L(\mu + r)$ of $U_\chi(C)$–modules that sends $\partial_1^i \otimes 1$ to $\partial_1^{[i+r]} \otimes 1$ can be extended to an isomorphism of $U_\chi(F_1)$–modules by letting $\partial_1^i \otimes (1 \otimes e)$ be mapped to $\partial_1^{i+r} \otimes (1 \otimes e)$. Thus, it follows from Schur's lemma that given an isomorphism $\varphi\colon L(\mu, E) \to L(\mu + r, E)$ of $U_\chi(F_1)$–modules, we may assume that $\varphi(\partial_1^i \otimes (1 \otimes e)) = \partial_1^{i+r} \otimes (1 \otimes e)$. However, $y(1 \otimes (1 \otimes e)) = 0$ implies $\varphi(y(1 \otimes (1 \otimes e))) = 0$. Hence $y\varphi(1 \otimes (1 \otimes e)) = 0$ which can only be possible if $r = 0$. □

7.4.2 Let $P_{F_1}(\mu, E)$ be the projective cover of $L(\mu, E)$ as a $U_\chi(F_1)$–module and consider the module $\tilde{P}_{F_1}(\mu, E) = U_\chi(W^n_{(0)}) \otimes_{U_\chi(F_1)} P_{F_1}(\mu, E)$. Since $P_{F_1}(\mu, E)$ is projective over $U_\chi(F_1)$, it follows that $\tilde{P}_{F_1}(\mu, E)$ is projective over $U_\chi(W^n_{(0)})$. For $a_E \neq 0$, we have

$$\begin{aligned}\operatorname{Hom}_{U_\chi(W^n_{(0)})}(\tilde{P}_{F_1}(\mu, E), L(\mu', E')) &\simeq \operatorname{Hom}_{U_\chi(F_1)}(P_{F_1}(\mu, E), L(\mu', E')) \\ &\simeq \begin{cases} K, & \text{if } \mu = \mu' \text{ and } E \simeq E', \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

Thus, $\tilde{P}_{F_1}(\mu, E) \simeq P(\mu, E)$ is the projective cover of $L(\mu, E)$ as a $U_\chi(W^n_{(0)})$–module.

Proposition. *We have $P_{F_1}(\mu, E) \simeq U_\chi(F_1) \otimes_{U_\chi(F_0)} (K_\mu \otimes P_E)$ if $a_E \neq 0$.*

Proof. For notational convenience we set $Q = U_\chi(F_1) \otimes_{U_\chi(F_0)} (K_\mu \otimes P_E)$. Since A commutes with ∂_1, we see that Q decomposes as a $U_\chi(A)$–module into a direct sum of all $\partial_1^r \otimes (K_\mu \otimes P_E)$. It follows that Q is isomorphic to $(P_E)^p$ as a $U_\chi(A)$–module. Thus, the radical $\operatorname{rad}_A Q$ of Q as a $U_\chi(A)$–module is isomorphic to $(\operatorname{rad} P_E)^p$ and hence to $U_\chi(F_1) \otimes_{U_\chi(F_0)} (K_\mu \otimes \operatorname{rad} P_E)$. In particular, $\operatorname{rad}_A Q$ is a $U_\chi(F_1)$–submodule of Q. We have

$$\begin{aligned} Q/\operatorname{rad}_A Q &\simeq U_\chi(F_1) \otimes_{U_\chi(F_0)} (K_\mu \otimes (P_E/\operatorname{rad} P_E)) \\ &\simeq U_\chi(F_1) \otimes_{U_\chi(F_0)} (K_\mu \otimes E) \\ &\simeq L(\mu, E). \end{aligned}$$

Thus, $\operatorname{rad}_A Q$ is a maximal $U_\chi(F_1)$–submodule. Note that every simple $U_\chi(F_1)$–module V decomposes as a $U_\chi(A)$–module into a direct sum of simple $U_\chi(A)$–modules. Indeed, if $V = L(\mu') \otimes E'$, then $V \simeq (E')^{\dim L(\mu')}$ as $U_\chi(A)$–modules. It follows that simple modules over $U_\chi(F_1)$ are semisimple over $U_\chi(A)$. Furthermore, every homomorphism $\phi\colon Q \to V$ of $U_\chi(F_1)$–modules may be considered as a homomorphism of $U_\chi(A)$–modules and hence $\operatorname{rad}_A Q \subset \ker \phi$. Thus, the module $\operatorname{rad}_A Q$ is contained in the intersection $\bigcap_{\phi, V} \ker \phi$ of all $\ker \phi$. Since $\operatorname{rad}_A Q$ is a maximal $U_\chi(F_1)$–module, we have $\operatorname{rad} Q \subset \operatorname{rad}_A Q$. But then $\bigcap_{\phi,V} \ker \phi \subset \operatorname{rad} Q$ implies $\operatorname{rad}_A Q = \operatorname{rad} Q$. Consequently, $\operatorname{rad} Q$ is maximal and $U_\chi(F_1)$ has a unique maximal submodule. We have a commutative diagram of $U_\chi(F_1)$–modules

$$\begin{array}{ccc} & & P_{F_1}(\mu, E) \\ & \overset{\varphi}{\swarrow} & \downarrow \\ Q & \xrightarrow[\pi]{} & L(\mu, E) \end{array}$$

where π is surjective and $\pi \circ \varphi$ is the canonical homomorphism from $P_{F_1}(\mu, E)$ to $L(\mu, E)$. It follows that φ is surjective because otherwise the image of φ would be contained in some maximal submodule of Q. This means that $\operatorname{im} \varphi \subset \operatorname{rad} Q$ and thus $\pi \circ \varphi = 0$ which is a contradiction to the fact that $\pi \circ \varphi$ is surjective. Now, the fact that φ is surjective implies $\dim P_{F_1}(\mu, E) \geq \dim Q = p \dim P_E$. Hence by the discussion before the proposition, we have

$$\dim P(\mu, E) \geq p^{\dim W^n_{(0)} - \dim F_1 + 1} \dim P_E. \tag{7.4}$$

On the other hand, Theorem 7.1.3 implies that the projective $U_\chi(B \times A)$–module $P(\mu) \otimes P_E$ is of dimension $(1 + \delta_{0\mu})p^{p-2} \dim P_E$ for $\mu \neq p - 1$. The induced module $Q(\mu, E)$ is, therefore, of dimension

$$\begin{aligned} \dim Q(\mu, E) &= (1 + \delta_{\mu 0})p^{\dim W^n_{(0)} - \dim(B \times A) + p - 2} \dim P_E \\ &= (1 + \delta_{\mu 0})p^{\dim W^n_{(0)} - \dim F_1 + 1} \dim P_E. \end{aligned}$$

Note that the assumption $a_E \neq 0$ implies $E \not\simeq K$. By Lemma 7.2.1, we see that the inequality in (7.4) is actually an equality for $\mu \neq 0, p - 1$ because in this case $Q(\mu, E) \simeq P(\mu, E)$. Furthermore, we have an isomorphism $Q(0, E) \simeq P(0, E) \oplus P(p - 1, E)$. Hence we always have equality in (7.4). But then $\dim P_{F_1}(\mu, E) = p \dim P_E = \dim Q$ and hence $P_{F_1}(\mu, E) \simeq Q$. $\square$

Remark. Since $h \in W^n_{(0)}$ is toral, it follows that every $U_\chi(W^n_{(0)})$–module decomposes into a direct sum of weight spaces relative to the action of h. These weight spaces are $U_\chi(W^n_{(0)})$–modules because h is central in $W^n_{(0)}$. Thus, h acts via scalar multiplication on every indecomposable $U_\chi(W^n_{(0)})$–module. In particular, h acts on $P(\mu, E)$ via multiplication by a_E. This means that the composition factors of $P(\mu, E)$ are of the form $L(\mu', E')$ with $a_E = a'_E$. For $n = 2$, this amounts to saying that the composition factors of $P(\mu, \lambda)$ are of the form $L(\mu', \lambda)$, cf. Corollary 7.3.2.

7.4.3 With the commutators (6.2)–(6.4) in mind, we proceed by considering the following restricted Lie subalgebra

$$F_2 = F_0 \oplus \bigoplus_{2\le r<p} \bigoplus_{2\le i,j\le n} Kx_1^r x_i\partial_j \quad \text{if } p>2,$$
$$F_2 = Kx_1\partial_1 \oplus A \oplus \bigoplus_{2\le i,j\le n} Kx_1x_i\partial_j \quad \text{if } p=2 \text{ and } n>2.$$

The condition $n>2$ for $p=2$ is imposed in order to avoid $F_2 = F_0$. Now, we want to express F_2 as a direct sum of F_0 and a unipotent ideal $f \subset F_2$ such that $\chi(f)=0$,

$$F_2 = F_0 \oplus f.$$

This is easy for $p>2$. Set

$$f = \bigoplus_{2\le r<p} \bigoplus_{2\le i,j\le n} Kx_1^r x_i\partial_j \quad \text{if } p>2.$$

Suppose that $p=2$. The subspace generated by all $x_1x_i\partial_j$ with $i,j\ge 2$ is isomorphic to $\mathfrak{gl}_{n-1}$ via the map that sends $x_1x_i\partial_j$ in E_{ij}. If $2\nmid n-1$ (i.e. 2 does not divide $n-1$), then $\mathfrak{gl}_{n-1} \simeq \mathfrak{sl}_{n-1} \oplus KI_{n-1}$, where I_{n-1} is the identity matrix of size $n-1$. This suggests

$$f = \left\{ \sum_{2\le i,j\le n} a_{ij}x_1x_i\partial_j \mid \sum_{i=2}^{n} a_{ii} = 0 \right\} \text{ if } p=2,\ n>2 \text{ and } 2\nmid n-1.$$

By (6.2)–(6.4), one sees that f is an ideal in F_2. (For $p=2$, it is not obvious that $[x_i\partial_j, \sum_{k,l} a_{kl}x_1x_k\partial_l]$ belongs to f for $\sum_k a_{kk}=0$, but it can be verified by carefully considering all the cases arising from (6.4).) Now, since f is unipotent and $\chi(f)=0$, it follows that the simple modules of $U_\chi(F_2)$ are just the simple modules of $U_\chi(F_0)$ with f acting trivially.

