On Extensions of Group C^*-Algebras

PhD thesis in mathematics

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Abstract

The thesis deals with the question of (non-)invertibility of C^*-extensions arising from group C^*-algebras. The first chapter outlines the basic preliminaries and sets the stage. In particualar the basics of C^*-extensions and group C^*-algebras are treated and more advanced theory such as semi-invertibility of extensions and weak containment of representations is explained.

In the second chapter we give a negative result showing that the reduced group C^*-algebra of an amalgamated free product of Abelian groups has non-invertible extensions by the compact operators.

The third chapter gives a positive result. We show that all extensions of the reduced group C^*-algebra of a free product of amenable groups by any stable and sigma-unital C^*-algebra are semi-invertible.

Resumé

Afhandlingen omhandler spørgsmålet om invertibilitet af ekstensioner af gruppe-C^*-algebraer. I det første kapitel gennemgåes den grundlæggende teori og baggrund. Specielt diskuteres de basale ting om ekstensioner af C^*-algebraer, men også mere avancerede emner som semiinvertibilitet bliver behandlet.

Det andet kapitel indeholder et negativt resultat, idet det vises, at den reducerede gruppe-C^*-algebra af et amalgameret frit produkt af Abelske grupper har ikke-invertible ekstensioner med de kompakte operatorer.

I det tredje kapitel er hovedresultatet positivt; det vises, at et frit produkt af amenable grupper har en reduceret gruppe-C^*-algebra, for hvilken alle ekstensioner med enhver stabil og sigma-unital C^*-algebra er semiinvertible.

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Preface

The present dissertation marks the conclusion of my work as a PhD student over the last four years. It is divided into 3 chapters. The first chapter serves as an introduction to the last two. The second chapter consists of a manuscript [Se] which is accepted for publication but has not appeared yet. The third chapter is [ST] written jointly with my advisor Klaus Thomsen.

The thesis focuses on the question of invertibility of C^*-extensions. This question has been studied since the 1970's where Brown, Douglas and Fillmore defined the semigroup of extensions of a separable commutative unital C^*-algebra by the compact operators $\mathbb{K}$ on a separable infinite-dimensional Hilbert space. See, e.g., [BDF]. Since then much has happened. There is now a theory (due to G.G. Kasparov) of extensions of a general (separable) C^*-algebra by any stable C^*-algebra. In the first section of the first chapter we review the basics of this general theory and draw connections to the more classical theory, where the role of the ideal is played by $\mathbb{K}$.

Since all the C^*-extensions in this thesis arise from group C^*-algebras there is also a section on groups and C^*-algebras. The main topic of that section is the notion of weak containment which has a C^*-algebraic as well as a group theoretic formulation. The connection between these are studied and a few consequences are derived.

The introductory chapter ends with an exposition of a couple of results needed in Chapter 3 which did not fit into the first two sections. In particular, we give a relatively detailed and self-contained account of a rather technical result of K. Thomsen and V. Manuilov which is fundamental for Chapter 3.

In Chapter 2 we establish the MF property of the reduced group C^*-algebra of an amalgamated free product of countable Abelian discrete groups. This result is then used to give a characterization of the amalgamated free products of Abelian groups for which the BDF semigroup of the reduced group C^*-algebra is a group. Along the way we get a tensor product factorization of the corresponding group von Neumann algebra. Towards the end of the chapter we apply the ideas from the first part to give a few more examples of groups with a reduced group C^*-algebra which is MF.

In Chapter 3 we prove that the unitary equivalence classes of extensions of the reduced group C^*-algebra of a free product of a countable collection of countable amenable groups by any σ-unital stable C^*-algebra, taken modulo extensions which split via an asymptotic homomorphism, form a group.

There is maybe some non-uniformity in the exposition due to the fact that Chapters 2 and 3 are written as individual papers, thus there may be some overlap between particularly the introductory sections of these chapters and Chapter 1. I have made no effort to prevent this and I hope that the reader will bear with me or maybe even appreciate the occasional repetition.

The thesis is written so as to be relatively self-contained up to a good background in operator algebras. The introductory chapter is intended to be more easily accessible than the articles in chapters two and three in the hope that the reader will have a pleasant and leisurely experience reading the background material.

I have not formally defined every notion appearing in this report, but I have tried to give a quick explanation whenever an object or notion appears for the first time. Should you encounter any unexplained terminology, a good place to look for an explanation is [B].

Finally, I feel the need to thank several people for their support. First and foremost my advisors Klaus and Steen for excellent supervision and for listening to half-finished and less than well-organized presentations of preliminary ideas.

A special thanks goes to Uffe Haagerup whose advice I have benefitted from several times through Steen, and whose questions and ideas after my part A exam ultimately led to the results which appear in chapter two.

I enjoyed the hospitality and company of Nate Brown at the Pennsylvania State University in the autumn and winter of 2010-2011. For that I am very grateful.

Last but not least, I would like to thank my fellow (PhD) students at IMF for taking my mind off my project from time to time. I hope I succeeded in returning the favour.

Jonas Andersen Seebach

Chapter 1

Introduction

This chapter is designed to give (some of) the necessary background for Chapters 2 and 3. It has 3 sections, two general on C^*-extensions and on group C^*-algebras repectively and one more specialized which focuses on providing some preliminary results needed in Chapter 3.

1.1 Basics of extensions

In this section we review the basics of the theory of C^*-extensions. In particular we discuss invertibility of these and relate the classical BDF-theory to the general theory of extensions by stable C^*-algebras. Towards the end we discuss and motivate the recent idea of semi-invertibility.

Definition and the Busby invariant

Everything in this subsection is very basic. The material covered is mostly from [JT]. Throughout A, B are C^*-algebras.

Definition 1.1.1. Let A, B be C^*-algebras. An extension of A by B is a short exact sequence of C^*-algebras

$$0 \to B \to E \to A \to 0 \tag{1.1.1}$$

i.e. each of the maps ($*$-homomorphisms) satisfies that the image of one map equals the kernel of the next. In particular, the second map is injective and the third is surjective.

We will now construct the so-called Busby invariant of an extension.

Fix an extension of A by B and let the second map in (1.1.1) be denoted by i and the third by p. We let $M(B)$ denote the multiplier algebra of B and let $Q(B) := M(B)/B$ be the generalized Calkin algebra. The canonical quotient map $M(B) \to Q(B)$ is called q. By the universal property of the multiplier algebra we get a unique $*$-homomorphism

$\tau : E \to M(B)$ that makes

$$\begin{tikzcd} B \arrow[r, "i"] \arrow[dr] & E \arrow[d] \\ & M(B) \end{tikzcd}$$

commute. Here the bottom arrow is the canonical inclusion.

Let $a \in A$ and suppose $e, f \in p^{-1}(a)$. Then $e - f \in i(B)$ and so $q(\tau(e - f)) = 0$. It follows that there is a well-defined map $\varphi : A \to Q(B)$ making the following diagram commute

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[d, "\tau"] & A \arrow[d, "\varphi"] \\ M(B) \arrow[r, "q"] & Q(B) \end{tikzcd}$$

It is easy to check that φ is in fact a $*$-homomorphism.

Definition 1.1.2. The *Busby invariant* of the extension (1.1.1) is the $*$-homomorphism $\varphi : A \to Q(B)$ making the following diagram with exact rows commute

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\tau"] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B \arrow[r] & M(B) \arrow[r, "q"] & Q(B) \arrow[r] & 0 \end{tikzcd}$$

To deserve the name of 'invariant' the Busby invariant should of course be invarant under a suitable notion of isomorphism. We say that two extensions of A by B respectively given by maps i_j, p_j and C^*-algebras E_j, $j = 1, 2$, as above, are *isomorphic* if there is a $*$-homomorphism (hence $*$-isomorphism) $\psi : E_1 \to E_2$ making

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d, "\psi"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & E_2 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

commute.

This notion of isomorphism of course defines an equivalence relation on the set of extensions of A by B. As it turns out the Busby invariant is a complete invariant for this relation, i.e., two extensions are isomorphic if and only if they have the same Busby invariant. Furthermore, any $*$-homomorphism $\varphi : A \to Q(B)$ gives rise to an extension in the following way: Define

$$E = \{(a, x) \in A \times M(B) \mid \varphi(a) = q(x)\}. \tag{1.1.2}$$

This is a C^*-algebra. Note that B embeds in E by $b \mapsto (0, b)$, that $(a, x) \mapsto a$ is a surjective $*$-homomorphism and that the image of the first map equals the kernel of the last. In the language of category theory this is just a concrete realization of the pullback of $(A, M(B))$ along (φ, q). Note that the Busby invariant of the above extension is exactly φ. From here it is a rather straight-forward verification that the Busby invariant is in fact a complete invariant with respect to isomorphism. See Theorem 3.1.4 of [JT].

This thesis deals almost exclusively with the notion of unitary equivalence of extensions which we now define.

Definition 1.1.3. Two extensions of A by B respectively given by maps i_j, p_j and algebras E_j, $j = 1, 2$, are *unitarily equivalent* if there is a unitary $u \in M(B)$ and a $*$-homomorphism (hence $*$-isomorphism) $\psi : E_1 \to E_2$ making

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{Ad}(u)"] & E_1 \arrow[r] \arrow[d, "\psi"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & E_2 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

commute.

Just as before unitary equivalence defines an equivalence relation on the set of extensions. As this equivalence is clearly weaker than isomorphism, we would like to describe the corresponding equivalence relation on the level of Busby invariants.

Lemma 1.1.4. *Let φ_i, $i = 1, 2$, be the Busby invariants of two extensions of A by B. If the two extensions are unitarily equivalent via $u \in M(B)$ then $\mathrm{Ad}\, q(u) \circ \varphi_1 = \varphi_2$. Conversely, if there is a unitary $u \in M(B)$ such that $\mathrm{Ad}\, q(u) \circ \varphi_1 = \varphi_2$, then the extensions are unitarily equivalent via u.*

Proof. Assume that the extensions are unitarily equivalent via $u \in M(B)$. Consider the extension (ι, E, p) defined by the map $\mathrm{Ad}(q(u^*)) \circ \varphi_2$ as in 1.1.2. Define a $*$-homomorphism $\xi : E_2 \to E$ by $\xi(e) = (p_2(e), u^* \tau_2(e) u)$, where $\tau_2 : E_2 \to M(B)$ is the canonical map. Then ξ makes

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{Ad}(u)"] & E_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{Ad}(u^*)"] & E_2 \arrow[r] \arrow[d, "\xi"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & E \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

commute and hence, forgetting the middle row, we see that the extension corresponding to φ_1 is isomorphic to the one defined by $\mathrm{Ad}(q(u^*)) \circ \varphi_2$. It follows that the two maps are equal.

Conversely, let (ι, E, p) be the extension corresponding to $\mathrm{Ad}(q(u)) \circ \varphi_1$. Then we have a map $\xi : E_1 \to E$ given by $\xi(e) = (p_1(e), u\tau_1(e)u^*)$, where as before $\tau_1 : E_1 \to M(B)$ is the canonical map. Since $\mathrm{Ad}(q(u)) \circ \varphi_1 = \varphi_2$, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{Ad}(u)"] & E_1 \arrow[r] \arrow[d, "\xi"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & E_2 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

This proves the lemma. □

Additive structure

We would like to make the unitary equivalence classes of extensions of A by B into an abelian semigroup and hence we need to define an addition satisfying the usual requirements of associativity and commutativity. In order for us to do this, we will from now on assume that our ideal, B, is stable, i.e., $B \simeq B \otimes \mathbb{K}$.

Consider the (right) Hilbert C^*-module

$$H_B := \{(b_n)_{n\in\mathbb{N}} \in B^{\mathbb{N}} \mid \sum_{n=1}^{\infty} b_n^* b_n \text{ converges in } B\}$$

over B with the obvious right action, pointwise algebraic operations and B-valued inner product

$$\langle (a_n), (b_n) \rangle = \sum_{n=1}^{\infty} a_n^* b_n.$$

Partition $\mathbb{N}$ into two infinite disjoint sets N_1, N_2 and take bijections $\varphi_i : N_i \to \mathbb{N}$, $i = 1, 2$. Define $V_i : H_B \to H_B$ by

$$V_i(b_n) = (c_n), \quad \text{where } c_n = \begin{cases} b_{\varphi_i(n)} & \text{if } n \in N_i \\ 0 & \text{if } n \notin N_i \end{cases},$$

for $i = 1, 2$. Then V_i is an adjointable B-module map with adjoint V_i^* given by

$$V_i^*(b_n) = (c_n), \quad \text{where } c_n = b_{\varphi_i^{-1}(n)}.$$

Note that the V_i's are isometries, that $V_1^* V_2 = V_2^* V_1 = 0$ and $V_1 V_1^* + V_2 V_2^* = 1$. Since the C^*-algebra of adjointable operators on H_B is isomorphic to the multiplier algebra of $B \otimes \mathbb{K}$ (Lemma 1.27 of [JT]) which by assumption (and the universal property of the multiplier algebra) is isomorphic to $M(B)$, we get isometries $v_1, v_2 \in M(B)$ such that

$$v_1^* v_2 = v_2^* v_1 = 0 \text{ and } v_1 v_1^* + v_2 v_2^* = 1. \tag{1.1.3}$$

Define a $*$-isomorphism[1] Θ from the 2×2-matrices over B, $M_2(B)$, to B by

$$\Theta(\{b_{ij}\}) = v_1 b_{11} v_1^* + v_1 b_{12} v_2^* + v_2 b_{21} v_1^* + v_2 b_{22} v_2^*.$$

This extends to a $*$-isomorphism from $M_2(M(B))$ to $M(B)$ which we also call Θ. By taking the image of v_i in $Q(B)$ we also get an isomorphism $\Theta' : M_2(Q(B)) \to Q(B)$ in the same way we obtained Θ.

We are now ready to define an addition on the set of $*$-homomorphisms from A to $Q(B)$, namely

$$\varphi + \psi := \Theta' \circ \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix} = \Theta' \circ (\varphi \oplus \psi),$$

for $*$-homomorphisms $\varphi, \psi : A \to Q(B)$.

There are now several things to check and we only discuss these briefly since there is really nothing to add to the exposition given in [JT].

Since tensoring with $M_2(\mathbb{C})$ is an exact functor, we get a commutative diagram with exact rows as follows

$$\begin{tikzcd} 0 \arrow[r] & M_2(B) \arrow[r] \arrow[d, "\Theta"] & M_2(M(B)) \arrow[r, "q\otimes \mathrm{id}_{M_2(\mathbb{C})}"] \arrow[d, "\Theta"] & M_2(Q(B)) \arrow[r] \arrow[d, "\Theta'"] & 0 \\ 0 \arrow[r] & B \arrow[r] & M(B) \arrow[r] & Q(B) \arrow[r] & 0 \end{tikzcd} \tag{1.1.4}$$

The first thing which calls for checking is whether our addition is well-defined on unitary equivalence classes of $*$-homomorphism. A trip around the right square in the diagram above establishes this.

The next thing on the list is if the addition is associative. Luckily this is the case up to unitary equivalence. See Lemma 3.2.3 of [JT]. The same is true for commutativity. Indeed, the unitary $v_2 v_1^* + v_1 v_2^*$ implements the equivalence of $\varphi+\psi$ and $\psi+\varphi$ for $*$-homomorphisms $\varphi, \psi : A \to Q(B)$. In other words we have an abelian semigroup structure on the unitary equivalence classes of extensions.

One may also wonder if our addition is affected by the choice of Θ. Again the answer is: Only up to unitary equivalence. More precisely, any other isomorphism from $M_2(B)$ to B built out of isometries from $M(B)$ satisfying algebraic relations as in (1.1.3) is unitarily equivalent to Θ. This is Lemma 1.3.9 of [JT].

Neutral element and invertibility

In this subsection we will finally construct the main object of study in this thesis, namely the semigroup $\mathrm{Ext}(A, B)$, where A, B are C^*-algebras with B stable.

Definition 1.1.5. Let A, B be C^*-algebras. A $*$-homomorphism $\varphi : A \to Q(B)$ is said to be *split* if there is a $*$-homomorphism $\Phi : A \to M(B)$ such that $q \circ \Phi = \varphi$.

[1]The inverse is given by $b \mapsto \{v_i^* b v_j\}$.

Clearly, any extension unitarily equivalent to an extension with split Busby invariant has itself split Busby invariant, so that

$$S := \{[\varphi] \mid \varphi \text{ is split }\}$$

is a subset of the semigroup of unitary equivalence classes of extensions constructed in the last section. Here the brackets mean the unitary equivalence class of $\varphi : A \to Q(B)$.

Actually, S is a subsemigroup. Indeed, if $\varphi, \psi : A \to Q(B)$ are split with $*$-homomorphic lifts Φ, Ψ respectively, then the $*$-homomorphism $\mathrm{Ad}(v_1) \circ \Phi + \mathrm{Ad}(v_2) \circ \Psi : A \to M(B)$ satisfies

$$q \circ (\mathrm{Ad}(v_1) \circ \Phi + \mathrm{Ad}(v_2) \circ \Psi) = \Theta' \circ \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix}$$

by (1.1.4). And so S is a subsemigroup. This leads to the following fundamental definition.

Definition 1.1.6. Let A, B be C^*-algebras with B stable. The quotient of the abelian semigroup of unitary equivalence classes of extensions of A by B with S is called the extension semigroup of A by B. We denote it by $\mathrm{Ext}(A, B)$.

By construction $\mathrm{Ext}(A, B)$ has a neutral element, namely any element arising from a split $*$-homomorphism (such as, e.g., the zero-homomorphism). Two $*$-homomorphisms $\varphi, \psi : A \to Q(B)$ define the same element in $\mathrm{Ext}(A, B)$ exactly when there are split $*$-homomorphisms, $\tau_i : A \to Q(B)$, $i = 1, 2$, and a unitary $u \in M(B)$ such that

$$\mathrm{Ad}(q(u)) \circ \Theta' \circ (\varphi \oplus \tau_1) = \Theta' \circ (\psi \oplus \tau_2).$$

Before discussing the interesting question of invertibility in the semigroup $\mathrm{Ext}(A, B)$, we digress to give an 'explanation' of our choice of the word *split* in the above.

In homological algebra terms an extension

$$0 \to B \to E \xrightarrow{p} A \to 0$$

is split if there is a morphism $s : A \to E$ such that $p \circ s = \mathrm{id}_A$. Consequently, the Busby invariant of a split extension is split. The converse also holds but we postpone the proof until Lemma 1.1.26.

Since we now have a semigroup $\mathrm{Ext}(A, B)$ with a neutral element e, it makes sense to talk about invertibility. An element $x \in \mathrm{Ext}(A, B)$ is said to be *invertible* if there is an element $y \in \mathrm{Ext}(A, B)$ such that $x + y = e$. The element y is then called the inverse of x. We may ask if the semigroup is in fact a group and if not, what elements in $\mathrm{Ext}(A, B)$ are invertible? We quote the following theorem due to W. Arveson and G. G. Kasparov and refer to Theorem 3.2.9 of [JT] for an excellent exposition of the proof.

Theorem 1.1.7. *Let A, B be C^*-algebras, B stable and A separable. Let $\varphi : A \to Q(B)$ be a $*$-homomorphism. The following are equivalent*

1. *The element of* $\mathrm{Ext}(A, B)$ *defined by φ is invertible.*

2. *There is a completely positive contraction $\Phi : A \to M(B)$, such that $q \circ \Phi = \varphi$.*

3. *There is a $*$-homomorphism $\pi : A \to M_2(M(B))$, such that $\varphi = q \circ \pi_{11}$, where $\pi_{11} : A \to M(B)$ is the map that takes any $a \in A$ to the $(1,1)$'th entry of $\pi(a)$.*

A $*$-homomorphism $\varphi : A \to Q(B)$ satisfying the equivalent conditions in the above theorem is said to be *semisplit.*

Theorem 1.1.7 shows in conjunction with a renowned lfting theorem of M. D. Choi and E. G. Effros that for a large class of C^*-algebras $\mathrm{Ext}(A, B)$ is a group. We quote the lifting theorem as follows and refer to pages 377-378 of [B] for a very nice proof.

Theorem 1.1.8. *Let A, C be C^*-algebras with A separable. Let $B \subseteq C$ be an ideal and $\varphi : A \to C/B$ be a nuclear completely positive contraction. Then there is a completely positive contraction $\Phi : A \to C$ such that $\pi \circ \Phi = \varphi$, where $\pi : C \to C/B$ is the quotient map.*

From these two fundamental theorems we obtain the following easy corollary.

Corollary 1.1.9. *Let A, B be C^*-algebras with B stable, A separable and nuclear. Then $\mathrm{Ext}(A, B)$ is a group.*

Proof. Since A is nuclear, the identity map is nuclear, so every $*$-homomorphism $\varphi : A \to Q(B)$ is as well. As $*$-homomorphisms are completely positive contractions, the corollary follows from the characterization of invertible elements and the lifting theorem. □

We will now leave the general considerations we have had so far, and start to discuss (non-)invertibility of extensions in the case where the role of the ideal is played by the compact operators $\mathbb{K}$ on a separable infinite-dimensional Hilbert space. Later on we will discuss another (weaker) invertibility notion than the one used to define $\mathrm{Ext}(A, B)$. This comes about by trivializing a larger subsemigroup of unitary equivalence classes of extensions than S above.

Extensions by $\mathbb{K}$

Fix a separable *unital* C^*-algebra A. We consider the extensions of A by the compact operators $\mathbb{K}$ on a separable infinite-dimensional Hilbert space H. Our present goal is to show that the question of invertibility in the semigroup $\mathrm{Ext}(A, \mathbb{K})$ may be formulated within another semigroup, i.e., the Brown-Douglas-Fillmore semigroup of unitary equivalence classes of unital, injective $*$-homomorphisms $A \to B(H)/\mathbb{K} := Q(\mathbb{K})$ which for many purposes is easier to handle.

We still let $q : B(H) \to Q(\mathbb{K})$ be the quotient map.

Lemma 1.1.10. *Every element of $\mathrm{Ext}(A, \mathbb{K})$ is represented by a unital $*$-homomorphism.*

Proof. Suppose $\varphi : A \to Q(\mathbb{K})$ is a $*$-homomorphism. If φ is not unital, then $\varphi(1)$ is a projection in $Q(\mathbb{K})$ and it lifts to a projection $P \in B(H)$ (see Corollary 1.1.13 below). The projection $P^\perp$ is of infinite rank. Pick a unital $*$-homomorphism $\psi : A \to B(P^\perp H)$. Then $\varphi' := \varphi + q \circ \psi : A \to Q(\mathbb{K})$ is a unital $*$-homomorphism. We claim that φ' defines the same element as φ in $\mathrm{Ext}(A, \mathbb{K})$ since $q \circ \psi$ is split. We briefly review the arguments needed to see this.

