Formal connections in deformation quantization

PhD Dissertation

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Abstract

The field of this thesis is deformation quantization, and we consider mainly symplectic manifolds equipped with a star product.

After reviewing basics in complex geometry, we introduce quantization, focusing on geometric quantization and deformation quantization. The latter is defined as a star product on a Poisson manifold that is in general non-commutative and corresponds to the composition of the quantized observables.

While in general it is difficult to express a star product globally on a curved manifold in an explicit way, we consider a case where this is possible, namely that of a Kähler manifold. Gammelgaard gave an explicit formula for a class of star products in this setting. We review his construction, which is combinatorial and based on a certain family of graphs and extend it, to provide the graph formalism with the notions of composition and differentiation.

We shall focus our attention on symplectic manifolds equipped with a family of star products, indexed by a parameter space. In this situation we can define a connection in the trivial bundle over the parameter space with fibres the formal smooth functions on the manifold, which relates the star products in the family and is called a formal connection. We study the question of classifying such formal connections. To each star product we can associate a certain cohomology class called the characteristic class. It turns out that a formal connection exists if and only if all the star products in the family have the same characteristic class, and that formal connections form an affine space over the derivations of the star products. Moreover, if the parameter space for the family of star products has trivial first cohomology, we obtain that any two flat formal connections are gauge equivalent via a self-equivalence of the family of star products.

Afterwards we study the problem of trivializing a formal connection, that is to define a differential operator on the manifold which makes any section of the bundle parallel with respect to the connection. To approach the problem we use the graph formalism described above to encode it in graph terms. This allows us to express the equations determining a trivialization of the formal connection completely in graph terms, and solving them amounts to finding a linear combination of graphs whose derivative is equal to a given expression. We shall also look at another approach to the problem that is more calculative. Moreover we use the graph formalism to give a set of recursive equations determining the formal connection for a given family of star products.

Dansk resumé

Denne afhandling beskæftiger sig med deformationskvantisering. Vi betragter hovedsageligt symplektiske mangfoldigheder med et stjerneprodukt.

Vi gennemgår det grundlæggende stof inden for kompleks geometri, og derefter introducerer vi kvantisering, med fokus på geometrisk kvantisering og deformationskvantisering. Det sidste er defineret ved at give et stjerneprodukt på en Poisson-mangfoldighed, hvilket som regel er ikke-kommutativt og svarer til sammensætning af de kvantiserede observable.

Helt generelt er det svært at udtrykke et stjerneprodukt eksplicit, men vi kigger på et tilfælde, hvor det er muligt, nemlig på en Kähler-mangfoldighed. Gammelgaard gav et eksplicit udtryk for et stjerneprodukt i dette tilfælde. Vi gennemgår hans konstruktion, som er kombinatorisk og baseret på en vis familie grafer, og vi udvider dette kombinatoriske sprog til at indeholde begreber som sammensætning af operatorer og differentiation.

Vi fokuserer vores analyse på de symplektiske mangfoldigheder, som er givet en familie stjerneprodukter, parametriserede af et parameterrum. I denne situation kan vi definere en forbindelse i det trivielle bundt over parameterrummet, hvis fibre er de formelle glatte funktioner på mangfoldigheden. Denne forbindelse kaldes for en formel forbindelse. Vi kigger på spørgsmålet om klassificering af sådanne formelle forbindelser. Vi kan vise, at en formel forbindelse findes, hvis og kun hvis alle stjerneprodukter i familien har den samme karakteristiske klasse, og at de formelle forbindelser udgør et affint rum over mængden bestående af derivationerne af disse stjerneprodukter.

Dernæst undersøger vi trivialiseringer af en formel forbindelse. En trivialisering er en differentialoperator, som gør ethvert snit af bundtet parallelt med hensyn til forbindelsen. Vi prøver at finde sådan en trivialisering ved hjælp af den ovennævnte grafformalisme. På den måde kan vi udtrykke problemet helt kombinatorisk med grafer. For at løse problemet skal man finde en linearkombination af grafer, som når den bliver afledt giver et bestemt udtryk. Vi skal også kigge på en anden metode, som er baseret på beregninger. Endeligt bruger vi vores grafformalisme til at give en række rekursive ligninger, som bestemmer den formelle forbindelse for en familie af stjerneprodukter.

Contents

Preface

This dissertation consists of the main results of my doctoral studies, carried out at Aarhus University in the years 2011–2014 under the supervision of Prof. Jørgen Ellegaard Andersen. The main topic is deformation quantization and formal connections. The dissertation extends and completes previous work that was presented in my progress report [Mas13].

I wish to take the opportunity to thank my supervisor Prof. Jørgen Ellegaard Andersen, for all the inputs, the many ideas that he proposed me and the countless useful discussions. A big thanks goes to all the people at QGM and at the Department of Mathematics at Aarhus University with whom I have interacted during my time here. I would like to thank in particular Florian Schätz for the very fruitful discussions and for pointing me at the paper [WX98]. The years I spent in Aarhus were – luckily for my sanity – not only Mathematics: I feel very lucky for having met so many amazing people whom now I consider friends in this town. Thank you all for being there!

Added in November 2014: This revised version includes the corrections that I have made after my thesis defence. I would like to thank Prof. Simone Gutt and Prof. Martin Schlichenmaier, who were the members of my evaluation committee, for their priceless and careful work of review of this thesis and for pointing out some mistakes that now hopefully have been fixed.

> *For the things we have to learn before we can do them, we learn by doing them.*
>
> – Aristotle, *Nicomachean Ethics*

From the next chapter, and for the rest of this thesis, the subject “I” will leave its place to the canonical “we”. The reason for this is mostly a matter of following an established custom, but also to signify that you, *reader* – as much as me – are involved in the process of studying and understanding the Mathematics that I have tried to put on paper. Let us then begin this mathematical journey together![1]

Paolo Masulli
Aarhus, August 2014

[1] You’re welcome to contact me with comments or to report some of the typos that unavoidably got unnoticed. You’re currently reading the version of this manuscript compiled on November 4, 2014.

1

Introduction

The main subject of this dissertation is *deformation quantization*, with a focus on *formal connections* on symplectic manifolds equipped with a family of star products.

Quantization and the Hitchin connection

Let us introduce the broad area in which we are working, namely quantization.

Edward Witten proposed in the paper [Wit89] that quantum Chern-Simons theory should form the two-dimensional part of a topological quantum field theory (TQFT) in $2+1$ dimensions. The study of geometric quantization of the moduli space M of flat $SU(n)$-connections on a surface Σ arose from there. This moduli space has a natural symplectic structure ω and admits a prequantum line bundle, which is a Hermitian line bundle $\mathcal{L}$ with a compatible connection, whose curvature is given by the symplectic form.

The Teichmüller space $\mathcal{T}$ of the surface Σ parametrizes complex structures on the moduli space, so for each point $\sigma \in \mathcal{T}$ and each natural number k, called the level of quantization, we have the quantum state space of geometric quantization, which is the space

$$Q_k(\sigma) = H^0(M_\sigma; \mathcal{L}^{\otimes k})$$

of holomorphic sections of the k-th tensor power of the prequantum line bundle. These form the fibres of a vector bundle Q over $\mathcal{T}$, called the Verlinde bundle, and it was shown independently by Hitchin in [Hit90] and Axelrod, Della Pietra and Witten in [ADPW91] that this bundle admits a natural projectively flat connection, which we shall call the *Hitchin connection*. Consequently, the quantum spaces associated with different complex structures are identified, as projective spaces, through the parallel transport of this connection.

