Simple 2-representations and Classification of Categorifications

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Abstract

We consider selfadjoint functors defined on categories of modules over finite dimensional algebras and classify those that satisfy some simple relations. In particular we classify selfadjoint idempotents and selfadjoint squareroots of a multiple of the identity functor. This is related to the theory of algebraic categorification which we review with the viewpoint that a genuine categorification is a 2-representation of a 2-category.

Resumé på dansk (Danish abstract)

Vi betragter selvadjungerede funktorer definerede på kategorien af moduler over en algebra af endelig dimension og klassificerer de funktorer, der opfylder visse simple relationer. Specielt klassificerer vi selvadjungerede idempotente funktorer og selvadjungerede kvadratrødder af et multiplum af identitetsfunktoren. Dette er relateret til teorien om kategorificeringer; vi giver et overblik over denne teori med det synspunkt at en ægte kategorificering er en 2-repræsentation af en 2-kategori.

INTRODUCTION

Since L. Crane [Cra95] and I. Frenkel [CF94] first wrote about *categorification* a lot of research has been done in this particular field of mathematics. The broad idea with categorification is the process of replacing a set with a category, i.e. to replace the set with something with more structure. The naïve hope is that the additional structure can be used to say something new about the set via the process of *decategorification*. Here we will deal with *algebraic categorification*, which means that we consider a module over some ring and replace the module with some abelian (or triangulated, or ...) category and replace the ring structure with an action of functors on this category.

Let us explain the idea of categorification with an example (a lot of the involved concepts will be explained more thoroughly later). Let $R = \mathbb{Q}[x]$ be the polynomial algebra over the rational numbers, and set $S = \mathrm{End}_{\mathbb{Q}}(R)$. Consider the maps $X, D \in S$ acting on $p \in R$ via

$$X(p) = x \cdot p, \quad D(p) = \frac{d}{dx}p$$

and denote by A the unital subalgebra of S generated by X and D. Since

$$D(X(p)) = D(xp) = D(x)p + xD(p) = p + X(D(p))$$

by the Leibniz rule, we see that the commutator

$$[D, X] = DX - XD = 1$$

is the unit in A. In fact A is isomorphic to $\mathbb{Q}\langle X, D\rangle/(DX - XD - 1)$, the (unital) $\mathbb{Q}$-algebra with generators X and D and one relation $DX = 1 + XD$. Notice that this relation is written with non-negative integral coefficients. A is called the Weyl algebra (or the first Weyl algebra) over $\mathbb{Q}$.

To categorify (modules over) A is now the process of finding a (not semisimple) abelian category $\mathcal{C}$ with exact endofunctors $F_X, F_D : \mathcal{C} \to \mathcal{C}$ (one for each generator) satisfying the same relation(s) as the generators:

$$F_D F_X = \mathbb{1}_{\mathcal{C}} \oplus F_X F_D$$

Here juxtaposition of functors means usual composition, $\mathbb{1}_{\mathcal{C}}$ denotes the identity functor on the category $\mathcal{C}$, and the direct sum takes place in the category of functors with natural transformations as morphisms.

By definition A acts on the polynomial algebra $R = \mathbb{Q}[x]$. This representation of A is sometimes referred to as the defining representation of A. R has a natural basis $\{x^n | n \geq 0\}$ in which both of the generators for A act integrally. In [Kho01] Khovanov categorifies this defining representation with help of nilCoxeter algebras.

Define the n'th nilCoxeter algebra

$$N_n = \mathbb{Q}\langle Y_1, \ldots Y_{n-1}\rangle / I_n$$

where I_n is the two-sided ideal generated by

$$\begin{array}{cc} Y_i^2 & \\ Y_iY_j - Y_jY_i, & |i-j| > 1 \\ Y_iY_{i+1}Y_i - Y_{i+1}Y_iY_{i+1} & \end{array}$$

We see that if we substituted the relation $Y_i^2 = 0$ with $Y_i^2 = 1$ we would get the group algebra of the symmetric group $\mathbb{Q}[S_n]$. There is a unique simple module over N_n call this L_n, as a rational vector space it is just $\mathbb{Q}$. Similarly to the proof for $\mathbb{Q}[S_n]$ one shows that N_n has a presentation

$$N_n \simeq \mathbb{Q}\langle\{Y_w | w \in S_n\}\rangle/\tilde{I}_n$$

where $\tilde{I}_n$ is given by the relations

$$Y_w Y_x = \begin{cases} Y_{wx} & \text{if } \ell(wx) = \ell(w) + \ell(x) \\ 0 & \text{if } \ell(wx) < \ell(w) + \ell(x) \end{cases}$$

Here $\ell(w)$ is the usual length function on S_n (the minimal number of simple transpositions used to write w). In particular the dimension of N_n is $n!$. Since the length function on S_n is the restriction of that from S_{n+1} one shows that the natural algebra homomorphism $N_n \to N_{n+1}$ mapping Y_i to Y_i is an injection of algebras.

If one defines $D_{n+1} = N_{n+1}$ as an N_n-N_{n+1}-bimodule and $X_n = N_{n+1}$ as an N_{n+1}-N_n-bimodule one shows that these are projective from both sides and as N_n-N_n-bimodules

$$D_{n+1} \underset{N_{n+1}}{\otimes} X_n \simeq N_n \oplus (X_{n-1} \underset{N_{n-1}}{\otimes} D_n)$$

This means that if we define $N = \bigoplus_{n\geq 0} N_n$ as the direct sum of algebras (it is a non-unital algebra, but the units in the various N_n are orthogonal idempotents), and define $X = \bigoplus_{n\geq 0} X_n$ and $D = \bigoplus_{n\geq 0} D_n$ (with the natural action of N) then

$$D \underset{N}{\otimes} X \simeq N \oplus (X \underset{N}{\otimes} D)$$

Define the category $\mathcal{C}$ as the category consisting of finite dimensional N-modules where N_n acts as 0 for almost all n. This is the direct sum of the categories $N_n\text{-mod}$. Define the functors

$$F_X, F_D : \mathcal{C} \to \mathcal{C}$$

by $F_V(M) = V \underset{N}{\otimes} M$ for $V \in \{X, D\}$ and $M \in \mathcal{C}$. The functor F_X is given by simultaneous induction and F_D is given by simultaneous restriction. These functors are then (well defined) exact endofunctors of the abelian category $\mathcal{C}$ satisfying

$$F_D F_X = \mathbb{1}_{\mathcal{C}} \oplus F_X F_D$$

The rational Grothendieck group $[\mathcal{C}]^{\mathbb{Q}}$ of $\mathcal{C}$ has a basis given by $\{[\mathrm{P}_n] | n \geq 0\}$ where $\mathrm{P}_n = N_n$ is the projective cover of L_n. If we define

$$\varphi : [\mathcal{C}]^{\mathbb{Q}} \to R = \mathbb{Q}[x]$$

by $\varphi([\mathrm{P}_n]) = x^n$, then φ is an isomorphism of A-modules where we act via $X[M] = [F_X(M)]$ and $D[M] = [F_D(M)]$ on the Grothendieck group.

The Grothendieck group also has another distinguished basis, namely $\{[\mathrm{L}_n] | n \geq 0\}$ and applying φ we get that $\varphi([\mathrm{L}_n]) = \frac{x^n}{n!}$ (since $[\mathrm{P}_n] = (\dim_{\mathbb{Q}} \mathrm{P}_n)[\mathrm{L}_n] = n![\mathrm{L}_n]$). The process of categorification has thus given us another basis of the A-module R in which both of the generators for A act integrally (in this simple example, obviously this basis could have been found without the help of categorification).

In the literature a lot of categorifications like the one above have been constructed, we do not try to list all of these, but let us briefly mention some examples. We refer the reader to the references for details.

In [CR08] Chuang and Rouquier proposes a theory of categorifications of $\mathfrak{sl}_2$-modules. They give examples of such categorications using the BGG category $\mathcal{O}$ and uses results obtained from categorifications to prove parts of Broué's abelian defect group conjecture.

In [MS07] Mazorchuk and Stroppel categorify the parabolic Hecke modules from [KL79] (in the description given in [Soe07]) using a certain subcategory of the BGG category $\mathcal{O}$.

In [FKS07] Frenkel, Khovanov and Stroppel categorify the finite-dimensional representations of the quantum group $U_q(\mathfrak{sl}_2)$ using blocs of the category of (graded) Harish-Chandra bimodules for $\mathfrak{gl}_n$.

In [KMS08] Khovanov, Mazorchuk and Stroppel categorify Specht (simple) modules for the symmetric groups using the parabolic category $\mathcal{O}$ (and action via translations functors).

In [MS08] Mazorchuk and Stroppel categorify right cell modules for Hecke algebras (of finite Weyl groups) using graded versions of (Serre) subcategories (related to parabolic subcategories) of the BGG category $\mathcal{O}$. In type A when one forgets the grading one obtains the categorification of the Specht modules from [KMS08]. In type A they also show that for right cells in a fixed two-sided cell the categories categorifying the right cells are in fact equivalent (inducing the isomorphism of the right cell modules).

All the above examples are examples of *algebraic* categorification where algebraic structures are replaced with algebraically constructed categories. There are also in the litterature a lot of geometric or topological categorications. We shall not be concerned with this kind of categorication here, but let us nonetheless mention a few examples.

In [Sus07] and [MS09] Sussan and Mazorchuk and Stroppel categorify the coloured Jones polynomial via functors on (derived categories of) the parabolic category $\mathcal{O}$. That is, they associate to each link a functor such that the functor acts on the Grothendieck group via the coloured Jones polynomial.

In [Lau10] Lauda uses a pictorial calculus to construct a 2-category which categorifies Lusztig's version of the quantum group $U_q(\mathfrak{sl}_2)$.

This project, however, takes a different approach. Instead of constructing new categorifications we ask wether it is possible to some extend to classify categorifications.

Our main results are the classification results in Chapter 4 (which in turn are the main results from [AM11]). This is joint work with Volodomyr Mazorchuk and was initiated during a visit at Uppsala University in the fall 2009.

A very general formulation of the problem would be: Given a ring Λ (satisfying some reasonable integrality conditions) can one classify all categorifications of all Λ-modules? Since this would mean that one also has to classify all Λ-modules this seams like an impossible task, so we are going to be more specific. We let $\Bbbk$ be an algebraically closed field and assume that Λ is a finite dimensional $\Bbbk$-algebra generated as a unital algebra by a single generator $a \in \Lambda$. Say $\Lambda = \Bbbk[x]/(f)$ and assume that $f \in \Bbbk[x]$ can be written as $f = h - g$ where $h, g \in \mathbb{N}[x]$ (i.e. we can rewrite the the relation $f(a) = 0$ (where $a = x + (f)$) as $h(a) = g(a)$, such that it only involves non-negative integral coefficients). For an endofunctor of an abelian category $F : \mathcal{C} \to \mathcal{C}$ it makes sense to ask if $h(F) \simeq g(F)$. Our main results are complete discussions of the cases $f = x^n - 1$, $f = x^2 - x$, $f = x^n$ and $f = x^2 - k$ (with $n, k \geq 2$). In the following A denotes a basic, finite dimensional $\Bbbk$-algebra. If $\varphi : A \to A$ is an algebra automorphism we denote by $F_\varphi : A\text{-mod} \to A\text{-mod}$ the functor that twists the action of A with φ^{-1}. We denote by $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$ the group of outer automorphisms of A.

Proposition. *Let $n \geq 2$ be a natural number. Isomorphism classes of endofunctors $F : A\text{-mod} \to A\text{-mod}$ satisfying $F^n \simeq \mathbb{1}_{A\text{-mod}}$ are in one-to-one correspondence with group homomorphisms from $\mathbb{Z}_n$ to $\mathrm{Out}(A)$. F is selfadjoint if and only $F^2 \simeq \mathbb{1}_{A\text{-mod}}$.*

Proposition. *Assume $F : A\text{-mod} \to A\text{-mod}$ is a selfadjoint functor satisfying $F^2 \simeq F$. Then A decomposes as $A = B \oplus C$ where B and C are unital (or zero) subalgebras of A, and F is isomorphic to the projection on $B\text{-mod}$ composed with the embedding into $A\text{-mod}$.*

Proposition. *Let $F : A\text{-mod} \to A\text{-mod}$ be a selfadjoint functor satisfying $F^k = 0$ for some $k \geq 1$. Then $F = 0$.*

Proposition. *Let $F : A\text{-mod} \to A\text{-mod}$ be a a selfadjoint functor satisfying $F^k \simeq F^m$ for some $1 \leq m < k$. Then A decomposes as $A = B \oplus C$ where B and C are unital or zero subalgebras of A. There are two possibe cases.*

(i) Assume $k - m$ is odd, then actually $F^2 \simeq F$ and F is the projection on $B\text{-mod}$ composed with the embedding into $A\text{-mod}$.

(ii) Assume $k - m$ is even, then there exists an algebra automorphism $\varphi : B \to B$ such that F acts on $C\text{-mod}$ as zero and on $B\text{-mod}$ as F_φ.

Proposition. *Assume there exists a selfadjoint functor $F : A\text{-mod} \to A\text{-mod}$ satisfying*

$$F^2 \simeq k\,\mathbb{1}_{A\text{-mod}} = \mathbb{1}_{A\text{-mod}} \oplus \ldots \oplus \mathbb{1}_{A\text{-mod}}$$

(k summands). Then actually $k = m^2$ for some natural number m, and there exists a selfadjoint functor $G : A\text{-mod} \to A\text{-mod}$ satisfying $G^2 \simeq \mathbb{1}_{A\text{-mod}}$, such that F is isomorphic to the direct sum of m copies of G. (Conversely each such direct sum produces a selfadjoint square root of $m^2\,\mathbb{1}_{A\text{-mod}}$.)

Before dwelling into that part we motivate the results with a review of the basic theory about (algebraic) categorification, ending with the description of categorification via 2-categories and their 2-representations. This culminates with a review of a recent article [MM10] by Mazorchuk and Miemietz about the 2-representation theory of 2-categories, and we propose a definition of when a 2-representation should be called simple. Finally we show that certain 2-representations are simple.

We have tried to place the necessary algebraic prerequisites (together with, occasionally brief, proofs) in the first chapter. Most of the results from this chapter will be used without further ado.

In the last chapter we also describe a possible direction for future exploration, which was in fact our motivation for the work done in Chapter 4.

Notation

We will denote by $\mathbb{Z}$ the integers and by $\mathbb{N}$ the set of non-negative integers (i.e. including 0). Also unsurprisingly we will denote by $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ the rational numbers, the real numbers and the complex numbers. If a ground field, say $\Bbbk$, is clear from the context we will denote by $\otimes$ the tensor product over $\Bbbk$: $\underset{\Bbbk}{\otimes}$, and by $\dim = \dim_\Bbbk$ the dimension as a $\Bbbk$-module.

Rings and algebras are associative and unital unless explicitly stated otherwise. If R is a ring (or even a semiring) and $n \geq 1$ we denote by $\mathrm{Mat}_n(R)$ the ring of $n \times n$ matrices with entries in R. By R^{opp} we denote the opposite ring and by $R^\times$ we denote the set of units in R.

Denote by $R\text{-Mod}$ the category of arbitrary left R-modules (in some fixed universe), and by $R\text{-mod}$ the category of finitely generated left modules. We also denote by $\text{mod-}R$ the category of finitely generated right R-modules.

If A and B are $\Bbbk$-algebras by an A-B-bimodule we mean a $\Bbbk$-module M, such that M is a left A-module and a right B-module with

$$a(mb) = (am)b$$

We assume that the action of $\Bbbk$ is the same from the left and from the right. It is clear that the category of A-B-bimodules is simply $A \otimes B^{\mathrm{opp}}\text{-mod} = A \underset{\Bbbk}{\otimes} B^{\mathrm{opp}}\text{-mod}$.

If M is a set we denote by $\mathrm{id} = \mathrm{id}_M$ the identity map on M.

If $\mathcal{C}$ is a category we denote by $\mathbb{1}_\mathcal{C}$ the identity functor on $\mathcal{C}$. If $\mathcal{C}$ is clear from the context we may ommit the subscript. If F is functor we denote by Id_F the identity natural transformation of F. If M is an object in $\mathcal{C}$ we freely write $M \in \mathcal{C}$. If also $N \in \mathcal{C}$ we write $\mathrm{Hom}_\mathcal{C}(M, N)$ for the morphisms from M to N.

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Tilst, July 2011 Troels Agerholm

Chapter 1

Basic Algebra

1.1 The Five Lemma

We recall here for easy reference a lemma from homological algebra.

Lemma 1.1 (The five lemma). *Given the commutative diagram (in any abelian category) with exact rows*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle j} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

If i is surjective and f and h are injective then g is injective.
If j is injective and f and h are surjective then g is surjective.
In particular if i, j, f and h are isomorphisms then g is an isomorphism.

1.2 Equivalence of Categories

Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is called an *equivalence of categories* whenever there exists a functor $G : \mathcal{D} \to \mathcal{C}$, and natural isomorphisms $\varepsilon : FG \to \mathrm{id}_{\mathcal{D}}$, $\eta : \mathrm{id}_{\mathcal{C}} \to GF$. $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if, and only if, it is fully faithful (i.e. it induces isomorphisms between homspaces) and essentially surjective (i.e. any object in $\mathcal{D}$ is isomorphic to an object of the form $F(M)$).

Definition 1.2. *Let* $\mathrm{P} \in \mathcal{C}$ *be a projective object in an abelian category. We say that* P *is a* projective generator *of* $\mathcal{C}$ *if for all* $M \in \mathcal{C}$ *there exists an index set* I *such that the direct sum* $\mathrm{P}^{(I)} \in \mathcal{C}$ *and such that there exists an epimorphism*

$$\mathrm{P}^{(I)} \to M \to 0$$

Remark 1.3. Usually one defines P to be a generator if $\mathrm{Hom}_{\mathcal{C}}(\mathrm{P}, \_)$ is faithful. If $\mathcal{C}$ has arbitrary direct sums the definitions are equivalent.

Theorem 1.4. *Let* $\mathcal{C}$ *be an abelian, noetherian category and* P *be a projective generator . Then* $A = \mathrm{End}_{\mathcal{C}}(\mathrm{P})^{opp}$ *is a left noetherian ring and the functor*

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{P}, \_) : \mathcal{C} \to A\text{-mod}$$

is an equivalence of categories. (Here $a \in A$ *acts on* $f \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{P}, M)$ *via* $af = f \circ a$*.)*

Proof in the case $\mathcal{C}$ is a category of modules. We begin by noticing that if $I \subseteq A$ is a left ideal, we can define the module

$$M = M(I) = \sum_{\varphi \in I} \operatorname{Im} \varphi$$

And since M is noetherian and P is projective we have that

$$I = \operatorname{Hom}_{\mathcal{C}}(\mathrm{P}, M(I))$$

The inclusion $\subseteq$ is clear, so let us prove $\supseteq$. Note that we have an epimorphism

$$\mathrm{P}^n \to M(I) \to 0$$

This is because $M(I)$ is noetherian and hence for any $(\varphi_i)_i \subseteq I$ the modules $M_m = \sum_{i=1}^m \operatorname{Im} \varphi_i$ must stabilize. Now if $\psi : \mathrm{P} \to M(I)$, we can factor $\psi = \sum_{i=1}^n \varphi_i \circ a_i = \sum_{i=1}^n a_i \varphi_i$ for some $\varphi_i \in I$ and $a_i \in A$. Thus $\psi \in I$ as we wanted to prove.

Let

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq A$$

be an ascending chain of left ideals in A.

Then $M_i = M(I_i)$ defines an ascending chain of submodules of P. Since this chain stabilizes the same is true for the chain of ideals (by the first argument in the proof). We see that A is a noetherian ring, and it follows that any $\operatorname{Hom}_{\mathcal{C}}(\mathrm{P}, M) \subseteq A$ is finitely generated. The functor

$$\mathbb{V} = \operatorname{Hom}_{\mathcal{C}}(\mathrm{P}, \_\,) : \mathcal{C} \to A\text{-mod}$$

is thus well defined. We claim it is an equivalence of categories.

It is enough to show that for all $X, Y \in \mathcal{C}$ the natural homomorphism

$$f_{X,Y} : \operatorname{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \operatorname{Hom}_A(\mathbb{V}(X), \mathbb{V}(Y))$$

is an isomorphism; and that for all $M \in A\text{-mod}$ there exist $X \in \mathcal{C}$ with $M \simeq \mathbb{V}(X)$.

We fix $Y \in \mathcal{C}$ and set

$$\mathcal{M} = \{X \in \mathcal{C} | f_{X,Y} \text{ is an isomorphism}\}$$

we have to show that $\mathcal{M} = \mathcal{C}$.

Note that $\mathrm{P} \in \mathcal{M}$ since

$$\operatorname{Hom}_{\mathcal{C}}(\mathrm{P}, Y) \longrightarrow \operatorname{Hom}_A(\mathbb{V}(\mathrm{P}), \mathbb{V}(Y)) = \operatorname{Hom}_A(A, \operatorname{Hom}_{\mathcal{C}}(\mathrm{P}, Y))$$

is just the natural isomorphism.

Next we note that since the functors $\operatorname{Hom}_{\mathcal{C}}(\_\,, Y)$ and $\operatorname{Hom}_A(\mathbb{V}(\_), \mathbb{V}(Y))$ both preserve finite direct sums, $\mathcal{M}$ is closed under finite direct sums.

Finally $\mathcal{M}$ is closed under the formation of cokernels. If namely

$$Q \to S \to T \to 0$$

is exact and $Q, S \in \mathcal{M}$ then the five lemma ensures that also $f_{X,T}$ is an isomorphism.

Now since P is a projective generator every object $X \in \mathcal{C}$ fits into an exact sequence

$$\mathrm{P}^m \to \mathrm{P}^n \to X \to 0$$

hence $\mathcal{M} = \mathcal{C}$. (Again the direct sums can be chosen to be finite since the modules are noetherian.)

It remains to show that $\mathbb{V}$ is essentially surjective, so let $M \in A\text{-mod}$. Choose a finite presentation (A is noetherian)

$$A^m \to A^n \to M \to 0$$

We have shown that

$$\operatorname{Hom}_{\mathcal{C}}(\mathrm{P}^m, \mathrm{P}^n) \simeq \operatorname{Hom}_A(A^m, A^n)$$

hence M is the image under $\mathbb{V}$ of the cokernel of a map $\mathrm{P}^m \to \mathrm{P}^n$ (here we use that $\mathbb{V}$ is exact since P is projective). $\square$

Remark 1.5. For a proof using limits instead of sums of modules (thereby proving the theorem in general) the reader should consult [Lam66]. The idea of the proof is the same as here, so only notation differs.

Proposition 1.6. *Let A and B be rings. Any functor $F : A\text{-mod} \to B\text{-mod}$ which preserves cokernels and is compatible with direct sums has the form $X \underset{A}{\otimes} \_$ for some B-A-bimodule X. In fact the module $X = F(A)$ (which is a right A-module from the right multiplication in A) works.*

Proof. We follow [Bas68, Chapter II, Theorem 2.3]

Let us make the last statement in the claim more precise. Let $X = F(A)$, then

$$A^{\mathrm{opp}} \simeq \mathrm{Hom}_A(A, A) \xrightarrow{F_A} \mathrm{Hom}_B(X, X)$$

where the map F_A is $a \mapsto F(\rho_a)$ (ρ_a is right multiplication with a in A), this gives X a structure as a *right* A-module, in such a way that it commutes with the right B-action (i.e. X is a bimodule).

More generally for any left A-module, M, we have

$$M \simeq \mathrm{Hom}_A(A, M) \xrightarrow{F_M} \mathrm{Hom}_B(X, F(M))$$

This (composed) map is A-linear (here the action of A on $\mathrm{Hom}_B(X, F(M))$ arises from our action of A on X).

Now we have a canonical isomorphism

$$\mathrm{Hom}_A(M, \mathrm{Hom}_B(X, F(M))) \simeq \mathrm{Hom}_B(X \underset{A}{\otimes} M, F(M))$$

Denote by g_M the image of F_M under this isomorphism. Because F_M is natural in M the same is true for g_M and thus we get a natural transformation of functors

$$g : X \underset{A}{\otimes} \_ \to F$$

Here both $X \underset{A}{\otimes} \_$ and F preserve cokernels and direct sums. Because we have the standard isomorphism

$$g_A : X \underset{A}{\otimes} A \simeq X = F(A)$$

we get as in the proof of **Theorem 1.4** that g is an isomorphism for all objects in $A\text{-mod}$. This was what we needed to proof. □

The proposition and the theorem says that when when considering exact functors of noetherian, abelian categories one can equivalently study bimodules. The correspondence is very well behaved with respect to most constructions, e.g. a decomposition of functors corresponds to a decomposition of bimodules, or more generally a natural transformations of functors corresponds to morphisms of bimodules. We will use these results without further ado.

1.3 Quotient Categories

We follow [And11] and [Soe00], see also [Gab62]. Let $\mathcal{C}$ be an abelian, artinian category (i.e. all objects have finite length). We specify a (full) subcategory by choosing which simple objects it contains. Let T be a subset of a set of representatives of isomorphism classes of simple objects in $\mathcal{C}$, and let $\mathcal{C}_T$ be the full subcategory consisting of all objects in $\mathcal{C}$ with all composition factors isomorphic to some element of T.

The subcategory $\mathcal{C}_T$ is a *Serre subcategory*, i.e. whenever

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

is a short exact sequence in $\mathcal{C}$ then M belongs to $\mathcal{C}_T$ precisely when both K and N belong to $\mathcal{C}_T$.

Given such a Serre subcategory it is possible to construct the *quotient category* $\mathcal{C}/\mathcal{C}_T$ as follows: We let the objects of $\mathcal{C}/\mathcal{C}_T$ be the objects of $\mathcal{C}$. The homomorphisms are more complicated to describe. For any $M \in \mathcal{C}$ we define

$$M^- = \text{ the smallest submodule of } M \text{ such that } M/M^- \in \mathcal{C}_T$$

and

$$M^+ = \text{ the largest submodule of } M \text{ with } M^+ \in \mathcal{C}_T$$

Now we define

$$\operatorname{Hom}_{\mathcal{C}/\mathcal{C}_T}(M,N) = \operatorname{Hom}_{\mathcal{C}}((M^- + M^+)/M^+, (N^- + N^+)/N^+)$$

Remark 1.7. The definition of $\mathcal{C}/\mathcal{C}_T$ is a special case of a more general construction where one defines

$$\operatorname{Hom}_{\mathcal{C}/\mathcal{C}_T}(M,N) = \varinjlim_{K,Q} \operatorname{Hom}_{\mathcal{C}}(K,Q)$$

where the limit is taken over all subobjects $K \subseteq M$ with $M/K \in \mathcal{C}_T$ (more precisely: the cokernel of the inclusion) and all quotients $\pi : M \twoheadrightarrow Q$ with $\ker \pi \in \mathcal{C}_T$.

This can be seen by checking that

$$\operatorname{Hom}_{\mathcal{C}}((M^- + M^+)/M^+, (N^- + N^+)/N^+) \simeq \operatorname{Hom}_{\mathcal{C}}(M^-, N/N^+)$$

$\mathcal{C}/\mathcal{C}_T$ is an abelian category and we have an exact functor $Q : \mathcal{C} \to \mathcal{C}/\mathcal{C}_T$. Note that if $M \in \mathcal{C}_T$ then $QM \simeq 0$ in $\mathcal{C}/\mathcal{C}_T$. The functor Q has the following universal property: If $\mathcal{D}$ is an abelian category and $F : \mathcal{C} \to \mathcal{D}$ with $FM \simeq 0$ for all $M \in \mathcal{C}_T$, then F factors uniquely as $F = F' \circ Q$ where $F' : \mathcal{C}/\mathcal{C}_T \to \mathcal{D}$ is an exact functor.

Definition 1.8. *Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor of abelian categories. We say that F is a* quotient functor *if the induced functor $\widetilde{F} : \mathcal{C}/\ker F \to \mathcal{D}$ is an equivalence of categories.*

Note $Q : \mathcal{C} \to \mathcal{C}/\mathcal{C}_T$ is a quotient functor, i.e. $\widetilde{Q} : \mathcal{C}/\ker Q \to \mathcal{C}/\mathcal{C}_T$ is an equivalence. This follows from the fact that $\ker Q = \mathcal{C}_T$.

Proposition 1.9. *Let (F, G) be a pair of adjoint functors between abelian, artinian categories $\mathcal{C}$ and $\mathcal{D}$. Assume that $\mathcal{M} \subseteq \mathcal{C}$ and $\mathcal{N} \subseteq \mathcal{D}$ are Serre subcategories and that $F\mathcal{M} \subseteq \mathcal{N}$ and $G\mathcal{N} \subseteq \mathcal{M}$, then (F, G) induces a pair of functors between the quotient categories, and this pair is also a pair of adjoint functors.*

Proof. We claim that $(FM)^+ = F(M^+)$ and $(FM)^- = F(M^-)$ and similarly for G. It follows that

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{C}/\mathcal{M}}(M, GN) &= \operatorname{Hom}_{\mathcal{C}}((M^- + M^+)/M^+, (G(N)^- + G(N)^+)/G(N)^+) \\
&= \operatorname{Hom}_{\mathcal{C}}((M^- + M^+)/M^+, G(N^- + N^+)/G(N^+)) \\
&= \operatorname{Hom}_{\mathcal{C}}((M^- + M^+)/M^+, G((N^- + N^+)/N^+)) \\
&\simeq \operatorname{Hom}_{\mathcal{D}}(F((M^- + M^+)/M^+), (N^- + N^+)/N^+) \\
&= \operatorname{Hom}_{\mathcal{D}/\mathcal{N}}(FM, N)
\end{aligned}$$

It remains to prove the claim. Let us first prove the claim in case $M^- = 0$. If $Q \subseteq FM$ is a submodule with $0 \neq Q \in \mathcal{N}$ then we have $0 \neq \operatorname{Hom}_{\mathcal{D}}(Q, FM) \simeq \operatorname{Hom}_{\mathcal{C}}(GQ, M)$, but $GQ \in \mathcal{M}$ so $GQ \subseteq M^- = 0$ which is a contradiction, hence $Q = 0$ and $(FM)^- = 0 = F0 = F(M^-)$ as claimed.