It is well-known from the rudiments of K-theory that the projection $1 \oplus 0$ is Murray von Neumann equivalent to $P \oplus P^\perp$ in the 2×2 matrices over $B(H)$. It follows that $1 \oplus 0 \oplus 0 \oplus 0$ is unitarily equivalent to $P \oplus P^\perp \oplus 0 \oplus 0$ in the square matrices over $B(H)$ of dimension 4, i.e., $\varphi' \oplus 0 \oplus 0 \oplus 0$ is unitarily equivalent to $\varphi \oplus q \circ \psi \oplus 0 \oplus 0$. By dividing H into four orthogonal infinite-dimensional closed subspaces, fixing the first subspace and considering the orthogonal complement to this, we get an isomorphism $M_4(B(H)) \simeq M_2(B(H))$ which leaves the top left corner of the matrices invariant (by composition of isomorphisms obtained in the same way we obtained Θ earlier). It follows that there is a unitary $U \in M_2(B(H))$ such that $\mathrm{Ad}(q \otimes \mathrm{id}_{M_2(\mathbb{C})}(U)) \circ (\varphi' \oplus 0) = \varphi \oplus s$ for some split $*$-homomorphism s. Applying Θ' proves the claim. □

Remark 1.1.11. The proof above only relies on the fact that we can lift projections from the Calkin algebra, thus the statement could be generalized to the case where the generalized Calkin algebra of the ideal has this property.

For the reader not familiar with the lifting of projections from the Calkin algebra, we include the following argument - others may want to skip the next proposition and corollary.

For $T \in B(H)$, $\sigma(T)$ is the spectrum of T and we define the essential spectrum of T, $\sigma_e(T)$, to be the spectrum of $q(T)$ in $Q(\mathbb{K})$.

It is obvious that $\sigma_e(T) \subseteq \sigma(T)$ for any $T \in B(H)$.

Proposition 1.1.12. *Let $T \in B(H)$ be normal. The set $\sigma(T)\backslash\sigma_e(T)$ is discrete. More precisely it is the set of isolated eigenvalues for T with finite multiplicity.*

Proof. Let $\lambda \in \sigma(T)\backslash\sigma_e(T)$. Note that $\sigma(T)\backslash\sigma_e(T)$ is open in $\sigma(T)$ and pick a continuous $f : \sigma(T) \to [0,1]$ satisfying $\mathrm{supp}\, f \subseteq \sigma(T)\backslash\sigma_e(T)$ and $f(x) = 1$ for every x in some open neighbourhood U of λ.

Assume that λ is not isolated in $\sigma(T)$, then the spectral projection P associated to U is infinite since there is a sequence of pairwise disjoint open sets inside U all containing an element from $\sigma(T)$. It follows that

$$0 < q(P) \leq q(f(T)) = f(q(T)) = 0, \tag{1.1.5}$$

which is absurd.

We have established that λ is isolated in the spectrum and hence is an eigenvalue. It cannot have infinite multiplicity because in that case the spectral projection E corresponding to λ could be used in place of P in (1.1.5).

If λ is an isolated eigenvalue of finite multiplicity of T, choose a continuous function f with values in $[0,1]$ so that $f(\lambda) = 1$ and $\mathrm{supp}\, f \cap \sigma(T) = \{\lambda\}$. Then the spectral projection associated to $\operatorname{supp} f$ is compact and dominates $f(T)$, whence $f(T)$ is compact. It follows that $f(q(T)) = 0$ and hence $\lambda \notin \sigma_e(T)$. $\square$

Corollary 1.1.13. *Any projection in $Q(\mathbb{K})$ lifts to a projection in $B(H)$.*

Proof. Let $r \in Q(\mathbb{K})$ be a projection. Take any lift $p' \in B(H)$ of r and note that $p := (p'^* + p')/2$ is a self-adjoint lift. Now the essential spectrum of p is contained in $\{0,1\}$ so we may find $x \in (0,1)\backslash\sigma(p)$ and it follows that the characteristic function $f = 1_{(x,\infty)} \in C(\sigma(p))$. The desired lift is $f(p)$. $\square$

After this detour, we continue our effort to describe $\operatorname{Ext}(A, \mathbb{K})$.

Proposition 1.1.14. *Every element of* $\operatorname{Ext}(A, \mathbb{K})$ *is represented by a unital, injective $*$-homomorphism from A to the Calkin algebra.*

Proof. Since A is separable and unital, there is a unital and injective $*$-homomorphism $\rho : A \to B(H)$. Let U be a unitary from the countable Hilbert space direct sum $\ell^2(H)$ of copies of H to H. Now, $\tau := \operatorname{Ad}(U) \circ (\bigoplus \rho) : A \to B(H)$ is unital, injective and has no nonzero compact operators in its range. It follows that $q \circ \tau$ is also injective and unital and it is obviously split. For any element in $\operatorname{Ext}(A, \mathbb{K})$ there is a unital homomorphism $\varphi : A \to Q(\mathbb{K})$ representing it, taking the direct sum with $q \circ \tau$ and applying Θ' we get the desired homomorphism. $\square$

It is a theorem of D. Voiculescu that the unital split homomorphism $q \circ \tau$ constructed above is unique up to unitary equivalence in the sense that if $\tau' : A \to B(H)$ is a unital $*$-homomorphism with $q \circ \tau'$ injective, then $q \circ \tau'$ and $q \circ \tau$ are unitarily equivalent via a unitary from $B(H)$. See Theorem II.8.4.29 of [B] or [Ar]. We may consider the unitary equivalence classes of the unital injective $*$-homomorphisms from A to $Q(\mathbb{K})$ as a semigroup called $\operatorname{Ext}(A)$ with the additive structure constructed in the subsection 'Additive structure'. This is the classical Brown-Douglas-Fillmore semigroup. Voiculescu also showed that the element $q \circ \tau$ acts as a neutral element in this semigroup. I.e., it is unitally absorbing c.f. [Th2]. See [Ar] for a proof. Now, in much the same way as is the case for the general extension semigroup, it can be shown that an element $[\varphi]$ in $\operatorname{Ext}(A)$ is invertible if and only if there is a *unital* completely positive lift of φ to $B(H)$. Again we refer to [Ar] for a proof.

We have a natural semigroup homomorphism from $\operatorname{Ext}(A)$ to $\operatorname{Ext}(A, \mathbb{K})$. The above proposition tells us that this map is surjective. It is not an isomorphism in general, see [MT4] for a result in that direction, but we have the following result.

Proposition 1.1.15. *An element in* $\operatorname{Ext}(A)$ *is invertible if and only if, it is invertible in* $\operatorname{Ext}(A, \mathbb{K})$. *In particular* $\operatorname{Ext}(A)$ *is a group if and only if* $\operatorname{Ext}(A, \mathbb{K})$ *is a group.*

Proof. Let $\varphi : A \to Q(\mathbb{K})$ be a unital injective $*$-homomorphism. If the corresponding element in $\operatorname{Ext}(A)$ is invertible, then it is obviously invertible in $\operatorname{Ext}(A, \mathbb{K})$. Assume that it is invertible in $\operatorname{Ext}(A, \mathbb{K})$. Then by Theorem 1.1.7 there is a completely positive contractive lift $\Phi : A \to B(H)$ of φ. If Φ is not unital then $1 - \Phi(1)$ is positive, as Φ is contractive.

Also $1 - \Phi(1) \in \mathbb{K}$ because φ is unital. Take any state $\psi : A \to \mathbb{C} \simeq \mathbb{C}1 \subseteq B(H)$, this is completely positive by the GNS construction and so the map $\Phi' : A \to B(H)$ given by

$$\Phi'(a) = \Phi(a) + \psi(a)(1 - \Phi(1)), \quad a \in A,$$

is a unital completely positive lift of φ. It follows by the remark above that $[\varphi]$ is invertible in $\mathrm{Ext}(A)$. $\square$

As a sidenote we mention that if $\mathrm{Ext}(A, \mathbb{K})$ is a group, a sufficient condition for the natural map considered above to be an isomorphism is that A has a character. This is a consequence of Voiculescu's theorem and the fact that the assumption ensures that lifts of unital extensions can be chosen to be unital.

It is generally a non-trivial task to prove the existence of non-invertible extensions. One way of attacking this problem is contained in the following result which is folk lore allegedly originating from [V1]. Recall that a separable C^*-algebra is MF in the sense of B. Blackadar and E. Kirchberg if it embeds in

$$\prod_{n\in\mathbb{N}} M_{k_n}(\mathbb{C}) \Big/ \sum_{n\in\mathbb{N}} M_{k_n}(\mathbb{C})$$

for some sequence of natural numbers k_n, $n \in \mathbb{N}$. See e.g. Definition V.4.3.4 and Theorem V.4.3.5 of [B].

Proposition 1.1.16. *Let A be an MF C^*-algebra. If $\mathrm{Ext}(A, \mathbb{K})$ is a group, then A is quasidiagonal.*

Proof. We have a sequence of natural numbers k_n, $n \in \mathbb{N}$ and an injective $*$-homomorphism

$$\tau : A \to \prod_{n\in\mathbb{N}} M_{k_n}(\mathbb{C}) \Big/ \sum_{n\in\mathbb{N}} M_{k_n}(\mathbb{C}).$$

Put $H = \bigoplus_n \mathbb{C}^{k_n}$. Since

$$\prod_{n\in\mathbb{N}} M_{k_n}(\mathbb{C}) \cap \mathbb{K} = \sum_{n\in\mathbb{N}} M_{k_n}(\mathbb{C}), \tag{1.1.6}$$

we have an embedding

$$\prod_{n\in\mathbb{N}} M_{k_n}(\mathbb{C}) \Big/ \sum_{n\in\mathbb{N}} M_{k_n}(\mathbb{C}) \subseteq B(H) \Big/ \mathbb{K}.$$

Thus τ defines an extension of A by $\mathbb{K}$. Let ρ be a section for the quotient map $\prod_n M_{k_n}(\mathbb{C}) \to \prod_n M_{k_n}(\mathbb{C}) \big/ \sum_n M_{k_n}(\mathbb{C})$. By 1.1.6 the map q from $B(H)$ to the Calkin algebra extends $\prod_n M_{k_n}(\mathbb{C}) \to \prod_n M_{k_n}(\mathbb{C}) \big/ \sum_n M_{k_n}(\mathbb{C})$. Let $\varphi : A \to B(H)$ be the contractive completely positive lift of τ which exists by assumption and Theorem 1.1.7 and let $P_n \in B(H)$

be the projection onto $\mathbb{C}^{k_n}$. Set $\varphi_n = \mathrm{Ad}(P_n) \circ \varphi$ and $\tau_n = \mathrm{Ad}(P_n) \circ \rho \circ \tau$ and note that φ_n is a completely positive contraction for each n. Note also that

$$\limsup_n \|\varphi_n(a) - \tau_n(a)\| = 0$$

for all $a \in A$, because $\tau = q \circ \rho \circ \tau$ and $q \circ \varphi = \tau$. In particular

$$\limsup_n \|\varphi_n(a)\| = \|\tau(a)\| = \|a\| \tag{1.1.7}$$

for all $a \in A$. Also note that for $a, b \in A$ and $n \in \mathbb{N}$

$$\begin{aligned} \|\tau_n(ab) - \tau_n(a)\tau_n(b)\| &= \|P_n\rho(\tau(ab))P_n - P_n\rho(\tau(a))P_n\rho(\tau(b))P_n\| \\ &= \|P_n\rho(\tau(ab))P_n - P_n\rho(\tau(a))\rho(\tau(b))P_n\| \\ &= \|P_n\left(\rho(\tau(ab)) - \rho(\tau(a))\rho(\tau(b))\right)P_n\| \end{aligned}$$

where the second equality follows from the fact that $\rho(\tau(a))$ is block diagonal. It follows that

$$\limsup_n \|\tau_n(ab) - \tau_n(a)\tau_n(b)\| = \|q\left(\rho(\tau(ab)) - \rho(\tau(a))\rho(\tau(b))\right)\| = \|\tau(ab) - \tau(a)\tau(b)\| = 0,$$

and hence by the triangle inequality

$$\lim_n \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = 0.$$

It follows from this, (1.1.7) and Voiculescu's characterization of quasidiagonality (Theorem V.4.2.14 of [B]) that A is quasidiagonal. $\square$

The above proposition is often used in conjunction with a result of Rosenberg (Proposition 1.2.5) stating that if the reduced group C^*-algebra of a discrete group is quasidiagonal, then it follows that the group is amenable. Specifically, to show that a group has a reduced group C^*-algebra which has non-invertible extensions by $\mathbb{K}$, it suffices to show that the reduced group C^*-algebra is MF (this is oftentimes hard) and that the group is not amenable (which is oftentimes known or easy).

Given a C^*-algebra A which has non-invertible extensions by $\mathbb{K}$, one may ask if there is another stable C^*-algbra B for which $\mathrm{Ext}(A, B)$ is a group. We have the following partial answer to this question which shows that in looking for non-invertible extensions of A, the ideal $\mathbb{K}$ is somehow the canonical starting point.

Proposition 1.1.17. *Let A, B be C^*-algebras with A separable and B unital. If* $\mathrm{Ext}(A, \mathbb{K})$ *is not a group then* $\mathrm{Ext}(A, \mathbb{K} \otimes B)$ *is not a group.*

Proof. In the proof all tensorproducts will be maximal.

Suppose

$$0 \to \mathbb{K} \to E \xrightarrow{p} A \to 0 \tag{1.1.8}$$

represents an element which is not invertible in $\mathrm{Ext}(A, \mathbb{K})$. Then by II.9.6.6 in [B]

$$0 \to \mathbb{K} \otimes B \to E \otimes B \to A \otimes B \to 0 \tag{1.1.9}$$

is exact, where the maps are just the identity on B tensored with the corresponding maps from the original extension.

We can embed A as a subalgebra in $A \otimes B$ via $\iota(a) = a \otimes 1$, $a \in A$. Let E_0 be the pullback of $(A, E \otimes B)$ along $(\iota, p \otimes \mathrm{id}_B)$ i.e.

$$E_0 = \{(a, x) \in A \times (E \otimes B) \mid \iota(a) = (p \otimes \mathrm{id}_B)(x)\}.$$

Then we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{K} \otimes B \arrow[r] \arrow[d, equal] & E_0 \arrow[r, "p'"] \arrow[d, "\chi"] & A \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & \mathbb{K} \otimes B \arrow[r] & E \otimes B \arrow[r] & A \otimes B \arrow[r] & 0
\end{tikzcd}$$

where p' and χ are just the projections on the first and second coordinate respectively.

We claim that the element of $\mathrm{Ext}(A, \mathbb{K} \otimes B)$ defined by the first row in the diagram above is not invertible. Indeed, if it was invertible there would by Theorem 1.1.7 and Lemma 1.1.26 exist a contractive completely positive section s for p'. Let ψ be a state on B then the map $(\mathrm{id}_E \otimes \psi) \circ \chi \circ s$ is a completely positive contraction by Corollary II.9.7.3 in [B] and it is a section for p contradicting that (1.1.8) is non-invertible. To see that the considered map is indeed a section for p, note first that $p \circ (\mathrm{id}_E \otimes \psi) = (\mathrm{id}_A \otimes \psi) \circ (p \otimes \mathrm{id}_B)$. Now for $a \in A$ we have

$$p \circ (\mathrm{id}_E \otimes \psi) \circ \chi \circ s(a) = (\mathrm{id}_A \otimes \psi) \circ (p \otimes \mathrm{id}_B) \circ \chi \circ s(a) = (\mathrm{id}_A \otimes \psi) \circ \iota(p'(s(a))) = a,$$

proving the claim. $\square$

Remark 1.1.18. From the above constructions we also get another non-invertible extension. Indeed, the extension (1.1.9) of $A \otimes_{\max} B$ by $\mathbb{K} \otimes B$ represents a non-invertible element of $\mathrm{Ext}(A \otimes_{\max} B, \mathbb{K} \otimes B)$. We leave the considerations needed in order to see this to the reader.

Semi-invertibility

We will now review another notion of invertibility of extensions. It comes about by replacing the subsemigroup of split extensions by another, larger subsemigroup which we will describe below. In other words we trivialize a larger class of extensions hence making it easier for an extension to be invertible. The definition of this semigroup is due to V. Manuilov and K. Thomsen.

We will need the notion of asymptotic homomorphisms and some basic properties of these. The material covered regarding this is from [Th4].

Definition 1.1.19. Let A, B be C^*-algebras. An asymptotic homomorphism π from A to B is a famililiy of maps $\pi_t : A \to B$, $t \in [1, \infty)$ such that $t \mapsto \pi_t(a)$ is continuous in the norm topology as a map from $[1, \infty)$ into B for every $a \in A$. Furthermore the family is required to satisfy

$$\lim_{t\to\infty} \|\pi_t(a)^* - \pi_t(a^*)\| = 0,$$
$$\lim_{t\to\infty} \|\pi_t(\lambda a + b) - (\lambda\pi_t(a) + \pi_t(b))\| = 0,$$
$$\lim_{t\to\infty} \|\pi_t(ab) - \pi_t(a)\pi_t(b)\| = 0$$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$.

It is a crucial fact that asymptotic homomorphisms are norm bounded.

Lemma 1.1.20. *Let A, B be C^*-algebras and $\pi : A \to B$ an asymptotic homomorphism. For every $a \in A$ the map*
$$t \mapsto \pi_t(a)$$
is bounded.

Proof. By considering the unitization $\pi_t'(a + \lambda 1) = \pi_t(a) + \lambda 1$, $a \in A$ and $\lambda \in \mathbb{C}$, where 1 denotes the unit in the unitizations of A, B respectively, we may assume that A, B are unital and that $\pi_t(1) = 1$ for all t. We leave it to the reader to check that this unitization yields a new asymptotic homomorphism.

The maps $\psi_t : M_2(A) \to M_2(B)$, $t \in [1, \infty)$ defined by

$$\psi_t\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} \pi_t(a_{11}) & \pi_t(a_{12}) \\ \pi_t(a_{21}) & \pi_t(a_{22}) \end{pmatrix}$$

are readily seen to constitute an asymptotic homomorphism.

Let $a \in A$ be a positive element of norm less than 1. Then

$$p = \begin{pmatrix} a & -\sqrt{(a - a^2)} \\ -\sqrt{(a - a^2)} & 1 - a \end{pmatrix}$$

is a projection.

Fix $\varepsilon > 0$. By the above we may pick $t_0 \in [1, \infty)$ so that $\|\psi_t(p) - \frac{1}{2}(\psi_t(p)^* + \psi_t(p))\| \leq \varepsilon$ and $\|\psi_t(p)^2 - \psi_t(p)\| \leq \varepsilon$ for t greater than t_0. It follows that for $t \geq t_0$ the imaginary part, y, of $\psi_t(p)$ has norm less than ε. Denote by x the real part of $\psi_t(p)$. Then

$$\psi_t(p)^2 - \psi_t(p) = x^2 - y^2 - x + i(xy + yx - y)$$

so

$$\|x^2 - x\| \leq \|x^2 - x - y^2\| + \varepsilon^2 \leq \|\psi_t(p)^2 - \psi_t(p)\| + \varepsilon^2 \leq \varepsilon + \varepsilon^2.$$

In particular $\|x\| \leq 1 + \varepsilon$ and so

$$\limsup_t \|\pi_t(a)\| \leq \limsup_t \|\psi_t(p)\| \leq 1.$$

From this it is clear that

$$\limsup_t \|\pi_t(b)\| \leq 4\|b\|,$$

for any $b \in A$, proving the lemma. $\square$

Remark 1.1.21. In the setting of Lemma 1.1.20 the statement can be improved to the effect that

$$\limsup_t \|\pi_t(a)\| \leq \|a\| \tag{1.1.10}$$

for $a \in A$. Indeed, π may be thought of as a $*$-homomorphism into the quotient

$$C_b([1,\infty), B)/C_0([1,\infty), B).$$

The inequality (1.1.10) is then nothing but the fact that $*$-homomorphisms are contractions.

Another basic fact is the following which shows that for many purposes it suffices to consider asymptotic homomorphisms given by equi-continuous families of maps.

Lemma 1.1.22. *Let A, B be C^*-algebras and $\pi : A \to B$ an asymptotic homomorphism. There is an equi-continuous asymptotic homomorphism $\pi' : A \to B$ satisfying*

$$\lim_{t\to\infty} \|\pi_t(a) - \pi'_t(a)\| = 0$$

for all $a \in A$.

Proof. Consider the quotient C^*-algebra

$$Q := C_b([1,\infty), B)/C_0([1,\infty), B).$$

Note that we may think of π as a $*$-homomorphism from A to Q.

From the Bartle-Graves selection theorem we obtain a continuous map $S : Q \to C_b([1,\infty), B)$ which is a section for the canonical quotient map going with Q. Put $\pi'_t(a) = (S \circ \pi)(a)(t)$ for $t \in [1,\infty)$ and $a \in A$.

It is easily checked that this map has the desired properties. $\square$

Definition 1.1.23. Let A, B be C^*-algebras. A $*$-homomorphism $\varphi : A \to Q(B)$ is said to be *asymptotically split* if there is an asymptotic homomorphism $\pi : A \to M(B)$ such that $q \circ \pi_t(a) = \varphi(a)$ for all $a \in A$ and $t \in [1,\infty)$.

Let A, B be C^*-algebras with B stable. By a repetition of the argument in 'Neutral element and invertibility' the unitary equivalence classes of extensions corresponding to the asymptotically split $*$-homomorphisms from A to $Q(B)$ constitute a subsemigroup of the unitary equivalence classes of all extensions, and so we obtain another semigroup with a neutral element by dividing out with this subsemigroup. Two $*$-homomorphisms φ, ψ define the same element in this semigroup, if there are asymptotically split homomorphisms τ_i, $i = 1, 2$, such that $\varphi \oplus \tau_1$ is unitarily equivalent to $\psi \oplus \tau_2$. An extension defining an invertible element in this semigroup is called *semi-invertible*.

Several natural questions arise in this setting. The first question is whether this new semigroup is different from $\mathrm{Ext}(A,B)$. In general it is; Haagerup and Thorbjrnsen have shown that $\mathrm{Ext}(C_r^*(\mathbb{F}_n),\mathbb{K})$ is not a group [HT] and Thomsen has shown that all extensions of $C_r^*(\mathbb{F}_n)$ by $\mathbb{K}$ are semi-invertible [Th1]. Here $C_r^*(\mathbb{F}_n)$ is the reduced group C^*-algebra of the free group $\mathbb{F}_n$ on $n \geq 2$ generators. See Chapter 3 for a more general result.

One may also wonder if this new semigroup is in fact always a group. This question has been answered in the negative by Manuilov and Thomsen [MT3].

Thirdly one may ask for a reason to even consider the quotient with this subsemigroup which may seem like an artificial object to study. We will try to provide some motivation for this in the guise of Proposition 1.1.25 below, which shows that to obtain a group of extensions from the semigroup of unitary equivalence classes of extensions in general, it is not only natural to trivialize the asymptotically split extensions, it is also necessary. The result in Proposition 1.1.25 is mentioned in the introduction of [MT5]. We need a little preparation first.

The following lemma of independent interest shows that asymptotically split extensions of a separable C^*-algebra A have lifts which are uniformly continuous in the sense that $t \mapsto \Phi_t(a)$ is uniformly continuous for all $a \in A$. We simply repeat the argument from Lemma 4.2 of [MT2].