Formal connections

On a Poisson manifold M, a *deformation quantization*, or *star product*, is a $\mathbb{C}[[h]]$-linear product in the space $C^\infty(M)[[h]]$ that is associative, gives to the pointwise product modulo h, and such that the part of degree 1 in h of its commutator is a multiple of the Poisson bracket.

Andersen in [And06] has studied the asymptotic relationship between Toeplitz operators and the Hitchin connection, and in [And12] he has extended his asymptotic analysis of the relationship between the Hitchin connection and the Toeplitz operators to higher orders, which led him to define the following notion.

Definition. If M is a symplectic manifold equipped with a smooth family of star products parametrized by a manifold $\mathcal{T}$, a *formal connection* on M is a connection in the bundle $\mathcal{T} \times C^\infty(M)[[h]] \to \mathcal{T}$ of the form:

$$D_V f = V[f] + A(V)(f), \tag{1.1}$$

where A is a smooth 1-form on $\mathcal{T}$ with values in differential operators on M such that $A = 0 \pmod{h}$, f is a smooth section of the bundle, V is any smooth vector field on $\mathcal{T}$, and $V[f]$ denotes the derivative of f along V.

The formal connection is *compatible* with the family of star products $\{\star_\sigma\}_{\sigma\in\mathcal{T}}$, if it is a derivation of the products at any point in the base space $\mathcal{T}$:

$$D_V(f \star_\sigma g) = D_V(f) \star_\sigma g + f \star_\sigma D_V(g), \tag{1.2}$$

for any $\sigma \in \mathcal{T}$ and any smooth functions f and g on M.

On a Kähler manifold M we always have the family of the so called Berezin-Toeplitz star products $\star^{BT}$, which can be constructed using the theory of Toeplitz operators.

Definition. Let M be a symplectic manifold with a family of compatible almost complex structures parametrized by a complex manifold $\mathcal{T}$, so that for any $\sigma \in \mathcal{T}$, the manifold M_σ is a Kähler manifold, and let $\{\star_\sigma^{BT}\}_{\sigma\in\mathcal{T}}$ be the associated family of Berezin-Toeplitz star products. A *formal Hitchin connection* on M is a formal connection that is compatible with this family of star products and that is flat.

Andersen in [And12] studied a particular formal Hitchin connection that is associated to the Hitchin connection from geometric quantization, and he showed that the projective flatness of the Hitchin connection implies the flatness of the formal Hitchin connection he defined. In [And12] and together with Gammelgaard [AG11], Andersen gave an explicit expression for the 1-form $\tilde{A}(V)$ for the formal Hitchin connection associated to the Hitchin connection:

$$\tilde{A}(V)(f) = -V[F]f + V[F] \star^{BT} f + h(E(V)(f) - H(V) \star^{BT} f), \tag{1.3}$$

where E is a 1-form on $\mathcal{T}$ with values in differential operators on M, H is a 1-form with values in $C^\infty(M)$ such that $H(V) = E(V)(1)$, and F indicated the Ricci potential for the family.

Andersen noticed that any formal Hitchin connection can be used to identify these deformation quantizations and obtain a mapping class group equivariant deformation quantization on the moduli space, provided that certain cohomology groups of the mapping class group vanish.

We aim to give a result of classification of formal connections. It turns out that a formal connection exists if the characteristic class of the star products in the family is constant in cohomology, which is one of our main results.

Theorem. *Fix ∇ a symplectic connection for (M,ω) and let $\{\star_\sigma\}_{\sigma\in\mathcal{T}}$ be a smooth family of natural star products on M parametrized by a manifold $\mathcal{T}$. There exists a compatible formal connection compatible with the family of star products if and only if the characteristic class of the star products is constant.*

To show this, we do a construction inspired by Fedosov's geometrical construction of a star product [Fed94], which ties the star products of the family together.

This result specializes to the case of a symplectic manifold equipped with a smooth family of compatible Kähler structures, where we take the family of Berezin-Toeplitz star products associated with them. In that situation we get the following result, granting the existence of a formal Hitchin connection up to order one in the formal parameter.

Theorem. *Let (M,ω) be a compact, symplectic manifold, and let $\mathcal{T}$ be a complex manifold parametrizing a family of compatible Kähler structures I_σ on M, for $\sigma\in\mathcal{T}$. The family of Berezin-Toeplitz star products associated with the family has constant characteristic class, and therefore it admits a formal connection.*

Classification of formal connections

Let (M,ω) be a symplectic manifold with a family of star products $\{\star_\sigma\}_{\sigma\in\mathcal{T}}$ that are parametrized by $\mathcal{T}$, and let us denote with $\mathrm{Der}_0(M,\star_\sigma)$ the space of the derivations of the star product $\star_\sigma$ that are trivial modulo h.

We shall see that the space of the formal connections compatible with the family of star products on M form an affine space over the space of 1-forms on $\mathcal{T}$ with values in $\mathrm{Der}_0(M,\star_\sigma)$.

Theorem. *Let M be a symplectic manifold equipped with a smooth family of star products $\{\star_\sigma\}_{\sigma\in\mathcal{T}}$ parametrized by $\mathcal{T}$. The space $\mathcal{F}(M,\star_\sigma)$ of the formal connections on M that are compatible with the family of star products is an affine space over the space of 1-forms on $\mathcal{T}$ with values in the derivations of the star product that are trivial modulo h, and it can then be written as:*

$$\mathcal{F}(M,\star_\sigma) = D_0 + \Omega^1(\mathcal{T},\mathrm{Der}_0(M,\star)),$$

for a fixed formal connection D_0.

Furthermore, if we assume that $H^1(M;\mathbb{R})$ vanishes, we have that all derivations of a star product on M are inner, therefore they are parametrized by an element in $\tilde{C}^\infty_h(M)$, the space of formal functions on M modulo the constant functions, which allows us to refine the previous result to the following.

Theorem. *If M satisfies that $H^1(M;\mathbb{R})=0$, the space $\mathcal{F}(M,\star_\sigma)$ of the formal connections on M that are compatible with the family of star products is an affine space over the 1-forms on $\mathcal{T}$ with values in $\tilde{C}^\infty_h(M)$:*

$$\mathcal{F}(M,\star_\sigma) \cong D_0 + \Omega^1(\mathcal{T},\tilde{C}^\infty_h(M)),$$

for a fixed formal connection D_0.

Gauge transformations of formal connections

We study gauge transformations in the space of formal connections $\mathcal{F}(M, \star_\sigma)$. The transformations we look at are differential self-equivalences of the family of star products, since the connections should still act as derivations when we transform them. If we assume that the parameter space $\mathcal{T}$ of the family of star products has trivial first cohomology, we obtain the following result.

Theorem. *Let M be a symplectic manifold with a family of star products $\{\star_\sigma\}_{\sigma\in\mathcal{T}}$ parametrized by a smooth manifold $\mathcal{T}$ such that $H^1(\mathcal{T};\mathbb{R}) = 0$. Let $D, D' \in \mathcal{F}(M, \star_\sigma)$ be formal connections for the family and let us assume that they are flat. Then they are gauge equivalent via a self-equivalence of the family of star products $P \in C^\infty(\mathcal{T}, \mathcal{D}_h(M))$, meaning that*

$$D'_V = P^{-1} D_V P, \tag{1.4}$$

for any vector field V on $\mathcal{T}$.

