On the Determinantal Complexity of the 2-hook-immanant

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July 2012

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He had bought a large map representing the sea,
Without the least vestige of land:
And the crew were much pleased when they found it to be
A map they could all understand.

- Lewis Carroll, *The Hunting of the Snark*

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Introduction

The determinant is a well-studied object and should be known to any student who has taken a course in basic linear algebra. The first encounter one has with the determinant is likely to be via the Laplace expansion formula,

$$\det_n(A) = \sum_{j=1}^{n} a_{1j}(-1)^{j+1} \det_{n-1}(A_{1j}),$$

where A_{1j} is obtained from A by removing the first row and the j'th column.

This might not strike many a freshman as an object of particular beauty, nor might the somewhat simpler formula

$$\det_n = \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \prod_{i=1}^{n} X_{i\sigma i}.$$

(Especially because not all freshmen would know what the symmetric group, S_n is.)

Nevertheless, the determinant is an object with quite a few extraordinary properties, which makes it interesting not only to mathematicians, who deal mainly with theory and abstraction, but also to e.g. theoretical computer scientists who (in theory) would like to do actual computations.

The permanent which appears strikingly similar to the determinant

$$\operatorname{per}_n = \sum_{\sigma \in S_n} \prod_{i=1}^{n} X_{i\sigma i}$$

is another polynomial which one should know from combinatorics if one has studied such subjects.

However, the permanent unlike the determinant seems very hard to compute because generally $\operatorname{per}_n(AB) \neq \operatorname{per}_n(A) \operatorname{per}_n(B)$, and so when computing the permanent we cannot use Gaussian elimination which is possible when computing the determinant.

Polya, [27], noted that

$$\mathrm{per}_2 = \det_2 \begin{pmatrix} X_{11} & X_{12} \\ -X_{21} & X_{22} \end{pmatrix}$$

and asked the question,

> *Given $n \geq 3$, is it possible to express the permanent,* $\mathrm{per}_n(\{X_{ij}\})$*, as* $\det_n(\{a_{ij}X_{ij}\})$*?, where $a_{ij} \in \{1, -1\}$?*

Szegö, [32], found the answer to be no, it is never possible when $n \geq 3$. Later Marcus and Minc, [24], found that for $n \geq 3$ there are no endomorphism of $\mathrm{Mat}_n(\mathbb{C})$, L, such that $\mathrm{per}_n = \det_n \circ L$.

In 1979 Leslie Valiant, [33], however, found that given any polynomial, f, (over any field) it is possible to find an affine map, L, such that $f = \det_m \circ L$ for some suitable $m \in \mathbb{N}$. Now, if we take $f = \mathrm{per}_n$ we can ask

> *What is the smallest integer, m, such that* $\mathrm{per}_n = \det_m \circ L$*, for some affine map, L?*

The problem is still not solved. Several lower bounds have been found over the years. Marcus and Minc's result of course gives us $m \geq n+1$, this was improved by von zur Gathen, [16], to $m \geq \sqrt{8/7}n$, and again by Cai, [9], to $m \geq \sqrt{2}n$. The best known lower bound was found Mignon and Ressayre, [25], and is quadratic, $m \geq n^2/2$. The smallest such m is called the determinantal complexity of the permanent, and it is conjectured that it grows faster than any polynomial in n.

Summary

Chapter 1

The first chapter is primarily a discussion of the subject of dual varieties. These are often useful when one wishes to give a lower bound of the determinantal complexity of a polynomial, as the dimension of the dual variety of the determinant is relatively low. (This is one of the intriguing features of this fascinating polynomial.)

An explicit computation of the dimension of the dual of the variety defined by the determinant is presented, along with a sketch of how one computes the dimension of the dual of the variety defined by the permanent.

Finally Katz' dimension formula for hypersurfaces is presented together with a discussion on how this can be helpful in finding lower bounds of determinantal complexities of general polynomials.

Chapter 2

In the second chapter several issues from theoretical computer science are discussed. In particular the complexity classes P and NP are presented, and their algebraic analogues VP and VNP are also defined.

A proof of Valiant's theorem which states that any polynomial is expressible as a determinant or a permanent of a matrix with entries that are constants or variables, is given, and several notions of determinantal complexities are defined.

Finally the polynomials known as immanants, of which the determinant and permanent are special cases, are presented along with a discussion on why these might be interesting in the research of the determinantal complexity of the permanent.

Chapter 3

In the third, and final chapter the main result, a quadratic lower bound of one particular immanant, is given, along with different ways of computing said immanant.

Appendix A

In the appendix there are some lines of code which can be used in Macaulay2 to compute the rank of the Hessian of the immanant which is the subject of chapter 3.

Acknowledgements

I would like to thank my advisor Niels Lauritzen not only for his help and support during my time as a Ph.D.-student, but also for the lectures he has given when I was just a lowly student. Thank you also for suggesting the subject of geometric complexity theory for my project, it has been quite interesting.

I would also like to express my gratitude to Nicolas Ressayre who was my host for half a year when I visited Département de Mathématiques at Université Montpellier II, and who presented me to several problems. Particularly I am grateful for having my attention turned to the subject of immanants.

A warm thank you is also due to my fellow (Ph.D.-)students who has made my everyday-life enjoyable, and who has a major part in creating a pleasant environment. Without you neither lunch nor Friday-bar would be any fun at all.

Finally I would like to thank everyone from the staff (scientific as well as administrative) at the Department of Mathematics at Aarhus University who has ever helped me or taught me something valuable in one way or another.

Dansk Resumé

For cirka 100 år siden fandt Polya ud af, at 2×2-permanenten kunne konstrueres ud fra determinanten af samme størrelse ved et simpelt fortegnsskift. Han spurgte om dette kunne lade sig gøre for mere generelle $n \times n$-permanenter. Szegö fandt ud af at svaret er nej; det er umuligt for alle $n \geq 3$.

Ikke desto mindre er det muligt at konstruere $n \times n$-permanenten som determinanten af en matrix med indgange, der enten er konstanter eller variable. Den determinantale kompleksitet af ($n \times n$-)permanenten er det mindste tal for hvilket der findes en matrix med affine indgange, således at determinanten af denne matrix er permanenten. Den determinantale kompleksitet af permanenten formodes at vokse hurtigere end noget polynomium i n. Den hidtil bedste nedre grænse for den determinantale kompleksitet af permanenten vokser kvadratisk.

I denne afhandling gennemgåes forskellige aspekter af determinanten og permanenten samt det mere generelle begrebe immananten. Specielt vises en kvadratisk nedre grænse af den immanant, der i en vis forstand ligger tættest på determinanten.

Chapter 1

Dual Varieties

In this chapter we shall give a brief introduction to the topic from algebraic geometry known as dual varieties. Given a hypersurface $X = V(f) \subseteq \mathbb{P}^n(\mathbb{C})$ the dimension of the dual variety, X^*, can be essential in giving a lower bound for the determinantal complexity of the polynomial f. (We shall define what is meant by determinantal complexity in chapter 2.) Thus, we will present Katz' dimension formula, which links the dimension of dual varieties to the rank of Hessian matrices. Furthermore, we shall investigate the dimension of the dual variety of the variety defined by the determinant.

We start at a very basic level by defining what we consider to be an algebraic variety - we follow the book [20].

The results of remaining parts of this chapter are taken from chapter 1 in the book [17] unless anything else is stated.

1.1 Basic Definitions and the Biduality Theorem

Algebraic Varieties

Let X be a topological space, and K an algebraically closed field. We shall primarily consider the field of complex numbers but in these opening definitions we will allow more arbitrary fields.

We define a space with functions in the following way:

Definition 1.1.1. A space with functions is a topological space, X, such that for each open subspace, $U \subseteq X$, we have a K-algebra of regular functions from U to K which we denote $K[U]$. These regular functions must satisfy the following two conditions

1. $f \in K[U]$ if and only if $f|_{U_a} \in K[U_a]$ for all $a \in J$, where $\bigcup_{a \in J} U_a = U$ is an open cover of U.

2. The set $D(f) = \{x \in U | f(x) \neq 0\}$ is open in U and $1/f$ is regular on $D(f)$ whenever f is regular on U

Example 1.1.2. One such space with functions could be any topological space X with $K[U]$ the continuous functions from $U \subseteq X$ to K (where in the topology on K points are closed, and the inversion function $i : K^* \to K$ is continuous).

Another example could be a closed, affine, algebraic set, i.e.,

$$X = V(I) = \{x \in \mathbb{A}^n | f(x) = 0 \text{ for all } f \in I\},$$

where $I \subseteq R = K[T_1, \ldots, T_n]$ is a radical ideal, here $K[U]$ is the set of rational functions, i.e., if $f : U \to K$ is rational then $f(x) = P(x)/Q(x)$ for some polynomials $P, Q \in R$. In particular $K[X] = R/I$.

In order to define what is meant by algebraic varieties we need to first define what affine varieties are, this requires the notion of morphisms:

Definition 1.1.3. A morphism between two spaces with functions, $\varphi : Y \to X$, is a continuous map that respect regularity, i.e., if U is an open subset of X, we get a K-algebra-homomorphism, $\varphi^* : K[U] \to K[\varphi^{-1}(U)]$, defined by $\varphi^*(f) = f \circ \varphi$. If a morphism is a bijection and the inverse is also an morphism then it is called an isomorphism.

This definition makes us consider the map

$$*_{Y,X} : \mathrm{Mor}(Y, X) \to \mathrm{Hom}_K(K[X], K[Y]),$$

which we use to define affine varieties:

Definition 1.1.4. If X is a space with functions such that $K[X]$ is finitely generated as a K-algebra, and for all Y the map $*_{Y,X}$ is a bijection, then we call X an affine variety.

The definition above is reasonable in the light of this proposition, though we shall omit the proof.

Proposition 1.1.5. *Any affine variety is isomorphic to $X = V(I)$ for some radical ideal $I \subseteq K[T_1, \ldots, T_n]$.*

We are now ready to define the more general notion of algebraic varieties.

Definition 1.1.6. An algebraic variety is a space with functions, X, such that there is some finite open covering, $X = \bigcup_{i=1}^{r} U_i$, where each U_i is an affine variety.

If $I \subseteq K[X_0, \ldots, X_n]$ is a homogeneous ideal, i.e., if I is generated by homogeneous polynomials, then we get a closed, projective set,

$$V(I) = \{[a] = [a_0, \ldots, a_n] \in \mathbb{P}^n | f(a) = 0 \text{ for all } f \in I\} \subseteq \mathbb{P}^n.$$

One can show that the set $U_i = V(I) \cap \{[a] \in \mathbb{P}^n | [a] = [a_0, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n]\}$ is an affine variety, and as $U_0 \cup \cdots \cup U_n$ is an open covering of $V(I)$ we have

Proposition 1.1.7. *Closed, projective sets are algebraic varieties.*

We now turn our attention towards dual varieties over the complex numbers.

Dual Varieties

If we consider a line $l = \mathbb{C} \cdot v \subseteq \mathbb{C}^{n+1}$ we have that the orthogonal complement, $l^{\perp} = \ker v^* \subseteq \mathbb{C}^{n+1}$, is a hyperplane of which we can take the projectivisation. This we also call a (projective) hyperplane.

Conversely given some projective hyperplane, $H \subseteq \mathbb{P}^n$, we shall consider H as a point in the projectivisation of $(\mathbb{C}^{n+1})^*$, namely as $[v^*] \in \mathbb{P}^{n*}$, where $\mathbb{C} \cdot v \subseteq \mathbb{C}^{n+1}$ is the line orthogonal to the linear version of H. Thus, the set of projective hyperplanes form a projective space which it is natural to think of as $\mathbb{P}^{n*}$.

Now, given some variety, $X = V(I)$, we may consider some point in the smooth locus, $x \in X_{sm}$, then we say that a hyperplane, H, containing x is tangent to X at x if $T_x X \subseteq T_x H$.

We denote by W_0 the set

$$W_0 = \{(x, H) \in \mathbb{P}^n \times \mathbb{P}^{n*} | x \in X_{sm}, \text{ and } H \text{ is tangent to } X \text{ at } x\},$$

and we may now define what the dual variety of X is.

Definition 1.1.8. Let $W = \overline{W_0}$. The dual variety of X, denoted X^*, is the image of W under the second projection, $\mathbb{P}^n \times \mathbb{P}^{n*} \ni (x, H) \overset{\pi_2}{\mapsto} H \in \mathbb{P}^{n*}$, i.e.,

$$X^* = \pi_2(W).$$

Note that if $X = V(f)$ is a hypersurface we have a well-defined map

$$\begin{aligned} \delta : X_{sm} &\to X^* \\ [y] &\mapsto [\nabla f(y)], \end{aligned}$$

and X^* is the closure of $\delta(X)$. We call δ the dual map.

The term *dual* variety is justified by the following theorem which we shall not prove. The proof is based on a clever use of Lagrangian subvarieties and can be found in its entirety in [17].

Theorem 1.1.9 (Biduality Theorem). $(X^*)^* = X$ *for any variety* $X \subseteq \mathbb{P}^n$

Whereas the proof is beyond our scope we shall spend a few words on the theorem's implications on varieties that are not hypersurfaces.

If there is an open subset of an irreducible projective variety, $U \subseteq X \subseteq \mathbb{P}^n$, such that U is the union of projective subspaces of dimension r, i.e., spaces that are projectivisations of $(r+1)$-dimensional linear subspaces, $W \subseteq \mathbb{C}^{n+1}$, then we say that X is ruled in projective spaces of dimension r.

Let us consider an irreducible variety, $X \subseteq \mathbb{P}^n$, of codimension $r+1$ for some $r \geq 0$, and take a smooth point $x \in X_{sm}$, furthermore, consider the set of projective hyperplanes tangent to X at x. Each such hyperplane, H, satisfy $T_xX \subseteq T_xH \subseteq V$ for some vector space V of dimension n.

Conversely, any hyperplane, $W \subseteq V$, containing T_xX corresponds to a projective hyperplane tangent to X at x. Thus, if we consider the space $L_x = \mathbb{P}\left((V/T_xX)^*\right)$ this will be a subset of X^*. The dimension of L_x is

$$\dim L_x = \dim V - \dim T_xX - 1 = n - (n - \operatorname{codim} X) - 1 = r,$$

and because the sets in $\{L_x | x \in X_{sm}\}$ cover an open subset of X^* this dual variety is ruled in projective spaces of dimension r. By the biduality theorem we can interchange the roles of X and X^*, and conclude

Corollary 1.1.10. *An irreducible variety, X, is ruled in projective spaces of dimension r if* $\operatorname{codim} X^* = r+1$.

The corollary tells us that the dual variety of a typical irreducible variety will be a hypersurface, since most varieties are not ruled in projective spaces of dimension r, for $r > 0$.

It is also a fact, which we shall not prove, that if X is irreducible, then X^* is irreducible as well (see [17, proposition 1.3]).

Now, if we have a variety which is not irreducible, $X = \bigcup_i X_i$, where each X_i is irreducible, and $X_i \nsubseteq X_j$ for $i \neq j$, we get $\left(\bigcup_i X_i\right)^* = \bigcup_i X_i^*$, and since we in general expect each X_i^* to be a hypersurface we should also expect this of X^*. Hence, generic reduced varieties have duals that are hypersurfaces.

Trivial counterexamples to this observation could be varieties $V(F) \subseteq \mathbb{P}^n$ for some $F \in \mathbb{C}[X_0, \ldots, X_k] \subseteq \mathbb{C}[X_0, \ldots, X_n]$ where $k < n$. Indeed if $a = [a_0, \ldots, a_n]$ is a zero of F then $a' = [a_0, \dots, a_k, b_{k+1}, \dots, b_n]$ is a zero of F too, and so we have that each point (with $a_i \neq 0$ for some $0 \leq i \leq k$) lie in a $(n-k)$-dimensional projective subspace of $\mathbb{P}^n$ which is also a subset of $V(F)$.

Projective transformations of such polynomials, i.e., $V\left(F \circ A^{-1}\right)$ for some $A \in \mathrm{GL}_{n+1}(\mathbb{C})$, and F as above, are naturally also ruled in projective spaces of positive dimension. This of course includes projective hyperplanes, as these are transformations of $V(X_0)$.

More generally projective subspaces of dimension k are obviously ruled in projective spaces of dimension k.

A non-trivial counterexample is the variety defined by the determinant, $V(\det_n) \subseteq \mathbb{P}^{n^2-1}$, which we shall investigate shortly.

It makes sense to define the dual of a variety in positive characteristics as well as in characteristic 0, however, the biduality theorem is not valid as seen in this example:

Example 1.1.11. Consider the polynomial $F = X_0^{p-1}X_2 - X_1^p \in \mathbb{F}_p[X_0, X_1, X_2]$, and the variety $\Gamma = V(F) \subseteq \mathbb{P}\left(K^3\right)$, i.e., the graph of the Frobenius map, $K \ni x \mapsto x^p \in K$, of some field $K \supseteq \mathbb{F}_p$. The hyperplanes (i.e. the lines) tangent to Γ can be found geometrically using the dual map, i.e., the gradient, $\nabla F = \left(-X_2X_0^{p-2}, 0, X_0^{p-1}\right)$, and we see that the dual variety, Γ^*, is the line $V\left(X_1^*\right)$.

This gives that $\left(\Gamma^*\right)^*$ is a point and, hence, not equal to Γ, showing that the biduality theorem is not true in this case.

Thus, as always one has to be careful when making a transition from the complex numbers to some field of positive characteristic.

1.2 The Determinant and Katz' Dimension Formula

We now turn to the determinant-variety as promised earlier. We shall not explicitly distinguish between matrices as points in $\mathrm{Mat}_n(\mathbb{C})$ and as points in $\mathbb{P}^{n^2-1}$, hopefully it will be clear from the context what is meant. Also, we shall identify $\left(\mathbb{P}^{n^2-1}\right)^*$ with $\mathbb{P}^{n^2-1}$ by sending a basis of $\mathrm{Mat}_n(\mathbb{C})^*$ to the dual basis in $\mathrm{Mat}_n(\mathbb{C})$.

Proposition 1.2.1. *Let $D = V(\det_n) \subseteq \mathbb{P}^{n^2-1}$ then D^* is isomorphic to the Segre product $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$, thus, D^* is of dimension $2n-2$.*

Proof. We shall as in example 1.1.11 use the dual map, and, thus, the gradient of the determinant. The gradient can be found by taking $(n-1)$-minors and multiplying with $(-1)^{i+j}$, when the minor is the determinant of the matrix obtained by removing the i'th row and the j'th column.

Note that if $A \in D$, then A is smooth if and only if $\nabla \det_n(A) \neq 0$ which is equivalent to $\operatorname{rank} A = n-1$. Recall from basic linear algebra that if we write $\nabla \det_n(A)$ as a square matrix in the natural way, then $\nabla \det_n(A)$ is the transpose of the adjoint matrix of A, $\operatorname{adj}(A)$. We have the identities

$$\operatorname{adj}(AB) = \operatorname{adj}(B)\operatorname{adj}(A), \text{ and } \operatorname{adj}(A)A = A\operatorname{adj}(A) = \det_n(A)I_n$$

for general matrices $A \in \mathrm{Mat}_n(\mathbb{C})$. Notice in particular that if $\det_n(A) = 1$ then $\operatorname{adj}(A) = A^{-1}$, and if $\det_n(A) = 0$ then $A\operatorname{adj}(A) = 0$.

Thus, returning to the case where $A \in D_{sm}$ we have that $\delta(A) = \operatorname{adj}(A)^t$ is a matrix of rank 1, since $A\operatorname{adj}(A)$ regarded as a linear transformation is the zero-map, hence, the (non-zero) image of $\operatorname{adj}(A)$ is contained in the 1-dimensional kernel of A.

On the other hand we have that any matrix of rank one is the adjoint of a matrix of rank $n-1$.

Indeed, consider a matrix, N, of rank 1, then we can find invertible matrices $G_1, G_2 \in \mathrm{SL}_n(\mathbb{C})$, and $0 \neq a \in \mathbb{C}$ (actually we may choose G_1 and G_2 to get $a = 1$ but this is unimportant) such that

$$G_1 N G_2 = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix} =: N'.$$

Now N' is the adjoint of the diagonal matrix, Δ, with diagonal $(0, a, 1, \dots, 1)$, and so we get that

$$N = G_1^{-1} N' G_2^{-1} = \operatorname{adj}(G_1)\operatorname{adj}(\Delta)\operatorname{adj}(G_2) = \operatorname{adj}(G_2 \Delta G_1).$$

Clearly $A = G_2 \Delta G_1$ is of corank one, thus, $A \in D_{sm}$.

Thus, we conclude that D^* contains the projectivisation of the set of matrices of rank 1, which we may identify as

$$\delta(D_{sm}) = V\left(\{X_{ii}X_{jj} - X_{ij}X_{ji} | 1 \leq i < j \leq n\}\right).$$

This shows that the dense subset $\delta(D_{sm})$ of D^* is also closed, hence, it is equal to D^*.