Let now M be arbitrary and note first that $F(M/M^-) \simeq F(M)/F(M^-) \in \mathcal{N}$ since F is exact. We see that $(F(M/M^-))^- = 0$, but clearly $(FM)^- \subseteq F(M^-)$ and

$$\begin{aligned}
\mathcal{N} \ni FM/(FM)^- &\simeq \big(FM/F(M^-)\big)/\big(F(M^-)/(FM)^-\big) \\
&\simeq \big(F(M/M^-)\big)/\big(F(M^-)/(FM)^-\big)
\end{aligned}$$

hence $F(M^-)/(FM)^- \subseteq (F(M/M^-))^- = 0$ and $(FM)^- = F(M^-)$.

The proof of $(FM)^+ = F(M^+)$ is entirely similar. □

1.4 2-categories

In this section we recall the basic definition of 2-categories.

1.4.1 Monoidal Categories

Let $\mathcal{M}$ be a category. We will define when a 5-tuple $(\otimes, I, \alpha, \lambda, \rho)$ should be called a *monoidal structure* on $\mathcal{M}$. Most prominently we require

$$\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$$

to be a bifunctor, and $I \in \mathcal{M}$ is an object called the *identity object*. As always we will write e.g. $M \otimes N$ instead of the awkward looking $\otimes(M, N)$. The last three ingredients are natural transformations. The first one is called the *associator* and is a natural isomorphism

$$\alpha : (\_ \otimes \_) \otimes \_ \to \_ \otimes (\_ \otimes \_)$$

Here $(\_ \otimes \_) \otimes \_$ should be interpreted as $\otimes(\otimes \times \mathbb{1})$ in light of the remark above. For any objects M, N, P we thus get a morphism

$$\alpha_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P)$$

The other natural transformations make sure that I acts as the identity from both right and left. Hence they are natural isomorphisms

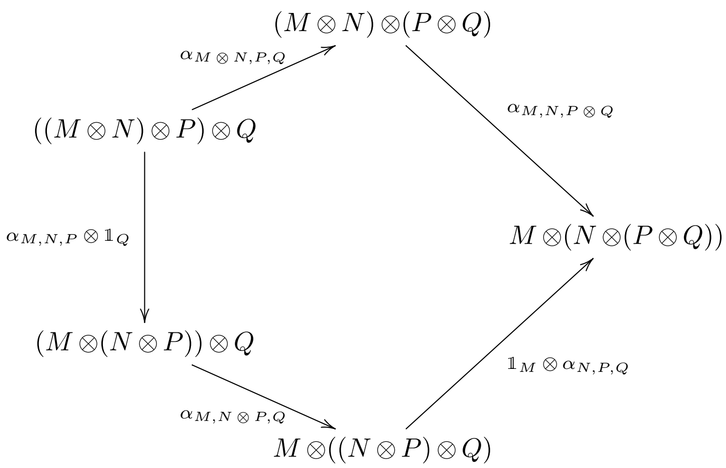
$$\lambda : I \otimes \_ \to \mathbb{1}$$

and

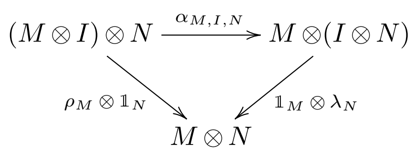
$$\rho : \_ \otimes I \to \mathbb{1}$$

(Here we use the fact that $\otimes$ is a bifunctor, such that for any object M we have functors $M \otimes \_$ and $\_ \otimes M$.)

These transformations are subject to the following conditions: The pentagon axiom: Let M,N,P,$Q \in \mathcal{M}$ be objects. The pentagon axiom states that the following diagram commutes.



The triangle axiom:



If a category admits a (fixed) monoidal structure, we call it a monoidal category and often we write just $(\mathcal{M}, \otimes)$, and suppress the rest of the structure. Sometimes a monoidal category is called a *tensor category*.

The category of categories (where the morphisms are functors) is a monoidal category where $\otimes = \times$ is the obvious generalisation of the cartesian product and $I = \mathbb{1}$ is the terminal category with one object and one morphism.

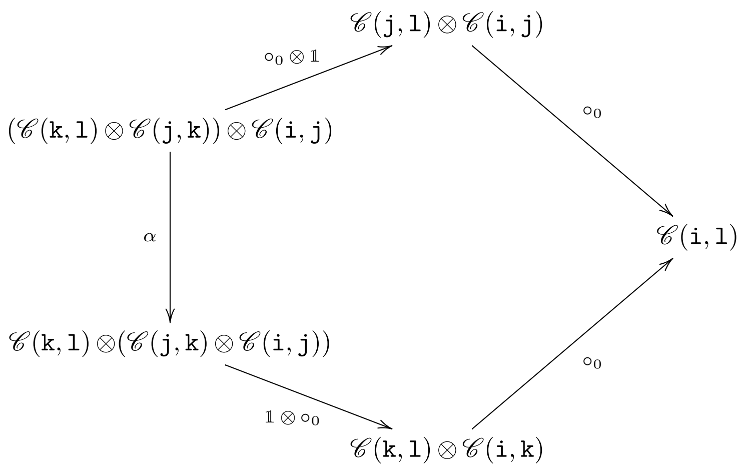
1.4.2 Enriched Categories

An *enriched* category is a category where the collection of homomorphisms between any two fixed objects carries additionally structure. E.g. for any $\Bbbk$-algebra, A, the category of A-modules is enriched over the category of $\Bbbk$-modules. Similarly any (pre-)additive category is enriched over the category of abelian groups.

Definition 1.10. *We say that $\mathscr{C}$ has the structure of an* enriched category *over the monoidal category $(\mathcal{M}, \otimes)$ if we have a proper class (set) of objects $\mathbf{Ob}(\mathscr{C})$, and for any objects $\mathtt{i}, \mathtt{j} \in \mathbf{Ob}(\mathscr{C})$ an object (sometimes called a hom-object) $\mathscr{C}(\mathtt{i}, \mathtt{j}) \in \mathcal{M}$. We want $\mathscr{C}$ to resemple a category, hence we require that the hom-objects are equipped with a (horizontal) composition, i.e. for any $\mathtt{i}, \mathtt{j}, \mathtt{k} \in \mathbf{Ob}(\mathscr{C})$ we have a morphism*

$$\circ_0 : \mathscr{C}(\mathtt{j}, \mathtt{k}) \otimes \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$$

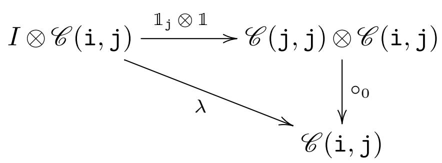
which we require to be associative in the sense that the following commutes (here also $\mathtt{l} \in \mathbf{Ob}(\mathscr{C})$)



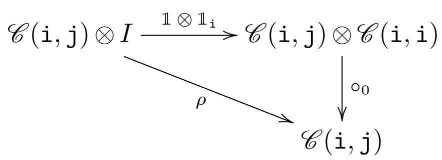
Additionally we want the composition to be unital specifically we require a morphism

$$\mathbb{1}_{\mathtt{i}} : I \to \mathscr{C}(\mathtt{i}, \mathtt{i})$$

such that the following commutes



and



Remark 1.11. If $\mathcal{M}$ is a concrete category (where the objects are sets) and if $\otimes$ is the usual cartesian product $\times$, or if the underlying set of $M \otimes N$ is (a quotient of) $M \times N$, then $\mathscr{C}$ is a category.

Let $\Bbbk$ be a field. The category of (arbitrary) $\Bbbk$-vector spaces $\Bbbk\text{-Mod}$ is a monoidal category with the usual tensor product $\_ \underset{\Bbbk}{\otimes} \_$. A category which is enriched over $\Bbbk\text{-Mod}$ is called a $\Bbbk$-linear category.

1.4.3 2-categories

Denote by $\mathscr{K}$ the category of small categories (in some fixed universe) and natural transformations. $\mathscr{C}$ is a (strict) 2-*category* if it is an enriched category over $(\mathscr{K}, \times)$. The enriched structure automatically makes $\mathscr{C}$ into a category. $\mathscr{C}$ consists of the following data:

a) 0-cells, $\mathbf{Ob}(\mathscr{C})$, often denoted by points or as $\mathtt{i}, \mathtt{j}, \ldots$, referred to as objects.

b) 1-cells: for any 0-cells $\mathtt{i}, \mathtt{j}$ a category $\mathscr{C}(\mathtt{i}, \mathtt{j}) \in \mathbf{Ob}(\mathscr{C})$. Objects in this category are denoted by arrows: $F, G, \ldots : \mathtt{i} \to \mathtt{j}$ and are often referred to as 1-morphisms.

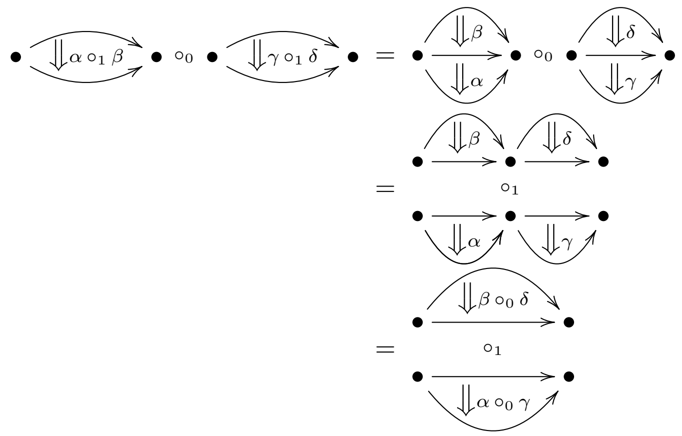
c) 2-cells, sometimes denoted by double arrows $\alpha, \beta, \ldots : F \Rightarrow G$. These are the morphisms in the categories $\mathscr{C}(\mathtt{i}, \mathtt{j})$, and are usually referred to as 2-morphisms.

Since $\mathscr{C}$ carries its own structure as a category in addition to the structure on the hom-objects, we write $\circ_1$ instead of $\circ$ for the (vertical) composition in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ in order to distinguish it from the horisontal composition $\circ_0$. We will often write the action of $\circ_0$ on 1-morphisms merely as $\circ$. The identity 1-morphism in $\mathscr{C}(\mathtt{i}, \mathtt{i})$ with respect to $\circ_0$ will be denoted by $\mathbb{1}_{\mathtt{i}}$. If $F \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ is a 1-morphism we denote the identity 2-morphism of F by Id_F.

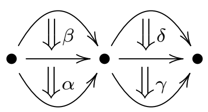
Now $\circ_0$ is a morphism in $\mathscr{K}$, i.e. a functor. Therefore we have the following coherence between the horizontal and vertical compositions. Let α, β, γ and δ be 2-morphisms. If the compositions make sense then

$$(\alpha \circ_1 \beta) \circ_0 (\gamma \circ_1 \delta) = (\alpha \circ_0 \gamma) \circ_1 (\beta \circ_0 \delta)$$

Or as a series of diagrams:



Because of this equality one often writes either of the diagrams as



Let us introduce some notation: Let F be a 1-morphism in the 2-category $\mathscr{C}$, say $F \in \mathscr{C}(\mathtt{i}, \mathtt{j})$. If α is a 2-morphism in $\mathscr{C}(\mathtt{j}, \mathtt{k})$ it is possible to make the composition $\alpha \circ_0 \mathrm{Id}_F$. We denote this 2-morphism by α_F. Similarly if β is a 2-morphism in $\mathscr{C}(\mathtt{k}, \mathtt{i})$ we define $F(\beta) = \mathrm{Id}_F \circ_0 \beta$. Sometimes we may abuse notation even further by denoting Id_F by F.

1.4.4 Morphisms of 2-categories: 2-functors

Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A 2-functor $\mathbb{V} : \mathscr{C} \to \mathscr{D}$ consists of three mappings, a mapping of objects, a mapping of 1-morphisms and a mapping of 2-morphisms. These mappings should preserve all of the structure of the 2-category $\mathscr{C}$ in the usual way. In particular the image of $\mathscr{C}$ under $\mathbb{V}$ in $\mathscr{D}$ becomes a 2-subcategory (i.e. it is a 2-category with respect to the induced structure).

If $\mathbb{W} : \mathscr{C} \to \mathscr{D}$ is a 2-functor (parallel to $\mathbb{V}$) a 2-natural transformation $\mathtt{N} : \mathbb{V} \to \mathbb{W}$ is given by a 1-morphism $\mathtt{N}_\mathtt{i} : \mathbb{V}(\mathtt{i}) \to \mathbb{W}(\mathtt{i})$ for any object $\mathtt{i} \in \mathscr{C}$ with the condition that for any 1-morphisms $F, G \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ and any 2-morphism $\alpha : F \to G$ we have

$$\begin{tikzcd} \mathbb{V}(\mathtt{i}) \arrow[r, bend left, "\mathbb{V}(F)"] \arrow[r, bend right, "\mathbb{V}(G)"'] \arrow[r, phantom, "\Downarrow \mathbb{V}(\alpha)"] & \mathbb{V}(\mathtt{j}) \arrow[r, "\mathtt{N}_\mathtt{j}"] & \mathbb{W}(\mathtt{j}) \end{tikzcd} = \begin{tikzcd} \mathbb{V}(\mathtt{i}) \arrow[r, "\mathtt{N}_\mathtt{i}"] & \mathbb{W}(\mathtt{i}) \arrow[r, bend left, "\mathbb{W}(F)"] \arrow[r, bend right, "\mathbb{W}(G)"'] \arrow[r, phantom, "\Downarrow \mathbb{W}(\alpha)"] & \mathbb{W}(\mathtt{j}) \end{tikzcd}$$

This should hold in the strictest sense, which means that we have an equality of 1-morphisms

$$\mathbb{W}(F) \circ \mathtt{N}_i = \mathtt{N}_j \circ \mathbb{V}(F)$$

for any $F \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, and with the notation from above we have an equality of 2-morphisms

$$\mathbb{W}(\alpha)_{\mathtt{N}_i} = \mathtt{N}_j \mathbb{V}(\alpha)$$

It is also possible to consider morphisms of 2-natural transformations. These are called *modifications* and are defined as follows. If $\mathtt{N}, \mathtt{M} : \mathbb{V} \to \mathbb{W}$ are parallel 2-natural transformations a modification $\mathfrak{M} : \mathtt{N} \to \mathtt{M}$ is given by a map which to each object $\mathtt{i}$ in $\mathscr{C}$ assigns a 2-morphism $\mathfrak{M}_\mathtt{i} : \mathtt{N}_\mathtt{i} \to \mathtt{M}_\mathtt{i}$ such that for any 1-morphisms F,$G \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ and any 2-morphism $\alpha : F \to G$ in $\mathscr{C}$ we have

$$\begin{tikzcd} \mathbb{V}(\mathtt{i}) \arrow[r, bend left, "\mathbb{V}(F)"] \arrow[r, bend right, "\mathbb{V}(G)"'] \arrow[r, phantom, "\Downarrow \mathbb{V}(\alpha)"] & \mathbb{V}(\mathtt{j}) \arrow[r, bend left, "\mathtt{N}_\mathtt{j}"] \arrow[r, bend right, "\mathtt{M}_\mathtt{j}"'] \arrow[r, phantom, "\Downarrow \mathfrak{M}_\mathtt{j}"] & \mathbb{W}(\mathtt{j}) \end{tikzcd} = \begin{tikzcd} \mathbb{V}(\mathtt{i}) \arrow[r, bend left, "\mathtt{N}_\mathtt{i}"] \arrow[r, bend right, "\mathtt{M}_\mathtt{i}"'] \arrow[r, phantom, "\Downarrow \mathfrak{M}_\mathtt{i}"] & \mathbb{W}(\mathtt{i}) \arrow[r, bend left, "\mathbb{W}(F)"] \arrow[r, bend right, "\mathbb{W}(G)"'] \arrow[r, phantom, "\Downarrow \mathbb{W}(\alpha)"] & \mathbb{W}(\mathtt{j}) \end{tikzcd}$$

again in the strictest sense.

1.5 Finitary 2-categories

We present here some definitions from [MM10].

Definition 1.12. *Let $\mathscr{C}$ be a 2-category. We say that $\mathscr{C}$ is* $\Bbbk$-finitary *if the following is satisfied:*

(I) The class of objects (0-cells) is a finite set

(II) For any objects $\mathtt{i}$,$\mathtt{j}$ the category $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is enriched over the category $\Bbbk$-mod *(it is* $\Bbbk$-linear*) with finitely many isomorphism classes of indecomposable objects. This category should also satisfy that given an idempotent $e : F \to F$ for some 1-morphism $F \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ there exists a 1-morphism $G \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ and 2-morphisms $r : F \to G$ and $s : G \to F$ such that $e = s \circ r$ and $\mathrm{Id}_G = r \circ s$. Additionally the (horizontal) composition of 1-morphisms should be biadditive.*

(III) The identity 1-morphism $\mathbb{1}_\mathtt{i} \in \mathscr{C}(\mathtt{i}, \mathtt{i})$ is indecomposable for any $\mathtt{i}$.

Definition 1.13. *A 2-ideal $\mathscr{I}$ in a $\Bbbk$-finitary 2-category $\mathscr{C}$ is a collection of 2-morphisms with the property: for any 2-morphisms $\alpha \in \mathscr{I}$ and $\alpha' \in \mathscr{C}$ also $\alpha \circ \alpha'$ and $\alpha \circ \alpha' \in \mathscr{I}$ for both $\circ = \circ_0$ and $\circ_1$, whenever any of the compositions make sense. We also require that for any 1-morphisms F, G*

$$\mathscr{I}(\mathtt{i}, \mathtt{j})(F, G) \subseteq \mathscr{C}(\mathtt{i}, \mathtt{j})(F, G)$$

is in fact a subspace.

Proposition 1.14. *Let $\mathscr{I}$ be a 2-ideal in the $\Bbbk$-finitary 2-category $\mathscr{C}$. The quotient 2-category $\mathscr{C}/\mathscr{I}$ defined with the same objects and 1-morphisms as $\mathscr{C}$, but where the 2-morphisms are given by the quotient*

$$\mathscr{C}(\mathtt{i}, \mathtt{j})(F, G)/\mathscr{I}(\mathtt{i}, \mathtt{j})(F, G)$$

is well-defined and is a 2-category.

We want our 1-morphisms to model functors and hence we define when a 1-morphism has an *adjoint*.

Definition 1.15. *A quadruple $(F, G, \varepsilon, \eta)$ in a 2-category $\mathscr{C}$ consisting of two anti-parallel 1-morphisms $F \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, $G \in \mathscr{C}(\mathtt{j}, \mathtt{i})$, and two 2-morphisms*

$$\varepsilon : F \circ G \to \mathbb{1}_{\mathtt{j}}, \qquad \eta : \mathbb{1}_{\mathtt{i}} \to G \circ F$$

such that

$$\varepsilon_F \circ_1 F(\eta) = \mathrm{Id}_F, \qquad G(\varepsilon) \circ_1 \eta_G = \mathrm{Id}_G$$

is called an (internal) adjunction *in $\mathscr{C}$. As usual (F, G) will be called an adjoint pair, and we say that F is left adjoint to G and that G is right adjoint to F.*

Remark 1.16. If $\mathscr{C}$ is the 2-category of small categories, functors and natural transformations an adjunction in $\mathscr{C}$ is nothing but a pair of adjoint functors with specified adjunction morphisms.

The following definition essentially just states that any 1-morphism should have a biadjoint.

Definition 1.17. *A $\Bbbk$-finitary 2-category is called* fiat *(i.e.* f*inatary with* i*nvolution and* a*djunc*t*ions) if there exists an involution $* : \mathscr{C} \to \mathscr{C}$ which is object preserving and an anti-automorphism of $\mathscr{C}$. In other words we require $*(\mathtt{i}) = \mathtt{i}$ for any 1-morphisms $\mathtt{i}$, and that $*$ is a functor for any objects $\mathtt{i}$,$\mathtt{j}$:*

$$* : \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{j}, \mathtt{i})$$

such that for any 1-morphism $F \in \mathscr{C}(\mathtt{i}, \mathtt{j})$: $F^* = *(F)$ we have $(F^*)^* = F$ (thereby justifying the term involution), and with

$$(F \circ G)^* = G^* \circ F^* \qquad \mathbb{1}_{\mathtt{i}}^* = \mathbb{1}_{\mathtt{i}}$$

for any composable 1-morphisms and any object $\mathtt{i}$. (We require the similar relations for 2-morphisms.)

In addition to the existence of an involution we also need adjunctions: for any 1-morphism F there should exist ε and η such that $(F, F^*, \varepsilon, \eta)$ is an adjunction.

Example 1.18. Consider the algebra of dual numbers $\mathbf{D} = \mathbb{C}[X]/(X^2)$ and the category of finitely generated $\mathbf{D}$-modules, $\mathscr{D} = \mathbf{D}\text{-mod}$. Denote by $\mathscr{C}$ the category of finite dimensional complex vector spaces. Consider the following picture

$$\mathscr{D} = \mathtt{i} \underset{\mathrm{Ind}=\mathbf{D} \underset{\mathbb{C}}{\otimes} \_}{\overset{\mathrm{Res}={}_{\mathbb{C}}\mathbf{D} \underset{\mathbf{D}}{\otimes} \_}{\rightleftarrows}} \mathtt{j} = \mathscr{C}$$

Define $F = \mathrm{Ind} \circ \mathrm{Res} = \mathbf{D} \underset{\mathbb{C}}{\otimes} \mathbf{D} \underset{\mathbf{D}}{\otimes} \_$. We calulate

$$F^2 = \mathbf{D} \underset{\mathbb{C}}{\otimes} \mathbf{D} \underset{\mathbf{D}}{\otimes} \mathbf{D} \underset{\mathbb{C}}{\otimes} \mathbf{D} \underset{\mathbf{D}}{\otimes} \_ = F \oplus F$$

since $\mathbf{D} \underset{\mathbf{D}}{\otimes} \mathbf{D} = \mathbb{C} \oplus \mathbb{C}$ considered as a $\mathbb{C}$-bimodule. Therefore it makes sense to denote by $\mathscr{S}_2$ the 2-category with one object $\mathtt{i}$ (identified with $\mathscr{D}$) and with 1-morphisms all direct sums of F and $\mathbb{1}_{\mathtt{i}} = \mathbb{1}_{\mathcal{D}}$, 2-morphisms are just all posible natural transformation of the 1-morphisms. The indecomposable 1-morphisms are F and $\mathbb{1}_{\mathtt{i}}$. F is selfadjoint (because $\mathbf{D}$ is a symmetric algebra) hence the identity may be used as the involution * and makes $\mathscr{S}_2$ into a fiat 2-category.

As an example where * is not the identity one could take the 2-category with two objects $\mathtt{i}$ and $\mathtt{j}$ as in the image above, and consider the restriction and induction functors. These are biadjoint since $\mathbf{D}$ is a symmetric algebra. We will later see further examples of fiat categories with non-trivial involutions.

1.6 Radicals of Artinian Rings

Let us in this section recall som basic facts about the Jacobson radical of an artinian ring and the radical of a module over such a ring. We follow to some extend [Pas04].

Let A be a left artinian ring (i.e. all descending chains of left ideals stabilize), and let M be a finitely generated A-module. Define

$$S(M) = \{N \subseteq M | M/N \text{ is a simple } A\text{-module}\}$$

Then

$$\mathrm{Rad}(M) = \bigcap_{N \in S(M)} N$$

is the intersection of all maximal A-submodules.

In particular we have the (Jacobson) radical of A, $\mathrm{Rad}(A)$, the intersection of all maximal left ideals in A.

If $\mathrm{L} = A/I$ is a simple A-module (with $I \subseteq A$ a maximal ideal) then clearly $\mathrm{Rad}(A) \subseteq I$ and hence $\mathrm{Rad}(A)\mathrm{L} = 0$. More generally $\mathrm{Rad}(A)$ annihilates any semisimple A-module.

The radical is a nilpotent ideal.

Proposition 1.19. *Let A be a left artinian ring and denote by $J = \mathrm{Rad}(A)$ its Jacobson radical. Then for some $k > 0$ we have $J^k = 0$.*

We need a simple lemma in the proof.

Lemma 1.20. *Let $x \in J$ then $1 - x$ is invertible.*

Proof. Consider the ideal $I = A(1-x) \subseteq A$. If I is a proper ideal then by Zorn's lemma there exists some maximal ideal $I \subseteq M \subseteq A$. Then $x \in J = \mathrm{Rad}(A) \subseteq M$ and $1 - x \in I \subseteq M$. It follows that $1 \in M$, a contradiction. Therefore $I = A$ and $1 - x$ has a left inverse.

Write the left inverse of $1 - x$ as $1 - y$. Then $1 = (1-y)(1-x) = 1 - y - x + yx$, hence $y = yx - x \in J$. It follows that also $1 - y$ has a left inverse. Therefore $1 - y$ is invertible with inverse $1 - x$. In particular $1 - x$ is invertible. □

Proof of ***Proposition 1.19****.* Consider the descending chain of left ideals

$$J \supseteq J^2 \supseteq J^3 \supseteq \dots$$

Since A is artinian we have $J^k = J^{k+1} = \dots$ for some $k > 0$. We claim $J^k = 0$. Assume to the contrary that $J^k \neq 0$. Since $J^k J = J^{k+1} = J^k \neq 0$ we may choose a minimal ideal $I \subseteq J$ such that $J^k I \neq 0$. For some $x \in I$ we have $J^k x \neq 0$. Then $J^k x \subseteq I$ is an ideal and

$$J^k(J^k x) = J^{2k} x = J^k x \neq 0$$

By the minimality of I we see that $J^k x = I$. Write $x = ax$ for some $a \in J^k \subseteq J$. Then $(1-a)x = 0$, but $1 - a$ has an inverse, and hence $x = 0$, a contradiction. □

Corollary 1.21. *The Jacobson radical consists of nilpotent elements.*

An ideal consisting of nilpotent elements is called a *nil* ideal.

Define the *nilradical* of A as

$$\mathrm{Nil}(A) = \sum_{I \in N(A)} I$$

where

$$N(A) = \{I \subseteq A | I \text{ is a two-sided nil ideal}\}$$

We want to prove that the nilradical and the Jacobson radical is the same (for an artinian ring). Let us first present a useful lemma.

Lemma 1.22. *Let $I \subseteq A$ be a nilpotent left ideal. Then IA is a nilpotent twosided ideal (hence a nil ideal).*

Proof. Choose $k > 0$ such that $I^k = 0$. Then $(IA)^k = I(AI)^{k-1}A = II^{k-1}A = I^kA = 0$. $\square$

In particular we see that $\mathrm{Rad}(A)A$ is a nil two-sided ideal. It follows that $\mathrm{Rad}(A) \subseteq \mathrm{Rad}(A)A \subseteq \mathrm{Nil}(A)$.

Proposition 1.23. *Let A be a left artinian ring. Then* $\mathrm{Rad}(A) = \mathrm{Nil}(A)$.

Proof. We need to prove that $\mathrm{Nil}(A) \subseteq \mathrm{Rad}(A)$. Let $I \in N(A)$ and $M \subseteq A$ be a maximal ideal. We are done if we show that $I \subseteq M$. If this is not the case then $M \subsetneq I + M \subseteq A$, and since M is maximal we see that $A = I + M$. Write $1 = x + m$ where $m \in M$ and $x \in I$. Then $m = 1 - x$ is a unit (since x is nilpotent), a contradiction. $\square$

Corollary 1.24. *The radical* $\mathrm{Rad}(A) \subseteq A$ *is a two-sided ideal.*

The next proposition is about radicals of A-modules.

Proposition 1.25. *If A is an artinian ring and* $\mathrm{Rad}(A)$ *its radical, then for any finitely generated left A-module, M, we have*

a) $\mathrm{Rad}(M/\mathrm{Rad}(M)) = 0$

b) If $\mathrm{Rad}(M) = 0$ *then M is semisimple.*

c) The module $M/\mathrm{Rad}(M)$ *is a semisimple module, in fact* $\mathrm{Rad}(M)$ *is the (unique) minimal submodule, $N \subseteq M$, with M/N semisimple.*

d) $\mathrm{Rad}(A)M \subseteq \mathrm{Rad}(M)$

Proof. First we prove a). Well clearly

$$\begin{aligned} S(M/\mathrm{Rad}(M)) &= \{N/\mathrm{Rad}(M) | \mathrm{Rad}(M) \subseteq N \subseteq M, N \in S(M)\} \\ &= \{N/\mathrm{Rad}(M) | N \in S(M)\} \end{aligned}$$

since any $N \in S(M)$ contains $\mathrm{Rad}(M)$. It follows that

$$\mathrm{Rad}(M/\mathrm{Rad}(M)) = \bigcap_{N \in S(M)} N/\mathrm{Rad}(M) = \Big(\bigcap_{N \in S(M)} N\Big)/\mathrm{Rad}(M) = \mathrm{Rad}(M)/\mathrm{Rad}(M) = 0$$

For the proof of b) let us assume that $\mathrm{Rad}(M) = 0$. Consider the collection of submodules of M given by

$$\mathcal{M} = \{N_1 \cap N_2 \cap \ldots \cap N_r | r \in \mathbb{N}, N_i \in S(M)\}$$

We see that $M = \cap_{i \in \emptyset} \in \mathcal{M}$, hence $\mathcal{M}$ is non-empty and has a smallest element (since M is finitely generated, hence artinian)

$$K = N_1 \cap N_2 \cap \ldots \cap N_r$$

with no submodules belonging to $\mathcal{M}$. We claim that $K = 0$. Assume to the contrary that $k \in K \setminus \{0\}$. Then $k \notin \mathrm{Rad}(M)$ and hence there exists $N \in S(M)$ with $k \notin N$. But then

$$K \supsetneq K \cap N \in \mathcal{M}$$

contradicting the definition of K. Because $K = 0$ we can define

$$\varphi : M \to M/N_1 \oplus \ldots \oplus M/N_r$$

given by the canonical projections on each summand. The kernel of φ is then $K = 0$ and each of M/N_i is simple. As M is a submodule of a semisimple module it is semisimple.

