On the asymptotic expansion of the curvature of perturbations of the L_2 connection

Abstract

We establish that the Hitchin connection is a perturbation of the L_2 connection. We notice that such a formulation of the Hitchin connection does not necessarily require the manifold in question possessing a rigid family of Kähler structures. We then proceed to calculate the asymptotic expansion of general perturbations of the L_2-connection, and see when under certain assumptions such perturbations are flat and projectively flat. During the calculations we also found an asymptotic expansion of the projection operator $\pi_\sigma^{(k)}$ which projects onto the holomorphic sections of the $k-$th tensor of prequantum line bundle.

Vi viser at Hitchin connectionen er en perturbation af L_2 connectionen. Vi bemærker, at en sådan formulering af Hitchin connectionen ikke nødvendigvis kræver, at den pågældende mangfoldighed besidder en rigid familie af Kähler strukturer. Vi fortsætter derefter med at udregne den asymptotiske udvidelse af generelle pertubationer af L_2 connectionen og ser, hvornår, under bestemte antagelser, sådanne pertubationer er flat og projectively flat. I løbet af udregningerne fandt vi desuden en asymptotisk udvidelse af projektionsoperatoren, $\pi_\sigma^{(k)}$, som projicerer ned på de holomorfe sektioner af den k'te tensor af prequantum linie bundtet.

Introduction

In his paper titled "Quantum field theory and the Jones polynomial" (Comm. Math. Phys., 121(3):351–399, 1989), Witten proposed that Chern-Simons theory should form the two dimensional part of $(2+1)$-dimensional TQFT, which in turn led to the study of geometric quantization of the moduli space $\mathcal{M}$ of flat $SU(n)$-connections on a surface Σ. This moduli space is *prequantizable* in the sense that it admits a *prequantum line bundle*. Given the surface Σ, the Teichmüller space $\mathcal{T}$ associated to Σ parametrizes the complex structures such that for every $\sigma \in \mathcal{T}$ and for every $k \in \mathbb{N}$, we have the quantum space of geometric quantization, which is the space, $H^{(k)}(\sigma) = H_\sigma^{(k)} = H^0\left(\mathcal{M}, \mathcal{L}^k\right)$, of holomorphic sections of the k-th tensor power of the prequantum line bundle. The spaces $H^{(k)}$ form the fibers of the Verlinde bundle over $\mathcal{T}$, and it was shown independently by Hitchin and Axelrod, Della Pietra and Witten that this bundle admits a natural projectively flat connection, called the *Hitchin connection*. As a result, there exists an identification, as projective spaces, of the quantum spaces associated with different complex structures, through the parallel transport of this connection.

We now give a brief description of the Hitchin connection as developed in [1] by Andersen. We start off with a compact symplectic (M, ω) equipped with a prequantum line bundle $\mathcal{L}$, further satisfying the condition that $H^1(M, \mathbb{R}) = 0$ and that there exists an $n \in \mathbb{Z}$ such that the first Chern class of (M, ω) is $n[\omega] \in H^2(M, \mathbb{Z})$. Now on $\mathcal{L}^k$, we have the smooth family of $\bar{\partial}$ operators $\nabla^{0,1}$ defined at $\sigma \in \mathcal{T}$ by

$$\nabla_\sigma^{0,1} = \frac{1}{2}(Id + iJ_\sigma)\nabla$$

For every $\sigma \in \mathcal{T}$, consider the finite dimensional subspace of $C^\infty\left(M, \mathcal{L}^k\right)$ given by $H_\sigma^{(k)} = H^0\left(M_\sigma, \mathcal{L}^k\right) = \left\{s \in C^\infty\left(M, \mathcal{L}^k\right) | \nabla_\sigma^{0,1} s = 0\right\}$. Our assumption is that these subspaces of holomorphic sections form a smooth finite rank subbundle $H^{(k)}$ of the trivial bundle $\mathcal{T} \times C^\infty\left(M, \mathcal{L}^k\right)$.

Let $\hat{\nabla}^t$ denote the trivial connection in the trivial bundle $\mathcal{T} \times C^\infty\left(M, \mathcal{L}^k\right)$ and let $\mathcal{D}\left(M, \mathcal{L}^k\right)$ denote the vector space of differential operators on $C^\infty\left(M, \mathcal{L}^k\right)$. For any $\mathcal{D}\left(M, \mathcal{L}^k\right)$-valued smooth 1-form u on $\mathcal{T}$, we have a connection $\hat{\nabla}$ in $\mathcal{T} \times C^\infty\left(M, \mathcal{L}^k\right)$ given by $\hat{\nabla}_V = \hat{\nabla}_V^t - u(V)$ for all vector fields V on $\mathcal{T}$.

Proposition : The connection $\hat{\nabla}$ in $\mathcal{T} \times C^\infty\left(M, \mathcal{L}^k\right)$ induces a connection in $H^{(k)}$ if and only if

$$\frac{i}{2} V [J] \nabla^{1,0} s + \nabla^{0,1} u (V) s = 0 \tag{1}$$

holds for all vector fields V on $\mathcal{T}$ and all smooth sections s on $H^{(k)}$.

This induced connection is called the *Hitchin Connection*.

We next make the assumption that M is endowed with a rigid family J of Kähler structures parametrised by the complex manifold $\mathcal{T}$. To find a u that satisfies the above equation we use the operator

$$\begin{aligned} \triangle_G : C^{\infty}\left(M, \mathcal{L}^k\right) &\xrightarrow{\nabla^{1,0}} C^{\infty}\left(M, T_{\sigma}^{(1,0)} M^* \otimes \mathcal{L}^k\right) \xrightarrow{G} C^{\infty}\left(M, T_{\sigma}^{(1,0)} M \otimes \mathcal{L}^k\right) \\ &\xrightarrow{\nabla^{1,0}} C^{\infty}\left(M, T_{\sigma}^{(1,0)} M^* \otimes T_{\sigma}^{(1,0)} M \otimes \mathcal{L}^k\right) \xrightarrow{Tr} C^{\infty}\left(M, \mathcal{L}^k\right) \end{aligned}$$

defined for $G \in C^{\infty}\left(M, S^2\left(T_{\sigma}^{(1,0)} M\right)\right)$, where G satisfies the relation, $V'[J] = G(V) \cdot \omega$, V' being the $(1,0)$ part of the vector field , and the relation concerning rigidity i.e., $\bar{\partial}_{\sigma}\left(G\left(V_{\sigma}\right)\right) = 0$. Under such circumstances it is proven in [1] that

$$u(V) = \frac{1}{2k+n} o(V) + V'[F] \tag{2}$$

where F is the Ricci potential and

$$o(V) = \frac{1}{2} \triangle_{G(\mathrm{V})} - \nabla_{G(V) dF} - n V'[F]$$

solve the equation (1).

The Hitchin connection as defined above, necessarily requires the manifold M to admit a rigid family of Kähler structures. During the course of this thesis however, we eschew the concept of rigidity by attacking the problem from another perspective.

We first begin by examining the relationship between the Hitchin connection (associated to the form u that solves (1)) and the he L_2 connection, denoted ∇^{L_2}, which has the property that for a given vector field V on $\mathcal{T}$, $\nabla_V^{L_2} s = \pi_{\sigma}^{(k)} V(s)$, where $\pi_{\sigma}^{(k)}$ denoted the projection onto holomorphic sections. In proposition 7 of chapter 5 of this thesis, we develop a new 1-form u_N given by the equation

$$u_N(V) s = -\frac{i}{2}\left(\nabla^{0,1}\right)^* P V'[J] \nabla^{1,0} s$$

where P is the parametrix to the operator $\nabla^{0,1}\left(\nabla^{0,1}\right)^{*}+\left(\nabla^{0,1}\right)^{*}\nabla^{0,1}$ (the Laplace Operator) and that for a sufficiently large k, this 1-form u_N, solve the equation (1).

We next consider the connection ∇^N, given by the equation $\nabla^N=\hat{\nabla}^t_V-u_N\left(V\right)$ and we prove in proposition 9, chapter 5, that this is nothing but the L_2 connection, itself. Further we prove that the relation between the L_2 connection and the Hitchin connection is given by the relation $\nabla^H-\nabla^{L_2}=-\pi_\sigma^{(k)}u$, where u is given by (2) (as developed by Andersen in [1]). Thus we see that the Hitchin connection is of the form ∇^M, where$\nabla^M=\nabla^{L_2}+\pi_\sigma^{(k)}g_k$ where g_k is a 1-form on $\mathcal{T}$ with values in $\mathcal{D}\left(M,\mathcal{L}^k\right)$.

The raison d'être behind proceeding in this way, i.e. express the Hitchin connection as $\nabla^H=\nabla^{L_2}-\pi_\sigma^{(k)}u$, is that given a connection of the form $\nabla^M=\nabla^{L_2}+\pi_\sigma^{(k)}g_k$ where g_k is a 1-form on $\mathcal{T}$ with values in $\mathcal{D}\left(M,\mathcal{L}^k\right)$, we hope to calculate the asymptotic expansion of the curvature $\mathcal{F}_{\nabla^M}$ and thus get a set of conditions as to when such a connection might be flat or *projectively flat* (since the Hitchin connection, we know is projectively flat).

We therefore carry out a curvature calculation of connections of the form ∇^M in section 5.1 to obtain the formula

$$\begin{aligned}\mathcal{F}_{\nabla^M}\left(X,Y\right)s &= \pi_\sigma^{(k)}\left[X\left[\pi_\sigma^{(k)}\right],Y\left[\pi_\sigma^{(k)}\right]\right]s+\pi_\sigma^{(k)}d\left(\pi_\sigma^{(k)}g_k\right)\left(X,Y\right)s\\ &+\pi_\sigma^{(k)}g_k\left(Y\right)X\left(\pi_\sigma^{(k)}\right)s-\pi_\sigma^{(k)}g_k\left(X\right)Y\left(\pi_\sigma^{(k)}\right)s\\ &+\left[\pi_\sigma^{(k)}g_k\left(X\right),\pi_\sigma^{(k)}g_k\left(Y\right)\right]s\end{aligned}$$

Given this result we express the 1-form g_k as an infinite series, given by $g_k=\sum_{l=0}^{\infty}g^{(l)}k^{-l}$, where each $g^{(l)}$ is a 1-form on $\mathcal{T}$ with values in $\mathcal{D}\left(M,\mathcal{L}^k\right)$. This is the first step to giving an asymptotic expansion of the curvature of ∇^M, denoted $\mathcal{F}_{\nabla^M}$, as given above. However in order to do so we first of all need an aymptotic expansion of the operator $\pi_\sigma^{(k)}$. This expansion is developed at the end of chapter 4 in this thesis in theorems 9 and 10. We state the theorems here for reference

Theorem : *There exist global operators* $D_l^{(k)}:C^\infty\left(M,\mathcal{L}^k\right)\to C^\infty\left(M,\mathcal{L}^k\right)$ *such that we have for all section* s *of* $\mathcal{L}^k$*, we have an asymptotic expansion*

$$\pi_\sigma^{(k)}s\sim s+\sum_{l=1}^{\infty}\frac{D_l^{(k)}\left(s\right)}{k^l}, i.e.,$$

$$\mid \pi_\sigma^{(k)} s - s - \sum_{l=1}^{N-1} \frac{D_l^{(k)}(s)}{k^l} \mid = O\left(\frac{1}{k^N}\right)$$

the norm being the C^m norm with respect to the norm on sections of $\mathcal{L}^k$ over M.

Theorem : *For a vector field V on $\mathcal{T}$, $V\left(\pi_\sigma^{(k)}\right) s$ has the asymptotic expansion*

$$V\left(\pi_\sigma^{(k)}\right) s \sim \sum_{l=1}^{\infty} \frac{V\left[D_l^{(k)}\right](s)}{k^l}, i.e.,$$

$$\mid V\left(\pi_\sigma^{(k)}\right) s - \sum_{l=1}^{N-1} \frac{V\left[D_l^{(k)}\right](s)}{k^l} \mid = O\left(\frac{1}{k^N}\right)$$

the norm being the C^m norm with respect to the norm on sections of $\mathcal{L}^k$ over M.

With the help of the above results we are indeed able to give an asymptotic expansion of the curvature $\mathcal{F}_{\nabla^M}$ in theorem 15. We state the theorem here for reference (the notation used is as developed thus far).

Theorem : *Keeping with the notation developed thus far, the asymptotic expansion for the curvature $\mathcal{F}_{\nabla^M}$ is given by the expression*

$$\mathcal{F}_{\nabla^M} = \sum_{n=0}^{\infty} \frac{T_n}{k^n}$$

where

$$T_0 = dg^{(0)} + g^{(0)} \wedge g^{(0)}$$

$$T_1 = dg^{(1)} + D_1 dg^{(0)} + A_1 + D_1 A_0 + B_1$$

with A_1 and A_0 being given by

$$A_1 = g^{(0)} \wedge g^{(1)} + g^{(1)} \wedge g^{(0)} + dD_1 \wedge g^{(0)} - g^{(0)} \wedge dD_1$$

and

$$A_0 = g^{(0)} \wedge g^{(0)},$$

and

$$T_n = dg^{(n)} + \sum_{i+j=n} D_i dg^{(j)}$$
$$+ A_n + \sum_{i+j=n} D_i A_j + B_n + \sum_{i+j=n} D_i B_j$$

where A_n for $n \geq 2$ is given by

$$A_n = \sum_{i+j=n} g^{(i)} \wedge g^{(j)} + \sum_{i+j=n} dD_i \wedge dD_j + \sum_{i+j=n} dD_i \wedge g^{(j)} - \sum_{i+j=n} g^{(i)} \wedge dD_j$$

and B_n $n > 0$ is given by

$$B_n = \sum_{i+j+r=n} g^{(i)} \wedge D_j g^{(r)}.$$

We then proceed to analyze the cases as to when the curvature may be flat. This is done in theorem 16 (under the assumption $g^{(0)} = 0$). In fact we get a set of necessary and sufficient condition in theorem 16 for the connection ∇^M to be flat. These conditions are given by equations (76) and (77) in the statement of theorem 16. We state the theorem here for reference

Theorem : *Given the setting and notation of theorem 15, under the additional assumption that $g^{(0)} = 0$, the necessary and sufficient conditions for the curvature $\mathcal{F}_{\nabla^M}$ to vanish are*

$$g^{(1)} \wedge dD_1 \wedge g^{(1)} + 2dD_1 \wedge dD_1 \wedge dD_1 = 0$$

and

$$\begin{aligned} 0 = & \sum_{i+j+r=n} g^{(i)} \wedge dD_j \wedge g^{(r)} + 2 \sum_{i+j+r=n} dD_i \wedge dD_j \wedge dD_r \\ & - \sum_{i+j+r+q=n} g^{(i)} \wedge D_j g^{(r)} \wedge g^{(q)} + \sum_{i+j+r+q=n} g^{(i)} \wedge D_j g^{(r)} \wedge dD_q \\ & + \sum_{i+j+r=n} g^{(i)} \wedge D_j \left(A_r + B_r\right) - \sum_{i+j+r+q=n} dD_i \wedge dD_j \wedge D_r g^{(q)} \\ & + \sum_{i+j+r+q=n} g^{(i)} \wedge dD_j \wedge D_r g^{(q)} - \sum_{i+j+p+q+r=n} g^{(i)} \wedge D_j g^{(p)} \wedge D_q g^{(r)} \end{aligned}$$

for $n > 3$.

We notice, in fact that the connection ∇^M may be made flat upto order 2 with no obstructions whatsoever (indeed the equations given above point out the obstructions to making the curvature $\mathcal{F}_{\nabla^M}$ flat to orders higher than 2). As a corollary (corollary 16.1), we prove that the connection ∇^M is flat up to order 2 if we let $g^{(1)}$ and $g^{(2)}$ satisfy the following conditions

$$g^{(1)}(X) = iX(D_1)$$

and

$$dg^{(2)} = 0.$$

We already know that *the Hitchin connection is projectively flat* and therefore for the final portion of thesis, we turn our attention to when the connection ∇^M may be projectively flat . Before doing so we establish the following notation - given the context and notation established thus far, for a given differential operator D, $f_{D,k}$ is a function on M such that $\pi_\sigma^{(k)} Ds = \pi_\sigma^{(k)} f_{D,k} s$. The case when ∇^M may be projectively flat is done is theorem 17 under the additional assumption that $g^{(i)}(X)$ are smooth functions on M for all i's. The rationale behind making this assumption is that the Hitchin connection, in case of rigidity is a perturbation of the L_2 connection, of precisely this form (this is remarked upon in the text in an exposition following proposition 9 in chapter 5). These conditions are given by equations (91) and (95). We state theorem here in the introduction for further reference

Theorem : *Given the setting and the notation of theorem 15, and under the additional assumptions that $g^{(i)}(X)$ are C^∞ functions on M for all, $i \geq 0$ and vector fields, X on $\mathcal{T}$, the conditions for projective flatness of the connection ∇^M are*

$$d_M dg^{(0)} = 0,$$

$$\begin{aligned} 0 = d_M dg^{(1)}(X,Y) + d_M\left(f_{D_1,k} dg^{(0)}(X,Y)\right) + \\ d_M\left(f_{X(D_1),k} g^{(0)}(Y) - f_{Y(D_1),k} g^{(0)}(X)\right) - \\ d_M\left(f_{g^{(0)}(X)Y(D_1),k} - f_{g^{(0)}(Y)X(D_1),k}\right) + \\ d_M\left(f_{g^{(0)}(X)D_1,k} g^{(0)}(Y) - f_{g^{(0)}(Y)D_1,k} g^{(0)}(X)\right) \end{aligned}$$

and

$$
\begin{aligned}
0 &= d_M dg^{(n)}(X,Y) + \sum_{i+j=n} d_M \left(f_{D_i,k} dg^{(j)}(X,Y) \right) + \\
&\sum_{i+j=n} d_M \left(f_{X(D_i)Y(D_j),k} - f_{Y(D_i)X(D_j),k} \right) + \\
&\sum_{i+j=n} d_M \left(f_{X(D_i),k} g^{(j)}(Y) - f_{Y(D_i),k} g^{(j)}(X) \right) - \\
&\sum_{i+j=n} d_M \left(f_{g^{(i)}(X)Y(D_j),k} - f_{g^{(i)}(Y)X(D_j),k} \right) + \\
&\sum_{i+l+m=n} d_M \left(f_{D_i,k} \left(f_{X(D_l)Y(D_m),k} - f_{Y(D_l)X(D_m),k} \right) \right) + \\
&\sum_{i+l+m=n} d_M \left(f_{D_i,k} \left(f_{X(D_l),k} g^{(m)}(Y) - f_{Y(D_l),k} g^{(m)}(X) \right) \right) - \\
&\sum_{i+l+m=n} d_M \left(f_{D_i,k} \left(f_{g^{(l)}(X)Y(D_m),k} - f_{g^{(l)}(Y)X(D_m),k} \right) \right) + \\
&\sum_{i+j+r=n} d_M (\left(f_{g^{(i)}(X)D_j,k} g^{(r)}(Y) - f_{g^{(i)}(Y)D_j,k} g^{(r)}(X) \right) + \\
&\sum_{i+l+m+p=n} d_M \left(f_{D_i,k} \left(f_{g^{(l)}(X)D_m,k} g^{(p)}(Y) - f_{g^{(l)}(Y)D_l,k} g^{(p)}(X) \right) \right)
\end{aligned}
$$

for all vector fields X and Y on $\mathcal{T}$. Here d_M is the exterior derivative on M.

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Contents

1 Complex Differential Geometry

We begin by considering a smooth manifold M of dimension $2m$ and let TM be the tangent bundle of M.

1.1 Almost Complex Structures

Definition 1 *A smooth section I of the endomorphism bundle* $\mathrm{End}\,(TM)$ *of the tangent bundle of M is called an* almost complex structure *if it satisfies the relation $I^2 = -Id$ where I denotes the identity morphism.*

$\lozenge$

Given such an almost complex structure I, we can think of TM as a complex vector bundle TM_I, where mutliplication by i is given by I. As a result we see that any almost complex manifold must necessarily be even dimensional and orientable. We now consider the complexified tangent bundle $TM_{\mathbb{C}} = TM \otimes \mathbb{C}$. The linear extentension of I induces a decomposition

$$TM_{\mathbb{C}} = T^{(1,0)}M_I + T^{(0,1)}M_I$$

where $T^{(1,0)}M_I$ and $T^{(0,1)}M_I$ are the eigenspaces of I with eigenvalues i and $-i$ respectively; the explicit projections being given by $\pi_I^{(1,0)} = \frac{1}{2}(Id - iI)$ and $\pi_I^{(0,1)} = \frac{1}{2}(Id + iI)$. We use the notation $X = X' + X''$ where $X' \in T^{(1,0)}M_I$ and $X'' \in T^{(0,1)}M_I$, or the decomposition of a vector field on M.

Further we see that the almost complex structure acts on the cotangent bundle TM^* in an obvious way

$$(I\alpha)\,X = \alpha\,(IX)$$

for every α, a 1-form X a vector field on M. As before we have a decomposition

$$TM_{\mathbb{C}}^* = T^{(1,0)}M_I^* + T^{(0,1)}M_I^*$$

into subbundles of eigenspaces. It is easily seen that $T^{(1,0)}M_I^*$ consists of forms vanishing on $T^{(0,1)}M_I$ and $T^{(0,1)}M_I^*$ consists of forms vanishing on $T^{(1,0)}M_I$.

Given the splittings of $TM_{\mathbb{C}}$ and $TM_{\mathbb{C}}^*$ obtained above, we now have splittings on the level of the tensor bundles into direct sums of eigensubbundles of $TM_{\mathbb{C}}$

and $TM_{\mathbb{C}}^*$. Let $\wedge^{p,q}TM_I^* = \wedge^p T^{(1,0)}M_I^* \otimes \wedge^q T^{(0,1)}M_I^*$, then we have the obvious decomposition

$$\wedge^k TM_{\mathbb{C}}^* = \bigoplus_{p+q=k} \wedge^{p,q} TM_I^*$$

which in turn induces a splitting of the complex valued differential forms,

$$\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

where $\Omega^{p,q}(M) = C^\infty(M, \wedge^{p,q}TM_I^*)$, the space of (p,q)-type complex differential forms. Similar splittings take place for other tensor bundles such as the symmetric powers $S^k(TM_{\mathbb{C}}^*)$ and $S^k(TM_{\mathbb{C}})$.