Lemma. *The weights of $U_\chi(f)$ with respect to $Kx_1\partial_1 \times A^0$ belong to $\mathbb{F}_p \times \tilde{\Lambda}$. Furthermore, the dimension of the weight space $U_\chi(f)_{\mu,\lambda}$ does not depend on $\mu \in \mathbb{F}_p$ and $\lambda \in \tilde{\Lambda}$.*

Proof. We consider $U_\chi(f)$ as a $U_\chi(F_0)$–module via the adjoint representation and we wish to compute the character of $U_\chi(f)$ with respect to $Kx_1\partial_1 \times A^0$. By using similar arguments as those in the proof of Theorem 7.3.1, we see that

$$\operatorname{ch} U_\chi(f) = \prod_{r=2}^{p-1}\prod_{i=2}^{n}\prod_{j=2}^{n}\sum_{s=0}^{p-1} e(r,\varepsilon_i-\varepsilon_j)^s \quad \text{if } p>2.$$

To handle the case where $p = 2$, $n > 2$ and $2 \nmid n-1$, we shall first determine a basis for f. Observe that if $\sum_i a_{ii} = 0$, then

$$\begin{aligned}\sum_{i=2}^{n} a_{ii}x_1x_i\partial_i &= \sum_{i=2}^{n-1} a_{ii}(x_1x_i\partial_i - x_1x_n\partial_n) + \left(\sum_{i=2}^{n-1} a_{ii} + a_{nn}\right) x_1x_n\partial_n \\ &= \sum_{i=2}^{n-1} a_{ii}(x_1x_i\partial_i - x_1x_n\partial_n).\end{aligned}$$

Thus, we obtain a basis consisting of all $x_1x_i\partial_i - x_1x_n\partial_n$ with $2 \le i \le n-1$ together with all $x_1x_i\partial_j$ with $i \ne j$. This implies

$$\operatorname{ch} U_\chi(f) = \left(\prod_{i\ne j}\sum_{s=0}^{1} e(1, \varepsilon_i - \varepsilon_j)^s\right)\sum_{s=0}^{1} e(1,0)^s \quad \text{if } p = 2,\ n > 2,\ \nmid n-1.$$

In all cases, we see that the weights of $U_\chi(f)$ belong to $\mathbb{F}_p \times \tilde{\Lambda}$. Next, suppose that

$$\operatorname{ch} U_\chi(f) = \sum_{\mu\in\mathbb{F}_p}\sum_{\lambda\in\tilde{\Lambda}} c_{\mu,\lambda}e(\mu, \lambda). \tag{7.5}$$

If $p > 2$, then by using similar arguments as those in the proof of Theorem 7.3.1, we obtain by multiplying both sides of (7.5) by $1 - e(r, \varepsilon_i - \varepsilon_j)$

$$\sum_{\mu\in\mathbb{F}_p}\sum_{\lambda\in\tilde{\Lambda}}(c_{\mu,\lambda} - c_{\mu-r,\lambda-(\varepsilon_i-\varepsilon_j)})e(\mu, \lambda) = 0,$$

for all $r > 1$ and $i, j \ge 2$. In particular, we have $c_{\mu,\lambda} = c_{\mu+1,\lambda}$. (Choose $r = p-1$ and $i = j$.) Thus, the coefficients having the same second index are equal. We have $c_{\mu,\lambda} = c_{\mu-r,\lambda-(\varepsilon_i-\varepsilon_j)} = c_{\mu,\lambda-(\varepsilon_i-\varepsilon_j)}$ hence by using $c_{\mu,\lambda} = c_{\mu,\lambda-(\varepsilon_i-\varepsilon_j)}$ sufficiently many times we conclude that all the coefficients are equal. Similarly, if $p = 2$, $n > 2$ and $2 \nmid n-1$, then we obtain by multiplying both sides of (7.5) by $1 - e(r, \varepsilon_i - \varepsilon_j)$ and $1 - e(1, 0)$

$$\sum_{\mu\in\mathbb{F}_p}\sum_{\lambda\in\tilde{\Lambda}}(c_{\mu,\lambda} - c_{\mu-1,\lambda-(\varepsilon_i-\varepsilon_j)})e(\mu, \lambda) = 0,$$

$$\sum_{\mu\in\mathbb{F}_p}\sum_{\lambda\in\tilde{\Lambda}}(c_{\mu,\lambda} - c_{\mu-1,\lambda})e(\mu, \lambda) = 0.$$

Here, it should be noted that $i \ne j$. The last equation implies that the coefficients having the same second index are equal whereas the first equation implies that all the coefficients are equal. This proves the lemma. (The character of $U_\chi(f)$ can be computed by using dimension arguments.) □

Proposition. *Let $\mu \in \mathbb{F}_p$ and let E be a simple $U_\chi(A)$–module such that $a_E \neq 0$. Then, $[P(\mu, E)] = [P(0, E)]$ if (1) $p > 2$ or (2) $p = 2$ and $n > 2$ and $2 \nmid n - 1$.*

Proof. It follows from Proposition 7.4.2 and the discussion preceding it that

$$P(\mu, E) \simeq U_\chi(W^n_{(0)}) \otimes_{U_\chi(F_0)} (K_\mu \otimes P_E).$$

Since $[P_E] = \sum_{\nu \in \Lambda} [Z(\nu) : E][Z(\nu)]$, this implies

$$[P(\mu, E)] = \sum_{\nu \in \Lambda} [Z(\nu) : E][U_\chi(W^n_{(0)}) \otimes_{U_\chi(F_0)} (K_\mu \otimes Z(\nu))].$$

The claim to be proved amounts to saying that the right-hand side is independent of μ. Set

$$\tilde{W}(\mu, \nu) = U_\chi(W^n_{(0)}) \otimes_{U_\chi(F_0)} (K_\mu \otimes Z(\nu)).$$

If $W(\mu, \nu) = U_\chi(F_2) \otimes_{U_\chi(F_0)} (K_\mu \otimes Z(\nu))$, then

$$\tilde{W}(\mu, \nu) = U_\chi(W^n_{(0)}) \otimes_{U_\chi(F_2)} W(\mu, E).$$

The induction takes simple $U_\chi(F_0)$–modules to simple $U_\chi(F_2)$–modules. Thus, it suffices to prove that $[W(\mu, \nu)]$ is independent of μ as a $U_\chi(F_0)$–module. We have an isomorphism of $U_\chi(F_0)$–modules

$$W(\mu, \nu) \simeq U_\chi(f) \otimes (K_\mu \otimes Z(\nu)),$$

where $U_\chi(f)$ is a $U_\chi(F_0)$–module via the adjoint representation. The very same arguments used in the proof of Theorem 7.3.1 imply that

$$[W(\mu, \nu)] = \sum_{\mu' \in \mathbb{F}_p} \sum_{\lambda' \in \tilde{\Lambda}} \dim U_\chi(f)_{\mu', \lambda'} [K_{\mu + \mu'} \otimes Z(\nu + \lambda')],$$

which together with Lemma 7.4.3 proves the proposition. □

As an immediate consequence of the above proposition, we obtain a generalization of Corollary 7.3.2.

Corollary. *If $n = 2$ and $p > 2$, then*

1. $[P(0, 0)] = 4p^{2p-5}[L(0, 0)] + 2p^{2p-5} \sum_{\mu'=1}^{p-2} [L(\mu', 0)]$,
2. $[P(\mu, 0)] = 2p^{2p-5}[L(0, 0)] + p^{2p-5} \sum_{\mu'=1}^{p-2} [L(\mu', 0)]$ *for* $\mu \neq 0$,
3. $[P(\mu, \lambda)] = p^{2p-5} \sum_{\mu'=0}^{p-1} [L(\mu', \lambda)]$ *for* $\lambda \neq 0$.

Remark. We have $\dim P(0, 0) = 2p^{2p-3}$ and $\dim P(\mu, \lambda) = p^{2p-3}$ if $(\mu, \lambda) \neq (0, 0)$ and $(\mu, \lambda) \neq (p - 1, 0)$ which is consistent with Proposition 7.2.2.

8 The Witt–Jacobson Lie algebra of rank 2

8.1 Notation

8.1.1 From now on, we will focus on the Witt–Jacobson Lie algebra W^2 of rank 2. If $0 \leq i, j \leq p-1$ and $k = 1, 2$, we set

$$e_{ijk} = x_1^i x_2^j \partial_k,$$

and we let $e_{ijk} = 0$ otherwise. We use (1.2) to obtain

$$[e_{rs1}, e_{ij1}] = (i-r)e_{i+r-1,j+s,1}, \tag{8.1}$$
$$[e_{rs1}, e_{ij2}] = -se_{i+r,j+s-1,1} + ie_{i+r-1,j+s,2}, \tag{8.2}$$
$$[e_{rs2}, e_{ij1}] = je_{i+r,j+s-1,1} - re_{i+r-1,j+s,2}, \tag{8.3}$$
$$[e_{rs2}, e_{ij2}] = (j-s)e_{i+r,j+s-1,2}. \tag{8.4}$$

The p–mapping is given by (1.3)

$$e_{ijk}^{[p]} = \begin{cases} e_{012}, & \text{if } (i,j,k) = (0,1,2), \\ e_{101}, & \text{if } (i,j,k) = (1,0,1), \\ 0, & \text{otherwise.} \end{cases}$$

In addition to the grading from Section 1.2.4, we have a new grading

$$W^2 = \bigoplus_{i=-1}^{p-1} W^2_{(i)},$$

where

$$W^2_{(-1)} = \bigoplus_{r=0}^{p-1} Ke_{r02} \quad \text{and} \quad W^2_{(0)} = \bigoplus_{r=0}^{p-1} Ke_{r01} \oplus \bigoplus_{r=0}^{p-1} Ke_{r12}.$$

As before, we set

$$N^+ = W^2_{(>0)} \quad \text{and} \quad N^- = W^2_{(<0)},$$
$$B^+ = W^2_{(\geq 0)} \quad \text{and} \quad B^- = W^2_{(\leq 0)}.$$

8.2 Induction

8.2.1 We keep the notation introduced in Section 7.3.2: The set of all $L(\lambda,\mu)$ with $(\lambda,\mu) \neq (p-1,0)$ forms a complete set of pairwise non–isomorphic simple modules of $U_\chi(W^2_{(0)})$. The composition factors of the corresponding projective covers $P(\lambda,\mu)$ are determined in Corollary 7.4.3. For every $(\lambda,\mu) \in \mathbb{F}_p^2$, we consider the induced module

$$V_{irr}(\lambda,\mu) = U_\chi(W^2) \otimes_{U_\chi(B^+)} L(\lambda,\mu).$$

As mentioned in Section 6.3.1, $V_{irr}(\lambda,\mu)$ has a unique maximal submodule and the set of heads $\mathcal{L}(\lambda,\mu)$ of $V_{irr}(\lambda,\mu)$ forms a complete set of pairwise non–isomorphic simple modules of $U_\chi(W^2)$. The main goal of this section is Theorem 8.2.5 which determines the composition factors of $V_{irr}(\lambda,\mu)$.