Readily the combination of a) and b) gives the first part of c). For the second statement we define

$$S'(M) = \{N \subseteq M | M/N \text{ is a semisimple } A\text{-module}\}$$

and we claim that

$$\operatorname{Rad}(M) = \bigcap_{N \in S'(M)} N$$

The inclusion $\supseteq$ is obvious, since $S(M) \subseteq S'(M)$ (also we just saw that $\operatorname{Rad}(M) \in S'(M)$). For the other inclusion assume that $m \in \operatorname{Rad}(M)$, i.e. for all maximal submodules $N \subseteq M$ we have $m \in N$. Given $N' \in S'(M)$ we must show that also $m \in N'$. Well M/N' is semisimple hence

$$M/N' \simeq \mathrm{L}_1 \oplus \ldots \oplus \mathrm{L}_r$$

denote the projection on the simple module L_i by π_i. Denote also the canonical projection $M \to M/N'$ by π. Then every $\pi_i \circ \pi$ is surjective, hence

$$\mathrm{L}_i \simeq M/\ker(\pi_i \circ \pi)$$

and $\ker(\pi_i \circ \pi) \in S(M)$ is a maximal submodule. Therefore any $\pi_i \circ \pi(m) = 0$ and we see that $\pi(m) = 0$, hence $m \in N'$.

We see that any $N' \in S'(M)$ satisfies $\operatorname{Rad}(M) \subseteq N'$, and since $M/\operatorname{Rad}(M)$ is semisimple the radical is in fact the minimal element in $S'(M)$.

Finally for the proof of d) we note that since $M/\operatorname{Rad}(M)$ is semisimple it is killed by $\operatorname{Rad}(A)$ as $\operatorname{Rad}(A)$ kills all simple modules. This shows readily that $\operatorname{Rad}(A)M \subseteq \operatorname{Rad}(M)$. □

1.7 Lifting of Idempotents

In this section we recall some results about lifting of orthogonal idempotents in Artin rings. We will only need them for finite dimensional algebras over a field, but the proofs in the general case are so short that we include them here for completeness. The results and proofs are adapted from [Lam76] and [Pas04].

In this section let A denote a left artinian ring.

Proposition 1.26 (Lifting of idempotents modulo nil ideals)**.** *Let $N \subseteq A$ be a two-sided nil ideal. If $x \in A/N$ is an idempotent then there exists an idempotent $e \in A$ such that $x = e + N$.*

Proof. Choose $a \in A$ such that $x = a + N$. Since $x^2 = x$ we have that $a^2 - a \in N$ hence it is nilpotent. Choose $k \geq 2$ such that $(a^2 - a)^k = 0$ (if $k = 1$ works, then $a^2 = a$ and we are done). From the binomial formula

$$(1 - a)^k = 1 - ad$$

for some $d \in A$ which can be expressed as a polynomial in a. It is thus clear that $ad = da$. In the factor ring A/N we get that

$$1 - ad + N = (1 - a)^k + N = (1 - a + N)^k = 1 - a + N \tag{1.1}$$

For the last equality we used that $a+N$ is idempotent hence also $(1+N)-(a+N)$ is idempotent. Furthermore we get

$$0 = (a - a^2)^k = a^k(1 - a)^k = a^k(1 - ad)$$

hence $a^k = a^k(ad)$. Clearly then since a and d commute

$$a^k = a^k(ad) = a^k(ad)^2 = \ldots = a^k(ad)^k = a^{2k}d^k$$

Define now $e = (ad)^k$, we claim that e is an idempotent with $x = e + N$. Well

$$e^2 = (ad)^{2k} = a^{2k}d^k d^k = a^k d^k = e$$

we used again that a and d commute. Also from (1.1)

$$e + N = (ad)^k + N = (ad + N)^k = (a + N)^k = x^k = x$$

as claimed. □

In order to lift a set of pairwise orthogonal idempotents (i.e. a set $\{f_i\}$ with $f_i f_j = \delta_{ij} f_i$ where δ_{ij} is the Kronecker delta) we need the following lemma.

Lemma 1.27. *Let $N \subseteq \mathrm{Rad}(A)$ be a two-sided ideal (necessarily nil since A is artinian). If $f \in A$ is an idempotent and $x = a + N \in A/N$ is an idempotent with both fa and $af \in N$ then there exists an idempotent $e \in A$ such that $e - a \in N$ and $ef = fe = 0$ (i.e. $\{e, f\}$ is a pair of orthogonal idempotents).*

Proof. The idempotent $x \in A/N$ can be lifted to an idempotent $g \in A$. Then $g - a \in N$ and hence $fg + N = fa + N = N$ as well as $gf + N = af + N = N$, so $gf, fg \in N$. Now $gf \in N$ means that gf is nilpotent and hence $1 - gf$ has an inverse $(1 - gf)^{-1} \in A$. Define

$$h = (1 - gf)^{-1} g (1 - gf)$$

Because $g^2 = g$ also $h^2 = h$, and

$$hf = (1 - gf)^{-1} g(f - gf^2) = (1 - gf)^{-1}(gf - g^2 f^2) = 0$$

since $g^2 f^2 = gf$. We also notice that

$$(1 - gf)h = g(1 - gf)$$

and hence

$$h - g = gfh - g^2 f$$

Clearly $g^2 f = g(gf) \in N$ and $gfh \in N$ since N is a two-sided ideal. We see that $h - g \in N$.

Now we are ready to define $e = (1 - f)h$. We see that

$$e^2 = (1 - f)h(1 - f)h = (1 - f)(h - hf)h = (1 - f)(h - 0)h = (1 - f)h^2 = (1 - f)h = e$$

and that

$$ef = (1 - f)hf = 0, \quad fe = (f - f^2)h = 0$$

Finally in A/N we see, using $h - g \in N$, that

$$e + N = h - fh + N = g - fg + N = g + N = x$$

such that e lifts x. □

We are now ready to prove the following proposition.

Proposition 1.28 (Lifting of orthogonal idempotents modulo the radical)**.** *Let $N \subseteq \mathrm{Rad}(A)$ be a two-sided ideal. A (finite or countable) set, $\{f_i | i \in I\}$, of mutually orthogonal idempotents in A/N can be lifted to a set of mutually orthogonal idempotents $\{e_i | i \in I\}$ in A.*

Proof. We do induction over the size of I. The base case is done allready. Assume the theorem to be true for $I = \{1, 2, \ldots, k\}$, we need to prove it for $I' = I \cup \{k + 1\}$. By induction we may choose $\{e_i | i \in I\}$ pairwise orthogonal with $f_i = e_i + N$. Write $f_{k+1} = a + N$. Define $f = e_1 + e_2 + \ldots + e_k$, then $f^2 = f$ and

$$af + N = (ae_1 + N) + (ae_2 + N) + \ldots + (ae_k + N) = f_{k+1} f_1 + f_{k+1} f_2 + \ldots + f_{k+1} f_k = 0$$

and similarly $fa + N = (f + N)(a + N) = (f_1 + \ldots + f_k) f_{k+1} = 0$. Thus both fa and $af \in N$. It follows from **Lemma 1.27** that there exists an idempotent $e_{k+1} = e$ such that $e - a \in N$ and such that $ef = 0 = fe$, i.e. e is a lift of f_{k+1} orthogonal to f. Multiplying with e_j, and using that $e_j f = e_j = f e_j$, we see that

$$ee_j = e(fe_j) = efe_j = 0 = e_j fe = (e_j f)e = e_j e$$

hence e is orthogonal to any e_j. This finishes the proof. □

1.8 The Robinson-Schensted Correspondence

We review briefly the Robinson-Schensted correspondence in terms of an algorithm. We refer the reader to other sources e.g. [Jan83] for details and further properties.

Let $(a_1, \dots, a_n)$ be a (finite) sequence of pairwise distinct integers (we only consider permutations of $\{1, \dots, n\}$ in our application). We want to define a bijection between the set of all such sequences (for fixed n) and the set of pairs (P, Q) of standard Young tableaux of the same shape, where Q is a Young tableau with n boxes filled with the numbers $1, \dots, n$, and P is filled with the numbers $a_1, \dots, a_n$.

We define (P_j, Q_j) inductively and define $P = P_n$ and $Q = Q_n$. Set $P_0 = Q_0 = \emptyset$, the empty tableau. Assume that the numbers $(a_1, \dots, a_{j-1})$ have been inserted and we have tableaux (P_{j-1}, Q_{j-1}) (of the same shape, and with Q a standard tableau filled with $1, \dots, j$). We need to explain how to construct P_j and Q_j by inserting a_j into P_{j-1} and j into Q_{j-1}. We begin by choosing the first row of P_{j-1}. If this row contains an element larger than a_j, we place a_j at the place where the smallest entry x greater than a_j was originally placed (we remove x), and we continue by choosing the next row and inserting x into this in the same way. If there is no entry greater than a_j we place a_j at the end of the row. When we have chosen an empty row we simply place our element in the first column of this row. Now we have constructed P_j and we simply add j to Q_{j-1} such that Q_j and P_j are of the same shape.

This process can be reverted, hence it is clear that it defines a bijection. We illustrate the algorithm with some examples.

If we begin with $(1, 2, \dots, n)$ we get the tableaux

$$(\emptyset,\emptyset),\ \left(\begin{array}{|c|}\hline 1\\\hline\end{array},\begin{array}{|c|}\hline 1\\\hline\end{array}\right),\ \left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array},\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\right),\dots,\left(\begin{array}{|c|c|c|c|c|}\hline 1&2&\dots&&n\\\hline\end{array},\begin{array}{|c|c|c|c|c|}\hline 1&2&\dots&&n\\\hline\end{array}\right)$$

The sequence $(n, n-1, \dots, 1)$ gives us

$$(\emptyset,\emptyset),\ \left(\begin{array}{|c|}\hline n\\\hline\end{array},\begin{array}{|c|}\hline 1\\\hline\end{array}\right),\ \left(\begin{array}{|c|}\hline n-1\\\hline n\\\hline\end{array},\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right),\dots,\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline \\\hline \\\hline n\\\hline\end{array},\begin{array}{|c|}\hline 1\\\hline 2\\\hline \\\hline \\\hline n\\\hline\end{array}\right)$$

The correspondence is usually used to describe permutations. Consider the symmetric group on n letters (say $\{1, 2, \dots, n\}$), S_n. We write an element $\sigma \in S_n$ in the usual 2-row notation

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$$

The permutation σ then corresponds to the list $(\sigma(1), \sigma(2), \dots, \sigma(n))$ which in turn corresponds to a pair of standard young tableaux as explained above.

Example 1.29. Consider the symmetric group on 3 letters. Then the identity element, $1 = (1)(2)(3)$, corresponds to the list $(1, 2, 3)$ which we have seen corresponds to the pair

$$\left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array},\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right)$$

The simple transposition $s = (12)$ corresponds to the list $(2, 1, 3)$. This gives us the tableaux

$$(\emptyset,\emptyset),\ \left(\begin{array}{|c|}\hline 2\\\hline\end{array},\begin{array}{|c|}\hline 1\\\hline\end{array}\right),\ \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array},\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right),\ \left(\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array},\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right)$$

The simple transposition $t = (23)$ corresponds to the list $(1, 3, 2)$, which yields

$$(\emptyset,\emptyset),\ \left(\begin{array}{|c|}\hline 1\\\hline\end{array},\begin{array}{|c|}\hline 1\\\hline\end{array}\right),\ \left(\begin{array}{|c|c|}\hline 1&3\\\hline\end{array},\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\right),\ \left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array},\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right)$$

The 3-cycle $(123) = st$ first gives the sequence $(2, 3, 1)$ which corresponds to

$$(\emptyset,\emptyset),\ \left(\begin{array}{|c|}\hline 2\\\hline\end{array},\begin{array}{|c|}\hline 1\\\hline\end{array}\right),\ \left(\begin{array}{|c|c|}\hline 2&3\\\hline\end{array},\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\right),\ \left(\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array},\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right)$$

The 3-cycle $(132) = ts$ is the sequence $(3, 1, 2)$, and via the algorithm we get

$$(\emptyset, \emptyset), \left(\begin{array}{|c|}\hline 3 \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline\end{array}\right), \left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right)$$

Finally the longest element, $w_0 = (13) = sts = tst$, corresponds to the list $(3, 2, 1)$ which in turn corresponds to

$$\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\right)$$

as we saw earlier.

If $\sigma \in S_n$ corresponds to the pair of tableaux (P, Q), we say that $P = P_\sigma = \alpha(\sigma)$ is the *left tableau* of σ. Similarly $Q = Q_\sigma = \beta(\sigma)$ is called the *right tableau* of σ. It is naturally to group the permutations which yields the same left tableau (or right tableau) and we say that these permutations is a (Kazhdan-Lusztig) *left (resp. right) cell*.

In the above example we see that $\{1\}$ and $\{w_0\}$ are both left and right cell. The sets $\{s, st\}$ and $\{t, ts\}$ are left cells, whereas $\{s, ts\}$ and $\{t, st\}$ are right cells. The sets $\{1\}$, $\{w_0\}$ and $\{s, t, st, ts\}$ which consists of all permutations corresponding to tableaux of a certain shape are called two-sided cells.

The cells are related to Hecke algebras as explained in [KL79], and can thus be defined for more general Coxeter groups.

CHAPTER 2

ALGEBRAIC CATEGORIFICATION

We define the notions of naïve, weak and genuine categorifications. Here we use the terminology from [Maz10] on which the presentation also rely heavily – both in form and in notation.

2.1 Grothendieck Groups and Categorifications

Definition 2.1. *Let $\mathcal{C}$ be an additive category (i.e. a category enriched over the category of abelian groups). The split* Grothendieck group *of $\mathcal{C}$, $[\mathcal{C}]_\oplus$, is the quotient of the free abelian group generated by symbols $[X]$ where X ranges over the objects of $\mathcal{C}$ modulo the relations $[X] = [X'] + [X'']$ whenever there exists an isomorphism $X = X' \oplus X''$.*

The split Grothendieck group $[\mathcal{C}]_\oplus$ is called the decategorification *of the additive category $\mathcal{C}$ because we forget the categorical structure.*

Remark 2.2. If $F : \mathcal{C} \to \mathcal{D}$ is an additive functor (i.e. it preserves the enriched structure) it obviously induces a group homomorphism between the Grothendieck groups.

If $\mathcal{C}$ is an abelian category then $\mathcal{C}$ is an additive category as well and we could consider its split Grothendieck group. This group is in some ways too big. In particular two objects having the same composition factors may not be identified in the split Grothendieck group. Therefore we also define the (non-split) Grothendieck group

Definition 2.3. *Let $\mathcal{C}$ be an abelian (or triangulated) category. The Grothendieck group of $\mathcal{C}$ is the quotient of the free abelian group generated by $[X]$ where X ranges over the objects of $\mathcal{C}$ modulo the relations $[X] = [X'] + [X'']$ whenever there exists a short exact sequence*

$$0 \to X' \to X \to X'' \to 0$$

If $\mathcal{C}$ is triangulated replace the short exact sequence with a distinguished triangle

$$X' \to X \to X'' \to X'[\![1]\!]$$

If $\mathcal{C}$ is an abelian (or triangulated) category we call its Grothendieck group the decategorification *of $\mathcal{C}$.*

Remark 2.4. If $F : \mathcal{C} \to \mathcal{D}$ is an exact functor between abelian categories it induces a group homomorphism of the Grothendieck groups.

If M is an abelian group and $\mathcal{C}$ is an abelian (or additive) category with decategorification isomorphic to M we say that *$\mathcal{C}$ categorifies M*.

A priori the Grothendieck groups has no other structure than the structure of abelian groups, so it seems as if we can only categorify abelian groups (or sets). In the following we will explain how one is able to categorify modules over more general rings. These rings are very often algebras over some field $\Bbbk$ or some ring $\mathbb{F}$ (with $\mathbb{F} = \mathbb{Z}$ as a special case). This leads to the definitions given in the following text.

Fix a commutative unital ring $\mathbb{F}$.

Definition 2.5. *Let $\mathcal{C}$ be a category with decategorification $\mathbf{C}$. The* $\mathbb{F}$-decategorification *of $\mathbb{F}$ is the $\mathbb{F}$-module $\mathbf{C}^{\mathbb{F}} := \mathbb{F} \underset{\mathbb{Z}}{\otimes} \mathbf{C}$.*

Example 2.6. Let $\mathbf{D}$ be the $\mathbb{C}$-algebra $\mathbb{C}[X]/(X^2)$ (the algebra of dual numbers), and consider the category of finitely generated $\mathbf{D}$-modules: $\mathbf{D}\text{-mod}$. This is an abelian category and its decategorification is $[\mathbf{D}\text{-mod}] \simeq \mathbb{Z}$ where a basis of the free abelian group is given by the image of the unique simple $\mathbf{D}$-module $\mathbb{C}$ (where X acts as 0 of course). The unique indecomposable projective module has image $[\mathbf{D}] = [\mathbb{C}] + [\mathbb{C}] = 2$, and is thus not a basis of $[\mathbf{D}\text{-mod}]$. However the $\mathbb{C}$-decategorification, $[\mathbf{D}\text{-mod}]^{\mathbb{C}} \simeq \mathbb{C}$, has each of $[\mathbf{D}]$ and $[\mathbb{C}]$ as basis. In particular this means that the map $\varphi : \mathbb{Z} \to [\mathbf{D}\text{-mod}]$ with $\varphi(1) = [\mathbf{D}]$ is injective but not surjective. On the other hand the map defined in the same way, but with the complexified Grothendieck group: $\varphi' : \mathbb{C} \to [\mathbf{D}\text{-mod}]^{\mathbb{C}}$, $\varphi(1) = [\mathbf{D}]$, is a bijection.

Definition 2.7. *An* $\mathbb{F}$-precategorification *of an $\mathbb{F}$-module M is a pair $(\mathcal{C}, \varphi)$ where $\mathcal{C}$ is a category with $\mathbb{F}$-decategorification $\mathbf{C}^{\mathbb{F}}$ and φ is an injective map $\varphi : M \to \mathbf{C}^{\mathbb{F}}$. If φ is an isomorphism we say that $(\mathcal{C}, \varphi)$ is an* $\mathbb{F}$-categorification.

In **Example 2.6** we see that $\mathbf{D}\text{-mod}$ gives a $\mathbb{Z}$-precategorification of the abelian group $\mathbb{Z}$, and that it gives a $\mathbb{C}$-categorification of the vector space $\mathbb{C}$.

2.2 Categorifications of Linear Maps and Modules over $\mathbb{F}$-algebras

Fix $\mathbb{F}$-modules M and N and $\mathbb{F}$-categorifications $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$. Write $\mathbf{M}^{\mathbb{F}}$ resp. $\mathbf{N}^{\mathbb{F}}$ for their $\mathbb{F}$-decategorifications

Definition 2.8. *Let $f : M \to N$ be an $\mathbb{F}$-linear map. An $\mathbb{F}$-categorification of f is a functor $F : \mathcal{M} \to \mathcal{N}$ such that F induces a morphism, $[F]$ of Grothendieck groups (i.e. F is additive, exact or triangulated), such that*

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathbf{M}^{\mathbb{F}} & \xrightarrow{\quad [F] \quad} & \mathbf{N}^{\mathbb{F}} \end{array}$$

Example 2.9. If $(\mathcal{M}, \varphi)$ is an $\mathbb{F}$-categorification of M, the identity functor $\mathbb{1}_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ categorifies the identity map $\mathrm{id}_M : M \to M$; this map however usually has more (non-isomorphic) categorifications. The zero-functor categorifies the zero-map.

The definition of categorifications of linear maps makes it possible to define categorifications of modules over algebras, where we replace $\mathbb{F}$-modules with categories and replace the action of an algebra with endofunctors of the categories. Let A be an associative $\mathbb{F}$-algebra generated by $\{a_i | i \in I\}$. If M is an A-module each a_i defines an $\mathbb{F}$-linear endomorphism of M, denote this by a_i^M.

Definition 2.10. *A* naïve $\mathbb{F}$-categorification *of the A-module M is a tuple $(\mathcal{M}, \varphi, \{F_i | i \in I\})$ where $(\mathcal{M}, \varphi)$ is an $\mathbb{F}$-categorification of M, and each F_i is an $\mathbb{F}$-categorification of a_i^M.*

If A is a unital ring we usually require that the identity element is among the chosen generators, and that it is categorified via the identity functor.

Example 2.11. Over fields some (finite dimensional) A-modules admit a trivial A-categorification. If M has a basis in which each a_i^M has integral, non-negative coefficients, one takes the direct sum of copies of $\mathbb{C}\text{-mod}$, one for each basis vector. This specifies a $\mathbb{C}$-categorification,

$(\mathcal{M}, \varphi)$, such that each a_i^M can be categorified via sums of copies of the identity functor $\mathbb{1}_{\mathbb{C}\text{-mod}} : \mathbb{C}\text{-mod} \to \mathbb{C}\text{-mod}$. However since the category $\mathcal{M} = \bigoplus \mathbb{C}\text{-mod}$ is a semisimple category this categorification contains no new structure, and this sort of categorification is rather trivial.

At this point we are still ignoring the fact that A is a ring with a multiplication. The following example leads to the idea of weak categorifications.

Example 2.12. Consider the algebra $A = \mathbb{C}[X]/(X^2 - 2X)$. The image of X in A (which we write simply as X) generates A as a $\mathbb{C}$-algebra. There are two non-isomorphic simple A-modules L_0 and L_2. Both are isomorphic to $\mathbb{C}$ as a vector space but X acts as a on L_a. Each of the simple modules has the semisimple category $\mathcal{C} = \mathbb{C}\text{-mod}$ as a $\mathbb{C}$-categorications. The functor $F_0 = 0$ gives a naïve categorification of L_0, and the functor $F_2 = \mathbb{1}_{\mathbb{C}\text{-mod}} \oplus \mathbb{1}_{\mathbb{C}\text{-mod}}$ gives a categorification of L_2. Notice also that $F_0 \circ F_0 = F_0 \oplus F_0$ and $F_2 \circ F_2 = F_2 \oplus F_2$, lifting the relation $X \cdot X = X + X$.

Alternatively put $\mathbf{D} = \mathbb{C}[X]/(X^2)$ and replace $\mathcal{C}$ above with $\mathbf{D}$-mod. Here also the functor $G_2 = \mathbf{D} \underset{\mathbb{C}}{\otimes} \_$ gives a naïve categorification of L_2. This categorification looks better than the previous one because $\mathbf{D}$-mod is not semisimple. We have also seen earlier that the relation $X^2 = 2X$ can be lifted to the relation $G_2 \circ G_2 \simeq G_2 \oplus G_2$.

In the examples considered so far there is an easy way of choosing a generating set of A and to see which relations to impose. In general there is no canonical way of doing this so the definition of a weak categorifation will have to be a little vaque. The broad idea is that since we categorify the action of a_i via functors we want to categorify relations between the a_i's via relations between functors. Usually this does not make sense at all, e.g. how should one interpret the relation $a^2 = \sqrt{2}a - i$? However if all relations can be written as polynomials in the a_i's where all coefficients are integers, they can be lifted to relations between functors in the following way:

- Write all relations such that only non-negative integers occur on each side of the equality.
- Replace all natural numbers with sums of 1's.
- Replace addition with direct sums of functors.
- Multiplication is replaced with composition of functors.
- 1's are replaced with the identity functor.
- Variables are replaced with their corresponding categorification (functor).

Remark 2.13. If we consider triangulated categories negative coefficients can also be handled via shift in the homological degree (i.e. by shifting position in complexes). This also suggests that one can replace subtraction with taking cones in the derived category, this approach is e.g. used in [MS09].

Very often the rings categorified has an involution in which case the involution should be categorified by taking the (bi-)adjoint functor.

A categorification $(\mathcal{M}, \varphi, \{F_i | i \in I\})$ of some A-module M will be called a *weak categorification* if we have a fixed interpretation of the defining relations in A like above.

2.3 Categorification and 2-categories

To weakly categorify an A-module, M, we need to replace M with some category, replace the action of A on M by some functors acting on the category, and replace relations in A with relations between these functors. Relation between functors are encoded via natural isomorphisms, so a weak categorification of M is some category $\mathcal{C}$ with some functors and some specified natural isomorphisms. This setup is conveniently encoded into a 2-category with one object (0-cell). We identify the unique 0-cell with the category which categorifies

M. The 1-cells are then the functors categorifying the action of A together with all direct sums of copies of these functors. As 2-cells we usually take all natural transformations of these functors (even though it would be enough to take only the natural transformations that specify the relations we need).

The following definition formalizes and generalizes this idea. By an additive 2-category we mean a category enriched over the category of additive categories.

Definition 2.14. *Let $\mathscr{C}$ be a an additive 2-category. The* Grothendieck category *of $\mathscr{C}$, denoted $[\mathscr{C}]$, is the category with the same objects as $\mathscr{C}$, but as morphisms from $\mathtt{i}$ to $\mathtt{j}$ we take the Grothendieck group of the category $\mathscr{C}(\mathtt{i},\mathtt{j})$. The composition of morphisms comes from the horizontal composition of 1-cells: If $F, G \in \mathscr{C}(\mathtt{i},\mathtt{j})$ are 1-cells we define the composition (of their images in the Grothendieck group) via*

$$[F] \circ [G] = [F \circ_0 G]$$

The identity morphism of $\mathtt{i}$ is $[\mathbb{1}_{\mathtt{i}}]$ and the associativity rule is seen to be satisfied. Since $\mathscr{C}$ is additive, $[\mathscr{C}]$ is enriched over the category of abelian groups.

The $\mathbb{F}$-decategorification of $\mathscr{C}$ is $[\mathscr{C}]^{\mathbb{F}} = \mathbb{F} \underset{\mathbb{Z}}{\otimes} [\mathscr{C}]$. This is the $\mathbb{F}$-linear category with the same objects as $[\mathscr{C}]$ but where the abelian groups of morphisms are tensored with $\mathbb{F}$, and the compositions are the induced maps between these tensorproducts.

Remark 2.15. In a commonly used notation we have $[\mathscr{C}](\mathtt{i},\mathtt{j}) = [\mathscr{C}(\mathtt{i},\mathtt{j})]$.

Definition 2.16 ([Maz10])**.** *Let $\mathcal{A}$ be an $\mathbb{F}$-linear category. A* (genuine) categorification *of $\mathcal{A}$ is a pair $(\mathscr{A}, \varphi)$ where $\mathscr{A}$ is an additive 2-category, and $\varphi : \mathcal{A} \to [\mathscr{A}]^{\mathbb{F}}$ is an equivalence of categories.*

Example 2.17. Consider the algebras $A = \mathbb{C}[X]/(X^2 - 2X)$ and $\mathbf{D} = \mathbb{C}[X]/(X^2)$ from before.

Note that A is in fact an $\mathbb{C}$-linear category with one object and morphisms given by A, here composition of morphisms corresponds to multiplication in A. We abuse notation and write also A for this category.

We saw that with $G_2 = \mathbf{D} \underset{\mathbb{C}}{\otimes} \_ : \mathbf{D}\text{-mod} \to \mathbf{D}\text{-mod}$ we get that $(G_2)^2 \simeq G_2 \oplus G_2$. We define the 2-category $\mathscr{C}$ with one object $\mathtt{i}$ identified with $\mathbf{D}$-mod, 1-morphisms are all functors isomorphic to $\mathbb{1}_{\mathbf{D}\text{-mod}}$, G_2 and any of their compositions or sums, and finally 2-morphisms are all natural transformations of such functors. Then we can define

$$\varphi : A \to [\mathscr{C}]^{\mathbb{F}}$$

given by mapping the unique object in A to the unique object in $[\mathscr{C}]^{\mathbb{F}}$, and with the action on elements determined by $\varphi(1) = [\mathbb{1}_{\mathbf{D}\text{-mod}}]$ and $\varphi(X) = [G_2]$. This gives an equivalence of categories and hence $\mathscr{C}$ categorifies the category A.

Remark 2.18. The definition contains as a special case categorifications of $\mathbb{F}$-algebras, when we as in the example consider an algebra to be a category with one object. Conversely if $\mathcal{A}$ is an $\mathbb{F}$-linear category with a unique object, $\mathtt{i}$, then $\mathcal{A}(\mathtt{i},\mathtt{i})$ is an $\mathbb{F}$-algebra.

2.4 Representations of 2-categories

We want to generalize the idea of categorifying modules to representations of 2-categories. We follow [Maz10] and [MM10] closely.

Let $\Bbbk$ be an algebraically closed field.

Recall that if A is a $\Bbbk$-algebra a representation of A is a $\Bbbk$-module, M, and a ringhomomorphism

$$\rho : A \to \mathrm{End}_{\Bbbk}(M)$$

If we consider A as a category with a unique object, $\mathtt{i}$, then ρ corresponds to a functor $\mathcal{M} : A \to \Bbbk\text{-mod}$, with $\mathcal{M}(\mathtt{i}) = M$ and with $\mathcal{M}(a) = \rho(a) : M \to M$. This can be generalized. If $\mathcal{A}$ is a $\Bbbk$-linear category by a module over $\mathcal{A}$ we mean a functor

$$\mathcal{M} : \mathcal{A} \to \Bbbk\text{-mod}$$

In order to define representations of $\Bbbk$-finitary 2-categories we introduce a certain 2-category which describes categories of modules over finite dimensional $\Bbbk$-algebras and functors between such categories. We define the 2-category $\mathfrak{R}_\Bbbk$ by

- 0-cells are all categories equivalent to a category of the form A-mod where A is a finite dimensional $\Bbbk$-algebra.
- 1-cells are all functors between the 0-cells.
- 2-cells are all natural transformations between 1-cells.

Definition 2.19. *Given a $\Bbbk$-linear 2-category $\mathscr{C}$ by a* 2-representation *of $\mathscr{C}$ we will mean a $\Bbbk$-linear 2-functor*

$$\mathbb{M} : \mathscr{C} \to \mathfrak{R}_\Bbbk$$

The collection of all 2-representations of $\mathscr{C}$ constitute a 2-category as follows. Denote by $\mathscr{C}$-$\mathfrak{mod}$ the 2-category given by

- 0-cells are 2-representations of $\mathscr{C}$, i.e. all 2-functors $\mathbb{M} : \mathscr{C} \to \mathfrak{R}_\Bbbk$
- 1-cells are 2-natural transformations of the 0-cells.
- 2-cells are modifications of the 1-cells.

This setup allows us to define categorifications of A-modules.