Any matrix of rank 1 can be constructed as the product vw^t, for non-zero column-vectors $v, w \in \mathbb{C}^n$. The vectors v and w are not unique, though; take some non-zero scalar, λ, then we may replace v by λv if we at the same time replace w by $\lambda^{-1}w$. Moving to projective space solves this problem, and if we consider the Segre-embedding

$$s_{k,m} : \mathbb{P}^{k-1} \times \mathbb{P}^{m-1} \hookrightarrow \mathbb{P}^{km-1}$$
$$([a_1, \dots, a_k], [b_1, \dots, b_m]) \mapsto \begin{bmatrix} a_1b_1 & \dots & a_1b_m \\ \vdots & \ddots & \vdots \\ a_kb_1 & \dots & a_kb_m \end{bmatrix},$$

with $k = m = n$ we get that the image of $s_{n,n}$ is exactly D^*. This concludes the proof. □

Proposition 1.2.1 shows that the hypersurface defined by the determinant is ruled in projective spaces of high dimension and in this way distinguishes the determinant geometrically from a generic polynomial.

The dual variety of a generic hypersurface might not be as easy to determine as the one of the determinant-variety. However, we may find the dimension of the dual variety using Katz' dimension formula. We shall only present a version of the formula which gives the dimension of dual varieties of hypersurfaces, though it can be generalised to varieties of any dimension.

Theorem 1.2.2 (Katz' Dimension Formula for Hypersurfaces). *Let $X = V(f) \subseteq \mathbb{P}^n$ be irreducible, and let r be the rank of the Hessian matrix*

$$\operatorname{Hes} f = \left(\frac{\partial^2 f}{\partial X_i \partial X_j} \right)_{0 \le i,j \le n}$$

evaluated at generic $x \in X$. The dimension of X^* is equal to $r - 2$.

Proof. We will consider the affine variety $Y = V(f) \subseteq \mathbb{A}^{n+1}$ rather than the projective version. By Y^* we shall denote the affine cone over X^*.

As in the projective case we have a dual map, which we once again may use $Y_{sm} \ni y \overset{\delta}{\mapsto} \nabla f(y) \in Y^*$. We recognise the Hessian of f as the Jacobian matrix of δ, $d_y\delta$, which is a map of tangent spaces

$$d_y\delta : T_yY \to T_{\delta(y)}Y^*.$$

Because this map is surjective for y in a dense subset of Y_{sm} (see, e.g, [18, Proposition 14.4]) we have $\dim Y^* = \dim d_y\delta(T_yY)$ for generic y. As T_yY is a hypersurface, i.e., it has codimension 1, $\dim Y^* = \dim d_y\delta(T_yY) = \operatorname{rank} d_y\delta - 1$, and so for the projective version we get

$$\dim X^* = \dim Y^* - 1 = r - 2,$$

showing the formula is true. □

In [25] Thierry Mignon and Nicolas Ressayre found a point $E_n \in \mathrm{Mat}_n(\mathbb{C})$ such that the permanent

$$\mathrm{per}_n = \sum_{\sigma \in S_n} \prod_{i=1}^{n} X_{i\sigma i},$$

vanishes at E_n, and the Hessian of the permanent evaluated at E_n is an invertible matrix. Hence, the variety defined by the permanent has a dual variety which is a hypersurface.

The point they considered is

$$E_n = \begin{pmatrix} 1-n & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \in V(\mathrm{per}_n).$$

Indeed, an easy computation gives that

$$\mathrm{Hes}\,\mathrm{per}_n(E_n) = \begin{pmatrix} 0 & A_n & A_n & \dots & A_n \\ A_n & 0 & B_n & \dots & B_n \\ A_n & B_n & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & B_n \\ A_n & B_n & \dots & B_n & 0 \end{pmatrix},$$

where $A_n = (n-2)!(J_n - I_n)$ (here J_n is the matrix with all entries equal to 1), and

$$B_n = (n-3)! \begin{pmatrix} 0 & n-2 & n-2 & \dots & n-2 \\ n-2 & 0 & -2 & \dots & -2 \\ n-2 & -2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & -2 \\ n-2 & -2 & \dots & -2 & 0 \end{pmatrix}.$$

If we combine this with the fact ([25, Lemma 3.7]) that (block-)matrices of the form

$$\begin{pmatrix} 0 & C & C & \dots & C \\ C & 0 & D & \dots & D \\ C & D & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & D \\ C & D & \dots & D & 0 \end{pmatrix}$$

are invertible whenever C and D are invertible matrices (or non-zero complex numbers), we get that $\operatorname{Hes}\operatorname{per}_n(E_n)$ is invertible. Hence, in the light of Katz' dimension formula Mignon and Ressayre proved the following

Proposition 1.2.3. $\dim V(\operatorname{per}_n)^* = n^2 - 2$, *i.e., the dual of the permanent-variety is a hypersurface in* $\left(\mathbb{P}^{n^2-1}\right)^*$.

We shall just present one last result concerning dual varieties before turning our attention elsewhere.

Let us consider an affine map, L_1,

$$\begin{aligned} L_1 : \mathbb{A}^{n_1} &\to \mathbb{A}^m \\ x &\mapsto Ax + b \end{aligned}$$

with $n_1 < m$, and a homogeneous polynomial, $d \in \mathbb{C}[X_1, \dots, X_m]$, and use these to construct a new polynomial, $p_1 \in \mathbb{C}[X_1, \dots, X_{n_1}]$, by composing the maps d and L_1;

$$p_1 = d \circ L_1.$$

Denote by p the homogenization of p_1, i.e.,

$$p(X_0, X_1, \dots, X_n) = X_0^{-e} d\left(A(X_1, \dots, X_{n_1})^t + bX_0\right)$$

where $e = \deg d - \deg p_1$ (if p_1 itself is already homogeneous we also put $p = p_1$). Hence, we have $p = d \circ L$ for some affine, possibly linear, map, $L : \mathbb{A}^n \to \mathbb{A}^m$ for $n = n_1 + 1 \leq m$. We now get

Proposition 1.2.4. *With notation as above, consider the projective varieties* $X = V(p) \subseteq \mathbb{P}^{n-1}$ *and* $Y = V(d) \subseteq \mathbb{P}^{m-1}$. *We have*

$$\dim X^* \leq \dim Y^*.$$

Proof. Let us define the following sets analogous to definition 1.1.8

$$W_0^X = \{(x, \nabla p(x)) | x \in X_{sm}\} \text{ and } W_0^Y = \{(y, \nabla d(y)) | y \in Y_{sm}\},$$

thus, (for $Z = X, Y$) Z^* is the second projection of the closure of W_0^Z.

Consider the map

$$\begin{aligned} \psi : W_0^X &\to W_0^Y \\ (x, \nabla p(x)) &\mapsto (Lx, \nabla d(Lx)), \end{aligned}$$

which is a well-defined morphism as $0 = p(x) = d(Lx)$, and the chain rule gives

$$\nabla p(x) = \nabla d(Lx) A,$$

hence, Lx is a smooth point in Y when x is a smooth point in X.

Now, assume that L is injective so that we may find a map $\Lambda : \mathbb{A}^m \to \mathbb{A}^n$ such that Λ restricted to $L(\mathbb{A}^n)$, is the inverse of L. This gives us a new morphism

$$\begin{aligned} \varphi : \psi(W_0^X) &\to W_0^X \\ (y, \nabla d(y)) &\mapsto (\Lambda y, \nabla p(\Lambda y)). \end{aligned}$$

Clearly ψ and φ are the inverses of the others, thus, $W^X = \overline{W_0^X}$ is (isomorphic to) a subvariety of $W^Y = \overline{W_0^Y}$, using the second projection gives $\dim X^* \leq \dim Y^*$.

If the affine map L is not injective but of rank $k < n$ we may find a matrix, $C \in \mathrm{GL}_n(\mathbb{C})$, such that $B = AC$ is a matrix in which the first k columns are non-zero and the last $n - k$ columns are zero, i.e., $B = \left(\begin{smallmatrix} B' & 0 \end{smallmatrix} \right)$ for some $m \times k$-matrix B' of rank k. Now consider the injective map

$$\begin{aligned} L' : \mathbb{A}^k &\to \mathbb{A}^m \\ x &\mapsto B'x + b, \end{aligned}$$

and the polynomial $p' = d \circ L'$: If we embed $\mathbb{A}^k$ into $\mathbb{A}^n$ then we see that p' is the restriction of $p \circ C$ to $\mathbb{A}^k$, hence, p' is homogeneous and we get in the same fashion as above that $\dim V(p')^* \leq \dim Y^*$.

Furthermore, X and $V(p \circ C)$ are isomorphic, thus, the same is true for X^* and $V(p \circ C)^*$. Note that $p \circ C \in \mathbb{C}[X_1, \ldots, X_k] \subseteq \mathbb{C}[X_1, \ldots, X_n]$, and so $V(p \circ C)$ is ruled in projective spaces of dimension $n - k$, meaning that when we consider the dual of $V(p \circ C)$ we might as well consider the dual of $V(p')$ - the only difference between the two dual varieties is that one lies in a $(n - 1)$-dimensional projective space whereas the other lies in a space of dimension $k-1$. In particular we have $\dim V(p \circ C)^* = \dim V(p')^*$, which gives

$$\dim X^* = \dim V(p \circ C)^* = \dim V(p')^* \leq \dim Y^*$$

which concludes the proof. □

Remark 1.2.5. The proposition in particular tells us that if we construct a homogeneous polynomial using the determinant and some affine map, $L : \mathbb{A}^n \to \mathbb{A}^{m^2}$, in the following way,

$$p = \det_m \circ L,$$

then $\dim V(p)^* \leq \dim V(\det_m)^* = 2m-2$, and so the size of the determinant, m, is required to be at least $\frac{\dim V(p)^*+2}{2}$.

We shall see later, that any polynomial can be constructed in this way, and so for generic polynomials, i.e., polynomials which define hypersurfaces whose dual varieties are hypersurfaces themselves, we then have that m is at least half of the number of variables used.

Chapter 2

The Determinant and the Permanent

In this chapter we shall, as the title indicates, investigate the two (families of) polynomials called the determinant and the permanent. To a mathematician the former is well-known and -studied, whereas, the latter is less likely to be known at all unless one has studied combinatorics, graph theory or complexity theory. This fact is the motivation to begin the chapter with an introduction to basic complexity theory by a presentation of some complexity classes which are important in theoretical computer science. In particular we will focus on the famous P versus NP-problem. Furthermore, we shall investigate Valiant's algebraic analogues VP and VNP which links complexity theory to polynomials. We shall also present Valiant's result about universality of the determinant and permanent; any polynomial can be realised as a determinant or permanent of a certain matrix.

[12], [33], [34], and in particular [8] are major references in these opening sections.

Once said basic notions are unravelled we will turn our attention to the determinantal complexity of polynomials. As the determinant is easily evaluated using Gaussian elimination, and any polynomial, p, can be realised as a determinant of some matrix, it might be feasible to use this determinant-construction to evaluate p if the matrix is not too large.

We shall also take a glance at Molmuley and Sohoni's notion of bounded determinantal complexity.

Finally we will investigate families of polynomials called immanants, which is a generalization of the determinant- and permanent-families.

2.1 Complexity Theory

P and NP

In complexity theory it is essential to have some notion of the size of a problem. E.g., if a problem is concerned with a graph, the number of vertices and edges could be one measure of the size of the graph. If a problem has something to do with matrices then size of the problem is likely to be related to the dimension of our space of matrices.

The P versus NP-problem asks the question if the two complexity classes P and NP are in fact equal. A very loose description (a precise one will follow briefly) of the class P could be that P is the set of problems that can be answered yes or no, and where an answer can be computed using a number of calculations that grows at most polynomially with the size of the problem.

NP on the other hand is the set of problems that can be answered yes or no, and if one has found a positive answer (a 'yes') to the problem, then it is possible to verify this answer in polynomial time.

It might not be obvious from these descriptions that P is a subset of NP, however, once we have seen the rigorous definitions this should be quite clear.

In order to define P and NP we first need to define what we mean by an alphabet, and by a deterministic Turing machine. The following definitions of Turing machines, P and NP etc. are taken from [12].

Definition 2.1.1. A finite alphabet is a finite non-empty set, Σ, we shall require that it contains at least two elements. By Σ^* we denote the set of strings over Σ, an element of Σ^* is called a word, and by a language, L over Σ we mean a set of words, i.e., a subset of Σ^*.

A deterministic Turing machine is a tuple, $(Q, \Gamma, \Sigma, \delta)$, where Q, Γ, and Σ are finite sets; the states, the symbols, and the input alphabet respectively. They must satisfy

1. Γ contains a blank element, denoted b.
2. $\Sigma \subseteq \Gamma \setminus \{b\}$.
3. Q must contain three special states; the initial state, q_0, the reject state, q_r, and the accept state, q_a. The latter two are called terminating states.

Finally

$$\delta : Q \setminus \{q_r, q_a\} \times \Gamma \to Q \times \Gamma \times \{-1, 1\}$$

is a function known as the transition function.

A Turing machine can be thought of as a formal version of an algorithm. The intuition in the definition of a Turing machine is to think of it as an infinite strip of squares each containing a symbol from Γ, furthermore, the strip is in some state from Q. The machine has a head which reads the symbols on one of the squares in the strip. Now, assume that the head is at a square containing the symbol s and the strip is in the state $q \neq q_r, q_a$, our transition function now determines what to do, let's say $\delta(q, s) = (q', s', n)$, then the head overwrites the s in the square changing it to s', and moves one to the left if $n = -1$ and one to the right if $n = 1$, the state of the strip is changed to q'.

Initially the strip has a word from Σ^* written on adjacent squares and the head is placed at the leftmost symbol in the word. All other squares have the blank symbol written on them, and the state of the strip is set to q_0.

The machine terminates if the strip gets into one of the terminating states. This might not happen in which case we say that the machine does not terminate.

The term *deterministic* is used because once the word is written on our strip, the actions of the head, and the changes of states are completely determined by the transition function.

One also has a notion of non-deterministic Turing machines, which resembles the deterministic version, but there is no transition function rather there is a transition relation

$$\delta \subseteq (Q \setminus \{q_r, q_a\} \times \Gamma) \times (Q \times \Gamma \times \{-1, 1\}).$$

We shall not trouble ourselves further with these non-deterministic Turing machines, though they can be used to define NP.

Example 2.1.2. Let us construct a Turing machine M_{w_0} which determines if a word contains a fixed substring, w_0, i.e., if we have the words $w = t_1 \ldots t_k, w_0 = u_1 \ldots u_l \in \{s_1, \ldots, s_n\}^*$ the machine shall determine if $w = uw_0v$ where $u, v \in \{s_1, \ldots, s_n\}^*$.

We need to specify the tuple $(Q, \Gamma, \Sigma, \delta)$; let

- $Q = \{q_0, q_r, q_a, q_1, \ldots, q_{l-1}\}$,
- $\Sigma = \{s_1, \ldots, s_n\}$,
- $\Gamma = \Sigma \cup \{b\}$, and

- the transition function is defined by

$$\delta(q_j, s) = \begin{cases} (q_0, s, 1) & \text{if } s \neq u_{j+1} \text{ and } s \neq u_1 \\ (q_1, s, 1) & \text{if } s \neq u_{j+1} \text{ and } s = u_1 \\ (q_{j+1}, s, 1) & \text{if } s = u_{j+1} \text{ and } j \neq l-1 \\ (q_a, s, 1) & \text{if } s = u_{j+1} \text{ and } j = l-1 \\ (q_r, s, 1) & \text{if } s = b. \end{cases}$$

What the machine does is very simple. It reads from the left to the right in the word w, and once it finds the first symbol of w_0 it remembers by changing the state to q_1. Once it has read the first $1 \leq k < l-1$ symbols of w_0 as consecutive symbols in w, it reaches the state q_k. If the next symbol the machine reads is u_{k+1} the state is changed to q_{k+1}, if the symbol is some other symbol from Σ the state is changed to q_0 (or q_1 if $s = u_1$), as we may start over from this new position. If the symbol is blank the machine has reached the end of w before without finding w_0 as a substring, and the machine terminates at the rejecting state. If the head at state q_{l-1} reads u_l, w_0 is a substring of w, and the machine terminates at the accepting state.

Since we always shift to the right, the transition function could just as well have changed the letters, s, it reads on the strip to b, or any other symbol from Γ, it doesn't matter because we never return to the same square twice it.

In order to compare different Turing machines we will define the worst case runtime (or just *time* for short) of a Turing machine.

In order to do so we set $\Sigma_n^* := \{w \in \Sigma^* | w = t_1 \ldots t_n\}$, for some alphabet $\Sigma = \{s_1, \ldots, s_r\}$.

Definition 2.1.3. If M is a Turing machine with input alphabet Σ we let $t_M(w) \in \mathbb{N} \cup \{\infty\}$ be the number of times the transition function is called when the Turing machine is applied during the evaluation of the word w. We define the (worst case run)time of M to be

$$T_M(n) = \max\{t_M(w) | w \in \Sigma_n^*\}.$$

We say that M runs in polynomial time if there are constants $k_1, k_2, k_3 \in \mathbb{N}$ such that $T_M(n) \leq k_1 n^{k_2}$ for all $n \geq k_3$. If this is the case we write $T_M(n) = O(n^{k_2})$

In example 2.1.2 the time of the Turing machine constructed is $T_{M_{w_0}}(k) = k+1$ when our input is $w = t_1 \ldots t_k$, and the machine runs in this time whenever it rejects a word. $k+1 \leq 2k$ for all $k \geq 1$ hence the machine runs in $O(k)$ which is polynomial time.

Let M denote some general Turing machine, we may define the accepted language, $L(M)$, as the set of words the machine accepts, i.e.,

$$L(M) = \{w \in \Sigma^* | M \text{ terminates at state } q_a \text{ when the input is } w\}.$$

In our example above we get $L(M_{w_0}) = \{uw_0v | u, v \in \Sigma^*\}$.

We may now define the class P.

Definition 2.1.4. The complexity class P is the set of languages, L, such that $L = L(M)$ for some Turing machine M which runs in polynomial time.

The definition of NP is a bit more cumbersome. Let Σ and Ω be two finite alphabets, then we call R a checking relation if $R \subseteq \Sigma^* \times \Omega^*$. Uniting the alphabets and adding a new symbol, s, we get a third alphabet $\Sigma \cup \Omega \cup \{s\}$. We define the language L_R as

$$L_R = \{wsv | (w, v) \in R\},$$

and say R is polynomial-time if L_R is in P.

Definition 2.1.5. The complexity class NP is the set of languages, L, over an alphabet, Σ, such that there is an alphabet, Ω, and a polynomial-time checking relation, $R \subseteq \Sigma^* \times \Omega^*$, along with an integer, k, satisfying that for all $w \in \Sigma^*$

$$w \in L \iff \exists v \in \Omega^* : |v| \leq |w|^k \text{ and } (w, v) \in R.$$

At this point it should be obvious that P is a subset of NP. Indeed, given some $L \in$ P over the alphabet Σ, we may now take the checking relation $R = \{(x, \lambda) | x \in L\} \subseteq \Sigma^* \times \Sigma^*$ where λ is the empty word. It should be clear that R is polynomial-time as any machine, M, such that $L = L(M)$ can be modified to M_s if we let s play the same role as the blank symbol, now, $L_R = L(M_s) \in$ P.

One of the Millennium Prize Problems stated by Clay Mathematics Institute in 2000 is to prove (or disprove) Cook's conjecture, P $\neq$ NP.

Example 2.1.6. Here are some examples of problems from P and NP

1. To determine if a $m \times n$-matrix has rank (at least) k is in P; using Gaussian elimination the rank can be computed in $O(m^2n)$ time.

2. Determining if a bipartite graph has a perfect matching is in P; a maximal flow algorithm can determine a maximal matching in time polynomial in the number of vertices and edges.

3. The subset sum problem is in NP, i.e., given a finite set of integers, S, determining the existence of a non-empty subset that adds to 0 is in NP; if one has a positive solution it can be verified using $O(|S|)$ additions.

Some problems can be used in order to solve others, i.e., there may exist a map, $\rho : \Sigma^* \to \Omega^*$, computable in polynomial time, and languages, $L_1 \in \Sigma^*$ and $L_2 \in \Omega^*$, such that $L_1 = \rho^{-1}(L_2)$. Clearly, if L_2 is in NP then we may verify a solution of any instance (word) in the problem (language) L_1 in polynomial time, hence, we say that L_1 can be reduced, or reduces to L_2.

Definition 2.1.7. A language, L, is said to be hard for the complexity class C if every language $L' \in C$ reduces to L.

A language, L, is C-complete if it is hard for C and lies in C.

It is a fact that there exist NP-complete problems. One such is the subset sum problem from example 2.1.6, however, it is far beyond our scope to prove this.

When we have a decision problem it sometimes makes sense to consider a related counting problem; instead of asking for existence of a subset that adds to zero in the subset sum problem, one might ask for the number of subsets which add to zero. It's not always a decision problem gives rise to a counting problem, e.g., the problem concerning the rank of a matrix in example 2.1.6 has no natural way to induce a counting problem. However, when one may lift a decision problem to a counting problem it should be clear that the counting problem is harder than the decision problem. Indeed, any solution to the counting problem gives the answer to the decision problem - if we count to 0 the decision problem should be answered no, else it should be answered yes.

#P is loosely defined to be the set of counting problems that are related to decision problems from NP. A more formal definition is

Definition 2.1.8. #P is the set of functions $\varphi : L \to \mathbb{N}$, such that

$$\varphi(x) = |\{y|(x, y) \in R\}|$$

for some language $L \subseteq \Sigma^*$, and some polynomial-time checking relation $R \subseteq \Sigma^* \times \Omega^*$.