8.2.2 For every $(\lambda,\mu) \in \mathbb{F}_p^2$, we let $y(\lambda,\mu)$ denote the restricted simple W_0^2–module having maximal vector of weight (λ,μ) with respect to e_{011}. In Appendix B, we show that for every $(\lambda,\mu) \in \mathbb{F}_p^2$ with $\mu - \lambda \neq p-1$ there exists a p–dimensional W_0^2–module $x(\lambda,\mu)$ such that the sequence

$$0 \to y(\mu+1,\lambda-1) \to x(\lambda,\mu) \to y(\lambda,\mu) \to 0 \tag{8.5}$$

is short exact. Set

$$Y(\lambda,\mu) = U_\chi(W^2) \otimes_{U_\chi(W_{\geq 0})} y(\lambda,\mu),$$
$$X(\lambda,\mu) = U_\chi(W^2) \otimes_{U_\chi(W_{\geq 0})} x(\lambda,\mu).$$

Theorem. *If $p>2$, then $Y(\lambda,\mu) \simeq \mathcal{L}(\lambda,\mu)$ if and only if $(\lambda,\mu) \neq (p-1,0)$. Furthermore, we have $[Y(p-1,0)] = [\mathcal{L}(0,0)] + [\mathcal{L}(p-1,p-1)]$.*

Proof. The first assertion is due to Holmes, see [10, Thm. 4.3]. See also Remark 8.2.2 below. The second assertion is a consequence of [10, Prop. 3.8] and [10, Prop. 3.10] which yield a short exact sequence

$$0 \to Y(p-1,p-1) \to Y(p-1,0) \to Y(0,0) \to 0.$$

Here the two outer modules are simple which proves the theorem. □

Remark. (1) Holmes' paper assumes that the characteristic is larger than 3. However, the results referred to in the proof of the theorem can be generalized to include the case $p = 3$, see Appendix D. (2) The maximal vectors in Holmes' paper are taken with respect to e_{012}. In order to transfer the results of [10] into our setting, one should note that, in the notation of [10], we have $Y(\lambda, \mu) \simeq Z^{\chi}(\mu, \lambda)$.

Lemma. *If $\mu - \lambda = p - 1$, then*

$$[X(\lambda, \mu)] = [Y(\lambda, \mu)].$$

If $\mu - \lambda \neq p - 1$, then

$$[X(\lambda, \mu)] = [Y(\lambda, \mu)] + [Y(\mu + 1, \lambda - 1)].$$

Proof. The first assertion follows from the construction of $X(\lambda, \mu)$, see Appendix B. The second assertion follows from sequence (8.5) and the fact that the induction functor is exact. □

8.2.3 We consider the restricted Lie subalgebra $W^2_+ \subset W^2$ given by

$$W^2_+ = W^2_{>0} \oplus Ke_{101} \oplus Ke_{011} \oplus Ke_{012}.$$

The simple modules of $U_\chi(Ke_{101} \oplus Ke_{012})$ are 1–dimensional and given by $K_{\lambda,\mu}$. Since $W^2_{>0}$ and Ke_{011} are unipotent and satisfy $\chi(W^2_{>0}) = \chi(Ke_{011}) = 0$, it follows from Lemma 6.1.3 that they annihilate every simple module over W^2_+. Thus, the set of all $K_{\lambda,\mu}$ forms a complete set of isomorphism classes of simple $U_\chi(W^+)$–modules.

8.2.4 We have an isomorphism

$$V_{irr}(\lambda, \mu) \simeq U_\chi(W^2) \otimes_{U_\chi(H \oplus W^2_{(>0)})} K_{\lambda,\mu}.$$

(This follows from Appendix C. Indeed, set $L = W^2$, $V = W^2_{(0)}$, $U = H$ and $I = W^2_{(>0)}$ and use the proposition in Appendix C.) The transitivity property implies

$$V_{irr}(\lambda, \mu) \simeq U_\chi(W^2) \otimes_{U_\chi(W^2_+)} (U_\chi(W^2_+) \otimes_{U_\chi(H \oplus W^2_{(>0)})} K_{\lambda,\mu}).$$

If we let $M(\lambda, \mu)$ denote the term inside the parenthesis, then $V_{irr}(\lambda, \mu) \simeq \mathrm{ind}^{U_\chi(W^2)}_{U_\chi(W^2_+)} M(\lambda, \mu)$. We consider a composition series of $M(\lambda, \mu)$

$$0 = M_0 \subset M_1 \subset \cdots \subset M_N = M(\lambda, \mu).$$

In view of Section 8.2.3, there exists for every i an isomorphism $M_i/M_{i-1} \simeq K_{\lambda_i,\mu_i}$ where $\lambda_i, \mu_i \in \mathbb{F}_p$. By inducing, we get a filtration of $V_{irr}(\lambda,\mu)$ with factors isomorphic to the p^3–dimensional modules

$$\begin{aligned}\tilde{X}(\lambda_i,\mu_i) &= \operatorname{ind}_{U_\chi(W^2_+)}^{U_\chi(W^2)} M_i / \operatorname{ind}_{U_\chi(W^2_+)}^{U_\chi(W^2)} M_{i-1} \\ &\simeq \operatorname{ind}_{U_\chi(W^2_+)}^{U_\chi(W^2)} K_{\lambda_i,\mu_i},\end{aligned}$$

where the isomorphism follows from the fact that the induction functor is exact.

Lemma (A)**.** *We have $\tilde{X}(\lambda_i,\mu_i) \simeq X(\lambda_i,\mu_i)$ for every λ_i and μ_i.*

Proof. This follows from the proposition in Appendix C. In the notation of the proposition, we have $L = W^2$, $V = W^2_0$, $U = W^2_+ \cap W^2_0$ and $I = W^2_{>0}$. □

The module $M(\lambda,\mu)$ has a weight space decomposition with respect to $Ke_{101} \oplus Ke_{012}$. (The weights of $M(\lambda,\mu)$ belong to $\mathbb{F}_p^2$, see e.g. Lemma 8.2.5.)

Lemma (B)**.** *If the formal character of $M(\lambda,\mu)$ is given by $\operatorname{ch} M(\lambda,\mu) = \sum_{\delta,\gamma\in\mathbb{F}_p} m_{\delta\gamma} e(\delta,\gamma)$, then $[V_{irr}(\lambda,\mu)] = \sum_{\delta,\gamma\in\mathbb{F}_p} m_{\delta\gamma}[X(\delta,\gamma)]$.*

Proof. On one side, we have $\operatorname{ch} M(\lambda,\mu) = \sum_{\delta,\gamma\in\mathbb{F}_p} m_{\delta\gamma} e(\delta,\gamma)$. On the other side, we have

$$\operatorname{ch} M(\lambda,\mu) = \sum_{i=1}^{N} \operatorname{ch} M_i/M_{i-1} = \sum_{i=1}^{N} e(\lambda_i,\mu_i).$$

It follows that $m_{\delta\gamma}$ is the number of times that (δ,γ) is equal to a (λ_i,μ_i). Therefore, we have

$$[V_{irr}(\lambda,\mu)] = \sum_{i=1}^{N}[X(\lambda_i,\mu_i)] = \sum_{\delta,\gamma\in\mathbb{F}_p} m_{\delta\gamma}[X(\delta,\gamma)],$$

which completes the proof. □

8.2.5 With Lemma 8.2.4 B in mind, we proceed to determine the character of $M(\lambda,\mu)$. If $p > 2$, then $M(\lambda,\mu)$ has a basis consisting of all

$$e_{202}^{r_2} e_{302}^{r_3} \cdots e_{p-1,0,2}^{r_{p-1}} \otimes 1, \quad 0 \le r_i \le p-1.$$

Thus

$$\operatorname{ch} M(\lambda,\mu) = \sum_{\substack{0\le r_i<p \\ \text{for all } i}} \operatorname{ch} K e_{202}^{r_2} e_{302}^{r_3} \cdots e_{p-1,0,2}^{r_{p-1}} \otimes 1,$$

where each summand is determined by the following lemma

Lemma. *For every tuple* $(r_2, r_3, \ldots, r_{p-1})$ *with* $0 \leq r_i < p$*, we have*

$$\operatorname{ch} K e_{202}^{r_2} e_{302}^{r_3} \cdots e_{p-1,0,2}^{r_{p-1}} \otimes 1 = e(\lambda + \sum_{\nu=2}^{p-1} r_\nu \nu, \mu + \sum_{\nu=2}^{p-1} r_\nu (p-1)).$$

Proof. We set $v^r = e_{202}^{r_2} e_{302}^{r_3} \cdots e_{p-1,0,2}^{r_{p-1}} \otimes 1$ for each $r = (r_2, r_3, \ldots, r_{p-1})$ with $0 \leq r_i < p$. Now, if M is a $U_\chi(W^2)$–module and $v \in M$ has weight (ξ, ζ), then $e_{\nu 02} v$ has weight $(\xi + \nu, \zeta + (p-1))$. In our case, this implies

$$e_{101} v^r = (\lambda + \sum_{\nu=2}^{p-1} r_\nu \nu) v^r \quad \text{and} \quad e_{012} v^r = (\mu + \sum_{\nu=2}^{p-1} r_\nu (p-1)) v^r.$$

Hence the claim. □

Theorem. *Let* $(\lambda, \mu) \in \mathbb{F}_p^2$.