This now leaves us in a position to define the ∂ and $\bar{\partial}$ operators, which we do using the exterior differential d and the projections $\pi_I^{p,q} : \Omega^{p+q}(M) \to \Omega^{p,q}(M)$,

$$\partial_I : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$$

where $\partial_I = \pi_I^{p+1,q} \circ d$ and,

$$\bar{\partial}_I : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$$

where $\bar{\partial}_I = \pi_I^{p,q+1} \circ d$.

We now turn our attention to defining the complex structure on M.

1.2 Complex Structures

Definition 2 *A* complex structure *on the manifold M comprises of a maximal atlas of smooth charts, $\varphi_i : U^i \to U^{i'} \subset \mathbb{C}\left(\equiv \mathbb{R}^2\right)$, with the added condition that* every transition function $\varphi_j \circ \varphi_i^{-1} : \varphi_i\left(U^{i'} \cap U^{j'}\right) \to \varphi_j\left(U^{i'} \cap U^{j'}\right)$ is holomorphic.

$\Diamond$

Remark Note that any complex manifold admits a natural and canonical almost complex structure on its tangent bundle. To see this, let $z^k = x^k + iy^k$ be he local coordinates, with the corresponding vector fields $X^k = \dfrac{\partial}{\partial x^k}$ and $Y^k = \dfrac{\partial}{\partial y^k}$. Then the almost complex structure is given by

$$I\left(X^k\right) = Y^k \qquad \text{and} \qquad I\left(Y^k\right) = -X^k$$

i.e.,

$$I\left(\frac{\partial}{\partial x^k}\right)=\frac{\partial}{\partial y^k} \qquad \text{and} \qquad I\left(\frac{\partial}{\partial y^k}\right)=-\frac{\partial}{\partial x^k}$$

Since the transition functions on M are holomorphic, the almost complex structure is independent of the coordinates chosen and therefore the tangent bundle becomes a complex vector bundle.

Definition 3 *An almost complex structure is said to be* integrable *if it is induced by a complex structure. We further define the* torsion tensor N_I *(also known as the* Nijenhuis tensor*), as the antisymmetric tensor on M given by*

$$N_I\left(X,Y\right)=\left[IX,\,IY\right]-\left[X,\,Y\right]-I\left[X,\,IY\right]-I\left[IX,\,Y\right]$$

for all vector fields X and Y on M.

♢

It is easy to see that an intgrable almost complex structure implies that the Nijenhuis tensor vanishes (i.e. an almost complex structure is torsion-free). However the converse statement is also true as stated in the celebrated *Newlander-Nirenberg theorem.*

Theorem 1 *(Newlander-Nirenberg Thoerem) Any torsion-free almost complex structure is induced by a unique complex structure.*

♢

There are several equivalent formulations of integrability, a few of which are listed in the following proposition (stated without proof) for future reference.

Proposition 1 *Let I be an almost complex structure on M. Then the following are eqivalent*

(i) The Nijenhuis tensor N_I vanishes

(ii) The bundle $T^{(1,0)}M_I$ is preserved by the Lie-bracket

(iii) The exterior differential decomposes as $d = \partial_I + \bar{\partial}_I$.

◊

Remark Property (iii) from proposition 1 implies the following identities

$$\partial_I^2 = 0 \qquad \bar{\partial}_I^2 = 0 \qquad \text{and} \qquad \partial_I \bar{\partial}_I = -\bar{\partial}_I \partial_I.$$

As a result we have the cochain complex

$$\Omega^{p,0}(M) \xrightarrow{\bar{\partial}_I} \Omega^{p,1}(M) \xrightarrow{\bar{\partial}_I} \Omega^{p,2}(M) \xrightarrow{\bar{\partial}_I} \dots$$

for every positive integer p, the cohomology, being denoted by $H_J^{p,q}(M)$ is called the *Dolbeault cohomogy of* M.

1.3 Symplectic Structure

Definition 4 *A manifold* M *is called a* symplectic manifold *if there is a defined on* M *a closed non-degenerate 2-form* ω, *i.e.*, $\omega \in \Omega^2(M)$ *such that*

(i) $d\omega = 0$

(ii) on each tangent tangent space T_mM, $m \in M$ *if* $\omega_m(X,Y) = 0$ *for all* $Y \in T_mM$ *then* $X = 0$.

◊

Remark The assumptions about ω say that its restrictions to each $m \in M$ makes the tangent space T_mM, into a symplectic vector space. Further by the theorem of Darboux, we know that all symplectic manifolds of the same dimensions are *locally the same.* We state the theorem for future reference.

Theorem 2 *(Darboux theorem) Let* ω_0 *and* ω_1 *be two non degenrate and closed forms of degree 2 on a 2n dimensional manifold* M *with* $\omega_0|_m = \omega_1|_m$ *for some* $m \in M$. *Then there exists a neighbourhood* U *of* m *with a diffeomorphism* $F : U \to F(U) \subset M$ *with* $F(m) = m$ *such that* $F^*\omega_1 = \omega_0$.

◊

We now turn to the concept of the *Hamiltonian vector fields.*

Definition 5 *Given a symplectic manifold* (M, ω) *and* $H \in C^{\infty}(M)$*. Then a vector field* X_H *on* M *is called a* Hamiltonian vector field *with energy fuction* H*, if we have* $i(X_H)\omega = dH$*. The triple* (M, ω, X_H) *is called a* Hamiltonian system*.*

$\lozenge$

Remark With these definitions in mind, we are now in a position to discuss the concept of the *Poisson structure.* Given a symplectic structure on a manifold, we can introduce the concept of the *Poisson bracket* on functions on M denoted $\{\cdot, \cdot\}$, given by the formula

$$\{f, g\} = -\omega(X_f, X_g)$$

It is easy to see that the Poisson bracket thus defined satisfies the Jacobi identity and the Leibniz rule.

Definition 6 *The* Poisson bracket *gives rise to a* Poisson structure *on a manifold* M*, wherein the* Poisson structure *is an antisymmetric biliner mapping* $\{\cdot, \cdot\} : C^{\infty}(M) \times C^{\infty}(M) \to C^{\infty}(M)$ *satisfying the Leibniz rule and the Jacobi identity. The triple* $(M, \omega, \{\cdot, \cdot\})$ *denotes a* Poisson manifold*.*

$\lozenge$

Remark It is easily seen that on a Poisson manifold we have the identity

$$X_{\{f,g\}} = [X_f, X_g].$$

Therefore we see that the association $f \to X_f$ is nothing but a Lie algebra homomorphism from the Lie Algebra of smooth functions equipped with the Poisson bracket to the Lie Algebra of Hamiltonian vector fields.

Definition 7 *Given a vector field* X *on* M*, we have the* divergence of X *as the unique function* δX *such that*

$$\mathcal{L}_X \omega^m = \delta X \omega^m$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to X.

$\Diamond$

Remark Since $\mathcal{L}_X\omega = 0$ for all locally Hamiltonian vector fields, and Hamiltonian vector fields in particular, we see that $\delta X = 0$ for all Hamiltonian vector fields. This now leaves us in a position to explore the relationship between the almost complex structure, the symplectic structure and the Riemannian metric.

Definition 8 *Given a symplectic manifold (M, ω), an almost complex structure I, is said to be* compatible *if the assignment $m \dashrightarrow g_m : T_mM \times T_mM \to \mathbb{R}$, given by*

$$g_m(u, v) = \omega(u, I_m v)$$

defines a Riemannian metric on M, *i.e., the bilinear form* g *must be symmetric and positive definite. The triple (ω, g, I) is called* compatible triple *if $g(\cdot,\cdot) = \omega(\cdot, I\cdot)$.*

$\Diamond$

Given a symplectic manifold (M, ω) and a Riemannian metric g on M, it is easily seen that there exists a canonical almost complex structure on which is compatible. In particular, since Riemannian metrics always exist, any symplectic manifold has compatible almost complex structures. Further it can be be shown that the set of almost complex structures on a symplectic manifold is path connected and indeed contractible.

Further it is easily seen that the symmetry of the metric g is equivalent to the I-invariance of ω, and therefore also of g. As a consequence both g and ω are of type $(1, 1)$.

Remark The metric g induces the usual isomorphism $i_g : TM \to TM^*$ given by

$$i_g(X)(Y) = g(X, Y)$$

for all vector fields X and Y on M. This can be related to the corresponding isomorphism i_ω by the equation

$$i_\omega = i_g \circ I.$$

Since the metric g and the symplectic form ω have type $(1,1)$, the isomorphisms interchange types. The inverse metric tensor $\tilde{g}$ defined by the equation

$$\tilde{g} = \left(i_g^{-1} \otimes i_g^{-1}\right)(g)$$

is the unique bivector field which satisfies

$$g \cdot \tilde{g} = \tilde{g} \cdot g = Id.$$

The relation between the Poisson tensor $\tilde{\omega}$ and the metric tensor $\tilde{g}$ is $\tilde{\omega} = I \cdot \tilde{g}$.

Remark We further note that the Riemannian metric g induces a Hermitian structure $h^{T^{(1,0)}}$ on the eigen subbundle $T^{(1,0)}$ by

$$h^{T^{(1,0)}}(X,\, Y) = g\left(X,\, \overline{Y}\right)$$

which in turn gives the canonical bundle[1] K a Hermitian structure h^K.[2]

1.4 Kähler Manifolds

Definition 9 *A* Kähler manifold *is a complex, symplectic, Riemannian manifold M with the added property that the symplectic form ω, the Rimannian metric g and the (integrable) almost complex structure I (arising from the complex structure) form a compatible triple (compatibility, as defined in definition 8).*

$\diamondsuit$

Definition 10 *The* Levi-Civita connection *on a Riemannian manifold (in our case Kähler manifold) M is the unique connection ∇ on the tangent bundle of M such that that it is torsion free, in the sense that,*

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

[1]The canonical bundle K_I is defined by the following equation,

$$K_I = \wedge^n T^{(1,0)} M_I^*$$

where $2n$ is the real dimension of M.

[2]In general we shall denote the Hermitian structure of a Hermitian vector bundle by h with the name of the bundle as a superscript. However for the sake of brevity when the relevant bundle is clear from the context, we shall drop the superscript. We shall be following the same procedure regarding connections, etc.

and compatible with the metric g, in the sense that

$$\nabla g = 0 \Leftrightarrow X\left[g\left(Y,Z\right)\right] = g\left(\nabla_X Y, Z\right) + g\left(Y, \nabla_X Z\right)$$

for all vector fields X, Y and Z.

$\diamondsuit$

Remark An important fact to note about Kähler manifolds is that the integrable almost complex structure is *parallel* also with respect to the Levi-Civita connection, i.e.,

$$\nabla I = 0 \Leftrightarrow \nabla_X\left(IY\right) = I\nabla_X Y.$$

But since the compatibility relation of the almost complex structure states that $g = \omega\left(\cdot, I\cdot\right)$, we have, due to the compatibility of the Riemannian metric g with the Levi-Civita connection ∇, and the parallelism of I with respect to the Levi-Civita connection, that ω is itself *parallel* with respect to the Levi-Civita connection. Further note that the parallelism of I implies that the Levi-Civita connection preserves the eigen subbundles $T^{(1,0)}M$ and $T^{(0,1)}M$ of $TM_{\mathbb{C}}$ and therefore as a result, induces a connection on $T^{(1,0)}M$ which is compatible with the Hermitian (and holomorphic) structure of $T^{(1,0)}M$.

Before we move on to curvature we state a propostion (without proof) of relating to some special coordinates for Kähler manifolds called *geodesic coordinates* (see [30])

Proposition 2 *Around any point p of a Kähler manifold M, there exist complex coordinates $z_1, \ldots, z_m$, such that the corresponding coordinate vector fields $Z^1, \ldots, Z^m$ (as defined in the previous section) satisfy*

$$g\left(Z^i, Z^j\right) = \delta_{jk} \qquad \text{and} \qquad \nabla Z^j = 0$$

at the point p. These coordinates are called geodesic coordinates.

$\diamondsuit$

Definition 11 *The* Kähler curvature *R of the manifold M, is the curvature corresponding to the Levi-Civita connection, given by,*

$$R\left(X,Y\right)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for all vector fields X, Y and Z. Clearly this is a 2-form with values in the endomorphism bundle of the tangent budle End(TM).

$\Diamond$

Remark Since I is parallel with respect to the Levi-Civita connection ∇, we have

$$\begin{aligned} R(X,Y)IZ &= \nabla_X \nabla_Y IZ - \nabla_Y \nabla_X IZ - \nabla_{[X,Y]} IZ \\ &= I\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right) \\ &= IR(X,Y)Z \end{aligned}$$

for all vector fields X, Y and Z. Therefore we see that the endomorphism part of the curvature preserves the type of the vector and as a result we conclude that R is $(1,1)$ form with values in $\mathrm{End}\left(T^{(1,0)}M\right) \oplus \mathrm{End}\left(T^{(0,1)}M\right)$.

Remark One can use the metric g to raise or lower the indices, as usual, and in particular, by lowering an index, the curvature can be viewed as a symmetric section of $\wedge^{1,1}TM_I^* \otimes \wedge^{1,1}TM_I^*$ called the *curvature tensor*. Alternatively, by raising the index we get the *curvature operator* which is an endomorphism of $\wedge^{1,1}TM_I^*$.

Definition 12 *The* Ricci tensor *denoted* r *is an* I*-invariant symmetric, bilinear form determined by*

$$r(X,Y) = Tr\left(Z \longmapsto R(Z,X)Y\right)$$

The associated antisymmetric $(1,1)$*-form* ρ *given by*

$$\rho(X,Y) = r(IX,Y)$$

is called the Ricci form.

$\Diamond$

Remark Using the symmetries of the Kähler curvature it can be shown that Ricci form is minus the image of the Kähler form under the image of the curvature operator, i.e.,

$$\rho = -R(\omega).$$

On any complex manifold, we know that closed forms are locally exact with respect to the $\partial\bar{\partial}$ operator, i.e., if we have a closed form $\alpha \in \Omega^{p,q}(M)$ and a contractible open subset $U \subset M$, then $\beta \in \Omega^{p-1,q-1}(U)$ such that $\alpha|_U = \partial\bar{\partial}\beta$.

But on Kähler manifolds , a global version of this is true (as is stated and proved in the following proposition from [6]).

Proposition 3 *For any exact form* $\alpha \in \Omega^{p,q}(M)$*, there exists a form* $\beta \in \Omega^{p-1,q-1}(M)$*, such that* $\alpha = 2i\partial\bar{\partial}\beta$.

◊

We can apply the above proposition to the Ricci form which we know is closed and therefore differs form its harmonic part ρ^H by a real, exact $(1,1)$-form. Consequently, we can write the Ricci-form as

$$\rho = \rho^H + 2i\partial\bar{\partial}F$$

where $F \in C^\infty(M)$ is a real function called the *Ricci potential.* Clearly if M is compact the Ricci potential is determined upto a constant, and therefore it is uniquely determined if we require that its average over M is 0.

We now turn our attention to the concept of prequantization and the prequantum line bundle. As in the previous section, let M denote a symplectic manifold of dimension $2n$ with symplectic form ω. Let Γ denote the group of symmetries acting on M by symplectomorphisms.

Definition 13 *A* prequantum line bundle *over a symplectic manifold* (M,ω)*, is a triple* $(\mathcal{L}, h^{\mathcal{L}}, \nabla^{\mathcal{L}})$*, where* $\mathcal{L}$ *is a line bundle over* M *with a Hermitian metric* $h^{\mathcal{L}}$ *and a compatible connection* $\nabla^{\mathcal{L}}$ *whose curvature is*

$$F_{\nabla^{\mathcal{L}}} = \frac{i}{2\pi}\omega$$

e.g.

$$\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} = \omega(X,Y)$$

for all vector fields X and Y on M. We say the symplectic manifold (M, ω) is prequantizable *if there exists a prequantum line bundle over it.*

$\Diamond$

Remark It is often natural to require that the action of the symmetry group Γ on M lift to an action on the prequantum line bundle $\mathcal{L}$ by means of bundle maps that preserve the Hermitian structure and the connection.

Remark Interestingly every symplectic manifold need not be prequantizable. We know that the real first Chern class of the prequantum line bundle is given by $\tilde{c}_1(\mathcal{L}) = [\omega]$. This leads us to the necessary condition for prequantizability, known as the *prequantum condition*,

$$[\omega] \in \operatorname{Im}\left(H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})\right).$$

This is also a sufficient condition (see [31]) for the existence of a prequantum line bundle. When inequivalent prequantum line bundles exist over M, they are parametrized by $H^1(M, U(1))$.

Remark If we choose an almost complex structure I on our manifold (M, ω) which is compatible to the symplectic structure (and the metric), this will give the manifold M, the structure of a Kähler manifold denoted M_I, and since ω is of the form (1,1), it follows that the connection on the prequantum line bundle $\mathcal{L}$ gives it a holomorphic structure.

2 Families of Kähler Structures

During the course of this chapter we shall study the families of Kähler structures on a symplectic manifold (which lies at the heart of our study of the Hitchin connection). For the rest of this section we shall use $\mathcal{T}$ to denote a smooth manifold that will parametrize the Kähler structures on our symplectic manifold denoted as usual as (M,ω). The reference for this chapter is [6] and [14].

2.1 Families of Kähler Structures

Definition 14 *A* family of Kähler Structures *on a symplectic manifold* (M,ω) *parametrized by* $\mathcal{T}$ *is a map*

$$I : \mathcal{T} \to C^{\infty}(M, End(TM))$$

which associates to every point $\sigma \in \mathcal{T}$ an integrable, ω-compatible almost complex structure on M. For the point σ the manifold M with its ω-compatible Kähler structure $I(\sigma)$ will be denoted M_σ (similar notation being used for the corresponding metric), however when the point σ is clear from the context we shall omit the use of the subscript.

$\lozenge$

Definition 15 *A family of Kähler structures is called* smooth *if the map I (as in definition 14) is smooth in the sense that it defines a smooth section of the pull back bundle $\pi_M^* End(TM)$ over $\mathcal{T} \times M$, where π_M is the canonical projection of $\mathcal{T} \times M$ onto M.*

$\lozenge$

2.2 Infinitisemal Deformations

Given a smooth family of Kähler structures I, we can can take its derivative along a vector field V on $\mathcal{T}$ to obtain the map

$$V[I] : T \to C^{\infty}(M, \mathrm{End}(TM))$$

Further we have the identity

$$I^2 = -Id \tag{3}$$

Differentiating (3), we have

$$V[I]I + IV[I] = 0. \tag{4}$$

Therefore from equation (4), we see that I anti-commutes with $V[I]$, thus implying that $V[I]_\sigma$ changes types on M_σ. Thus we have a decomposition

$$V[I] = V[I]' + V[I]'' \tag{5}$$

where

$$V[I]'_\sigma \in C^\infty\left(M, T^{(1,0)}M_\sigma \otimes T^{(0,1)}M^*_\sigma\right)$$

and

$$V[I]''_\sigma \in C^\infty\left(M, T^{(0,1)}M_\sigma \otimes T^{(1,0)}M^*_\sigma\right).$$

We see that

$$V[g](X,Y) = \omega(X, V[I]Y)$$

since ω is of type $(1,1)$ and g is symmetric, we have that

$$V[g] \in C^\infty\left(M, S^2\left(T^{(1,0)}M^*_\sigma\right) \oplus S^2\left(T^{(0,1)}M^*_\sigma\right)\right).$$

We now define a bivector field $\tilde{G}(V) \in C^\infty(M, TM_{\mathbb{C}} \otimes TM_{\mathbb{C}})$ by the relation

$$V[I] = \tilde{G}(V) \cdot \omega$$

for any vector field V on $\mathcal{T}$. Further we define

$$G(V) \in C^\infty\left(M, T^{(1,0)}M_\sigma \otimes T^{(1,0)}M_\sigma\right)$$

such that

$$\tilde{G}(V) = G(V) + \bar{G}(V) \tag{6}$$

where $\bar{G}(V) \in C^\infty\left(M, T^{(0,1)}M_\sigma \otimes T^{(0,1)}M_\sigma\right)$ for all real vector fields V on $\mathcal{T}$. We see that $\tilde{G}$ and G are nothing but 1-forms with values in $C^\infty(TM_{\mathbb{C}} \otimes TM_{\mathbb{C}})$ and $C^\infty\left(T^{(1,0)}M_\sigma \otimes T^{(1,0)}M_\sigma\right)$ respectively.

Remark We make the further assumption $\mathcal{T}$ is a complex manifold and I is a holomorphic mapping (the family of Kähler structures being called *holomorphic*). Conecretely this means that

$$V'[I]_\sigma = V[I]'_\sigma \text{ and } V''[I]_\sigma = V[I]''_\sigma$$

where V' is the $(1,0)$ part of V and V'' is the $(0,1)$ of V.

Under the assumptions of the preceding remark we notice,

$$V'[I] = G(V) \cdot \omega$$

and that $G(V) = G(V')$. Since $V[g] = \omega \cdot V[I]$, we have

$$V[g] = \omega \cdot V[I] = \omega \cdot \tilde{G}(V) \cdot \omega \tag{7}$$

and from this it is clear that $\tilde{G}$ takes values in $C^{\infty}\left(M, S^2\left(TM_{\mathbb{C}}\right)\right)$ and G in $C^{\infty}\left(M, S^2\left(T^{(1,0)}M_{\sigma}\right)\right)$.

Finally we need the variation of the Levi-Civita connection which is the tensor field

$$V[\nabla] \in C^{\infty}\left(M,\ S^2\left(T^{(1,0)}M_{\sigma}^{*}\right) \otimes T^{(1,0)}M_{\sigma}\right).$$

We state the formula for this variation without proof (for further reference see [6]).