A reduction from φ_1 to φ_2 in #P is a pair of functions, $\psi : \Sigma_1^* \to \Sigma_2^*$ and $f : \mathbb{N} \to \mathbb{N}$, computable in polynomial time, such that

$$\varphi_1 = f \circ \varphi_2 \circ \psi,$$

and we say that φ is #P-complete if any other function in #P reduces to φ.

Once again we have as a fact that there exist #P-complete problems (or functions) but it is beyond our scope to prove this. One example of such a problem is the counting problem related to the subset sum problem.

This is probably not very surprising (if we accept that there are #P-complete problems) seeing that the subset sum problem is NP-complete. What is surprising, however, is the following theorem given by Leslie Valiant in [34].

Theorem 2.1.9. *Computing the permanent of (0,1)-matrices, $A \in \mathrm{Mat}_n(\{0,1\})$, is # P-complete.*

The #P-completeness of the permanent is surprising because the related decision problem lies in P and is therefore perceived to be easy; the problem is the one of determining the existence of perfect matchings in bipartite graphs. It would be natural to expect that #P-complete problems should arise from NP-complete problems, or at least problems that aren't known to lie in P, because a problem which is #P-complete it is NP-hard.

Here is the way we count perfect matchings using the permanent: If we have a bipartite graph with vertex-set $X \cup Y$, such that all edges go between X and Y, we will use the vertices of X to index the rows of the adjacency matrix, A, and the vertices of Y to index the columns. (Usually the rows and columns of an adjacency matrix will be index by the entire vertex-set, this, however, is unnecessary when the graph is bipartite.) The entry a_{ij} is 1 if there is an edge between $x_i \in X$ and $y_j \in Y$ and 0 else. An obvious requirement for a perfect matching to exist is that $|X| = |Y|$, in which case a perfect matching can be thought of as a bijection from X to Y, or a permutation of $[n] := \{1, \ldots, n\}$ where $n = |X|$. If there is a perfect matching this must then correspond to some $\sigma \in S_n$ such that $a_{1\sigma 1} \cdots a_{n\sigma n} = 1$, and the total number of perfect matchings must then be

$$\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma i}$$

which we recognise as the permanent of the adjacency matrix.

At present time the best methods to compute the permanent of a matrix are based on Ryser's formula,

$$\mathrm{per}_n(X) = \sum_{S \subseteq [n]} (-1)^{n-|S|} \prod_{i=1}^{n} \sum_{j \in S} X_{ij}, \tag{2.1}$$

which involves $O(2^n n^2)$ additions and multiplications.

One proof of Ryser's formula is based on a version of the inclusion-exclusion formula;

$$\bigcup_{i=1}^{n+1} B_i = \sum_{k=1}^{n+1} (-1)^{k+1} \sum_{1 \le i_1 < \cdots < i_k \le n+1} B_{i_1} \cap \cdots \cap B_{i_k},$$

where on the right-hand side we have a 'sum of sets', i.e., a formal $\mathbb{Z}$-combination of the elements in the sets. A $\mathbb{Z}$-combination of elements is considered to be a regular set if all of the coefficients are 0 or 1, and vice versa. (E.g., if we add or subtract the sets $A = \{a, b\}$ and $B = \{a, c\}$ we get $A+B = 2a+b+c$ and $A-B = 0a+b-c = b-c$, whereas the union is of course $A \cup B = \{a, b, c\}$ which we will think of as $a + b + c$.)

For $i = 1, \ldots, n$ we define $B_i := \{f : [n] \to [n] | f^{-1}(\{i\}) = \emptyset\}$, let $B_{n+1} = S_n$ and note that

- $B := \bigcup_{i=1}^{n+1} B_i = \{f : [n] \to [n]\}$,
- $\bigcap_{i=1}^{n} B_i = \emptyset$, and
- $B_{n+1} \cap B_i = \emptyset$ for all $i < n + 1$.

We now rearrange the sum to get

$$\begin{aligned} S_n = B_{n+1} &= B + \sum_{k=1}^{n-1} (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le n} B_{i_1} \cap \cdots \cap B_{i_k} \\ &= \sum_{k=0}^{n-1} (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le n} B \cap B_{i_1} \cap \cdots \cap B_{i_k}, \end{aligned}$$

where we interpret the case when the index k is 0 to be just B. Counting like this we see that the permanent can be expressed as follows

$$\mathrm{per}_n = \sum_{k=0}^{n-1} (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le n} \sum_{f \in B \cap B_{i_1} \cap \cdots \cap B_{i_k}} X_f,$$

where $X_f = \prod_{i=1}^{n} X_{if(i)}$. However, the sum of monomials, X_f, for which $i_1, \ldots, i_k \notin f([n])$, can also be expressed

$$\sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{i=1}^{n} \sum_{j \ne i_1, \ldots, i_k} X_{ij},$$

and so

$$\mathrm{per}_n = \sum_{k=0}^{n-1}(-1)^k \sum_{1\leq i_1<\cdots<i_k\leq n} \prod_{i=1}^{n} \sum_{j\neq i_1,\ldots,i_k} X_{ij}.$$

(This is the compact version of how Ryser's formula is originally stated in [28, corollary 4.2].) If we instead focus on sets, S, and monomials, X_f, such that $f([n]) = S$, we see that the first summation can be interpreted as summing over sets of size $n-k$, the second summation specifies the elements of S, and the last summation is then to be substituted by $\sum_{j\in S} X_{ij}$, i.e., we get Ryser's formula as stated in (2.1).

That the computation of the permanent is #P-complete leads us to the next subject where we use polynomials to investigate complexity theory.

VP and VNP

We now turn to Valiant's algebraic analogue of P and NP, VP and VNP. Most definitions and results are taken from [8].

We choose to work over the complex numbers even though our ground-field isn't important, at least as long as the characteristic is not 2.

A family of polynomials, $(f_n)_{n\geq 1}$, is a sequence of polynomials, and by $v(f_n)$ we denote the number of variables needed to write down a formula for f_n. When it is convenient we will assume that $f \in \mathbb{C}[X_1, \ldots, X_{v(f)}]$, at other times we will rather consider multi-index variables, e.g., X_{ij}.

Definition 2.1.10. A p-family is a family of polynomials, $(f_n) = (f_n)_{n\geq 1}$, such that $\deg f_n$ and $v(f_n)$ are both polynomially bounded, i.e., there are constants, c_d and c_v, such that $\deg f_n = O(n^{c_d})$ and $v(f_n) = O(n^{c_v})$.

A family of polynomials, (f_n), is p-computable if $f_n(A)$ can be evaluated in polynomial (in n) time at any point $A \in \mathbb{C}^{v(f_n)}$.

Examples of p-families could be (f_n) with $f_n = X_1 \cdots X_n$, and (g_n) with $g_n = X_1 + \cdots + X_n$, the determinant, $\det = (\det_n)$, and the permanent, $\mathrm{per} = (\mathrm{per}_n)$. The first 3 are also p-computable (with regards to the determinant we may use Gaussian elimination), the permanent is not known to be p-computable as this would indicate P=NP.

X^{k^n}, where $k \in \mathbb{N}$ is a constant greater than 1, is not a p-family as the degree is not polynomially bounded, but it is p-computable by use of repeated squaring.

The polynomial

$$h_n = \prod_{\emptyset\neq T\subseteq[n]} \sum_{i\in T} X_i$$

which is related to the subset sum decision problem is of degree $2^n - 1$, hence (h_n) is not a p-family. It is not known to be p-computable either, if it was we would again have P=NP.

Definition 2.1.11. Let $F = (f_n)$ be a p-family.

1. F lies in VP if and only if F is p-computable.
2. F lies in VNP if and only if there exists a family $G = (g_n)$ in VP which satisfies $t(n) := v(g_n) - v(f_n) \geq 0$, and if $t(n) = 0$ then $g_n = f_n$, if $t(n) > 0$ then
$$f_n(X_1, \dots, X_{v(f_n)}) = \sum_{e \in \{0,1\}^{t(n)}} g_n(X_1, \dots, X_{v(f_n)}, e),$$
for all $n > 0$.

We call families from VNP p-definable. The permanent is an example of a family which lies in VNP (and is not known to lie in VP), which we shall see. If $F \in$ VP we may take $G = F$, to see that $F \in$ VNP, hence, VP $\subseteq$ VNP. It is not known if VP $\subsetneq$ VNP, but it is Valiant's conjecture that there is strict inclusion, which seems plausible as equality would imply P=NP.

Recall that we have reductions in the P and NP set up, sometimes we are also be able to reduce the task of computing one polynomial to computing another, these reductions are called projections.

Definition 2.1.12. Let f and g be polynomials in $\mathbb{C}[X_1, \dots, X_n]$ and $\mathbb{C}[X_1, \dots, X_m]$ respectively. We say that f is a projection of g or g projects to f, if there are constants and variables $a_1, \dots, a_m \in \mathbb{C} \cup \{X_1, \dots, X_n\}$ such that $f = g(a_1, \dots, a_m)$.

Let $F = (f_n)$, and $G = (g_n)$ be p-families. We say that F is a p-projection of G if f_n is a projection of $g_{t(n)}$ for some $t : \mathbb{N} \to \mathbb{N}$ which is polynomially bounded. We write $F \leq G$, or $F \leq^t G$.

Clearly if $G \in$ VP and F is a p-projections of G then F is a p-family, and F is also p-computable; if we want to compute $f_n(A)$ we may do this by computing $g_{t(n)}(A')$ for suitable A' and a polynomially bounded function t. As g_m is evaluated in polynomial time this also holds for $f_n(A)$. We conclude that $F \in$ VP

Now, let $F \in$ VNP, and assume $H \leq^s F$, i.e., $h_n = f_{s(n)}(a_1, \dots, a_{v(f_{s(n)})})$. It follows that $H \in$ VNP. Indeed, assume $G \in$ VP is a family of polynomials such that the conditions of definition 2.1.11.2 are satisfied, i.e. $t(n) = v(g_n) - v(f_n) \geq 0$, and when there is inequality we have
$$f_n = \sum_{e \in \{0,1\}^{t(n)}} g_n(X_1, \dots, X_{v(f_n)}, e).$$

Define $Q = (q_n)$ by

$$q_n = g_{s(n)}(a_1, \ldots, a_{v(f_{s(n)})}, X_{v(f_{s(n)})+1}, \ldots, X_{v(g_{s(n)})}),$$

and $u(n) = v(q_n) - v(h_n)$. We now have that $Q \leq^s G$, hence, $Q \in \mathrm{VP}$. Furthermore, notice how the variables used in q_n are $X_1, \ldots, X_{v(h_n)}$, and $X_{v(f_{s(n)})+1}, \ldots, X_{v(g_{s(n)})}$, hence,

$$u(n) = v(h_n) + v(g_{s(n)}) - v(f_{s(n)}) - v(h_n) = t(s(n)) \geq 0.$$

If $u(n) > 0$ then

$$\begin{aligned} h_n = f_{s(n)}(a_1, \ldots, a_{v(f_{s(n)})}) &= \sum_{e \in \{0,1\}^{t(s(n))}} g_{s(n)}(a_1, \ldots, a_{v(f_{s(n)})}, e) \\ &= \sum_{e \in \{0,1\}^{u(n)}} q_n(X_1, \ldots, X_{v(h_n)}, e). \end{aligned}$$

If $u(n) = 0$ then $g_{s(n)} = f_{s(n)}$, implying $q_n = f_{s(n)}(a_1, \ldots, a_{v(f_{s(n)})}) = h_n$.

We have

Proposition 2.1.13. VP *and* VNP *are closed under p-projections.*

In view of this proposition it makes sense to define VNP-completeness:

Definition 2.1.14. A family of polynomials, $F \in \mathrm{VNP}$, is called VNP-complete if $G \leq F$ for any other family $G \in \mathrm{VNP}$.

One such VNP-complete family is the permanent.

Theorem 2.1.15. *The permanent* $\mathrm{per} = (\mathrm{per}_n)$ *lies in* VNP, *and is* VNP*-complete.*

Proof. We shall only prove the first part and skip the part about completeness. A complete proof can be found in [8, Chapter 21].

Let us consider the polynomials in $\mathbb{C}[X_{ij}, Y_{ij}]_{1 \leq i,j \leq n}$ given by

$$\begin{aligned} \alpha_n &= \prod_{\substack{1 \leq i,j,k,l \leq n \\ i=k \text{ iff } j \neq l}} (1 - Y_{ij}Y_{kl}) \\ \beta_n &= \prod_{i=1}^{n} \sum_{j=1}^{n} Y_{ij} \\ \pi_n &= \prod_{i=1}^{n} \sum_{j=1}^{n} X_{ij}Y_{ij} \\ g_n &= \alpha_n \beta_n \pi_n. \end{aligned}$$

In order to get per $\in$ VNP, we will show

$$\text{per}_n(X) = \sum_{e \in \{0,1\}^{n^2}} g_n(X, e).$$

We think of $e \in \{0,1\}^{n^2}$ as square-matrices, and see that

1. $\alpha_n(e) = 0$ when e has more than one non-zero entry in a row or column,
2. $\beta_n(e) = 0$ when there is a row without ones.

This implies that $g_n(X, e) = 0$ unless e is a permutation matrix, e_σ, in which case

$$\pi_n(X, e_\sigma) = \prod_{i=1}^{n} \sum_{j=1}^{n} X_{ij} e_{ij} = \prod_{i=1}^{n} X_{i\sigma i} e_{i\sigma i} = \prod_{i=1}^{n} X_{i\sigma i},$$

and we conclude

$$\sum_{e \in \{0,1\}^{n^2}} g_n(X, e) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} X_{i\sigma i} = \text{per}_n(X).$$

It remains to show $G = (g_n) \in$ VP. However, α_n is a product of less than n^4 factors, each of which is computable in constant time, β_n is a product of n factors each consisting of n terms, hence, we can compute β_n in $O(n^2)$ time, and the same is true for π_n. These observations also show that $\deg g_n$ is polynomially bounded, and as $v(g_n) = 2n^2$ G is a p-computable p-family, hence, $G \in$ VP. $\square$

An essential part of the proof of completeness of the permanent is that it, like the determinant, is universal in the sense that any polynomial can be expressed as the permanent of some quadratic matrix. We shall present a method from [33] to construct one such matrix which works for both the permanent and the determinant, however, it can be done in other ways and the construction used to prove the VNP-completeness of the permanent in [8] differs significantly, though there are also obvious similarities.

2.2 Universality

Let $f \in \mathbb{C}[X_1, \ldots, X_n]$ be the polynomial we want to express as a determinant. The matrix constructed depends on our formula defining f, e.g., if $f = X^2 + 2X + 1$ we will get the matrix

$$A = \begin{pmatrix} 1 & X & & X & \\ & 1 & X & & \\ 1 & & 1 & & \\ & & & 1 & 2 \\ 1 & & & & 1 \end{pmatrix},$$

which satisfy $\operatorname{per}_5 A = \det_5 A = f$. However, we also have that $f = (X+1)^2$ and based on this formula we would construct the matrix

$$B = \begin{pmatrix} & X+1 & \\ & 1 & X+1 \\ 1 & & \end{pmatrix},$$

which clearly also satisfy $f = \det_3 B = \operatorname{per}_3 B$. It is worth noting that this construction only proves the universality of the determinant and the permanent, whereas the example shows it is likely that we will not find the smallest matrix which works. In this particular case we could let $C = \left(\begin{smallmatrix} X+1 & \\ & X+1 \end{smallmatrix}\right)$ to get $\operatorname{per}_2 C = \det_2 C = f$.

In order to explain the construction let us formally define what is meant by a formula.

Definition 2.2.1. A formula of size 0 is a constant or a variable. If φ_1 and φ_2 are formulae of size s_1 and s_2 respectively then $\varphi_1 \square \varphi_2$ is a formula of size $s_1 + s_2 + 1$ whenever $\square \in \{+, \cdot\}$. By a formula we mean a formula of size k, for some $k \in \mathbb{N}$.

If f is a polynomial then the expressions size of f is the smallest k, such that f can be represented by a formula of size k. In other words, the expression size of a polynomial is the smallest number of additions and multiplications one needs in order to write down the polynomial.

Now, let φ be a formula of size k representing our polynomial f. From φ we shall construct a weighted, directed graph, G_φ, such that the adjacency matrix, A_{G_φ}, satisfies $\det A_{G_\varphi} = \operatorname{per} A_{G_\varphi} = f$, and the entries of A_{G_φ} are polynomials in $\mathbb{C}[X_1, \ldots, X_n]$ of degree at most 1. (One can easily get the extra restriction that A_{G_φ} has entries which are constants or variables, and even though this is not very important to us we will discuss this case later.)

In order to construct G_φ we first construct a (weighted, directed) graph with one sink and one source, G_φ^r, $r \in \{0, 1\}$.

G_φ^r is constructed recursively; suppose first $\varphi = \varphi_1 \square \varphi_2$ for two smaller formulae φ_1, and φ_2, and $\square \in \{+, \cdot\}$, then we have two graphs $G_{\varphi_1}^{r_1}$ and $G_{\varphi_2}^{r_2}$.

If $\square = +$ and $r_1 = r_2$, then set $r = r_1$, and identify the sources and the sinks of the two graphs to get G_φ^r.

If $\Box = +$ and $r_1 \neq r_2$, then assume without loss of generality that $r_2 = 0$. Add an extra node to the graph $G_{\varphi_2}^{r_2}$, and make an edge of weight 1 from the old sink to the new node, set $r_2 = 1$ and proceed as the case $\Box = +$ and $r_1 = r_2$.

If $\Box = \cdot$, then set $r = r_1 + r_2 \mod 2$, and identify the source of $G_{\varphi_2}^{r_2}$ with the sink of $G_{\varphi_1}^{r_1}$.

Now, in order to start our construction, suppose that $\varphi = \Xi \in \mathbb{C} \cup \{X_1, \ldots, X_n\}$ is a formula of size 0, then our graph is G_φ^1 with just two nodes, the source and the sink, where the edge between these is given the weight Ξ.

Notice that

1. G_φ^r is acyclic.

2. If $r = 0$ then all paths from source to sink will have even length (the length of a path is the number of edges), and if $r = 1$ then all paths from source to sink will be of odd length.

3. Every term in the polynomial, f, represented by φ can be found as the weight of some path (the weight of a path is the product of the weights of occurring edges) from source to sink, and the weight of each such path is a term in f.

In the graph G_φ^r the nodes which are not the source or the sink are called internal nodes. In order to get from G_φ^r to G_φ we add loops, i.e., edges from a node to itself, with weight 1 to all internal nodes, and if $r = 1$ we identify the source and the sink of G_φ^r, if $r = 0$ we add an edge from the sink to the source with weight 1.

In a graph a cycle-cover is a union of vertex-disjoint cycles such that each vertex lies in one of the cycles. The weight of a cycle-cover is the product of weights of edges used.

Notice that a cycle cover of G_φ, which unlike G_φ^r is not acyclic, consists of a unique cycle of odd length containing the node(s) which used to be the source and sink (which might coincide) and some loops, hence, the weights of the cycle-covers correspond to terms in f, and the sum of these weights of cycle-covers will equal f. We have now almost proven

Theorem 2.2.2. *Assume that f is of expression-size s, and φ is a formula of minimal size s defining f. There is a square matrix, A, of size $t \leq s + 2$ with entries that are polynomials of degree at most 1 such that $f = \det_t A = \mathrm{per}_t A$.*

Proof. With the notation above the theorem the matrix is defined to be $A = A_{G_\varphi}$, the adjacency matrix of G_φ.

We have already discussed that f is the sum of weights of cycle-covers of G_φ, however, if we number the vertices in G_φ, a cycle-cover can be considered to be a permutation of $[m]$, where m is the number of vertices in G_φ. These permutations, σ, are all cycles of odd length, and so $\operatorname{sgn}\sigma = 1$.

Now, if we take some arbitrary permutation, $\tau \in S_m$, and consider $A_\tau := \prod_{i=1}^{m} a_{i\tau i}$ then A_τ is non-zero if and only if there is an edge between vertices i and τi in G_φ for all $i \in [m]$, in other words if and only if $\{(i, \tau i) | i \in [m]\}$ is a cycle-cover of G_φ. Hence, we get

$$f = \sum_{C \text{ cycle-cover}} \text{weight of } C = \sum_{\sigma \in S_m} A_\sigma,$$

which we recognise as the permanent of A, furthermore, it can be identified as the determinant of A because $A_\sigma \neq 0$ only if $\operatorname{sgn}\sigma = 1$.

It remains to show that the size of A is at most $s + 2$. But notice how, in the construction of G_φ^r, we get an internal node only if we fuse two graphs by making an addition or a multiplication of two formulae, hence, the number of internal nodes are at most s. Bearing in mind the source and sink we get the result. □

If one insists on making f as a projection of the permanent and determinant in the sense of definition 2.1.12, i.e., where the entries of our matrix are constants or variables, then one must consider what happens when we add two formulae of size 0. If we want to construct $\varphi = X + 1$ the method above would suggest we make the graph G_φ^1 as in figure 2.1, which is then turned into the graph G_φ shown in the same figure. The resulting adjacency matrix would be $(X + 1) \notin \mathrm{Mat}_1(\mathbb{C} \cup \{X\})$.