1. *If* $p > 3$*, then*

$$[V_{irr}(\lambda, \mu)] = 4p^{p-4}[\mathcal{L}(0,0)] + 4p^{p-4}[\mathcal{L}(p-1, p-1)] + 2p^{p-4} \sum_{(\sigma,\tau) \in \Omega} [\mathcal{L}(\sigma, \tau)] + p^{p-4} \sum_{\tau - \sigma = p-1} [\mathcal{L}(\sigma, \tau)],$$

where Ω *is the set of all* $(\sigma, \tau) \in \mathbb{F}_p^2$ *with* $\tau - \sigma \neq p - 1$ *and* $(\sigma, \tau) \notin \{(0,0), (p-1, p-1), (p-1, 0)\}$.

2. *If* $p = 3$ *and* $\mu - \lambda = p - 1$*, then*

$$[V_{irr}(\lambda, \mu)] = [\mathcal{L}(1,0)] + [\mathcal{L}(2,1)] + [\mathcal{L}(0,2)].$$

3. *If* $p = 3$ *and* $\mu - \lambda \neq p - 1$*, then*

$$[V_{irr}(\lambda, \mu)] = 2[\mathcal{L}(0,0)] + 2[\mathcal{L}(2,2)] + [\mathcal{L}(1,1)] + [\mathcal{L}(0,1)] + [\mathcal{L}(1,2)].$$

Proof. It follows from Lemma 8.2.5 that

$$\begin{aligned} \operatorname{ch} M(\lambda, \mu) &= \sum_{\substack{0 \leq r_i < p \\ \text{for all } i}} e(\lambda + \sum_{\nu=2}^{p-1} r_\nu \nu, \mu + \sum_{\nu=2}^{p-1} r_\nu (p-1)) \\ &= e(\lambda, \mu) \prod_{\nu=2}^{p-1} \sum_{r_\nu = 0}^{p-1} e(\nu, p-1)^{r_\nu}, \end{aligned}$$

where in the second equality we used the multiplication rule in (6.1). Since $e(\eta)^p = 1$ for all $\eta \in \mathbb{F}_p^2$ the formula $0 = 1-e(\eta)^p = (1-e(\eta))(1+e(\eta)+\cdots+e(\eta)^{p-1})$ implies $(1-e(\nu,p-1))\operatorname{ch} M(\lambda,\mu) = 0$ for all $\nu = 2,3,\ldots,p-1$. Now, suppose that

$$\operatorname{ch} M(\lambda,\mu) = \sum_{\sigma,\tau\in\mathbb{F}_p} c_{\sigma,\tau} e(\sigma,\tau).$$

By multiplying both sides by $1-e(\nu,p-1)$, we obtain

$$\sum_{\sigma,\tau\in\mathbb{F}_p} (c_{\sigma,\tau} - c_{\sigma-\nu,\tau+1}) e(\sigma,\tau) = 0,$$

which means $c_{\sigma,\tau} = c_{\sigma-\nu,\tau+1}$ for all σ,τ and $\nu = 2,3,\ldots,p-1$. In particular, we have $c_{\sigma,\tau} = c_{\sigma-2,\tau+1}$ for $\nu = 2$ and $c_{\sigma,\tau} = c_{\sigma+1,\tau+1}$ for $\nu = p-1$. Furthermore, if the characteristic is greater than 3 then $c_{\sigma-2,\tau+1} = c_{\sigma+1,\tau+2}$. (Set $\nu = p-3$ and replace σ and τ by $\sigma-2$ and $\tau+1$, respectively.) It follows that $c_{\sigma+1,\tau+1} = c_{\sigma+1,\tau+2}$ for all σ and τ. Thus, the coefficients having the same first index σ are equal. By using $c_{\sigma,\tau} = c_{\sigma+1,\tau+1}$ sufficiently many times, we conclude that all the coefficients are equal. Now, a dimension argument yields that

$$\operatorname{ch} M(\lambda,\mu) = p^{p-4} \sum_{\sigma,\tau\in\mathbb{F}_p} e(\sigma,\tau),$$

which together with Lemma 8.2.4 B yields

$$[V_{irr}(\lambda,\mu)] = p^{p-4} \sum_{\sigma,\tau\in\mathbb{F}_p} [X(\sigma,\tau)].$$

By Lemma 8.2.2 and Theorem 8.2.2, we have

$$\begin{aligned}[V_{irr}(\lambda,\mu)] &= p^{p-4}\sum_{\tau-\sigma\neq p-1} [X(\sigma,\tau)] + p^{p-4}\sum_{\tau-\sigma=p-1} [X(\sigma,\tau)] \\ &= p^{p-4}\sum_{\tau-\sigma\neq p-1} ([Y(\sigma,\tau)] + [Y(\tau+1,\sigma-1)]) \\ &\quad + p^{p-4}\sum_{\tau-\sigma=p-1} [\mathcal{L}(\sigma,\tau)].\end{aligned}$$

Since $\tau-\sigma \neq p-1$ if and only if $(\sigma-1)-(\tau+1) \neq p-1$, we have

$$[V_{irr}(\lambda,\mu)] = 2p^{p-4}\sum_{\tau-\sigma\neq p-1} [Y(\sigma,\tau)] + p^{p-4}\sum_{\tau-\sigma=p-1} [\mathcal{L}(\sigma,\tau)].$$

The assertion follows from Theorem 8.2.2.

If the characteristic is 3 then $M(\lambda,\mu)$ is 3–dimensional. The weight space $M(\lambda,\mu)_{(\lambda,\mu)}$ is nonzero, in fact we have $M(\lambda,\mu)_{(\lambda,\mu)} = K(1\otimes 1)$. Therefore, $c_{\lambda,\mu} = 1$ and it follows from the formula $c_{\sigma,\tau} = c_{\sigma+1,\tau+1}$ that $c_{\lambda+1,\mu+1}$ and $c_{\lambda+2,\mu+2}$ are both equal to 1. We conclude that

$$\operatorname{ch} M(\lambda,\mu) = \sum_{\sigma=0}^{2} e(\lambda+\sigma,\mu+\sigma),$$

and thus

$$[V_{irr}(\lambda,\mu)] = \sum_{\sigma=0}^{2}[X(\lambda+\sigma,\mu+\sigma)].$$

If $\mu-\lambda = p-1$ then $(\lambda,\mu)\in\{(1,0),(2,1),(0,2)\}$ and it follows from Lemma 8.2.2 and Theorem 8.2.2 that

$$\begin{aligned}[V_{irr}(\lambda,\mu)] &= \sum_{\sigma=0}^{2}[Y(\lambda+\sigma,\mu+\sigma)]\\ &= [\mathcal{L}(1,0)]+[\mathcal{L}(2,1)]+[\mathcal{L}(0,2)].\end{aligned}$$

If $\mu-\lambda\neq p-1$ then $(\lambda,\mu)\in\{(0,0),(1,1),(2,2),(2,0),(0,1),(1,2)\}$ and we have

$$\begin{aligned}[V_{irr}(\lambda,\mu)] = &\sum_{\sigma=0}^{2}([Y(\lambda+\sigma,\mu+\sigma)]+[Y(\mu+\sigma+1,\lambda+\sigma-1)])\\ &= [Y(0,0)]+[Y(1,1)]+[Y(2,2)]+[Y(2,0)]\\ &\quad +[Y(0,1)]+[Y(1,2)].\end{aligned}$$

Thus, by Theorem 8.2.2, we get

$$[V_{irr}(\lambda,\mu)] = 2[\mathcal{L}(0,0)]+2[\mathcal{L}(2,2)]+[\mathcal{L}(1,1)]+[\mathcal{L}(0,1)]+[\mathcal{L}(1,2)].$$

This completes the proof of the theorem. □

8.2.6 In this section, we describe the head of $V_{irr}(\lambda,\mu)$. Note that $X(\lambda,\mu)$ has a basis consisting of elements of the form $e_{001}^{s_1}e_{002}^{s_2}e_{102}^{s_3}\otimes 1$.

Lemma. *The map $\phi\colon L(\lambda,\mu)\to X(\lambda,\mu)$ that sends $e_{001}^{i}\otimes 1$ to $e_{001}^{i}\otimes 1$ is a homomorphism of $U_\chi(B^+)$–modules.*

Proof. The element $e_{rs1}e_{001}^i$ acts on a $U_\chi(B^+)$–module as

$$\sum_{k=0}^{\min(i,r)} (-1)^k \binom{r}{k} \frac{i!}{(i-k)!} e_{001}^{i-k} e_{r-k,s,1}.$$

This can easily be seen by using induction on i. Thus, in order to prove the compatibility of ϕ with the action of e_{rs1} it suffices to consider how e_{rs1} acts on $1\otimes 1$ in $L(\lambda,\mu)$ and $X(\lambda,\mu)$, respectively. But e_{rs1} acts by the same scalar on these elements; λ if $(r,s)=(1,0)$ and 0 otherwise.