Lemma 1 *For all vector fields X, Y, Z on M, we have*

$$2g\left(V[\nabla]_X Y, Z\right) = \nabla_X\left(V[g]\right)(Y,Z) + \nabla_Y\left(V[g]\right)(X,Z) - \nabla_Z\left(V[g]\right)(X,Y)$$

$\Diamond$

2.3 Holomorphic Families of Kähler Structures

We now turn our attention to the necessary and sufficient conditions to ensure the holomorphicity of the family of Kähler structures. The reference for the results in this section is [14]. Let us start this section by giving an alternative characterization of holomorphic families of Kähler structures from the one developed above. As in the remark in section 2.2, we start by assuming that $\mathcal{T}$ is a complex manifold, I is a holomorphic mapping and the the family of Kähler structures is holomorphic. Let J be an integrable almost complex structure on $\mathcal{T}$ induced by the complex structure. We now get an almost complex structure $\hat{I}$ on $\mathcal{T} \times M$ defined by

$$\hat{I}(V \oplus X) = JV \oplus IX \tag{8}$$

where $V \oplus X \in T_{(\sigma,p)}(\mathcal{T} \times M)$.

Proposition 4 (Andersen, Gammelgaard and Lauridsen) *The family I is holomorphic iff the complex structure $\hat{I}$ defined by equation* (8) *is integrable.*

Proof: To say that $\hat{I}$ is integrable is the same as saying that the Nijenhuis tensor for $\hat{I}$ vanishes. We further know that J is integrable so the Nijenhuis tensor of J vanishes when evaluated on vector tangent to $\mathcal{T}$ and similarly Nijenhuis tensor of I vanishes when evaluated on vectors tangent to M since I is a family of integrable almost complex structues on M. So let V and X be vector fields on $\mathcal{T}$ and M respectively. We have $[V, IX] = V[I]X$, and so we have

$$\begin{aligned} N_{\hat{I}}(V', X) &= [JV', IX] - [V', X] - \hat{I}[JV', X] - \hat{I}[V', IX] \\ &= i[V', IX] - \hat{I}[V', IX] \\ &= iV'[I]X - IV'[I]X \\ &= 2i\pi^{0,1}V'[I]X \end{aligned} \tag{9}$$

and similarly we show that $N_{\hat{I}}(V'', X) = -2\pi^{1,0}V'[I]X$ and therefore the Nijemhuis tensor vanishes iff

$$\pi^{0,1}V'[I]X = 0 \qquad \text{and} \qquad \pi^{1,0}V''[I]X = 0$$

which completes the proof of the proposition.

◊

Lemma 2 (Andersen, Gammelgaard and Lauridsen) *If I is holomorphic family of Kähler structures, then*

$$W''V'[I] = \frac{i}{2}[V'[I], W''[I]]$$

for any vector fields V and W on $\mathcal{T}$ such that V' and W'' commute.

Proof: The holomorphicity of I implies that $V'[I]\pi^{1,0} = 0$, and therefore we have

$$W''V'[I] = \frac{i}{2}V'[I]W''[I] \tag{10}$$

by differentiating along W''. In a similar fashion, by differentiating the relation $W''[I]\pi^{0,1} = 0$, we have

$$V'W''[I] = -\frac{i}{2}W''[I]V'[I] \tag{11}$$

By adding equations (10) and (11) and using the fact that V' and W'' commute, we prove the assertion in the lemma.

◊

2.4 Rigid Families of Kähler Structures

We now turn our attention to the the extremely important concept of *rigidity.*

Definition 16 (Andersen) *A family of Kähler structures I is said to be rigid if*

$$\nabla_{X''} G(V) = 0 \tag{12}$$

for all vector fields X on M and V on $\mathcal{T}$ ($G(V) \in C^{\infty}\left(M, T^{(1,0)}M_{\sigma} \otimes T^{(1,0)}M_{\sigma}\right)$, see section 2).

$\Diamond$

Remark We therefore have that the family I is rigid if $G(V)$ is a *holomorphic section* of $S^2\left(T^{(1,0)}M\right)$, for any vector field V on $\mathcal{T}$.

2.5 Families of Ricci Potential

We end this chapter with a short discussion on families of Ricci potential. Throughout this discussion we shall assume that our manifold M is compact and of real dimension $2n$. For each $\sigma \in \mathcal{T}$ we have a corresponding Ricci potential $F_{\sigma} \in C^{\infty}(M)$ satisfying

$$\rho_{\sigma} = \rho_{\sigma}^{H} + 2i\partial_{\sigma}\bar{\partial}_{\sigma}F_{\sigma}$$

where ρ_{σ}^{H} is the unique harmonic part of the Ricci form.

Since we assume that M is compact the function F_{σ} uniquely determined upto a constant, we can fix a unique Ricci potential by demanding that it have 0 average on our manifold, i.e.

$$\int_M F_{\sigma}\omega^n = 0.$$

With this normalization the Ricci potentials define a smooth function $\hat{F} \in C^{\infty}(\mathcal{T}\times M)$, which can be be interpreted as a smooth map $F : \mathcal{T} \to C^{\infty}(M)$.

In general we say a smooth map $\hat{F} \in C^{\infty}(\mathcal{T} \times M)$ is a *smooth family of Ricci potentials* if it satisfies

$$\rho_{\sigma} = \rho_{\sigma}^{H} + 2i\partial_{\sigma}\bar{\partial}_{\sigma}F_{\sigma} \tag{13}$$

for any $\sigma \in \mathcal{T}$.

We further make an additional assumption that there exists an $m \in \mathbb{Z}$ such that the real first Chern class of (M, ω) is given by

$$\tilde{c}_1(M, \omega) = m\left[\frac{\omega}{2\pi}\right]. \tag{14}$$

But we know that the real first Chern class is represented by $\dfrac{\rho}{2\pi}$ and consequently we have

$$\rho = m\omega + 2i\partial\bar{\partial}F. \tag{15}$$

Since the Kähler form is harmonic. The following lemma (stated without proof, see for reference [14]) gives a useful identity involving the variation of the Ricci potential, given our assumptions.

Lemma 3 (Andersen) *Suppose that M is a compact, symplectic manifold which satisfies $H^1(M, \mathbb{R}) = 0$ and $\tilde{c}_1(M, \omega) = m\left[\frac{\omega}{2\pi}\right]$, and let I be a holomorphic family Kähler structures on M. Then any family of Ricci potentials satisfies*

$$4i\bar{\partial}V'[F] = \delta(V'[I]) + 2dF \cdot V'[I]$$

for any vector field V on $\mathcal{T}$.

◊

Remark Using lemma 3 , we can express the divergence of $V[I]$ in terms of the Ricci potential. By lemma 3 we have

$$4iV'X''[F] = -2(V'[I]V)[F] + 4iX''V'[F] = \delta(V'[I])X \tag{16}$$

Conjugating equation (16), we have

$$-4iV''X'[F] = \delta(V''[I])X \tag{17}$$

The result follows by adding the above identities.

3 Hitchin Connection

3.1 Setup for the construction of the Hitchin Connection

In this chapter we will detail the construction of the Hitchin connection as given in [1]. We begin with the assumption that the manifold M is a compact (complex) Kähler manifold with symplectic form ω. We further assume that the manifold M is *prequantizable*. Recall that for a manifold M to be prequantizable means that there exists a *prequantum line bundle* $\left(\mathcal{L}, h^{\mathcal{L}}, \nabla^{\mathcal{L}}\right)$ over M where $\mathcal{L}$ is a line bundle over M with a Hermitian metric $h^{\mathcal{L}}$ and a compatible connection $\nabla^{\mathcal{L}}$ with curvature $\frac{i}{2\pi}\omega$ (see definition 13). Indeed having assumed our manifold to be prquantizable, we fix the prequantum line bundle $\mathcal{L}$ over it.

We next assume that the smooth manifold $\mathcal{T}$ smoothly parametrizes the Kähler structures on M (see definitions 14 and 15) by means of the map

$$I : \mathcal{T} \rightarrow C^{\infty}\left(M, \mathrm{End}\left(TM\right)\right).$$

As in section 2.2 we make the further assumption that $\mathcal{T}$ is a complex manifold and I is a holomorphic map.

Remark For the duration of this chapter, for the sake of brevity, we shall adopt the usual notation along the lines established in chapter 2, by denoting $I\left(\sigma\right)$ as I_{σ}.

Let V be a vector field on $\mathcal{T}$. Further as in section 2.2, we define a bivector fields $\tilde{G}\left(V\right) \in C^{\infty}\left(M, TM_{\mathbb{C}} \otimes TM_{\mathbb{C}}\right)$ and $G\left(V\right) \in C^{\infty}\left(M, T^{(1,0)}M_{\sigma} \otimes T^{(1,0)}M_{\sigma}\right)$ by the relations

$$V\left[I\right] = \tilde{G}\left(V\right) \cdot \omega.$$

and

$$\tilde{G}\left(V\right) = G\left(V\right) + \bar{G}\left(V\right)$$

where $\bar{G}\left(V\right) \in C^{\infty}\left(M, T^{(0,1)}M_{\sigma} \otimes T^{(0,1)}M_{\sigma}\right)$ for all real vector fields V on $\mathcal{T}$.

We now consider the k-th tensor power of the prequantum line bundle $\mathcal{L}$, denoted $\mathcal{L}^k$. For every $\sigma \in \mathcal{T}$ consider the operator $\nabla^{(0,1)}_{\sigma}$ given by

$$\nabla^{(0,1)}_{\sigma} = \frac{1}{2}\left(Id + iI_{\sigma}\right)\nabla$$

where ∇ is the connection induced on $\mathcal{L}^k$ by the connection on $\mathcal{L}$. This defines a $\bar{\partial}$ operator on $\mathcal{L}^k$.

Consider now the finite dimensional subspace of $C^\infty\left(M,\mathcal{L}^k\right)$ given by

$$H_\sigma^{(k)} = H^0\left(M_\sigma,\mathcal{L}^k\right) = \left\{s \in C^\infty\left(M,\mathcal{L}^k\right) \mid \nabla_\sigma^{0,1} s = 0\right\}.$$

We further make the assumption that these subspaces of holomorphic sections form a smooth finite rank subbundle $H^{(k)}$ of the trivial bundle $\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty\left(M,\mathcal{L}^k\right)$.

We denote by $\hat{\nabla}^t$ the trivial connection in the trivial bundle $\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty\left(M,\mathcal{L}^k\right)$. Let $\mathcal{D}\left(M,\mathcal{L}^k\right)$ denote the vector space of differential operators on $C^\infty\left(M,\mathcal{L}^k\right)$. For any smooth 1-form u on $\mathcal{T}$ with values in $\mathcal{D}\left(M,\mathcal{L}^k\right)$ we have a connection $\hat{\nabla}$ on $\mathcal{H}^{(k)}$ given by

$$\hat{\nabla}_V = \hat{\nabla}_V^t - u\left(V\right) \tag{18}$$

for any vector field V on $\mathcal{T}$.

We now state and prove the lemma from [1] that defines the condition for the existence of the Hitchin connection.

Lemma 4 (Andersen) *The connection $\hat{\nabla}$ in $\mathcal{H}^{(k)}$ induces a connection in $H^{(k)}$ iff*

$$\frac{i}{2}V'\left[I\right]\nabla^{1,0}s + \nabla^{0,1}u\left(V\right)s = 0 \tag{19}$$

for all vector fields V on $\mathcal{T}$ and all smooth sections s of $H^{(k)}$.

Proof: Let s be a section of the subbundle $H^{(k)}$. Then $\hat{\nabla}_V s$ is a section of $\mathcal{H}^{(k)}$ for V, a vector field on $\mathcal{T}$. Applying the $\nabla^{0,1}$ operator to $\hat{\nabla}_V s$, we have

$$\begin{aligned}\nabla_\sigma^{0,1}\left(\left(\hat{\nabla}_V\left(s\right)\right)_\sigma\right) &= \nabla_\sigma^{0,1}\left(V\left[s\right]_\sigma\right) - \nabla_\sigma^{0,1}\left(\left(u\left(V\right)s\right)_\sigma\right)\\ &= -\frac{i}{2}\left(V\left[I\right]\nabla^{1,0}s\right)_\sigma - \nabla_\sigma^{(0,1)}\left(\left(u\left(V\right)s\right)_\sigma\right)\end{aligned}$$

where the second equation above holds since

$$\nabla_\sigma^{0,1}\left(V\left[s\right]_\sigma\right) + \frac{i}{2}\left(V\left[I\right]\nabla^{1,0}s\right)_\sigma = 0$$

Therefore $\hat{\nabla}$ preserves the subbundle $H^{(k)}$ iff (19) holds.

$\diamondsuit$

Remark Note that $V''[I]\nabla_\sigma^{1,0}s = 0$, where V'' is the $(0,1)$-part of the vector field V on $\mathcal{T}$. Therefore $u(V'') = 0$ solves equation (19) along the anti-holomorphic directions, i.e., the $(0,1)$-part of the trivial connection $\hat{\nabla}_V^t$ (see (18)) induces a $\bar{\partial}$-operator on $H^{(k)}$ and hence makes it a holomorphic vector bundle over $\mathcal{T}$.

We now consider the bivector field G from the previous section. Recall (from section 2.2) that $G(V) \in C^\infty\left(M, S^2\left(T^{(1,0)}M_\sigma\right)\right)$. Therefore $G(V)$ induces the bundle map

$$G(V) : T^{(1,0)}M_\sigma^* \to T_\sigma^{(1,0)}M$$

Corresponding to $G(V)$ we construct the operator

$$\Delta_{G(V)} : C^\infty\left(M, \mathcal{L}^k\right) \to C^\infty\left(M, \mathcal{L}^k\right)$$

defined by

$$\begin{aligned}\Delta_{G(V)} : C^\infty\left(M, \mathcal{L}^k\right) &\xrightarrow{\nabla_\sigma^{1,0}} C^\infty\left(M, T^{(1,0)}M_\sigma^* \otimes \mathcal{L}^k\right) \xrightarrow{G(V)\otimes Id} C^\infty\left(M, T^{(1,0)}M_\sigma \otimes \mathcal{L}^k\right) \\ &\xrightarrow{\nabla_\sigma^{1,0}\otimes Id + Id\otimes\nabla_\sigma^{1,0}} C^\infty\left(M, T^{(1,0)}M_\sigma^* \otimes T^{(1,0)}M_\sigma \otimes \mathcal{L}^k\right) \xrightarrow{Tr} C^\infty\left(M, \mathcal{L}^k\right)\end{aligned}$$

Now let F be the Ricci potential (see section 1.4 for reference). We now have the following theorem.

3.2 The Hitchin Connection

We now construct an ansatz for a 1-form u that solves (19).

Theorem 3 (Andersen) *Suppose I is a rigid family of Kähler structures on the compact symplectic prequantizable manifold manifold (M, ω), with the property that there exists an $n \in \mathbb{Z}$ such that the first Chern class of (M, ω) is $n[\omega] \in H^2(M, \mathbb{Z})$ and $H^1(M, \mathbb{R}) = 0$. Then the 1-form u given by*

$$u(V) = \frac{1}{2k+n}o(V) + V'[F] \tag{20}$$

where

$$o(V) = \frac{1}{2}\Delta_{G(V)} - \nabla_{G(V)dF} - V'[F] \tag{21}$$

solves equation (18) *for all* k *such that* $2k+n \neq 0$.

Lemma 5 (Andersen) *Assume that the first Chern class of* (M,ω) *is* $n[\omega] \in H^2(M,\mathbb{Z})$. *For any* $\sigma \in \mathcal{T}$ *and for and* $G \in H^0(M_\sigma, S^2(T_\sigma))$ *we have the following formula*

$$\begin{aligned}\nabla_\sigma^{0,1}\left(\Delta_G(s) - 2\nabla_{GdF_\sigma}(s)\right) = &-i(2k+n)\,G\omega\nabla_\sigma^{0,1}(s)\\ &- ikTr\left(-2G\partial_\sigma F\omega + \nabla_\sigma^{0,1}(G)\,\omega\right)s,\end{aligned}$$

for all $s \in H^0(M_\sigma, \mathcal{L}^k)$.

Proof: Consider the effect of applying the operator $\nabla_\sigma^{0,1}$ to the operator $\Delta_G(s)$. We have

$$\begin{aligned}\nabla_\sigma^{0,1}(\Delta_G(s)) &= Tr\left(\nabla^{0,1}\nabla^{1,0}G\nabla^{1,0}(s)\right)\\ &= Tr\left(\nabla^{1,0}\nabla^{0,1}G\nabla^{1,0}(s)\right)\\ &- ikTr\left(\omega G\nabla^{1,0}(s)\right) - iTr\left(\rho_\sigma G\nabla^{1,0}(s)\right)\end{aligned} \tag{22}$$

where $\rho_\sigma \in \Omega^{1,1}(M_\sigma)$ is the Ricci form ($\rho_\sigma^H \in \Omega^{1,1}(M_\sigma)$ being the harmonic part of the Ricci form). Now G is holomorphic by our assumptions, therefore (22) becomes

$$\begin{aligned}\nabla_\sigma^{0,1}(\Delta_G(s)) = &-ikTr\left(\nabla^{1,0}(G\omega s)\right)\\ &- ikTr\left(\omega G\nabla^{1,0}(s)\right) - iTr\left(\rho_\sigma G\nabla^{1,0}(s)\right)\end{aligned} \tag{23}$$

Now notice that $\nabla(\omega) = 0$ due to our assmption that (M,ω) is Kähler. Therefore we can further write (23) as

$$\begin{aligned}\nabla_\sigma^{0,1}(\Delta_G(s)) = &-ikTr\left(\nabla^{1,0}(G)\,\omega\right)\otimes s\\ &- 2ikTr\left(\omega G\nabla^{1,0}(s)\right) - iTr\left(\rho_\sigma G\nabla^{1,0}(s)\right)\end{aligned} \tag{24}$$

Finally notice that the assumption $c_1(M,\omega) = n\omega$ implies that $\rho_\sigma^H = n\omega$ which in turn implies

$$\rho_\sigma = n\omega + 2i\partial_\sigma\bar{\partial}_\sigma F_\sigma$$

Putting this in equation (24), one proves the lemma.

◊

We next state (without proof) a lemma that gives us the formula for the variation of the Ricci-form.

Lemma 6 (Andersen) *For any smooth vector field V on $\mathcal{T}$ we have that*

$$\left(V'[\rho]\right)^{1,1} = -\frac{1}{2}\partial Tr\left(\nabla^{1,0}\left(G\left(V\right)\right)\omega\right).$$

$\diamondsuit$

Lemma 7 (Andersen) *Given $H^1(M,\mathbb{R}) = 0$, we have the following relation*

$$2i\bar{\partial}_\sigma\left(V'[F]_\sigma\right) = \frac{1}{2}Tr\left(2G\left(V\right)\partial\left(F\right)\omega - \nabla^{1,0}\left(G\left(V\right)\right)\omega\right)_\sigma$$

Proof: We know by definition, that

$$\rho = \rho^H + 2id\bar{\partial}F.$$

Further we have assumed that $\rho^H = n\omega$. Hence we have

$$V'[\rho] = -dV'[I]\,dF + 2id\bar{\partial}V'[F]$$

Therefore we have

$$2i\partial\bar{\partial}V'[F] = \left(V'[\rho]\right)^{1,1} + \partial V'[I]\,\partial F.$$

But from lemma 6, we know that

$$\left(V'[\rho]\right)^{1,1} = -\frac{1}{2}\partial Tr\left(\nabla^{1,0}\left(G\left(V\right)\right)\omega\right).$$

Therefore we have that

$$\partial_\sigma\left(\frac{1}{2}Tr\left(2G\left(V\right)\partial F\omega - \nabla^{1,0}\left(G\left(V\right)\right)\omega\right)_\sigma - 2i\bar{\partial}_\sigma V'[F]_\sigma\right) = 0$$

But we know from lemma 5 that

$$\bar{\partial}_\sigma\left(\frac{1}{2}Tr\left(2G\left(V\right)\partial F\omega - \nabla^{1,0}\left(G\left(V\right)\right)\omega\right)_\sigma - 2i\bar{\partial}_\sigma V'[F]_\sigma\right) = 0$$

Thus in fact $\frac{1}{2}Tr\left(2G\left(V\right)\partial F\omega - \nabla^{1,0}\left(G\left(V\right)\right)\omega\right)_\sigma - 2i\bar{\partial}_\sigma V'[F]_\sigma$ is a closed 1-form on M. But our assumption for the manifold is that $H^1(M,\mathbb{R}) = 0$. Therefore $\frac{1}{2}Tr\left(2G\left(V\right)\partial F\omega - \nabla^{1,0}\left(G\left(V\right)\right)\omega\right)_\sigma - 2i\bar{\partial}_\sigma V'[F]_\sigma$ is also an exact form, but then in fact it vanishes, since it is also of type $(0,1)$ on M_σ.

$\diamondsuit$

From lemmas 5 and 7 we conclude that

$$u(V) = \frac{1}{2k+n}\left\{\frac{1}{2}\Delta_{G(V)} - \nabla_{G(V)dF} + 2kV'[F]\right\}$$

solves (19) thus proving theorem 3.

$\diamondsuit$

4 Toeplitz Operators and Star Products

4.1 Definition

During the course of this chapter we shall present a review of the concepts of Toepltiz operators and other related concepts. We continue the notations of the previous chapters and assume the manifold M to be as in the last chapter, i.e. prequantizable, with a fixed prequantum line bundle $\mathcal{L}$ (and $\mathcal{L}^k$ denoting the k-th tensor power of $\mathcal{L}$).

Let us begin by considering the space $C^\infty(M)$. For each $f \in C^\infty(M)$ we consider the *prequantum* operator

$$M_f^{(k)} : C^\infty\left(M, \mathcal{L}^k\right) \to C^\infty\left(M, \mathcal{L}^k\right)$$

given by

$$M_f^{(k)}(s) = f \cdot s$$

where $s \in H^0\left(M, \mathcal{L}^k\right)$ ($H^0\left(M, \mathcal{L}^k\right)$ as in last chapter).

