Infinite Dimensional Spherical Analysis and Harmonic Analysis for Groups acting on Homogeneous Trees

PhD Thesis

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July 2012

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Abstract

In this thesis, we study groups of automorphisms for homogeneous trees of countable degree by using an inductive limit approach. The main focus is the thourough discussion of two Olshanski spherical pairs consisting of automorphism groups for a homogeneous tree and a homogeneous rooted tree, respectively. We determine the spherical functions, discuss their positive definiteness, and make realizations of the corresponding spherical representations. We turn certain double coset spaces into semigroups and use this to make a complete classification of a certain class of unitary representations of the groups, the so-called irreducible tame representations. We prove the existence of irreducible non-tame representations by constructing a compactification of the boundary of the tree - an object which until now has not played any role in the analysis of automorphism groups for trees which are not locally finite. Finally, we discuss conditionally positive definite functions on the groups and use the generalized Bochner-Godement theorem for Olshanski spherical pairs to prove Levy-Khinchine formulas for both of the considered pairs.

Dansk resume (Danish abstract)

I denne afhandling gennemføres med udgangspunkt i en tilgang baseret på induktive grænser af topologiske grupper et studie af automorfier for homogene træer af tællelig grad. Fokus er en grundig diskussion af to konkrete Olshanski-sfæriske par, der består af automorfigrupper for et homogent træ både med og uden rod. Vi bestemmer de sfæriske funktioner, diskuterer, hvorvidt disse er positivt definitte, og giver realisationer af de tilhørende sfæriske repræsentationer. Vi udstyrer visse dobbeltsideklasserum med en semigruppestruktur og giver ved brug heraf en fuldstændig klassifikation af en væsentlig type af unitære repræsentationer af de betragtede grupper, de såkaldte irreducible, tamme repræsentationer. Vi beviser eksistensen af irreducible, ikke-tamme repræsentationer ved konstruktion af en kompaktifikation af træets rand, der i den eksisterende litteratur ingen rolle har spillet i analysen af automorfigrupper for homogene træer, der ikke er lokalt endelige. Endelig diskuterer vi betinget positivt definitte funktioner på de betragtede grupper og beviser ved brug af den generaliserede Bochner-Godement-sætning for Olshanski-sfæriske par Levy-Khinchine-formler for begge de betragtede par.

Introduction

Homogeneous trees and their automorphisms come up naturally in a wide range of mathematical areas and have been the center of attention for mathematical research for years. The foundation for a renewed interest in the field was laid by the publication of the lecture notes [Se] in the 1970s, and in the following years the development of harmonic analysis for groups of automorphisms of locally finite, homogeneous trees was initiated. The main instigator was Cartier whose papers [Ca1] and [Ca2] were the first to succesfully develop a theory of spherical functions for such groups. During the last part of the 1970s, Olshanski took up the gauntlet by providing an extensive insight into the representation theory, cf. the paper [O2]. During recent years, the study has continued from a number of different perspectives (cf. [KuV] and [CKS]), but has reached a stage where the harmonic analysis for groups acting on locally finite, homogeneous trees is quite well-understood.

However, automorphism groups for homogeneous trees which are not locally finite have been mostly neglected by the mathematical society. In the paper [O3] from 1982, Olshanski obtained a complete classification of the irreducible representations for such groups, but since then not much development has taken place. At its current state, the insight into harmonic analysis of these groups is by no means satisfactory.

Meanwhile, the study of inductive limits of locally compact groups has been a hot topic in harmonic analysis for the past decades, and the theory in the area has been rapidly developing. A large number of families of well-known and heavily investigated groups are parametrized by some dimension parameter, and this has made the study of inductive limits of such groups natural. The most notable and well-understood examples are classical matrix groups for which inductive limits have been studied in several different cases (for example, the reader could consult [O1], [Fa1] and [Ra2] and their bibliographies). A totally different example is the infinite symmetric group which has been thoroughly studied in the papers [O5] and [KOV].

In his paper [O1] from 1990, Olshanki greatly facilitated this study by developing a general theory of inductive limits of locally compact groups. The relation between unitary representations of the inductive limit group and of the underlying locally compact groups was investigated, and it was proved that every irreducible unitary representation of the inductive limit may in a natural way be approximated by irreducible unitary representations of the underlying groups. Furthermore, pairs of groups were studied, and the classical notions of a Gelfand pair and spherical functions were generalized by the introduction of what is today known as an Olshanski spherical pair.

The purpose of this thesis is to make use of the theory of Olshanski spherical pairs in a new setting and hereby bring forward the development of the harmonic analysis of automorphism groups for homogeneous trees which are not locally finite. The inductive limit approach has turned out to be of invaluable use in the study of the infinite symmetric group which - from a number of different perspectives - share similarities with such automorphism groups. Despite this fact, inductive limits of automorphism groups have never been considered. This thesis fills the void.

The thesis will center around the study of two Olshanski spherical pairs. One consists of groups of automorphisms for a homogeneous tree which is not locally finite while the other is built from groups of automorphisms of a rooted tree with similar characteristics. Our main emphasis will be firstly to gain insight into the spherical functions and representations for the pairs and to completely characterize a class of irreducible unitary representations for the groups, the so-called tame representations. Secondly, we approach the main complication arising from the fact that the tree is not locally finite, namely the non-existence of certain measures on the boundary of the tree. We solve the problem by constructing a compactification of the boundary with the required properties and show how the resulting representation theory differs from what is known from the locally finite case.

The thesis is divided into two parts. Chapters 1 and 2 contain the background material which is necessary for the later chapters. Since the reader is not expected to be familiar with the harmonic analysis for automorphism groups for locally finite, homogeneous trees, the treatment of this topic is fairly detailed. The purpose is to familiarize the reader with the geometry of the tree and the reasoning applied in the area and to make the thesis as self-contained as possible. Readers with the necessary background may skip large parts of these chapters, most notably the proofs. The main material of the thesis is contained in chapters 3-8 which give a detailed description of the most important results of my research.

Chapter 1 is devoted to a short presentation of the theory of general Olshanski spherical pairs and their spherical functions. We define such pairs and give a short discussion of their topological properties. We will focus on this new concept as a generalization of the well-known notion of a Gelfand pair. Hence, we prove that some of the well-known properties of unitary representations in the case of Gelfand pairs are "‘inherited"’ in the case of Olshanski spherical pairs. Finally, we generalize the notion of a spherical function to this setting and motivate it by proving that the definition is "‘the right one"’ in the sense that important properties of spherical functions for Gelfand pairs are carried over to this new situation.

In chapter 2 we develop the parts of the harmonic analysis for automorphism groups for locally finite, homogeneous trees which will be needed in the remaining chapters. We give detailed definitions of the basic concepts and show how the group of automorphisms can be turned into a Hausdorff topological group which is locally compact if the tree is locally finite. This leads to the construction of a certain Gelfand pair which is the center of attention. The main point is to find the spherical functions for this pair, to determine which are positive definite, and to find realizations of the corresponding spherical representations. This is done in great detail and involves the study of the Laplace operator on the tree and certain measures and functions on the boundary. New proofs have been provided in certain places, and details are given in places where the existing literature fails to do this.

Chapter 3 centers around the construction and initial study of the first of the two Olshanski spherical pairs which are the focus in this thesis, namely a pair (G, K) built from groups of automorphisms for a homogeneous tree of countable degree. The main complication of the construction involves an appropriate extension of automorphisms of certain locally finite, homogeneous subtrees to the "‘big"’ tree. We prove that the pair is really an Olshanski spherical pair and discuss its topological properties. The main part of the chapter is the discussion of the spherical functions for the pair. These are determined, and we investigate their positive definiteness and give concrete realizations of the corresponding spherical representations. We finish the chapter by observing that a new proof for a well-known fact on positive definiteness of certain functions on the free group with countably many generators is provided by the content of this chapter.

Chapter 4 returns to the locally finite case to develop harmonic analysis for a certain

Gelfand pair consisting of automorphisms for a homogeneous rooted tree. Surprisingly, this naturally occurring pair has not been studied in the existing litterature. It provides the foundation for the study of our second Olshanski spherical pair in later chapters. Again the main focus is the construction of the spherical functions (which are all positive definite) and the corresponding spherical representations. These representations are studied in some detail, and their relation to the spherical representations of chapter 2 is discussed.

Chapter 5 is concerned with the construction and initial study of the second of the two Olshanski spherical pairs which are the main focus of this thesis - a pair which consists of automorphism groups for a homogeneous rooted tree of countable degree. As in chapter 3 the construction involves the extension of automorphisms of locally finite subtrees to the "‘big"’ tree, but it is way more technical and requires a lot of preliminary work. We prove that the construction gives rise to an Olshanski spherical pair, determine the spherical functions (which are all positive definite), and give realizations of the spherical representations. The latter are investigated in greater detail.

The objective of chapter 6 is to continue the study of the groups from chapter 3 and 5 by making a complete classification of a big class of unitary representations, namely the so-called irreducible tame representations. We make use of the inducing construction to create a family of unitary representations and prove that these exhaust the class of irreducible tame representations. The strategy centers around the construction of certain semigroups of partial automorphisms of the tree and is based on a further development of ideas presented by Olshanski in [O4] for the infinite symmetric group. Much of the chapter is devoted to the construction of the semigroups and to their representations and to show how representations of the semigroups are related to tame representations of the group. It turns out that this relation and insight into the semigroups make a complete classification possible.

Chapter 7 has a two-fold purpose. Firstly, it is a continuation of the discussion of the previous chapter which encourages the study of non-tame representations. Secondly, it approaches the problem of non-existence of certain invariant measures on the boundary - a feature which is the main complication arising from the fact that the tree is not locally finite. It turns out that these two questions are related. We replace the central role of the boundary in the analysis of the locally finite tree by a certain compactification which possesses the required measures. It turns out that this attempt to imitate the ideas from the locally finite case produces a family of non-tame representations and hence proves the existence of such objects. The representations are studied in greater detail, and the differences from the locally finite case are discussed.

Chapter 8 makes a small digression and studies conditionally positive definite functions for the two Olshanski spherical pairs which are the focus of this thesis. The main objective is to prove Levy-Khinchine formulas for both pairs. We recall the basic facts on conditionally positive definite functions and their relation to cocycles. We construct a certain function ψ on the group of chapter 3 which by a study of an extension of the Haagerup cocycle is proved to be conditionally positive definite and pure. This leads to the Levy-Khinchine decomposition formulas in which the functions ψ plays a central role.

The thesis has been written at the conclusion of my time as a ph.d. student at the Faculty of Science at Aarhus University. As such, it contains a detailed account of the main results of the research project I have carried out. This project has been concerned with two major topics: the study of Olshanski spherical pairs and the discussion of automorphism groups for homogeneous trees. As pointed out above, the main idea of this thesis is to apply an inductive limit approach to the study of automorphism groups. In this way both areas are reflected in the thesis.

The use of inductive limit methods in the analysis of automorphism groups is by no means exhausted, and the insight into the groups considered here are not yet complete. A number of interesting questions remain unsolved. Along the way, I have included a number of remarks which point out open problems that have caught my attention during my research and that should be approached in the future.

On purpose, the proofs and explanations in the thesis are quite detailed. The idea is to make the thesis as self-contained as possible and to make the material accessible to people who are not completely familiar with either Olshanski spherical pairs or automorphism groups for homogeneous trees.

Finally, I would like to express my sincere gratitude to my supervisor, Bent Ørsted, without whom this project would never have reached the point where it is today. He has not only turned my attention in the direction of this interesting topic, his immense knowledge and superior insight have been a real inspiration and a great source of help. Furthermore, I warmly thank Jacques Faraut and Marek Bozejko whose hospitality and helpfulness I have enjoyed on two separate occassions during my project. Fruitful discussions and interesting ideas have been invaluable results of my visits in Paris and Wroclaw.

Emil Axelgaard
Aarhus, July 2012

Chapter 1

Olshanski Spherical Pairs and Spherical Functions

The purpose of this chapter is to develop the basic abstract theory of Olshanski spherical pairs. The results obtained here form important background information and provide a foundation for the study of concrete Olshanski spherical pairs in chapters 3 and 5. In section 1, we define such a pair (G, K) as an inductive limit of an increasing sequence of Gelfand pairs and mention how this gives rise to a toplogical group. In section 2, we prove that the usual results concerning K-invariant vectors for unitary representations known from the theory of Gelfand pairs are still true in this more general situation. In section 3, we generalize the notion of a spherical function and motivate the definition by proving that the basic characterization of positive definite spherical functions for Gelfand pairs are still true in this more general setting. Finally, we make a short discussion of the relation between spherical functions for an Olshanski spherical pair and spherical functions for the underlying Gelfand pairs.

1.1 Olshanski spherical pairs

Let $G_1 \subseteq G_2 \subseteq \ldots \subseteq G_n \subseteq \ldots$ be an increasing sequence of locally compact groups such that, for each n, G_n has the topology induced from G_{n+1} and G_n is a closed subgroup of G_{n+1}. Furthermore, assume that, for each n, K_n is a compact subgroup of G_n such that $K_n = G_n \cap K_{n+1}$ and such that (G_n, K_n) is a Gelfand pair, i.e. the Banach algebra $L^1(K_n \backslash G_n / K_n)$ of K_n-biinvariant integrable functions on G_n, is commutative under convolution. Define

$$G = \bigcup_{n=1}^{\infty} G_n \quad \text{and} \quad K = \bigcup_{n=1}^{\infty} K_n$$

G is clearly a group with the natural multiplication, and if we endow it with the inductive limit topology, it follows by [TSH, Theorem 2.7] that G is a topological group. If e is the neutral element in G, $\{e\}$ is a closed subset of G_n for all n and so a closed subset of G. By [Fo, Corollary 2.3], this means that G is Hausdorff. However, it is in general not locally compact, cf. Remark 1.1.1.

K is clearly a subgroup of G, and the relation $K_n = K_{n+1} \cap G_n$ shows that $K \cap G_n$ is closed in G_n for all n. Hence, K is a closed subgroup of G which in general is not compact, cf. Remark 1.1.1.

We say that (G, K) is an *Olshanski spherical pair*.

Remark 1.1.1 It is an easy consequence of the definition of the inductive limit topology that the group inversion in G is continuous - even if the groups were not assumed to be locally compact. This assumption is, however, essential in establishing the continuity of the group mulitplication in G. [TSH, Example 1.2] contains the discussion of a case where the group multiplication in the inductive limit of topological groups is not continuous in the inductive limit topology. However, [TSH, Part 1] introduces a new topology, the Bamboo-Shoot topology, on the inductive limit which makes the group multiplication continuous in a more general set-up. For locally compact groups, the topologies coincide.

It is not difficult to provide examples of the above construction where G turns out not to be locally compact (for example our main examples of Olshanski spherical pairs which are considered in chapters 3 and 5). In [HSTH, Proposition 6.5], it is, however, shown that G is a locally compact group if and only if there exists n such that G_n is an open subgroup of G.

Furthermore, it is not difficult to see that K is compact if and only if $K = K_n$ for some n. Indeed, if K is compact and $K \neq K_n$ for all n, we can find a subsequence $\{n_k\}$ of $\mathbb{N}$ such that

$$K_{n_1} \subsetneq K_{n_2} \subsetneq \ldots K_{n_k} \subsetneq \ldots$$

Let $x_1 \in K_{n_1}$ and $x_k \in K_{n_k} \setminus K_{n_{k-1}}$ for $n \geq 2$. Put $A_n = \{x_j | j \geq n\}$. The intersection $A_n \cap G_k$ is - by the identity $K_j = G_j \cap K_{j+1}$ - finite and hence closed in G_k for all k. This means that A_n is a closed subset of K and so compact. But $\cap_{n=1}^{\infty} A_n = \emptyset$ - contradicting the finite intersection property which should hold in the Hausdorff space G. Hence, $K = K_n$ for some n. On the other hand, if $K = K_n$ for some n, the continuity of the inclusion map from G_n into G immediately shows that K is compact.

Plenty of examples of Olshanski spherical pairs from the world of classical matrix groups are known and have been studied in great detail. Examples are found in [O1], [Fa1] and [Ra2]. We will not list these examples here since our focus will be Olshanski spherical pairs which are of a very different nature. Our main example arises from automorphism groups for homogeneous trees and will be discussed in great detail in chapters 3 and 5.

1.2 Olshanski spherical pairs and unitary representations

The above introduced concept of an Olshanski spherical pair obviously generalizes the well-known concept of a Gelfand pair. Fortunately, some of the most important results concerning the relation between irreducibility of a unitary representation of G and the space of K-invariant vectors remain true in this more general situation. Propositions 1.2.1 and 1.2.2 are natural generalizations of well-known results for Gelfand pairs (cf. [Fa2, Propositions I.3 and I.5]).

We begin by introducing some standard notation.

For a unitary representation π of G with representation space $\mathcal{H}$, we let $\mathcal{H}^K$ and $\mathcal{H}^{K_n}$ be the subspaces of K- and K_n-invariant vectors, respectively. It is clear that $\mathcal{H}$ and $\mathcal{H}^{K_n}$ are closed, and we denote by P and P_n the corresponding orthogonal projections. Since

$$\mathcal{H}^{K_1} \supseteq \mathcal{H}^{K_2} \supseteq \ldots \supseteq \mathcal{H}^{K_n} \supseteq \ldots$$

and

$$\mathcal{H}^K = \bigcap_{n=1}^{\infty} \mathcal{H}^{K_n},$$

it is well-known that P_n converges strongly to P.

Since the Gelfand-Naimark-Segal construction applied to a continuous, K-bi-invariant, positive definite function provides us with a unitary representation with a K-invariant cyclic vector, it is natural to ask when such representations are irreducible. The following proposition reveals that this is the case if $\mathcal{H}^K$ is "‘small"’.

Proposition 1.2.1 *Let (G, K) be an Olshanski spherical pair, and let π be a unitary representation of G with representation space $\mathcal{H}$ and with a cyclic vector $v \in \mathcal{H}^K$. If dim $\mathcal{H}^K = 1$, π is irreducible.*

Proof. Let $\mathcal{U}$ be a closed, invariant subspace.

First we assume that $P(\mathcal{U}) = \{0\}$. In this case, $\mathcal{U}$ is orthogonal to $\mathcal{H}^K$ and hence to $v \in \mathcal{H}^K$. This means that for all $x \in G$ and $u \in \mathcal{U}$

$$\langle \pi(x)v, u\rangle = \langle v, \pi(x^{-1})u\rangle = 0$$

since $\pi(x^{-1})u \in \mathcal{U}$ by the invariance of $\mathcal{U}$. Hence, $\mathcal{U}$ is orthogonal to the dense subspace spanned by the $\pi(x)v$'s for $x \in G$ and so to $\mathcal{H}$. This shows that $\mathcal{U} = \{0\}$.

Now assume that $P(\mathcal{U}) \neq \{0\}$. Since $\mathcal{H}^K$ has dimension 1, this means that $P(\mathcal{U}) = \mathcal{H}^K$. If we by ν_n denote the normalized Haar measure on K_n, it is well-known that

$$P_n w = \int_{K_n} \pi(k)w \; \nu_n(dk) \tag{1.1}$$

for all $w \in \mathcal{H}$ and all n. By [Fo, Theorem A3.1], $P_n w \in \overline{\text{span}\,\{\pi(k)w \mid k \in K_n\}}$ which shows that $P_n(\mathcal{U}) \subseteq \mathcal{U}$ for all n. Since P_n converges strongly to P, we see that $\mathcal{H}^K = P(\mathcal{U}) \subseteq \mathcal{U}$, and so $v \in \mathcal{U}$. The fact that v is cyclic now means that $\mathcal{U} = \mathcal{H}$.

This shows that π is irreducible. □

In light of Proposition 1.2.1, an obvious question is whether irreducibility automatically implies that $\mathcal{H}^K$ is "‘small"’, i.e. has dimension at most 1. This is the case as it is the content of the following proposition:

Proposition 1.2.2 *Let (G, K) be an Olshanski spherical pair, and let π be an irreducible unitary representation of G. Then dim $\mathcal{H}^K \leq 1$.*

Proof. Assume that $\mathcal{H}^K \neq \{0\}$, and let $n \in \mathbb{N}$. It is easy to see that the σ-algebra on K_n induced by the Borel σ-algebra on G_n is just the Borel σ-algebra on K_n. Hence, we may extend the normalized Haar measure ν_n on K_n to a finite Borel measure μ_n on G_n by defining

$$\mu_n(A) = \nu_n(A \cap K_n)$$

for all Borel subsets A of G_n. It is not difficult to check that μ_n is a Radon measure, and it is obviously K-biinvariant.

Let $x \in G_n$, and denote by δ_x the corresponding Dirac measure which is a Radon measure on G_n. For all Borel sets A in G_n and $k \in K_n$, the convolution $\mu_n * \delta_x * \mu_n$ satisfies the identity

$$\begin{aligned}(\mu_n * \delta_x * \mu_n)(kA) &= \int \mu_n(kAy^{-1}) \; (\delta_x * \mu_n)(dy) = \int \mu_n(Ay^{-1}) \; (\delta_x * \mu_n)(dy) \\ &= (\mu_n * \delta_x * \mu_n)(A)\end{aligned}$$

which shows that $\mu_n * \delta_x * \mu_n$ is invariant under K_n from the left. Similarly, it is seen that $\mu_n * \delta_x * \mu_n$ is invariant under K_n from the right, and so it is a K_n-biinvariant, finite Radon measure. Since (G_n, K_n) is a Gelfand pair, it follows by [W, Theorem 8.1.7] that the algebra of regular K_n-biinvariant complex measures is commutative under convolution which shows that

$$\mu_n * \delta_x * \mu_n * \mu_n * \delta_y * \mu_n = \mu_n * \delta_y * \mu_n * \mu_n * \delta_x * \mu_n$$

for all $x, y \in G_n$. If we by μ and ν denote finite Radon measures on G_n and if $w \in \mathcal{H}$, we have the obvious identities (recall (1.1) which describes the projection P_n as an integral)

$$\pi(x)w = \int_{G_n} \pi(g)w \; \delta_x(dg) \quad \text{and} \quad P_n w = \int_{G_n} \pi(g)w \; \mu_n(dg)$$

and

$$\int_{G_n} \pi(g)w \; (\mu * \nu)(dg) = \int_{G_n} \pi(g)(\int_{G_n} \pi(h)w \; \nu(dh)) \; \mu(dg)$$

They reveal that

$$P_n\pi(x)P_n\pi(y)P_n = P_n\pi(x)P_nP_n\pi(y)P_n = P_n\pi(y)P_nP_n\pi(x)P_n = P_n\pi(y)P_n\pi(x)P_n$$

For all $k \geq 0$, we have the inclusion $\mathcal{H}^{K_{n+k}} \subseteq \mathcal{H}^{K_n}$ which means that $P_nP_{n+k} = P_{n+k}$. Hence, we observe that for all $k \geq 0$ and all $w \in \mathcal{H}$

$$\begin{aligned} P_n\pi(x)P_n\pi(y)P_{n+k}w &= P_n\pi(x)P_n\pi(y)P_nP_{n+k}w = P_n\pi(y)P_n\pi(x)P_nP_{n+k}w \\ &= P_n\pi(y)P_n\pi(x)P_{n+k}w \end{aligned}$$

which by the strong convergence of the P_m's and the continuity of the operators means that $P_n\pi(x)P_n\pi(y)P = P_n\pi(y)P_n\pi(x)P$.

Similarly, it is true that $P_{n+k}P_n = P_{n+k}$ for all $k \geq 0$ which by an analogous argument means that $P\pi(x)P_n\pi(y)P = P\pi(y)P_n\pi(x)P$ - an identity which is true for all $n \in \mathbb{N}$.

Finally, the strong convergence and the continuity of the operators show that

$$(P\pi(x)P)(P\pi(y)P) = P\pi(x)P\pi(y)P = P\pi(y)P\pi(x)P = (P\pi(y)P)(P\pi(x)P)$$

Let $\mathcal{A}$ be the norm closed algebra generated by the operators $P\pi(x)P$ with $x \in G$. Since this is just the closure of the set of all polynomial expressions in the $P\pi(x)P$'s, the above shows that $\mathcal{A}$ is a commutative Banach *-algebra. Similarly, we observe that $\mathcal{H}^K$ is invariant under $\mathcal{A}$ since it is invariant under the operators generating $\mathcal{A}$.

The inclusion map from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$, the Banach *-algebra of bounded operators on $\mathcal{H}$, is a *-representation of $\mathcal{A}$. We claim that $\mathcal{H}^K$ is an irreducible subspace. Indeed, let $\mathcal{U} \neq \{0\}$ be a closed $\mathcal{A}$-invariant subspace of $\mathcal{H}^K$, and denote by $\mathcal{U}^\perp$ the othogonal complement in $\mathcal{H}^K$. Let $u \in \mathcal{U}$, $u \neq 0$. For $v \in \mathcal{U}^\perp$, the invariance of $\mathcal{U}$ implies that

$$\langle \pi(x)u, v \rangle = \langle \pi(x)Pu, Pv \rangle = \langle P\pi(x)Pu, v \rangle = 0$$

Since π is irreducible, u is a cyclic vector for π, and so v is orthogonal to $\mathcal{H}$. Hence, $v = 0$ which means that $\mathcal{U} = \mathcal{H}^K$.

Since $\mathcal{A}$ is a commutative Banach *-algebra and $\mathcal{H}^K$ is an irreducible subspace of the considered *-representation of $\mathcal{A}$, we see by [VD, Lemma 6.2.4] that $\dim \mathcal{H}^K = 1$.

This finishes the proof. □

Propositions 1.2.1 and 1.2.2 will be invaluable tools in the discussion of spherical functions in the next section.

1.3 Spherical functions

It is a natural desire to try to generalize the notion of a spherical function for Gelfand pairs to the more general concept of Olshanski spherical pairs. Fortunately, this is possible. As is seen by Theorem 1.3.4, the following definition is the "‘right"’ one in the sense that the most important properties of spherical functions for Gelfand pairs carry over to this more general situation:

Definition 1.3.1 Let (G, K) be an Olshanski spherical pair. A K-biinvariant, continuous function φ on G, $\varphi \neq 0$, is said to be spherical for (G, K) if for all $x, y \in G$.

$$\lim_{n\to\infty} \int_{K_n} \varphi(xky)\ \nu_n(dk) = \varphi(x)\varphi(y) \tag{1.2}$$

where ν_n is the normalized Haar measure on K_n.

It should be noticed that the integrals in Definition 1.3.1 - by definition of the inductive limit topology - make sense for any continuous function φ on G and for any n. Furthermore, it is obvious that the constant function 1 is always a spherical function for (G, K).

The following properties of spherical functions are rather obvious:

Lemma 1.3.2 *Let (G, K) be an Olshanski spherical pair, and let φ be a spherical function for (G, K). Then*

1. $\varphi(e) = 1$
2. $\|\varphi\|_\infty = 1$ *if φ is bounded*

Proof. Choose $x \in G$ such that $\varphi(x) \neq 0$. By the K-biinvariance of φ, we observe that

$$\int_{K_n} \varphi(kx)\ \nu_n(dk) = \int_{K_n} \varphi(x)\ \nu_n(dk) = \varphi(x)$$

for all n. Since φ is spherical, this means that $\varphi(x)\varphi(e) = \varphi(x)$ which proves 1.

To prove 2., we observe that for all $x, y \in G$ and all $n \in \mathbb{N}$

$$\left| \int_{K_n} \varphi(xky)\ \nu_n(dk) \right| \leq \|\varphi\|_\infty$$

if φ is bounded. Since φ is spherical, this means that $\|\varphi\|_\infty^2 \leq \|\varphi\|_\infty$ which shows that $\|\varphi\|_\infty \leq 1$. Equality follows by 1. □

The spherical functions for several Olshanski spherical pairs arising from classical matrix groups are known, cf. [Ra2] and [Fa1]. In chapters 3 and 5, we determine the spherical functions for our main examples of Olshanski spherical pairs arising from automorphism groups for homogeneous trees.

Define $\mathcal{P}_1(K\backslash G/K)$ to be the set consisting of all continuous, K-biinvariant, positive definite functions φ on G with $\varphi(e) = 1$. This is clearly a convex subset of the vector space consisting of all complex-valued functions on G. In general, not all spherical functions for an Olshanski spherical pair are bounded, let alone positive definite, cf. chapter 3. For $\varphi \in \mathcal{P}_1(K\backslash G/K)$, the equation (1.2) is, however, intimately related with the question of irreducibility of the corresponding representation arising from the Gelfand-Naimark-Segal construction and of extremality of φ in the convex set $\mathcal{P}_1(K\backslash G/K)$. This is the content of

Theorem 1.3.4 which generalizes a well-known result for Gelfand pairs (cf. [Fa2, Proposition I.4 and I.6]).

First we need the following general and well-known fact whose proof may be found in [Fo, Theorem 3.25]:

LEMMA 1.3.3 *Let G be a topological group and K a closed subgroup. Let $\varphi \in \mathcal{P}_1(K\backslash G/K)$. φ is extremal in the convex set $\mathcal{P}_1(K\backslash G/K)$ if and only if the representation π associated to φ by the Gelfand-Naimark-Segal construction is irreducible.*

We are now ready to state and prove the main theorem:

THEOREM 1.3.4 *Let (G, K) be an Olshanski spherical pair. For a function $\varphi \in \mathcal{P}_1(K\backslash G/K)$, the following are equivalent:*

1. *φ is a spherical function for (G, K).*
2. *The representation π associated to φ by the Gelfand-Naimark-Segal construction is irreducible.*
3. *φ is an extremal point in $\mathcal{P}_1(K\backslash G/K)$*

PROOF. By Lemma 1.3.3, it is enough to show that 1. and 2. are equivalent. So we assume that φ is a spherical function for (G, K), and let v be a K-invariant cyclic unit vector in $\mathcal{H}$, the representation space of π, such that $\varphi(x) = \langle v, \pi(x)v\rangle$ for all $x \in G$. By the integral expresson in (1.1) and the strong convergence of the projections P_n, we observe that

$$\begin{aligned}\left\langle \pi(x)v, \overline{\varphi(y)}v \right\rangle &= \varphi(y)\left\langle v, \pi(x^{-1})v\right\rangle = \varphi(y)\varphi(x^{-1}) = \lim_{n\to\infty}\int_{K_n}\varphi(x^{-1}ky)\ \nu_n(dk)\\ &= \lim_{n\to\infty}\int_{K_n}\langle \pi(x)v, \pi(ky)v\rangle\ \nu_n(dk) = \lim_{n\to\infty}\langle \pi(x)v, P_n\pi(y)v\rangle = \langle \pi(x)v, P\pi(y)v\rangle\end{aligned}$$

for all $x, y \in G$. Since v is cyclic, this means that $P\pi(y)v = \overline{\varphi(y)}v$ for all $y \in G$ which by the same fact has as a consequence that P maps $\mathcal{H}$ into the 1-dimensional, closed subspace spanned by v. Since $\mathcal{H}^K \neq \{0\}$, this means that $\dim \mathcal{H}^K = 1$, and so π is irreducible by Proposition 1.2.1.

Now assume that π is irreducible, and let $x, y \in G$. The calculations

$$\begin{aligned}\langle v, \pi(x)P\pi(y)v\rangle &= \lim_{n\to\infty}\left\langle \pi(x^{-1})v, P_n\pi(y)v\right\rangle\\ &= \lim_{n\to\infty}\int_{K_n}\left\langle \pi(x^{-1})v, \pi(k)\pi(y)v\right\rangle\ \nu_n(dk)\\ &= \lim_{n\to\infty}\int_{K_n}\varphi(xky)\ \nu_n(dk)\end{aligned}$$

show that it suffices to prove that $\varphi(x)\varphi(y) = \langle v, \pi(x)P\pi(y)v\rangle$.

By Proposition 1.2.2, $\mathcal{H}^K$ has dimension 1, i.e. $\mathcal{H}^K = \mathbb{C}v$. This shows that

$$Pw = \langle w, v\rangle\, v$$

for all $w \in \mathcal{H}$, and so

$$\langle w, P\pi(y)v\rangle = \langle Pw, \pi(y)v\rangle = \langle w, v\rangle\,\langle v, \pi(y)v\rangle = \left\langle w, \overline{\varphi(y)}v\right\rangle$$

This means that $P\pi(y)v = \overline{\varphi(y)}v$ and so

$$P\pi(x)P\pi(y)v = \overline{\varphi(y)}P\pi(x)v = \overline{\varphi(x)\varphi(y)}v$$

Hence, we see that

$$\begin{aligned}\langle v, \pi(x)P\pi(y)v\rangle &= \langle Pv, \pi(x)P\pi(y)v\rangle = \langle v, P\pi(x)P\pi(y)v\rangle \\ &= \varphi(x)\varphi(y)\langle v, v\rangle = \varphi(x)\varphi(y)\end{aligned}$$

which finishes the proof. □

Theorem 1.3.4 shows that one way to find irreducible unitary representations of G is by determining the positive definite spherical functions for the pair (G, K). Recall that a unitary representation of G is said to be *spherical* for the pair (G, K) if it is irreducible and has a non-zero, K-invariant vector. By the Gelfand-Naimark-Segal construction, there is a bijective correspondance between equivalence classes of spherical representations and the set of positive definite spherical functions. Hence, one way to check that a set of spherical representations is exhaustive is by finding all positive definite spherical functions. In chapters 3 and 5, we find these functions in our main examples and make realizations of the corresponding spherical representations. However, Theorem 1.3.4 shows that the set of such functions is of interest from multiple perspectives. We emphasize this importance by denoting the set of positive definite spherical functions for (G, K) by Ω, and we will refer to it as *the spherical dual* of (G, K). As we shall see below in the generalized Bochner-Godement theorem for Olshanski spherical pairs (Theorem 1.3.6), the functions in Ω are in a way the building blocks of all K-biinvariant, continuous, positive definite functions on G.

Remark 1.3.5 It is a natural question to ask whether a spherical function for the Olshanski spherical pair (G, K) is the pointwise limit of spherical functions for the Gelfand pairs (G_n, K_n). Olshanski has solved this problem in the case of a positive definite spherical function.

In [O1, Theorem 22.10], it is proved - in this general situation - that every such function is the uniform limit on compact sets of positive definite spherical functions for the pairs (G_n, K_n) (notice that this type of convergence makes sense since an argument similar to the one in Remark 1.1.1 shows that every compact set $C \subseteq G$ is a subset of G_n for some n). This and the relation in Theorem 1.3.4 between spherical representations and positive definite spherical functions is used to show that every irreducible representation of G may be approximated - in a way defined in [O1] - by irreducible representations of the G_n's. This is the content of [O1, Theorem 22.9].

Unfortunately, the general question has not yet been answered. Olshanski's proof clearly fails to deal with this situation since it heavily relies on the relation between spherical functions and extremal points in convex sets of Theorem 1.3.4. Nobody has, however, constructed spherical functions for (G, K) that are not pointwise limits of spherical functions for (G_n, K_n). An important open problem in the abstract theory of Olshanski spherical pairs is to provide a satisfactory answer to this question.

A famous result is the Bochner-Godement theorem which states that for a Gelfand pair (G, K) all K-biinvariant, continuous, positive definite functions on G may be decomposed as an integral over the spherical dual with respect to some finite measure (see [W, Theorem 9.3.4] or [Fa2, Theorem II.1]). Recently, Rabaoui has generalized this theorem by using Choquet theory to prove that the same is true for Olshanski spherical pairs (see [Ra1]). The precise statement of the theorem is as follows:

Theorem 1.3.6 *Let* (G, K) *be the Olshanski spherical pair corresponding to the increasing sequence of Gelfand pairs* $\{(G_n, K_n)\}_{n=1}^{\infty}$. *Assume that* G_n *is second countable for all* n. *Let* Ω *be the spherical dual for* (G, K), *and equip it with the topology of uniform convergence on compact sets. For every* K*-biinvariant, continuous, positive definite function* φ *on* G, *there exists a unique positive, finite Borel measure on* Ω *such that*

$$\varphi(g) = \int_{\Omega} \tau(g)\, \mu(d\tau)$$

for all $g \in G$.

This theorem will be the main tool in our study of conditionally positive definite functions in chapter 8.

The aim of the main chapters 3-8 is to make the abstract theory of this chapter concrete by making an extensive study of two Olshanski spherical pairs. In both cases, the spherical functions will be determined and realizations of the spherical representations will be given and scrutinized. The generalized Bochner-Godement theorem will be applied to give general formulas for continuous, biinvariant, positive definite functions and to establish Levy-Khinchine formulas which give integral decompositions of continuous, biinvariant, conditionally positive definite functions. Along the way, we will clarify the connection with a number of open problems in the general theory of Olshanski spherical pairs.

Chapter 2

Harmonic Analysis for Groups Acting on Homogeneous Trees

This chapter provides the necessary background information on groups acting on homogeneous trees and will be an important foundation for our main chapters 3-8. We develop harmonic analysis and representation theory for groups consisting of automorphisms of locally finite, homogeneous tree. The main emphasis will be on finding the spherical functions for certain Gelfaind pairs, determine which are positive definite and find realizations of the spherical representations. The reader is not assumed to be familiar with trees and their automorphisms. Hence, the purpose of the chapter is two-fold. Firstly, it goes through a number of results which will be needed later on. Secondly, a detailed and careful treatment should make the reader more comfortable with reasoning focusing on trees. In section 1, we introduce the basic concepts of this chapter, notably the concept of a tree and an automorphism, and study different types of automorphisms. Furthermore, we turn the group of automorphisms into a topological group and prove that it in some cases is locally compact. Section 2 contains the construction of the boundary of a tree and a description of its most basic properties. In section 3 we construct the Gelfand pair that will be the center of our attention. Section 4 is devoted to the spherical functions for this Gelfand pair, and in section 5 we give realizations of the spherical representations. The main results are Theorems 2.4.6 and 2.5.2.

2.1 Trees and automorphisms

A (non-directed) graph is a pair $(\mathfrak{X}, \mathfrak{C})$ consisting of a set $\mathfrak{X}$ whose elements are known as the *vertices* of the graph, and a set $\mathfrak{C}$ consisting of two-element subsets of $\mathfrak{X}$ - known as the *edges* of the graph. Two elements $x, y \in \mathfrak{X}$ are said to be *neighbours* if $\{x, y\} \in \mathfrak{C}$, and the cardinality of $\{y \in \mathfrak{X} \mid \{x, y\} \in \mathfrak{C}\}$ is known as the *degree* of x. A *path* of length n in $(\mathfrak{X}, \mathfrak{C})$ is a finite sequence $x_0, x_1, \ldots, x_n$ of vertices such that $\{x_k, x_{k+1}\} \in \mathfrak{C}$ for all k, and a path $x_0, x_1, \ldots, x_n$ is known as a *chain* if $x_k \neq x_{k+2}$ for all k. A *circuit* is a chain $x_0, x_1, \ldots, x_n$ such that $x_0 = x_n$.

Similarly, sequences $x_0, x_1, \ldots, x_n \ldots$ and $\ldots, x_0, x_1, \ldots, x_n, \ldots$, of vertices are said to be an *infinite chain starting at* x_0 and a *doubly infinite chain*, respectively, if $\{x_n, x_{n+1}\} \in \mathfrak{C}$ and $x_n \neq x_{n+2}$ for all n.

The graph $(\mathfrak{X}, \mathfrak{C})$ is said to be *connected* if there for all $x, y \in \mathfrak{X}$ exists a path $x_0, \ldots, x_n$ with $x_0 = x$ and $x_n = y$. A *tree* is a connected graph without circuits. A *subtree* of a tree $(\mathfrak{X}\, \mathfrak{C})$ is a connected graph $(\mathfrak{Y}, \mathfrak{D})$ such that $\mathfrak{Y} \subseteq \mathfrak{X}$ and $\mathfrak{D} = \{\{x, y\} \in \mathfrak{C} \mid x, y \in \mathfrak{Y}\}$. Note that $(\mathfrak{Y}, \mathfrak{D})$ is again a tree since a circuit in $(\mathfrak{Y}, \mathfrak{D})$ would again be a circuit in $(\mathfrak{X}, \mathfrak{C})$.

Let $(\mathfrak{X}, \mathfrak{C})$ be a tree, and let $x, y \in \mathfrak{X}$. It is easy to see that the connectedness and the lack of circuits show that there exists a unique chain $x_0, \ldots, x_n$ with $x_0 = x$ and $x_n = y$. We denote this chain by $[x, y]$. If we define $d(x, y) = n$, it is not difficult to check that d is a metric on $\mathfrak{X}$ which induces the discrete topology on $\mathfrak{X}$. We will denote d the natural metric and always regard $\mathfrak{X}$ as a metric space equipped with this metric.

A bijective map $g : \mathfrak{X} \to \mathfrak{X}$ is said to be an *automorphism* if it has the property that x and y are neighbours if and only if $g(x)$ and $g(y)$ are neighbours. An automorphism preserves chains and so is an isometry of the metric space $\mathfrak{X}$. Conversely, a surjective isometry is obviously an automorphism. Hence, a map from $\mathfrak{X}$ into $\mathfrak{X}$ is an automorphism if and only if it is a surjective isometry.

The automorphism property is obviously preserved under composition, and the identity map on $\mathfrak{X}$ is of course an automorphism. Furthermore, for any automorphism g it follows by definition that the inverse map g^{-1} is also an automorphism. Hence, the set $\mathrm{Aut}(\mathfrak{X})$ of automorphisms of a tree is a group under composition.

The automorphisms may in a natural way be split into three mutually disjoint sets. This is the content of Proposition 2.1.2 below. For the proof, we need the following lemma:

Lemma 2.1.1 *Let g be an automorphism of a tree $(\mathfrak{X}, \mathfrak{C})$, and let $x \in \mathfrak{X}$. Let $x_0 = x, x_1, \ldots, x_n = g(x)$ be the chain from x to $g(x)$, and assume that $n > 0$. If $g(x_1) \neq x_{n-1}$, the chain can be extended to a doubly infinite chain $\ldots, x_{-1}, x_0, x_1, \ldots$ such that $g(x_k) = x_{k+n}$ for all $k \in \mathbb{Z}$.*

Proof. Define $x_{n+1} = g(x_1)$. By assumption, $x_{n+1} \neq x_{n-1}$, and since g is an automorphism, $d(x_n, x_{n+1}) = 1$. Hence, $x_0, \ldots, x_{n+1}$ is a chain. The chain property tells that $x_0 \neq x_2$, and so it follows by injectivity of g that $g(x_2) \neq g(x_0) = x_n$. This means that $x_0, \ldots, x_{n+2}$ is a chain if we define $x_{n+2} = g(x_2)$. In this way, we may inductively define $x_{k+n} = g(x_k)$ for $k \geq 1$ to obtain an infinite chain $x_0, x_1, \ldots, x_n, \ldots$ with the property that $g(x_k) = x_{n+k}$.

Using the same procedure on the chain $x_n, \ldots, x_0 = g^{-1}(x_n)$ for which $g^{-1}(x_{n-1}) \neq x_1$, we may extend it to an inifite chain $x_n, \ldots, x_0, x_{-1}, \ldots$ such that $g^{-1}(x_k) = x_{k-n}$ for all k. The chain $\ldots, x_{-1}, x_0, x_1, \ldots$ now has the required property. □

Proposition 2.1.2 *Let g be an automorphism of a tree $(\mathfrak{X}, \mathfrak{C})$. Then g has exactly one of the following properties:*

1. *There exists a vertex $x \in \mathfrak{X}$ such that $g(x) = x$.*
2. *There exists an edge $\{a, b\} \in \mathfrak{C}$ such that $g(a) = b$ and $g(b) = a$.*
3. *There exists a doubly infinite chain $\ldots, x_0, x_1, \ldots, x_n, \ldots$ and an integer $k \in \mathbb{N}$ such that $g(x_n) = x_{n+k}$ for all $n \in \mathbb{Z}$.*

Proof. Let $k = \min \{d(x, g(x)) |\ x \in \mathfrak{X}\}$, and let $x \in \mathfrak{X}$ be a vertex such that $d(x, g(x)) = k$. If $k = 0$, g has property 1. If $k = 1$ and $g^2(x) = x$, g has property 2. with $a = x$ and $b = g(x)$. If $k = 1$ and $g^2(x) \neq x$, it follows by Lemma 2.1.1 that we may extend the chain $x_0 = x, x_1 = g(x)$ to a doubly inifite chain $\ldots, x_0, x_1, \ldots, x_n, \ldots$ such that 3. is satisfied. If $k \geq 2$ and $x_0 = x, \ldots, x_k = g(x)$ is the chain from x to $g(x)$, it is by the minimality of k true that $g(x_1) \neq x_{k-1}$ since $d(x_1, x_{k-1}) = k - 2 < k$. Hence, Lemma 2.1.1 shows that we may extend this chain to a doubly infinite chain $\ldots, x_0, x_1, \ldots, x_n, \ldots$ such that 3. is satisfied. So g has at least one of the studied properties.

To see that it cannot have more than one, we assume that g has properties 1. and 2. We may without loss of generality assume that $d(x, b) = d(x, a) + 1$. Since g is an isometry, we see that $d(x, a) = d(x, b) + 1$ which is of course a contradiction.

Now assume that g has properties 1. and 3. Obviously, we may choose n such that x_n is the unique vertex in the doubly infinite chain with minimal distance to x. g is an isometry, and so $d(x, x_n) = d(x, x_{n+k})$ which contradicts the uniqueness of n since $k \geq 1$.

Finally, assume that g has properties 2. and 3. It is obviously impossible that both a and b are vertices in the doubly infinite chain $\ldots, x_0, x_1, \ldots, x_n, \ldots$, so we may without loss of generality assume that $d(b, x_n) = d(a, x_n) + 1$ for all n. g is an isometry, and so we see that $d(a, x_{n+k}) = d(b, x_{n+k}) + 1$ for all n which is a contradiction.

This finishes the proof. □

An automorphism g is said to be a *rotation* if it has property 1., an *inversion* if it has property 2. and a *k-step translation* along the chain $\ldots, x_0, x_1, \ldots, x_n, \ldots$ if it has property 3.

We now want to turn the group of automorphisms $\mathrm{Aut}(\mathfrak{X})$ of a tree $(\mathfrak{X}, \mathfrak{C})$ into a topological group. To do this, we endow $\mathrm{Aut}(\mathfrak{X})$ with the compact-open topology. Recall that this topology has as a subbase the collection of sets $V(K, U) = \{g \in \mathrm{Aut}(\mathfrak{X}) \mid g(K) \subseteq U\}$ where $K \subseteq \mathfrak{X}$ is compact and $U \subseteq \mathfrak{X}$ is open. Since $\mathfrak{X}$ is discrete, K runs through all finite subsets of $\mathfrak{X}$ and U through all subsets of $\mathfrak{X}$.

Observe that $W \subseteq \mathrm{Aut}(\mathfrak{X})$ is open if and only if we for all $g \in W$ can find a finite set $F \subseteq \mathfrak{X}$ such that $U_F(g) = \{h \in \mathrm{Aut}(\mathfrak{X}) \mid h(x) = g(x) \text{ for all } x \in F\} \subseteq W$. Indeed, let $W \subseteq \mathrm{Aut}(\mathfrak{X})$ be open, and let g in W. There exist finite subsets $K_1, \ldots, K_n$ of $\mathfrak{X}$ and subsets $U_1, \ldots, U_n$ of $\mathfrak{X}$ such that $g \in \bigcap_{i=1}^{n} V(K_i, U_i) \subseteq W$. Put $F = \cup_{i=1}^{n} K_i$ which is a finite subset of $\mathfrak{X}$. It is obvious that $U_F(g) \subseteq \bigcap_{i=1}^{n} V(K_i, U_i) \subseteq W$. Conversely, let $W \subseteq \mathrm{Aut}(\mathfrak{X})$ be a set such that for all $g \in W$ there exists a finite subset $F = \{x_i\}_{i=1}^{n}$ of $\mathfrak{X}$ such that $U_F(g) \subseteq W$. Put $K_i = \{x_i\}$ which is compact, and $U_i = \{g(x_i)\}$ which is open. Then $g \in \bigcap_{i=1}^{n} V(K_i, U_i) = U_F(g) \subseteq W$. Hence, W is open in the compact-open topology.

Since this characterization shows that the sets $U_F(g)$ are trivially open, $\mathrm{Aut}(\mathfrak{X})$ is obviously a Hausdorff space. Actually, $\mathrm{Aut}(\mathfrak{X})$ is a Hausdorff topological group. To show continuity of the group multiplication, let $g, h \in \mathrm{Aut}(\mathfrak{X})$, let $F \subseteq \mathfrak{X}$ be finite and consider the set $U_F(gh)$. Put $F' = \{h(x) \mid x \in F\}$. If $(\tilde{g}, \tilde{h}) \in U_{F'}(g) \times U_F(h)$, we see that $\tilde{g}\tilde{h} \in U_F(gh)$. Hence, the group multiplication is continuous. Similarly, if $g \in \mathrm{Aut}(\mathfrak{X})$ and $F \subseteq \mathfrak{X}$ is finite, we define $F' = \{g^{-1}(x) \mid x \in F\}$. If $\tilde{g} \in U_{F'}(g)$, $x \in F$ and $y = g^{-1}(x)$, we observe that $\tilde{g}^{-1}(x) = \tilde{g}^{-1}(g(y)) = \tilde{g}^{-1}(\tilde{g}(y)) = y = g^{-1}(x)$, and so $\tilde{g}^{-1} \in U_F(g^{-1})$. Hence, the inversion is also continuous, and $\mathrm{Aut}(\mathfrak{X})$ is a topological group.

Remark 2.1.3 It should be observed that the set $U_F(g)$ is closed for all finite $F \subseteq \mathfrak{X}$ and $g \in \mathrm{Aut}(\mathfrak{X})$. Indeed, if $h \in \mathrm{Aut}(\mathfrak{X}) \setminus U_F(g)$, there exists $x \in F$ such that $h(x) \neq g(x)$. Put $F' = \{x\}$. Then $U_{F'}(h) \subseteq \mathrm{Aut}(\mathfrak{X}) \setminus U_F(g)$. Since $U_F(g)$ is also open, and the sets of this type constitute a basis for the topology, $\mathrm{Aut}(\mathfrak{X})$ is totally disconnected.

Remark 2.1.4 An important property of the topology on $\mathrm{Aut}(\mathfrak{X})$ is that the natural action $(g, x) \mapsto g(x)$ is a continuous map from $\mathrm{Aut}(\mathfrak{X}) \times \mathfrak{X}$ to $\mathfrak{X}$, i.e. that $\mathrm{Aut}(\mathfrak{X})$ is a topological transformation group of $\mathfrak{X}$. This is true since the neighbourhood $U_{\{x\}}(g) \times \{x\}$ is mapped into $\{g(x)\}$ for all $g \in \mathrm{Aut}(\mathfrak{X})$ and $x \in \mathfrak{X}$ (the continuity is actually a consequence of general properties of the compact-open topology since $\mathfrak{X}$ is a locally compact Hausdorff space).

If we restrict our attention to a certain class of trees, the automorphism group $\mathrm{Aut}(\mathfrak{X})$ becomes a locally compact group. A tree $(\mathfrak{X}, \mathfrak{C})$ is said to be *locally finite* if for all vertices $x \in \mathfrak{X}$ the set $\{y \in \mathfrak{X} \mid d(x, y) = 1\}$ is finite. It is said to be *homogeneous* if $\{y \in \mathfrak{X} \mid d(x, y) = 1\}$ has the same cardinality for all $x \in \mathfrak{X}$. This common cardinality is known as the *degree* of the tree.

For a locally finite tree $(\mathfrak{X}, \mathfrak{C})$, the group $\mathrm{Aut}(\mathfrak{X})$ is locally compact. To see this, fix a vertex $x \in \mathfrak{X}$, and consider the subgroup $K_x = \{g \in \mathrm{Aut}(\mathfrak{X}) \mid g(x) = x\}$ of rotations of x. Since $K_x = U_{\{x\}}(e)$ where e is the identity map, it is an open subgroup. However, K_x is also compact. Indeed, let H be the set of functions $f : \mathfrak{X} \to \mathfrak{X}$ such that $f(\mathfrak{D}_n) = \mathfrak{D}_n$ for all n where $\mathfrak{D}_n = \{y \in \mathfrak{X} \mid d(x, y) = n\}$, and endow H with the compact-open topology. Note that $\mathfrak{D}_n$ is finite for all n since $(\mathfrak{X}, \mathfrak{C})$ is locally finite. Put $k_n = |\mathfrak{D}_n|$. We may in an obvious way identify H with $P = \prod_{n=1}^{\infty} S(k_n)$ - an infinite product of symmetric groups. If we equip P with the product topology of the discrete topologies, the identification is between H and P as topological spaces. Indeed, let $f \in H$, and let $\{s_n\}$ be the corresponding element in P. If $F \subseteq \mathfrak{X}$ is finite, and $m = \max\{d(y, x) \mid y \in F\}$, the open set $\prod_{n=1}^{m}\{s_n\} \times \prod_{n=m+1}^{\infty} S(k_n)$ corresponds to a subset of $V_F(f) = \{h \in H \mid h(x) = f(x) \text{ for all } x \in F\}$ containing f. Conversely, a set $\prod_{n=1}^{k}\{s_n\} \times \prod_{n=k+1}^{\infty} S(k_n)$ corresponds exactly to the set $V_F(f)$ where $F = \bigcup_{i=1}^{k} \mathfrak{D}_i$ which is finite.

Since P by Tychonoff's theorem is compact, H is a compact space. Automorphisms are isometries, and so it is obvious that $K_x \subseteq H$. If $f \in H \setminus K_x$, f is not a surjective isometry, and since it is surjective, there exist $x_1, x_2 \in \mathfrak{X}$ such that $d(f(x_1), f(x_2)) \neq d(x_1, x_2)$. If $F = \{x_1, x_2\}$, it is true that $V_F(f) \subseteq H \setminus K_x$, and so K_x is a closed subset of H. This means that K_x is compact in the compact-open topology and so is a compact subset of $\mathrm{Aut}(\mathfrak{X})$.

The above considerations show that K_x is a compact neighbourhood of e in $\mathrm{Aut}(\mathfrak{X})$ which shows that $\mathrm{Aut}(\mathfrak{X})$ is a locally compact group.

Remark 2.1.5 If $(\mathfrak{X}, \mathfrak{C})$ is not locally finite, K_x is in general not compact. For instance, consider the tree in which one vertex x has infinite degree and the remaining vertices have degree 1. If y is a vertex with degree 1, it is not difficult to see that the orbit $K_x(y) = \{g(y) \mid g \in K_x\}$ consists of all vertices of degree 1. Hence, $K_x(y)$ is infinite and so non-compact in the discrete space $\mathfrak{X}$. This contradicts the continuity of the natural action of $\mathrm{Aut}(\mathfrak{X})$ on $\mathfrak{X}$, cf. Remark 2.1.4.

Similarly, the automorphism group $\mathrm{Aut}(\mathfrak{X})$ is in general not locally compact if the tree is not locally finite. Again, the tree above serves as an example. If the identity e has a compact neighbourhood, it follows by the definition of the topology that there exists a finite set $F \subseteq \mathfrak{X}$ such that $U_F(e)$ is compact (recall that $U_F(e)$ is closed by Remark 2.1.3). If $y \notin F$ is a vertex with degree 1, it is, however, easy to see that the orbit $(U_F(e))(y)$ is infinite and hence not compact - contradicting the compactness of $U_F(e)$.

2.2 The boundary of a tree

An important feature of a tree $(\mathfrak{X}, \mathfrak{C})$ is that we - in a rather obvious way - may construct its boundary. It turns out that the boundary has a natural topology and plenty of naturally occurring Borel measures and that the automorphism group acts on the boundary. These observations greatly facilitate the harmonic analysis for $\mathrm{Aut}(\mathfrak{X})$.

Let $(\mathfrak{X}, \mathfrak{C})$ be a tree. We define a binary relation on the set of infinite chains in $(\mathfrak{X}, \mathfrak{C})$ by declaring two infinite chains $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ to be equivalent if there exists $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $x_n = y_{n+k}$ for all $n \geq N$. This relation is clearly an equivalence relation, and we denote by Ω the set of equivalence classes. We will refer to Ω as the *boundary* of the tree $(\mathfrak{X}, \mathfrak{C})$. It is easy to see that for any $x \in \mathfrak{X}$ we may regard Ω as the set of infinite chains $\{x_n\}_{n=0}^{\infty}$ with $x_0 = x$. If $\omega \in \Omega$ and $x \in \mathfrak{X}$, we denote by $[x, \omega)$ the infinite chain $\{x_n\}_{n=0}^{\infty}$ corresponding to ω with $x_0 = x$ and refer to it as the chain from x to ω.

We now want to define a Hausdorff topology on Ω. To do this, we fix $x \in \mathfrak{X}$ and declare a subset $U \subseteq \Omega$ to be open if for each $\omega \in U$ there exists a vertex $y \in [x, \omega)$ such that $\Omega(x, y) = \{\omega' \in \Omega \mid [x, y] \subseteq [x, \omega')\} \subseteq U$. This obviously defines a topology on Ω which clearly does not depend on the choice of x. The sets $\Omega(x, y)$ are of course open for all $y \in \mathfrak{X}$, and since the topology does not depend on the choice of x, this is true for all $x \in \mathfrak{X}$. This shows that the topology is Hausdorff.

The sets $\Omega(x, y)$ are also closed. To see this, let $x_0 = x, \ldots, x_n = y$ be the chain from x to y, and observe that the complement of $\Omega(x, y)$ is the union of the sets $\Omega(x, z)$ where $z \notin [x, y]$ is neighbour of some x_k with $k \leq n-1$. Hence, the topology has a basis consisting of open and closed sets, and so Ω is totally disconnected.

Assume that $(\mathfrak{X}, \mathfrak{C})$ is locally finite, and fix $x \in \mathfrak{X}$. For $n \geq 1$, define the finite set $\mathfrak{D}_n = \{y \in \mathfrak{X} \mid d(x, y) = n\}$, and consider $P = \prod_{n=1}^{\infty} \mathfrak{D}_n$. If we endow P with the product topology of the discrete topologies on the $\mathfrak{D}_n$'s, P is by Tychonoff's theorem a compact space. We may in an obvious way regard Ω as a subset of P. It follows immediately by the definition of the topologies involved that the topology on Ω is the topology inherited from P. For $\{x_n\} \in P \setminus \Omega$, there exists an n such that x_{n-1} and x_n are not neighbours in the tree. Since $\prod_{i=1}^{n} \{x_i\} \times \prod_{i=n+1}^{\infty} \mathfrak{D}_i \subseteq P \setminus \Omega$, Ω is a closed subset of P. Hence, the boundary Ω is compact for any locally finite tree $(\mathfrak{X}, \mathfrak{C})$, and the sets $\Omega(x, y)$ with $x, y \in \mathfrak{X}$ are open and compact sets.

If $x_0, x_1, \ldots, x_n, \ldots$ is an infinite chain and $g \in \mathrm{Aut}(\mathfrak{X})$, $g(x_0), g(x_1), \ldots, g(x_n), \ldots$ is clearly also an infinite chain, so $\mathrm{Aut}(\mathfrak{X})$ acts on the set of infinite chains in $(\mathfrak{X}, \mathfrak{C})$. Since this action obviously preserves the equivalence relation introduced above, $\mathrm{Aut}(\mathfrak{X})$ acts on Ω. We denote this action by $(g, \omega) \mapsto g \cdot \omega$. It is continuous. Indeed, let $x_0, x_1, \ldots, x_n, \ldots$ be an infinite chain, let $\omega \in \Omega$ be the corresponding equivalence class, and let $g \in \mathrm{Aut}(\mathfrak{X})$. Put $F = \{x_0, x_1, \ldots, x_n\}$. If $(g', \omega') \in U_F(g) \times \Omega(x_0, x_n)$, we observe that $g' \cdot \omega' \in \Omega(g(x_0), g(x_n))$. Hence, $\mathrm{Aut}(\mathfrak{X})$ is a topological transformation group of the compact Hausdorff space Ω.

The boundary and the corresponding action of $\mathrm{Aut}(\mathfrak{X})$ will be the central tool in the study of spherical functions and representations in sections 2.4 and 2.5.

2.3 The Gelfand pair (G, K)

Let $(\mathfrak{X}, \mathfrak{C})$ be a locally finite, homogeneous tree of degree $q + 1 \geq 3$ (note that for $q = 0$ the tree consists of only 1 edge and for $q = 1$ the tree is very simple. For instance, it only has one doubly infinite chain. We exclude these simple cases from the considerations below). We will construct a certain Gelfand pair which will be the center of attention for the remainder of this chapter.

To do this, we choose a vertex $o \in \mathfrak{X}$ and keep it fixed for the remainder of this discussion. Let G be a closed subgroup of the locally compact group $\mathrm{Aut}(\mathfrak{X})$, and assume that it acts transitively on $\mathfrak{X}$ and that the subgroup $K = K_o \cap G$ acts transitively on Ω (since the tree is homogeneous, $\mathrm{Aut}(\mathfrak{X})$ is an example of a group satisfying these conditions). The above observations imply that G is a locally compact group and that K is a compact subgroup. We fix a Haar measure on G and denote the group algebra of G by $L^1(G)$ and the closed subalgebra of K-biinvariant L^1-functions by $L^1(K \backslash G / K)$.

Remark 2.3.1 It is actually sufficient to assume that the closed subgroup G acts transitively on $\mathfrak{X}$ and Ω. The transitive action of K on Ω will then be a consequence of these assumptions. This follows by [FN, Proposition 1.10.1] which states that a closed, non-compact subgroup $H \subseteq \mathrm{Aut}(\mathfrak{X})$ acting transitively on Ω satisfies that there exists a vertex $x \in \mathfrak{X}$ such that $H \cap K_x$ acts transitively on Ω. Since G acts transitively on $\mathfrak{X}$, it follows by the continuity of the action of G on $\mathfrak{X}$ (cf. Remark 2.14.4) that G is non-compact, and so $G \cap K_x$ acts transitively on Ω for some $x \in \mathfrak{X}$. It is now an easy consequence of the transitive action of G on $\mathfrak{X}$ that the same is true for K.