In particular, Andersen showed that the formal Hitchin connection associated to the Hitchin connection is flat whenever the Hitchin connection in geometric quantization is projectively flat. This implies the following corollary.

Corollary. *Let $\mathcal{T}$ be a smooth manifold with $H^1(\mathcal{T};\mathbb{R}) = 0$. If there exists a formal Hitchin connection D in the bundle C_h on $\mathcal{T}$, then it is unique up to gauge equivalence.*

The formal Hitchin connection at low orders

As mentioned above, a formal connection compatible with a family of star products is required to be a derivation of them. In the case our main example of a formal connection, namely the formal Hitchin connection defined by Andersen, which is associated to the Hitchin connection in geometric quantization, we know that it is a derivation of the Berezin-Toeplitz star product, which was proved by Andersen [And12] by using a correspondence between geometric and deformation quantization through Toeplitz operators, but this result of course assumes the existence of a Hitchin connection in geometric quantization, which puts several requirements on the objects involved in the construction, among the others the fact that the family of Kähler structures has to be *holomorphic* and *rigid*. We shall see that these are quite strong requirements.

On the other hand, the explicit expression (1.3) that Andersen obtained makes sense in a more general situation, and therefore we can ask whether that expression in general gives a derivation of the Berezin-Toeplitz star product. This is a difficult question, because it involves the coefficients of the star product, which are in general difficult to understand. But we can give an affirmative answer if we only look at low degrees in the formal parameter.

Moreover it is easy to check that the same expression (1.3) defines a flat formal connection, therefore we have the conclusion that the expression obtained by Andersen gives a formal Hitchin connection up to order one.

Proposition. *Let M be a symplectic manifold with a family of compatible Kähler structures parametrized by a complex manifold $\mathcal{T}$. Then the expression* (1.3) *defines a formal* *connection that, up to order one in the formal parameter, is a derivation of the family of Berezin-Toeplitz star products on M, and that is flat up to order one in the formal parameter. Therefore it defines up to order one a formal Hitchin connection in the sense of our definition above.*

A combinatorial approach to deformation quantization

Deformation quantization on Kähler manifolds was studied initially, among the others, by Berezin in [Ber74], where he gave integral formulae for a star product though with strong assumptions on the Kähler manifold.

A later important result in deformation quantization for this type of manifold is due to Karabegov, who showed in [Kar96] the existence of a star products in this setting, and gave a classification of formal deformation quantizations with separation of variables by showing that they bijectively correspond to closed formal $(1,1)$-forms on the manifold. Such a form ω, called a *Karabegov form* can be written as:

$$\omega = \omega_{-1}\frac{1}{h} + \omega_0 + \omega_1 h + \omega_2 h^2 + \dots, \tag{1.5}$$

where each ω_i is a closed 2-form on M of type $(1,1)$, and ω_{-1} is the symplectic form of M.

Around the same time Schlichenmaier [Sch00] gave a geometric but implicit construction of a deformation quantization on any compact Kähler manifold, by means of the asymptotic expansion of products of Toeplitz operators in geometric quantization, which is used to uniquely identify the star product.

In the symplectic setting a construction of star product is due to Fedosov, who in the paper [Fed94] constructed a canonical star product on any symplectic manifold.

The first result giving an explicit construction of a deformation quantization on a general Kähler manifold is due to Reshetikhin and Takhtajan in [RT00] and is based on interpreting Berezin's integral formulae formally and studying their asymptotical behaviour. The resulting explicit formula expresses the star product combinatorially in terms of certain graphs that get interpreted as differential operators.

Gammelgaard in [Gam14] gave an explicit combinatorial formula for deformation quantizations a Kähler manifold, which expresses the star products with separation of variables that were classified by Karabegov as a linear combination of differential operators associated to graphs, in a way that the coefficients of the linear combination are easily computable. To do so he defined a class of graphs denoted $\mathcal{A}_2$ and to each graph he associated a partition function Γ^ω_G, which encodes the way the star product differentiates the two function of which we wish to compute the star product.

Theorem (Gammelgaard)**.** *On a Kähler manifold M, the unique formal deformation quantization with Karabegov form ω is given by the local formula*

$$f \star^\omega g = \sum_{G\in\mathcal{A}_2} \frac{1}{|\mathrm{Aut}(G)|}\Gamma^\omega_G(f,g)h^{w(G)},$$

for any functions f and g on M, where Γ^{ω}_{G} is the partition function associated to the graph G.

In this thesis we apply Gammelgaard's work to give a combinatorial interpretation of the formal Hitchin connection associated to the Hitchin connection of geometric quantization and to express the problem of trivializing it in a combinatorial way. The graph language we develop allows us to make sense of composition and differentiation of graphs, and thereby interpret the differential equation that has to be solved (in local coordinates) as a relation between linear combinations of graphs. For instance, given two differential operators A_1, A_2 and their expressions as a linear combination of graphs, we can express their composition through an operation on graphs that is called *fusion*. That is, the composed differential operators $A_1 A_2$ has a graph expression given obtained by fusing the graphs for the two initial operators.

Compatible formal connection in graph language

We can use this graph language to express what it means for a formal connection to be compatible with a family of star products. We get a result which gives us a set of recursive equations that a 1-form A has to satisfy in order for the formal connection $D_V = V + A(V)$ to be compatible with a family of Karabegov star products.

Proposition. *The formal* 1*-form A defines a formal connection if and only if it satisfies the following equations for any $k \geq 1$, for any vector field V on $\mathcal{T}$ and any two smooth functions f and g.*

$$A^k(V)(fg) - A^k(V)(f)g - fA^k(V)(g) = \sum_{G \in \mathcal{L}^c_{2,k}} \frac{1}{C(G)} V[\Lambda_G](f,g)$$

$$- \sum_{i=1}^{k} \Bigg(A^{k-i}(V)\Big(\sum_{G \in \mathcal{L}^c_{2,i}} \frac{1}{C(G)} \Lambda_G(f,g)\Big)$$

$$- \sum_{G \in \mathcal{L}^c_{2,i}} \frac{1}{C(G)} \Lambda_G(A^{k-i}(V)(f), g) - \sum_{G \in \mathcal{L}^c_{2,i}} \frac{1}{C(G)} \Lambda_G(f, A^{k-i}(V)(g)) \Bigg).$$

The formal Hitchin connection in graph language

Using similar considerations we could prove the following result, which expresses the formal Hitchin connection associated to the Hitchin connection from geometric quantization completely in terms of graphs. In a similar fashion to the previous result, we identify a certain class of graphs denoted $\mathcal{L}^c_{2,1}$ that encode the ways that the star products coefficients differentiate the arguments, and one more graph denoted G_0, that corresponds to the Laplace operator. To each of them we associate a partition function Λ_G. The theorem shows how we can use these partition functions to express the formal Hitchin connection.