It is clear that these operators acting on $C^\infty\left(M, \mathcal{L}^k\right)$ also act on $\mathcal{H}^{(k)}$, but since f is merely smooth, the operators of the form M_f need not preserve the subbundle $H^0\left(M, \mathcal{L}^k\right)$. Our aim however, is to have operators that preserve the holomorphic subbundle $H^0\left(M, \mathcal{L}^k\right)$.

Consider the pre-Hilbert space structure on $C^\infty\left(M, \mathcal{L}^k\right)$, i.e. the integral

$$\langle s_1, s_2 \rangle = \frac{1}{m!} \int_M (s_1, s_2)\, \omega^m \tag{25}$$

(recall that the prequantum line bundle $\mathcal{L}$ comes equipped with a Hermitian structure $(\cdot, \cdot)$). Now recall that $\mathcal{H}^{(k)}$ is defined to be

$$\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty\left(M, \mathcal{L}^k\right)$$

Therefore we can think of the pre-Hilbert space structure on $C^\infty\left(M, \mathcal{L}^k\right)$ given by $\langle \cdot, \cdot \rangle$ in equation (25) as a pre-Hilbert space structure on $\mathcal{H}^{(k)}$. This in turn, induces a pre-Hilbert space structure on the finite rank subbundle $H^{(k)}$ of $\mathcal{H}^{(k)}$. And this in turn induces the operator norm $\| \cdot \|$ on $\mathrm{End}\left(H^{(k)}\right)$.

Since $H_\sigma^{(k)}$ is a finite dimensional subspace of $C^\infty\left(M, \mathcal{L}^k\right) = \mathcal{H}_\sigma^{(k)}$ therefore it is closed. As a result we have the orthogonal projection

$$\pi_\sigma^{(k)} : \mathcal{H}_\sigma^{(k)} \to H_\sigma^{(k)}.$$

Since $H^{(k)}$ is a smooth subbundle of $\mathcal{H}^{(k)}$, the projections $\pi_\sigma^{(k)}$ form a smooth map $\pi^{(k)}$ from $\mathcal{T}$ to the space of bounded operatoes on the L_2-completion of $C^\infty\left(M, \mathcal{L}^k\right)$.

To see this consider a local frame $(s_1, \cdots, s_{\text{Rank } H^{(k)}})$ of $H^{(k)}$. Let

$$h_{ij} = \langle s_1, s_2 \rangle$$

where $\langle s_1, s_2 \rangle$ is defined is (25). Let h_{ij}^{-1} denote the inverse of the matrix h_{ij}. Then

$$\pi_\sigma^{(k)} = \sum_{i,j} \langle s, (s_i)_\sigma \rangle \left(h_{ij}^{-1}\right)_\sigma (s_j)_\sigma$$

Definition 17 *For a smooth function $f \in C^\infty(M)$, the corresponding* Toeplitz operator $T_{f,\sigma}^{(k)}$ *is the operator*

$$T_{f,\sigma}^{(k)} : \mathcal{H}_\sigma^{(k)} \to \mathcal{H}_\sigma^{(k)}$$

defined by

$$T_{f,\sigma}^{(k)}(s) = \pi_\sigma^{(k)}(fs)$$

for any element $s \in \mathcal{H}_\sigma^{(k)}$ and any point $\sigma \in \mathcal{T}$.

$\diamondsuit$

We now prove a small lemma but fundamental lemma relating to the adjoint of the operator ∇_X where $X \in C^\infty(M, TM_{\mathbb{C}})$.

Lemma 8 (Andersen) *The adjoint of ∇_X, denoted $(\nabla_X)^*$, is given by*

$$(\nabla_X)^* = -\nabla_{\bar{X}} - \delta \bar{X},$$

for any complex vector field $X \in C^\infty(M, TM_{\mathbb{C}})$.

Proof: Recall from definition 7, that the divergence $\delta\bar{X}$ is the unique function satisfying

$$\mathcal{L}_{\bar{X}}\omega^n = \delta\bar{X}\omega^n.$$

Therefore we have

$$\mathcal{L}_{\bar{X}}\left(s_1, s_2\right)\omega^n = \left(\nabla_{\bar{X}}s_1, s_2\right)\omega^n + \left(s_1, \nabla_X s_2\right)\omega^n + \left(s_1, s_2\right)\delta\bar{X}\omega^n \tag{26}$$

for smooth sections $s_1, s_2 \in C^\infty\left(M, \mathcal{L}^k\right)$. By Cartan's formula for the Lie derivative, the expression in equation (26) is an exact expression.

Applying integration and the Stoke's theorem to (26), we get

$$\left\langle \left(\nabla_X\right)^* s_1, s_2\right\rangle = -\left\langle \nabla_{\bar{X}}s_1, s_2\right\rangle - \left\langle \delta\bar{X}s_1, s_2\right\rangle$$

which is what we seek.

$\Diamond$

Proposition 5 (Andersen) *If $X \in C^\infty\left(M, T^{(1,0)}M_\sigma\right)$ is a smooth section of the holomorphic tangent bundle on M, then*

$$\pi_\sigma^{(k)}\nabla_X = -T_{\delta(X),\sigma}^{(k)}.$$

Proof: Let $s_1 \in C^\infty\left(M, \mathcal{L}^k\right)$ and $s_2 \in H^0\left(M, \mathcal{L}^k\right)$. Then we have

$$X\left(s_1, s_2\right) = \left(\nabla_X s_1, s_2\right) + \left(s_1, \nabla_{\bar{X}}s_2\right)$$

But we have

$$\left(s_1, \nabla_{\bar{X}}s_2\right) = 0$$

Therefore

$$X\left(s_1, s_2\right) = \left(\nabla_X s_1, s_2\right)$$

Taking the Lie derivative along X of (s_1, s_2), we have

$$d\left(\left(s_1, s_2\right) i_X\omega^n\right) = \left(\nabla_X s_1, s_2\right)\omega^n + \left(s_1, s_2\right)\delta X\omega^n$$

Integrating over M, we have,

$$0 = \left\langle \nabla_X s_1, s_2\right\rangle + \left\langle \delta X s_1, s_2\right\rangle$$

which implies the proposition.

$\Diamond$

4.2 Star Products and Deformation Quantization

During the course of this section we will define star products and deformation quantizations with separation of variables and investigate briefly some of their properties. We begin by considering V, a Hausdorff topological vector space. The references for this section are [17],[18] and [19]

Definition 18 *The elements of the space of formal Laurent series with finite principal part* $V\left[\nu^{-1},\nu\right]]$ *are* formal vectors. *Let* $v(m)$, $m\in\mathbb{R}$ *be a family of vectors in* V *that admits an asymptotic expansion as* m *tends to* ∞, $v(m)\sim\sum_{r\geq r_0}\frac{v_r}{m^r}$. *The* formalizer $\mathbb{F}$ *is a operator that acts on* $v(m)$ *such that*

$$\mathbb{F}:v(m)\mapsto\sum_{r\geq r_0}\nu^r v_r.$$

◊

Now we turn our attention to our manifold M. Let $U\subset M$. We denote by $\mathcal{F}(U)$ the space of all *formal* complex valued functions on U and let $\mathcal{F}(M)=\mathcal{F}$. Let $\mathbb{K}=\mathbb{C}\left[\nu^{-1},\nu\right]]$ be the set of all formal complex numbers.

Definition 19 *A* deformation quantization *on* (M,ω) *is an associative* $\mathbb{K}$*-algebra structure on* $\mathcal{F}$, *with a product called* star product, *denoted* $\star$, *such that for* $f=\sum\nu^j f_j$ *and* $g=\sum\nu^k g_k\in\mathcal{F}$ *we have the following formula*

$$f\star g=\sum_r\nu^r\sum_{i+j+k}C_i\left(f_j,g_k\right)$$

where C_r *for* $r=0,1,\cdots$ *is a sequence of bilinear mappings*

$$C_r:C^\infty(M)\times C^\infty(M)\to C^\infty(M)$$

such that $C_0(\varphi,\psi)=\varphi\psi$ *and* $C_1(\varphi,\psi)-C_1(\psi,\varphi)=i\{\varphi,\psi\}$ *for* $\varphi,\psi\in C^\infty(M)$. *If all* C_r *are bidifferential operators, then the deformation quantization is called* differential.

◊

Remark Given an open set $U\subset M$, a deformation quantization $\mathcal{F}$ can be localized to U (with the corresponding star product also denoted $\star$).

Definition 20 *Two deformation quantizations* $(\mathcal{F}, \star_1)$ *and* $(\mathcal{F}, \star_2)$ *are said to be* equivalent *if there exists an isomorphism* $B : (\mathcal{F}, \star_1) \to (\mathcal{F}, \star_2)$ *of algebras such that* B *is of the form*

$$B = 1 + \nu B_1 + \nu^2 B_2 + \cdots$$

where B_k *are endomorphisms of* $C^{\infty}(M)$*.*

◊

Remark Given $f \in \mathcal{F}$, we have the operators *left* and *right multiplication* operators denoted by L_f and R_f, such that

$$L_f g = R_g f = f \star g$$

where $f, g \in \mathcal{F}$. It is clear that the associativity of the star product $\star$ corresponds to the commutativity of L_f with R_g for all $f, g \in \mathcal{F}$.

We now turn to the concept of deformation quantization with separation of variables.

Definition 21 *A deformation quantization* $(\mathcal{F}, \star)$ *is said to be a* deformation quantization with separation of variables *if for any open set* $U \subset M$ *and any holmorphic function* a *and antiholmorphic function* b *on* U *the operators* L_a *and* R_b *are pointwise multiplication operators by* a *and* b *respectively, i.e.,* $L_a = a$ *and* $R_b = b$*.*

◊

Definition 22 *A formal form* $\tilde{\omega}$ *given by*

$$\tilde{\omega} = \left(\frac{1}{\nu}\right)\omega + \sum_{r \geq 0} \nu^r \omega_r$$

is called the formal deformation *of the form* $\left(\frac{1}{\nu}\right)\omega$ *if the forms* ω_r, $r \geq 0$ *are closed but not necessarily nondegenerate* $(1,1)$*-forms on* M*.*

◊

We now give a brief description of how to construct the star product $\star$ with separation of variables on M corresponding to the formal form $\tilde{\omega}$. The discussion follows closely the discussion from [17].

Consider an arbitrary contractible coordinate chart U of M with holomorphic coordinates $\{z^k\}$. Let $\Phi = \sum_{r\geq -1} \nu^r \Phi_r$, be a formal potential of the $\tilde{\omega}$ on U, i.e., $\tilde{\omega} = -i\partial\bar{\partial}\Phi$. The star product corresponding to $\tilde{\omega}$ is such that on U we have

$$L_{\partial\Phi/\partial z^k} = {}^{\partial\Phi}/_{\partial z^k} + {}^{\partial}/_{\partial z^k}$$

and

$$R_{\partial\Phi/\partial \bar{z}^l} = {}^{\partial\Phi}/_{\partial \bar{z}^l} + {}^{\partial}/_{\partial \bar{z}^l}.$$

Let $L(U)$ be the set of all left multiplication operators on U. $L(U)$ is completely described as the set of all formal differential operators commuting with the pointwise multiplication operators by antiholomorphic coordinates $R_{\bar{z}^l} = \bar{z}^l$ and the operators $R_{\partial\Phi/\partial\bar{z}^l}$ defined above. One can then construct the star product on U from the knowledge of $L(U)$ and since the local star products agree on the intersections, they define a global star product $\star$ on M.

We now state 2 theorems of Schlichenmaier's that relates the operator T_{fg} with the operator $T_f T_g$.

Theorem 4 *(Schlichenmaier) There exists a unique (formal) star product on* $C^\infty(M)$

$$f \star g = \sum_{l=0}^{\infty} \nu^l c^{(l)}(f,g)$$

where $c^{(l)}(f,g) \in C^\infty(M)$, *such that for* $f, g \in C^\infty(M)$ *and for every* $N \in \mathbb{M}$ *we have with suitable constants* $K_N(f,g)$ *for all* k

$$\| T^{(k)}_{f,\sigma} T^{(k)}_{g,\sigma} - \sum_{0\leq l\leq N} \left(\frac{1}{k}\right)^j T^{(k)}_{c^{(l)}(f,g),\sigma} \| = K_N(f,g) \left(\frac{1}{k}\right)^N.$$

$\Diamond$

Theorem 5 (Bordemann Meinrenken Schlichenmaier)

(i) For every $f \in C^{\infty}(M)$ there is $C > 0$ such that

$$\| f \|_{\infty} + \frac{C}{k} \leq \| T^{(k)}_{f,\sigma} \| \leq \| f \|_{\infty}$$

In particular, $lim_{k \to \infty} \| T^{(k)}_{f,\sigma} \| = \| f \|_{\infty}$, $\| \cdot \|_{\infty}$ is the sup-norm of f on M.

(ii) For every $f, g \in C^{\infty}(M)$

$$\| mi \left[T^{(k)}_{f,\sigma}, T^{(k)}_{g,\sigma} \right] - T^{(k)}_{\{f,g\}} \| = O\left(\frac{1}{m}\right)$$

as $k \to \infty$.

(iii) For every $f, g \in C^{\infty}(M)$

$$\| T^{(k)}_{f,\sigma} T^{(k)}_{g,\sigma} - T^{(k)}_{fg,\sigma} \| = O\left(\frac{1}{m}\right)$$

as $k \to \infty$.

$\Diamond$

4.3 Further Properties

We use the notations from [19]. As earlier V is a vecor field on the parametrizing manifold $\mathcal{T}$. We begin by restating the setting from [19].

We begin by considering the dual line bundle $\tau : \mathcal{L}^* \to M$, with metric $\tilde{h}$ induced by the metric h on $\mathcal{L}$. We restrict our attention to the the S^1-principal bundle X defined by,

$$X = \left\{ \alpha \in \mathcal{L}^* \mid \tilde{h}(\alpha) = 1 \right\}.$$

Let $\tilde{\Omega}$ be defined as the S^1-invariant volume form on X such that the following holds

$$\int_X \tau^*(f)\, \tilde{\Omega} = \int_M f \Omega$$

for all $f \in C^{\infty}(M)$.

We next identify the sections of $\mathcal{L}^k$ with the k-homogenous functions on $\mathcal{L}^*$ by means of the mapping

$$\Theta_k : s \to \psi_s \tag{27}$$

where

$$\psi_s(\alpha) = \alpha^{\otimes k}(s(x))$$

for all $\alpha \in X$, $x \in M$ and $s \in C^\infty(M, \mathcal{L}^k)$. In fact this is an isometry between the L_2 sections of $\mathcal{L}^k$ and the k-th weight space of the S^1 action on $L_2\left(X, \tilde{\Omega}\right)$.

For every $\sigma \in \mathcal{T}$, we consider the Hardy space $\mathbb{H}_\sigma$, a closed subset of square integrable functions on X, which extends over the unit disc bundle D in $\mathcal{L}^*$ holomorphically ($D = \left\{\alpha \in \mathcal{L}^* \mid \tilde{h}(\alpha) < 1\right\}$), with respect the complex structure induced from the one on M_σ. The orthogonal projection from $L_2(X)$ to $\mathbb{H}_\sigma$ is called the Szegö projection and is denoted Π_σ.

The Hardy space $\mathbb{H}_\sigma$ splits up into weight spaces $\mathbb{H}_\sigma = \sum_m \mathbb{H}_\sigma^{(m)}$ where $\mathbb{H}_\sigma^{(k)}$ is the k-th weight space of the the S^1 action on $L_2\left(X, \tilde{\Omega}\right)$. The isomorphism in (27) restricts to an isometry $H^0\left(M_\sigma, \mathcal{L}^k\right) \cong \mathbb{H}_\sigma^{(k)}$. Denote the Bergman projection (orthogonal projection) onto $\mathbb{H}_\sigma^{(k)}$ by $\pi_\sigma^{(k)}$ and the Bergman kernel by $B_\sigma^{(k)}$ (note that $B_\sigma^{(k)} \in C^\infty(X \times X)$).

4.3.1 Coherent states

As usual let $x \in M$. Let $\alpha \in \mathcal{L}^* - \{0\}$ ('$-\{0\}$' means the zero section removed). For every $\sigma \in \mathcal{T}$, let $e_{\alpha,\sigma}^{(k)}$ be the coherent state, i.e.,

$$\begin{aligned} \left\langle s, e_{\alpha,\sigma}^{(k)} \right\rangle &= \alpha(s) \\ &= \alpha^{\otimes k}(s(x)) \end{aligned} \tag{28}$$

for all $s \in H^0\left(M_\sigma, \mathcal{L}^k\right)$ ($\langle .,. \rangle$ being the hermitian scalar product on $L_2\left(\mathcal{L}^k\right)$, antilinear in the second argument).

Now for $s \in C^\infty\left(M, \mathcal{L}^k\right)$, we have

$$\begin{aligned} \pi_\sigma^{(k)}(\psi_s)\,\alpha &= \psi_{\pi_\sigma^{(k)} s}(\alpha) \\ &= \int_X B_\sigma^{(k)}(\alpha, \beta)\, \psi_s(\beta)\, \tilde{\Omega}(\beta). \end{aligned}$$

In fact we have that $B_\sigma^{(k)}(\alpha,\beta) = \left\langle e^{(k)}_{\beta,\sigma}, e^{(k)}_{\alpha,\sigma} \right\rangle = \psi_{e^{(k)}_{\beta,\sigma}}(\alpha)$ (see page 11 of [19]).

Since the Bergman kernel decays faster than any power of k (see page 13 of [19]), it follows that

$$\mid \left\langle e^{(k)}_{\alpha_1,\sigma_1}, e^{(k)}_{\alpha_2,\sigma_2} \right\rangle \mid = O\left(k^{-N}\right)$$

for all $N \in \mathbb{N}$,$\alpha_i \in \mathcal{L}^*_{x_i}$, $x_1 \neq x_2$ and $\sigma_i \in \mathcal{T}$, $i = 1, 2$. As a result we only need the expression for the Bergman kernel $B_\sigma^{(k)}$ near the diagonal.

4.3.2 Near diagonal expansion for the Bergman kernel

We now recall the near diagonal expansion of the Bergman kernel from [19].

We fix an arbitrary point $x_0 \in M$, having chosen a $\sigma \in \mathcal{T}$. Let $U \subset M$ be a contractible neighbourhood of x_0 with local coordinates $\left\{z^k\right\}$. Let s be a local holomorphic frame of $\mathcal{L}^*$ over U. Let $\alpha(x)$ be a smooth section of X over U given by

$$\alpha(x) = \frac{s(x)}{\sqrt{\left(\tilde{h}(s(x))\right)}}.$$

If we set $\Phi(x) = \log \tilde{h}(s(x))$, we can express $\alpha(x)$ as

$$\alpha(x) = \exp\left(-\frac{\Phi(x)}{2}\right) s(x)$$

It follows from the fact the $\mathcal{L}$ is a prequantum line bundle that Φ is a potential of ω on U.

Let $\widetilde{\Phi} \in C^\infty\left(U \times \overline{U}\right)$ be an almost analytic extension of the potential Φ from the diagonal of $U \times \overline{U}$ in the sense of Hörmander (see page 16 of [19]). It is interesting to note that we may choose $(1/2)\left(\widetilde{\Phi}(x,y) + \overline{\widetilde{\Phi}(y,x)}\right)$ instead of $\tilde{\Phi}(x,y)$. Therefore we may choose $\widetilde{\Phi}$ such that $\widetilde{\Phi}(y,x) = \overline{\widetilde{\Phi}(x,y)}$. Denote by $\chi \in C^\infty\left(U \times \bar{U}\right)$ the function given by

$$\chi(x,y) = \tilde{\Phi}(x,y) - \frac{1}{2}(\Phi(x) + \Phi(y)).$$

It is clear that $\chi(x,x) = 0$. Further we can assume that $\mathrm{Re}(\chi(x,y)) < 0$ (see lemma 5.5 in [19]). Lastly we have that the function $y \mapsto \chi(x_0,y)$ has a non-degenerate critical point at $y = x_0$. Having recalled the setup we can now state theorem 5.6 form [19] that gives us the near diagonal expansion of the Bergman kernel.

Theorem 6 *There exists an asymtotic expansion of the Bergman kernel on $U \times U$ as $m \to \infty$, of the form*

$$B_{\sigma}^{(k)}\left(\alpha\left(x\right),\alpha\left(y\right)\right) \sim k^{n} e^{k\chi(x,y)} \sum_{r \geqq 0} \left(\frac{1}{k}\right)^{r} \tilde{b}_{r}\left(x,y\right) \tag{29}$$

such that

(i) for any compact $E \subset U \times U$ and $N \in \mathbb{N}$

$$\sup_{(x,y)\in E} \mid B_{\sigma}^{(k)}\left(\alpha\left(x\right),\alpha\left(y\right)\right) - k^{n} e^{k\chi(x,y)} \sum_{r=0}^{N-1} \left(\frac{1}{k}\right)^{r} \tilde{b}_{r}\left(x,y\right) \mid = O\left(k^{n-N}\right)$$

(ii) $\tilde{b}_{r}\left(x,y\right)$ is an almost analytic extension of $b_{r}\left(x\right)$ from the diagonal of $U \times U$, where b_{r}, $r \geqq 0$, are as defined in [32].

$\lozenge$

4.3.3 Asymptotic expansion of π_{σ}

Notation For the sake of brevity, for the rest of this subsection, whenever the point $\sigma \in \mathcal{T}$ and the tensor power k of the bundle $\mathcal{L}^{k}$ are clear, we shall simply use π to denote the projection $\pi_{\sigma}^{(k)}$.