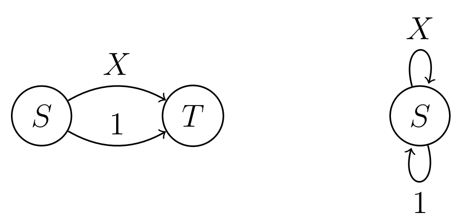


Figure 2.1: The graphs G_φ^1 and G_φ, where $\varphi = X + 1$

Thus, to make a proper projection we should 'add' the graphs of size 0-formulae by considering the formula $\Xi_1 + (1 \cdot \Xi_2)$ rather than $\Xi_1 + \Xi_2$, where $\Xi_1, \Xi_2 \in \mathbb{C} \cup \{X_1, \ldots, X_n\}$. I.e., we make one of our size 0-formulae into a size 1-formula by multiplying with 1 before adding. If we do this with $\Xi_1 = X$ and $\Xi_2 = 1$ we get the graphs in figure 2.2.

Notice how we manage to keep the size of the graph to 3, which keeps us within the bound $s + 2$. This would not be true if we had allowed $r = 0$ when fusing two

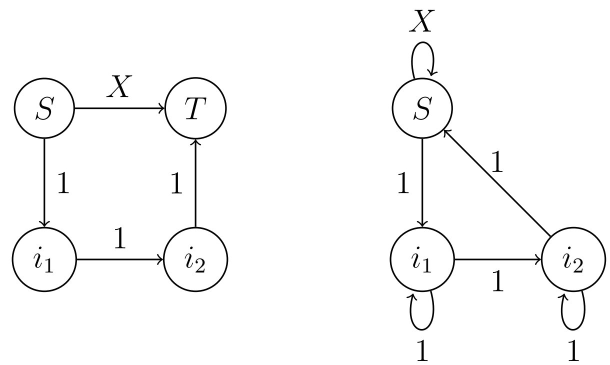


Figure 2.2: The graphs G^1_φ and G_φ, where $\varphi = X + (1 \cdot 1)$

graphs with $r_1 \neq r_2$ by an addition, i.e., if we had made the the graph $G^0_{X+(1\cdot 1)}$ (where paths from source to sink would be of even length) rather than $G^1_{X+(1\cdot 1)}$. Had we done this then our graph would have size 4, as the sink (which would be the vertex in the lower right corner of the graph to the left in figure 2.2) would not be merged with the source to become one node in the final graph.

All in all we have proven

Corollary 2.2.3. *If $f \in \mathbb{C}[X_1, \ldots, X_n]$ is a polynomial of expression size s we can find a square matrix, $A \in \mathrm{Mat}_t(\mathbb{C} \cup \{X_1, \ldots, X_n\})$, for some $t \leq s + 2$, such that $f = \det_t A = \mathrm{per}_t A$.*

The universality of the determinant leads us to the next question, if p is a polynomial, how large an integer, m, is needed to have p as a projection of $\det_m$.

2.3 Determinantal Complexity

Standard Determinantal Complexity

In the light of the universality of the determinant, and because the determinant is easily computed using Gaussian elimination one might get tempted to try and compute polynomials via the determinant. I.e., if we consider some $f \in \mathbb{C}[X_1, \ldots, X_n]$, which is tiresome to compute, and we find an m, which is not too large, such that f is a projection of $\det_m$, or if $f = \det_m A$ for some A, with entries that are polynomials in $\mathbb{C}[X_1, \ldots, X_n]$ of degree at most one, then it might be a short cut to compute this determinant rather than f directly.

In particular if we have a p-family, $F = (f_n)$, which we would like to be able to compute efficiently, it would be good to know if $F \leq \det$, or, if we are less strict, if $f_n = \det_{t(n)} \circ L_n$ for some polynomially growing map, $t : \mathbb{N} \to \mathbb{N}$, and an affine map, $L_n : \mathbb{A}^{v(f_n)} \to \mathbb{A}^{t(n)^2}$. If either of these cases hold then F can be computed in polynomial time.

We define the determinantal complexity of polynomials in the following way.

Definition 2.3.1. Let f be a polynomial.

1. The determinantal complexity of f is the smallest natural number, m, such that $f = \det_m A$ for some A with entries that are affine. We denote this integer by $\mathrm{dc}(f)$.

2. We furthermore define the projectional determinantal complexity, $\mathrm{dc}_p(f)$, to be the smallest integer, m, such that f is a projection of $\det_m$ in the sense of definition 2.1.12.

Clearly $\mathrm{dc}(f) \leq \mathrm{dc}_p(f)$ and we might have strict inequality. Indeed, consider $f = \mathrm{per}_2 = X_{11}X_{22} + X_{12}X_{21}$ then $f = \det_2 \left(\begin{smallmatrix} X_{11} & X_{12} \\ -X_{21} & X_{22} \end{smallmatrix} \right)$, hence, $\mathrm{dc}(\mathrm{per}_2) = 2$ (naturally the determinantal complexity is forced to be at least the degree of the polynomial). However, this is not a p-projection, and clearly no projection of $\det_2$ to per_2 exists, instead we get

$$\mathrm{per}_2 = \det_3 \begin{pmatrix} & X_{11} & X_{12} \\ X_{22} & -1 & \\ X_{21} & & -1 \end{pmatrix},$$

i.e., $\mathrm{dc}_p(\mathrm{per}_2) = 3$.

The determinantal complexity of polynomials in at most 3 variables or of degree at most 2 is known. The following theorem corresponds to [25, proposition 2.2, and theorem 1.4].

Theorem 2.3.2. *Assume q is a quadratic form of rank r, and that f is a polynomial such that either $f \in \mathbb{C}[X, Y, Z]$ is homogeneous, or $f \in \mathbb{C}[X_1, \ldots, X_n]$ is of degree 1, then*

- $\mathrm{dc}(q) = \begin{cases} 2 & \text{if } r \leq 4 \\ \lceil \frac{r+1}{2} \rceil & \text{else,} \end{cases}$

- $\mathrm{dc}(f) = \deg f$

The proof is omitted in its entirety, I just note that in the case where $f \in \mathbb{C}[X, Y]$ is homogeneous, one may first consider $g = f(X, 1) \in \mathbb{C}[X]$ and factorise this using the roots $\beta_1, \ldots \beta_k$; $g = \alpha \prod_{i=1}^{k}(X - \beta_i)$. Now, let $d = \deg f$, and $s = \deg f - k$, then $f = Y^d g(X/Y) = \alpha Y^s \prod_{i=1}^{k}(X - \beta_i Y)$, and so

$$f = \alpha \det_d \begin{pmatrix} YI_s & & & \\ & (X - \beta_1 Y) & & \\ & & \ddots & \\ & & & (X - \beta_k Y) \end{pmatrix}.$$

Here we of course use that $\mathbb{C}$ is algebraically closed which actually is unnecessary.

In fact in [25] they construct a square matrix of size d with determinant f, and with entries which are zero or linear polynomials in the variables X and Y, with coefficients taken from the set $\{a_0, \ldots, a_d, 0, -1, 1\}$, where $f = \sum_{i=1}^{d} a_i X^i Y^{d-i}$.

However, when we involve more than 3 variables or increase the degree of our (homogeneous) polynomial the question of determinantal complexity gets more cumbersome. The following problem was presented to me by Nicolas Ressayre:

Problem 2.3.3. We have that

$$Z^3 + XT^2 + X^2Y = \det_5 \begin{pmatrix} 1 & & Y & & \\ X & T & & Z & \\ & 1 & T & & \\ & & Z & & X \\ & & & -1 & Z \end{pmatrix},$$

and so the determinantal complexity (standard as well as projectional) of $Z^3 + XT^2 + X^2Y$ is at most 5. It is also known from [3, proposition 4.3] that $\mathrm{dc}(Z^3 + XT^2 + X^2Y) > 3$, and that $Z^3 + XT^2 + X^2Y$ is essentially the only polynomial defining a cubic surface in $\mathbb{P}^3$ which has determinantal complexity greater than 3. Now, is $\mathrm{dc}(Z^3 + XT^2 + X^2Y)$ equal to 4 or 5?

Even though this might not look very scary it certainly is not an easy task either, and I have not found a way to determine the accurate determinantal complexity of the polynomial. It goes to show that determining the determinantal complexity of a given polynomial is not something which is just done overnight.

It is an open question if the determinant is VNP-complete. (It is not even known if the determinant is VP-complete, though, it is known to be complete in the class VQP $\supseteq$ VP with respect to what is called quasi-polynomial projections. We shall not go into details about this here but refer to [7]).

If we take Valiant's conjecture, VP $\subsetneq$ VNP, to be true then the determinant is not VNP-complete.

If Valiant's conjecture is false, i.e, if VP $=$ VNP then Cook's conjecture, P $\subsetneq$ NP, is false as well, hence, Cook's conjecture also suggest that the determinant is not VNP-complete.

In order to prove or disprove VNP-completeness of the determinant it is enough to show per $\leq$ det or per $\not\leq$ det respectively. This can be translated into the question of how fast $\mathrm{dc}_p(\mathrm{per}_n)$ grows with n. If this growth is polynomial, then per $\leq$ det, which means that given $F \in$ VNP we have $F \leq$ det, because $F \leq$ per by completeness of the permanent, and using transitivity of $\leq$ we get $F \leq$ det.

On the other hand if $\mathrm{dc}_p(\mathrm{per}_n)$ grows faster than any polynomial in n, then the permanent is not a p-projection of the determinant, i.e., the determinant is not VNP-complete. Note that non-VNP-completeness of the determinant is not enough to prove Valiant's conjecture, unless one also proves that the determinant is complete in VP, i.e., $F \leq$ det for all $F \in$ VP.

Rather than study $\mathrm{dc}_p(\mathrm{per}_n)$ it might be clever to study the determinantal complexity $\mathrm{dc}(\mathrm{per}_n)$. There are several reasons for this.

First and foremost, $\mathrm{dc}(\mathrm{per}_n) \leq \mathrm{dc}_p(\mathrm{per}_n)$ as we have noticed, hence, showing lower bounds of the growth of the determinantal complexity implies the same lower bounds of the p-projectional determinantal complexity.

Furthermore, we get the added flexibility of allowing affine projections so that one does not have to worry about, e.g., removing signs (or, more generally, constants) from variables as in the example above with $\mathrm{per}_2 = \det_2 \left(\begin{smallmatrix} X_{11} & X_{12} \\ -X_{21} & X_{22} \end{smallmatrix} \right)$.

Finally if one should find that $\mathrm{dc}(\mathrm{per}_n) = O\left(n^k\right)$ for some $k \in \mathbb{N}$, then this is enough to show that the permanent is computable in polynomial time, since each coordinate of the affine projection, $L_n : \mathbb{A}^{n^2} \to \mathbb{A}^{\mathrm{dc}(\mathrm{per}_n)^2}$, is computable in $O\left(n^2\right)$ time. This of course would disprove Valiant's as well as Cook's conjectures.

One disadvantage of investigating the determinantal complexity rather than the projectional determinantal complexity of the permanent is that, if $\mathrm{dc}(\mathrm{per}_n) \in O(n^k)$ for some $k \in \mathbb{N}$ then this in it self will not enough to prove VNP-completeness of the determinant. However, if Valiant's conjecture is true then this disadvantage will have no consequences, and so it is a reasonable price to pay.

The following is the main result in [25], and it is the best known lower bound of the determinantal complexity of the permanent.

Theorem 2.3.4. *The determinantal complexity of the permanent* per $= (\mathrm{per}_n)$ *grows at least quadratically with n. Indeed, we have the lower bound*

$$\mathrm{dc}(\mathrm{per}_n) \geq \frac{n^2}{2}.$$

Proof. By proposition 1.2.3 we have that $\dim V(\mathrm{per}_n)^* = n^2 - 2$ and by remark 1.2.5 we have that

$$\mathrm{dc}(\mathrm{per}_n) \geq \frac{\dim V(\mathrm{per}_n)^* + 2}{2} = \frac{n^2}{2}.$$

□

Bounded Determinantal Complexity

This section is primarily a elaboration of the article [21], in particular lemma 2.3.9 along with theorem 2.3.10 and corollary 2.3.11 are results which are taken from said article, namely lemma 2.4.1, and theorems 1.0.3 and 1.0.1 respectively.

In order to allow ourselves the added flexibility of affine projections compared to projections in the sense of definition 2.1.12, we have already skipped from the question 'what is the projectional determinantal complexity of a given family of polynomials?' to the question 'what is its the determinantal complexity?'. Now we want to give ourselves even more elbowroom.

Consider an n-dimensional vector space, V, over the complex numbers, then for $f \in \mathbb{C}[X_1, \dots, X_n] \cong \mathbb{C}[V]$ and $M \in \mathrm{End}(V)$ we define $(M.f)(x) := (f \circ M)(x)$.

Now, assume f is homogeneous of degree d, i.e., $f \in S^d(V^*) \subseteq \mathbb{C}[X_1, \dots, X_n]$, as in [21] we define the function $\varepsilon = \varepsilon_{V,d} : S^d(V^*) \to \mathbb{N}$ on f to be the smallest natural number, m, such that $Y^{m-d} f \in \mathrm{End}(\mathrm{Mat}_m(\mathbb{C})).\det_m$. (Here Y is a new variable such that $\mathbb{C}[X_1 \dots, X_n, Y] \cong \mathbb{C}[V \oplus \mathbb{C}]$.)

It should be clear that it is always possible to find such an m using the universality of the determinant:

If $\mathrm{dc}(f)^2 \geq n+1$ we may simply linearise the map the affine map $L(X) = AX + b$, where $f = \det_m \circ L$, i.e., define $L'(X, Y) = AX + bY$ then $Y^{m-d} f = \det_m \circ L'$. Notice that L' induces an endomorphism $M \in \mathrm{End}(\mathrm{Mat}_m(\mathbb{C}))$ as we can embed $V \oplus \mathbb{C}$ into $\mathrm{Mat}_m(\mathbb{C})$ whenever $m^2 \geq \mathrm{dc}(f)^2 \geq n + 1$.

If $\mathrm{dc}(f)^2 < n + 1$ we may certainly also find an m such that $m^2 \geq n + 1$ and an affine map $L : \mathbb{A}^n \to \mathbb{A}^{m^2}$ such that $f = \det_m \circ L$, and in this case we may proceed as above.

Proposition 2.3.5. *If $f \in \mathbb{C}[X_1, \dots, X_n]$ is a degree d polynomial of determinantal complexity at least $\sqrt{n+1}$ then $\varepsilon(f) = \mathrm{dc}(f)$.*

Proof. We have already seen that $\mathrm{dc}(f) \geq \varepsilon(f)$ as we, given an affine map, L, such that $f = \det_m \circ L$, may construct an endomorphism, M, of $\mathrm{Mat}_m(\mathbb{C})$ such that $Y^{m-d} f = M.\det_m$.

The other inequality is also easy. If $Y^{m-d}f = M.\det_m$ then we first embed $\mathbb{C}^n$ into the 'first' natural n-dimensional subspace of $\mathrm{Mat}_m(\mathbb{C})$ by a linear map, φ, then we define $L : \mathbb{A}^n \to \mathrm{Mat}_m(\mathbb{C}) \cong \mathbb{A}^{m^2}$ by

$$L(X) = M(\varphi(X), 1, \ldots, 1).$$

This gives

$$f(X) = 1^{m-d}f(X) = \det_m(M(\varphi(X), 1, \ldots, 1)) = (\det_m \circ L)(X),$$

hence, $\mathrm{dc}(f) \leq \varepsilon(f)$. □

Remark 2.3.6. Note that if f is a linear polynomial in $n > 1$ variables then $\mathrm{dc}(f) = 1$ but we cannot construct an endomorphism, $M \in \mathrm{End}(\mathrm{Mat}_1(\mathbb{C})) \cong \mathbb{C}$, such that $f = \sum_{i=1}^n a_i X_i = M.\det_1 = M.X_1$. Indeed, it is necessary that our determinantal complexity is large enough for us to be able to embed $\mathbb{C}^{n+1}$ into $\mathrm{Mat}_{\mathrm{dc}(f)}(\mathbb{C})$ in order for ε and dc to coincide.

However, as our interest in determinantal complexity is mostly concerned with families of polynomials, $F = (f_n)$, and as $\deg f_n$ is usually going to grow at least as $\sqrt{v(f_n)}$, we have that $\mathrm{dc}(f_n) \geq \deg(f_n) \geq \sqrt{v(f_n)}$ where it is most likely that the first inequality is strict. Thus, it is not important for us to distinguish between the functions ε and dc in this case.

We now define a new kind of determinantal complexity called bounded determinantal complexity. Recall that $\mathrm{GL}(V)$ acts on $\mathbb{C}[V]$ by $g.f = f \circ g^{-1}$, for $g \in \mathrm{GL}(V)$, and $f \in \mathbb{C}[V]$.

Definition 2.3.7. For a homogeneous polynomial f of degree d the bounded determinantal complexity of f, denoted $\overline{\mathrm{dc}}(f)$, is the smallest integer m such that $[Y^{m-d}f] \in \overline{\mathrm{GL}(\mathrm{Mat}_m(\mathbb{C})).[\det_m]} \subseteq \mathbb{P}(S^m(\mathrm{Mat}_m(\mathbb{C})^*))$

This notion of bounded determinantal complexity gives us, as promised earlier, even more elbowroom than the regular determinantal complexity as any polynomial of the form $M.\det_m$ can be approximated by a series of polynomials $g_t.\det_m$, where $M \in \mathrm{End}(\mathrm{Mat}_m(\mathbb{C}))$ and $g_t \in \mathrm{GL}(\mathrm{Mat}_m(\mathbb{C}))$, i.e., we have

$$\mathbb{P}(\mathrm{End}(\mathrm{Mat}_m(\mathbb{C})).\det_m) \subseteq \overline{\mathrm{GL}(\mathrm{Mat}_m(\mathbb{C})).[\det_m]}$$

and when considering $\overline{\mathrm{dc}}$ rather than dc we precisely allow ourselves to take limits. Hence, $\overline{\mathrm{dc}}(f) \leq \varepsilon(f) = \mathrm{dc}(f)$ for $f \in S^n(V^*)$ that are of sufficiently high determinantal complexity compared to $v(f)$.

For the rest of this section we shall assume that $\varepsilon(f) = \mathrm{dc}(f)$.

In order to give a lower bound on the bounded determinantal complexity of a polynomial we again make use of Katz' dimension formula (theorem 1.2.2).

Let $V = \mathbb{C}^m$, and take an irreducible polynomial, $f \in S^d(V^*)$. Recall that Katz' dimension formula gives the dimension of $V(f)^*$ as $\operatorname{rank} \operatorname{Hes} f(x) - 2$ for generic points $x \in V(f)$. Assume that $\dim V(f)^* = k$, then for any $(k+3)$-dimensional subspace of V, and for any point $x \in V(f)$ the determinant of $\operatorname{Hes} f(x)$ restricted to this subspace will vanish. Hence, if $F \subseteq V$ is a subspace of dimension $k+3$ then $\det(\operatorname{Hes} f|_F)$ is a homogeneous polynomial of degree $(d-2)(k+3)$ which vanishes at $V(f)$.

Using the assumption that f is irreducible we have that $V(f) \subseteq V(\det(\operatorname{Hes} f|_F))$ implies f divides $\det(\operatorname{Hes} f|_F)$.

We conclude

Lemma 2.3.8. *Let $f \in S^d(V^*)$ be irreducible. Then $\dim V(f)^* \leq k$ if and only if f divides $\det(\operatorname{Hes} f|_F)$ for any subspace $F \subseteq V$ of dimension $k+3$.*

Now we want to introduce a certain variety. Consider the set

$$\mathrm{Dual}^0_{k,d,n} \subseteq \mathbb{P}(S^d(V^*)),$$

defined by the conditions $[f] \in \mathrm{Dual}^0_{k,d,n}$ if and only if $f \in S^d(V^*)$ is irreducible and the dimension of $V(f)^* \subseteq \mathbb{P}(V^*)$ is at most k. We define our variety to be

$$\mathrm{Dual}_{k,d,n} = \overline{\mathrm{Dual}^0_{k,d,n}}.$$

As we have already seen in proposition 1.2.1 $\dim V(\det_n)^* = 2n-2$, hence, $[\det_n] \in \mathrm{Dual}_{2n-2,n,n^2}$.

In [21] they show that $[\det_n]$ is a smooth point, and that $\overline{\mathrm{GL}(\mathrm{Mat}_n(\mathbb{C})).[\det_n]}$ is an irreducible component of $\mathrm{Dual}_{2n-2,n,n^2}$, along with the following lemma:

Lemma 2.3.9. *Let $n < N$, and set $V = \mathbb{C}^n$ and $W = \mathbb{C}^N$. Consider an irreducible polynomial $f \in S^d(V^*)$ along with a linear inclusion $V^* \oplus \mathbb{C} \cdot Y \subseteq W^*$ and let $g = Y^{m-d} f \in S^m(W^*)$. If $[f] \in \mathrm{Dual}_{k,d,n} \setminus \mathrm{Dual}_{k-1,d,n}$ then $[g] \in \mathrm{Dual}_{k,m,N} \setminus \mathrm{Dual}_{k-1,m,N}$.*

The proof of the lemma will follow shortly.