Theorem. *Let M be a symplectic manifold with a family of Kähler structures which is parametrized by a complex manifold $\mathcal{T}$, and let F denote the Ricci potential for the family of complex structures. Let D_V be the formal Hitchin connection (associated to the Hitchin* *connection of geometric quantization) in the bundle $C^\infty(M)[[h]] \times \mathcal{T}$ over $\mathcal{T}$, which is expressed in the form $D_V = V + \tilde{A}(V)$, for $\tilde{A}$ as in* (1.3). *Then $\tilde{A}$ can be completely expressed in terms of graphs via the equations:*

$$
\begin{aligned}
\tilde{A}^1(V)(f) &= \sum_{G\in\mathcal{L}^c_{2,1}} \frac{1}{C(G)}\Lambda_G(V[F],f) - \frac{i}{2}V[\Lambda_{G_0}](f) + \sum_{G\in\mathcal{L}^c_{2,1}} V[\Lambda_G](F,f),\\
\tilde{A}^k(V)(f) &= \sum_{G\in\mathcal{L}^c_{2,k}} \frac{1}{C(G)}\Lambda_G(V[F],f)\\
&\qquad + \sum_{G\in\mathcal{L}^c_{2,k-1}} \frac{1}{C(G)}\Lambda_G(-\frac{1}{2}\Delta_{\tilde{G}(V)}(F) + \frac{n}{2}V[F],f), \qquad \text{for } k\geq 2.
\end{aligned}
$$

Trivialization of the formal Hitchin connection

When we have a formal connection in the bundle $\mathcal{T} \times C^\infty(M)[[h]]$ over $\mathcal{T}$, we can look at a *formal trivialization* of the connection: that is a smooth map $P\colon \mathcal{T} \to \mathcal{D}_h(M)$ which modulo h is the identity, and which satisfies $D_V(P(f)) = 0$ for all vector fields V on $\mathcal{T}$ and all $f \in C^\infty(M)[[h]]$.

Andersen and Gammelgaard could produce a formal trivialization of the formal Hitchin connection associated to the Hitchin connection of geometric quantization up to order one. In this thesis we show an approach to solve this problem at higher and possible all degrees, by formulating it in the graph-theoretical language mentioned above. The following result shows how we can write the equation determining P as a system of recursive equations of graphs.

Theorem. *Let D_V be the formal Hitchin connection (associated to the Hitchin connection of geometric quantization) in the bundle $C^\infty(M)[[h]] \times \mathcal{T}$ over $\mathcal{T}$, which is expressed in the form $D_V = V + \tilde{A}(V)$. A smooth map $P\colon \mathcal{T} \to \mathcal{D}_h(M)$ is a formal trivialization for D if it satisfies the following recursive sequence of relations expressed in graph theoretical language:*

$$
\begin{aligned}
P_0 &= \textit{empty graph}\\
V[P_k] &= -\tilde{A}^1(V)(P_{k-1}) - \tilde{A}^2(V)(P_{k-2}) - \cdots - \tilde{A}^k(V)(P_0)\\
&= -\sum_{G\in\mathcal{L}^c_{2,1}} \frac{1}{C(G)}\Lambda_G(V[F],P_{k-1}) + \frac{i}{2}V[\Lambda_{G_0}](P_{k-1}) - \frac{1}{2}\sum_{G\in\mathcal{L}^c_{2,1}} V[\Lambda_G](F,P_{k-1})\\
&\quad - \sum_{G\in\mathcal{L}^c_{2,2}} \frac{1}{C(G)}\Lambda_G(V[F],P_{k-2}) - \sum_{G\in\mathcal{L}^c_{2,1}} \frac{1}{C(G)}\Lambda_G(-\frac{1}{2}\Delta_{\tilde{G}(V)}(F) + \frac{n}{2}V[F],P_{k-2})\\
&\quad - \dots\\
&\quad - \sum_{G\in\mathcal{L}^c_{2,k}} \frac{1}{C(G)}\Lambda_G(V[F],\mathrm{Id}) - \sum_{G\in\mathcal{L}^c_{2,k-1}} \frac{1}{C(G)}\Lambda_G(-\frac{1}{2}\Delta_{\tilde{G}(V)}(F) + \frac{n}{2}V[F],\mathrm{Id}).
\end{aligned}
$$

for $k \geq 1$.

Trivialization on abelian varieties

An interesting example is that of principally polarised abelian variety, which are manifolds written in the form
$$M = V/\Lambda,$$
where V is a real vector space with a symplectic form ω fulfilling some additional conditions, and Λ is a discrete lattice of maximal rank. They are symplectic manifolds, and as usual we can consider a complex manifold $\mathcal{T}$ parametrizing the Kähler structures I_σ on M that are compatible with ω for $\sigma \in \mathcal{T}$.

Andersen studied this case in [And05] and later in [AB11] together with Blaavand, and gave an expression for a formal Hitchin connection for these manifolds, which is particularly simple:

$$D_V f = V[f] - \frac{h}{4}\Delta_{\tilde{G}(V)}(f).$$

The equations that we obtain for the trivialization problem simplify in this case, and this allows us to find a trivialization of the formal connection to all degrees, which has the following form:

$$P = \sum_{k\in\mathbb{N}} h^k \frac{\Delta^k}{4^k k!} = \exp\left(h\frac{\Delta}{4}\right).$$

Comparing the result obtained here with the formulae in [And05] and [AB11], we see that they match, taking into account the different normalizations.

1.1 Organization of the thesis

The thesis is organized as follows:

Chapter 2 provides a brief exposition of basic notions of complex differential geometry that we are going to need in what follows. It is mostly meant as a reference and to establish notation.

Chapter 3 introduces quantization and gives some physical background for its study. In this chapter we give an overview of quantization in general, but at the same time we give also a more detailed treatment of geometric quantization following the approach used by Andersen in [And12] and introduce Toeplitz operators. We also introduce polarizations and discuss some of their properties.

In **Chapter 4** we switch to a different approach to quantization, namely deformation quantization, which is our main focus in this thesis. Deformation quantization is based on the definition of certain products called *star products*, which we shall introduce and study in detail. In particular we focus our attention to a certain type of star products, namely those that are differential and with separation of variables. We review Karabegov's classification of such products and define the Berezin-Toeplitz star product, which is a particular product that is in some sense natural to consider on a Kähler manifold.

Chapter 5 is devoted to the construction of the Hitchin connection. This connection was first defined by Hitchin [Hit90], but here we review the construction proposed by Andersen in [And12]. To do so we introduce different objects, and discuss their properties.

In the following **Chapter 6** we go back to star products and focus our attention to differential products with separation of variables on a Kähler manifold. Unlike the general situation with star products, those belonging to this class can be described explicitly in a combinatorial fashion by using graphs. We review Gammelgaard's construction [Gam14] of this combinatorial description, which allows to represent each star product of Karabegov's classification as a linear combination of differential operators associated to certain graphs.

Chapter 7 is about formal connections and contains some of the main results of this thesis. The formal Hitchin connection was introduced by Andersen translating the Hitchin connection to the setting of deformation quantization. Andersen's work is mainly focused on this particular formal connection since it is related to the problem of quantization, but actually formal connections can be studied independently from this, forgetting the fact that they originate from this problem. Following this approach we give a description of the space of formal connections and give a necessary and sufficient condition on the star product for the existence of formal connections compatible with it. Moreover, if the parameter space for the family of star products has trivial first cohomology, we show that any two flat formal connections are gauge equivalent via a self-equivalence of the family of star products.