Theorem 7 *Let $K \subset \mathbb{R}^{n}$ be a compact set, X an open neighbourhood of K and m a positive integer. If $u \in C_{0}^{2m}$, $f \in C_{0}^{3m+1}$ and $\mathrm{Im} f \geq 0$ in X, $\mathrm{Im} f = 0$, $f'\left(x_{0}\right) = 0$, $\det f''\left(x_{0}\right) \neq 0$, $f' \neq 0$ in K-$\{x_{0}\}$, then*

$$\mid \int u\left(x\right) e^{ikf(x)} dx - e^{ikf(x_{0})} \left(\det\left(kf''\left(x_{0}\right)/2\pi i\right)\right)^{-\frac{1}{2}} \sum_{j<m} k^{-j} L_{j} u \mid$$
$$\leq C k^{-m} \sum_{|\alpha| \leq 2m} \sup \mid D^{\alpha} u \mid \tag{30}$$

with $k > 0$. Here C is bounded when f stays in a bounded set in C_{0}^{3m+1} and $\mid x - x_{0} \mid / \mid f'\left(x_{0}\right) \mid$ has a uniform bound. With

$$G_{(x_{0})}\left(x\right) = f\left(x\right) - f\left(x_{0}\right) - \left\langle f''\left(x_{0}\right)\left(x - x_{0}\right), x - x_{0}\right\rangle / 2$$

which vanishes to third order at x_0 we have

$$L_j u = \sum_{\nu-\mu=j}\sum_{2\nu\geqq 3\mu} i^{-j}2^{-\nu}\left\langle f''(x_0)^{-1}D, D\right\rangle^{\nu}\left(G^{\mu}_{x_0}u\right)(x_0)/\mu!\nu!.$$

Here in keeping with the notation from [16], $G^{\mu}_{x_0}$ stands for the μ-th power of the function G_{x_0}, and D stands for the column vector whose $j-th$ entry is the operator $-i\dfrac{\partial}{\partial x_j}$, $0 \leq j \leq n$.

◊

Remark It is further proved in [16] that L_j is a distribution of order $2j$ supported at x_0

Theorem 8 (Hörmander) *Given the setting of theorems 6 and 7, there exists an asymptotic expansion of $\psi_{\pi(s)}(\alpha)(x_0)$ given by*

$$\psi_{\pi(s)}(\alpha)(x_0) \sim \sum_{r\geq 0}\left(\frac{1}{k^r}\right)D_r(F)$$

where $x_0 \in M$, U is a coordinate neighbourhood around x_0, $x, y \in U$, D_r (for $r \geq 0$) are distributions of R-th order supported at x_0 and $D_0 = \delta_{x_0}$, δ_{x_0} being the Dirac measure, $F(y) = b(x_0, y, k)\phi(y)$, $b(x,y,k) \in S^0((U\times U)\times\mathbb{R})$ is a symbol such that it has the asymptotic expansion $b \sim \sum_{r=0}^{\infty}\left(1/m^r\right)\tilde{b}_r$ and $\phi(y) = \psi_s(\alpha(y))$.

Proof: We begin by considering the expansion of the Toeplitz operator $\pi(s)$. We know that

$$\pi(\psi_s)(\alpha) = \psi_{\pi s}(\alpha) = \int_X B(\alpha,\beta)\psi_s(\beta)\widetilde{\Omega}(\beta)$$

As per our discussion in the subsection on coherent states, we need only consider the expansion of the Bergman kernel near the diagonal. Therefore using the notation of theorem 6 and the discussion immediately preceding it, we have

$$\pi(\psi_s)(\alpha)(x_0) \sim \int_U B(\alpha(x_0), \alpha(y))\psi_s(\alpha)(y)\Omega(y) \tag{31}$$

But by (27) and (28), we can write the integral above (denoted by I) as

$$I(x_0) = \int_U B(\alpha(x_0), \alpha(y))\phi(y)\Omega(y) \tag{32}$$

where $\phi(y) = \langle \alpha^{\otimes k}, s \rangle (y)$.

Now as in [19], choose a symbol $b(x, y, k) \in S^0((U \times U) \times \mathbb{R})$ such that it has the asymptotic expansion $b \sim \sum_{r=0}^{\infty} (1/m)^r \tilde{b}_r$. Therefore we have from theorem 6, the expansion of the Bergman kernel near the diagonal is asymptotically equal to $k^n e^{k\chi(x,y)} b(x, y, k)$. Therefore the intgral $I(x)$ in equation (32) asymptotically becomes

$$I(x_0) \sim \int_U k^n e^{k\chi(x_0,y)} b(x_0, y, k) \phi(y) \Omega(y) \tag{33}$$

Treating $b(x_0, y, k) \phi(y)$ as functions of the argument y we can write $b(x_0, y, k) \phi(y)$ as

$$b(x_0, y, k) \phi(y) = F(y)$$

Similary we can regard $\chi(x_0, y)$ as a function of the argument y, and write $\chi(x_0, y) = \hat{\chi}(y)$. Therefore the integral $I(x)$ in equation (33), becomes

$$\begin{aligned} I(x_0) &\sim \int_U k^n e^{k\hat{\chi}(y)} F(y) \Omega(y) \\ &= k^n \int_U e^{k\hat{\chi}(y)} F(y) \Omega(y). \end{aligned} \tag{34}$$

We now follow closely the discussion on pages 15 - 17 of [19]. We notice that we can assume, that within U, $\mathrm{Re}(\hat{\chi}) \leq 0$ (with equality holding for $y = x_0$). Further as has been pointed out, the function $y \mapsto \chi(x_0, y)$ has only one non-degenerate critical point at $y = x_0$ within U. Therefore as in [19], we can apply theorem 7 to the integral in (34). Therefore our integral admits an asymptotic expansion

$$\begin{aligned} I(x_0) &\sim k^n \sum_{r \geq 0} \left(\frac{1}{k^{n+r}} \right) D_r(F) \\ &= \sum_{r \geq 0} \left(\frac{1}{k^r} \right) D_r(F), \end{aligned} \tag{35}$$

where D_r, $r \geq 0$, are distributions of R-th order supported at x_0 and $D_0 = c_n \delta_{x_0}$; c_n being a nonzero constant and δ_{x_0} being the Dirac measure. We can further normalize D_0 to be just $D_0 = \delta_{x_0}$ (for reference see theorem 2.3 from [23] for the explicit homotopy map which makes this possible).

$\Diamond$

We next consider, in the same vein the asymptotic expansion of the term $\pi V(s)$.

Corollary 8.1 *Given the setting in theorem 8, we have*

$$\psi_{\pi(V(s))}(\alpha)(x_0) \sim \sum_{r\geq 0}\left(\frac{1}{k^r}\right) D_r(F_1)$$

where $x_0 \in M$*,* U *is a coordinate neighbourhood around* x_0*,* $x, y \in U$*,* D_r *(for* $r \geq 0$*) are distributions of r-th order supported at* x_0 *and* $D_r = \delta_{x_0}$*,* δ_{x_0} *being the Dirac measure,* $F_1(y) = b(x_0, y, k)\phi_1(y)$*,* $b(x, y, k) \in S^0((U \times U) \times \mathbb{R})$ *is a symbol such that it has the asymptotic expansion* $b \sim \sum_{r=0}^{\infty}(1/m^r)\tilde{b}_r$ *and* $\phi_1(y) = \psi_{V(s)}(\alpha)(y)$*.*

$\Diamond$

We now shift to working withing a local coordinate neighbourhood U of x_0 and a local trivialization of the bundle $\mathcal{L}^k$. Applying theorem 7, to the integral (33), and using the notation from theorem 8 and the homotopy map given in [23] we obtain the asymptotic expansion, given by [3]

$$\pi_\sigma^{(k)} s(x_0) \sim s(x_0) + \sum_{l=1}^{\infty}\frac{\tilde{D}_{l,x_0}^{(k)}(s)(x_0)}{k^l} \tag{36}$$

where

$$\tilde{D}_{l,x_0}^{(k)}(s) = \sum_{\nu-\mu=l}\sum_{2\nu\geq 3\mu} i^{-l}2^{-\nu}\left\langle\left(\frac{\hat{\chi}''(x_0)}{i}\right)^{-1} D, D\right\rangle^{\nu}\frac{G_{(x_0)}^{\mu}bs}{\mu!\nu!}(x_0)$$

and

$$G_{(x_0)}(y) = \hat{\chi}(y) - \langle\hat{\chi}''(x_0)(y - x_0), y - x_0\rangle/2$$

Notice that within the coordinate neighbourhood U, $\tilde{D}_{l,x_0}$ is smooth in x_0 (χ as defined in theorem 6 is smooth on $U \times U$ and the Hessian $\hat{\chi}''(x_0)$ at each $x_0 \in U$, is non vanishing as is proved in [19] and in [32]). Therefore comparing equations (35) and (36) we have that within the coordinate neighbourhood U, there are operators $D_{l,U}^{(k)} \in D(U, \mathcal{L}^k)$ such that

$$\tilde{D}_{l,x_0}^{(k)}(s) = D_{l,U}^{(k)}(s)(x_0).$$

[3] with the asymptotics in equation (36) is in C^m in $x_0 \in M$ with respect to the norm on $\mathcal{L}^k$ over M

If we take another coordinate neighbourhood U' of x_0 and repeat the process above, we get an operator $D^{(k)}_{l,U'}$ satisfying the properties above in the neighbourhood U'. Therefore on the intersection $U' \cap U$, $D^{(k)}_{l,U} = D^{(k)}_{l,U'}$. Therefore we can define a global operators

$$D^{(k)}_l : C^\infty \left(M, \mathcal{L}^k\right) \to C^\infty \left(M, \mathcal{L}^k\right)$$

such that for every $x_0 \in M$ we have $(D_l s)(x_0) = D_{l,x_0}(s)$ (note that the section s can easily be considered to be a global section by the use of a bump function). Therefore we now have an asymptotic expansion of $\pi^k s$ with the asymptotics in C^m in $x_0 \in M$ with respect to the norm on $\mathcal{L}^k$ over M. We now differentiate (36) along a vector field V on $\mathcal{T}$ to get the following asymtotic expansion

$$V\left(\pi^{(k)}_\sigma\right) s \sim \sum_{l=1}^{\infty} \frac{V\left[D^{(k)}_l\right](s)}{k^l} \tag{37}$$

Therefore we have the following theorems.

Theorem 9 *There exist global operators* $D^{(k)}_l : C^\infty \left(M, \mathcal{L}^k\right) \to C^\infty \left(M, \mathcal{L}^k\right)$ *such that we have for all section s of* $\mathcal{L}^k$*, we have an asymptotic expansion*

$$\pi^{(k)}_\sigma s \sim s + \sum_{l=1}^{\infty} \frac{D^{(k)}_l(s)}{k^l}, \textit{i.e.},$$

$$\mid \pi^{(k)}_\sigma s - s - \sum_{l=1}^{N-1} \frac{D^{(k)}_l(s)}{k^l} \mid = O\left(\frac{1}{k^N}\right)$$

the norm being the C^m *norm with respect to the norm on sections of* $\mathcal{L}^k$ *over* M*.*

◊

Theorem 10 *For a vector field* V *on* $\mathcal{T}$*,* $V\left(\pi^{(k)}_\sigma\right) s$ *has the asymptotic expansion*

$$V\left(\pi^{(k)}_\sigma\right) s \sim \sum_{l=1}^{\infty} \frac{V\left[D^{(k)}_l\right](s)}{k^l}, \textit{i.e.},$$

$$\mid V\left(\pi^{(k)}_\sigma\right) s - \sum_{l=1}^{N-1} \frac{V\left[D^{(k)}_l\right](s)}{k^l} \mid = O\left(\frac{1}{k^N}\right)$$

the norm being the C^m *norm with respect to the norm on sections of* $\mathcal{L}^k$ *over* M*.*

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Remark Note that, in view of corllary 8.1 the dependence of $D_l^{(k)}$ on k is a mild one, in the sense that $D_l^{(k)}$ acts as the restriction of an operator $\tilde{D}_l : C^\infty(\mathcal{L}^*) \to C^\infty(\mathcal{L}^*)$. Therefore we shall suppress the k dependence of $D_l^{(k)}$ and instead use the notation D_l for the rest of the thesis.

Remark Suppose we have two operators

$$A_1^{(k)} : C^\infty\left(M, \mathcal{L}^k\right) \to C^\infty\left(M, \mathcal{L}^k\right)$$

and

$$A_2^{(k)} : C^\infty\left(M, \mathcal{L}^k\right) \to C^\infty\left(M, \mathcal{L}^k\right)$$

where each $A_i^{(k)}$ for $i = 1, 2$ is given by

$$A_i^{(k)} s(x) = \int_{y \in M} K_i^{(k)}(x, y)\, s(y) \frac{\omega^n(y)}{n!}$$

where $K_i^{(k)}(x, y)$ is the integral kernel for $A_i^{(k)}$. Now suppose we have the kernel expansion for K_i, like we do for the Bergman kernel (as given in theorem 6) we can evaluate the composition $A_1^{(k)}$ and $A_2^{(k)}$ as follows

$$\begin{aligned} A_2 \circ A_1 s(x) &= \int\limits_{y \in M} K_2^{(k)}(x, y)\, A_1 s(y) \tfrac{\omega^n(y)}{n!} \\ &= \int\limits_{(y,z) \in M \times M} K_2^{(k)}(x, y)\, K_2^{(k)}(y, z) \tfrac{\omega^n(y)}{n!} \tfrac{\omega^n(z)}{n!} \end{aligned}$$

By considering the asymptotic expansion of the integral in the variable y and then considering the asymptotic expansion in the variable z, it is easily seen that the composition of the two expansions in terms of the distributions (as given in theorem 7) gives the expansion of the compostion of the operators in terms of the distributions. We will use this in the calculations of theorem 15.

5 New ansatz for the Hitchin connection

For the rest of this chapter, we shall assume that (M, ω) is a compact sympectic manifold.

Recall that a *prequantum line bundle* over M is the triple $(\mathcal{L}, (\cdot,\cdot), \nabla)$, where $\mathcal{L}$ is a complex line bundle with a Hermitian structure $(\cdot,\cdot)$ and a compatible connection ∇ whose curvature is given by $F_\nabla = \frac{i}{2\pi}\omega$.

We say that a symplectic manifold M is *prequantizable* if there exists a prequantum line bundle over it.

As in chapter 3, we parametrize the Kähler structures on the manifold M with the smooth manifold $\mathcal{T}$, i.e., we have a smooth map $I : \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$, such that $(M,\ \omega,\ I_\sigma)$ is a Kähler manifold for every $\sigma \in \mathcal{T}$. In fact, we assume further that $\mathcal{T}$ is a complex manifold and I is holomorphic. Let V be a vector field $\mathcal{T}$ and V' be the $(1,0)$ part of the vector field. For every $\sigma \in \mathcal{T}$ we consider the finite dimensional subspace of $C^\infty\left(M, \mathcal{L}^k\right)$ given by

$$H_\sigma^{(k)} = H^0\left(M_\sigma, \mathcal{L}^k\right) = \left\{s \in C^\infty\left(M, \mathcal{L}^k\right) \mid \nabla_\sigma^{0,1} s = 0\right\}$$

We make the assumption these subspaces of holomorphic sections form a smooth finite rank subbundle $H^{(k)}$ of the trivial bundle $\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty\left(M, \mathcal{L}^k\right)$.

We denote by $\hat{\nabla}^t$ the trivial connection in the trivial bundle $\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty\left(M, \mathcal{L}^k\right)$. Let $\mathcal{D}\left(M, \mathcal{L}^k\right)$ denote the vector space of differential operators on $C^\infty\left(M, \mathcal{L}^k\right)$. For any smooth 1-form u on $\mathcal{T}$ with values in $\mathcal{D}\left(M, \mathcal{L}^k\right)$ we have a connection $\hat{\nabla}$ on $\mathcal{H}^{(k)}$ given by

$$\hat{\nabla}_V = \hat{\nabla}_V^t - u(V)$$

for any vector field V on $\mathcal{T}$.

We restate for further reference, without proof, the lemma 4.

Lemma (Andersen) *The connection $\hat{\nabla}$ in $\mathcal{H}^{(k)}$ induces a connection in $H^{(k)}$ iff*

$$\frac{i}{2} V'[I]\, \nabla^{1,0} s + \nabla^{0,1} u(V)\, s = 0$$

for all vector fields V on $\mathcal{T}$ and all smooth sections s of $H^{(k)}$.

$\Diamond$

We further restate, for future reference, the 1-form $u(V)$ that satisfies (19), as obtained in [1] and theorem 3 of this thesis.

Proposition 6 (Andersen) *The smooth 1-form u given by*

$$u(V) = \frac{1}{2k+n}\left\{\frac{1}{2}\Delta_{G(V)} - \nabla_{G(V)dF} + 2kV'[F]\right\}$$

where F is the Ricci potential and $\triangle_G$ is an operator corresponding to the symmetric 2-tensor G (defined in section 2.4), given by $Tr\left(\nabla^{1,0}G\nabla^{1,0}(s)\right)$, solves (19) *under the assumption of rigidity.*

$\Diamond$

Next we notice that $-\frac{i}{2}V'[I]\nabla^{1,0}s \in Ker\nabla^{0,1}$. Further we know that

$$\frac{Ker\nabla^{0,1}}{Im\nabla^{0,1}} = H^{0,1}\left(M, \mathcal{L}^k\right).$$

We now state the Kodaira-Serre vanishing theorem that we shall use to establish that $-\frac{i}{2}V'[I]\nabla^{1,0}s \in Im\nabla^{0,1}$.

Theorem 11 *(Kodaira-Serre Vanishing Thoerem) If is a positive line bundle on M, then there exists a k_0 such that for any $k \geq k_0$,*

$$H^{0,q}\left(M,\ \mathcal{L}^k\right) = 0$$

for any $q > 0$.

$\Diamond$

Since $\mathcal{L}$ is a positive line bundle on M we can apply the Kodaira-Serre vanishing theorem to $\mathcal{L}^k$ for a sufficiently large k, to conclude that $H^{0,1}\left(M, \mathcal{L}^k\right) = 0$ for such a k. This in turn allows us to conclude that $-\frac{i}{2}V'[I]\nabla^{1,0}s \in Im\nabla^{0,1}$, under the assumption that k is sufficiently large.

We now consider the complex

$$\Omega^0\left(M,\mathcal{L}^k\right) \xrightarrow{\nabla^{0,1}} \Omega^{0,1}\left(M,\mathcal{L}^k\right) \xrightarrow{\nabla^{0,1}} \Omega^{0,2}\left(M,\mathcal{L}^k\right) \xrightarrow{\nabla^{0,1}} \dots$$

where $\mathcal{L}^k = \mathcal{L}^{\otimes k}$. Define the operator $\Delta = \nabla^{0,1}\left(\nabla^{0,1}\right)^* + \left(\nabla^{0,1}\right)^*\nabla^{0,1}$. We define a partial inverse P to Δ,

$$P : \Omega^{0,1}\left(M,\mathcal{L}^k\right) \to \Omega^{0,1}\left(M,\mathcal{L}^k\right)$$

such that $\Delta P = Id - \pi_\sigma^{(k)}$ (where $\pi_\sigma^{(k)}$ is the orthogonal projection from $C^\infty\left(M,\mathcal{L}^k\right)$ to $H^0\left(M,\mathcal{L}^k\right)$). P is therefore a parametrix to Δ.

Proposition 7 *The smooth 1-form u_N on $\mathcal{T}$ with values in $\mathcal{D}\left(M,\mathcal{L}^k\right)$ given by*

$$u_N\left(V\right)s = -\frac{i}{2}\left(\nabla^{0,1}\right)^* PV'\left[I\right]\nabla^{1,0}s \tag{38}$$

satifies (19).

Proof: Consider the effect of applying the operator $\nabla^{0,1}$ to the given 1-form u_N

$$\nabla^{0,1}u_N\left(V\right)s = \nabla^{0,1}\left(\nabla^{0,1}\right)^* P\left(-\frac{i}{2}V'\left[I\right]\nabla^{1,0}s\right)$$

From the Kodaira-Serre Vanishing theorem (theorem 1) we know that $-\frac{i}{2}V'\left[I\right]\nabla^{1,0}s \in Im\,\nabla^{0,1}$ and therefore we have that,

$$\begin{aligned}\nabla^{0,1}u_N\left(V\right)s &= \left(Id - \pi_\sigma^{(k)}\right)\left(-\frac{i}{2}V'\left[I\right]\nabla^{1,0}s\right)\\ &= -\frac{i}{2}V'\left[I\right]\nabla^{1,0}s\end{aligned}$$

Thus proving the proposition.

$\Diamond$

We now turn our attention to exploring the relationship between the Hitchin connection (as established in [1]), the L_2 connection (defined by the property

$\nabla_V^{L_2} s = \pi_\sigma^{(k)} V(s))$ and the connection ∇^N we have as a result of the the ansatz u_N satisfying (19), defined by $\nabla_V^N = \hat{\nabla}_V^t - u_N(V)$ (see equation (18)).

Proposition 8 $\nabla^H - \nabla^{L_2} = -\pi_\sigma^{(k)} u$, *with the notations as established before.*

Proof: Recall that by lemma 4, proposition 3 and (18), we have that

$$\nabla_V^H = \hat{\nabla}_V^t - u(V) \tag{39}$$

But notice also that since the Hitchin connection is a connection in $H^{(k)}$, we have that

$$\nabla_V^H s = \pi_\sigma^{(k)} \nabla_V^H s \tag{40}$$

Combining the equations (39) and (40), we establish that

$$\begin{aligned} \nabla_V^H s &= \pi_\sigma^{(k)} \nabla_V^H s \\ &= \pi_\sigma^{(k)} (V - u(V)) s \\ &= \pi_\sigma^{(k)} V[s] - \pi_\sigma^{(k)} u(V) s \end{aligned}$$

Finally notice that $\pi_\sigma^{(k)} V[s]$ is nothing but $\nabla_V^{L_2} s$, thus proving the proposition.

$\diamond$

Proposition 9 *Given the notations established before, we have that* $\nabla^N = \nabla^{L_2}$.

Proof: Since ∇^N is a connection in $H_\sigma^{(k)}$, we have for a section $s \in H_\sigma^{(k)}$, the identity

$$\pi_\sigma^{(k)} s = s \tag{41}$$

Differentiating the above with respect to a vector field V on $\mathcal{T}$, we have that

$$V\left[\pi_\sigma^{(k)}\right] s + \pi_\sigma^{(k)} V[s] = V[s]$$

But this in turn implies

$$V\left[\pi_\sigma^{(k)}\right] s = \left(Id - \pi_\sigma^{(k)}\right) V[s]. \tag{42}$$

We further notice that

$$V[s] = \pi_\sigma^{(k)} V[s] + \left(Id - \pi_\sigma^{(k)}\right) V[s] \tag{43}$$

Notice further that $\left(Id - \pi_\sigma^{(k)}\right) V[s] \in Im\left(\nabla^{0,1}\right)^*$ and that $\pi_\sigma^{(k)} u_N(V) = 0$.