Similarly, the element $e_{rs2}e_{001}^i$ acts on a $U_\chi(B^+)$–module as

$$\sum_{k=0}^{\min(i,r)} (-1)^k \binom{r}{k} \frac{i!}{(i-k)!} e_{001}^{i-k} e_{r-k,s,2}.$$

The claim follows since e_{rs2} annihilates $1\otimes 1$ for $s\neq 0$. □

The isomorphism

$$\mathrm{Hom}_{U_\chi(B^+)}(L(\lambda,\mu), X(\lambda,\mu)) \simeq \mathrm{Hom}_{U_\chi(W^2)}(V_{irr}(\lambda,\mu), X(\lambda,\mu))$$

induces a homomorphism $\tilde{\phi}$ that maps every $e_{002}^{r_0}e_{102}^{r_1}\cdots e_{p-1,0,2}^{r_{p-1}}e_{001}^{s}\otimes 1$ in $V_{irr}(\lambda,\mu)$ to $e_{002}^{r_0}e_{102}^{r_1}\cdots e_{p-1,0,2}^{r_{p-1}}e_{001}^{s}\otimes 1$ in $X(\lambda,\mu)$. Since by the PBW theorem, every $e_{002}^{a_0}e_{102}^{a_1}e_{001}^{b}e_{202}^{a_2}\cdots e_{p-1,0,2}^{a_{p-1}}$ can be written as a linear combination of monomials of the form $e_{002}^{r_0}e_{102}^{r_1}\cdots e_{p-1,0,2}^{r_{p-1}}e_{001}^{s}$ we have

$$\tilde{\phi}(e_{002}^{a_0}e_{102}^{a_1}e_{001}^{b}e_{202}^{a_2}\cdots e_{p-1,0,2}^{a_{p-1}}\otimes 1) = e_{002}^{a_0}e_{102}^{a_1}e_{001}^{b}e_{202}^{a_2}\cdots e_{p-1,0,2}^{a_{p-1}}\otimes 1.$$

This is clearly a surjection. Furthermore, since all e_{r02} commute, the kernel of $\tilde{\phi}$ consists exactly of those elements with $a_i>0$ for some $2\le i\le p-1$; in particular we have $\dim(\ker\tilde{\phi}) = p^3(p^{p-2}-1) = p^{p+1}-p^3$.

Remark. The existence of the homomorphism $V_{irr}(\lambda,\mu)\to X(\lambda,\mu)$ means that the head of $V_{irr}(\lambda,\mu)$ is the head of $X(\lambda,\mu)$.

8.3 Projective indecomposable modules of W^2

8.3.1 This section is devoted to determining the projective indecomposable modules and to computing their composition factors for the algebras

$U_\chi(W^2)$. For every $\lambda, \mu \in \mathbb{F}_p$ with $(\lambda,\mu) \neq (p-1,0)$, we let $\mathcal{P}(\lambda,\mu)$ denote the projective cover of $\mathcal{L}(\lambda,\mu)$. Moreover, we set

$$V^+_{proj}(\lambda,\mu) = U_\chi(W^2) \otimes_{U_\chi(B^+)} P(\lambda,\mu),$$
$$V^-_{proj}(\lambda,\mu) = U_\chi(W^2) \otimes_{U_\chi(B^-)} P(\lambda,\mu).$$

The dimension of $V^\pm_{proj}(\lambda,\mu)$ can be computed by use of Remark 7.4.3. We have $\dim V^-_{proj}(0,0) = 2p^{2p^2-p-3}$ and $\dim V^-_{proj}(\lambda,\mu) = p^{2p^2-p-3}$ otherwise. Likewise, $\dim V^+_{proj}(0,0) = 2p^{3p-3}$ and $\dim V^+_{proj}(\lambda,\mu) = p^{3p-3}$ for $(\lambda,\mu) \neq (0,0)$. If $\mathfrak{s} \subset W^2$ is a restricted Lie subalgebra of W^2 and V is a module over $\mathfrak{s}$, we set

$$D(U_\chi(W^2) \otimes_{U_\chi(\mathfrak{s})} V) = (U_{-\chi}(W^2) \otimes_{U_{-\chi}(\mathfrak{s})} V^*)^*.$$

Nakano shows in [17, Thm. 1.3.5] that the projective modules $\mathcal{P}(\lambda,\mu)$ admit $V^\pm_{proj}(\lambda,\mu)$ filtrations. The following theorem is due to Nakano, see [17, Thm. 1.3.6].

Theorem (Reciprocity Theorem)**.** *Let $\lambda,\mu,\lambda',\mu' \in \mathbb{F}_p$ such that $(\lambda,\mu) \neq (p-1,0)$ and $(\lambda',\mu') \neq (p-1,0)$. Then*

$$[\mathcal{P}(\lambda,\mu) : V^-_{proj}(\lambda',\mu')] = [DV_{irr}(\lambda',\mu') : \mathcal{L}(\lambda,\mu)].$$

The following lemma allows us to compute $DV_{irr}(\lambda,\mu)$.

Lemma. *We have for every $\lambda,\mu \in \mathbb{F}_p$ with $(\lambda,\mu) \neq (p-1,0)$*

$$DV_{irr}(\lambda,\mu) = V_{irr}(\lambda,\mu).$$

Proof. The dual of an induced module is an induced module. In fact, we have

$$DV_{irr}(\lambda,\mu) \simeq U_\chi(W^2) \otimes_{U_\chi(B^+)} (L(\lambda,\mu) \otimes K_{\delta_{W^2}-\delta_{B^+}});$$

here $(\delta_{W^2} - \delta_{B^+})(x) = \operatorname{tr}(\operatorname{ad}_{W^2} x - \operatorname{ad}_{B^+} x)$ for all $x \in B^+$ and $K_{\delta_{W^2}-\delta_{B^+}}$ is the vector space K endowed with the twisted action $\delta_{W^2} - \delta_{B^+}$, see [14, Sec. 1.5]. Now, if $x^{[p]} = 0$ then $(\operatorname{ad}_{W^2} x)^p = 0$ and hence $\operatorname{tr}(\operatorname{ad}_{W^2} x) = 0$. It follows that $\operatorname{tr}(\operatorname{ad}_{W^2} x) = 0$ for all $x \neq e_{101}$ and $x \neq e_{012}$. (The same applies for B^+ instead of W^2.) Now, the commutators

$$[e_{101}, e_{ij1}] = (i-1)e_{ij1},$$
$$[e_{101}, e_{ij2}] = ie_{ij2},$$
$$[e_{012}, e_{ij1}] = je_{ij1},$$
$$[e_{012}, e_{ij2}] = (j-1)e_{ij2}$$

show that $\mathrm{tr}(\mathrm{ad}_{W^2} x) = 0$ for $x = e_{101}, e_{012}$. Hence, $\mathrm{ad}_{W^2} x$ is traceless for all $x \in W^2$. Similarly, the commutators above show that $\mathrm{tr}(\mathrm{ad}_{B^+} x) = 0$ for $x = e_{101}, e_{012}$. Thus, $\mathrm{ad}_{B^+} x$ is traceless for all $x \in B^+$ and

$$\begin{aligned} DV_{irr}(\lambda, \mu) &\simeq U_\chi(W^2) \otimes_{U_\chi(B^+)} (L(\lambda, \mu) \otimes K_0) \\ &\simeq U_\chi(W^2) \otimes_{U_\chi(B^+)} L(\lambda, \mu). \end{aligned}$$

This completes the proof. □

Proposition. *If $p > 2$ and $(\lambda, \mu) \in \mathbb{F}_p^2$ with $(\lambda, \mu) \neq (p-1, 0)$, then*

1. $\dim \mathcal{P}(\lambda, \mu) = 4p^{2p^2-5}$ *for* $(\lambda, \mu) \in \{(0,0), (p-1, p-1)\}$.
2. $\dim \mathcal{P}(\lambda, \mu) = p^{2p^2-5}$ *for* $\mu - \lambda = p - 1$.
3. $\dim \mathcal{P}(\lambda, \mu) = 2p^{2p^2-5}$ *otherwise.*

Proof. We have

$$\begin{aligned} [\mathcal{P}(\lambda, \mu)] &= \sum_{(\lambda', \mu') \neq (p-1, 0)} [\mathcal{P}(\lambda, \mu) : V^-_{proj}(\lambda', \mu')][V^-_{proj}(\lambda', \mu')] \\ &= \sum_{(\lambda', \mu') \neq (p-1, 0)} [DV_{irr}(\lambda', \mu') : \mathcal{L}(\lambda, \mu)][V^-_{proj}(\lambda', \mu')] \\ &= \sum_{(\lambda', \mu') \neq (p-1, 0)} [V_{irr}(\lambda', \mu') : \mathcal{L}(\lambda, \mu)][V^-_{proj}(\lambda', \mu')]. \end{aligned}$$

Therefore, we get

$$\dim \mathcal{P}(\lambda, \mu) = \sum_{(\lambda', \mu') \neq (p-1, 0)} [V_{irr}(\lambda', \mu') : \mathcal{L}(\lambda, \mu)] \dim V^-_{proj}(\lambda', \mu').$$

The claim follows from Theorem 8.2.5. □

8.3.2 In this section, we will compute the Cartan invariants of $U_\chi(W^2)$. As in Theorem 8.2.5, we let Ω denote the set of all (σ, τ) in $\mathbb{F}_p^2$ with $\tau - \sigma \neq p - 1$ and $(\sigma, \tau) \notin \{(0,0), (p-1, p-1), (p-1, 0)\}$.