Lemma 1.1.24. *Let A, B be C^*-algebras with A separable. Let $\varphi : A \to B$ be an asymptotic homomorphism. There is an increasing continuous function $r : [1,\infty) \to [1,\infty)$ satisfying* $\lim_{t\to\infty} r(t) = \infty$ *such that the asymptotic homomorphism given by $\varphi_{r(t)}$, $t \in [1,\infty)$ is uniformly continuous, i.e., such that $t \mapsto \varphi_{r(t)}(a)$ is uniformly continuous for all $a \in A$.*

Proof. Suppose φ is equi-continuous. Let F_i, $i \in \mathbb{N}$, be an increasing sequence of finite sets with dense union in A. For each $n \in \mathbb{N}$ there is a $\delta_n > 0$ such that

$$\|\varphi_t(a) - \varphi_s(a)\| \leq \tfrac{1}{n}$$

whenever $t, s \in [1, n]$ with $|t - s| \leq \delta_n$ and $a \in F_n$. Choose an increasing sequence $k_n \in \mathbb{N}$, $n \in \mathbb{N}$, so that $\delta_{n+2} \geq \frac{1}{k_n}$ and put $N_0 = 1$ and $N_j = 1 + \sum_{n=1}^{j} k_n$ for $j = 1, 2, \ldots$. Let $r : [1,\infty) \to [1,\infty)$ be given by

$$r(t) = j + \frac{t - N_{j-1}}{k_j}, \quad t \in [N_{j-1}, N_j].$$

Clearly, r is continuous, increasing and goes to ∞ as $t \to \infty$. Let $a \in A$ and $\varepsilon > 0$. By equi-continuity and definition of the F_n's, there is an $n \geq \frac{3}{\varepsilon}$ and a $b \in F_n$ such that $\|\varphi_t(a) - \varphi_t(b)\| \leq \frac{\varepsilon}{3}$ for all $t \in [1,\infty)$. Choose $1 \geq \delta > 0$ so that

$$\|\varphi_{r(x)}(b) - \varphi_{r(y)}(b)\| \leq \tfrac{\varepsilon}{3}$$

whenever $x, y \in [1, N_n + 1]$ and $|x - y| \leq \delta$.

Suppose $x, y \in [1,\infty)$ with $|x - y| \leq \delta$. Then if $x, y \in [1, N_n + 1]$ it follows from the above that $\|\varphi_{r(x)}(a) - \varphi_{r(y)}(a)\| \leq \varepsilon$, so we may assume that $x, y \in [N_j, N_{j+2}]$ for a $j \geq n$

(since $\delta \leq 1$). Then $|r(x) - r(y)| \leq \max\{\frac{1}{k_{j+1}}, \frac{1}{k_{j+2}}\} \leq \frac{1}{k_j} \leq \delta_{j+2}$, but $r(x), r(y) \in [1, j+2]$ so this implies that

$$\|\varphi_{r(x)}(b) - \varphi_{r(y)}(b)\| \leq \tfrac{1}{j+2} \leq \tfrac{1}{n} \leq \tfrac{\varepsilon}{3},$$

and so also in this case $\|\varphi_{r(x)}(a) - \varphi_{r(y)}(a)\| \leq \varepsilon$. This proves the claim when φ is equi-continuous. If φ is just an asymptotic homomorphism take the equi-continuous asymptotic homomorphism φ' from Lemma 1.1.22 corresponding to φ and find r as above for this. Then $\varphi_{r(t)}$, $t \in [1, \infty)$ will do the job. We leave the details to the reader. □

When B is a C^*-algebra, recall the definition of H_B given in the beginning of 'Additive structure'. We let $\mathcal{L}(H_B)$ denote the C^*-algebra of adjointable operators on H_B and $\mathcal{K}(H_B)$ denote the closed two-sided ideal in $\mathcal{L}(H_B)$ generated by operators of the form $T_{x,y}(z) = x\langle y, z\rangle$, $x, y, z \in H_B$. Also remember that thinking of B as a Hilbert C^*-module over itself, we have $\mathcal{L}(B) = M(B)$ and $\mathcal{K}(B) = B$. We refer to [JT] for the proofs of any of the facts just mentioned.

Proposition 1.1.25. *Let A, B be C^*-algebras with A separable and B stable. Suppose $\varphi : A \to Q(B)$ is asymptotically split. It follows that φ represents the neutral element in any group quotient of the semigroup of unitary equivalence classes of extensions of A by B.*

Proof. Let Φ_t be a uniformly continuous asymptotic homomorphism such that $q \circ \Phi_t = \varphi$ for all t. The existence of this is guaranteed by Lemma 1.1.24. Let (t_n), $n \in \mathbb{N}$, be a sequence of numbers in $[1, \infty)$ converging monotonuously to ∞ such that $t_n - t_{n-1}$ converges to 0 as $n \to \infty$ and define $\Phi^\infty : A \to \mathcal{L}(H_B)$ by

$$\Phi^\infty(a)(b_i) = (\Phi_{t_i}(a)b_i).$$

Let $P : \mathcal{L}(H_B) \to \mathcal{L}(H_B)/\mathcal{K}(H_B)$ be the quotient map. Then $P \circ \Phi^\infty =: \varphi^\infty$ is a $*$-homomorphism because Φ is an asymptotic homomorphism and $t_n \to \infty$ as $n \to \infty$. For instance, to check multiplicativity note that for $a, a' \in A$ we have

$$\|\varphi^\infty(aa') - \varphi^\infty(a)\varphi^\infty(a')\| = \limsup_n \|\Phi_{t_n}(aa') - \Phi_{t_n}(a)\Phi_{t_n}(a')\| = 0.$$

Let $\Psi^\infty : A \to \mathcal{L}(H_B)$ be defined by

$$\Psi^\infty(a)(b_i) = (\Phi_1(a)b_1, \Phi_{t_1}(a)b_2, \Phi_{t_2}(a)b_3, \dots)$$

this also defines a $*$-homomorphism by composition with P. In fact, for $a \in A$

$$\|\varphi^\infty(a) - P \circ \Psi^\infty(a)\| = \limsup_n \|\Phi_{t_n}(a) - \Phi_{t_{n-1}}(a)\| = 0$$

because Φ is a uniformly continuous asymptotic homomorphism and $t_n - t_{n-1} \to 0$ as $n \to \infty$.

Adapting the procedure in 'Additive structure' we may find a sequence $v_i \in M(B)$, $i \in \mathbb{N}$ of orthogonal isometries ($v_i^* v_j = 0$ for $i \neq j$) such that

$$\sum_i v_i v_i^* = 1$$

in the strict topology. See Lemma 1.3.2 of [JT] for details. This gives a unitary $U : H_B \to B$ given by

$$U((b_i)) = \sum_i v_i b_i$$

for $(b_i) \in H_B$.

The unitary U implements an isomorphism of $\mathcal{L}(H_B)$ and $M(B) = \mathcal{L}(B)$ taking $\mathcal{K}(H_B)$ to B and so we also get an induced isomorphism $\iota : \mathcal{L}(H_B)/\mathcal{K}(H_B) \to Q(B)$.

Consider the following adjointable operators: Let $W_1 : B \to B \oplus H_B$ and $W_2 : H_B \to B \oplus H_B$ be the inclusions on the respective factors and $V : B \oplus H_B \to H_B$ be the unitary given by $V((b, (b_i)) = (b, b_1, b_2, \dots)$. Define $u_1 := UVW_1 \in M(B)$ and $u_2 := UVW_2U^* \in M(B)$. These are clearly orthogonal isometries such that $u_1u_1^* + u_2u_2^* = 1$ and hence we can use these to define the addition of extensions. Let $a \in A$, then

$$\begin{aligned}\|q(u_1)\varphi(a)q(u_1)^* &+ q(u_2)(\iota \circ \varphi^\infty)(a)q(u_2)^* - (\iota \circ \varphi^\infty)(a)\| \\ &= \|P(VW_1\Phi_1(a)W_1^*V^* + VW_2\Phi^\infty(a)W_2^*V^* - \Psi^\infty(a))\| = 0,\end{aligned}$$

since

$$VW_1\Phi_1(a)W_1^*V^* + VW_2\Phi^\infty(a)W_2^*V^* = \Psi^\infty(a)$$

for $a \in A$.

It follows that $\varphi + \iota \circ \varphi^\infty = \iota \circ \varphi^\infty$ in the semigroup of unitary equivalence classes of extensions, proving the claim. □

We owe a proper explanation of the use of the word 'split'. This is given in the simple lemma below. We have postponed the proof until now in order for us to be able to give a unified exposition of several related results.

Lemma 1.1.26. *Let A, B be C^*-algebras and let $\varphi : A \to Q(B)$ be a $*$-homomorphism. It follows that φ is split/semi-split/asymptotically split if and only if the extension*

$$0 \to B \to E \xrightarrow{p} A \to 0$$

with Busby invariant φ splits in the homological algebra sense via a $*$-homomorphism/completely positive contraction/asymptotic homomorphism $s : A \to E$.

Proof. One direction is clear in all instances. Suppose we have a lift Φ of φ which is a $*$-homomorphism/completely positive contraction/asymptotic homomorphism. Let $\xi : E \to M(B)$ be the canonical map extending the inclusion $B \subseteq E$. Let $a \in A$ and $e \in E$ be such that $p(e) = a$. Then $\Phi(a) - \xi(e) \in B$ since

$$q(\Phi(a) - \xi(e)) = \varphi(a) - \varphi(a) = 0.$$

Put

$$s(a) = e + (\Phi(a) - \xi(e)).$$

This is a well-defined map from A to E. Indeed, if $e' \in E$ satisfies $p(e') = a$, then

$$e - e' - (\xi(e) - \xi(e')) = e - e' - \xi(e - e') = 0,$$

as $e - e' \in B$ and $\xi|_B = \mathrm{id}_B$. Note also that

$$p(s(a)) = a$$

for all $a \in A$.

We need only see that s is a map of the desired type. Clearly, if Φ is linear, so is s. It is also easy to see that s respects the involution if Φ does.

Note that $\xi(s(a)) = \Phi(a)$ so that

$$s(ab) = s(a)s(b) + \Phi(ab) - \xi(s(a)s(b)) = s(a)s(b) + \Phi(ab) - \Phi(a)\Phi(b) \tag{1.1.11}$$

for all $a, b \in A$.

This proves that s is multiplicative if Φ is. If Φ is an asymptotic homomorphism simply put a t as a subscript on Φ and s in the definition of s. Then continuity, asymptotic linearity and asymptotic respect of the involution is clear. The asymptotic multiplicativity follows by (1.1.11).

If Φ is a completely positive contraction, it respects the involution and hence so does s. To check complete positivity let $M = (m_{ij}) \in M_n(A)$. Then by (1.1.11)

$$\begin{aligned} s_n(M^*M) &= \left(\sum_k s(m_{ki})^* s(m_{kj})\right) + \left(\sum_k \Phi(m_{ki}^* m_{kj})\right) - \left(\sum_k \Phi(m_{ki})^* \Phi(m_{kj})\right) \\ &= s_n(M)^* s_n(M) + (\Phi_n(M^*M) - \Phi_n(M)^*\Phi_n(M)). \end{aligned}$$

Here the subscript n means that we take the induced map on the $n \times n$-matrices. The first summand is clearly positive and the last summand is positive by Kadison's inequality (Proposition II.6.9.14 of [B]) because Φ_n is a positive contraction.

It also follows that s satisfies Kadison's inequality which forces it to be contractive (II.6.9.15 of [B]). □

1.2 Groups and C^*-algebras

In this section we review the basics of group C^*-algebras. We restrict attention to the case of countable discrete groups since it suffices for our purposes. Everything is well-known, but many proofs are included for completeness.

Group C^*-algebras

Let G be a discrete group. There are (at least) two natural C^*-algebras associated to G, one universal and one concrete. We first outline a construction of the universal algebra. Let $\mathbb{C}[G]$ be the algebraic group algebra. This is a $*$-algebra via the involution defined by

$$\left(\sum_i \gamma_i g_i\right)^* = \sum_i \overline{\gamma_i} g_i^{-1}, \quad \sum_i \gamma_i g_i \in \mathbb{C}[G].$$

If π is a unitary representation of G on some Hilbert space, then π extends to an algebraic $*$-homomorphism on $\mathbb{C}[G]$. We define a norm on $\mathbb{C}[G]$ by

$$\left\|\sum_i \gamma_i g_i\right\| := \sup_\pi \left\|\pi\left(\sum_i \gamma_i g_i\right)\right\|, \tag{1.2.1}$$

where the supremum is taken over the set of all cyclic unitary representations of G.

Definition 1.2.1. For a discrete group G the full group C^*-algebra $C^*(G)$ is the completion of $\mathbb{C}[G]$ in the norm defined in (1.2.1).

Since every unitary representation is a direct sum of cyclic representations, $C^*(G)$ enjoys the universal property that every unitary representation of G extends to a unique $*$-homomorphism on $C^*(G)$.

Recall that the left regular representation is the unitary representation λ of G on $\ell^2(G)$ given by

$$(\lambda(g)f)(h) = f(g^{-1}h)$$

for $g, h \in G$ and $f \in \ell^2(G)$.

Definition 1.2.2. The reduced group C^*-algebra $C^*_r(G)$ of a discrete group G is the completion of $\mathbb{C}[G]$ in the norm induced by the left regular representation λ. I.e., $C^*_r(G) = C^*(\lambda(G)) \subseteq B(\ell^2(G))$.

Observe that both $C^*(G)$ and $C^*_r(G)$ are unital C^*-algebras.

The following is fundamental.

Theorem 1.2.3. *Let G be a countable discrete group and $H \leq G$ a subgroup. The following holds:*

1. *The inclusion $H \subseteq G$ extends to an inclusion $C^*(H) \subseteq C^*(G)$ making $C^*(H)$ a unital subalgebra of $C^*(G)$.*
2. *The inclusion $H \subseteq G$ extends to an inclusion $C^*_r(H) \subseteq C^*_r(G)$ making $C^*_r(H)$ a unital subalgebra of $C^*_r(G)$.*
3. *The characteristic function 1_H of H on G extends to a conditional expectation both on the reduced and the full group C^*-algebra level.*

Proof. Obviously we have a unital homomorphism $\pi : C^*(H) \to C^*(G)$ extending the inclusion $H \subseteq G$.

Since $C^*(H)$ is separable, there is, by 3.7.2 in [Pe], a state ψ on $C^*(H)$ which has faithful GNS-representation π_ψ. By restriction to $H \subseteq C^*(H)$, we may think of ψ as a positive definite function on H (since states are completely positive). Extend this positive definite function to G by setting it equal to 0 outside H. If $g_1, \dots, g_n \in gH$, then the matrix $(\psi(g_i^{-1}g_j)) \in M_n(\mathbb{C})$ is positive since ψ is positive definite on H. If $g_1, \dots, g_n \in G$, we may assume that they are ordered by left cosets and so $(\psi(g_i^{-1}g_j)) \in M_n(\mathbb{C})$ is a block diagonal matrix with positive blocks, hence positive. This shows that ψ is positive definite on G. By the group version of the GNS construction, ψ determines a cyclic unitary representation φ of G. The representation $\varphi|_H$ is unitarily equivalent to the unitary representation of H determined by π_ψ by uniqueness of the GNS representation. It follows that on the level of C^*-algebras $\varphi \circ \pi$ is unitarily equivalent to π_ψ, and hence π is injective.

In the reduced case, note that

$$\ell^2(G) = \bigoplus_{Hg} \ell^2(Hg),$$

and that this is a decomposition where the subspaces are invariant under the action of left translation by elements from H. There is an obvious unitary intertwining the action of $\mathbb{C}[H] \subseteq \mathbb{C}[G] \subseteq C_r^*(G)$ on $\ell^2(Hg)$ with the action of $\mathbb{C}[H] \subseteq C_r^*(H)$ on $\ell^2(H)$. It follows that for $x \in \mathbb{C}[H]$, x has the same norm in $C_r^*(H)$ as in $C_r^*(G)$.

For the statement on conditional expectations, let 1_H denote the characteristic function of H on G and define a map $E : \mathbb{C}[G] \to \mathbb{C}[H]$ by extending the characteristic function 1_H by linearity. Theorem 2.5.11 of [BO] tells us that E extends to unital completely positive maps on $C^*(G)$ and $C_r^*(G)$ respectively if and only if 1_H is a positive definite function on G. The constant function taking the value 1 on H is positive definite on H and so, by the argument above, 1_H is positive definite on G. $\square$

It is sometimes useful to picture the conditional expectation in the reduced case to be the map taking an (infinite) $G \times G$-matrix to the canonical $H \times H$ submatrix. A more precise description is perhaps to consider the projection P onto the subspace $\ell^2(H)$ in $\ell^2(G)$ and let $a \in C_r^*(G)$ act on $\ell^2(H)$ via $Pa|_{\ell^2(H)}$. The validity of this picture is easily checked on $\mathbb{C}[G]$.

Another well-known fact is the following.

Lemma 1.2.4. *Let G be a discrete group. There is a faithful tracial state τ on $C_r^*(G)$ given by*

$$\tau(T) = \langle T1_{\{1\}}, 1_{\{1\}} \rangle.$$

Proof. One minute of hard thinking establishes the trace property on $\mathbb{C}[G]$ and hence on $C_r^*(G)$. Assume that $T \in C_r^*(G)$ with $\tau(T^*T) = 0$. Let $(T_n) \subseteq \mathbb{C}[G]$ be a sequence such that $T_n \to T$ in norm. Now for each $n \in \mathbb{N}$, $T_n = \sum_i \gamma_{in} g_{in}$ for suitable scalars γ_{in} and elements of G, g_{in} and

$$\tau(T_n^*T_n) = \left\| \sum_i \gamma_{in} 1_{\{g_{in}\}} \right\|^2 = \sum_i |\gamma_{in}|^2 \to \tau(T^*T) = 0.$$

So since

$$\|T_n 1_{\{g\}}\|^2 = \sum_i |\gamma_{in}|^2$$

for any $g \in G$, we see that (T_n) converges strongly to 0 and so $T = 0$. Indeed, let $\varepsilon > 0$ and $x \in \ell^2(G)$. Then

$$x = \sum_{g \in G} \gamma_g 1_{\{g\}}.$$

Pick a finite set $F \subseteq G$ such that $x_F = \sum_{g \in F} \gamma_g 1_{\{g\}}$ satisfies

$$\|x - x_F\| < \frac{\varepsilon}{\sup_n \|T_n\|}.$$

By the above we may choose $N \in \mathbb{N}$ so that $n \geq N$ implies that $\|T_n x_F\| < \varepsilon$. It follows that for $n \geq N$, $\|T_n x\| < 2\varepsilon$. □

We end this section by discussing a fundamental result of Rosenberg which is used in almost every argument for the existence of non-invertible extensions by $\mathbb{K}$.

Proposition 1.2.5. *Let G be a countable discrete group. If $C_r^*(G)$ is quasidiagonal then G is amenable.*

Proof. By Voiculescu's characterization of quasidiagonality we have unital completely positive maps $\varphi_n : C_r^*(G) \to M_{k_n}(\mathbb{C})$ which are asymptotically multiplicative and asymptotically norm-preserving. These can be extended to all of $B(\ell^2(G))$ by the Arveson Extension Theorem, Theorem 1.6.1. of [BO]. These extensions induce a unital completely positive map $\varphi : B(\ell^2(G)) \to \prod_{n\in\mathbb{N}} M_{k_n}(\mathbb{C}) \Big/ \sum_{n\in\mathbb{N}} M_{k_n}(\mathbb{C})$ with $C_r^*(G)$ in the multiplicative domain (see Proposition 1.5.7. of [BO]). Let $\omega \in \beta\mathbb{N}\backslash\mathbb{N}$ be a character on $\ell^\infty(\mathbb{N})$ which is not evaluation at any $n \in \mathbb{N}$. Then

$$\tau((a_n)) := \omega((\mathrm{tr}_{k_n}(a_n))$$

defines a trace on $\prod_{n\in\mathbb{N}} M_{k_n}(\mathbb{C}) \Big/ \sum_{n\in\mathbb{N}} M_{k_n}(\mathbb{C})$. To see that τ is well-defined, we need to show that $\ker(\omega)$ contains any sequence that is eventually 0 and so by continuity all sequences converging to 0.

If $x \in \ell^\infty(\mathbb{N})$ is a sequence of 0's and 1's, $\omega(x) = \omega(x)^2$, so $\omega(x)$ is either 1 or 0. In particular if $x = 1_{\{n\}}$ and $\omega(x) = 1$ then by Urysohn's Lemma there is an $1 \geq f \geq 0$ in $\ell^\infty(\mathbb{N})$ with $f(n) = 0$ and $\omega(f) > 0$ since ω is not evaluation at n. Then

$$\|f + 1_{\{n\}}\| \geq \omega(f + 1_{\{n\}}) > 1$$

but the norm equals 1. It follows that $\omega(1_{\{n\}}) = 0$ for all $n \in \mathbb{N}$.

Now, $m = \tau \circ \varphi$ is a state which, when restricted to $\ell^\infty(G)$, is in fact an invariant mean on this algebra. To see this let $\psi \in \ell^\infty(G)$ and $g \in G$. Note that if $g\psi$ is the translation of ψ by g then $g\psi = \lambda(g)\psi\lambda(g)^*$ as operators on $\ell^2(G)$, so since $C_r^*(G)$ is contained in the multiplicative domain of φ and τ is a trace

$$\begin{aligned} m(g\psi) &= \tau(\varphi(\lambda(g)\psi\lambda(g)^*)) = \tau(\varphi(\lambda(g))\varphi(\psi)\varphi(\lambda(g)^*)) \\ &= \tau(\varphi(\lambda(g)^*)\varphi(\lambda(g))\varphi(\psi)) = \tau \circ \varphi(\psi) = m(\psi) \end{aligned}$$

which was the desired conclusion. □

Weak containment

In this section we discuss the notion of weak containment for groups as well as for C^*-algebras and relate the notions. The material is mostly from [BHV] and [Di].

Definition 1.2.6. Let σ be a unitary representation of G on $\mathcal{H}_\sigma$. A function $\varphi : G \to \mathbb{C}$ is called a positive definite function associated to σ if there is a $\xi \in \mathcal{H}_\sigma$ such that

$$\varphi(g) = \langle \sigma(g)\xi, \xi \rangle$$

for all $g \in G$.

Definition 1.2.7. Let σ, π be unitary representations of G. We say that σ is weakly contained in π and write $\sigma \prec \pi$ if every positive definite function associated to σ can be approximated uniformly on finite sets by sums of positive definite functions associated to π. I.e., for $x \in \mathcal{H}_\sigma$, $\varepsilon > 0$ and every finite $F \subseteq G$ there are $y_1, \ldots, y_n \in \mathcal{H}_\pi$ such that

$$\left| \langle \sigma(g)x, x \rangle - \sum_{i=1}^{n} \langle \pi(g)y_i, y_i \rangle \right| < \varepsilon,$$

for every $g \in F$.