Now recall that $\nabla_V^N s = V[s] - u_N(V)s$. But given equations (42) and (43), we have that

$$\nabla_V^N s = \nabla_V^{L_2} s + \left(V\left[\pi_\sigma^{(k)}\right] - u_N(V)\right)s \tag{44}$$

Now consider the equation $\left(\pi_\sigma^{(k)}\right)^2 = \pi_\sigma^{(k)}$. Differentiating with respect to a vector field V on $\mathcal{T}$, we have that

$$V\left[\pi_\sigma^{(k)}\right] = V\left[\pi_\sigma^{(k)}\right]\pi_\sigma^{(k)} + \pi_\sigma^{(k)} V\left[\pi_\sigma^{(k)}\right] \tag{45}$$

But since $\left(\pi_\sigma^{(k)}\right)^2 = \pi_\sigma^{(k)}$, the equation (45) yields that

$$\pi_\sigma^{(k)} V\left[\pi_\sigma^{(k)}\right] = \pi_\sigma^{(k)} V\left[\pi_\sigma^{(k)}\right]\pi_\sigma^{(k)} + \pi_\sigma^{(k)} V\left[\pi_\sigma^{(k)}\right] \tag{46}$$

And as a result we have $\pi_\sigma^{(k)} V\left[\pi_\sigma^{(k)}\right]\pi_\sigma^{(k)} = 0$. Therefore since $s \in H_\sigma^{(k)}$, we have that $\pi_\sigma^{(k)}\left(\left(V\left[\pi_\sigma^{(k)}\right] - u_N(V)\right)s\right) = 0$; and ∇^N and ∇^{L_2} are connections in $H_\sigma^{(k)}$, this necessarily implies that $\left(V\left[\pi_\sigma^{(k)}\right] - u_N(V)\right)s = 0$, hence proving the assertion of the proposition.

◊

We now temporarily turn to certain cases when the Hitchin connection is known to exist, i.e. the case when the manifold M admits a rigid family of Kähler structures. Recall firstly the result from proposition 8 which states that

$$\nabla^H = \nabla^{L_2} - \pi_\sigma^{(k)} u$$

Let g_k be a 1-form with values in $\mathcal{D}\left(M, \mathcal{L}^k\right)$, satisfying the relation $g_k = -u$. Further let g_k admit the asymptotic expansion

$$g_k = \sum_{l=0}^{\infty} g^{(l)} k^{-l}$$

where $g^{(l)}$ are 1-forms with values in $\mathcal{D}\left(M, \mathcal{L}^k\right)$.

Recall that $u(V)$ is given by

$$u(V) = \frac{1}{2k+n}\left\{\frac{1}{2}\Delta_{G(V)} - \nabla_{G(V)dF} + 2kV'[F]\right\}$$

Thus we have that

$$u(V) = \frac{1}{2k+n}\left\{\frac{1}{2}\Delta_{G(V)} - \nabla_{G(V)dF} - nV'[F]\right\} + V'[F]$$

Thus $g^{(0)}(V)$ is nothing but $-V'[F]$. Now let

$$o(v) = \frac{1}{2k+n}\left\{\frac{1}{2}\Delta_{G(V)} - \nabla_{G(V)dF} - nV'[F]\right\}.$$

Therefore we have

$$\begin{aligned} o(v) &= \frac{1}{2k+n}\left\{\frac{1}{2}\Delta_{G(V)} - \nabla_{G(V)dF} - nV'[F]\right\} \\ &= \frac{1}{2k}\left(1+\frac{n}{2k}\right)^{-1}\left\{\frac{1}{2}\Delta_{G(V)} - \nabla_{G(V)dF} - nV'[F]\right\} \end{aligned}$$

Expanding the power series we have that the first order term $\hat{u}^{(1)}$ is nothing but

$$\hat{u}^{(1)}(V) = \frac{1}{2}\left\{\frac{1}{2}\Delta_{G(V)} - \nabla_{G(V)dF} - nV'[F]\right\}$$

Now applying $\pi_\sigma^{(k)}$ to $\hat{u}^{(1)}$, we have that

$$\pi_\sigma^{(k)}\hat{u}^{(1)}(V) = \frac{1}{2}\left\{\frac{1}{2}\pi_\sigma^{(k)}\Delta_{G(V)} - \pi_\sigma^{(k)}\nabla_{G(V)dF} - \pi_\sigma^{(k)}nV'[F]\right\} \tag{47}$$

We now state two theorems from [2] that help us to simply the above equation.

Theorem 12 *(Andersen and Gammelgaard) If $X \in C^\infty\left(M, T^{(1,0)}M\right)$ is a smooth section of the holomorphic tangent bundle on M, then we have*

$$\pi_\sigma^{(k)}\nabla_X s = -\pi_\sigma^{(k)}(\delta X)\, s$$

for any smooth section $s \in C^\infty\left(M, \mathcal{L}^k\right)$.

Theorem 13 *(Andersen and Gammelgaard) If $B \in C^\infty\left(M, S^2\left(T^{(1,0)}M\right)\right)$ is a symmetric bivector field, then the operator Δ_B satisfies*

$$\pi_\sigma^{(k)}\Delta_B = 0$$

for all smooth sections $s \in C^\infty\left(M, \mathcal{L}^k\right)$.

Notice firstly that in equation (47), $G(V)\,dF \in C^{\infty}\left(M, T^{(1,0)}M\right)$. Therefore for the above two theorems, we have that

$$\pi_{\sigma}^{(k)}\hat{u}^{(1)}(V) = \frac{1}{2}\left\{\pi_{\sigma}^{(k)}\delta\left(G(V)\,dF\right) - \pi_{\sigma}^{(k)} nV'[F]\right\} \tag{48}$$

Therefore we have that $g^{(1)}(V)$ is given by

$$g^{(1)}(V) = \frac{1}{2}\{-\delta\left(G(V)\,dF\right) + nV'[F]\}$$

Indeed we have for all $g^{(l)}$, since the $l-th$ order term, $\hat{u}^{(l)}$ is given by the equation

$$\hat{u}^{(l)}(V) = \frac{(-1)^{l-1} n^{l-1}}{2^{l}}\left\{\frac{1}{2}\Delta_{G(V)} - \nabla_{G(V)dF} - nV'[F]\right\}$$

and proceeding as above, we have

$$g^{(l)}(V) = \frac{(-1)^{l-1} n^{l-1}}{2^{l}}\{-\delta\left(G(V)\,dF\right) + nV'[F]\}$$

For the next sections we shall be dealing with connections of the form

$$\nabla^{M} = \nabla^{L_2} + \pi_{\sigma}^{(k)} g_k,$$

where g_k is a $\mathcal{D}\left(M, \mathcal{L}^{k}\right)$-valued 1 form. Further we denote the curvature of the connection ∇^{M} as $\mathcal{F}_{\nabla^{M}}$.[4] We do not however assume that the manifold M admits a rigid family of Kähler structures.

5.1 Calculation of curvature of connections of the form ∇^{M}

We note firstly that by the definition of the curvature of a connection, we can express the curvature of the connection ∇^{M}, denoted $\mathcal{F}_{\nabla^{M}}$, as

$$\mathcal{F}_{\nabla^{M}}(X,Y)\,s = \nabla_X^{M}\nabla_Y^{M}s - \nabla_Y^{M}\nabla_X^{M}s - \nabla_{[X,Y]}^{M}s.$$

[4]We note from proposition 8 that this covers the case of the Hitchin connection, ∇^{H} in the case when it exists.

Remark We know that in any local coordinate system U around a point $\sigma \in \mathcal{T}$ (with coordinates given by $(x_1, \ldots, x_m)$ where m is the dimension of $\mathcal{T}$), we have that

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$$

for all $1 \leq i,\ j \leq m$. We can always choose our vector fields X and Y to be such that in a local coordinate patch around the point σ,

$$X = \frac{\partial}{\partial x_i} \quad \text{and} \quad Y = \frac{\partial}{\partial x_j},$$

for $i \neq j$ (if we choose $i = j$, the Lie bracket is trivially 0). Then we have, within the coordinate neighbourhood U, $[X, Y] = 0$. As a result we have

$$\mathcal{F}_{\nabla^M}(X, Y)\, s = \nabla_X^M \nabla_Y^M s - \nabla_Y^M \nabla_X^M s. \tag{49}$$

We begin by calculating the curavture of the L_2 connection.

Proposition 10 *For vector field X, Y on $\mathcal{T}$, such that $[X, Y] = 0$, the curvature of the L_2 connection, ∇^{L_2}, given by $\mathcal{F}_{\nabla^{L_2}}$ is be given by*

$$\mathcal{F}_{\nabla^{L_2}}(X, Y)\, s = \pi_\sigma^{(k)} \left[X\left(\pi_\sigma^{(k)}\right), Y\left(\pi_\sigma^{(k)}\right)\right] s.$$

Proof: Notice firstly that given vector fields X, Y on $\mathcal{T}$, such that $[X, Y] = 0$, we have that the curvature of the L_2 connection, given by $\mathcal{F}_{\nabla^{L_2}}$ can be given by

$$\begin{aligned} \mathcal{F}_{\nabla^{L_2}}(X, Y)\, s &= \pi_\sigma^{(k)} X\left(\pi_\sigma^{(k)} Y(s)\right) - \pi_\sigma^{(k)} Y\left(\pi_\sigma^{(k)} X(s)\right) \\ &= \pi_\sigma^{(k)} X\left(\pi_\sigma^{(k)}\right) Y(s) - \pi_\sigma^{(k)} Y\left(\pi_\sigma^{(k)}\right) X(s) \end{aligned} \tag{50}$$

Now note that for our holomorphic section s, we have that $\pi_\sigma^{(k)} s = s$. Therefore differentiating we have

$$X\left(\pi_\sigma^{(k)}\right) s + \pi_\sigma^{(k)} X(s) = X(s) \tag{51}$$

which implies

$$X\left(\pi_\sigma^{(k)}\right) s = \left(-\pi_\sigma^{(k)} + Id\right) X(s) \tag{52}$$

Further recall that $\left(\pi_\sigma^{(k)}\right)^2 = \pi_\sigma^{(k)}$. Differentiating, we have

$$X\left(\pi_\sigma^{(k)}\right)\pi_\sigma^{(k)} + \pi_\sigma^{(k)} X\left(\pi_\sigma^{(k)}\right) = X\left(\pi_\sigma^{(k)}\right) \tag{53}$$

But (53), implies that $\pi_\sigma^{(k)} X\left(\pi_\sigma^{(k)}\right)\pi_\sigma^{(k)} = 0$.

So from equations (52) and (53) and the fact that $\pi_\sigma^{(k)} s = s$, we have that

$$\begin{aligned}\pi_\sigma^{(k)} X\left(\pi_\sigma^{(k)}\right) Y\left(\pi_\sigma^{(k)}\right) s &= \pi_\sigma^{(k)} X\left(\pi_\sigma^{(k)}\right)\left(Id - \pi_\sigma^{(k)}\right) Y(s) \\ &= \pi_\sigma^{(k)} X\left(\pi_\sigma^{(k)}\right) Y(s)\end{aligned} \tag{54}$$

Therefore we now have,

$$\mathcal{F}_{\nabla^{L_2}}(X,Y)\,s = \pi_\sigma^{(k)}\left[X\left(\pi_\sigma^{(k)}\right), Y\left(\pi_\sigma^{(k)}\right)\right] s. \tag{55}$$

$\Diamond$

Theorem 14 *For the connection ∇^M given by the equation $\nabla^M = \nabla^{L_2} + \pi_{(0)} g_k$, the curvature $\mathcal{F}_{\nabla^M}$, is given by*

$$\mathcal{F}_{\nabla^M}(X,Y)\,s = \mathcal{F}_{\nabla^{L_2}}(X,Y)\,s + \mathcal{C}(X,Y)\,s, \tag{56}$$

where $\mathcal{F}_{\nabla^{L_2}}(X,Y)\,s = \pi_\sigma^{(k)}\left[X\left(\pi_\sigma^{(k)}\right), Y\left(\pi_\sigma^{(k)}\right)\right] s$ as proven in proposition 10, $\mathcal{C}(X,Y)\,s$ (henceforth called the correction term) is given by

$$\begin{aligned}\mathcal{C}(X,Y)\,s = {} & \pi_\sigma^{(k)} d\left(\pi_\sigma^{(k)} g_k\right)(X,Y)\,s \\ & + \pi_\sigma^{(k)} g_k(Y)\, X\left(\pi_\sigma^{(k)}\right) s - \pi_\sigma^{(k)} g_k(X)\, Y\left(\pi_\sigma^{(k)}\right) s \\ & + \left[\pi_\sigma^{(k)} g_k(X), \pi_\sigma^{(k)} g_k(Y)\right] s\end{aligned}$$

for all X, Y vector fields on $\mathcal{T}$ and smooth sections s of $H^{(k)}$.

Proof: We begin the proof by considering the term $\nabla_X^M \nabla_Y^M s$ from (49).

$$\begin{aligned}\nabla_X^M \nabla_Y^M s &= \left(\nabla_X^{L_2} + \pi_\sigma^{(k)} g_k(X)\right)\left(\nabla_Y^{L_2} + \pi_\sigma^{(k)} g_k(Y)\right) s \\ &= \nabla_X^{L_2}\nabla_Y^{L_2} s + \nabla_X^{L_2}\pi_\sigma^{(k)} g_k(Y)\,s \\ &\quad + \pi_\sigma^{(k)} g_k(X)\,\nabla_Y^{L_2} s + \pi_\sigma^{(k)} g_k(X)\,\pi_\sigma^{(k)} g_k(Y)\,s\end{aligned}$$

As a result of equation (49) and the calculations above, we can write the curvature $\mathcal{F}_{\nabla^M}(X,Y)\,s$ as

$$\begin{aligned}\mathcal{F}_{\nabla^M}(X,Y)\,s &= \nabla_X^{L_2}\nabla_Y^{L_2}s + \nabla_X^{L_2}\pi_\sigma^{(k)}g_k(Y)\,s + \pi_\sigma^{(k)}g_k(X)\,\nabla_Y^{L_2}s\\ &+\pi_\sigma^{(k)}g_k(X)\,\pi_\sigma^{(k)}g_k(Y)\,s - \nabla_Y^{L_2}\nabla_X^{L_2}s - \nabla_Y^{L_2}\pi_\sigma^{(k)}g_k(X)\,s\\ &-\pi_\sigma^{(k)}g_k(Y)\,\nabla_X^{L_2}s - \pi_\sigma^{(k)}g_k(Y)\,\pi_\sigma^{(k)}g_k(X)\,s\end{aligned}$$

But given our assumption that $[X,Y]=0$, we have that,

$$\mathcal{F}_{\nabla^{L_2}}(X,Y)\,s = \nabla_X^{L_2}\nabla_Y^{L_2}s - \nabla_Y^{L_2}\nabla_X^{L_2}s.$$

Therefore we can write $\mathcal{F}_{\nabla^M}(X,Y)\,s$ as

$$\begin{aligned}\mathcal{F}_{\nabla^M}(X,Y)\,s &= \mathcal{F}_{\nabla^{L_2}}(X,Y)\,s + \nabla_X^{L_2}\pi_\sigma^{(k)}g_k(Y)\,s + \pi_\sigma^{(k)}g_k(X)\,\nabla_Y^{L_2}s\\ &+\pi_\sigma^{(k)}g_k(X)\,\pi_\sigma^{(k)}g_k(Y)\,s - \nabla_Y^{L_2}\pi_\sigma^{(k)}g_k(X)\,s\\ &-\pi_\sigma^{(k)}g_k(Y)\,\nabla_X^{L_2}s - \pi_\sigma^{(k)}g_k(Y)\,\pi_\sigma^{(k)}g_k(X)\,s\end{aligned} \tag{57}$$

Now we define $\mathcal{C}(X,Y)\,s$ to be

$$\begin{aligned}\mathcal{C}(X,Y)\,s &= \nabla_X^{L_2}\pi_\sigma^{(k)}g_k(Y)\,s + \pi_\sigma^{(k)}g_k(X)\,\nabla_Y^{L_2}s + \pi_\sigma^{(k)}g_k(X)\,\pi_\sigma^{(k)}g_k(Y)\,s\\ &-\nabla_Y^{L_2}\pi_\sigma^{(k)}g_k(X)\,s - \pi_\sigma^{(k)}g_k(Y)\,\nabla_X^{L_2}s - \pi_\sigma^{(k)}g_k(Y)\,\pi_\sigma^{(k)}g_k(X)\,s\end{aligned} \tag{58}$$

Because of equations (57) and (58), we can write

$$\mathcal{F}_{\nabla^M}(X,Y)\,s = \mathcal{F}_{\nabla^{L_2}}(X,Y)\,s + \mathcal{C}(X,Y)\,s$$

Now recall that $\nabla_X^{L_2}s = \pi_\sigma^{(k)}X\,[s]$. Therefore we have,

$$\pi_\sigma^{(k)}g_k(X)\,\nabla_Y^{L_2}s = \pi_\sigma^{(k)}g_k(X)\,\pi_\sigma^{(k)}Y\,[s] \tag{59}$$

$$\pi_\sigma^{(k)}g_k(Y)\,\nabla_X^{L_2}s = \pi_\sigma^{(k)}g_k(Y)\,\pi_\sigma^{(k)}X\,[s]\,. \tag{60}$$

Next consider the terms of the form $\nabla_X^{L_2}\pi_\sigma^{(k)}g_k(Y)\,s$ appearing in (58). Evaluating $\nabla_X^{L_2}\pi_\sigma^{(k)}g_k(Y)\,s$, we have

$$\begin{aligned}\nabla_X^{L_2}\pi_\sigma^{(k)}g_k(Y)\,s &= \pi_\sigma^{(k)}X\left[\pi_\sigma^{(k)}g_k(Y)\,s\right]\\ &= \pi_\sigma^{(k)}X\left[\pi_\sigma^{(k)}\right]g_k(Y)\,s + \pi_\sigma^{(k)}X\,(g_k(Y)\,s)\end{aligned} \tag{61}$$

From equations (58), (59), (60) and (61), we have

$$\begin{aligned}\mathcal{C}(X,Y)\,s &= \pi_\sigma^{(k)}X\left[\pi_\sigma^{(k)}\right]g_k(Y)\,s + \pi_\sigma^{(k)}X\,(g_k(Y)\,s) + \pi_\sigma^{(k)}g_k(X)\,\pi_\sigma^{(k)}Y\,[s]\\ &- \pi_\sigma^{(k)}Y\left[\pi_\sigma^{(k)}\right]g_k(X)\,s - \pi_\sigma^{(k)}Y\,(g_k(X)\,s) - \pi_\sigma^{(k)}g_k(Y)\,\pi_\sigma^{(k)}X\,[s]\\ &+ \pi_\sigma^{(k)}g_k(X)\,\pi_\sigma^{(k)}g_k(Y)\,s - \pi_\sigma^{(k)}g_k(Y)\,\pi_\sigma^{(k)}g_k(X)\,s\end{aligned}$$

But notice that

$$\pi_\sigma^{(k)} X \left(g_k (Y) s\right) - \pi_\sigma^{(k)} g_k (Y) \pi_\sigma^{(k)} X [s] = \pi_\sigma^{(k)} X \left(g_k (Y)\right) s + \pi_\sigma^{(k)} g_k (Y) \left(Id - \pi_\sigma^{(k)}\right) X [s]$$

But we know that $\left(Id - \pi_\sigma^{(k)}\right) X [s] = X \left[\pi_\sigma^{(k)}\right] s$. Therefore we have that

$$\pi_\sigma^{(k)} X \left(g_k (Y) s\right) - \pi_\sigma^{(k)} g_k (Y) \pi_\sigma^{(k)} X [s] = \pi_\sigma^{(k)} X \left(g_k (Y)\right) s + \pi_\sigma^{(k)} g_k (Y) X \left[\pi_\sigma^{(k)}\right] s$$

Thus we have that

$$\begin{aligned} \mathcal{C} (X,Y) s = {} & \pi_\sigma^{(k)} X \left[\pi_\sigma^{(k)}\right] g_k (Y) s + \pi_\sigma^{(k)} X \left(g_k (Y)\right) s + \pi_\sigma^{(k)} g_k (Y) X \left(\pi_\sigma^{(k)}\right) s \\ & - \pi_\sigma^{(k)} Y \left[\pi_\sigma^{(k)}\right] g_k (X) s - \pi_\sigma^{(k)} Y \left(g_k (X)\right) s - \pi_\sigma^{(k)} g_k (X) Y \left(\pi_\sigma^{(k)}\right) s \\ & + \pi_\sigma^{(k)} g_k (X) \pi_\sigma^{(k)} g_k (Y) s - \pi_\sigma^{(k)} g_k (Y) \pi_\sigma^{(k)} g_k (X) s \end{aligned}$$

Note further that $\pi_\sigma^{(k)} X \left[\pi_\sigma^{(k)}\right] g_k (Y) s + \pi_\sigma^{(k)} X \left(g_k (Y)\right) s = \pi_\sigma^{(k)} \left(X \left[\pi_\sigma^{(k)} g_k (Y)\right] s\right)$ (since $\left(\pi_\sigma^{(k)}\right)^2 = \pi_\sigma^{(k)}$). Therefore we have that

$$\begin{aligned} \mathcal{C} (X,Y) s = {} & \pi_\sigma^{(k)} d \left(\pi_\sigma^{(k)} g_k\right) (X,Y) s \\ & + \pi_\sigma^{(k)} g_k (Y) X \left(\pi_\sigma^{(k)}\right) s - \pi_\sigma^{(k)} g_k (X) Y \left(\pi_\sigma^{(k)}\right) s \\ & + \left[\pi_\sigma^{(k)} g_k (X), \pi_\sigma^{(k)} g_k (Y)\right] s \end{aligned}$$

thus completing the proof of the theorem.