Chapter 8 gives account for the trivialization problem for the formal Hitchin connection associated to the Hitchin connection of geometric quantization. We show two different approaches to the problem: the first one is based on the graph language of Chapter 6, where we try making sense of what it means to "differentiate" a graph, in a fashion corresponding to differentiation of the associated partition function. This way we are able to formulate the problem purely in graph terms. The second approach is more calculative and gives us as a by-product some relations that the coefficients of the star products satisfy. Moreover we give a solution to the trivialization problem at all degrees for a class of abelian varieties.

2

Complex geometry

This chapter introduces briefly some basic notions of complex geometry that will be needed in the rest of the dissertation. Even if it is expected that the reader is familiar with most of the content of this chapter, its presence can serve as a reference and to establish notation. For a more extended treatment the reader is referred for example to [Wel08], [BN06], [DS08], [Mor07], and [Esp12].

2.1 Symplectic manifolds

Definition 2.1. A *symplectic manifold* (M, ω) is the data of a smooth manifold M of even dimension together with a non-degenerate differential 2-form ω, called *symplectic form*.

Recall that non-degenerate means that, at each point $p \in M$, the form ω_p is non-degenerate, and hence a symplectic form ω defines an isomorphism $i_\omega \colon TM \to T^*M$ between the tangent and cotangent bundles, given by contraction in the first entry: $X \mapsto i_X\omega$. We can use this isomorphism to define the bivector field:

$$\tilde{\omega} = -(i_\omega^{-1} \otimes i_\omega^{-1})(\omega),$$

which satisfies the identity $\omega \cdot \tilde{\omega} = \tilde{\omega} \cdot \omega = \mathrm{Id}$, where the dot indicated contraction of tensors in their entries closest to the dot, which is relevant when working with non-symmetric tensors. For example when we write $\omega \cdot \tilde{\omega}$ we mean that the right-most entry of ω is contracted with the left-most one of $\tilde{\omega}$.

It is inherent in the definition of differential form that ω is skew-symmetric, and since it is invertible at each point, this implies that the dimension of M is even, since skew-symmetric matrices in odd dimension are singular. A symplectic manifold of dimension m gets a canonical orientation from ω^m, and a canonical volume form that is usually normalized as $\frac{\omega^m}{m!}$.

On a symplectic manifold we have two important families of vector fields: the symplectic and the Hamiltonian ones, as we shall see in the following definitions.

Definition 2.2. Let (M, ω) be a symplectic manifold and X a vector field on it. We say that X is *symplectic* if its (local) flow preserves ω or, in other words:

$$\mathcal{L}_X\omega = 0.$$

Note that by the Cartan identity we have:

$$\mathcal{L}_X\omega = i_X d\omega + d(i_X\omega) = d(i_X\omega),$$

since ω is a closed form. It follows that, for a symplectic vector field X, the form $i_X\omega$ is closed.

If $f \in C^\infty(M)$ is a smooth function on M, then its differential df is a section of the cotangent bundle, and if we compose it with the inverse of i_ω we get a section of the tangent bundle, or a vector field, which plays an very important role in symplectic geometry:

Definition 2.3. Let (M, ω) be a symplectic manifold and f a smooth function on it. The *Hamiltonian vector field* of f, denoted X_f, is characterized by:

$$i_{X_f}\omega = df.$$

Generally we say that a vector field X on M is *Hamiltonian* is $i_X\omega$ is exact.

Saying that X is Hamiltonian is of course equivalent to requiring $\omega(X_f, Y) = df(Y) = Yf$ satisfied for any vector field Y. This can be expressed in terms of $\tilde{\omega}$ by requiring $X_f = df \cdot \tilde{\omega}$. The set of the Hamiltonian vector fields on M is denoted by $\Gamma_{\mathrm{ham}}(TM)$.

Clearly, if X is Hamiltonian, then it is symplectic, since the exact form $i_X\omega$ is closed. The space of symplectic vector fields modulo the Hamiltonian vector fields is then isomorphic to the space of closed one-forms modulo the exact ones, and so to $H^1(M; \mathbb{R})$. We have also, by the Poincaré Lemma, that any symplectic vector field locally is Hamiltonian.

2.1.1 Physical motivation

Let us here attempt to informally give a motivation for studying such a geometric structure, namely its use in encoding the time evolution of a classical mechanical system.

Symplectic manifolds arise naturally in classical mechanics as the representation of the phase space of a closed mechanical system. The phase space is a set of observables of a physical system, where each point normally represents a specific position and specific momentum. The energy of the system, which is preserved, is represented by a real-valued differentiable smooth function $H\colon M \to \mathbb{R}$ called *Hamiltonian*. The symplectic form and the Hamiltonian allow one to obtain a vector field describing the flow of the system, and this vector field is precisely the Hamiltonian vector field associated to H, i.e. the vector field V_H satisfying $dH = \omega(V_H, \cdot)$. It is immediate to check that the preservation of the Hamiltonian along flow lines corresponds to the form ω being alternating, $dH(V_H) = \omega(V_H, V_H) = 0$. In the physical interpretation, we want the form ω not to vary along the flow lines V_H, i.e. that its Lie derivative is zero. By the Cartan's formula this is equivalent to ω being closed.

2.2 Poisson structure

Definition 2.4. A *Poisson algebra* A is an algebra (over $\mathbb{R}$ or $\mathbb{C}$) equipped with a bilinear map

$$\{\cdot,\cdot\}\colon A \times A \to A,$$

called *Poisson bracket*, which is anti-symmetric and satisfies:

- the Jacobi identity: $\{x,\{y,z\}\} + \{y,\{z,x\}\} + \{z,\{x,y\}\} = 0$,
- the derivation rule: $\{x,yz\} = \{x,y\}z + y\{x,z\}$,

for any $x, y, z \in A$.

Definition 2.5. A *Poisson manifold* is a smooth manifold M together with a Poisson bracket on the algebra of smooth functions $C^\infty(M)$.

It is easy to check that, if (M,ω) is a symplectic manifold, then the assignment

$$\{f,g\} = df \cdot \tilde{\omega} \cdot g = -\omega(X_f, X_g) = \omega(X_g, X_f), \tag{2.1}$$

for $f, g \in C^\infty(M)$ defines a Poisson structure on M.

Remark 2.6. The tensor $\tilde{\omega}$ is the Poisson tensor of this structure, and the Jacobi identity corresponds to the vanishing of $d\omega$.

In general for a Poisson manifold one can see that the derivation rule for the bracket implies that the bracket is given by an anti-symmetric bivector field, called the *Poisson tensor*, as in (2.1). If the Poisson structure is induced by a symplectic structure, then the Poisson tensor is non-degenerate, but in general it can be degenerate. In fact the Poisson tensor is non-degenerate if and only if the Poisson structure is induced by a symplectic structure.

On a Poisson manifold, the following identity holds for any smooth functions f and g:

$$[X_f, X_g] = X_{\{f,g\}},$$

and therefore the map

$$\begin{aligned} C^\infty(M) &\to \Gamma_{\mathrm{ham}}(TM) \\ f &\mapsto X_f, \end{aligned}$$

sending a smooth function on M to its Hamiltonian vector field defines a homomorphism of Lie algebras from the smooth functions on M (equipped with the Poisson bracket) and the Hamiltonian vector fields (with the commutator).