Since we only work with countable discrete groups, the above approximation property is equivalent to the existence of a sequence of sums of positive definite functions converging pointwise.

We need the following technicality.

Lemma 1.2.8. *Let σ, π be unitary representations of G. Assume that $T \in \mathcal{B}(\mathcal{H}_\sigma, \mathcal{H}_\pi)$ satisfies $T\sigma(g) = \pi(g)T$ for all $g \in G$.*

It follows that $(\operatorname{Ker} T)^\perp$ and $\overline{\operatorname{Ran} T}$ are closed, invariant subspaces for σ and π respectively and that the subrepresentation of σ defined by $(\operatorname{Ker} T)^\perp$ is unitarily equivalent to the subrepresentation of π corresponding to $\overline{\operatorname{Ran} T}$.

Proof. The invariance properties are clear.

First note that $T^*\pi(g) = \sigma(g)T^*$ for all $g \in G$, so that

$$T^*T\sigma(g) = T^*\pi(g)T = \sigma(g)T^*T$$

and hence, $|T|$ commutes with $\sigma(g)$ for all $g \in G$.

Let U be the partial isometry from the polar decomposition of T, i.e., $T = U|T|$. It is a fact that $U : (\operatorname{Ker} T)^\perp \to \overline{\operatorname{Ran} T}$ is a unitary.

For $g \in G$ and $x \in \mathcal{H}_\sigma$ we have

$$\pi(g)U|T|x = T\sigma(g)x = U\sigma(g)|T|x,$$

which proves the claim since $(\operatorname{Ker} T)^\perp = (\operatorname{Ker} |T|)^\perp = \overline{\operatorname{Ran} |T|}$. □

The next lemma is crucial. It looks like a standard reduction, but the proof is relatively involved.

Lemma 1.2.9. *Let σ, π be unitary representations of G. Assume $V \subseteq \mathcal{H}_\sigma$ satisfies that $\{\sigma(g)x \mid g \in G, x \in V\}$ is total in $\mathcal{H}_\sigma$.*

If each of the positive definite functions $g \mapsto \langle \sigma(g)x, x\rangle$, $x \in V$, can be approximated uniformly on finite sets by sums of positive definite functions associated to π then $\sigma \prec \pi$.

Proof. Let $X \subseteq \mathcal{H}_\sigma$ be the set of vectors, $x \in \mathcal{H}_\sigma$, such that $g \mapsto \langle \sigma(g)x, x\rangle$ can be approximated uniformly on finite subsets of G by sums of positive definite functions associated to π. We prove the lemma by showing that X is a closed subspace of $\mathcal{H}_\sigma$.

Note that X is invariant under the action of G and that it is stable under scalar multiplication.

If $x \in X$ and $g_1, g_2 \in G$ it follows that $y = \sigma(g_1)x + \sigma(g_2)x \in X$. Indeed, the positive definite function, ψ, corresponding to y satisfies

$$\psi(g) = \sum_{k,m=1}^{2} \langle \sigma(g_k^{-1} g g_m)x, x\rangle, \quad g \in G.$$

Given $\varepsilon > 0$ and a finite set $C \subseteq G$ the set $g_k^{-1}Cg_m$ is finite for $k, m = 1, 2$. Since $x \in X$ we get $x_1, \ldots, x_n \in \mathcal{H}_\pi$ such that

$$\left| \langle \sigma(g)x, x\rangle - \sum_{i=1}^{n} \langle \pi(g)x_i, x_i\rangle \right| \leq \varepsilon$$

for all $g \in \bigcup_{k,m}^{2} g_k^{-1}Cg_m$.

Put

$$\zeta(g) = \sum_{i=1}^{n} \langle \pi(g)(\pi(g_1)x_i + \pi(g_2)x_i), \pi(g_1)x_i + \pi(g_2)x_i\rangle.$$

Then

$$|\psi(g) - \zeta(g)| = \left| \sum_{k,m=1}^{2} \left(\langle \sigma(g_k^{-1} g g_m) x, x \rangle - \sum_{i=1}^{n} \langle \pi(g_k^{-1} g g_m) x_i, x_i \rangle \right) \right| \leq 4\varepsilon$$

for all $g \in C$.

It is straight-forward that X is a closed subset of $\mathcal{H}_\sigma$. Indeed, let $x \in \overline{X}$, since X is closed under scalar multiplication we may assume that $\|x\| = 1$. Let $C \subseteq G$ be finite, $\varepsilon > 0$ and let $y \in X$ be such that $\|x - y\| < \varepsilon$. Then for $g \in G$

$$|\langle \sigma(g)x, x \rangle - \langle \sigma(g)y, y \rangle| \leq |\langle \sigma(g)x, x - y \rangle| + |\langle \sigma(g)(y - x), y \rangle| < \varepsilon + \varepsilon(1 + \varepsilon). \quad (1.2.2)$$

Let $x_1, \ldots x_n \in \mathcal{H}_\pi$ be such that

$$|\langle \sigma(g)y, y \rangle - \sum_{i=1}^{n} \langle \pi(g)x_i, x_i \rangle| < \varepsilon \quad (1.2.3)$$

when $g \in C$.

The conclusion now follows from (1.2.2) and (1.2.3).

The hard part of the proof is to show that X is closed under addition. To see this let $x_1, x_2 \in X$ and put

$$\mathcal{H}_i := \overline{\operatorname{span}}\{\sigma(g)x_i \mid g \in G\}, \quad i = 1, 2.$$

By the above $\mathcal{H}_i \subseteq X$, $i = 1, 2$.

Let $L = \overline{\mathcal{H}_1 + \mathcal{H}_2}$ and note that this is an invariant subspace of $\mathcal{H}_\sigma$. Let $P : L \to \mathcal{H}_1^\perp$ be the orthogonal projection in L and observe that $P(\mathcal{H}_2)$ is dense in $\mathcal{H}_1^\perp$ in L. The subspace $\mathcal{H}_1^\perp$ is invariant because $\mathcal{H}_1$ is, so for $y = z + z' \in L$ with $z \in \mathcal{H}_1, z' \in \mathcal{H}_1^\perp$

$$P\sigma(g)y = \sigma(g)z' = \sigma(g)Py$$

for $g \in G$.

Restricting P to $\mathcal{H}_2$ we see from Lemma 1.2.8 that the subrepresentation of σ defined by $\mathcal{H}_1^\perp$ is equivalent to the subrepresentation corresponding to the orthogonal complement of $\operatorname{Ker} P$ in $\mathcal{H}_2$. From this we see that $\mathcal{H}_1^\perp \subseteq X$ since $\mathcal{H}_2 \subseteq X$. Put

$$y = P(x_1 + x_2) \in \mathcal{H}_1^\perp \quad \text{and} \quad z = (I - P)(x_1 + x_2) \in \mathcal{H}_1.$$

Then $y, z \in X$ and for $g \in G$

$$\langle \sigma(g)(x_1 + x_2), x_1 + x_2 \rangle = \langle \sigma(g)(y + z), y + z \rangle = \langle \sigma(g)y, y \rangle + \langle \sigma(g)z, z \rangle,$$

so that $x_1 + x_2 \in X$.

This finishes the proof since $V \subseteq X$ implies $X = \mathcal{H}_\sigma$. □

Having established Lemma 1.2.9 we get a few easy corollaries.

Corollary 1.2.10. *Let $\sigma_1, \sigma_2, \pi_1, \pi_2$ be unitary representations of G. Assume that $\sigma_i \prec \pi_i$, $i = 1, 2$. Then $\sigma_1 \otimes \sigma_2 \prec \pi_1 \otimes \pi_2$.*

Proof. For each $x_1 \in \mathcal{H}_{\sigma_1}$, $x_2 \in \mathcal{H}_{\sigma_2}$ and $g \in G$, we have

$$\langle (\sigma_1 \otimes \sigma_2)(g)(x_1 \otimes x_2), x_1 \otimes x_2 \rangle = \langle \sigma_1(g)x_1, x_1) \rangle \langle \sigma_2(g)x_2, x_2 \rangle.$$

The function defined by the righthand side can clearly be approximated uniformly on finite sets by sums of positive definite functions associated to $\pi_1 \otimes \pi_2$, so since the set of simple tensors is total in $\mathcal{H}_{\sigma_1} \otimes \mathcal{H}_{\sigma_2}$ we are done by Lemma 1.2.9. $\square$

Corollary 1.2.11. *Let λ denote the left regular representation of G. If σ is a unitary representation such that $\sigma \prec \lambda$, then $\sigma \otimes \pi \prec \lambda$ for any unitary representation π of G.*

Proof. By Fell's absorption principle, see, e.g., Theorem 2.5.5 [BO], and Corollary 1.2.10

$$\sigma \otimes \pi \prec \lambda \otimes \pi \sim \lambda^{(\dim \pi)} \prec \lambda.$$

The equivalence, $\sim$, being unitary equivalence. $\square$

We will now discuss weak containment from the C^*-algebraic viewpoint.

For a representation $\sigma : G \to \mathcal{U}(\mathcal{H}_\sigma)$ we let h_σ denote the corresponding $*$-homomorphism from the full group C^*-algebra $C^*(G)$ into $B(\mathcal{H}_\sigma)$.

Our goal will be to establish the following theorem.

Theorem 1.2.12. *Let σ, π be unitary representations of the discrete group G. Then $\sigma \prec \pi$ if and only if $\ker(h_\pi) \subseteq \ker(h_\sigma)$.*

We will more or less proceed as in [Di].

Lemma 1.2.13. *Every positive definite function is bounded. The action of such a function on $\ell^1(G)$ is a positive functional.*

Proof. Let φ be a positive definite function. Then

$$\begin{pmatrix} \varphi(1) & \varphi(h) \\ \varphi(h^{-1}) & \varphi(1) \end{pmatrix}$$

is positive, so that $\varphi(h^{-1}) = \overline{\varphi(h)}$. By taking the determinant it follows that

$$|\varphi(h)| \leq \varphi(1), \quad \forall h \in G.$$

That $\varphi(f^* * f) \geq 0$ for $f \in \ell^1(G)$, is an immediate consequence of the fact that

$$\sum_{i,j}^{n} \alpha_i \overline{\alpha_j} \varphi(g_j^{-1} g_i) \geq 0$$

for any choice of $n \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$ and $g_i \in G$, $i = 1, \ldots, n$. $\square$

The first ingredient in the proof of Theorem 1.2.12 is the following theorem. In our discrete setup it is a mere observation, but the corresponding statement for locally compact groups is relatively deep.

Let P_1 denote the positive definite functions φ on G such that $\varphi(1) = 1$.

Theorem 1.2.14. *Let $(\varphi_i)_{i\in I}$ be a net in P_1 and $\varphi \in P_1$. Then (φ_i) converges pointwise to φ if and only if (φ_i) converges to φ in the weak* topology on $\ell^\infty(G) = \ell^1(G)^*$.*

We will now proceed to discuss the relationship between positive definite functions associated to a representation of a group and the positive forms associated with the corresponding $*$-representation.

Recall that $\ell^1(G) \subseteq C^*(G)$.

The following lemma is of a completely general nature. It concerns the relationship between a Banach $*$-algebra and its enveloping C^*-algebra, but we only treat it in our setting.

Lemma 1.2.15. *Let $i : \ell^1(G) \to C^*(G)$ be the inclusion. If φ is a positive functional on $\ell^1(G)$, there is a positive functional τ on $C^*(G)$ such that $\varphi = \tau \circ i$. The map $\varphi \mapsto \tau$ is a norm-preserving bijection from the set of positive functionals on $\ell^1(G)$ to the set of positive functionals on $C^*(G)$.*

If M is a bounded set of positive functionals on $\ell^1(G)$ the restriction of the above map to M is a weak homeomorphism onto its image.*

Proof. Let φ be a positive functional on $\ell^1(G)$ and let $f \in \mathbb{C}[G] \subseteq \ell^1(G)$. Then by the CBS inequality

$$|\varphi(f)|^2 \leq \varphi(1)\varphi(f^* * f) \leq \|\varphi\|^2\|i(f)\|^2,$$

since $\|i(f)\| = \sup \psi(f^* * f)^{1/2}$, where the sup is taken over positive functionals of norm less than 1 by the GNS construction and definition of the norm on $C^*(G)$. The inequality above implies that there is a functional τ on $C^*(G)$ such that $\varphi = \tau \circ i$ and $\|\tau\| \leq \|\varphi\|$. That τ is positive follows by approximation since for $a \in C^*(G)$ we have a sequence $(f_n) \subseteq \ell^1(G)$ such that $i(f_n) \to a$ so that

$$\tau(a^*a) = \lim_n \tau(i(f_n)^* i(f_n)) = \lim_n \varphi(f_n^* * f_n) \geq 0.$$

That τ is unique is clear by density of $i(\ell^1(G))$ in $C^*(G)$ and by restriction of positive functionals on $C^*(G)$ to $\ell^1(G)$ we see that the map above, $\varphi \mapsto \tau$, is indeed a bijection. To see that the map is norm-preserving consider $f \in \ell^1(G)$

$$|\varphi(f)| = |\tau(i(f))| \leq \|\tau\|\|f\|,$$

since i is a $*$-homomorphism. The other inequality was established earlier.

Let $M \subseteq \ell^1(G)^*$ be norm-bounded by $K \in \mathbb{N}$. Consider a net (φ_j) in M converging weak* to $\varphi \in M$. Fix $a \in C^*(G)$ and $\varepsilon > 0$ and choose $f \in \ell^1(G)$ such that $\|i(f) - a\| <$

ε/K along with a j_0 such that $\|\varphi(f) - \varphi_j(f)\| < \varepsilon$ whenever $j \geq j_0$. Then if τ_j is the image of φ_j and τ of φ respectively

$$\|\tau_j(a) - \tau(a)\| \leq \|\tau_j(i(f)) - \tau_j(a)\| + \|\tau_j(i(f)) - \tau(i(f))\| + \|\tau(a) - \tau(i(f))\| < 3\varepsilon$$

when $j \geq j_0$.

Clearly (φ_j) will converge weak* to φ if τ_j converge weak* to τ. □

The next major step towards Theorem 1.2.12 is the following.

Theorem 1.2.16. *Let A be a unital C^*-algebra and σ, π be $*$-representations of A. The following are equivalent*

1. $\ker(\sigma) \subseteq \ker(\pi)$.
2. *Every positive form of A associated to π, i.e., functions of the type $x \mapsto \langle \pi(x)\xi, \xi\rangle$ is a weak* limit of linear combinations of positive forms associated to σ.*
3. *Every state of A associated to π, i.e., functions as in* 2. *which are states, is a weak* limit of states which are finite sums of positive forms associated with σ.*

Proof. That 3. implies 2. is clear.

If 2. is true and $a \in \ker(\sigma)$ then every positive form associated with σ vanishes on a^*a. It follows that

$$\langle \pi(a^*a)\xi, \xi\rangle = 0$$

for all $\xi \in \mathcal{H}_\pi$ so that $\pi(a) = 0$ and hence 1. holds.

The last implication is somewhat harder. Assume that 1. holds. We will establish 3.

Note that each state of A associated with π vanishes on $\ker(\sigma)$. By restricting to a subspace (the essential subspace) of $\mathcal{H}_\sigma$ we may assume that σ is non-degenerate, i.e., if $\xi \in \mathcal{H}_\sigma \backslash \{0\}$, then there is an $a \in A$ such that $\sigma(a)\xi \neq 0$.

Furthermore, by going to the quotient with $\ker(\sigma)$, we may assume that σ is injective and we may thus identify A with a sub-C^*-algebra of $B(\mathcal{H}_\sigma)$ acting non-degenerately on $\mathcal{H}_\sigma$. Then if $x \in A$ is self-adjoint and $\varphi(x) \geq 0$ for every state φ associated with σ, then $\sigma(x) \geq 0$ and this implies that x is positive since σ is injective.

We complete the proof by showing that the weak* closed convex hull, C, of the set of states associated with σ, is the set of all states on A. Assume to the contrary that this is not the case. Then there is a state τ on A not in C and by the Hahn-Banach Theorem there is a weak* continuous functional $\omega : A^* \to \mathbb{C}$ and a $\gamma \in \mathbb{R}$ such that

$$\mathrm{Re}(\omega(\tau)) > \gamma > \mathrm{Re}(\omega(\rho)) \tag{1.2.4}$$

for all $\rho \in C$.

It is a well-known fact that there is an $a \in A$ such that $\omega(\eta) = \eta(a)$ for all $\eta \in A^*$. Let $b = \mathrm{Re}(a)$ then by positivity and (1.2.4) we have $\rho(\gamma \cdot 1 - b) \geq 0$ for all $\rho \in C$. By the above this implies that $\gamma \cdot 1 - b \geq 0$ so that

$$\tau(\gamma \cdot 1 - b) = \gamma - \mathrm{Re}(\omega(\tau)) \geq 0$$

contradicting (1.2.4).

□

Remark 1.2.17. The conclusion in the above theorem also holds if A is non-unital but we have no need of this.

We are now ready to prove Theorem 1.2.12.

Proof of Theorem 1.2.12. The proof is basically a question of using the above results in the right order.

That $\ker(h_\pi) \subseteq \ker(h_\sigma)$ is by Theorem 1.2.16 equivalent to the fact that every state on $C^*(G)$ associated to h_σ is a weak* limit of states which are sums of positive forms associated with h_π. This, in turn, is by Lemma 1.2.15 equivalent to the condition that every state associated with h_σ restricted to $\ell^1(G)$ is the weak* limit of the restriction to $\ell^1(G)$ of states which are sums of positive forms associated to h_π.

Note that for $f \in \ell^1(G)$ and $\xi \in \mathcal{H}_\sigma$ we have

$$\langle h_\sigma(f)\xi, \xi\rangle = \sum_{g\in G} f(g)\langle \sigma(g)\xi, \xi\rangle,$$

so that the function in $\ell^\infty(G) = \ell^1(G)^*$, corresponding to the functional on $\ell^1(G)$ arising from a positive form associated with h_σ by restriction, is actually nothing but the positive definite function associated with σ determined by ξ. The same is of course true if we replace σ by π.

In other words the above requirement is by Theorem 1.2.14 met if and only if the positive definite functions associated with σ taking the value 1 at 1 can be approximated uniformly on finite sets by positive definite functions taking the value 1 at 1 which are sums of positive definite functions associated with π.

This last requirement is equivalent to the definition of weak containment given in Definition 1.2.7 by Lemma 1.2.18 below. □

We used the following elementary lemma above. The uninteresting proof is included for completeness.

Lemma 1.2.18. *Let σ, π be unitary representations of G. Then $\sigma \prec \pi$ if and only if every positive definite function associated to σ defined by a unit vector can be approximated uniformly on finite sets by functions, taking the value 1 at 1, which are sums of positive definite functions associated to π.*

Proof. Clearly, the condition in the lemma implies that $\sigma \prec \pi$.

For the converse assume that $\sigma \prec \pi$. Let φ be a positive definite function associated with σ defined by a unit vector, $C \subseteq G$ be finite and $1/2 > \varepsilon > 0$. Choose $K \geq 1$ such that $\varphi(C)$ is contained in the ball of radius K centered at 0. By assumption we may find positive definite functions $\psi_1, \ldots, \psi_n$ associated with π such that

$$\left|\varphi(g) - \sum_{i=1}^{n} \psi_i(g)\right| \leq \frac{\varepsilon}{K+\varepsilon},$$

when $g \in C \cup \{1\}$.

Put

$$\tilde{\psi}_i(g) = \frac{\psi_i(g)}{\sum_{i=1}^n \psi_i(1)} \quad g \in G.$$

Then $\tilde{\psi}_i$ is a positive definite function associated with π for each $i = 1, \dots, n$ and for $g \in C$

$$\begin{aligned}\left|\varphi(g) - \sum_{i=1}^n \tilde{\psi}_i(g)\right| &\leq \varepsilon + \left|\sum_{i=1}^n \psi_i(g) - \sum_{i=1}^n \tilde{\psi}_i(g)\right| \\ &= \varepsilon + \left|1 - \sum_{i=1}^n \psi_i(1)\right| \cdot \left|\sum_{i=1}^n \psi_i(1)\right|^{-1} \cdot \left|\sum_{i=1}^n \psi_i(g)\right| \\ &\leq \varepsilon + \frac{\varepsilon}{K+\varepsilon} \cdot 2 \cdot (K+\varepsilon).\end{aligned}$$

Clearly the sum of the $\tilde{\psi}_i$ take 1 to 1. □

Lastly we draw a line to amenability for discrete groups. Recall that a group is amenable if the canonical map $C^*(G) \to C^*_r(G)$ is an isomorphism. By Theorem 1.2.12 this is equivalent to saying that the left regular representation of G weakly contains any other representation. Corollary 1.2.11 says that this happens if and only if the left regular representation weakly contains the trivial one-dimensional representation.

1.3 Special preliminaries

In this section we collect a few results of a more specialized nature not fitting into the first couple of sections of this chapter but needed in Chapter 3. The first result is well-known and easily established.

Lemma 1.3.1. *Let A be a unital AF-algebra. The unitary group of A is path-connected in the norm topology.*

Proof. If A has finite dimension the lemma is Corollary 2.1.4 of [RLL] since A is isomorphic to a direct sum of full matrix algebras. Assume that A is infinite-dimensional and let $u \in A$ be a unitary. Since A is the inductive limit of finite-dimensional algebras we may choose f in a finite-dimensional subalgebra A' containing the unit of A such that $\|f - u\| < 1$. It follows from Proposition 2.1.11 of [RLL] that f is invertible and homotopic to u in the group of invertible elements of A.

In A' the element f must also be invertible and hence it is in the path-component of 1 by Proposition 2.1.8 in [RLL]. It follows that u is homotopic to 1 in the invertible elements of A and so also in the unitaries by Proposition 2.1.8 of [RLL]. □

The following very plausible corollary can sometimes be useful.

Corollary 1.3.2. *If k is a compact operator on a separable Hilbert space H such that $1+k$ is unitary, then $1+k$ can be connected to 1 by a norm-continuous path of unitaries in $1+\mathbb{K}$.*

Proof. The statement is trivial in the finite-dimensional case, so assume H to be infinite-dimensional.

Choose a unitary path $s_t + k_t$, $t \in [0,1]$ in $\mathcal{U}(\mathbb{C}1+\mathbb{K})$ connecting $1+k$ to 1. Note that

$$(1-|s_t|^2)1 = \overline{s_t}k_t + s_t k_t^* + k_t^* k_t$$

for each $t \in [0,1]$. This entails that $|s_t| = 1$ for every $t \in [0,1]$. In particular, $1+\overline{s_t}k_t$ is a unitary for each t and this is the desired path if we can show that $t \mapsto s_t$ is continuous.

Assume that $t \mapsto s_t$ is not continuous at t_0. For some $\varepsilon > 0$ there is a sequence $(t_n) \subseteq [0,1]$ converging to t_0 such that $|s_{t_n} - s_{t_0}| \geq \varepsilon$ for all n. Since (s_{t_n}) is a sequence in the compact set S^1 we may assume that it is actually convergent to $\zeta \in S^1$ satisfying $|\zeta - s_{t_0}| \geq \varepsilon$. Now

$$k_{t_n} = (k_{t_n} + s_{t_n}1) - s_{t_n}1 \to k_{t_0} + s_{t_0}1 - \zeta 1$$

in norm, but the right-hand side is not compact. □

We move on to show a rather technical result from [MT2] which is the very foundation of Chapter 3. We give a self-contained and fairly detailed proof.