We may combine these observations in the following way: If $f \in S^d(V^*)$ is a homogeneous, irreducible polynomial defining a variety with dual of dimension k, and $\overline{\mathrm{dc}}(f) = m$ then by the lemma $[g] = [Y^{m-d} f] \in \mathrm{Dual}_{k,m,N} \setminus \mathrm{Dual}_{k-1,m,N}$ for some suitable N, e.g., $N = m^2$. But at the same time $[g] \in \overline{\mathrm{GL}(\mathrm{Mat}_m(\mathbb{C})).[\det_m]} \subseteq \mathrm{Dual}_{2m-2,m,m^2}$. Now, as we have the natural filtration

$$\mathrm{Dual}_{\kappa,\delta,\nu} \subseteq \mathrm{Dual}_{\kappa+\rho,\delta,\nu},$$

for any $\rho \in \mathbb{N}$, we may conclude that $k \leq 2m-2$, i.e., we have:

Theorem 2.3.10. *Let $f \in S^d(V^*)$ be a homogeneous, irreducible polynomial, then*

$$\overline{\mathrm{dc}}(f) \geq \frac{\dim V(f)^* + 2}{2}.$$

In particular this shows

Corollary 2.3.11.

$$\overline{\mathrm{dc}}(\mathrm{per}_n) \geq \frac{n^2}{2}$$

We now turn to proving the lemma.

Proof of lemma 2.3.9. That f is irreducible and lies in $\mathrm{Dual}_{k,d,n} \setminus \mathrm{Dual}_{k-1,d,n}$ means that $\dim V(f)^* = k$, and since $V(g)^* \cong V(f)^* \cup V(Y)^*$ or $V(g)^* \cong V(f)^*$ (depending on what $m-d$ is), we also have $\dim V(g)^* = k$ as the dual of a projective hyperspace is just a point. However, g is certainly not irreducible, hence, a bit more ingenuity is needed to get $[g] \in \mathrm{Dual}_{k,m,N} \setminus \mathrm{Dual}_{k-1,m,N}$.

First, denote the set of reducible polynomials in $S^r(V^*)$ by M, and notice that M is closed. Indeed, if $h = h_1 h_2 \in S^r(V^*)$ where $0 < d_1 := \deg h_1 < r$, then h is in the image of the morphism

$$\begin{aligned} \varphi_{r,d_1} : S^{d_1}(V^*) \times S^{r-d_1}(V^*) &\to S^r(V^*) \\ (p,q) &\mapsto p \cdot q. \end{aligned}$$

Now we get that

$$M = \bigcup_{i=1}^{\lfloor \frac{r}{2} \rfloor} \varphi_{r,i}\left(S^i(V^*) \times S^{r-i}(V^*)\right),$$

hence, M is closed. (For additional details see [14, corollary 14.3].) We will use that irreducibility is an open condition in the space of homogeneous polynomials of degree m.

Recall Katz' dimension formula in the version stated in lemma 2.3.8. We see that it is enough to investigate if g divides the polynomials $\det(\mathrm{Hes}\, g|_F)$ for all spaces

$F = \mathbb{C}^{k+3} \subseteq \mathbb{C}^N$, as this condition defines an open subset of $\mathrm{Dual}_{k,m,N} \setminus \mathrm{Dual}_{k-1,m,N}$. (We should also check the existence of a space, $H \subseteq W$, of dimension $k+2$ such that g does not divide $\det(\operatorname{Hes} g|_H)$, however, this space exists as we may find such a space, $H \subseteq V$, which satisfies that f does not divide $\det(\operatorname{Hes} f|_H)$.)

Now, pick an ordered basis $\langle v_1, \dots, v_n, u, w_{n+2}, \dots, w_N \rangle$ such that V is spanned by $\{v_1, \dots, v_n\}$, and $\ker Y = \operatorname{span}\{v_1, \dots, v_n, w_{n+2}, \dots, w_N\}$.

In this basis we find $\operatorname{Hes} g$ to be to following block diagonal-matrix with block structure $(n, 1, N-n-1)$

$$\begin{pmatrix} Y^{m-d} \operatorname{Hes} f & & \\ & (m-d)(m-d-1)Y^{m-d-2} f & \\ & & 0 \end{pmatrix}.$$

Because $\dim V(f)^* = k$ then for arbitrary $F = \operatorname{span}\{e_1, \dots, e_{k+3}\}$, where $e_i \in V$, we have f divides $\det_{k+3}(\operatorname{Hes} f|_F)$ and so f along with g also divides $\det_{k+3}(\operatorname{Hes} g|_F)$ for such F, because this determinant is $Y^D \det_{k+3}(\operatorname{Hes} f|_F)$ for $D = (m-d)(k+3)$.

If we consider $\operatorname{Hes} g|_F$ for some F that has non-trivial intersection with $W/(V \oplus \mathbb{C} \cdot u)$, then our determinant vanishes (and so it most certainly is divisible by g).

Finally, if one of our spanning vectors of F is u and the remaining vectors come from V, then $\det_{k+3}(\operatorname{Hes} g|_F)$ is certainly also divisible by g, as we get

$$\det_{k+3}(\operatorname{Hes} g|_F) = Y^{m-d-2} f \det_{k+2}\left(Y^{m-d} \operatorname{Hes} f|_{F'}\right)$$

where $F' = F/\mathbb{C} \cdot u$.

We see that g satisfy a condition that defines an open set of our space, and so $[g] \in \mathrm{Dual}_{k,m,N} \setminus \mathrm{Dual}_{k-1,m,N}$ □

One final result from the article which I think is worth mentioning is an example of a family of polynomials Λ_n such that $\overline{\mathrm{dc}}(\Lambda_n) < \mathrm{dc}(\Lambda_n)$.

Let X be a $(2n+1) \times (2n+1)$-matrix of indeterminates. Then $X_A = X - X^t$ is the anti-symmetric part of X (up to a factor 1/2), and we denote by $\mathrm{Pf}_i(X_A)$ the Pfaffian of the matrix one gets when deleting the i'th row and column from X_A, then Λ_n is defined as follows:

$$\Lambda_n = \sum_{i,j} (X_{ij} + X_{ji}) \,\mathrm{Pf}_i(X_A) \,\mathrm{Pf}_j(X_A).$$

We omit all details as to why this polynomial satisfies $\overline{\mathrm{dc}}(\Lambda_n) < \mathrm{dc}(\Lambda_n)$, and instead turn our attention towards a certain type of polynomials which generalises the determinant and the permanent.

2.4 General Immanants

Recall that a (finite dimensional) representation of a group, G, is a homomorphism of groups, ρ, and a (finite dimensional) vector space, V, such that $\rho : G \to \mathrm{GL}(V)$. We say that ρ is irreducible if there is no subspace $V' \subsetneq V$ of positive dimension such that $\rho(G)(V') \subseteq V'$. When we have a representation, ρ, we define the character $\chi_\rho : G \to \mathbb{C}$ by $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$. This is well defined as $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, hence, a change of basis in V will not affect this function. We call χ_ρ irreducible if the representation ρ is irreducible. Furthermore a character is a class function, i.e., if g and g' are conjugate in G via h then $\chi_\rho(g) = \chi_\rho(g')$, as we have

$$\mathrm{Tr}\left(\rho(g)\right) = \mathrm{Tr}\left(\rho(h)\rho(g'h^{-1})\right) = \mathrm{Tr}\left(\rho(g'h^{-1})\rho(h)\right) = \mathrm{Tr}\left(\rho(g'h^{-1}h)\right) = \mathrm{Tr}\left(\rho(g')\right).$$

Obviously the set of class functions is a vector space, furthermore, it is equipped with an inner product if G is finite (which we shall assume from this point on) defined by

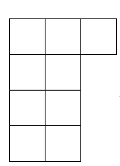
$$\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)},$$

for class functions $f_1, f_2 : G \to \mathbb{C}$. The set of irreducible characters of G is an orthonormal basis with respect to this inner product. Furthermore, the character of any representation of a group is a $\mathbb{N}$-linear combination of the irreducible characters of the group. One may read the book [19] for more representation theory of finite groups

The representation theory of the symmetric group is very well-developed, and it is a commonly known fact that there is a one to one correspondence between the set of irreducible representations of S_n and partitions of n, $\lambda \vdash n$, i.e., sequences of integers, $\lambda = (\lambda_1 \ \ldots \ \lambda_k)$, such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$, and $n = \sum_i \lambda_i$.

E.g., the sign-representation $\mathrm{sgn} : S_n \to \mathrm{GL}(\mathbb{C})$ corresponds to the constant partition $\lambda = (1 \ \ldots \ 1)$, whereas the trivial representation $1_{S_n} : S_n \to \mathrm{GL}(\mathbb{C})$ corresponds to the partition $\lambda = (n)$.

Partitions $\lambda = (\lambda_1 \ \ldots \ \lambda_k)$ are sometimes visualised by Young diagrams, which are rows of boxes where the number of boxes in the i'th row equals λ_i. E.g., if we consider the partition $\lambda = (3\ 2\ 2\ 2)$ we get a Young diagram that looks like



.

We can fill the boxes with the integers $1, \ldots, n$ to get a Young tableau

$$\begin{array}{|c|c|c|}\hline 1 & 5 & 6 \\ \hline 3 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & 9 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & 8 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}.$$

A partition of the type

$$\lambda = (k \ \underbrace{1 \ \ldots \ 1}_{\text{r times}}) =: (k \ 1^r)$$

is called a hook-partition, or a k-hook-partition, as the corresponding Young diagram resembles a hook

$$\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & \ \\ \hline \ & \multicolumn{4}{c}{} \\ \cline{1-1} \ & \multicolumn{4}{c}{} \\ \cline{1-1} \ & \multicolumn{4}{c}{} \\ \cline{1-1} \ & \multicolumn{4}{c}{} \\ \cline{1-1} \ & \multicolumn{4}{c}{} \\ \cline{1-1} \ & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array}$$

(here $k = 5, r = 6$, and $n = k + r = 11$).

Young tableaux are helpful if you want to compute the irreducible character of some representation of S_n arising from a partition, λ. In order to find the character you consider a certain couple of subgroups of S_n called the row-stabiliser and the column-stabiliser. These are defined via an arbitrary Young tableau, T, of shape λ. The row-stabiliser, denoted $R(T)$, is the set of permutations $\sigma \in S_n$ such that $\sigma(j)$ and j are in the same row of T for all $j = 1, \ldots, n$. The column-stabiliser, $C(T)$, is defined similarly except of course $\sigma \in C(T)$ if and only if $\sigma(j)$ and j are in the same column. We now consider the sign-representation of $C(T)$, and the trivial representation of $R(T)$ and induce their characters to the characters $\psi_{C(T)}$ and $\psi_{R(T)}$ of S_n. This is done by the Frobenius formula:

$$\psi_H(\sigma) = \sum_{\overline{\tau} \in S_n/H} \widehat{\chi}_H \left(\tau\sigma\tau^{-1}\right),$$

where H is one of the groups $R(T)$ or $C(T)$, $\widehat{\chi}_{R(T)}(\sigma) := 1_{R(T)}(\sigma)$ and $\widehat{\chi}_{C(T)}(\sigma)$ is $\operatorname{sgn}(\sigma)$ for $\sigma \in C(T)$ and 0 else. It turns out that

$$\left\langle \psi_{C(T)}, \psi_{R(T)} \right\rangle = 1,$$

thus, there is a unique irreducible character that occurs in both of the induced characters, this is the character corresponding to λ and we denote it χ_λ.

Example 2.4.1. If we consider the partition $\mu = (1^n)$ we have that $C(T_\mu) = S_n$ and $R(T_\mu) = \{\mathrm{id}\}$, hence, when we induce the characters $\mathrm{sgn}_{C(T_\mu)}$ and $1_{R(T_\mu)}$ to S_n we get $\psi_{C(T_\mu)} = \mathrm{sgn}$, and $\psi_{R(T_\mu)}$ is the regular character, i.e., the one that takes the value $|S_n|$ at the identity and 0 elsewhere. As sgn is itself an irreducible character we get $\chi_{(1^n)} = \mathrm{sgn}$ - as was mentioned already

Similarly if we take the partition $\nu = (n)$ we get $\psi_{C(T_\nu)}$ to be the regular character and $\psi_{R(T_\nu)} = 1_{S_n}$, thus, $\chi_{(n)} = 1_{S_n}$.

Finally if we consider the 2-hook-partition, $\lambda = (2\ 1^{n-2})$, an easy computation yields $\psi_{C(T_\lambda)}(\sigma) = \mathrm{sgn}\,\sigma\,\mathrm{fix}_n\,\sigma$, where $\mathrm{fix}_n : S_n \to \mathbb{N}$ is the function that counts fixed points of the permutation σ, i.e., $\mathrm{fix}_n\,\sigma = |\{i \in [n] \mid \sigma(i) = i\}|$, and $\psi_{R(T_\lambda)}$ takes the value $n!/2$ on the identity, $(n-2)!$ on transpositions, and zero else. Now, using the inner product one gets that $\mathrm{sgn}\,\mathrm{fix}_n - \mathrm{sgn}$ is an irreducible character and that it occurs in both $\psi_{C(T_\lambda)}$ and $\psi_{R(T_\lambda)}$, thus, $\chi_\lambda(\sigma) = \mathrm{sgn}\,\sigma(\mathrm{fix}_n\,\sigma - 1)$.

The last partition in example 2.4.1 corresponds to the representation where

$$V = \left\{ \sum_{i=1}^n a_i e_i \middle| \sum_{i=1}^n a_i = 0 \right\} \subseteq \bigoplus_{i=1}^n \mathbb{C}e_i = \mathbb{C}^n.$$

The action of S_n via ρ on V is

$$\sigma.f_i = \rho(\sigma)(e_i - e_n) = \mathrm{sgn}\,\sigma(e_{\sigma(i)} - e_{\sigma(n)}) = \mathrm{sgn}\,\sigma(f_{\sigma(i)} - f_{\sigma(n)}),$$

where $f_i := e_i - e_n$ for $i = 1, \ldots, n-1$ is a basis of V and $f_n := 0$. Note that, if we ignore the $\mathrm{sgn}\,\sigma$-part of the representation for a moment, then the trace of $\rho(\sigma)$ will get a positive contribution $(+1)$ for every i such that $\sigma(i) = i$ and exactly one negative negative contribution (-1 - from the row $\sigma(n)$ in $\rho(\sigma)$), all in all this adds up to the number of points fixed by σ minus 1. Thus, brining the sgn-part of the representation back into consideration we get that the character of this representation is the one mentioned in the example.

If we have a function, $f : S_n \to \mathbb{C}$, we may consider the polynomial

$$\mathrm{im}_f = \sum_{\sigma \in S_n} f(\sigma) \prod_{i=1}^n X_{i\sigma i}.$$

In the particular case where f is an irreducible character, χ_λ, of some representation of S_n we get what is called an immanant (a term that dates back to the paper [23] from 1934 by Littlewood and Richardson):

$$\mathrm{im}_{\chi_\lambda} = \mathrm{im}_\lambda := \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \prod_{i=1}^n X_{i\sigma i}.$$

Note that characters of representations of S_n are integer-valued, hence, we can also consider the immanant-polynomials over fields of positive characteristic if we should have any desire to do so.

Immanants are interesting to us, because one might hope that they can help bridge the gap between the determinant and the permanent which are the immanants corresponding to the 'extreme' partitions (1^n) and (n). To be specific; rather than make a projection from the determinant directly to the permanent it could be more feasible to take several steps by finding a series of intermediate immanants, $\det = \mathrm{im}_{\lambda_0}, \mathrm{im}_{\lambda_1}, \ldots, \mathrm{im}_{\lambda_r} = \mathrm{per}$, and projections $\pi_1, \ldots, \pi_r$ such that $\mathrm{im}_{\lambda_k} = \mathrm{im}_{\lambda_{k-1}}(\pi_k)$, for $k = 1, \ldots, r$. Now if the number of projections is reasonably small, e.g., bounded by a constant, and each projection is polynomial in size we get that the determinantal complexity of the permanent is polynomially bounded.

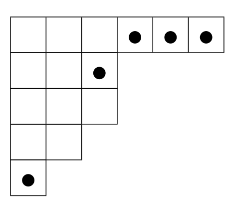
Indeed, Bürgisser proves in [6] that immanants corresponding to hook-partitions $(k\ 1^r) \vdash n$, and rectangular partitions $(k^r) \vdash n$ are VNP-complete if k grows at least as fast as cn^δ for some $c, \delta > 0$.

The result was generalised by Brylinski & Brylinski in [4]. If we consider the partition $\lambda = (\lambda_1\ \ldots\ \lambda_r)$, and define the separation of λ to be

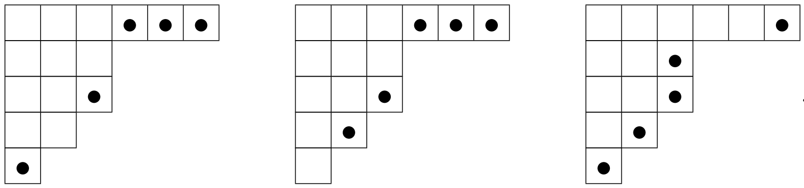
$$s = \max\left\{\lambda_i - \lambda_{i+1} | i = 1, \ldots, r\right\},$$

where $\lambda_{r+1} := 0$, then the corresponding immanant, im_λ, projects to per_s. Thus, because the separation of a k-hook-partition is $\lambda_1 - \lambda_2 = k - 1$, and that of the rectangular partition (k^r) is $\lambda_r - \lambda_{r+1} = k$, we get that if k grows sufficiently fast with n we get VNP-complete families of immanants as the permanent can be found using these projections.

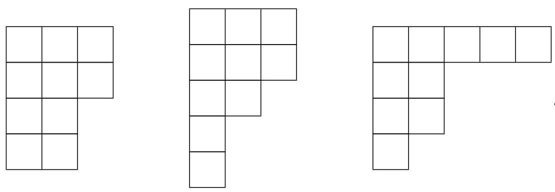
In order to present the theorem from [4], where they also specify what the projections are, we need to introduce the terms vertical and horizontal strips in a Young diagram. By a horizontal strip we mean a set of boxes such that they are all the bottommost in their column, and if a box lies in the strip then all boxes to its right in the same row will also be in the strip. A vertical strip is defined similarly; boxes in a vertical strip must be the rightmost in their respective rows, and if a box lies in the strip then all boxes below in the same column is in the strip as well. Below we see an example of a set of boxes which is neither a vertical strip (as there are several boxes in the first row), nor is it a horizontal strip (as not all boxes are the bottommost in their columns)



Here's an examples of two horizontal and one vertical strips of size 5 in a Young diagram of shape (6 3 3 2 1):



Note that 5 is the maximal size of a vertical strip in the diagram of this shape, as there are 5 rows, furthermore, a strip of maximal size is unique. However, we could increase the size of the horizontal strip by 1, as the number of columns (the width) of the diagram is 6. It should be clear that if we remove a strip (vertical or horizontal) from a Young diagram then we get a new Young diagram, the three examples above will look like



Now we write $\mu \leq_\ell \lambda$ and $\mu \leq'_\ell \lambda$ respectively when the partition μ corresponds to some diagram one get when removing a horizontal or vertical strip of size ℓ from a diagram of shape λ. The result by the Brylinskis can then be stated as:

Theorem 2.4.2. *Let $\lambda = (\lambda_1 \ \dots \ \lambda_r) \vdash n$, and fix some $\ell_1 \leq \lambda_1, \ell_2 \leq r$. The immanant im_λ projects to $\ell_1! \sum_{\mu \leq_{\ell_1} \lambda} \mathrm{im}_\mu$ and $\ell_2! \sum_{\mu \leq'_{\ell_2} \lambda} \mathrm{im}_\mu$.*

For $A_i \in \mathrm{Mat}_{(n-\ell_i)}(\mathbb{C})$ the projections are

1. $\ell_1! \sum_{\mu \leq_{\ell_1} \lambda} \mathrm{im}_\mu(A_1) = \mathrm{im}_\lambda \begin{pmatrix} A_1 & 0 \\ 0 & J \end{pmatrix}$, *where $J \in \mathrm{Mat}_{\ell_1}(\mathbb{C})$ is the square matrix with all entries equal to 1, and*

2. $\ell_2! \sum_{\mu \leq'_{\ell_2} \lambda} \mathrm{im}_\mu(A_2) = \mathrm{im}_\lambda \begin{pmatrix} A_2 & 0 \\ 0 & E \end{pmatrix}$, *where $E \in \mathrm{Mat}_{\ell_2}(\mathbb{C})$ is given by*

$$E = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ -1 & 1 & 2 & \ddots & \vdots \\ 1 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & \ell_2 - 1 \\ (-1)^{\ell_2 - 1} & \dots & 1 & -1 & 1 \end{pmatrix}.$$

For a detailed proof of the theorem I recommend reading the original paper by the Brylinskis. It consists of a few steps, that I shall summarise:

The fundamental part of the proof is based on the following formula that dictates how to compute immanants of block diagonal matrices (with the diagonal blocks $A \in \mathrm{Mat}_k(\mathbb{C})$, and $B \in \mathrm{Mat}_r(\mathbb{C})$, and $\lambda \vdash n = k + r$):

$$\mathrm{im}_\lambda \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sum_{\mu \vdash k, \nu \vdash r} c^\lambda_{\mu,\nu}\, \mathrm{im}_\mu(A)\, \mathrm{im}_\nu(B),$$

here $c^\lambda_{\mu,\nu}$ are the Littlewood-Richardson coefficients.