$\diamondsuit$

5.2 Asymptotic expression for the curvature of the connection ∇^M

We now return to the question of giving an expression for the curvature $\mathcal{F}_{\nabla^M}$ of the connection ∇^M. Recall that,

$$\begin{aligned} \mathcal{F}_{\nabla^M} (X,Y) s = {} & \pi_\sigma^{(k)} \left[X \left[\pi_\sigma^{(k)}\right], Y \left[\pi_\sigma^{(k)}\right]\right] s + \pi_\sigma^{(k)} d \left(\pi_\sigma^{(k)} g_k\right) (X,Y) s \\ & + \pi_\sigma^{(k)} g_k (Y) X \left(\pi_\sigma^{(k)}\right) s - \pi_\sigma^{(k)} g_k (X) Y \left(\pi_\sigma^{(k)}\right) s \\ & + \left[\pi_\sigma^{(k)} g_k (X), \pi_\sigma^{(k)} g_k (Y)\right] s \end{aligned}$$

We next write the 1-form g_k in the above equation as

$$g_k = \sum_{l=0}^{\infty} g^{(l)} k^{-l}$$

where $g^{(l)}$ are 1-forms with values in $\mathcal{D}\left(M, \mathcal{L}^k\right)$.

Theorem 15 *Keeping with the notation developed thus far, the asymptotic expansion for the curvature $\mathcal{F}_{\nabla^M}$ is given by the expression*

$$\mathcal{F}_{\nabla^M} = \sum_{n=0}^{\infty} \frac{T_n}{k^n}$$

where

$$T_0 = dg^{(0)} + g^{(0)} \wedge g^{(0)}$$

$$T_1 = dg^{(1)} + D_1 dg^{(0)} + A_1 + D_1 A_0 + B_1$$

with A_1 and A_0 being given by

$$A_1 = g^{(0)} \wedge g^{(1)} + g^{(1)} \wedge g^{(0)} + dD_1 \wedge g^{(0)} - g^{(0)} \wedge dD_1$$

and

$$A_0 = g^{(0)} \wedge g^{(0)},$$

and

$$\begin{aligned} T_n = {} & dg^{(n)} + \sum_{i+j=n} D_i dg^{(j)} \\ & + A_n + \sum_{i+j=n} D_i A_j + B_n + \sum_{i+j=n} D_i B_j \end{aligned}$$

where A_n for $n \geq 2$ is given by

$$\begin{aligned} A_n = {} & \sum_{i+j=n} g^{(i)} \wedge g^{(j)} + \sum_{i+j=n} dD_i \wedge dD_j + \\ & \sum_{i+j=n} dD_i \wedge g^{(j)} - \sum_{i+j=n} g^{(i)} \wedge dD_j \end{aligned}$$

and B_n $n > 0$ is given by

$$B_n = \sum_{i+j+r=n} g^{(i)} \wedge D_j g^{(r)}.$$

Proof: We begin by examing the n-th order term in the expansion of $\pi_\sigma^{(k)} g_k(X)\, \pi_\sigma^{(k)} g_k(Y)\, s$. As before let $\pi_\sigma^{(k)} g_k(Y)\, s = s_1$. Thus we have the expansion

$$\pi_\sigma^{(k)} g_k(X)\, s_1 = g_k(X)\, s_1 + \sum_{l=1}^{\infty} \frac{D_l\left(g_k(X)\, s_1\right)(x_0)}{k^l} \tag{62}$$

But s_1 itself admits the expansion

$$s_1 = \pi_\sigma^{(k)} g_k(Y)\, s = \sum_{l=1}^{\infty} \frac{g^{(l)}(Y)\, s + \sum_{i+j=l} D_i g^{(j)}(Y)\, s}{k^l}$$

Thus we see that the n-th order term $\left(\pi_\sigma^{(k)} g_k(X)\, \pi_\sigma^{(k)} g_k(Y)\, s\right)_n$, in the asymptotic expansion of $\pi_\sigma^{(k)} g_k(X)\, \pi_\sigma^{(k)} g_k(Y)\, s$ is given by

$$\begin{aligned}\left(\pi_\sigma^{(k)} g_k(X)\, \pi_\sigma^{(k)} g_k(Y)\, s\right)_n &= \sum_{i+j=n} g^{(i)}(X)\, g^{(j)}(Y)\, s + \sum_{i+j+r=n} g^{(i)}(X)\, D_j g^{(r)}(Y)\, s \\ &+ \sum_{i+j+r=n} D_i\left(g^{(j)}(X)\, g^{(r)}(Y)\, s\right) \\ &+ \sum_{i+j+r+m=n} D_i\left(g^{(j)}(X)\, D_r g^{(m)}(Y)\, s\right)\end{aligned} \tag{63}$$

Next we turn to the terms of the form $\pi_\sigma^{(k)} X \left[\pi_\sigma^{(k)}\right] Y \left[\pi_\sigma^{(k)}\right] s$. Upon examination that the n-th order term, $\left(\pi_\sigma^{(k)} X \left[\pi_\sigma^{(k)}\right] Y \left[\pi_\sigma^{(k)}\right] s\right)_n$ in the asymptotic examination is given by

$$\left(\pi_\sigma^{(k)} X \left[\pi_\sigma^{(k)}\right] Y \left[\pi_\sigma^{(k)}\right] s\right)_n = \sum_{i+j=n} X(D_i)\, Y(D_j)\, s + \sum_{i+j+r=n} D_i\left(X(D_j)\, Y(D_r)\, s\right) \tag{64}$$

Next we turn to the case of terms of the form $\pi_\sigma^{(k)} X \left[\pi_\sigma^{(k)}\right] g_k(Y)\, s$. As before let $g_k(Y)\, s$ be denoted s^1. Therefore, upon examination of the asymptotic expansion of both s^1 and $X\left[\pi_\sigma^{(k)}\right] s^1$ the n-th order term $\left(\pi_\sigma^{(k)} X \left[\pi_\sigma^{(k)}\right] g_k(Y)\, s\right)_n$ can be given by

$$\left(\pi_\sigma^{(k)} X \left[\pi_\sigma^{(k)}\right] g_k(Y)\, s\right)_n = \sum_{i+j=n} X(D_i)\, g^{(j)}(Y)\, s + \sum_{i+j+r=n} D_i\left(X(D_j)\, g^{(r)}(Y)\, s\right) \tag{65}$$

We next turn to the terms of the $\pi_\sigma^{(k)} X\left(g_k\left(Y\right)\right)s$. And evaluating the asymptotic expansion, we see that the n-th order term $\left(\pi_\sigma^{(k)} X\left(g_k\left(Y\right)\right)s\right)_n$ is given by

$$\left(\pi_\sigma^{(k)} X\left(g_k\left(Y\right)\right)s\right)_n = X\left(g^{(n)}Y\right)s + \sum_{i+j=n} D_i\left(X\left(g^{(j)}Y\right)s\right) \tag{66}$$

Finally we turn to the terms of the form $\pi_\sigma^{(k)} g_k\left(Y\right)X\left(\pi_\sigma^{(k)}\right)s$. As before we examine the asymptotic expansions to get the expression for the n-th term $\left(\pi_\sigma^{(k)} g_k\left(Y\right)X\left(\pi_\sigma^{(k)}\right)s\right)_n$,

$$\left(\pi_\sigma^{(k)} g_k\left(Y\right)X\left(\pi_\sigma^{(k)}\right)s\right)_n = \sum_{i+j=n} g^{(i)}\left(Y\right)X\left(D_j\right)s + \sum_{i+j+r=n} D_i\left(g^{(j)}\left(Y\right)X\left(D_r\right)s\right) \tag{67}$$

Therefore from equations (63), (64), (65), (66) and (67), we can deduce that the n-th order term in the asymptotic expansion of $\mathcal{F}_{\nabla^M}\left(X,Y\right)s$, denoted $T_n\left(X,Y\right)$

is given by

$$
\begin{aligned}
T_n(X,Y)s = & \sum_{i+j=n} g^{(i)}(X)\, g^{(j)}(Y)\, s + \sum_{i+j+r=n} g^{(i)}(X)\, D_j g^{(r)}(Y)\, s \\
& + \sum_{i+j+r=n} D_i \left(g^{(j)}(X)\, g^{(r)}(Y)\, s \right) \\
& + \sum_{i+j+r+m=n} D_i \left(g^{(j)}(X)\, D_r g^{(m)}(Y)\, s \right) \\
& - \sum_{i+j=n} g^{(i)}(Y)\, g^{(j)}(X)\, s - \sum_{i+j+r=n} g^{(i)}(Y)\, D_j g^{(r)}(X)\, s \\
& - \sum_{i+j+r=n} D_i \left(g^{(j)}(Y)\, g^{(r)}(X)\, s \right) \\
& - \sum_{i+j+r+m=n} D_i \left(g^{(j)}(Y)\, D_r g^{(m)}(X)\, s \right) \\
& + \sum_{i+j=n} X(D_i)\, Y(D_j)\, s + \sum_{i+j+r=n} D_i \left(X(D_j)\, Y(D_r)\, s \right) \\
& - \sum_{i+j=n} Y(D_i)\, X(D_j)\, s - \sum_{i+j+r=n} D_i \left(Y(D_j)\, X(D_r)\, s \right) \\
& + \sum_{i+j=n} X(D_i)\, g^{(j)}(Y)\, s + \sum_{i+j+r=n} D_i \left(X(D_j)\, g^{(r)}(Y)\, s \right) \\
& - \sum_{i+j=n} Y(D_i)\, g^{(j)}(X)\, s - \sum_{i+j+r=n} D_i \left(Y(D_j)\, g^{(r)}(X)\, s \right) \\
& + \sum_{i+j=n} g^{(i)}(Y)\, X(D_j)\, s + \sum_{i+j+r=n} D_i \left(g^{(j)}(Y)\, X(D_r)\, s \right) \\
& - \sum_{i+j=n} g^{(i)}(X)\, Y(D_j)\, s - \sum_{i+j+r=n} D_i \left(g^{(j)}(X)\, Y(D_r)\, s \right) \\
& + \sum_{i+j=n} D_i \left(dg^{(j)}(X,Y)\, s \right) \\
& + dg^{(n)}(X,Y)\, s
\end{aligned} \tag{68}
$$

Let $A_n(X,Y)$ be given by the equation

$$
\begin{aligned}
A_n(X,Y) = & \sum_{i+j=n} \left[g^{(i)}(X), g^{(j)}(Y) \right] + \sum_{i+j=n} \left[X[D_i], Y[D_j] \right] \\
& + \sum_{i+j=n} \left(X[D_i]\, g^{(j)}(Y) - Y[D_i]\, g^{(j)}(X) \right) \\
& + \sum_{i+j=n} \left(g^{(i)}(Y)\, X[D_j] - g^{(i)}(X)\, Y[D_j] \right)
\end{aligned} \tag{69}
$$

Therefore we can write

$$A_n = \sum_{i+j=n} g^{(i)} \wedge g^{(j)} + \sum_{i+j=n} dD_i \wedge dD_j + \sum_{i+j=n} dD_i \wedge g^{(j)} - \sum_{i+j=n} g^{(i)} \wedge dD_j \tag{70}$$

Let $B_n(X, Y)$ be given by the equation

$$B_n(X, Y) = \sum_{i+j+r=n} \left(g^{(i)}(X) D_j g^{(r)}(Y) - g^{(i)}(Y) D_j g^{(r)}(X) \right) \tag{71}$$

Therefore we can write

$$B_n = \sum_{i+j+r=n} g^{(i)} \wedge D_j g^{(r)} \tag{72}$$

Therefore from equations (70) and (72), we have that

$$T_n = dg^{(n)} + \sum_{i+j=n} D_i dg^{(j)} + A_n + \sum_{i+j=n} D_i A_j + B_n + \sum_{i+j=n} D_i B_j \tag{73}$$

In particular we have that

$$T_0 = g^{(0)} \wedge g^{(0)} + dg^{(0)} \tag{74}$$

Further, we have that

$$T_1 = dg^{(1)} + D_1 dg^{(0)} + A_1 + D_1 A_0 + B_1 \tag{75}$$

where

$$A_1 = g^{(0)} \wedge g^{(1)} + g^{(1)} \wedge g^{(0)} + dD_1 \wedge g^{(0)} - g^{(0)} \wedge dD_1$$

and

$$A_0 = g^{(0)} \wedge g^{(0)}$$

Thus proving the theorem

$\Diamond$

In the subsequent part of the thesis, we explore the conditions under which the connection ∇^M is flat.

Theorem 16 *Given the setting and notation of theorem 15, under the additional assumption that $g^{(0)} = 0$, the necessary and sufficient conditions for the curvature $\mathcal{F}_{\nabla^M}$ to vanish are*

$$g^{(1)} \wedge dD_1 \wedge g^{(1)} + 2dD_1 \wedge dD_1 \wedge dD_1 = 0 \tag{76}$$

and

$$\begin{aligned}
0 = & \sum_{i+j+r=n} g^{(i)} \wedge dD_j \wedge g^{(r)} + 2 \sum_{i+j+r=n} dD_i \wedge dD_j \wedge dD_r \\
& - \sum_{i+j+r+q=n} g^{(i)} \wedge D_j g^{(r)} \wedge g^{(q)} + \sum_{i+j+r+q=n} g^{(i)} \wedge D_j g^{(r)} \wedge dD_q \\
& + \sum_{i+j+r=n} g^{(i)} \wedge D_j \left(A_r + B_r\right) - \sum_{i+j+r+q=n} dD_i \wedge dD_j \wedge D_r g^{(q)} \\
& + \sum_{i+j+r+q=n} g^{(i)} \wedge dD_j \wedge D_r g^{(q)} - \sum_{i+j+p+q+r=n} g^{(i)} \wedge D_j g^{(p)} \wedge D_q g^{(r)}
\end{aligned} \tag{77}$$

for $n > 3$.

Proof: Note that under the assumption that $g^{(0)} = 0$ the condition for the first order term T_1 to vanish is merely $dg^{(1)} = 0$, but $dg^{(1)}(X,Y) = 0$ merely implies that locally the 1-form $g^{(1)}$ is exact, i.e., there exists a 0-form $\tilde{g}^{(1)}$ such that for any vector field X on $\mathcal{T}$, we have that $g^{(1)}(X) = X\left[\tilde{g}^{(1)}\right]$.

We now turn to the second order term T_2. Note that under the assumptions of the theorem, the expression for T_2 is given by,

$$T_2 = dg^{(2)} + g^{(1)} \wedge g^{(1)} + dD_1 \wedge dD_1 + dD_1 \wedge g^{(1)} - g^{(1)} \wedge dD_1$$

We thus have

$$\begin{aligned}
dT_2 &= dg^{(1)} \wedge g^{(1)} - g^{(1)} \wedge dg^{(1)} + ddD_1 \wedge dD_1 - dD_1 \wedge ddD_1 \\
&\quad + ddD_1 \wedge g^{(1)} - dD_1 \wedge dg^{(1)} - dg^{(1)} \wedge dD_1 + g^{(1)} \wedge ddD_1 \\
&= 0
\end{aligned} \tag{78}$$

Thus allowing us to solve for $g^{(2)}$ such that $T_2 = 0$.

So far therefore under the assumptions of the theorem, we have not encountered any obstructions to the curvature of the connection ∇^M vanishing to order 2. Let us now turn to order to the order 3 term T_3. As in the earlier cases we assume that we have obtained solutions for $g^{(1)}$ and $g^{(2)}$, such that the first and second order terms, T_1 and T_2 vanish. Therefore we have that $dg^{(1)} = 0$ and that

$$dg^{(2)} = -A_2 \tag{79}$$

where $A_2 = g^{(1)} \wedge g^{(1)} + dD_1 \wedge dD_1 + dD_1 \wedge g^{(1)} - g^{(1)} \wedge dD_1$. Therefore we have

$$\begin{aligned} T_3 &= dg^{(3)} + D_1 dg^{(2)} + D_2 dg^{(1)} + A_3 + B_3 + D_1 A_2 \\ &= dg^{(3)} + D_1 \left(dg^{(2)} + A_2 \right) + A_3 + B_3 \\ &= dg^{(3)} + A_3 + g^{(1)} \wedge D_1 g^{(1)} \end{aligned}$$

Thus we have that

$$\begin{aligned} dT_3 &= dA_3 - g^{(1)} \wedge dD_1 \wedge g^{(1)} \\ &= -g^{(1)} \wedge dg^{(2)} + dg^{(2)} \wedge g^{(1)} - dD_1 \wedge dg^{(2)} - dg^{(2)} \wedge dD_1 \\ &- g^{(1)} \wedge dD_1 \wedge g^{(1)} \end{aligned} \tag{80}$$

From equation (79), we have that $dg^{(2)} = -A_2$, and $A_2 = g^{(1)} \wedge g^{(1)} + dD_1 \wedge dD_1 + dD_1 \wedge g^{(1)} - g^{(1)} \wedge dD_1$. Substituting these in equation (80), we have that

$$dT_3 = g^{(1)} \wedge dD_1 \wedge g^{(1)} + 2dD_1 \wedge dD_1 \wedge dD_1$$

Therefore, we encounter our first obstruction to the curvature of ∇^M vanishing, namely, for the curvature $\mathcal{F}_{\nabla^M}$ to vanish up to third order, $g^{(1)}$ must necessarily satisfy the equation

$$g^{(1)} \wedge dD_1 \wedge g^{(1)} + 2dD_1 \wedge dD_1 \wedge dD_1 = 0 \tag{81}$$

We now turn to the general n-th order term. As before, we assume that the $T_k = 0$ for all $k < n$. Thus we have that

$$\begin{aligned} T_n &= dg^{(n)} + \sum_{i+j=n} D_i dg^{(j)} \\ &+ A_n + \sum_{i+j=n} D_i A_j + B_n + \sum_{i+j=n} D_i B_j \\ &= dg^{(n)} + A_n + B_n + \sum_{i+j=n} D_i \left(dg^{(j)} + A_j + B_j \right) \\ &= dg^{(n)} + A_n + B_n \end{aligned}$$

where

$$\begin{aligned} A_n = &\sum_{i+j=n, i>0, j>0} g^{(i)} \wedge g^{(j)} + \sum_{i+j=n, i>0, j>0} dD_i \wedge dD_j + \\ &\sum_{i+j=n, i>0, j>0} dD_i \wedge g^{(j)} - \sum_{i+j=n, i>0, j>0} g^{(i)} \wedge dD_j \end{aligned}$$

and

$$B_n = \sum_{i+j+r=n, i>0, j>0, r>0} g^{(i)} \wedge D_j g^{(r)}$$

Given that we assume $g^{(0)} = 0$, we shall assume for the rest of this proof that all indices i and j are for $g^{(i)}$ and D_j are greater than 0. Therefore we have that

$$dT_n = dA_n + dB_n \tag{82}$$

We first consider the term dB_n.

$$\begin{aligned}
dB_n = & \sum_{i+j+r=n} dg^{(i)} \wedge D_j g^{(r)} - \sum_{i+j+r=n} g^{(i)} \wedge dD_j \wedge g^{(r)} \\
& - \sum_{i+j+r=n} g^{(i)} \wedge D_j dg^{(r)} \\
& = - \sum_{i+j+r=n} g^{(i)} \wedge dD_j \wedge g^{(r)} + \sum_{i+j+r=n} g^{(i)} \wedge D_j \left(A_r + B_r\right) \\
& - \sum_{i+j+r+q=n} g^{(i)} \wedge g^{(j)} \wedge D_r g^{(q)} - \sum_{i+j+r+q=n} dD_i \wedge dD_j \wedge D_r g^{(q)} \\
& - \sum_{i+j+r+q=n} dD_i \wedge g^{(j)} \wedge D_r g^{(q)} + \sum_{i+j+r+q=n} g^{(i)} \wedge dD_j \wedge D_r g^{(q)} \\
& - \sum_{i+j+p+q+r=n} g^{(i)} \wedge D_j g^{(p)} \wedge D_q g^{(r)}
\end{aligned} \tag{83}$$

We now turn to the term dA_n. We have

$$dA_n = \sum_{i+j=n} dg^{(i)} \wedge g^{(j)} - g^{(i)} \wedge dg^{(j)} - dD_i \wedge dg^{(j)} - dg^{(i)} \wedge dD_j \tag{84}$$

Recalling the formulae for A_k and B_k and keeping in mind our assumption that $T_k = 0$, i.e., $dg^{(k)} + A_k + B_k = 0$ for all $k < n$, we have that

$$\begin{aligned}
dA_n = & - \sum_{i+j=n} A_i \wedge g^{(j)} - \sum_{i+j=n} B_i \wedge g^{(j)} \\
& + \sum_{i+j=n} g^{(i)} \wedge A_j + \sum_{i+j=n} g^{(i)} \wedge B_j \\
& + \sum_{i+j=n} dD_i \wedge A_j + \sum_{i+j=n} dD_i \wedge B_j \\
& + \sum_{i+j=n} A_i \wedge dD_j + \sum_{i+j=n} B_i \wedge dD_j \\
& = 2 \sum_{i+j+r=n} g^{(i)} \wedge dD_j \wedge g^{(r)} + 2 \sum_{i+j+r=n} dD_i \wedge dD_j \wedge dD_r \\
& + \sum_{i+j+r+q=n} g^{(i)} \wedge g^{(j)} \wedge D_r g^{(q)} - \sum_{i+j+r+q=n} g^{(i)} \wedge D_j g^{(r)} \wedge g^{(q)} \\
& + \sum_{i+j+r+q=n} dD_i \wedge g^{(j)} \wedge D_r g^{(q)} + \sum_{i+j+r+q=n} g^{(i)} \wedge D_j g^{(r)} \wedge dD_q
\end{aligned} \tag{85}$$

Therefore we have that

$$\begin{aligned}
dT_n &= dA_n + dB_n \\
&= \sum_{i+j+r=n} g^{(i)} \wedge dD_j \wedge g^{(r)} + 2 \sum_{i+j+r=n} dD_i \wedge dD_j \wedge dD_r \\
&- \sum_{i+j+r+q=n} g^{(i)} \wedge D_j g^{(r)} \wedge g^{(q)} + \sum_{i+j+r+q=n} g^{(i)} \wedge D_j g^{(r)} \wedge dD_q \\
&+ \sum_{i+j+r=n} g^{(i)} \wedge D_j \left(A_r + B_r\right) - \sum_{i+j+r+q=n} dD_i \wedge dD_j \wedge D_r g^{(q)} \\
&+ \sum_{i+j+r+q=n} g^{(i)} \wedge dD_j \wedge D_r g^{(q)} - \sum_{i+j+p+q+r=n} g^{(i)} \wedge D_j g^{(p)} \wedge D_q g^{(r)}
\end{aligned} \tag{86}$$

Therefore the necessary condition for the solution of $T_n = 0$ to exist is

$$\begin{aligned}
0 = & \sum_{i+j+r=n} g^{(i)} \wedge dD_j \wedge g^{(r)} + 2 \sum_{i+j+r=n} dD_i \wedge dD_j \wedge dD_r \\
&- \sum_{i+j+r+q=n} g^{(i)} \wedge D_j g^{(r)} \wedge g^{(q)} + \sum_{i+j+r+q=n} g^{(i)} \wedge D_j g^{(r)} \wedge dD_q \\
&+ \sum_{i+j+r=n} g^{(i)} \wedge D_j \left(A_r + B_r\right) - \sum_{i+j+r+q=n} dD_i \wedge dD_j \wedge D_r g^{(q)} \\
&+ \sum_{i+j+r+q=n} g^{(i)} \wedge dD_j \wedge D_r g^{(q)} - \sum_{i+j+p+q+r=n} g^{(i)} \wedge D_j g^{(p)} \wedge D_q g^{(r)}
\end{aligned}$$

Thus proving the theorem.