The following is a basic lemma, showing that in working with equi-continuous maps, homotopies can be chosen equi-continuous as well. We let $IB := C[0,1] \otimes B$ and $\mathrm{ev}_t : IB \to B$ be the $*$-homomorphism that evaluates at $t \in [0,1]$.

Lemma 1.3.3. *Let A, B be C^*-algebras and $\pi, \psi : A \to B$ equi-continuous asymptotic homomorphisms connected by a homotopy $\Phi : A \to IB$, i.e., Φ is an asymptotic homomorphism with $\mathrm{ev}_0 \circ \Phi = \pi$ and $\mathrm{ev}_1 \circ \Phi = \psi$. There is then an equi-continuous asymptotic homomorphism $\Phi' : A \to IB$ with $\mathrm{ev}_0 \circ \Phi' = \pi$ and $\mathrm{ev}_1 \circ \Phi' = \psi$.*

Proof. By Lemma 1.1.22 we have an equi-continuous asymptotic homomorphism $\tilde{\Phi} : A \to IB$ such that

$$\lim_t \|\Phi_t(a) - \tilde{\Phi}_t(a)\| = 0$$

for all $a \in A$.

Put

$$\Phi'_t(a)(s) = \begin{cases} (1-3s)\pi_t(a) + 3s\tilde{\Phi}_t(a)(0) & \text{for } s \in [0, \frac{1}{3}] \\ \tilde{\Phi}_t(a)(3s-1) & \text{for } s \in [\frac{1}{3}, \frac{2}{3}] \\ (3-3s)\tilde{\Phi}_t(a)(1) + (3s-2)\psi_t(a) & \text{for } s \in [\frac{2}{3}, 1] \end{cases}$$

for $t \in [1, \infty)$ and $a \in A$.

This map is readily seen to be the desired homotopy. □

We are finally ready to prove what we set out to do. The theorem is stated in greater generality than we will need, but it makes no difference for the proof.

Theorem 1.3.4. *Let A, B be C^*-algebras, A separable and B σ-unital. Let $\lambda : A \to Q(B)$ be an equi-continuous asymptotic homomorphism such that there is a homotopy $\Phi' : A \to IQ(B)$ with $\mathrm{ev}_0 \circ \Phi' = \lambda$ and $\mathrm{ev}_1 \circ \Phi'$ an asymptotically split asymptotic homomorphism. There is then an equi-continuous asymptotic homomorphism $\delta : A \to M(B)$ such that $\lambda_t = q_B \circ \delta_t$ for all t, i.e., λ is asymptotically split.*

Proof. The proof consists of a number of constructions needed to define the asymptotic homomorphism we are after.

Let $\psi' : A \to M(B)$ be the asymptotic homomorphism satisfying $\mathrm{ev}_1 \circ \Phi' = q_B \circ \psi'$. By Lemma 1.1.22 we have an equi-continuous asymptotic homomorphism $\psi : A \to M(B)$, such that $\lim_{t\to\infty} \psi_t(a) - \psi'_t(a) = 0$ in norm for each $a \in A$. It is not hard to see that ψ and ψ' are homotopic and hence we get an equi-continuous asymptotic homomorphism $\Phi : A \to IQ(B)$ which is a homotopy from λ to $q_B \circ \psi$.[2]

Define a continuous (by equi-continuity of Φ) map $\Lambda : A \to C_b([1,\infty), IQ(B))$ by $\Lambda(a)(t) = \Phi_t(a)$ and let $Q : C_b([1,\infty), IM(B)) \to C_b([1,\infty), IQ(B))$ be the surjective $*$-homomorphism given by $Q(f)(t) = \mathrm{id}_{C[0,1]} \otimes q_B(f(t))$, $t \in [1,\infty)$. The Bartle-Graves selection theorem provides a continuous section S for Q. Put

$$\mu_t^s(a) = \mathrm{ev}_s(S \circ \Lambda(a)(t)) + s(\psi_t(a) - \mathrm{ev}_1(S \circ \Lambda(a)(t)))$$

for all $a \in A$, $t \in [1,\infty)$ and $s \in [0,1]$. We note that the following holds for the maps μ_t^s, $t \in [1,\infty)$, $s \in [0,1]$

(m0) μ_t^s, $t \in [1,\infty)$, $s \in [0,1]$ is an equi-continuous family of maps.

(m1) μ_t^1, $t \in [1,\infty)$ is an asymptotic homomorphism.

(m2) $q_B \circ \mu_t^0(a) = \lambda_t(a)$ for all $t \in [1,\infty)$ and $a \in A$.

(m3) $q_B \circ \mu_t^-$, $t \in [1,\infty)$ is an asymptotic homomorphism into $IQ(B)$.

Indeed, (m0) is easily verified via the continuity of Λ and S. (m1) is satisfied by definition. By definition of the various maps S, Q and Λ we have

$$q_B(\mu_t^0(a)) = \mathrm{ev}_0(Q \circ S \circ \Lambda(a)(t)) = \mathrm{ev}_0(\Lambda(a)(t)) = \lambda_t(a) \tag{1.3.1}$$

for all $a \in A$ and $t \in [1,\infty)$, which establishes (m2). To see that (m3) holds, note that for $t \in [1,\infty)$ and $a \in A$

$$q_B(\psi_t(a) - \mathrm{ev}_1(S \circ \Lambda(a)(t))) = q_B(\psi_t(a)) - q_B(\psi_t(a)) = 0,$$

so that $q_B \circ \mu_t^s(a) = \Phi_t(a)(s)$ for all $t \in [1,\infty)$, $s \in [0,1]$ and $a \in A$.

There is a sequence of continuous functions $f_i : [1,\infty) \to [0,1]$, $i = 1, 2, \ldots$ such that

[2]We are a little brief here; the homotopy between ψ and ψ' is given by $s\psi + (1-s)\psi'$, $s \in [0,1]$ and we implicitly use that homotopy is an equivalence relation. Both facts are standard to establish.

(m6) $f_0 = 0$,

(m7) $f_n \leq f_{n+1}$ for all n,

(m8) for each $n \in \mathbb{N}$ there is an $m_n \in \mathbb{N}$ such that $f_i(t) = 1$ for all $i \geq m_n$ and $t \in [1, n+1]$,

(m9) for each $a \in A$, $\max_i \|\mu_t^{f_i(t)}(a) - \mu_t^{f_{i+1}(t)}(a)\| \to 0$ as $t \to \infty$.

We offer the following construction of the maps f_i:

Let F_n, $n \in \mathbb{N}$ be an increasing family of finite subsets of A such that $\overline{\cup_n F_n} = A$. Fix $n \in \mathbb{N}$. Since $(s,t) \mapsto \mu_t^s(a)$ is continuous for each $a \in A$, there is a $\delta_n > 0$ such that

$$\|\mu_t^s(a) - \mu_t^{s'}(a)\| \leq \frac{1}{n},$$

when $|s - s'| < \delta_n$ for all $a \in F_n$ and $t \in [1, n+1]$.

Choose $N_n \in \mathbb{N}$ such that $\frac{1}{N_n} < \delta_n$ and consider the N_n continuous functions $f_{n1}, \ldots, f_{nN_n}$ given by

$$f_{nk}(t) = \begin{cases} 1 & \text{for } t \in [1, n + \frac{k}{N_n}] \\ -t + 1 + n + \frac{k}{N_n} & \text{for } t \in [n + \frac{k}{N_n}, n + 1 + \frac{k}{N_n}] \\ 0 & \text{for } t \geq n + 1 + \frac{k}{N_n} \end{cases}$$

In the above construction we may of course assume that $\delta_n \geq \delta_{n+1}$ for all $n \in \mathbb{N}$.

The sequence we are after is $0, f_{11}, f_{12}, \ldots, f_{1N_1}, f_{21}, f_{22}, \ldots, f_{2N_2}, f_{31}, \ldots$. Clearly the conditions (m6) - (m8) are satisfied.

To see that also (m9) holds let $a' \in A$ and $\varepsilon > 0$ be given. Choose $a \in \cup_n F_n$ such that $\|\mu_t^s(a) - \mu_t^s(a')\| \leq \varepsilon$ for all $(s,t) \in [0,1] \times [1, \infty)$, then $a \in F_k$ for some $k \in \mathbb{N}$. Choose $K \in \mathbb{N}$ such that $\frac{1}{K} \leq \varepsilon$. Now if $t \geq \max\{k, K\}$ then $t \in [N, N+1]$ for some integer $N \geq \max\{k, K\}$ and hence $\max_i |f_i(t) - f_{i+1}(t)| < \delta_N$ by construction of the f_i's. It follows that

$$\max_i \left\| \mu_t^{f_i(t)}(a') - \mu_t^{f_{i+1}(t)}(a') \right\| < 3\varepsilon.$$

Of course we can arrange that $m_n < m_{n+1}$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$ define compact sets $L_n, S_n \subseteq M(B)$ as follows

$$L_n = \{\mu_t^s(a) \mid s \in [0,1],\ a \in F_n,\ t \in [1, n+1]\}$$

and

$$\begin{aligned} S_n = &\{\mu_t^s(a) + \alpha\mu_t^s(b) - \mu_t^s(a + \alpha b) \mid s \in [0,1],\ a, b \in F_n, \\ &\quad t \in [1, n+1],\ \alpha \in \mathbb{C},\ |\alpha| \leq n\} \\ &\cup \{\mu_t^s(ab) - \mu_t^s(a)\mu_t^s(b) \mid s \in [0,1],\ a, b \in F_n,\ t \in [1, n+1]\} \\ &\cup \{\mu_t^s(a^*) - \mu_t^s(a)^* \mid s \in [0,1],\ a \in F_n,\ t \in [1, n+1]\}. \end{aligned}$$

Here the sets $F_n \subseteq A$ are chosen as above.

By Corollary II.4.2.5 and Proposition II.4.3.2 of [B] there is an increasing, almost idempotent (in the sense of [B]) quasi-central approximate unit for B. It follows that we may find elements $X_i^n \in B$, $i = 0, 1, \ldots, n = 1, 2, \ldots$ such that $(X_i^n)_{i\geq 0}$ is a decreasing sequence with $X_i^n = 0$ when $i \geq m_n$ for each n, and $0 \leq X_i^n \leq 1$ for all i, n and for each n

(i) $X_i^n X_{i+1}^n = X_{i+1}^n$ for all i,

(ii) $\|X_i^n x - x\| \leq \frac{1}{n} + \|q_B(x)\|$ for $i = 0, 1, \ldots, m_n - 1$ and $x \in S_n$,

(iii) $\|X_i^n y - yX_i^n\| \leq \frac{1}{n}$ for all i and $y \in L_n$.

Furthermore, we may assume that $X_i^{n+1} X_k^n = X_k^n$ for all k and all $i < m_{n+1}$.

Indeed, for each n and each $i < m_n$, X_n^i is an element of the aforementioned approximate unit. Then (i) and the extra assumption $X_i^{n+1} X_k^n = X_k^n$ for all k, i, can be fulfilled thanks to the fact that our approximate unit is almost idempotent, (ii) can be satisfied by Lemma 1.5.4 in [Pe] whereas (iii) is just the quasi-centrality of the approximate unit.

For any n pick numbers $t_0, t_1, \ldots, t_{m_n+1} \in [n, n+1]$ such that $n = t_0 < t_1 < t_2 < \cdots < t_{m_n+1} = n+1$ and define for each $i = 0, 1, 2, \ldots, m_n$ a norm-continuous path $X(t, i)$, $t \in [n, n+1]$ by putting $X(t, i) = X_i^n$ for $t \leq t_i$, letting $X(t, i)$ run through a parametrization of the straight line between X_i^n and X_i^{n+1} as t runs through $[t_i, t_{i+1}]$ and putting $X(t, i) = X_i^{n+1}$ for $t \geq t_{i+1}$. For $i > m_n$ put $X(t, i) = 0$ for all $t \in [n, n+1]$. Then

(m10) $X(t, i)X(t, i+1) = X(t, i+1)$ for all $t \in [n, n+1]$ and all $i = 0, 1, \ldots$,

(m11) $\|X(t, i)x - x\| \leq \frac{1}{n} + \|q_B(x)\|$ for all $t \in [n, n+1]$, $x \in S_n$ and all $i = 0, 1, \ldots, m_n - 1$,

(m12) $\|X(t, i)y - yX(t, i)\| \leq \frac{1}{n}$ for all $y \in L_n$, $t \in [n, n+1]$ and $y \in L_n$.

For $t \in [1, \infty)$ set $\Lambda_0^t = \sqrt{1 - X(t, 0)}$ and $\Lambda_j^t = \sqrt{X(t, j-1) - X(t, j)}$ for $j \geq 1$. Note that (m10) ensures that

$$|i - j| \geq 2 \;\Rightarrow\; \Lambda_j^t \Lambda_i^t = 0 \tag{1.3.2}$$

for all $t \in [1, \infty)$. Note also that for every $t \in [1, \infty)$

$$\sum_{j=0}^{\infty} \left(\Lambda_j^t\right)^2 = 1. \tag{1.3.3}$$

We are finally ready to define a candidate for our asymptotic homomorphism δ. Indeed, for $t \in [1, \infty)$ define $\delta_t : A \to M(B)$ by

$$\delta_t(a) = \sum_{j=0}^{\infty} \Lambda_j^t \mu_t^{f_j(t)}(a) \Lambda_j^t.$$

Observe that the sum is finite for any t as $X(t, i)$ vanishes for i big. Note also that $t \mapsto \delta_t(a)$ is norm-continuous for any $a \in A$.

Combining the facts $\Lambda_j^t \in B$, $j = 1, 2, \ldots$ and $q_B(\Lambda_0^t) = 1$ for all $t \in [1, \infty)$ with (m2) and (m6) we see that

$$q_B \circ \delta_t(a) = q_B \circ \mu_t^0(a) = \lambda_t(a)$$

for all $a \in A$ and $t \in [1, \infty)$.

It remains to show that δ is actually an equi-continuous asymptotic homomorphism. To do this we will need the following estimates

$$\left\| \sum_{j=0}^{\infty} \Lambda_j^t g_j(t) \Lambda_{j+i}^t \right\| \leq \sup_j \|g_j(t)\| \tag{1.3.4}$$

valid for any $t \in [1, \infty)$, any sequence of functions $g_j : [1, \infty) \to M(B)$, $j = 1, 2, \ldots$ and any $i = 0, 1, 2, \ldots$. These estimates immediately establish equi-continuity of δ by (m0).

To prove (1.3.4) let i and such a sequence of functions be given and fix $t \in [1, \infty)$. Then by the CBS inequality ([B], Proposition II.7.1.4)

$$\begin{aligned} \left\| \sum_{j=0}^{\infty} \Lambda_j^t g_j(t) \Lambda_{j+i}^t \right\|^2 &= \left\| \sum_{j=0}^{N} \Lambda_j^t g_j(t) \Lambda_{j+i}^t \right\|^2 \\ &\leq \left\| \sum_{j=0}^{N} \Lambda_j^t g_j(t) g_j(t)^* \Lambda_j^t \right\| \left\| \sum_{j=0}^{N} \left(\Lambda_{j+i}^t\right)^2 \right\| \\ &\leq \sup_j \|g_j(t)\|^2. \end{aligned}$$

We only prove that δ is asymptotically multiplicative since linearity and self-adjointness follows similarly.

To finish the proof it suffices by equi-continuity of δ to consider $a, b \in F_n$ and show that $\delta_t(a)\delta_t(b) - \delta_t(ab) \to 0$ in norm as $t \to \infty$.

By (1.3.2) we have

$$\begin{aligned} \delta_t(a)\delta_t(b) = &\sum_{j=0}^{\infty} \Lambda_j^t \mu_t^{f_j(t)}(a) \left(\Lambda_j^t\right)^2 \mu_t^{f_j(t)}(b) \Lambda_j^t \\ &+ \sum_{j=0}^{\infty} \Lambda_j^t \mu_t^{f_j(t)}(a) \Lambda_j^t \Lambda_{j+1}^t \mu_t^{f_{j+1}(t)}(b) \Lambda_{j+1}^t \\ &+ \sum_{j=0}^{\infty} \Lambda_{j+1}^t \mu_t^{f_{j+1}(t)}(a) \Lambda_{j+1}^t \Lambda_j^t \mu_t^{f_j(t)}(b) \Lambda_j^t \end{aligned}$$

for any t.

By approximating the squareroot function on $[0, 1]$ with polynomials and invoking (m12) we see that $\lim_t \sup_j \|\Lambda_{j+i}^t \mu_t^{f_j(t)}(b) - \mu_t^{f_j(t)}(b) \Lambda_{j+i}^t\| = 0$ for $i = -1, 0, 1$. Using this, the fact that $\sup_{s,t} \|\mu_t^s(a)\| < \infty$ and (1.3.4), we see that the above expression is asymptotically equivalent to

$$\sum_{j=0}^{\infty}\Lambda_j^t\mu_t^{f_j(t)}(a)\mu_t^{f_j(t)}(b)\left(\Lambda_j^t\right)^2\Lambda_j^t+\sum_{j=0}^{\infty}\Lambda_j^t\mu_t^{f_j(t)}(a)\mu_t^{f_{j+1}(t)}(b)\Lambda_j^t\Lambda_{j+1}^t\Lambda_{j+1}^t$$
$$+\sum_{j=1}^{\infty}\Lambda_j^t\mu_t^{f_j(t)}(a)\mu_t^{f_{j-1}(t)}(b)\Lambda_j^t\left(\Lambda_{j-1}^t\right)^2.$$

Invoking (m9) and (1.3.4) again, we see that this expression is again asymptotically equivalent to

$$\sum_{j=0}^{\infty}\Lambda_j^t\mu_t^{f_j(t)}(a)\mu_t^{f_j(t)}(b)\left(\Lambda_j^t\right)^2\Lambda_j^t+\sum_{j=0}^{\infty}\Lambda_j^t\mu_t^{f_j(t)}(a)\mu_t^{f_j(t)}(b)\Lambda_j^t\left(\Lambda_{j+1}^t\right)^2$$
$$+\sum_{j=1}^{\infty}\Lambda_j^t\mu_t^{f_j(t)}(a)\mu_t^{f_j(t)}(b)\Lambda_j^t\left(\Lambda_{j-1}^t\right)^2,$$

which in turn, via (m11), (m3), (m8), (m1) and (1.3.4), is equivalent to

$$\sum_{j=0}^{\infty}\Lambda_j^t\mu_t^{f_j(t)}(ab)\left(\Lambda_j^t\right)^2\Lambda_j^t+\sum_{j=0}^{\infty}\Lambda_j^t\mu_t^{f_j(t)}(ab)\Lambda_j^t\left(\Lambda_{j+1}^t\right)^2$$
$$+\sum_{j=1}^{\infty}\Lambda_j^t\mu_t^{f_j(t)}(ab)\Lambda_j^t\left(\Lambda_{j-1}^t\right)^2.$$

Indeed, for $\varepsilon>0$, $\|\mu_t^1(a)\mu_t^1(b)-\mu_t^1(ab)\|\le\varepsilon$ for t big by (m1). By (m3)

$$\sup_j\|q_B(\mu_t^{f_j(t)}(a)\mu_t^{f_j(t)}(b)-\mu_t^{f_j(t)}(ab))\|\le\varepsilon$$

for t big. It follows from (m11) that

$$\|(X(t,j-1)-X(t,j))(\mu_t^{f_j(t)}(a)\mu_t^{f_j(t)}(b)-\mu_t^{f_j(t)}(ab))\|\le\varepsilon \tag{1.3.5}$$

for n big, $t\in[n,n+1]$, and $j<m_n$. And so by use of the C^*-identity, we may replace $X(t,j-1)-X(t,j)$ by Λ_j^t in (1.3.5). Putting all of this together remembering (m8) we have the desired equivalence. We leave the remaining details.

By (1.3.2) and (1.3.3)

$$\Lambda_0^t=\Lambda_0^t\left(\left(\Lambda_0^t\right)^2+\left(\Lambda_1^t\right)^2\right)\text{ and }\Lambda_j^t=\Lambda_j^t\left(\left(\Lambda_{j-1}^t\right)^2+\left(\Lambda_j^t\right)^2+\left(\Lambda_{j+1}^t\right)^2\right)$$

for $j\ge1$, which by the above shows that $\delta_t(a)\delta(b)$ is asymptotically equivalent to

$$\sum_{j=0}^{\infty}\Lambda_j^t\mu_t^{f_j(t)}(ab)\Lambda_j^t=\delta_t(ab),$$

as we wanted to prove. □

Chapter 2

On reduced amalgamated free products of C^*-algebras and the MF-property

This chapter consists of [Se].

2.1 Introduction

Since Anderson [An] found the first example of a C^*-algebra with non-invertible extensions by the compact operators $\mathbb{K}$ on a separable Hilbert space, several new examples of C^*-algebras with this property have been discovered. Here invertibility is in the sense of Brown, Douglas and Fillmore, see, e.g., [Ar]. Most notably Haagerup and Thorbjørnsen [HT] showed that there is a non-invertible extension of $C_r^*(\mathbb{F}_n)$ by $\mathbb{K}$ where $C_r^*(\mathbb{F}_n)$ is the reduced group C^*-algebra of the free group $\mathbb{F}_n$ on $n \in \{2, 3, \cdots\} \cup \{\infty\}$ generators, thus providing the first 'non-artificial' example of such an algebra.

This paper grew out of an investigation of the extensions of the reduced group C^*-algebras of the so-called torus knot groups which are the one-relator groups with presentations $\langle a_1, a_2 \mid a_1^k a_2^{-m} \rangle$, $k, m \in \mathbb{N}$. Somehow these groups seemed to be the natural next step upwards from the free group case as they only have one relation and can be realized as an amalgamated free product of copies of $\mathbb{Z}$.

The main result of the paper is an inclusion of the group C^*-algebra of an amalgamated free product of Abelian groups into an algebra which is MF in the sense of Blackadar and Kirchberg, [BK]. Since the MF property passes to C^*-subalgebras this result gives the first examples of reduced amalgamated free products of C^*-algebras with amalgamation over an infinite-dimensional C^*-subalgebra which are MF, c.f., the recent work of Li and Shen [LS]. See [V] for more on reduced amalgamated free products and their relation to free probability.

Since almost none of the treated groups are amenable, we get the existence of non-invertible extensions of the corresponding reduced group C^*-algebras by $\mathbb{K}$ as easy corollaries.

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2.2 Preliminaries

Most of the paper is concerned with the reduced group C^*-algebras of discrete groups and we recall some standard facts.