Using the orthogonality relations on the following identities

$$\mathrm{im}_{\nu_1}(J) = \ell_1! \langle \chi_{\nu_1}, 1_{S_{n-\ell_1}} \rangle, \text{ and } \mathrm{im}_{\nu_2}(E) = \ell_2! \langle \chi_{\nu_2}, \mathrm{sgn} \rangle \text{ (for } \nu_i \vdash n - \ell_i,\ i = 1, 2),$$

one gets that

$$\mathrm{im}_\lambda \begin{pmatrix} A_1 & 0 \\ 0 & J \end{pmatrix} = \ell_1! \sum_{\mu \vdash n-\ell_1} c^\lambda_{\mu,(\ell_1)}\, \mathrm{im}_\mu(A_1)$$

$$\mathrm{im}_\lambda \begin{pmatrix} A_2 & 0 \\ 0 & E \end{pmatrix} = \ell_2! \sum_{\mu \vdash n-\ell_2} c^\lambda_{\mu,(1^{\ell_2})}\, \mathrm{im}_\mu(A_2).$$

The proof is completed by showing that $c^\lambda_{\mu,(\ell_1)} = c^\lambda_{\mu,(1^{\ell_2})} = 1$ if μ is obtained by removing a horizontal and vertical strip respectively from λ, and $c^\lambda_{\mu,(\ell_1)} = c^\lambda_{\mu,(1^{\ell_2})} = 0$ else.

A Young diagram, T, contains a unique horizontal and vertical strip of maximal size, the maximal horizontal strip consists of all the boxes that are the bottommost in their respective columns, whereas, the vertical strip consist of the rightmost boxes in each row. Thus, if T is of type $\lambda = (\lambda_1 \ \ldots \ \lambda_{r_0})$ and the separation is found to be $s = \lambda_i - \lambda_{i+1}$, and $i < r_0$ we apply the maximal 'vertical strip-projection' from theorem 2.4.2, i.e., the one where $\ell_2 = r_0$, on im_λ, hence, we get

$$r_0!\, \mathrm{im}_{\lambda_{(1)}} \left(X_{(1)}\right) = \mathrm{im}_\lambda \begin{pmatrix} X_{(1)} & 0 \\ 0 & E_{(0)} \end{pmatrix}, \text{ where } \lambda_{(1)} \leq'_{r_0} \lambda.$$

Let $\lambda_{(1)} = (\lambda_{1,1} \ \ldots \ \lambda_{1,r_1})$ then the separations of λ and $\lambda_{(1)}$ are the same. We continue to do maximal 'vertical strip-projections' on $\mathrm{im}_{\lambda_{(1)}}, \ldots, \mathrm{im}_{\lambda_{(k-1)}}$ until we get

$$r_0! \cdots r_{k-1}!\, \mathrm{im}_{\lambda_{(k)}} \left(X_{(k)}\right) = \mathrm{im}_\lambda \begin{pmatrix} X_{(k)} & & & \\ & E_{(k-1)} & & \\ & & \ddots & \\ & & & E_{(0)} \end{pmatrix}$$

with $\lambda_{(k)} = (\lambda_{k,1} \ \dots \ \lambda_{k,r_k})$ such that $s = \lambda_{k,r_k}$. Now we do $r_k - 1$ successive maximal 'horizontal strip-projections' which gives us a partition $\mu = (s)$, i.e., we have made a projection from im_λ to the s-permanent.

This proofs the following corollary.

Corollary 2.4.3. *Let* $(\mathrm{im}_{\lambda^{(n)}})$ *be a p-family of immanants such that the separation of* $\lambda^{(n)}$ *is greater than* cn^δ *for some* $c, \delta > 0$ *and* $n \gg 0$*, then* $(\mathrm{im}_{\lambda^{(n)}})$ *is VNP-complete.*

This might lead to optimism amongst those who think there is a suitable projection from the determinant to the permanent, however, the next chapter might be a cure for that optimism.

Chapter 3

Main Result

In this chapter we shall focus on the immanant corresponding to the partition $(2\ 1^{n-2})$, which is defined for $n \geq 2$. In particular we will show a quadratic lower bound of the determinantal complexity of $\mathrm{im}_{(2\ 1^{n-2})}$, furthermore we shall present an algorithm due to Barvinok, see [1], that computes this immanant in polynomial time. When we consider the restriction of the 2-hook-immanant to the group of invertible matrices there is a significant improvement due to Bürgisser, see [5], which we shall briefly discuss. Finally we shall present a third method to compute $\mathrm{im}_{(2\ 1^{n-2})}(A)$ which in general does not run in polynomial time but works well if the number of non-zero diagonal entries of A is low, say $O(\log n)$.

We shall primarily work over the complex numbers, but we will also take a peak at the determinantal complexity of $\mathrm{im}_{(2\ 1^{n-2})}$ in positive characteristics.

3.1 On the determinantal complexity of the 2-hook-immanant

Recall that the 2-hook-immanant is the polynomial

$$\mathrm{im}_n := \mathrm{im}_{(2\ 1^{n-2})} = \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \, (\mathrm{fix}_n \, \sigma - 1) \prod_{i=1}^{n} X_{i\sigma i},$$

where $\mathrm{fix}_n : S_n \to \mathbb{N}$ is defined by $\mathrm{fix}_n \, \sigma = |\{i \in [n] | \sigma i = i\}|$.

Seeing that the 2-hook-partition is the one that deviates the least from (1^n) we could expect that the polynomial im_n should be close or somehow related to the determinant $\det_m = \mathrm{im}_{(1^m)}$.

And indeed, we do have that we can project from im_n to $\det_{n-2}$ as

$$2\det_{n-2}(A) = \operatorname{im}_n \begin{pmatrix} A & & \\ & 1 & 1 \\ & 1 & 1 \end{pmatrix}$$

which is just a special case of the projections from theorem 2.4.2. An interesting question is now if we can project the other way, i.e., what is the determinantal complexity of the 2-hook-immanant?

The main result is the following.

Theorem 3.1.1. *Both the determinantal complexity,* $\operatorname{dc}(\operatorname{im}_n)$*, and the bounded determinantal complexity,* $\overline{\operatorname{dc}}(\operatorname{im}_n)$*, of the 2-hook-immanant are bounded below by* $\frac{n^2-n+2}{2}$.

The theorem is a direct consequence of remark 1.2.5 and theorem 2.3.10 when Katz' dimension formula is applied to this lemma:

Lemma 3.1.2. *Let* $n > 1$*. Consider the matrix*

$$A_n = \begin{pmatrix} \frac{n-3}{n-1} & 1 & \\ 1 & 1 & \\ & & I_{n-2} \end{pmatrix} \in \operatorname{Mat}_n(\mathbb{C}).$$

For $n \neq 3$ *we have* $\operatorname{rank} \operatorname{Hes} \operatorname{im}_n(A_n) = n^2 - n + 2$.

Proof. The entries of $\operatorname{Hes} \operatorname{im}_n$ are of the form

$$\frac{\partial^2 \operatorname{im}_n}{\partial X_{ij} \partial X_{kl}}.$$

Obviously we have

$$\frac{\partial^2 \operatorname{im}_n}{\partial X_{ij} \partial X_{il}} = \frac{\partial^2 \operatorname{im}_n}{\partial X_{ij} \partial X_{kj}} = 0,$$

since each monomial in im_n contains exactly one variable with first index equal to r and exactly one variable with second index equal to s for $1 \leq r, s \leq n$.

If we evaluate at A_n then

$$\frac{\partial^2 \operatorname{im}_n}{\partial X_{ij} \partial X_{kl}}(A_n) = \sum_{\substack{\sigma \in S_n \\ \sigma i = j, \sigma k = l}} \operatorname{sgn} \sigma (\operatorname{fix} \sigma - 1) \prod_{r \neq i,k} a_{r\sigma r}.$$

If $i, k > 2$ we get

$$\frac{\partial^2 \operatorname{im}_n}{\partial X_{ii} \partial X_{kk}}(A_n) = \operatorname{sgn} \operatorname{id}(\operatorname{fix}_n \operatorname{id} - 1) \prod_{r \neq i,k} a_{rr} + \operatorname{sgn}(1\ 2)(\operatorname{fix}_n(1\ 2) - 1) \prod_{r \neq i,k} a_{r(1\ 2)(r)}$$
$$= (n-1)\frac{n-3}{n-1} - (n-3) = 0,$$

as all permutations different from id and $(1\ 2)$ that fix i and k will have a non-fixed point $r > 2$ and $a_{r\sigma r} = 0$.

If we take $i, j > 2$ to be two distinct numbers we see that $\prod_{r \neq i,k} a_{r\sigma r} = 0$ unless $k = j$ and $\sigma(k) = i$, as a permutation of length at least 3, $(i\ j\ m\ \ldots)$, will pick out the entry $a_{jm} = 0$ in A. The same is true for permutations which are not just a transposition, e.g., $(i\ j)(m\ q \ldots)$ will pick out $a_{mq} = 0$. However, if $k = j$ and $i = l$ then

$$\begin{aligned}\frac{\partial^2 \operatorname{im}_n}{\partial X_{ij} \partial X_{ji}}(A_n) =& \operatorname{sgn}(i\ j)(\operatorname{fix}_n(i\ j) - 1) \prod_{r \neq i,j} a_{r(i\ j)(r)} \\ &+ \operatorname{sgn}((1\ 2)(i\ j))(\operatorname{fix}_n((1\ 2)(i\ j)) - 1) \prod_{r \neq i,j} a_{r(1\ 2)(i\ j)(r)} \\ =& -(n-3)\frac{n-3}{n-1} + n - 5 \\ =& \frac{-4}{n-1}.\end{aligned}$$

Now suppose $2 = i = l < j = k$, then $\prod_{r \neq 2,j} a_{r\sigma r} = 0$ for any $\sigma \neq (2\ j)$, which can be seen by the same argument with permutations containing multiple cycles as before. We get

$$\begin{aligned}\frac{\partial^2 \operatorname{im}_n}{\partial X_{2j} \partial X_{j2}}(A_n) =& \operatorname{sgn}(2\ j)(\operatorname{fix}_n(2\ j) - 1) \prod_{r \neq 2,j} a_{r(2\ j)(r)} \\ =& -(n-3)\frac{n-3}{n-1} = \frac{(n-3)^2}{1-n}.\end{aligned}$$

If $i = j = 2 \leq k = l$ then the only interesting permutation is the identity and

$$\frac{\partial^2 \operatorname{im}_n}{\partial X_{22} \partial X_{kk}}(A_n) = (n-1)\frac{n-3}{n-1} = n - 3.$$

If $i = l = 1 < j = k$ then we are only concerned about the permutation $(1\ j)$; we get

$$\frac{\partial^2 \operatorname{im}_n}{\partial X_{1j} \partial X_{j1}}(A_n) = \operatorname{sgn}(1\ j)(n-3) = 3 - n.$$

If $\{i,j\}=\{1,2\}$ and $k=l$ then

$$\frac{\partial^2 \operatorname{im}_n}{\partial X_{12}\partial X_{kk}}(A_n)=\operatorname{sgn}(1\ 2)(n-3)=3-n.$$

Finally if $i=j=1<k=l$, we get

$$\frac{\partial^2 \operatorname{im}_n}{\partial X_{11}\partial X_{kk}}(A_n)=n-1.$$

Using that A_n is symmetric and that the immanant is invariant under transposition we have

$$\frac{\partial^2 \operatorname{im}_n}{\partial X_{kl}\partial X_{ij}}(A_n)=\frac{\partial^2 \operatorname{im}_n}{\partial X_{ij}\partial X_{kl}}(A_n)=\frac{\partial^2 \operatorname{im}_n}{\partial X_{ji}\partial X_{lk}}(A_n).$$

With all this in mind the following is sufficient to describe the Hessian matrix of im_n evaluated at A_n:

$$\frac{\partial^2 \operatorname{im}_n}{\partial X_{ij}\partial X_{kl}}(A_n)=\begin{cases} n-1 & \text{if } i=j=1<k=l \\ 3-n & \text{if } \{i,j\}=\{1,2\},\ \text{and } k=l>2 \\ & \quad\text{or if } i=l=1<j=k \\ n-3 & \text{if } i=j=2<k=l \\ \frac{(n-3)^2}{1-n} & \text{if } i=l=2<j=k \\ \frac{-4}{n-1} & \text{if } 2<i=l<j=k \\ 0 & \text{else.} \end{cases} \tag{3.1}$$

Thus, in the row of the Hessian indexed by $ij\neq 12,21,rr$ for any $r=1,\dots,n$ the only non-zero value is found in column ji and this value is either $3-n$, $\frac{(n-3)^2}{n-1}$, or $\frac{-4}{n-1}$. By symmetry we see that the only non-zero value in column ji is in row ij.

Let $M[C,C']$ denote the matrix obtained from a matrix M by only including entries m_{rs} if and only if $(r,s)\in C\times C'$ for some sets of indices C and C'. It remains to describe the last $n+2$ rows (and columns) indexed by

$$S:=\{11,12,21,22,33,\dots,nn\}\subseteq\{ij|1\leq i,j\leq n\}$$

in our Hessian matrix. By the list (3.1) above we get

$$\operatorname{Hes}\operatorname{im}_n(A_n)[S,S]=\begin{pmatrix} 0 & 0 & 0 & n-1 & n-1 & \dots & n-1 \\ 0 & 0 & 3-n & 0 & 3-n & \dots & 3-n \\ 0 & 3-n & 0 & 0 & 3-n & \dots & 3-n \\ n-1 & 0 & 0 & 0 & n-3 & \dots & n-3 \\ n-1 & 3-n & 3-n & n-3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & 3-n & 3-n & n-3 & 0 & \dots & 0 \end{pmatrix}.$$

This is a matrix of rank 4, as the first four rows 11,12,21, and 22 are linearly independent, and $\frac{n-3}{n-1}$ times row 11 plus rows 12, 21, and 22 equals row rr for any $3 \leq r \leq n$. Notice that $\operatorname{Hes}\operatorname{im}_n(A_n)[S, S^c] = \operatorname{Hes}\operatorname{im}_n(A_n)[S^c, S]^t = 0$.

Combining these observations we get

$$\begin{aligned}\operatorname{rank}\operatorname{Hes}\operatorname{im}_n(A_n) &= \operatorname{rank}\operatorname{Hes}\operatorname{im}_n(A_n)[S,S] + \operatorname{rank}\operatorname{Hes}\operatorname{im}_n(A_n)[S^c,S^c] \\ &= 4 + n^2 - (n+2) = n^2 - n + 2\end{aligned}$$

which proves the lemma. □

Proof of Theorem 3.1.1. First, if $n \neq 3$, notice that $A_n \in V(\operatorname{im}_n)$, and that A_n is a smooth point as

$$\frac{\partial \operatorname{im}_n}{\partial X_{11}}(A_n) = n - 1.$$

Apply Katz' dimension formula, theorem 1.2.2, to obtain $\dim V(\operatorname{im}_n)^* \geq n^2 - n$. Using remark 1.2.5 in the standard way we obtain the promised quadratic lower bound of the determinantal complexity

$$\operatorname{dc}(\operatorname{im}_n) \geq \frac{n^2 - n + 2}{2}.$$

Likewise, from theorem 2.3.10 we get the same bound on the bounded determinantal complexity:

$$\overline{\operatorname{dc}}(\operatorname{im}_n) \geq \frac{\dim V(\operatorname{im}_n)^* + 2}{2} \geq \frac{n^2 - n + 2}{2}.$$

For completeness we consider the case $n = 3$: Take the point

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \in V(\operatorname{im}_3),$$

the rank of the Hessian at this (smooth) point is 9, again we may apply Katz' dimension formula, to get $\dim V(\operatorname{im}_3)^* = 7$ thus

$$\overline{\operatorname{dc}}(\operatorname{im}_3), \operatorname{dc}(\operatorname{im}_3) \geq \frac{9}{2} > \frac{3^2 - 3 + 2}{2}.$$

□

Though I have not found a series of points, $B_n \in \operatorname{Mat}_n(\mathbb{C})$, that improves the lower bound of the rank of the Hessian compared to that in lemma 3.1.2, I see no reason the bound cannot be improved, probably even to n^2. At least for small n it is possible to find better points:

Example 3.1.3. Consider the matrices

$$B_4 = \begin{pmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 \end{pmatrix}, \; B_5 = \begin{pmatrix} 1 & 2 & 0 & 0 & 2 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 4 & -1 & -2 & 0 \\ 1 & 2 & 2 & 3 & 0 \\ 1 & 1 & 2 & 2 & 1 \end{pmatrix}$$

$$B_6 = \begin{pmatrix} 1 & 0 & -1 & 2 & 3 & 1 \\ 0 & -2 & 3 & 1 & 2 & -1 \\ 0 & -2 & 1 & 2 & 0 & 1 \\ -1 & -1 & 2 & 3 & 0 & -2 \\ 1 & 1 & 1 & -2 & 3 & 0 \\ 0 & -7 & 14 & 7 & 7 & 10 \end{pmatrix}, \; B_7 = \begin{pmatrix} 1 & 2 & 1 & 3 & 1 & 0 & 95 \\ 2 & 3 & 1 & 2 & 1 & 2 & 0 \\ 1 & 2 & 1 & 1 & 2 & 3 & 95 \\ 2 & 3 & 1 & 2 & 3 & 1 & 190 \\ 3 & 2 & 2 & 0 & 2 & 0 & 285 \\ 1 & 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 3 & 2 & 1 & 2 & 3 & -227 \end{pmatrix}$$

We have $\operatorname{rank} \operatorname{Hes} \operatorname{im}_n(B_n) = n^2$ for $n = 4, 5, 6, 7$. In the appendix there are some lines of code for Macaulay2 which can be used to do the actual calculations.

As mentioned I see no reason why it should not be possible to find points like these giving full rank to the Hessian for general n. However, the problem is of course to find points that are sufficiently easy to analyse when one computes the Hessian. For a geometer it might be interesting to further investigate the maximal rank of Hessian matrices simply to find out if $V(\operatorname{im}_n)^*$ is a hypersurface or not, because if this is not a hypersurface then $V(\operatorname{im}_n)$ is somewhat peculiar as it will be ruled in projective spaces of positive dimension.

It should be mentioned that the four points, B_4, B_5, B_6, and B_7, found in example 3.1.3 are not related to each other but rather picked at random from the zero-set of the immanants. Again, I refer to the appendix for a way to find points at which the immanant vanishes.

The determinantal complexity of the 2-hook-immanant in positive characteristics

Even though some of the representation theory for the symmetric group breaks down when we move from the complex numbers to positive characteristics it still makes sense to consider the immanants as polynomials over fields $K \supseteq \mathbb{F}_p$, because characters of the symmetric group are integer-valued. E.g., $\operatorname{im}_3 \in \mathbb{F}_2[X_{ij}]$ is the third Hamilton-polynomial,

$$\operatorname{im}_3 = 2X_{11}X_{22}X_{33} - (X_{12}X_{23}X_{31} + X_{13}X_{21}X_{32}) = X_{12}X_{23}X_{31} + X_{13}X_{21}X_{32},$$

which we may write as a 3×3-determinant:

$$\mathrm{im}_3 = \det_3 \begin{pmatrix} 0 & X_{12} & X_{13} \\ X_{21} & 0 & X_{23} \\ X_{31} & X_{32} & 0 \end{pmatrix}.$$

Now, as the immanants also exist over more general fields than the complex numbers we may examine their determinantal complexities over these fields also.

One should note that in the construction of Valiant's projections in theorem 2.2.2 and corollary 2.2.3 we do not use any special properties of the complex numbers which other commutative rings might not have. I.e., the notion of determinantal complexity can be carried over when working with much more general rings, and in particular when working over polynomial rings over fields of positive characteristics.

Once again the Hessian matrix plays an important role in order to prove lower bounds.

If we use the chain rule twice on the polynomial f defined by

$$\begin{tikzcd} \mathbb{A}^n \arrow[r, "L"] \arrow[rr, dashed, bend right, "f=\det_m \circ L"'] & \mathbb{A}^{m^2} \arrow[r, "\det_m"] & K \end{tikzcd},$$

where $L(X) = AX + B$ is some affine map, we get

$$\operatorname{Hes} f = A^t \operatorname{Hes} \det_m A,$$

thus, $\operatorname{rank} \operatorname{Hes} f \leq \operatorname{rank} \operatorname{Hes} \det_m$. If $x \in V(f)$ then $L(x) \in V(\det_m)$ and then $\operatorname{rank} \operatorname{Hes} \det_m(L(x)) \leq 2m$, and so the determinantal complexity of f is at least $\frac{1}{2} \operatorname{rank} \operatorname{Hes} f(x)$.

Remark 3.1.4. When the characteristic of our field is at least 3 we get the same lower bound of the determinantal complexity of the 2-hook-immanant as in the complex case.