2.3 Almost complex manifolds

An almost complex structure is an endomorphism of the tangent bundle that plays a similar role of the imaginary unit for the complex numbers.

Definition 2.7. Let M be a smooth manifold of even dimension $2m$. An *almost complex structure* $I \in C^\infty(M, \mathrm{End}(TM))$ on M is a smooth section of the endomorphism bundle of TM such that $I^2 = -\mathrm{Id}$.

This structure turns the tangent bundle TM into a complex vector bundle TM_I where multiplication by i is given by the endomorphism I.

The *complexified tangent bundle* of M is the complexification of TM:

$$TM_{\mathbb{C}} = TM \otimes_{\mathbb{R}} \mathbb{C},$$

and the almost complex structure induces a natural decomposition of the complexified tangent bundle:

$$TM_{\mathbb{C}} = T'M_I \oplus T''M_I,$$

where the two summands are the eigenspaces of the endomorphism I respectively for the eigenvalues i and $-i$:

$$T'M_I = \ker(I - i\,\mathrm{Id}), \qquad T''M_I = \ker(I + i\,\mathrm{Id}).$$

A section of the first subspace are said to be a vector fields *of type* $(1,0)$, and a section of the second subspace is a vector field *of type* $(0,1)$.

The decomposition is explicitly given by the projections to the two subspaces:

$$\pi_I^{1,0} = \frac{1}{2}(\mathrm{Id} - iI), \qquad \pi_I^{0,1} = \frac{1}{2}(\mathrm{Id} + iI).$$

Let us introduce the notation

$$X = X_I' + X_I''$$

for the decomposition of a vector field X on M.

The conjugation map on $TM_{\mathbb{C}}$ identifies $T'M_I$ and $T''M_I$ as real vector bundles. We can act on the cotangent bundle TM^* via I by:

$$(I\alpha)X = \alpha(IX),$$

for a vector field X and a covector α, and we get a decomposition of the cotangent bundles similarly to above:

$$TM_{\mathbb{C}}^* = T'M_I^* \oplus T''M_I^*,$$

where the two summands are again the eigenspaces for i and $-i$. One can see with an easy computation that $T'M_I^*$ is the subbundle of $TM_{\mathbb{C}}^*$ consisting of the forms that vanish on $T''M_I$. As we have splittings for the complexified tangent and cotangent bundle, we obtain a splitting of the tensor bundles of M, as direct sums of the tensor products of the subbundles corresponding to the eigenvalues of i. If we denote

$$\Lambda^{p,q}(TM_I^*) = \Lambda^p(T'M_I^*) \otimes \Lambda^q(T''M_I^*),$$

then we can write:

$$\Lambda^k(TM_I^*) = \bigoplus_{p+q=k} \Lambda^{p,q}(TM_I^*).$$

This induces a splitting of complex-valued differential k-forms:

$$\Omega^k(M) = \bigoplus_{p+q=k} \Omega_I^{p,q}(M),$$

where $\Omega_I^{p,q}(M) = C^\infty(M, TM_I^*)$ is the space of complex-valued differential forms of type (p,q). Given a k-form α, we write its component of type (p,q) as $\alpha^{(p,q)}$.

We have projection maps

$$\pi^{(p,q)} \colon \Omega^{p+q}(M) \to \Omega_I^{p,q}(M),$$

which can be composed with the differential

$$d \colon \Omega^{p+q}(M) \to \Omega^{p+q+1}(M)$$

in the appropriate degree and form the operators:

$$\begin{aligned} \partial_I \colon \Omega_I^{p,q}(M) &\to \Omega_I^{p+1,q}(M) \qquad & \partial_I &= \pi^{(p+1,q)} \circ d, \\ \bar{\partial}_I \colon \Omega_I^{p,q}(M) &\to \Omega_I^{p,q+1}(M) \qquad & \bar{\partial}_I &= \pi^{(p,q+1)} \circ d. \end{aligned}$$

2.4 Complex structures

A *complex structure* on a space M is a maximal atlas of smooth charts $\varphi_j \colon U_j \to U_j' \subset \mathbb{C}^m$, such that the transition maps:

$$\varphi_{kj} = \varphi_k \circ \varphi_j^{-1} \colon \varphi_j(U_k \cap U_j) \to \varphi_k(U_k \cap U_j)$$

are holomorphic, meaning that each component of the transition map is holomorphic in each coordinate.

One can see immediately that, if a manifold M admits a complex structure, it also admits an almost complex structure: if we have local holomorphic coordinates $z^k = x^k + iy^k$ with corresponding coordinate vector fields X^k and Y^k, the almost complex structure is given by:

$$I(X^k) = Y^k, \qquad I(Y^k) = -X^k,$$

which is independent of the coordinate chosen because the transition maps are holomorphic. Because of this the tangent bundle is a complex vector bundle. Note also that the coordinates for the tangent bundle have holomorphic transition maps, therefore the bundle TM_I is a holomorphic vector bundle.

If an almost complex structure is induced by a complex structure, it is called *integrable*. It turns out that this property for an almost complex structure is equivalent to a tensorial condition, which is quite surprising. If one considers the expression:

$$N_I(X,Y) := [IX, IY] - [X,Y] - I[IX,Y] - I[X,IY], \tag{2.2}$$

it can be verified that it defines an anti-symmetric tensor on M, which is called the *Nijenhuis tensor* or *torsion* of I. An easy computation shows that, if I is an integrable almost complex structure, then it is *torsion-free*, in the sense that N_I vanishes. The opposite implication is a famous result by Newlander and Nirenberg [NN57].

Theorem 2.8 (Newlander-Nirenberg). *Any torsion-free almost complex structure is induced by a unique complex structure.*

The following proposition gives equivalent statements to the integrability of an almost complex structure:

Proposition 2.9. *Let I be an almost complex structure on M. The following statements are equivalent:*

1. *The structure is torsion-free, i.e. the Nijenhuis tensor N_I vanishes.*
2. *The bundle $T'M_I$ is preserved by the Lie bracket.*
3. *The exterior differential is decomposed as $d = \partial_I + \bar{\partial}_I$.*

Note that the third equivalent property implies the following identities:

$$\partial_I^2 = 0 \qquad \bar{\partial}_I^2 = 0 \qquad \partial_I \bar{\partial}_I = -\bar{\partial}_I \partial_I.$$

2.5 Compatible almost complex structure

Definition 2.10. Let (M, ω) be a symplectic manifold and let I be an almost complex structure on M. We say that ω and I are *compatible* if

$$g(X, Y) := \omega(X, IY) \tag{2.3}$$

defines a Riemannian metric on M, or, in other words, the bilinear form g is symmetric and positive definite.

Similarly, if on a manifold M we have three structures: a Riemannian metric g, an almost complex structure I and a symplectic form ω, then the triple (I, g, ω) is said to be compatible if it satisfies the relation (2.3), and in this case each of the three structures is determined by the other two.

If we assume that g is a symmetric bilinear form, then we have the following:

$$\omega(IX, IY) = g(IX, Y) = g(Y, IX) = \omega(Y, I^2X) = -\omega(Y, X) = \omega(X, Y),$$

i.e. ω is I-invariant. A similar computation shows that the converse is also true, therefore we have that g is symmetric if and only if ω is I-invariant, and equivalently, if g is I-invariant. A consequence of this is that both g and ω have type $(1, 1)$.