If G is a discrete group we let $C^*_r(G)$ denote the reduced group C^*-algebra associated to G, i.e., the unital C^*-subalgebra of $B(\ell^2(G))$ generated by the left-regular representation. We let $L(G) := C^*_r(G)''$ be the group von Neumann algebra. The algebra $L(G)$, and hence also $C^*_r(G)$, is endowed with a faithful tracial state which can be realized as the vector state corresponding to any $1_{\{g\}} \in \ell^2(G)$, $g \in G$, where $1_{\{g\}}$ is the characteristic function corresponding to the singleton $\{g\}$.

If H is a subgroup of G, then $C^*_r(H)$ is a unital subalgebra of $C^*_r(G)$. If we have a family of discrete groups G_i, $i \in I$ with a common subgroup H we denote by $\star_H G_i$ their amalgamated free product. If there is no amalgamation (i.e., if $H = \{1\}$), we simply write $\star G_i$ for the free product.

We will need the following elementary observation on reduced group C^*-algebras.

Lemma 2.2.1. *Let G_i, $i \in \mathbb{N}$ be discrete groups such that $G_i \subseteq G_{i+1}$ for all i and let G be the (direct) union of the G_i. Then we have an isomorphism of C^*-algebras*

$$\lim_k C^*_r(\cup_{i=1}^k G_i) \simeq C^*_r(G),$$

where the inductive limit is taking with respect to the natural inclusions $C^*_r(\cup_{i=1}^k G_i) \to C^*_r(\cup_{i=1}^{k+1} G_i)$.

Proof. We have the standard inclusions of C^*-algebras corresponding to the inclusion of subgroups which for each k make the following diagram commute

$$\begin{tikzcd} C^*_r(\cup_{i=1}^k G_i) \arrow[r] \arrow[d] & C^*_r(G) \\ C^*_r(\cup_{i=1}^{k+1} G_i) \arrow[ur] & \end{tikzcd}$$

The induced $*$-homomorphism $\varphi : \lim_k C^*_r(\cup_{i=1}^k G_i) \to C^*_r(G)$ is readily seen to be an isomorphism. $\square$

It is a standard fact that if a C^*-algebra A is MF but not quasidiagonal, then A has an extension by $\mathbb{K}$ which is not invertible in the sense of Brown, Douglas and Fillmore. By a result of Rosenberg (see Theorem V.4.2.13 in [B] for an elegant proof) $C_r^*(G)$ is not quasidiagonal if G is a non-amenable group. Thus to prove the existence of a non-invertible extension of $C_r^*(G)$ by $\mathbb{K}$, i.e., that the extension semigroup, $\mathrm{Ext}(C_r^*(G))$, is not a group, one needs only establish the MF-property of the algebra and realize that the group is not amenable.

2.3 The result

Our main result will be a consequence of several lemmas which we prove below. The first lemma is a mere observation.

Lemma 2.3.1. *Let A, B be unital C^*-algebras with a surjective $*$-homomorphism $\pi : A \to B$ and a state $\varphi : B \to \mathbb{C}$. Let $\tilde{\varphi} = \varphi \circ \pi$, then the GNS-representation corresponding to $\tilde{\varphi}$ is unitarily equivalent to $\pi_\varphi \circ \pi$ where π_φ is the GNS-representation corresponding to φ.*

Proof. It suffices to show that $(\pi_\varphi \circ \pi, H_\varphi, 1_B)$ is a GNS-triple for $\tilde{\varphi}$. Clearly, 1_B is a cyclic vector and

$$\tilde{\varphi}(a) = \varphi(\pi(a)) = \langle \pi_\varphi(\pi(a))1_B, 1_B\rangle$$

when $a \in A$. This proves the claim. □

We briefly introduce a notion from harmonic analysis that will prove useful to us.

For a unitary representation σ of a (discrete) group G, we let h_σ denote the $*$-homomorphism on the full group C^*-algebra, $C^*(G)$, that extends σ.

Definition 2.3.2. If σ, τ are unitary representations of the discrete group G, we say that σ is *weakly contained* in τ and write $\sigma \prec \tau$ if $\ker h_\tau \subseteq \ker h_\sigma$.

We refer to the books [Di] and [BHV] for the basics of this concept.

The following result which is interesting in its own right, is an important ingredient in our proof. The result is in fact an immediate consequence of the so-called 'continuity of induction' due to J.M.G. Fell applied to the trivial representation of the subgroup H, see, e.g., [BHV] Theorem F.3.5. We give a self-contained proof below.

Proposition 2.3.3. *Let G be a discrete group with a normal subgroup H. The canonical quotient map $q : G \to G/H$ extends to a $*$-homomorphism $\pi : C_r^*(G) \to C_r^*(G/H)$ if and only if H is amenable.*

Proof. Suppose $\pi : C_r^*(G) \to C_r^*(G/H)$ extends q. We have a natural inclusion $\iota : C_r^*(H) \to C_r^*(G)$ and since $\pi \circ \iota(h) = 1$ for all $h \in H$ we get a $*$-homomorphism $\pi \circ \iota : C_r^*(H) \to \mathbb{C}$ that sends all group elements to 1. In other words the trivial representation of H is weakly contained in the left regular representation of H and consequently, H is amenable.

Conversely assume that H is amenable. We wish to find a $*$-homomorphism $\pi : C_r^*(G) \to C_r^*(G/H)$ that will make

$$\begin{tikzcd} C^*(G) \arrow[r] \arrow[d] & C_r^*(G/H) \\ C_r^*(G) \arrow[ur, dashed] & \end{tikzcd}$$

commute.

By Proposition 8.5 of [P] we may find a net of unit vectors $(f_i) \subseteq \ell^2(H)$ such that

$$1 = \lim (f_i * \tilde{f}_i)(h)$$

for all $h \in H$. The function $\tilde{f}_i$ is given by $\tilde{f}_i(h) = \overline{f_i(h^{-1})}$, $h \in H$ and $*$ denotes convolution. Define a net of unit vectors in $\ell^2(G)$ by

$$g_i(k) = \begin{cases} f_i(k) & \text{if } k \in H \\ 0 & \text{if } k \notin H \end{cases}$$

Then

$$1_H(k) = \lim (g_i * \tilde{g}_i)(k)$$

for all $k \in G$. We wish to define a state ψ on $C_r^*(G)$ which is equal to 1_H on G. To see that this is possible let $\sum_n \gamma_n k_n \in \mathbb{C}[G]$, then

$$\begin{aligned} \left\langle \left(\sum_n \gamma_n k_n \right) g_i, g_i \right\rangle &= \sum_n \gamma_n \sum_{k \in G} g_i(k_n^{-1} k) \overline{g_i(k)} \\ &= \sum_n \gamma_n \sum_{k \in G} g_i(k) \overline{g_i(k_n k)} \\ &= \sum_n \gamma_n (g_i * \tilde{g}_i)(k_n^{-1}) \\ &\to \sum_n \gamma_n 1_H(k_n). \end{aligned}$$

Since g_i is a unit vector in $\ell^2(G)$ for each i, it follows from the CBS-inequality that 1_H may be extended to a map on $C_r^*(G)$. This extension is clearly a state.

We may consider the canonical map $\mu : C^*(G) \to C_r^*(G)$. By composition we get a state $\varphi = \psi \circ \mu$ on $C^*(G)$ which is nothing but the state on $C^*(G)$ which equals 1_H on G. The GNS-representation of this state is unitarily equivalent to the map in the top row of the diagram above and so the previous lemma tells us that the map we are looking for is the GNS-representation of ψ. $\square$

Note that if $H = G$, Proposition 2.3.3 reduces to a well-known characterization of amenability.

Like Proposition 2.3.3 above, the following result is also due to Fell.

Lemma 2.3.4. *Let λ denote the left regular representation of the discrete group G. If σ is a unitary representation such that $\sigma \prec \lambda$, then $\sigma \otimes \pi \prec \lambda$ for any unitary representation π of G.*

Proof. By Fell's absorption principle, see, e.g., Theorem 2.5.5 of [BO], and Proposition F.3.2 of [BHV]

$$\sigma \otimes \pi \prec \lambda \otimes \pi \sim \lambda^{(\dim \pi)} \prec \lambda.$$

The equivalence, $\sim$, being unitary equivalence. ☐

For a discrete Abelian group G we will let $\hat{G}$ denote the unitary dual of G. That is, $\hat{G}$ is the set of unitary equivalence classes of irreducible unitary representations of G. These are by Schur's Lemma all one-dimensional and so $\hat{G}$ may be identified with the character space of $C^*(G)$, i.e., $C^*(G) \simeq C(\hat{G})$. The unitary dual is a compact Hausdorff group in the topology of pointwise convergence on G (= the weak* topology from $C^*(G)$) under pointwise multiplication.

Lemma 2.3.5. *Let G be a discrete Abelian group. If G is countable then $\hat{G}$ is metrizable.*

Proof. This is all very standard. If G is countable, $C(\hat{G}) = C^*(G)$ is separable. Take a dense sequence f_n, $n \in \mathbb{N}$ in $C(\hat{G})$ and define a metric d on $\hat{G}$ by

$$d(x, y) = \sum_n \frac{1}{2^n \|f_n\|} |f_n(x) - f_n(y)|.$$

One easily checks, by use of Urysohn's Lemma, that d is indeed a metric. To see that d induces the right topology on $\hat{G}$ let $x_\lambda \to x$ in $\hat{G}$ and let $\varepsilon > 0$, then

$$d(x_\lambda, x) \leq \sum_{n=1}^{N} \frac{1}{2^n \|f_n\|} |f_n(x_\lambda) - f_n(x)| + \tfrac{\varepsilon}{2}$$

for a suitable N independent of λ and so $\lim_\lambda d(x_\lambda, x) = 0$. The identity map on $\hat{G}$ is then a continuous bijection from a compact space to a Hausdorff space, hence a homeomorphism. ☐

Remark 2.3.6. The converse of Lemma 2.3.5 is of course true but we will not need it in the following.

Theorem 2.3.7. *Let G_i, $i \in I$ be a finite or countably infinite collection of countable discrete Abelian groups with a common subgroup H. Let G denote the amalgamated free product $\star_H G_i$ of the G_i's with amalgamation over H. Then we have an injective $*$-homomorphism*

$$C_r^*(G) \to C_r^*(\star(G_i/H)) \otimes L^\infty(\hat{H}),$$

where the algebra of equivalence classes of measurable essentially bounded functions is with respect to Haar measure on $\hat{H}$.

Proof. Since $G/H = \star(G_i/H)$, Proposition 2.3.3 gives a unital $*$-homomorphism $\rho : C_r^*(G) \to C_r^*(\star(G_i/H))$. Consequently, ρ determines a unitary representation of G which is weakly contained in the left regular representation.

It is a classical result in harmonic analysis that the restriction map gives a homeomorphic isomorphism of compact Hausdorff groups

$$\hat{G}_i/\{\omega \in \hat{G}_i \mid \omega(h) = 1,\ h \in H\} \to \hat{H}.$$

See, e.g., Theorem 4.39 in [F].

Now, since a surjective continuous map between compact metric spaces admits a Borel section (see for instance [BR] for an elegant proof of this classical fact), we have, by Lemma 2.3.5 a Borel section for the quotient map

$$\hat{G}_i \to \hat{G}_i/\{\omega \in \hat{G}_i \mid \omega(h) = 1,\ h \in H\}.$$

Let $e_i : \hat{H} \to \hat{G}_i$ be the Borel map obtained by composition of the maps just considered, i.e., $e_i(\omega)$ is a (measurable) choice of an extension of ω to a character on all of G_i.

Consider the group homomorphism σ_i from G_i to the unitary group of $L^\infty(\hat{H})$ given by sending $g \in G_i$ to the function

$$\omega \mapsto (e_i(\omega))(g)$$

for $\omega \in \hat{H}$.

Since $\sigma_i(h) = \sigma_j(h)$ for all i, j and $h \in H$, we get an induced unitary representation σ of G and by Lemma 2.3.4 we then get a $*$-homomorphism $\psi : C_r^*(G) \to C_r^*(\star(G_i/H)) \otimes L^\infty(\hat{H})$ by considering the tensor product of the unitary representation corresponding to ρ with σ.

We show that ψ is injective by considering the trace. Indeed, consider the tensor product of the standard trace τ on $C_r^*(\star(G_i/H))$ and the normalized Haar measure m on $L^\infty(\hat{H})$. By composing this tensor product with ψ we get a trace on $C_r^*(G)$. We claim that this trace is equal to the canonical faithful trace on $C_r^*(G)$ which will ensure the injectivity of ψ. To see that this is true, it suffices to show that

$$\tau \otimes m(\psi(g)) = \begin{cases} 0 & \text{if } g \neq 1 \\ 1 & \text{if } g = 1 \end{cases}$$

for $g \in G \subseteq C_r^*(G)$.

Clearly, the equation above is satisfied for $g = 1$. The equation is obviously also satisfied if $\rho(g) \neq 1$.

If $\rho(g) = 1$ and $g \neq 1$, then $g \in H \backslash \{1\}$ so by the Gelfand-Raikov Theorem (Theorem 3.34 of [F]) there is $\omega_0 \in \hat{H}$ such that $\omega_0(g) \neq 1$ and so by invariance of Haar measure

$$\tau \otimes m(\psi(g)) = m(\sigma(g)) = \int_{\hat{H}} \omega(g)\, dm(\omega) = \omega_0(g) \int_{\hat{H}} \omega(g)\, dm(\omega),$$

which implies that $\tau \otimes m(\psi(g)) = 0$. $\square$

The representation σ in the proof of Theorem 2.3.12 may seem like a somewhat mysterious object but in concrete cases it may have a nice description as the following example shows.

Example 2.3.8. Consider the torus knot group $\Gamma_{k,m} = \langle a_1, a_2 \mid a_1^k a_2^{-m} \rangle$ from the introduction. This can be realized as the amalgamated free product $\mathbb{Z} \star_{\mathbb{Z}} \mathbb{Z}$ where the subgroup embeds in the first factor by multiplication by k and in the second by multiplication by m. Let for $r \in \mathbb{N}$, $\varphi_r : \mathbb{T} = \hat{\mathbb{Z}} \to \mathbb{C}$ be given by $\varphi_r(e^{it}) = e^{it/r}$, $t \in [0, 2\pi)$ and consider the unitary representation ζ of $\langle a_1, a_2 \mid a_1^k a_2^{-m} \rangle$ on $L^2(\mathbb{T})$ given by

$$\zeta(a_1) = \varphi_k \quad \text{and} \quad \zeta(a_2) = \varphi_m$$

where the functions φ_r are identified with the multiplication operators they induce on $L^2(\mathbb{T})$. This naturally occuring representation is exactly (a choice of) the representation σ in the proof above.

The following result is an immediate consequence of Theorem 2.3.7 and [HLSW].

Theorem 2.3.9. *Let G_i, $i \in I$ be a finite or countably infinite collection of countable discrete Abelian groups with a common subgroup H. Then $C_r^*(\star_H G_i)$ is MF.*

Proof. Since the class of MF algebras is stable under inductive limits it suffices to show the claim when I is finite by Lemma 2.2.1.

Suppose I is finite. Then

$$C_r^*(\star(G_i/H)) = \star_{\mathbb{C}} C_r^*(G_i/H),$$

where the right-hand side is the reduced free product with respect to the standard traces on the group C^*-algebras, furthermore this right-hand side is MF by Theorem 3.3.3 of [HLSW]. From Theorem 2.3.7 we get an inclusion

$$C_r^*(\star_H G_i) \subseteq C_r^*(\star(G_i/H)) \otimes L^\infty(\hat{H}).$$

It follows from this and Proposition 3.3.6 of [BK] that $C_r^*(\star_H G_i)$ is MF since abelian C^*-algebras are nuclear and MF and C^*-subalgebras of MF algebras are MF. $\square$

As mentioned in the introduction the above result is related to the theory of C^*-extensions via the following lemma.

Lemma 2.3.10. *Let I be a possibly infinite set with $|I| \geq 2$, G_i, $i \in I$ be a collection of countable discrete groups with a common normal subgroup H of index greater than or equal to 2 in each G_i. Then $\star_H G_i$ is amenable if and only if I has cardinality 2 and H is amenable with index 2 in each G_i.*

Proof. If I has cardinality $|I| = 2$, H is amenable and has index 2 in both groups G_i, we see that $\star_H G_i$ is an extension of amenable groups whence $\star_H G_i$ is amenable.

Obviously amenability of $\star_H G_i$ forces H to be amenable too.

If the cardinality of I is greater than or equal to 3 then $\star(G_i/H)$ contains a subgroup isomorphic to one of the following groups

$$\mathbb{Z}_{p_1} \star \mathbb{Z}_{p_2} \star \mathbb{Z}_{p_3},\ \mathbb{Z} \star \mathbb{Z}_{p_1},\ \mathbb{F}_2,$$

where each integer $p_i \geq 2$.

Each of these groups is non-amenable. The first one by Proposition 14.2 of [P], the second one has the non-amenable group (again by Proposition 14.2 [P]) $\mathbb{Z}_3 \star \mathbb{Z}_{p_1}$ as homomorphic image and hence cannot itself be amenable. Finally everyone knows that $\mathbb{F}_2$ is not amenable. From this it follows that $\star(G_i/H)$ and hence $\star_H G_i$ is not amenable when we are dealing with 3 or more groups.

If $|I| = 2$, and at least one of the indices is strictly greater than 2, $\star(G_i/H)$ contains a subgroup isomomorphic to one of the following

$$\mathbb{F}_2,\ \mathbb{Z}_{p_1} \star \mathbb{Z},\ \mathbb{Z}_{p_1} \star \mathbb{Z}_{p_2},\ \mathbb{Z}_2 \star (\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \langle a, b, c \mid a^2 = b^2 = c^2 = cbc^{-1}b^{-1} = 1 \rangle,$$

where $p_1 \geq 2$ and $p_2 \geq 3$.

We have already noted that the first three groups are non-amenable. The last group is non-amenable because the subgroup generated by the elements $abab$ and $acac$ is isomorphic to $\mathbb{F}_2$. We leave the tedious but straightforward argument to the reader. This completes the proof. □

Corollary 2.3.11. *Let I be a countable or finite set with $|I| \geq 2$, G_i, $i \in I$ a collection of countable, discrete Abelian groups with a common subgroup H which has index greater than or equal to 2 in each G_i. Then the BDF semigroup of extensions by $\mathbb{K}$, $\mathrm{Ext}(C_r^*(\star_H G_i))$, is a group if and only if $2 = |I| = |G_i/H|$ for both $i \in I$.*

Proof. Combine Theorem 2.3.9 and Lemma 2.3.10 with the fact that $\mathrm{Ext}(C_r^*(\star_H G_i))$ is a group if $C_r^*(\star_H G_i)$ is nuclear, see, e.g., [Ar]. □

The proof of Theorem 2.3.7 has some von Neumann algebra flavour to it. Indeed, in the proof we use a measurable section of a certain surjective map. In the example following the proof this corresponds to taking k'th roots in $\mathbb{C}$ which cannot possibly be done continuously.

Exploring this von Neumann aspect we get an isomorphism on the level of von Neumann algebras. A result which may be known to experts. The precise statement is as follows.

Theorem 2.3.12. *Let G_i, $i \in I$ be a finite or countably infinite collection of countable discrete Abelian groups with a common subgroup H. Set $\star_H G_i = G$. Then we have an isomorphism of von Neumann algebras*

$$L(G) \xrightarrow{\sim} L(\star(G_i/H)) \bar{\otimes} L^\infty(\hat{H}),$$

where the tensor product is the spatial tensor product of von Neumann algebras.

Proof. The injective $*$-homomorphism from Theorem 2.3.7 extends to a normal $*$-homomorphism $\psi : L(G) \to L(\star(G_i/H))\bar{\otimes}L^\infty(\hat{H})$ on the von Neumann algebra level.

Injectivity of ψ follows in exactly the same way as it did in the proof of Theorem 2.3.7 since the trace is also faithful on the von Neumann algebra level.

For surjectivity of ψ note first that the algebra $1 \otimes L^\infty(\hat{H})$ is in the image of ψ. Indeed, combining the classical theorems of Stone-Weierstrass and Gelfand-Raikov, we see that the algebra generated by $\psi(H)$ is norm dense in $1 \otimes C(\hat{H})$ which in turn is strongly dense in $1 \otimes L^\infty(\hat{H}) \subseteq B(\ell^2(\star(G_i/H)) \otimes L^2(\hat{H}))$. On the other hand, for any $b \in \star(G_i/H) \subseteq L(\star(G_i/H))$ there is a unitary $c \in L^\infty(\hat{H})$ such that $b \otimes c$ is in the image of ψ. It follows that the image of ψ contains anything of the form $a \otimes b$ where $a \in L(\star(G_i/H))$ and $b \in L^\infty(\hat{H})$ and thus ψ is surjective. □

2.4 More Examples of MF Algebras

The ideas from the last section can be used to give some new examples of reduced group C^*-algebras with the MF property.

The simple strategy above is basically reducing the MF question of amalgamated free products of groups to the same question with no amalgamation. More precisely given a collection of discrete groups (G_i) with a common normal subgroup H the idea is to find a nuclear algebra with the MF property (i.e., an NF algebra) A such that

$$C_r^*(\star_H G_i) \subseteq C_r^*(\star(G_i/H)) \otimes A,$$

and then from this deduce that the left-hand side is MF if $C_r^*(\star(G_i/H))$ is.

Executing the strategy, of course, requires a candidate for the algebra A and in the same breath a candidate for the map realizing the inclusion. This is where the real work lies. In the following we will discuss this line of attack in the case of a tower of groups.

Proposition 2.4.1. *Suppose $H \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n$ are discrete groups with H amenable and normal in each G_i. Then we have an inclusion of C^*-algebras*

$$C_r^*(\star_H G_i) \subseteq C_r^*(\star(G_i/H)) \otimes_{\min} C_r^*(G_n).$$

Proof. The proof is very similar to that of Theorem 2.3.7 so we will be brief. Note that the free product of the group inclusions gives a map $\star_H G_i \to G_n \subseteq C_r^*(G_n)$. This representation tensored with the representation $\star_H G_i \to (\star_H G_i)/H = \star(G_i/H) \subseteq C_r^*(\star(G_i/H))$ is weakly contained in the left regular representation (Proposition 2.3.3 and Lemma 2.3.4) and so gives a map between the C^*-algebras in question. The injectivity of this map is established by considering the trace once again. □

To use this observation to get new examples of MF algebras, we need to know that each $C_r^*(G_i/H)$ is an ASH algebra so that we can invoke [HLSW] and we need to know that $C_r^*(G_n)$ is NF. The last condition is related to the well known (and somewhat notorious) conjecture of Rosenberg stating that $C_r^*(G)$ is quasidiagonal for any countable discrete amenable group G. Indeed, if $C_r^*(G_n)$ is NF it must be quasidiagonal by Voiculescu's characterization of quasidiagonality and the Choi-Effros Lifting Theorem and G_n must be amenable.

In other words to get concrete examples, G_n must be an amenable group for which Rosenberg's conjecture holds. Unfortunately Rosenberg's conjecture has not, to the knowledge of the author, been established in very many cases. Obvious classes for which the conjecture holds are the abelian and the finite groups. Another concrete example based on what we have done so far is the following.