Definition 2.20. *Let $\mathcal{A}$ be a $\Bbbk$-linear category and let $\mathcal{M}$ be an $\mathcal{A}$-module. A* categorification *of $\mathcal{M}$ is a 4-tuple $(\mathscr{A}, \mathbb{M}, \varphi, \psi)$, such that*

- *$\mathscr{A}$ is a $\Bbbk$-linear 2-category and φ is an equivalence of categories $\varphi : \mathcal{A} \to [\mathscr{A}]^\Bbbk$ (such that $\mathscr{A}$ is a categorification of $\mathcal{A}$).*
- *$\mathbb{M} \in \mathscr{A}$-$\mathfrak{mod}$ is a representation of $\mathscr{A}$ such that for any objects $\mathtt{i}, \mathtt{j} \in \mathscr{A}$ and any 1-cell $F \in \mathscr{A}(\mathtt{i}, \mathtt{j})$ the functor*
$$\mathbb{M}(F) : \mathbb{M}(\mathtt{i}) \to \mathbb{M}(\mathtt{j})$$
is an exact functor (recall that $\mathbb{M}(\mathtt{i})$ has the form $A_\mathtt{i}$-mod where $A_\mathtt{i}$ is a finite dimensional $\Bbbk$-algebra $A_\mathtt{i}$).
- *$\psi = (\psi_\mathtt{i})_{\mathtt{i} \in \mathscr{A}}$ is a collection of isomorphisms*
$$\psi_\mathtt{i} : \mathcal{M}(\varphi^{-1}(\mathtt{i})) \to [\mathbb{M}(\mathtt{i})]^\Bbbk$$
such that for any objects $\mathtt{i}, \mathtt{j} \in \mathscr{A}$ and any 1-cells $F \in \mathscr{A}(\mathtt{i}, \mathtt{j})$ we have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}(\varphi^{-1}[\mathtt{i}]) & \xrightarrow{\mathcal{M}(\varphi^{-1}([F]))} & \mathcal{M}(\varphi^{-1}[\mathtt{j}]) \\ \downarrow \psi_\mathtt{i} & & \downarrow \psi_\mathtt{j} \\ [\mathbb{M}(\mathtt{i})]^\Bbbk & \xrightarrow{[\mathbb{M}(F)]} & [\mathbb{M}(\mathtt{j})]^\Bbbk \end{array}$$

If the $\psi_\mathtt{i}$ are not isomorphisms but merely monomorphisms we talk about a precategorification of $\mathcal{M}$.

Let us explain the definition in the special case where $\mathcal{A} = A$ is an algebra, i.e. has only one object, say $\mathtt{i}$. Then $\mathcal{M}$ is given by an A-module $M = \mathcal{M}(\mathtt{i})$, where the action of $a \in A$ is given by $\mathcal{M}(a)$. Also $[\mathscr{A}]^\Bbbk$ up to isomorphism has a unique object, hence we may assume that $\mathscr{A}$ has a unique object, which we denote also by $\mathtt{i}$. Now $\mathbb{M}$ is a 2-functor

$$\mathbb{M} : \mathscr{A} \to \mathfrak{R}_\Bbbk$$

i.e. it is given by $\mathbb{M}(\mathtt{i}) = A_{\mathtt{i}}\text{-mod}$ for some finite dimensional $\Bbbk$-algebra $A_{\mathtt{i}}$ together with exact endofunctors of this category and natural transformations of these functors. If $a \in A$ is chosen such that for an $F \in \mathscr{A}(\mathtt{i}, \mathtt{i})$ we have $[F] = \varphi(a)$, then

$$\begin{array}{ccc} M & \xrightarrow{\quad a^M \quad} & M \\ \downarrow{\scriptstyle \psi_{\mathtt{i}}} & & \downarrow{\scriptstyle \psi_{\mathtt{i}}} \\ [A_{\mathtt{i}}\text{-mod}]^{\Bbbk} & \xrightarrow{\ [\mathbb{M}(F)]\ } & [A_{\mathtt{i}}\text{-mod}]^{\Bbbk} \end{array}$$

This means that $(A_{\mathtt{i}}\text{-mod}, \psi_{\mathtt{i}})$ is a weak categorification of M.

2.5 The BGG category $\mathcal{O}$ in Type A

Let $\mathfrak{sl}_n$ be the semisimple Lie algebra over the complex numbers given by complex, traceless $n \times n$ matrices (and the commutator as Lie bracket). This algebra comes with the natural triangular decomposition

$$\mathfrak{sl}_n = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

where the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{sl}_n$ is the subset of diagonal matrices and $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) is the subset of strictly upper (resp. lower) triangular matrices. Associated to this fixed decomposition we have the BGG (Bernstein-Gelfand-Gelfand) category $\mathcal{O} = \mathcal{O}(\mathfrak{sl}_n)$ from [BGG76] (the reader can find an introduction to the theory of this category in [Hum08]). If we denote by $U(\mathfrak{sl}_n)$ the enveloping algebra of $\mathfrak{sl}_n$ (the quotient of the tensor algebra $T(\mathfrak{sl}_n) = \oplus_i (\mathfrak{sl}_n)^{\otimes i}$ with the two-sided ideal generated by $x \otimes y - y \otimes x - [x, y]$, $x, y \in \mathfrak{sl}_n$) then $\mathcal{O}$ is the full subcategory of the category of $U(\mathfrak{sl}_n)$-modules given by

$$\mathrm{Ob}(\mathcal{O}(\mathfrak{sl}_n)) = \left\{ M \in U(\mathfrak{sl}_n)\text{-mod} \;\middle|\; \begin{array}{c} M \text{ is finitely generated} \\ M \text{ is a weight module, i.e. } M = \oplus_{\lambda \in \mathfrak{h}^*} M_\lambda \\ M \text{ is locally } U(\mathfrak{n}^+)\text{-finite} \end{array} \right\}$$

Here for each weight $\lambda \in \mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ we define for a $U(\mathfrak{sl}_n)$-module M the λ-weight space corresponding to the weight λ

$$M_\lambda = \{m \in M | \forall h \in \mathfrak{h} : hm = \lambda(h)m\}$$

This is the intersection of eigenspaces for all $h \in \mathfrak{h}$ corresponding to the eigenvalues $\lambda(h)$.

The requirement that M is locally $U(\mathfrak{n}^+)$-finite means that for each $m \in M$ the vector space $U(\mathfrak{n}^+)m$ is finite dimensional.

Let $\lambda \in \mathfrak{h}^*$ and define $\mathbb{C}_\lambda$ as a $U(\mathfrak{b})$-module where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ such that $\mathfrak{h}$ acts via λ and $\mathfrak{n}^+$ acts as 0. The Verma module

$$\mathrm{M}(\lambda) = U(\mathfrak{sl}_n) \underset{U(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda$$

lies in $\mathcal{O}$. It has simple head which we denote by $\mathrm{L}(\lambda)$. All simple modules in $\mathcal{O}$ arise this way.

The Weyl group of $\mathfrak{sl}_n$ is the symmetric group S_n. It acts naturally on $\mathfrak{h}$ by permuting the diagonal entries, and this actions induces an action on $\mathfrak{h}^*$. Define the weight $\rho : \mathfrak{h} \to \mathbb{C}$ by

$$\rho(\mathbf{diag}(t_1, \ldots, t_n)) = \frac{1}{2} \sum_{j=1}^{n} (n + 1 - 2j) t_j$$

Then we have the Jantzen dot-action of S_n on $\mathfrak{h}^*$ given by

$$w_\bullet \lambda = w(\lambda + \rho) - \rho$$

where $w \in S_n$ and $\lambda \in \mathfrak{h}^*$. If we define the root system $\Phi = \{\pm\alpha_{ij} | 1 \leq i < j \leq n\}$ where

$$\alpha_{ij}(\mathbf{diag}(t_1, \ldots, t_n)) = t_i - t_j$$

then ρ is half the sum of positive roots $\Phi^+ = \{\alpha_{ij} | 1 \leq i < j \leq n\}$.

The category $\mathcal{O}$ decomposes into blocks (which may decompose further)

$$\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*/(W\bullet)} \mathcal{O}_\lambda$$

Here $\mathcal{O}_\lambda$ is the full subcategory of $\mathcal{O}$ where the objects only have composition factors of the form $\mathrm{L}(w\bullet\lambda)$ for some $w \in W$. It is convenient to parametrize Verma modules, simple modules and indecomposable projective modules in a certain block by Weyl group elements. Whenever λ is clear from the context we freely write $\mathrm{M}_x = \mathrm{M}(x\bullet\lambda)$ for $x \in W$. Similarly we write $\mathrm{L}_x = \mathrm{L}(x\bullet\lambda)$ and $\mathrm{P}_x = \mathrm{P}(x\bullet\lambda)$ (the projective cover of L_x), all lie in $\mathcal{O}_\lambda$.

We will consider the (indecomposable) principal block, this is the block corresponding to $\lambda = 0$, i.e. the block that contains the trivial 1-dimensional module $\mathrm{L}(0) = \mathbb{C}$. The weight 0 is regular meaning that $|S_n\bullet 0| = n!$, therefore a basis (with no repetitions) for the Grothendieck group of $\mathcal{O}_0$ is given by $\{[\mathrm{L}_w] | w \in S_n\}$, but since $\mathcal{O}_0$ has finite homological dimension we have another basis namely $\{[\mathrm{P}_w] | w \in S_n\}$, and the images of the Verma modules $\{[\mathrm{M}_w | w \in S_n\}$ also give a basis (use e.g. that the Verma modules as well as their simple quotients are highest weight modules where the weight space corresponding to the highest weight is 1-dimensional). This gives three isomorphisms of abelian groups

$$\mathbb{Z}[S_n] \to [\mathcal{O}_0]$$

It turns out that the isomorpism $\varphi : \mathbb{Z}[S_n] \to [\mathcal{O}_0]$ given by $\varphi(w) = [\mathrm{M}_w]$ is the appropriate choice for our application.

Tensoring (over $\mathbb{C}$) with a finite dimensional $U(\mathfrak{sl}_n)$-module, E, preserves the category $\mathcal{O}$. That is it gives an exact endofunctor

$$E \underset{\mathbb{C}}{\otimes} \_ : \mathcal{O} \to \mathcal{O}$$

It follows that $E \underset{\mathbb{C}}{\otimes} \_$ also defines an endofunctor of each of the blocks $\mathcal{O}_\lambda$, since by composing with the natural inclusions and projections of the blocks we get functors between the blocks, in particular Jantzen's translation functors arise in this way. Functors isomorphic to direct summands of tensoring with a finite dimensional module are called projective functors. The funcor $E^* \underset{\mathbb{C}}{\otimes} \_$ (where $E^* = \mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C})$ is the usual dual module) is biadjoint to $E \underset{\mathbb{C}}{\otimes} \_$. In particular the projective functors have biadjoints that are also projective functors.

In [BG80] they classify projective functors this is the content of the following theorem.

Theorem 2.21. *Any projective functor on the category $\mathcal{O}$ decomposes uniquely into a direct sum of indecomposable exact projective functors. A projective functor $F : \mathcal{O}_0 \to \mathcal{O}_0$ is uniquely determined by its value on the projective Verma module $\mathrm{M}_e = \mathrm{M}(0) = \mathrm{P}_e$, the resulting module is a projective module, and the decomposition of this module into indecomposable projectives corresponds to the decomposition of the functor. Furthermore if also $G : \mathcal{O}_0 \to \mathcal{O}_0$ is projective then we have an isomorphism of vector spaces*

$$\mathrm{Nat}(F, G) \simeq \mathrm{Hom}_{\mathcal{O}}(F\mathrm{M}_e, G\mathrm{M}_e) \tag{2.1}$$

Here $\mathrm{Nat}(F, G)$ is the space of natural transformations between the functors F and G.

We will write θ_w for the unique projective functor mapping P_e to P_w. If $s \in S_n$ is a simple transposition we have the wall-crossing functor $\theta_s : \mathcal{O}_0 \to \mathcal{O}_0$ mapping P_e to P_s. The following proposition describes in particular how this functor acts on the Grothendieck group.

Proposition 2.22. *Let $s \in S_n$ be a simple transposition and let $w \in W$ with $\ell(ws) > \ell(w)$. There are short exact sequences*

$$0 \to \mathrm{M}_w \to \theta_s \mathrm{M}_w \to \mathrm{M}_{ws} \to 0$$

and

$$0 \to \mathrm{M}_w \to \theta_s \mathrm{M}_{ws} \to \mathrm{M}_{ws} \to 0$$

This means that θ_s acts as right multiplication with $s+e$ when we identify the Grothendieck group with $\mathbb{Z}[S_n]$ via φ. In particular we see that θ_s^2 acts as right multiplication with $(s+e)^2 = 2(e+s)$. Since θ_s is projective (thus uniquely defined via its action on M_e) it follows that $\theta_s^2 \simeq \theta_s \oplus \theta_s$.

Since $st = ts$ is equivalent to $(e+s)(e+t) = (e+t)(e+s)$ and $sts = tst$ is equivalent to

$$(e+s)(e+t)(e+s) + (e+t) = (e+t)(e+s)(e+t) + (e+s)$$

one gets that

$$\theta_s\theta_t \simeq \theta_t\theta_s, \qquad \text{if } st = ts$$

and

$$\theta_s\theta_t\theta_s \oplus \theta_t \simeq \theta_t\theta_s\theta_t \oplus \theta_s, \qquad \text{if } sts = tst$$

Denote by $\mathfrak{P} = \mathfrak{P}_0$ the category of projective endofunctors of $\mathcal{O}_0$ with natural transformations as morphisms. The theorem shows that $\{[\theta_w] | w \in S_n\}$ is a basis for the (split) Grothendieck group of $\mathfrak{P}$, and composition of functors induces a ring structure on this group. We have just seen that

$$\psi : \mathbb{Z}[S_n]^{\mathrm{opp}} \to [\mathfrak{P}]_\oplus$$

given by $\psi(e+s) = [\theta_s]$ is a homomorphism of rings (since $\mathbb{Z}[S_n]$ has a presentation with generators $e+s$ and the above relations). Since both rings are free $\mathbb{Z}$-modules of the same rank it is natural to ask if ψ is an isomorphism. This is in fact the case since any θ_w can be obtained by composition wall crossing functors and taking direct summands (proving that ψ is surjective, hence bijective). For an easy argument notice that $\theta_s \mathrm{M}_w$ surjects on M_{ws} if $\ell(ws) > \ell(w)$. Hence if $w = s_1 \dots s_k$ is a reduced expression (i.e. $k = \ell(w)$) then $P = \theta_{s_k} \dots \theta_{s_1} \mathrm{M}_e$ surjects onto M_w and hence the projective cover P_w of M_w occurs as a direct summand in P. (One may also show using the ingredients given above that any other indecomposable module occuring as a direct summands in P has the form P_x for some $x \neq w$ arising by removing some of the s_i from the expression of w. This gives the setting for an inductive proof for the surjectivity of ψ.)

Let us draw the diagram that illustrates that θ_s categorifies the map on $\mathbb{Z}[S_n]$ which is right multiplication with $e+s$.

$$\begin{array}{ccc} \mathcal{O}_0 & \xrightarrow{\theta_s} & \mathcal{O}_0 \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ \mathbb{Z}[S_n] & \xrightarrow{\cdot(e+s)} & \mathbb{Z}[S_n] \end{array}$$

The elements $e+s$ generates $\mathbb{Z}[S_n]$ as a unital ring, and therefore we see that the projective functors acting on $\mathcal{O}_0$ categorifies the right regular $\mathbb{Z}[S_n]$-module.

Now we define a 2-category with one object `i` identified with $\mathcal{O}_0$ and with 1-morphisms given by projective functors, and 2-morphisms given by natural transformation. When taking Grothendieck groups we get the right regular representation of $\mathbb{Z}[S_n]$. Let us give an illustration copied from [Maz10, page 23]

$$\mathcal{O}_0 \circlearrowleft \mathfrak{P} \mapsto [\mathcal{O}_0] \circlearrowleft [\mathfrak{P}]_\oplus \simeq \mathbb{Z}[S_n] \circlearrowleft \mathbb{Z}[S_n]$$

The 2-category comes along with its defining representation (meaning that we map `i` to $\mathcal{O}_0$ and do not change neither functors nor natural transformations). This is a genuine categorification of the right regular $\mathbb{Z}[S_n]$-module. This example can be generalized to any semisimple complex Lie algebra (giving a categorification of the right regular representation of the group ring of the corresponding Weyl group).

CHAPTER 3

CELL 2-REPRESENTATIONS OF FINITARY 2-CATEGORIES

This chapter is a brief overview of the definitions and some of the main results from [MM10], in order to give the basic fundament for Section 3.5 and motivate the calculations made therein. The aim of this chapter is not to prove anything, and we refer to [MM10] (together with its references) for thorough proofs. However some proofs are short and give a good understanding of the theory; these are to some extend included here.

Given an algebra A there are two natural ways to construct A-modules. The first one is via generators and relations, i.e. we fix a set $\{a_i | i \in I\}$ that generates A as an algebra; an abelian group M is then an A-module if we have group endomorphisms $a_i^M \in \mathrm{End}_{\mathbb{Z}}(M)$ which satisfies the same realations as the various a_i's. Another approach is via cokernels of homomorphisms of free A-modules. Such a presentation of M can be obtained by fixing surjections $\psi : A^n \to M$ and $\varphi : A^m \to \ker \psi$. This is usually written as the exact sequence

$$A^m \to A^n \to M \to 0$$

Now $M \simeq \mathrm{coker}\,\varphi$ (when φ is considered as a morphism between A^n and A^m).

Inspired by these methods it is possible to construct 2-representations of 2-categories.

The approach with generators and relations is used in e.g. [Rou08]. In [MM10] they follow the approach with cokernels.

3.1 The Principal 2-representations

Let $\mathscr{C}$ be a $\Bbbk$-finitary 2-category.

If $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ are objects we define the category $\overline{\mathscr{C}}(\mathtt{i}, \mathtt{j})$ by

- Objects are diagrams $F \xrightarrow{\alpha} G$ where $F, G \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ are 1-morphisms and $\alpha : F \to G$ is a 2-morphism.
- Morphisms are equivalence classes of commutative diagrams

$$\begin{array}{ccc} F & \xrightarrow{\alpha} & G \\ \beta\downarrow & & \downarrow\beta' \\ F' & \xrightarrow[\alpha']{} & G' \end{array}$$

where $F \xrightarrow{\alpha} G$ and $F' \xrightarrow{\alpha'} G'$ are objects, and $\beta : F \to F'$ and $\beta' : G \to G'$ are 2-morphisms. The equivalence classes are defined by the ideal generated by diagrams for which there exists a 2-morphism $\delta : G \to F'$ with $\alpha'\delta = \beta'$:

$$\begin{array}{ccc} F & \xrightarrow{\alpha} & G \\ {\scriptstyle\beta}\big\downarrow & \overset{\delta}{\swarrow} & \big\downarrow{\scriptstyle\beta'} \\ F' & \xrightarrow[\alpha']{} & G' \end{array}$$

- Composition is defined by placing diagrams on top of each other.

It can be shown that $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ is an abelian category and that it is equivalent to the category of finitely generated modules over a finite dimensional $\Bbbk$-algebra. In the equivalence $F \xrightarrow{\alpha} G$ corresponds to the cokernel of α. For details of this see [Fre66]. This means that $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ is an object of $\mathfrak{R}_\Bbbk$. If $F \in \mathscr{C}(\mathtt{i},\mathtt{j})$ is a 1-cell we denote by P_F the object $\mathtt{0} \to F$ in $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$. This is in fact a projective object.

Given $\mathtt{i} \in \mathscr{C}$ we can define the $\mathtt{i}$'th principal 2-representation $\mathbf{P}_\mathtt{i}$ of $\mathscr{C}$. This is done as follows

- For $\mathtt{j} \in \mathscr{C}$ let $\mathbf{P}_\mathtt{i}(\mathtt{j}) = \overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$.
- If $\mathtt{j},\mathtt{k} \in \mathscr{C}$ and $H \in \mathscr{C}(\mathtt{j},\mathtt{k})$ we define a functor, $\mathbf{P}_\mathtt{i}(H)$ from $\mathbf{P}_\mathtt{i}(\mathtt{j}) = \overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ to $\mathbf{P}_\mathtt{i}(\mathtt{k}) = \overline{\mathscr{C}}(\mathtt{i},\mathtt{k})$. First on objects via

$$\mathbf{P}_\mathtt{i}(H)\left(F \xrightarrow{\alpha} G \right) = \left(H \circ F \xrightarrow{H(\alpha)} H \circ G \right)$$

(recall that $H(\alpha) = \mathrm{Id}_H \circ_0 \alpha$). On morphisms the functor is given by

$$\mathbf{P}_\mathtt{i}(H) \left(\begin{array}{ccc} F & \xrightarrow{\alpha} & G \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\beta'} \\ F' & \xrightarrow[\alpha']{} & G' \end{array} \right) = \left(\begin{array}{ccc} H \circ F & \xrightarrow{H(\alpha)} & H \circ G \\ {\scriptstyle H(\beta)}\big\downarrow & & \big\downarrow{\scriptstyle H(\beta')} \\ H \circ F' & \xrightarrow[H(\alpha')]{} & H \circ G' \end{array} \right)$$

- Finally if $\gamma : H \to H'$ is a 2-morphism between $H, H' \in \mathscr{C}(\mathtt{i},\mathtt{j})$ we define a natural transformation $\mathbf{P}_\mathtt{i}(\gamma) : \mathbf{P}_\mathtt{i}(H) \to \mathbf{P}_\mathtt{i}(H')$ via left horizontal composition, i.e. if we write $X = F \xrightarrow{\alpha} G$ we define

$$\mathbf{P}_\mathtt{i}(\gamma)_X : \mathbf{P}_\mathtt{i}(H)(X) \to \mathbf{P}_\mathtt{i}(H')(X)$$

via

$$\begin{array}{ccc} H \circ F & \xrightarrow{H(\alpha)} & H \circ G \\ {\scriptstyle\gamma_F}\big\downarrow & & \big\downarrow{\scriptstyle\gamma_G} \\ H' \circ F & \xrightarrow[H'(\alpha)]{} & H' \circ G \end{array}$$

We have the following universal property of the principal 2-representations.

Proposition 3.1 ([MM10, Proposition 5]). *Let $\mathbf{M} : \mathscr{C} \to \mathfrak{R}_\Bbbk$ be a 2-representation, $\mathtt{i} \in \mathscr{C}$ and let $M \in \mathbf{M}(\mathtt{i})$.*

a) Define for $\mathtt{j} \in \mathscr{C}$ the functor $\Psi_\mathtt{j}^M : \mathbf{P}_\mathtt{i}(\mathtt{j}) \to \mathbf{M}(\mathtt{j})$ via

$$\Phi_j^M \left(F \xrightarrow{\alpha} G \right) = \operatorname{coker} \mathbf{M}(\alpha)_M$$

Then $\Phi^M = (\Phi_\mathtt{j}^M)_{\mathtt{j}\in\mathscr{C}}$ is the unique morphism from $\mathbf{P}_\mathtt{i}$ to $\mathbf{M}$ mapping $\mathrm{P}_{\mathbb{1}_\mathtt{i}}$ to M.

b) The correspondence $M \mapsto \Phi^M$ is functorial.

3.2 Cells and Cell 2-representations

Let $\mathscr{C}$ be a fiat 2-category.

Let $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ and let $\mathcal{C}_{\mathtt{i},\mathtt{j}}$ be the full subcategory of $\mathscr{C}(\mathtt{i}, \mathtt{j})$ generated by a complete (finite) list of non-isomorphic indecomposable objects. Define $\mathcal{C} = \cup_{\mathtt{i},\mathtt{j}}\mathcal{C}_{\mathtt{i},\mathtt{j}}$, i.e. the objects of $\mathcal{C}$ is a complete list of non-isomorphic indecomposable 1-morphisms from $\mathscr{C}$. We want to define equivalence relations that partition the 1-morphisms into cells. Let $F \in \mathcal{C}_{\mathtt{i},\mathtt{j}}$ and $G \in \mathcal{C}_{\mathtt{k},\mathtt{l}}$ and define

$$F \leq_R G \iff \exists H \in \mathscr{C}(\mathtt{j}, \mathtt{l}) : G | H \circ F$$

Here $G|H \circ F$ means that G occurs as a direct summand of $H \circ F$, and this forces $\mathtt{i} = \mathtt{k}$. We also define

$$F \leq_L G \iff \exists H \in \mathscr{C}(\mathtt{k}, \mathtt{i}) : G | F \circ H$$

This forces $\mathtt{j} = \mathtt{l}$. Finally define

$$F \leq_{LR} G \iff \exists H_1 \in \mathscr{C}(\mathtt{k}, \mathtt{i}), H_2 \in \mathscr{C}(\mathtt{j}, \mathtt{l}) : G | H_2 \circ F \circ H_1$$

Notice that if $G|H \circ F$ then $G^*|(H \circ F)^* = F^* \circ H^*$. Hence if $F \leq_R G$ then $G^* \leq_L F^*$, in other words we see that * swaps $\leq_R$ and $\leq_L$. It is also apparent that * preserves $\leq_{LR}$.

For $S \in \{L, R, LR\}$ we define

$$F \sim_S G \iff F \leq_S G \text{ and } G \leq_S F$$

This defines three equivalence relations on $\mathcal{C}$. Given $F \in \mathcal{C}$ we denote by $\mathcal{R}_F$ the equivalence class under $\sim_R$ containing F, the right cell of F. Similarly we define the left cell $\mathcal{L}_F$ as the equivalence class under $\sim_L$ containing F, and the two-sided cell $\mathcal{LR}_F$ as the equivalence class under $\sim_{LR}$. Clearly a two-sided cell is the disjoint union of the right cells it contains (intersects), as well as the disjoint union of the left cells it contains.

Let us give another characterisation of $\leq_L$. If $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ the isomorphism classes of simple modules in $\mathbf{P}_{\mathtt{i}}(\mathtt{j})$ are indexed by the objects of $\mathcal{C}_{\mathtt{i},\mathtt{j}}$. Given $F \in \mathcal{C}_{\mathtt{i},\mathtt{j}}$ denote by L_F the unique simple quotient of P_F. Let also $G \in \mathcal{C}$, say $G \in \mathcal{C}_{\mathtt{j},\mathtt{k}}$, such that it makes sense to consider $G\mathrm{L}_F$.

Lemma 3.2 ([MM10, Lemma 12]).

$$G\mathrm{L}_F \neq 0 \iff G^* \leq_L F$$

Proof. Since $\mathbf{P}_{\mathtt{i}}(\mathtt{j}) = \overline{\mathscr{C}}(\mathtt{i}, \mathtt{k})$ is just the category of finitely generated modules over some finite dimensional $\Bbbk$-algebra we can simply calculate. We see that $G\mathrm{L}_F \neq 0$ exactly when there exists $H \in \mathcal{C}$ such that

$$\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{k})}(\mathrm{P}_H, G\mathrm{L}_F) \neq 0$$

Now $\mathrm{P}_H = H\mathrm{P}_{\mathbb{1}_\mathtt{i}}$ and by adjunction we get

$$0 \neq \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{k})}(H\mathrm{P}_{\mathbb{1}_\mathtt{i}}, G\mathrm{L}_F) \simeq \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(G^* \circ H\mathrm{P}_{\mathbb{1}_\mathtt{i}}, \mathrm{L}_F)$$

This in turn is equivalent to $\mathrm{P}_F = F\mathrm{P}_{\mathbb{1}_\mathtt{i}}$ being a direct summand of $G^* \circ H\mathrm{P}_{\mathbb{1}_\mathtt{i}}$, which is the same as $F|G^* \circ H$. ☐

In a similar vein the following results are obtained.

Proposition 3.3 ([MM10, Lemma 13, Corollary 14, Lemma 15]).

a) Let $F, H \in \mathcal{C}$. We have

$$\exists G \in \mathcal{C} : [G\mathrm{L}_F : \mathrm{L}_H] \neq 0 \iff H \leq_R F$$

b) Let $F, G, H \in \mathcal{C}$. If L_H occurs in either the head or the socle of $G\mathrm{L}_F$ then $H \in \mathcal{R}_F$.

c) Let $F \in \mathcal{C}_{\mathtt{i},\mathtt{j}}$. There is (up to scalar) a unique non-trivial homomorphism from $\mathrm{P}_{\mathbb{1}_\mathtt{i}}$ to $F^*\mathrm{L}_F$. It follows that $F^*\mathrm{L}_F \neq 0$.

Sketch of proof. We see that with the notation from before

$$[GL_F : L_H] \neq 0 \iff 0 \neq \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{k})}(P_H, GL_F) \simeq \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{k})}(G^* \circ HP_{\mathbb{1}_\mathtt{i}}, L_F)$$

This happens if and only if $F|G^* \circ H$. Thus a) follows.

Assume that L_H occurs in the socle (if L_H is in the head the proof is similar) of GL_F. Then $[GL_F : L_H] \neq 0$ and $H \leq_R F$. By adjunction also

$$0 \neq \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{k})}(L_H, GL_F) \simeq \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(G^* L_H, L_F)$$

such that $[G^* L_H : L_F] \neq 0$ and then $F \leq_R H$. We see that $H \in \mathcal{R}_F$ which proves b).

Finally for the proof of c) adjunction yields

$$\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{i})}(P_{\mathbb{1}_\mathtt{i}}, F^* L_F) \simeq \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(P_F, L_F) \simeq \Bbbk$$ □

3.2.1 Serre 2-subrepresentations

Let $\mathbf{M}$ be a 2-representation of $\mathscr{C}$.

Definition 3.4. *For any* $\mathtt{i}$ *let* $\mathbf{N}(\mathtt{i})$ *be a Serre subcategory of* $\mathbf{M}(\mathtt{i})$. *If for any* $\mathtt{j} \in \mathscr{C}$ *and any* $F \in \mathscr{C}(\mathtt{i},\mathtt{j})$ *have that* $F\mathbf{N}(\mathtt{i}) \subseteq \mathbf{N}(\mathtt{j})$ *we say that* $\mathbf{N}$ *is a* Serre 2-subrepresentation *of* $\mathbf{M}$.

Notice that $\mathbf{N}$ is in fact a 2-representation of $\mathscr{C}$, such that the label makes sense.

If $\mathbf{N}$ is a Serre 2-subrepresentation of $\mathbf{M}$ we define $(\mathbf{M}/\mathbf{N})(\mathtt{i}) = \mathbf{M}(\mathtt{i})/\mathbf{N}(\mathtt{i})$ via the construction in Section 1.3. General nonsense shows that $\mathbf{M}/\mathbf{N}$ defines a 2-representation of $\mathscr{C}$.