Since we do not need the fact that we may weaken our assumptions in this way, we will not give a proof of [FN, Proposition 1.10.1]. Instead, we refer to [FN].

It turns out that the transitive action of K on Ω implies that (G, K) is a Gelfand pair:

Proposition 2.3.2 *Let G and K be as above. The pair (G, K) is a Gelfand pair, i.e. the Banach algebra $L^1(K\backslash G/K)$ is commutative under convolution.*

Proof. Let $g \in G$. Since g is an isometry, it is true that $d(o, g^{-1}(o)) = d(g(o), o)$. K acts transitively on Ω, and every element in Ω can be regarded as an infinite chain starting at o. Hence, K acts transitively on the sets $\{x \in \mathfrak{X} \mid d(o, x) = n\}$ for all $n \geq 0$. This means that there exists $k \in K$ such that $(kg^{-1})(o) = g(o)$, and so $h = g^{-1}kg^{-1} \in K$. Since $g^{-1} = hgk^{-1}$, we observe that $g^{-1} \in KgK$. By Gelfand's lemma [W, Proposition 8.1.3], this shows that (G, K) is a Gelfand pair. □

This immediately implies the following:

Corollary 2.3.3 *Let G be as above. Then G is unimodular.*

Proof. This immediately follows from Proposition 2.3.2 and [W, Lemma 8.1.5]. □

Remark 2.3.4 If we add the assumption that G contains a subgroup acting faithfully and transitively on $\mathfrak{X}$, one may observe that the transitive action of K on Ω is actually equivalent to the commutativity of $L^1(K\backslash G/K)$. This is claimed in [FN, Section 2.4]. Hence, we can not weaken our assumptions on K without destroying the Gelfand pair property in this case. Since we do not need this in the sequel, we will omit a proof of this fact.

Sections 2.4 and 2.5 are devoted to the study of spherical pairs and representations for the Gelfand pair (G, K).

2.4 Spherical functions for the Gelfand pair (G, K)

Let (G, K) be the Gelfand pair considered above. The purpose of this section is to determine the spherical functions for this pair. To do this, we begin by observing some basic consequences of our assumptions on G and K.

Since G acts transitively on $\mathfrak{X}$, the image of the map $g \mapsto g(o)$ is $\mathfrak{X}$. Since K is the isotropy group corresponding to o, this map induces a bijection from the coset space G/K onto $\mathfrak{X}$. Hence, we may regard a function f on G which is right invariant under K as a function $\tilde{f}$ on $\mathfrak{X}$. Conversely, we may regard every function on $\mathfrak{X}$ as a function on G which is right invariant under K.

Let f be a K-biinvariant function on G and regard f as a function $\tilde{f}$ on $\mathfrak{X}$. If $x, y \in \mathfrak{X}$ satisfy that $d(o, x) = d(o, y)$, it follows by the transitive action of K on Ω that there exists $k \in K$ such that $k(x) = y$. If $g \in G$ satisfy that $g(o) = x$, it follows by the left-invariance of f that $f(kg) = f(g)$. Hence, $\tilde{f}(x) = \tilde{f}(y)$, and so $\tilde{f}$ is constant on the sets $\{z \in \mathfrak{X} \mid d(o, z) = n\}$ for $n \geq 0$. A function on $\mathfrak{X}$ with this property is said to be *radial*. Hence, every K-biinvariant function corresponds to a radial function on $\mathfrak{X}$. Conversely, every radial function corresponds to a K-biinvariant function in exactly the same way.

Since spherical functions for the pair (G, K) are K-biinvariant, we will study such functions by considering radial functions on $\mathfrak{X}$. To do this, we introduce the Laplace operator L which is a linear operator on the vector space consisting of complex-valued functions on $\mathfrak{X}$. For $f : \mathfrak{X} \to \mathbb{C}$ we define Lf to be the complex-valued function on $\mathfrak{X}$ whose value in x is the average of the values of f in the neighbours of x, i.e.

$$(Lf)(x) = \frac{1}{q+1} \sum_{\{x,y\} \in \mathfrak{C}} f(y)$$

The spherical functions may be determined using the Laplace operator since it turns out that such functions are closely related with radial eigenfunctions of L. The following proposition characterizes the spherical functions of (G, K) as radial functions on $\mathfrak{X}$.

PROPOSITION 2.4.1 *Let G and K be as above. Let f be a K-biinvariant function on G, and let $\tilde{f}$ be the corresponding radial function on $\mathfrak{X}$. Then f is a spherical function for (G, K) if and only if $\tilde{f}$ is a eigenfunction for the Laplace operator L with $\tilde{f}(o) = 1$.*

PROOF. Assume that f is a spherical function. As is well-known, this means that $f(e) = 1$, and so $\tilde{f}(o) = 1$. Let $\mu \in \mathbb{C}$ be the common value of $\tilde{f}$ on the set $\{y \in \mathfrak{X} \mid d(o, y) = 1\}$, and consider a vertex $x \in \mathfrak{X}$. By the transitive action of G on $\mathfrak{X}$, we choose $g, h \in G$ such that $g(o) = x$ and $h(o) = y$ is a neighbour of x. If $k \in K$, we observe that $d((gkg^{-1}h)o, x) = 1$. If $y_1, \dots, y_{q+1}$ are the neighbours of x, and if k_i - by the transitive action of K on Ω - is chosen such that $(k_i g^{-1}h)(o) = g^{-1}(y_i)$ for all i, we observe that

$$\begin{aligned}\left\{k \in K \mid (gkg^{-1}h)o = y_i\right\} &= \left\{k \in K \mid (kg^{-1}h)o = g^{-1}(y_i)\right\} \\ &= k_i \left\{k \in K \mid (kg^{-1}h)(o) = (g^{-1}h)(o)\right\}\end{aligned}$$

for all i. Since the union of these disjoint open sets is K, they all have measure $(q+1)^{-1}$ under the normalized Haar measure on K. This observation and the fact that f is spherical shows that

$$\begin{aligned}\mu\tilde{f}(x) = \tilde{f}(x)\tilde{f}((g^{-1}h)(o)) = f(g)f(g^{-1}h) &= \int_K f(gkg^{-1}h)\, dk \\ = \int_K \tilde{f}((gkg^{-1}h)(o))\, dk &= \frac{1}{q+1} \sum_{i=1}^{q+1} \tilde{f}(y_i) = (L\tilde{f})(x)\end{aligned}$$

where dk is the normalized Haar measure on K. This shows that $\tilde{f}$ is an eigenfunction of L corresponding to the eigenvalue μ.

Conversely, assume that $\tilde{f}$ is an eigenfunction of L corresponding to the eigenvalue μ and that $\tilde{f}(o) = 1$. To simplify the notation, we write $f(n)$ for $\tilde{f}(x)$ with $d(o, x) = n \geq 0$ (which is independent of the choice of x). We now observe that

$$\mu = \mu\tilde{f}(o) = (L\tilde{f})(o) = \frac{1}{q+1}(q+1)f(1) = f(1)$$

since all neighbours of o have distance 1 to o. Similarly, let $x \in \mathfrak{X}$ satisfy that $d(o, x) = k \geq 1$. By the eigenfunction property of $\tilde{f}$, we now have that

$$f(1)f(k) = \mu\tilde{f}(x) = (L\tilde{f}(x)) = \frac{1}{q+1}f(k-1) + \frac{q}{q+1}f(k+1)$$

where we for the last equality have used that x has q neighbours of distance $k+1$ to o and one neighbour of distance $k-1$ to o. Hence, we get the identity

$$f(k+1) = \frac{q+1}{q} f(1)f(k) - \frac{1}{q} f(k-1) \tag{2.1}$$

for all $k \geq 1$.

f is constant on the disjoint double cosets KgK for $g \in G$ which are open since K is an open subset of G. Hence, f is continuous. Since $\tilde{f} \neq 0$, the same is true for f, and so it is left to prove that

$$\int_K f(gkh)\, dk = f(n)f(m) \tag{2.2}$$

for all $g \in G$ with $d(g(o), o) = n$ and $h \in G$ with $d(g(o), o) = m$.

If $n = 0$, we see that $g(o) = o$, and so $g \in K$. The K-biinvariance of f and normalization of dk then shows that

$$\int_K f(gkh)\, dk = \int_K f(h)\, dk = f(h) = f(n)f(m)$$

where we for the last equality have used that $f(n) = f(0) = \tilde{f}(o) = 1$. In a similar way, we see that (2.2) is satisfied if $m = 0$.

If $n \geq 1$ and $m = 1$, we denote by $y_1, \ldots, y_{q+1}$ the neighbours of $g(o)$. Since $d(h(o), o) = 1$, we may argue as in the first part of the proof to see that

$$\int_K f(gkh)\, dk = \frac{1}{q+1} \sum_{i=1}^{q+1} \tilde{f}(y_i) = (L\tilde{f})(g(o)) = \mu \tilde{f}(g(o)) = f(1)f(n) = f(m)f(n)$$

where we have used that $\mu = f(1)$.

Let $n \geq 1$ and $m \geq 2$. We should now split into two cases: $n \leq m$ and $n > m$. The general calculations are very tedious and not in any way illuminating, so we will only consider the case where $n = 2$. This illustrates the idea of the proof. We leave it to the reader to perform the calculations in the general case.

We denote by $y_1, \ldots, y_k$ the vertices with distance m to o (notice that $k = (q+1)q^{m-1}$) and observe that the transitive action of K on Ω implies that there for each i exists $k_i \in K$ such that $k_i(h(o)) = y_i$. We see that

$$\{k \in K \mid k(h(o)) = y_i\} = k_i \{k \in K \mid k(h(o)) = h(o)\},$$

and since the sets on the left hand side constitute a partition of K into open disjoint sets, we see that they have measure $(q-1)^{-1} q^{1-m}$ under the normalized Haar measure on K. If we denote by $x_1, \ldots, x_k$ the vertices of distance m to $g(o)$, we may choose the numbering such that $g(y_i) = x_i$ for all i. We now see that

$$\begin{aligned}
\int_K f(gkh)\, dk &= \sum_{i=1}^{k} \frac{1}{(q+1)q^{m-1}} \tilde{f}(x_i) \\
&= \frac{q^m}{(q+1)q^{m-1}} f(m+2) + \frac{(q-1)q^{m-2}}{(q+1)q^{m-1}} f(m) + \frac{q^{m-2}}{(q+1)q^{m-1}} f(m-2) \\
&= \frac{q}{q+1} f(m+2) + \frac{q-1}{q(q+1)} f(m) + \frac{1}{q(q+1)} f(m-2)
\end{aligned}$$

where we have used that q^m of the x_i's have distance $m+2$ to o, that $(q-1)q^{m-2}$ of the x_i's have distance m to o and that q^{m-2} of the x_i's have distance $m-2$ to o (this is true since $m \geq 2$). Using (2.1) numerous times, we now get

$$\begin{aligned}\int_K f(gkh)\,dk &= f(1)f(m+1) - \frac{2}{q+1}f(m) + \frac{q-1}{q(q+1)}f(m) + \frac{1}{q}f(1)f(m-1)\\ &= -\frac{1}{q}f(m) + f(1)(f(m+1) + \frac{1}{q}f(m-1))\\ &= -\frac{1}{q}f(m) + f(1)(\frac{q+1}{q}f(1)f(m)) = f(m)(\frac{q+1}{q}f(1)f(1) - \frac{1}{q}f(0))\\ &= f(m)f(2)\end{aligned}$$

This finishes the proof in this case. □

The above proposition implies that in order to determine the spherical functions for the pair (G, K), we should find the normalized radial eigenfunctions of L.

We begin by identifying some basic eigenfunctions which turn out to be - in some way to be made precise below - the building blocks for all eigenfunctions of L. To do this, we fix a boundary point $\omega \in \Omega$ and introduce an equivalence relation on $\mathfrak{X}$. If $x, y \in \mathfrak{X}$ are vertices and $[x, \omega) = \{x_n\}_{n=0}^{\infty}$, there exists a smallest $N \geq 0$ such that $\{x_n\}_{n=N}^{\infty} \subseteq [y, \omega)$. We say that x and y are ω-equivalent if $d(x, x_N) = d(y, x_N)$. This relation is obviously reflexive and symmetric. It is also transitive. Indeed, let $x, y, z \in \mathfrak{X}$ be vertices such that x and y are ω-equivalent and y and z are ω-equivalent. Let $[y, \omega) = \{y_n\}_{n=0}^{\infty}$, and let $N \geq 0$ and $M \geq 0$ be the smallest numbers such that $y_N \in [x, \omega)$ and $y_M \in [z, \omega)$. We may without loss of generality assume that $N \leq M$. If $M > N$, y_M is the first vertex in $[z, \omega)$ which also belongs to $[x, \omega)$. Since $d(z, y_M) = d(y, y_M) = M - N + d(y, y_N) = M - N + d(x, y_N) = d(x, y_M)$, x and z are ω-equivalent. If $M = N$ and t is the first vertex in $[z, \omega)$ such that $t \in [x, \omega)$, we see that $d(x, t) + d(t, y_N) = d(x, y_N) = d(y, y_N) = d(z, y_N) = d(z, t) + d(t, y_N)$ which shows that $d(x, t) = d(z, t)$. Hence, x and z are ω-equivalent. This means that the relation is also transitive and so is an equivalence relation. The equivalence classes of this relation is known as *the horocycles of* ω.

Let $[o, \omega) = \{x_n\}_{n=0}^{\infty}$ and extend it to a doubly infinite chain $\{x_n\}_{n=-\infty}^{\infty}$. It is easy to see that this chain consists of exactly one representative of each of the horocycles corresponding to ω. We denote by H_n the horocycle corresponding to x_n. If $x, y \in \mathfrak{X}$ satisfy that $x \in H_n$ and $y \in H_m$, we define $\delta(x, y, \omega) = m - n$ (notice that this does obviously not depend on the choice of the doubly infinite chain). We will study the function P defined on $\mathfrak{X} \times \mathfrak{X} \times \Omega$ given by

$$P(x, y, \omega) = q^{\delta(x,y,\omega)}$$

Let $\omega \in \Omega$ and $z \in \mathbb{C}$ be fixed, and consider the function f_z on $\mathfrak{X}$ defined by $f_z(x) = P^z(o, x, \omega)$ for $x \in \mathfrak{X}$. If we put

$$\mu(z) = \frac{q^z + q^{1-z}}{q+1},$$

this function is an eigenfunction of L corresponding to the eigenvalue $\mu(z)$. Indeed, for $x \in \mathfrak{X}$ with $x \in H_n$, a neighbour y of x will always satisfy that $y \in H_{n+1}$ or $y \in H_{n-1}$. The first possibility holds for exactly one neighbour while the last is true for the q remaining neighbours. Hence, $f_z(y) = q^z f_z(x)$ for one neighbour y while $f_z(y) = q^{-z} f_z(x)$ if y is one of the remaining

q neighbours. This shows that

$$(Lf_z)(x) = \frac{1}{q+1}(q^z f_z(x) + qq^{-z} f_z(x)) = \mu(z) f_z(x)$$

We see that $Lf_z = \mu(z)f_z$.

Note that elementary calculations show that the map $z \mapsto \mu(z)$ from $\mathbb{C}$ to $\mathbb{C}$ is surjective. Hence, we have found eigenfunctions corresponding to any eigenvalue $w \in \mathbb{C}$.

Using the function P, we are able to construct even more eigenfunctions for L. To do this, fix $x \in \mathfrak{X}$ and consider the function ψ from Ω to $\mathbb{C}$ defined by $\psi(\omega) = P(o, x, \omega)$ for all $\omega \in \Omega$. Let $[o, x] = \{x_k\}_{k=0}^n$. For $0 \leq k \leq n$, ψ takes the value q^{2k-n} on the open set $\Omega(o, y)$ where $y \notin [o, x]$ is a neighbour of x_k. Since the union of all these open sets is Ω, we have seen that ψ is a locally constant - and hence continuous - function assuming only finitely many values.

Let λ be a finitely additive complex measure on Ω defined on the algebra $\mathscr{A}$ generated by the sets $\Omega(o, y)$ where $y \in \mathfrak{X}$. For $z \in \mathbb{C}$, consider the function $\mathscr{P}_z\lambda$ given by

$$(\mathscr{P}_z\lambda)(x) = \int P^z(o, x, \omega)\ \lambda(d\omega)$$

for all $x \in \mathfrak{X}$. By the above observations, this definition makes sense. $\mathscr{P}_z\lambda$ is an eigenfunction of L corresponding to the eigenvalue $\mu(z)$. To see this, let $x \in \mathfrak{X}$, and let $x_1, \ldots, x_{q+1}$ be the neighbours of x. We have seen above that we may choose finitely many non-empty, disjoint sets $E_j \in \mathscr{A}$ such that the functions $\omega \mapsto P(o, x, \omega)$ and $\omega \mapsto P(o, x_i, \omega)$ are all constant on the sets E_j for all j and all i. If we choose $\omega_i \in E_i$ for all i, we observe that

$$\begin{aligned}(L(\mathscr{P}_z\lambda))(x) &= \frac{1}{q+1}\sum_i \int P^z(o, x_i, \omega)\ \lambda(d\omega) = \frac{1}{q+1}\sum_i\sum_j \lambda(E_i)P^z(o, x_i, \omega_j) \\ &= \sum_j \lambda(E_j)\frac{1}{q+1}\sum_i P^z(o, x_i, \omega_j) = \sum_j \lambda(E_j)\mu(z)P^z(o, x, \omega_j) \\ &= \mu(z)(\mathscr{P}_z\lambda)(x)\end{aligned}$$

since we have already seen that the maps $y \mapsto P^z(o, y, \omega_j)$ are eigenfunctions corresponding to the eigenvalue $\mu(z)$.

The map $\mathscr{P}_z\lambda$ is known as *the Poisson transformation* of the finitely additive complex measure λ, and the map $\mathscr{P}_z$ is known as *the Poisson transform.*

Remark 2.4.2 It should be noticed that $\mathscr{A}$ consists of all finite unions of sets of the type $\Omega(o, y)$ for $y \in \mathfrak{X}$. To see this, let $x \in \mathfrak{X}$, and consider the chain $x_0 = o, \ldots, x_n = x$ from o to x. The complement of $\Omega(o, x)$ is clearly just the union of the finitely many sets $\Omega(o, y)$ where $y \notin [o, x]$ is a neighbour of some x_k for $k \leq n - 1$. Furthermore, two sets $\Omega(o, y)$ and $\Omega(o, z)$ are either disjoint or satisfy that $\Omega(o, y) \subseteq \Omega(o, z)$ or $\Omega(o, z) \subseteq \Omega(o, y)$, and so their intersection is either empty or a set of the same type. This shows that the complement of a finite union of sets of the type $\Omega(o, y)$ is again such a finite union, and so $\mathscr{A}$ exactly consists of the sets claimed above.

Actually, it follows by the above observation that every element in $\mathscr{A}$ may be written as a finite *disjoint* union of sets of the considered type.

The following theorem states the amazing fact that we have now found all eigenfunctions of the Laplace operator L, i.e. every eigenfunction of L is the Poisson transform of some finitely additive complex measure λ.

THEOREM 2.4.3 *Let f be an eigenfunction of the Laplace operator L corresponding to the eigenvalue $\mu \in \mathbb{C}$. Let $z \in \mathbb{C}$, $z \neq \frac{k\pi i}{\log q}$ for all $k \in \mathbb{Z}$, satisfy that $\mu = \mu(z)$. Then there exists a unique finitely additive complex measure λ on Ω defined on the algebra $\mathscr{A}$ generated by the sets $\Omega(o, x)$ with $x \in \mathfrak{X}$ such that $f = \mathscr{P}_z\lambda$.*

REMARK 2.4.4 The assumption that $z \neq \frac{k\pi i}{\log q}$ for all $k \in \mathbb{Z}$ does not mean that some eigenfunctions are not covered by the statement in the theorem. If $z = \frac{k\pi i}{\log q}$, $\mu(z)$ is either 1 or -1. We can, however, cover these eigenvalues by other choices of z. For instance, $\mu(1) = 1$ and $\mu(1+\frac{\pi i}{\log q}) = -1$. Hence, this restriction does not impose any restriction on the corresponding eigenvalue.

PROOF. Let $(\mathfrak{Y}, \mathfrak{D})$ be a finite subtree of $(\mathfrak{X}, \mathfrak{E})$ with $o \in \mathfrak{Y}$. We say that a vertex $x \in \mathfrak{Y}$ is an interior point if it has degree $q + 1$ in $(\mathfrak{Y}, \mathfrak{D})$. Otherwise, we say that it is a boundary point, and we denote the set of such vertices by $\partial\mathfrak{Y}$. A function φ on $\mathfrak{Y}$ is said to be an eigenfunction of L corresponding to the eigenvalue μ if $(L\varphi)(x) = \mu\varphi(x)$ for all interior points $x \in \mathfrak{Y}$.

Let $y \in \mathfrak{Y}$ be a boundary point, and choose $\omega \in \Omega(o, y)$ such that $[o, \omega) \cap \mathfrak{Y} = [o, y]$. It is clear that y is in the corresponding horocycle $H_{d(o,y)}$. For $x \in \mathfrak{Y}$, we have by construction of ω that $x \in H_{d(o,y)-d(x,y)}$. Hence, we see that

$$P(o, x, \omega) = q^{d(o,y)-d(x,y)}$$

for all $x \in \mathfrak{Y}$. We now define $\varphi_y : \mathfrak{Y} \to \mathbb{C}$ by putting $\varphi_y(x) = q^{z(d(o,y)-d(x,y))}$ for all $x \in \mathfrak{Y}$. We have seen that φ_y is just the restriction to $\mathfrak{Y}$ of a μ-eigenfunction of L, and so φ_y is an eigenfunction for L on $\mathfrak{Y}$.

We claim that $F = \{\varphi_y \mid y \in \partial\mathfrak{Y}\}$ is a basis for the vector space consisting of μ-eigenfunctions for L on $\mathfrak{Y}$. To show this, we start by proving that the φ_y's are linearly independent, and we do this by induction in $|\mathfrak{Y}|$.

If $|\mathfrak{Y}| = 1$, the only vertex in $\mathfrak{Y}$ is o, and F only contains the constant function 1.

Now assume that F is linearly independent for all subtrees containing o with n vertices, and consider a subtree $(\mathfrak{Y}, \mathfrak{D})$ with $|\mathfrak{Y}| = n + 1$ and $o \in \mathfrak{Y}$. Since $\mathfrak{Y}$ is finite, there exists a boundary point $s \in \mathfrak{Y}$ with degree 1 in $(\mathfrak{Y}, \mathfrak{D})$ such that $s \neq o$. Let $\mathfrak{Z} = \mathfrak{Y} \setminus \{s\}$ and $\mathfrak{E} = \{\{x, y\} \in \mathfrak{C} \mid x, y \in \mathfrak{Z}\}$. Then $(\mathfrak{Z}, \mathfrak{E})$ is clearly a subtree with n vertices and with $o \in \mathfrak{Z}$. Denote by y_1 the only neighbour of s in $(\mathfrak{Y}, \mathfrak{D})$, and note that $y \in \partial\mathfrak{Z}$. Let $y_2, \ldots, y_m$ be the remaining boundary points of $(\mathfrak{Z}, \mathfrak{E})$.

We first consider the case where y_1 is an interior point in $(\mathfrak{Y}, \mathfrak{D})$ and observe that $F = \{\varphi_s, \varphi_{y_2}, \ldots, \varphi_{y_m}\}$. Furthermore, we see that $d(o, s) = d(o, y_1) + 1$ and $d(x, s) = d(x, y_1) + 1$ for all $x \in \mathfrak{Z}$. Hence, it follows by definition that the restriction of φ_s to $\mathfrak{Z}$ is exactly φ_{y_1}. By the induction hypothesis, this means that the restrictions of φ_s, $\varphi_{y_2}, \ldots \varphi_{y_m}$ to $\mathfrak{Z}$ are linearly independent, and so F is linearly independent.

We now consider the case where $y_1 \in \partial\mathfrak{Y}$ and observe that $F = \{\varphi_s, \varphi_{y_1}, \ldots, \varphi_{y_m}\}$. Assume that

$$0 = c_0\varphi_s + \sum_{i=1}^{m} c_i\varphi_{y_i}$$

with $c_i \in \mathbb{C}$. As seen above, the restrictions of φ_s and φ_{y_1} to $\mathfrak{Z}$ coincide, and so it follows by the induction hypothesis that $c_0 + c_1 = 0$ and that $c_i = 0$ for $i \geq 2$. Hence, we have the relation

$$0 = c_0\varphi_s + c_1\varphi_{y_1} \tag{2.3}$$

We observe that $\varphi_s(s) = q^{zd(o,s)}$ and that $\varphi_{y_1}(s) = q^{z(d(o,s)-1-1)} = q^{-2z}q^{zd(o,s)}$. If we evaluate in s in the relation (2.3), we see that $c_0 + c_1 q^{-2z} = 0$. Since we also have that $c_0 + c_1 = 0$, we see that $c_0 = c_1 = 0$, unless $q^{-2z} = 1$. But this is only the case if $z = \frac{k\pi i}{\log q}$ for some $k \in \mathbb{Z}$, and this case has been excluded by assumption. Hence, we see that F is also linear independent in this case.

To see that F is a basis for the set of μ-eigenfunctions for L on $\mathfrak{Y}$, we only need to see that this space is spanned by $|\partial\mathfrak{Y}|$ such eigenfunctions. To prove this, we again use induction in $|\mathfrak{Y}|$. The case $|\mathfrak{Y}| = 1$ is trivial since there are no interior points, and so all functions on $\mathfrak{Y}$ are μ-eigenfunctions for L. Hence, the considered space is obviously 1-dimensional in this case.

We now assume that our claim is true if $|\mathfrak{Y}| = n$ and consider a subtree $(\mathfrak{Y}, \mathfrak{D})$ with $|\mathfrak{Y}| = n+1$. Let s be given as above, and consider again the subtree $(\mathfrak{Z}, \mathfrak{E})$ with n vertices and $o \in \mathfrak{Z}$. Let $y_1, \ldots, y_m$ be constructed as above. By the induction hypothesis, we can find μ-eigenfunctions $f_1', \ldots, f_m'$ for L on $\mathfrak{Z}$ spanning the set of all such functions (note that $m = |\partial\mathfrak{Z}|$). Let φ be a μ-eigenfunction for L on $\mathfrak{Y}$. There exists $c_1, \ldots, c_m$ such that

$$\varphi(x) = \sum_{i=1}^{m} c_i f_i'(x)$$

for all $x \in \mathfrak{Z}$.

We first consider the case where y_1 is an interior point in $(\mathfrak{Y}, \mathfrak{D})$. For all i, f_i' may be extended to a μ-eigenfunction f_i for L on $\mathfrak{Y}$ by defining

$$f_i(s) = (q+1)\mu f_i'(y_1) - \sum_{\{y_1,x\}\in\mathfrak{D}, x\neq s} f_i'(x)$$

Since φ is a μ-eigenfunction for L on $\mathfrak{Y}$, we have the following

$$\begin{aligned}\varphi(s) &= (q+1)\mu\varphi(y_1) - \sum_{\{y_1,x\}\in\mathfrak{D}, x\neq s} \varphi(x) \\ &= \sum_{i=1}^{m} c_i((q+1)\mu f_i(y_1) - \sum_{\{y_1,x\}\in\mathfrak{D}, t\neq s} f_i(x)) = \sum_{i=1}^{m} c_i f_i(s)\end{aligned}$$

Since $|\partial\mathfrak{Y}| = m$, this finishes the proof in this case.

Consider now the case where $y_1 \in \partial\mathfrak{Y}$. In this case, we may for all i extend f_i' to a μ-eigenfunction f_i for L on $\mathfrak{Y}$ by defining $f_i(s) = 0$. Consider the function f_{m+1} on $\mathfrak{Y}$ defined by $f_{m+1}(s) = 1$ and $f_{m+1}(x) = 0$ for $x \neq s$. Since $y_1 \in \partial\mathfrak{Y}$, f_{m+1} is a μ-eigenfunction for L. We now see that

$$\varphi(x) = \sum_{i=1}^{m} c_i f_i(x) + \varphi(s) f_{m+1}(x)$$

for all $x \in \mathfrak{Y}$, and since $|\partial\mathfrak{Y}| = m+1$, this finishes the proof of the fact that F is a basis for the set of μ-eigenfunctions for L on $\mathfrak{Y}$.

Now return to the given eigenfunction f for L on $\mathfrak{X}$. For all $n \geq 0$, consider the subtree $(\mathfrak{Y}_n, \mathfrak{D}_n)$ where $\mathfrak{Y}_n = \{x \in \mathfrak{X} \mid d(o,x) \leq n\}$ and $\mathfrak{D}_n = \{\{x,y\} \in \mathfrak{C} \mid x, y \in \mathfrak{Y}_n\}$. This is clearly a finite subtree with $o \in \mathfrak{D}_n$ and with $\partial\mathfrak{Y}_n = \{x \in \mathfrak{X} \mid d(o,x) = n\}$. Since the restriction of f to $\mathfrak{Y}_n$ is a μ-eigenfunction for L on $\mathfrak{Y}_n$, we can by the above find unique coefficients $\lambda_y \in \mathbb{C}$ such that

$$f(x) = \sum_{y\in\partial\mathfrak{Y}_n} \lambda_y \varphi_y(x)$$

for all $x \in \mathfrak{Y}_n$. If $y \in \mathfrak{X}$ satisfies that $d(o,y) = n$ and $t \in \mathfrak{X}$ is a neighbour with $d(o,t) = n+1$, φ_y and the restriction of φ_t to $\mathfrak{Y}_n$ coincide. Hence, we see that

$$f(x) = \sum_{t \in \partial\mathfrak{Y}_{n+1}} \lambda_t \varphi_t(x) = \sum_{y \in \partial\mathfrak{Y}_n} \Big(\sum_{t \in \partial\mathfrak{Y}_{n+1}, \{t,y\} \in \mathfrak{E}} \lambda_t \Big) \varphi_y(x)$$

for $x \in \mathfrak{Y}_n$. By uniqueness of the coefficients, $\lambda_y = \sum_{t \in \partial\mathfrak{Y}_{n+1}, \{t,y\} \in \mathfrak{E}} \lambda_t$.

We now define $\lambda(\Omega(o,y)) = \lambda_y$ for all $y \in \mathfrak{X}$. Using the above observation and the structure of the sets in $\mathscr{A}$ (cf. Remark 2.4.2), it is easy to see that λ extends to a finitely additive complex measure on $\mathscr{A}$. For $x \in \mathfrak{Y}_n$, we have seen above that $\omega \mapsto P^z(o,x,\omega)$ is constantly equal to $\varphi_y(x)$ on $\Omega(o,y)$ for all $y \in \partial\mathfrak{Y}_n$. Hence, we see that

$$f(x) = \sum_{y \in \partial\mathfrak{Y}_n} \lambda_y \varphi_y(x) = \int P^z(o,x,\omega)\, \lambda(d\omega)$$

which shows that f is just the Poisson transform of λ.

To prove uniqueness, we assume that $f = \mathscr{P}_z\lambda$ and see as above that

$$f(x) = \int P^z(o,x,\omega)\, \lambda(d\omega) = \sum_{y \in \partial\mathfrak{Y}_n} \lambda(\Omega(o,y)) \varphi_y(x)$$

for all $x \in \mathfrak{Y}_n$. Hence, it follows by the uniqueness that $\lambda(\Omega(o,y)) = \lambda_y$, and so Remark 2.4.2 now shows that λ is uniquely given.

This finishes the proof. □

To find the spherical functions for the pair (G, K), we have seen above that we should find the *radial* eigenfunctions for L. This means that we should investigate which finitely additive complex measures on $\mathscr{A}$ give rise to eigenfunctions with this property.

To do this, we define

$$\nu(\Omega) = 1, \quad \nu(\Omega(o,x)) \frac{1}{(q+1)q^{d(o,x)-1}}$$

for all $x \in \mathfrak{X}$ with $d(o,x) \geq 1$. Observe that $\sum_{d(o,x)=n} \nu(\Omega(o,x)) = 1$ for $n \geq 1$ since there are $(q+1)q^{n-1}$ vertices with distance n to o. By definition, the sets $\Omega(o,x)$ with $x \in \mathfrak{X}$ constitute a basis for the topology on Ω, and since there are only finitely many vertices with a given distance to o, $\mathfrak{X}$ is countable, and so this basis is countable. Hence, the σ-algebra generated by these sets is the Borel σ-algebra. Using the fact that the sets $\Omega(o,x)$ are open and compact in Ω, it is easy to check that the conditions in the standard extension theorem [Gr, Theorem 17.1] are satisfied, and so ν extends uniquely to a Borel probability measure on Ω. Since $\nu(k \cdot \Omega(o,x)) = \nu(\Omega(o,k(x))) = \nu(\Omega(o,x))$ for all $k \in K$, ν is K-invariant.

The restriction of ν to $\mathscr{A}$ is of course a finitely additive complex measure, and it turns out that it is exactly the scalar multiples of this measure which by the Poisson transform give rise to radial eigenfunctions of L.

PROPOSITION 2.4.5 *Let f be a complex-valued function on $\mathfrak{X}$, and let $\mu \in \mathbb{C}$. Let $z \in \mathbb{C}$, $z \neq \frac{k\pi i}{\log q}$ for all $k \in \mathbb{Z}$, satisfy that $\mu = \mu(z)$. Then f is a radial eigenfunction for L corresponding to the eigenvalue μ if and only if f is a scalar multiple of the function $\mathscr{P}_z\nu$.*

PROOF. Assume that f is a radial μ-eigenfunction for L. By Theorem 2.4.3, we can find a finitely additive complex measure λ such that $\mathscr{P}_z\lambda = f$. Let $E = \Omega(o,x)$ for some $x \in \mathfrak{X}$. We observe that $k \cdot E = \Omega(o,k(x))$ for $k \in K$, and so we may define the map $k \mapsto \lambda(k \cdot E)$.

This map is constant on the open disjoint sets $U_y = \{k \in K \mid k(x) = y\}$ for $y \in \mathfrak{X}$ with $d(o,y) = d(o,x)$. Hence, it is continuous. Since any $E \in \mathscr{A}$ is just the disjoint union of sets of the type $\Omega(o,x)$ (cf. Remark 2.4.2), $k \mapsto \lambda(k \cdot E)$ is a sum of continuous maps and hence continuous. This means that we may define a finitely additive compex measure λ^K by the prescription

$$\lambda^K(E) = \int_K \lambda(k \cdot E)\, dk,$$

where dk is the normalized Haar measure on K. λ^K is clearly K-invariant. Since K acts transitively on Ω, a K-invariant finitely additive complex measure τ with $\tau(\Omega) = 1$ must satisfy that $\tau(\Omega(o,x)) = \frac{1}{(q+1)q^{d(o,x)-1}}$ for all $x \neq o$, and so $\tau = \nu$ on $\mathscr{A}$ (cf. Remark 2.4.2). Since a K-invariant finitely additive complex measure τ with $\tau(\Omega) = 0$ is constantly equal to 0, this shows that $\lambda^K = \lambda(\Omega)\nu$.

Let $x \in \mathfrak{X}$. For $\omega \in \Omega$, let $y \in \mathfrak{X}$ satisfy that $[x,\omega) \cap [o,\omega) = [y,\omega)$. Then $\delta(o,x,\omega) = d(o,y) - d(x,y)$. For $k \in K$, it is true that $[k(x), k\cdot\omega) \cap [o, k\cdot\omega) = [k(y), k\cdot\omega)$, and so $\delta(o,k(x),k\cdot\omega) = d(o,k(y)) - d(k(x),k(y)) = d(o,y) - d(x,y) = \delta(x,y,\omega)$. Hence, we see that $P^z(o,k(x),k\cdot\omega) = P^z(o,x,\omega)$.

For all $y \in \mathfrak{X}$ with $d(o,y) = d(o,x)$, define $A_y = \Omega(o,y)$. Since $\omega \mapsto P^z(o,x,\omega)$ is constant on the sets A_y, the map $\omega \mapsto P^z(o,x,k^{-1}\cdot\omega)$ is constant on the sets $k \cdot A_y$ for all $k \in K$.

If we choose $\omega_y \in A_y$ for all y, these two observations and the fact that $f(k(x)) = f(x)$ for all $k \in K$ show that

$$\begin{aligned}
f(x) &= \int_K f(k(x))\, dk = \int_K \int_\Omega P^z(o,k(x),\omega)\, \lambda(d\omega)\, dk \\
&= \int_K \int_\Omega P^z(o,x,k^{-1}\cdot\omega)\, \lambda(d\omega)\, dk \\
&= \int_K \sum_{d(o,y)=d(o,x)} \lambda(k\cdot A_y) P^z(o,x,\omega_y)\, dk = \sum_{d(o,y)=d(o,x)} P^z(o,x,\omega_y)\lambda^K(A_y) \\
&= \lambda(\Omega) \sum_{d(o,y)=d(o,x)} P^z(o,x,\omega_y)\nu(A_y) = \lambda(\Omega) \int_\Omega P^z(o,x,\omega)\, \nu(d\omega) \\
&= \lambda(\Omega)(\mathscr{P}_z\nu)(x)
\end{aligned}$$

which proves one way of the theorem.

Conversely, assume that $f = c\mathscr{P}_z\nu$ for $c \in \mathbb{C}$. We have seen above that f is a μ-eigenfunction for L, so it is left to shows that it is radial. If $x, y \in \mathfrak{X}$ satisfy that $d(o,x) = d(o,y)$, it follows by the transitive action of K on Ω that there exists $k \in K$ such that $y = k(x)$. Hence, we observe that

$$\begin{aligned}
f(y) = f(k(x)) &= c\int_\Omega P^z(o,k(x),\omega)\, \nu(d\omega) = c\int_\Omega P^z(o,x,k^{-1}\cdot\omega)\, \nu(d\omega) \\
&= c\int_\Omega P^z(o,x,\omega)\, \nu(d\omega) = f(x)
\end{aligned}$$

since ν is K-invariant.

This finishes the proof. □

We observe that $P(o,o,\omega) = 1$ for all $\omega \in \Omega$. Hence, $(\mathscr{P}_z\nu)(o) = \nu(\Omega) = 1$ for all $z \in \mathbb{C}$. If we combine this fact with Propositions 2.4.1 and 2.4.5 above, we have proved the following theorem which is the main result of this section:

Theorem 2.4.6 *The spherical functions for the pair (G,K) are the functions f_z on G given by*

$$f_z(g) = (\mathscr{P}_z\nu)(g(o)) = \int_\Omega P^z(o, g(o), \omega)\, \nu(d\omega)$$

for all $g \in G$, where $z \in \mathbb{C}$, $z \neq \frac{k\pi i}{\log q}$ for $k \in \mathbb{Z}$.

If f is a spherical function for the pair (G,K), f corresponds to a normalized radial eigenfunction $\tilde{f}$ for L. Since $\tilde{f}$ is radial, it is true for all $x \in \mathfrak{X}$ that $\tilde{f}(x)$ only depends on $d(o,x)$. Hence, we may regard $\tilde{f}$ and so f as a function on $\mathbb{N}\cup\{0\}$, and we will in the following abuse notation and write $f(n)$ for $f(g)$ with $d(o,g(o)) = n$.

It turns out that we can inductively calculate the values of f without considering the integral in Theorem 2.4.6. This is the content of the following proposition in which we have made use of the above notation.

Proposition 2.4.7 *Let f be a spherical function for the pair (G,K), and assume that f corresponds to a radial eigenfunction for L with eigenvalue μ. Then $f(0) = 1$, $f(1) = \mu$ and*

$$f(n) = \frac{q+1}{q} f(1) f(n-1) - \frac{1}{q} f(n-2)$$

for all $n \geq 2$.

Proof. By Proposition 2.4.1, f corresponds to a normalized function, and so $f(0) = 1$. Since f corresponds to a radial eigenfunction for L, we observe that

$$\mu = \mu f(o) = (Lf)(o) = \frac{1}{q+1}(q+1)f(1) = f(1)$$

For $n \geq 2$, we observe that any point with distance $n-1$ to o has q neighbours with distance n and 1 neighbour with distance $n-2$. Hence, the eigenfunction property shows that

$$f(1)f(n-1) = \mu f(n-1) = (Lf)(n-1) = \frac{1}{q+1}(qf(n) + f(n-2))$$

which finishes the proof. □

Remark 2.4.8 It should be observed that the proposition above implies that there exists at most one spherical function corresponding to a radial μ-eigenfunction for L for a given eigenvalue μ. Since we have in Theorem 2.4.6 found a spherical function corresponding to all possible eigenvalues (since the map $z \mapsto \mu(z)$ is surjective), this shows that there exists exactly one spherical function corresponding to all possible eigenvalues. This means that if $\mu(z_1) = \mu(z_2)$, the spherical functions in Theorem 2.4.6 corresponding to z_1 and z_2 are identical.

2.5 Positive definite spherical functions and spherical representations for the pair (G,K)

Our next goal is to investigate which spherical functions are positive definite and to find the corresponding irreducible spherical representations arising from the Gelfand-Naimark-Segal construction. We will find a criterion for positive definiteness in terms of the eigenvalue μ

corresponding to a spherical function. Recall that there is a bijective correspondance between the spherical functions and the corresponding eigenvalues (cf. Remark 2.4.8)

Let f be a positive definite spherical function for (G, K) corresponding to the eigenvalue μ. Choose by the transitive action $g \in G$ such that $d(o, g(o)) = 1$. Since we also have that $d(o, g^{-1}(o)) = 1$, Proposition 2.4.7 reveals that $f(g^{-1}) = f(g) = \mu$. By the positive definiteness, we see that

$$\overline{\mu} = \overline{f(g)} = f(g^{-1}) = \mu,$$

and so μ is real. Furhermore, the positive definiteness means that

$$|\mu| = |f(g)| \leq f(e) = 1$$

Hence, we see that $\mu \in [-1, 1]$.

If $\mu = 1$, it follows by Proposition 2.4.7 that the corresponding spherical function is (of course) the constant function 1 which is certainly positive definite. The corresponding spherical representation is as always the trivial representation.

If $\mu = -1$, it follows by Proposition 2.4.7 that $f(g) = (-1)^{d(g(o),o)}$ for all $g \in G$. If $\mathfrak{X}^+ = \{x \in \mathfrak{X} \mid d(o, x) \text{ is even}\}$ and $\mathfrak{X}^- = \mathfrak{X} \setminus \mathfrak{X}^+$, it is for $g \in G$ clearly true that $g(\mathfrak{X}^+) = \mathfrak{X}^+$ or $g(\mathfrak{X}^+) = \mathfrak{X}^-$. Hence, f is multiplicative, and so it is positive definite. We define $\pi(g)w = (-1)^{d(g(o),o)}w$ for all w in the one-dimensional Hilbert space $\mathcal{H}$ and all $g \in G$. Since f is multiplicative, π is a group homomorphism from G into the group of unitary operators on $\mathcal{H}$. The fact that $g \mapsto \pi(g)v$ is constant on the open set $\{h \in G \mid h_0(o) = h(o)\}$ for $h_0 \in G$ and $v \in \mathcal{H}$ shows that π is an irreducible unitary representation of G. Furthermore, if h is a unit vector, h is K-invariant and cyclic, and $f(g) = \langle h, \pi(g)h\rangle$ for all $g \in G$. By the essential uniqueness of the representation in the Gelfand-Naimark-Segal construction, π is a realisation of the spherical representation corresponding to f.

To consider the cases $\mu \in (-1, 1)$, we need to observe that $\omega \mapsto P(x, y, \omega)$ for $x, y \in \mathfrak{X}$ is the Radon-Nikodym derivative for some naturally occuring Borel measures on Ω. To see this, we let $x \in \mathfrak{X}$ and define $\nu_x(\Omega) = 1$ and $\nu_x(\Omega(x, z)) = \frac{1}{(q+1)q^{d(x,z)-1}}$ for all $z \neq x$. As it was the case for ν, ν_x extends uniquely to a Borel measure on Ω, and we observe that $\nu = \nu_o$. Since Ω has a countable basis for the topology, it even follows by [Co, Proposition 7.2.3] that ν_x is a Radon measure.

Fix $x, y \in \mathfrak{X}$ and define a Borel measure τ on Ω by $\tau(E) = \int_E P(x, y, \omega)\ \nu_x(d\omega)$ for any Borel set E. Consider $z \in \mathfrak{X}$ with $d(o, z) > \max\{d(o, x), d(o, y)\}$. For $\omega \in \Omega(o, z)$, a moment of thought shows that $P(x, y, \omega) = q^{d(x,z)-d(y,z)}$. This means that $\tau(\Omega(o, z)) = q^{d(x,z)-d(y,z)}\nu_x(\Omega(o, z))$.

We now observe that the choice of z implies that $\Omega(o, z) = \Omega(x, z) = \Omega(y, z)$. Using this, we see that

$$\begin{aligned}\nu_y(\Omega(o, z)) &= \frac{1}{(q+1)q^{d(y,z)-1}} = q^{d(x,z)-d(,yz)}\frac{1}{(q+1)q^{d(x,z)-1}} \\ &= q^{d(x,z)-d(y,z)}\nu_x(\Omega(o, z)) = \tau(\Omega(o, z))\end{aligned}$$

This shows that τ is a probability measure, and since the sets $\Omega(o, z)$ with z as above constitute a countable basis for the topology on Ω and so generate the Borel σ-algebra, we see that $\tau = \nu_y$. Hence, ν_y is absolutely continuous with respect to ν_x with Radon-Nikodym derivative $\omega \mapsto P(x, y, \omega)$.

If $g \in G$ satisfies that $g^{-1}(x) = y$, we observe that $g \cdot \Omega(y, z) = \Omega(x, g(z))$ for all $z \in \mathfrak{X}$. Since $d(y, z) = d(x, g(z))$, this shows that $\nu_y(\Omega(y, z)) = \nu_x(g \cdot \Omega(y, z))$, and so we see as above that $\nu_y(E) = \nu_x(g \cdot E)$ for all Borel subsets E of Ω.

Since $\omega \mapsto P(x,y,\omega)$ is continuous and the set $\{g \in G \mid g^{-1}(x) = z\}$ is open in G, we see that the map $(g,\omega) \mapsto P(x,g^{-1}(x),\omega)$ from $G \times \Omega$ into $(0,\infty)$ is continuous. Combining this with the above observations, we have shown that ν_x is a strongly quasi-invariant probability measure on Ω.

We now observe that

$$\mu(\frac{1}{2}+it) = \frac{q^{\frac{1}{2}}(q^{it}+q^{-it})}{q+1} = \frac{2q^{\frac{1}{2}}\Re(q^{it})}{q+1}$$

which shows that $\mu(z)$ runs through all values of the interval $[-\frac{2q^{\frac{1}{2}}}{q+1}, \frac{2q^{\frac{1}{2}}}{q+1}] \subseteq (-1,1)$ (recall that $q \geq 2$) as t runs through $\mathbb{R}$. Putting $z = \frac{1}{2}+it$ for some $t \in \mathbb{R}$, we will show that the spherical function corresponding to the eigenvalue $\mu(z)$ is positive definite.

This is done using the fact that we have a strongly quasi-invariant measure ν on Ω, and so we can in a standard way construct a unitary representation π of G on $L^2(\nu)$. Indeed, for $g \in G$ and $\varphi \in L^2(\nu)$, we define

$$(\pi(g)\varphi)(\omega) = P^z(o,g(o),\omega)\varphi(g^{-1}\cdot\omega)$$

for all $\omega \in \Omega$. Then π is a unitary representation (cf. [Fo, section 3.1]). Furhermore, if we denote by $\mathbf{1}$ the constant function 1 on Ω and by f the spherical function corresponding to the eigenvalue $\mu(z)$, we observe that

$$\langle \mathbf{1}, \pi(g)\mathbf{1}\rangle = \int_\Omega P^z(o,g(o),\omega)\,\nu(d\omega) = f(g),$$

and so f is positive definite.

Furthermore, we observe that $P(o,k(o),\omega) = 1$ for all $k \in K$ and $\omega \in \Omega$, and so $\mathbf{1}$ is a K-invariant vector in $L^2(\nu)$. We will show that $\mathbf{1}$ is also cyclic.

To do this, we consider the vector space $\mathcal{K}(\Omega)$ spanned by all indicator functions for sets of the form $\Omega(o,x)$ with $x \in \mathfrak{X}$. This is known as the space of *cylindrical functions*. Since the sets $\Omega(o,x)$ are both open and closed, cylindrical functions are continuous functions on Ω. Hence, we see that $\mathcal{K}(\Omega) \subseteq L^2(\nu)$.

The space $\mathcal{K}(\Omega)$ actually consists of all continuous functions on Ω taking only finitely many values. Indeed, for such a function φ the sets $B_a = \varphi^{-1}(\{a\})$ are open and closed and so also compact for all $a \in \mathbb{C}$. Since B_a is open, it may be written as a union of sets of the form $\Omega(o,x)$ which are open. By compactness, it may then be written as a finite union of such sets, and by Remark 2.4.2 we may even choose this union to be disjoint. This shows that φ is cylindrical.

Let U be an open set, and let $\epsilon > 0$. Since the sets $\Omega(o,x)$ constitute a countable basis for the topology, we may write U as a countable union of sets of this type. Using the observations of Remark 2.4.2, we may even assume this union to be disjoint. By countable additivity of ν, we may find a finite union V of pairwise disjoint sets of the type $\Omega(o,x)$ such that $V \subseteq U$ and $\nu(U \setminus V) < \epsilon^2$. If we by χ_U and χ_V denote the corresponding indicator functions, we observe that $\chi_V \in \mathcal{K}(\Omega)$ and that $\|\chi_U - \chi_V\|_2 < \epsilon$.

Since ν is a Radon measure, it is outer regular, and so we may for a Borel set $E \subseteq \Omega$ and an $\epsilon > 0$ find an open set $U \supseteq E$ such that $\nu(U \setminus E) < \epsilon^2$ which means $\|\chi_U - \chi_E\|_2 < \epsilon$.

Combining this with the fact that the simple L^2-functions are dense in $L^2(\nu)$, we see that $\mathcal{K}(\Omega)$ is dense in $L^2(\nu)$.

We now observe that $\mathcal{K}(\Omega) = \bigcup_{n=0}^{\infty}\mathcal{K}_n(\Omega)$ where for $n \geq 0$ $\mathcal{K}_n(\Omega)$ is the set spanned by indicator functions for sets of the type $\Omega(o,x)$ for $x \in \mathfrak{X}$ with $d(o,x) \leq n$. We claim that

$\mathcal{K}_n(\Omega)$ is spanned by the functions $\omega \mapsto P(o,x,\omega)$ where $x \in \mathfrak{X}$ has $d(o,x) \leq n$. For simplicity, we will denote the function $\omega \mapsto P(o,x,\omega)$ by P_x for all $x \in \mathfrak{X}$.

We will prove this by induction. For $n = 0$, it is obvious since $\mathcal{K}_n(\Omega)$ consists of all constant functions, and P_o is the constant function 1.

Assume that it is true for some $n \geq 0$, and consider the indicator function $\chi_{\Omega(o,x)}$ with $d(o,x) = n+1$. Let $x_0 = o, \ldots, x_{n+1} = x$ be the chain from o to x. We have observed that P_x is constant on the sets $\Omega(o,y)$ where $y \notin [o,x]$ is a vertex that is neighbour of some x_k. Furthermore, it is easy to see that P_x takes the same value on $\Omega(o,y)$ and $\Omega(o,z)$ if $y,z \in \mathfrak{X}$ are such neighbours of the same x_k. Hence, P_x is constant on $\Omega(o,x)$, and so we may choose $c \in \mathbb{C} \setminus \{0\}$ such that $P_x - c\chi_{\Omega(o,x)}$ is constant on $\Omega(o,x_n)$. This means that $P_x - c\chi_{\Omega(o,x)} \in \mathcal{K}_n(\Omega)$ and so is a linear combination of the P_y's for $y \in \mathfrak{X}$ with $d(o,y) \leq n$. This shows that $\chi_{\Omega(o,x)}$ belongs to the span of the P_y's for $y \in \mathfrak{X}$ with $d(o,y) \leq n+1$. Since the above also shows that this span is contained in $\mathcal{K}_{n+1}(\Omega)$, this finishes the induction proof.

The conclusion is that $\mathcal{K}(\Omega)$ is spanned by the P_x's with $x \in \mathfrak{X}$. Since $\pi(g)\mathbf{1} = P_{g(o)}$, and since G acts transitively on $\mathfrak{X}$, the fact that $\mathcal{K}(\Omega)$ is dense in $L^2(\nu)$ shows that $\mathbf{1}$ is a cyclic vector for π.

It now follows by the essential uniqueness of the spherical representation associated to f by the Gelfand-Naimark-Segal construction that this representation is equivalent with π. Hence, π is a realisation of the spherical representation corresponding to f which is irreducible. This means that π is also irreducible.

We have now proved that the spherical functions corresponding to eigenvalues in $[-\frac{2q^{\frac{1}{2}}}{q+1}, \frac{2q^{\frac{1}{2}}}{q+1}]$ are also positive definite, and we have found realisations of the corresponding spherical representations. Left are the cases $\mu \in (\frac{2q^{\frac{1}{2}}}{q+1}, 1)$ and $\mu \in (-1, -\frac{2q^{\frac{1}{2}}}{q+1})$.

To consider these cases, we need to introduce an intertwining operator for all $z \in \mathbb{C}$ with $z \neq \frac{k\pi i}{\log q}$ for $k \in \mathbb{Z}$. Fix such a z, let $\varphi \in \mathcal{K}(\Omega)$, and let $n \geq 0$ be given such that $\varphi \in \mathcal{K}_n(\Omega)$. Using that $z \neq \frac{k\pi i}{\log q}$ for $k \in \mathbb{Z}$, we may argue as above to see that the P_x^z's with $x \in \mathfrak{X}$ and $d(o,x) \leq n$ span $\mathcal{K}_n(\Omega)$, and so we can find constants $a_x \in \mathbb{C}$ such that $\varphi = \sum_{d(o,x)\leq n} a_x P_x^z$. We calculate the following:

$$(\mathscr{P}_{1-z}(\varphi\, d\nu))(y) = \int_\Omega P_y^{1-z}\varphi\, d\nu = \sum_{d(o,x)\leq n} a_x \int_\Omega P_y^{1-z} P_x^z\, d\nu = \sum_{d(o,x)\leq n} a_x \int_\Omega (\frac{d\nu_x}{d\nu_y})^z\, d\nu_y$$

For the last equality, we have used that $P_x = \frac{d\nu_x}{d\nu}$ and $P_y = \frac{d\nu_y}{d\nu}$, and so it is easy to see that $P_x P_y^{-1} = \frac{d\nu_x}{d\nu_y}$.

By symmetry, we may of course change the roles of x and y in the last integral, and so we see that

$$\begin{aligned}(\mathscr{P}_{1-z}(\varphi d\nu))(y) &= \sum_{d(o,x)\leq n} a_x \int_\Omega (\frac{d\nu_y}{d\nu_x})^z\, d\nu_x = \sum_{d(o,x)\leq n} a_x \int_\Omega P_x^{1-z} P_y^z\, d\nu \\ &= (\mathscr{P}_z(I_z\varphi\, d\nu))(y)\end{aligned}$$

where $I_z\varphi = \sum_{d(o,x)\leq n} a_x P_x^{1-z} \in \mathcal{K}_n(\Omega)$.

We have seen that there exists a function $I_z\varphi \in \mathcal{K}_n(\Omega)$ such that

$$\mathscr{P}_{1-z}(\varphi\, d\nu) = \mathscr{P}_z(I_z\varphi\, d\nu) \tag{2.4}$$

This condition and the continuity requirement even determines $I_z\varphi$ uniquely. Indeed, by Theorem 2.4.3 and the fact that all Poisson transformations are eigenfunctions, the Poisson transform $\mathscr{P}_z$ is injective. Hence, $I_z\varphi$ is uniquely determined ν-almost everywhere. Since non-empty open sets have positive measure under ν, the continuity of $I_z\varphi$ means that it is uniquely determined.

We denote the operator I_z from $\mathcal{K}(\Omega)$ into $\mathcal{K}(\Omega)$ uniquely defined by the relation (2.4) *an intertwining operator relative to the eigenvalue* $\mu(z)$ (note that I_{1-z} is an intertwining operator related to the same eigenvalue), and observe that I_z is clearly linear. The above considerations show that $I_z P_x^z = P_x^{1-z}$, and since these functions span $\mathcal{K}(\Omega)$, $I_z I_{1-z}$ and $I_{1-z}I_z$ are both the identity operator. Hence, the intertwining operators are bijective.

We now define φ_o to be the constant function 1 on Ω. For $x \neq o$, we denote by $p(x)$ the neighbour of x in the chain $[o, x]$, and we define

$$\varphi_x = q^{d(o,x)}\chi_{\Omega(o,x)} - q^{d(o,p(x))}\chi_{\Omega(o,p(x))}$$

if $d(o, x) \geq 2$ and

$$\varphi_x = q^{d(o,x)}\chi_{\Omega(o,x)} - \frac{q}{q+1}\chi_{\Omega(o,p(x))}$$

if $d(o, x) = 1$.

For all $x \in \mathfrak{X}$, we see that $\varphi_x \in \mathcal{K}_{d(o,x)}(\Omega)$. The following lemma reveals that the functions φ_x are actually a collection of eigenfunctions of I_z which span $\mathcal{K}(\Omega)$.

Lemma 2.5.1 *The functions φ_x with $x \in \mathfrak{X}$ and $d(o, x) \leq n$ span $\mathcal{K}_n(\Omega)$ for all $n \geq 0$.*

Furthermore, φ_o is an eigenfunction for I_z corresponding to the eigenvalue 1, while φ_x is an eigenfunction for I_z corresponding to the eigenvalue

$$\frac{q^{1-z} - q^{z-1}}{q^z - q^{-z}} q^{-(2z-1)d(o,p(x))}$$

for $x \neq o$.

Proof. By an induction proof similar to the one used to prove that the P_x's with $x \in \mathfrak{X}$ span $\mathcal{K}(\Omega)$, it follows immediately that $\mathcal{K}_n(\Omega)$ is spanned by the φ_x's with $d(o, x) \leq n$. Hence, the first claim in the lemma follows.

To prove the second claim, we immediately observe that $\varphi_o = P_o^z$ and so $I_z\varphi_o = P_o^{1-z} = \varphi_o$. Let $x \neq o$, and let $n = d(o, x) \geq 1$. It follows by the earlier observations that there exists $c' \in \mathbb{C} \setminus \{0\}$ such that $\chi_{\Omega(o,x)} - c'P_x^z \in K_{n-1}(\Omega)$, and so $\varphi_x - cP_x^z \in K_{n-1}(\Omega)$ for some $c \in \mathbb{C}\setminus\{0\}$. This means that $I_z\varphi_x - cP_x^{1-z} \in K_{n-1}(\Omega)$. Hence, we see that $\varphi_x = c_1\chi_{\Omega(o,x)} + \psi_1$ and $I_z\varphi_x = c_2\chi_{\Omega(o,x)} + \psi_2$ for $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ and $\psi_1, \psi_2 \in K_{n-1}(\Omega)$. This shows that we can find $c_3 \in \mathbb{C} \setminus \{0\}$ and $\psi_3 \in K_{n-1}(\Omega)$ such that $I_z\varphi_x = c_3\varphi_x + \psi_3$.

Let $y \in \mathfrak{X}$ be such that $d(o, y) = n - 1$. If $y \neq p(x)$, it is trivial that $\int_{\Omega(y)} \varphi_x \, d\nu = 0$, and if $y = p(x)$, we see that

$$\int_{\Omega(y)} \varphi_x \, d\nu = \frac{q^n}{(q+1)q^{n-1}} - \frac{q^{n-1}}{(q+1)q^{n-2}} = 0$$

for $n \geq 2$ and

$$\int_{\Omega(y)} \varphi_x \, d\nu = \frac{q}{q+1} - \frac{q}{q+1} = 0$$

for $n = 1$. Hence, the fact that $P_t^z, P_t^{1-z} \in K_{n-1}(\Omega)$ for all t with $d(o, t) \leq n - 1$ shows that $\mathscr{P}_z(\varphi_x \, d\nu)$ and $\mathscr{P}_{1-z}(\varphi_x \, d\nu)$ are identically 0 on $\mathfrak{X}_{n-1} = \{t \in \mathfrak{X} \mid d(o, t) \leq n - 1\}$. This implies that

$$(\mathscr{P}_z(\psi_3 \, d\nu))(t) = (\mathscr{P}_z(I_z\varphi_x \, d\nu))(t) - c_3(\mathscr{P}_z(\varphi_x \, d\nu))(t) = (\mathscr{P}_{1-z}(\varphi_x \, d\nu))(t) = 0$$

for all $t \in \mathfrak{X}$ with $d(o,t) \leq n-1$. By the proof of Theorem 2.4.3, the restriction of $\mathscr{P}_z(\psi_3\ d\nu)$ to $\mathfrak{X}_{n-1}$ can be written as linear combination of some basis for the space of $\mu(z)$-eigenfunctions on $\mathfrak{X}_{n-1}$ with coefficients $\int_{\Omega(y)} \psi_3\ d\nu$ for all $y \in \mathfrak{X}$ with $d(o,y) = n-1$. This tells us that $\int_{\Omega(y)} \psi_3\ d\nu = 0$ for all $y \in \mathfrak{X}$ with $d(o,y) = n-1$. But $\psi_3 \in K_{n-1}(\Omega)$, and so we see that $\psi_3 = 0$. Hence, we see that $I_z\varphi_x = c_3\varphi_x$, i.e. φ_x is an eigenfunction for I_z.