Indeed, let $p > 2$ be a prime number, if $p \nmid (n-1)(n-3)$ then

$$\mathrm{dc}(\mathrm{im}_n) \geq \frac{n^2 - n + 2}{2}.$$

In order to show this it suffices to find a point $A_n \in V(\mathrm{im}_n)$ such that $\operatorname{rank} \operatorname{Hes}(A_n) = n^2 - n + 2$, but as in lemma 3.1.2 we may take

$$A_n = \begin{pmatrix} \frac{n-3}{n-1} & 1 & \\ 1 & 1 & \\ & & I_{n-2} \end{pmatrix} \in \mathrm{Mat}_n(\mathbb{F}_p)$$

which makes sense as $n-1 \neq 0$ in $\mathbb{F}_p$, and the computations from the proof of the lemma carries over. Furthermore, as $p > 2$ and $n-3 \neq 0$ in $\mathbb{F}_p$ the non-zero entries in the characteristic zero case are also non-zero in our positive characteristic case.

In characteristic 2 the permanent and determinant-problem does not exist, as the two polynomials are the same, thus, I have not prioritised to study the determinantal complexity of the immanant over such fields very thoroughly. There is, however, a quadratic lower bound also in this case.

Proposition 3.1.5. *Let $n = 2m+1$ be odd, let $J = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Mat}_2(\mathbb{F}_2)$, and consider the block diagonal-matrix*

$$E_n = \begin{pmatrix} J & & & \\ & \ddots & & \\ & & J & \\ & & & 1 \end{pmatrix} \in \mathrm{Mat}_n(\mathbb{F}_2).$$

$E_n \in V(\mathrm{im}_n)$, and the rank of $\mathrm{Hes}\,\mathrm{im}_n(E_n)$ *is in $\Omega(n^2)$, thus,* $\mathrm{dc}(\mathrm{im}_n) = \Omega(n^2)$.

Proof. First of all note that a monomial $X_{1\sigma 1} \cdots X_{n\sigma n}$ occur in im_n (with coefficient 1) if and only if $\mathrm{fix}_n\,\sigma$ is even, thus, $\mathrm{im}_n(E_n) = 0$ as the only permutation for which $E_{1\sigma 1} \cdots E_{n\sigma n} \neq 0$ is $\sigma = \mathrm{id}$ which has $n = 2m+1$ fixed points. Now it suffices to find $\Omega(n^2)$ rows in $\mathrm{Hes}\,\mathrm{im}_n(E_n)$ with exactly on non-zero entry. In order to do so, take $K = 2k$ and $L = 2l$ for $1 \leq k < l \leq m$ and consider

$$h(i,j) := \frac{\partial^2 \mathrm{im}_n}{\partial X_{KL} \partial X_{ij}}(E_n).$$

We want to find all i, j, σ such that $1 \leq i, j \leq n$ and $\sigma \in S_n$ satisfy $\sigma i = j$, $\sigma K = L$, $\mathrm{fix}_n\,\sigma$ is even, and $\prod_{r \neq K,i} E_{r\sigma r} = 1$.

If we focus on the last part, $\prod_{r \neq K,i} E_{r\sigma r} = 1$, we get that i must be equal to L as there is no possibilities for σ such that $E_{L\sigma L} = 1$ when $\sigma L \neq L$ (bear in mind that $\sigma K = L$ and $K \neq L$).

Let us analyse our possible choices of j. If we take $j = K$ our only choice of permutation such that $\prod_{r \neq K,L} E_{r\sigma r} = 1$ is $\sigma = (K\ L)$, but $\mathrm{fix}_n(K\ L) = n-2$ is odd and, thus, the monomial $X_{11} \cdots X_{K-1K-1} X_{KL} X_{K+1K+1} \cdots X_{L-1L-1} X_{LK} X_{L+1L+1} \cdots X_{nn}$ does not occur in im_n.

We see that we must take $j \neq K$ in order to get something non-zero, furthermore, as $\sigma j \neq j$ we must have that j is odd and $\sigma j = j+1$ (otherwise $E_{j\sigma j} \neq 1$). Now, $j+1$ is even, and unless $j+1 = K$ we get $\sigma(j+1) = j+1 = \sigma j$, if we require

$E_{j+1\sigma(j+1)} \neq 0$. This is absurd, thus, $j+1 = K$ and σ must equal the 3-cycle $(K\ L\ K-1)$.

Because $\prod_{r\neq K,L} E_{r\sigma r} = 1$, and $\prod_{r=1}^{n} X_{r\sigma r}$ occurs in im_n with coefficient 1, we get that $h(i,j) = 1$ if and only if $i = L$ and $j = K-1$. This means that in rows indexed by KL we have exactly one non-zero entry. The number of such rows is equal to the number of pairs $1 \leq k < l \leq m = \frac{n-1}{2}$ which is $\binom{m}{2} = \frac{n^2-4n+3}{8} = \Omega(n^2)$. $\square$

I think it is worth noting that the projection from the 2-hook-immanant to the determinant we have in characteristic 0 does not exist in characteristic 2 because

$$\mathrm{im}_n \begin{pmatrix} A & & \\ & 1 & 1 \\ & 1 & 1 \end{pmatrix} = 2\det_{n-2}(A) = 0.$$

Likewise I suspect that if one did come up with a clever series of projections that went from the determinant via some intermediate immanants to the permanent in characteristic 0, then in characteristic p it would probably break down somewhere as n grows larger than p. For instance if we apply the second projection from theorem 2.4.2 on a $k+1$-hook-immanant to obtain the k-permanent we also get

$$\mathrm{im}_{(k+1\ 1^r)} \begin{pmatrix} A & & & & & \\ & 1 & 1 & 0 & \dots & 0 \\ & -1 & 1 & 2 & \ddots & \vdots \\ & 1 & -1 & \ddots & \ddots & 0 \\ & \vdots & \ddots & \ddots & 1 & r \\ & (-1)^r & \dots & 1 & -1 & 1 \end{pmatrix} = (r+1)!\,\mathrm{per}_k(A) = 0$$

when working over fields of characteristic $p \leq r+1$. Thus, asymptotically the permanent cannot be computed via these projections of hook-immanants, assuming that r is growing along with k. More generally, if we in characteristic 0 make a projection where we remove a vertical strip of size at least p then the copy of this projection will be the zero-projection in characteristic p.

3.2 Ways of computing the 2-hook-immanant

An Exponential Time Algorithm

One approach to computing $\mathrm{im}_n(A)$ is based on the multilinearity of (general) immanants, along with a property that is special for the 2-hook-immanant.

The idea is developed from the observation that $\operatorname{im}_n(A) = -\det_n(A)$ if $A = (a_{ij})$ satisfy $a_{11} = \cdots = a_{nn} = 0$.

If we define $A_\sigma := \prod_{i=1}^n a_{i\sigma i}$ for $\sigma \in S_n$, and $A = (a_{ij}) \in \operatorname{Mat}_n(\mathbb{C})$ (the characteristic of the field really isn't important, but let's stick to the complex numbers) we get that the defining formula of the immanant is

$$\operatorname{im}_n(A) = \sum_{\sigma \in S_n} \operatorname{sgn}\sigma(\operatorname{fix}_n \sigma - 1)A_\sigma = \sum_{r=0}^{n}(r-1)\sum_{\sigma \in F(r)} \operatorname{sgn}\sigma A_\sigma,$$

where $F(r) := \operatorname{fix}_n^{-1}(\{r\}) \subseteq S_n$ is the set of permutations of $[n]$ with exactly r fixed points.

Note that if $A_\sigma = 0$ for any $\sigma \in F(r)$ for all but one value of r, say r_0, then

$$\operatorname{im}_n(A) = (r_0 - 1)\sum_{\sigma \in F(r_0)} \operatorname{sgn}\sigma A_\sigma = (r_0 - 1)\sum_{\sigma \in S_n} \operatorname{sgn}\sigma A_\sigma = (r_0 - 1)\det_n A.$$

Matrices, M, which have a special structure, namely

$$M = P\begin{pmatrix} T & B \\ B' & M_0 \end{pmatrix} P^{-1}, \tag{3.2}$$

where T is a (lower or upper) triangular $r_0 \times r_0$-matrix, B or B' is the 0-matrix, $M_0 \in \operatorname{Mat}_{n-r_0}(\mathbb{C})$ has a diagonal with all entries equal to zero, and P is a permutation matrix, all satisfy $M_\sigma = 0$ for $\sigma \notin F(r_0)$.

If we combine this observation with the multilinearity of the immanant we get for $A = (a^1, \ldots, a^n) = (a^i_j)$, with $a^i \in \mathbb{C}^n$, and $a^i_j \in \mathbb{C}$ that

$$\operatorname{im}_\lambda(A) =$$

$$\operatorname{im}_\lambda \begin{pmatrix} \vdots & & \vdots & a^i_1 & \vdots & & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & & \vdots & a^i_{i-1} & \vdots & & \vdots \\ a^1 & \cdots & a^{i-1} & 0 & a^{i+1} & \cdots & a^n \\ \vdots & & \vdots & a^i_{i+1} & \vdots & & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & & \vdots & a^i_n & \vdots & & \vdots \end{pmatrix} + \operatorname{im}_\lambda \begin{pmatrix} \vdots & & \vdots & 0 & \vdots & & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & & \vdots & 0 & \vdots & & \vdots \\ a^1 & \cdots & a^{i-1} & a^i_i & a^{i+1} & \cdots & a^n \\ \vdots & & \vdots & 0 & \vdots & & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & & \vdots & 0 & \vdots & & \vdots \end{pmatrix},$$

hence, if we take $\lambda = (2\ 1^{n-2})$, computing $\operatorname{im}_n(A)$ can be reduced to computing two immanants of matrices that are simpler, as they are closer to the nice structure as in (3.2).

Now, if we walk along the diagonal of A splitting the immanant into two immanants every time we reach a non-zero entry we will end up with 2^k immanants, where k is the number of non-zero diagonal entries. Each of these immanants corresponds to $\mathrm{im}_n(M)$ for some M with structure as in (3.2), hence, each of these are easily computed using the determinant.

Example 3.2.1. Let us consider the 4×4-matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{14} \\ \vdots & \ddots & \vdots \\ a_{41} & \dots & a_{44} \end{pmatrix}.$$

We first split it into two matrices by considering the first column as the sum $(a_{11}, 0, 0, 0)^t + (0, a_{21}, a_{31}, a_{41})^t$:

$$A \rightsquigarrow A_{(0)} = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \ A_{(1)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

We now proceed with the second columns

$$A_{(0)} \rightsquigarrow A_{(0,0)} = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \ A_{(0,1)} = \begin{pmatrix} 0 & 0 & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & 0 & a_{33} & a_{34} \\ a_{41} & 0 & a_{43} & a_{44} \end{pmatrix},$$

$$A_{(1)} \rightsquigarrow A_{(1,0)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}, \ A_{(1,1)} = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}.$$

We continue until we get 16 matrices:

$$A_{(0,0,0,0)} = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{pmatrix}, \ A_{(0,0,0,1)} = \begin{pmatrix} 0 & a_{12} & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

$$A_{(0,0,1,0)} = \begin{pmatrix} 0 & a_{12} & 0 & a_{14} \\ a_{21} & 0 & 0 & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}, \ A_{(0,0,1,1)} = \begin{pmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & 0 & a_{44} \end{pmatrix},$$

$$A_{(0,1,0,0)} = \begin{pmatrix} 0 & 0 & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & 0 & 0 & a_{34} \\ a_{41} & 0 & a_{43} & 0 \end{pmatrix}, \ A_{(0,1,0,1)} = \begin{pmatrix} 0 & 0 & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & 0 & 0 & 0 \\ a_{41} & 0 & a_{43} & a_{44} \end{pmatrix},$$

$$A_{(0,1,1,0)} = \begin{pmatrix} 0 & 0 & 0 & a_{14} \\ a_{21} & a_{22} & 0 & a_{24} \\ a_{31} & 0 & a_{33} & a_{34} \\ a_{41} & 0 & 0 & 0 \end{pmatrix}, \ A_{(0,1,1,1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix},$$

$$A_{(1,0,0,0)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & 0 & a_{34} \\ 0 & a_{42} & a_{43} & 0 \end{pmatrix}, \ A_{(1,0,0,1)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ 0 & 0 & a_{23} & 0 \\ 0 & a_{32} & 0 & 0 \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

$$A_{(1,0,1,0)} = \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} \\ 0 & 0 & 0 & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & 0 & 0 \end{pmatrix}, \ A_{(1,0,1,1)} = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 \\ 0 & a_{42} & 0 & a_{44} \end{pmatrix},$$

$$A_{(1,1,0,0)} = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & a_{43} & 0 \end{pmatrix}, \ A_{(1,1,0,1)} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix},$$

$$A_{(1,1,1,0)} = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A_{(1,1,1,1)} = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix}.$$

If we define $\ell : \{0,1\}^4 \to \mathbb{N}$ by $\ell(e_1, \dots, e_4) = \sum_i e_i$ we get

$$\mathrm{im}_4(A) = \sum_{e \in \{0,1\}^4} \mathrm{im}_4(A_e) = \sum_{e \in \{0,1\}^4} (\ell(e) - 1) \det{}_4(A_e).$$

Using this method to compute the general 2-hook-immanant is of course a rather tedious job if k (the number of non-zero diagonal entries) is large, but if $k \leq c \log n$, for some constant c, then the method will run using $O(n^{3+c})$ arithmetic operations, as we only need to compute at most $2^k \leq 2^{c \log n} = n^c$ determinants all of which can be computed in $O(n^3)$ time. If, e.g., $a_{11} = 0$ in example 3.2.1 then the last eight immanants would not appear.

A Polynomial Time Algorithm

To my knowledge, the (asymptotically) best algorithm to compute $\mathrm{im}_n(A)$ for some general $A \in \mathrm{Mat}_n(\mathbb{C})$ was made by Barvinok in [1]. Actually he made an algorithm to compute general immanants but we shall focus only on the 2-hook-immanant.

Theorem 3.2.2 (Barvinok's Algorithm). *Given $A \in \mathrm{Mat}_n(\mathbb{C})$ the 2-hook-immanant can be evaluated at A using $O(n^{10})$ calculations.*

We shall only present how the algorithm works, and leave the details from [1] to the reader.

Some notation: Set $V := \mathbb{C}^n$ with basis $\{e_i\}$, $i = 1, \ldots, n$, and let $\mathrm{GL}(V)$ act on $V^{\otimes n}$ by

$$G.(v_1 \otimes \cdots \otimes v_n) = (Gv_1) \otimes \cdots \otimes (Gv_n).$$

Define $f \in V^{\otimes n}$ to be the element

$$f = \sum_{\sigma \in S_n} \operatorname{sgn} \sigma (\mathrm{fix}_n \sigma - 1) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}$$

Consider the vector space L spanned by the set $\{G.f | G \in \mathrm{GL}(V)\}$, and let $\{f_t\}$ be a basis of L. The dimension of L is $O(n^4)$. Let $a_t \in \mathbb{C}$ be the coordinates of f in the basis $\{f_t\}$, i.e.,

$$f = \sum_t a_t f_t.$$

Let e_{ij}, $1 \leq i, j \leq n$, be the standard basis of the vector space $\mathrm{End}(V)$, i.e., e_{ij} is the rank one operator that sends e_j to e_i, and denote by $E_{ij}(\alpha) \in \mathrm{End}(V)$ the following operators

$$E_{ij}(\alpha) = \begin{cases} I + \alpha e_{ij} & \text{if } i \neq j \\ I + (\alpha - 1) e_{ii} & \text{else,} \end{cases}$$

where $\alpha \in \mathbb{C}$, and $I \in \mathrm{End}(V)$ is the identity operator.

Finally define $\gamma_t = \langle e_1 \otimes \cdots \otimes e_n, f_t \rangle$, and, given $A \in \mathrm{Mat}_n(\mathbb{C})$, let $G \in \mathrm{End}(V)$ be the operator with matrix representation A (in the basis $\{e_i\}$). Note that

$$\mathrm{im}_n(A) = \langle e_1 \otimes \cdots \otimes e_n, G^* f \rangle .$$

The algorithm goes as follows: Factor G^* into elementary operators of the type $E_{ij}(\alpha)$ using no more than $O(n^2)$ factors. I.e., $G^* = \prod_{k=1}^m E_{i_k j_k}(\alpha_k)$, for some $m = O(n^2)$. Basically this goes like Gaussian elimination. Now compute $G^* f = \sum \beta_t f_t$ by using the factorisation of G^* just found on $\sum a_t f_t$. Finish by computing $\sum \beta_t \gamma_t$ as we have

$$\mathrm{im}_n(A) = \left\langle e_1 \otimes \cdots \otimes e_n, \sum \beta_t f_t \right\rangle = \sum \beta_t \gamma_t .$$

The time required to carry out the algorithm is determined by the second step in which we compute $G^* f = \sum \beta_t f_t$, which should not be hard to verify as the other steps are (essentially) Gaussian elimination and taking an inner product. This second step consists of $O(n^2)$ linear operations on vectors in a vector space of dimension $O(n^4)$, i.e., each will need $O(n^8)$ additions and multiplications, hence, we are required to do $O(n^{10})$ calculations in total.

One note on the factorisation of G^*. In [1] Barvinok allows n^3 factors, however this is not necessary. If we for a moment do not distinguish between operators in $\mathrm{End}(V)$ and matrices from $\mathrm{Mat}_n(\mathbb{C})$, then we see that a matrix $A \in \mathrm{Mat}_n(\mathbb{C})$ of rank r can be factored as

$$A = R^{-1} \prod_{k=r+1}^{n} E_{kk}(0) C^{-1},$$

where R is a series of row-operations that brings A to reduced row-echelon form, i.e., $RA = H$ with H on reduced row-echelon form. We may now apply a series of column-operations, C, on H to get at diagonal matrix, D, with ones in the first r diagonal-entries and zeroes in the last $n - r$. The length of both sequences R and C can be bounded by $O(n^2)$ as interchanging rows/columns i and j can be done by the series of operations

$$E_{jj}(-1) E_{ij}(1) E_{ji}(-1) E_{ij}(1),$$

thus, we need only use $O(1)$ operations to change each entry of A (and H) to get to H (and D) by R (and C respectively).

An Improved Polynomial Time Algorithm for Invertible Matrices

Bürgisser has in [5] found a way to evaluate the 2-hook-immanant at $A \in \mathrm{GL}_n(\mathbb{C})$ using $O(n^5 \log n)$ calculations.

In order to do so he considers an irreducible polynomial representation of $\mathrm{GL}_n(\mathbb{C})$,

$$D_\lambda : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(V) \cong \mathrm{GL}_{d_\lambda}(\mathbb{C})$$

where λ is a general but non-constant partition into n parts, i.e., $\lambda = (\lambda_1 \ \dots \ \lambda_n) \neq (k \ \dots \ k)$. It should be noted that $\lambda \in \mathbb{N}^n$, thus, we now consider sequences $\lambda_1 \geq \dots \geq \lambda_n$ with $\lambda_i \geq 0$ rather than $\lambda_i > 0$. In particular the 2-hook-immanant which is our main concern will arise from the partition $(2 \ 1^{n-2} \ 0)$. Now, consider the chain of subgroups

$$\mathrm{GL}_n \supset \mathrm{GL}_{n-1} \times \mathbb{C}^* \supset \dots \supset \mathrm{GL}_{n-k} \times (\mathbb{C}^*)^k \supset \dots \supset (\mathbb{C}^*)^n,$$

and consider restrictions of D_λ to these subgroups successively. Each time we make a restriction we may split V into subspaces, terminating with d_λ one-dimensional spaces, that are spanned by weight vectors, i.e., vectors, v, that satisfy

$$D_\lambda \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} v = t_1^{w_1} \cdots t_n^{w_n} v,$$

for some $w \in \mathbb{N}^n$. Such a basis of weight vectors is called a Gelfand-Tsetlin basis and is unique up to ordering and scaling.

One may visualise the splitting behaviour of V by a layered graph, in which each node in layer k is a pair $(\mu, w) \in \mathbb{N}^k \times \mathbb{N}^{n-k}$, where μ is a partition into k parts and w satisfies $|w| + |\mu| = |\lambda|$. In the top (n'th) layer we have a node $(\lambda, -)$ representing V. In the next layer we have nodes representing each of the vector spaces we get in the decomposition of V by restricting D_λ to $\mathrm{GL}_{n-1} \times \mathbb{C}^*$, and so on until we reach the bottom (first) layer where we have nodes for the spaces of dimension one.

We make an edge between a node in layers k and $k-1$ if the space, V_{k-1}, represented by the lower node is a part of the decomposition of the space, V_k, after restricting D_λ from $\mathrm{GL}_{n-k} \times \mathbb{C}^k$ to $\mathrm{GL}_{n-k-1} \times \mathbb{C}^{k+1}$. The number of paths from a node in layer i to some node in layer $j < i$, tells us what the multiplicity of corresponding jth layer space is in the ith layer space after restriction from $\mathrm{GL}_i \times \mathbb{C}^{n-i}$ to $\mathrm{GL}_j \times \mathbb{C}^{n-j}$. We define $\mathrm{mult}(\lambda)$ by looking at all pairs of nodes two layers apart and take the maximum number of paths between such pairs.