Example 2.4.2. Let S_∞ be the group of permutations on $\mathbb{N}$ with finite support. It has a normal subgroup A_∞ consisting of the finitely supported even permutations. By Lemma 2.3.10 the group $S_\infty \star_{A_\infty} S_\infty$ is amenable and clearly $C^*(S_\infty)$ is NF (it is in fact AF), so by Proposition 2.4.1 $S_\infty \star_{A_\infty} S_\infty$ has a group C^*-algebra which is MF and the discussion above tells us that $S_\infty \star_{A_\infty} S_\infty$ satisfies Rosenberg's conjecture. Note that we do not have to invoke the results of [HLSW] since the quotient $S_\infty \star_{A_\infty} S_\infty / A_\infty$ is the infinite dihedral group which is amenable and residually finite and hence has a group C^*-algebra which is MF. See details below.

We will call a group MF if its reduced group C^*-algebra is MF. Using standard terminology from group theory a group G is then residually MF if there are MF groups G_n, $n \in \mathbb{N}$, such that $G \subseteq \prod_n G_n$.

Proposition 2.4.3. *Suppose G is a discrete group and that there is a sequence of discrete groups $(G_n)_{n\in\mathbb{N}}$ and surjective homomorphisms $\varphi_n : G \to G_n$ satisfying that $\ker \varphi_n$ is amenable for each n and that*

$$\bigcap_{n\in I} \ker \varphi_n = \{1\},$$

whenever I is an infinite subset of $\mathbb{N}$.

It follows that we have an embedding

$$\pi : C_r^*(G) \to \prod_{n=1}^{\infty} C_r^*(G_n) \Big/ \sum_{n=1}^{\infty} C_r^*(G_n).$$

Proof. Lemma 2.3.3 ensures the existence of the $*$-homomorphism π. As usual the injectivity follows by considering the trace.

Fix a character ω on $\ell^\infty(\mathbb{N})$ such that ω is not evaluation at any point $n \in \mathbb{N}$. Note that $\ker(\omega)$ contains any sequence that is eventually 0 and so by continuity all sequences converging to 0. In fact if $x \in \ell^\infty(\mathbb{N})$ is a sequence of 0's and 1's, $\omega(x) = \omega(x)^2$, so $\omega(x)$ is either 1 or 0. In particular if $x = 1_{\{n\}}$ and $\omega(x) = 1$ then by Urysohn's Lemma there is an $1 \geq f \geq 0$ in $\ell^\infty(\mathbb{N})$ with $f(n) = 0$ and $\omega(f) > 0$ since ω is not evaluation at n. Then

$$\|f + 1_{\{n\}}\| \geq \omega(f + 1_{\{n\}}) > 1$$

but the norm equals 1. It follows that $\omega(1_{\{n\}}) = 0$ for all $n \in \mathbb{N}$.

Let τ_n denote the standard trace on $C_r^*(G_n)$. The equation

$$\tau((x_n)) = \omega((\tau_n(x_n)))$$

defines a trace on $\prod_{n=1}^{\infty} C_r^*(G_n)$ which drops to

$$\prod_{n=1}^{\infty} C_r^*(G_n) \Big/ \sum_{n=1}^{\infty} C_r^*(G_n)$$

since the ideal is killed by τ.

We consider the trace $\tau \circ \pi$ on $C_r^*(G)$. Let $g \in G$. Then

$$\tau \circ \pi(g) = \tau((\varphi_n(g))) = \omega(\tau_n(\varphi_n(g))), \tag{2.4.1}$$

where φ_n is the quotient map $G \to G_n$.

Since $(\tau_n(\varphi_n(g)))$ is a sequence of 0's and 1's (2.4.1) equals 0 or 1 by the reasoning above. If (2.4.1) is 1 then $\tau_n(\varphi_n(g)) = 1$ for infinitely many n which in turn implies that $\varphi_n(g) = 1$ for infinitely many n and so $g = 1$ by assumption. In short

$$\tau \circ \pi(g) = \begin{cases} 1 & \text{if } g = 1 \\ 0 & \text{if } g \neq 1 \end{cases}.$$

It follows that $\tau \circ \pi$ is exactly the usual faithful trace on $C_r^*(G)$ and so π is injective. □

Corollary 2.4.4. *Suppose G is an amenable residually MF group. It follows that G is MF and hence satisfies Rosenberg's conjecture.*

Proof. Let G_n, $n \in \mathbb{N}$ be MF groups such that $G \subseteq \prod_n G_n$. We may assume that each G_n is amenable and hence $\prod_n^k G_n$ is MF. Now, the maps $G \subseteq \prod_n G_n \to \prod_n^k G_n$ satisfies the hypothesis of Proposition 2.4.3. An application of Corollary 3.4.3 of [BK] shows that G is MF. □

For instance all amenable residually finite groups satisfy Rosenberg's conjecture as can also easily be observed since their group C^*-algebras are even residually finite dimensional.

In the setting of Proposition 2.4.1 with G_n amenable and, say residually finite, we can conclude that the group $\star_H G_i$ is MF. We end the exposition by giving an example of an MF group which is a free product of non-abelian groups with infinite amalgamation.

Example 2.4.5. Consider the set

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & I & z \\ 0 & 0 & 1 \end{pmatrix} \middle| x, z \in \mathbb{Z}^n,\ y \in \mathbb{Z} \right\}$$

for $n \in \mathbb{N}$. This is a (non-Abelian) group under matrix multiplication called the discrete Heisenberg group. It has a normal subgroup isomorphic to $\mathbb{Z}^{n+1}$ consisting of the matrices with $x = 0$ in the above presentation. Taking the quotient with this, we get $\mathbb{Z}^n$, so in particular G is amenable. Let for $k \in \mathbb{N}$, G_k be the normal subgroup of G given by

$$G_k = \left\{ \begin{pmatrix} 1 & kx & ky \\ 0 & I & kz \\ 0 & 0 & 1 \end{pmatrix} \middle| x, z \in \mathbb{Z}^n,\ y \in \mathbb{Z} \right\}.$$

This has finite index in G and it follows that G is residually finite and hence satisfies Rosenberg's conjecture. Furthermore $G \star_{G_k} G$ is an MF group for each k, note here that the subgroup we amalgamate over is infinite and non-Abelian.

The infinite dihedral group provides another example of a residually finite amenable group and the Baumslag-Solitar groups $B(1, m) := \langle a, b \mid a^{-1}ba = b^m \rangle$, $m \in \mathbb{N}$ also have these properties and so we could form suitable amalgamated free products of these to get more MF groups.

Chapter 3

Extensions of the reduced group C^*-algebra of a free product of amenable groups

This chapter is the paper [ST] written jointly with Klaus Thomsen. Nothing has been changed.

3.1 Introduction

The stock of examples of C^*-algebras for which the semi-group of extensions by the compact operators is not a group is still growing. The latest newcomers consist of a series of reduced free products of nuclear C^*-algebras, cf. [HLSW]. This stresses the necessity of finding a way to handle the many extensions without inverses. In joint work with Vladimir Manuilov the second-named author has proposed a way to amend the definition of the semi-group of extensions of a C^*-algebra by a stable C^*-algebra in such a way that nothing is changed in the case of nuclear algebras where the usual theory already works perfectly, and such that at least some of the extensions which fail to have inverses in the usual sense become invertible in the new, slightly weaker sense. This new semi-group grew out of investigations of the relation between the E-theory of Connes and Higson and the theory of C^*-extensions, [MT1]. The change consists merely in trivializing not only the split extensions, but also the asymptotically split extensions; those for which there are asymptotic homomorphisms consisting of right inverses for the quotient map, cf. [MT1]. When an extension can be made asymptotically split by addition of another extension we say that the extension is semi-invertible, and the resulting group of semi-invertible extensions, taken modulo asymptotically split extensions, is an abelian group with a close connection to the E-theory of Connes and Higson, [CH]. In some, but not all cases where the usual semi-group of extensions is not a group the alternative definition does give a group; i.e. all extensions are semi-invertible, cf. [MT1],[Th1],[MT3]. Specifically, in [MT1] this was shown to be the case when the quotient is a suspended C^*-algebra and in [Th1] when the quotient is the reduced group C^*-algebra $C^*_r(\mathbb{F}_n)$ of a free group with finitely many generators, and the ideal is the C^*-algebra $\mathbb{K}$ of compact operators. This gave the first example of a unital C^*-algebra for which all extensions by the compact operators are semi-invertible, but not all invertible; by the result of Haagerup and Thorbjørnsen, [HT], there are non-invertible extensions of $C^*_r(\mathbb{F}_n)$ by $\mathbb{K}$ when $n \geq 2$. The purpose of the present note is to show that the situation in [Th1] is not exceptional at all. This is done by showing that all extensions of a reduced group C^*-algebra $C^*_r(G)$ by any stable σ-unital C^*-algebra are semi-invertible when G is the free product of a countable collection of discrete countable and amenable groups. The basic idea of the proof is identical to that employed in [Th1]. The crucial improvement over the argument from [Th1] is that the explicitly given homotopy of representations of $\mathbb{F}_n$ from [C] is replaced by results of Dadarlat and Eilers from [DE]. The pairing in the first variable of the usual extension group Ext^{-1} with KK-theory and Cuntz' results on K-amenability from [C] remain key ingredients.

In [Th1] the inverse of an extension, modulo asymptotically split extensions, could be taken to be invertible in the usual sense, i.e. to admit a completely positive contractive splitting. This turns out to be possible also in the more general situation considered here, and as a consequence it follows that the obvious map from the usual KK-theory group $\mathrm{Ext}^{-1}(C^*_r(G), B)$ to the group of all extensions, taken modulo asymptotically split extensions, is surjective. By combining results of Cuntz, Tu and Thomsen it follows that $C^*_r(G)$ satisfies the universal coefficient theorem of Rosenberg and Schochet, and from this it follows easily that the map is also injective. Hence the group of extensions of $C^*_r(G)$ by B, taken modulo the asymptotically split extensions, can be calculated from K-theory by use of the UCT.

3.2 The results

For a C^*-algebra B we let $M(B)$ denote the multiplier algebra of B and $Q(B)$ denotes the generalized Calkin algebra $M(B)/B$.

The main result of the paper is the following.

Theorem 3.2.1. *Let $(G_i)_{i\in\mathbb{N}}$ be a countable collection of discrete countable amenable groups and let $G = \star_i G_i$ be their free product. Let B be a stable σ-unital C^*-algebra. For every extension $\varphi : C^*_r(G) \to Q(B)$ there is an invertible extension $\varphi' : C^*_r(G) \to Q(B)$ such that $\varphi \oplus \varphi'$ is asymptotically split.*

More explicitly the conclusion is that there is an extension φ', a completely positive contraction $\psi : C^*_r(G) \to M(B)$ and an asymptotic $*$-homomorphism $\pi = (\pi_t)_{t\in[1,\infty)} : C^*_r(G) \to M(B)$, in the sense of Connes and Higson, cf. [CH], such that $\varphi' = q_B \circ \psi$ and $\varphi \oplus \varphi' = q_B \circ \pi_t$ for all $t \in [1, \infty)$, where $q_B : M(B) \to Q(B)$ is the quotient map.

If only one of the G_i's in Theorem 3.2.1 is non-trivial or if $G = \mathbb{Z}_2 \star \mathbb{Z}_2$, the conclusion of the theorem is trivial and can be improved because G is then amenable. It seems very plausible that such cases are exceptional; indeed it follows from [HLSW] that there is a non-invertible extension of $C_r^*(G)$ by the compact operators whenever G is the free product of finitely many non-trivial groups each of which is either abelian or finite and $G \neq \mathbb{Z}_2 \star \mathbb{Z}_2$.

As in [Th1] we will prove Theorem 3.2.1 by use of results from [MT2]. Recall that two extensions $\varphi, \varphi' : A \to Q(B)$ are *strongly homotopic* when there is a $*$-homomorphism $A \to C[0,1] \otimes Q(B)$ giving us φ when we evaluate at 0 and φ' when we evaluate at 1. By Lemma 4.3 of [MT2] it suffices then to establish the following

Theorem 3.2.2. *Let $(G_i)_{i \in \mathbb{N}}$ be a countable collection of discrete countable amenable groups and let $G = \star_i G_i$ be their free product. Let B be a stable σ-unital C^*-algebra. For every extension $\varphi : C_r^*(G) \to Q(B)$ there is an invertible extension $\varphi' : C_r^*(G) \to Q(B)$ such that $\varphi \oplus \varphi'$ is strongly homotopic to a split extension.*

We proceed to give a couple of definitions and list some lemmas needed in the proof of Theorem 3.2.2.

The following notion from harmonic analysis proves very useful.

Definition 3.2.3. Let A be a C^*-algebra and φ, ψ be $*$-representations of A on some Hilbert spaces. Then φ is weakly contained in ψ if $\ker \psi \subseteq \ker \varphi$.

If σ, π are unitary representations of a locally compact group then σ is weakly contained in π if and only if the representation of the full group C^*-algebra corresponding to σ is weakly contained in the representation corresponding to π. An equivalent definition of weak containment in this case, is that every positive definite function associated to σ can be approximated uniformly on compact subsets by finite sums of positive definite functions associated to π. See sections 3.4 and 18.1 of [Di] for details.

A proof of the following lemma can be found in [BHV].

Lemma 3.2.4. *Let σ, π be unitary representations of a locally compact group. Assume that σ is weakly contained in the left regular representation λ. It follows that $\sigma \otimes \pi$ is weakly contained in λ.*

For any discrete group G we denote in the following the canonical surjective $*$-homomorphism $C^*(G) \to C_r^*(G)$ from the full to the reduced group C^*-algebra by μ.

Besides the main results of [C] we shall also need the following technical lemma which is a slight reformulation of Cuntz' definition of K-amenability. See [C] for the proof.

Lemma 3.2.5. *Let H be an infinite-dimensional separable Hilbert space and let G be a countable discrete K-amenable group. Then there exist $*$-homomorphisms $\sigma, \sigma_0 : C_r^*(G) \to B(H)$ such that $\sigma \circ \mu, h_\tau \oplus \sigma_0 \circ \mu : C^*(G) \to B(H)$ are unital, $\sigma \circ \mu(a) - (h_\tau \oplus \sigma_0 \circ \mu)(a) \in \mathbb{K}$ for all $a \in C^*(G)$, and $[\sigma \circ \mu, h_\tau \oplus \sigma_0 \circ \mu] = 0$ in $KK(C^*(G), \mathbb{C})$, where $h_\tau : C^*(G) \to \mathbb{C} \subseteq B(H)$ is the $*$-homomorphism going with the trivial one-dimensional representation τ of G and $\mathbb{K}$ is the ideal of compact operators on H.*

The next lemma gives us the appropriate substitute for the homotopy of representations of $\mathbb{F}_n$ which was a crucial tool in [Th1]. The proof is easy, thanks to the results of Dadarlat and Eilers in [DE].

Lemma 3.2.6. *Let $(G_i)_{i\in\mathbb{N}}$ be a countable collection of discrete countable amenable groups and let $G = \star_i G_i$ be their free product. Let $\mu : C^*(G) \to C^*_r(G)$ be the canonical surjection and let $h_\tau : C^*(G) \to \mathbb{C}$ be the character corresponding to the trivial one-dimensional representation of G. There are then a separable infinite-dimensional Hilbert space H, $*$-homomorphisms $\sigma, \sigma_0 : C^*_r(G) \to B(H)$ and a path $\zeta_s : C^*(G) \to B(H), s \in [0,1]$, of unital $*$-homomorphism such that*

a) $\zeta_0 = \sigma \circ \mu$;

b) $\zeta_1 = h_\tau \oplus \sigma_0 \circ \mu$;

c) $\zeta_s(a) - \zeta_0(a) \in \mathbb{K},\ a \in C^*(G), s \in [0,1]$, and

d) $s \mapsto \zeta_s(a)$ is continuous for all $a \in C^*(G)$.

Proof. Being amenable G_i has the Haagerup Property. See the discussion in 1.2.6 of [CCJJV]. It follows then from Propositions 6.1.1 and 6.2.3 of [CCJJV] that also G has the Haagerup Property. Since the Haagerup Property implies K-amenability by [Tu] we conclude that G is K-amenable. We can therefore pick $*$-homomorphisms $\sigma, \sigma_0 : C^*_r(G) \to B(H)$ as in Lemma 3.2.5. By adding the same unital and injective $*$-homomorphism to σ and σ_0 we can arrange that both σ and σ_0 are injective and have no non-zero compact operator in their range. Since $\mu|_{C^*_r(G_i)} : C^*(G_i) \to C^*_r(G_i)$ is injective it follows then that $\sigma \circ \mu|_{C^*(G_i)}$ and $(h_\tau \oplus \sigma_0 \circ \mu)|_{C^*(G_i)}$ are admissible in the sense of Section 3 of [DE] for each i. Thus Theorem 3.12 of [DE] applies to show that there is a norm-continuous path $u^i_s, s \in [1, \infty)$, of unitaries in $1 + \mathbb{K}$ such that

$$\lim_{s\to\infty} \left\| \sigma \circ \mu|_{C^*(G_i)}(a) - u^i_s \left(h_\tau \oplus \sigma_0 \circ \mu\right)|_{C^*(G_i)}(a) {u^i_s}^* \right\| = 0$$

for all $a \in C^*(G_i)$ and

$$\sigma \circ \mu|_{C^*(G_i)}(a) - u^i_s \left(h_\tau \oplus \sigma_0 \circ \mu\right)|_{C^*(G_i)}(a) {u^i_s}^* \in \mathbb{K}$$

for all $a \in C^*(G_i)$ and all $s \in [1, \infty)$. Since the unitary group of $1 + \mathbb{K}$ is connected in norm there are therefore norm-continuous paths of unital $*$-homomorphisms $\zeta^j_s : C^*(G_j) \to B(H),\ s \in [0,1],\ j \in \mathbb{N}$, such that

aj) $\zeta^j_0 = \sigma \circ \mu|_{C^*(G_j)}$;

bj) $\zeta^j_1 = \sigma_0 \circ \mu|_{C^*(G_j)} \oplus h_\tau|_{C^*(G_j)}$;

cj) $\zeta^j_s(a) - \zeta^j_0(a) \in \mathbb{K},\ a \in C^*(G_j),\ s \in [0,1]$, for each j. The universal property of the free product construction gives us then a path of unital $*$-homomorphisms $\zeta_s : C^*(G) \to B(H), s \in [0,1]$, with the stated properties, a)-d). $\square$

Our proof of Theorem 3.2.2 uses the notion of absorbing $*$-homomorphisms.

Definition 3.2.7. Let A and B be separable C^*-algebras with B stable. A $*$-homomorphism $\pi : A \to M(B)$ is said to be absorbing if it holds that for any $*$-homomorphism $\varphi : A \to M(B)$ there is a sequence of unitaries $U_n \in M(B), n \in \mathbb{N}$, such that

$$\lim_n \|U_n (\pi(a) \oplus \varphi(a)) U_n^* - \pi(a)\| = 0$$

for all $a \in A$ and

$$U_n (\pi(a) \oplus \varphi(a)) U_n^* - \pi(a) \in B,$$

for all $n \in \mathbb{N}$ and all $a \in A$.

We will also need the notion of a *unitally* absorbing $*$-homomorphism which is defined similarly, but with A and π both unital, and π is only required to absorb unital morphisms. We refer to [Th2] for the precise statement and for the proof of the fact that (unitally) absorbing homomorphisms exist.

We shall also need the following lemma which is a unital version of Lemma 2.2 in [Th3]. The proof is the same.

Lemma 3.2.8. *Let A be a separable unital C^*-algebra, $D \subseteq A$ a unital nuclear C^*-subalgebra and B a stable separable C^*-algebra. Let $\pi : A \to M(B)$ be a unitally absorbing $*$-homomorphism. It follows that $\pi|_D : D \to M(B)$ is unitally absorbing.*

Lemma 3.2.9. *In the setting of Theorem 3.2.1 it holds that every extension $\varphi : C^*(G) \to Q(B)$ of $C^*(G)$ by B is invertible. If φ is unital, it is invertible in the semi-group of unitary equivalence classes of unital extensions, modulo the unital split extensions.*

Proof. Assume first that φ is unital. For each $i \in \mathbb{N}$ the C^*-algebra $C_r^*(G_i) = C^*(G_i)$ is nuclear and hence the unital extensions $\varphi_i = \varphi|_{C^*(G_i)} : C^*(G_i) \to Q(B)$ are all invertible. There are therefore unital extensions $\psi_i : C^*(G_i) \to Q(B)$ and $*$-homomorphisms $\pi_i : C^*(G_i) \to M(B)$ such that $\varphi_i \oplus \psi_i = q_B \circ \pi_i,\ i \in \mathbb{N}$. Let $\omega_i : C^*(G_i) \to \mathbb{C}$ denote the $*$-homomorphism corresponding to the trivial unitary representation of G_i. By replacing π_i with $\pi_i + \omega_i \pi_i(1)^\perp$ we may assume that π_i is unital. The universal property of the free product gives us a unital extension $\psi = \star_i \psi_i : C^*(G) \to Q(B)$ and a unital $*$-homomorphism $\pi = \star_i \pi_i : C^*(G) \to M(B)$. Since $\varphi \oplus \psi = q_B \circ \pi$, this completes the proof of the unital case.

Now let φ be a general extension. Again consider $\varphi_i = \varphi|_{C^*(G_i)} : C^*(G_i) \to Q(B)$. Then $\varphi_i(1) = \varphi_j(1) = p$ for all $i, j \in \mathbb{N}$, so the extensions $\tilde{\varphi}_i := \varphi_i + \omega_i p^\perp$ are all unital. As above we get $\psi : C^*(G) \to Q(B)$ and a unital $*$-homomorphism $\pi : C^*(G) \to M(B)$ such that $(\star_i \tilde{\varphi}_i) \oplus \psi = q_B \circ \pi$. Since $\star_i \tilde{\varphi}_i$ and $\varphi \oplus \left(\star_i \omega_i p^\perp\right)$ are equal in $\mathrm{Ext}(C^*(G), B)$ this completes the proof. $\square$

The next lemma will allow us to focus the proof of Theorem 3.2.2 to the case where B is separable.