It only remains to calculate the corresponding eigenvalue c_3. To do this we observe, that $P_x^z - q^{-z}P_{p(x)}^z$ is identically equal to 0 on $\Omega \setminus \Omega(x)$ and constantly equal to $q^{nz} - q^{-z}q^{(n-1)z} = q^{(n-1)z}(q^z - q^{-z})$ on $\Omega(x)$. Furthermore, we see that $q^z - q^{-z} \neq 0$ by the assumption on z. Putting $\tau = \mathscr{P}_z(I_z\varphi_x\ d\nu) = \mathscr{P}_{1-z}(\varphi_x\ d\nu)$ and remembering that $\tau(p(x)) = 0$, we see that

$$\begin{aligned}
\frac{1}{q^z - q^{-z}} q^{-(n-1)z}\tau(x) &= \frac{1}{q^z - q^{-z}} q^{-(n-1)z}(\tau(x) - q^{-z}\tau(p(x))) \\
&= \frac{1}{q^z - q^{-z}} q^{-(n-1)z} \int_\Omega I_z\varphi_x (P_x^z - q^{-z}P_{p(x)}^z)\ d\nu \\
&= \frac{1}{q^z - q^{-z}} q^{-(n-1)z}(q^z - q^{-z})q^{(n-1)z} \int_{\Omega(x)} I_z\varphi_x\ d\nu \\
&= c_3 \int_{\Omega(x)} \varphi_x\ d\nu
\end{aligned}$$

Using that $\tau = \mathscr{P}_{1-z}(\varphi_x\ d\nu)$, we see in exactly the same way that

$$\tau(x) = (q^{1-z} - q^{-(1-z)})q^{(n-1)(1-z)} \int_{\Omega(x)} \varphi_x\ d\nu$$

Since $\int_{\Omega(x)} \varphi_x\ d\nu = \frac{q^n - q^{n-1}}{(q+1)q^{n-1}} \neq 0$ if $d(o,x) \geq 2$ and $\int_{\Omega(x)} \varphi_x\ d\nu = \frac{q - \frac{q}{q+1}}{q+1} \neq 0$ if $d(o,x) = 1$, this shows that the eigenvalue c_3 has the desired expression. $\square$

Using the intertwining operator I_z, we are now able to prove that the spherical functions corresponding to an eigenvalue $\mu \in (-1, \frac{-2q^{\frac{1}{2}}}{q+1}) \cup (\frac{2q^{\frac{1}{2}}}{q+1}, 1)$ is positive definite. We begin by observing that $z \mapsto \mu(z)$ maps $\left\{ s + \frac{k\pi i}{\log q} \;\middle|\; k = 0,1, s \in (0, \frac{1}{2}) \right\}$ onto this set, and so we fix $z = s + \frac{k\pi i}{\log q}$ with $s \in (0, \frac{1}{2})$ and $k = 0, 1$. Let f be the spherical function corresponding to the eigenvalue $\mu(z)$. We observe that the intertwining operator I_z is well-defined and has eigenfunctions and -values as stated in Lemma 2.5.1.

Let $g, h \in G$. An easy computation shows that $\omega \mapsto P(o, g(o), \omega)P(o, h(o), g^{-1} \cdot \omega)$ is a continuous Radon-Nikodym derivative for $\nu_{g(h(o))}$ with respect to ν. This shows that

$$P(o, g(h(o)), \omega) = P(o, g(o), \omega)P(o, h(o), g^{-1} \cdot \omega) \tag{2.5}$$

for all $\omega \in \Omega$. Replacing h with g and g with g^{-1} and remembering that $P(o, o, \omega) = 1$ for all $\omega \in \Omega$, we see that

$$P(o, g(o), g \cdot \omega) = P(o, g^{-1}(o), \omega)^{-1}$$

for all $\omega \in \Omega$. Using these facts, we see that

$$\begin{aligned}
f(g^{-1}h) = (\mathscr{P}_z(\nu))(g^{-1}(h(o))) &= \int_\Omega P^z(o, g^{-1}(h(o)), \omega)\, \nu(d\omega) \\
&= \int_\Omega P^z(o, g^{-1}(o), \omega) P^z(o, h(o), g \cdot \omega)\, \nu(d\omega) \\
&= \int_\Omega P^z(o, g^{-1}(o), g^{-1} \cdot \omega) P^z(o, h(o), \omega)\, \nu_{g(o)}(d\omega) \\
&= \int_\Omega P^z(o, g^{-1}(o), g^{-1} \cdot \omega) P^z(o, h(o), \omega) P(o, g(o), \omega)\, \nu(d\omega) \\
&= \int_\Omega P^{1-z}(o, g(o), \omega) P^z(o, h(o), \omega)\, \nu(d\omega) = \int_\Omega (I_z P^z_{g(o)}) \overline{P^z_{h(o)}}\, d\nu
\end{aligned}$$

since $P^z_{h(o)}$ is real by the choice of z. This shows that for $c_1, \dots, c_n \in \mathbb{C}$ and $g_1, \dots, g_n \in G$

$$\sum_{i,j} c_i \overline{c_j} f(g_i^{-1} g_j) = \int_\Omega (I_z \varphi) \overline{\varphi}\, d\nu = \langle I_z \varphi, \varphi \rangle_{L^2(\nu)}$$

where $\varphi = \sum_{i=1}^n c_i P^z_{g_i(o)} \in \mathcal{K}(\Omega)$.

Let $x, y \in \mathfrak{X}$ be given such that the eigenvalues of I_z corresponding to the eigenfunctions φ_x and φ_y are different. We claim that φ_x and φ_y are orthogonal in $L^2(\nu)$. To see this, we first assume that $x = o$, and so we have that $y \neq o$. In the proof of Lemma 2.5.1, we have seen that $\int_\Omega \varphi_y\, d\nu = 0$, and so φ_x and φ_y are ortogonal in $L^2(\nu)$. Now assume that $x \neq o$ and that $y \neq o$. Since x and y correspond to different eigenvalues, it must be true that $d(o, p(x)) \neq d(o, p(y))$. We assume without loss of generality that $d(o, p(x)) < d(o, p(y))$. If $\Omega(o, p(x)) \cap \Omega(o, p(y)) = \emptyset$, we clearly have that $\varphi_x \varphi_y = 0$, and so φ_x and φ_y are orthogonal in $L^2(\nu)$. If $\Omega(o, p(x)) \cap \Omega(o, p(y)) \neq \emptyset$, we have that $\Omega(o, p(x)) \cap \Omega(o, p(y)) = \Omega(o, p(y))$. Hence, we see that

$$\varphi_x \varphi_y = c_1 \varphi_y \chi_{\Omega(o,x)} - c_2 \varphi_y$$

for $c_1, c_2 \in \mathbb{C}$. If $\Omega(o, x) \cap \Omega(o, p(y)) = \emptyset$, the first term vanishes, and so $\int_\Omega \varphi_x \varphi_y\, d\nu = 0$. Otherwise, $\Omega(o, x) \cap \Omega(o, p(y)) = \Omega(o, p(y))$ which reveals that $\varphi_x \varphi_y = (c_1 - c_2)\varphi_y$, and so φ_x and φ_y are also orthogonal in this case. This proves that eigenfunctions from Lemma 2.5.1 corresponding to different eigenvalues are ortogonal.

Furthermore, we observe that our choice of z implies that the eigenvalues of I_z from Lemma 2.5.1 are positive.

Since $\varphi \in \mathcal{K}(\Omega)$, it follows by Lemma 2.5.1 that φ can be written as a linear combination of eigenfunctions of I_z. By the above observation, we may write $\varphi = \sum_{j=1}^m \psi_j$ where the ψ_j's are ortogonal eigenfunctions of I_z corresponding to positive eigenvalues b_j. This shows that

$$\sum_{i,j} c_i \overline{c_j} f(g_i^{-1} g_j) = \langle I_z \varphi, \varphi \rangle_{L^2(\nu)} = \sum_{j=1}^m b_j \left\| \psi_j \right\|^2_{L^2(\nu)} \geq 0$$

which proves that f is positive definite.

The above observations even give a new realisation of the unitary representation associated to f by the Gelfand-Naimark-Segal construction. To see this, we define

$$\langle \varphi, \psi \rangle_z = \langle I_z \varphi, \psi \rangle_{L^2(\nu)}$$

for $\varphi, \psi \in \mathcal{K}(\Omega)$. $\langle \cdot, \cdot \rangle_z$ is clearly a sesquilinear form on $\mathcal{K}(\Omega)$. The considerations above show that this form is even positive and so Hermitian (cf. [Fo, Corollary A1.2]). If $\langle \varphi, \varphi \rangle_z = 0$ and $\varphi = \sum_{j=1}^m \psi_j$ as above, we see that

$$\sum_{j=1}^{m} b_j \, \|\psi_j\|^2_{L^2(\nu)} = 0$$

which means that $\psi_j = 0$ for all j, and so $\varphi = 0$. This shows that $\langle \cdot, \cdot \rangle_z$ is an inner product on $\mathcal{K}(\Omega)$.

Denote by $\mathcal{H}_z$ the completion of $\mathcal{K}(\Omega)$ with respect to this inner product, by $\mathcal{H}$ the Hilbert space associated to f by the Gelfand-Naimark-Segal construction and by $V \subseteq \mathcal{H}$ the dense subspace spanned by the left translates of f. For $c_1, \ldots, c_n \in \mathbb{C}$ and $g_1, \ldots, g_m \in G$, we put $v = \sum_{i=1}^n c_i L(g_i) f \in V$ and $\varphi = \sum_{i=1}^n c_i P^z_{g_i(o)} \in \mathcal{K}(\Omega)$. The above observations and the definition of the inner product $\langle \cdot, \cdot \rangle$ in V show that

$$\langle v, v \rangle = \sum_{i,j} c_i \overline{c_j} f(g_i^{-1} g_j) = \langle I_z \varphi, \varphi \rangle = \langle \varphi, \varphi \rangle_z$$

This means that we may (well-)define a map $T : V \to \mathcal{K}(\Omega)$ by the condition

$$T(\sum_{i=1}^{n} c_i L(g_i) f) = \sum_{i=1}^{n} c_i P^z_{g_i(o)}$$

for all $c_1, \ldots, c_n \in \mathbb{C}$ and $g_1, \ldots, g_n \in G$. T is clearly linear, isometric and surjective, and so it extends to a unitary operator from $\mathcal{H}$ onto $\mathcal{H}_z$ which we also denote by T. The representation π associated to f by the Gelfand-Naimark-Segal construction may by T be translated into an equivalent unitary representation π_z of G on $\mathcal{H}_z$.

We recall that $\pi(g)$ on V is just the translation operator $L(g)$ for $g \in G$. This shows that for $\varphi = \sum_{i=1}^n c_i P^z_{g_i(o)} \in \mathcal{K}(\Omega)$, $g \in G$ and $\omega \in \Omega$

$$\begin{aligned}(\pi_z(g)(\varphi))(\omega) &= ((T\pi(g)T^{-1})(\varphi))(\omega) = (T(\sum_{i=1}^{n} c_i L(g g_i) f))(\omega) = \sum_{i=1}^{n} c_i P^z_{(g g_i)(o)}(\omega) \\ &= P^z_{g(o)}(\omega) \sum_{i=1}^{n} c_i P^z_{g_i(o)}(g^{-1} \cdot \omega) = P^z_{g(o)}(\omega) \varphi(g^{-1} \cdot \omega)\end{aligned}$$

where we have used the identity (2.5). Hence, we see that π_z on $\mathcal{K}(\Omega)$ is the same operator as we studied in the case where $z = \frac{1}{2} + it$ with $t \in \mathbb{R}$.

We collect the above observations in the following theorem in which the positive definite spherical functions for the pair (G, K) are listed, and in which we have found realisations of all spherical representations of G. This is the main result of this section.

Theorem 2.5.2 *Let (G, K) be the Gelfand pair introduced above. The positive definite spherical functions for (G, K) are the functions*

$$f_z(g) = (\mathscr{P}_z \nu)(g(o)) = \int_{\Omega} P^z(o, g(o), \omega) \, \nu(d\omega)$$

for all $g \in G$, where $z = \frac{1}{2} + it$ with $t \in \mathbb{R}$ or $z = s + \frac{ik\pi}{\log q}$ with $s \in [0, \frac{1}{2})$ and $k = 0, 1$.

If $z = 0$, a realisation of the corresponding spherical representation from the Gelfand-Naimark-Segal construction is the trivial representation.

If $z = \frac{i\pi}{\log q}$, a realisation π of the corresponding spherical representation from the Gelfand-Naimark-Segal construction is given by $\pi(g)v = (-1)^{d(o,g(o))}v$ for all $g \in G$ and $v \in \mathcal{H}$ where $\mathcal{H}$ is a one-dimensional Hilbert space.

In the remaining cases, the spherical representation from the Gelfand-Naimark-Segal is equivalent to a representation π on the completion of the space $\mathcal{K}(\Omega)$ of cylindrical functions on Ω with respect to a certain inner product. In all cases, $\pi(g)$ is given by

$$(\pi(g)\varphi)(\omega) = P^z_{g(o)}(\omega)\varphi(g^{-1}\cdot\omega)$$

for all $g \in G$, $\omega \in \Omega$ and $\varphi \in \mathcal{K}(\Omega)$.

Remark 2.5.3 For later analogy, we observe that the set of positive definite spherical function on (G,K) by the above observations - using the eigenvalues of the corresponding eigenfunctions for the Laplace operator - might be parametrized by the interval $[-1,1]$.

Remark 2.5.4 One should observe that all the unitary representations of the group G may actually be determined if we add the assumption that G contains "‘sufficiently many"’ rotations (we will not state the exact condition here since it is rather technical. It can be found in [N]. One should, however, notice that the full automorphism group $\mathrm{Aut}(\mathfrak{X})$ is an example of a group with the required properties). The ideas in this classification is due to Olshanski, cf. [O2].

A subtree $(\mathfrak{Y},\mathfrak{D})$ of $(\mathfrak{X},\mathfrak{C})$ is said to be complete if it consists of a single vertex or if all vertices have degree 1 or $q+1$. If π is an irreducible unitary representation of G, one may prove that there exists a finite complete subtree $(\mathfrak{Y},\mathfrak{D})$ of $(\mathfrak{X},\mathfrak{C})$ for which there exists a non-zero vector which is invariant under the group $K(\mathfrak{Y}) = \{g \in G \mid g(x) = x \text{ for all } x \in \mathfrak{Y}\}$. Let $(\mathfrak{J},\mathfrak{E})$ be a minimal finite complete subtree with this property. We will refer to $(\mathfrak{J},\mathfrak{E})$ as a minimal tree for π. π is said to be *spherical* if $|\mathfrak{J}| = 1$, *special* if $|\mathfrak{J}| = 2$ and cuspidal if $|\mathfrak{J}| > 2$.

It is not difficult to see that this concept of a spherical representation coincides with the one used in the abstract setting for the pair (G,K). Hence, we have found all spherical representations in Theorem 2.5.2 above. In [FN, Theorem 3.2.6], it is proved that there exist exactly two non-equivalent special representations and that these are L^2, but not L^1.

To deal with the cuspidal representations with minimal tree $(\mathfrak{J},\mathfrak{E})$, we consider the maximal proper subtrees $(\mathfrak{J}_1,\mathfrak{E}_1),\ldots,(\mathfrak{J}_n,\mathfrak{E}_n)$ of $(\mathfrak{J},\mathfrak{E})$ and the subgroup $\tilde{K}(\mathfrak{J}) = \{g \in G \mid g(\mathfrak{J}) = \mathfrak{J}\}$. We denote by $F(\mathfrak{J})$ the set of equivalence classes of all irreducible unitary representations of $\tilde{K}(\mathfrak{J})$ which are trivial on $K(\mathfrak{J})$ and which for all i do not have any $K(\mathfrak{J}_i)$-invariant vectors. For $\pi \in F(\mathfrak{J})$, we denote by $\mathrm{Ind}(\pi)$ the induced representation of π from $\tilde{K}(\mathfrak{J})$ to G. It now follows by [FN, Theorem 3.3.14] that the map $\pi \mapsto \mathrm{Ind}(\pi)$ is a bijection from $F(\mathfrak{J})$ onto the set of equivalence classes of cuspidal representations of G with minimal tree $(\mathfrak{J},\mathfrak{E})$. Furthermore, it follows by [FN, Corollary 3.3.3] that the matrix coefficients of such a representation have compact support.

We will not go through the details in this classification of the irreducible representations of G since it would make this thesis too lengthy and since we do not need it. The reader is referred to [FN] and [O2].

Remark 2.5.5 Other Gelfand pairs consisting of groups of automorphisms of a locally finite, homogeneous tree have been studied. One important example is the subgroup B_ω of $\mathrm{Aut}(\mathfrak{X})$ consisting of all rotations fixing a given point $\omega \in \Omega$. This group may also be characterized as the stabilizer of the ω-horicycles. In [FN, Section 1.8] it is proved that this is an amenable group. The spherical functions for the pair (B_ω, K) where K is a certain compact subgroup of B_ω are found in [N]. Furthermore, [N] classifies the irreducible unitary representations of B_ω. This is again based on the ideas of Olshanski in [O2].

The material covered in this section only deals with locally finite, homogeneous trees and their automorphism groups. The purpose of the remaining chapters is to combine ideas and results of chapters 1 and 2 to consider groups of automorphisms of homogeneous trees of countable order.

Chapter 3

An Olshanski Spherical Pair Consisting of Automorphism Groups for a Homogeneous Tree

This chapter is devoted to the construction and initial study of the first of the two Olshanski spherical pairs which will be the center of attention for the remainder of this thesis. The pair arises from Gelfand pairs consisting of automorphisms of locally finite, homogeneous trees of increasing degree and is a natural extension of the considerations of chapter 2. The main emphasis will be the determination of the spherical functions for the pair. We clarify which of those are positive definite and give realizations of the corresponding spherical representations. The main results are Theorem 3.2.1 and Corollary 3.2.3. We begin in section 1 by constructing the Olshanski spherical pair which will be the center of attention in this chapter. In section 2, we determine the spherical functions for this and find out which of these are positive definite while section 3 deals with the spherical representations of the group. In section 4, we apply the results developed in this chapter to prove positive definiteness of certain functions on the free group with countably many generators. Along the way, we discuss several open questions which are as yet unanswered.

3.1 The Olshanski spherical pair (G, K)

The purpose of this chapter is to make use of the theory of Olshanski spherical pairs in the study of groups of automorphisms of a homogeneous tree of countable degree. The main idea making this possible is the extension of automorphisms of certain locally finite, homogeneous subtrees. This will be discussed below.

Let $(\mathfrak{X}, \mathfrak{C})$ be a homogeneous tree of countable degree, and denote by d the natural metric on $\mathfrak{X}$. We fix a vertex $o \in \mathfrak{X}$. For each vertex $x \in \mathfrak{X}$, we number its neighbours by choosing a bijection $\tau_x : \mathbb{N} \to \{y \in \mathfrak{X} \mid d(x,y) = 1\}$ such that - for $x \neq o$ - $\tau_x(1)$ is contained in the chain from o to x. We fix these bijections $\{\tau_x\}_{x\in\mathfrak{X}}$ for the remainder of this discussion.

For each $n \geq 2$, we define $\mathfrak{X}_n \subseteq \mathfrak{X}$ by the condition that $x \in \mathfrak{X}_n$ if the chain $x_0, x_1, \dots, x_k$ with $x_0 = o$ and $x_k = x$ satisfies that $x_{j+1} \in \tau_{x_j}(\{1, 2, \dots, n+1\})$ for all $j \in \{0, \dots, k-1\}$. Furthermore, we put $\mathfrak{C}_n = \{\{x,y\} \in \mathfrak{C} \mid x, y \in \mathfrak{X}_n\}$. It is now evident that $(\mathfrak{X}_n, \mathfrak{C}_n)$ is a locally finite, homogeneous subtree of $(\mathfrak{X}, \mathfrak{C})$ of degree $n+1$, that

$$o \in \mathfrak{X}_2 \subseteq \mathfrak{X}_3 \subseteq \ldots \subseteq \mathfrak{X}_n \subseteq \ldots$$

and that

$$\mathfrak{X} = \bigcup_{n=2}^{\infty} \mathfrak{X}_n$$

We denote by G_∞ and G_n the automorphism groups for $\mathfrak{X}$ and $\mathfrak{X}_n$, respectively. Endowed with the compact-open topology, all are Hausdorff topological groups, and the G_n's are even locally compact groups.

For each n, we embed G_n in G_∞ in the following way: Let $g \in G_n$. We want to extend g to an automorphism $\tilde{g}$ of $\mathfrak{X}$, so we consider $x \in \mathfrak{X}$. Let x' be the unique vertex in $\mathfrak{X}_n$ with minimal distance to x and construct the chain $x_0 = x', x_1, \dots, x_{k-1}, x_k = x$ from x' to x. For each j, there exists $l_j \geq 2$ such that $x_{j+1} = \tau_{x_j}(l_j)$, and we inductively define y_j by putting $y_0 = g(x')$ and $y_{j+1} = \tau_{y_j}(l_j)$ for $j \in \{0, \dots k-1\}$. Put $\tilde{g}(x) = y_k$. $\tilde{g}$ is clearly an extension of g to $\mathfrak{X}$ taking values in $\mathfrak{X}$, and it is not difficult to see that it is actually an automorphism of $\mathfrak{X}$.

We will refer to $\tilde{g}$ as the standard extension of g.

Fortunately, the algebraic and topological structures are preserved under the map $g \mapsto \tilde{g}$:

Lemma 3.1.1 *For each n, the map $\varphi_n : G_n \to G_\infty$, $g \mapsto \tilde{g}$, is an injective group homomorphism from G_n into G_∞ which is a homeomorphism onto its image.*

Proof. Let $g, h \in G_n$, and consider $x \in \mathfrak{X}$. Let $x', x_0, x_1, \dots, x_{k-1}, x_k$ and l_j be as above. If we inductively define y_j by putting $y_0 = h(g(x'))$ and $y_{j+1} = \tau_{y_j}(l_j)$ for $j \in \{0, \dots k-1\}$, we by definition have that $(\widetilde{hg})(x) = y_k$.

Now put $z_0 = g(x')$ and $z_{j+1} = \tau_{z_j}(l_j)$ for $j \in \{0, \dots k-1\}$. Then $\tilde{g}(x) = z_k$. Since $l_0 > n+1$ and $l_j > 1$ for $j \geq 1$, $z_j \notin \mathfrak{X}_n$ for $j \geq 1$. This shows that $z_0 = g(x')$ is the unique vertex in $\mathfrak{X}_n$ with minimal distance to $\tilde{g}(x)$. Hence, $\tilde{h}(\tilde{g}(x)) = w_k$ if we define $w_0 = h(g(x'))$ and $w_{j+1} = \tau_{w_j}(l_j)$ for $j \in \{0, \dots k-1\}$. But $y_j = w_j$ for all j and so $(\widetilde{hg})(x) = \tilde{h}(\tilde{g}(x))$.

This shows that φ_n is a group homomorphism. Since $\tilde{g}$ is an extension of g for all $g \in G_n$, the kernel obviously only consists of the identity and so φ_n is injective.

To prove the continuity, we let $F = \{x_1, \dots, x_k\} \subseteq \mathfrak{X}$ be a finite set and let $g \in G_n$. Put $U_F(\tilde{g}) = \{f \in G_\infty \mid f(x) = \tilde{g}(x) \text{ for all } x \in F\}$. We define x'_j as the unique vertex in $\mathfrak{X}_n$ with minimal distance to x_j and let $F' = \{x'_1, \dots, x'_k\}$. If we consider an automorphism $h \in V_{F'}(g) = \{f \in G_n \mid f(x) = g(x) \text{ for all } x \in F'\}$, it is an immediate consequence of the definition of $\tilde{h}, \tilde{g}$ that $\tilde{h} \in U_F(\tilde{g})$. This proves that the group homomorphism is even continuous.

Finally, let $F \subseteq \mathfrak{X}_n$ be finite and $g \in G_n$. If $h \in G_n$ satisfies that $\tilde{h} \in U_F(\tilde{g})$, it is of course also true that $h \in V_F(g)$. This shows that φ_n is a homeomorphism onto its image. □

For the remainder of this discussion, we will identify G_n with its image under φ_n and so regard it as a subgroup of G_∞. Accordingly, we will not distinguish between g and its standard extension for $g \in G_n$.

An immediate observation is that for $g \in G_n$ and $k \geq n$, $g(\mathfrak{X}_k) = \mathfrak{X}_k$. Indeed, if $x \in \mathfrak{X}_k$ and $x', x_0, \dots, x_k$ and l_j is defined as above, it must by definition of $\mathfrak{X}_k$ be true that $l_j \in \{2, \dots, k+1\}$ for all j. Since $g(x') \in \mathfrak{X}_n$, this reveals that $g(x) \in \mathfrak{X}_k$. That we have equality is an obvious consequence of the fact that $\mathfrak{X}_k$ is locally finite and homogeneous.

The above observation makes us realize that the restriction of g to $\mathfrak{X}_k$ is an automorphism of $\mathfrak{X}_k$, and it is an immediate consequence of the definition that its extension to $\mathfrak{X}$ using the above procedure is again g. This shows that

$$G_2 \subseteq G_3 \subseteq \ldots \subseteq G_n \subseteq \ldots$$

For all n, we now define $K_n = \{g \in G_n \mid g(o) = o\}$ which is a compact subgroup of G_n. Furthermore, we put $K_\infty = \{g \in G_\infty \mid g(o) = o\}$. Consider

$$G = \bigcup_{n=2}^{\infty} G_n \subseteq G_\infty, \; K = \bigcup_{n=2}^{\infty} K_n \subseteq K_\infty$$

G is obviously a subgroup of G_∞ and consists of all standard extensions of automorphisms of the considered subtrees. Similarly, K is a subgroup of K_∞, and $K = \{g \in G \mid g(o) = o\}$. We equip G with the inductive limit topology.

The following observation is immediate, but important:

Proposition 3.1.2 (G, K) *is an Olshanski spherical pair, and the inductive limit topology on* G *is stronger than the topology inherited from* G_∞.

Proof. $\{G_n\}$ is an increasing sequence of locally compact groups, and since all G_n's have the induced topology from G_∞, the topology on G_n is the one induced by G_{n+1}.

We want to prove that G_n is a closed subgroup of G_{n+1}. Let $\{g_\lambda\} \subseteq G_n$ be a net which converges to $g \in G_\infty$. If $x \in \mathfrak{X}_n$, it follows by the definition of the topology on G_∞ that there exists λ such that $g(x) = g_\lambda(x) \in \mathfrak{X}_n$. Since $\mathfrak{X}_n$ is locally finite and homogeneous, this shows that $g(\mathfrak{X}_n) = \mathfrak{X}_n$, and so the restriction of g to $\mathfrak{X}_n$ is an automorphism of $\mathfrak{X}_n$. To show that $g \in G_n$, we must prove that g is the standard extension h of this restriction.

Let $x \in \mathfrak{X}$, and let $x' \in \mathfrak{X}_n$ be the unique vertex with minimal distance to x. By definition of the topology on G_∞, there exists λ such that g and g_λ coincide on $\{x, x'\}$. Since g_λ is the standard extension of its restriction to $\mathfrak{X}_n$, it follows by definition of standard extensions that $g(x) = h(x)$. This shows that G_n is a closed subgroup of G_∞ and hence also of G_{n+1}.

By Proposition 2.3.2, (G_n, K_n) is a Gelfand pair for all n since G_n and K_n by homogeneity of $(\mathfrak{X}_n, \mathfrak{C}_n)$ act transitively on $\mathfrak{X}_n$ and the boundary of $(\mathfrak{X}_n, \mathfrak{C}_n)$, respectively. The identity $K_n = G_n \cap K_{n+1}$ is obvious, and so (G, K) is an Olshanski spherical pair.

The last claim in the proposition is an immediate consequence of the definition of the inductive limit topology and the fact that the topology on G_n is the one inherited from G_∞. □

Remark 3.1.3 The inductive limit topology on G is actually strictly stronger than the one inherited from G_∞. To see this, let $x_2 \in \mathfrak{X}_2$ and define inductively $x_n = \tau_{x_{n-1}}(n+1)$ for $n \geq 3$. x_n is by definition the unique neighbour of x_{n-1} which is contained in $\mathfrak{X}_n \setminus \mathfrak{X}_{n-1}$. Consider the set $U = \{g \in G \mid g(x_n) = x_n \text{ for all } n \in \mathbb{N}\}$. By definition of the standard extension, $U \cap G_n = \{g \in G_n \mid g(x_k) = x_k \text{ for all } k \in \{2, \ldots, n\}\}$ which is open in G_n. Hence, U is open in the inductive limit topology.

However, U is not open in the topology inherited from G_∞. Indeed, if $F \subseteq \mathfrak{X}$ is finite, there exists n such that $F \subseteq \mathfrak{X}_n$, and so we can find $g \in G_{n+2}$ such that $g(x) = x$ for all $x \in F$ and such that $g(x_{n+1}) \neq x_{n+1}$ (g can be chosen such that it fixes every vertex x which satisfies that the chain from o to x does not pass through x_{n+1} and $\tau_{x_n}(n+3)$ and such that $g(x_{n+1}) = \tau_{x_n}(n+3)$). Hence, $\{g \in G \mid g(x) = x \text{ for all } x \in F\} \nsubseteq U$ which shows that U is not open in the topology inherited from G_∞.

Remark 3.1.4 The subgroup G is actually dense in G_∞. Indeed, let $g \in G_\infty$ and $F \subseteq \mathfrak{X}$ be a finite set. Choose $N \geq 0$ such that $F \cup g(F) \subseteq \mathfrak{X}_N$. It is now easy to see - for instance by induction in the number of elements in F - that there exists $h \in G_N$ such that $h(x) = g(x)$ for all $x \in F$. This proves that G is dense in G_∞.

Remark 3.1.5 The idea of studying the group G may be compared to the approach which has been applied to infinite symmetric groups. The analysis of such groups revolves around two groups, namely the group $S(\infty) = \bigcup_{n=1}^{\infty} S(n)$ of finite permutations of the set $\mathbb{N}$ of natural numbers and the group $\tilde{S}(\infty)$ of all permutations of $\mathbb{N}$. The group $\tilde{S}(\infty)$ is usually equipped with a topology which is strictly weaker than the discrete topology, and it is easily seen that $S(\infty)$ is dense in $\tilde{S}(\infty)$. However, $S(\infty)$ is the inductive limit of discrete groups, and so the inductive limit topology is the discrete one. Both groups have been studied and their relations have been examined closely, cf. [L], [KOV], [O4] and [O5].

The situation considered in this chapter is similar. The group G_∞ is the full automorphism group and may be compared to the group $\tilde{S}(\infty)$. The group G contains all automorphisms of the considered subtrees (seen as automorphisms of the "'big"' tree) and may in this regard be seen as the analogue of $S(\infty)$. The topological structures also share some similarity in the fact that the inductive limit topology is strictly stronger than the topology inherited from the "'full"' group.

In this chapter, we will be concerned only with the group G (as is the case for the group $S(\infty)$ in the papers [KOV] and [O5]) while we discuss the relation with G_∞ in chapter 6. The relation between $S(\infty)$ and $\tilde{S}(\infty)$ is the subject of the paper [O4].

We will study the pair (G, K) from a number of different perspectives. The remainder of this chapter is devoted to the spherical functions and representations for the pair. In chapter 6, we will investigate the relation between representations of the group G_∞ and its dense subgroup G which leads to the concept of a tame representation. Finally, conditionally positive definite functions and cocycles will be the center of attention in chapter 8.

Remark 3.1.6 Even though by far the most research in the area of trees and automorphisms has centered on locally finite, homogeneous trees, the idea of considering homogeneous trees of infinite degree is not completely new. In the important paper [O3], Olshanski classifies all irreducible representations of the automorphism group for a homogeneous tree of infinite degree (regardless of the cardinality). The main strategy is to apply ideas from the analysis of the infinite symmetric group. However, since the publication of [O3] not much development has taken place. We take up the gauntlet of extending the theory by applying the theory of Olshanski spherical pairs - an approach which has not been applied before.

This is in stark contrast to the infinite symmetric group which has been treated from a number of different perspectives. The paper [O3] is heavily inspired by the approach used by Liebermann in [L] which contains the classification of the irreducible representations of the infinite symmetric group. In the papers [O4], [O5] and [KOV], the study centers, however, around the inductive limit approach which is completely different. Our objective is to extend the study of automorphism groups for homogeneous trees of infinite degree in a similar way, and we will along the way use ideas from the analysis of the infinite symmetric group in our study of the pair (G, K).

3.2 Spherical functions for (G, K)

Our main objective is to determine the spherical functions corresponding to the Olshanski spherical pair (G, K). Using our knowledge of locally finite, homogeneous trees, we will do this in Theorem 3.2.1 below. The spherical functions turn out to be much simpler than they are for the Gelfand pairs (G_n, K_n).

Theorem 3.2.1 *A function $\varphi : G \to \mathbb{C}$ is spherical for the pair (G,K) if and only if there exists a constant $a \in \mathbb{C}$ such that*

$$\varphi(g) = a^{d(g(o),o)} \tag{3.1}$$

for all $g \in G$.

Remark 3.2.2 The spherical functions in Theorem 3.2.1 were already found by Olshanski in [O3] as an important part of his classification of all irreducible representations of the automorphism group for a homogeneous tree of infinite degree. His approach is, however, completely different from the one used here.

Proof. Assume that φ is a spherical function on G. Since G_n acts transitively on $\mathfrak{X}_n$ for each n, G acts transitively on $\mathfrak{X}$. Furthermore, the relation $K = G \cap K_\infty$ reveals that K is the isotropy group corresponding to $o \in \mathfrak{X}$. This means that the map $g \mapsto g(o)$ induces a bijection from G/K onto $\mathfrak{X}$, and since φ is K-right-invariant, we may in the natural way regard φ as a function $\tilde{\varphi}$ on $\mathfrak{X}$.

If $x_1, x_2 \in \mathfrak{X}$ are vertices such that $d(o,x_1) = d(o,x_2)$, there exists n such that $x_1, x_2 \in \mathfrak{X}_n$. Hence, there exists $k \in K_n \subseteq K$ such that $k(x_1) = x_2$. If $g \in G$ satisfies that $g(o) = x_1$, the K-left-invariance of φ reveals that

$$\tilde{\varphi}(x_2) = \varphi(kg) = \varphi(g) = \tilde{\varphi}(x_1),$$

and so $\tilde{\varphi}(x)$ does only depend on the distance of x to o. Hence, $\tilde{\varphi}$ is radial. Abusing the notation, we will write $\varphi(n)$ for $\tilde{\varphi}(x)$ with $d(o,x) = n$.

Let $n \geq 1$, and find $g \in G$ such that $d(o, g(o)) = n$. There exists l such that $g \in G_l$, and we may choose $h \in G_l$ such that $d(h(o), o) = 1$. Let $m \geq l$. For $k \in K_m$, $d((gkh)(o), g(o)) = d((kh)(o), o) = d(h(o), o) = 1$. Furthermore, if $y_1, \ldots, y_{m+1}$ are the neighbours of $g(o)$ in $\mathfrak{X}_m$ and $A_j = \{k \in K_m \mid (gkh)(o) = y_j\}$, we observe that $A_j = k_{ij} A_i$ for all i,j where $k_{ij} \in K_m$ is an automorphism (which exists) for which $(k_{ij} g^{-1})(y_i) = g^{-1}(y_j)$. This shows that the normalized Haar measure on K_m takes the same value on the open sets A_j whose union is K_m. Since we may choose our numbering such that $d(o, y_1) = n-1$ and $d(o, y_j) = n+1$ for $j \neq 1$, we observe that

$$\int_{K_m} \varphi(gkh)\, dk = \frac{1}{m+1}\varphi(n-1) + \frac{m}{m+1}\varphi(n+1)$$

The fact that φ is spherical now makes us realize that

$$\varphi(n+1) = \varphi(1)\varphi(n) \tag{3.2}$$

φ is spherical, and so it follows by Lemma 1.3.2 that $\varphi(0) = \varphi(e) = 1$. Hence, (3.2) also holds for $n = 0$. Induction now shows that $\varphi(n) = a^n$ where $a = \varphi(1)$. This concludes the proof of the "‘only if"’-part of the theorem.

Conversely, let φ be given by the condition (3.1). We observe that $\varphi(e) = 1$, so φ is non-zero and clearly K-biinvariant. Furthermore, since $\{h \in G \mid h(o) = g(o)\}$ is open in the topology inherited from G_∞ for all $g \in G$, φ is continuous in this topology and hence by Propositon 3.1.2 in the inductive limit topology.

By Remark 2.4.8 and Proposition 2.4.7, we may for all n choose a spherical function φ_n for the pair (G_n, K_n) such that (with the usual abuse of notation) $\varphi_n(1) = a$. We will by induction in k prove that for all $k \geq 0$ $\varphi_n(k)$ converges to $\varphi(k)$ as $n \to \infty$, i.e. that φ_n converges pointwise to φ. By Proposition 2.4.7, $\varphi_n(0) = 1$ for all n, and so the convergence is obvious in this case. It follows by assumption that $\varphi_n(1) = a$ for all n, and so the convergence is also true for $k = 1$.

Now consider $k \geq 2$, and assume that $\varphi_n(j)$ converges to $\varphi(j)$ for $j \leq k$. By Proposition 2.4.7, the relation

$$\varphi_n(k) = \frac{n+1}{n} a\varphi_n(k-1) - \frac{1}{n}\varphi_n(k-2)$$

holds for all n. By the induction hypothesis, this means that $\varphi_n(k)$ converges to $a\varphi(k-1) = \varphi(k)$ as $n \to \infty$.

Let $g, h \in G$, and let m satisfy that $g, h \in G_m$. Let $\epsilon > 0$. Since φ_n is spherical for all n, it is for $n \geq m$ true that

$$\left| \int_{K_n} \varphi(gkh)\, dk - \varphi(g)\varphi(h) \right| \leq \int_{K_n} |\varphi(gkh) - \varphi_n(gkh)|\, dk + |\varphi_n(g)\varphi_n(h) - \varphi(g)\varphi(h)| \quad (3.3)$$

where dk is the normalized Haar measure on K_n. Since $d((gkh)(o), g(o)) = d(h(o), o)$, the triangle inequality for the metric d reveals that $d(o, (gkh)(o))$ takes only finitely many values. Hence, we may use that φ_n converges pointwise to φ and that $\varphi(gkh) - \varphi_n(gkh)$ only depends on $d(o, (gkh)(o))$ to find $N \geq m$ such that for all $n \geq N$ $|\varphi_n(g)\varphi_n(h) - \varphi(g)\varphi(h)| < \frac{\epsilon}{2}$ and $|\varphi(gkh) - \varphi_n(gkh)| < \frac{\epsilon}{2}$ for all $k \in K_n$. (3.3) and the fact that the Haar measure dk is normalized then shows that

$$\left| \int_{K_n} \varphi(gkh)\, dk - \varphi(g)\varphi(h) \right| < \epsilon$$

for $n \geq N$. Hence, φ is a spherical function. □

An immediate consequence of Theorem 3.2.1 is that we are able to determine the spherical dual for the pair (G, K):

Corollary 3.2.3 *A function $\varphi : G \to \mathbb{C}$ is a positive definite spherical function for the pair (G, K) if and only if there exists a constant $a \in [-1, 1]$ such that*

$$\varphi(g) = a^{d(g(o), o)} \quad (3.4)$$

for all $g \in G$.

Proof. Let φ be a positive definite spherical function for (G, K). Since φ is spherical, we can find $a \in \mathbb{C}$ such that (3.4) is satisfied. The fact that $d(g^{-1}(o), o) = d(g(o), o)$ for $g \in G$ shows that $\overline{\varphi(g)} = \varphi(g^{-1}) = \varphi(g)$ for all $g \in G$, and so a is real. Finally, the positive definiteness shows that $|\varphi(g)| \leq |\varphi(e)| = 1$ for all $g \in G$ which shows that a is as claimed in the corollary.

Conversely, assume that φ satisfies (3.4). By Theorem 3.2.1, φ is a spherical function for the pair (G, K), and we may - as in the proof of this theorem - for each n choose a spherical function φ_n for the pair (G_n, K_n) such that $\varphi_n(1) = a$ and such that φ is the pointwise limit of the φ_n's. By Theorem 2.5.2, φ_n is positive definite for all n, and so the same is true for the pointwise limit φ. □

Remark 3.2.4 In light of Remark 1.3.5, it should be observed that the proof of Theorem 3.2.1 above shows that every spherical function for (G, K) is the pointwise limit of spherical functions for the pairs (G_n, K_n). The convergence is even uniform on compact sets (note that this concept makes sense by Remark 1.3.5). Indeed, let $P \subseteq G$ be compact. Since the action of G on $\mathfrak{X}$ by Remark 2.1.4 is continuous in the inherited topology on G from G_∞, it follows by Proposition 3.1.2 that it is continuous when G is equipped with the inductive limit topology. Hence, the set $P(o)$ is a compact subset of $\mathfrak{X}$, so it is finite. Since $\varphi_n(g)$ and $\varphi(g)$ only depend on $d(g(o), o)$ for $g \in G_n$, this shows that the convergence is uniform on P.

This proves that in this case the answer to the question of Remark 1.3.5 is affirmative and further strengthens the conjecture that the answer to the open problem of approximation of spherical functions for an Olshanski spherical pair by spherical functions for the underlying Gelfand pairs is positive. A satisfying general answer to this open problem has, however, not yet been given.

In the proof of Corollary 3.2.3, we observed that every positive definite spherical function for (G, K) is a pointwise limit of positive definite spherical functions for (G_n, K_n), and the argument above shows that the convergence is uniform on compact sets. The existence of such functions was guaranteed by the theorem of Olshanski mentioned in Remark 1.3.5. Hence, we have provided a concrete example of this abstract theorem.

Finally, one should observe that by Remarks 2.4.8 and 2.5.3 the spherical functions and positive definite spherical functions for (G_n, K_n) are naturally parametrized by $\mathbb{C}$ and $[-1, 1]$, respectively. Theorem 3.2.1 and Corollary 3.2.3 reveal that these parametrizations are in a natural way inherited by the spherical and positive definite spherical functions for (G, K). Actually, it follows by the proofs and the above remarks that the spherical functions for (G_n, K_n) corresponding to a given parameter converges uniformly on compact sets to the spherical function for (G, K) corresponding to the same parameter.

Remark 3.2.5 As seen from Theorem 3.2.1, the spherical functions for (G, K) have a certain multiplicative property. Surprisingly, this is a well-known fact from the study of spherical functions for a large number of Olshanski spherical pairs arising from classical matrix groups. This indicates that there is some kind of similarity between spherical functions in these cases which have very different natures.

To make this precise, we begin by observing that the double coset KgK for $g \in G$ consists of all $h \in G$ such that $d(h(o), o) = d(g(o), o)$. Indeed, it is obvious that $d(h(o), o) = d(g(o), o)$ if $h \in KgK$. Conversely, if $d(h(o), o) = d(g(o), o)$ for $h \in G$, we can find n such that $g, h \in G_n$ and $k_1 \in K_n$ such that $(k_1 g)(o) = h(o)$. This means that $k_2 = h^{-1} k_1 g \in K_n$, and so $h = k_1 g k_2^{-1} \in KgK$. Hence, the double coset consists of the elements with the claimed property.

This structure of the double cosets means that we may in an obvious way identify the space $K \backslash G / K$ of such double cosets with $\mathbb{N} \cup \{0\}$ which - equipped with addition - is a commutative monoid. This semigroup structure is inherited by $K \backslash G / K$. K-biinvariant functions on G may naturally be regarded as functions on the commutative monoid $K \backslash G / K$. Theorem 3.2.1 now states that a non-zero, K-biinvariant function on G is spherical if and only if it is a multiplicative function on $K \backslash G / K$. This is the multiplicative property we mentioned above. It should be observed that a K-biinvariant function in this case is automatically continuous since $K \cap G_n = K_n$ is open in G_n for all n which means that K is open in G. Hence, a K-biinvariant function is constant on the disjoint open sets KgK for $g \in G$ and so continuous.

In [O1, Section 23.12], Olshanski studies a large class of Olshanski spherical pairs (G, K) arising from classical matrix groups and shows that we may equip the double coset space $K \backslash G / K$ with a structure as a commutative monoid. Furthermore, he proves exactly the same relation between multiplicative and spherical functions as we did above, i.e. that a non-zero, continuous, K-biinvariant function is spherical if and only if the corresponding function on $K \backslash G / K$ is multiplicative. Hence, the spherical functions have a similar characterization in these two cases even though the groups are of two different, unrelated types.

This surprising similarity between these very different cases suggests that there is something deeper and more general going on. At a first glance, the semigroup structures are not in any way related in the two cases, but there might, however, be a connection. It is still an open problem to study these semigroup structures in an attempt to identify the reasons for this similarity from an abstract point of view.

Remark 3.2.6 In light of Remark 2.5.5, it should be observed that the automorphism groups studied in [N] give rise to an Olshanski spherical pair if we - as above - let the degree of the tree tend to infinity. The spherical functions for this pair are not yet known. It is still an open problem to consider the harmonic analysis of this pair in detail.

Having determined the spherical dual for the pair (G, K), we may now apply the Bochner-Godement theorem for Olshanski spherical pairs in Theorem 1.3.6 to determine all positive definite, K-biinvariant functions on G. This is the content of Corollary 3.2.7 below.

Corollary 3.2.7 *A function $\varphi : G \to \mathbb{C}$ is positive definite and K-biinvariant if and only if there exists a finite Borel measure μ on $[-1, 1]$ such that*

$$\varphi(g) = \int_{[-1,1]} a^{d(g(o),o)}\ \mu(da)$$

If this is the case, the measure μ is unique.

Proof. Let Ω be the spherical dual for (G, K) equipped with the topology of uniform convergence on compact sets. Denote by φ_a the positive definite spherical function corresponding to the parameter $a \in [-1, 1]$ in Corollary 3.2.3. By an argument similar to the one from Remark 3.2.4, it is easy to see that a net $\{a_\lambda\} \subseteq [-1, 1]$ converges to $a \in [-1, 1]$ if and only if φ_{a_λ} converges to φ_a uniformly on compact sets. Hence, Ω may as a topological space be identified with the interval $[-1, 1]$.

Let $n \geq 2$. Since $\mathfrak{X}_n$ is locally finite, there exists for all $m \geq 0$ finitely many $y \in \mathfrak{X}_n$ such that $d(o, y) = m$. Hence, $\mathfrak{X}_n$ is countable. For each $m \geq 1$, denote by $\mathfrak{X}_n^m$ the direct product of n copies of $\mathfrak{X}_n$. The set $\mathfrak{X}_n^m \times \mathfrak{X}_n^m$ is countable. For each $m \geq 1$ and each pair $(x, y) \in \mathfrak{X}_n^m \times \mathfrak{X}_n^m$ with $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$, we choose an automorphism $g_{x,y} \in G_n$ with the property that $g(x_i) = y_i$ for all i if such an automorphism exists. Put $F_x = \{x_1, \ldots, x_m\}$ for $x \in \mathfrak{X}_n^m$, and define $C_n = \{U_{F_x}(g_{x,y})\}$ which is a countable set. By definition of the topology on G_n, C_n is a basis for the topology on G_n, and so G_n is second countable.

Let φ be a positive definite, K-biinvariant function. It follows by Remark 3.2.5 that φ is automatically continuous. Hence, Theorem 1.3.6 provides the existence of a finite Borel measure $\tilde{\mu}$ on Ω such that

$$\varphi(g) = \int_{\Omega} \varphi_a(g)\ \tilde{\mu}(d\varphi_a)$$

for all $g \in G$. If we translate $\tilde{\mu}$ to $[-1, 1]$, we get a measure μ with the desired properties.

The other direction is obvious since φ_a is positive definite and K-biinvariant for all a.

The uniqueness of μ is a direct consequence of Theorem 1.3.6. □

The integral representation of all K-biinvariant, positive definite functions in Corollary 3.2.7 will be the central tool in our proof of a Levy-Khinchine decomposition formula for the pair (G, K) in chapter 8.

3.3 Spherical representations

Having determined the positive definite spherical functions in Corollary 3.2.3, we want to make concrete realizations of the spherical representations for the pair (G, K). All such representations arise by the Gelfand-Naimark-Segal construction from a positive definite spherical function.

The case $a = 1$ of course corresponds to the trivial representation of G.

In the case $a = -1$, we define $\pi(g)v = (-1)^{d(g(o),o)}v$ for all v in the one-dimensional Hilbert space $\mathcal{H}$ and all $g \in G$. If $\mathfrak{X}^+ = \{x \in \mathfrak{X} \mid d(o,x) \text{ is even}\}$ and $\mathfrak{X}^- = \mathfrak{X} \setminus \mathfrak{X}^+$, it is clearly true that $g(\mathfrak{X}^+) = \mathfrak{X}^+$ or $g(\mathfrak{X}^+) = \mathfrak{X}^-$. Hence, π is a group homomorphism from G into the group of unitary operators on $\mathcal{H}$. Since $g \mapsto \pi(g)v$ is constant on the - by Proposition 3.1.2 open - sets $\{h \in G \mid h_0(o) = h(o)\}$ for $h_0 \in G$ and $v \in \mathcal{H}$, π is an irreducible unitary representation of G. Furthermore, if h is a unit vector, h is K-invariant and cyclic, and $\varphi(g) = \langle h, \pi(g)h\rangle$ for all $g \in G$. By the essential uniqueness of the representation in the Gelfand-Naimark-Segal construction, π is a realisation of the spherical representation corresponding to φ.

The remaining cases are - as in chapter 2 - more complicated. Let $a \in (-1, 1)$, and let φ be the positive definite spherical function given by (3.4). Let V be a vector space with basis $\{v_x\}_{x\in\mathfrak{X}}$. Define a sesquilinear form $H : V \times V \to \mathbb{C}$ by the condition that $H(v_x, v_y) = a^{d(x,y)}$ for all $x, y \in \mathfrak{X}$. Since a is real, H is clearly Hermitian.

We now put $w_o = v_o$ and $w_x = (1 - a^2)^{-\frac{1}{2}}(v_x - av_{\tau_x(1)})$ for all $x \in \mathfrak{X} \setminus \{o\}$ (notice that $1 - a^2$ is positive). Observe that

$$H(w_o, w_o) = a^{d(o,o)} = 1$$

and

$$\begin{aligned} H(w_x, w_x) &= (1 - a^2)^{-1}(a^{d(x,x)} - 2aa^{d(x,\tau_x(1))} + a^2a^{d(\tau_x(1),\tau_x(1))}) \\ &= (1 - a^2)^{-1}(1 - 2a^2 + a^2) = 1 \end{aligned}$$

for $x \neq o$.

Furthermore, it is true that

$$H(w_o, w_x) = (1 - a^2)^{-\frac{1}{2}}(a^n - aa^{n-1}) = 0$$

for $x \in \mathfrak{X} \setminus \{o\}$ with $d(o, x) = n$. Similarly, for $x, y \in \mathfrak{X} \setminus \{o\}$ for which y is not contained in the chain from o to x and x is not contained in the chain from o to y, we see that

$$H(w_x, w_y) = (1 - a^2)^{-1}(a^n - aa^{n-1} - aa^{n-1} + a^2a^{n-2}) = 0$$

where $n = d(x, y)$. Finally, for $x, y \in \mathfrak{X} \setminus \{o\}$ for which $x \neq y$ and y is contained in the chain from o to x, we obtain the following:

$$H(w_x, w_y) = (1 - a^2)^{-1}(a^n - aa^{n+1} - aa^{n-1} + a^2a^n) = 0$$

where $n = d(x, y)$. Hence, we see that $\{w_x\}_{x\in\mathfrak{X}}$ is an orthonormal system in V for the Hermitian form H, and so it is linearly independent.

$\{w_x\}_{x\in\mathfrak{X}}$ is actually a basis. Indeed, let $x \in \mathfrak{X}$ and let $x_0 = o, x_1, \ldots, x_n = x$ be the chain from o to x. Put $a_j = a^{n-j}$ for all $j \in \{0, \ldots, n\}$. Then $v_x = a_0w_{x_0} + \sum_{j=1}^{n} a_j(1 - a^2)^{\frac{1}{2}}w_{x_j}$ which shows that $\{w_x\}_{x\in\mathfrak{X}}$ span V.

The existence of a basis for V which is an orthonormal system for the Hermitian form H now immediately implies that H is actually an inner product on V.

Let $\mathcal{H}$ be the completion of the inner product space V, and denote the inner product by $\langle \cdot, \cdot \rangle$. Since $V = \text{span}\,\{w_x\}_{x \in \mathfrak{X}}$ is dense in $\mathcal{H}$, $\{w_x\}_{x \in \mathfrak{X}}$ is an orthonormal basis for $\mathcal{H}$.

For all $g \in G$, define a linear operator $\pi(g)$ on V by the condition that $\pi(g)v_x = v_{g(x)}$ for all $x \in \mathfrak{X}$. The isometry property of g means that $H(\pi(g)v_x, \pi(g)v_y) = H(v_x, v_y)$ for all $x, y \in \mathfrak{X}$. Furthermore, the surjectivity of g shows that the image of $\pi(g)$ is V. Hence, $\pi(g)$ extends to a unitary operator on $\mathcal{H}$, and the map $\pi : G \to U(\mathcal{H})$, the group of unitary operators on $\mathcal{H}$, is clearly a group homomorphism. Finally, fix $x \in \mathfrak{X}$, and let $g_0 \in G$. If $g \in \{h \in G \mid h(x) = g_0(x)\}$ which is open in the topology on G induced by G_∞ and so by Proposition 3.1.2 in the inductive limit topology on G, it is true that $\pi(g)v_x = \pi(g_0)v_x$. This shows that $g \mapsto \pi(g)v_x$ is continuous. Since $\{v_x\}_{x \in \mathfrak{X}}$ is a basis for V which is dense in $\mathcal{H}$, this proves that π is a unitary representation of G.

v_o is clearly a K-invariant unit vector. Since

$$\text{span}\,\{g \in G \mid \pi(g)v_o\} = V,$$

v_o is even a cyclic vector. It is clearly true that

$$\varphi(g) = \langle v_o, \pi(g)v_o \rangle\,,$$

and so it follows by the essential uniqueness of the representation in the Gelfand-Naimark-Segal construction that π is equivalent to the representaion obtained by applying this construction to φ. This shows that π is irreducible and a realization of the spherical representation corresponding to φ.

The above representations constitute the complete family of spherical representations for the pair (G, K).

Remark 3.3.1 The above construction of the spherical representations in the case $a \in (-1, 1)$ differs dramatically from the construction in section 2.5 of the representations associated to the positive definite spherical function for (G_n, K_n) corresponding to eigenvalues in $(-1, 1)$. In some of these cases, the representation space was just $L^2(\nu_n)$ where ν_n was a K_n-invariant probability measure on the boundary Ω_n of $(\mathfrak{X}_n, \mathfrak{C}_n)$.

Inspired by this, it is natural to try to construct realizations of the spherical representations for (G, K) on some L^2-space related to the boundary and to obtain an integral representation for the spherical functions as in Theorem 2.4.6 for the pairs (G_n, K_n). Since the boundary of $(\mathfrak{X}, \mathfrak{C})$ does clearly not possess a K-invariant measure, the task is to find an object which should replace the boundary. An approach could be to attack this problem using the techniques of [KOV, Section 2] where the infite symmetric group is considered. The idea should be to construct continuous projection maps $s_n : \Omega_{n+1} \to \Omega_n$ for all n such that the image measure $s_n(\nu_{n+1}) = \nu_n$ for all n. This creates a projective system, and the corresponding projective limit could - in the product topology - be regarded as some compactification of the boundary. The Kolmogorov consistency theorem would provide the existence of a measure on this object.

In chapter 7, we construct such a compactification of the boundary, and it plays an important role in our study of the second of our two main examples of Olshanski spherical pairs. The corresponding representation gives rise to a family of new interesting representations. Unfortunately, it does not play a role in the analysis of the group G since this group does not act on it. The group G_n acts on the boundary Ω_n for all n, and so we may - using the idea of [KOV, Section 2] - extend this action to the projective limit of the sets Ω_n if the projection maps satisfy that $s_n(g \cdot \omega) = g \cdot s_n(\omega)$ for all $g \in G_n$ and all $\omega \in \Omega_{n+1}$, i.e. if the projection maps are equivariant with respect to the action of G_n. Unfortunately, the projection maps mentioned above do not have this property, and so G does not act on the projective limit. Hence, the compactification of the boundary may not be used to write up an integral representation of the spherical functions in the case of a homogeneous tree of countable degree.

It is still an open problem whether reasonable integral representations of the spherical functions for the pair (G, K) can be obtained and whether the spherical representations may be realised on some L^2-space arising from the boundary.

Remark 3.3.2 Since the Gelfand-Naimark-Segal construction shows that there is a bijective correspondance between the set of equivalence classes of spherical representations for (G, K) and the spherical dual for (G, K), we have found realizations of all spherical representations for (G, K). In chapter 6, we extend our class of unitary representations of G by giving a complete classification of a family of representations which will be known as *tame*. These are the representations of G which arise by restriction of representations of G_∞. More representations of the group G are, however, not known.

As mentioned in Remark 2.5.4, a complete classification of the irreducible representations can be obtained for the Gelfand pairs (G_n, K_n). It is an open problem to get a more complete picture of the representation theory of G. The methods of Olshanski applied in the locally finite case are certainly not applicable here, so new techniques need to be developed.

We finish this chapter with the observation that an interesting fact on free groups is an easy consequence of our results. The analysis of G will be continued in chapters 6 and 8.

3.4 An application

We finish this chapter by observing that the results developed here provide a new proof of a well-known, non-trivial fact concerning positive definiteness of functions on free groups. Let F be the free group with countably many generators, and let $B \subseteq F$ be a free countable set of generators. Recall that we define the length $|x|$ of $x \in F$ as the number of factors in $B \cup B^{-1}$ which are needed to write x as a reduced word. Let $t \in [-1, 1]$, and define $\varphi : F \to \mathbb{C}$ by

$$\varphi(x) = t^{|x|}$$

for $x \in F$. In [H], it is proved that φ is positive definite. Another proof which covers a much larger class of functions (known as Haagerup functions) can be found in [DF]. We will prove that the positive definiteness of φ is a consequence of Corollary 3.2.3 above.

For the proof, we recall that we to a group H and a generating subset S may associate the *Cayley graph* which is the graph $(\mathfrak{Y}, \mathfrak{D})$ for which $\mathfrak{Y} = H$ and for which $x, y \in H$, $x \neq y$, are neighbours if an only if there exists $s \in S \cup S^{-1}$ such that $y = xs$. The fact that S is generating clearly means that $(\mathfrak{Y}, \mathfrak{D})$ is a connected graph.

We now state and prove the desired result:

Corollary 3.4.1 *Let F be a free group with countably many generators, let $B = \{b_n\}_{n=1}^{\infty}$ be a free subset generating F, and denote by $|x|$ the length of $x \in F$. For $t \in [-1, 1]$, the function $\varphi : F \to \mathbb{C}$ defined by*

$$\varphi(x) = t^{|x|}$$

for $x \in F$ is positive definite.

Proof. Let $(\mathfrak{X}, \mathfrak{C})$ be the Cayley graph associated to G using B as the set of generators. As seen above, $(\mathfrak{X}, \mathfrak{C})$ is a connected graph, and it is obvious that the fact that B is a free subset implies that there are no circuits in $(\mathfrak{X}, \mathfrak{C})$. Hence, $(\mathfrak{X}, \mathfrak{C})$ is a tree which clearly is homogeneous with countable degree.

Denote by F_n the subgroup generated by $b_1, \dots, b_n$, and observe that $F = \bigcup_{n=1}^{\infty} F_n$. Let o be the vertex in $\mathfrak{X}$ corresponding to the neutral element $e \in F$. As above, we will for each $x \in \mathfrak{X}$ choose bijections $\tau_x : \mathbb{N} \to \{y \in \mathfrak{X} \mid d(x,y) = 1\}$ with the requirements introduced in the beginning of this section. If $x \in F_n$, we do make this choice such that $\tau_x(\{1, \dots, 2n\}) = \{xb_1, xb_1^{-1}, \dots, xb_n, xb_n^{-1}\}$ which is clearly possible. It now follows by construction that $\mathfrak{X}_{2n-1} = F_n$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$, and let $x \in F_n$. The map $z \mapsto xz$ is a bijection from F_n onto F_n and so a bijection from $\mathfrak{X}_{2n-1}$ onto $\mathfrak{X}_{2n-1}$. It follows by definition of the edges in $(\mathfrak{X}_{2n-1}, \mathfrak{C}_{2n-1})$ that this map is even an automorphism of the tree $(\mathfrak{X}_{2n-1}, \mathfrak{C}_{2n-1})$. Hence, we have in a natural way a map from F_n into G_{2n-1} which is clearly an injective group homomorphism. In this way, we may regard F_n as a subgroup of G_{2n-1} and so of G.