Definition 3.5. *Let* $S \in \{L, R, LR\}$. *A subset* $\mathcal{I} \subseteq \mathcal{C}$ *is said to be an* ideal *with respect to* $\leq_S$ *if*

$$F \in \mathcal{I} \text{ and } G \leq_S F \Rightarrow G \in \mathcal{I}$$

$\mathcal{J}$ *is said to be an* coideal *with respect to* $\leq_S$ *if*

$$F \in \mathcal{J} \text{ and } F \leq_S G \Rightarrow G \in \mathcal{J}$$

Let $\mathcal{J}$ be a coideal with respect to $\leq_{LR}$. Given $\mathtt{i} \in \mathscr{C}$ and denote by $\mathbf{M}_\mathcal{J}(\mathtt{i})$ the Serre subcategory of $\mathbf{M}(\mathtt{i})$ generated by all simple modules L with

$$\mathcal{J} \subseteq \mathrm{Ann}_\mathcal{C}(L) := \{F \in \mathcal{C} | FL = 0\}$$

Proposition 3.6 ([MM10, Lemma 10]). $\mathbf{M}_\mathcal{J} : \mathscr{C} \to \mathfrak{R}_\Bbbk$ *is a 2-representation via restriction. Therefore it is a Serre 2-subrepresentation of* $\mathbf{M}$.

Sketch of proof. We need to prove that any $\mathbf{M}_\mathcal{J}(\mathtt{i})$ is stable under the action of 1-morphisms from $\mathscr{C}$. It is enough to let $L \in \mathbf{M}_\mathcal{J}(\mathtt{i})$ be a simple object, let $F \in \mathcal{C}_{\mathtt{i},\mathtt{j}}$ and prove that FL is also in $\mathbf{M}_\mathcal{J}(\mathtt{i})$. Given $G \in \mathcal{J}$ we need to show that G kills any composition factor of FL. By exactness of G it is enough to show that $G \circ FL = 0$. Well $G \circ F$ decomposes into a direct sum of indecomposables H_i, and all of them satisfies $G \leq_L H_i$, hence also $G \leq_{LR} H_i$. We see that any $H_i \in \mathcal{J}$ and then $H_i L = 0$. □

It is also possible to define *Serre filtrations* of 2-representations.

3.2.2 Right Cell 2-representations

Given an $\mathtt{i} \in \mathscr{C}$ we may choose a rightcell $\mathcal{R}$ in $\mathcal{C}$ such that $\mathcal{R} \cap \mathcal{C}_{\mathtt{i},\mathtt{j}} \neq \emptyset$ for some $\mathtt{j} \in \mathscr{C}$. We have the following proposition

Proposition 3.7 ([MM10, Proposition 17]).

a) There is a unique submodule $K = K_\mathcal{R}$ *of* $P_{\mathbb{1}_\mathtt{i}}$ *satisfying*

i) Every composition factor of $P_{\mathbb{1}_\mathtt{i}}/K$ *is killed by all* $F \in \mathcal{R}$.

ii) The head of K *is simple of the form* $\mathrm{L}_{G_{\mathcal{R}}}$ *for some* $G_{\mathcal{R}} \in \mathcal{C}$ *and* $F\mathrm{L}_{G_{\mathcal{R}}} \neq 0$ *for all* $F \in \mathcal{R}$.

b) If $F \in \mathcal{R}$ *then* $F\mathrm{L}_{G_{\mathcal{R}}}$ *has a unique simple quotient namely* L_F.

c) Both $G_{\mathcal{R}}$ *and* $G_{\mathcal{R}}^*$ *belong to* $\mathcal{R}$

d) If $F \in \mathcal{R}$ *then both* $F^* \leq_L G_{\mathcal{R}}$ *and* $F \leq_R G_{\mathcal{R}}^*$.

The main idea in the proof of this proposition (for which we again refer to [MM10]) is to fix $F \in \mathcal{R}$ and define K as the minimal submodule of $\mathrm{P}_{\mathbb{1}_{\mathtt{i}}}$ such that all composition factors of the quotient is killed by F. Using that $\mathcal{R}$ is a right cell one shows that all $G \in \mathcal{R}$ kill the quotient.

Corollary 3.8 ([MM10, Lemma 26])**.** *If* $F \in \mathcal{C}$ *then* $F \sim_{LR} F^*$.

Proof. Let $\mathcal{R} = \mathcal{R}_F$, then $F \sim_R G_{\mathcal{R}}$ and therefore $F^* \sim_L G_{\mathcal{R}}^*$. Since $G_{\mathcal{R}}^* \sim_R G_{\mathcal{R}}$ the chain is completed. □

Since $F\mathrm{L}_{G_{\mathcal{R}}}$ is indecomposable with head L_F for any $F \in \mathcal{R}$ we have a short exact sequence

$$0 \longrightarrow \ker_F \longrightarrow \mathrm{P}_F \longrightarrow F\mathrm{L}_{G_{\mathcal{R}}} \longrightarrow 0$$

We define

$$\ker_{\mathcal{R},\mathtt{j}} = \bigoplus_{F \in \mathcal{R} \cap \mathcal{C}_{\mathtt{i},\mathtt{j}}} \ker_F, \qquad \mathrm{P}_{\mathcal{R},\mathtt{j}} = \bigoplus_{F \in \mathcal{R} \cap \mathcal{C}_{\mathtt{i},\mathtt{j}}} \mathrm{P}_F, \qquad Q_{\mathcal{R},\mathtt{j}} = \bigoplus_{F \in \mathcal{R} \cap \mathcal{C}_{\mathtt{i},\mathtt{j}}} F\mathrm{L}_{G_{\mathcal{R}}}$$

By [MM10, Lemma 20] $\ker_{\mathcal{R},\mathtt{j}}$ is stable under any endemorphism of $\mathrm{P}_{\mathcal{R},\mathtt{j}}$.

Define $\mathbf{C}_{\mathcal{R}}(\mathtt{j})$ as the full subcategory of $\mathbf{P}_{\mathtt{i}}(\mathtt{j})$ consisting of all modules M for which there exists a resolution

$$X_1 \to X_0 \to M \to 0$$

where $X_0, X_1 \in \mathrm{add}(Q_{\mathcal{R},\mathtt{j}})$. This category is in fact a equivalent to the category of modules over an algebra ([MM10, Lemma 21]).

Theorem 3.9 ([MM10, Theorem 22])**.** *Via restriction of* $\mathbf{P}_{\mathtt{i}}$ *the above definition yields a* 2*-representation* $\mathbf{C}_{\mathcal{R}} : \mathscr{C} \to \mathfrak{R}_{\Bbbk}$, *the* right cell 2-representation *corresponding to* $\mathcal{R}$.

The next theorem describes homomorphisms from cell 2-representations. We still consider the right cell $\mathcal{R}$, and we choose $\mathtt{i} \in \mathscr{C}$ such that $G_{\mathcal{R}} \in \mathcal{C}_{\mathtt{i},\mathtt{i}}$. Choose a projective presentation of $\mathrm{L}_{G_{\mathcal{R}}}$

$$F\mathrm{P}_{\mathbb{1}_{\mathtt{i}}} \to G_{\mathcal{R}}\mathrm{P}_{\mathbb{1}_{\mathtt{i}}} \to \mathrm{L}_{G_{\mathcal{R}}}$$

where the first map is $\mathbf{P}_{\mathtt{i}}(\alpha) : F\mathrm{P}_{\mathbb{1}_{\mathtt{i}}} \to G_{\mathcal{R}}\mathrm{P}_{\mathbb{1}_{\mathtt{i}}}$ where $\alpha : F \to G_{\mathcal{R}}$ is a 2-cell.

Theorem 3.10 ([MM10, Theorem 24])**.** *Let* $\mathbf{M} : \mathscr{C} \to \mathfrak{R}_{\Bbbk}$ *be a* 2*-representation. Denote by* $\Theta = \Theta_{\mathcal{R}}^{\mathbf{M}}$ *the functor we get by taking the cokernel of the natural transformation* $\mathbf{M}(\alpha)$.

a) The functor $\Theta : \mathbf{M}(\mathtt{i}) \to \mathbf{M}(\mathtt{i})$ *is right exact.*

b) If $\Psi : \mathbf{C}_{\mathcal{R}} \to \mathbf{M}$ *is a morphism then* $\Psi(\mathrm{L}_{G_{\mathcal{R}}}) \in \Theta(\mathbf{M}(\mathtt{i}))$.

c) If $M \in \Theta(\mathbf{M}(\mathtt{i}))$ *there exists a unique morphism* $\Psi^M : \mathbf{C}_{\mathcal{R}} \to \mathbf{M}$ *with* $\Psi(\mathrm{L}_{G_{\mathcal{R}}}) = M$.

d) The correspondence $\Theta(\mathbf{M}(\mathtt{i})) \ni M \mapsto \Psi^M$ *is functorial.*

3.2.3 Regular Cells

Definition 3.11. *Let $\mathcal{Q}$ be a two-sided cell. We say that $\mathcal{Q}$ is* regular *if it does not contain two different right cells which can be compared with respect to $\leq_R$. A one-sided cell is called regular if the two sided cell it is contained in is regular.*

Notice that $\mathcal{Q}$ is regular precisely when all its left cells are non-comparable with respect to $\leq_L$ by **Corollary 3.8**.

If $\mathcal{Q}$ is a regular two-sided cell which contains a right cell $\mathcal{R}$ and a left cell $\mathcal{L}$ then $\mathcal{R} \cap \mathcal{L} \neq \emptyset$ ([MM10, Proposition 28]): If $F \in \mathcal{R}$ and $G \in \mathcal{L}$ then for some H, K we have $G|H \circ F \circ K$. Hence for some summand N of $H \circ F$ we have $G|N \circ K$ and therefore $N \leq_L G$ and $F \leq_R N$. Now because $F \sim_{LR} G$ we see that $N \in \mathcal{Q}$. Since $\mathcal{Q}$ is regular $N \in \mathcal{R} \cap \mathcal{L}$.

If any right cell $\mathcal{R}$ and left cell $\mathcal{L}$ in the regular two-sided cell $\mathcal{Q}$ intersect in precisely one point we say that $\mathcal{Q}$ (and any of the one-sided cells it contains) is *strongly regular*.

The structure of strongly regular right cells is particularly nice.

Proposition 3.12 ([MM10, Proposition 30]). *Let $\mathcal{R}$ be a strongly regular right cell. Then*

a) $G_{\mathcal{R}} \simeq G^*_{\mathcal{R}}$, and if $F \in \mathcal{R}$ satisfies $F \simeq F^*$ then $F = G_{\mathcal{R}}$.

b) If $F \in \mathcal{R}$ and $G \sim_L F$ with $G^* \simeq G$ then $G\mathrm{L}_F \neq 0$. Further the only module occuring in either the head or socle of $G\mathrm{L}_F$ is L_F.

Example 3.13 (Cells in the BGG category $\mathcal{O}_0$). Recall the 2-category defined in Section 2.5. From **Theorem 2.21** (and the fact that $\mathcal{O}_0$ is equivalent to $A\text{-mod}$ where A is the finite dimensional $\mathbb{C}$-algebra $\mathrm{End}_{\mathcal{O}}(\oplus_w \mathrm{P}_w)^{\mathrm{opp}}$) it follows that $\mathcal{O}_0$ is $\mathbb{C}$-finitary. Since $\mathfrak{P}$ is also stable under the formation of adjoint functors we have a fiat category. The cells are described via the Robinson–Schensted correspondence (see [KL79], the projective functors corresponds to elements in the Hecke algebra of S_n): A two-sided cell is given by $\{\theta_w\}_w$ where w runs through all permutation corresponding to tableaux of a fixed shape. For right cells we require that the right tableau (including its content) is fixed, and similarly for the left cells. It is easily seen that all cells are strongly regular. In Section 1.8 we calculated the cells in the case $n = 3$.

3.3 The 2-category of a Two-sided Cell

Let $\mathcal{Q} \subseteq \mathcal{C}$ be a two-sided cell and let $\mathcal{R} \subseteq \mathcal{Q}$ be a right cell. We want to replace $\mathscr{C}$ with a smaller 2-category in such a way that it still contains all information about the cell 2-representation $\mathbf{C}_{\mathcal{R}}$.

Denote by $\mathscr{I}_{\mathcal{Q}}$ the 2-ideal of $\mathscr{C}$ consiting of all 2-morphisms which factor through a direct sum of 1-morphisms from the set $\{F | F \not\leq_{LR} \mathcal{Q}\}$. In particular for any F with $F \not\leq_{LR} \mathcal{Q}$ we see that $\mathscr{I}_{\mathcal{Q}}$ contains the identity 2-morphism Id_F. It now makes sense to consider the quotient 2-category $\mathscr{C}/\mathscr{I}_{\mathcal{Q}}$.

Lemma 3.14 ([MM10, Lemma 31]). *The 2-ideal $\mathscr{I}_{\mathcal{Q}}$ annihilates the cell 2-representation $\mathbf{C}_{\mathcal{R}}$. It follows that $\mathbf{C}_{\mathcal{R}}$ is a 2-representation of $\mathscr{C}/\mathscr{I}_{\mathcal{Q}}$.*

Proof. We need to prove that if $\alpha : G \to F_1 \oplus \ldots \oplus F_n \to G'$ is a 2-morphism where any $F_i \not\leq_{LR} \mathcal{Q}$ and $X \in \mathbf{C}_{\mathcal{R}}(\mathtt{i})$ then $\mathbf{C}_{\mathcal{R}}(\alpha)_X = 0$. By the 5-lemma we may assume that $X = F\mathrm{L}_{G_{\mathcal{R}}}$ for some $F \in \mathcal{R}$. Then

$$\mathbf{C}_{\mathcal{R}}(\alpha)_X : G \circ F\mathrm{L}_{G_{\mathcal{R}}} \to \bigoplus_{i=1}^{n} F_i \circ F\mathrm{L}_{G_{\mathcal{R}}} \to G' \circ F\mathrm{L}_{G_{\mathcal{R}}}$$

We claim that the middle term is 0. Assume to the contrary that some $F_i \circ F\mathrm{L}_{G_{\mathcal{R}}} \neq 0$. Then from our alternative characterisation of $\leq_L$ we get that $(F_i \circ F)^* \leq_L G_{\mathcal{R}}$, or equivalently $F_i \circ F \leq_R G^*_{\mathcal{R}}$. It follows that $F_i \leq_{LR} G^*_{\mathcal{R}} \in \mathcal{Q}$, which is a contradiction. □

We now define the 2-category associated to $\mathcal{Q}$, $\mathscr{C}_{\mathcal{Q}}$, as the full 2-subcategory of $\mathscr{C}/\mathscr{I}_{\mathcal{Q}}$ generated by the 1-morphisms (isomorphic to) $\mathbb{1}_{\mathtt{i}}$ and F where $\mathtt{i} \in \mathscr{C}$ and $F \in \mathcal{Q}$. If $\mathcal{Q}$ is strongly regular then $\mathcal{Q}$ is in fact also a two-sided cell for $\mathscr{C}$ (even though there are possibly fewer 1-morphisms to use in the relation) by [MM10, Proposition 32]. As promised the 2-category $\mathscr{C}_{\mathcal{Q}}$ can be used to study the cell 2-representations.

Proposition 3.15 ([MM10, Corollary 33])**.** *Let $\mathcal{Q} \subseteq \mathcal{C}$ be a strongly regular two-sided cell and let $\mathcal{R}$ be a right cell contained in $\mathcal{Q}$. The restriction of $\mathbf{C}_{\mathcal{R}}$ from $\mathscr{C}$ to $\mathscr{C}_{\mathcal{Q}}$ is in fact the cell 2-representation of $\mathscr{C}_{\mathcal{Q}}$ belonging to $\mathcal{R}$.*

As usual we refer to [MM10] for a proof. The idea of the proof is to use the morphism from the cell 2-representation of $\mathscr{C}_{\mathcal{Q}}$ from **Theorem 3.10** corresponding to $M = \mathrm{L}_{G_{\mathcal{R}}}$.

Because of these results we may assume that $\mathscr{C} = \mathscr{C}_{\mathcal{Q}}$.

The following proposition is about the combinatorics of the cell 2-representation.

Proposition 3.16 ([MM10, Proposition 34])**.** *Let $F, G \in \mathcal{Q}$.*

a) We have $H^* \circ F \simeq m_{F,H} G$ where $m_{F,H} \in \mathbb{N}$ and $\{G\} = \mathcal{L}_{H^*} \cap \mathcal{R}_F$. Also $m_{F,F} \neq 0$.

b) If F and H belong to the same right cell then $m_{F,H} = m_{H,F}$.

c) If $H = H^*$ and belongs to the same right cell as F then $F \circ H \simeq m_{H,H} F$ and $H \circ F^* \simeq m_{H,H} F^*$.

d) If $G^* = G \in \mathcal{R}_F$, and $H = H^*$ belongs to the left cell of F then

$$m_{F,F} m_{G,G} = m_{F^*,F^*} m_{H,H}$$

Sketch of proof. For a) assume that G is an indecomposable summand of $H^* \circ F$. By definition then G is (the unique element) in the right cell of F and the left cell of H^*. We have seen that $F\mathrm{L}_{G_{\mathcal{R}_F}} \neq 0$, hence by adjunction $F^* \circ F \neq 0$, and $m_{F,F} \neq 0$.

If $F \sim_R H$ then $\mathcal{L}_{F^*} = \mathcal{L}_{H^*}$, and the functor $G_{\mathcal{R}_F} \sim_R F$, hence $F^* \sim_L G^*_{\mathcal{R}_F} = G_{\mathcal{R}_F} \in \mathcal{L}_{H^*} \cap \mathcal{R}_F$. Hence $H^* \circ F$ is a multiple of a selfadjoint functor, hence selfadjoint. Therefore $m_{H,F} = m_{F,H}$, which proves b).

For c) notice that if $H^* = H \sim_R F$ then $F \in \mathcal{L}_F \cap \mathcal{R}_H$ and $F \circ H$ is a multiple of F. say $F \circ H = kF$. Then $k \neq 0$ by **Proposition 3.7** and

$$k^2 F = kF \circ H = F \circ H \circ H = F \circ H^* \circ H = m_{H,H} F \circ H = m_{H,H} kF$$

For the claim in d) we calculate

$$m_{F,F} m_{G,G} F = m_{F,F} F \circ G = F \circ F^* \circ F = m_{F^*,F^*} H \circ F = m_{F^*,F^*} m_{H,H} F \qquad \square$$

3.4 Simple 2-representations

In [MM10] they define the so called *abelian envelope*, $\hat{\mathscr{C}}$, of the 2-category $\mathscr{C}$. It is a 2-category that resembles $\overline{\mathscr{C}}$ The motivation is that it would have been nice if $\overline{\mathscr{C}}$ was a 2-category, however it seems to contain to many objects and is only a bicategory (a 2-category where the axioms only hold up to isomorphisms). The abelian envelope is needed in the following definition.

Definition 3.17 ([MM10])**.** *Let $\mathbf{M} : \mathscr{C} \to \mathfrak{R}_{\Bbbk}$ be a 2-representation. We say that $M \in \mathbf{M}(\mathtt{i})$* generates $\mathbf{M}$ *if for any $X, Y \in \mathbf{M}(\mathtt{j})$ there exists $F, G \in \hat{\mathscr{C}}(\mathtt{i}, \mathtt{j})$ such that $FM \simeq X$ and $GM \simeq Y$, and such that the evaluation map*

$$\mathrm{Hom}_{\hat{\mathscr{C}}(\mathtt{i},\mathtt{j})}(F, G) \to \mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(FM, GM)$$

is surjective. If such an M exists we say that $\mathbf{M}$ is cyclic.

We say that $\mathbf{M}$ is quasi-simple *if it it non-trivial and generated by a simple module (in particular if $\mathbf{M}$ is quasi-simple, then it is cyclic).*

If $\mathbf{M}$ is quasi-simple we say that it is strongly simple *if it is generated by any simple module.*

Both the principal and the cell 2-representations are cyclic.

Proposition 3.18 ([MM10, Proposition 40]).

a) Let $\mathtt{i} \in \mathscr{C}$*. The* $\mathtt{i}$*'th principal* 2*-representation* $\mathbf{P}_{\mathtt{i}}$ *is generated by* $\mathrm{P}_{\mathbb{1}_{\mathtt{i}}}$.

b) Let $\mathcal{R}$ *be a right cell. The cell* 2*-representation* $\mathbf{C}_{\mathcal{R}}$ *is generated by* $\mathrm{L}_{G_{\mathcal{R}}}$.

If $\mathcal{R} \subseteq \mathcal{Q}$ is a strongly regular right cell contained in a two-sided cell then the 2-category $\mathscr{C}_{\mathcal{Q}}$ can be used to decide if the cell 2-representation $\mathbf{C}_{\mathcal{R}}$ is strongly simple or not.

Proposition 3.19 ([MM10, Proposition 41]). *Let* $\mathcal{Q}$ *be a strongly regular two-sided cell, and let* $\mathcal{R} \subseteq \mathcal{Q}$ *be a right cell. The* 2*-representation* $\mathbf{C}_{\mathcal{R}}$ *is strongly simple if and only if its restriction to* $\mathscr{C}_{\mathcal{Q}}$ *is.*

The following theorem is the main result of [MM10].

Theorem 3.20 ([MM10, Theorem 43]). *Let* $\mathcal{Q}$ *be a strongly regular two-sided cell. Assume that the function*

$$\mathcal{Q} \ni F \mapsto m_{F,F} \in \mathbb{N}$$

is constant on each of the left cells of $\mathcal{Q}$*.*

a) Let $\mathcal{R}$ *be a right cell in* $\mathcal{Q}$ *then the cell* 2*-representation* $\mathbf{C}_{\mathcal{R}}$ *is strongly simple.*

b) All of the cell 2*-representations belonging to right cells of* $\mathcal{Q}$ *are equivalent.*

Example 3.21. In the example with the BGG category $\mathcal{O}_0$ in type A it is possible to prove that the condition of the theorem is satisfied. This means that all cell 2-representations corresponding to right cells in a fixed two-sided cell are equivalent. The cell 2-representations actually categorify the cell modules from [MS08] and the setup given in [MM10] thus reproves the results from [MS08] mentioned in the introduction.

What we want to do here is to propose a third notion of simplicity. We look at the image of a 2-representation.

Let $\mathbf{M} : \mathscr{C} \to \mathfrak{R}_{\Bbbk}$ be a 2-representation. Consider the 2-category $\mathscr{D} = \mathbf{M}(\mathscr{C})$ given by

- 0-cells are all $\mathbf{M}(\mathtt{i})$ where $\mathtt{i} \in \mathscr{C}$ is a 0-cell.
- 1-cells $\mathscr{D}(\mathbf{M}(\mathtt{i}), \mathbf{M}(\mathtt{i}))$ are all $\mathbf{M}(F)$ where $F \in \mathscr{C}(\mathtt{i}, \mathtt{j})$.
- 2-cells from $\mathbf{M}(F)$ to $\mathbf{M}(G)$ are all $\mathbf{M}(\alpha)$ where $\alpha : F \to G$ is a 2-cell.

This is in fact a $\Bbbk$-finitary 2-category since $\mathbf{M}$ is a 2-functor.

For each 0-cell $\mathtt{i} \in \mathscr{C}$ the category $\mathbf{M}(\mathtt{i})$ is equivalent to a category of modules over some finite dimensional $\Bbbk$-algebra.

Definition 3.22. *Let* $\mathbf{M} : \mathscr{C} \to \mathfrak{R}_{\Bbbk}$ *be a (non-trivial)* 2*-representation. We say that* $\mathbf{M}$ *is* almost simple*, if for any non-zero* 2*-ideal* $\mathscr{I} \subseteq \mathbf{M}(\mathscr{C})$ *in the image of* $\mathbf{M}$ *there exists a* 1*-cell* $F \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ *such that* $\mathrm{Id}_{\mathbf{M}(F)} \in \mathscr{I}$*, but* $\mathbf{M}(F) \not\simeq 0$*. If any non-zero* 2*-ideal contains* $\mathrm{Id}_{\mathbf{M}(F)}$ *for all* F *(hence all* 2*-cells from the image of* $\mathbf{M}$*) we say that* $\mathbf{M}$ *is* simple*.*

We see that if $\mathbf{M}$ is a simple 2-representation and $\mathscr{I} \subseteq \mathscr{D} = \mathbf{M}(\mathscr{C})$ is a non-zero 2-ideal then the quotient $\mathscr{D}/\mathscr{I}$ is a 2-category in which all objects, 1-morphisms and 2-morphisms are identified with 0.

3.5 An Example [MM10, Section 7.3]

Recall that a self-injective $\Bbbk$-algebra is an algebra such that the regular A-module ${}_AA$ is injective. Notice that for such an algebra any indecomposable projective A-module is injective. Recall also that such an algebra is called weakly symmetric, if for any indecomposable projective A-module the top and socle are isomorphic (i.e. the Nakayama permutation, mapping a simple module to the socle of its projective cover, is the identity).

Finally recall that an algebra A is called *connected* if it cannot be written as $A = B \oplus C$ the direct sum of two unital algebras B and C. Equivalently for any non-trivial idempotent $e \in A$, $e \neq 0, 1$, we have $A \neq eAe \oplus (1-e)A(1-e)$.

We use the notation $\otimes = \underset{\Bbbk}{\otimes}$.

Let A be a finite dimensional self-injective, weakly symmetric $\Bbbk$-algebra.

Clearly since A is finite dimensional, we may write $A = A_1 \oplus \ldots \oplus A_k$ where each A_i is a connected algebra.

Definition 3.23. *A functor $F : A_i\text{-mod} \to A_j\text{-mod}$ is called* projective *if it is isomorphic to a direct sum of functors of the form $A_je \otimes fA_i \underset{A_i}{\otimes} \_$ where $e \in A_j$ and $f \in A_i$ are idempotents.*

We see that the composition of two projective functors is again a projective functor. Therefore it makes sense to define the following 2-category. Denote by $\mathscr{C}_A$ the $\Bbbk$-finitary 2-category with

- 0-cells are $\mathtt{1}, \mathtt{2}, \ldots \mathtt{k}$ (each identified with a category $\mathtt{i} \leftrightarrow A_i\text{-mod}$).
- 1-cells from $\mathtt{i}$ to $\mathtt{j}$ are generated by projective functors from $A_i\text{-mod}$ to $A_j\text{-mod}$ and, if $\mathtt{i} = \mathtt{j}$, the identity functor.
- 2-cells are natural transformations of the 1-cells.

We begin by calculating the adjoint functors of $A_je \otimes fA_i \underset{A_i}{\otimes} \_$. The right adjoint of this functor is $\mathrm{Hom}_{A_j}(A_je \otimes fA_i, \_)$, now we calculate:

$$\begin{aligned}
\mathrm{Hom}_{A_j}(A_je \otimes fA_i, M) &\simeq \mathrm{Hom}_{\Bbbk}(fA_i, \mathrm{Hom}_{A_j}(A_je, M)) \\
&\simeq \mathrm{Hom}_{\Bbbk}(fA_i, eM) \\
&\simeq \mathrm{Hom}_{\Bbbk}(fA_i, eA_j \underset{A_j}{\otimes} M) \\
&\simeq \mathrm{Hom}_{\Bbbk}(fA_i, \Bbbk) \otimes eA_j \underset{A_j}{\otimes} M \\
&\simeq (fA_i)^* \otimes eA_j \underset{A_j}{\otimes} M
\end{aligned}$$

Now $(fA_i)^*$ is an indecomposable injective A_i-module, and since A, hence also A_i, is self-injective this module is projective. Thus it is an indecomposable projective. Since A, hence also A_i, is weakly symmetric its simple socle and simple top are isomorphic. It follows that $(fA_i)^* \simeq A_if$. Therefore $\mathrm{Hom}_{A_j}(A_je \otimes fA_i, \_)$ is isomorphic to the projective functor $A_if \otimes eA_j \underset{A_j}{\otimes} \_$ which in turn has $\mathrm{Hom}_{A_i}(A_if \otimes eA_j, \_) \simeq A_je \otimes fA_i \underset{A_i}{\otimes} \_$ as right adjoint. We see that the functors are biadjoint.

This proves that the 2-category $\mathscr{C}_A$ is not only $\Bbbk$-finitary but acutally fiat, hence we may consider its cell 2-representations.

Remark 3.24. The reader may check that the following makes sense in greater generality, and in fact the results from [MM10] hold (mutatis mutandis) if we do not assume that * maps F to a *biadjoint* of F but only to a *right adjoint* (then * is not an involution but only of finite order). In this setup we may assume that A is merely a self-injective finite dimensional $\Bbbk$-algebra.

We want to compute the left, right and two-sided cells of a given indecomposable projective functor.

Given idempotents $e \in A_j$ and $f \in A_i$ we introduce the notation F^{ji}_{ef} for the projective functor from A_i-mod to A_j-mod that tensors with $A_j e \otimes f A_i$. Notice that if $F^{sr}_{gh} \circ F^{ji}_{ef} \neq 0$ then $j = r$. If $j = r$ we see that

$$F^{sj}_{gh} \circ F^{ji}_{ef} = (\dim_{\Bbbk} hA_j e) F^{si}_{gf}$$

This means that

$$\{F | F \leq_R F^{ji}_{ef}\} = \{F^{si}_{gf} | s \in \{1, \dots, k\}, g \in A_s : g^2 = g\}$$

and from this describtion we see that also

$$\mathcal{R}^i_f = \mathcal{R}_{F^{ji}_{ef}} = \{F^{si}_{gf} | s \in \{1, \dots, k\}, g \in A_s : g^2 = g\}$$

In the same way it follows that

$$\mathcal{L}^j_e = \mathcal{L}_{F^{ji}_{ef}} = \{F^{jr}_{eh} | r \in \{1, \dots, k\}, h \in A_s : h^2 = h\}$$

Hence all projective functors is in the same two-sided cell. If some $\mathbb{1}_{\mathtt{i}}$ is projective (e.g. if $A_{\mathtt{i}} = \mathbb{C}$) this functor occurs in the two-sided cell of projective functors. On the other hand if $\mathbb{1}_{\mathtt{i}}$ is not a projective functor then $\{\mathbb{1}_{\mathtt{i}}\}$ is a two-sided cell.

Let $\mathcal{Q}$ be the two-sided cell of projective functors, and write $\mathscr{C} = \mathscr{C}_A$. The ideal $\mathscr{I}_{\mathcal{Q}}$ is zero and it can easily be seen that $\mathscr{C}_{\mathcal{Q}}$ is (equivalent to) the whole 2-category $\mathscr{C}$. This 2-category comes with its defining 2-representation where $\mathtt{i}$ is mapped to A_i-mod and each projective functor acts as itself. Given a right cell $\mathcal{R} \subseteq \mathcal{Q}$ this is in fact the cell 2-representation corresponding to $\mathcal{R}$ as one can prove using **Theorem 3.10** (for details we refer again to [MM10]).