From the bilinear form g we can define as usual an isomorphism

$$i_g \colon TM \to TM^*,$$

and one can check that i_g and i_ω are related by: $i_\omega = i_g \circ I$. From the fact that g and ω have type $(1, 1)$ it follows that these two isomorphisms exchange types. As done for ω, we can define the inverse metric tensor by

$$\tilde{g} = (i_g^{-1} \otimes i_g^{-1})(g),$$

which gives a symmetric bivector field satisfying the relation $g \cdot \tilde{g} = \tilde{g} \cdot g$. This bivector field is related to the bivector field associated to ω by the relation $\tilde{\omega} = I \cdot \tilde{g}$.

2.6 First Chern Class

Looking at the equation (2.3), we see that for each compatible almost complex structure on M we can uniquely identify a Riemannian metric on the symplectic manifold. It can also be proven that, on a symplectic manifold, the space of compatible almost complex structures is non-empty and contractible (see for example [MS95] for more details.

Definition 2.11. On an almost complex manifold M with almost complex structure I, the *canonical line bundle* K_I is defined by

$$K_I = \Lambda^m T' M_I^*.$$

The *first Chern class* of a symplectic manifold (M, ω) can then be defined by

$$c_1(M, \omega) = c_1(M, I) = -c_1(K_I) \in H^2(M; \mathbb{Z}), \tag{2.4}$$

i.e. the opposite of the first Chern class of the canonical line bundle, for any almost complex structure I compatible with ω.

The first Chern class is an element of $H^2(M; \mathbb{Z})$, and from it we can obtain the *second Stiefel-Whitney class* by reducing modulo 2: it is denoted $w_2(M)$ and is a class in $H^2(M; \mathbb{Z}/2)$. The second Stiefel-Whitney class does not depend on the symplectic structure and is thereby a topological invariant of M.

Let L be any complex line bundle on M. If we take the image of the first Chern class of L under the homomorphism

$$H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{R}),$$

we obtain the *real first Chern class* of L, which is denoted $\tilde{c}_1(M, \omega)$. If ∇ is any connection on L, then we have that:

$$\tilde{c}_1(L) = \frac{i}{2\pi}[F_\nabla],$$

where F_∇ denotes the curvature of the connection.

2.7 Kähler structure

In this section we review Kähler manifolds, where we have three structures interplaying with each other: a symplectic, a Riemannian and a complex structure.

Definition 2.12. A *Kähler manifold* is a symplectic manifold (M, ω) together with a compatible almost complex structure I that is integrable.

The metric $g(X, Y) = \omega(X, IY)$ that we obtain because of compatibility is called the *Kähler metric*. The form ω takes the name of *Kähler form*.

Note that, to make a symplectic manifold into a Kähler manifold amounts to choosing an integrable and compatible almost complex structure. We will later consider the situation of a manifold with a fixed symplectic structure and a family of Kähler structures on it, which will be determined by a family of integrable and compatible almost complex structures on it.

As the manifold has a Riemannian metric, we can consider the Levi-Civita connection ∇ on it, which is the unique connection in the tangent bundle satisfying the following:

1. ∇ is *torsion-free*, i.e. $\nabla_X Y - \nabla_Y X - [X, Y] = 0$ for all vector fields X, Y.
2. ∇ is compatible with the metric, in the sense that $\nabla g = 0$, or

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

for any vector fields X, Y and Z.

Remark 2.13. Let us note an important property of Kähler manifolds: the complex structure I is parallel with respect with the Levi-Civita connection, in the sense that $\nabla I = 0$, or:

$$\nabla_X(IY) = I\nabla_X Y, \tag{2.5}$$

for any smooth vector fields X and Y. Therefore the Kähler form ω is also parallel with respect to this connection, since it satisfies (2.3) that relates it with I.

Another consequence of (2.5) is that the Levi-Civita connection preserves types, and therefore it preserves the subbundles $T'M_I$ and $T''M_I$ of the complexified tangent bundle. We can then restrict the connection ∇ to a connection on $T'M_I$ that is compatible with the Hermitian and the holomorphic structures, which we will denote with the same symbol when it is clear which bundle we are looking at.

2.7.1 Kähler coordinates

We are interested in studying quantization on Kähler manifolds also because the presence of this structure guarantees the existence of certain coordinates that have very good properties, which simplify greatly the computations. The following result establishes the existence of such coordinates, which take the name of *geodesic coordinates* or *Kähler coordinates*. The reader is referred to [Wel08] for a proof and further discussion.

Proposition 2.14. *Let M be a Kähler manifold of complex dimension m, and let $p \in M$ be a point. Then there exist a neighbourhood of p where there are complex coordinates $z^1, \dots, z^m$ such that the corresponding coordinate vector fields $Z^1, \dots, Z^m$ satisfy the following relations:*

$$g(Z^i, \bar{Z}^j) = \delta_{ij} \qquad \nabla Z^i = 0, \tag{2.6}$$

at the point p.

2.7.2 Kähler curvature

On a Kähler manifold we have the *Kähler curvature*, which corresponds to the usual notion of curvature of the Levi-Civita connection ∇: the curvature tensor of the complex bilinear extension of ∇ to $TM_{\mathbb{C}}$ is the complex trilinear extension of the usual curvature tensor R of M to $TM_{\mathbb{C}}$:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.7}$$

The Kähler curvature is then a 2-form with values in the endomorphism bundle of the tangent bundle $\mathrm{End}(TM)$. The tensor R satisfies the usual symmetries, but now, more generally, for complex vector fields X, Y, Z, U, V:

$$R(X,Y) = -R(Y,X),$$
$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0,$$

and:

$$\langle R(X,Y)U,V\rangle = -\langle R(X,Y)V,U\rangle,$$
$$\langle R(X,Y)U,V\rangle = \langle R(U,V)X,Y\rangle.$$

Let us note that the fact that the almost complex structure I on a Kähler manifold is parallel (2.5) implies that the curvature commutes with it:

$$R(X,Y)IZ = IR(X,Y)Z,$$

and therefore the curvature endomorphism preserves types. If we combine this equation with the symmetries above, then we get:

$$\langle R(X,Y)IU,IV\rangle = \langle R(IX,IY)U,V\rangle = \langle R(X,Y)U,V\rangle. \tag{2.8}$$

Moreover for the subbundles $T'M_I$ and $T''M_I$ we have that:

$$R(X,Y)(T'M_I) \subseteq T'M_I, \qquad R(X,Y)(T''M_I) \subseteq T''M_I.$$

2.8 Divergence

Let M be a Kähler manifold, and X a vector field on it. The *divergence* δX of X is defined in terms of inner multiplication with the volume form $\frac{\omega^m}{m!}$ by the following formula:

$$\delta X \frac{\omega^m}{m!} = d\left(i_X \frac{\omega^m}{m!}\right).$$

While the divergence of X does not depend on the Levi-Civita connection, one can show that it is possible to compute it with the formula:

$$\delta X = \mathrm{Tr}\,\nabla X.$$

Nevertheless the divergence is independent of the complex and the Riemannnian structure. The formula above can be generalized to tensors as follows, when $X_1, \ldots, X_n$ are vector fields on M in the following way:

$$\delta(X_1 \otimes \cdots \otimes X_n) = \delta(X_1)X_2 \otimes \cdots X_n + \sum_j X_2 \otimes \cdots \otimes \nabla_{X_1} X_j \otimes \cdots \otimes X_n.$$

This map

$$\delta\colon C^\infty(M, TM^n) \to C^\infty(M, TM^{n-1})$$

is also called divergence and does depend on the Riemannian and complex structure.