Let $t \in [-1, 1]$. By Corollary 3.2.3, the map $\psi : G \to \mathbb{C}$ defined by the relation

$$\psi(g) = t^{d(g(o),o)}$$

is positive definite on G. Hence, the restriction to F_n is also positive definite. If $x \in F_n$ and $g \in G_{2n-1}$ denote the corresponding automorphism, we clearly have that $d(g(o), o) = |x|$. This shows that the restrictions of ψ and φ to F_n coincide. Since $F = \cup_{n=1}^{\infty} F_n$, this implies that φ is positive definite. $\square$

Remark 3.4.2 The idea of Corollary 3.4.1 is to study the free group by associating it with a tree on which the elements of the group act as automorphisms. A similar idea may be used to study other objects. One important example is the case of p-adic numbers $\mathbb{Q}_p$ where p is a prime number, cf. [Sa]. In two different ways, it is possible to construct a locally finite, homogeneous tree of degree $p+1$ from $\mathbb{Q}_p$, and these constructions make it possible to study different groups using the harmonic analysis for groups acting on such trees.

One construction is discussed in [FN, Appendix A]. The set $\mathfrak{X}$ of vertices is defined to be the set of equivalence classes corresponding to an equivalence relation on the set of lattices in a 2-dimensional vector space over $\mathbb{Q}_p$. One may define an integer-valued metric on this set. If we define the set of edges $\mathfrak{C}$ using this metric, it is possible to show that $(\mathfrak{X}, \mathfrak{C})$ is a locally finite, homogeneous tree of degree $p+1$ and that the boundary is homeomorphic to the one-point compactification of $\mathbb{Q}_p$. Since an element $GL(2, \mathbb{Q}_p)$ acts in a natural way on the set of lattices, it is possible to identify the group $PGL(2, \mathbb{Q}_p) = GL(2, \mathbb{Q}_p)/ \{\alpha I \mid \alpha \in \mathbb{Q}_p \setminus \{0\}\}$ with a closed subgroup of the automorphism group $\mathrm{Aut}(\mathfrak{X})$. Hence, the representation theory and harmonic analysis of $PGL(2, \mathbb{Q}_p)$ may be studied by the methods developed in the study of groups acting on trees. We refer to [FN, Appendix A] for the details in the construction above.

Another construction is discussed in [F-T]. Here the set $\mathfrak{X}$ of vertices is just the set of closed balls in $\mathbb{Q}_p$ with radius p^k for $k \in \mathbb{Z}$. Two balls are said to be neighbours if one is contained in the other and if the radius of the "‘big"’ ball is p times the radius of the "‘small"’ ball. This defines a set $\mathfrak{C}$ of edges. One may prove that $(\mathfrak{X}, \mathfrak{C})$ is a homogeneous tree of degree $p+1$ and that the boundary is homeomorphic to the one-point compactification of $\mathbb{Q}_p$. One may also prove that the group $\mathrm{Isom}(\mathbb{Q}_p)$ of isometries of $\mathbb{Q}_p$ may be identified with a closed subgroup of $\mathrm{Aut}(\mathfrak{X})$ (actually with the group of rotations fixing a point on the boundary which is mentioned in Remark 2.5.5). Hence, the representation theory and harmonic analysis of $\mathrm{Isom}(\mathbb{Q}_p)$ may also be studied using the methods developed in the study of groups acting on trees. We refer to [F-T] for a detailed discussion of the construction of $(\mathfrak{X}, \mathfrak{C})$.

Using the ideas of this section, we may consider the automorphism groups for locally finite, homogeneous trees as the degree of the tree tends to infinity. In the above situation, this corresponds to the prime number p tending to infinity, and so the ideas of this chapter may be applied to investigate the groups $PGL(2, \mathbb{Q}_p)$ and $\mathrm{Isom}(\mathbb{Q}_p)$ as $p \to \infty$. Such a study has not yet been carried out.

Remark 3.4.3 Let H be a group, and assume that $l : H \to \mathbb{N} \cup \{0\}$ is a map which we will refer to as the length function. If $t \in [-1, 1]$, it has in a number of different contexts been studied whether the function $\varphi : H \to \mathbb{C}$ defined by

$$\varphi(h) = t^{l(h)}$$

for $h \in H$ is positive definite. For instance, Corollary 3.4.1 shows that this is the case if H is the free group with countably many generators and l is the usual length function. Corollary 3.2.3 shows that it is also true if $H = G$ is the inductive limit of automorphism groups as above if we define the length $l(g) = d(g(o), o)$ for $g \in G$.

Another example is the case where H is the symmetric group S_n and where the length $l(x)$ for $x \in S_n$ is the minimal number with the property that there exist simple transpositions $y_1, \ldots, y_{l(x)}$ such that $x = y_1 \ldots y_{l(x)}$. The idea in this definition of length is that the length is the number of basic "‘building blocks"’ needed to build x. In [BS], it is proved that φ is also positive definite in this situation (actually, more general operator-valued functions are considered).

A similar idea can be used to define another length function on the automorphism group $H = \mathrm{Aut}(\mathfrak{X})$ for a homogeneous tree $(\mathfrak{X}, \mathfrak{C})$ of degree at least 3. It follows by [FN, Proposition 1.3.4] that every automorphism $h \in \mathrm{Aut}(\mathfrak{X})$ may be written as the product of step-1 translations. Hence, step-1 translations are the basic "‘building blocks"’ of the automorphism group, and we may define $l(h)$ to be the minimal number of such translations needed to "‘build"’ h. It is not yet known whether the function φ is positive definite in this situation.

Chapter 4

The Spherical Functions for a Gelfand Pair Consisting of Automorphism Groups for a Homogeneous Rooted Tree

The purpose of this chapter is to develop the theory of spherical functions and representations for a Gelfand pair consisting of groups of automorphisms of a locally finite, homogeneous rooted tree. The main emphasis will be on constructing a family of spherical representations for the pair and - by considering the corresponding spherical functions - show that these exhaust all spherical representations for the pair. In some regard, the theory developed here may be seen as an equivalent for rooted trees to the material in chapter 2, and it will form the foundation for the study of our second Olshanski spherical pair in chapter 5. We will need a fair amount of basic definitions, notation and elementary observations. These will be collected in section 1. In section 2, we construct a family of spherical representations for the considered pair, and in section 3 we determine all spherical functions and show that the representations from section 2 exhaust all spherical representations. Section 4 is devoted to the decomposition into direct sums of irreducible representations of the restriction of the spherical representations to a certain subgroup and contain observations which will be needed in chapters 5 and 7. Finally, section 5 determines the relation between the spherical representations found in this chapter and the spherical representations of chapter 2.

4.1 Definitions, notation and preliminary observations

A rooted tree is a tree $(\mathfrak{X}, \mathfrak{C})$ together with a fixed vertex $o \in \mathfrak{X}$. The vertex o will be known as *the root* of the tree. Let $(\mathfrak{X}, \mathfrak{C})$ be a rooted tree which is locally finite and homogeneous of degree $q + 1 \geq 3$, and let $o \in \mathfrak{X}$ be its root. This will be kept fixed in the following.

An automorphism of the tree $(\mathfrak{X}, \mathfrak{C})$ which fixes the root o will be known as *an automorphism of the rooted tree* $(\mathfrak{X}, \mathfrak{C})$. As seen in chapter 2, the group $\mathrm{Aut}_o(\mathfrak{X})$ of such automorphisms form a compact subgroup of the group $\mathrm{Aut}(\mathfrak{X})$ of all automorphisms of the tree $(\mathfrak{X}, \mathfrak{C})$.

Let K be a closed subgroup of $\mathrm{Aut}_o(\mathfrak{X})$. By the above, K is a compact group. Define $\mathfrak{M}_n = \{x \in \mathfrak{X} \mid d(o, x) = n\}$ for $n \geq 0$. Since every automorphism is an isometry, K clearly acts on each $\mathfrak{M}_n$, and it is an immediate consequence of the definition of the topology on K that this action is continuous. We will assume that the action on $\mathfrak{M}_n$ is transitive for each n.

An example of a group satisfying this assumption is the full group of automorphisms of the rooted tree.

For a vertex $x \in \mathfrak{X}$, we denote by K_x the subgroup of K consisting of automorphisms fixing x. It is immediate by the definition of the topology that K_x is both closed and open and so a compact and open subgroup of K.

For a vertex $x \in \mathfrak{M}_n$ with $n \geq 1$, we define $p(x) \in \mathfrak{X}$ to be the unique vertex satisfying that $d(x, p(x)) = 1$ and that $p(x) \in \mathfrak{M}_{n-1}$. We will refer to the map $p : \mathfrak{X} \setminus \{o\} \to \mathfrak{X}$ as *the canonical projection.*

Denote by Ω the boundary of $(\mathfrak{X}, \mathfrak{C})$ which in its usual topology is a compact Hausdorff space. *A rooted chain* is an infinite chain $x_0, x_1, x_2, \ldots$ in $\mathfrak{X}$ with $x_0 = o$. Recall that Ω may be identified with the set of rooted chains. For $x \in \mathfrak{X}$ with $d(o, x) = n$, we denote by $\Omega(x)$ the set of $\omega \in \Omega$ for which the corresponding rooted chain $x_0, x_1, x_2, \ldots$ satisfies that $x_n = x$.

We define on Ω a metric δ which induces the right topology. Indeed, let $\omega, \omega' \in \Omega$ and assume that $\omega \neq \omega'$. Let $x_0, x_1, x_2, \ldots$ and $y_0, y_1, y_2, \ldots$ be the rooted chains corresponding to ω and ω', respectively, and let $n = \min\{m \geq 0 \mid x_m \neq y_m\}$. Define $\delta(\omega, \omega') = \frac{1}{n}$, and $\delta(\omega, \omega) = 0$.

If $\omega, \omega', \omega'' \in \Omega$ and $\delta(\omega, \omega') \geq \max\{\delta(\omega, \omega''), \delta(\omega', \omega'')\}$, it is clear that $\delta(\omega, \omega') = \delta(\omega, \omega'')$ or $\delta(\omega, \omega') = \delta(\omega', \omega'')$. Hence, δ satisfies the triangle inequality and so is a metric on Ω. Since the open ball $b(\omega, \frac{1}{n})$ with center ω and radius $\frac{1}{n}$ is clearly just $\Omega(x_n)$, δ induces the already given topology on Ω.

Recall that the group K by the map $(g, \omega) \mapsto g \cdot \omega$ acts continuosly on Ω. The injectivity of an automorphism shows that $\delta(g \cdot \omega, g \cdot \omega') = \delta(\omega, \omega')$ for all $\omega, \omega' \in \Omega$ and all $g \in K$, and so δ is K-invariant.

We fix once and for all $\omega \in \Omega$, and let $x_0, x_1, x_2, \ldots$ be the corresponding rooted chain. This will be kept fixed for the remainder of this chapter. We define for $n \in \mathbb{N}$

$$A_n = \left\{ \tau \in \Omega \;\middle|\; \delta(\omega, \tau) = \frac{1}{n} \right\}$$

and put $A_\infty = \{\omega\}$. Furthermore, we define B_n to be the open ball with center ω and radius $\frac{1}{n}$ for $n \in \mathbb{N}$. Finally, we declare K^ω to be the isotropy group of ω, i.e.

$$K^\omega = \{g \in K \mid g \cdot \omega = \omega\}$$

Since $K^\omega = \bigcap_{n=0}^{\infty} K_{x_n}$ is an intersection of closed sets, it is a closed, and hence compact, subgroup of K.

We now construct a map $m : \mathfrak{X} \to \mathbb{N}$ by defining $m(x_n) = q - 1$ for $n \geq 1$ and $m(y) = q$ for all $y \in \mathfrak{X}$ satisfying that $y \neq x_n$ for all $n \geq 1$. We make two assumptions on K to ensure that K is "‘big enough"’. Firstly, for each $n \geq 1$ and each $y \in \mathfrak{M}_{n-1}$ there exists a labeling $z_1, z_2, \ldots, z_{m(y)}$ of the vertices in $p^{-1}(y) \setminus \{x_n\}$ and an automorphism $k \in K^\omega$ such that $k(z_i) = z_{i+1}$ for $i < m(y)$ and $k(z_{m(y)}) = z_1$ and such that $k(z) = z$ for all $z \in \mathfrak{M}_n$ with $p(z) \neq y$. Secondly, for all $y \in \mathfrak{M}_n \setminus \{x_n\}$, there exists a labeling $z_1, z_2, \ldots, z_{m(y)-1}$ of the vertices in $p^{-1}(p(y)) \setminus \{x_n, y\}$ and an automorphism $k \in K^\omega$ such that $k(z_i) = z_{i+1}$ for $i < m(y) - 1$ and $k(z_{m(y)-1}) = z_1$ and such that $k(z) = z$ for all $z \in \mathfrak{M}_n$ with $p(z) \neq p(y)$. Observe that k automatically satisfies that $k(y) = y$. These conditions are of course satisfied by the full group of automorphisms of the rooted tree.

It is an easy observation that the assumptions on K imply that K acts transitively on Ω. This is part of the content of the following lemma:

Lemma 4.1.1 *Let K be as above. Then the action of K on Ω is transitive, and the orbits of the action of K^ω on Ω are exactly the sets A_n with $n \in \mathbb{N}$ and $n = \infty$.*

Proof. Let $\tau, \tau' \in \Omega$, and let $y_0, y_1, y_2, \ldots$ and $z_0, z_1, z_2, \ldots$ be the rooted chains corresponding to τ and τ', respectively. For $n \geq 0$, define

$$K_n = \{g \in K \mid g(y_n) = z_n\}$$

It follows immediately by the above characterization of the topology of $\mathrm{Aut}(\mathfrak{X})$ that K_n is a closed subset of K. Hence, each K_n is compact. Furthermore, it is clear that $K_0 \supseteq K_1 \supseteq \ldots \supseteq K_n \supseteq \ldots$. Since the transitive action of K on $\mathfrak{M}_n$ implies that $K_n \neq \emptyset$ for $n \geq 1$, the finite intersection property means that $\bigcap_{n=0}^{\infty} K_n \neq \emptyset$. Let $g \in \bigcap_{n=0}^{\infty} K_n$. g evidently satisfies that $g \cdot \tau = \tau'$ which proves the first part of the lemma.

To prove the second part, we begin by observing that the K^ω-orbit $\mathcal{O}(\omega)$ of ω is clearly A_∞. Now let $\tau \in \Omega$ satisfy that $\tau \neq \omega$, let $n \in \mathbb{N}$ be given such that $\tau \in A_n$, and let $y_0, y_1, y_2 \ldots$ be the corresponding rooted chain. The K-invariance of δ immediately implies that $\delta(g \cdot \tau, \omega) = \delta(\tau, \omega)$ for all $g \in K^\omega$ and so the K^ω-orbit $\mathcal{O}(\tau)$ of τ satisfies that $\mathcal{O}(\tau) \subseteq A_n$.

For the other inclusion, let $\tau' \in A_n$, assume that $\tau' \neq \tau$ and denote by $y_0, y_1, y_2, \ldots$ and $z_0, z_1, z_2, \ldots$ the rooted chains corresponding to τ and τ', respectively. Let m be the smallest number such that $y_m \neq z_m$. We observe that $m \geq n$. By our assumption on K, we may find an automorphism $g_0 \in K^\omega$ such that $g_0(y_m) = z_m$. Now assume that we have constructed automorphisms $g_1, \ldots, g_l$ such that $(g_l g_{l-1} \ldots g_0)(y_{m+l}) = z_{m+l}$ and hence also $(g_l g_{l-1} \ldots g_0)(y_i) = z_i$ for $i < m + l$. By our assumption, we may find $g_{l+1} \in K^\omega$ such that $g_{l+1}((g_l g_{l-1} \ldots g_0)(y_{m+l+1})) = z_{m+l+1}$. In this way we inductively define a sequence $\{g_l\} \subseteq K^\omega$ such that $(g_l g_{l-1} \ldots g_0)(y_{m+l}) = z_{m+l}$ for all l. By compactness, the sequence $\left\{\prod_{i=0}^{l} g_i\right\}$ has a convergent subsequence with limit $g \in K^\omega$. It follows clearly by the definition of the topology on K that $g(y_l) = z_l$ for all l and so $g \cdot \tau = \tau'$. This proves that $A_n \subseteq \mathcal{O}(\tau)$ and so finishes the proof of the proposition. □

Since K acts transitively on Ω by Lemma 4.1.1, and since K^ω is the isotropy group for ω, we may use the bijection $gK^\omega \mapsto g \cdot \omega$ to identify the coset space K/K^ω with Ω. By [W, Theorem 1.6.7], this map is even a homeomorphism, so it is an identification of topological spaces. We may now identifiy a right-K^ω-invariant function φ on K with a function $\tilde{\varphi}$ on Ω. Furthermore, the description of the orbits of the action of K^ω on Ω in Lemma 4.1.1 means that we may identify a K^ω-biinvariant function φ on K^ω with a function $\tilde{\varphi}$ on Ω which is constant on the sets A_n. Conversely, every such function may be identified with a K^ω-biinvariant function on K.

The following proposition is an important observation:

Proposition 4.1.2 *The pair (K, K^ω) is a Gelfand pair.*

Proof. Let $g \in K$. Since δ is K-invariant, we have that $\delta(g^{-1} \cdot \omega, \omega) = \delta(\omega, g \cdot \omega)$. Hence, there exists an n such that $g^{-1} \cdot \omega, g \cdot \omega \in A_n$. By Lemma 4.1.1, this implies that there exists $k \in K^\omega$ such that $(kg) \cdot \omega = g^{-1} \cdot \omega$ which means that $k_1 = gkg \in K^\omega$. This leads to the conclusion that $g^{-1} = kgk_1^{-1} \in K^\omega g K^\omega$. By [W, Proposition 8.1.3], this implies that (K, K^ω) is a Gelfand pair. □

The remainder of this chapter will revolve around the study of spherical functions and representations for the pair (K, K^ω).

4.2 Construction of spherical representations for (K, K^ω)

The objective of this section is to construct a family $\{\pi_n\}_{n=0}^{\infty}$ of spherical representations for the pair (K, K^ω).

To do this, we let π_0 be the one-dimensional trivial representation which is of course spherical. For the remaining representations, let $n \geq 1$ be fixed, and consider $\ell^2(\mathfrak{M}_n)$ which is a $(q+1)q^{n-1}$-dimensional Hilbert space. Since the counting measure on $\mathfrak{M}_n$ is a K-invariant Radon measure, it follows by [Fo, Section 3.1] that we may construct a unitary representation $\tilde{\pi}_n$ by defining $(\tilde{\pi}_n(g)f)(x) = f(g^{-1}(x))$ for $x \in \mathfrak{M}_n$, $f \in \ell^2(\mathfrak{M}_n)$ and $g \in K$ (The fact that this definition defines a unitary representation is actually easier checked in this than in the general case. Indeed, $\tilde{\pi}_n$ is clearly a group homomorphism from K into the group of unitary operators on $\ell^2(\mathfrak{M}_n)$, and the continuity in the strong operator topology follows by the fact that the map $g \mapsto \tilde{\pi}_n(g)f$ with $f \in \ell^2(\mathfrak{M}_n)$ is constant on the open sets $\{g \in K \mid g(x) = g_0(x) \text{ for all } x \in \mathfrak{M}_n\}$ for $g_0 \in K$ and so locally constant).

Now consider the subspace

$$\mathcal{H}_n = \left\{ f \in \ell^2(\mathfrak{M}_n) \;\middle|\; \sum_{x \in p^{-1}(\{y\})} f(x) = 0 \text{ for all } y \in \mathfrak{M}_{n-1} \right\}$$

Since the automorphism property means that $p(g^{-1}(x)) = g^{-1}(p(x))$ and $\left|p^{-1}(\{g^{-1}(y)\})\right| = \left|p^{-1}(\{y\})\right|$ for $g \in K$ and $x, y \in \mathfrak{X}$, we see that

$$\sum_{x \in p^{-1}(\{y\})} (\tilde{\pi}_n(g)f)(x) = \sum_{x \in p^{-1}(\{y\})} f(g^{-1}(x)) = \sum_{x \in p^{-1}(\{g^{-1}(y)\})} f(x) = 0$$

for $y \in \mathfrak{M}_{n-1}$, $f \in \mathcal{H}_n$ and $g \in K$. Hence, $\mathcal{H}_n$ is invariant under $\tilde{\pi}_n$.

We denote by π_n the subrepresentation of $\tilde{\pi}_n$ corresponding to the invariant subspace $\mathcal{H}_n$.

Let $n \geq 2$. We observe that $|\mathfrak{M}_{n-1}| = (q+1)q^{n-2}$ and denote the elements by $y_1, \ldots, y_{(q+1)q^{n-2}}$. Furthermore, we note that $\left|p^{-1}(\{y_i\})\right| = q$ for $i \in \left\{1, \ldots, (q+1)q^{n-2}\right\}$ and denote the vertices in $p^{-1}(\{y_i\})$ by $x_1^i, \ldots, x_q^i$. For $i \in \left\{1, \ldots (q+1)q^{n-2}\right\}$ and $j \in \{1, \ldots, q-1\}$, we define a function $f_j^i \in \mathcal{H}_n$ by the conditions that $f_j^i(x_j^i) = 1$, $f_j^i(x_q^i) = -1$ and that $f_j^i(x) = 0$ for all other vertices $x \in \mathfrak{M}_n$. The set $\left\{f_j^i\right\}$ is clearly linearly independent and spans $\mathcal{H}_n$ and so is a basis for $\mathcal{H}_n$. Hence, the representation π_n has dimension $(q+1)(q-1)q^{n-2}$.

Since the space $\mathcal{H}_1$ consists of all functions f on $\mathfrak{M}_1$ with the property that $\sum_{x \in \mathfrak{M}_1} f(x) = 0$ and since $|\mathcal{M}_1| = q+1$, it is seen in a similar way that π_1 is a q-dimensional representation.

A key observation is that the representation π_n is irreducible for all $n \geq 0$. This is the content of Proposition 4.2.1 below:

Proposition 4.2.1 *The representation π_n is an irreducible representation of K for all n.*

Proof. Let $n \geq 1$. Let $\mathcal{M}$ be a non-trivial subspace of $\mathcal{H}_n$ which is invariant under π_n. Let $f \in \mathcal{M}$ satisfy that $f \neq 0$, and let $x \in \mathfrak{M}_n$ be given such that $f(x) \neq 0$. By the transitive action of K on $\mathfrak{M}_n$, there exists $g \in K$ such that $g(x) = x_n$. Hence, $(\pi(g)f)(x_n) = f(x) \neq 0$. Hence, we may assume that $f(x_n) \neq 0$.

Let $y \in \mathfrak{M}_{n-1}$. By assumption, there exists a labeling $z_1^y, z_2^y, \ldots, z_{m(y)}^y$ of the vertices in $p^{-1}(y) \setminus \{x_n\}$ and an automorphism $k_y \in K^\omega$ such that $k_y(z_i^y) = z_{i+1}^y$ for $i < m(y)$, $k(z_{m(y)}^y) = z_1^y$ and $k_y(z) = z$ for $z \in \mathfrak{M}_n$ with $p(z) \neq y$. Since $f \in \mathcal{H}_n$, we have that

$$(\pi(k_y)f + \pi(k_y^2)f + \ldots + \pi(k_y^{m(y)})f)(z_i^y) = 0 \tag{4.1}$$

if $y \neq x_{n-1}$ and
$$(\pi(k_y)f + \pi(k_y^2)f + \ldots + \pi(k_y^{m(y)})f)(z_i^y) = -f(x_n) \tag{4.2}$$
if $y = x_{n-1}$. Define $h \in \mathcal{H}_n$ to be the function given by the conditions that $h(x_n) = 1$, that $h(z) = -\frac{1}{m(x_{n-1})}$ if $p(z) = x_{n-1}$ and $z \neq x_n$ and that $h(z) = 0$ if $p(z) \neq x_{n-1}$. Since $k_y(z) = z$ if $p(z) \neq y$, we may use (4.1) and (4.2) and a suitable scaling to see that $h \in \mathcal{M}$.

Since $\mathcal{M}^\perp$ is also a non-trivial π_n-invariant subspace, we have that $h \in \mathcal{M}^\perp$. However, we know that $h \neq 0$, and so this is a contradiction. Hence, π_n is irreducible. Since π_0 is just the trivial 1-dimensional representation, this finishes the proof. □

An immediate consequence of Proposition 4.2.1 is that the representation π_n is spherical for the pair (K, K^ω) for all $n \geq 0$:

Corollary 4.2.2 *The representation π_n is a spherical representation for the pair (K, K^ω) for all n.*

Proof. Let $n \geq 1$. Let $h \in \mathcal{H}_n$ be the function from the proof of Proposition 4.2.1. Since $k^{-1}(x_n) = x_n$, $k^{-1}(x_{n-1}) = x_{n-1}$ and $p(k^{-1}(x)) = k^{-1}(p(x))$ for all $x \in \mathfrak{X}$ and all $k \in K^\omega$, we observe that $\pi(k)h = h$ for all $k \in K^\omega$ and so h is non-zero and K^ω-invariant. Since π_n is irreducible by Proposition 4.2.1, it is spherical for the pair (K, K^ω). Since π_0 is just the trivial representation which is clearly spherical for (K, K^ω), this finishes the proof. □

Remark 4.2.3 For later purposes, we observe that the function h in the proof of Corollary 4.2.2 is actually K_{x_n}-invariant.

It is of interest to determine the positive definite spherical functions corresponding to the spherical representations π_n, $n \geq 0$.

To do this, we begin by observing that the function φ_0 constantly equal to 1 is the spherical function corresponding to π_0.

Let $n = 1$, and let $h \in \mathcal{H}_n$ be the function of the proof of Proposition 4.2.1. We observe that
$$\|h\|^2 = 1 + \sum_{i=1}^{q} \frac{1}{q^2} = 1 + \frac{1}{q}$$
Hence, the spherical function φ_1 corresponding to the spherical representation π_1 is given by
$$\varphi_1(g) = \frac{1}{1+\frac{1}{q}} \langle \pi(g)h, h \rangle$$
for $g \in K$. If $g \in K_{x_1}$, we clearly have that $\varphi_1(g) = 1$. If $g \notin K_{x_1}$, we see that
$$\varphi_1(g) = \frac{1}{1+\frac{1}{q}}\left(-\frac{2}{q} + (q-1)\frac{1}{q^2}\right) = \frac{1}{1+\frac{1}{q}} \frac{-q-1}{q^2} = \frac{-q-1}{q(q+1)} = -\frac{1}{q}$$
If we by 1_C denote the indicator function for a set C, this means that the spherical function φ_1 is given by
$$\varphi_1 = 1_{K_{x_1}} - \frac{1}{q} 1_{K \setminus K_{x_1}}$$
and the corresponding function $\tilde{\varphi_1}$ on Ω satisfies that
$$\tilde{\varphi_1} = 1_{B_1} - \frac{1}{q} 1_{A_1}$$

Let $n \geq 2$ and let $h \in \mathcal{H}_n$ be the function of the proof of Proposition 4.2.1. We observe that

$$\|h\|^2 = 1 + \sum_{i=1}^{q-1} \frac{1}{(q-1)^2} = 1 + \frac{1}{q-1}$$

Hence, the spherical function φ_n corresponding to the spherical representation π_n is given by

$$\varphi_n(g) = \frac{1}{1+\frac{1}{q-1}} \langle \pi(g)h, h\rangle$$

for $g \in K$. If $g \in K_{x_n}$, we clearly have that $\varphi_n(g) = 1$. If $g \in K_{x_{n-1}} \setminus K_{x_n}$, we see that

$$\varphi_n(g) = \frac{1}{1+\frac{1}{q-1}}(-\frac{2}{q-1} + (q-2)\frac{1}{(q-1)^2}) = \frac{1}{1+\frac{1}{q-1}} \frac{-q}{(q-1)^2} = \frac{-q}{q(q-1)} = -\frac{1}{q-1}$$

If $g \notin K_{x_{n-1}}$, we clearly have that $\langle \pi(g)h, h\rangle = 0$ which means that $\varphi_n(g) = 0$. This implies that the spherical function φ_n is given by

$$\varphi_n = 1_{K_{x_n}} - \frac{1}{q-1} 1_{K_{x_{n-1}} \setminus K_{x_n}}$$

and the corresponding function $\tilde{\varphi_n}$ on Ω satisfies that

$$\tilde{\varphi_n} = 1_{B_n} - \frac{1}{q-1} 1_{A_n}$$

The objective of the following section is to prove that the set of spherical functions calculated above is complete, i.e. contains all spherical functions for the pair (K, K^ω).

4.3 The spherical functions for (K, K^ω)

It turns out that $\{\pi_n\}_{n=0}^\infty$ is the set of all spherical representations for the pair (K, K^ω). To see this, we will find all the spherical functions for (K, K^ω).

Theorem 4.3.1 *The spherical functions for the pair (K, K^ω) are the functions φ_n, $n \geq 0$, where*

$$\varphi_0 \equiv 1,$$

$$\varphi_1 = 1_{K_{x_1}} - \frac{1}{q} 1_{K \setminus K_{x_1}}$$

and

$$\varphi_n = 1_{K_{x_n}} - \frac{1}{q-1} 1_{K_{x_n} \setminus K_{x_{n-1}}}$$

for $n \geq 2$.

The spherical functions are all positive definite.

Proof. We have already seen that the functions φ_n with $n \geq 0$ are all spherical. Furthermore, their positive definiteness follows by the compactness of K and [VD, Theorem 6.5.1]. Hence, it is left to show that there are no more spherical functions.

To do this, let φ be a spherical function, and let $\tilde{\varphi}$ be the corresponding function on Ω which is constant on the sets A_n with $n \in \mathbb{N}$ and $n = \infty$. To simplify things, we will abuse the notation and denote the value of $\tilde{\varphi}$ on A_n by $\tilde{\varphi}(n)$. Since $\varphi \neq 0$, there exists a minimal $n \in \mathbb{N} \cup \{\infty\}$ such that $\tilde{\varphi}(n) \neq 0$.

We begin by observing that $n \neq \infty$. Indeed, if this was the case φ would just - since every spherical function is normalized at the identity - be the indicator function for the subgroup K^ω. Hence, the continuity of φ implies that K^ω is open. We recall that $K^\omega = \bigcap_{n=1}^\infty K_{x_n}$. It is an immediate consequence of the transitivity of the action of K on $\mathfrak{M}_n$ that K_{x_n} has measure $(q+1)^{-1}q^{1-n}$ under the normalized Haar measure on K for $n \geq 1$. This proves that K^ω has measure 0. However, this contradicts the fact that K^ω is open and non-empty. Hence, $n \neq \infty$.

Now let $m > n$, and choose by the transitive action of K on Ω automorphisms $g, h \in K$ such that $g \cdot \omega \in A_n$ and $h \cdot \omega \in A_m$. Since φ is spherical, we have that

$$\tilde{\varphi}(m)\tilde{\varphi}(n) = \int_{K^\omega} \varphi(hkg)\, dk = \int_{K^\omega} \tilde{\varphi}(n)\, dk = \tilde{\varphi}(n)$$

where we have used that $(hkg) \cdot \omega \in A_n$ for all $k \in K^\omega$. This shows that $\tilde{\varphi}(m) = 1$ for $m > n$.

We now assume that $n = 1$. It is clear that $(gkg) \cdot \omega \in A_1$ for all $k \in K^\omega$ with the property that $(kg)(x_1) \neq g^{-1}(x_1)$ and that $(gkg) \cdot \omega \in B_1$ for all $k \in K^\omega$ with the property that $(kg)(x_1) = g^{-1}(x_1)$. Since it is a consequence of the assumptions that K^ω acts transitively on $p^{-1}(\{x_0\}) \setminus \{x_1\}$, it follows by the translation invariance of the normalized Haar measure on K^ω that $(gkg) \cdot \omega \in A_1$ on an open subset of K^ω of measure $\frac{q-1}{q}$ and that $(gkg) \cdot \omega \in B_1$ on an open subset of measure $\frac{1}{q}$. By the previous observation that $\tilde{\varphi}(m) = 1$ for $m > n$, this implies that

$$(\tilde{\varphi}(1))^2 = \int_{K^\omega} \varphi(gkg)\, dk = \frac{1}{q} + \frac{q-1}{q}\tilde{\varphi}(1)$$

Solving this equation, we see that $\tilde{\varphi}(1) = 1$ or $\tilde{\varphi}(1) = -\frac{1}{q}$ which means that $\tilde{\varphi} \equiv 1$ or $\tilde{\varphi} = 1_{B_1} - \frac{1}{q}1_{A_1}$. This shows that $\varphi = \varphi_0$ or that $\varphi = \varphi_1$.

Now assume that $n \geq 2$. Exactly as above, we see that

$$(\tilde{\varphi}(n))^2 = \int_{K^\omega} \varphi(gkg)\, dk = \frac{1}{q-1} + \frac{q-2}{q-1}\tilde{\varphi}(n)$$

Solving this equation, we see that $\tilde{\varphi}(n) = 1$ or $\tilde{\varphi}(n) = -\frac{1}{q-1}$ which means that $\tilde{\varphi} = 1_{B_{n-1}}$ or $\tilde{\varphi} = 1_{B_n} - \frac{1}{q-1}1_{A_n}$. If $\tilde{\varphi} = 1_{B_{n-1}}$, the same considerations as above show that

$$0 = (\tilde{\varphi}(n-1))^2 = \frac{q-1}{q}\tilde{\varphi}(n-1) + \frac{1}{q} = \frac{1}{q}$$

if $n = 2$ and that

$$0 = (\tilde{\varphi}(n-1))^2 = \frac{q-2}{q-1}\tilde{\varphi}(n-1) + \frac{1}{q-1} = \frac{1}{q-1}$$

if $n > 2$. In both cases we have a contradiction, and so $\tilde{\varphi} = 1_{B_n} - \frac{1}{q-1}1_{A_n}$. Hence, we see that $\varphi = \varphi_n$ which finishes the proof. □

The bijective correspondance between positive definite spherical functions and spherical representations for the pair (K, K^ω) now implies the following corollary.

Corollary 4.3.2 *The set of spherical representations for the pair* (K, K^ω) *is* $\{\pi_n\}_{n=0}^\infty$.

Remark 4.3.3 In light of Remark 2.5.4, it is of interest to notice that we may actually classify all irreducible unitary representations of K if we make some further assumption on K. Again we will not state the complete further requirement on K since it is rather technical.

It may be found in [N]. It should, however, be observed that the full group $\mathrm{Aut}_o(\mathfrak{X})$ of automorphisms of the rooted tree $(\mathfrak{X}, \mathfrak{C})$ satisfies this further condition and so all irreducible unitary representations of this group may be completely classified.

As in Remark 2.5.4, the classification is based on the fact that every unitary representation of K has a minimal tree. Hence, the unitary representations of K may again be divided into spherical, special and cuspidal representations. We shall see that the irreducible spherical representations are exactly the representations which are spherical for the pair (K, K^ω) and which have been found in Corollary 4.3.2. Furthermore, we will observe that K does not possess any special representations.

Assume that π is an irreducible spherical representation of K. Let $x \in \mathfrak{X}$ be given such that the projetion $P(x)$ on the subspace $\mathcal{H}(x)$ of K_x-invariant vectors is non-zero, and let $n = d(o, x)$. By the transitive action of K on $\mathfrak{M}_n$, there exists $g \in K$ such that $g(x) = x_n$. It is easy to check that $\pi(g)(\mathcal{H}(x)) = \mathcal{H}(x_n)$ which proves that $\mathcal{H}_{x_n} \neq 0$. Hence, there exists a non-zero vector which is invariant under $K_{x_n} \supseteq K^\omega$ which means that π is spherical for the pair (K, K^ω). By Corollary 4.3.2, this implies that $\pi = \pi_n$ for some n. Since Remark 4.2.3 shows that π_n has a K_{x_n}-invariant vector for each $n \geq 0$ and so is an irreducible spherical representation, we see that the irreducible spherical reprentations of K are the representations π_n with $n \geq 0$.

Now assume that π is an irreducible special representation of K, and let $e \in \mathfrak{C}$ be an edge such that $P(e) \neq 0$. Let $a, b \in \mathfrak{X}$ be the vertices of e and assume without loss of generality that $d(o, a) < d(o, b)$. It is now clear that $K_b = K(e)$ and so $\mathcal{H}(b) \neq 0$. This contradicts the fact that π is special. Hence, K does not have any irreducible special representations.

To classify the cuspidal representations, we need the above mentioned further assumption on K. The methods developed by Olshanski in [O2] apply directly to this case and gives a complete classification of the irreducible cuspidal representations of K, and the results are identical to the ones mentioned in Remark 2.5.4. We will not go through the details and refer to [O2], [FN] and [N] for the ideas behind this classification.

The spherical representations will be discussed in greater detail in the following section.

4.4 Restriction of spherical representations for the pair (K, K^ω) to K^ω

The set of spherical representations for the pair (K, K^ω) consists exactly of the irreducible unitary representations with the property that their restrictions to K^ω contain the trivial representation as a subrepresentation. We are, however, able to write up the complete decompositions of these restrictions into direct sums of irreducible representations of K^ω. This is the content of Theorem 4.4.1 below. The irreducible representations of K^ω turning up in our decompositions may seem to be of less importance, but they will play a crucial role in our study of representations in relation to one of our main examples of Olshanski spherical pairs in chapters 5 and 7.

To make the decomposition, consider for $n \geq 1$ the subspace

$$\mathcal{V}_n = \left\{ f \in \mathcal{H}_n \;\middle|\; f(x_n) = 0,\ f(y) = 0 \text{ for all } y \notin p^{-1}(\{x_{n-1}\}) \right\}$$

The facts that $g(x_n) = x_n$, $g(x_{n-1}) = x_{n-1}$ and $g(p(y)) = p(g(y))$ for all $g \in K^\omega$ immediately imply that $\mathcal{V}_n$ is K^ω-invariant. We denote by σ_n the corresponding subrepresentation of the restriction of π_n to K^ω. With the notation of section 4.2, we choose i such that $y_i = x_{n-1}$ and we choose our numbering such that $x_q^i \neq x_n$. It is now obvious that the functions f_j^i with

$x_j^i \neq x_n$ constitute a basis for $\mathcal{V}_n$ and so σ_n is a $q-1$-dimensional representation for $n = 1$ and a $q-2$-dimensional representation for $n \geq 2$.

For $n \geq 2$, we futhermore consider the subspace

$$\mathcal{U}_n = \left\{ f \in \mathcal{H}_n \;\middle|\; f(y) = 0 \text{ for all } y \in p^{-1}(\{x_{n-1}\}) \right\}$$

The same facts as above prove that $\mathcal{U}_n$ is also K^ω-invariant. We denote by λ_n the corresponding subrepresentation of the restriction of π_n to K^ω. With the notation of section 4.2, we observe that the functions f_j^i with i chosen such that $y_i \neq x_{n-1}$ constitute a basis for $\mathcal{U}_n$. Hence, the representation λ_n is $(q-1)((q+1)q^{n-2}-1)$-dimensional.

Finally, we denote for $n \geq 1$ by $\mathcal{W}_n$ the 1-dimensional subspace spanned by the K^ω-invariant function of the proof of Corollary 4.2.2 and denote by α_n the corresponding subrepresentation of the restriction of π_n to K^ω, i.e. α_n is equivalent to the trivial representation of K^ω.

We are now in a position to prove the following:

Theorem 4.4.1 *The representations α_n, σ_n and λ_n are irreducible for all n. The restriction of π_n to K^ω is irreducible for $n = 0$, equivalent to $\alpha_n \oplus \sigma_n$ for $n = 1$ and equivalent to $\alpha_n \oplus \sigma_n \oplus \lambda_n$ for $n \geq 2$.*

Proof. The irreducibility of α_n is obvious while the irreducibility of λ_n and σ_n may be proved in the same way as the irreducibility of π_n in Proposition 4.2.1 - now using both of our assumptions on K^ω.

To prove the second statement, let $n \geq 1$ and assume that $f \in \mathcal{W}_n^\perp$, the ortogonal complement of $\mathcal{W}_n$ in $\mathcal{H}_n$. We observe that

$$\sum_{y \in p^{-1}(\{x_{n-1}\}) \setminus \{x_n\}} f(y) = c f(x_n)$$

where $c = q$ for $n = 1$ and $c = q - 1$ for $n \geq 2$. The fact that $f \in \mathcal{H}_n$ means, however, that

$$\sum_{y \in p^{-1}(\{x_{n-1}\}) \setminus \{x_n\}} f(y) = -f(x_n)$$

which implies that $f(x_n) = 0$. Hence, it is easy to see that

$$\mathcal{W}_n^\perp = \{ f \in \mathcal{H}_n \mid f(x_n) = 0 \}$$

This proves that the restriction of π_1 is equivalent to $\alpha_1 \oplus \sigma_1$. For $n \geq 2$, we clearly have that $\mathcal{V}_n$ is the orthogonal complement of $\mathcal{U}_n$ in $W_n^\perp$ which implies that the restriction of π_n is equivalent to $\alpha_n \oplus \sigma_n \oplus \lambda_n$. Since the irreducibility of the restriction of π_0 is trivial, this finishes the proof. □

A pattern similar to the one observed in Theorem 4.4.1 will turn up again in Theorem 5.4.1 which deals with spherical representations in the case of a homogeneous rooted tree of countable degree. In a sharp contrast, we will in chapter 7 encounter a family of representations which at a first glance and by construction may look quite similar to the spherical representations. However, they have the very strong property that the restriction to K^ω remains irreducible.

4.5 The K-types for spherical representations of certain subgroups of $\mathrm{Aut}(\mathfrak{X})$

An important aspect in the analysis of spherical representations with respect to a Gelfand pair is the question of decomposition of the restriction to the compact subgroup of such representations into a direct sum of irreducible representations. Having discussed irreducible representations of K in this chapter, we are able to give a satisfactory answer to this question for a number of the spherical representations for the pair (G,K) found in chapter 2. We finish our discussion of the locally finite case by providing this answer. The ideas and results will be of importance in our discussion of a compactification of the boundary for a homogeneous rooted tree of countable degree.

We consider a closed subgroup G of $\mathrm{Aut}(\mathfrak{X})$ with the property that the subgroup

$$K = \{g \in G \mid g(o) = o\}$$

satisfies the assumptions of the previous sections. Since K acts transitively on Ω by Lemma 4.1.1, it follows by [FN, Lemma 4.1] that the pair (G,K) is a Gelfand pair for which the spherical representations are determined in section 2.5. We consider the Borel probability measure ν on Ω which was studied in chapter 2 and denote the corresponding L^2-space by $L^2(\Omega)$. As seen in chapter 2, the spherical representations corresponding to eigenvalues in the interval $[-\frac{2q^{\frac{1}{2}}}{q+1}, \frac{2q^{\frac{1}{2}}}{q+1}]$ may be realised on $L^2(\Omega)$ as described in section 2.5 with $z = \frac{1}{2} + it$ and $t \in \mathbb{R}$. The function P satisfies that $P(g,\tau) = 1$ for all $g \in K$ and all $\tau \in \Omega$. Hence, the restriction of each of these spherical representations to K is one common representation π on $L^2(\Omega)$ given by

$$(\pi(g)f)(\tau) = f(g^{-1} \cdot \tau)$$

for all $f \in L^2(\Omega)$ and all $\tau \in \Omega$. We will decompose this into a direct sum of irreducible representations of K.

To do this, we consider - as in chapter 2 - the spaces $\mathcal{K}(\Omega)$ and $\mathcal{K}_n(\Omega)$ with $n \geq 0$ of continuous functions on Ω taking only finitely many values. We recall that $\mathcal{K}(\Omega)$ is dense in $L^2(\Omega)$ and that $K(\Omega) = \bigcup_{n=0}^{\infty} \mathcal{K}_n(\Omega)$. The map $1_x \mapsto (q+1)^{\frac{1}{2}} q^{\frac{n-1}{2}} 1_{\Omega(x)}$ clearly extends to an isometric isomorphism T_n of $\ell^2(\mathfrak{M}_n)$ onto $\mathcal{K}_n(\Omega)$ for each $n \in \mathbb{N}$. Since $\pi(g)1_{\Omega(x)} = 1_{\Omega(g(x))}$ for all $g \in K$ and $x \in \mathfrak{X}$, this map satsifies the relation that $\pi(g)T_n = T_n\tilde{\pi}_n(g)$. Hence, we may - with some abuse of notation - regard $\tilde{\pi}_n$ and π_n as subrepresentations of π and $\mathcal{H}_n$ as a subspace of $L^2(\Omega)$ for all $n \geq 0$.

We now observe that $\mathcal{H}_0 = \mathcal{K}_0(\Omega)$. If we by $f(y)$ denote the common value on $\Omega(y)$, we observe that $\mathcal{H}_n$ consists of all functions $f \in \mathcal{K}_n(\Omega)$ with the property that

$$\sum_{y \in p^{-1}(\{x\})} f(y) = 0$$

for all $y \in \mathfrak{M}_{n-1}$. This implies that $\mathcal{H}_n$ is the orthogonal complement of $\mathcal{K}_{n-1}(\Omega)$ in $\mathcal{K}_n(\Omega)$ for all $n \in \mathbb{N}$. Hence, it follows by induction that $\mathcal{K}_n(\Omega) = \bigoplus_{k=0}^{n} \mathcal{H}_k$ for all $n \geq 0$. Since $\mathcal{K}(\Omega)$ is dense in $L^2(\Omega)$, this has as consequence that π is equivalent to $\bigoplus_{n=0}^{\infty} \pi_n$.

We have proved the following proposition:

Proposition 4.5.1 *The common restriction to K of the representations in the unitary principal series for G is the direct sum of the spherical representations for the pair (K, K^ω), i.e. $\pi = \bigoplus_{n=0}^{\infty} \pi_n$.*

We will return to this decomposition when we construct non-tame representations in chapter 7 and will consider a similar result in the case of a homogeneous rooted tree of countable degree. Here Ω will be replaced by a compactification of the boundary, and it turns out that a similar decomposition will produce a family of new representations.

Chapter 5

An Olshanski Spherical Pair Consisting of Automorphism Groups for a Homogeneous Rooted Tree

In this chapter, we construct the second of the two Olshanski spherical pairs which are the focus of this thesis and determine the spherical representations and functions for the pair. The main results are Theorem 5.3.1 and Corollary 5.3.7. The material of the chapter may be seen as an analogue to the results of chapter 3 for a homogeneous rooted tree of countable degree. We will continue the study of the pair in chapters 6-8. In section 1, we carry out the rather technical construction of the relevant groups and show that they constitute an Olshanski spherical pair. Section 2 is devoted to the construction of a family of spherical representations for the pair while the purpose of section 3 is to study spherical functions and in this way show that the constructed family of spherical representations is exhaustive. We finish off by pointing out how the results concerning our pair fit into the structure seen in relation to other Olshanski spherical pairs.

5.1 An Olshanski spherical pair consisting of automorphism groups for a rooted, homogeneous tree of infinite degree

We begin by going through the construction of the Olshanski spherical pair which we will study in this chapter. The ideas are the same as applied in the construction of the pair (G, K) in chapter 3, but the details are much more technical in this case. The reason for this might not be clear, but will be justified in chapter 7, cf. Remark 7.1.2.

Let $(\mathfrak{X}, \mathfrak{C})$ be a homogeneous tree of countable degree, and denote by d the natural metric on $\mathfrak{X}$ and by p the canonical projection of $\mathfrak{X}$. We fix a vertex $o \in \mathfrak{X}$. For each vertex $x \in \mathfrak{X}$, we number its neighbours in $p^{-1}(\{x\})$ by choosing a bijection $\tau_x : \mathbb{N} \cup \{0\} \to p^{-1}(\{x\})$. We fix these bijections $\{\tau_x\}_{x\in\mathfrak{X}}$ for the remainder of this discussion.

We choose any natural number $a \geq 3$. For each $n \geq 0$, we define $\mathfrak{X}_n \subseteq \mathfrak{X}$ by the condition that $x \in \mathfrak{X}_n$ if the chain $x_0, x_1, \ldots, x_k$ with $x_0 = o$ and $x_k = x$ satisfies that $x_1 \in \tau_o(\{0, 1, \ldots, a^{2^n} - 1\})$ and that $x_{j+1} \in \tau_{x_j}(\{0, 1, \ldots, a^{2^n} - 2\})$ for all $j \in \{1, \ldots, k-1\}$. Furthermore, we put $\mathfrak{C}_n = \{\{x, y\} \in \mathfrak{C} \mid x, y \in \mathfrak{X}_n\}$. It is now evident that $(\mathfrak{X}_n, \mathfrak{C}_n)$ is a locally finite, homogeneous subtree of $(\mathfrak{X}, \mathfrak{C})$ of degree a^{2^n}, that

$$o \in \mathfrak{X}_0 \subseteq \mathfrak{X}_1 \subseteq \ldots \subseteq \mathfrak{X}_n \subseteq \ldots$$

and that

$$\mathfrak{X} = \bigcup_{n=0}^{\infty} \mathfrak{X}_n$$

We define

$$\mathfrak{M}_k^n = \{x \in \mathfrak{X}_n \mid d(o,x) = k\}$$

and

$$\mathfrak{M}_k = \{x \in \mathfrak{X} \mid d(o,x) = k\}$$

for all $n, k \geq 0$.

For each $n \geq 0$, we will define a projection map q_n from $\mathfrak{X}_{n+1}$ onto $\mathfrak{X}_n$ with the following properties:

1. $q_n(x) = x$ for all $x \in \mathfrak{X}_n$.

2. $q_n(x) \in \mathfrak{M}_k^n$ if $x \in \mathfrak{M}_k^{n+1}$

3. If $x, y \in \mathfrak{X}_{n+1}$ satisfy that $d(x,y) = 1$, we have that $d(q_n(x), q_n(y)) = 1$.

4. For $x \in \mathfrak{M}_k^n$ with $k \geq 1$, the set $q_n^{-1}(\{x\})$ has cardinality $a^{2^n}(a^{2^n}+1)^{k-1}$.

We do this by defining q_n on $\mathfrak{M}_k^{n+1}$ and use induction in k.

As a starting point, we define q_n on $\mathfrak{M}_0^{n+1}$ by putting $q_n(o) = o$. Now let $x \in \mathfrak{M}_1^{n+1}$. We may find $l_x \in \left\{0, \ldots, a^{2^{n+1}} - 1\right\}$ such that $x = \tau_o(l_x)$. We write $l_x = m_x a^{2^n} + r_x$ with $m_x, r_x \in \left\{0, \ldots, a^{2^n} - 1\right\}$ and observe that $\tau_o(r_x) \in \mathfrak{X}_n$. We define $q_n(x) = \tau_o(r_x)$.

We now assume that q_n has been defined on $\mathfrak{M}_k^{n+1}$ for some $k \geq 1$. Let $x \in \mathfrak{M}_{k+1}^{n+1}$. We may find $l_x \in \left\{0, \ldots, a^{2^{n+1}} - 2\right\}$ such that $x = \tau_{p(x)}(l_x)$. We write $l_x = m_x(a^{2^n} - 1) + r_x$ with $m_x \in \left\{0, \ldots, a^{2^n}\right\}$ and $r_x \in \left\{0, \ldots, a^{2^n} - 2\right\}$ and observe that the fact that $q_n(p(x)) \in \mathfrak{X}_n$ implies that $\tau_{q_n(p(x))}(r_x) \in \mathfrak{X}_n$. We define $q_n(x) = \tau_{q_n(p(x))}(r_x)$.

It follows immediately by the definition and induction that q_n has properties 1. and 2. above while property 3. is built into the construction. To see that property 4. is satisfied, we consider $x \in \mathfrak{M}_k^n$ for $k \geq 1$ and use induction in k. If $k = 1$, it follows by the definition that $q_n^{-1}(\{x\})$ has cardinality a^{2^n}. Now let $k \geq 2$ and assume that the claim is true for all vertices in $\mathfrak{M}_{k-1}^n$. This assumption implies that $q_n^{-1}(p(x))$ has cardinality $a^{2^n}(a^{2^n}+1)^{k-2}$. It is clear by the definition that $q_n^{-1}(\{x\})$ contains exactly $(a^{2^n}+1)$ vertices for every vertex in $q_n^{-1}(p(x))$ and so has cardinality $a^{2^n}(a^{2^n}+1)^{k-1}$. This proves that q_n has property 4.

We denote by K_∞ and K_n the groups of automorphisms of $\mathfrak{X}$ and $\mathfrak{X}_n$ as rooted trees, respectively. Endowed with the compact-open topology, all are Hausdorff topological groups, and the K_n's are even compact groups.

For each n, we embed K_n in K_∞ in the following way: Let $g \in K_n$. We want to extend g to an automorphism $\tilde{g}$ of $\mathfrak{X}$ such that $\tilde{g}(\mathfrak{X}_k) = \mathfrak{X}_k$ and $q_k \circ \tilde{g} = \tilde{g} \circ q_k$ on $\mathfrak{X}_{k+1}$ for $k \geq n$. To do this, it suffices to extend g to an automorphism g_1 of $\mathfrak{X}_{n+1}$ such that $q_n \circ g_1 = g_1 \circ q_n$ since an inductive application of this procedure produces an extension $\tilde{g} \in K_\infty$ with the required properties. We will construct g_1 by extending g from $\mathfrak{M}_k^n$ to $\mathfrak{M}_k^{n+1}$ and use induction in k.

To do this, we begin by observing that $\mathfrak{M}_0^n = \mathfrak{M}_0^{n+1}$, and so the extension in the induction start is trivial. Let $x \in \mathfrak{M}_1^{n+1}$ and find $l_x \in \left\{0, a^{2^{n+1}} - 1\right\}$ such that $x = \tau_o(l_x)$. We write $l_x = m_x a^{2^n} + r_x$ with $m_x, r_x \in \{0, \ldots, a^{2^n} - 1\}$. Similarly, we find $s_x \in \{0, \ldots, a^{2^n} - 1\}$ such that $g(q_n(x)) = g(\tau_o(r_x)) = \tau_o(s_x)$. We now define $g_1(x) = \tau_o(m_x a^{2^n} + s_x)$. It is clear that this definition gives a bijection from $\mathfrak{M}_1^{n+1}$ onto $\mathfrak{M}_1^{n+1}$, and we observe that $q_n(g_1(x)) = q_n(\tau_o(m_x a^{2^n} + s_x)) = \tau_o(s_x) = g(q_n(x))$. If $x \in \mathfrak{X}_n$, we see that $m_x = 0$ and so $g_1(x) = \tau_o(s_x) = g(x)$ which means that g_1 extends g.

Now let $k \geq 1$, and assume that g_1 has been defined as an extension of g on $\mathfrak{M}_k^{n+1}$ which is a bijection from $\mathfrak{M}_k^{n+1}$ onto $\mathfrak{M}_k^{n+1}$ and which satisfies that $q_n(g_1(y)) = g_1(q_n(y))$ for all $y \in \mathfrak{M}_k^{n+1}$. Let $x \in \mathfrak{M}_{k+1}^{n+1}$, and find $l_x \in \left\{0, \ldots, a^{2^{n+1}} - 2\right\}$ such that $x = \tau_{p(x)}(l_x)$. We write $l_x = m_x(a^{2^n} - 1) + r_x$ with $m_x \in \{0, \ldots, a^{2^n} + 1\}$ and $r_x \in \{0, \ldots, a^{2^n} - 2\}$. Similarly, we find $s_x \in \{0, \ldots, a^{2^n} - 2\}$ such that $g(q_n(x)) = g(\tau_{q_n(p(x))}(r_x)) = \tau_{g(q_n(p(x)))}(s_x)$. We now define $g_1(x) = \tau_{g_1(p(x))}(m_x(a^{2^n} - 1) + s_x)$. By the induction hypothesis, it is clear that this definition gives a bijection from $\mathfrak{M}_{k+1}^{n+1}$ onto $\mathfrak{M}_{k+1}^{n+1}$, and we observe that $q_n(g_1(x)) = q_n(\tau_{g_1(p(x))}(m_x(a^{2^n} - 1) + s_x)) = \tau_{q_n(g_1(p(x)))}(s_x) = \tau_{g(q_n(p(x)))}(s_x) = g(q_n(x))$. If $x \in \mathfrak{X}_n$, we see that $m_x = 0$ and so $g_1(x) = \tau_{g_1(p(x))}(s_x) = \tau_{g(q_n(p(x)))}(s_x) = g(x)$ which means that g_1 extends g.

By induction, we get a bijection g_1 from $\mathfrak{X}_{n+1}$ onto $\mathfrak{X}_{n+1}$ which extends g and which satisfies that $g_1 \circ q_n = q_n \circ g_1$. It is an immediate consequence of the construction that g_1 is actually an automorphism of $\mathfrak{X}_{n+1}$.

By an inductive application of this procedure, g extends to an automorphism $\tilde{g}$ of $\mathfrak{X}$. We will refer to $\tilde{g}$ as the standard extension of g.

Fortunately, the algebraic and topological structures are preserved under the map $g \mapsto \tilde{g}$:

Lemma 5.1.1 *For each n, the map $\psi_n : K_n \to K_\infty$, $g \mapsto \tilde{g}$, is an injective group homomorphism from K_n into K_∞ which is a homeomorphism onto its image.*

Proof. Let $g, h \in K_n$. By the inductive definition of $\tilde{g}$, $\tilde{h}$ and $\widetilde{gh}$, the relation $\widetilde{gh} = \tilde{g}\tilde{h}$ is a consequence of the relation $(gh)_1 = g_1 h_1$. Hence, we will establish the latter.

Let $x \in \mathfrak{X}_{n+1}$ and find $k \geq 0$ such that $x \in \mathfrak{M}_k^{n+1}$. We will prove that $(gh)_1(x) = g_1(h_1(x))$ by induction in k. If $k = 0$, the relation is trivial. Assume that $k = 1$, and let m_x be given as in the definition of $g_1(x)$ above. We let $s_x \in \{0, \ldots, a^{2^n} - 1\}$ be given such that $(gh)(q_n(x)) = \tau_o(s_x)$. By definition, we have that $(gh)_1(x) = \tau_o(m_x a^{2^n} + s_x)$. If we let $t_x \in \{0, \ldots, a^{2^n} - 1\}$ be given such that $h(q_n(x)) = \tau_o(t_x)$, we similarly have that $h_1(x) = \tau_o(m_x a^{2^n} + t_x)$. Since $q_n(h_1(x)) = h(q_n(x))$, we observe that $g(q_n(h_1(x))) = (gh)(q_n(x)) = \tau_o(s_x)$ which by definition of g_1 implies that $g_1(h_1(x)) = \tau_o(m_x a^{2^n} + s_x) = (gh)_1(x)$.

Now let $k \geq 1$, and assume that $(gh)_1(y) = g_1(h_1(y))$ for all $y \in \mathfrak{M}_k^{n+1}$. Let $x \in \mathfrak{M}_{k+1}^{n+1}$, and let m_x be given as in the definition of $g_1(x)$ above. We let $s_x \in \{0, \ldots, a^{2^n} - 2\}$ be given such that $(gh)(q_n(x)) = \tau_{(gh)(q_n(p(x)))}(s_x)$. By definition, we have that $(gh)_1(x) = \tau_{(gh)_1(p(x)}(m_x(a^{2^n} - 1) + s_x)$. If we let $t_x \in \{0, \ldots, a^{2^n} - 2\}$ be given such that $h(q_n(x)) = \tau_{h(q_n(p(x)))}(t_x)$, we similarly have that $h_1(x) = \tau_{h_1(p(x))}(m_x(a^{2^n} - 1) + t_x)$. As above, we observe that $g(q_n(h_1(x))) = (gh)(q_n(x)) = \tau_{(gh)(q_n(p(x)))}(s_x) = \tau_{g(q_n(p(h_1(x))))}(s_x)$, and since $g_1(p(h_1(x))) = g_1(h_1(p(x)))$, the definition of g_1 shows that $g_1(h_1(x)) = \tau_{g_1(h_1(p(x)))}(m_x(a^{2^n} - 1) + s_x)$. The induction hypothesis now implies that $(gh)_1(x) = g_1(h_1(x))$.

This shows that ψ_n is a group homomorphism. Since $\tilde{g}$ is an extension of g for all $g \in K_n$, the kernel obviously only consists of the identity and so ψ_n is injective.

To prove the continuity, we let $F \subseteq \mathfrak{X}$ be a finite set and let $g \in K_n$. Put $U_F(\tilde{g}) = \{f \in K_\infty \mid f(x) = \tilde{g}(x) \text{ for all } x \in F\}$. We choose $k \geq 0$ such that $d(o, x) \leq k$ for all $x \in F$, and put $F' = \{x \in \mathfrak{X} \mid d(o,x) \leq k\}$. If $h \in V_{F'}(g) = \{f \in K_n \mid f(x) = g(x) \text{ for all } x \in F'\}$, it is an immediate consequence of the (in two steps) inductive definition of $\tilde{h}, \tilde{g}$ that $\tilde{h} \in U_F(\tilde{g})$. This proves that the group homomorphism ψ_n is continuous.

Finally, let $F \subseteq \mathfrak{X}_n$ be finite and $g \in K_n$. If $h \in K_n$ satisfies that $\tilde{h} \in U_F(\tilde{g})$, it is of course also true that $h \in V_F(g)$. This shows that φ_n is a homeomorphism onto its image. □

For the remainder of this discussion, we will identify K_n with its image under ψ_n and so regard it as a subgroup of K_∞. Accordingly, we will not distinguish between g and its standard extension for $g \in K_n$.

We immediately observe that the inductive construction of $\tilde{g}$ implies that

$$K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n \subseteq \ldots$$

We denote by Ω_n the boundary of the tree $(\mathfrak{X}_n, \mathfrak{C}_n)$ and by Ω_∞ the boundary of the tree $(\mathfrak{X}, \mathfrak{C})$. We may regard Ω_n as a subset of Ω_∞ by identifying $\omega \in \Omega_n$ with a given rooted chain with the point in Ω_∞ corresponding to the same rooted chain. Under this identification, we have that

$$\Omega_0 \subseteq \Omega_1 \subseteq \ldots \subseteq \Omega_n \subseteq \ldots$$

We define $\Omega = \bigcup_{n=0}^\infty \Omega_n$ and observe that it is an immediate consequence of the definition of the topology on Ω_∞ that Ω is dense in Ω_∞. We denote by δ the metric on Ω_∞ as defined in section 4.1.