We see that $\mathcal{Q}$ is strongly regular since both left and right cells are indexed by pairs (e, j) where $e \in A_j$ is an idempotent, and

$$\mathcal{R}^i_f \cap \mathcal{L}^j_e = \{F^{ji}_{ef}\}$$

Additionally the function $F \mapsto m_{F,F}$ is given by

$$F^{ji}_{ef} \mapsto \dim_{\Bbbk} eA_j e$$

In particular it is constant on the left cells. Therefore the cell 2-representation is strongly simple by **Theorem 3.20**. We can prove more than that.

Lemma 3.25. *The cell 2-representation of $\mathscr{C} = \mathscr{C}_A$ corresponding to a right cell in $\mathcal{Q}$ is simple.*

Remark 3.26. The idea of the proof is that a 2-ideal is closed under all possible compositions, both vertical and horizontal. This means that if a 2-ideal is non-zero it contains a lot of 2-morphisms. More specifically we take a non-zero 2-morphism and by composing with identity 2-morphisms on appropiate 1-morphisms we can generate other 2-morphisms. We use one special feature of the projective functors, namely that they are given by a restriction to $\Bbbk$-mod composed by an induction. Therefore the method unfortunately does not obviously generalize to the general setup. We give more details in the following proof.

Proof. Assume $\mathscr{I}$ is a non-zero 2-ideal in $\mathscr{C}$, and let $\alpha \in \mathscr{I}$ be a non-zero 2-morphism, say $\alpha : F \to G$. Each of F and G decomposes into a sum of indecomposables, say $F = \bigoplus F_i$ and $G = \bigoplus G_j$. This means that we have natural inclusions $F_i \to F$ and projections $G \to G_j$, these are natural transformations, and because $\mathscr{I}$ is a 2-ideal we see that each of the compositions

$$F_i \longrightarrow F \xrightarrow{\alpha} G \longrightarrow G_j$$

lies in $\mathscr{I}$. Also α is the sum of all these, hence one of them must be non-zero. This shows that we may assume that $\alpha : F \to G$ is a natural transformation between indecomposables.

Let us assume that $F = F^{ji}_{ef}$ and that $G = F^{ji}_{e'f'}$, here $e, e' \in A_j$ and $f, f' \in A_i$ are primitive idempotens.

Given $H = F^{st}_{gh}$ we need to show that Id_H lies in $\mathscr{I}$.

Consider the sum of all non-isomorphic indecomposable projective functors from A_τ-mod to A_σ-mod

$$\mathcal{P}_{\sigma\tau} := \bigoplus_{\varepsilon,\varphi} \left(A_\sigma \varepsilon \otimes \varphi A_\tau \underset{A_\tau}{\otimes} \_ \right) = \left(\bigoplus_{\varepsilon} A_\sigma \varepsilon \right) \otimes \left(\bigoplus_{\varphi} \varphi A_\tau \right) \underset{A_\tau}{\otimes} \_ = A_\sigma \otimes A_\tau \underset{A_\tau}{\otimes} \_$$

where we sum over all primitive idempotents $\varepsilon \in A_\sigma$ and $\varphi \in A_\tau$.

Consider also the identity natural transformation on this functor

$$\gamma_{\sigma\tau} = \mathrm{Id}_{\mathcal{P}_{\sigma\tau}} = \mathrm{Id}_{A_\sigma \otimes A_\tau \underset{A_\tau}{\otimes} \_}$$

The functor

$$\mathcal{P}_{sj} \circ F^{ji}_{ef} \circ \mathcal{P}_{it} = A_s \otimes (A_j \underset{A_j}{\otimes} A_j e \otimes f A_i \underset{A_i}{\otimes} A_i) \otimes A_t \underset{A_t}{\otimes} \_ = m(A_s \otimes A_t \underset{A_t}{\otimes} \_)$$

is the direct sum of $m = \dim_{\Bbbk} A_j e \otimes f A_i$ copies of the functor $A_s \otimes A_t \underset{A_t}{\otimes} \_$. Similarly we get

$$\mathcal{P}_{sj} \circ F^{ji}_{e'f'} \circ \mathcal{P}_{it} = m'(A_s \otimes A_t \underset{A_t}{\otimes} \_)$$

with $m' = \dim_{\Bbbk} A_j e' \otimes f' A_i$.

Calculating the composition of the identities on $\mathcal{P}_{sj}$ and on $\mathcal{P}_{it}$ and our given α we get a natural transformation

$$\tilde{\alpha} := \gamma_{sj} \circ_0 \alpha \circ_0 \gamma_{it} : (A_s \otimes A_t \underset{A_t}{\otimes} \_)^{\oplus m} \to (A_s \otimes A_t \underset{A_t}{\otimes} \_)^{\oplus m'}$$

and this 2-morphism is part of our 2-ideal $\mathscr{I}$.

Choosing bases for $A_j e \otimes f A_i$ and $A_j e' \otimes f' A_i$ as vector spaces over $\Bbbk$ we get bases for the bimodules corresponding to the 1-cells $\mathcal{P}_{sj} \circ F^{ji}_{ef} \circ \mathcal{P}_{it}$ and $\mathcal{P}_{sj} \circ F^{ji}_{e'f'} \circ \mathcal{P}_{it}$ simply by tensoring with the identities $1 \in A_j$ and $1 \in A_j$. Calculating the effect of $\tilde{\alpha}$ in these bases we get an $m' \times m$ matrix with entries in $\Bbbk$. Since α is non-zero one of the entries must be non-zero, and after composing with inclusion and projection we get a natural isomorphism proportional to the identity transformation of $A_s \otimes A_t \underset{A_t}{\otimes} \_$.

More specifically let $(x_k)_{k=1}^m$ be a basis of $A_j e \otimes f A_i$ and $(y_l)_{l=1}^{m'}$ a basis of $A_j e' \otimes f' A_i$ as $\Bbbk$-modules. The natural transformation α of the corresponding functors induces a homomorphism of the bimodules, we abuse notation, and call this α as well. Write

$$\alpha(x_k) = \sum_{l=1}^{m'} \alpha_{kl} y_l$$

with each $\alpha_{kl} \in \Bbbk$.

Since α is non-zero one of the matrix elements α_{kl} is non-zero.

The bimodule corresponding to the functor $\mathcal{P}_{sj} \circ F^{ji}_{ef} \circ \mathcal{P}_{it}$ is

$$A_s \otimes A_j e \otimes f A_i \otimes A_t = \bigoplus_{k=1}^{m} A_s \otimes x_k \Bbbk \otimes A_t$$

and has a basis as an A_s-A_t-bimodule given by

$$\{1 \otimes x_k \otimes 1 | k = 1, \ldots, m\}$$

Similarly the the bimodule corresponding to the functor $\mathcal{P}_{sj} \circ F^{ji}_{e'f'} \circ \mathcal{P}_{it}$ has a basis as an A_s-A_t-bimodule

$$\{1 \otimes y_k \otimes 1 | k = 1, \ldots, m'\}$$

We calculate the effect of the bimodule map induced by $\tilde{\alpha}$ calculated in these bases:

$$\tilde{\alpha}(1 \otimes x_k \otimes 1) = 1 \otimes \alpha(x_k) \otimes 1 = \sum_{l=1}^{m'} \alpha_{kl} 1 \otimes y_l \otimes 1$$

Hence we recover the matrix $(\alpha_{kl})_{k,l}$ (here the entries are elements of $\Bbbk$ but should be considered as elements of the $\Bbbk$-algebra $A_s \otimes A_t$), at least one of these entries, $\alpha_{\kappa\lambda}$, is non-zero.

Composing the inclusion of the κ'th copy of the functor $A_s \otimes A_t \underset{A_t}{\otimes} \_$ with $\tilde{\alpha}$ and further with the projection on the λ'th copy of $A_s \otimes A_t \underset{A_t}{\otimes} \_$ we get the natural transformation

$$\alpha_{\kappa\lambda} \operatorname{Id}_{A_s \otimes A_t \underset{A_t}{\otimes} \_} : A_s \otimes A_t \underset{A_t}{\otimes} \_ \to A_s \otimes A_t \underset{A_t}{\otimes} \_$$

in our 2-ideal $\mathscr{I}$. Rescaling we see that $\operatorname{Id}_{A_s \otimes A_t \underset{A_t}{\otimes} \_} \in \mathscr{I}$.

Now it is only left to notice that the functor $H = A_s g \otimes h A_t$ is a direct summand of $A_s \otimes A_t \underset{A_t}{\otimes} \_$, hence also the identity transformation $\operatorname{Id}_H \in \mathscr{I}$. □

In [MM10] they prove the following proposition.

Proposition 3.27 ([MM10, Proposition 46]). *Let $\mathscr{C}$ be a fiat category, and $\mathcal{R} \subseteq \mathcal{Q}$ a right cell contained in a two-sided cell of $\mathscr{C}$. For $\mathtt{i} \in \mathscr{C}$ choose a finite dimensional $\Bbbk$-algebra, $A_{\mathtt{i}}$, such that $\mathbf{C}_{\mathcal{R}}(\mathtt{i}) \simeq A_{\mathtt{i}}$-mod, and define $A = \oplus_{\mathtt{i} \in \mathscr{C}} A_{\mathtt{i}}$. Assume that the assumption in **Theorem 3.20** is satisfied. Then A is a weakly symmetric self-injective $\Bbbk$-algebra and $\mathbf{C}_{\mathcal{R}}$ gives rise to a 2-functor*

$$\mathbf{C}_{\mathcal{R}} : \mathscr{C}_{\mathcal{Q}} \to \mathscr{C}_A$$

From **Lemma 3.25** and the proposition we get the following corollary.

Corollary 3.28. *Let $\mathscr{C}$ be as in the proposition. The cell 2-representation $\mathbf{C}_{\mathcal{R}}$ is simple.*

Chapter 4

Classification of Categorifications

This chapter is a presentation of some of the results from [AM11], this is joint work with V. Mazorchuk, but he should not be blamed for the details given here.

We consider weak categorifications of modules over a unital, associative algebra Λ. It is natural to ask whether, given Λ and a set of generators $\{a_i | i \in I\} \subseteq \Lambda$, one can classify all weak categorifications of all Λ-modules (up to e.g. equivalence of categories). It seems hopeless for general rings (cf. wild algebras) to classify all modules so here we will have to deal only with the very basic examples.

We fix an algebraically closed field $\Bbbk$. In this chapter we will use the notation $\otimes = \underset{\Bbbk}{\otimes}$ and $\dim = \dim_{\Bbbk}$. All Grothendieck groups will be $\Bbbk$-modules, i.e. we write $[\mathcal{C}] = [\mathcal{C}]^{\Bbbk}$ given an abelian category $\mathcal{C}$.

Let Λ be a unital, finite dimensional, associative $\Bbbk$-algebra generated by a single element $a \in \Lambda$. Since Λ is finite dimensional we may choose $f \in \Bbbk[X] \setminus \{0\}$ (of minimal degree) with $f(a) = 0$. A weak categorification of a Λ-module is an abelian category, $\mathcal{C}$ with an exact endofunctor $F : \mathcal{C} \to \mathcal{C}$ satisfying some resonable version of $f(F) \simeq 0$. We will restrict our setup to noetherian, $\Bbbk$-linear categories with a projective generator, hence we may assume that $\mathcal{C} = A\text{-mod}$ for some finite dimensional $\Bbbk$-algebra. All functors considered are tacitly assumed to be $\Bbbk$-linear. In order to simplify things even more we want to categorify the trivial involution (i.e. the identity mapping) of Λ via taking the biadjoint functor: We require F to be a selfadjoint functor. In order to make sense to the isomorphism $f(F) \simeq 0$, we assume $f \in \mathbb{Z}[X]$ has integral coefficients and we rewrite $f(a) = 0$ as $g(a) = h(a)$ where $g, h \in \mathbb{N}[X]$. For any exact functor G and any natural number n we define

$$nG = \begin{cases} G \oplus \cdots \oplus G & n > 0 \\ 0 & n = 0 \end{cases}$$

If we also replace 1 with the identity functor $\mathbb{1}$, it makes sense to require

$$g(F) \simeq h(F) \tag{4.1}$$

where $+$ is replaced with $\oplus$.

Recall that a finite dimensional $\Bbbk$-algebra is called basic if any simple module has dimension 1, or equivalently if any indecomposable projective occur with multiplicity 1 in the regular module ${}_AA$. To see this equivalence write ${}_AA = \bigoplus_i \mathrm{P}_i^{n_i}$ where the P_i's are the non-isomorphic projective indecomposables occurring with multiplicities n_i. Denote by L_i the simple head of

P_i, then we see that

$$\begin{aligned}\dim_{\Bbbk} \mathrm{L}_i &= \dim_{\Bbbk} \mathrm{Hom}_A(A, \mathrm{L}_i) \\ &= \dim_{\Bbbk} \bigoplus_j \mathrm{Hom}_A(\mathrm{P}_j, \mathrm{L}_i)^{n_j} \\ &= \sum_j n_j \dim_{\Bbbk} \mathrm{Hom}_A(\mathrm{P}_j, \mathrm{L}_i) \\ &= n_i \dim_{\Bbbk} \mathrm{End}_A(\mathrm{L}_i) \\ &= n_i\end{aligned}$$

where we used that $\mathrm{End}_A(\mathrm{L}_i)$ is a finite dimensional division algebra over $\Bbbk$ hence has dimension 1.

Because of the following result we only consider basic algebras.

Proposition 4.1. *Let A be a finite dimensional $\Bbbk$-algebra. There exists a finite dimensional $\Bbbk$-algebra, B, such that B is a basic algebra, with A and B being morita equivalent, i.e. there exists an equivalence of categories*

$$F : A\text{-mod} \to B\text{-mod}$$

Proof. Let $\mathrm{P}_1, \dots, \mathrm{P}_n$ be a complete list of pairwise non-isomorphic, finitely generated indecomposable projective A-modules. Then $\mathrm{P} = \mathrm{P}_1 \oplus \ldots \oplus \mathrm{P}_n$ is a projective generator for A-mod, and if we define

$$B = \mathrm{End}_A(\mathrm{P})^{\mathrm{opp}}$$

then B is a basic algebra (the projective indecomposables are of the form $\mathrm{Hom}_A(\mathrm{P}, \mathrm{P}_i)$), and the functor

$$\mathbb{V} = \mathrm{Hom}_A(\mathrm{P}, \_\,) : A\text{-mod} \to B\text{-mod}$$

is an equivalence of categories, since P is a projective generator. □

In this chapter by an algebra, unless otherwise specified, we will mean a finite dimensional, unital, associative, basic $\Bbbk$-algebra. By a module we will always mean a finitely generated (hence finite dimensional) one.

4.1 Functors as Linear Operators on the Grothendieck Group

Let A be an algebra and assume $F : A\text{-mod} \to A\text{-mod}$ is an exact (e.g. selfadjoint) functor. Then F induces a map $[F] : [A\text{-mod}] \to [A\text{-mod}]$ of the Grothendieck group. Fix a complete set of non-isomorphic indecomposable projective modules $\mathrm{P}_1, \dots, \mathrm{P}_n$ and denote by L_i the simple head of P_i. Since A is finite dimensional all (finitely generated) modules have finite length, and hence $([\mathrm{L}_1], \dots, [\mathrm{L}_n])$ is a basis for $[A\text{-mod}]$. The matrix for $[F]$ in this basis we denote by M_F, i.e.

$$(M_F)_{ij} = \dim_{\Bbbk} \mathrm{Hom}_A(\mathrm{P}_i, F\mathrm{L}_j)$$

If F satisfies the isomorphism (4.1), then on the Grothendieck group we get

$$g([F]) = [g(F)] = [h(F)] = h([F])$$

and also

$$g(M_F) = M_{g(F)} = M_{h(F)} = h(M_F)$$

This means that the matrix M_F satisfies a similar equation as the one F satisfies. Hence it can often be convenient to solve a matrix equation before trying to solve the isomorphism involving the functor. Sometimes this is even enough as will be seen in the following section.

4.2 Categorifications via Semisimple Categories

The category $A\text{-mod}$ is semisimple whenever every module is semisimple. In particular an indecomposable projective P is semisimple, and hence simple (and of dimension 1 since A is assumed to be basic). Let $\mathrm{P}_i = \mathrm{L}_i$, $i = 1, \ldots, n$, be the (non-isomorphic) simple (and projective) A-modules. Now we get an isomorphism of $\Bbbk$-algbras:

$$A^{\mathrm{opp}} \simeq \mathrm{End}_A({}_AA) \simeq \mathrm{End}_A(\oplus_{i=1}^n \mathrm{L}_i) \simeq \bigoplus_{i=1}^n \mathrm{End}_A(\mathrm{L}_i) \simeq \bigoplus_{i=1}^n \Bbbk$$

Then also $A \simeq \oplus_{i=1}^n \Bbbk$, and $A\text{-mod} \simeq \oplus_{i=1}^n \Bbbk\text{-mod}$.

Endofunctors of $A\text{-mod}$ are particularly easy to describe, up to isomorphism they correspond to $n \times n$ matrices with non-negative integral entries.

Recall that for each $n \in \mathbb{N}$ there is a unique (up to isomorphism) exact, $\Bbbk$-linear endofunctor of $\Bbbk\text{-mod}$ mapping $\Bbbk$ to $\Bbbk^n$, namely the sum of n copies of the identity functor. The easiest way to see this is probably using the equivalence between $\Bbbk\text{-mod}$ and the category with objects $\mathbb{N}$ and where morphisms from i to j are $j \times i$ matrices over $\Bbbk$: here the considered endofunctors are uniquely determined by the value on 1. Consequently since any functor $F : A\text{-mod} \to A\text{-mod}$ decomposes into functors between the various $\Bbbk\text{-mod}$, we see that F is given by a matrix $M_F \in \mathrm{Mat}_n(\mathbb{N})$ (with the obvious correspondence).

It is natural to ask when the functor F corresponding to the matrix M_F is selfadjoint. To this end notice that if F is selfadjoint then in the notation from above

$$\begin{aligned}
(M_F)_{ij} &= [F\mathrm{P}_j : \mathrm{L}_i] \\
&= [F\mathrm{L}_j : \mathrm{L}_i] \\
&= \dim_\Bbbk \mathrm{Hom}_A(\mathrm{P}_i, F\mathrm{L}_j) \\
&= \dim_\Bbbk \mathrm{Hom}_A(F\mathrm{P}_i, \mathrm{L}_j) \\
&= \dim_\Bbbk \mathrm{Hom}_A(F\mathrm{L}_i, \mathrm{L}_j) \\
&= [F\mathrm{L}_i : \mathrm{L}_j] \\
&= [F\mathrm{P}_i : \mathrm{L}_j] \\
&= (M_F)_{ji}
\end{aligned}$$

We used that $F\mathrm{L}_i$ is a semisimple module. This means that M_F is a symmetric matrix. On the other hand if M_F is symmetric then F is selfadjoint (as can be seen by a direct calculation, which we have essentially just done).

Let $g, h \in \mathbb{N}[X]$, then it makes sense to calculate $g(F)$ and $h(F)$ using the conventions introduced above. Now since $M_{g(F)} = g(M_F)$ it is clear that F satisfies $g(F) \simeq h(F)$ if and only if $g(M_F) = h(M_F)$. This means that weakly categorifying via semisimple categories is just a matter of solving matrix equations. If we impose the demand that F is selfadjoint we should solve the matrix equation with symmetric matrices.

Example 4.2. One might think that if F is selfadjoint then M_F is always symmetric. Let us give a counter example to this statement. Consider the BGG category $\mathcal{O}(\mathfrak{sl}_2)_0$ in rank 1 and the translation through the wall θ_s. This category has two simple modules L_e and $\mathrm{L}_s = \mathrm{M}_s$. Using the short exact sequences

$$0 \longrightarrow \mathrm{M}_e \longrightarrow \theta_s\mathrm{M}_s = \theta_s\mathrm{M}_e \longrightarrow \mathrm{M}_s \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{L}_s \longrightarrow \mathrm{M}_e \longrightarrow \mathrm{L}_e \longrightarrow 0$$

one sees that $\theta_s\mathrm{L}_e = 0$, but $\theta_s\mathrm{L}_s$ has a filtration involving L_s twice and L_e once. The functor θ_s is selfadjoint with

$$M_{\theta_s} = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$$

which is not symmetric.

4.3 Group Actions on Module Categories

Let G be a group.

Definition 4.3. *A* weak group action *of G on the abelian category $\mathcal{C}$ is a collection of endofunctors $\{F_g | g \in G\}$ of $\mathcal{C}$ with*

$$F_g \circ F_h \simeq F_{gh}$$

for any $g, h \in G$, and with $F_1 \simeq \mathbb{1}_{\mathcal{C}}$. If all isomorphisms can be replaced with equalities we say that the group action is strong.

Two group actions $\{F_g | g \in G\}$ and $\{F'_g | g \in G\}$ are called equivalent if $F_g \simeq F'_g$ for any $g \in G$.

Let A be an algebra, and consider the category $\mathcal{C} = A\text{-mod}$. We define the group of automorphisms

$$\mathrm{Aut}(A) = \{\varphi : A \to A | \varphi \text{ is an algebra automorphism}\}$$

Given $s \in A^\times$ we define the *inner automorphism* belonging to s, $\varphi_s : A \to A$, via

$$\varphi_s(a) = sas^{-1}$$

which is easily seen to lie in $\mathrm{Aut}(A)$. The set

$$\mathrm{Inn}(A) = \{\varphi_s \in \mathrm{Aut}(A) | s \in A^\times\} \subseteq \mathrm{Aut}(A)$$

is a normal subgroup and we define the group of *outer automorphisms* as the quotient

$$\mathrm{Out}(A) = \mathrm{Aut}(A) / \mathrm{Inn}(A)$$

Proposition 4.4. *Equivalence classes of weak group actions of a group G on A-mod are in one-to-one correspondence with group homomorphisms $G \to \mathrm{Out}(A)$.*

Corollary 4.5. *Selfadjoint endofunctors $F : A\text{-mod} \to A\text{-mod}$ with $F^2 \simeq \mathbb{1}_{A\text{-mod}}$ are, up to isomorphism, in one-to-one correspondence with group homomorphisms from S_2 to $\mathrm{Out}(A)$. In particular if $\Lambda = \Bbbk[a]/(a^2 - 1)$ with the trivial involution $a^* = a$, the class of weak categorifications of finite dimensional Λ-modules corresponds to pairs (A, φ) where A is an algebra and $\varphi \,\mathrm{Inn}(A) \in \mathrm{Out}(A)$ with $\varphi^2 \in \mathrm{Inn}(A)$ (the correspondence will become clear in the proof of the proposition).*

Proof of the corollary. If $F^2 \simeq \mathbb{1}$ then F is a selfadjoint autoequivalence. Thus it gives a weak group action of $\mathbb{Z}_2$ on A-mod. Conversely if $\{F_1, F_s\}$ is a weak group action of $S_2 = \{1, s\}$ then $F_1 \simeq \mathbb{1}$ and hence $F_s^2 \simeq F_{s^2} = F_1 \simeq \mathbb{1}$ and F is a selfadjoint autoequivalence. □

Before we prove the proposition we explore the action of $\mathrm{Aut}(A)$ on A-mod.

Given an element $\varphi \in \mathrm{Aut}(A)$ we define the A-A-bimodule ${}^{\varphi}A$ via

$$a \cdot x \cdot b = \varphi(a)xb, \qquad x \in {}^{\varphi}A, a, b \in A$$

where on the right side of the equation we use the multiplication in A.

Proposition 4.6. *Let $\mathcal{C} = A\text{-mod}$.*

a) $F_\varphi = {}^{\varphi^{-1}}A \underset{A}{\otimes} \_$ *is an autoequivalence of $\mathcal{C}$ for any $\varphi \in \mathrm{Aut}(A)$.*

b) $F_\varphi \circ F_\psi \simeq F_{\varphi \circ \psi}$ *for any $\varphi, \psi \in \mathrm{Aut}(A)$.*

c) Every autoequivalence of A-mod is isomorphic to some F_φ.

d) F_φ *is isomorphic to $\mathbb{1}_{\mathcal{C}}$ exactly when $\varphi \in \mathrm{Inn}(A)$.*

e) F_φ *is selfadjoint if and only if $\varphi^2 \in \mathrm{Inn}(A)$.*

Remark 4.7. If we define $F'_\varphi(M) = {}^{\varphi^{-1}}M$ where for $x \in {}^{\varphi^{-1}}M$ and $a \in A$ we act as (on the right we use the old action of A on M)

$$a \cdot x = \varphi^{-1}(a)x$$

then $F_\varphi \simeq F'_\varphi$ for all $\varphi \in \mathrm{Aut}(A)$ and the isomorphism in b) can be replaced by equalities (i.e. $\mathrm{Aut}(A)$ acts on $\mathcal{C}$ with a strong group action).

Proof. Notice first that

$$\begin{array}{rccc} f: & {}^{\varphi^{-1}}A \underset{A}{\otimes} {}^{\psi^{-1}}A & \to & {}^{(\varphi\circ\psi)^{-1}}A \\ & a \otimes b & \mapsto & \psi^{-1}(a)b \end{array}$$

is an isomorphism of A-A-bimodules:

$$\begin{aligned} f(x \cdot (a \otimes b) \cdot y) &= f(\varphi^{-1}(x)a \otimes by) \\ &= \psi^{-1}(\varphi^{-1}(x)a)by \\ &= (\varphi \circ \psi)^{-1}(x)\psi^{-1}(a)by \\ &= x \cdot \psi^{-1}(a)b \cdot y \\ &= x \cdot f(a \otimes b) \cdot y \end{aligned}$$

Therefore $F_\varphi \circ F_\psi \simeq F_{\varphi\circ\psi}$. In particular F_φ is an autoequivalence:

$$F_\varphi \circ F_{\varphi^{-1}} \simeq F_{\varphi\circ\varphi^{-1}} = F_{\mathrm{id}} = \mathbb{1}_{\mathcal{C}}$$

since obviously

$${}^{\mathrm{id}}A = A$$

as A-A-bimodules. This proves a) and b).

For the proof of c) assume $F : A\text{-mod}$ is an autoequivalence. Since F is an equivalence it is exact and hence maps projectives to projectives: If P is projective then

$$\mathrm{Hom}_A(FP, \_) \simeq \mathrm{Hom}_A(P, F^{-1}\_) \simeq \mathrm{Hom}_A(P, \_) \circ F^{-1}$$

is a composition of exact functors, hence exact. We only used that F has a biadjoint functor (and that this biadjoint is necessarily exact).

If M is indecomposable then

$$\mathrm{End}_A(M) \simeq \mathrm{End}_A(FM)$$

as rings, hence both rings are local. Therefore also FM is indecomposable. (The projection onto a non-zero summand of FM is an idempotent $e \in \mathrm{End}_A(FM)$. Both e and $1 - e$ are non-zero, non-units, hence they each generate an ideal inside the unique maximal ideal in $\mathrm{End}_A(FM)$, so $1 = 1 - e + e$ belongs to the unique maximal ideal, contradiction.)

It follows that F maps indecomposable projectives to indecomposable projectives. The same is true for the quasi-inverse functor of F, and therefore F permutes the indecomposable projectives. We see that $FA \simeq {}_AA$ as left A-modules since A is a basic algebra. We fix an isomorphism $\alpha : {}_AA \to F_AA$ of A-modules.

Recall the isomorphism of algebras

$$\begin{array}{rccc} \beta: & A^{\mathrm{opp}} & \to & \mathrm{End}_A({}_AA) \\ & a & \mapsto & \rho_a \\ & f(1) & \leftarrow\!\shortmid & f \end{array}$$

here $\rho_a(b) = ba$ is right multiplication with a. Notice that $\rho_a(bc) = (bc)a = b(ca) = b\rho_a(c)$. Write $\cdot$ for the multiplication in A^{opp}, then

$$\beta(b \cdot a) = \beta(ab) = \rho_{ab} = \rho_b \circ \rho_a = \beta(b) \circ \beta(a)$$

Thus β is a well-defined algebra homomorphism.

We want to define an algebra automorphism $\varphi : A \to A$, and we consider the A-linear map

$$ {}_AA \xrightarrow{\alpha} F_AA \xrightarrow{F\rho_a} F_AA \xrightarrow{\alpha^{-1}} {}_AA $$

We define for $a \in A$

$$ \varphi(a) = \beta^{-1}(\alpha^{-1} \circ F\rho_a \circ \alpha) \in A $$

We claim that $\varphi : A \to A$ is an algebra automorphism and that $F \simeq F_\varphi$.

For the first claim we calculate

$$ \begin{aligned} \varphi(ab) &= \beta^{-1}(\alpha^{-1} \circ F\rho_{ab} \circ \alpha) \\ &= \beta^{-1}(\alpha^{-1} \circ F(\rho_b \circ \rho_a) \circ \alpha) \\ &= \beta^{-1}(\alpha^{-1} \circ F\rho_b \circ F\rho_a \circ \alpha) \\ &= \beta^{-1}(\alpha^{-1} \circ F\rho_b \circ \alpha \circ \alpha^{-1} \circ F\rho_a \circ \alpha) \\ &= \beta^{-1}(\alpha^{-1} \circ F\rho_b \circ \alpha) \cdot \beta^{-1}(\alpha^{-1} \circ F\rho_a \circ \alpha) \\ &= \varphi(b) \cdot \varphi(a) = \varphi(a)\varphi(b) \end{aligned} $$

Again we have used juxtaposition for the multiplication in A and $\cdot$ for the multiplication in A^{opp}. We see that φ is an algebra homomorphism. To prove that it is an automorphism it is enough to show that it is injective. If $\varphi(a) = 0$, then in particular $\alpha^{-1} \circ F\rho_a \circ \alpha$ is the 0-map. It follows that also $F\rho_a = 0$. Let G denote the quasi inverse functor of F. Then also $GF\rho_a = 0$ in $\text{End}_A(GFA) \simeq \text{End}_A(A)$, but this means that also $\rho_a = 0$ (since $GF \simeq \mathbb{1}$), and then $a = 0$.