Furthermore we may find the weights of the elements of the Gelfand-Tsetlin basis by successive use of the betweenness condition, which states that the nodes (μ^k, w^k) and (μ^{k+1}, w^{k+1}) are connected if and only if $\mu_j^{k+1} \geq \mu_j^k \geq \mu_{j+1}^{k+1}$, and $w^k = (e, w^{k+1})$, for $k < n$ and $e = |\mu^{k+1}| - |\mu^k|$.

Now, the weight of an element in our Gelfand-Tsetlin basis, which is represented by $(\mu^1, w^1) = ((\mu_1), (w_2, \ldots, w_n)) \in \mathbb{N} \times \mathbb{N}^{n-1} = \mathbb{N}^n$ is simply $w = (w_1 \ldots, w_n) := (\mu_1, w_2 \ldots, w_n)$. (A sketch of such a layered graph can be found in figure 3.1 at the end of the chapter.)

When we have a Gelfand-Tsetlin basis it is a fact that

$$\mathrm{im}_\lambda(A) = \sum_{w(v)=(1,\ldots,1)} D_\lambda(A)_{v,v}$$

where $w(v)$ is the weight of the (Gelfand-Tsetlin basis) vector v. Hence, if we can compute the entries of $D_\lambda(A)$ with respect to the Gelfand-Tsetlin basis we may find the immanant by just adding the diagonal entries indexed by vectors of weight $(1, \ldots, 1)$.

This is the basic idea of Bürgisser's algorithm, and in order to do so he needs a few auxiliary algorithms, one of them we saw in a slightly different version in Barvinok's algorithm, namely, we need to factor an invertible matrix A into a product of $O(n^2)$ matrices $E_{ij}(\alpha)$, with $j = i \pm 1$, times some diagonal matrix Δ:

$$A = E_{i_1 j_1}(\alpha_1) \cdots E_{i_k j_k}(\alpha_k)\Delta \;\text{ for some } k = O(n^2).$$

Another algorithm is used to compute $e^{tJ}u$ for a nilpotent Jordan-matrix $J \in \mathrm{Mat}_r(\mathbb{C})$, $t \in \mathbb{C}$, and $u \in \mathbb{C}^r$ in $O(r \log r)$ operations, using that e^{tJ} is a Toeplitz matrix, i.e., a matrix $A = (a_{ij})$ such that a_{ii+k} is constant for fixed $1-n \le k \le n-1$, when we vary i. This algorithm is a building stone in showing that if $M \in \mathrm{Mat}_n(\mathbb{C})$ is a block matrix with r^2 blocks, $M_{ij} \in \mathrm{Mat}_{n_i \times n_j}(\mathbb{C})$, such that $M_{ij} = 0$ except if $j = i - 1$, then

- M is conjugate to a direct sum of nilpotent Jordan blocks of size at most r via PS for some block diagonal matrix S and a permutation matrix P.
- $e^{tM}u$ can be computed in $O\left(\sum_{i=1}^r n_i^2 + n \log r\right)$ operations when $t \in \mathbb{C}$ and $u \in \mathbb{C}^n$.

Finally we need to consider the one-parameter subgroups $(i \ne j)$

$$F_{ij} : \mathbb{C} \to \mathrm{GL}_{d_\lambda},\; t \mapsto D_\lambda(E_{ij}(t)).$$

Note that this is indeed a one-parameter subgroup because $E_{ij}(t+s) = E_{ij}(t)E_{ij}(s)$. It is not hard to prove that $F_{ij}(t) = e^{tF'(0)}$, that $F'_{ij}(0)$ is nilpotent, and that $F'_{ij}(0)$ maps a weight vector of weight $w \in \mathbb{Z}^n$ to one of weight $w + e_i - e_j$.

Bürgisser now proves

Theorem 3.2.3. *If $\lambda = (\lambda_1 \ \ldots \ \lambda_n)$ is a non-constant partition into n parts, and $D_\lambda : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(V) \overset{*}{\cong} \mathrm{GL}_{d_\lambda}(\mathbb{C})$ (where we have chosen the Gelfand-Tsetlin basis to make $*$) is a polynomial representation then*

$$\mathrm{GL}_m \times \mathbb{C}^{d_\lambda} \to \mathbb{C}^{d_\lambda} \qquad (A, v) \mapsto D_\lambda(A)v$$

can be computed using $O(n^2(\mathrm{mult}(\lambda) + \log|\lambda|)d_\lambda)$ arithmetic operations.

We shall omit the proof. However, from this theorem we easily get the corollary giving the promised bound on the computational complexity of the 2-hook-immanant at invertible matrices:

Corollary 3.2.4. *The 2-hook-immanant can be evaluated at $A \in \mathrm{Mat}_n(\mathbb{C})$ using $O(n^5 \log n)$ arithmetic operations.*

Proof. In order to compute the immanant it is enough to compute the diagonal entries of $D_\lambda(A)$ that are indexed by Gelfand-Tsetlin basis vectors of weight $(1, \ldots, 1)$. There are $n-1$ such vectors because the betweenness condition tells us that we can pass from the top node $((2\ 1^{n-2}\ 0), -)$ to $((1), (1, \ldots, 1))$ in our layered graph in $n-1$ different ways:

At the top level we have node $((2\ 1^{n-2}\ 0), -)$ and at the next we have the nodes $((2\ 1^{n-2}), (0))$, $((2\ 1^{n-3}\ 0), (1))$, $((1^{n-1}), (1))$, and $((1^{n-2}\ 0), (2))$, hence, we can either pass to the second or third of these if we want to end up at $((1), (1, \ldots, 1))$. At the node with the constant partition the betweenness condition dictates that all descending nodes must contain the constant partition, and so we no longer have a choice in where to go.

Now, from $((2\ 1^{n-3}\ 0), (1))$ we may move on to $((2\ 1^{n-4}\ 0), (1, 1))$, $((1^{n-2}), (1, 1))$ or two other nodes which will give non-constant weights. As we have already discussed constant partitions we need only consider what happens if we pass on to $(2\ 1^{n-4}\ 0)$.

Repeating the argument gives us that apart from nodes with constant partitions a path from the top node to $((1), (1, \ldots, 1))$ can pass through nodes $((2\ 1^{n-i}\ 0), (1, \ldots, 1))$ until we get to $((2\ 0), (1, \ldots, 1))$ at which point we are forced to move to $((1), (1, \ldots, 1))$. Thus, there is a path from the top level to the node $((1), (1, \ldots, 1))$ at the bottom level for each point at which we may choose to depart from a node $((2\ 1^k\ 0), (1, \ldots, 1))$ to a node $((1^{k+1}), (1, \ldots, 1))$. This we can do each time we move down one layer, hence, we get $n-1$ paths as there are n layers.

Now, for the 2-hook-partition $\mathrm{mult}(\lambda) = 2$, which vanishes in the O-notation, furthermore $d_\lambda = n^2 - 1$, as this is the number of semi-standard tableaux of shape λ, thus, we need to make $n-1$ multiplications like the one in theorem 3.2.3 each can be computed in $O(n^4 \log n)$ operations, hence, the combined time required (as adding the proper diagonal entries will not dominate the multiplications) is $O(n^5 \log n)$. $\square$

We shall finish the chapter with a sketch of the layered graph representing the splitting behaviour when we consider the partition $\lambda = (2\ 1^{n-2}\ 0)$.

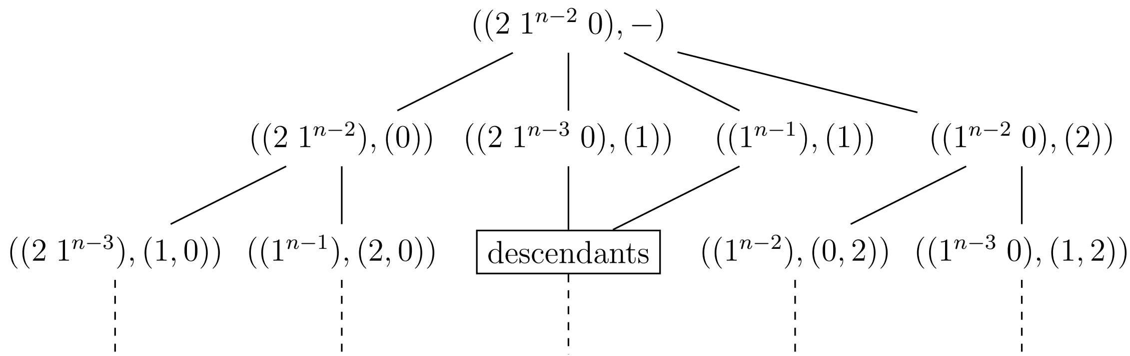


Figure 3.1: A sketch of the layered graph arising from the partition $(2\ 1^{n-2}\ 0)$

The (only) child of the node $x = ((1^{n-1}), (1))$ is $((1^{n-2}), (1,1))$ which is also a child of $y = ((2\ 1^{n-3}\ 0), (1))$, hence the edges from nodes x and y point to the same descendants.

The dashed edges off course represent that the graph continues with more layers. The nodes in these layers (and the nodes in the block 'descendants') are in principle similar to nodes which have been specified, except that we start with a smaller n and non-empty weights.

The graph will perhaps also convince the reader that $\mathrm{mult}(\lambda) = 2$ which was claimed in the proof of corollary 3.2.4.

Appendix A

Computing the rank of Hessian matrices using Macaulay2

Recall that we in example 3.1.3 considered the matrices

$$B_4 = \begin{pmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 \end{pmatrix},\ B_5 = \begin{pmatrix} 1 & 2 & 0 & 0 & 2 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 4 & -1 & -2 & 0 \\ 1 & 2 & 2 & 3 & 0 \\ 1 & 1 & 2 & 2 & 1 \end{pmatrix}$$

$$B_6 = \begin{pmatrix} 1 & 0 & -1 & 2 & 3 & 1 \\ 0 & -2 & 3 & 1 & 2 & -1 \\ 0 & -2 & 1 & 2 & 0 & 1 \\ -1 & -1 & 2 & 3 & 0 & -2 \\ 1 & 1 & 1 & -2 & 3 & 0 \\ 0 & -7 & 14 & 7 & 7 & 10 \end{pmatrix},\ B_7 = \begin{pmatrix} 1 & 2 & 1 & 3 & 1 & 0 & 95 \\ 2 & 3 & 1 & 2 & 1 & 2 & 0 \\ 1 & 2 & 1 & 1 & 2 & 3 & 95 \\ 2 & 3 & 1 & 2 & 3 & 1 & 190 \\ 3 & 2 & 2 & 0 & 2 & 0 & 285 \\ 1 & 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 3 & 2 & 1 & 2 & 3 & -227 \end{pmatrix},$$

and that we have $\operatorname{rank} \operatorname{Hes} \operatorname{im}_n(B_n) = n^2$ for $n = 4, 5, 6, 7$. This can be verified using the following commands in Macaulay2:

```
n= insert number n = 4,5,6 or 7
k=n^2
Per=permutations(n)
p=#Per
L= for i from 0 to n-1 list i
Per1=Per -( i=1; while i<p+1 list L do i=i+1)
```

```
07:  fix = l-> (for x in l when 1==1 list (if x=!=0 then continue; x))
08:  FIX= l-> #fix(l)-1
09:  sgn0 = l -> for i from 0 to n-2 when 1==1 list (L=for j from i+1
      to n-1 when 1==1 list (if l_i<l_j then continue; i,j);#L)
10:  sgn=l->(-1)^(sum(sgn0(l)))
11:  character= (for i from 0 to #Per-1 when 1==1 list
      sgn(Per_i)*FIX(Per1_i))
12:  R=QQ[x_1..x_k]
13:  M=genericMatrix(R,n,n)
14:  monoR = l-> for i from 0 to n-1 when 1==1 list
      (transpose(M^{i}))^{l_i}
15:  07:  MonoR= l-> product monoR(l)
16:  LR= (i=0; while i<#Per list MonoR(Per_i)*character_i do i=i+1)
17:  im=sum(LR)
18:  A=matrix{ insert matrix B_n }
19:  evA=map(QQ,R,A)
20:  evA(im)
21:  rank evA jacobian transpose jacobian im
```

Here the first four lines establish a few constants and a list of permutations that will make it out for the symmetric group.
Line 5 gives a list $\{0, 1, \ldots, n-1\}$ to be used in
line 6 which provides a list of lists, `Per1`, in which zeroes correspond to fixed points of permutations.
Line 7 discards the non-fixed points from line 6, and line 8 counts the remaining fixed points and subtracts 1, giving one part of the character function.
Line 9 makes lists of inversions of our permutations, and line 10 gives us the sgn-part of the character.
Line 11 computes the value of the character at each of the permutations.
Lines 12 and 13 construct a polynomial ring with n^2 variables and a matrix of indeterminates.
Lines 14, 15, and 16 construct monomials equivalent to $(\operatorname{fix}\sigma - 1)\prod_{i=1}^{n} X_{i\sigma i}$.
Line 17 sums the monomials and constructs our immanant im_n.
Line 18 and 19 constructs the points specified (note that the matrix in line 18 must be written as a $1 \times n^2$-array), and makes a map to evaluate functions from our polynomial ring to the rational numbers at the point specified.
Line 20 and 21 finally evaluates the immanant (showing it vanishes at the point), and returns the rank of the Hessian of the immanant.

In order to find points at which the immanant vanishes it can be helpful to use

```
evB=map(R,R,B)
evB(im)
```

where B is some $1 \times n^2$-array with $n^2 - 1$ random (integer/rational) entries and the last entry an indeterminate, e.g. x_k. This is likely to return a polynomial $ax_k - b$ (you may be unfortunate to just get a constant, i.e., it may happen that $a = 0$). Now make the array A similar to B but with the indeterminate entry replaced by b/a, A will now be a vanishing point of im_n, and we may, thus, find the Hessian at A and compute its rank.

I have written a package for Macaulay2 which is available via dropbox:

`https://www.dropbox.com/s/vm9btp07etd3w4q/Immanant.m2`

The code of the package is as follows (I have chosen to clean these lines of all comments which can be found in the file, furthermore due to the margin not being extremely wide some lines have been broken into 2 or 3 shorter lines these lines are in the methods '`monomialVariables`', and '`involution`'):

```
newPackage(
     "Immanant",
     Reload => true
     )

export (immanant)

export {immanantV2}

export {genericHessian}

export {genericHessianV2}

export {genericRank}

export {genericRankV2}

export {genericZero}

immanant = method()
```

```
symGroup = method()
symGroup ZZ := n-> permutations(n)

monomialVariables = method()
monomialVariables (ZZ,Matrix) := (n,A) -> for x in symGroup(n)
              list (for i from 0 to n-1 list ((A^{i})_{x_i}) )

monomial = method()
monomial List := l-> for x in l list product x

L = method()
L ZZ := n-> for i from 0 to n-1 list i

dif = method()
dif ZZ := n-> symGroup(n) -( for j from 1 to n! list L(n) )

keepFixed=method()
keepFixed List := x -> for e in x list (if e==0 then e else continue)

totalFixed = method()
totalFixed List := x-> #keepFixed(x)-1

involution = method()
involution (ZZ,List) := (n,x) ->  for i from 0 to n-2
           list (I := for j from i+1 to n-1
           list (if x_i<x_j then continue; i,j);#I)

signature = method()
signature (ZZ,List) := (n,x) -> (-1)^(sum(involution(n,x)))

weightedMonomials = method()
weightedMonomials (ZZ,Ring) := (n,R) -> (
     d := dif(n);
     l := symGroup(n);
     s := for x in l list signature(n,x);
     f := for x in d list totalFixed(x);
```

```
     character := for i to #l-1 list s_i*f_i;
     M := genericMatrix(R,n,n);
     x1 := monomialVariables(n,M);
     xn := monomial(x1);
     wM := for i to #l-1 list (character_i*xn_i)
     )

immanant (ZZ,Ring) := (n,R)-> (sum(weightedMonomials(n,R)))_(0,0)

immanantMatrixFormat = method()
immanantMatrixFormat (ZZ,Ring) := (n,R) -> sum(weightedMonomials(n,R))

immanantV2 = method()
immanantV2 ZZ := n-> (
     k := n^2;
     R := QQ[vars(0..(k-1))];
     immanant(n,R)
     )

genericHessian = method()
genericHessian (ZZ,Ring):= (n,R) -> (
     k := n^2;
     M := mutableMatrix genericMatrix(R,1,k);
     for i from 1 to k-1 do M_(0,i) = random(ZZ);
     m := matrix M;
     evalm := map(R,R,m);
     imn := immanantMatrixFormat(n,R);
     Affine := evalm(imn);
     z := matrix mutableMatrix(R,1,k);
     evalz := map(R,R,z);
     M_(0,0) = (-1)*evalz(Affine)_(0,0);
     a := (jacobian Affine)_(0,0);
     if a=!=0 then for i from 1 to n-1 do M_(0,i*n)= a*M_(0,i*n);
     N := matrix M;
     evalN := map(R,R,N);
     evalN(jacobian transpose jacobian imn)
     )
```

```
genericHessianV2 = method()
genericHessianV2 ZZ:= n -> (
     k := n^2;
     R := QQ[vars(0..(k-1))];
     M := mutableMatrix genericMatrix(R,1,k);
     for i from 1 to k-1 do M_(0,i) = random(ZZ);
     m := matrix M;
     evalm := map(R,R,m);
     imn := immanantMatrixFormat(n,R);
     Affine := evalm(imn);
     z := matrix mutableMatrix(R,1,k);
     evalz := map(R,R,z);
     M_(0,0) = (-1)*evalz(Affine)_(0,0);
     a := (jacobian Affine)_(0,0);
     if a=!=0 then for i from 1 to n-1 do M_(0,i*n)= a*M_(0,i*n);
     N := matrix M;
     evalN := map(R,R,N);
     evalN(jacobian transpose jacobian imn)
     )

genericRank = method()
genericRank (ZZ,Ring) := (n,R) -> (
     k := n^2;
     M := mutableMatrix genericMatrix(R,1,k);
     for i from 1 to k-1 do M_(0,i) = random(ZZ);
     m := matrix M;
     evalm := map(R,R,m);
     imn := immanantMatrixFormat(n,R);
     Affine := evalm(imn);
     z := matrix mutableMatrix(R,1,k);
     evalz := map(R,R,z);
     M_(0,0) = (-1)*evalz(Affine)_(0,0);
     a := (jacobian Affine)_(0,0);
     if a=!=0 then for i from 1 to n-1 do M_(0,i*n)= a*M_(0,i*n);
     N := matrix M;
     evalN := map(R,R,N);
     rank evalN(jacobian transpose jacobian imn)
     )
```

```
genericRankV2 = method()
genericRankV2 ZZ := n -> (
     k := n^2;
     R := QQ[vars(0..(k-1))];
     M := mutableMatrix genericMatrix(R,1,k);
     for i from 1 to k-1 do M_(0,i) = random(ZZ);
     m := matrix M;
     evalm := map(R,R,m);
     imn := immanantMatrixFormat(n,R);
     Affine := evalm(imn);
     z := matrix mutableMatrix(R,1,k);
     evalz := map(R,R,z);
     M_(0,0) = (-1)*evalz(Affine)_(0,0);
     a := (jacobian Affine)_(0,0);
     if a=!=0 then for i from 1 to n-1 do M_(0,i*n)= a*M_(0,i*n);
     N := matrix M;
     evalN := map(R,R,N);
     rank evalN(jacobian transpose jacobian imn)
     )

genericZero = method()
genericZero (ZZ,Ring):= (n,R) -> (
     k := n^2;
     M := mutableMatrix genericMatrix(R,1,k);
     for i from 1 to k-1 do M_(0,i) = random(ZZ);
     m := matrix M;
     evalm := map(R,R,m);
     imn := immanantMatrixFormat(n,R);
     Affine := evalm(imn);
     z := matrix mutableMatrix(R,1,k);
     evalz := map(R,R,z);
     M_(0,0) = (-1)*evalz(Affine)_(0,0);
     a := (jacobian Affine)_(0,0);
     if a=!=0 then for i from 1 to n-1 do M_(0,i*n)= a*M_(0,i*n);
     N := matrix M
     )
```

```
beginDocumentation()
document {
     Key => Immanant
     }
document {
     Key => {(immanant,ZZ,Ring),immanant},
     Usage => "immanant(n,R)",
     Inputs => {"n", "R"}
     }
document {
     Key => {(immanantV2,ZZ),immanantV2,},
     Usage => "ImmanantV2(n)",
     Inputs => {"n"}
     }
document {
     Key => {(genericHessian,ZZ,Ring),genericHessian},
     Usage => "genericHessian(n,R)",
     Inputs => {"n", "R"}
     }
document {
     Key => {(genericHessianV2,ZZ),genericHessianV2},
     Usage => "genericHessianV2(n)",
     Inputs => {"n"}
     }
document {
     Key => {(genericRank,ZZ,Ring),genericRank},
     Usage => "genericRank(n,R)",
     Inputs => {"n", "R"}
     }
document {
     Key => {(genericRankV2,ZZ),genericRankV2},
     Usage => "genericRankV2(n)",
     Inputs => {"n"}
     }
document {
     Key => {(genericZero,ZZ,Ring),genericZero},
     Usage => "genericZero(n,R)",
     Inputs => {"n", "R"}
```

```
     }
TEST ///
    assert ( ideal( immanant(2,QQ[x_1..x_4])) ==
    permanents(2,genericMatrix(QQ[x_1..x_4])))
///
```

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