We now fix a point $\omega \in \Omega_0$, and denote by $x_0, x_1, \ldots$ the corresponding rooted chain in $(\mathfrak{X}_0, \mathfrak{C}_0)$. This boundary point will be kept fixed for the remainder of this chapter. For all $n \geq 0$, we define $K_n^\omega = \{g \in K_n \mid g \cdot \omega = \omega\}$ which is a compact subgroup of K_n. Furthermore, we put $K_\infty^\omega = \{g \in K_\infty \mid g \cdot \omega = \omega\}$. Consider

$$K = \bigcup_{n=0}^{\infty} K_n \subseteq K_\infty, \;\; K^\omega = \bigcup_{n=0}^{\infty} K_n^\omega \subseteq K_\infty^\omega$$

K is obviously a subgroup of K_∞ and consists of all standard extensions of automorphisms of the considered subtrees. Similarly, K^ω is a subgroup of K_∞^ω, and $K^\omega = \{g \in K \mid g \cdot \omega = \omega\}$. We equip K with the inductive limit topology.

The following observation is immediate, but important:

PROPOSITION 5.1.2 *(K, K^ω) is an Olshanski spherical pair, and the inductive limit topology on K is stronger than the topology inherited from K_∞.*

PROOF. $\{K_n\}$ is an increasing sequence of compact groups, and since all K_n's have the induced topology from K_∞, the topology on K_n is the one induced by K_{n+1}.

We want to prove that K_n is a closed subgroup of K_{n+1}. Let $\{g_\lambda\} \subseteq K_n$ be a net which converges to $g \in K_{n+1}$. If $x \in \mathfrak{X}_n$, it follows by the definition of the topology on K_{n+1} that there exists λ such that $g(x) = g_\lambda(x) \in \mathfrak{X}_n$. Since $\mathfrak{X}_n$ is locally finite and homogeneous, this shows that $g(\mathfrak{X}_n) = \mathfrak{X}_n$, and so the restriction h of g to $\mathfrak{X}_n$ is an automorphism h of $\mathfrak{X}_n$ which fixes o. To show that $g \in K_n$, we must prove that g is the standard extension $\tilde{h}$ of this restriction.

To see this, we observe that a consequence of the definition of the topology on K_n is that for each $k \geq 0$ there exists λ such that $g(x) = g_\lambda(x)$ for all $x \in \mathfrak{X}_{n+1}$ with $d(o,x) \leq k$. Hence, $h(x) = g_\lambda(x)$ for all $x \in \mathfrak{X}_n$ with $d(o,x) \leq k$, and so it follows by definition of the standard extension that $h_1(x) = g_\lambda(x) = g(x)$ for all $x \in \mathfrak{X}_{n+1}$ with $d(o,x) \leq k$. Since this holds for all k, the restriction of g to $\mathfrak{X}_{n+1}$ coincides with h_1 and so the identity $g = \tilde{h}$ follows by the fact that $g \in K_{n+1}$. This shows that K_n is a closed subgroup of K_{n+1}.

We have seen that (K_n, K_n^ω) is a Gelfand pair for all n since K_n satisfies the assumptions of chapter 4. The identity $K_n^\omega = K_n \cap K_{n+1}^\omega$ is obvious, and so (K, K^ω) is an Olshanski spherical pair.

The last claim in the proposition is an immediate consequence of the definition of the inductive limit topology and the fact that the topology on K_n is the one inherited from K_∞. □

Remark 5.1.3 The inductive limit topology on K is actually strictly stronger than the one inherited from K_∞. To see this, define

$$U = \{g \in K \mid g(x) = x \text{ for all } x \in \mathfrak{M}_1\}$$

It is an immediate consequence of the definition of the standard extension that

$$U \cap K_n = \{g \in K_n \mid g(x) = x \text{ for all } x \in \mathfrak{M}_1^n\}$$

for all $n \geq 0$, and this set is of course open in K_n. Hence, U is open in the inductive limit topology.

However, U is not open in the topology inherited from K_∞. Indeed, if $F \subseteq \mathfrak{X}$ is finite, there exists $n \geq 0$ such that $F \subseteq \mathfrak{X}_n$, and so we can find $g_0 \in K_{n+1}$ such that $g_0(x) = x$ for all $x \in F$ and such that $g_0(x) \neq x$ for some $x \in \mathfrak{M}_1^{n+1}$ (g_0 may be chosen such that it fixes all vertices in $\mathfrak{X}_n$ and interchanges two vertices in $\mathfrak{M}_1^{n+1} \backslash \mathfrak{M}_1^n$). Hence, $\{g \in K \mid g(x) = x \text{ for all } x \in F\} \nsubseteq U$ which shows that U is not open in the topology inherited from K_∞.

Remark 5.1.4 The subgroup K is actually dense in K_∞. Indeed, let $g \in K_\infty$ and $F \subseteq \mathfrak{X}$ be a finite set. Choose $N \geq 0$ such that $F \cup g(F) \subseteq \mathfrak{X}_N$. It is now easy to see - for instance by induction in the number of elements in F - that there exists $h \in K_N$ such that $h(x) = g(x)$ for all $x \in F$. This proves that K is dense in K_∞.

Remark 5.1.5 In light of Remark 3.1.5, one should observe that the analysis of the group K is comparable to the situation for the inifinite symmetric group. Here K plays the role of the group $S(\infty)$ of all finite permutations of $\mathbb{N}$ while K_∞ may be seen as an analogue of $\tilde{S}(\infty)$, the group of all permutations of $\mathbb{N}$. This similarity is discussed further in Remark 3.1.5.

In this chapter, we will be concerned only with the group K. The relation between the groups K and K_∞ will be the subject of chapter 7.

We finish this section by observing that each K_n satisfies the conditions of chapter 4. Hence, K_n acts transitively on Ω_n according to Lemma 4.1.1. This implies that K acts transitively on Ω. Since K^ω is just the isotropy group of ω, we may as in chapter 4 identify each K^ω-right-invariant function on K with a function on Ω.

Furthermore, we define $A_n = \left\{\tau \in \Omega \,\middle|\, \delta(\omega, \tau) = \frac{1}{n}\right\}$ and $B_n = \left\{\tau \in \Omega \,\middle|\, \delta(\omega, \tau) < \frac{1}{n}\right\}$ for $n \geq 1$, and we put $A_\infty = \{\omega\}$. For $n \geq 1$ and $j \geq 0$ we define A_n^j and B_n^j to be the corresponding sets with Ω replaced by Ω_j. We observe that $A_n = \bigcup_{j=0}^\infty A_n^j$ and $B_n = \bigcup_{j=0}^\infty B_n^j$. Since the orbits of the action of K_j^ω on Ω_j by Lemma 4.1.1 are A_n^j with $n \geq 1$ and $n = \infty$, the orbits of the action of K^ω on Ω are the sets A_n with $n \geq 1$ and $n = \infty$. Hence, a K^ω-biinvariant function φ on K may be regarded as a function $\tilde{\varphi}$ on Ω which is constant on the sets A_n with $n \geq 1$ and $n = \infty$.

With the notation firmly established, we will turn our attention to spherical representations for the pair (K, K^∞).

5.2 Spherical representations for the pair (K, K^ω)

In this section, we will construct a family $\{\pi_n\}_{n=0}^{\infty}$ of spherical representations for the pair (K, K^ω).

To do this, we let π_0 be the one-dimensional trivial representation which is of course spherical. For the remaining representations, let $n \geq 1$ be fixed, and consider $\ell^2(\mathfrak{M}_n)$ which is an infinite dimensional Hilbert space. We define $(\pi_n(g)f)(x) = f(g^{-1}(x))$ for $x \in \mathfrak{M}_n$, $f \in \ell^2(\mathfrak{M}_n)$ and $g \in K$. It is trivial to check that this is a unitary representation of K (for the continuity of the map $g \mapsto \pi(g)f$ with $f \in \ell^2(\mathfrak{M}_n)$, it suffices to prove continuity of the restriction of this map to K_m for each m, and by definition of the standard extension this restriction is constant on the set $\{g \in K_m \mid g(x) = g_0(x) \text{ for all } x \in \mathfrak{M}_n^m\}$ which is open in K_m).

Remark 5.2.1 The representations π_n are actually also continuous in the strong operator topology if we equip K with the induced topology from K_∞. Indeed, this is clearly true for π_0. For $n \geq 1$, let $x \in \mathfrak{M}_n$ and denote by $f_x \in \ell^2(\mathfrak{M}_n)$ the function given by the conditions that $f_x(x) = 1$ and $f_x(y) = 0$ for $y \neq x$. For $g_0 \in K$, the map $g \mapsto \pi(g)f_x$ is constant on the open set $\{g \in K \mid g(x) = g_0(x)\}$ which implies that it is continuous. Since $\{f_x\}_{x\in\mathfrak{M}_n}$ is an othonormal basis for $\ell^2(\mathfrak{M}_n)$, this implies the continuity of the map $g \mapsto \pi(g)f$ for all $f \in \ell^2(\mathfrak{M}_n)$.

A key observation is that the representation π_n is irreducible for all $n \geq 0$. This is the content of Proposition 5.2.3 below. For the proof, we need the following important lemma which will be used repeatedly throughout the remaining chapters:

Lemma 5.2.2 *For $n \geq 0$, let G_n be a locally compact group, and assume that*

$$G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n \subseteq \ldots$$

Define $G = \bigcup_{n=0}^{\infty} G_n$, and endow G with the inductive limit topology. Let π and ρ be unitary representations of G with representation spaces $\mathcal{H}^\pi$ and $\mathcal{H}^\rho$, respectively. Assume that there exist increasing sequences $\{\mathcal{H}_n^\pi\}$ and $\{\mathcal{H}_n^\rho\}$ of closed, non-zero subspaces of $\mathcal{H}^\pi$ and $\mathcal{H}^\rho$, respectively, such that $\mathcal{H}_n^\pi$ and $\mathcal{H}_n^\rho$ are G_n-invariant for all $n \geq 0$, and such that the subrepresentations π_n and ρ_n of the G_n-restrictions of π and ρ corresponding to $\mathcal{H}_n^\pi$ and $\mathcal{H}_n^\rho$ are irreducible. Furthermore, assume that $\bigcup_{n=0}^{\infty} \mathcal{H}_n^\pi$ and $\bigcup_{n=0}^{\infty} \mathcal{H}_n^\rho$ are both dense in $\mathcal{H}^\pi$ and $\mathcal{H}^\rho$, respectively. Then the following holds:

1. *π and ρ are irreducible unitary representations of G.*

2. *If π_n and ρ_n are non-equivalent for all $n \geq 0$, π and ρ are non-equivalent.*

Proof. To prove 1., we consider $T \in \mathcal{B}(\mathcal{H}^\pi)$, the algebra of bounded operators on $\mathcal{H}^\pi$, and assume that $T\pi(g) = \pi(g)T$ for all $g \in G$. Denote by P_n^π the orthogonal projection onto $\mathcal{H}_n^\pi$, and consider the operator T_n which is the restriction of $P_n^\pi T$ to $\mathcal{H}_n^\pi$. Then $T_n \in \mathcal{B}(\mathcal{H}_n^\pi)$, and since $\mathcal{H}_n^\pi$ is invariant under G_n, we have that $\pi(g)T_n = T_n\pi(g)$ for all $g \in G_n$. By irreducibility of π_n and Schur's Lemma, this implies that $T_n = \lambda_n I$ for some $\lambda_n \in \mathbb{C}$.

For $h \in \mathcal{H}_n^\pi$ with $h \neq 0$, this means that $P_{n+1}Th = \lambda_{n+1}h$. Applying P_n on both sides yields the equality $\lambda_n h = \lambda_{n+1}h$, and so there exists $\lambda \in \mathbb{C}$ such that $T_n = \lambda I$ for all $n \geq 0$. With $h \in \mathcal{H}_m^\pi$ for some $m \geq 0$, this implies that $P_nTh = \lambda h$ for all $n \geq m$. By denseness of $\bigcup_{n=0}^{\infty} \mathcal{H}_n^\pi$, P^j converges strongly to I which implies that $Th = \lambda h$. Another application of the denseness means that $T = \lambda I$ which by Schur's Lemma implies that π is irreducible.

To prove 2., we assume that π_n and ρ_n are non-equivalent for all $n \geq 0$. Let T be a bounded operator from $\mathcal{H}^\pi$ into $\mathcal{H}^\rho$ such that $T\pi(g) = \rho(g)T$ for all $g \in G$. Let $n \geq 0$. Denote by Q_n the orthogonal projection onto the closed subspace $\mathcal{H}_n^\rho$, and define T_n to be the restricion of $Q_n T$ to the subspace $\mathcal{H}_n^\pi$. We observe that T_n is a bounded operator from $\mathcal{H}_n^\pi$ into $\mathcal{H}_n^\rho$, and since $\mathcal{H}_n^\rho$ is G_n-invariant, we see that $\rho(g)T_n = T_n\pi(g)$ for all $g \in G_n$. By Schur's Lemma and irreducibility of π_n and ρ_n, this implies that $T_n = 0$.

Let $h \in \mathfrak{H}_m^\pi$ for some $m \geq 0$. We have seen that $Q_n Th = 0$ for all $n \geq 0$. By denseness of $\bigcup_{n=0}^\infty \mathcal{H}_n^\rho$, the projections Q_n converge strongly to the identity on $\mathcal{H}_\rho$. This implies that $Th = 0$. The denseness of $\bigcup_{n=0}^\infty \mathcal{H}_n^\pi$ now reveals that $T = 0$, and so π and ρ are non-equivalent. □

We are now able to prove irreducibility of the representations π_n with $n \geq 0$.

Proposition 5.2.3 *The representation π_n is an irreducible representation of K for all n.*

Proof. The representation π_0 is trivially irreducible.

Let $n \geq 1$. Denote by π_n^j the n'th spherical representation for the pair (K_j, K_j^ω) from Corollary 4.3.2 and denote by $\mathcal{H}_n^j$ the corresponding representation space which is a subspace of $l^2(\mathfrak{M}_n^j)$. We may embed $\mathcal{H}_n^j$ isometrically into $\ell^2(\mathfrak{M}_n)$ by defining $f(x) = 0$ for all $x \in \mathfrak{M}_n \setminus \mathfrak{M}_n^j$ and all $f \in \mathcal{H}_n^j$. If we denote this embedding by S_j, we clearly have that $\pi_n(g)S_j = S_j\pi_n^j(g)$ for all $g \in K_j$, and so we may regard $\mathcal{H}_n^j$ as a closed subspace of $\ell^2(\mathfrak{M}_n)$ and π_n^j as a subrepresentation of the restriction of π_n to K_j.

We clearly have that
$$\mathcal{H}_n^0 \subseteq \mathcal{H}_n^1 \subseteq \ldots \subseteq \mathcal{H}_n^j \subseteq \ldots$$

Furthermore, we observe that $\bigcup_{j=0}^\infty \mathcal{H}_n^j$ is dense in $\ell^2(\mathfrak{M}_n)$. Indeed, let $x \in \mathfrak{M}_n$ and find $N \geq 0$ such that $x \in \mathfrak{M}_n^N$. Define for all $j \geq 0$ $m_j = a^{2^j} - 1$ if $n = 1$ and $m_j = a^{2^j} - 2$ if $n \geq 2$. For $j \geq N$, consider the function $f_x^j \in \mathcal{H}_n^j$ given by the conditions that $f_x^j(x) = 1$, that $f_x^j(y) = -\frac{1}{m_j}$ for all $y \in (p^{-1}(\{p(x)\}) \cap \mathfrak{M}_n^j) \setminus \{x\}$ and that $f_x^j(y) = 0$ for $y \in \mathfrak{M}_n^j$ with $p(y) \neq p(x)$. Finally, denote by f_x the indicator function for the set $\{x\}$. We clearly have that
$$\|f_x - f_x^j\| = \sum_{y \in (p^{-1}(\{p(x)\}) \cap \mathfrak{M}_n^j) \setminus \{x\}} \frac{1}{m_j^2} = \frac{1}{m_j}$$

Since $m_j \to \infty$ as $j \to \infty$, and since $\{f_x\}_{x \in \mathfrak{M}_n}$ constitute an orthonormal basis for $\ell^2(\mathfrak{M}_n)$, this proves that $\bigcup_{j=0}^\infty \mathcal{H}_n^j$ is dense in $\ell^2(\mathfrak{M}_n)$.

The irreducibility of π_n is now an immediate consequence of 1. in Lemma 5.2.2. □

Remark 5.2.4 The representation π_n with $n \geq 1$ is the natural representation arising from the action of K on $\mathfrak{M}_n$. By Proposition 5.2.4, π_n is irreducible. This is in sharp contrast to the case of a locally finite, homogeneous tree for which the natural representation arising from the same action was not irreducible, cf. section 4.2. To get irreducibility and so the spherical representation for the given pair, we had to pass to a certain subspace by imposing an extra condition on the considered functions in $\ell^2(\mathfrak{M}_n)$.

This phenomenon is not unique and turns up on a number of occassions. Another example will be considered in section 7.2 where the representation arising from the natural action of the group G_∞ (see chapter 3) on the set of oriented edges is treated. Again the representation is irreducible in the case of a homogeneous tree of countable degree while this not the case if the tree is locally finite.

An immediate consequence of Proposition 5.2.3 is that the representation π_n is spherical for the pair (K, K^ω) for all $n \geq 0$:

Corollary 5.2.5 *The representation π_n is a spherical representation for the pair (K, K^ω) for all n.*

Proof. Let $n \geq 1$. Consider the indicator function $f_{x_n} \in \ell^2(\mathfrak{M}_n)$ of the set $\{x_n\}$. Since $g^{-1}(x_n) = x_n$ for all $g \in K^\omega$, we have that $\pi_n(g)f_{x_n} = f_{x_n}$ for all $g \in K^\omega$. Since π_n is irreducible by Proposition 5.2.3, it is spherical for the pair (K, K^ω). Since π_0 is just the trivial representation which is clearly spherical for (K, K^ω), this finishes the proof. □

In light of chapter 1, it is of interest to determine the positive definite spherical functions corresponding to the spherical representations π_n, $n \geq 0$. We will as in section 1 define

$$K_x = \{g \in K \mid g(x) = x\}$$

We begin by observing that the function φ_0 constantly equal to 1 is the spherical function corresponding to π_0. Now let $n \geq 1$. As seen in the proof of Corollary 5.2.5, a K^ω-invariant unit vector in $\ell^2(\mathfrak{M}_n)$ is the indicator function f_{x_n} for the set $\{x_n\}$. Hence, the positive definite spherical function φ_n corresponding to π_n is given by

$$\varphi_n(g) = \langle \pi_n(g) f_{x_n}, f_{x_n} \rangle = 1_{K_{x_n}}$$

The corresponding function $\tilde{\varphi_n}$ on Ω is given by

$$\tilde{\varphi_n} = 1_{B_n}$$

In the next section, we will prove that this list of spherical functions for the pair (K, K^ω) is actually exhaustive.

5.3 The spherical functions for (K, K^ω)

It turns out that $\{\pi_n\}_{n=0}^\infty$ is the set of all spherical representations for the pair (K, K^ω). To see this, we will find all the spherical functions for (K, K^ω).

Theorem 5.3.1 *The spherical functions for the pair (K, K^ω) are the functions φ_n, $n \geq 0$, where*

$$\varphi_n = 1_{K_{x_n}}$$

The spherical functions are all positive definite.

Proof. We have already seen that the functions φ_n with $n \geq 0$ are all spherical and positive definite. Hence, it is left to show that there are no more spherical functions.

To do this, let φ be a spherical function, and let $\tilde{\varphi}$ be the corresponding function on Ω which is constant on the sets A_n with $n \in \mathbb{N}$ and $n = \infty$. To simplify things, we will as in the proof of Theorem 4.3.1 abuse the notation and denote the value of $\tilde{\varphi}$ on A_n by $\tilde{\varphi}(n)$. Since $\varphi \neq 0$, there exists a minimal $n \in \mathbb{N} \cup \{\infty\}$ such that $\tilde{\varphi}(n) \neq 0$.

We begin by observing that $n \neq \infty$. Indeed, if this was the case φ would just - since every spherical function is normalized at the neutral element - be the indicator function for the subgroup K^ω. Hence, the continuity of φ implies that K^ω is open in the inductive limit topology. However, $K^\omega \cap K_n = K_n^\omega$ for all $n \geq 0$, and this set is by the proof of Theorem 4.3.1 not open in K_n. This contradicts the fact that K^ω is open in the inductive limit topology. Hence, $n \neq \infty$.

Now let $m > n$, and choose by the transitive action of K on Ω automorphisms $g, h \in K$ such that $g \cdot \omega \in A_n$ and $h \cdot \omega \in A_m$. Since φ is spherical, we have that

$$\tilde{\varphi}(m)\tilde{\varphi}(n) = \lim_{j\to\infty} \int_{K_j^\omega} \varphi(hkg)\, \alpha_j(dk) = \lim_{j\to\infty} \int_{K_j^\omega} \tilde{\varphi}(n)\, \alpha_j(dk) = \tilde{\varphi}(n)$$

where we by α_j have denoted the normalized Haar measure on K_j and where we have used that $(hkg) \cdot \omega \in A_n$ for all $k \in K^\omega$. This shows that $\tilde{\varphi}(m) = 1$ for $m > n$.

Now choose $N \geq 0$ such that $g \in K_N$ and let $j \geq N$. It is clear that $(gkg) \cdot \omega \in A_n$ for all $k \in K_j^\omega$ such that $(kg)(x_n) \neq g^{-1}(x_n)$ and that $(gkg) \cdot \omega \in B_n$ for all $k \in K_j^\omega$ such that $(kg)(x_n) = g^{-1}(x_n)$. Since K_j^ω acts transitively on $p^{-1}(\{x_{n-1}\}) \setminus \{x_n\}$, it follows by the translation invariance of α_j that $(gkg) \cdot \omega \in A_n$ on a subset of K_j^ω of measure $\frac{a^{2^j}-3}{a^{2^j}-2}$ if $n \geq 2$ and of measure $\frac{a^{2^j}-2}{a^{2^j}-1}$ if $n = 1$ and that $(gkg) \cdot \omega \in B_1$ on a subset of measure $\frac{1}{a^{2^j}-2}$ if $n \geq 2$ and of measure $\frac{1}{a^{2^j}-1}$ if $n \geq 1$. By the previous observation that $\tilde{\varphi}(m) = 1$ for $m > n$, this implies that

$$(\tilde{\varphi}(n))^2 = \lim_{j\to\infty} \int_{K_j^\omega} \varphi(gkg)\, \alpha_j(dk) = \tilde{\varphi}(n)$$

Since $\tilde{\varphi}(n) \neq 0$, this means that $\tilde{\varphi}(n) = 1$. This proves that $\varphi = \varphi_{n-1}$. $\square$

Remark 5.3.2 In light of Remarks 1.3.5 and 3.2.4, we once again observe that the spherical functions for the pair (K, K^ω) are all uniform limits on compact sets of spherical functions for the pairs (K_n, K_n^ω). Hence, this example - like the example of chapter 3 and the matrix groups in [O1] - suggests that the conjecture of this being the case for all Olshanski spherical pairs is true. As pointed out in Remark 1.3.5, it is, however, still an open problem to prove this conjecture. An investigation of the validity of this conjecture should form an important part of a further development of the abstract theory of Olshanski spherical pairs. With the rich source of examples now available, the foundation for such a general theory is constantly increasing. We refer to Remarks 1.3.5 and 3.2.4 for a further discussion of the question.

One should observe that all spherical functions for the pair (K, K^ω) are positive definite. For an Olshanski spherical pair, this is in general not true, cf. the example of chapter 3. It is not even true for Gelfand pairs, cf. the example of chapter 2. For a Gelfand pair consisting of compact groups, it is, however, a general fact, cf. [VD, Theorem 6.5.1]. The Olshanski spherical pair (K, K^ω) arises from Gelfand pairs of this nature. If the above conjecture is true, it will be an immediate consequence that all spherical functions for an Olshanski spherical pair arising from Gelfand pairs consisting of compact groups will be positive definite. After all, they will be pointwise limits of positive definite functions as in the case considered here.

Remark 5.3.3 One should observe that for the pairs (G, K) and (K, K^ω) it is true that the product of spherical functions is again a spherical function. A similar property has been observed in a rich number of examples arising from classical matrix groups, cf. [O1]. A general theorem concerning this observation has not been established, but the rich source of examples suggest that something general must be going on.

It should be observed that a Gelfand pair is also an Olshanski spherical pair. The stability under multiplication of the set of spherical functions is, however, not a feature shared by all Gelfand pairs - just consider the Gelfand pairs of chapters 2 and 4. Hence, we can not expect to prove a general theorem for all Olshanski spherical pairs. As soon as we consider a "‘real"’ limit of Gelfand pairs, the multiplicative property seems to come into play. The search for a further assumption on an Olshanski spherical pair which makes the multiplicative property inevitable should play an important role in the further development of the abstract theory of Olshanski spherical pairs.

In light of Remark 3.2.5 concerning monoid structures of double coset spaces, we make the following observation. It is an easy consequence of the transitive action of K^ω on A_n that for all $g \in K$ with $g \cdot \omega \in A_n$

$$K^\omega g K^\omega = \{h \in K \mid h \cdot \omega \in A_n\}$$

By this observation, we may identify $K^\omega \backslash K / K^\omega$ with $\mathbb{N} \cup \{\infty\}$ which is a commutative semigroup under the operation $\cdot$ given by

$$n \cdot m = \min\{n, m\}$$

for all $n, m \in \mathbb{N} \cup \{\infty\}$. Here, ∞ is a neutral element, and so this semigroup is a monoid.

In light of the above characterization of the spherical functions for the pair (K, K^ω), we have the following corollary:

Corollary 5.3.4 *Let φ be a non-zero, continuous, K^ω-biinvariant function on K. Then φ is spherical if and only if the corresponding function on $K^\omega \backslash K / K^\omega$ is multiplicative with respect to the monoid structure introduced above.*

Remark 5.3.5 In Remark 3.2.5, we noticed that for the pair (G, K) a commutative monoid structure may also be imposed on $K \backslash G / K$ in a way such that a non-zero, continuous, K-biinvariant function is spherical for the pair (G, K) if and only if the corresponding function on $K \backslash G / K$ is multiplicative with respect to this struture. A similar result has been observed by Olshanski for a number of Olshanski spherical pairs arising from classical matrix groups in [O1]. The pair (K, K^ω) and Corollary 5.3.4 further suggest that a general phenomenon occurs, but it is still an open problem to make any general observations on the existence of such monoid structures on double coset spaces.

In light of Remark 5.3.3, it should be observed that the existence of a monoid structure on the double coset space such that a non-zero, continuous, biinvariant function is spherical if and only if the corresponding function on the double coset space is multiplicative with respect to this structure immediately implies that the product of spherical functions is again spherical. In Remark 5.3.3, we observed that this property is not shared by all Olshanski spherical pairs, but seems to be inevitable if the Olshanski spherical pair is a "‘real"’ limit. The task for a further study of monoid structures on double coset spaces of Olshanski spherical pairs should thus revolve around a further assumption on the pair which secures the existence of such a structure.

With our knowledge of all positive definite spherical functions for the pair (K, K^ω), we may now apply the generalized Bochner-Godement theorem in Theorem 1.3.6 to obtain a decomposition of all continuous, K^ω-biinvariant, positive definite functions.

Corollary 5.3.6 *A function $\varphi : K \to \mathbb{C}$ is continuous, positive definite and K^ω-biinvariant if and only if there exists a sequence $\{a_n\}_{n=0}^\infty \in \ell^2(\mathbb{N} \cup \{0\})$ such that*

$$\varphi(g) = \sum_{n=0}^{\infty} a_n 1_{K_{x_n}}(g) \tag{5.1}$$

for all $g \in K$.

If this is the case, the sequence $\{a_n\}_{n=0}^\infty$ is unique.

Proof. Let $\Omega = \left\{1_{K_{x_n}}\right\}_{n=0}^\infty$ be the spherical dual for (K, K^ω) equipped with the topology of uniform convergence on compact sets. Since uniform convergence on compact set implies pointwise convergence, it is immediate that this is just the discrete topology. Hence, Ω may as a topological space be identified with the discrete space $\mathbb{N} \cup \{0\}$. Furthermore, it follows by the proof of Corollary 3.2.7 that the groups K_n with $n \geq 0$ are all second countable.

Let φ be a continuous, K^ω-biinvariant, positive definite function on K. Theorem 1.3.6 provides the existence of a finite Borel measure μ on Ω such that

$$\varphi(g) = \int_\Omega 1_{K_{x_n}}(g)\ \mu(d1_{K_{x_n}})$$

for all $g \in K$. Since the countable set Ω is discrete, we get a sequence $\{a_n\}_{n=0}^\infty$ with the desired properties.

The other direction is obvious since $1_{K_{x_n}}$ is positive definite and K^ω-biinvariant for all $n \geq 0$ and since the restriction to K_m of a function φ defined by (5.1) is continuous by the fact that the sets $K_{x_n} \cap K_m$ are open in K_m for all $m, n \geq 0$.

The uniqueness of $\{a_n\}_{n=0}^\infty$ is a direct consequence of Theorem 1.3.6. □

In chapter 8 we will make a similar decomposition of continuous, K^ω-biinvariant, conditionally positive definite functions on K. Again the spherical functions for the pair (K, K^ω) play an important role.

Combining Theorem 5.3.1 with section 1.3, we immediately see that the family of spherical representations constructed in section 5.2 constitute the complete list of such representations. This observation is the content of the following Corollary 5.3.7.

Corollary 5.3.7 *The set of spherical representations for the pair (K, K^ω) is $\{\pi_n\}_{n=0}^\infty$.*

Corollary 5.3.7 contains the first part of a classification of the irreducible representations of K. The purpose of chapter 6 is to extend this classification by getting hold on a much larger class of representations, the so-called tame representations.

5.4 Decomposition of spherical representations for the pair (K, K^ω)

In Theorem 4.4.1, we decomposed the restrictions to K_n^ω of the spherical representations for the pair (K_n, K_n^ω) into direct sums of irreducible representations. In this section, we do similar considerations for the pair (K, K^ω). It turns out that for the spherical representations the pattern is similar to what we found in Theorem 4.4.1. The discussion will be continued in section 7.4.

As a starting point, we define for $n \geq 1$

$$\mathcal{V}_n = \left\{f \in \ell^2(\mathfrak{M}_n) \;\middle|\; f(x_n) = 0,\ f(y) = 0 \text{ for all } y \notin p^{-1}(\{x_{n-1}\})\right\}$$

which is a closed subspace of $\ell^2(\mathfrak{M}_n)$. The facts that $g(x_n) = x_n$, $g(x_{n-1}) = x_{n-1}$ and $g(p(y)) = p(g(y))$ for all $g \in K^\omega$ immediately imply that $\mathcal{V}_n$ is K^ω-invariant. We denote by σ_n the corresponding subrepresentation of the restriction of π_n to K^ω. This representation is clearly infinite-dimensional.

For $n \geq 2$, we futhermore consider the closed subspace

$$\mathcal{U}_n = \{f \in \ell^2(\mathfrak{M}_n) \mid f(y) = 0 \text{ for all } y \in p^{-1}(\{x_{n-1}\})\}$$

The same facts as above prove that $\mathcal{U}_n$ is also K^ω-invariant. We denote by λ_n the corresponding subrepresentation of the restriction of π_n to K^ω. This representation is also infinite-dimensional.

Finally, we denote for $n \geq 1$ by $\mathcal{W}_n$ the 1-dimensional subspace spanned by the K^ω-invariant function $1_{\{x_n\}}$ and denote by α_n the corresponding subrepresentation of the restriction of π_n to K^ω, i.e. α_n is equivalent to the trivial representation of K^ω.

We are now in a proposition to prove the following:

Theorem 5.4.1 *The representations α_n, σ_n and λ_n are irreducible for all n. The restriction of π_n to K^ω is irreducible for $n = 0$, equivalent to $\alpha_n \oplus \sigma_n$ for $n = 1$ and equivalent to $\alpha_n \oplus \sigma_n \oplus \lambda_n$ for $n \geq 2$.*

Proof. The irreducibility of α_n is obvious for all $n \geq 1$. To prove irreducibility of the remaining representations, we define for $k \geq 0$ and $n \geq 1$ $\mathcal{V}_n^k$ to be the subspace of $\mathcal{H}_n^k$ corresponding to $\mathcal{V}_n$ of section 4.4. Similarly, we define for $k \geq 0$ and $n \geq 2$ $\mathcal{U}_n^k$ to be the subspace of $\mathcal{H}_n^k$ corresponding to $\mathcal{U}_n$ of section 4.4.

For all n, we observe that

$$\mathcal{V}_n^0 \subseteq \mathcal{V}_n^1 \subseteq \ldots \subseteq \mathcal{V}_n^k \subseteq \ldots,$$

that

$$\mathcal{U}_n^0 \subseteq \mathcal{U}_n^1 \subseteq \ldots \subseteq \mathcal{U}_n^k \subseteq \ldots,$$

that $\bigcup_{k=0}^\infty \mathcal{V}_n^k \subseteq \mathcal{V}_n$ and that $\bigcup_{k=0}^\infty \mathcal{U}_n^k \subseteq \mathcal{U}_n$. As in the proof of Proposition 5.2.3, we observe that $\bigcup_{k=0}^\infty \mathcal{V}_n^k$ and $\bigcup_{k=0}^\infty \mathcal{U}_n^k$ are dense in $\mathcal{V}_n$ and $\mathcal{U}_n$, respectively. Since $\mathcal{V}_n^k$ and $\mathcal{U}_n^k$ are K_k^ω-invariant and the corresponding subrepresentations of the restrictions to K_k^ω of σ_n and λ_n, respectively, are irreducible, it follows by Lemma 5.2.2 that σ_n and λ_n are irreducible for all n.

To prove the second statement, it suffices to make the easy observation that $\ell^2(\mathfrak{M}_n) = \mathcal{W}_n \oplus \mathcal{V}_n \oplus \mathcal{U}_n$ for $n \geq 2$ and $\ell^2(\mathfrak{M}_1) = \mathcal{W}_1 \oplus \mathcal{V}_1$. Since the irreducibility of the restriction of π_0 is trivial, this finishes the proof. □

Theorem 5.4.1 shows that the decomposition of the restriction to K^ω of the spherical representations for the pair (K, K^ω) follow a pattern similar to the one observed for the pairs (K_n, K_n^ω) in Theorem 4.4.1. In section 7.4 we shall consider a new family of representations which by construction are quite similar to the spherical representations. By restriction to K^ω, they do, however, behave completely different from what we have seen above.

Chapter 6

Tame Representations of a Group Consisting of Automorphisms of a Homogeneous Rooted Tree

This chapter is devoted to the study of a class of irreducible unitary representations of the group K from chapter 5, namely representations which arise by restriction of irreducible unitary representations of the bigger group K_∞. These will be known as irreducible tame representations. It turns out that these may be completely classified, and the main task is to carry out this classification using ideas developed by Olshanski in [O4]. In section 1, we use the inducing construction to create a family of irreducible tame representations, and the remainder of the chapter is devoted to the proof of the fact that this list is exhaustive. The main tool is a family of semigroups which will be constructed in section 2. Section 3 deals with the connection between tame representations and representations of these semigroups while section 4 focuses on the classification of irreducible representations of the semigroups. In section 5, we combine the previous observations to obtain a complete classification of all irreducible tame representations of K. The main theorem is Theorem 6.5.1. The question of the possible existence of non-tame representations will be considered in chapter 7.

6.1 Construction of irreducible unitary representations of K_∞

The study of irreducible representations of the group K is difficult in the sense that we are not able to classify all such representations. However, these difficulties do not arise when studying the group K_∞ which has K as a dense subgroup (remember that the topology of K is strictly stronger than the topology inherited from K_∞). By restriction of unitary representations of K_∞, we obtain a family of unitary representations of K which as a consequence may be completely classified. Such representations arising by restriction of representations of K_∞ are known as tame representations. Clearly, an irreducible tame representation arises by restriction of an irreducible representation of K_∞. Conversely, it follows by denseness of K in K_∞ and continuity that the restriction of an irreducible representation is an irreducible tame representation. Since we can completely classfify the irreducible representations of K_∞, the same is true for the irreducible tame representations. The purpose of this chapter is to continue the study of representations of K which we started in chapter 5 by giving a more convenient definition of tame representations and by carrying out the classification of all irreducible such representations. We will follow and modify an idea of Olshanski developed in [O4] where it is used to classify all irreducible tame representations of the infinite symmetric group. Our main results are collected in Theorem 6.5.1.

Since there is a bijective correspondance between the set of irreducible tame representations of K and the set of irreducible representations of K_∞, one should notice that the results of Theorem 6.5.1 contain a complete classification of the irreducible representations of K_∞. This has already been obtained in [O3] by Olshanski. The semigroup approach applied here differs, however, completely from the one used in [O3].

We begin by using the inducing construction of representations to construct irreducible unitary representations of K_∞ which by restriction give rise to irreducible tame representations of K. These representations are of course continuous in the topology inherited from K_∞ and so in the inductive limit topology on K, and by construction they may be extended to K_∞.

For this purpose, we introduce some notation. A subtree $\mathfrak{J} \subseteq \mathfrak{X}$ with the property that $o \in \mathfrak{J}$ will be referred to as a *rooted subtree* of $(\mathfrak{X}, \mathfrak{C})$. We fix a finite rooted subtree $\mathfrak{J}$. We define

$$\tilde{K}_\infty(\mathfrak{J}) = \{g \in K_\infty \mid g(\mathfrak{J}) = \mathfrak{J}\}$$

and

$$K_\infty(\mathfrak{J}) = \{g \in K_\infty \mid g(x) = x \text{ for all } x \in \mathfrak{J}\}$$

The sets $\tilde{K}_\infty(\mathfrak{J})$ and $K_\infty(\mathfrak{J})$ are clearly subgroups of K_∞ which are both open and closed. Furthermore, $K_\infty(\mathfrak{J})$ is a normal subgroup of $\tilde{K}_\infty(\mathfrak{J})$, and we observe that the quotient group $\tilde{K}_\infty(\mathfrak{J})/K_\infty(\mathfrak{J})$ is clearly isomorphic as a topological group to the discrete, compact group $\mathrm{Aut}_o(\mathfrak{J})$ consisting of all automorphisms of $\mathfrak{J}$ fixing o. Hence, we may identify the set of unitary representations of $\tilde{K}_\infty(\mathfrak{J})$ which are trivial on $K_\infty(\mathfrak{J})$ with the set of all unitary representations of $\mathrm{Aut}_o(\mathfrak{J})$. By using this fact, we may regard every irreducible unitary representation of $\mathrm{Aut}_o(\mathfrak{J})$ as a unitary representation of the open subgroup $\tilde{K}_\infty(\mathfrak{J})$ of K_∞, and so we may for every such representation make use of the inducing construction to produce a unitary representation of K_∞. In what follows, we will induce all irreducible representations of the groups $\mathrm{Aut}_o(\mathfrak{J})$ with $\mathfrak{J}$ a finite rooted subtree of $(\mathfrak{X}, \mathfrak{C})$ and consider the properties of these representations.

Let σ be a unitary representation of $\tilde{K}_\infty(\mathfrak{J})$ which is trivial on $K_\infty(\mathfrak{J})$. Denote by $\mathcal{H}_\sigma$ the representation space of σ. Since $\tilde{K}_\infty(\mathfrak{J})$ is open and closed in K_∞, it follows by definition of the topology that $K_\infty/\tilde{K}_\infty(\mathfrak{J})$ is a discrete topological space. Hence, the counting measure is a K_∞-invariant Radon measure on $K_\infty/\tilde{K}_\infty(\mathfrak{J})$, and so we may use the inducing construction of [Fo, Section 6.1] to construct the induced representation $\pi_{\mathfrak{J},\sigma}$. Choose a set of representatives $\{g_\alpha\}_{\alpha \in A}$ for the left cosets in $K_\infty/\tilde{K}_\infty(\mathfrak{J})$. The corresponding representation space $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$ is the set of functions $f : K_\infty \to \mathcal{H}_\sigma$ with the properties that

$$f(gh) = \sigma(h^{-1})(f(g)) \tag{6.1}$$

for all $h \in \tilde{K}_\infty(\mathfrak{J})$ and all $g \in K_\infty$ and

$$\sum_{\alpha \in A} \|f(g_\alpha)\|_\sigma^2 < \infty$$

(note that this condition by (6.1) does not depend on the choice of the representatives $\{g_\alpha\}_{\alpha \in A}$) and the inner product $\langle \cdot, \cdot \rangle_{\pi_{\mathfrak{J},\sigma}}$ on $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$ is given by

$$\langle f_1, f_2 \rangle_{\pi_{\mathfrak{J},\sigma}} = \sum_{\alpha \in A} \langle f_1(g_\alpha), f_2(g_\alpha) \rangle_\sigma$$

for $f_1, f_2 \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$. The representation $\pi_{\mathfrak{J},\sigma}$ is given by the relation

$$(\pi_{\mathfrak{J},\sigma}(g)f)(x) = f(g^{-1}x)$$

for all $f \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$ and $g, x \in K_\infty$.

We may give another realization of the representation $\pi_{\mathfrak{J},\sigma}$. Denote by $\mathcal{I} = \mathcal{I}_{\mathfrak{J}}$ the set of embeddings of $\mathfrak{J}$ into $\mathfrak{X}$ fixing o, i.e. $\mathcal{I}$ is the set of maps $h : \mathfrak{J} \to \mathfrak{X}$ with the properties that $h(o) = o$ and that x and y are neighbours in $\mathfrak{J}$ if and only if $h(x)$ and $h(y)$ are neighbours in $\mathfrak{X}$. We observe that $\mathrm{Aut}_o(\mathfrak{J})$ acts on $\mathcal{I}$ by the map $\mathrm{Aut}_o(\mathfrak{J}) \times \mathcal{I} \to \mathcal{I}$, $(g, h) \mapsto h \circ g^{-1}$. This gives rise to an action $(g, h) \mapsto g \cdot h$ of $\tilde{K}_\infty(\mathfrak{J})$ on $\mathcal{I}$ which is trivial on $K_\infty(\mathfrak{J})$.

Similarly, K_∞ acts on $\mathcal{I}$ by the map $K_\infty \times \mathcal{I} \to \mathcal{I}$, $(g, h) \mapsto g \bullet h = g \circ h$.

Define $\mathcal{H}_{\tau_{\mathfrak{J},\sigma}}$ to be the set of functions $f : \mathcal{I} \to \mathcal{H}_\sigma$ with the properties that

$$\sum_{h \in \mathcal{I}} \|f(h)\|_\sigma^2 < \infty \tag{6.2}$$

and

$$f(g \cdot h) = \sigma(g) f(h) \tag{6.3}$$

for all $g \in \tilde{K}_\infty(\mathfrak{J})$ and all $h \in \mathcal{I}$. It is evident that $\mathcal{H}_{\tau_{\mathfrak{J},\sigma}}$ is a vector space. Furthermore, it is an obvious consequence of (6.2) that $\sum_{h \in \mathcal{I}} \langle f_1(h), f_2(h) \rangle_\sigma$ converges absolutely for all $f_1, f_2 \in \mathcal{H}_{\tau_{\mathfrak{J},\sigma}}$, and so we may define an inner product on $\mathcal{H}_{\tau_{\mathfrak{J},\sigma}}$ by defining

$$\langle f_1, f_2 \rangle_{\tau_{\mathfrak{J},\sigma}} = \sum_{h \in \mathcal{I}} \langle f_1(h), f_2(h) \rangle_\sigma$$

for $f_1, f_2 \in \mathcal{H}_{\tau_{\mathfrak{J},o}}$. Standard arguments show that this turns $\mathcal{H}_{\tau_{\mathfrak{J},\sigma}}$ into a Hilbert space.

For $g \in K_\infty$, $f \in \mathcal{H}_{\tau_{\mathfrak{J},\sigma}}$ and $h \in \mathcal{I}$, we define

$$(\tau_{\mathfrak{J},\sigma}(g)f)(h) = f(g^{-1} \bullet h)$$

It is easy to check that this defines a unitary representation $\tau_{\mathfrak{J},\sigma}$ of K_∞ on $\mathcal{H}_{\tau_{\mathfrak{J},\sigma}}$.

We now define a unitary operator U between $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$ and $\mathcal{H}_{\tau_{\mathfrak{J},\sigma}}$ intertwining $\pi_{\mathfrak{J},\sigma}$ and $\tau_{\mathfrak{J},\sigma}$. For f in $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$ and $h \in \mathcal{I}$, we choose $g \in K_\infty$ such that $g(x) = h(x)$ for all $x \in \mathfrak{J}$ and define

$$(Uf)(h) = \frac{1}{|\mathrm{Aut}_o(\mathfrak{J})|^{\frac{1}{2}}} f(g)$$

By (6.1) and the fact that σ is trivial on $K_\infty(\mathfrak{J})$, this definition does not depend on the choice of g. In this way, we have defined a map $Uf : \mathcal{I} \to \mathcal{H}_\sigma$. Keeping the notation from above, we get for $g_0 \in \tilde{K}_\infty(\mathfrak{J})$ that

$$(Uf)(g_0 \cdot h) = (Uf)(h \circ g_0^{-1}) = \frac{1}{|\mathrm{Aut}_o(\mathfrak{J})|^{\frac{1}{2}}} f(g g_0^{-1}) = \sigma(g_0) \frac{1}{|\mathrm{Aut}_o(\mathfrak{J})|^{\frac{1}{2}}} f(g) = \sigma(g_0)((Uf)(h))$$

Furthermore, we observe that

$$\sum_{h \in \mathcal{I}} \|(Uf)(h)\|_\sigma^2 = \frac{1}{|\mathrm{Aut}_o(\mathfrak{J})|} |\mathrm{Aut}_o(\mathfrak{J})| \sum_{\alpha \in A} f(g_\alpha) = \sum_{\alpha \in A} f(g_\alpha) < \infty$$

Hence, U is an isometry from $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$ to $\mathcal{H}_{\tau_{\mathfrak{J},\sigma}}$ which is clearly linear. Furthermore, for $f_0 \in \mathcal{H}_{\tau_{\mathfrak{J},\sigma}}$, we may define

$$f(g) = |\mathrm{Aut}_o(\mathfrak{J})|^{\frac{1}{2}} f_0(g_{|\mathfrak{J}})$$

for $g \in K_\infty$. As above, it is easy to see that $f_0 \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$ and that $Uf = f_0$. This proves that U is unitary. Finally, with the notation from above we have for $g_0 \in K_\infty$ that

$$(\tau_{\mathfrak{J},\sigma}(g_0)Uf)(h) = \frac{1}{|\mathrm{Aut}_o(\mathfrak{J})|^{\frac{1}{2}}} f(g_0^{-1}g) = (U\pi_{\mathfrak{J},\sigma}(g_0)f)(h)$$

and so U establishes an equivalence between $\pi_{\mathfrak{J},\sigma}$ and $\tau_{\mathfrak{J},\sigma}$.

This proves that $\tau_{\mathfrak{J},\sigma}$ is another realization of the induced representation of σ to K_∞. In the following, we will abuse notation and denote both representations by $\pi_{\mathfrak{J},\pi}$ and both representation spaces by $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$. It will be clear from the context which of the realizations is used.

It turns out that the key to studying irreducible unitary representations of K_∞ and so irreducible tame representations of K consists of the subspaces of invariant vectors under certain subgroups. For a finite rooted subtree $\mathfrak{J}' \subseteq \mathfrak{X}$, we consider the open and closed subgroup

$$K_\infty(\mathfrak{J}') = \{g \in K_\infty \mid g(x) = x \text{ for all } x \in \mathfrak{J}'\}$$

and define

$$\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}') = \{h \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}} \mid \pi_{\mathfrak{J},\sigma}(g)h = h \text{ for all } g \in K_\infty(\mathfrak{J}')\}$$

which is a closed subspace of $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$. Furthermore, we define $\mathcal{I}(\mathfrak{J}') \subseteq \mathcal{I}$ to consist of all $h \in \mathcal{I}$ with the property that $h(\mathfrak{J}) \subseteq \mathfrak{J}'$, and we define $\mathcal{H}'_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}')$ to be the set of $f \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$ with the property that $f(h) = 0$ for all $h \notin \mathcal{I}(\mathfrak{J}')$. This is clearly also a closed subspace of $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$. It turns out that the spaces $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}')$ and $\mathcal{H}'_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}')$ coincide and this gives a more convenient description of the spaces $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}')$:

Lemma 6.1.1 *Let $\mathfrak{J}' \subseteq \mathfrak{X}$ be a finite rooted subtree of $(\mathfrak{X}, \mathfrak{C})$. Then*

1. $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}') = \mathcal{H}'_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}')$
2. *the projection onto the subspace $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}')$ is the map $f \mapsto f \cdot 1_{\mathcal{I}(\mathfrak{J}')}$*

Proof. Let $g \in K_\infty(\mathfrak{J}')$. Since $g^{-1} \bullet h = h$ for $h \in \mathcal{I}(\mathfrak{J}')$ and $g^{-1} \bullet h \notin \mathcal{I}(\mathfrak{J}')$ for $h \notin \mathcal{I}(\mathfrak{J}')$, it follows that $\pi_{\mathfrak{J},\sigma}(g)f = f$ for $f \in \mathcal{H}'_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}')$. Hence, we see that $\mathcal{H}'_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}') \subseteq \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}')$.

Now assume that $f \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}')$, and let $h \notin \mathcal{I}(\mathfrak{J}')$. We see that $f(g^{-1} \bullet h) = f(h)$ for all $g \in K(\mathfrak{J}')$. Since the degree of $\mathfrak{X}$ is infinite and since $h \notin \mathcal{I}(\mathfrak{J}')$, the orbit $\{g^{-1} \bullet h \mid g \in K(\mathfrak{J}')\}$ is infinite. The condition (6.2) now implies that $f(h) = 0$ and so $f \in \mathcal{H}'_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}')$ which establishes the inclusion $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}') \subseteq \mathcal{H}'_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}')$. This proves 1.

To prove 2., let $f \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$ and observe that $f = f \cdot 1_{\mathcal{I}(\mathfrak{J}')} + f \cdot 1_{\mathcal{I}(\mathfrak{J}')^c}$. We clearly have that $f \cdot 1_{\mathcal{I}(\mathfrak{J}')}, f \cdot 1_{\mathcal{I}(\mathfrak{J}')^c} \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$, and 1. even shows that $f \cdot 1_{\mathcal{I}(\mathfrak{J}')} \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}')$ and $f \cdot 1_{\mathcal{I}(\mathfrak{J}')^c} \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}')^\perp$. This proves 2. □

Remark 6.1.2 1. in Lemma 6.1.1 above makes it possible to immediately determine which of the spaces $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}')$ are the zero space and which are not. If $\mathfrak{J}'$ does not contain a rooted subtree which is isomorphic to $\mathfrak{J}$ as a rooted tree, the set $\mathcal{I}(\mathfrak{J}')$ is clearly empty and so $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}') = \{0\}$. Conversely, if such a subtree of $\mathfrak{J}'$ exists, $\mathcal{I}(\mathfrak{J}')$ is non-empty. Hence, we may choose $h_0 \in \mathcal{I}(\mathfrak{J}')$. If we furthermore choose $x \in \mathcal{H}_\sigma \setminus \{0\}$ and (well)-define $f(g \cdot h_0) = \sigma(g)x$ for $g \in \tilde{K}_\infty(\mathfrak{J})$ and $f(h) = 0$ if $h \in \mathcal{I}$ does not belong to the $\tilde{K}_\infty(\mathfrak{J})$-orbit of h_0 (which is finite), we clearly have that $h \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}')$. Hence, we see that $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}') \neq \{0\}$.

Thus, Lemma 6.1.1 implies that $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}') \neq \{0\}$ if and only if $\mathfrak{J}'$ contains a rooted subtree which is isomorphic to $\mathfrak{J}$ as a rooted tree.

Putting $\mathfrak{J}' = \mathfrak{J}$ in Lemma 6.1.1 gives a useful characterization of the space $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J})$ of $K_\infty(\mathfrak{J})$-invariant vectors. This enables us to prove certain useful properties of the representation $\pi_{\mathfrak{J},\sigma}$. These are collected in the following Lemma 6.1.3.

Lemma 6.1.3 *The representation $\pi_{\mathfrak{J},\sigma}$ has the following properties:*

1. *The closed subspace $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J})$ is invariant under $\tilde{K}_\infty(\mathfrak{J})$ and the corresponding subrepresentation of $\tilde{K}_\infty(\mathfrak{J})$ is equivalent to σ and so irreducible.*

2. *The closed subspace $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J})$ is cyclic.*

Proof. Since $g^{-1} \bullet h \notin \mathcal{I}(\mathfrak{J})$ for $g \in \tilde{K}_\infty(\mathfrak{J})$ and $h \notin \mathcal{I}(\mathfrak{J})$, it follows by 1. in Lemma 6.1.1 that $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J})$ is invariant under $\tilde{K}_\infty(\mathfrak{J})$.

Now denote by $h_0 \in \mathcal{I}(\mathfrak{J})$ the natural embedding of $\mathfrak{J}$ into $\mathfrak{X}$ and construct $U : \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}) \to \mathcal{H}_\sigma$ by defining $Uf = |\mathrm{Aut}_o(\mathfrak{J})|^{\frac{1}{2}} f(h_0)$ for $f \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J})$. For $h \in \mathcal{I}(\mathfrak{J})$, we may write $h = g \cdot h_0$ for some $g \in \tilde{K}_\infty(\mathfrak{J})$. Combining this with the fact that $g_1 \cdot h_0 = g_2 \cdot h_0$ if and only if $g_1, g_2 \in \tilde{K}_\infty(\mathfrak{J})$ belong to the same coset in $\tilde{K}_\infty(\mathfrak{J})/K_\infty(\mathfrak{J})$, we see that U is a linear isometry which by the construction of Remark 6.1.2 is surjective. Hence, U is a unitary operator which by the relation (6.3) intertwines $\pi_{\mathfrak{J},\sigma}$ and σ. This establishes the equivalence in 1.

To prove 2., let $h \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$. Denote by $\mathcal{P}$ the set of rooted subtrees $\mathfrak{J}'$ of $\mathfrak{X}$ which are isomorphic to $\mathfrak{J}$ as rooted subtrees. For each $\mathfrak{J}' \in \mathcal{P}$, we choose $g_{\mathfrak{J}'} \in K_\infty$ such that $g_{\mathfrak{J}'}(\mathfrak{J}) = \mathfrak{J}'$. It follows by 2. in Lemma 6.1.1 that $h \cdot 1_{\mathcal{I}(\mathfrak{J}')} \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$ for all $\mathfrak{J}' \in \mathcal{P}$, and so we may define $h_{\mathfrak{J}'} = \pi_{\mathfrak{J},\sigma}(g_{\mathfrak{J}'}^{-1})(h \cdot 1_{\mathcal{I}(\mathfrak{J}')}) \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$ for all $\mathfrak{J}' \in \mathcal{P}$. By 1. in Lemma 6.1.1, it is immediate that $h_{\mathfrak{J}'} \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J})$ for all $\mathfrak{J}' \in \mathcal{P}$. It is now evident by (6.2) that the net

$$\left\{ \sum_{\mathfrak{J}' \in F} \pi_{\mathfrak{J},\sigma}(g_{\mathfrak{J}'}) h_{\mathfrak{J}'} \;\middle|\; F \subseteq \mathcal{P} \text{ is finite} \right\}$$

converges to h in $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$. This proves 2. □

Using Lemma 6.1.3, we are now able to prove that the representation $\pi_{\mathfrak{J},\sigma}$ is irreducible.

Proposition 6.1.4 *The unitary representation $\pi_{\mathfrak{J},\sigma}$ is irreducible.*

Proof. Let V be a closed invariant subspace of $\mathcal{H}_{\pi_{\mathfrak{J}},\sigma}$. Since V is invariant, it follows by [Fo, Proposition 3.4] that the orthogonal projection P onto V commutes with $\pi_{\mathfrak{J},\sigma}(g)$ for all $g \in K_\infty$. For $h \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J})$ and $g \in K_\infty(\mathfrak{J})$, this implies that

$$\pi_{\mathfrak{J},\sigma}(g) Ph = P\pi_{\mathfrak{J},\sigma}(g) h = Ph$$

This shows that $Ph \in \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J})$. Since $V^\perp$ is also a closed, invariant subspace, we see that

$$\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}) = (\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}) \cap V) \oplus (\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}) \cap V^\perp)$$

By 1. in Lemma 6.1.3, $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J})$ is invariant under the restriction of $\pi_{\mathfrak{J},\sigma}$ to $\tilde{K}_\infty(\mathfrak{J})$, and the corresponding subrepresentation is irreducible. Since $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}) \cap V$ and $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}) \cap V^\perp$ are closed invariant subspaces for this representation, we have one of the two identities $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}) \cap V = \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J})$ and $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}) \cap V^\perp = \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J})$. Hence, $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}) \subseteq V$ or $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J}) \subseteq V^\perp$. However, $\mathcal{H}_{\pi_{\mathfrak{J},\sigma}}(\mathfrak{J})$ is cyclic for $\pi_{\mathfrak{J},\sigma}$ by 2. of Lemma 6.1.3, and since V and $V^\perp$ are invariant and closed, this implies that $V = \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$ or $V^\perp = \mathcal{H}_{\pi_{\mathfrak{J},\sigma}}$. This proves irreducibility of $\pi_{\mathfrak{J},\sigma}$. □

By Proposition 6.1.4, the inducing construction described above provides us with a family of irreducible representations of K_∞. It turns out that it is easy to point out when these are equivalent.

Proposition 6.1.5 *Two representations $\pi_{\mathfrak{J}_1,\sigma_1}$ and $\pi_{\mathfrak{J}_2,\sigma_2}$ are unitarily equivalent if and only if $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are isomorphic as rooted trees and σ_1 and σ_2 are unitarily equivalent (under identification of the isomorphic groups $\tilde{K}(\mathfrak{J}_1)$ and $\tilde{K}(\mathfrak{J}_2)$).*

Proof. Assume that $\pi_{\mathfrak{J}_1,\sigma_1}$ and $\pi_{\mathfrak{J}_2,\sigma_2}$ are unitarily equivalent. Since the spaces $\mathcal{H}_{\pi_{\mathfrak{J}_1,\sigma_1}}(\mathfrak{J}_1)$ and $\mathcal{H}_{\pi_{\mathfrak{J}_2,\sigma_2}}(\mathfrak{J}_2)$ are both non-zero by Remark 6.1.2, it follows by the same remark that $\mathfrak{J}_1$ contains a finite rooted subtree which is isomorphic to $\mathfrak{J}_2$ as a rooted tree and that $\mathfrak{J}_2$ contains a finite rooted subtree which is isomorphic to $\mathfrak{J}_1$ as a rooted tree. This implies that $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are isomorphic as rooted trees.

Using this, we may now choose $g \in K_\infty$ such that $g(\mathfrak{J}_1) = \mathfrak{J}_2$. It is obvious that the map $h \mapsto ghg^{-1}$ defines an isomorphism (as topological groups) from $\tilde{K}(\mathfrak{J}_1)$ onto $\tilde{K}(\mathfrak{J}_2)$. By 1. in Lemma 6.1.3, the subrepresentation of the restriction of $\pi_{\mathfrak{J}_1,\sigma_1}$ to $\tilde{K}(\mathfrak{J}_1)$ corresponding to the subspace $\mathcal{H}_{\pi_{\mathfrak{J}_1,\sigma_1}}(\mathfrak{J}_1)$ is equivalent to σ_1. By unitary equivalence, the subrepresentation of the restriction of $\pi_{\mathfrak{J}_1,\sigma_1}$ to $\tilde{K}(\mathfrak{J}_2)$ corresponding to the subspace $\mathcal{H}_{\pi_{\mathfrak{J}_1,\sigma_1}}(\mathfrak{J}_2)$ is equivalent to σ_2. However, it is obvious that $\pi_{\mathfrak{J}_1,\sigma_1}(g)(\mathcal{H}_{\pi_{\mathfrak{J}_1,\sigma_1}}(\mathfrak{J}_1)) = \mathcal{H}_{\pi_{\mathfrak{J}_1,\sigma_1}}(\mathfrak{J}_2)$ and so the restriction of $\pi_{\mathfrak{J}_1,\sigma_1}(g)$ is a unitary operator from $\mathcal{H}_{\pi_{\mathfrak{J}_1,\sigma_1}}(\mathfrak{J}_1)$ onto $\mathcal{H}_{\pi_{\mathfrak{J}_1,\sigma_1}}(\mathfrak{J}_2)$ which satisfies that

$$\pi_{\mathfrak{J}_1,\sigma_1}(ghg^{-1})\pi_{\mathfrak{J}_1,\sigma_1}(g) = \pi_{\mathfrak{J}_1,\sigma_1}(g)\pi_{\mathfrak{J}_1,\sigma_1}(h)$$

for all $h \in \tilde{K}(\mathfrak{J}_1)$. This proves that σ_1 and σ_2 are equivalent under the above identification of the isomorphic groups $\tilde{K}(\mathfrak{J}_1)$ and $\tilde{K}(\mathfrak{J}_2)$.