◊

Corollary 16.1 *Under the setting of theorem 16, if we let $g^{(1)}$ and $g^{(2)}$ satisfy the conditions*

$$g^{(1)}(X) = iX(D_1)$$

and

$$dg^{(2)} = 0$$

then the connection ∇^M is flat upto order 2.
Proof: Clearly, if $g^{(1)}(X) = iX(D_1)$, we have $dg^{(1)} = 0$, therefore $T_1 = 0$. Now consider T_2. Recall that T_2 is given by

$$T_2 = dg^{(2)} + g^{(1)} \wedge g^{(1)} + dD_1 \wedge dD_1 + dD_1 \wedge g^{(1)} - g^{(1)} \wedge dD_1$$

Clearly $dg^{(2)} = 0$ and $g^{(1)}(X) = iX(D_1)$ satisfy $T_2 = 0$, hence completing the proof.

◊

We now turn our attention to the conditions that ∇^M is projectively flat. Before proceeding however we establish the following notation. Given the context and notation established thus far, for a given differential operator D, $f_{D,k}$ is a function on M such that $\pi_\sigma^{(k)} Ds = \pi_\sigma^{(k)} f_{D,k} s$.

Theorem 17 *Given the setting and the notation of theorem 15, and under the additional assumptions that $g^{(i)}(X)$ are C^∞ functions on M for all, $i \geq 0$ and vector fields, X on $\mathcal{T}$, the conditions for projective flatness of the connection ∇^M are*

$$d_M dg^{(0)} = 0,$$

$$\begin{aligned} 0 = {} & d_M dg^{(1)}(X,Y) + d_M \left(f_{D_1,k} dg^{(0)}(X,Y) \right) + \\ & d_M \left(f_{X(D_1),k} g^{(0)}(Y) - f_{Y(D_1),k} g^{(0)}(X) \right) - \\ & d_M \left(f_{g^{(0)}(X)Y(D_1),k} - f_{g^{(0)}(Y)X(D_1),k} \right) + \\ & d_M \left(f_{g^{(0)}(X)D_1,k} g^{(0)}(Y) - f_{g^{(0)}(Y)D_1,k} g^{(0)}(X) \right) \end{aligned}$$

and

$$\begin{aligned} 0 = {} & d_M dg^{(n)}(X,Y) + \sum_{i+j=n} d_M \left(f_{D_i,k} dg^{(j)}(X,Y) \right) + \\ & \sum_{i+j=n} d_M \left(f_{X(D_i)Y(D_j),k} - f_{Y(D_i)X(D_j),k} \right) + \\ & \sum_{i+j=n} d_M \left(f_{X(D_i),k} g^{(j)}(Y) - f_{Y(D_i),k} g^{(j)}(X) \right) - \\ & \sum_{i+j=n} d_M \left(f_{g^{(i)}(X)Y(D_j),k} - f_{g^{(i)}(Y)X(D_j),k} \right) + \\ & \sum_{i+l+m=n} d_M \left(f_{D_i,k} \left(f_{X(D_l)Y(D_m),k} - f_{Y(D_l)X(D_m),k} \right) \right) + \\ & \sum_{i+l+m=n} d_M \left(f_{D_i,k} \left(f_{X(D_l),k} g^{(m)}(Y) - f_{Y(D_l),k} g^{(m)}(X) \right) \right) - \\ & \sum_{i+l+m=n} d_M \left(f_{D_i,k} \left(f_{g^{(l)}(X)Y(D_m),k} - f_{g^{(l)}(Y)X(D_m),k} \right) \right) + \\ & \sum_{i+j+r=n} d_M (\left(f_{g^{(i)}(X)D_j,k} g^{(r)}(Y) - f_{g^{(i)}(Y)D_j,k} g^{(r)}(X) \right) + \\ & \sum_{i+l+m+p=n} d_M \left(f_{D_i,k} \left(f_{g^{(l)}(X)D_m,k} g^{(p)}(Y) - f_{g^{(l)}(Y)D_l,k} g^{(p)}(X) \right) \right) \end{aligned}$$

for all vector fields X and Y on $\mathcal{T}$. Here d_M is the exterior derivative on M.

Proof: We begin by examining the first order term. Recall that T_1 is given by the expression

$$T_1 = dg^{(1)} + D_1 dg^{(0)} + A_1 + D_1 A_0 + B_1.$$

But notice that in this cas $A_0 = 0$. Now consider the asymptotic expansion for the curvature

$$\mathcal{F}_{\nabla^M}(X,Y)\,s = \sum_{n=0}^{\infty} \frac{T_n(X,Y)\,s}{k^n}$$

Applying the projection operator $\pi_\sigma^{(k)}$ to both sides of the above equation, we have

$$\pi_\sigma^{(k)} \mathcal{F}_{\nabla^M}(X,Y)\,s = \sum_{n=0}^{\infty} \frac{\pi_\sigma^{(k)} T_n(X,Y)\,s}{k^n}$$

But notice that $\pi_\sigma^{(k)} \mathcal{F}_{\nabla^M}(X,Y)\,s = \mathcal{F}_{\nabla^M}(X,Y)\,s$. Therefore we have that

$$\mathcal{F}_{\nabla^M}(X,Y)\,s = \sum_{n=0}^{\infty} \frac{\pi_\sigma^{(k)} T_n(X,Y)\,s}{k^n}$$

Therefore, for $n = 1$, we have that

$$\begin{aligned} \pi_\sigma^{(k)} T_1(X,Y)\,s = \pi_\sigma^{(k)} dg^{(1)}(X,Y)\,s + \pi_\sigma^{(k)} D_1 dg^{(0)}(X,Y)\,s \\ + \pi_\sigma^{(k)} A_1(X,Y)\,s + \pi_\sigma^{(k)} B_1(X,Y)\,s \end{aligned} \tag{87}$$

Now since D_1 is a differential operator, we can say that $\pi_\sigma^{(k)} D_1 = \pi_\sigma^{(k)} f_{D_1,k}$, where $f_{D_1,k}$ is a function on M. Since $A_1(X,Y)\,s$ is given by

$$\begin{aligned} A_1(X,Y)\,s = g^{(0)} \wedge g^{(1)}(X,Y)\,s + g^{(0)} \wedge g^{(1)}(X,Y)\,s + \\ dD_1 \wedge g^{(0)}(X,Y)\,s - g^{(0)} \wedge dD_1(X,Y)\,s \end{aligned}$$

and we know that all $g^{(i)}(X)$ are functions on M, we have that

$$A_1(X,Y)\,s = dD_1 \wedge g^{(0)}(X,Y)\,s - g^{(0)} \wedge dD_1(X,Y)\,s$$

Applying $\pi_\sigma^{(k)}$ to both sides we have that

$$\begin{aligned} \pi_\sigma^{(k)} A_1(X,Y)\,s = \pi_\sigma^{(k)} \left(f_{X(D_1),k}\, g^{(0)}(Y) - f_{Y(D_1),k}\, g^{(0)}(X) \right) s \\ - \pi_\sigma^{(k)} \left(f_{g^{(0)}(X)Y(D_1),k} - f_{g^{(0)}(Y)X(D_1),k} \right) s \end{aligned} \tag{88}$$

And since $B_1(X,Y)\,s$ is given by

$$B_1(X,Y)\,s = g^{(0)} \wedge D_1 g^{(0)}(X,Y)\,s$$

We have that

$$\begin{aligned} \pi_\sigma^{(k)} B_1(X,Y)\,s = \pi_\sigma^{(k)} f_{g^{(0)}(X)D_1,k}\, g^{(0)}(Y)\,s - \\ \pi_\sigma^{(k)} f_{g^{(0)}(Y)D_1,k}\, g^{(0)}(X)\,s \end{aligned} \tag{89}$$

Thus we have that $\pi_\sigma^{(k)} T_1(X,Y)\,s = \pi_\sigma^{(k)} t_1(X,Y)\,s$ where $t_1(X,Y)$ is a function on M given by

$$\begin{aligned} t_1(X,Y) = {} & dg^{(1)}(X,Y) + f_{D_1,k} dg^{(0)}(X,Y) + \\ & \left(f_{X(D_1),k} g^{(0)}(Y) - f_{Y(D_1),k} g^{(0)}(X)\right) - \\ & \left(f_{g^{(0)}(X)Y(D_1),k} - f_{g^{(0)}(Y)X(D_1),k}\right) + \\ & f_{g^{(0)}(X)D_1,k} g^{(0)}(Y) - f_{g^{(0)}(Y)D_1,k} g^{(0)}(X) \end{aligned} \tag{90}$$

Since the connection is projectively flat, we have that $d_M t_1(X,Y) = 0$ where d_M is the exterior derivative on M. Thus we have the condition that

$$\begin{aligned} 0 = {} & d_M dg^{(1)}(X,Y) + d_M\left(f_{D_1,k} dg^{(0)}(X,Y)\right) + \\ & d_M\left(f_{X(D_1),k} g^{(0)}(Y) - f_{Y(D_1),k} g^{(0)}(X)\right) - \\ & d_M\left(f_{g^{(0)}(X)Y(D_1),k} - f_{g^{(0)}(Y)X(D_1),k}\right) + \\ & d_M\left(f_{g^{(0)}(X)D_1,k} g^{(0)}(Y) - f_{g^{(0)}(Y)D_1,k} g^{(0)}(X)\right) \end{aligned} \tag{91}$$

We now turn our attention to the general term T_n. Recall that T_n is given by the expression

$$\begin{aligned} T_n = {} & dg^{(n)} + \sum_{i+j=n} D_i dg^{(j)} \\ & + A_n + \sum_{i+j=n} D_i A_j + B_n + \sum_{i+j=n} D_i B_j \end{aligned}$$

where A_n for $n > 1$ is given by

$$\begin{aligned} A_n = {} & \sum_{i+j=n} g^{(i)} \wedge g^{(j)} + \sum_{i+j=n} dD_i \wedge dD_j + \\ & \sum_{i+j=n} dD_i \wedge g^{(j)} - \sum_{i+j=n} g^{(i)} \wedge dD_j \end{aligned}$$

and B_n for $n > 1$ is given by

$$B_n = \sum_{i+j+r=n} g^{(i)} \wedge D_j g^{(r)}.$$

Proceeding as before and applying $\pi_\sigma^{(k)}$ to $B_n(X,Y)\,s$ we have that

$$\pi_\sigma^{(k)} B_n(X,Y)\,s = \pi_\sigma^{(k)} \sum_{i+j+r=n} \left(f_{g^{(i)}(X)D_j,k} g^{(r)}(Y) - f_{g^{(i)}(Y)D_j,k} g^{(r)}(X)\right) s \tag{92}$$

And applying $\pi_\sigma^{(k)}$ to $A_n(X,Y)s$ we have that

$$\begin{aligned}\pi_\sigma^{(k)} A_n(X,Y)s = &\sum_{i+j=n}\left(f_{X(D_i)Y(D_j),k} - f_{Y(D_i)X(D_j),k}\right)s+ \\ &\sum_{i+j=n}\left(f_{X(D_i),k}g^{(j)}(Y) - f_{Y(D_i),k}g^{(j)}(X)\right)s- \\ &\sum_{i+j=n}\left(f_{g^{(i)}(X)Y(D_j),k} - f_{g^{(i)}(Y)X(D_j),k}\right)s\end{aligned} \tag{93}$$

Thus we have that $\pi_\sigma^{(k)} T_n(X,Y)s = \pi_\sigma^{(k)} t_n(X,Y)s$ where $t_n(X,Y)$ is a function on M given by

$$\begin{aligned}t_n(X,Y) = &\, dg^{(n)}(X,Y) + \sum_{i+j=n} f_{D_i,k}dg^{(j)}(X,Y) + \\ &\sum_{i+j=n}\left(f_{X(D_i)Y(D_j),k} - f_{Y(D_i)X(D_j),k}\right) + \\ &\sum_{i+j=n}\left(f_{X(D_i),k}g^{(j)}(Y) - f_{Y(D_i),k}g^{(j)}(X)\right) - \\ &\sum_{i+j=n}\left(f_{g^{(i)}(X)Y(D_j),k} - f_{g^{(i)}(Y)X(D_j),k}\right) + \\ &\sum_{i+l+m=n} f_{D_i,k}\left(f_{X(D_l)Y(D_m),k} - f_{Y(D_l)X(D_m),k}\right) + \\ &\sum_{i+l+m=n} f_{D_i,k}\left(f_{X(D_l),k}g^{(m)}(Y) - f_{Y(D_l),k}g^{(m)}(X)\right) - \\ &\sum_{i+l+m=n} f_{D_i,k}\left(f_{g^{(l)}(X)Y(D_m),k} - f_{g^{(l)}(Y)X(D_m),k}\right) + \\ &\sum_{i+j+r=n}\left(f_{g^{(i)}(X)D_j,k}g^{(r)}(Y) - f_{g^{(i)}(Y)D_j,k}g^{(r)}(X)\right) + \\ &\sum_{i+l+m+p=n} f_{D_i,k}\left(f_{g^{(l)}(X)D_m,k}g^{(p)}(Y) - f_{g^{(l)}(Y)D_l,k}g^{(p)}(X)\right)\end{aligned} \tag{94}$$

Since the connection is projectively flat, we have that $d_M t_n(X,Y) = 0$ where

d_M is the exterior derivative on M. Thus we have the condition that

$$
\begin{aligned}
0 = {} & d_M dg^{(n)}(X,Y) + \sum_{i+j=n} d_M \left(f_{D_i,k} dg^{(j)}(X,Y) \right) + \\
& \sum_{i+j=n} d_M \left(f_{X(D_i)Y(D_j),k} - f_{Y(D_i)X(D_j),k} \right) + \\
& \sum_{i+j=n} d_M \left(f_{X(D_i),k} g^{(j)}(Y) - f_{Y(D_i),k} g^{(j)}(X) \right) - \\
& \sum_{i+j=n} d_M \left(f_{g^{(i)}(X)Y(D_j),k} - f_{g^{(i)}(Y)X(D_j),k} \right) + \\
& \sum_{i+l+m=n} d_M \left(f_{D_i,k} \left(f_{X(D_l)Y(D_m),k} - f_{Y(D_l)X(D_m),k} \right) \right) + \\
& \sum_{i+l+m=n} d_M \left(f_{D_i,k} \left(f_{X(D_l),k} g^{(m)}(Y) - f_{Y(D_l),k} g^{(m)}(X) \right) \right) - \\
& \sum_{i+l+m=n} d_M \left(f_{D_i,k} \left(f_{g^{(l)}(X)Y(D_m),k} - f_{g^{(l)}(Y)X(D_m),k} \right) \right) + \\
& \sum_{i+j+r=n} d_M (\left(f_{g^{(i)}(X)D_j,k} g^{(r)}(Y) - f_{g^{(i)}(Y)D_j,k} g^{(r)}(X) \right) + \\
& \sum_{i+l+m+p=n} d_M \left(f_{D_i,k} \left(f_{g^{(l)}(X)D_m,k} g^{(p)}(Y) - f_{g^{(l)}(Y)D_l,k} g^{(p)}(X) \right) \right)
\end{aligned}
\tag{95}
$$

which proves our theorem.

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References

[1] J E Andersen, *Hitchin's Connection, Toeplitz Operators and Symmetry Invariant Deformation Quantization.* Quantum Topology, Vol. 3, No. 3/4, 2012, p. 293-325.

[2] J E Andersen, N L Gammelgaard, *Hitchin's projectively flat connection, Toeplitz operators and the asymptotic expansion of TQFT curve operators.* Grassmannians, Moduli Spaces and Vector Bundles. ed. / David A. Ellwood; Emma Previato. American Mathematical Society, 2011. p. 1-24 (Clay Mathematics Proceedings, Vol. 14).

[3] J E Andersen, N L Gammelgaard, M R Lauridsen *Hitchin's connection in metaplectic quantization.* Quantum Topology, vol. 3, nr. 3/4, s. 293-325.

[4] F A Berezin, *Quantization.* Izv. Akad. Nauk SSSR Ser. Mat., 38:1116-1175, 1974

[5] R Berndt, *An introduction to symplectic geometry*, AMS, 2000

[6] A L Besse, *Einstein manifolds.* volume 10 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3).* Springer-Verlag, Berlin, 1987

[7] A Böchter, B Silbermann, *Analysis of Toeplitz Operators.* Springer Monographs in Mathematics (2nd ed.), Springer Verlag, 2006

[8] M Bordemann, E Meinrenken, M Schlichenmaier, *Toeplitz qunatization of Kähler manifolds and* $gl(N)$, $N \to \infty$ *limits.* Comm. Math. Phys., 165(2): 281-296, 1994.

[9] L Charles, *Berezin-Toeplitz operators, a semi-classical approach.* Comm. Math. Phys. 239, (2003), no. 1-2, p. 1–28.

[10] J Dupont, *Curvature and Characteristic Classes*, Lecture Notes in Mathematics, Springer, 1978

[11] J Dupont, *Fibre Bundles and Chern-Weil Theory*, Aarhus Univeristy, 2003

[12] O Forster, *Lectures on Riemann Surfaces*, Springer, 1981.

[13] P Griffiths, J Harris, *Principles of algebraic geometry.* Wiley, New York, 1978.

[14] N L Gammelgaard, *Kähler quantization and Hitchin Connections.* Thesis

[15] N Hitchin, *Flat connections and geometric quantization.* Comm. Math. Phys., 131, (2), (1990), 347-380

[16] L Hörmander, *The Analysis of Linear Partial Differential Operators I, Distribution Theory and Fourier Analysis.* Springer Verlag, New York, 1983

[17] A V Karabegov, *Deformation quantizations with separation of variables on a Kähler manifold.* Comm. Math. Phys., 180(3):745 - 755, 1996.

[18] A V Karabegov, *Cohomological classification of deformation quantizations with separation of variables.* Lett. Math. Phys., 43(4):347-357, 1998.

[19] A V Karabegov, M Schlichenmaier, *Identification of Berezin-Toeplitz deformation qunatization.* J. Reine Angew. Math, **540** (2001), 49-76

[20] K Kodaira *Complex manifolds and deformation of complex structures,* vol. 283 of *Grundlehren der Mathematischen Wissenschaften [Fundamentals Principles of Mathematical Sciences].* Springer Verlag, New York, 1986. Translated from Japanese by Kazuo Akao, With an appendix by Daisuke Fujiwara.

[21] Zhiquin Lu *On the lower order terms of the expansion of the aymptotic expansion of Zelditch.* arXiv:math/9811126v2 [math.DG] 31 Jan 1999

[22] Xiaonan Ma, G Marinescu, *Berezin Toeplitz quantization and its expansion.* arXiv:math/1203.4201v1 [math.DG].

[23] A Melin, J Sjöstrand, *Fourier integral operators with complex valued phase functions.* Lect. Notes Math, **459**, 120-223 (1975)

[24] Newlander, L Nirenberg, *Complex analytic coordinates in almost complex manifolds.* Ann. of Math. (2), 65:319-404, 1957

[25] M Rosenblum, J Rovnyak, *Hardy Classes and Operator Theory,* Oxford University Press (1985).

[26] M Schlichenmaier, *Berezin Toeplitz quantization and conformal field theory.* Thesis

[27] M Schlichenmaier, *Deformation quantization of compact Kähler manifolds by Berezin-Toeplitz quantization, In Conférence Moshé Flato 1999, Vol. II (Dijon).* Math. Phys. Stud., **22**, Kluwer Acad. Publ., Dodrecht, (2000), 289-306

[28] M Schlichenmaier, *Berezin Toeplitz quantization and Berezin Transform.* In *Long time behaviour of classical and quantum systems (Bologna, 1999).* Ser. Concr. Appl. Math., **1**, World Sci. Publishing, River Edge, NJ (2001), 217-287

[29] A C da Silva, *Lectures on Symplectic Geometry*, Springer, 2001

[30] R O Wells, *Differential Analysis on Complex Manifolds.* volume 65 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1980.

[31] N M J Woodhouse, *Geometric Quantization.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1992. Oxford Science Publications.

[32] S Zelditch, *Szegö kernels and a theorem of Tian.* Int. Math. Res. Not. **6** (1998), 317 - 331

[33] F Zheng, *Complex Differential Geometry.* AMS, 2000.