Theorem. *Let $p > 2$ and let $(\lambda, \mu) \in \mathbb{F}_p^2$ such that $(\lambda, \mu) \neq (p-1, 0)$. Then*

1. *If $(\lambda, \mu) \in \{(0,0), (p-1, p-1)\}$, then*

$$\begin{aligned} [\mathcal{P}(\lambda, \mu)] &= 16p^{2p^2-10}[\mathcal{L}(0,0)] + 16p^{2p^2-10}[\mathcal{L}(p-1, p-1)] \\ &\quad + 8p^{2p^2-10} \sum_{(\sigma, \tau) \in \Omega} [\mathcal{L}(\sigma, \tau)] + 4p^{2p^2-10} \sum_{\tau - \sigma = p-1} [\mathcal{L}(\sigma, \tau)]. \end{aligned}$$

2. If $\mu - \lambda = p - 1$, then

$$[\mathcal{P}(\lambda,\mu)] = 4p^{2p^2-10}[\mathcal{L}(0,0)] + 4p^{2p^2-10}[\mathcal{L}(p-1,p-1)] + 2p^{2p^2-10}\sum_{(\sigma,\tau)\in\Omega}[\mathcal{L}(\sigma,\tau)] + p^{2p^2-10}\sum_{\tau-\sigma=p-1}[\mathcal{L}(\sigma,\tau)].$$

3. Otherwise, we have

$$[\mathcal{P}(\lambda,\mu)] = 8p^{2p^2-10}[\mathcal{L}(0,0)] + 8p^{2p^2-10}[\mathcal{L}(p-1,p-1)] + 4p^{2p^2-10}\sum_{(\sigma,\tau)\in\Omega}[\mathcal{L}(\sigma,\tau)] + 2p^{2p^2-10}\sum_{\tau-\sigma=p-1}[\mathcal{L}(\sigma,\tau)].$$

Proof. Suppose that $p > 3$. Then, by Theorem 8.2.5, the composition factors of $V_{irr}(\lambda,\mu)$ do not depend on λ and μ. Thus $[V_{irr}(\lambda,\mu)] = [V_{irr}(0,0)]$ for all $\lambda,\mu \in \mathbb{F}_p$ with $(\lambda,\mu) \neq (p-1,0)$. We have

$$\begin{aligned}[\mathcal{P}(\lambda,\mu)] &= \sum_{(\lambda',\mu')\neq(p-1,0)}[\mathcal{P}(\lambda,\mu) : V_{irr}(\lambda',\mu')][V_{irr}(\lambda',\mu')] \\ &= \sum_{(\lambda',\mu')\neq(p-1,0)}[\mathcal{P}(\lambda,\mu) : V_{irr}(\lambda',\mu')][V_{irr}(0,0)],\end{aligned}$$

which implies first

$$\sum_{(\lambda',\mu')\neq(p-1,0)}[\mathcal{P}(\lambda,\mu) : V_{irr}(\lambda',\mu')] = \frac{\dim \mathcal{P}(\lambda,\mu)}{\dim V_{irr}(0,0)},$$

and then

$$[\mathcal{P}(\lambda,\mu)] = \frac{\dim \mathcal{P}(\lambda,\mu)}{\dim V_{irr}(0,0)}[V_{irr}(0,0)].$$

Together with Theorem 8.2.5 and Proposition 8.3.1, this yields the claim for $p > 3$. Now, suppose that $p = 3$. Corollary 7.4.3 determines the composition factores of $P(\lambda,\mu)$. By inducing over $U_\chi(B^+)$, we obtain

$$\begin{aligned}[V^+_{proj}(0,0)] &= 12[V_{irr}(0,0)] + 6[V_{irr}(1,0)], \\ [V^+_{proj}(1,0)] &= 6[V_{irr}(0,0)] + 3[V_{irr}(1,0)], \\ [V^+_{proj}(\lambda,\mu)] &= 3[V_{irr}(0,\mu)] + 3[V_{irr}(1,\mu)] + 3[V_{irr}(2,\mu)] \text{ for } \mu \neq 0.\end{aligned}$$

The crucial observation now is that $[V^+_{proj}(\lambda,\mu)] = [V^+_{proj}(1,0)]$ if $\mu \neq 0$. This is a consequence of Theorem 8.2.5 which implies that exactly one of the terms $[V_{irr}(0,\mu)]$, $[V_{irr}(1,\mu)]$ and $[V_{irr}(2,\mu)]$ is equal to $[V_{irr}(1,0)]$ while the remaining terms are equal to $[V_{irr}(0,0)]$. Thus, $[V^+_{proj}(0,0)] = 2[V^+_{proj}(\lambda,\mu)]$ for every pair $(\lambda,\mu) \neq (0,0)$. We will use this observation in a moment, but first, we have

$$\begin{aligned}[V^+_{proj}(1,0)] = 12[\mathcal{L}(0,0)] + 12[\mathcal{L}(2,2)] + 6[\mathcal{L}(1,1)] + 6[\mathcal{L}(0,1)] \\ + 6[\mathcal{L}(1,2)] + 3[\mathcal{L}(1,0)] + 3[\mathcal{L}(2,1)] + 3[\mathcal{L}(0,2)].\end{aligned}$$

Now, for each pair $(\lambda',\mu') \neq (p-1,0)$ we set $a_{\lambda',\mu'} = [\mathcal{P}(\lambda,\mu) : V_{irr}(\lambda',\mu')]$. Then

$$\begin{aligned}[\mathcal{P}(\lambda,\mu)] &= a_{0,0}[V^+_{proj}(0,0)] + \sum_{\substack{(\lambda',\mu')\neq(p-1,0)\\(\lambda',\mu')\neq(0,0)}} a_{\lambda',\mu'}[V^+_{proj}(\lambda',\mu')] \\ &= 2a_{0,0}[V^+_{proj}(1,0)] + \sum_{\substack{(\lambda',\mu')\neq(p-1,0)\\(\lambda',\mu')\neq(0,0)}} a_{\lambda',\mu'}[V^+_{proj}(1,0)].\end{aligned}$$

This implies first

$$\sum_{\substack{(\lambda',\mu')\neq(p-1,0)\\(\lambda',\mu')\neq(0,0)}} a_{\lambda',\mu'} + 2a_{0,0} = \frac{\dim \mathcal{P}(\lambda,\mu)}{\dim V^+_{proj}(1,0)},$$

and then

$$[\mathcal{P}(\lambda,\mu)] = \frac{\dim \mathcal{P}(\lambda,\mu)}{\dim V^+_{proj}(1,0)}[V^+_{proj}(1,0)].$$

The claim follows from Proposition 8.3.1 and the discussion above. □

A Orbits of height 0 characters

There is a natural action of the general linear group GL_n on the truncated polynomial algebra

$$B^n = K[X_1, X_2, \dots, X_n]/(X_1^p, X_2^p, \dots, X_n^p).$$

This action is given by

$$gx_i = \sum_{j=1}^{n} g_{ji} x_j,$$

for every $g = (g_{ij})$ in GL_n. (Here x_i denotes the image of X_i in B^n.) Furthermore, GL_n acts on the set of derivations W^n of B^n and the set of all characters of W^n. These actions are given by

$$\begin{aligned}(gD)(x) &= D(g^{-1}x),\\ (g\varphi)(D) &= \varphi(g^{-1}D),\end{aligned}$$

for all $D \in W^n$, $\varphi \in (W^n)^*$ and $x \in B^n$. The formula $D = \sum_{i=1}^{n} D(x_i)\partial_i$ implies, in particular,

$$\begin{aligned}g^{-1}\partial_i &= \sum_{j=1}^{n} (g^{-1}\partial_i)(x_j)\partial_j\\ &= \sum_{j=1}^{n} \partial_i(gx_j)\partial_j\\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} \partial_i(g_{kj}x_k)\partial_j\\ &= \sum_{j=1}^{n} g_{ij}\partial_j.\end{aligned}$$

Next, let $\varphi_1, \varphi_2, \dots, \varphi_n$ denote the dual basis of W_{-1}^n with respect to the basis $\partial_1, \partial_2, \dots, \partial_n$; that is $\varphi_i(\partial_j) = \delta_{ij}$, where δ_{ij} denotes Kronecker's delta.

It follows from the above computation that

$$\begin{aligned} g\varphi_i &= \sum_{j=1}^{n} (g\varphi_i)(\partial_i)\varphi_j \\ &= \sum_{j=1}^{n} \varphi_i(g^{-1}\partial_j)\varphi_j \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} \varphi_i(g_{jk}\partial_k)\varphi_j \\ &= \sum_{j=1}^{n} g_{ji}\varphi_j. \end{aligned}$$

Thus, g acts on the space $(W^n_{-1})^*$ precisely as it acts on K^n. Now, any $g \in GL_n$ induces an isomorphism of the Lie algebra W^n in the following way:

$$g(D) = g \circ D \circ g^{-1} \quad \text{for all} \quad D \in W^n.$$

This induces a natural inclusion of GL_n into $\mathrm{Aut}(W^n)$, see e.g. [21, Sec. 2.3]. Thus, we obtain the following proposition.

Proposition. *Under the automorphism group of W^n, all height 0 characters are conjugate to a character φ of height 0 such that $\varphi(\partial_1) \neq 0$ and $\varphi(\partial_i) = 0$ for all $i > 1$.*

B Irreducible representations of W_0^2

We use the setting described in Part II. In particular, we have a character $\chi \in (W^n)^*$ of height 0 such that $\chi(\partial_1) \neq 0$ and $\chi(\partial_i) = 0$ for all $i \neq 1$. There is an isomorphism of Lie algebras $W_0^n \simeq \mathfrak{gl}_n$ that sends $x_i\partial_j$ into the $n \times n$ matrix E_{ij} with 1 in the (i,j)th position and 0 elsewhere. In particular, this defines for $n = 2$ a triangular decomposition

$$W_0^2 = Ke_{011} \oplus (Ke_{101} \oplus Ke_{012}) \oplus Ke_{102}.$$

Each $(\lambda, \mu) \in \mathbb{F}_p^2$ defines a 1–dimensinal module $K_{\lambda,\mu}$ over $U_\chi(Ke_{101} \oplus Ke_{012})$ such that e_{101} and e_{012} act by multiplication by λ and μ, respectively. Since Ke_{011} is unipotent and $\chi(e_{011}) = 0$, this module can be extended to a module over $U_\chi(Y) = U_\chi(Ke_{101} \oplus Ke_{012} \oplus Ke_{011})$ by letting e_{011} act trivially. Now, the set of all $K_{\lambda,\mu}$ is a complete set of pairwise non–isomorphic simple modules of $U_\chi(Y)$. The simple modules of $U_\chi(W_0^2)$ are, therefore, homomorphic images of the p–dimensional induced module

$$x(\lambda, \mu) = U_\chi(W_0^2) \otimes_{U_\chi(Y)} K_{\lambda,\mu}.$$

Set $x^0 = 1 \otimes 1$ and $x^{i+1} = e_{102}x^i$ for every $i = 0, 1, \ldots, p-1$. Then $x^0, x^1, \ldots, x^{p-1}$ form a basis for $x(\lambda, \mu)$. We have

$$e_{101}x^i = (\lambda + i)x^i \quad \text{and} \quad e_{012}x^i = (\mu - i)x^i.$$

Furthermore, for every $i > 0$

$$e_{011}x^i = i(\mu - \lambda - i + 1)x^{i-1}.$$

It follows that $x(\lambda, \mu)$ is simple if $\mu - \lambda = p - 1$. If $\mu - \lambda \neq p - 1$, we obtain a maximal submodule generated by $x^{[\mu-\lambda]+1}, x^{[\mu-\lambda]+2}, \ldots, x^{p-1}$. (Here we use the notation introduced in Section 3.1.1.) For every $\lambda, \mu \in \mathbb{F}_p$, we let $y(\lambda, \mu)$ denote the head of $x(\lambda, \mu)$.