Conversely, assume that $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are isomorphic as rooted subtrees and assume that σ_1 and σ_2 are equivalent under the identification of $\tilde{K}_\infty(\mathfrak{J}_1)$ with $\tilde{K}(\mathfrak{J}_2)$. Let $F : \mathfrak{J}_1 \to \mathfrak{J}_2$ be an isomorphism from $\mathfrak{J}_1$ onto $\mathfrak{J}_2$ as rooted trees, and extend it to an automorphism $g \in K_\infty$. Let V be a unitary equivalence between σ_1 and σ_2 under the identification $h \mapsto g^{-1}hg$ of $\tilde{K}(\mathfrak{J}_2)$ with $\tilde{K}(\mathfrak{J}_1)$. We leave it to the reader to check that the map $U : \mathcal{H}_{\pi_{\mathfrak{J}_1,\sigma_1}} \to \mathcal{H}_{\pi_{\mathfrak{J}_2,\sigma_2}}$ defined by

$$(Uf)(h) = V(f(h \circ F))$$

for all $f \in \mathcal{H}_{\pi_{\mathfrak{J}_1,\sigma_1}}$ and $h \in \mathcal{I}_{\mathfrak{J}_2}$ is a unitary equivalence between $\pi_{\mathfrak{J}_1,\sigma_1}$ and $\pi_{\mathfrak{J}_2,\sigma_2}$. □

The objective of the remaining sections is to prove that we hereby have constructed all irreducible unitary representations of K_∞ and hence by restriction all irreducible tame representations of K. This is the content of Theorem 6.5.1.

6.2 Construction of semigroups

We now construct a family of semigroups related to the group K_∞ which will be the key tool in our characterization of all irreducible representations of K_∞. Our semigroups may be divided into two groups: one consisting of a family of finite semigroups $\{\Gamma(\mathfrak{J})\}$ indexed by the set of all finite rooted subtrees $\mathfrak{J} \subseteq \mathfrak{X}$ and another consisting of a single infinite semigroup $\Gamma(\mathfrak{X})$. The important feature of the semigroups is that we to every unitary representation π of K_∞ may associate representations of the semigroups $\Gamma(\mathfrak{J})$ and $\Gamma(\mathfrak{X})$, cf. Theorem 6.3.3 and Theorem 6.3.7. The representations of $\Gamma(\mathfrak{J})$ may be used to give a complete characterization of π while the representation of $\Gamma(\mathfrak{X})$ is just an extension of π since K_∞ is a subsemigroup of $\Gamma(\mathfrak{X})$.

For the construction of our semigroups, we need the following definition:

Definition 6.2.1 Let $\mathfrak{Y}$ be either $\mathfrak{X}$ or a finite rooted subtree of $\mathfrak{X}$. A partial automorphism γ of $\mathfrak{Y}$ is an isomorphism between two rooted subtrees of $\mathfrak{Y}$ with the property that $\gamma(o) = o$. We denote by $\Gamma(\mathfrak{Y})$ the set of all partial automorphisms of $\mathfrak{Y}$.

Let $\mathfrak{Y}$ be as in Definition 6.2.1. We will equip $\Gamma(\mathfrak{Y})$ with a multiplication which turns it into a semigroup with a unity and a zero-element. To do this, let $\gamma_1 : \mathfrak{G}_1 \to \mathfrak{J}_1$ and $\gamma_2 : \mathfrak{G}_2 \to \mathfrak{J}_2$ be two partial automorphisms of $\mathfrak{Y}$. We observe that $\gamma_1^{-1}(\mathfrak{J}_1 \cap \mathfrak{G}_2)$ and $\gamma_2(\mathfrak{J}_1 \cap \mathfrak{G}_2)$ are finite rooted subtrees of $\mathfrak{Y}$. We define $\gamma = \gamma_2\gamma_1$ to be the isomorphism from $\gamma_1^{-1}(\mathfrak{J}_1 \cap \mathfrak{G}_2)$ onto $\gamma_2(\mathfrak{J}_1 \cap \mathfrak{G}_2)$ given by the relation $\gamma(x) = \gamma_2(\gamma_1(x))$ for all $x \in \gamma_1^{-1}(\mathfrak{J}_1 \cap \mathfrak{G}_2)$. Clearly, $\gamma \in \Gamma(\mathfrak{Y})$.

The multiplication in $\Gamma(\mathfrak{Y})$ is evidently associative and so turns $\Gamma(\mathfrak{Y})$ into a semigroup. The identity map on $\mathfrak{Y}$ is a unity 1, and the identity map on the subtree $\{o\}$ is a zero element 0 in $\Gamma(\mathfrak{Y})$.

Finally, we define an involution on $\Gamma(\mathfrak{Y})$. For $\gamma \in \Gamma(\mathfrak{Y})$, we denote by γ^* the inverse map of γ. It is immediate that the map $\gamma \mapsto \gamma^*$ is an involution on $\Gamma(\mathfrak{Y})$, i.e. satisfies the relations $(\gamma_1\gamma_2)^* = \gamma_2^*\gamma_1^*$ and $(\gamma_1^*)^* = \gamma_1$ for $\gamma_1, \gamma_2 \in \Gamma(\mathfrak{Y})$.

If $\mathfrak{Y}_1 \subseteq \mathfrak{Y}_2$, we define a natural map $p_{\mathfrak{Y}_2,\mathfrak{Y}_1} : \Gamma(\mathfrak{Y}_2) \to \Gamma(\mathfrak{Y}_1)$ in the following way: for a partial automorphism $\gamma : \mathfrak{G} \to \mathfrak{J}$ of $\mathfrak{Y}_2$ we observe that $\mathfrak{G}' = \mathfrak{Y}_1 \cap \gamma^{-1}(\mathfrak{Y}_1 \cap \mathfrak{J})$ and $\mathfrak{J}' = \gamma(\mathfrak{G}')$ are finite rooted subtrees of $\mathfrak{Y}_1$, and so we may define $p_{\mathfrak{Y}_2,\mathfrak{Y}_1}(\gamma) \in \Gamma(\mathfrak{Y}_1)$ to be the restriction of γ to $\mathfrak{G}'$. We observe that these maps satisfy the relation that $p_{\mathfrak{Y}_3,\mathfrak{Y}_1} = p_{\mathfrak{Y}_2,\mathfrak{Y}_1} \circ p_{\mathfrak{Y}_3,\mathfrak{Y}_2}$ if $\mathfrak{Y}_1 \subseteq \mathfrak{Y}_2 \subseteq \mathfrak{Y}_3$ which implies that under these circumstances it is true that $p_{\mathfrak{Y}_3,\mathfrak{Y}_1}(\gamma_1) = p_{\mathfrak{Y}_3,\mathfrak{Y}_1}(\gamma_2)$ if $p_{\mathfrak{Y}_3,\mathfrak{Y}_2}(\gamma_1) = p_{\mathfrak{Y}_3,\mathfrak{Y}_2}(\gamma_2)$. For every finite rooted subtree $\mathfrak{J}$ of $\mathfrak{X}$, we will denote the map $p_{\mathfrak{X},\mathfrak{J}}$ by $p_{\mathfrak{J}}$.

The maps $p_{\mathfrak{Y}_2,\mathfrak{Y}_1}$ are clearly not semigroup homomorphisms. However, they have the following basic properties whose immediate proof is left to the reader:

Lemma 6.2.2 *Assume that $\mathfrak{Y}_1 \subseteq \mathfrak{Y}_2$. The map $p_{\mathfrak{Y}_2,\mathfrak{Y}_1}$ preserves the involution, identity and zero element.*

Using the above maps, we will endow $\Gamma(\mathfrak{Y})$ with a topology. This is done by declaring a subset $U \subseteq \Gamma(\mathfrak{Y})$ to be open if there for every $\gamma_0 \in U$ exists a finite rooted subtree $\mathfrak{J} \subseteq \mathfrak{Y}$ such that $p_{\mathfrak{Y},\mathfrak{J}}^{-1}(\{p_{\mathfrak{Y},\mathfrak{J}}(\gamma_0)\}) \subseteq U$. Since the observations above show that $p_{\mathfrak{Y},\mathfrak{J}}^{-1}(\{p_{\mathfrak{Y},\mathfrak{J}}(\gamma_0)\}) \subseteq p_{\mathfrak{Y},\mathfrak{G}}^{-1}(\{p_{\mathfrak{Y},\mathfrak{G}}(\gamma_0)\})$ if $\mathfrak{G} \subseteq \mathfrak{J}$, this defines a topology on $\Gamma(\mathfrak{Y})$.

The semigroup $\Gamma(\mathfrak{Y})$ is clearly finite if $\mathfrak{Y}$ is a finite rooted subtree, and since $p_{\mathfrak{Y},\mathfrak{Y}}$ is the identity map, the topology on $\Gamma(\mathfrak{Y})$ is just the discrete topology in this case. However, the semigroup $\Gamma(\mathfrak{X})$ is infinite, and the topology is obviously not discrete in this case.

We observe that it is an immediate consequence of the definition of the topology that the maps $p_{\mathfrak{Y}_2,\mathfrak{Y}_1}$ are all continuous. The topology on $\Gamma(\mathfrak{X})$ is actually the initial topology with respect to the maps $p_{\mathfrak{J}}$ for $\mathfrak{J}$ a finite rooted subtree of $\mathfrak{X}$ (having endowed the semigroups $\Gamma(\mathfrak{J})$ with the discrete topology).

The following lemma whose proof is an immediate consequence of the definition of the topology shows that there is a suitable connection between the topology and the semigroup structure on $\Gamma(\mathfrak{Y})$.

Lemma 6.2.3 *The involution on $\Gamma(\mathfrak{Y})$ is continuous and the multiplication is separately continuous.*

Remark 6.2.4 The multiplication is of course jointly continuous if $\mathfrak{Y}$ is finite, and so $\Gamma(\mathfrak{Y})$ is a topological semigroup in this case. However, this is clearly not the case for the infinite semigroup $\Gamma(\mathfrak{X})$.

We now observe that K_∞ is related to the semigroup $\Gamma(\mathfrak{X})$ in the way that it may be regarded as the subgroup of $\Gamma(\mathfrak{X})$ consisting of partial automorphisms whose domain and range are all of $\mathfrak{X}$, i.e. K_∞ may be seen as the subgroup

$$\{\gamma \in \Gamma(\mathfrak{X}) \mid \gamma\gamma^* = \gamma^*\gamma = 1\}$$

Furthermore, it is evident by the definitions that the topology on K_∞ is just the topology inherited from $\Gamma(\mathfrak{X})$.

Surprisingly, the finite semigroups $\Gamma(\mathfrak{J})$ with $\mathfrak{J}$ finite are also related to the groups K and K_∞. To see this, we define - as we did in the case of the group K_∞ - for a finite rooted subtree $\mathfrak{J}$ of $\mathfrak{X}$ the open and closed (in both topologies) subgroup

$$K(\mathfrak{J}) = \{g \in K \mid g(x) = x \text{ for all } x \in \mathfrak{J}\}$$

Since K and K_∞ by the above observation may be regarded as subsets of $\Gamma(\mathfrak{X})$, the map $p_\mathfrak{J}$ sends K and K_∞ into $\Gamma(\mathfrak{J})$. It turns out that we under this map obtain an identification of $\Gamma(\mathfrak{J})$ with the double coset spaces $K(\mathfrak{J})\backslash K/K(\mathfrak{J})$ and $K_\infty(\mathfrak{J})\backslash K_\infty/K_\infty(\mathfrak{J})$, respectively. This is the content of the following Proposition 6.2.5. It shows that we may regard the finite semigroup $\Gamma(\mathfrak{J})$ as these double coset spaces equipped with a suitable semigroup structure and a suitable involution.

Proposition 6.2.5 *Let $\mathfrak{J}$ be a finite rooted subtree of $\mathfrak{X}$. The restriction of $p_\mathfrak{J}$ to K (resp. K_∞) is constant on each double coset in $K(\mathfrak{J})\backslash K/K(\mathfrak{J})$ (resp. $K_\infty(\mathfrak{J})\backslash K_\infty/K_\infty(\mathfrak{J})$) and sets up a bijection between $K(\mathfrak{J})\backslash K/K(\mathfrak{J})$ (resp. $K_\infty(\mathfrak{J})\backslash K_\infty/K_\infty(\mathfrak{J})$) and $\Gamma(\mathfrak{J})$.*

A moment of thought makes it easy to see that the double cosets in $K(\mathfrak{J})\backslash K/K(\mathfrak{J})$ (resp. $K_\infty(\mathfrak{J})\backslash K_\infty/K_\infty(\mathfrak{J})$) consist exactly of automorphisms which have the same image under $p_\mathfrak{J}$ and that the restriction of $p_\mathfrak{J}$ to K (resp. K_∞) is surjective. A formal proof is, however, very tedious to write out and not in any way - considering the geometric simplicity of the question - illuminating. Hence, we omit it. We leave it to the reader to write out the details. A proof may be using the facts that K_∞ as a set may be identified with $\prod_{x\in\mathfrak{X}} S(\infty)$ (where we by $S(\infty)$ denote the symmetric group corresponding to the set $\mathbb{N}$), that K may be identified with a certain subset of this product, and that a similar statement for $S(\infty)$ has been proved in [O4, Lemma 4.13].

Remark 6.2.6 The idea of analysing groups by equipping certain double coset spaces with semigroup structures has been used extensively in the literature. We have already encountered two examples in chapters 3 and 5. The analysis of the infinite symmetric group is approached by a similar idea in [O4] and [O5], and the idea has been applied to a number of classical matrix groups in [O1]. A matrix group in the field $\mathbb{F}_q$ has been studied by a similar approach in the paper [Du].

A consequence of Proposition 6.2.5 is that the restriction to K of $p_\mathfrak{J}$ is surjective for every finite rooted subtree $\mathfrak{J}$. Combining this fact with the definition of the topology of $\Gamma(\mathfrak{X})$ immediately yields the following corollary:

Corollary 6.2.7 *K is a dense subset of $\Gamma(\mathfrak{X})$.*

It turns out that the study of representations of the finite semigroups $\Gamma(\mathfrak{J})$ with $\mathfrak{J}$ a finite rooted subtree enables us to determine all irreducible unitary representations of K_∞. The principal tool in this approach is the fact that we from every such representation of K_∞ may construct representations of $\Gamma(\mathfrak{Y})$ for all possible choices of $\mathfrak{Y}$. This construction is the content of the following section.

6.3 Tame representations of K and corresponding representations of the semigroups $\Gamma(\mathfrak{Y})$

As we prove in Theorem 6.3.3, all unitary representations of K_∞ may be extended to a representation of the semigroup $\Gamma(\mathfrak{X})$. Hence, tame representations may also be defined as representations of K which may be extended to $\Gamma(\mathfrak{X})$. However, we will in Definition 6.3.1 give a much more convenient definition of the concept of a tame reprsentation of K which in Theorem 6.3.3 will turn out to be equivalent to our previous concept.

For the definition, we need some notation. We consider the finite subtrees

$$\mathfrak{S}_n = \{x \in \mathfrak{X}_n \mid d(o,x) \leq n\}$$

which have the property that $\bigcup_{n=0}^{\infty} \mathfrak{S}_n = \mathfrak{X}$. Evidently, for every finite rooted subtree $\mathfrak{J}$ of $\mathfrak{X}$ there exists n such that $\mathfrak{J} \subseteq \mathfrak{S}_n$. Hence, the subgroups $K(\mathfrak{S}_n)$ constitute a local base for the topology on K inherited from K_∞ at the identity e.

For a unitary representation π of K with representation space $\mathcal{H}_\pi$, we denote by $\mathcal{H}_\pi(\mathfrak{J})$ the closed subspace of all vectors invariant under $K(\mathfrak{J})$, and the corresponding orthogonal projection will be denoted by $P_{\mathfrak{J}}$. For simplicity, the subspace $\mathcal{H}_\pi(\mathfrak{S}_n)$ will be denoted by $\mathcal{H}_\pi^n$ and the correspondong orthogonal projections will be referred to as P_n. We observe that this defines an increasing family of closed subspaces. Finally, we define

$$\mathcal{H}_\pi^\infty = \bigcup_{n=0}^{\infty} \mathcal{H}_\pi^n = \bigcup_{\mathfrak{J}} \mathcal{H}_\pi(\mathfrak{J}),$$

where the last union is over all finite rooted subtrees $\mathfrak{J} \subseteq \mathfrak{X}$.

Using this notation, we will now define the concept of a tame representation:

Definition 6.3.1 A unitary representation π of K is said to be *tame* if the subspace $\mathcal{H}_\pi^\infty$ is dense in $\mathcal{H}_\pi$.

Remark 6.3.2 This approach to the concept of a tame representation is not new. It has been applied widely, most notably in the analysis of the infinite symmetric group, cf. [O4] and [O5]. It also appears in the paper [Du] for a matrix group in the field $\mathbb{F}_q$.

Remarkably, tame representations of K are exactly the ones arising by restriction of unitary representations of K_∞ - in line with our earlier claim. Even more surprisingly, they are the ones arising by restriction of representations of the semigroup $\Gamma(\mathfrak{X})$. Hence, the task of characterizing tame representations of K is equivalent to the one of characterizing unitary representations of K_∞ or semigroup representations of $\Gamma(\mathfrak{X})$. This is the content of the following theorem.

Theorem 6.3.3 *Let π be a unitary representation of K. The following three conditions are equivalent:*

1. *π is tame.*

2. π *may be extended to a unitary representation of* K_∞*.*

3. π *may be extended to a representation of the semigroup* $\Gamma(\mathfrak{X})$*.*

Remark 6.3.4 It follows by Corollary 6.2.7 that K is dense in $\Gamma(\mathfrak{X})$. Hence, the continuity requirement of a representation immediately implies that the extension τ to $\Gamma(\mathfrak{X})$ of a tame representation π in Theorem 6.3.3 is unique. Of course this also means that the same is true for its extension to K_∞. Furthermore, the fact that $K_\infty = \{\gamma \in \Gamma(\mathfrak{X}) \mid \gamma\gamma^* = \gamma^*\gamma = 1\}$ means that the restriction to K of a representation of $\Gamma(\mathfrak{X})$ is always unitary. This implies that there are bijective correspondances between the sets of tame representations of K, unitary representations of K_∞ and representations of the semigroups $\Gamma(\mathfrak{X})$. As a consequence, the study of these three concepts is essentially the same.

Furthermore, the extension of an irreducible tame representation of K to $\Gamma(\mathfrak{X})$ is clearly irreducible. Conversely, if τ is an irreducible representation of $\Gamma(\mathfrak{X})$ with representation space $\mathcal{H}_\tau$, its restriction to K is also irreducible. Indeed, let $\mathcal{M} \subseteq \mathcal{H}_\tau$ be a closed subspace invariant under K and denote by P the orthogonal projection onto $\mathcal{M}$. For $\gamma \in \Gamma(\mathfrak{X})$, we may by Corollary 6.2.7 choose a net $\{\gamma_\lambda\} \subseteq K$ which converges to γ. For $v \in \mathcal{M}$ and $w \in \mathcal{H}_\tau$, we see that

$$\begin{aligned}\langle \tau(\gamma)v, w\rangle &= \lim_\lambda \langle \tau(\gamma_\lambda)v, w\rangle = \lim_\lambda \langle P\tau(\gamma_\lambda)v, w\rangle = \lim_\lambda \langle \tau(\gamma_\lambda)v, Pw\rangle \\ &= \langle \tau(\gamma)v, Pw\rangle = \langle P\tau(\gamma)v, w\rangle\end{aligned}$$

This proves that $P\tau(\gamma)v = \tau(\gamma)v$, and so $\tau(\gamma)v \in \mathcal{M}$. Hence, τ is irreducible.

This shows that the sets of irreducible tame representations of K, of irreducible representations of K_∞ and of irreducible representations of $\Gamma(\mathfrak{X})$ are in bijective correspondance.

Proof. First assume 3. Since

$$K_\infty = \{\gamma \in \Gamma(\mathfrak{X}) \mid \gamma\gamma^* = \gamma^*\gamma = 1\},$$

the restriction of a representation of $\Gamma(\mathfrak{X})$ to K_∞ is a unitary representation. Furthermore, the topology on K_∞ is just the topology inherited by $\Gamma(\mathfrak{X})$ which proves 2.

Now assume 2., and let $h \in \mathcal{H}_\pi$. We will abuse notation and also denote the extended representation by π. Let $\epsilon > 0$. By continuity of π, we may find n such that

$$\|h - \pi(g)h\|_\pi < \frac{\epsilon}{2}$$

for all $g \in K_\infty(\mathfrak{S}_n)$ and so for all $g \in K(\mathfrak{S}_n)$. Since π is unitary and $K(\mathfrak{S}_n)$ is a subgroup of K_∞, it follows by the Alaoglu-Birkhoff mean ergodic theorem [BR, Proposition 4.3.4] that P_n is in the strong closure of the convex hull of $\pi(K(\mathfrak{S}_n))$. Hence, there exists an N, automorphisms $g_1, \dots, g_N \in K(\mathfrak{S}_n)$ and scalars $c_1, \dots, c_N \in [0,1]$ with $\sum_{k=1}^N c_k = 1$ such that

$$\left\| \sum_{k=1}^N c_k \pi(g_k)h - P_n h \right\|_\pi < \frac{\epsilon}{2}$$

This implies that

$$\begin{aligned}\|h - P_n h\|_\pi &\leq \left\| h - \sum_{k=1}^N c_k \pi(g_k)h \right\|_\pi + \left\| \sum_{k=1}^N c_k \pi(g_k)h - P_n h \right\|_\pi \\ &< \sum_{k=1}^N c_k \|h - \pi(g_k)h\|_\pi + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

This proves 1.

Now assume 1., i.e. that π is tame. We will complete the proof by proving 3., and so we will extend π to a representation τ of the semigroup $\Gamma(\mathfrak{X})$. To do this, consider a finite rooted subtree $\mathfrak{Z} \subseteq \mathfrak{X}$ with the property that $\mathcal{H}_\pi(\mathfrak{Z}) \neq 0$. Let $\gamma \in \Gamma(\mathfrak{Z})$. By Proposition 6.2.5, we may choose $g \in K$ such that $p_{\mathfrak{Z}}(g) = \gamma$. If $\tilde{g} \in K$ is another automorphism satisfying that $p_{\mathfrak{Z}}(\tilde{g}) = \gamma$, it follows by Proposition 6.2.5 that we may find $k_1, k_2 \in K(\mathfrak{Z})$ such that $\tilde{g} = k_1 g k_2$. For $h \in \mathcal{H}_\pi(\mathfrak{Z})$, this implies that

$$P_{\mathfrak{Z}}\pi(\tilde{g})h = P_{\mathfrak{Z}}\pi(k_1)\pi(g)\pi(k_2)h = \pi(k_1)P_{\mathfrak{Z}}\pi(g)h = P_{\mathfrak{Z}}\pi(g)h$$

since $P_{\mathfrak{Z}}$ and $\pi(k_1)$ commute by [Fo, Proposition 3.4]. Hence, we may (well)-define a map $\tau_{\mathfrak{Z}}$ from $\Gamma(\mathfrak{Z})$ into $\mathcal{B}(\mathcal{H}_\pi(\mathfrak{Z}))$, the algebra of bounded operators on $\mathcal{H}_\pi(\mathfrak{Z})$, by

$$\tau_{\mathfrak{Z}}(\gamma) = P_{\mathfrak{Z}}\pi(g)_{|\mathcal{H}_\pi(\mathfrak{Z})} \tag{6.4}$$

Clearly, $\tau_{\mathfrak{Z}}(\gamma)$ is a contraction, i.e. $\|\tau_{\mathfrak{Z}}(\gamma)\| \leq 1$. Furthermore, we observe that

$$\begin{aligned} \tau_{\mathfrak{Z}}(\gamma^*) &= P_{\mathfrak{Z}}\pi(g^{-1})_{|\mathcal{H}_\pi(\mathfrak{Z})} = (P_{\mathfrak{Z}}\pi(g)^* P_{\mathfrak{Z}})_{|\mathcal{H}_\pi(\mathfrak{Z})} \\ &= (P_{\mathfrak{Z}}\pi(g)P_{\mathfrak{Z}})^*_{|\mathcal{H}_\pi(\mathfrak{Z})} = (P_{\mathfrak{Z}}\pi(g)_{|\mathcal{H}_\pi(\mathfrak{Z})})^* = \tau_{\mathfrak{Z}}(\gamma)^* \end{aligned} \tag{6.5}$$

Finally, we see that for a finite rooted subtree $\mathfrak{Z}' \subseteq \mathfrak{X}$ and $\mathfrak{Z}' \subseteq \mathfrak{Z}$

$$p_{\mathfrak{Z},\mathfrak{Z}'}(\gamma) = p_{\mathfrak{Z},\mathfrak{Z}'}(p_{\mathfrak{Z}}(g)) = p_{\mathfrak{Z}'}(g)$$

and so

$$\tau_{\mathfrak{Z}'}(p_{\mathfrak{Z},\mathfrak{Z}'}(\gamma)) = P_{\mathfrak{Z}'}\pi(g)_{|\mathcal{H}_\pi(\mathfrak{Z}')} = P_{\mathfrak{Z}'}P_{\mathfrak{Z}}\pi(g)_{|\mathcal{H}_\pi(\mathfrak{Z}')} = P_{\mathfrak{Z}'}\tau_{\mathfrak{Z}}(\gamma)_{|\mathcal{H}_\pi(\mathfrak{Z}')} \tag{6.6}$$

since $\mathcal{H}_\pi(\mathfrak{Z}') \subseteq \mathcal{H}_\pi(\mathfrak{Z})$.

$\mathcal{H}_\pi^\infty$ is by assumption dense in $\mathcal{H}_\pi$, and since $\{\mathcal{H}_\pi^m\}_{m=n}^\infty$ form an increasing sequence, it follows by basic Hilbert space theory that we may choose an orthonormal basis $\{e_j\}_{j\in J}$ for $\mathcal{H}_\pi$ with the property that there exists an increasing sequence $\{J_m\}_{m=n}^\infty$ of subsets of J such that $J = \bigcup_{m=n}^\infty J_m$ and $\{e_j\}_{j\in J_m}$ is an orthonormal basis for $\mathcal{H}_\pi^m$ for each $m \geq n$.

Now let $\gamma \in \Gamma(\mathfrak{X})$, and let $h \in \mathcal{H}_\pi^\infty$. Choose n minimal such that $\mathcal{H}_\pi^n \neq \{0\}$. For $k \geq n$ such that $h \in \mathcal{H}_\pi^k$, we write

$$\tau_{\mathfrak{S}_k}(p_{\mathfrak{S}_k}(\gamma))h = \sum_{j\in J_k} a_j^k e_j$$

By the relation (6.6), we observe for $m \geq k$ that

$$\sum_{j\in J_k} a_j^k e_j = \tau_{\mathfrak{S}_k}(p_{\mathfrak{S}_k}(\gamma))h = P_k\tau_{\mathfrak{S}_m}(\gamma)h = P_k(\sum_{j\in J_m} a_j^m e_j) = \sum_{j\in J_k} a_j^m e_j$$

which implies that $a_j^k = a_j^m$ for $j \in J_k$. By this independence fact, we may for $j \in J$ choose $k \geq n$ such that $j \in J_k$ and such that $h \in \mathcal{H}_\pi^k$ and define $a_j = a_j^k$. By the contraction property of $\tau_{\mathfrak{S}_k}(p_{\mathfrak{S}_k}(\gamma))$, it is true that $\sum_{j\in J_k} |a_j|^2 \leq \|h\|^2$ for all $k \geq n$ with $h \in \mathcal{H}_\pi^k$, and so we deduce that $\sum_{j\in J} |a_j|^2 \leq \|h\|^2$. Hence, we may define a (clearly linear) operator $\tau(\gamma)$ on $\mathcal{H}_\pi^\infty$ by

$$\tau(\gamma)h = \sum_{j\in J} a_j e_j$$

which by the above observation is a contraction. By denseness of $\mathcal{H}_\pi^\infty$, we may uniquely extend $\tau(\gamma)$ to a linear contraction on $\mathcal{H}_\pi$ which by construction has the property that

$$P_k\tau(\gamma)_{|\mathcal{H}_\pi^k} = \tau_{\mathfrak{S}_k}(p_{\mathfrak{S}_k}(\gamma)) \tag{6.7}$$

for all $k \geq n$. More generally, for a finite rooted subtree $\mathfrak{J} \subseteq \mathfrak{X}$ with $\mathcal{H}_\pi(\mathfrak{J}) \neq \{0\}$, we may choose $k \geq n$ such that $\mathfrak{J} \subseteq \mathfrak{S}_k$. Combining (6.6) and (6.7), this makes us realize that

$$\begin{aligned} P_{\mathfrak{J}}\tau(\gamma)_{|\mathcal{H}_\pi(\mathfrak{J})} &= P_{\mathfrak{J}}P_k\tau(\gamma)_{|\mathcal{H}_\pi(\mathfrak{J})} = P_{\mathfrak{J}}\tau_{\mathfrak{S}_k}(p_{\mathfrak{S}_k}(\gamma))_{|\mathcal{H}_\pi(\mathfrak{J})} \\ &= \tau_{\mathfrak{J}}(p_{\mathfrak{S}_k,\mathfrak{J}}(p_{\mathfrak{S}_k}(\gamma))) = \tau_{\mathfrak{J}}(p_{\mathfrak{J}}(\gamma)) \end{aligned} \tag{6.8}$$

In this way, we have defined a map τ from $\Gamma(\mathfrak{X})$ into $\Gamma(\mathcal{H}_\pi)$, the semigroup of linear contractions on $\mathcal{H}_\pi$. For $g \in K$, it follows by (6.7) and (6.4) that

$$P_k\tau(g)_{|\mathcal{H}_\pi^k} = \tau_{\mathfrak{S}_k}(p_{\mathfrak{S}_k}(g)) = P_k\pi(g)_{|\mathcal{H}_\pi^k}$$

for all $k \geq n$. By denseness of $\mathcal{H}_\pi^\infty$, the projections P_k converge strongly to the identity, and so $\tau(g)$ and $\pi(g)$ agree on $\mathcal{H}_\pi^\infty$. By contintuity, this implies that $\tau(g) = \pi(g)$, and so τ is an extension of π. As a consequence, $\tau(1) = I$.

By (6.7), Lemma 6.2.2 and (6.5), we see that for $\gamma \in \Gamma(\mathfrak{X})$

$$\begin{aligned} P_k\tau(\gamma^*)_{|\mathcal{H}_\pi^k} &= \tau_{\mathfrak{S}_k}(p_{\mathfrak{S}_k}(\gamma^*)) = \tau_{\mathfrak{S}_k}(p_{\mathfrak{S}_k}(\gamma))^* = (P_k\tau(\gamma)_{|\mathcal{H}_\pi^k})^* \\ &= ((P_k\tau(\gamma)P_k)^*)_{|\mathcal{H}_\pi^k} = (P_k\tau(\gamma)^*)_{|\mathcal{H}_\pi^k} \end{aligned}$$

for all $k \geq n$. By strong convergence of the P_k's, this implies that $\tau(\gamma^*)$ and $\tau(\gamma)$ agree on $\mathcal{H}_\pi^\infty$ which by continuity and denseness implies that $\tau(g^*) = \tau(g)^*$.

Now let $\gamma \in \Gamma(\mathfrak{X})$ and consider a net $\{\gamma_\lambda\} \subseteq \Gamma(\mathfrak{X})$ which converges to γ. For $h, v \in \mathcal{H}_\pi^k$, it follows by (*) that

$$\langle \tau(\gamma_\lambda)h, v\rangle_\pi = \langle \tau(\gamma_\lambda)h, P_k v\rangle_\pi = \langle P_k\tau(\gamma_\lambda)h, v\rangle_\pi = \langle P_k\tau(\gamma)h, v\rangle_\pi = \langle \tau(\gamma)h, v\rangle_\pi$$

for λ such that $p_{\mathfrak{S}_k}(\gamma) = p_{\mathfrak{S}_k}(\gamma_\lambda)$. By definition of the topology on $\Gamma(\mathfrak{X})$, this implies that the net $\{\langle \tau(g_\lambda)h, v\rangle\}$ converges to $\langle \tau(g)h, v\rangle$ which by denseness of $\mathcal{H}_\pi^\infty$ implies that $\{\tau(g_\lambda)\}$ converges weakly to $\tau(g)$. Hence, τ is continuous.

Finally, we must show that

$$\tau(\gamma_1\gamma_2) = \tau(\gamma_1)\tau(\gamma_2) \tag{6.9}$$

for all $\gamma_1, \gamma_2 \in \Gamma(\mathfrak{X})$. To see this, fix $\gamma_1 \in K$. Since τ extends π, (6.9) is true for all $\gamma_2 \in K$. Since the multiplication in $\Gamma(\mathfrak{X})$ is separately continuous, both sides of (6.9) are continuous in γ_2. Since they agree on K which by Corollary 6.2.7 is dense in $\gamma(\mathfrak{X})$, (6.9) holds for all $\gamma_2 \in \Gamma(\mathfrak{X})$. The proof is completed by fixing $\gamma_2 \in \Gamma(\mathfrak{X})$ and repeating the argument for γ_1.

This proves that τ is a representation of $\Gamma(\mathfrak{X})$ extending π. □

By extending a tame reprentation of K to K_∞, we get the following corollary (which may actually easily be directly proved by using the denseness of $\mathcal{H}_\pi^\infty$ and the fact that the subgroups $K(\mathfrak{S}_n)$ constitute a local base at the identity for the topology on K inherited by K_∞).

Corollary 6.3.5 *Let π be a tame representation of K. Then π is continuous in the topology on K inherited from K_∞.*

Theorem 6.3.3 proves that every tame representation π of K in a natural way is associated with a unique representation of the semigroup $\Gamma(\mathfrak{X})$. However, in a similar way it is also closely related to representations of the semigroups $\Gamma(\mathfrak{J})$ for all finite rooted subtrees $\mathfrak{J} \subseteq \mathfrak{X}$ with $\mathcal{H}_\pi(\mathfrak{J}) \neq \{0\}$. In fact, the maps $\tau_{\mathfrak{J}}$ constructed in the proof of Theorem 6.3.3 are such representations. This is the content of Theorem 6.3.7.

For the proof, we need the following Lemma 6.3.6 and some notation. For $\mathfrak{J}$ as above, we denote by $\epsilon_{\mathfrak{J}} \in \Gamma(\mathfrak{X})$ the partial automorphism of $\mathfrak{X}$ which is just the identity on $\mathfrak{J}$.

LEMMA 6.3.6 *Let π be a tame representation of K, and denote by τ its extension to $\Gamma(\mathfrak{X})$ from Theorem 6.3.3. For a finite rooted subtree $\mathfrak{J} \subseteq \mathfrak{X}$ with $\mathcal{H}_\pi(\mathfrak{J}) \neq \{0\}$, it is true that*

$$\tau(\epsilon_{\mathfrak{J}}) = P_{\mathfrak{J}}$$

PROOF. Let $\mathfrak{J}$ be as in the lemma. Clearly, $\epsilon_{\mathfrak{J}}$ is idempotent and self-adjoint, i.e. $\epsilon_{\mathfrak{J}}^2 = \epsilon_{\mathfrak{J}} = \epsilon_{\mathfrak{J}}*$. Hence, the fact that τ is a representation implies that $\tau(\epsilon_{\mathfrak{J}})$ is an orthogonal projection. Define $\mathcal{U}(\mathfrak{J}) = \tau(\epsilon_{\mathfrak{J}})(\mathcal{H}_\pi)$. We will prove that $\mathcal{U}(\mathfrak{J}) = \mathcal{H}_\pi(\mathfrak{J})$.

Let $g \in K(\mathfrak{J})$. Then $g\epsilon_{\mathfrak{J}} = \epsilon_{\mathfrak{J}}$ and so $\tau(g)\tau(\epsilon_{\mathfrak{J}}) = \tau(\epsilon_{\mathfrak{J}})$. This implies that $\mathcal{U}(\mathfrak{J}) \subseteq \mathcal{H}_\pi(\mathfrak{J})$ and so $P_{\mathfrak{J}}\tau(\epsilon_{\mathfrak{J}}) = \tau(\epsilon_{\mathfrak{J}})$.

By the relation (6.8), we see that

$$\tau(\epsilon_{\mathfrak{J}})_{|\mathcal{H}_\pi(\mathfrak{J})} = P_{\mathfrak{J}}\tau(\epsilon_{\mathfrak{J}})_{|\mathcal{H}_\pi(\mathfrak{J})} = \tau_{\mathfrak{J}}(p_{\mathfrak{J}}(\epsilon_{\mathfrak{J}})) = \tau_{\mathfrak{J}}(p_{\mathfrak{J}}(1)) = P_{\mathfrak{J}}\tau(1)_{|\mathcal{H}_\pi(\mathfrak{J})} = I_{|\pi(\mathfrak{J})}$$

This proves that $\mathcal{H}_\pi(\mathfrak{J}) \subseteq \mathcal{U}(\mathfrak{J})$ which finishes the proof. □

THEOREM 6.3.7 *Let π be a tame representation of K. For each finite rooted subtree $\mathfrak{J} \subseteq \mathfrak{X}$ with $\mathcal{H}_\pi(\mathfrak{J}) \neq \{0\}$, the map $\tau_{\mathfrak{J}}$ constructed in the proof of Theorem 6.3.3 is a representation of $\Gamma(\mathfrak{J})$ on $\mathcal{H}_\pi(\mathfrak{J})$.*

PROOF. Let $\mathfrak{J}$ be as in the theorem. By (6.5), $\tau_{\mathfrak{J}}$ commutes with the involution, and it is an immediate consequence of the definition of $\tau_{\mathfrak{J}}$ that $\tau_{\mathfrak{J}}(1)$ is the identity on $\mathcal{H}_\pi(\mathfrak{J})$. Furthermore, the fact that $\Gamma(\mathfrak{J})$ is discrete makes continuity considerations superfluous. Hence, we are left with the multiplicativity of $\tau_{\mathfrak{J}}$.

To prove this, we observe that every partial automorphism of $\mathfrak{J}$ is also a partial automorphism of $\mathfrak{X}$. This gives rise to a natural imbedding $\gamma \mapsto \tilde{\gamma}$ of $\Gamma(\mathfrak{J})$ into $\Gamma(\mathfrak{X})$ which is clearly multiplicative. Furthermore, we construct a natural imbedding $A \mapsto \tilde{A}$ of $\mathcal{B}(\mathcal{H}_\pi(\mathfrak{J}))$ into $\mathcal{B}(\mathcal{H}_\pi)$ by defining $\tilde{A} = AP_{\mathfrak{J}}$. For $\gamma \in \Gamma(\mathfrak{J})$, we now observe by (6.8) and Lemma 6.3.6 that

$$\widetilde{\tau_{\mathfrak{J}}(\gamma)} = \tau_{\mathfrak{J}}(\gamma)P_{\mathfrak{J}} = P_{\mathfrak{J}}\tau(\tilde{\gamma})P_{\mathfrak{J}} = \tau(\epsilon_{\mathfrak{J}})\tau(\tilde{\gamma})\tau(\epsilon_{\mathfrak{J}}) = \tau(\tilde{\gamma})$$

For $\gamma_1, \gamma_2 \in \Gamma(\mathfrak{J})$, this yields the identity

$$\tau_{\mathfrak{J}}(\gamma_1)\tau_{\mathfrak{J}}(\gamma_2)P_{\mathfrak{J}} = \widetilde{\tau_{\mathfrak{J}}(\gamma_1)}\widetilde{\tau_{\mathfrak{J}}(\gamma_2)} = \tau(\tilde{\gamma_1})\tau(\tilde{\gamma_2}) = \tau(\widetilde{\gamma_1\gamma_2}) = \tau_{\mathfrak{J}}(\gamma_1\gamma_2)P_{\mathfrak{J}}$$

which shows that $\tau_{\mathfrak{J}}$ is multiplicative. □

We finish this section by a technical lemma which will be needed in the following sections. Let $\mathfrak{J}$ be as in Theorem 6.3.7 and consider a finite rooted subtree $\mathfrak{J}' \subseteq \mathfrak{J}$. We denote by $\epsilon_{\mathfrak{J},\mathfrak{J}'} \in \Gamma(\mathfrak{J})$ the partial automorphism of $\mathfrak{J}$ which is just the identity on $\mathfrak{J}'$. Along the lines of Lemma 6.3.6, we observe that the representation $\tau_{\mathfrak{J}}$ has the following important property:

LEMMA 6.3.8 *Let π be a tame representation of K, and let $\mathfrak{J}$ and $\mathfrak{J}'$ be as above. Denote by $\tau_{\mathfrak{J}}$ the corresponding representation of $\Gamma(\mathfrak{J})$ from Theorem 6.3.3. Then $\tau_{\mathfrak{J}}(\epsilon_{\mathfrak{J},\mathfrak{J}'})$ is the orthogonal projection onto the closed subspace $\mathcal{H}_\pi(\mathfrak{J}')$ of $\mathcal{H}_\pi(\mathfrak{J})$.*

PROOF. Let τ be the extension of π to a representation of $\Gamma(\mathfrak{X})$ from Theorem 6.3.3. Since it is clearly true that $\epsilon_{\mathfrak{J},\mathfrak{J}'} = p_{\mathfrak{J}}(\epsilon_{\mathfrak{J}'})$, we use (6.8) and Lemma 6.3.6 to see that

$$\tau_{\mathfrak{J}}(\epsilon_{\mathfrak{J},\mathfrak{J}'}) = (P_{\mathfrak{J}}\tau(\epsilon_{\mathfrak{J}'}))_{|\mathcal{H}_\pi(\mathfrak{J})} = (P_{\mathfrak{J}}P_{\mathfrak{J}'})_{|\mathcal{H}_\pi(\mathfrak{J})} = P_{\mathfrak{J}'|\mathcal{H}_\pi(\mathfrak{J})}$$

This finishes the proof. □

With the connection between tame representations of K and representations of the finite semigroups $\Gamma(\mathfrak{J})$, it is of natural interest to study general representations of the latter. This will be the focus of the following section.

6.4 Representations of the finite semigroups $\Gamma(\mathfrak{J})$

Using Theorems 6.3.3 and 6.3.7, we see that we to every tame representation π of K may associate representations of $\Gamma(\mathfrak{X})$ and $\Gamma(\mathfrak{J})$ for finite rooted subtrees $\mathfrak{J} \subseteq \mathfrak{X}$ with $\mathcal{H}_\pi(\mathfrak{J}) \neq \{0\}$. These representations may be used to give a complete classification of all irreducible tame representations of K. To do this, we need to develop some theory on representations of the finite semigroups $\Gamma(\mathfrak{J})$.

Fix a finite rooted subtree $\mathfrak{J} \subseteq \mathfrak{X}$ for the remainder of this section. For a finite rooted subtree $\mathfrak{J}' \subseteq \mathfrak{J}$, we denote by $\delta_{\mathfrak{J}'} \in \Gamma(\mathfrak{J})$ the partial automorphism of $\mathfrak{J}$ which is just the identity on $\mathfrak{J}'$. With this notation, we observe that $\delta_{\mathfrak{J}} = 1$, that $\delta_{\{o\}} = 0$ and that - with the notation of section 6.3 - $\delta_{\mathfrak{J}'} = \epsilon_{\mathfrak{J},\mathfrak{J}'}$.

Clearly, $\delta_{\mathfrak{J}'}^* = \delta_{\mathfrak{J}'} = \delta_{\mathfrak{J}'}^2$, i.e. it is self-adjoint and idempotent. Conversely, if $\delta \in \Gamma(\mathfrak{J})$ is self-adjoint and idempotent, the first condition tells that the domain and range of δ must coincide and the second reveals that δ must be the identity on its domain. Hence, $\delta = \delta_{\mathfrak{J}'}$ where we by $\mathfrak{J}'$ denote the domain of δ.

Using this observation, we may introduce a partial ordering $\leq$ on the set of self-adjoint idempotents in $\Gamma(\mathfrak{J})$ by declaring that $\delta_{\mathfrak{J}_1} \leq \delta_{\mathfrak{J}_2}$ if and only if $\mathfrak{J}_1 \subseteq \mathfrak{J}_2$.

Fix for the remainder of this section a representation τ of $\Gamma(\mathfrak{J})$ with representation space $\mathcal{H}_\tau$. For every finite rooted subtree $\mathfrak{J}' \subseteq \mathfrak{J}$, the fact that $\delta_{\mathfrak{J}'}$ is self-adjoint and idempotent implies that the same is true for $\tau(\delta_{\mathfrak{J}'})$. Hence, it is a projection. We denote by $\mathcal{H}_\tau(\mathfrak{J}')$ its range.

The following important observation is immediate.

Lemma 6.4.1 *Let $\gamma \in \Gamma(\mathfrak{J})$, and let $\mathfrak{J}'$ be its domain. If $\mathcal{H}_\tau(\mathfrak{J}') = \{0\}$, it is true that $\tau(\gamma) = 0$.*

Proof. Since $\mathcal{H}_\tau(\mathfrak{J}') = \{0\}$, we have that $\tau(\delta_{\mathfrak{J}'}) = 0$. Since $\gamma^*\gamma = \delta_{\mathfrak{J}'}$, this implies that $\tau(\gamma)^*\tau(\gamma) = 0$, and for $h \in \mathcal{H}_\tau$ a consequence of this is that

$$\|\tau(\gamma)h\|_\tau^2 = \langle \tau(\gamma)h, \tau(\gamma)h \rangle_\tau = \langle \tau(\gamma)^*\tau(\gamma)h, h \rangle_\tau = 0$$

This shows that $\tau(\gamma) = 0$. □

Now assume that τ is irreducible. Studying the representation τ, it turns out that it is of special interest to consider finite rooted subtrees $\mathfrak{Y} \subseteq \mathfrak{J}$ with the property that $\mathcal{H}_\tau(\mathfrak{Y}) \neq \{0\}$ while $\mathcal{H}_\tau(\mathfrak{Y}') = \{0\}$ for all finite rooted subtrees $\mathfrak{Y}' \subsetneq \mathfrak{Y}$. We will refer to such a subtree as a *basic subtree for τ*. Since $\mathcal{H}_\tau(\mathfrak{J}) = \mathcal{H}_\tau$, the existence of such a subtree is immediate. Observe that for a basic subtree $\mathfrak{Y}$ we may in a natural way regard $\mathrm{Aut}_o(\mathfrak{Y})$ as a subsemigroup of $\Gamma(\mathfrak{J})$. One reason for the importance of basic subtrees is the following lemma which establishes a connection between τ and an irreducible unitary representation of $\mathrm{Aut}_o(\mathfrak{Y})$ or - equivalently - an irreducible unitary representation of $\tilde{K}_\infty(\mathfrak{Y})$ which is trivial on $K_\infty(\mathfrak{Y})$.

Lemma 6.4.2 *Assume that τ is an irreducible representation of $\Gamma(\mathfrak{J})$. Let $\mathfrak{Y} \subseteq \mathfrak{J}$ be a basic subtree for τ. Then $\mathcal{H}_\tau(\mathfrak{Y})$ is invariant under $\mathrm{Aut}_o(\mathfrak{Y})$), and the corresponding representation of $\mathrm{Aut}_o(\mathfrak{Y})$ is irreducible and unitary.*

Proof. Since $\gamma = \delta_{\mathfrak{Y}}\gamma$ for all $\gamma \in \mathrm{Aut}_o(\mathfrak{Y})$, it follows that $\tau(\gamma) = \tau(\delta_{\mathfrak{Y}})\tau(\gamma)$ and so $\mathcal{H}_\tau(\mathfrak{Y})$ is invariant under $\mathrm{Aut}_o(\mathfrak{Y})$.

Since $\Gamma(\mathfrak{J})$ is finite, it follows by [O4, Lemma 1.10] and the irreducibility of τ that $\mathcal{H}_\tau$ is finite dimensional. Hence, it follows by 1.5 in [O4] that the algebra $\mathcal{B}(\mathcal{H}_\tau)$ of bounded operators on $\mathcal{H}_\tau$ is spanned by the operators $\tau(\gamma)$ with $\gamma \in \Gamma(\mathfrak{J})$.

Now let A be a bounded operator on $\mathcal{H}_\tau(\mathfrak{Y})$. We may regard $A\tau(\delta_{\mathfrak{Y}})$ as a bounded operator on $\mathcal{H}_\tau$. Hence, there exists $\gamma_1, \ldots, \gamma_n \in \Gamma(\mathfrak{J})$ and $a_1, \ldots, a_n \in \mathbb{C}$ such that

$$A\tau(\delta_{\mathfrak{Y}}) = \sum_{j=1}^{n} a_j \tau(\gamma_j)$$

Multiplying by $\tau(\delta_{\mathfrak{Y}})$ on both sides and using that A takes values in $\mathcal{H}_\tau(\mathfrak{Y})$, this implies that

$$A\tau(\delta_{\mathfrak{Y}}) = \sum_{j=1}^{n} a_j \tau(\delta_{\mathfrak{Y}}\gamma_j\delta_{\mathfrak{Y}})$$

However, it follows by Lemma 6.4.1 and the fact that $\mathfrak{Y}$ is basic for τ that $\tau(\delta_{\mathfrak{Y}}\gamma_j\delta_{\mathfrak{Y}}) \neq 0$ only if the domain of $\delta_{\mathfrak{Y}}\gamma_j\delta_{\mathfrak{Y}}$ is all of $\mathfrak{Y}$, i.e. only if $\delta_{\mathfrak{Y}}\gamma_j\delta_{\mathfrak{Y}} \in \mathrm{Aut}_o(\mathfrak{Y})$. Hence, A may be written as a linear combination of operators $\tau(\gamma)_{|\mathcal{H}_\tau(\mathfrak{Y})}$ with $\gamma \in \mathrm{Aut}_o(\mathfrak{Y})$. It now follows by 1.5 in [O4] that $\mathcal{H}_\tau(\mathfrak{Y})$ is irreducible under $\mathrm{Aut}_o(\mathfrak{Y})$.

Finally, it is evident that $\tau(\gamma)\tau(\gamma)^* = \tau(\delta_{\mathfrak{Y}}) = \tau(\gamma)^*\tau(\gamma)$ for $\gamma \in \mathrm{Aut}_o(\mathfrak{Y})$, and since $\tau(\delta_{\mathfrak{Y}})$ is the identity on $\mathcal{H}_\tau(\mathfrak{Y})$, this implies that the required representation is unitary. □

The usefulness of this observation is underlined by Lemma 6.4.4 which states that an irreducible representation of $\Gamma(\mathfrak{J})$ is uniquely determined by a basic subtree $\mathfrak{Y}$ and the corresponding irreducible unitary representation of $\mathrm{Aut}_o(\mathfrak{Y})$.

For its proof, we need the following general lemma:

Lemma 6.4.3 *Let S be a topological semigroup, and let τ and τ' be representations of S with representation spaces $\mathcal{H}_\tau$ and $\mathcal{H}_{\tau'}$. Suppose that there exist cyclic vectors $h \in \mathcal{H}_\tau$ and $h' \in \mathcal{H}_{\tau'}$ such that*

$$\langle \tau(s)h, h\rangle_\tau = \langle \tau'(s)h', h'\rangle_{\tau'} \tag{6.10}$$

for all $s \in S$. Then τ and τ' are equivalent.

Proof. For $s_1, s_2 \in S$, the assumption (6.10) implies that

$$\langle \tau(s_1)h, \tau(s_2)h\rangle_\tau = \langle \tau(s_2^* s_1)h, h\rangle_\tau = \langle \tau'(s_2^* s_1)h', h'\rangle_{\tau'} = \langle \tau'(s_1)h', \tau'(s_2)h'\rangle_{\tau'}$$

This clearly shows that we may well-define a linear isometry T from span $\{\tau(s)h \mid s \in S\}$ onto span $\{\tau'(s)h' \mid s \in S\}$ by the condition that $T\tau(s)h = \tau'(s)h$ for all $s \in S$. Since h and h' are cyclic for τ and τ', this extends to a unitary operator from $\mathcal{H}_\tau$ onto $\mathcal{H}_{\tau'}$ which we by abuse of notation also denote by T.

For $s, s' \in S$, we have that

$$T\tau(s)\tau(s')h = T\tau(ss')h = \tau'(ss')h' = \tau'(s)\tau'(s')h' = \tau'(s)T\tau(s')h$$

Since h is cyclic for τ, this implies that $T\tau(s)$ and $\tau'(s)T$ agree on a dense subspace of $\mathcal{H}_\tau$. By continuity, this shows that $T\tau(s) = \tau'(s)T$ proving the equivalence. □

Lemma 6.4.4 *Let τ and τ' be irreducible representations of $\Gamma(\mathfrak{J})$ with representation spaces $\mathcal{H}_\tau$ and $\mathcal{H}_{\tau'}$, respectively. Assume that $\mathfrak{Y} \subseteq \mathfrak{J}$ is a basic subtree for τ and τ' and that the corresponding irreducible unitary representations of* $\mathrm{Aut}_o(\mathfrak{Y})$ *from Lemma 6.4.2 are equivalent. Then τ and τ' are equivalent.*

Proof. Let T be a unitary operator from $\mathcal{H}_\tau(\mathfrak{Y})$ onto $\mathcal{H}_{\tau'}(\mathfrak{Y})$ establishing the equivalence of the assumptions. Let $h \in \mathcal{H}_\pi(\mathfrak{Y})$ be non-zero. For $\gamma \in \Gamma(\mathfrak{J})$, we observe by Lemma 6.4.1 and the fact that $\mathfrak{Y}$ is basic that $\tau(\delta_\mathfrak{Y}\gamma\delta_\mathfrak{Y}) \neq 0$ (resp. $\tau'(\delta_\mathfrak{Y}\gamma\delta_\mathfrak{Y}) \neq 0$) only if $\delta_\mathfrak{Y}\gamma\delta_\mathfrak{Y} \in \mathrm{Aut}_o(\mathfrak{Y})$. Hence, we see that

$$\begin{aligned}\langle \tau(\gamma)h, h\rangle_\tau &= \langle \tau(\gamma)\tau(\delta_\mathfrak{Y})h, \tau(\delta_\mathfrak{Y})h\rangle_\tau = \langle \tau(\delta_\mathfrak{Y}\gamma\delta_\mathfrak{Y})h, h\rangle_\tau = \langle T\tau(\delta_\mathfrak{Y}\gamma\delta_\mathfrak{Y})h, Th\rangle_{\tau'} \\ &= \left\langle \tau'(\delta_\mathfrak{Y}\gamma\delta_\mathfrak{Y})Th, Th\right\rangle_{\tau'} = \left\langle \tau'(\gamma)Th, Th\right\rangle_{\tau'}\end{aligned}$$

By irreducibility of τ and τ', h and Th are cyclic for τ and τ', respectively. Lemma 6.4.3 now tells that τ and τ' are equivalent. □

Now fix a finite rooted subtree $\mathfrak{Y} \subseteq \mathfrak{J}$ and an irreducible unitary representation π of $\mathrm{Aut}_o(\mathfrak{Y})$ with representation space $\mathcal{H}_\pi$. In light of Lemma 6.4.4, it is of interest to ask whether there exists an irreducible representation τ of $\Gamma(\mathfrak{J})$ with $\mathfrak{Y}$ as a basic subtree and with π as the unitary representation of $\mathrm{Aut}_o(\mathfrak{Y})$ arising from Lemma 6.4.2. The answer turns out to be affirmative as we will see below.

We will construct a representation $\tau = \tau_{\mathfrak{Y},\pi}$ of $\Gamma(\mathfrak{J})$ with the required property. To do this, we denote by $\mathcal{I}_\mathfrak{Y}$ the set of injective maps $h : \mathfrak{Y} \to \mathfrak{J}$ with the property that $h(o) = o$ and that $h(x)$ and $h(y)$ are neighbours in $\mathfrak{J}$ if and only if x and y are neighbours in $\mathfrak{Y}$. Clearly, $\mathcal{I}_\mathfrak{Y}$ is finite. We define $\mathcal{H}_\tau$ to be the set of functions $f : \mathcal{I}_\mathfrak{Y} \to \mathcal{H}_\pi$ with the property that

$$f(h \circ g^{-1}) = \pi(g)(f(h)) \tag{6.11}$$

for all $g \in \mathrm{Aut}_o(\mathfrak{Y})$ and all $h \in \mathcal{I}_\mathfrak{Y}$. $\mathcal{H}_\tau$ is clearly a vector space, and we equip it with an inner product $\langle \cdot, \cdot\rangle_\tau$ by defining

$$\langle f_1, f_2\rangle_\tau = \sum_{h\in\mathcal{I}_\mathfrak{Y}} \langle f_1(h), f_2(h)\rangle_\pi$$

Standard arguments show that $\mathcal{H}_\tau$ is a Hilbert space. If we for $\gamma \in \Gamma(\mathfrak{J})$ by $\mathrm{dom}(\gamma)$ denote the domain of γ, we make the definition that

$$(\tau(\gamma)f)(h) = \begin{cases} f(\gamma^* \circ h) & \text{for } h(\mathfrak{Y}) \subseteq \mathrm{dom}(\gamma^*) \\ 0 & \text{otherwise}\end{cases}$$

for all $h \in \mathcal{I}_\mathfrak{Y}$, $f \in \mathcal{H}_\tau$ and $\gamma \in \Gamma(\mathfrak{J})$. Elementary considerations prove that $\tau(\gamma)f \in \mathcal{H}_\tau$ for all $f \in \mathcal{H}_\tau$ and for all $\gamma \in \Gamma(\mathfrak{J})$ and that $\tau(\gamma)$ is a linear contraction on $\mathcal{H}_\tau$ for all $\gamma \in \Gamma(\mathfrak{J})$. Furthermore, it is immediate that $\tau(\gamma_1\gamma_2) = \tau(\gamma_1)\tau(\gamma_2)$ for all $\gamma_1, \gamma_2 \in \Gamma(\mathfrak{J})$ and that $\gamma(1) = I$. Finally, we have for $\gamma \in \Gamma(\mathfrak{J})$ that

$$\begin{aligned}\langle \tau(\gamma^*)f_1, f_2\rangle_\tau &= \sum_{h\in\mathcal{I}_\mathfrak{Y}, h(\mathfrak{Y})\subseteq \mathrm{dom}(\gamma)} \langle f_1(\gamma \circ h), f_2(h)\rangle_\pi \\ &= \sum_{h\in\mathcal{I}_\mathfrak{Y}, h(\mathfrak{Y})\subseteq \mathrm{dom}(\gamma)} \langle f_1(\gamma \circ h), f_2(\gamma^* \circ \gamma \circ h)\rangle_\pi \\ &= \sum_{h\in\mathcal{I}_\mathfrak{Y}, h(\mathfrak{Y})\subseteq \mathrm{dom}(\gamma^*)} \langle f_1(h), f_2(\gamma^* \circ h)\rangle_\pi \\ &= \langle f_1, \tau(\gamma)f_2\rangle_\tau\end{aligned}$$

for all $f_1, f_2 \in \mathcal{H}_\tau$ which proves $\tau(\gamma^*) = \tau(\gamma)^*$. Since $\Gamma(\mathfrak{J})$ is discrete, the map τ is automatically weakly continuous, and so τ is a representation of $\Gamma(\mathfrak{J})$.

For a finite rooted subtree $\mathfrak{J}' \subseteq \mathfrak{J}$ and for $f \in \mathcal{H}_\tau$, it is evident that $\tau(\delta_{\mathfrak{J}'})f = f$ if and only if f is concentrated on the set $\{h \in \mathcal{I}_{\mathfrak{Y}} \mid h(\mathfrak{Y}) \subseteq \mathfrak{J}'\}$ and so $\mathcal{H}_\tau(\mathfrak{J}')$ consists exactly of the functions f with this property. If $\mathfrak{J}' \subsetneq \mathfrak{Y}$ the only function with this property is the 0-function and so $\mathcal{H}_\tau(\mathfrak{J}') = \{0\}$ for such rooted subtrees $\mathfrak{J}'$. Furthermore, $\mathcal{H}_\tau(\mathfrak{Y})$ consists exactly of the functions in $\mathcal{H}_\tau$ that are concentrated on $\mathrm{Aut}_o(\mathfrak{Y})$.

Now fix $x \in \mathcal{H}_\pi$. We define $f_x : \mathcal{I}_{\mathfrak{Y}} \to \mathcal{H}_\pi$ by declaring that $f_x(h) = |\mathrm{Aut}_o(\mathfrak{Y})|^{-\frac{1}{2}}\, \pi(h^{-1})x$ for all $h \in \mathrm{Aut}_o(\mathfrak{Y})$ and $f_x(h) = 0$ for $h \notin \mathrm{Aut}_o(\mathfrak{Y})$. Evidently, $f_x \in \mathcal{H}_\tau$, and since it is concentrated on $\mathrm{Aut}_o(\mathfrak{Y})$, it follows that $f_x \in \mathcal{H}_\tau(\mathfrak{Y})$. The fact that $f_x \neq 0$ for $x \neq 0$ now implies that $\mathcal{H}_\tau(\mathfrak{Y}) \neq \{0\}$, and so $\mathfrak{Y}$ is a basic subtree for τ.

The map $T : \mathcal{H}_\pi \to \mathcal{H}_\tau(\mathfrak{Y})$, $x \mapsto f_x$, is clearly a linear isometry. Since $f \in \mathcal{H}_\pi(\mathfrak{Y})$ by the condition (6.11) satisfies that $f = f_{|\mathrm{Aut}_o(\mathfrak{Y})|^{\frac{1}{2}} f(e_{\mathfrak{Y}})}$ where we by $e_{\mathfrak{Y}}$ denote the identity in $\mathrm{Aut}_o(\mathfrak{Y})$, it follows that T is even a unitary operator. Finally, it follows by defintition of τ and T that $T\pi(g) = \tau(g)T$ for all $g \in \mathrm{Aut}_o(\mathfrak{Y})$ which proves that the subrepresentation of the restriction to $\mathrm{Aut}_o(\mathfrak{Y})$ of τ corresponding to the subspace $\mathcal{H}_\tau(\mathfrak{Y})$ is equivalent to π.