To prove that $F \simeq F_\varphi$ we notice that $F \simeq FA \underset{A}{\otimes} \_$ where the structure on FA as a right A-module arises from the right multiplication in A. It is enough to prove that

$$ f : {}^{\varphi^{-1}}A \to FA $$

given by $f(x) = \alpha \circ \varphi(x)$ is an isomorphism of A-A-bimodules. It is obviously bijective, so we want to show that it respects the structure of the involved bimodules. Let $a \in {}^{\varphi^{-1}}A$, and let $b \in A$. We need to prove that

$$ f(b \cdot a) = b \cdot f(a) \quad \text{and} \quad f(a \cdot b) = f(a) \cdot b $$

(Here the dots denote the action of A on the A-A-bimodule.) We calculate

$$ f(b \cdot a) = f(\varphi^{-1}(b)a) = \alpha(b\varphi(a)) = b\alpha\varphi(a) = bf(a) = b \cdot f(a) $$

and

$$ \begin{aligned} f(a \cdot b) = f(ab) &= \alpha\varphi(ab) = \alpha(\varphi(a)\varphi(b)) \\ &= \alpha(\rho_{\varphi(b)}(\varphi(a))) = (\alpha \circ \beta(\varphi(b)) \circ \varphi)(a) \\ &= ((F\rho_b) \circ \alpha \circ \varphi)(a) = (F\rho_b)(\alpha\varphi(a)) \\ &= (F\rho_b)(f(a)) = f(a) \cdot b \end{aligned} $$

This proves c).

To prove d) consider first the inner automorphism $\varphi = \varphi_{s^{-1}} : A \to A$ given by

$$ \varphi(a) = s^{-1}as $$

for some $s \in A^\times$. Let $\psi = \varphi^{-1} = \varphi_s$. We claim that the map

$$ \begin{array}{rccc} \alpha : & A & \to & {}^{\psi}A \\ & a & \mapsto & sa \end{array} $$

is an isomorphism of A-A-bimodules. Given $a, b, x \in A$ we see that

$$ a \cdot \alpha(x) \cdot b = (sas^{-1})(sx)b = s(axb) = \alpha(axb) $$

It follows that $F_\varphi \simeq \mathbb{1}$.

Conversely assume that $F_\varphi \simeq \mathbb{1}$. Then A and ${}^{\varphi^{-1}}A$ are isomorphic as A-A-bimodules. Write $\psi = \varphi^{-1}$, and choose an isomorphism of bimodules

$$f : A \to {}^{\psi}A$$

Define $s = f(1)$. We claim that $s \in A^\times$ and that $\psi = \varphi_s$. Choose $x \in A$ such that $f(x) = 1$. Then

$$1 = f(x) = f(x \cdot 1) = x \cdot f(1) = \psi(x)f(1) = \psi(x)s$$

and

$$1 = f(x) = f(1 \cdot x) = f(1) \cdot x = f(1)x = sx$$

hence $s \in A^\times$.

We also see that for any $a \in A$

$$\psi(a)s = a \cdot f(1) = f(a) = f(1)a = sa$$

hence $\psi(a) = sas^{-1} = \varphi_s(a)$. It follows that $\varphi = \psi^{-1} = \varphi_{s^{-1}}$ is an inner automorphism.

Finally we prove e). Since F_φ has an quasi inverse functor $F_{\varphi^{-1}}$ this functor is also the (bi-)adjoint of F_φ. It follows that F_φ is selfadjoint if and only if $F_\varphi \simeq F_{\varphi^{-1}}$. This happens precisely when

$$F_{\varphi^2} = F_\varphi \circ F_\varphi \simeq F_\varphi \circ F_{\varphi^{-1}} \simeq F_{\mathrm{id}} = \mathbb{1}$$

We have just seen that this is true if and only if φ^2 is an inner automorphism. □

Corollary 4.8. *If $F : \mathcal{C} \to \mathcal{C}$ has a biadjoint functor then F maps projective modules to projective modules.*

We return to the proof of the proposition.

*Proof of **Proposition 4.4**.* Let $\{F_g | g \in G\}$ be a weak group action on A-mod. For any $g \in G$ we have an autoequivalence $F_g : A\text{-mod} \to A\text{-mod}$, hence there exists an automorphism $\varphi_g : A \to A$ with $F_g \simeq F_{\varphi_g}$. Notice that φ_g is determined up to inner automorphisms. We thus get a well defined map $G \to \mathrm{Out}(A)$ which maps g to $\varphi_g \,\mathrm{Inn}(A)$. Since

$$F_{gh} \simeq F_g \circ F_h \simeq F_{\varphi_g} \circ F_{\varphi_h} \simeq F_{\varphi_g \circ \varphi_h}$$

we see that $\varphi_{gh} \,\mathrm{Inn}(A) = (\varphi_g \circ \varphi_h) \,\mathrm{Inn}(A)$. Thus the map is a group homomorphism.

It is also clear that inequivalent group actions yield different group homomorphisms. Finally we have also seen that any group homomorphism $G \to \mathrm{Out}(A)$ gives a weak group action. This finishes the proof. □

Let us finish our section about weak group actions with some corollaries.

Corollary 4.9. *If $\mathrm{Inn}(A) = 1$ every weak group action is equivalent to a strong group action.*

Proof. When $\mathrm{Inn}(A)$ is trivial for any F_g there exists a unique φ_g such that $F_g \simeq F_{\varphi_g}$, and therefore $\{F_g | g \in G\}$ and $\{F_{\varphi_g} | g \in G\}$ are equivalent group actions. In the light of **Remark 4.7** the group action $\{F_{\varphi_g} | g \in G\}$ is equivalent to a strong group action. □

Corollary 4.10. *Let $n \geq 2$ be a natural number. Isomorphism classes of endofunctors $F : A\text{-mod} \to A\text{-mod}$ satisfying $F^n \simeq \mathbb{1}_{A\text{-mod}}$ are in one-to-one correspondence with group homomorphisms from $\mathbb{Z}_n$ to $\mathrm{Out}(A)$. F is selfadjoint if and only $F^2 \simeq \mathbb{1}_{A\text{-mod}}$.*

4.4 Idempotent Selfadjoint Functors

In this section we consider $\Lambda = \Bbbk[a]/(a^2 - a)$ with the trivial involution. A module over this algebra is weakly categorified via a category $A\text{-mod}$ and a selfadjoint functor $F : A\text{-mod} \to A\text{-mod}$ satisfying $F^2 \simeq F$.

Let us first recall some algebra. Let B and C be finite dimensional unital $\Bbbk$-algebra. Define A as the product ring

$$A = B \oplus C$$

We see that the unit in A decomposes $1 = e + f$ such that $e \in B$ and $f \in C$ are the units in B and C. We have a decomposition (an equivalence of categories)

$$B\text{-mod} \oplus C\text{-mod} \simeq A\text{-mod} \tag{4.2}$$

where an A-module M is mapped to (eM, fM) and a pair (M, N) of a B-module and a C module is mapped to $M \oplus N$ with the obvious action. We skip the cumbersome notation with ordered pairs and simply write $(M, N) = M \oplus N$, whenever M is a B-module and N is a C-module. The projection $\mathrm{p}_B : A\text{-mod} \to B\text{-mod}$ is given by $M \mapsto eM$, similarly for C. Denote by F the functor that projects onto $B\text{-mod}$ and then embeds into $A\text{-mod}$, i.e. $FM = eM \oplus 0$.

We see that

$$\begin{aligned}
\mathrm{Hom}_A(FM, N) &= \mathrm{Hom}_A(eM \oplus 0, N) \\
&\simeq \mathrm{Hom}_A(eM \oplus 0, eN \oplus fN) \\
&\simeq \mathrm{Hom}_B(eM, eN) \oplus \mathrm{Hom}_C(0, fN) \\
&= \mathrm{Hom}_B(eM, eN) \oplus \mathrm{Hom}_C(fM, 0) \\
&\simeq \mathrm{Hom}_A(eM \oplus fM, eN \oplus 0) \\
&= \mathrm{Hom}_A(M, FN)
\end{aligned}$$

that is, F is selfadjoint. It is also obvious that $F^2 \simeq F$.

This example is actually the whole story as the next proposition tells us.

Proposition 4.11. *Assume $F : A\text{-mod} \to A\text{-mod}$ is a selfadjoint functor satisfying $F^2 \simeq F$. Then A decomposes as $A = B \oplus C$ where B and C are unital (or zero) subalgebras of A, and F is isomorphic to the projection on $B\text{-mod}$ composed with the embedding into $A\text{-mod}$ both arising from the equivalence in* (4.2).

Remark 4.12. Notice that the special cases $F = 0$ or $F = \mathbb{1}$ are both covered by the proposition.

Proof. We procede in steps. Given a simple A-module L, define $X_{\mathrm{L}} = F\mathrm{L}$.

Step I. Assume that $X_{\mathrm{L}} \neq 0$. We claim that X_{L} decomposes $X_{\mathrm{L}} = \mathrm{L} \oplus Y_{\mathrm{L}}$ where Y_{L} is a module with $FY_{\mathrm{L}} = 0$. To see this we calculate

$$\begin{aligned}
0 &\neq \mathrm{Hom}_A(X_{\mathrm{L}}, X_{\mathrm{L}}) \\
&= \mathrm{Hom}_A(F\mathrm{L}, F\mathrm{L}) \\
&\simeq \mathrm{Hom}_A(\mathrm{L}, F^2\mathrm{L}) \\
&\simeq \mathrm{Hom}_A(\mathrm{L}, F\mathrm{L}) \\
&= \mathrm{Hom}_A(\mathrm{L}, X_{\mathrm{L}})
\end{aligned}$$

such that L is in the socle of X_{L}. Similarly we get

$$\begin{aligned}
0 &\neq \mathrm{Hom}_A(X_{\mathrm{L}}, X_{\mathrm{L}}) \\
&= \mathrm{Hom}_A(F\mathrm{L}, F\mathrm{L}) \\
&\simeq \mathrm{Hom}_A(F^2\mathrm{L}, \mathrm{L}) \\
&\simeq \mathrm{Hom}_A(F\mathrm{L}, \mathrm{L}) \\
&= \mathrm{Hom}_A(X_{\mathrm{L}}, \mathrm{L})
\end{aligned}$$

We see that L is also in the top of X_{L}. In particular L is a compositionfactor of X_{L}. In the Grothendieck group we write

$$[X_{\mathrm{L}}] = [\mathrm{L}] + z$$

for some $z \in [A\text{-mod}]$. Since F is selfadjoint it is both left and right exact and hence induces a linear endomorphism, $[F]$, of the Grothendieck group. We calculate

$$[X_{\mathrm{L}}] = [F\mathrm{L}] = [F][\mathrm{L}] = [F^2][\mathrm{L}] = [F][F\mathrm{L}] = [F][X\mathrm{L}] = [F]([\mathrm{L}] + z) = [X_{\mathrm{L}}] + [F]z$$

and see that $[F]z = 0$. In particular [L] cannot be involved in z and it follows that the composition factor multiplicity $[X_{\mathrm{L}} : \mathrm{L}]$ equals 1. Now L has multiplicity 1 but occurs both in the socle and the top of X_{L}, therefore it must split off $X_{\mathrm{L}} = \mathrm{L} \oplus Y_{\mathrm{L}}$. Also $[Y_{\mathrm{L}}] = z$ hence $[FY_{\mathrm{L}}] = [F][Y_{\mathrm{L}}] = 0$. Therefore $FY_{\mathrm{L}} = 0$.

Step II. If $X_{\mathrm{L}} \neq 0$ then $X_{\mathrm{L}} = \mathrm{L}$. We need to prove that $Y_{\mathrm{L}} = 0$. Assume to the contrary that $Y_{\mathrm{L}} \neq 0$ and choose a simple submodule $\mathrm{L}' \subset Y_{\mathrm{L}}$. Then because F is exact and $FY_{\mathrm{L}} = 0$ we have $F\mathrm{L}' = 0$. Now we calculate

$$\begin{aligned} 0 &\neq \mathrm{Hom}_A(\mathrm{L}', Y_{\mathrm{L}}) \\ &\simeq \mathrm{Hom}_A(\mathrm{L}', \mathrm{L} \oplus Y_{\mathrm{L}}) \\ &\simeq \mathrm{Hom}_A(\mathrm{L}', X_{\mathrm{L}}) \\ &= \mathrm{Hom}_A(\mathrm{L}', F\mathrm{L}) \\ &\simeq \mathrm{Hom}_A(F\mathrm{L}', \mathrm{L}) \\ &\simeq \mathrm{Hom}_A(0, \mathrm{L}) = 0 \end{aligned}$$

which is a contradiction. We used that L is not a composition factor of Y_{L}, hence $\mathrm{L}' \not\simeq \mathrm{L}$ and $\mathrm{Hom}_A(\mathrm{L}', \mathrm{L}) = 0$.

Step III. In this step we investigate the action of F on projective modules. We have seen that the action of F on simple modules is such that $F\mathrm{L} \in \{\mathrm{L}, 0\}$. Let all the non-isomorphic simple A-modules be $\mathrm{L}_1, \ldots, \mathrm{L}_n$, ordered such that there exists $k \in \{1, 2, \ldots, n\}$ with

$$F\mathrm{L}_j = \begin{cases} \mathrm{L}_j & \text{if } j \leq k \\ 0 & \text{if } j > k \end{cases}$$

Since F is selfadjoint it maps projectives to projectives. Write P_i for the projective cover of L_i. $F\mathrm{P}_i$ is a projective module and we want to determine which indecomposable summands it has. We calculate

$$\mathrm{Hom}_A(F\mathrm{P}_i, \mathrm{L}_j) = \mathrm{Hom}_A(\mathrm{P}_i, F\mathrm{L}_j) = \begin{cases} \Bbbk & \text{if } i = j \leq k \\ 0 & \text{otherwise} \end{cases}$$

We see that

$$F\mathrm{P}_j = \begin{cases} \mathrm{P}_j & \text{if } j \leq k \\ 0 & \text{if } j > k \end{cases}$$

It follows that if $i \leq k$ then all composition factors of P_i have the form L_j for some $j \leq k$, and for $i > k$ all composition factors of P_i have the form L_j for some $j > k$.

Step IV. In this step we show that A decomposes into a direct sum of unital subalgebras. We have just seen that if $i \leq k < j$ then

$$\mathrm{Hom}_A(\mathrm{P}_i, \mathrm{P}_j) = 0 = \mathrm{Hom}_A(\mathrm{P}_j, \mathrm{P}_i) \tag{4.3}$$

It is then apparent that we have isomorphisms of algebras (recall that we assume A to be basic, hence $A = \mathrm{P}_1 \oplus \ldots \oplus \mathrm{P}_n$)

$$\begin{aligned} A &\simeq \mathrm{End}_A(A)^{\mathrm{opp}} \\ &\simeq \mathrm{End}_A(\mathrm{P}_1 \oplus \ldots \oplus \mathrm{P}_n)^{\mathrm{opp}} \\ &\simeq \mathrm{End}_A(\mathrm{P}_1 \oplus \ldots \oplus \mathrm{P}_k)^{\mathrm{opp}} \oplus \mathrm{End}_A(\mathrm{P}_{k+1} \oplus \ldots \oplus \mathrm{P}_n)^{\mathrm{opp}} \end{aligned}$$

For the last isomorphism we used (4.3). The decomposition is clear if we define

$$B = \mathrm{End}_A(\mathrm{P}_1 \oplus \ldots \oplus \mathrm{P}_k)^{\mathrm{opp}}$$

and

$$C = \mathrm{End}_A(\mathrm{P}_{k+1} \oplus \ldots \oplus \mathrm{P}_n)^{\mathrm{opp}}$$

Step V. In the final step we show that F is indeed the projection on B-mod. Since F is selfadjoint we have an isomorphism (natural in M)

$$\mathrm{Hom}_A(FM, FM) \simeq \mathrm{Hom}_A(M, F^2M)$$

denote by η_M the image of id_{FM} under this isomorphism. This defines a natural transformation

$$\mathbb{1} \xrightarrow{\eta} F^2 \simeq F$$

This transformation is non-zero when applied to any L_i with $i \leq k$, hence it is an isomorphism $\eta_{\mathrm{L}_i} : \mathrm{L}_i \to F^2\mathrm{L}_i \simeq F\mathrm{L}_i$. Let us prove by induction over the (Jordan-Hölder) lenght of M that η_M is an isomorphism. Consider the diagram with short exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{L} & \longrightarrow & M & \longrightarrow & M/\mathrm{L} & \longrightarrow & 0 \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \eta_{\mathrm{L}}} & & \downarrow{\scriptstyle \eta_M} & & \downarrow{\scriptstyle \eta_{M/\mathrm{L}}} & & \downarrow{\scriptstyle 0} \\ 0 & \longrightarrow & F^2\mathrm{L} & \longrightarrow & F^2M & \longrightarrow & F^2(M/\mathrm{L}) & \longrightarrow & 0 \end{array}$$

Here $\mathrm{L} \subseteq M$ is a simple submodule of M, hence η_{L} is an isomorphism, and since $N = M/\mathrm{L}$ has fewer composition factors than M we see that also η_N is an isomorphism by induction. The 5-lemma ensures that η is in fact an isomorphism on all B-modules. It follows that when we restrict our functors to B-mod then η becomes an isomorphism between $\mathbb{1}_{B\text{-mod}}$ and $F_{|B\text{-mod}}$. Restricted to C-mod the functor F is clearly just the 0-functor. Therefore up to isomorphism F is the projection on B-mod composed with the embedding into A-mod. □

4.5 More Functors Showing Cyclic Patterns

In this section we let $k \geq 1$ be a natural number and begin with considering $\Lambda = \mathbb{k}[a]/(a^k)$. Given a $\mathbb{k}$-categorification of a Λ-module there is a unique way of making this into a weak categorification, namely by taking $F = 0$.

Proposition 4.13. *Let $F : A\text{-mod} \to A\text{-mod}$ be a selfadjoint functor satisfying $F^k = 0$. Then $F = 0$.*

Proof. We do induction over k. The first step $k = 1$ is obvious. We will need the case $k = 2$ as well, so assume $F^2 = 0$. Assume that $F \neq 0$, we want to reach a contradiction. Since $F \neq 0$ and F is exact there exists some simple module L such that $F\mathrm{L} \neq 0$. However $F^2\mathrm{L} = 0$, hence

$$0 \neq \mathrm{Hom}_A(F\mathrm{L}, F\mathrm{L}) \simeq \mathrm{Hom}_A(\mathrm{L}, F^2\mathrm{L}) \simeq 0$$

which is a contradiction.

Assume now $k > 2$. Then $2(k-1) > k$ and hence $(F^2)^{k-1} = F^{2(k-1)} = 0$. Since F^2 is a selfadjoint functor it follows by induction that $F^2 = 0$. We have allready seen that this implies $F = 0$. □

Let now also $1 \leq m < k$ be a natural number and consider $\Lambda = \mathbb{k}[a]/(a^k - a^m)$.

Proposition 4.14. *Let $F : A\text{-mod} \to A\text{-mod}$ be a a selfadjoint functor satisfying $F^k \simeq F^m$. Then A decomposes as $A = B \oplus C$ where B and C are unital or zero subalgebras of A. There are two possibe cases.*

a) *Assume $k - m$ is odd, then actually $F^2 \simeq F$ and F is the projection on B-mod composed with the embedding into A-mod.*

b) *Assume $k - m$ is even, then there exists an algebra automorphism $\varphi : B \to B$ such that F acts on C-mod as zero and on B-mod as F_φ.*

Remark 4.15. We see that the descriptions in the proposition actually give rise to selfadjoint functors satisfying the isomorphism.

Proof. We procede in steps.

Step I. We see that

$$F^m \simeq F^k = F^{m+(k-m)}$$

and by composing with F^{k-m} we get

$$F^{m+(k-m)} \simeq F^{m+2(k-m)}$$

hence for all $j \in \mathbb{N}$ we get

$$F^m \simeq F^{m+j(k-m)}$$

Choose j such that $s = j(k - m) > m$. We know that

$$F^m \simeq F^{m+s}$$

and by composing with F^{s-m} we get

$$F^s \simeq F^{2s}$$

Since F^s is selfadjoint **Proposition 4.11** tells us that A decomposes as claimed. We also get that F^s is the projection onto B-mod composed with the embedding into A-mod.

Step II. We claim that F acts as zero on C-mod. More precisely let $M \in C$-mod we want to prove that $FM = 0$. Notice that for the functor F^s we already have that $F^s M = 0$, so let us use downwards induction and prove that $F^i M = 0$ for any i. As mentioned $i = s$ is clear, as well as any $i > s$. Let $1 \leq i < s$ and assume that $F^i M \neq 0$. Then

$$0 \neq \mathrm{Hom}_A(F^i M, F^i M) \simeq \mathrm{Hom}_A(M, F^{2i} M)$$

However $2i > i$ and hence $F^{2i} M = 0$ by induction. This is a contraction and hence $F^i M = 0$.

Step III. In this step we prove that F preserves B-mod and hence that F decomposes as a functor $F_{|B\text{-mod}}$ acting on B-mod and the zero functor acting on C-mod. Let $N \in B$-mod be a B-module and $M \in C$-mod be any C-module, consider them as A-modules via the natural embedding. We want to prove that FN is a B-module (i.e. C acts on FN as 0). By adjunction we get

$$\mathrm{Hom}_A(M, FN) \simeq \mathrm{Hom}_A(FM, N) = 0$$

Where we used that $FM = 0$. In particular the projection of FN onto C-mod is 0 (take e.g. as M the projection of FN).

Step IV. We have seen that F decomposes as a (selfadjoint) functor $F_{|B\text{-mod}} : B\text{-mod} \to B\text{-mod}$ and the zero functor on C-mod. Also since F^s is isomorphic to the projection on B-mod we get that $F^s_{|B\text{-mod}} \simeq \mathbb{1}_{B\text{-mod}}$. It follows from **Proposition 4.6** that there exists an algebra automorphism $\varphi : B\text{-mod} \to B\text{-mod}$ with $F_{|B\text{-mod}} \simeq F_\varphi$. The proposition also tells us that since F_φ is selfadjoint then $\varphi^2 \in \mathrm{Inn}(B)$. This in turn forces $F^2_{|B\text{-mod}} \simeq \mathbb{1}_{B\text{-mod}}$, which finishes the proof in case $m - k$ is even.

Step V. Assume that $m - k$ is odd. Then the isomorphisms

$$F^k_{|B\text{-mod}} \simeq F^m_{|B\text{-mod}}, \quad F^2_{|B\text{-mod}} \simeq \mathbb{1}_{B\text{-mod}}$$

combine into $F_{|B\text{-mod}} \simeq \mathbb{1}_{B\text{-mod}}$. □

4.6 Functorial Square Roots

Let $k \geq 2$ be a natural number and consider $\Lambda = \Bbbk[a]/(a^2 - k)$.

This section will be devoted to give a proof of the following proposition.

Proposition 4.16. *Assume there exists a selfadjoint functor $F : A\text{-mod} \to A\text{-mod}$ satisfying*

$$F^2 \simeq k\,\mathbb{1}_{A\text{-mod}} = \mathbb{1}_{A\text{-mod}} \oplus \ldots \oplus \mathbb{1}_{A\text{-mod}}$$

(k summands). Then actually $k = m^2$ for some natural number m, and there exists a selfadjoint functor $G : A\text{-mod} \to A\text{-mod}$ satisfying $G^2 \simeq \mathbb{1}_{A\text{-mod}}$, such that F is isomorphic to the direct sum of m copies of G. (Conversely each such direct sum produces a selfadjoint square root of $m^2\,\mathbb{1}_{A\text{-mod}}$.)

The proof is quite long and rather technical. The idea, however, is fairly simple. We begin with solving the corresponding matrix equation, then using the solution for the matrix equation we prove the proposition in the case where A is a semisimple algebra. Finally we combine this with the lifting of idempotents modulo the radical, to prove the theorem for A in general.

Recall that a *permutaion matrix* is a quadratic matrix with entries in the set $\{0, 1\}$, such that each column and each row contains exactly one non-zero entry. Such a matrix permutes the standard basis vectors and corresponds thus to a unique permutation. A permutation matrix is clearly invertible and its inverse corresponds to the inverse permutation.

Lemma 4.17. *Let $M \in \mathrm{Mat}_n(\mathbb{N})$ be a matrix with $M^2 = kI_n$ where I_n is the $n \times n$ identity matrix. There exists a permutation matrix $P \in \mathrm{Mat}_n(\{0, 1\})$ such that the matrix PMP^{-1} is a block diagonal matrix with blocks of the following forms*

$$(m), \quad m \in \mathbb{N}, m^2 = k$$

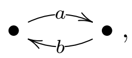
or

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \quad a, b \in \mathbb{N}, ab = k$$

Proof. It is possible to prove this lemma by induction in n (see [AM11]), but here we give a proof via combinatorics. We refer to [vLW01] for a thorough introduction to the field of combinatorics. Since M is a matrix with non-negative integral entries we may construct a finite oriented graph whose incidence matrix is M. That is we denote by Γ the graph with vertices $\{1, 2, \ldots, n\}$ and with $M_{i,j}$ edges directed from i to j. Conjugating M with a permutation matrix corresponds simply to renumbering the vertices. We need to prove that if $M^2 = kI$ then the graph corresponding to M has as connectedness components only graphs of the form

$m^2 = k$

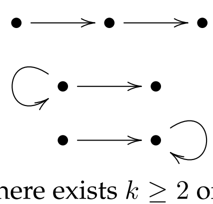
or



$ab = k$

Here the labels indicate multiple arrows.

The (i, j)'th entry of M^2 is the number of oriented paths from i to j of length two. Since this number is zero unless $i = j$ we see that Γ does not contain any subgraphs of the forms

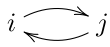


Now let i be a vertex of Γ. Since there exists $k \geq 2$ oriented paths from i to i of length two there must exist a vertex j with an edge from i to j and an edge from j to i.

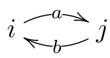
If $i = j$ we have a simple loop an no other vertices are connected to i (from the analysis above). It is thus clear that the connectedness component of Γ that contains i is

where $m^2 = k$.

If $i \neq j$, then Γ contains



as a subgraph. Hence no other vertices are connected to i or j and there are no simple loops based at i or j (again from the analysis above). We see that the connectedness component of Γ containing i is



and that $ab = k$.

This completes the proof. □

Corollary 4.18. *The claim in* ***Proposition 4.16*** *is true when* $A = \bigoplus \Bbbk\text{-mod}$ *is a semisimple algebra.*

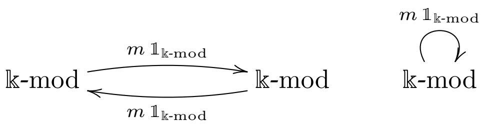
Proof. Recall that isomorphism classes of exact functors $F : A\text{-mod} \to A\text{-mod}$ satisfying $F^2 \simeq k\,\mathbb{1}_{A\text{-mod}}$ corresponds bijectively to matrices $M_F \in \mathrm{Mat}_n(\mathbb{N})$ with $M_F^2 = kI_n$ (here n is the dimension of A, i.e. the number of isomorphism classes of simple A-modules). After a permutation of the simple modules the lemma shows that M_F is a block diagonal matrix with blocks

$$(m), \quad \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

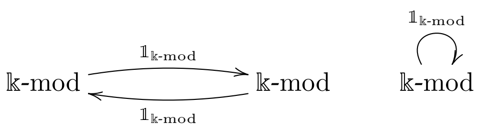
with $m^2 = k$ and $ab = k$. If we also impose the condition that F is selfadjoint, then M_F is a symmetric matrix, hence $a = b$. This means that M_F is a block diagonal matrix with blocks

$$(m), \quad \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$$

We see that indeed k is a square and that F is build up from the following components:



Define G to be the functor build in a similar way but from the components



Then $F \simeq mG$ and $G^2 \simeq \mathbb{1}_{A\text{-mod}}$ as claimed. □

In the proof we saw that the matrix M_F corresponding to F is symmetric when A is a semisimple algebra. Let us show that this is also the case in general

Lemma 4.19. *Let F be as in* ***Proposition 4.16*** *and let $\mathrm{L}_1, \ldots, \mathrm{L}_n$ be a complete list of pairwise non-isomorphic simple A-modules. Let M_F be the matrix of $[F]$ on the Grothendieck group of $A\text{-mod}$ with respect to the basis $([\mathrm{L}_1], \ldots, [\mathrm{L}_n])$. There exists a permutation matrix P such that PM_FP^{-1} is a block diagonal matrix with blocks*

$$(m), \quad \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$$

with $m^2 = k$. In particular k is a square.

Proof. We allready know that the matrix M_F satisfies $M_F^2 = kI_n$, and hence we may choose a permutation matrix P such that M_F is block diagonal with blocks

$$(m), \qquad \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

with $m^2 = k$ and $ab = k$. If only blocks of the form (m) occurs we are done. Assume there exists a block of the other form. This means that there exists $i \neq j$ such that in the Grothendieck group

$$a[\mathrm{L}_i] = [F][\mathrm{L}_j] = [F\mathrm{L}_j], \qquad b[\mathrm{L}_j] = [F][\mathrm{L}_i] = [F\mathrm{L}_i]$$

In particular, since $a \neq 0$ and L_i is the only composition factor of $F\mathrm{L}_j$, we have an inclusion

$$\mathrm{L}_i \subseteq F\mathrm{L}_j$$

Applying the exact functor F we get an inclusion

$$F\mathrm{L}_i \subseteq FF\mathrm{L}_j \simeq k\mathrm{L}_j = \mathrm{L}_j \oplus \ldots \oplus \mathrm{L}_j$$

Since $F\mathrm{L}_i$ is a submodule of a semisimple module it is semisimple itself and thus $F\mathrm{L}_i \simeq b\mathrm{L}_j$.

From selfadjointness of F we get

$$\begin{aligned} a &= [F\mathrm{L}_j : \mathrm{L}_i] \\ &= \dim_{\Bbbk} \mathrm{Hom}_A(F\mathrm{L}_j, \mathrm{L}_i) \\ &= \dim_{\Bbbk} \mathrm{Hom}_A(\mathrm{L}_j, F\mathrm{L}_i) \\ &= \dim_{\Bbbk} \mathrm{Hom}_A(\mathrm{L}_j, b\mathrm{L}_j) \\ &= b \dim_{\Bbbk} \mathrm{End}_A(\mathrm{L}_j) \\ &= b \end{aligned}$$

Hence $k = ab = a^2$. □

We also proved the following corollary

Corollary 4.20. *Let $G : A\text{-mod} \to A\text{-mod}$ be an exact functor with $G^2 \simeq k\,\mathbb{1}_{A\text{-mod}}$, and let $A\text{-mod}^{ss}$ be the full subcategory of $A\text{-mod}$ consisting of semisimple modules. Then G preserves $A\text{-mod}^{ss}$.*

To prove **Proposition 4.16** in general we need to examine the functor F. Recall that if we define the A-A-bimodule $V = FA$ where the structure as a right A-module comes from the right multiplication in A then

$$F\_ \simeq V \underset{A}{\otimes} \_$$

are isomorphic functors. The claim in the proposition is equivalent to proving that V as a bimodule is isomorphic to the direct sum of m copies of the bimodule corresponding to G. To prove that V splits up as wanted we construct m mutually orthogonal idempotent endomorphisms of V considered as an A-A-bimodule.