Theorem. *There are p^2 distinct (up to isomorphism) simple $U_\chi(W_0^2)$–modules. They are represented by $\{y(\lambda,\mu) \mid \lambda,\mu \in \mathbb{F}_p\}$. We have $y(\lambda,\mu) \simeq x(\lambda,\mu)$ if and only if $\mu - \lambda = p-1$. Furthermore, we have* $\dim y(\lambda,\mu) = [\mu-\lambda]+1$.

Remark (A)**.** If $\mu - \lambda = p - 1$, then by using weight considerations, we have a short exact sequence

$$0 \to y(\mu+1,\lambda-1) \to x(\lambda,\mu) \to y(\lambda,\mu) \to 0.$$

Remark (B)**.** Since $W_{>0}$ is a unipotent ideal in $W_{\geq 0}$, any $U_\chi(W_0)$–module can be extended to $U_\chi(W_{\geq 0})$ by letting $U_\chi(W_{>0})$ act trivially. Furthermore, the simple modules for $U_\chi(W_{\geq 0})$ are just the simple modules for $U_\chi(W_0)$ with $U_\chi(W_{>0})$ acting trivially. In particular, this induces a $U_\chi(W_{\geq 0})$–module structure on $x(\lambda,\mu)$ and $y(\lambda,\mu)$.

C Induction and inflation

Let L be a restricted Lie algebra with a p–mapping $[p]$ and let $\varphi \in L^*$. Furthermore, let $I \subset L$ be an ideal such that $\varphi(I) = 0$ and $x^{[p]} \in I$ for all $x \in I$. It follows that L/I is a restricted Lie algebra with $\bar{\varphi} \in (L/I)^*$

$$\bar{\varphi}(x + I) = \varphi(x) \text{ for all } x \in L.$$

Every $U_{\bar{\varphi}}(L/I)$–module M becomes a $U_\varphi(L)$–module via $x \cdot v = (x + I) \cdot v$ for all $x \in L$ and $v \in M$. If M is simple over $U_{\bar{\varphi}}(L/I)$, then it is simple over $U_\varphi(L)$. Furthermore, two modules are isomorphic over $U_{\bar{\varphi}}(L/I)$ if and only if they are isomorphic over $U_\varphi(L)$. If I is unipotent, then it follows from Lemma 6.1.3 that I annihilates every simple $U_\varphi(L)$–module. We then obtain a one–to–one correspondence between the set of isomorphism classes of simple $U_{\bar{\varphi}}(L/I)$–modules and the set of isomorphism classes of simple $U_\varphi(L)$–modules. Suppose that L has a restricted Lie subalgebra U such that

$$L = U \oplus I \quad \text{as vector spaces.}$$

Forget for a moment that I is unipotent. If we identify L/I with U and $\bar{\varphi}$ with $\varphi_{|U}$, we get an embedding of the set of isomorphism classes of $U_\varphi(U)$–modules in the set of isomorphism classes of $U_\varphi(L)$–modules. If M is a $U_\varphi(U)$–module, we let $\inf_{U_\varphi(U)}^{U_\varphi(L)} M$ denote a representative of M as a $U_\varphi(L)$–module. Note that I acts as 0 on this module. Now, if I is unipotent, then $\inf_{U_\varphi(U)}^{U_\varphi(L)}$ defines a one–to–one correspondence between the isomorphism classes of simple $U_\varphi(U)$–modules and the isomorphism classes of simple $U_\varphi(L)$–modules.

Proposition. *Let L be a restricted Lie algebra and let $\varphi \in L^*$. Furthermore, let $U \subset V$ and I be restricted Lie subalgebras such that $\varphi(I) = 0$, $[V, I] \subset I$ and $V \cap I = 0$. Then, we have an isomorphism of $U_\varphi(L)$–modules*

$$\operatorname{ind}_{U_\varphi(U\oplus I)}^{U_\varphi(L)} \inf_{U_\varphi(U)}^{U_\varphi(U\oplus I)} M \simeq \operatorname{ind}_{U_\varphi(V\oplus I)}^{U_\varphi(L)} \inf_{U_\varphi(V)}^{U_\varphi(V\oplus I)} \operatorname{ind}_{U_\varphi(U)}^{U_\varphi(V)} M,$$

for every $U_\varphi(U)$–module M.

Proof. The homomorphism $M \to U_\varphi(V) \otimes_{U_\varphi(U)} M$ of $U_\varphi(U)$–modules that maps x into $1 \otimes x$ can be considered as a homomorphism of $U_\varphi(U \oplus I)$–modules
$$\inf_{U_\varphi(U)}^{U_\varphi(U\oplus I)} M \to \inf_{U_\varphi(V)}^{U_\varphi(V\oplus I)} \operatorname{ind}_{U_\varphi(U)}^{U_\varphi(V)} M,$$
since I acts as 0 on both sides. We also have a homomorphism

$$U_\varphi(V) \otimes_{U_\varphi(U)} M \to U_\varphi(L) \otimes_{U_\varphi(V\oplus I)} (U_\varphi(V) \otimes_{U_\varphi(U)} M)$$

of $U_\varphi(U \oplus I)$–modules that maps $v \otimes x$ into $1 \otimes v \otimes x$. The composite of these two homomorphisms gives a homomorphism

$$M \to U_\varphi(L) \otimes_{U_\varphi(V\oplus I)} (U_\varphi(V) \otimes_{U_\varphi(U)} M)$$

of $U_\varphi(U \oplus I)$–modules which maps x into $1 \otimes 1 \otimes x$ and which extends to a homomorphism of $U_\varphi(L)$–modules

$$U_\varphi(L) \otimes_{U_\varphi(U\oplus I)} M \to U_\varphi(L) \otimes_{U_\varphi(V\oplus I)} (U_\varphi(V) \otimes_{U_\varphi(U)} M).$$

This homomorphism is, in fact, an isomorphism because it sends a PBW basis to a PBW basis. □

D Irreducible representations of W^2

The simple W^n–modules with height at most 1 were computed by Holmes in [10]. Holmes' paper assumes, however, that the characteristic p of the ground field is larger than 3. The main purpose of this section is to generalize some of these results to include the case $p = 3$.

We use the setting described in Part II. In particular, we have a character $\chi \in (W^2)^*$ of height 0 such that $\chi(e_{001}) \neq 0$ and $\chi(e_{ijk}) = 0$ for all $(i, j, k) \neq (0, 0, 1)$. We assume that $p = 3$. Since all simple $U_\chi(W^2_{(\geq 0)})$–modules are 3–dimensional, any finite dimensional $U_\chi(W^2)$–module has dimension equal to a multiple of 3. There are, however, no $U_\chi(W^2)$–modules of dimension 3 because a 3–dimensional $U_\chi(W^2)$–module M would induce a homomorphism $\rho\colon W^2 \to \mathfrak{gl}(M)$ from an 18–dimensional Lie algebra to a 9–dimensional Lie algebra. This is impossible because the kernel of ρ would be the entire Lie algebra W^2 (as $\ker \rho \neq 0$ is an ideal in the simple Lie algebra W^2) which is clearly in contradiction with the fact that $e_{001}M \neq 0$. (Recall that $e_{001}^3 x = \chi(e_{001})^3 x = x$ for all $x \in M$.) It follows that all $U_\chi(W^2)$–modules of dimension 9 are simple since otherwise there would exist a composition factor of dimension 3.

For every $(\lambda, \mu) \in \mathbb{F}_p^2$, we let $y(\lambda, \mu)$ denote the simple $U_\chi(W^2_{\geq 0})$–modules introduced in Appendix B and we consider the $9([\mu - \lambda] + 1)$–dimensional module

$$Y(\lambda, \mu) = U_\chi(W^2) \otimes_{U_\chi(W^2_{\geq 0})} y(\lambda, \mu).$$

The foregoing discussion leads to the following lemma.

Lemma. *$Y(\lambda, \lambda)$ is simple for all $\lambda \in \mathbb{F}_3$.*

It follows from Appendix B that we have a basis for $y(\lambda, \mu)$ consisting of the cosets represented by $x^0, x^1, \ldots, x^{[\mu-\lambda]}$. For the sake of clarity, this basis will be denoted by $x^0_{\lambda,\mu}, x^1_{\lambda,\mu}, \ldots, x^{[\mu-\lambda]}_{\lambda,\mu}$. The next proposition is proved for $p > 5$ by Holmes, see [10, Prop. 3.10(1) & Prop. 3.8].