We denote by $\mathcal{P}$ the finite set of rooted subtrees $\mathfrak{J}'$ of $\mathfrak{J}$ which are isomorphic to $\mathfrak{Y}$ as rooted subtrees. For $\mathfrak{J}' \in \mathcal{P}$, we refer to $\{h \in \mathcal{I}_{\mathfrak{Y}} \mid h(\mathfrak{Y}) = \mathfrak{J}'\}$ as $\mathcal{I}_{\mathfrak{Y}}(\mathfrak{J}')$. Clearly, $\mathcal{I}_{\mathfrak{Y}} = \bigcup_{\mathfrak{J}' \in \mathcal{P}} \mathcal{I}_{\mathfrak{Y}}(\mathfrak{J}')$. For fixed $f \in \mathcal{H}_\tau$, we observe that $f \cdot 1_{\mathcal{I}_{\mathfrak{Y}}(\mathfrak{J}')} \in \mathcal{H}_\tau$ for all $\mathfrak{J}' \in \mathcal{P}$. For $\mathfrak{J}' \in \mathcal{P}$, we choose an isomorphism $\gamma_{\mathfrak{J}'} \in \Gamma(\mathfrak{J})$ from $\mathfrak{Y}$ onto $\mathfrak{J}'$, and we observe that $f_{\mathfrak{J}'} = \tau(\gamma^*_{\mathfrak{J}'})(f \cdot 1_{\mathcal{I}_{\mathfrak{Y}}(\mathfrak{J}')}) \in \mathcal{H}_\tau(\mathfrak{Y})$. It is now evident that

$$f = \sum_{\mathfrak{J}' \in \mathcal{P}} \tau(\gamma_{\mathfrak{J}'}) f_{\mathfrak{J}'}$$

which proves that $\mathcal{H}_\tau(\mathfrak{Y})$ is cyclic for τ.

Finally, we let V be a closed, τ-invariant subspace of $\mathcal{H}_\tau$, and we denote by P the corresponding orthogonal projection. By invariance, P commutes with $\tau(\delta_{\mathfrak{Y}})$ and so we get for $h \in \mathcal{H}_\tau(\mathfrak{Y})$ that

$$Ph = P\tau(\delta_X)h = \tau(\delta_X)Ph$$

implying that $Ph \in \mathcal{H}_\tau(\mathfrak{Y})$. We may now repeat the proof of Proposition 6.1.4 to see that τ is an irreducible representation of $\Gamma(\mathfrak{J})$.

This proves that $\tau = \tau_{\mathfrak{Y},\pi}$ has the required properties, and so for every finite rooted subtree $\mathfrak{Y} \subseteq \mathfrak{J}$ and every irreducible unitary representation π of $\mathrm{Aut}_o(\mathfrak{Y})$ there exists an irreducible representation of $\Gamma(\mathfrak{J})$ with $\mathfrak{Y}$ as a basic subtree and for which the unitary representation from Lemma 6.4.2 is π.

Combining this with Lemma 6.4.4, we easily deduce that we have now constructed all irreducible representations of τ.

Theorem 6.4.5 *The representations $\tau_{\mathfrak{Y},\pi}$ with $\mathfrak{Y} \subseteq \mathfrak{J}$ a finite rooted subtree and π a unitary representation of $\mathrm{Aut}_o(\mathfrak{Y})$ exhaust all irreducible representations of $\Gamma(\mathfrak{J})$.*

Proof. Let τ be an irreducible unitary representation of $\Gamma(\mathfrak{J})$. Choose a basic subtree $\mathfrak{Y}$ for τ and denote by π the irreducible unitary representation of $\mathrm{Aut}_o(\mathfrak{Y})$ from Lemma 6.4.2. It now immediately follows by Lemma 6.4.4 that τ is equivalent to $\tau_{\mathfrak{Y},\pi}$. □

Combining the material of this section with the content of section 6.1, we obtain in the next section the complete classification of the irreducible tame representations of K.

6.5 Classification of irreducible tame representations of K

With the complete classification of the irreducible representations of $\Gamma(\mathfrak{J})$ for all finite rooted subtrees $\mathfrak{J} \subseteq \mathfrak{X}$ in Theorem 6.4.5, we may now state and prove our main theorem 6.5.1 which gives a complete classification of all irreducible tame representations of K - or what according to Theorem 6.3.3 is equivalent, all irreducible unitary representations of K_∞. It turns out that we have already in section 6.1 constructed all these representations, namely the representations $\pi_{\mathfrak{J},\sigma}$ with $\mathfrak{J} \subseteq \mathfrak{X}$ a finite rooted subtree and σ an irreducible representation of $\mathrm{Aut}_o(\mathfrak{J})$. We will abuse the notation and also denote the corresponding tame representation of K by $\pi_{\mathfrak{J},\sigma}$. By Proposition 6.1.4 and Remark 6.3.4, this representation remains irreducible.

For the proof of the main theorem, we need to introduce some terminology which is closely related to the concepts introduced in section 6.4. For a tame representation π of K, we say that a finite rooted subtree $\mathfrak{J} \subseteq \mathfrak{X}$ is *a basic subtree for* π if $\mathcal{H}_\pi(\mathfrak{J}) \neq \{0\}$ and if $\mathcal{H}_\pi(\mathfrak{J}') = \{0\}$ for all finite rooted subtrees $\mathfrak{J}' \subsetneq \mathfrak{J}$. Since π is tame, it is obvious that there exists a basic subtree for every tame representation of K.

Theorem 6.5.1 *The representations $\pi_{\mathfrak{J},\sigma}$ with $\mathfrak{J} \subseteq \mathfrak{X}$ a finite rooted subtree and σ an irreducible unitary representation of $\mathrm{Aut}_o(\mathfrak{J})$ exhaust all irreducible tame representations of K. The representations $\pi_{\mathfrak{J}_1,\sigma_1}$ and $\pi_{\mathfrak{J}_2,\sigma_2}$ are equivalent if and only if $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are isomorphic as rooted trees and σ_1 and σ_2 are equivalent.*

Proof. The last statement immediately follows by Proposition 6.1.5 and the denseness of K in K_∞. Hence, we are left with a proof of the fact that every irreducible tame representation of K is equivalent to some $\pi_{\mathfrak{J},\sigma}$ with $\mathfrak{J}$ and σ as in the theorem.

Let π be an irreducible tame representation of K with representation space $\mathcal{H}_\pi$. Since π is tame, we may choose a basic subtree $\mathfrak{J} \subseteq \mathfrak{X}$ for π. By Theorem 6.3.7, we may consider the associated representation $\tau_\mathfrak{J}$ of the semigroup $\Gamma(\mathfrak{J})$ which has representation space $\mathcal{H}_\pi(\mathfrak{J})$.

$\tau_\mathfrak{J}$ is irreducible. To see this, we recall that π is irreducible, and so it follows by 1.5 in [O4] that span$\{\pi(g) \mid g \in K\}$ is weakly dense in the algebra $\mathcal{B}(\mathcal{H}_\pi)$ of bounded operators on $\mathcal{H}_\pi$. For a bounded operator A on $\mathcal{H}_\pi(\mathfrak{J})$, this implies that $AP_\mathfrak{J}$ which is a bounded operator on $\mathcal{H}_\pi$ may be approximated weakly by linear combinations of the operators $\pi(g)$ with $g \in K$. However, for $x, y \in \mathcal{H}_\pi(\mathfrak{J})$ and $g \in K$ we see that

$$\langle \pi(g)x, y\rangle_\pi = \langle \pi(g)x, P_\mathfrak{J} y\rangle_\pi = \langle P_\mathfrak{J}\pi(g)x, y\rangle_\pi = \langle \tau_\mathfrak{J}(p_\mathfrak{J}(g))x, y\rangle_\pi$$

which proves that A may be approximated weakly by linear combinations of the operators $\tau_\mathfrak{J}(\gamma)$ with $\gamma \in \Gamma(\mathfrak{J})$. By 1.5 in [O4], this implies that $\tau_\mathfrak{J}$ is irreducible.

Since $\mathfrak{J}$ is a basic subtree for π, it follows by Lemma 6.3.8 that $\mathfrak{J}$ is also a basic subtree for the semigroup representation $\tau_\mathfrak{J}$. Hence, it follows by Lemma 6.4.2 that the restriction σ of $\tau_\mathfrak{J}$ to $\mathrm{Aut}_o(\mathfrak{J})$ is an irreducible unitary representation. By the construction of the representation $\tau_{\mathfrak{J},\sigma}$ and Lemma 6.4.4, we deduce that $\tau_\mathfrak{J} = \tau_{\mathfrak{J},\sigma}$.

We now observe that $\gamma \in \Gamma(\mathfrak{J})$ is either in $\mathrm{Aut}_o(\mathfrak{J})$ or has a domain which is strictly contained in $\mathfrak{J}$. Hence, Lemma 6.4.1 implies that

$$\tau_\mathfrak{J}(\gamma) = \begin{cases} \sigma(\gamma) & \text{for } \gamma \in \mathrm{Aut}_o(\mathfrak{J}) \\ 0 & \text{otherwise} \end{cases}$$

Now let $h \in \mathcal{H}_\pi(\mathfrak{J})$ be non-zero. By the above observation, we deduce for $g \in K$ that

$$\begin{aligned}\langle \pi(g)h, h\rangle_\pi = \langle \pi(g)h, P_\mathfrak{J} h\rangle_\pi &= \langle P_\mathfrak{J}\pi(g)h, h\rangle_\pi = \langle \tau_\mathfrak{J}(p_\mathfrak{J}(g))h, h\rangle_\pi \\ &= \begin{cases} \langle \sigma(p_\mathfrak{J}(g))h, h\rangle_\pi & \text{for } p_\mathfrak{J}(g) \in \mathrm{Aut}_o(\mathfrak{J}) \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

Since π is irreducible, h is cyclic for π. This together with Lemma 6.4.3 shows that π is uniquely determined by the basic subtree $\mathfrak{J}$ and the representation σ which is the restriction of the associated representation $\tau_\mathfrak{J}$ to $\mathrm{Aut}_o(\mathfrak{J})$.

Finally, we observe by Remark 6.1.2 that $\mathfrak{J}$ is a basic subtree for $\pi_{\mathfrak{J},\sigma}$. Furthermore, it follows immediately by definition of the semigroup representations of Theorem 6.3.7 (cf. (6.4)) that the restriciton to $\mathrm{Aut}_o(\mathfrak{J})$ of the representation of $\Gamma(\mathfrak{J})$ corresponding to $\pi_{\mathfrak{J},\sigma}$ is equivalent to σ. Hence, the above considerations imply that π is equivalent to $\pi_{\mathfrak{J},\sigma}$.

This finishes the proof. □

An immediate consequence of the proof is the following:

Corollary 6.5.2 *Let π be an irreducible tame representation of K with representation space $\mathcal{H}_\pi$. For every finite rooted subtree $\mathfrak{J} \subseteq \mathfrak{X}$, the space $\mathcal{H}_\pi(\mathfrak{J})$ is finite-dimensional.*

Proof. Let $\mathfrak{J} \subseteq \mathfrak{X}$ be a finite rooted subtree with $\mathcal{H}_\pi(\mathfrak{J}) \neq \{0\}$. Using the arguments in the proof of Theorem 6.5.1, we see that $\tau_\mathfrak{J}$ is irreducible. Since $\Gamma(\mathfrak{J})$ is finite and $\mathcal{H}_\pi(\mathfrak{J})$ is the representation space of $\tau_\mathfrak{J}$, it follows by [O4, Lemma 1.10] that $\mathcal{H}_\pi(\mathfrak{J})$ is finite-dimensional. □

Another consequence of our previous work is that tame representations of K behave nicely in the way that they may be completely decomposed as a direct sum of irreducible representations of K. This is the content of the following Theorem 6.5.3.

Theorem 6.5.3 *Let π be a tame representation of K. Then π can be decomposed into a direct sum of irreducible representations.*

Proof. Since π is tame, we may choose a finite rooted subtree $\mathfrak{J} \subseteq \mathfrak{X}$ such that $\mathcal{H}_\pi(\mathfrak{J}) \neq \{0\}$. By Theorem 6.3.7, we may consider the associated representation $\tau_\mathfrak{J}$ of $\Gamma(\mathfrak{J})$ which has $\mathcal{H}_\pi(\mathfrak{J})$ as representation space.

Since $\Gamma(\mathfrak{J})$ is finite, it follows by [O4, Lemma 1.10] that $\tau_\mathfrak{J}$ decomposes into a direct sum of irreducible representations. In particular, there exists a closed subspace $\mathcal{H} \subseteq \mathcal{H}_\pi(\mathfrak{J})$ which is invariant and irreducible under $\Gamma(\mathfrak{J})$.

We now consider the cyclic span $\mathcal{H}_\sigma \subseteq \mathcal{H}_\pi$ of $\mathcal{H}$ under the representation π. This is a closed, π-invariant subspace of $\mathcal{H}_\pi$, and we denote by σ the corresponding subrepresentation of π. We will prove that $\mathcal{H}_\sigma \cap \mathcal{H}_\pi(\mathfrak{J}) = \mathcal{H}$.

The inclusion $\supseteq$ is obvious. Now consider an element $h_0 \in \mathcal{H}_\sigma \cap H_\pi(\mathfrak{J})$ which may be approximated by linear combinations of vectors $\pi(g)h$ with $g \in K$ and $h \in \mathcal{H}$. Since $h_0 \in \mathcal{H}_\pi(\mathfrak{J})$, this implies that it may be approximated by linear combinations of vectors $P_\mathfrak{J}\pi(g)h$ with $g \in K$ and $h \in \mathcal{H}$. We observe that

$$P_\mathfrak{J}\pi(g)h = \tau_\mathfrak{J}(p_\mathfrak{J}(g))h \in \mathcal{H}$$

for $g \in K$ and $h \in \mathcal{H}$, and since $\mathcal{H}$ is closed, this implies that $h_0 \in \mathcal{H}$. This proves the inclusion $\subseteq$ and so the desired equality.

This equality implies that $\mathcal{H} = \mathcal{H}_\sigma(\mathfrak{J})$. Using this, we now prove that σ is irreducible. To do this, consider a closed, invariant subspace $\mathcal{V} \subseteq \mathcal{H}_\sigma$. Since $\mathcal{H} = \mathcal{H}_\sigma(\mathfrak{J})$, we may argue as in the proof of Proposition 6.1.4 to see that

$$\mathcal{H} = (\mathcal{H} \cap \mathcal{V}) \oplus (\mathcal{H} \cap \mathcal{V}^\perp)$$

Now let $h \in \mathcal{H} \cap \mathcal{V}$, and consider $g \in K$. By invariance of $\mathcal{V}$, $\pi(g)h \in \mathcal{V}$. Furthermore, it is well-known by [Fo, Theorem 3. that the invariance of $\mathcal{V}$ implies that $P_\mathfrak{J}$ maps $\mathcal{V}$ into $\mathcal{V}$. Hence, it follows that $\tau_\mathfrak{J}(p_\mathfrak{J}(g))h = P_\mathfrak{J}\pi(g)h \in \mathcal{V}$. Since $\mathcal{H}$ is $\tau_\mathfrak{J}$-invariant and the restriction of $p_\mathfrak{J}$ to K by Proposition 6.2.5 is surjective, this shows that $\mathcal{H} \cap \mathcal{V}$ is a closed, $\tau_\mathfrak{J}$-invariant subspace of $\mathcal{H}$. By irreducibility of $\mathcal{H}$, this implies that $\mathcal{H} = \mathcal{H} \cap \mathcal{V}$ or $\mathcal{H} = \mathcal{H} \cap \mathcal{V}^\perp$. Hence, $\mathcal{H} \subseteq \mathcal{V}$ or $\mathcal{H} \subseteq \mathcal{V}^\perp$. Since $\mathcal{H}$ by construction is cyclic for σ, the invariance of $\mathcal{V}$ and $\mathcal{V}^\perp$ implies that $\mathcal{V} = \mathcal{H}_\sigma$ or $\mathcal{V}^\perp = \mathcal{H}_\sigma$. This proves that σ is irreducible.

The existence of an irreducible subrepresentation of π may be combined with standard arguments using Zorn's Lemma to prove that π decomposes into a direct sum of irreducible representations. □

One may ask how the spherical representations $\{\pi_n\}_{n=0}^\infty$ for the pair (K, K^ω) considered in chapter 5 fit into this chapter. For $n \geq 1$, the definition of the representation π_n on the Hilbert space $\ell^2(\mathfrak{M}_n)$ in section 5.2 may without any modifications be extended to the group K_∞. In light of Remark 5.2.1, the extended map π_n is continuous in the topology on K_∞ and so is a unitary representation of K_∞. According to Theorem 6.3.3, this proves that π_n is tame. Being the trivial representation, π_0 is of course also tame. Theorem 6.5.1 now reveals that for each $n \geq 0$ there exists a finite rooted subtree $\mathfrak{J}_n \subseteq \mathfrak{X}$ and an irreducible unitary representation σ_n of $\mathrm{Aut}_o(\mathfrak{J}_n)$ such that π_n is equivalent to $\pi_{\mathfrak{J}_n,\sigma_n}$.

Fortunately, Proposition 6.5.4 below shows that this is really the case.

Proposition 6.5.4 *Let $n \geq 0$ and denote by $\mathfrak{J}_n$ the finite rooted subtree $\{x_k\}_{k=0}^n$ and by σ_n the trivial representation of the group $\mathrm{Aut}_o(\mathfrak{J}_n)$. Then π_n is equivalent to $\pi_{\mathfrak{J}_n,\sigma_n}$.*

Proof. Let $n \geq 1$ and consider the second realization of the representation $\pi_{\mathfrak{J}_n,\sigma_n}$ in section 6.1. For $y \in \mathfrak{M}_n$, we denote by h_y the unique map in $\mathcal{I}_{\mathfrak{J}_n}$ with $h_y(x_n) = y$. We clearly have the identity $\mathcal{I}_\mathfrak{J} = \{h_y \mid y \in \mathfrak{M}_n\}$ and so $\mathcal{I}_{\mathfrak{J}_n}$ is in bijective correspondance with the set $\mathfrak{M}_n$.

Since σ_n is the trivial representation, we may put $\mathcal{H}_\sigma = \mathbb{C}$. We observe that $\tilde{K}(\mathfrak{J}_n) = K(\mathfrak{J}_n)$ which implies that the condition (6.3) on functions in $\mathcal{H}_{\pi_{\mathfrak{J}_n,\sigma_n}}$ is trivially satisfied by all functions $h : \mathcal{I}_{\mathfrak{J}_n} \to \mathbb{C}$. Hence, the condition (6.2) shows that $\mathcal{H}_{\pi_{\mathfrak{J}_n,\sigma_n}}$ is isomorphic to $\ell^2(\mathfrak{M}_n)$. It is immediate by the definiton of the representations that π_n and $\pi_{\mathfrak{J}_n,\sigma_n}$ are equivalent under this identification of the representation spaces.

Since $\tilde{K}(\mathfrak{J}_0) = K$ and σ_0 is the trivial representation, $\pi_{\mathfrak{J}_0,\sigma_0}$ is also trivial and so equivalent to π_0. □

Remark 6.5.5 The method used here may also be applied to the group G to obtain similar results, among those another proof of the classification of the irreducible unitary representations of $\mathrm{Aut}(\mathfrak{X})$ obtained by Olshanski in [O3]. The main difference is to consider finite subtrees of $\mathfrak{X}$ instead of the finite rooted subtrees considered above and modify the definition of tame representations, $\Gamma(\mathfrak{J})$ and $\Gamma(\mathfrak{X})$ accordingly. We leave it to the reader to make the necessary modifications.

With the classification of all irreducible tame representations of K and the observation that the irreducible representations considered in chapter 5 are all tame, one may ask whether we are done, i.e. whether all irreducible representations are tame. This turns - unfortunately - out not be the case as we shall see in the next chapter where the analysis of the action of K on a compactification of its boundary produces a family of irreducible representations which are not tame.

Chapter 7

A Compactification of the Boundary of a Homogeneous Tree of Countable Degree and the Existence of Non-tame Representations

Much of the analysis of the automorphism group for a locally finite, homogeneous rooted tree in chapter 4 depends on the action of the group on the boundary of the tree and the existence of invariant measures on the boundary. Such measures do not exist if the tree is not locally finite which makes the analysis of the group K from chapter 5 more complicated. This chapter deals with this case and solves the difficulties by carrying out the construction of a certain compactication of the boundary on which the desired measure exists. The resulting representation theory produces a family of irreducible representations which turn out not to be tame - proving the existence of non-tame irreducible representations. Section 1 is devoted to the construction of the compactification of the boundary which is the centrepiece for the remainder of the chapter. In section 2 it is shown how the action of the group on this compactification gives rise to a new family of representations, and in section 3 it is proved that these are non-tame. We finish the chapter in section 4 by showing that the representations remain irreducible even when they are restricted to the much smaller subgroup K^{ω}.

7.1 A compactification of the boundary

Consider the homogeneous tree $(\mathfrak{X}, \mathfrak{C})$ and the group K from chapter 5. We will keep the notation from that chapter. With the classification of all irreducible tame representations in chapter 6, one may ask whether non-tame irreducible representations exist. The objective of this section is to prove that the answer is affirmative, and so the representation theory of the group K is much more complicated than it is for the group K_{∞}. This is similar to the case of the infinite symmetric group for which the representation theory of the group $\tilde{S}(\infty)$ of all permutations of $\mathbb{N}$ is well-understood, cf. [L], while it is way more difficult for the group $S(\infty)$ of all finite permutations, cf. [KOV] and [O5]. The goal of the chapter is not to get a complete and satisfactory insight into the representation theory of K. We will only make a considerable contribution by proving the existence of non-tame representations and so underline the differences between the groups K and K_∞.

One of the main tools in the analysis of the automorphism group of a locally finite homogeneous tree in chapters 3 and 4 is the action of the automorphism group on the boundary of the tree and the existence of invariant measures. The action of the boundary survives when we pass from locally finite trees to trees of countable degree. Unfortunately, the same cannot be said for the measure. Even with these difficulties, we will use the action on the boundary to analyse K. The idea is to replace the boundary by a compactification. The action on the boundary may be extended to this larger object, and a K-invariant measure exists. This section is devoted to the construction of the compactification. We refer to Remark 3.3.1 for a discussion of the problem which originally inspired the author to construct this new object.

We define for each n a map $s_n : \Omega_{n+1} \to \Omega_n$ in the following way: let $\tau \in \Omega_{n+1}$ be given, and consider the corresponding rooted chain $y_0, y_1, \ldots,$ in $\mathfrak{X}_{n+1}$. By properties 1.-3. of the map q_n in section 5.1, it follows that $q_n(y_0), q_n(y_1), \ldots$ is a rooted chain in $\mathfrak{X}_n$. We define $s_n(\tau) \in \Omega_n$ to be the corresponding element in the boundary.

We clearly have that $s_n(\tau) = \tau$ for all $\tau \in \Omega_n$, and it is an immediate consequence of the definition of the topology on Ω_n and Ω_{n+1} that s_n is continuous.

For $n < m$, we define $r_{n,m} = s_n \circ s_{n+1} \circ \ldots \circ s_{m-1}$ which is a continuous map from Ω_m onto Ω_n, and we denote by $r_{n,n}$ the identity map on Ω_n. The sets Ω_n with $n \geq 0$ together with the maps $r_{n,m}$ with $n \leq m$ constitute a projective system, and we denote by $\mathfrak{K} = \lim_{\leftarrow} \Omega_n$ the corresponding projective limit. By definition, we have that

$$\mathfrak{K} = \left\{ (\tau_n) \in \prod_{n=0}^{\infty} \Omega_n \;\middle|\; s_n(\tau_{n+1}) = \tau_n \text{ for all } n \geq 0 \right\}$$

and we equip $\mathfrak{K}$ with the product topology. For $n \geq 0$, we denote by r_n the projection map from $\mathfrak{K}$ onto Ω_n, and these maps are of course continuous.

For $n \geq 0$ and $x \in \mathfrak{M}_n^k$, we define by $\Omega_n(x)$ the set of $\tau \in \Omega_n$ with the property that the corresponding rooted chain $y_0, y_1, \ldots,$ in Ω_n satisfies that $y_k = x$. As is well-known, the sets $\Omega_n(x)$ for $x \in \mathfrak{X}_n$ constitute a countable basis for the topology on Ω_n. Hence, the sets $\mathfrak{K} \cap \prod_{n=0}^{\infty} U_n$ for which there exists a $K \geq 0$ such that $U_n = \Omega_n(y_n)$ with $y_n \in \mathfrak{X}_n$ for $n \in \{0, \ldots, K\}$ and such that $U_n = \Omega_n$ for $n > K$, constitute a countable basis for the topology on $\mathfrak{K}$. With U_n, K and y_n as above, we observe that $\mathfrak{K} \cap \prod_{n=0}^{\infty} U_n = r_K^{-1}(\bigcap_{n=0}^{K} r_{n,K}^{-1}(\Omega_n(y_n)))$. It follows by the definition of the maps $r_{n,K}$ that $r_{n,K}^{-1}(\Omega_n(y_n))$ may be written as a finite union of sets of the form $\Omega_K(x)$ with $x \in \mathfrak{X}_K$. Since the intersection of such sets is either the empty set or again on this form, we have seen that $\bigcap_{n=0}^{K} r_{n,K}^{-1}(\Omega_n(y_n))$ may be written as a union of such sets. This shows that the sets $r_n^{-1}(\Omega_n(x))$ with $x \in \mathfrak{X}_n$ and $n \geq 0$ constitute a countable basis for the topology on $\mathfrak{K}$. We even see that the sets $r_n^{-1}(\Omega_n(x))$ with $x \in \mathfrak{X}_n$ and $n \geq N$ for some fixed N constitute a countable basis for the topology on $\mathfrak{K}$.

Furthermore, we see that $\mathfrak{K}$ is a compact Hausdorff space. Indeed, let $\{\tau_n\} \in \prod_{n=0}^{\infty} \Omega_n$ satisfy the condition that there exists i such that $\tau_i \neq r_{i,i+1}(\tau_{i+1})$. By the Hausdorff property, we may find open subsets U and V of Ω_n such that $\tau_i \in U$ and $r_{n,n+1}(\tau_{i+1}) \in V$ and such that $U \cap V = \emptyset$. Define $W_i = U$, $W_{i+1} = r_{n,n+1}^{-1}(V)$ and $W_n = \Omega_n$ for $n \neq i$ and $n \neq i+1$ and observe that each W_n is open in Ω_n. We now see that $\{\tau_n\}_{n=0}^{\infty} \in \prod_{n=0}^{\infty} W_n \subseteq \prod_{n=0}^{\infty} \Omega_n \setminus \mathfrak{K}$ which proves that $\mathfrak{K}$ is a closed subset of $\prod_{n=0}^{\infty} \Omega_n$. Since this product is compact by Tychonoff's theorem, this proves the compactness of $\mathfrak{K}$. The Hausdorff property is immediate.

We may introduce a natural action of K on $\mathfrak{K}$. To do this, let $g \in K$ and $\{\tau_i\}_{i=0}^{\infty} \in \mathfrak{K}$, and find $n \geq 0$ such that $g \in K_n$. Define $\tau_i' = g \cdot \tau_i$ for $i \geq n$ and $\tau_i' = r_{i,n}(\tau_n')$ for $i < n$, and observe that the relation $q_k \circ g = g \circ q_k$ implies that $\tau_i' = r_{i,i+1}(\tau_{i+1}')$ for all $i \geq 0$. This shows that $\{\tau_i'\}_{i=0}^{\infty} \in \mathfrak{K}$, and so we may (well-)define $g \cdot \{\tau_i\}_{i=0}^{\infty} = \{\tau_i'\}_{i=0}^{\infty}$. This clearly defines an action of K on $\mathfrak{K}$.

To prove continuity of this action, we observe that [MS, Lemma 5.5] reveals that the product topology on $K \times \mathfrak{K}$ is the inductive limit of the product topologies on $K_n \times \mathfrak{K}$. Hence, the action is continuous if only its restriction to $K_n \times \mathfrak{K}$ is continuous for all n. However, this immediately follows by the continuity of the action of K_n on Ω_m with $m \geq n$ and the fact that the sets $r_m^{-1}(\Omega_m(x))$ with $x \in \mathfrak{X}_m$ and $m \geq n$ constitute a countable basis for the topology on $\mathfrak{K}$.

By ν_n, we denote the unique Borel probability measure on Ω_n with the property that

$$\nu(\Omega_n(x)) = \frac{1}{a^{2^n}(a^{2^n}-1)^{d(o,x)-1}}$$

for $x \in \mathfrak{X}_n$ with $x \neq o$. This measure has been studied in chapters 2 and 4. By property 4. of the map q_n from section 5.1 and by the definition of $r_{n,n+1}$, we see that the following holds for $x \in \mathfrak{M}_k^n$ with $k \geq 1$:

$$\nu_{n+1}(r_{n,n+1}^{-1}(\Omega_n(x))) = a^{2^n}(a^{2^n}+1)^{k-1}\frac{1}{a^{2^{n+1}}(a^{2^{n+1}}-1)^{k-1}} = \frac{1}{a^{2^n}(a^{2^n}-1)^{k-1}}$$

which proves that the push-forward measure $r_{n,n+1}(\nu_{n+1})$ which is a Borel probability measure, is just the measure ν_n. Since the measure ν_n is clearly K_n-invariant, it follows by [Bour, Theorem 4.2] that there exists a unique K-invariant Borel probability measure ν on $\mathfrak{K}$ such that $\nu_n = r_n(\nu)$.

Remark 7.1.1 The compact Hausdorff space $\mathfrak{K}$ may be seen as some sort of compactification of the boundary Ω_∞ of the tree $(\mathfrak{X}, \mathfrak{C})$. To explain this, let $\tau \in \Omega_\infty$ and denote by $y_0, y_1, \ldots$ the corresponding rooted chain in $(\mathfrak{X}, \mathfrak{C})$. For $n \geq 0$, we define $\tau_n \in \Omega_n$ in the following way: for $m \geq 0$ choose $k > n$ such that $y_m \in \mathfrak{M}_m^k$ and define $z_m = (q_n \circ \ldots \circ q_{m-1})(y_m) \in \mathfrak{M}_m^n$. By property 1. of the maps q_j, this definition is clearly independent of the choice of k. Furthermore, it follows by properties 1., 2. and 3. that $z_0, z_1, \ldots,$ is a rooted chain in $(\mathfrak{X}_n, \mathfrak{C}_n)$. We denote by $\tau_n \in \Omega_n$ the corresponding boundary point.

It is evident by the definition $s_n(\tau_{n+1}) = \tau_n$ for all $n \geq 0$ and so $(\tau_n) \in \mathfrak{K}$. We define a map $\varphi : \Omega_\infty \to \mathfrak{K}$ by defining $\varphi(\tau) = (\tau_n)$.

Let $\tau_1, \tau_2 \in \Omega_\infty$ with corresponding rooted chains $y_0, y_1, \ldots,$ and $z_0, z_1, \ldots,$ respectively, and assume that $\tau_1 \neq \tau_2$. This implies that we may find $k \geq 1$ such that $y_k \neq z_k$. If we choose $m \geq 0$ such that $y_k, z_k \in \mathfrak{M}_k^m$, it follows by definition that $r_m(\varphi(\tau_1)) \neq r_m(\varphi(\tau_2))$. This shows that $\varphi(\tau_1) \neq \varphi(\tau_2)$. Hence, φ is injective.

The map φ is even continuous. Indeed, let $\tau \in \Omega_\infty$, let $y_0, y_1, \ldots$ be the corresponding rooted chain in $(\mathfrak{X}, \mathfrak{C})$, and assume for $n \geq 0$ and $x \in \mathfrak{X}_n$ that $\varphi(\tau) \in r_n^{-1}(\Omega_n(x))$. Put $m = d(o, x)$. It now follows by defintion of φ that $\varphi(\Omega_\infty(y_m)) \subseteq r_n^{-1}(\Omega_n(x))$. Since the sets $r_n^{-1}(\Omega_n(x))$ with $n \geq 0$ and $x \in \mathfrak{X}$ constitute a basis for the topology on $\mathfrak{K}$, this implies continuity of φ.

However, one should observe that φ is not a homeomorphism onto its image. Indeed, let $\tau \in \Omega_0$ be a boundary point. By the construction of the maps s_n, we may for each $n \geq 1$ choose a boundary point $\tau_n \in \Omega_n \setminus \Omega_{n-1}$ such that $s_n(\tau_n) = \tau$ and such that the intersection of the corresponding rooted chain and $\mathfrak{X}_{n-1}$ only contains the vertex o. By this last property, the rooted chains of τ and τ_n do only intersect in the vertex o and this shows that the sequence $\{\tau_n\}$ does not converge to τ in Ω_∞.

It is, however, true that $\{\varphi(\tau_n)\}$ converges to $\varphi(\tau)$ in $\mathfrak{K}$. To see this, we observe that $r_n(\varphi(\tau_m)) = r_{n,m}(\tau_m) = \tau = r_n(\varphi(\tau))$ for $m > n$. Since the sets $r_n^{-1}(\Omega_n(x))$ with $n \geq 0$ and $x \in \mathfrak{X}_n$ constitute a basis for the topology on $\mathfrak{K}$, this implies that the sequence $\{\varphi(\tau_n)\}$

converges to $\varphi(\tau)$. This proves that φ is not a homeomorphism onto its image and so the topology on Ω_∞ is strictly stronger than the initial topology on Ω_∞ with respect to the map φ.

Since the sets $r_n^{-1}(\Omega_n(x))$ with $n \geq 0$ and $x \in \mathfrak{X}_n$ constitute a basis for the topology on $\mathfrak{K}$, it is evident that $\varphi(\Omega_\infty)$ is dense in $\mathfrak{K}$. Furthermore, one should observe that for $g \in K_n$ and $m \geq n$ we have that $q_m \circ g = g \circ q_m$ and so $\varphi(g \cdot \tau) = g \cdot \varphi(\tau)$ for all $g \in K$ and $\tau \in \Omega_\infty$. Hence, the action on $\mathfrak{K}$ may be seen as an extension of the natural action on Ω_∞ if we identify Ω_∞ with its image under φ.

Finally, we observe that $\varphi(\Omega_\infty)$ is a ν-null set. Indeed, let $x \in \mathfrak{M}_1$ and choose $m \geq 0$ such that $x \in \mathfrak{X}_m$. By definition of the map φ, we have the inclusion

$$\varphi(\Omega_\infty(x)) \subseteq \bigcap_{n=m}^{\infty} r_n^{-1}(\Omega_n(x))$$

The sequence $\left\{r_n^{-1}(\Omega_n(x))\right\}_{n=m}^{\infty}$ of subsets of $\mathfrak{K}$ is decreasing and have the property that

$$\nu(r_n^{-1}(\Omega_n(x))) = \nu_n(\Omega_n(x)) = \frac{1}{a^{2^n}}$$

and so $\varphi(\Omega_\infty(x))$ is a null set. Since

$$\varphi(\Omega_\infty) = \bigcup_{x \in \mathfrak{M}_1} \varphi(\Omega_\infty(x))$$

and $\mathfrak{M}_1$ is countable, this proves that $\varphi(\Omega_\infty)$ is a ν-null set.

In contrast to the case of a locally finite tree, the boundary Ω_∞ is not compact, and we may not equip it with a K-invariant probability measure. Since a great part of the analysis of the groups K_n carried out in chapter 4 is related to its action on the boundary of the tree and the existence of a K-invariant probability measure on Ω_n, similar considerations may at a first glance not be carried out for the group K. However, the above considerations show that we may equip Ω_∞ with a weaker topology and consider a natural compactification of the boundary, namely the space $\mathfrak{K}$. The natural action on Ω_∞ may be extended to an action on its compactification $\mathfrak{K}$ which is continuous, and we may construct a K-invariant measure on this new space, namely the measure ν. This makes it natural to suggest that we - even with the complications which arise at a first glance - may consider representation theory arising from the action of the group K on the boundary of the tree. The role of the boundary Ω_∞ will just be played by its compactification $\mathfrak{K}$. In this sense, $\mathfrak{K}$ may be seen as a replacement of Ω_∞ in the analysis of K.

It is evident that Ω_∞ with its natural topology does not possess a K-invariant Borel measure with respect to its natural topology. The fact that $\varphi(\Omega_\infty)$ is a null set with respect to ν shows that we may not use the above construction to create a non-trivial K-invariant Borel measure on Ω_∞ with its new, weaker topology. Hence, we cannot restrict the analysis of the action of K on its boundary to the natural boundary Ω_∞. We have to deal with the complete compactification $\mathfrak{K}$.

Remark 7.1.2 The reader should be aware of the fact that the quite technical constructions of chapter 5 are all made with the sole purpose of making the definition of the compactification $\mathfrak{K}$ possible. The construction works because of the important fact that for every $n \geq 0$ and $x \in \mathfrak{X}_n$, the number $p^{-1}(\{x\}) \cap \mathfrak{X}_n$ divides $p^{-1}(\{x\}) \cap \mathfrak{X}_{n+1}$. This ensures the relation $\nu_n = r_{n,n+1}(\nu_{n+1})$ which is necessary for the existence of the measure ν. The tedious extension of an automorphism $g \in K_n$ to $\mathfrak{X}$ which was described in chapter 5 serves to make it possible to extend the action of K on Ω_∞ to the compactification $\mathfrak{K}$.

Remark 7.1.3 The group G from chapter 3 acts - as described in chapter 2 - on the boundary Ω_∞ of the tree. If the tree is locally finite, we have even seen in chapter 2 that the natural invariant measure on the boundary is strongly quasiinvariant for the action of the complete automorphism group. One may ask whether G also acts on the compactification $\mathfrak{K}$.

Unfortunately, the answer is negative. The natural definition of an action of G on $\mathfrak{K}$ only works if the relation $s_n(g \cdot \omega) = g \cdot s_n(\omega)$ holds for all $n \geq 0$, all $\omega \in \Omega_{n+1}$ and all $g \in G_n$. However, this is not the case. This is regretful since the original inspiration for the construction of the compactification $\mathfrak{K}$ was the desire to establish an integral representation of the spherical functions for the pair (G, K), cf. Remark 3.3.1. Such a representation has, however, not yet been established.

The remainder of this chapter is devoted to the study of the representation theory which arises from the action of K on the compactification $\mathfrak{K}$ of its boundary.

7.2 A new family of representations of K

Since K acts continuously on the compactfication $\mathfrak{K}$ of the boundary Ω_∞, and since $\mathfrak{K}$ has a K-invariant measure, we may consider the natural representation π of K on $L^2(\mathfrak{K})$. For each $f \in L^2(\mathfrak{K})$, $g \in K$ and $\{\tau_n\} \in \mathfrak{K}$, we define

$$(\pi(g)f)(\{\tau_n\}) = f(g^{-1} \cdot \{\tau_n\})$$

The K-invariance of ν implies that $\pi(g)$ is a unitary operator of $L^2(\mathfrak{K})$ for each $g \in K$. It follows by the compactness of the K_n's and [Fo, Section 3.1] that the restriction of π to each K_n is continuous in the strong operator topology. Hence, π is a unitary representation of K.

This section is devoted to the study of π and its decomposition into a direct sum of irreducible representations. The irreducible representations turning up will be of special interest and are the center of attention in the last sections of the chapter.

Even though the representation π might be seen as an analogue to the representations of the underlying groups K_n studied in Proposition 4.5.1, it behaves in a totally different way. It turns out that it is not equivalent to the direct sum of all the spherical representations for the pair (K, K^ω) as it was the case in Proposition 4.5.1. On the contrary, it might be written as the direct sum of the representations in a family of non-equivalent representations which - with one trivial exception - are not spherical for the pair (K, K^ω) and not tame. This is the content of Theorem 7.2.3 below.

To introduce this family of representations, we observe that the map $f \mapsto f \circ r_n$ embeds $L^2(\Omega_n)$ isometrically into $L^2(\mathfrak{K})$. This follows by the fact that $\nu_n = r_n(\nu)$. Hence, we may regard $L^2(\Omega_n)$ as a closed subspace of $L^2(\Omega)$ consisting of functions whose value at $\{\tau_k\} \in \mathfrak{K}$ only depend on τ_n. Similarly, the relation $\nu_n = s_n(\nu_{n+1})$ implies that we - by the map $f \mapsto f \circ s_n$ - may embed $L^2(\Omega_n)$ isometrically into $L^2(\Omega_{n+1})$, and since $f \circ s_n \circ r_{n+1} = f \circ r_n$ for $f \in L^2(\Omega)$, it follows that

$$L^2(\Omega_0) \subseteq L^2(\Omega_1) \subseteq \ldots \subseteq L^2(\Omega_n) \subseteq \ldots \tag{7.1}$$

As in the proof of Proposition 5.2.3, we denote by $\mathcal{H}_n^j$ the representation space for the n'the spherical representation π_n^j of the group K_j. By Proposition 4.5.1, we see that

$$L^2(\Omega_j) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^j \tag{7.2}$$

Hence, we may regard the spaces $\mathcal{H}_n^j$ with $n \geq 0$ and $j \geq 0$ as closed subspaces of $L^2(\mathfrak{K})$.

Let $n \geq 1$, let $j \geq 0$, and let $f \in \mathcal{H}_n^j$. We clearly have that the function $f \circ s_j$ is constant on the sets $\Omega_{j+1}(x)$ with $x \in \mathfrak{M}_n^{j+1}$. Let $x \in \mathfrak{M}_{n-1}^{j+1}$. By definition of q_j, we have for all $y \in p^{-1}(q_j(x)) \cap \mathfrak{X}_j$ the identity

$$\left| q_j^{-1}(y) \cap p^{-1}(x) \cap \mathfrak{X}_{j+1} \right| = a^{2^j} + 1$$

if $n \geq 2$ and

$$\left| q_j^{-1}(y) \cap p^{-1}(x) \cap \mathfrak{X}_{j+1} \right| = a^{2^j}$$

if $n = 1$. This implies that $f \circ s_j \in \mathcal{H}_n^{j+1}$. Hence, we have proved the inclusions

$$\mathcal{H}_n^0 \subseteq \mathcal{H}_n^1 \subseteq \ldots \subseteq \mathcal{H}_n^j \subseteq \ldots \tag{7.3}$$

for $n \geq 1$, and they are obvious in the case $n = 0$.

We now define $\mathcal{H}_n = \overline{\bigcup_{j=0}^{\infty} \mathcal{H}_n^j}$. (7.3) and the pairwise orthogonality of the spaces $\mathcal{H}_n^j$ for fixed j immediately reveal that the closed subspaces $\mathcal{H}_n$ are pairwise orthogonal. Furthermore, it is an easy consequence of the invariance of $\mathcal{H}_n^j$ under π_n^j and (7.3) that $\bigcup_{j=0}^{\infty} \mathcal{H}_n^j$ and so $\mathcal{H}_n$ is invariant under π. For $n \geq 0$, we denote by ρ_n the subrepresentation corresponding to the subspace $\mathcal{H}_n$.

The space $\mathcal{H}_0^j$ consists of all constant functions on Ω_j. Hence, $\mathcal{H}_0$ is the subspace of all constant functions on $\mathfrak{K}$, and so ρ_0 is just the trivial representation of K which is of course spherical for the pair (K, K^ω). However, it turns out that the representations ρ_n are not spherical for the pair (K, K^ω) if $n \geq 1$. Furthermore, the ρ_n's are non-equivalent for all $n \geq 0$. This is the content of the following proposition.

Proposition 7.2.1 *The representations ρ_n, $n \geq 0$, are non-equivalent. Furthermore, ρ_n is not spherical for the pair (K, K^ω) if $n \geq 1$.*

Proof. To see that the representations are non-equivalent, we observe that the isometric embedding S_j of $L^2(\Omega_j)$ into $L^2(\mathfrak{K})$ satisfies the relation $\pi(g)S_j = S_j \pi_n^j(g)$ for all $g \in K_j$. This proves that $\mathcal{H}_n^j$ is a K_j-invariant subspace of $L^2(\mathfrak{K})$ and that π_n^j is equivalent to the subrepresentation of the restriction of ρ_n to K_j corresponding to $\mathcal{H}_n^j$. The non-equivalence of the representations ρ_n with $n \geq 0$ now immediately follows by the non-equivalence of the representations π_n^j with $n \geq 0$ and Lemma 5.2.2.

To prove the second statement of the proposition, let $n \geq 1$. Assume that ρ_n is spherical for $n \geq 1$ and denote by $f \in \mathcal{H}_n$ a K^ω-invariant, non-zero function. Denote by P^j the ortogonal projection onto $\mathcal{H}_n^j$. By denseness of $\bigcup_{n=0}^{\infty} \mathcal{H}_n^j$ in $\mathcal{H}_n$, the P^j's converge strongly to the identity on $\mathcal{H}_n$. Hence, the sequence $\{P^j f\}$ converges to f. For $g \in K_j^\omega$, the K_j-invariance of $\mathcal{H}_n^j$ implies that

$$\rho_n(g) P^j f = P^j \rho_n(g) f = P^j f$$

Hence, $P^j f$ is a K_j^ω-invariant function in $\mathcal{H}_n^j$.

Now denote by m_j the map m of section 4.1 for the tree $(\mathfrak{X}_j, \mathfrak{C}_j)$. By [Fa2, Proposition I.5], the space of K_j^ω-invariant functions in $\mathcal{H}_n^j$ is 1-dimensional. In the proof of Corollary 4.2.2, we identified one K_j^ω-invariant function f_j. Hence, there exists $c_j \in \mathbb{C}$ such that $P^j f = c_j f_j$, i.e. $(P^j f)((\tau_k)) = c_j$ if $\tau_j \in \Omega_j(x_n)$, $(P^j f)((\tau_k)) = -\frac{c_j}{m_j(x_{n-1})}$ if $\tau_j \in \Omega_j(x_{n-1}) \setminus \Omega_j(x_n)$, and $(P^j f)((\tau_k)) = 0$ if $\tau_j \notin \Omega_j(x_{n-1})$. Furthermore, the $L^2(\mathfrak{K})$-convergence to f of the sequence

$\{P^j f\}$ implies the existence of a subsequence $\{P^{j_k} f\}$ which converges almost everywhere to f.

Let $n \geq 2$. The function $P^{j_k} f$ is constantly 0 on the set $\mathfrak{K} \setminus r_{j_k}^{-1}(\Omega_{j_k}(x_{n-1}))$ which by the relation $\nu_{j_k} = r_{j_k}(\nu)$ has measure $\frac{a^{2^{j_k}}(a^{2^{j_k}}-1)^{n-2}-1}{a^{2^{j_k}}(a^{2^{j_k}}-1)^{n-2}}$ under ν. Since $\left\{\mathfrak{K} \setminus r_{j_k}^{-1}(\Omega_{j_k}(x_{n-1})\right\})$ is an increasing sequence, this shows that $P^{j_k} f$ converges to 0 almost everywhere. Hence, $f = 0$ which is a contradiction.

For the case $n = 1$, we observe that

$$\|f_j\|^2 = \frac{1}{a^{2^j}}(1 + (a^{2^j} - 1)\frac{1}{(a^{2^j} - 1)^2}) = \frac{1}{a^{2^j} - 1}$$

The fact that $|c_j| \, \|f_j\| = \left\|P^j f\right\| \leq 1$ now implies that $|c_j| \leq (a^{2^j} - 1)^{\frac{1}{2}}$ and so

$$\left|(P^{j_k} f)((\tau_i))\right| \leq \frac{(a^{2^{j_k}} - 1)^{\frac{1}{2}}}{(a^{2^{j_k}} - 1)} = \frac{1}{(a^{2^{j_k}} - 1)^{\frac{1}{2}}}$$

for $(\tau_i) \in \mathfrak{K} \setminus r_{j_k}^{-1}(\Omega_{j_k}(x_1))$. This set has - by the relation $\nu_{j_k} = r_{j_k}(\nu)$ - measure $\frac{a^{2^{j_k}}-1}{a^{2^{j_k}}}$. Since the sets $\mathfrak{K} \setminus r_{j_k}^{-1}(\Omega_{j_k}(x_1))$ are increasing in k, this implies that $P^{j_k} f$ converges to 0 almost everywhere. Hence, we see that $f = 0$ which is a contradiction. □

REMARK 7.2.2 The fact that ρ_n is non-spherical for $n \geq 1$ actually follows as an easy corollary of Theorem 7.4.1 below. We have, however, included a direct proof for completeness.

The non-equivalent representations ρ_n are exactly the building blocks for the representation π as we prove in the following theorem.

THEOREM 7.2.3 *The unitary representation π is equivalent to the direct sum of representations ρ_n with $n \geq 0$, i.e. $\pi = \bigoplus_{n=0}^{\infty} \rho_n$.*

PROOF. We have already seen that the closed subspaces $\mathcal{H}_n$ in $L^2(\mathfrak{K})$ are pairwise orthogonal. Furthermore, we recall that the closed subspaces $L^2(\Omega_n)$ by (7.1) form an increasing sequence in $L^2(\mathfrak{K})$. We now observe that $\bigcup_{n=0}^{\infty} L^2(\Omega_n)$ is dense in $L^2(\mathfrak{K})$, i.e. that

$$\overline{\bigcup_{n=0}^{\infty} L^2(\Omega_n)} = L^2(\mathfrak{K})$$

Indeed, we have seen that $\mathfrak{K}$ is a second countable compact Hausdorff space. By [Co, Proposition 7.2.3], this implies that the finite measure ν is a Radon measure. Hence, it is enough to show that $1_U \in \overline{\bigcup_{n=0}^{\infty} L^2(\Omega_n)}$ for an open set $U \subseteq \mathfrak{K}$. However, we may write $U = \bigcup_{k=1}^{\infty} r_{n_k}^{-1}(\Omega_{n_k}(x_k))$ for some $n_k \geq 0$ and some $x_k \in \mathfrak{X}_{n_k}$. This implies that it is enough to show that $1_{\bigcup_{k=1}^{m} r_{n_k}^{-1}(\Omega_{n_k}(x_k))} \in \overline{\bigcup_{n=0}^{\infty} L^2(\Omega_n)}$ for all $m \geq 1$. To see that this is true, we define $N = \max_{k=1}^{m} n_k$, and observe that $\bigcup_{k=1}^{m} r_{n_k}^{-1}(\Omega_{n_k}(x_k)) = \bigcup_{k=1}^{M} r_N^{-1}(\Omega_N(y_k))$ for some $y_k \in \mathfrak{X}_m$ and some $M \geq 1$. Since two sets of the form $\Omega_N(y_k)$ are either disjoint or satisfies some inclusion relation, we may even assume that the sets $\Omega_N(y_k)$ are pairwise disjoint. Since $1_{\bigcup_{k=1}^{M} r_N^{-1}(\Omega_N(y_k))} = \sum_{k=1}^{M} 1_{r_N^{-1}(\Omega_N(y_k))}$, and since $1_{r_N^{-1}(\Omega_N(y_k))} \in L^2(\Omega_N)$, this implies the denseness of $\bigcup_{n=0}^{\infty} L^2(\Omega_n)$ in $L^2(\mathfrak{K})$.

It is now a consequence of (7.2) and the denseness of $\bigcup_{n=0}^{\infty} L^2(\Omega_n)$ that $L^2(\mathfrak{K}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$. This finishes the proof. □

Whether the representations ρ_n with $n \geq 0$ are "'new"' representations or covered by the observations of chapter 6 will be the focus of the following section.

7.3 The existence of non-tame representations

In light of chapter 6, a natural question is whether the representations ρ_n are tame or, equivalently, whether they can be extended continuously to unitary representations of the bigger group K_∞. It turns out that this is not the case as long as we exclude the trivial representation ρ_0 (which is of course tame), and so they are not covered by the complete classification of all irreducible tame representations in Theorem 6.5.1. They are new representations and prove the existence of non-tame representations.

Theorem 7.3.2 is a consequence of the following Proposition 7.3.1.

PROPOSITION 7.3.1 *For $n \geq 1$, the representation ρ_n is not continuous in the topology inherited by K_∞.*

PROOF. Let $n \geq 1$ and choose $y \in (p^{-1}(x_{n-1}) \cap \mathfrak{X}_0) \setminus \{x_n\}$. Consider the function $f \in \mathcal{H}_n^0$ with the property that $f(x_n) = 1$, $f(y) = -1$ and $f(x) = 0$ for all $x \in \mathfrak{M}_n^0$ with $x \notin \{x_n, y\}$. We regard f as an element in $\mathcal{H}_n \subseteq L^2(\mathfrak{K})$. By construction of the map q_m, we may for $m \geq 0$ choose $g_m \in K_{m+1}$ such that $g_m(x) = x$ for all $x \in \mathfrak{X}_m$, such that $g_m(q_m^{-1}(x_n) \cap (\mathfrak{X}_{m+1} \setminus \mathfrak{X}_m)) = q_m^{-1}(y) \cap (\mathfrak{X}_{m+1} \setminus \mathfrak{X}_m)$ and such that $g_m(q_m^{-1}(y) \cap (\mathfrak{X}_{m+1} \setminus \mathfrak{X}_m)) = q_m^{-1}(x_n) \cap (\mathfrak{X}_m \setminus \mathfrak{X}_{m-1})$. By the first of these three properties, it follows that the sequence $\{g_m\}_{m=0}^{\infty} \subseteq K$ converges to the identity e in the topology of K_∞.

Let $m \geq 1$. It follows by the property 4. of the map q_m that

$$
\begin{aligned}
(q_0 \circ q_1 \circ \dots q_m)^{-1}(\{z\}) \cap (\mathfrak{M}_n^{m+1} \setminus \mathfrak{M}_n^m) &= \prod_{j=0}^{m} a^{2^j}(a^{2^j}+1)^{n-1} - \prod_{j=0}^{m-1} a^{2^j}(a^{2^j}+1)^{n-1} \\
&= (a^{2^m}(a^{2^m}+1)^{n-1} - 1) \prod_{j=0}^{m-1} a^{2^j}(a^{2^j}+1)^{n-1}
\end{aligned}
$$

for all $z \in \mathfrak{M}_n^0$. Using this, the relation $\nu_{m+1} = r_{m+1}(\nu)$ and the fact that $f, \pi(g_m)f \in \mathcal{H}_n^{m+1}$, we see that

$$
\begin{aligned}
\|\pi(g_m)f - f\|^2 &= 2 \cdot \frac{(a^{2^m}(a^{2^m}+1)^{n-1} - 1) \prod_{j=0}^{m-1} a^{2^j}(a^{2^j}+1)^{n-1}}{a^{2^{m+1}}(a^{2^{m+1}} - 1)^{n-1}} \cdot 4 \\
&= 8 \frac{\prod_{j=0}^{m-1} a^{2^j}}{a^{2^{m+1}}} \frac{(a^{2^m}(a^{2^m}+1)^{n-1} - 1) \prod_{j=0}^{m-1}(a^{2^j}+1)^{n-1}}{(a^{2^{m+1}} - 1)^{n-1}} \\
&= 8 \frac{a^{\sum_{j=0}^{m-1} 2^j}}{a^{2^m}} \frac{(a^{2^m}(a^{2^m}+1)^{n-1} - 1) \prod_{j=0}^{m-1}(a^{2^j}+1)^{n-1}}{a^{2^m}(a^{2^m}+1)^{n-1}(a^{2^m} - 1)^{n-1}} \\
&= 8a^{-1} \frac{a^{2^m}(a^{2^m}+1)^{n-1} - 1}{a^{2^m}(a^{2^m}+1)^{n-1}} \left(\frac{\prod_{j=0}^{m-1}(a^{2^j}+1)}{a^{2^m} - 1}\right)^{n-1}
\end{aligned}
\tag{7.4}
$$

We have that

$$
\frac{\prod_{j=0}^{m-1}(a^{2^j}+1)}{a^{2^m} - 1} \geq \frac{\prod_{j=0}^{m-1} a^{2^j}}{a^{2^m} - 1} = \frac{a^{2^m} a^{-1}}{a^{2^m} - 1} \geq a^{-1}
$$

which proves that the last factor in (7.4) dominates a^{1-n}. Since the factor $\frac{a^{2^m}(a^{2^m}+1)^{n-1}-1}{a^{2^m}(a^{2^m}+1)^{n-1}}$ in (7.4) converges to 1 as m goes to infinity, this shows that $\|\pi(g_m)f - f\|^2$ does not converge to 0 as m goes to infinity.

This proves that π is not continues in the topology on K inherited from K_∞. $\square$

Combining Proposition 7.3.1 and Corollary 6.3.5, we have established the fact that the representations ρ_n are non-tame for $n \geq 1$.

THEOREM 7.3.2 *For $n \geq 1$, the representation ρ_n is not tame.*

The last section of this chapter is devoted to a certain remarkable irreducibility property of the representations ρ_n.

7.4 Restriction of ρ_n to K^ω

In section 5.4 we discussed the decomposition of the restriction to K^ω of the spherical representations $\{\pi_n\}_{n=0}^{\infty}$ for the pair (K, K^ω). One consequence of Theorem 5.4.1 is that for $n \geq 1$ these representations do not remain irreducible under this restriction. It turns out that the pattern is completely different for the representations $\{\rho_n\}_{n=0}^{\infty}$. This is the content of the following Theorem 7.4.1 which contains the answer of a classical problem in representation theory - a problem which has been studied for locally finite homogeneous trees, cf. [St].

THEOREM 7.4.1 *The restriction of ρ_n to K^ω is irreducible for all $n \geq 0$.*

PROOF. Let $n \geq 2$, and consider $f \in \mathcal{H}_n^N$ for $N \geq 0$. For $k \geq N$, we define $f_k = f \cdot 1_{\mathfrak{K} \setminus r_k^{-1}(\Omega_k(x_{n-1}))}$. By the above identification, we may for all $k \geq 0$ regard $\mathcal{U}_n^k$ as a subpace of $\mathcal{H}_n^k$ in $L^2(\mathfrak{K})$. Since $f \in \mathcal{H}_n^k$ for $k \geq N$, we observe that $f_k \in \mathcal{U}_n^k$. Furthermore, we clearly have that f is bounded by some $M \geq 0$. Hence, we see that

$$\|f - f_k\|^2 \leq M^2 \nu(r_k^{-1}(\Omega_k(x_{n-1}))) = M^2 \nu_k(\Omega_k(x_{n-1})) = \frac{M^2}{a^{2^k}(a^{2^k}-1)^{n-2}}$$

which proves that $f \in \overline{\bigcup_{k=0}^{\infty} \mathcal{U}_n^k}$. This implies that $\mathcal{H}_n = \overline{\bigcup_{k=0}^{\infty} \mathcal{U}_n^k}$. Since it follows by section 4.4 that the subrepresentation of the restriction to K_k^ω of ρ_n corresponding to the subspace $\mathcal{U}_n^k$ is irreducible, it follows by Lemma 5.2.2 that the restriction of ρ_n to K^ω is also irreducible.

We are left with the case $n = 1$. Consider $f \in \mathcal{H}_1^N$ for $N \geq 0$. Choose $y \in \mathfrak{M}_1^0$ with $y \neq x_1$. For $k \geq N$, we define

$$f_k = f \cdot 1_{\mathfrak{K} \setminus r_k^{-1}(\Omega_k(x_1))} + f(x_1) \cdot 1_{\Omega_k(y)}$$

As above, we may for all $k \geq 0$ regard $\mathcal{V}_1^k$ as a subpace of $\mathcal{H}_1^k$ in $L^2(\mathfrak{K})$. Since $f \in \mathcal{H}_1^k$ for $k \geq N$, we observe that $f_k \in \mathcal{V}_1^k$. Furthermore, we see that

$$\|f - f_k\|^2 = \frac{2\,|f(x_1)|^2}{a^{2^k}}$$

which proves that $f \in \overline{\bigcup_{k=0}^{\infty} \mathcal{V}_1^k}$. This implies that $\mathcal{H}_1 = \overline{\bigcup_{k=0}^{\infty} \mathcal{V}_1^k}$. Since it follows by section 4.4 that the subrepresentation of the restriction to K_k^ω of ρ_1 corresponding to the subspace $\mathcal{V}_1^k$ is irreducible, it follows by Lemma 5.2.2 that the restriction of ρ_1 to K^ω is also irreducible.

Since ρ_0 is just the trivial representations of K, its restriction to K^ω is of course irreducible. This finishes the proof. □

REMARK 7.4.2 The construction of the non-tame representations $\{\rho_n\}_{n=0}^{\infty}$ and their properties as they are discussed in this chapter will be our only dig into the theory of non-tame representations of K. The most important observation is of course the existence of such representations. A number of the classical questions in representation theory remain, however, unanswered. It is an important open problem to further the development of this theory and provide the necessary answers. The infinite symmetric group $S(\infty)$ of all finite permutations of $\mathbb{N}$ does - as pointed out numerous times - share plenty of properties with K, and the work on this group has been a great source of inspiration in the work presented here. Since the development of the representation theory of $S(\infty)$ has progressed much further, much of the work in this area should be helpful in the future research into the group K.

Chapter 8

Conditionally Positive Definite Functions and Cocycles

This chapter deals with continuous, conditionally positive definite, biinvariant functions for the pairs (G, K) and (K, K^ω) from chapters 3 and 5. The main purpose is to state and prove Levy-Khinchine decomposition formulas in both cases which is done in Theorem 8.3.1 and Theorem 8.4.1. Section 1 contains some background material on conditionally positive definite functions and their relation with cocycles. In section 2, we consider a natural continuous, K-biinvariant function on G and use the corresponding cocycle to prove that it is conditionally positive definite. In section 3, we prove a Levy-Khinchine decomposition formula for the pair (G, K) and observe that all K-biinvariant, conditionally positive function on G may be built solely from the function studied in section 2 and the spherical functions for the pair (G, K). In section 4, we prove a Levy-Khinchine decomposition formula for the pair (K, K^ω) and observe that the spherical functions are the only necessary building blocks in this case.