Let us rephrase the fact that F is selfadjoint and the isomorphism $F^2 \simeq k\,\mathbb{1}$ in terms of properties of the module V. In the following we will use the notation

$$\mathrm{Hom}_{A \otimes A^{\mathrm{opp}}}(M, N) = \{f : M \to N | f \text{ is a morphism between } A\text{-}A\text{-bimodules}\}$$

and keep the notation

$$\mathrm{Hom}_A(M, N) = \{f : M \to N | f \text{ is a morphism of left } A\text{-modules}\}$$

Given an A-B-bimodule X we recall that we have a natural isomorphism (the general version of Frobenius reciprocity)

$$\begin{array}{ccc} \mathrm{Hom}_A(X \underset{B}{\otimes} M, N) & \simeq & \mathrm{Hom}_B(M, \mathrm{Hom}_A(X, N)) \\ f & \mapsto & [m \mapsto [x \mapsto f(x \otimes m)]] \\ [x \otimes m \mapsto g(m)(x)] & \leftarrow & g \end{array}$$

for any A-module N and any B-module M.

Applied to the A-A-bimodule V we get

$$\mathrm{Hom}_A(V \underset{A}{\otimes} M, N) \simeq \mathrm{Hom}_A(M, \mathrm{Hom}_A(V, N))$$

for any A-modules M and N. This isomorphism is natural in both M and N. This means that the functor

$$V \underset{A}{\otimes} \_$$

has as right adjoint the functor

$$\mathrm{Hom}_A(V, \_)$$

However since $F\_ \simeq V \underset{A}{\otimes} \_$ and F is assumed to be selfadjoint, we get that these two functors are in fact isomorphic

$$V \underset{A}{\otimes} \_ \simeq \mathrm{Hom}_A(V, \_)$$

In particular $\mathrm{Hom}_A(V, \_)$ is exact and corresponds to the functor

$$\mathrm{Hom}_A(V, A) \underset{A}{\otimes} \_$$

where the structere as a right A-module comes from right multiplication of A. These facts combined means that as A-A-bimodules we have an isomorphism

$$V \simeq \mathrm{Hom}_A(V, A)$$

We proved the "only if" part of the next lemma.

Lemma 4.21. *Let $F : A\text{-mod} \to A\text{-mod}$ be an exact functor. Define the A-A-bimodule V via $V = FA$, and assume that the functor given by $\mathrm{Hom}_A(V, \_)$ (the right adjoint of F) is exact. Then F is selfadjoint if and only if there exists an isomorphism of A-A-bimodules*

$$V = \mathrm{Hom}_A(V, A)$$

Proof. We show the "if" part. We need to prove that we have a natural isomorphism

$$\mathrm{Hom}_A(V, N) \simeq V \underset{A}{\otimes} N$$

From the assumption we get a natural isomorphism

$$\mathrm{Hom}_A(V, A) \underset{A}{\otimes} N \simeq V \underset{A}{\otimes} N$$

hence it is enough to prove that we have an isomorphism

$$\mathrm{Hom}_A(V, N) \simeq \mathrm{Hom}_A(V, A) \underset{A}{\otimes} N$$

which is natural in N. There is an obvious candidate for a morphism from the right to the left, namely mapping $f \otimes n$ to $v \mapsto f(v)n$, this morphism is natural in N and actually also in V. Since $\mathrm{Hom}_A(V, \_)$ is exact, V is projective as a left A-module. We prove that

$$\varphi_P : \mathrm{Hom}_A(P, A) \underset{A}{\otimes} N \to \mathrm{Hom}_A(P, N)$$

given by $\varphi_P(f \otimes n)(p) = f(p)n$ is an isomorphism for any projective A-module P.

Let $P = A$, φ_A is the natural isomorphism

$$\mathrm{Hom}_A(A, A) \underset{A}{\otimes} N \simeq A \underset{A}{\otimes} N \simeq N \simeq \mathrm{Hom}_A(A, N)$$

since $\varphi_A(\mathrm{id} \otimes n)(1) = \mathrm{id}(1)n = n$.

Given a projective A-module, P, we choose Q such that $Q \oplus P \simeq A^m$. Now also φ_{A^m} is an isomorphism and restriction gives an isomorphism

$$\varphi_P : \mathrm{Hom}_A(P, A) \underset{A}{\otimes} N \to \mathrm{Hom}_A(P, N)$$

since both

$$\mathrm{Hom}_A(\_\,, A) \underset{A}{\otimes} N$$

and

$$\mathrm{Hom}_A(\_\,, N)$$

preserves direct sums. □

We can describe the left A-module $V \underset{A}{\otimes} A$.

Lemma 4.22. *The module $V \underset{A}{\otimes} A$ is projective as a left A-module. It is isomorphic to m copies of A.*

Proof. We know that the matrix M_F is of the form mP for some permutation matrix P, say $P = P_\sigma$, where σ has order 2. This means that $F\mathrm{L}_j$ has a unique composition factor $\mathrm{L}_{\sigma(j)}$, it has multiplicity m. Then by adjunction

$$\dim \mathrm{Hom}_A(F\mathrm{P}_i, \mathrm{L}_j) = \dim \mathrm{Hom}_A(\mathrm{P}_i, F\mathrm{L}_j) = \begin{cases} m & i = \sigma(j) \\ 0 & \text{otherwise} \end{cases}$$

We get that $F\mathrm{P}_i \simeq m\mathrm{P}_{\sigma(i)}$. Since $A = \oplus \mathrm{P}_i$ the lemma follows. □

Let us now examine the isomorphism $F^2 \simeq k\,\mathbb{1}$. In terms of A-A-bimodules this means that

$$V \underset{A}{\otimes} V \simeq kA := A^k$$

since

$$F^2 M \simeq F(V \underset{A}{\otimes} M) \simeq V \underset{A}{\otimes} (V \underset{A}{\otimes} M) \simeq (V \underset{A}{\otimes} V) \underset{A}{\otimes} M$$

and since $A \underset{A}{\otimes} \_$ corresponds to the identity functor on A-mod.

We saw in the proof of **Lemma 4.21** that

$$V \underset{A}{\otimes} V \simeq \mathrm{Hom}_A(V, A) \underset{A}{\otimes} V \simeq \mathrm{Hom}_A(V, V) = \mathrm{End}_A(V)$$

as A-A-bimodules. Here we used that F is selfadjoint for the first isomorphism and for the second we used that V is a projective left A-module (because the right adjoint of F is biadjoint to F hence exact) and the isomorphism in the proof of the lemma.

It follows that also

$$kA \simeq \mathrm{Hom}_A(V, V)$$

as A-A-bimodules.

As allready explained we want to find idempotents in the ring

$$\begin{aligned} \mathrm{Hom}_{A \otimes A^{\mathrm{opp}}}(V, V) &= \{f \in \mathrm{Hom}_A(V, V) | \forall a \in A \forall v \in V : f(va) = f(v)a\} \\ &= \{f \in \mathrm{Hom}_A(V, V) | \forall a \in A \forall v \in V : (af)(v) = (fa)(v)\} \\ &= \{f \in \mathrm{Hom}_A(V, V) | \forall a \in A : af = fa\} \end{aligned}$$

This is the subspace of $\mathrm{Hom}_A(V, V)$ of elements on which the right and left action of A coincide. Taking the same invariants on the A-A-bimodule kA we get

$$\begin{aligned} \mathrm{Hom}_{A \otimes A^{\mathrm{opp}}}(V, V) &\simeq \{(a_1, \dots, a_k) \in kA | \forall a \in A : a(a_1, \dots, a_k) = (a_1, \dots, a_k)a\} \\ &= \{(a_1, \dots, a_k) \in kA | \forall a \in A : (aa_1, \dots, aa_k) = (a_1 a, \dots, a_k a)\} \\ &= \{(a_1, \dots, a_k) \in kA | \forall i \forall a \in A : aa_i = a_i a\} \\ &= \{(a_1, \dots, a_k) \in kA | \forall i : a_i \in Z(A)\} \\ &= kZ(A) \end{aligned}$$

This is an isomorphism of not only $\Bbbk$-modules but actually of (right and left) $Z(A)$-modules.

We want to lift the idempotents from a corresponding bimodule over the semisimple ring $\bar{A} = A/\operatorname{Rad}(A)$. However a priori there is no reason to expect that $V/\operatorname{Rad}(A)V$ should be an $\bar{A}$-$\bar{A}$-bimodule, so let us prove that this is in fact the case.

Write $V_l = \operatorname{Rad}({}_AV)$ for the radical of V considered as a left A-module. Similarly we write $V_r = \operatorname{Rad}(V_A)$ for the radical when we consider V as a right A-module.

Lemma 4.23. *The left and the right radical of V coincide, i.e. $V_r = V_l$.*

Proof. Notice that by general facts (**Lemma 1.25**) about artinian algebras we have that $\operatorname{Rad}(A)V \subseteq V_l$ and that $V\operatorname{Rad}(A) \subseteq V_r$.

Notice also that

$$V/(V\operatorname{Rad}(A)) \simeq V \underset{A}{\otimes} A/V \underset{A}{\otimes} \operatorname{Rad}(A) \simeq V \underset{A}{\otimes} A/\operatorname{Rad}(A)$$

Since $A/\operatorname{Rad}(A)$ is a semisimple left A-module and $F\_ \simeq V \underset{A}{\otimes} \_$ maps preserves the category of semisimple A-modules, we get that $V/(V\operatorname{Rad}(A))$ is a semisimple left A-module. Hence $V_l \subseteq V\operatorname{Rad}(A)$, and as a result we have the following chain of submodules of V:

$$\operatorname{Rad}(A)V \subseteq V_l \subseteq V\operatorname{Rad}(A) \subseteq V_r$$

Now define $F' : \text{mod-}A \to \text{mod-}A$ by

$$F'(X) = X \underset{A}{\otimes} V$$

such that F' is tensoring with V from the right. Since $V \underset{A}{\otimes} V \simeq kA$ as A-A-bimodules we see that $(F')^2 \simeq k\,\mathbb{1}_{\text{mod-}A}$. Now F' is an exact functor since ${}_AV$ as a left A-module is projective. It follows that F' preserves the category of semisimple right A-modules, and therefore we see that

$$V/\operatorname{Rad}(A)V \simeq A \underset{A}{\otimes} V/\operatorname{Rad}(A) \underset{A}{\otimes} V \simeq A/\operatorname{Rad}(A) \underset{A}{\otimes} V = F'(A/\operatorname{Rad}(A))$$

Now $A/\operatorname{Rad}(A)$ is a semisimple right A-module and hence also $F'(A/\operatorname{Rad}(A)) \simeq V/\operatorname{Rad}(A)V$ is semisimple by **Corollary 4.20**. We see that $V_r \subseteq \operatorname{Rad}(A)V$. We have proved that

$$V_r \subseteq \operatorname{Rad}(A)V \subseteq V_l \subseteq V\operatorname{Rad}(A) \subseteq V_r \qquad \square$$

Because of the lemma we may define $R = V_r = V_l$ and refer to it as the radical of V. The module

$$\bar{V} = V/R = V/\operatorname{Rad}(A)V = V/(V\operatorname{Rad}(A))$$

is an $\bar{A}$-$\bar{A}$-bimodule.

If $f : V \to V$ is a morphism of A-A-bimodules, then

$$f(R) = f(\operatorname{Rad}(A)V) = \operatorname{Rad}(A)f(V) \subseteq \operatorname{Rad}(A)V = R$$

hence f induces a morphism

$$\bar{f} : \bar{V} \to \bar{V}$$

which is clearly a morphism of $\bar{A}$-$\bar{A}$-bimodules. If we define

$$\Phi : \operatorname{Hom}_{A \otimes A^{\text{opp}}}(V, V) \to \operatorname{Hom}_{\bar{A} \otimes \bar{A}^{\text{opp}}}(\bar{V}, \bar{V})$$

via $\Phi(f) = \bar{f}$ then Φ becomes a homomorphism of algebras.

Lemma 4.24. *The radical of the algebra $\operatorname{Hom}_{A \otimes A^{opp}}(V, V)$ contains the kernel of Φ (thus it is possible to lift idempotents modulo the kernel).*

Proof. If $\Phi(f) = 0$ then $f(V) \subseteq R = \mathrm{Rad}(A)V$. Let $h \in \mathrm{Hom}_{A \otimes A^{\mathrm{opp}}}(V, V)$ then

$$(h \circ f)^n(V), (f \circ h)^n(V) \subseteq (\mathrm{Rad}(A))^n V$$

Since $\mathrm{Rad}(A)$ is a nilpotent ideal we see that $h \circ f$ and $f \circ h$ are nilpotent, and hence that f generates a nil two-sided ideal. It follows that f is contained in the radical. □

It is easy to show that the natural candidates for morphisms are inverse isomorphisms of the following $\bar{A}$-$\bar{A}$-bimodules

$$\begin{aligned} \bar{V} \underset{\bar{A}}{\otimes} \bar{V} &\simeq V \underset{A}{\otimes} V / (V \underset{A}{\otimes} V \,\mathrm{Rad}(A)) \\ &\simeq V \underset{A}{\otimes} V / \mathrm{Rad}(A) V \underset{A}{\otimes} V \\ &\simeq kA / \mathrm{Rad}(A) kA \\ &\simeq k(A / \mathrm{Rad}(A)) \\ &\simeq k\bar{A} \end{aligned}$$

Now $\bar{A}$ is a semisimple ring and $H := \overline{V} \underset{\bar{A}}{\otimes} \_$ is a functor with $H^2 \simeq \mathbb{1}_{\bar{A}\text{-mod}}$ and therefore by **Corollary 4.18** we get that $\overline{V} \underset{\bar{A}}{\otimes} \bar{A} \simeq m\bar{A}$. By comparing dimensions we see that the inclusion

$$k\bar{A} = kZ(\bar{A}) \simeq \mathrm{Hom}_{\bar{A} \otimes \bar{A}^{\mathrm{opp}}}(\overline{V}, \overline{V}) \subseteq \mathrm{End}_{\bar{A}}(\overline{V}) \simeq \mathrm{End}_{\bar{A}}(m\bar{A}) \simeq \mathrm{Mat}_m(\bar{A})$$

is an equality.

The matrix-algebra $\mathrm{Mat}_m(\bar{A})$ has m mutually orthogonal idempotents given by matrices with exactly one non-zero entry, namely a single 1 in the diagonal.

Remark 4.25. It is not obvious that these idempotents are in fact in the image of Φ. Even though the short exact sequence of A-A-bimodules

$$0 \to R \to V \to V/R \to 0$$

induces a short exact sequence by applying the exact functor $\mathrm{Hom}_A(V, \_)$

$$0 \to \mathrm{Hom}_A(V, R) \to \mathrm{Hom}_A(V, V) \to \mathrm{Hom}_A(V, V/R) \to 0$$

then when we take A-invariants we usually only get an exact sequence

$$0 \to \mathrm{Hom}_{A \otimes A^{\mathrm{opp}}}(V, R) \to \mathrm{Hom}_{A \otimes A^{\mathrm{opp}}}(V, V) \to \mathrm{Hom}_{A \otimes A^{\mathrm{opp}}}(V, V/R)$$

(Here the first term is the kernel of Φ.) Therefore when determining the image of Φ, one cannot just take A-invariants of the quotient

$$\mathrm{Hom}_A(V, V) / \mathrm{Hom}_A(V, R) \simeq V \underset{A}{\otimes} V / V \underset{A}{\otimes} R \simeq k\bar{A}$$

to get that $\dim \mathrm{Im}\, \Phi = k \dim \bar{A}$ (which would mean that Φ is surjective).

We want to show that the subalgebra $\mathrm{Im}\, \Phi$ contains the idempotents from $\mathrm{Mat}_m(\bar{A})$.

Consider the center of A, $Z(A)$, and its image in $\bar{A}$.

Since $\mathrm{Rad}(Z(A))$ is just the nilradical of $Z(A)$ i.e. the set of nilpotent elements ($Z(A)$ is a commutative ring, hence any nilpotent element generates a two-sided ideal) we see that

$$\mathrm{Rad}(Z(A)) = Z(A) \cap \mathrm{Rad}(A)$$

The inclusion $\supseteq$ is clear because $\mathrm{Rad}(A)$ consists of nilpotent elements. For the inclusion the other way notice that any element $z \in \mathrm{Rad}(Z(A))$ is nilpotent and that

$$AzA = Az = zA$$

is a twosided nil ideal in A since $z \in Z(A)$. $\mathrm{Rad}(A)$ is the sum of all twosided nil ideals hence $Az \subseteq \mathrm{Rad}(A)$ and $z \in \mathrm{Rad}(A)$.

It follows that we have an inclusion of (commutative) algebras

$$\overline{Z(A)} = Z(A)/\operatorname{Rad}(Z(A)) \to A/\operatorname{Rad}(A) = \bar{A}$$

Consider the following diagram (Ψ is defined in the same way as Φ).

$$\begin{array}{ccccc} kZ(A) & \longrightarrow & \mathrm{Hom}_{A \otimes A^{\mathrm{opp}}}(V,V) & \xrightarrow{\Phi} & \mathrm{Hom}_{\bar{A} \otimes \bar{A}^{\mathrm{opp}}}(\overline{V},\overline{V}) \simeq \mathrm{Mat}_m(\overline{A}) \\ \big\downarrow & & \big\downarrow i & & \big\| \\ kA & \longrightarrow & \mathrm{Hom}_A(V,V) & \xrightarrow{\Psi} & \mathrm{Hom}_{\bar{A}}(\overline{V},\overline{V}) \end{array}$$

Here the composition $\Psi \circ i$ corresponds to the embedding $Z(A) \to A$ composed with factoring out the radical. It follows that Φ corresponds to the inclusion

$$\overline{Z(A)} = Z(A)/\operatorname{Rad}(A) \cap Z(A) \to A/\operatorname{Rad}(A) = \bar{A}$$

In other words the image of Φ is the subalgebra of $\mathrm{Mat}_m(\bar{A})$ corresponding to this inclusion, i.e. $\mathrm{Im}\,\Phi \simeq \mathrm{Mat}_n(\overline{Z(A)})$, thus the image actually contains the considered idempotents.

*Proof of **Proposition 4.16**.* Using **Proposition 1.28** we can lift the idempotents from $\mathrm{Mat}_n(\overline{Z(A)})$ to m mutually orthogonal idempotents in $\mathrm{Hom}_{A \otimes A^{\mathrm{opp}}}(V,V)$. This means that the functor F decomposes as

$$F = F_1 \oplus F_2 \oplus \ldots \oplus F_m$$

If we define the functor $\Pi : A\text{-mod} \to \bar{A}\text{-mod}$ by $\Pi(M) = M/\operatorname{Rad}(A)M$ then the following diagram commutes (up to isomorphism of functors)

$$\begin{array}{ccc} A\text{-mod} & \xrightarrow{V \otimes_A -} & A\text{-mod} \\ \Pi \big\downarrow & & \big\downarrow \Pi \\ \bar{A}\text{-mod} & \xrightarrow[\overline{V} \otimes_{\bar{A}} -]{} & \bar{A}\text{-mod} \end{array}$$

The decomposition of $F \simeq V \underset{A}{\otimes} \_$ corresponds to the decomposition $\overline{V} \underset{\bar{A}}{\otimes} \_ \simeq \bar{F}_1 \oplus \ldots \oplus \bar{F}_m$ given by **Corollary 4.18**.

We still denote by $\mathrm{L}_1, \ldots, \mathrm{L}_n$ a complete set of non-isomorphic simple A-modules. Also let $\mathrm{L} = \mathrm{L}_1 \oplus \ldots \oplus \mathrm{L}_n$. Since A is a basic algebra we have the following short exact sequence of A-modules

$$0 \to \operatorname{Rad}(A) \to A \to \mathrm{L} \to 0$$

Applying the exact functor $V \underset{A}{\otimes} \_$ we get the sequence

$$0 \to V \underset{A}{\otimes} \operatorname{Rad}(A) \to V \underset{A}{\otimes} A \to V \underset{A}{\otimes} \mathrm{L} \to 0$$

i.e.

$$0 \to R \to V \to F\mathrm{L} \to 0$$

Clearly an $\bar{A}$-module carries the natural structure of an A-module, and the $\bar{A}$-module $\Pi(\mathrm{L})$ is isomorphic to L as A-modules. This means that $F\mathrm{L}$ is just $\Pi(V)$. As a left module $V \simeq mA$ and hence $\Pi(V) \simeq m\mathrm{L}$. Each of the $\bar{F}_i$ permutes the simple modules of $\bar{A}$, i.e. $\bar{F}_i(\Pi(\mathrm{L})) \simeq \Pi(\mathrm{L})$. It follows that $F_i\mathrm{L} \simeq \mathrm{L}$ as A-modules, in other words the matrix M_{F_i} is a permutation matrix.

Since $F^2 \simeq k\,\mathbb{1}_{A\text{-mod}}$ we get that

$$k\,\mathbb{1}_{A\text{-mod}} \simeq F^2 \simeq (F_1 \oplus \ldots \oplus F_m)^2 \simeq \bigoplus_{i,j} F_i \circ F_j$$

This means that all of the $M_{F_i \circ F_j} = M_{F_i} M_{F_j}$ are diagonal permutation matrices, i.e. $M_{F_i \circ F_j} = I_n$ is the identity matrix for any i, j. By the Krull-Schmidt theorem we get that any $F_i \circ F_j \simeq \mathbb{1}_{A\text{-mod}}$ in particular $F_i^2 \simeq \mathbb{1}_{A\text{-mod}}$ is a selfadjoint equivalence. We also see that $F_i \simeq F_i^{-1} \simeq F_j^{-1} \simeq F_j$ for any i, j.

□

4.7 Future Problems

The research presented here was originally motivated from the example in Section 2.5 where we saw that the BGG category $\mathcal{O}_0$ categorified the (right) regular representation of $\mathbb{Z}[S_n]$.

An intriguing question is to what extend this categorification of the integral group ring $\mathbb{Z}[S_n]$ is unique, i.e. which properties should one impose to categorifications of rings that would make the category $\mathcal{O}_0$, up to categorical equivalence, the only choice.

In the example we have in particular the selfadjoint wall-crossing functors θ_s for each simple transposition $s \in S_n$. These are functors satisfying $\theta_s^2 \simeq \theta_s \oplus \theta_s$. This means that the category $\mathcal{O}_0$ weakly categorifies a module over the ring $\Lambda = \mathbb{Z}[a]/(a^2 - 2a)$. A natural starting point for exploring the mentioned uniqueness problem would be to classify weak categorification of Λ-modules. This means that we would like to understand selfadjoint functors $F : A\text{-mod} \to A\text{-mod}$ (on categories of modules over a finite dimensional complex algebra), which satisfy the relation $F^2 \simeq F \oplus F$. Unfortunately this problem seems much harder than the similar problems handled in Chapter 4. For semisimple algebras we can solve the problem after solving the corresponding matrix problem. The first few baby steps could be the following proposition.

Proposition 4.26. *Let $F : A\text{-mod} \to A\text{-mod}$ be a selfadjoint functor on the category of finitely generated modules over some $\Bbbk$-algebra A. Assume that $F^2 \simeq F \oplus F$. Fix a complete list of non-isomorphic simple A-modules $\mathrm{L}_1, \ldots, \mathrm{L}_n$, and let $M = M_F \in \mathrm{Mat}_n(\mathbb{N})$ be the matrix of $[F]$ calculated in the corresponding basis in the Grothendieck group. Then there exists a permutation matrix P such that the matrix PMP^{-1} is a block matrix*

$$\begin{pmatrix} D & 0 \\ X & 0 \end{pmatrix}$$

where X is some (perhaps rectangular) matrix with non-negative integral entries and D is a (square) block diagonal matrix with blocks of the form

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad (2)$$

If D contains a block of the first form then the corresponding columns of X are identical.

If A is a semisimple algebra then all entries of X are 0.

Sketch of proof. Let L be a simple A-module with $F\mathrm{L} \neq 0$. Then by adjointness

$$0 \neq \mathrm{Hom}_A(F\mathrm{L}, F\mathrm{L}) \simeq \mathrm{Hom}_A(\mathrm{L}, F^2\mathrm{L}) \simeq \mathrm{Hom}_A(\mathrm{L}, F\mathrm{L} \oplus F\mathrm{L})$$

This means that L occurs in the socle of $F\mathrm{L}$ and hence $[F\mathrm{L} : \mathrm{L}] \neq 0$. After permuting the simple modules we may write

$$M = \begin{pmatrix} I + Y & 0 \\ X & 0 \end{pmatrix}$$

where I is the identity matrix and X and Y are matrices with non-negative integral entries. Further $Y^2 = I$. This means that for any i

$$\sum_i Y_{ik} Y_{ki} = 1$$

such that there exists a unique k such that $Y_{ik} Y_{ki} \neq 0$. This defines a permutation σ with $\sigma^2 = 1$ and we may write $Y = P_\sigma + Y'$ (where P_σ is the permutation matrix corresponding to σ). Since both Y and P_σ square to the identity (and P_σ is invertible) it follows that $Y' = 0$. Since σ is a product of disjoint 2-cycles, after permuting the simple objects once again we see that Y is block diagonal with blocks

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (1)$$

The rest of the proof is just simple calculations. □

Bibliography

[AM11] T. Agerholm and V. Mazorchuk. On selfadjoint functors satisfying polynomial relations. *Journal of Algebra*, 330(1):448 – 467, 2011.

[And11] H.H. Andersen. Quotient categories of modular representations. In *Representation Theory of Algebraic Groups and Quantum Groups*, volume 284 of *Progress in Mathematics*, pages 1–16. Birkhäuser Boston, 2011. 10.1007/978-0-8176-4697-4_1.

[Bas68] H. Bass. *Algebraic* K*-theory*. W.A. Benjamin, 1968.

[BG80] J. Bernstein and S. Gelfand. Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras. *Compositio Math.*, 41(2):245–285, 1980.

[BGG76] J. Bernstein, I.M. Gelfand, and S.I. Gelfand. On some category of $\mathfrak{g}$-modules. *Functional Analysis and its Applications*, 10(2):1–8, 1976.

[CF94] L. Crane and I.B. Frenkel. Four dimensional topological quantum field theory, Hopf categories, and the canonical bases. *J.MATH.PHYS.*, 35:5136, 1994.

[CR08] J. Chuang and R. Rouquier. Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification. *Annals of Mathematics. Second Series*, 167:245–298, 2008.

[Cra95] L. Crane. Clock and Category; Is quantum gravity algebraic. *J.MATH.PHYS.*, 36:6180, 1995.

[FKS07] I. Frenkel, M. Khovanov, and C. Stroppel. A categorification of finite-dimensional irreducible representations of quantum $\mathfrak{sl}(2)$ and their tensor products. *Selecta Mathematica - New Series*, 12(3-4):379–431, April 2007.

[Fre66] P. Freyd. Representations in abelian categories. *in: Proc. Conf. Categorical Algebra*, pages 95–120, 1966.

[Gab62] P. Gabriel. Des catégories abéliennes. *Bulletin de la Société Mathématique de France*, 90:323–448, 1962.

[Hum08] J.E. Humphreys. *Representations of Semisimple Lie Algebras in the BGG category $\mathcal{O}$ (Graduate Studies in Mathematics)*. American Mathematical Society, August 2008.

[Jan83] J.C. Jantzen. *Einhüllende Algebren halbeinfacher Lie–Algebren*. Springer, 1983.

[Kho01] M. Khovanov. NilCoxeter algebras categorify the Weyl algebra. *COMM.ALGEBRA 29*, 11:5033, 2001.

[KL79] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Inventiones Mathematicae*, 53:165–184, 1979. 10.1007/BF01390031.

[KMS08] M. Khovanov, V. Mazorchuk, and C. Stroppel. A categorification of integral Specht modules. *PROC.AMS*, 136:1163, 2008.

[Lam66] T. Lam. The Category of Noetherian Modules. *Proc. Natl. Acad. Sci. USA*, 55(5):1038–1040, 1966.

[Lam76] J. Lambek. *Lectures on Rings and Modules*. Chelsea Publishing Co., New York, 1976.

[Lau10] A.D. Lauda. A categorification of quantum $\mathfrak{sl}_2$. *Advances in Mathematics*, 225:3327–3424, 2010.

[Maz10] V. Mazorchuk. Lectures on algebraic categorification, 2010. arxiv:1011.0144.

[MM10] V. Mazorchuk and V. Miemietz. Cell 2-representations of finitary 2-categories. *arXiv:1011.3322v1, To appear in Compositio Mathematica*, 2010.

[MS07] V. Mazorchuk and C. Stroppel. Translation and shuffling of projectively presentable modules and a categorification of a parabolic Hecke module. *Trans. Amer. Math. Soc.*, 314(1):97–182, 2007.

[MS08] V. Mazorchuk and C. Stroppel. Categorification of (induced) cell modules and the rough structure of generalised Verma modules. *Advances in Mathematics*, 219(4):1363–1426, 2008.

[MS09] V. Mazorchuk and C. Stroppel. A combinatorial approach to functorial quantum $\mathfrak{sl}(k)$ knot invariants. *Amer. J. Math.*, 131(6):1679–1713, 2009.

[Pas04] D.S. Passman. *A Course in Ring Theory* . American Mathematical Society, 2004.

[Rou08] R. Rouquier. 2-Kac-Moody algebras, 2008.

[Soe00] W. Soergel. On the relation between intersection cohomology and representation theory in positive characteristic. *J. Pure Appl. Algebra*, 152:311–335, 2000.

[Soe07] W. Soergel. Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln "uber Polynomringen. *JIMJ*, 6:501, 2007.

[Sus07] J. Sussan. Category $\mathcal{O}$ and $\mathfrak{sl}(k)$ link invariants. *arXiv:math/0701045v1 [math.QA]*, 2007.

[vLW01] J. H. van Lint and R. M. Wilson. *A Course in Combinatorics*. Cambridge University Press, December 2001.

[Zei10] D. Zeilberger. $\binom{5}{2}$ proofs that $\binom{n}{k} \leq \binom{n}{k+1}$ if $k < n/2$. *arXiv:1003.1273v1 [math.CO]*, 2010.