Proposition. *We have the following short exact sequence*

$$0 \to Y(2,2) \to Y(2,0) \to Y(0,0) \to 0.$$

Proof. The map $\varphi\colon y(2,0) \to Y(0,0)$ that sends $x^0_{2,0}$ to $e_{001} \otimes x^0_{0,0}$ is a homomorphism of $U_\chi(W^2_{\geq 0})$. It induces a $U_\chi(W^2)$–homomorphism $\tilde{\varphi}\colon Y(2,0) \to Y(0,0)$ such that $\tilde{\varphi}(1 \otimes x^0_{2,0}) = \varphi(x^0_{2,0})$. This homomorphism is surjective as $Y(0,0)$ is simple and $\tilde{\varphi} \neq 0$. Likewise, we have a homomorphism $\psi\colon y(2,2) \to Y(2,0)$ that sends $x^0_{2,2}$ to $e_{002} \otimes x^0_{2,0} + e_{001} \otimes x^1_{2,0}$ and which extends to a $U_\chi(W^2)$–homomorphism $\tilde{\psi}\colon Y(2,2) \to Y(2,0)$ by $\tilde{\psi}(1 \otimes x^0_{2,2}) = \psi(x^0_{2,2})$. We have

$$\begin{aligned}
\tilde{\varphi} \circ \tilde{\psi}(1 \otimes x^0_{2,0}) &= \tilde{\varphi}(e_{002} \otimes x^0_{2,0} + e_{001} \otimes x^1_{2,0}) \\
&= e_{002}e_{001} \otimes x^0_{0,0} + e_{001}e_{102}e_{001} \otimes x^0_{0,0} \\
&= e_{002}e_{001} \otimes x^0_{0,0} + e^2_{001} \otimes e_{102}x^0_{0,0} - e_{001}e_{002} \otimes x^0_{0,0} \\
&= 0.
\end{aligned}$$

Thus, $\operatorname{im}\tilde{\psi} \subset \ker\tilde{\varphi}$ and by dimension arguments one sees that the inclusion is actually an equality. This proves the proposition. □

Theorem. *The set of all $Y(\lambda,\mu)$ with $(\lambda,\mu) \in \mathbb{F}_3^2$ and $(\lambda,\mu) \neq (2,0)$ forms a complete set of pairwise non–isomorphic simple modules of $U_\chi(W^2)$.*

Proof. Let E be a simple module over $U_\chi(W^2)$. For some $\lambda,\mu \in \mathbb{F}_3$ there exists a monomorphism

$$\kappa\colon y(\lambda,\mu) \to E_{|U_\chi(W^2_0)}$$

of $U_\chi(W^2_0)$–modules. It follows that κ will extend to a homomorphism of $U_\chi(W^2)$–modules from $Y(\lambda,\mu)$ to E. In other words, every simple $U_\chi(W^2)$–module is a homomorphic image of some $Y(\lambda,\mu)$. Since there are exactly eight isomorphism classes of simple $U_\chi(W^2)$–modules, it suffices to prove that all $Y(\lambda,\mu)$ are simple, cf. Theorem 6.3.3 and 6.3.1.

Set $r = [\mu-\lambda]$. We may assume that $\lambda \neq \mu$ so that $1 \leq r \leq 2$. Furthermore, we assume that $(\lambda,\mu) \neq (2,0)$. Since $\chi(e_{002}) = 0$ and $\chi(e_{102}) = 0$, it follows that the trivial module is the only simple module (up to isomorphism) over the unipotent subalgebra $Ke_{002} \oplus Ke_{102}$, see [13, Prop. 3.2]. In particular, every nonzero submodule V of $Y(\lambda,\mu)$ contains a nonzero vector annihilated by e_{002} and e_{102}. Since e^3_{002} annihilates $Y(\lambda,\mu)$, we have

$$\{x \in Y(\lambda,\mu) \mid e_{002}x = 0\} = \bigoplus_{\substack{0\leq i\leq 2\\ 0\leq j\leq r}} Ke^2_{002}e^i_{001} \otimes x^j_{\lambda,\mu}.$$

A straightforward computation shows that

$$e_{102}(e_{002}^2 e_{001}^i \otimes x_{\lambda,\mu}^j) = e_{002}^2 e_{001}^i \otimes x_{\lambda,\mu}^{j+1}.$$

Thus,

$$\{x \in Y(\lambda,\mu) \mid e_{002}x = 0 \text{ and } e_{102}x = 0\} = \bigoplus_{0\le i\le 2} K e_{002}^2 e_{001}^i \otimes x_{\lambda,\mu}^r.$$

Now, V contains a nonzero vector which is annihilated by both e_{002} and e_{102}. Since, in addition, V is a direct sum of weight spaces with respect to e_{101} and e_{012} this implies that V contains an element of the form $e_{002}^2 e_{001}^i \otimes x_{\lambda,\mu}^r$. Recall that e_{001}^3 acts as identity on every $U_\chi(W^2)$–module. Thus, by applying e_{001} sufficiently many times, we see that $e_{002}^2 \otimes x_{\lambda,\mu}^r \in V$. We compute

$$\begin{aligned} e_{022}(e_{002}^2 \otimes x_{\lambda,\mu}^r) &= (e_{002}e_{022} - 2e_{012})e_{002} \otimes x_{\lambda,\mu}^r \\ &= (e_{002}^2 e_{022} + 2e_{002}(1-2e_{012})) \otimes x_{\lambda,\mu}^r \\ &= 2e_{002} \otimes (1-2e_{012})x_{\lambda,\mu}^r \\ &= 2e_{002} \otimes (1-2\lambda)x_{\lambda,\mu}^r, \end{aligned}$$

where, in the last equality, we used the fact that $x_{\lambda,\mu}^r$ has weight (μ,λ). It follows that if $\lambda \neq 2$, then $e_{002} \otimes x_{\lambda,\mu}^r \in V$. Furthermore, we have

$$\begin{aligned} e_{021}(e_{002} \otimes x_{\lambda,\mu}^r) &= (e_{002}e_{021} - 2e_{011}) \otimes x_{\lambda,\mu}^r \\ &= -2 \otimes e_{011}x_{\lambda,\mu}^r \\ &= -2r(\mu-\lambda-r+1) \otimes x_{\lambda,\mu}^{r-1} \\ &= -2r \otimes x_{\lambda,\mu}^{r-1}. \end{aligned}$$

Hence $1 \otimes x_{\lambda,\mu}^{r-1} \in V$ if $\lambda \neq 2$. Since $y(\lambda,\mu)$ is simple, the submodule generated by $1 \otimes x_{\lambda,\mu}^{r-1}$ must contain $1 \otimes y(\lambda,\mu)$. But then $V = Y(\lambda,\mu)$, proving the irreducibility of $Y(\lambda,\mu)$ for $\lambda \neq 2$. It only remains to consider the module $Y(2,1)$, but we shall prove that $Y(\lambda,\mu)$ is simple for all (λ,μ) with $r = 2$. To this end, assume that $r = 2$ and recall that $e_{002}^2 \otimes x_{\lambda,\mu}^2 \in V$. We have in $U_\chi(W^2)$

$$\begin{aligned} e_{021}e_{002}^2 &= (e_{002}e_{021} - 2e_{011})e_{002} \\ &= e_{002}^2 e_{021} - 2e_{002}e_{011} - 2e_{002}e_{011} + 2e_{001} \\ &= e_{002}^2 e_{021} - 4e_{002}e_{011} + 2e_{001}, \end{aligned}$$

which first implies

$$e_{021}(e_{002}^2 \otimes x_{\lambda,\mu}^2) = -4e_{002} \otimes e_{011}x_{\lambda,\mu}^2 + 2e_{001} \otimes x_{\lambda,\mu}^2,$$

and then

$$e_{021}^2(e_{002}^2 \otimes x_{\lambda,\mu}^2) = 8(1 \otimes e_{011}^2 x_{\lambda,\mu}^2).$$

An easy computation shows that $e_{011}x_{\lambda,\mu}^i = i(\mu - \lambda - i + 1) \otimes x_{\lambda,\mu}^{i-1}$ for $i > 0$. This implies $e_{011}^2 x_{\lambda,\mu}^2 = 4x_{\lambda,\mu}^0$. But then $1 \otimes x_{\lambda,\mu}^0 \in V$ and therefore $Y(\lambda, \mu)$ is simple for $r = 2$. In particular, we conclude that $Y(2, 1)$. □

List of Symbols

K	1
p, $[p]$	1
$K[X]$	1
$U_\chi(L)$	2
B^n	3
x_i	3
W^n	3
∂_i	3
W_i^n	4
$\mathrm{ht}(\chi)$	5
e_i	9
$\Lambda(\chi)$	9
$V_\chi(\lambda)$	9
S	10
$[\mu]$	15
$\mathrm{Ext}^i_{U_\chi(L)}(M,-)$	16
mod_R	51
$k\langle X,Y\rangle$	51
Q_s	53

J_χ 53

$\mathrm{ch}\, M$ 59

$\mathrm{ind}_{U_\chi(\mathfrak{h})}^{U_\chi(L)} -$ 60

$W_{(i)}^n$ 61

$N^\pm$ 62

$B^\pm$ 62

A 62

B 62

$A^0,\ A^\pm$ 63

ε_j 63

Λ 63

$Z(\lambda)$ 63

H 63

J 63

$L(\mu, E)$ 65

$L(\mu)$ 65

δ_L 67

$P(\mu)$ 68

$Q(\mu, E)$ 69

$P(\mu, E)$ 69

$P_H(\mu, E)$ 70

$\tilde{\Lambda}$ 72

$L(\mu, \lambda)$ 75

$P(\mu, \lambda)$ 75

$Q(\mu, \lambda)$ 75

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C 76

h 76

y 76

F_0 77

F_1 77

a_E 77

$P_{F_1}(\mu, E)$ 78

$\tilde{P}_{F_1}(\mu, E)$ 78

F_2 80

e_{ijk} 83

$V_{irr}(\lambda, \mu)$ 84

$\mathcal{L}(\lambda, \mu)$ 84

$Y(\lambda, \mu)$ 84

$X(\lambda, \mu)$ 84

W_+^2 85

$M(\lambda, \mu)$ 85

$\mathcal{P}(\lambda, \mu)$ 91

$V_{proj}^{\pm}(\lambda, \mu)$ 91

$x(\lambda, \mu)$ 97

$y(\lambda, \mu)$ 97

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