8.1 Conditionally positive definite functions and cocycles

Using the generalized Bochner theorem of Rabaoui, cf. Theorem 1.3.6, and our construction of all spherical functions for the pair (G, K), we have in chapter 3 determined all K-biinvariant, positive definite functions on G. A class of related functions are the so-called conditionally positive definite functions which have been studied in a variety of settings. For our Olshanski spherical pair (G, K), one essential question in this area is to decompose all K-biinvariant, conditionally positive definite function by formulating a Levy-Khinchine formula which gives an integral representation of every such function. The purpose of this section is to state and prove such a formula.

Conditionally positive definite functions and Levy-Khinchine decomposition formulas play an important role for limit theorems for independent and identically distributed random variables, cf. [GK]. This has prompted an extensive reasearch into the area - both for concrete groups and from an abstract point of view. Main references are [Gu], [De] and [KaV]. The most famous occurrence might be in the important theorem of Schoenberg which we state as Theorem 8.3.2 below. Conditionally positive definite functions defined on automorphism groups for locally finite homogeneous trees have been studied in [KuV]. In recent years, conditionally positive definite functions have been examined in the framework of Olshanski spherical pairs. An example may be found in the paper [Boua]. We will continue this idea in this chapter by studying the Olshanski spherical pairs (G, K) and (K, K^ω) of the previous chapters.

Conditionally positive definite functions are closely related to the so-called cocycles which are again related to unitary representations. This intimate relation has been studied in a number of papers, cf. [Gu], [De] and [KaV], and will play an important role in our work below. It turns out that an important ingredient in the decomposition of our Levy-Khinchine formula is the natural conditionally positive definite function $\psi : g \mapsto -d(o, g(o))$ which is related to the well-known Haagerup cocycle. Along the way, we prove that the associated representation is irreducible and so ψ is *pure* or *indecomposable*. This is in sharp contrast to the case of a locally finite tree for which ψ and the Haagerup cocycle have been studied in [KuV]. In that paper an explicit decomposition of ψ is found, and this proves that ψ is not pure in the locally finite case.

We begin by recalling the basic facts about conditionally positive definite functions and their relation with cocycles. For further details, we refer to [KaV]. For the remainder of this section, we denote by T a topological group with neutral element e.

Definition 8.1.1 A function $\psi : T \to \mathbb{C}$ is said to be *conditionally positive definite* if $\psi(t^{-1}) = \overline{\psi(t)}$ for all $t \in T$ and if

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \psi(t_j^{-1} t_i) \geq 0$$

for all $t_1, \ldots, t_n \in T$, all $c_1, \ldots, c_n \in \mathbb{C}$ with $\sum_{i=1}^{n} c_i = 0$ and all $n \in \mathbb{N}$.

A conditionally positive definite function ψ is said to be *normalized* if $\psi(e) = 0$.

Clearly, the set of continuous, conditionally positive functions form a convex cone in the vector space of all complex-valued functions on T. Functions belonging to extreme rays in this convex cone will be known as *pure* or *indecomposable*. Furthermore, the continuous, normalized, conditionally positive functions form a subcone. In the case of T being locally compact, more may be said about the structure of these cones. See [KaV] for more details.

A positive definite function φ is of course conditionally positive definite, but not normalized unless $\varphi = 0$. We may, however, from a positive definite function φ construct a number of new conditionally positive definite functions. Indeed, for every real constant $c \in \mathbb{R}$, the function $\psi : T \to \mathbb{C}$ given for all $t \in T$ by $\psi(t) = \varphi(t) + c$ is conditionally positive definite. ψ is of course normalized if and only $c = -\varphi(e)$ and so we may from every positive definite function construct a normalized, conditionally positive definite function.

The function ψ constructed above is bounded. Remarkably, the converse is also true as the following proposition states:

Proposition 8.1.2 *Let ψ be a conditionally positive definite function on T. Then ψ is bounded if and only if there exists a positive definite function φ on T and a real constant $c \in \mathbb{R}$ such that $\psi(t) = \varphi(t) + c$ for all $t \in T$.*

The proof may be found in [KaV].

Using ideas very similar to the Gelfand-Naimark-Segal construction, it turns out that continuous, conditionally positive definite functions are closely related to unitary representations and their cocycles. We will give a short description of this connection. We begin by giving the definition of a cocycle in a unitary representation.

Definition 8.1.3 Let π be a unitary representation of T with representation space $\mathcal{H}_\pi$. A *cocycle in* π is a continuous map $\beta : T \to \mathcal{H}_\pi$ with the property that

$$\beta(t_1 t_2) = \beta(t_1) + \pi(t_1)\beta(t_2)$$

for all $t_1, t_2 \in T$.

A heavy amount of terminology is used in relation to the study of cocycles. For a vector $h \in \mathcal{H}_\pi$, it is evident that the map $\beta : T \to \mathcal{H}_\pi$ given for all $t \in T$ by $\beta(t) = \pi(t)v - v$ is a cocycle. Such a cocycle will be known as *trivial.* A cocycle β is *total* if $\overline{\operatorname{span}\{\beta(t) \mid t \in T\}} = \mathcal{H}_\pi$, and it is *real* if $\Im \langle \beta(t_1), \beta(t_2)\rangle_\pi = 0$ for all $t_1, t_2 \in T$. Two cocycles β_1, β_2 in representations π_1 and π_2, respectively, are said to be equivalent if there exists a unitary operator $U : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ with the properties that $\pi_2(t)U = U\pi_1(t)$ and $\beta_2(t) = U\beta_1(t)$ for all $t \in T$.

In analogy with Proposition 8.1.2, we note the following interesting observation. A proof may be found in [Gu].

Proposition 8.1.4 *A cocycle $\beta : T \to \mathcal{H}_\pi$ in a unitary representation π is bounded, i.e. $\sup_{t\in T} \|\beta(t)\|_\pi < \infty$, if and only if β is trivial.*

The intimate relation between cocycles and continuous, conditionally positive definite functions arise from the fact that it is very easy to construct such a function from a real cocycle β. This is the content of the following Lemma 8.1.5.

Lemma 8.1.5 *Let π be a unitary representation of T with representation space $\mathcal{H}_\pi$, and let β be a real cocycle in π. Let $\psi_\beta : T \to \mathbb{C}$ be given by the condition that $\psi_\beta(t) = -\frac{1}{2}\|\beta(t)\|_\pi^2$ for all $t \in T$. Then ψ_β is a continuous, real, normalized, conditionally positive definite function.*

Proof. It is immediate that ψ_β is continuous and real. To prove that it is conditionally positive definite, we observe that

$$\beta(e) = \beta(ee) = \beta(e) + \pi(e)\beta(e) = 2\beta(e)$$

which implies that $\beta(e) = 0$. Furthermore, this shows that for $t \in T$

$$\beta(t) + \pi(t)\beta(t^{-1}) = \beta(tt^{1-}) = \beta(e) = 0$$

which means that

$$\pi(t)\beta(t^{-1}) = -\beta(t) \tag{8.1}$$

for all $t \in T$. Using (8.1) and the fact that β is real, we get for $n \in \mathbb{N}$, $t_1, \ldots, t_n \in T$ and $c_1, \ldots, c_n \in \mathbb{C}$ with $\sum_{i=1}^n c_i = 0$ that

$$\begin{aligned}
\sum_{i,j=1}^n c_i\overline{c_j}\psi_\beta(t_j^{-1}t_i) &= -\frac{1}{2}\sum_{i,j=1}^n c_i\overline{c_j}\left\langle \beta(t_j^{-1}) + \pi(t_j^{-1})\beta(t_i), \beta(t_j^{-1}) + \pi(t_j^{-1})\beta(t_i)\right\rangle_\pi \\
&= -\frac{1}{2}\sum_{i,j=1}^n c_i\overline{c_j}\left\|\beta(t_j^{-1})\right\|_\pi^2 - \frac{1}{2}\sum_{i,j=1}^n c_i\overline{c_j}\|\beta(t_i)\|_\pi^2 - \sum_{i,j=1}^n c_i\overline{c_j}\Re\left\langle \pi(t_j)\beta(t_j^{-1}), \beta(t_i)\right\rangle_\pi \\
&= -\frac{1}{2}\sum_{j=1}^n \overline{c_j}\left\|\beta(t_j^{-1})\right\|_\pi^2 \sum_{i=1}^n c_i - \frac{1}{2}\sum_{i=1}^n c_i\|\beta(t_i)\|_\pi^2 \overline{\sum_{j=1}^n c_j} + \sum_{i,j=1}^n c_i\overline{c_j}\Re\langle \beta(t_j), \beta(t_i)\rangle_\pi \\
&= \left\|\sum_{i=1}^n \overline{c_i}\beta(t_i)\right\|_\pi^2 \geq 0
\end{aligned}$$

which proves that ψ_β is conditionally positive definite. Finally, the fact that $\beta(e) = 0$ implies that ψ_β is normalized. □

It turns out that all continuous, real, normalized, conditionally positive definite functions arise from a real cocycle as in Lemma 8.1.5. The cocycle may actually be chosen to be total as it is seen in the following Lemma 8.1.6. The proof is based on ideas very similar to the Gelfand-Naimark-Segal construction.

Lemma 8.1.6 *Let $\psi : T \to \mathbb{C}$ be a continuous, real, normalized, conditionally positive definite function. There exists a Hilbert space $\mathcal{H}_\pi$, a unitary representation π of T on $\mathcal{H}_\pi$ and a total and real cocycle β in π such that $\psi(t) = -\frac{1}{2}\|\beta(t)\|_\pi^2$ for all $t \in T$.*

Proof. For $t \in T$, we denote by $1_{\{t\}}$ the indicator function on T corresponding to the set $\{t\}$. We define $\mathcal{H}_0$ to be the vector space of finitely supported functions on T whose integral with respect to the counting measure are 0, i.e.

$$\mathcal{H}_0 = \left\{ \sum_{i=1}^n c_i 1_{\{t_i\}} \;\middle|\; n \in \mathbb{N}, t_1, \ldots, t_n \in T, c_1, \ldots, c_n \in \mathbb{C} \text{ with } \sum_{i=1}^n c_i = 0 \right\}$$

We may (well-)define a positive sesquilinear form $\langle \cdot, \cdot \rangle_0$ on $\mathcal{H}_0$ by the condition that

$$\left\langle \sum_{i=1}^n c_i 1_{\{t_i\}}, \sum_{j=1}^m d_j 1_{\{s_j\}} \right\rangle_0 = \sum_{i,j} c_i \overline{d_j} \psi(s_j^{-1} t_i)$$

for $\sum_{i=1}^n c_i 1_{\{t_i\}}, \sum_{j=1}^m d_j 1_{s_j} \in \mathcal{H}_0$. Consider the subspace

$$\mathcal{N} = \{ f \in \mathcal{H}_0 \mid \langle f, f \rangle_0 = 0 \}$$

We define $\mathcal{H} = \mathcal{H}_0 / \mathcal{N}$ and we denote by $[f]$ the coset in $\mathcal{H}$ corresponding to $f \in \mathcal{H}_0$. The sesquilinear form $\langle \cdot, \cdot \rangle_0$ induces an inner product on $\mathcal{H}$. We denote by $\mathcal{H}_\pi$ the completion of $\mathcal{H}$ with respect to this inner product.

For $t \in T$ and $\sum_{i=1}^n c_i 1_{\{t_i\}} \in \mathcal{H}_0$, we now (well-)define

$$\pi_0(t)(\sum_{i=1}^n c_i 1_{\{t_i\}}) = \sum_{i=1}^n c_i 1_{\{t t_i\}}$$

Clearly, $\pi_0(t)$ is a surjective linear map on $\mathcal{H}_0$ which preserves $\langle \cdot, \cdot \rangle_0$. $\pi_0(t)$ induces a linear isometry on $\mathcal{H}$, and so we may by continuity extend it to a unitary operator $\pi(t)$ on $\mathcal{H}_\pi$. For t, t', it is evident that $\pi(tt') = \pi(t)\pi(t')$ on $\mathcal{H}$ and so by continuity it holds on all of $\mathcal{H}_\pi$. If $f \in \mathcal{H}$ and $\{t_\lambda\} \subseteq T$ is a net converging to some $t \in T$, one may easily use the continuity of ψ and the fact that T is a topological group to check that $\pi(t_\lambda) f$ converges to $\pi(t) f$ in $\mathcal{H}_\pi$. By denseness of $\mathcal{H}$ in $\mathcal{H}_\pi$, this proves that π is continuous in the strong operator topology. Hence, π is a unitary representation of T on $\mathcal{H}_\pi$.

We now define $\beta : T \to \mathcal{H}_\pi$ by the condition that $\beta(t) = \left[1_{\{t\}} - 1_{\{e\}} \right] \in \mathcal{H}$ for all $t \in T$. Again, it is easy to use the contunuity of ψ and the fact that T is a topological group to check that β is continuous. Furthermore, we see that for $t_1, t_2 \in T$

$$\beta(t_1 t_2) = \left[1_{\{t_1 t_2\}} - 1_{\{e\}} \right] = \left[1_{\{t_1\}} - 1_{\{e\}} \right] + \left[1_{\{t_1 t_2\}} - 1_{\{t_1\}} \right] = \beta(t_1) + \pi(t_1)\beta(t_2)$$

which proves that β is a cocycle in π. Clearly, $\mathcal{H} = \operatorname{span}\{\beta(t) \mid t \in T\}$ and so β is total. Since ψ is real and

$$\langle \beta(t), \beta(s) \rangle_\pi = \psi(s^{-1}t) - \psi(t) - \psi(s^{-1}) + \psi(e) = \psi(s^{-1}t) - \psi(t) - \psi(s^{-1}) \tag{8.2}$$

for $t, s \in T$, β is real.

Finally, (8.2) and the fact that ψ is hermitian and real implies that

$$-\frac{1}{2}\,\|\beta(t)\|_\pi^2 = -\frac{1}{2}(\psi(e) - \psi(t) - \psi(t^{-1})) = -\frac{1}{2}(-2\psi(t)) = \psi(t)$$

for all $t \in T$ which finishes the proof. □

Combining Lemma 8.1.5 and Lemma 8.1.6, one may easily prove the first statement in the following important Theorem 8.1.7 about the relationship between real, total cocycles and continuous, real, normalized, conditionally positive definite functions - a result which is similar to the relation between continuous, positive definite functions and unitary representations in [Fo, Proposition 3.15 and Theorem 3.20]. The proof of the second is based on the same ideas as the proof of [Fo, Theorem 3.25]. The details may be found in [Gu] and [De].

Theorem 8.1.7 *With the notation of Lemma 8.1.5, the map $\beta \mapsto \psi_\beta$ induces a bijection between the set of equivalence classes of real, total cocycles and the set of continuous, real, normalized, conditionally positive definite functions on T. The function ψ_β is pure if and only if β is a cocycle in an irreducible representation.*

In sections 8.2 and 8.3, we will consider the pair (G, K) and K-biinvariant, conditionally positive definite functions on G. We will construct a continuous, real, normalized, conditionally positive definite function which is K-biinvariant, and we will - by constructing its corresponding cocycle of Theorem 8.1.7 - prove that it is pure. In section 8.3, we will prove a Levy-Khinchine formula for this pair and show that all K-biinvariant, conditionally positive definite functions may be built from this particular function and all spherical functions for the pair (G, K).

8.2 The Haagerup cocycle and its extension to G

Consider the group G_∞ of all automorphisms of the tree $(\mathfrak{X}, \mathfrak{C})$. We define a function $\psi : G_\infty \to \mathbb{C}$ by the condition that $\psi(g) = -d(o, g(o))$ for all $g \in G_\infty$. This function will be kept fixed for the remainder of this section. We will prove that it is a continuous, real, normalized, conditionally positive definite function. To do this, we will construct its corresponding cocycle β from Theorem 8.1.7 and show that - with the notation of Lemma 8.1.5 - $\psi = \psi_\beta$.

It turns out that the construction of β is based on a well-known cocycle for the free group which is known as the Haagerup cocycle and which has been studied in [H]. To see this, consider the free group Γ in countably many generators and choose a countable set A of generators. We denote by e the neutral element in Γ and by $|x|$ the length of $x \in \Gamma$ with respect to the generating set A. As in chapter 3, we may regard the tree $(\mathfrak{X}, \mathfrak{C})$ as the Cayley tree for Γ corresponding to A. Hence, we may identify $\mathfrak{X}$ with Γ in a way such that for $x, y \in \Gamma$ it is true that $\{x, y\} \in \mathfrak{C}$ if and only if $x^{-1}y \in A \cup A^{-1}$. We may even choose the identification such that $o \in \mathfrak{X}$ corresponds to $e \in \Gamma$. Furthermore, for $y \in \Gamma$ the map $x \mapsto yx$ is an automorphism of $(\mathfrak{X}, \mathfrak{C})$ and so we may also regard Γ as a subgroup of G_∞. We observe that the inherited topology from G_∞ on Γ is the discrete topology, i.e. the usual topology on Γ.

To construct the cocycle β corresponding to ψ, we start by defining it on the subgroup Γ of G_∞. We consider

$$\Lambda = \left\{(x, y) \in \Gamma \times \Gamma \;\middle|\; x^{-1}y \in A\right\}$$

and

$$\Lambda^- = \left\{(x,y) \in \Gamma \times \Gamma \;\middle|\; x^{-1}y \in A^{-1}\right\} = \{(x,y) \in \Gamma \times \Gamma \mid (y,x) \in \Lambda\}$$

We define $\mathcal{H}_\pi = \ell^2(\Lambda)$, and we denote by $e_{(x,y)}$ the indicator function corresponding to the set $\{(x,y)\}$ with $(x,y) \in \Lambda$. The set $\left\{e_{(x,y)}\right\}_{(x,y)\in\Lambda}$ clearly constitute an orthonormal basis for the Hilbert space $\mathcal{H}_\pi$. For $(x,y) \in \Lambda^{-1}$, we define $e_{(x,y)} = -e_{(y,x)}$.

For $(x,y) \in \Lambda$ and $z \in \Gamma$, we have that $(zx, zy) \in \Lambda$. Hence, we may define $\pi(z)e_{(x,y)} = e_{(zx,zy)}$. Since the map $(x,y) \mapsto (zx,zy)$ is a bijection from Λ onto Λ, this by linearity and continuity extends to a unitary operator $\pi(z)$ on $\mathcal{H}_\pi$. The map π is clearly a group homomorphism from Γ into the group of unitary operators on $\mathcal{H}_\pi$. Since the topology on Γ is discrete, π is a unitary representation of Γ on $\mathcal{H}_\pi$.

We will now define β which is a cocycle in π. To do this, consider $z \in \Gamma$. We may uniquely write z as a reduced word $z = a_1a_2\ldots a_n$ with $a_1,\ldots,a_n \in A \cup A^{-1}$. We define

$$\beta(z) = \sqrt{2}(e_{(e,a_1)} + e_{(a_1,a_1a_2)} + \ldots + e_{(a_1a_2\ldots a_{n-1},a_1a_2\ldots a_n)})$$

and so β is a continuous map from Γ into $\mathcal{H}_\pi$. By definition, it is true that $e_{(x,xa)} + e_{(xa,xaa^{-1})} = 0$ for $x \in \Gamma$ and $a \in A \cup A^{-1}$. This implies that $\beta(z_1z_2) = \beta(z_1) + \pi(z_1)\beta(z_2)$ for $z_1, z_2 \in \Gamma$ and so β is a cocycle in π. This cocycle on the free group Γ is known as the Haagerup cocycle and has been considered in [H]. It is real, and since $\beta(xa) - \beta(x) = e_{(x,xa)}$ for $x \in \Gamma$ and $a \in A$, it is also total.

Finally, we observe that

$$-\frac{1}{2}\left\|\beta(z)\right\|_\pi^2 = -\frac{1}{2}2\left|z\right| = \psi(z)$$

where the last equality uses the identification of Γ with a subgroup of G_∞. By Lemma 8.1.5, this proves that the restriction of ψ to Γ is a continuous, real, normalized, conditionally positive definite function. The corresponding cocycle from Theorem 8.1.7 is the Haagerup cocycle β.

We will now extend π to a unitary representation of G_∞ and β to a cocycle in this extended representation π such that β is the cocycle corresponding to ψ. To do this, let $g \in G_\infty$. By the identification of $\mathfrak{X}$ with Γ, we may regard $g(o)$ as an element of Γ. Since Γ may also be identified with a subgroup of G_∞, we denote by $x_g \in \Gamma$ the automorphism corresponding to $g(o) \in \Gamma$. Clearly, $x_g(o) = g(o)$ and so $k_g = x_g^{-1}g \in K_\infty$. Hence, we may write $g = x_gk_g$ with $x_g \in \Gamma$ and $k_g \in K_\infty$. Furthermore, this decomposition is unique since $K_\infty \cap \Gamma = \{e\}$. By definition of the topology on G_∞, the map $g \mapsto x_g$ is a continuous map from G_∞ onto Γ. Hence, we may extend the Haagerup cocycle β to a continuous map from G_∞ into $\mathcal{H}_\pi$ by defining

$$\beta(g) = \beta(x_g)$$

Since $\{x,y\} \in \mathfrak{C}$ for $(x,y) \in \Lambda$, it is also true that $\{g(x), g(y)\} \in \mathfrak{C}$. Hence, $(g(x), g(y)) \in \Lambda \cup \Lambda^-$, and so we may extend π to G_∞ by defining

$$\pi(g)e_{(x,y)} = e_{(g(x),g(y))} \tag{8.3}$$

for all $(x,y) \in \Lambda$. Clearly, $\pi(g)$ extends by linearity and continuity to a unitary operator $\pi(g)$ on $\mathcal{H}_\pi$. Since the relation (8.3) also holds for $(x,y) \in \Lambda^-$, it is true that $\pi(g_1g_2) = \pi(g_1)\pi(g_2)$ for $g_1, g_2 \in G_\infty$. Finally, it follows by the definition of the topology on G_∞ that $g \mapsto \pi(g)e_{(x,y)}$ is continuous which implies that π is continuous in the strong operator topology. Hence, we have extended π to a unitary representation of G_∞.

We will prove that the extended map β is a cocycle in π. To see this, let $g, h \in G_\infty$ and write with the above notation $g = x_g k_g$, $h = x_h k_h$ and $k_g x_h = x_{k_g x_h} k_{k_g x_h}$. By definition, β is right-invariant under K_∞ and so

$$\beta(gh) = \beta(x_g k_g x_h) = \beta(x_g x_{k_g x_h}) = \beta(x_g) + \pi(x_g)\beta(x_{k_g x_h}) = \beta(g) + \pi(x_g)\beta(x_{k_g x_h}) \quad (8.4)$$

Here we have used that β is a cocycle in π on Γ. Now consider the chain $x_0, x_1, \dots, x_n$ with $x_0 = o$ and $x_n = k_g(x_h(o))$. We have that $x_{k_g x_h}(o) = x_n$, and so it follows by definition that

$$\beta(x_{k_g x_h}) = \sqrt{2}(e_{(x_0,x_1)} + e_{(x_1,x_2)} + \dots + e_{(x_{n-1},x_n)})$$

However, we also see that

$$\pi(k_g)\beta(x_h) = \sqrt{2}(e_{(x_0,x_1)} + e_{(x_1,x_2)} + \dots + e_{(x_{n-1},x_n)})$$

and so $\beta(x_{k_g x_h}) = \pi(k_g)\beta(x_h)$. Inserting this in (8.4), we have that

$$\beta(gh) = \beta(g) + \pi(x_g)\pi(k_g)\beta(x_h) = \beta(g) + \pi(g)\beta(h)$$

which proves that β is a cocycle in π. Since β's restriction to Γ is total, the same is of course true for the extended cocycle, and it is an immediate consequence of the definition of the extension that β is also real. Since $\psi(g) = \psi(x_g)$ for all $g \in G_\infty$, we finally have that $\psi(g) = -\frac{1}{2}\|\beta(g)\|_\pi^2$ for all $g \in G$ and so ψ is a conditionally positive definite function which is obviously continuous, real and normalized and whose corresponding cocycle as in Theorem 8.1.7 is β.

It turns out that ψ is even pure. To see this, we will exploit Theorem 8.1.7 and prove that the representation π is irreducible.

Fix an edge $\{o, x\} \in \mathfrak{E}$, and consider the group $\mathrm{Aut}(\{o, x\})$ of automorphisms of the tree $\{o, x\}$. This is clearly the cyclic group of order 2 whose generator we denote by t and so there exists exactly two irreducible unitary representations of $\{o, x\}$. We will consider one of those, namely the one-dimensional unitary representation σ which is defined by the condition that $\sigma(t) = -1$. Furthermore, we define the subgroups

$$\tilde{G}_\infty(\{o, x\}) = \{g \in G_\infty \mid g(\{o, x\}) = \{o, x\}\}$$

and

$$G_\infty(\{o, x\}) = \{g \in G_\infty \mid g(o) = o \text{ and } g(x) = x\}$$

and observe that $\mathrm{Aut}(\{o, x\})$ is isomorphic to the quotient group $\tilde{G}_\infty(\{o, x\})/G_\infty(\{o, x\})$. As in chapter 6, we will regard σ as an irreducible unitary representation of $G_\infty(\{o, x\})$ which is trivial on $G_\infty(\{o, x\})$, and we denote by π_σ the representation of G_∞ induced by σ. If we by $\{g_\alpha\}_{\alpha \in A}$ denote a set of representatives for the equivalence classes in $G_\infty/\tilde{G}_\infty(\{o, x\})$, we recall that the representation space $\mathcal{H}_{\pi_\sigma}$ of π_σ consists of functions $f : G_\infty \to \mathbb{C}$ with the properties that

$$f(gh) = \sigma(h) f(g)$$

for all $g \in G_\infty$ and all $h \in \tilde{G}_\infty(\{o, x\})$ and that

$$\sum_{\alpha \in A} |f(g_\alpha)|^2 < \infty$$

and the inner product $\langle \cdot, \cdot \rangle_{\pi_\sigma}$ on $\mathcal{H}_{\pi_\sigma}$ is given by

$$\langle f_1, f_2 \rangle_{\pi_\sigma} = \sum_{\alpha \in A} f_1(g_\alpha) \overline{f_2(g_\alpha)}$$

for all $f_1, f_2 \in \mathcal{H}_{\pi_\sigma}$. The representation π_σ is defined by the condition that

$$(\pi_\sigma(g) f)(h) = f(g^{-1}h)$$

for all $g, h \in G_\infty$ and $f \in \mathcal{H}_{\pi_\sigma}$.

Now consider $(a,b) \in \Lambda \cup \Lambda^{-1}$, and observe that $\{a,b\} \in \mathfrak{C}$. We define $f_{(a,b)} : G_\infty \to \mathbb{C}$ by the condition that

$$f_{(a,b)}(g) = \begin{cases} 1 & \text{if } g(o) = a \text{ and } g(x) = b \\ -1 & \text{if } g(o) = b \text{ and } g(x) = a \\ 0 & \text{otherwise} \end{cases}$$

and observe that $f_{(a,b)} \in \mathcal{H}_{\pi_\sigma}$. Clearly, the set $\left\{ f_{(a,b)} \right\}_{(a,b) \in \Lambda}$ consitute an orthonormal set in $\mathcal{H}_{\pi_\sigma}$. Since $\{a,b\}_{(a,b)\in\Lambda} = \mathfrak{C}$, the set is even an orthonormal basis for $\mathcal{H}_{\pi_\sigma}$. Hence, we may define a unitary operator $U : \mathcal{H}_\pi \to \mathcal{H}_{\pi_\sigma}$ by the condition that $Ue_{(a,b)} = f_{(a,b)}$ for all $(a,b) \in \Lambda$. Since $Ue_{(a,b)} = f_{(a,b)}$ also for $(a,b) \in \Lambda^{-1}$, it is true that

$$\pi_\sigma(g) U e_{(a,b)} = \pi_\sigma(g) f_{(a,b)} = f_{(g(a),g(b))} = U e_{(g(a),g(b))} = U \pi(g) e_{(a,b)}$$

for all $(a,b) \in \Lambda$ and all $g \in G_\infty$, and so it follows by linearity and continuity that $\pi_\sigma(g) U = U \pi(g)$ for all $g \in G_\infty$. This proves π and π_σ are equivalent.

Since it is a consequence of Remark 6.5.5 that π_σ is irreducible, we have proved that π is also irreducible. The cocycle corresponding to ψ as in Theorem 8.1.7 is a cocycle in π, and so it follows by Theorem 8.1.7 that ψ is a pure conditionally positive definite function.

We collect these observations in the following Theorem 8.2.1:

Theorem 8.2.1 *The function $\psi : G_\infty \to \mathbb{C}$ given by $\psi(g) = -d(o, g(o))$ for all $g \in G_\infty$ is a continuous, real, normalized, conditionally positive definite function which is pure. The corresponding cocycle as in Theorem 8.1.7 is the extended Haagerup cocycle β whose corresponding representation π is irreducible.*

Remark 8.2.2 In [KuV], the function ψ has been considered for the case where $(\mathfrak{X}, \mathfrak{C})$ is a locally finite, homogeneous tree of order $q+1$. Using an extension of the Haagerup cocycle which is similar to the one constructed above, it is proved that ψ is a continuous, real, normalized, conditionally positive definite function, and the corresponding cocycle β and representation π as in Theorem 8.1.7 are constructed using an approach similar to the one above. However, an explicit decomposition of ψ is given which proves that ψ is not pure in the locally finite case.

This is, however, not surprising. Indeed, $\Lambda \cup \Lambda^{-1}$ may clearly be identified with the set $\mathcal{E}$ of oriented edges in $(\mathfrak{X}, \mathfrak{C})$ on which G_∞ acts continuously (when $\mathcal{E}$ is endowed with the discrete topology). Hence, we may consider the natural representation of G_∞ on $\ell^2(\mathcal{E})$. A function in $\ell^2(\mathcal{E})$ is said to be *even* if $f((a,b)) = f((b,a))$ for all $\{a,b\} \in \mathfrak{C}$. Similarly, f is said to be odd if $f((a,b)) = -f((b,a))$ for all $\{a,b\} \in \mathfrak{C}$. We denote by $\mathcal{V}^+$ the closed subspace of even functions in $\ell^2(\mathcal{E})$ and by $\mathcal{V}^-$ the closed subspace of odd functions in $\ell^2(\mathcal{E})$. Evidently, $\mathcal{V}^+$ and $\mathcal{V}^-$ are invariant subspaces of $\ell^2(\mathcal{E})$, and - using the identification of $\Lambda \cup \Lambda^{-1}$ with $\mathcal{E}$ - it is not hard to see that $\ell^2(\mathcal{E}) = \mathcal{V}^+ \oplus \mathcal{V}^-$. It is easy to check that the representation π is equivalent to the subrepresentation corresponding to $\mathcal{V}^-$. As seen above, this representation is irreducible if $(\mathfrak{X}, \mathfrak{C})$ is of countable order.

If $(\mathfrak{X}, \mathfrak{C})$ is locally finite of order $q+1$, this is, however, not the case. The subspace

$$\mathcal{H}^- = \left\{ f \in \mathcal{V}^- \;\middle|\; \sum_{\{a,b\} \in \mathfrak{C}} f((a,b)) = 0 \text{ for all } a \in \mathfrak{X} \right\} \tag{8.5}$$

is clearly invariant, and so π is not irreducible. By Theorem 8.1.7, this is in agreement with the fact that ψ is not pure in this case. As seen in [FN, Theorem 3.2.6], the subrepresentation corresponding to $\mathcal{H}^-$ is actually irreducible, and one of only two irreducible *special* representations, cf. Remark 2.5.4. The other one is the subrepresentation corresponding to the subspace $\mathcal{H}^+$ whose definition is obtained from (8.5) by replacing $\mathcal{V}^-$ with $\mathcal{V}^+$.

It should be noticed that the existence of exactly two irreducible special representations is a feature shared by the case where $(\mathfrak{X}, \mathfrak{C})$ has countable order. This is seen by the discussion of irreducible representations in Remark 6.5.5 from which it follows that the only two irreducible special representations may be obtained by inducing the two irreducible representations of $\mathrm{Aut}(\{o, x\})$ from $\tilde{G}_\infty(\{o, x\})$ to G_∞. These are equivalent to the representations corresponding to the subspaces $\mathcal{V}^+$ and $\mathcal{V}^-$, respectively. The extra condition in (8.5) to get irreducibility disappears in the case of a tree of countable order. A similar pattern has been observed before, cf. Remark 5.2.4.

Finally, one should notice that if one naively considers the limit for $q \to \infty$ of the decomposition of ψ in [KuV, Theorem 5], the second term vanishes, and we are left with the first term which is a conditionally positive definite function corresponding to the "‘negative"’ special representation. This suggests that in the case of a tree of countable order, ψ should be pure and correspond to a cocycle in the "‘negative"’ special representation. As seen above, these naive suggestions turn out to be correct.

Having proved that ψ is conditionally positive definite and pure, the next section is devoted to the proof of the fact that it plays a key role as a building block for all K-biinvariant, conditionally positive definite functions.

8.3 A Levy-Khinchine formula for (G, K)

In the previous section, we have proved that the function ψ is conditionally positive definite. The same may of course be said of its restriction to G. We will abuse the notation and for the rest of this section ψ will refer to this restriciton. Clearly, ψ is K-biinvariant. Another class of K-biinvariant, conditionally posive definite functions are the positive definite spherical functions for the pair (G, K). The objective of this section is to prove that this family together with ψ constitute the building blocks from which we may construct all K-biinvariant, conditionally positive definite functions on G. This is the content of Theorem 8.3.1 which gives a Levy-Khinchine formula for the pair (G, K).

Theorem 8.3.1 *Let $\varphi : G \to \mathbb{C}$ be a K-biinvariant function. Then φ is conditionally positive definite if and only if there exist constants $C_0 \in \mathbb{R}$, $C_1 \geq 0$ and a Radon measure μ on $[-1, 1)$ with*

$$\int_{[-1,1)} (1-c)\, \mu(dc) < \infty \tag{8.6}$$

such that

$$\varphi(g) = C_0 - C_1 d(o, g(o)) - \int_{[-1,1)} (1 - c^{d(o,g(o))})\, \mu(dc) \tag{8.7}$$

for all $g \in G$. In this case, the constants C_0 and C_1 and the measure μ are all unique.

For the proof which is inspired by ideas in [BCR] and [Boua], we need the following famous result of Schoenberg, cf. [Sc, page 527] and [BCR, Theorem 3.2.2]. We recall that a complex-valued function φ on a group is said to be *negative definite* if and only $-\varphi$ is conditionally positive definite.

Theorem 8.3.2 *Let T be a group and $\varphi : T \to \mathbb{C}$ be a function on T. Then T is negative definite if and only if $e^{-t\varphi}$ is positive definite for all $t > 0$.*

Furthermore, we need for the uniqueness part of the theorem the following technical lemma:

Lemma 8.3.3 *Let μ_1 and μ_2 be Radon measures on $[-1, 1)$ with the property that*

$$\int_{[-1,1)} (1 - c)\, \mu_j(dc) < \infty$$

for $j = 1, 2$. Assume that

$$\int_{[-1,1)} (1 - c^n)\, \mu_1(dc) = \int_{[-1,1)} (1 - c^n)\, \mu_2(dc) \tag{8.8}$$

for all $n \geq 0$. Then

$$\int_{[-1,1)} c^m(1 - c^n)\, \mu_1(dc) = \int_{[-1,1)} c^m(1 - c^n)\, \mu_2(dc)$$

for all $n, m \geq 0$.

Proof. Let $j = 1, 2$. Denote by λ_n the normalized Haar measure on K_n. Since the map $g \mapsto c^{d(o,g(o))}$ with $c \in [-1, 1)$ by Theorem 3.2.1 is spherical for the pair (G, K), it is true that

$$\lim_{n\to\infty} \int_{K_n} c^{d(o,gkh(o))}\, \lambda_n(dk) = c^{d(o,g(o))} c^{d(o,h(o))} \tag{8.9}$$

for all $g, h \in G$ and all $c \in [-1, 1)$. We begin by proving that

$$\lim_{n\to\infty} \int_{K_n} \int_{[-1,1)} (1 - c^{d(o,gkh(o))})\, \mu_j(dc)\, \lambda_n(dk) = \int_{[-1,1)} (1 - c^{d(o,g(o))} c^{d(o,h(o))})\, \mu_j(dc) \tag{8.10}$$

for all $g, h \in G$.

To do this, we fix $g, h \in G$. Let $n \geq 0$. It follows by the proof of Corollary 3.2.7 that K_n is second countable, and the same is clearly true for $[-1, 1)$. Hence, it follows by [Co, Proposition 7.6.2] that the Borel σ-algebra on $K_n \times [-1, 1)$ is the product of the Borel σ-algebras on K_n and $[-1, 1)$. The non-negative function $(k, c) \mapsto 1 - c^{d(o,gkh(o))}$ is clearly continuous on $K_n \times [-1, 1)$ and so measurable with respect to the product of the Borel σ-algebras on K_n and $[-1, 1)$. Since μ_j is a Radon measure and $[-1, 1)$ is σ-compact, μ_j is σ-finite. Hence, it follows by the classical version of Tonelli's theorem that

$$\int_{K_n} \int_{[-1,1)} (1 - c^{d(o,gkh(o))})\, \mu_j(dc)\, \lambda_n(dk) = \int_{[-1,1)} \int_{K_n} (1 - c^{d(o,gkh(o))})\, \lambda_n(dk)\, \mu_j(dc) \tag{8.11}$$

We observe that for all $n \geq 0$ the function $k \mapsto d(o, gkh(o))$ is bounded on K_n by some fixed $m \in \mathbb{N}$ (which is independent of n). Since λ_n is a probability measure this implies that the function $c \mapsto \int_{K_n}(1 - c^{d(o,gkh(o))})\ \lambda_n(dk)$ on $[-1, 1)$ for all $n \geq 0$ is bounded by the function $f : [-1, 1) \to [0, \infty)$ given by

$$f(c) = \begin{cases} 1 - c^m & \text{if } c \in [0, 1) \\ 1 - c & \text{if } c \in [-1, 0) \end{cases}$$

By assumption, the function $c \mapsto 1-c$ on $[-1, 1)$ is integrable with respect to μ_j. Furthermore, we see that $1 - c^m = (1 - c)\sum_{i=0}^{m-1} c^j$. Since $c \mapsto \sum_{j=0}^{m-1} c^j$ is bounded on $[-1, 1)$, this shows that also the function $c \mapsto 1 - c^m$ on $[-1, 1)$ is integrable with respect to μ_j. Hence, f is μ_j-integrable. Using this, (8.9) and dominated convergence, we see that

$$\lim_{n\to\infty} \int_{[-1,1)} \int_{K_n} (1 - c^{d(o,gkh(o))})\ \lambda_n(dk)\ \mu_j(dc) = \int_{[-1,1)} (1 - c^{d(o,g(o))}c^{d(o,h(o))})\ \mu_j(dc) \quad (8.12)$$

Combining (8.11) and (8.12), we get (8.10).

(8.10) and the assumption (8.8) now imply that

$$\int_{[-1,1)} (1 - c^{d(o,g(o))}c^{d(o,h(o))})\ \mu_1(dc) = \int_{[-1,1)} (1 - c^{d(o,g(o))}c^{d(o,h(o))})\ \mu_2(dc) \quad (8.13)$$

Subtracting the assumption (8.8) (with $n = d(o, g(o))$) from (8.13), we see that

$$\int_{[-1,1)} c^{d(o,g(o))}(1 - c^{d(o,h(o))})\ \mu_1(dc) = \int_{[-1,1)} c^{d(o,g(o))}(1 - c^{d(o,h(o))})\ \mu_2(dc)$$

If we for $n, m \geq 0$ choose $g, h \in G$ with $n = d(o, g(o))$ and $m = d(o, h(o))$, this finishes the proof. □

Proof of Theorem 8.3.1. We first assume that φ is given by the expression in (8.7) for all $g \in G$. Observe that the integrand in (8.7) is integrable for all $g \in G$ by the assumption in (8.6) since $1-c^n = (1-c)\sum_{i=0}^{n-1} c^j$ for all $n \geq 1$ and $c \mapsto \sum_{j=0}^{n-1} c^j$ is bounded on $[-1, 1)$. Since C_0 is real, the first term is conditionally positive definite. The fact that $C_1 \geq 0$ and Theorem 8.2.1 imply that the second term has the same property. Finally, the map $g \mapsto c^{d(o,g(o))}$ is for all $c \in [-1, 1)$ positive definite by Corollary 3.2.3, and so the map $g \mapsto c^{d(o,g(o))}-1$ is conditionally positive definite. Hence, the same holds for the third term and so φ is conditionally positive definite.

Now assume that φ is conditionally positive definite. Since $\varphi(g^{-1}) = \overline{\varphi(g)}$ for all $g \in G$, it follows that $C_0 = \varphi(e)$ is real. Define $\psi : G \to \mathbb{C}$ by the condition that $\psi(g) = \varphi(g) - C_0$ for all $g \in G$. Clearly, ψ is also conditionally positive definite. Hence, $-\psi$ is negative definite, and so it follows by Theorem 8.3.2 that $e^{t\psi}$ is positive definite for all $t > 0$. Since $\psi(e) = 0$ and ψ is K-biinvariant, it follows by Corollary 3.2.7 that for all $n \in \mathbb{N}$ there exists a Borel probability measure ν_n on $[-1, 1]$ such that

$$e^{\frac{1}{n}\psi(g)} = \int_{[-1,1]} c^{d(o,g(o))}\ \nu_n(dc) \quad (8.14)$$

for all $g \in G$.

By differentiability, we observe that

$$\lim_{n\to\infty} \frac{e^{\frac{1}{n}\psi(g)} - 1}{\frac{1}{n}} = \psi(g)$$

for all $g \in G$. This implies that for each $g \in G$ there exists a constant $C_g > 0$ such that

$$\left| \frac{e^{\frac{1}{n}\psi(g)} - 1}{\frac{1}{n}} \right| \leq C_g$$

for all $n \in \mathbb{N}$. Using (8.14) and the fact that ν_n is a probability measure, this shows that

$$\int_{[-1,1]} n(1 - c^{d(o,g(o))})\, \nu_n(dc) = \left| \int_{[-1,1]} (c^{d(o,g(o))} - 1) n\, \nu_n(dc) \right| = \left| \frac{e^{\frac{1}{n}\psi(g)} - 1}{\frac{1}{n}} \right| \leq C_g \tag{8.15}$$

for all $n \in \mathbb{N}$ and $g \in G$.

For each $n \in \mathbb{N}$, we consider the Borel measure τ_n on $[-1, 1]$ given by

$$\tau_n(A) = \int_A n(1 - c)\, \nu_n(dc)$$

for all Borel sets $A \subseteq [-1, 1]$. By choosing $g_0 \in G$ with $d(o, g_0(o)) = 1$, we see by (8.15) that $\tau_n([-1, 1]) \leq C_{g_0}$ for all $n \in \mathbb{N}$. Furthermore, by Theorem [Co, Proposition 7.2.3] the measures τ_n are Radon measures for all $n \in \mathbb{N}$. By the Riesz representation theorem [Ru, Theorem 6.19], the set of regular complex measures $\mathscr{M}([-1, 1])$ on $[-1, 1]$ is isometrically isomorphic to the continuous dual of the normed vector space $\mathscr{C}([-1, 1])$ consisting of all continuous functions on the compact space $[-1, 1]$ (with the supremum norm). By the Banach-Alaoglu theorem [T, Theorem 1.48], this implies that the closed unit ball in $\mathscr{M}([-1, 1])$ is compact in the weak-* topology. Since $\mathscr{M}([-1, 1])$ by [T, Proposition 1.43] is a topological vector space in the weak-* topology, the closed ball of radius C_{g_0} is also compact in this topology. Hence, there exists a subsequence n_j of $\mathbb{N}$ and a regular complex measure τ on $[-1, 1]$ such that τ_{n_j} converges to τ in the weak-* topology, i.e.

$$\lim_{j\to\infty} \int_{[-1,1]} f(c)\, \tau_{n_j}(dc) = \int_{[-1,1]} f(c)\, \tau(dc)$$

for all $f \in \mathscr{C}([-1, 1])$. As a special case, this implies that $\int_{[-1,1]} f(c)\, \tau(dc) \geq 0$ for $f \in \mathcal{C}([-1, 1])$ with $f \geq 0$ and so it follows by the Riesz representation theorem [Ru, Theorem 6.19] that τ is a positive Radon measure on $[-1, 1]$.

Now fix $g \in G$. Consider the function $f : [-1, 1] \to \mathbb{C}$ given by the condition that

$$f(c) = \begin{cases} \frac{c^{d(o,g(o))}-1}{1-c} & \text{if } c \neq 1 \\ -d(o, g(o)) & \text{if } c = 1 \end{cases}$$

By differentiability, $\lim_{c\to 1^-} \frac{c^{d(o,g(o))}-1}{1-c} = -d(o, g(o))$ and so $f \in \mathbb{C}([-1, 1])$. Hence, we see that

$$\begin{aligned} \psi(g) &= \lim_{j\to\infty} \frac{e^{\frac{1}{n_j}\psi(g)} - 1}{\frac{1}{n_j}} = \lim_{j\to\infty} \int_{[-1,1]} n_j(c^{d(o,g(o))} - 1)\, \nu_{n_j}(dc) \\ &= \lim_{j\to\infty} \int_{[-1,1]} f(c) n_j(1 - c)\, \nu_{n_j}(dc) = \lim_{j\to\infty} \int_{[-1,1]} f(c)\, \tau_{n_j}(dc) \\ &= \int_{[-1,1]} f(c)\, \tau(dc) \end{aligned} \tag{8.16}$$

Furthermore, we observe that

$$\int_{[-1,1]} f(c)\, \tau(dc) = \tau(\{1\}) f(1) + \int_{[-1,1)} \frac{c^{d(o,g(o))} - 1}{1 - c}\, \tau(dc) \tag{8.17}$$

We now define a Borel measure μ on $[-1,1)$ by the condition that

$$\mu(A) = \int_A \frac{1}{1-c}\, \tau(dc)$$

for all Borel sets $A \subseteq [-1,1)$, and by finiteness of τ we clearly have that

$$\int_{[-1,1)} (1-c)\, \mu(dc) < \infty$$

Furthermore, it follows by [Co, Proposition 7.2.3] and [Fo, Proposition 2.23] that μ is a Radon measure. If we put $C_1 = \tau(\{0\}) \geq 0$, we combine (8.16) and (8.17) to get

$$\varphi(g) - C_0 = \psi(g) = -C_1 d(o, g(o)) - \int_{[-1,1)} (1 - c^{d(o,g(o))})\, \mu(dc)$$

This finishes the existence part of the proof.

To prove uniqueness, we assume that φ is given by the expression (8.7). We immediately observe that $C_0 = \varphi(e)$ which proves uniqueness of C_0. Now choose for $n \geq 1$ $g_n \in G$ such that $d(o, g_n(o)) = n$. We then have that

$$\frac{\varphi(e) - \varphi(g_n)}{n} = C_1 + \int_{[-1,1)} \frac{1-c^n}{n}\, \mu(dc)$$

Clearly, $\lim_{n\to\infty} \frac{1-c^n}{n} = 0$ for all $c \in [-1,1)$. Furthermore, we see that

$$\left|\frac{1-c^n}{n}\right| = \left|\frac{(1-c)\sum_{j=0}^{n-1} c^j}{n}\right| \leq (1-c)\frac{n}{n} = 1-c$$

for all $c \in [-1,1)$, and since $c \mapsto 1-c$ is μ-integrable by assumption, it follows by dominated convergence that $\lim_{n\to\infty} \int_{[-1,1)} \frac{1-c^n}{n}\, \mu(dc) = 0$. Hence, we see that $C_1 = \lim_{n\to\infty} \frac{\varphi(e)-\varphi(g_n)}{n}$ which proves uniqueness of C_1.

To prove uniqueness of μ, we consider - in light of the previous observations - Radon measures μ_1 and μ_2 on $[-1,1)$ with the properties that

$$\int_{[-1,1)} (1 - c^{d(o,g(o))})\, \mu_1(dc) = \int_{[-1,1)} (1 - c^{d(o,g(o))})\, \mu_2(dc)$$

for all $g \in G$ and

$$\int_{[-1,1)} (1-c)\, \mu_j(dc) < \infty$$

for $j = 1, 2$. It follows by Lemma 8.3.3 that

$$\int_{[-1,1)} c^{d(o,g(o))}(1-c^n)\, \mu_1(dg) = \int_{[-1,1)} c^{d(o,g(o))}(1-c^n)\, \mu_2(dg) \tag{8.18}$$

for all $g \in G$ and all $n \geq 0$. The functions $c \mapsto 1 - c^n$ are μ_j-integrable for $j = 1, 2$ and $n \geq 0$, and so we define for $n \geq 0$ and $j = 1, 2$ the finite Borel measure μ_j^n on $[-1,1]$ by the condition that

$$\mu_j^n(A) = \int_{A\setminus\{1\}} (1-c^n)\, \mu_j(dc)$$

for all Borel sets $A \subseteq [-1,1]$. It now follows by (8.18) that

$$\int_{[-1,1]} c^{d(o,g(o))}\, \mu_1^n(dc) = \int_{[-1,1]} c^{d(o,g(o))}\, \mu_2^n(dc)$$

for all $g \in G$ and all $n \geq 0$. Hence, it follows by the uniqueness part of Theorem 1.3.6 that $\mu_1^n = \mu_2^n$ for all $n \geq 0$ and so

$$\int_A (1 - c^n)\, \mu_1(dc) = \int_A (1 - c^n)\, \mu_2(dc) \tag{8.19}$$

for all Borel sets $A \subseteq [-1, 1)$.

Now let $m \geq 1$ and consider a Borel set $A \subseteq (-\frac{1}{m}, \frac{1}{m})$. We observe that

$$|1 - c^n| = (1 - c) \left| \sum_{j=0}^{n-1} c^j \right| \leq (1 - c) \sum_{j=0}^{n-1} (\frac{1}{m})^j \leq \frac{1}{1 - \frac{1}{m}} (1 - c)$$

for all $c \in A$ and all $n \geq 1$. Since $\lim_{n\to\infty}(1 - c^n)1_A(c) = 1_A(c)$ for all $c \in [-1, 1)$, it follows by dominated convergence and (8.19) that $\mu_1(A) = \mu_2(A)$. By basic measure theory, this implies that $\mu_1(A) = \mu_2(A)$ for all Borel sets $A \subseteq (-1, 1)$. Since (8.19) with $A = \{-1\}$ and n odd implies that $2\mu_1(\{-1\}) = 2\mu_2(\{-1\})$, we see that $\mu_1 = \mu_2$.

This finishes the proof. □

From the Levy-Khinchine decomposition in Theorem 8.3.1, we see that every K-biinvariant, conditionally positive definite function may be constructed using the positive definite spherical functions for the pair (G, K) - which are all bounded - and only one unbounded function, namely the function ψ. Hence, the decomposition in (8.7) consists of a bounded and an unbounded part.

Finally, we immediately deduce the following Corollary 8.3.4:

Corollary 8.3.4 *Let $\varphi : G \to \mathbb{C}$ be continuous, K-biinvariant and conditionally positive definite. Then φ is real and bounded from above.*

8.4 A Levy-Khinchine formula for (K, K^ω)

In analogue with the previous section, we will also consider continuous, conditionally positive definite functions on K which are K^ω-biinvariant. It turns out that these are all bounded and may be built solely from the spherical functions for the pair (K, K^ω) (which are all positive definite). This will be clear from the Levy-Khinchine decomposition formula for the pair (K, K^ω) in Theorem 8.4.1 which we will prove using methods and ideas similar to the ones used in the proof of Theorem 8.3.1.

Theorem 8.4.1 *Let $\varphi : K \to \mathbb{C}$ be a function. Then φ is continuous, K^ω-biinvariant and conditionally positive definite if and only if there exist a constant $C \in \mathbb{R}$ and a sequence $\{a_n\}_{n=1}^\infty \in \ell^1(\mathbb{N})$ of non-negative numbers such that*

$$\varphi(g) = C - \sum_{n=1}^{\infty} a_n 1_{K_{x_n}^c}(g) \tag{8.20}$$

for all $g \in K$. If this is the case, the constant C and the sequence $\{a_n\}_{n=1}^\infty$ are both unique.

Proof. We first assume that φ is given by the expression in (8.20) for all $g \in K$. Since the function $1_{K_n^c}$ is clearly K^ω-biinvariant for all $n \geq 1$, φ is K^ω-biinvariant. Let $\{g_\lambda\} \subseteq K$ be a net converging to $g \in K$ in the topology inherited by K_∞. If $g \notin K^\omega$, it follows by the definition of the topology that there exists λ_0 such that $\left\{1_{K^c_{x_n}}(g_\lambda)\right\}_{n=1}^\infty = \left\{1_{K^c_{x_n}}(g)\right\}_{n=1}^\infty$ for all $\lambda \geq \lambda_0$. Hence, $\varphi(g_\lambda) = \varphi(g)$ for all $\lambda \geq \lambda_0$. If $g \in K^\omega$, it is a consequence of the definition of the topology that for each $N \geq 1$, there exists a λ_0 such that $1_{K^c_{x_n}}(g_\lambda) = 0$ for all $n \leq N$ and all $\lambda \geq \lambda_0$. Hence, $\varphi(g) = C = \lim_\lambda \varphi(g_\lambda)$. This proves that φ is continuous in the topology inherited by K_∞ and so in the inductive limit topology.

Finally, we observe that the fact that $C \in \mathbb{R}$ implies that the first term in (8.20) is conditionally positive definite. Furthermore, the map $1_{K_{x_n}}$ is for all $n \geq 1$ positive definite by Theorem 5.3.1, and so the map $-1_{K^c_{x_n}} = 1_{K_{x_n}} - 1$ is conditionally positive definite. Hence, the fact that the numbers a_n are all non-negative implies that the second term is also conditionally positive definite and so the same is true for φ.

Now assume that φ is continuous, K^ω-biinvariant and conditionally positive definite. Since $\varphi(g^{-1}) = \overline{\varphi(g)}$ for all $g \in K$, it follows that $C = \varphi(e)$ is real. Define $\psi : K \to \mathbb{C}$ by the condition that $\psi(g) = \varphi(g) - C$ for all $g \in G$. Clearly, ψ is also conditionally positive definite. Hence, $-\psi$ is negative definite, and so it follows by Theorem 8.3.2 that $e^{t\psi}$ is positive definite for all $t > 0$. Since $\psi(e) = 0$ and ψ is continuous and K-biinvariant, it follows by Corollary 5.3.6 that for all $m \in \mathbb{N}$ there exist a sequence $\{a_n^m\}_{n=0}^\infty$ of non-negative real numbers with $\sum_{n=0}^\infty a_n^m = 1$ such that

$$e^{\frac{1}{m}\psi(g)} = \sum_{n=0}^\infty a_n^m 1_{K_{x_n}}(g) \tag{8.21}$$

for all $g \in K$.

By differentiability, we observe that

$$\lim_{m\to\infty} \frac{e^{\frac{1}{m}\psi(g)} - 1}{\frac{1}{m}} = \psi(g) \tag{8.22}$$

for all $g \in K$. This implies that for each $g \in K$ there exists a constant $C_g > 0$ such that

$$\left| \frac{e^{\frac{1}{m}\psi(g)} - 1}{\frac{1}{m}} \right| \leq C_g$$

for all $m \in \mathbb{N}$. Using (8.21) and the fact that $\sum_{n=0}^\infty a_n^m = 1$, we see that

$$\begin{aligned} \sum_{n=1}^\infty m a_n^m 1_{K^c_{x_n}}(g) &= \sum_{n=0}^\infty \frac{a_n^m(1 - 1_{K_{x_n}}(g))}{\frac{1}{m}} = \left| \frac{\sum_{n=0}^\infty a_n^m 1_{K_{x_n}}(g) - 1}{\frac{1}{m}} \right| \\ &= \left| \frac{e^{\frac{1}{m}\psi(g)} - 1}{\frac{1}{m}} \right| \leq C_g \end{aligned} \tag{8.23}$$

for all $m \in \mathbb{N}$ and $g \in G$.

We now put $b_n^m = m a_n^m$ for $n, m \geq 1$ and consider the sequences $\{b_n^m\}_{n=1}^\infty \in \ell^1(\mathbb{N})$ for $m \geq 1$. By choosing $g_0 \in K \setminus K_{x_1}$ in (8.23), we see that $\sum_{n=1}^\infty b_n^m \leq C(g_0)$ for all $m \geq 1$. It is well-known that $\ell^1(\mathbb{N})$ may be isometrically embedded into the dual $\ell^\infty(\mathbb{N})^*$. By the Banach-Alaoglu theorem [T, Theorem 1.48], the closed unit ball in $\ell^\infty(\mathbb{N})^*$ is compact in the weak-* topology. Since $\ell^\infty(\mathbb{N})^*$ by [T, Proposition 1.43] is a topological vector space in the weak-* topology, the closed ball of radius C_{g_0} is also compact in this topology. Hence, there exists a convergent subnet $\left\{\{b_n^\lambda\}_{n=1}^\infty\right\}_{\lambda \in A}$ of the sequence $\{\{b_n^m\}_{n=1}^\infty\}_{m=1}^\infty \in \ell^\infty(\mathbb{N})^*$. Let $\Lambda \in \ell^\infty(\mathbb{N})^*$ denote its limit.

By weak-* convergence, we observe that $\lim_\lambda \sum_{n=1}^\infty b_n^\lambda c_n = \Lambda(\{c_n\})$ for all $\{c_n\} \in \ell^\infty(\mathbb{N})$. Denote for $n \geq 1$ by $e_n \in \ell^\infty(\mathbb{N})$ the sequence whose n'th entry is 1 while all other entries are 0. Define for $n \geq 1$ $a_n = \Lambda(e_n)$. Since $\lim_\lambda b_n^\lambda = \Lambda(e_n) = a_n$, we see that $a_n \geq 0$ for all $n \geq 1$.

Let $g \in K$. By the above, we see that $\lim_\lambda \sum_{n=1}^\infty b_n^\lambda 1_{K_{x_n}^c}(g) = \Lambda(\left\{1_{K_{x_n}^c(g)}\right\}_{n=1}^\infty)$. Combining this, the limit in (8.22) and the identity in (8.23), we see that

$$\psi(g) = -\Lambda(\left\{1_{K_{x_n}^c}(g)\right\}) \tag{8.24}$$

Now define f_1 to be the sequence whose entries are all 1 and define for $n \geq 2$ $f_n = f_1 - \sum_{i=1}^{n-1} e_n$. For $n \geq 1$, choose an automorphism $g_n \in K_0$ such that $g_n(x) = x$ for all $x \in \mathfrak{X}_0$ with $d(o,x) \leq n$ and $g_n(x_{n+1}) \neq x_{n+1}$. It is a consequence of the definition of the topology in K_0 that $e = \lim_{n\to\infty} g_n$. Since ψ is continuous in the inductive limit topology, its restriction to K_0 is continuous. Hence, we see that $0 = \psi(e) = \lim_{n\to\infty} \psi(g_n)$. By construction of g_n, we see that $\left\{1_{K_{x_m}^c}(g_n)\right\}_{m=1}^\infty = f_{n+1}$. By (8.24), this implies that $\lim_{n\to\infty} \Lambda(f_n) = 0$. Since $\sum_{k=1}^n a_k = \Lambda(e_1 + \ldots + e_n) = \Lambda(f_1) - \Lambda(f_{n+1})$, this proves that $\{a_n\} \in \ell^1(\mathbb{N})$.

Finally, for $g \in K \setminus K^\omega$ there exists $n \geq 1$ such that $\left\{1_{K_{x_m}^c}(g)\right\}_{m=1}^\infty = f_n$. Since $a_j = \Lambda(f_j) - \Lambda(f_{j+1})$ for $j \geq 1$, we see that

$$\psi(g) = -\Lambda(f_n) = \lim_{k\to\infty}(-\Lambda(f_n) + \Lambda(f_{k+1})) = -\lim_{k\to\infty}\sum_{j=n}^k a_j = -\sum_{j=n}^\infty a_j = -\sum_{j=1}^\infty a_j 1_{K_{x_j}^c}(g)$$

For $g \in K^\omega$, we have that $1_{K_{x_n}^c}(g) = 0$ for all $n \geq 1$, and so it is by (8.24) immediate that $\psi(g) = -\sum_{n=1}^\infty a_n 1_{K_n^c}(g)$. Since $\varphi = C + \psi$, we see that φ satisfies (8.20).

For the uniqueness, assume that φ is written as in (8.20). Choose for $n \geq 1$ an automorphism $g_n \in K_{x_{n-1}} \cap K_{x_n}^c$. We now see that $C = \varphi(e)$ and that $a_n = \varphi(g_n) - \varphi(g_{n+1})$ for all $n \geq 1$ which proves uniqueness of C and the sequence $\{a_n\}_{n=1}^\infty$. □

As is seen by the theorem, all continuous, K^ω-biiinvariant, conditionally positive definite functions may be constructed using only the spherical functions (K, K^ω), and they are all bounded. There is no unbounded part in the Levy-Khinchine decomposition (8.20). Observe that the restriction of the function $\psi : g \mapsto -d(o, g(o))$ to K is just the 0-function and so the study of this particular function is completely irrelevant for the pair (K, K^ω).

Finally, we deduce the following corollary:

Corollary 8.4.2 *Let $\varphi : K \to \mathbb{C}$ be continuous, K^ω-biinvariant and conditionally positive definite. Then φ is real and bounded.*

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