Lemma 3.2.10. *Let A be a separable C^*-algebra. Suppose that for any stable separable C^*-algebra B, it holds that for every extension $\varphi : A \to Q(B)$ there is an invertible extension $\varphi' : A \to Q(B)$ such that $\varphi \oplus \varphi'$ is strongly homotopic to a split extension. Then this conclusion also holds with the separable and stable B replaced by any stable and σ-unital C^*-algebra.*

Proof. Consider $D \otimes \mathbb{K}$, where D is σ-unital, along with an extension

$$0 \to D \otimes \mathbb{K} \to E \xrightarrow{p} A \to 0.$$

Since A is separable and $D \otimes \mathbb{K}$ σ-unital we can find a separable C^*-subalgebra $E_0' \subseteq E$ such that $p(E_0') = A$ and such that E_0' contains an approximate unit for $D \otimes \mathbb{K}$. There is then a separable C^*-subalgebra D_0 of D such that $(D \otimes \mathbb{K}) \cap E_0' \subseteq D_0 \otimes \mathbb{K}$. Similarly, there is a separable C^*-subalgebra $D_1 \subseteq D$ such that $D_0 \subseteq D_1$ and $E_0' \left(D_0 \otimes \mathbb{K}\right) \subseteq D_1 \otimes \mathbb{K}$. In fact, we can recursively find a sequence $\{D_n\}$ of separable C^*-subalgebras of D such that $E_0' \left(D_n \otimes \mathbb{K}\right) \subseteq D_{n+1} \otimes \mathbb{K}$. Then $D_\infty = \overline{\bigcup_n D_n}$ and $E_0 = C^*\left(E_0', D_\infty \otimes \mathbb{K}\right) \subseteq E$ are separable. Furthermore $D_\infty \otimes \mathbb{K} = \ker p|_{E_0}$ by construction and $p(E_0) = A$ so that we have an extension

$$0 \to D_\infty \otimes \mathbb{K} \to E_0 \xrightarrow{p} A \to 0$$

of separable C^*-algebras. Since $D_\infty \otimes \mathbb{K}$ contains an approximate unit for $D \otimes \mathbb{K}$ the inclusion $D_\infty \otimes \mathbb{K} \subseteq D \otimes \mathbb{K}$ extends to an injective $*$-homomorphism $M\left(D_\infty \otimes \mathbb{K}\right) \subseteq M(D \otimes \mathbb{K})$ and we get an embedding $\iota : Q(D_\infty \otimes \mathbb{K}) \to Q(D \otimes \mathbb{K})$.

Now, by construction the Busby invariant φ of the original extension has the form $\varphi = \iota \circ \varphi'$, where $\varphi' : A \to Q(D_\infty \otimes \mathbb{K})$ is the Busby invariant of the last extension. By assumption there is an invertible extension $\psi' : A \to Q(D_\infty \otimes \mathbb{K})$ such that $\varphi' \oplus \psi'$ is strongly homotopic to a split extension (by $D_\infty \otimes \mathbb{K}$). It follows that $\varphi \oplus \iota \circ \psi'$ is strongly homotopic to a split extension by $D \otimes \mathbb{K}$. Note that $\iota \circ \psi'$ is invertible. ☐

Proof of Theorem 3.2.2. By Lemma 3.2.10 we can assume that B is separable.

In order to control the images of the unit for the extensions we consider, we need a result of Skandalis which we first describe. Note that the unital inclusion $i : \mathbb{C} \to C^*(G)$ has a left-inverse $h_\tau : C^*(G) \to \mathbb{C}$ given by the trivial one-dimensional representation τ. Therefore the map

$$i^* : \mathrm{Ext}^{-1}(C^*(G), SB) \to \mathrm{Ext}^{-1}(\mathbb{C}, SB) = K_0(B)$$

is surjective. We put this into the six-term exact sequence of Skandalis, 10.11 in [S], whose proof can be found in [MT4]. Using the notation from [MT4] we obtain the following commuting diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^{-1}_{\mathrm{unital}}(C^*(G), B) \arrow[r] & \mathrm{Ext}^{-1}(C^*(G), B) \arrow[r] & K_0(Q(B)) \\
 & \mathrm{Ext}^{-1}_{\mathrm{unital}}(C^*_r(G), B) \arrow[r] \arrow[u, "\mu^*"] & \mathrm{Ext}^{-1}(C^*_r(G), B) \arrow[r] \arrow[u, "\mu^*"] & K_0(Q(B)) \arrow[u, equal]
\end{tikzcd} \tag{3.2.1}$$

Recall that G is K-amenable as observed in the proof of Lemma 3.2.6. By [C] this implies that $\mu^* : \operatorname{Ext}^{-1}(C_r^*(G), B) \to \operatorname{Ext}^{-1}(C^*(G), B)$ is an isomorphism.

Let $\varphi : C_r^*(G) \to Q(B)$ be a unital extension. Let $\pi_1 : C_r^*(G) \to Q(B)$ be a unitally absorbing split extension (whose existence is guaranteed by [Th2] since both $C_r^*(G)$ and B are separable) and set $\varphi' = \varphi \oplus \pi_1$. It follows from Lemma 3.2.9 and a diagram chase in (3.2.1) that there is an invertible unital extension $\varphi'' : C_r^*(G) \to Q(B)$ such that

$$[\varphi' \circ \mu \oplus \varphi'' \circ \mu] = 0 \tag{3.2.2}$$

in $\operatorname{Ext}^{-1}_{\text{unital}}(C^*(G), B)$. Since $C^*(G_i)$ is nuclear $\mu|_{C^*(G_i)} : C^*(G_i) \to C_r^*(G_i), i \in \mathbb{N}$, is a $*$-isomorphism and it follows from Lemma 3.2.8 that $\pi_1|_{C_r^*(G_i)} : C_r^*(G_i) \to Q(B)$ is unitally absorbing for each $i \in \mathbb{N}$. Hence $\pi_1 \circ \mu|_{C^*(G_i)} : C^*(G_i) \to Q(B)$ is a unitally absorbing split extension. It follows therefore from (3.2.2) that $(\varphi' \circ \mu \oplus \varphi'' \circ \mu)|_{C^*(G_i)}$ is a unitally split extension for each i. As in the proof of Lemma 3.2.9 this implies that $\varphi' \circ \mu \oplus \varphi'' \circ \mu$ is unitally split. There is therefore a unitary representation $\gamma : G \to M(B)$ such that

$$q_B \circ h_\gamma = \varphi' \circ \mu \oplus \varphi'' \circ \mu, \tag{3.2.3}$$

where $h_\gamma : C^*(G) \to M(B)$ is the $*$-homomorphism defined by γ.

Consider the homotopy ζ_s from Lemma 3.2.6. Let $\nu_s : G \to B(H)$ be the unitary representation defined by ζ_s so that $\zeta_s = h_{\nu_s}$. It follows from the property a) of Lemma 3.2.6 that ν_0 is weakly contained in the left-regular representation of G and from b) that ν_1 is a direct sum $\tau \oplus \lambda_0$ where λ_0 is a representation of G which is weakly contained in the left-regular representation of G. Consider the unitary representations

$$\gamma \otimes \nu_s : G \to M(B) \otimes B(H) \subseteq M(B \otimes \mathbb{K}),\ s \in [0,1].$$

Then $q_{B\otimes\mathbb{K}} \circ h_{\gamma\otimes\nu_s} : C^*(G) \to Q(B \otimes \mathbb{K}),\ s \in [0,1]$, is a norm-continuous path of extensions. Note that

$$q_{B\otimes\mathbb{K}} \circ h_{\gamma\otimes\nu_1} = q_{B\otimes\mathbb{K}} \circ h_{\gamma\otimes\tau} \oplus q_{B\otimes\mathbb{K}} \circ h_{\gamma\otimes\lambda_0} = (\varphi' \oplus \varphi'') \circ \mu \oplus q_{B\otimes\mathbb{K}} \circ h_{\gamma\otimes\lambda_0}.$$

Since $\gamma \otimes \nu_0$ and $\gamma \otimes \lambda_0$ are weakly contained in the left-regular representation of G by Lemma 3.2.4 it follows from an argument almost identical with one used in [Th1] that each $q_{B\otimes\mathbb{K}} \circ h_{\gamma\otimes\nu_s}$ factors through $C_r^*(G)$ and hence the family $q_{B\otimes\mathbb{K}} \circ h_{\gamma\otimes\nu_s},\ s \in [0,1]$ defines a strong homotopy connecting the split extension $q_{B\otimes\mathbb{K}} \circ h_{\gamma\otimes\nu_0} : C_r^*(G) \to Q(B \otimes \mathbb{K})$ to the direct sum $\varphi' \oplus \varphi'' \oplus q_{B\otimes\mathbb{K}} \circ h_{\gamma\otimes\lambda_0}$. For completeness we include the argument: Let $s \in [0,1]$ and $x = \sum_j c_j g_j \in \mathbb{C}G$, where $c_j \in \mathbb{C}$ and $g_j \in G$. Then

$$h_{\gamma\otimes\nu_s}(x) = \sum_j c_j \gamma(g_j) \otimes \nu_0(g_j) + \sum_j c_j \gamma(g_j) \otimes \Delta(g_j), \tag{3.2.4}$$

where $\Delta(g_j) = \nu_s(g_j) - \nu_0(g_j)$. Note that $\Delta(g_j) \in \mathbb{K}$ by c). Since ν_0 is weakly contained in the left regular representation we can use Lemma 3.2.4 to conclude that $\left\| \sum_j c_j \gamma(g_j) \otimes \nu_0(g_j) \right\| \leq$

$\|x\|_{C_r^*(G)}$ and hence

$$\left\| q_{B\otimes\mathbb{K}}\left(\sum_j c_j\gamma\left(g_j\right)\otimes\nu_0\left(g_j\right)\right)\right\| \le \|x\|_{C_r^*(G)} .$$

To handle the second term in (3.2.4) note that $M(B)\otimes\mathbb{K}/B\otimes\mathbb{K}\simeq Q(B)\otimes\mathbb{K}$ so

$$\left\| q_{B\otimes\mathbb{K}}\left(\sum_j c_j\gamma\left(g_j\right)\otimes\Delta\left(g_j\right)\right)\right\| = \left\|\sum_j c_j\left(\varphi'\oplus\varphi''\right)\left(g_j\right)\otimes\Delta\left(g_j\right)\right\|_{Q(B)\otimes\mathbb{K}} .$$

Since $\varphi'\oplus\varphi'' : C_r^*\left(G\right)\to Q(B)$ is injective (because φ' contains the unitally absorbing split extension π_1) and $\left(\varphi'\oplus\varphi''\right)\otimes\mathrm{id}_{\mathbb{K}}$ isometric,

$$\left\|\sum_j c_j\left(\varphi'\oplus\varphi''\right)\left(g_j\right)\otimes\Delta\left(g_j\right)\right\|_{Q(B)\otimes\mathbb{K}} = \left\|\sum_j c_j\lambda\left(g_j\right)\otimes\Delta\left(g_j\right)\right\|_{C_r^*(G)\otimes\mathbb{K}} .$$

And

$$\begin{aligned}\left\|\sum_j c_j\lambda\left(g_j\right)\otimes\Delta\left(g_j\right)\right\|_{C_r^*(G)\otimes\mathbb{K}} &= \\ \left\|\sum_j c_j\lambda\left(g_j\right)\otimes\nu_s\left(g_j\right) - \sum_j c_j\lambda\left(g_j\right)\otimes\nu_0\left(g_j\right)\right\| &\le 2\left\|x\right\|_{C_r^*(G)} ,\end{aligned}$$

by Fell's absorbtion principle or Lemma 3.2.4. Inserting these estimates into (3.2.4) yield the conclusion that

$$\left\|q_{B\otimes\mathbb{K}}\circ h_{\gamma\otimes\nu_s}(x)\right\| \le 3\left\|x\right\|_{C_r^*(G)} ,$$

proving that $q_{B\otimes\mathbb{K}}\circ h_{\gamma\otimes\nu_s}$ factors through $C_r^*(G)$ as claimed.

It remains to reduce the general case of a possibly non-unital extension to the case of a unital extension. Let $\varphi : C_r^*(G)\to Q(B)$ be an arbitrary extension. From Lemma 3.2.9 and K-amenability we get an invertible extension $\varphi' : C_r^*(G)\to Q(B)$ such that $[\varphi\circ\mu\oplus\varphi'\circ\mu] = 0$ in $\mathrm{Ext}^{-1}(C^*(G), B)$. In particular,

$$p = \left(\varphi\circ\mu\oplus\varphi'\circ\mu\right)(1)$$

is a projection which represents 0 in $K_0(Q(B))$. Since $K_0(M(B)) = 0$ we see $[1-p]+[p] = [1] = 0$ in $K_0(Q(B))$ so we find that also $p^\perp = 1-p$ represents 0 in $K_0(Q(B))$. Since $M_k(Q(B))\simeq Q(B)$ for all k this implies that

$$p\oplus 1 \ \sim\ 0\oplus 1 \text{ and } p^\perp\oplus 1 \ \sim\ 0\oplus 1$$

in $M_2(Q(B))$, where $\sim$ is Murray-von Neumann equivalence. It follows that

$$p\oplus 1\oplus 0 \ \sim\ 1\oplus 0\oplus 0 \text{ and } p^\perp\oplus 0\oplus 1 \ \sim\ 0\oplus 1\oplus 1$$

in $M_3(Q(B))$. So there is a unitary $w \in M_4(Q(B))$ contained in the connected component of the unit in the unitary group of $M_4(Q(B))$ such that

$$w\,(p \oplus 1 \oplus 0 \oplus 0)\,w^* = 1 \oplus 0 \oplus 0 \oplus 0.$$

Let $\chi : C_r^*(G) \to Q(B)$ be a unital split extension. It follows that

$$w\,((\varphi \circ \mu \oplus \varphi' \circ \mu) \oplus \chi \circ \mu \oplus 0 \oplus 0)\,w^* = \psi_0 \oplus 0 \oplus 0 \oplus 0 \tag{3.2.5}$$

for some unital extension $\psi_0 : C^*(G) \to Q(B)$. It follows from (3.2.5) that ψ_0 factors through $C_r^*(G)$, i.e. there is a unital extension $\psi : C_r^*(G) \to Q(B)$ such that $\psi_0 = \psi \circ \mu$. Via an isomorphism $M_4(Q(B)) \simeq M_2(Q(B))$ which leaves the upper lefthand corner unchanged, we see that there is an invertible extension $\varphi'' : C_r^*(G) \to Q(B)$ and a unitary u in the connected component of 1 such that

$$\operatorname{Ad} u \circ (\varphi \oplus \varphi'') = \psi \oplus 0$$

as $*$-homomorphisms $C_r^*(G) \to Q(B)$. Since ψ is unital the first part of the proof gives us an invertible (unital) extension $\psi' : C_r^*(G) \to Q(B)$ such that $\psi \oplus \psi'$ is strongly homotopic to a split extension. Since

$$\varphi \oplus \varphi'' \oplus \psi' = \operatorname{Ad}(u^* \oplus 1) \circ (\psi \oplus 0 \oplus \psi'),$$

we conclude that $\varphi \oplus \varphi'' \oplus \psi'$ is strongly homotopic to a split extension. Note that $\varphi'' \oplus \psi'$ is invertible. ☐

Let A be a separable C^*-algebra and B a stable σ-unital C^*-algebra. Following [MT1] we let $\operatorname{Ext}^{-1/2}(A, B)$ denote the group of unitary equivalence classes of semi-invertible extensions of A by B. There is then an obvious map

$$\operatorname{Ext}^{-1}(A, B) \to \operatorname{Ext}^{-1/2}(A, B)$$

which in [Th1] was shown to be an isomorphism when $B = \mathbb{K}$ and $A = C_r^*(\mathbb{F}_n)$. We can now extend this conclusion as follows.

Theorem 3.2.11. *Let $(G_i)_{i \in \mathbb{N}}$ be a countable collection of discrete countable amenable groups and let $G = \star_i G_i$ be their free product. Let B be a stable σ-unital C^*-algebra. It follows that $C_r^*(G)$ satisfies the UCT and that the natural map $\operatorname{Ext}^{-1}(C_r^*(G), B) \to \operatorname{Ext}^{-1/2}(C_r^*(G), B)$ is an isomorphism.*

Proof. It follows from Theorem 3.2.1 that the map $\operatorname{Ext}^{-1}(C_r^*(G), B) \to \operatorname{Ext}^{-1/2}(C_r^*(G), B)$ is surjective. To conclude that the map is also injective note that the six-term exact sequence of K-theory arising from an asymptotically split extension has trivial boundary maps and the resulting group extensions are split. Hence the injectivity of the map we consider will follow if we can show that $C_r^*(G)$ satisfies the UCT. Since G is K-amenable $C_r^*(G)$ is KK-equivalent to $C^*(G)$, cf. [C], so we may as well show that $C^*(G)$ satisfies the

UCT. We do this in the following three steps: Since the class of C^*-algebras which satisfies the UCT is closed under countable inductive limits we need only show that $C^*\left(\star_{i\leq n}G_i\right)$ satisfies the UCT. Next observe that it follows from [Th3] that an amalgamated free product $A\star_{\mathbb{C}} B$ of unital separable C^*-algebras A and B is KK-equivalent to the mapping cone of the inclusion $\mathbb{C} \subseteq A \oplus B$. Thus $A\star_{\mathbb{C}} B$ will satisfy the UCT when A and B do. Since

$$C^*\left(\star_{i\leq n}G_i\right) \simeq C^*\left(G_1\right)\star_{\mathbb{C}} C^*\left(G_2\right)\star_{\mathbb{C}}\cdots\star_{\mathbb{C}} C^*(G_n)$$

we can apply this observation $n-1$ times to conclude that $C^*\left(\star_{i\leq n}G_i\right)$ satisfies the UCT if each $C^*(G_i)$ does. And this follows from [Tu] because G_i is amenable. $\square$

Bibliography

[An] J. Anderson. *A C^*-algebra A for which* $\mathrm{Ext}(A)$ *is not a group*. Ann. of Math. **107**, 455-458. (1978).

[Ar] W. Arveson. *Notes on extensions of C^*-algebras*. Duke Math. J. **44**(2), 329-355. (1977).

[BR] L. W. Bagget and A. Ramsay. *A Functional Analytic Proof of a Selection Lemma.* Can. J. Math., 32(2), 441-448. (1980).

[BHV] B. Bekka, P. de la Harpe and A. Valette. *Kazhdan's Property (T).* Cambridge University Press. (2008).

[B] B. Blackadar. *Operator Algebras - Theory of C^*-Algebras and von Neumann Algebras*. Encyclopaedia of Mathematical Sciences 122. Springer. (2006).

[BK] B. Blackadar and E. Kirchberg. *Generalized inductive limits of finite-dimensional C^*-algebras*. Math. Ann. **307**(3), 343-380. (1997).

[BD] E. F. Blanchard and K. J. Dykema. *Embeddings of Reduced Free Products of Operator Algebras*. Pac. J. of Math. **199**, 1-19. (2001).

[BDF] L. G. Brown, R. G. Douglas and P. A. Fillmore. *Extensions of C^*-Algebras and K-Homology*. Ann. of Math. **105**(2), 265-324. (1977).

[BO] N. P. Brown and N. Ozawa. *C^*-Algebras and Finite-Dimensional Approximations.* Graduate Studies in Mathematics 88, AMS. (2008).

[CCJJV] P-A. Cherix, M. Cowling, P. Jolissaint, P. Julg and A. Valette, *Groups with the Haagerup Property*, Birkhäuser Verlag, (2001).

[CH] A. Connes and N. Higson, *Déformations, morphismes asymptotiques et K-théories bivariante*, C.R. Acad. Sci. Paris Sér I Math. **311**, 101-106. (1990).

[C] J. Cuntz. *K-theoretic amenability for discrete groups*. J. Reine u. Angew. Math. **344**, 180-195. (1983).

[DE] M. Dadarlat and S. Eilers. *Asymptotic Unitary Equivalence in KK-theory.* K-theory **23**, 305-322. (2001).

[Di] J. Dixmier. *C^*-algebras*. North-Holland Publishing Company. (1977).

[F] G. B. Folland. *A Course in Abstract Harmonic Analysis*. Studies in Advanced Mathematics. CRC Press. (1995).

[HT] U. Haagerup and S. Thorbjørnsen. *A new application of random matrices:* $\mathrm{Ext}(C_r^*(\mathbb{F}_2))$ *is not a group*. Ann. of Math. **162**, 711-775. (2005).

[HLSW] D. Hadwin, J. Li, J. Shen and L. Wang. *Reduced Free products of Unital AH algebras and Blackadar and Kirchberg's MF Algebras*. arXiv:0812.0189. (2008).

[HS] D. Hadwin and J. Shen. *Some Examples of Blackadar and Kirchberg's MF Algebras*. arXiv:0806.4712. (2008).

[JT] K. K. Jensen and K. Thomsen. *Elements of KK-theory*. Birkhäuser. (1991).

[LS] Q. Li and J. Shen. *On The MF Property of Reduced Amalgamated Free products of UHF Algebras*. arXiv:1005.4170. (2010).

[MT1] V. Manuilov and K. Thomsen. *The Connes-Higson construction is an isomorphism*. J. Func. Anal. **213**, 154-175. (2004).

[MT2] V. Manuilov and K. Thomsen. *E-theory is a special case of KK-theory*. Proc. London Math. Soc. **88**, 455-478. (2004).

[MT3] V. Manuilov and K. Thomsen. *On the lack of inverses to C^*-extensions related to property T groups*. Can. Math. Bull. **50**, 268-283. (2007).

[MT4] V. Manuilov and K. Thomsen. *The group of unital C^*-extensions* in *C^*-algebras and elliptic theory*, 151-156, Trends Math., Birkhäuser Verlag, Basel. (2006).

[MT5] V. Manuilov and K. Thomsen. *Semi-invertible extensions of C^*-algebras*. arXiv:1005.1999. (2010).

[Mu] G. J. Murphy. *C^*-algebras and Operator Theory*. Academic Press. 1990.

[Pe] G. K. Pedersen. *C^*-Algebras and their Automorphism Groups*. Academic Press. (1979).

[P] J-P. Pier. *Amenable Locally Compact Groups*. John Wiley & Sons. (1984).

[RLL] M. Rørdam, F. Larsen and N. J. Laustsen. *An Introduction to K-theory for C^*-algebras*. London Mathematical Society Student Texts 49. Cambridge University Press. 2000.

[Se] J. A. Seebach. *On reduced amalgamated free products of C^*-algebras and the MF-property*. arXiv:1004.3721. (2010). To appear in the Indiana University Mathematics Journal.

[ST] J. A. Seebach and K. Thomsen. *Extensions of the reduced group C^*-algebra of a free product of amenable groups*. Adv. Math. **223**(5), 1845-1854. (2010).

[Ser] J-P. Serre. *Trees*. Springer Verlag. (1980).

[S] G. Skandalis, *Kasparov's bivariant K-theory and applications*, Expo.Math. **9**, 193-250. (1991).

[Th1] K. Thomsen, *All extensions of $C_r^*(\mathbb{F}_n)$ are semi-invertible*, Math. Ann. **342**, 273-277. (2008).

[Th2] K. Thomsen, *On absorbing extensions*, Proc. Amer. Math. Soc. **129**, 1409-1417. (2001).

[Th3] K. Thomsen, *On the KK-theory and the E-theory of amalgamated free products of C^*-algebras*, J. Funct. Anal. **201**, 30-56. (2003).

[Th4] K. Thomsen. *Asymptotic Equivariant E-theory*. Lecture notes.

[Tu] J.L. Tu, *La conjecture de Baum-Connes pour les feuilletages moyennables*, K-theory **17**, 215-264. (1999).

[V] D. Voiculescu. *Operations on Certain Non-Commutative Operator-Valued Random Variables*. Asterisque **232**, 243-275. (1995).

[V1] D. Voiculescu. *Around quasidiagonal operators*. Integral Equations Operator Theory, **17**(1), 137-149. (1993).