The divergence can be further extended to sections of the endomorphism bundle of the tangent bundle: if α is a 1-form in $\Omega^1(M)$ and X is a vector field, then we define

$$\delta(X \otimes \alpha) = \delta(X)\alpha + \nabla_X \alpha,$$

obtaining a map

$$\delta\colon C^\infty(M, \mathrm{End}(TM)) \to \Omega^1(M).$$

2.9 The Ricci curvature and the Ricci potential

The Ricci curvature will play an important role in what follows.

Definition 2.15. The *Ricci tensor* r is a symmetric bilinear form defined by:

$$r(X,Y) = \mathrm{Tr}\left[Z \mapsto R(Z,X)Y\right].$$

To the Ricci tensor we can associate an endomorphism of the tangent bundle by raising an index: if we still denote the endomorphism by r, we have: $\langle r(X), Y\rangle := r(X,Y)$. Note that the Ricci tensor is I-invariant, as follows from the following proposition.

Proposition 2.16. *Let M be a Kähler manifold, and let us consider an orthonormal frame of the form: $(X_1, IX_1, \dots, X_m, IX_m)$. Then the Ricci tensor can be written as follows:*

$$r(X,Y) = \sum_{i=1}^{m} R(X_i, IX_i)IX$$

Proof. Let us compute, for vector fields X and Y:

$$\begin{aligned}
r(X,Y) &= \sum \langle R(X_i,X)Y, X_i\rangle + \sum \langle R(IX_i,X)Y, IX_i\rangle \\
&= \sum \langle R(X_i,X)IY, IX_i\rangle - \sum \langle R(IX_i,X)IY, X_i\rangle \\
&= \sum \langle R(X,X_i)IX_i, IY\rangle - \sum \langle R(IX_i,X)X_i, IY\rangle \\
&= -\sum \langle R(X_i,IX_i)X, IY\rangle \\
&= \sum \langle R(X_i,IX_i)IX, Y\rangle
\end{aligned}$$

□

Therefore we can conclude that $r(IX, IY) = r(X,Y)$. As a consequence, in correspondence of the Ricci tensor we can then define the *Ricci form* ρ, which is a real $(1,1)$-form:

$$\rho(X,Y) = r(IX,Y).$$

The Ricci form is skew-symmetric. Because of the symmetries of the Kähler curvature we have that the Ricci form is related to the Kähler form as follows:

$$\rho = -R(\omega),$$

where we denote by R the *curvature operator*, that is an endomorphism of $\Lambda^{(1,1)}TM_I^*$ obtained from the Kähler curvature by raising an index.

Recall that on a complex manifold M closed forms are locally exact with respect to the operator $\partial\bar{\partial}$, meaning that, if $\alpha \in \Omega^{(p,q)}(M)$ is a closed form of type (p,q)

and $U \subseteq M$ is an open contractible subset, then there exists $\beta \in \Omega^{p-1,q-1}(U)$ such that:

$$\alpha|_U = \partial\bar{\partial}\beta,$$

by the complex Poincaré Lemma.

When we are in the Kähler setting and M is compact, then we have a global version of this statement that can be proved by using Hodge theory.

Let now assume that M is a compact Kähler manifold. If α and β are differential forms on M, the metric g induces a pointwise inner product which we denote by $g(\alpha, \beta)$. This can be used to define an inner product on forms in the following way:

$$\langle \alpha, \beta \rangle = \int_M g(\alpha, \beta) \frac{\omega^m}{m!}. \tag{2.9}$$

In Hodge theory one defines the *Hodge star operator*, which is the unique bundle isomorphism

$$*\colon \Lambda^k TM^* \to \Lambda^{2m-k} TM^*$$

that satisfies:

$$\alpha \wedge *\beta = g(\alpha, \beta) \frac{\omega^m}{m!}$$

for any differential forms α and β on M.

If d^* and $\bar{\partial}^*$ denote the adjoints of respectively d and $\bar{\partial}$ with respect to this inner product, we have that they can be expressed in terms of the Hodge star operator in the following way:

$$d^* = - * d * \qquad \text{and} \qquad \bar{\partial}^* = - * \bar{\partial} * . \tag{2.10}$$

The operators Δ and $\bar{\Box}$ can then be expressed in terms of these operators:

$$\Delta = dd^* + d^*d \qquad \text{and} \qquad \bar{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}. \tag{2.11}$$

The forms in the kernel of Δ (respectively $\bar{\Box}$) are called Δ-harmonic ($\bar{\Box}$-harmonic).

The following result shows that a form on a Kähler manifold is Δ-harmonic if and only if it is $\bar{\Box}$-harmonic. The reader is referred to [Wel08] for a proof.

Theorem 2.17. *If M is a Kähler manifold, then the operators Δ and $\bar{\Box}$ are related by: $\Delta = 2\bar{\Box}$.*

If we use this result we can easily compute the Laplacian of a function on a Kähler manifold by:

$$\Delta f = -2i\delta X'_f. \tag{2.12}$$

To obtain this formula we can observe that the divergence of vector field is related to the adjoint of the differential by the formula:

$$\delta X = -d^* i_g(X), \tag{2.13}$$

that holds for any vector field X. Therefore we can compute:

$$\delta X'_f = -i\delta I X'_f = id^*\bar{\partial} f = i(\partial^* + \bar{\partial}^*)\bar{\partial} f = i\bar{\partial}^*\bar{\partial} f = i\bar{\Box} f = \frac{i}{2}\Delta f,$$

where we use the fact that $\partial^*\bar{\partial} = 0$ when the manifold is Kähler. This computation proves the formula (2.12) above.

An immediate calculation shows that the Kähler form is harmonic, and this is proved via the definition of the star operator and (2.10).

Remark 2.18. The fact that every cohomology class in $H^k(M, \mathbb{C})$ is represented in a unique way by a harmonic form is a classical result proved, for example, in Wells [Wel08]. The same holds for cohomology classes in $H^{p,q}(M, \mathbb{C})$, and therefore we have a well defined notion of *harmonic part* of a closed form $\alpha \in \Omega^k(M)$ (respectively $\Omega^{p,q}(M)$), which is the unique harmonic representative of the cohomology class $[\alpha]$ in $H^k(M, \mathbb{C})$ (respectively $H^{p,q}(M, \mathbb{C})$.

As a consequence, by using harmonic representative one can get the following Hodge decomposition of the cohomology:

$$H^k(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M, \mathbb{C}). \tag{2.14}$$

The following proposition is also proved by Hodge theory techniques (see Besse [Bes87]).

Proposition 2.19. *If α is an exact form in $H^{p,q}(M, \mathbb{C})$, then there exists a form β in $H^{p-1,q-1}(M, \mathbb{C})$ such that*

$$\alpha = 2i\partial\bar{\partial}\beta.$$

The Ricci form ρ is a real, closed $(1,1)$-form on M, and therefore its difference from its harmonic part ρ^H is a real, exact $(1,1)$-form. We can then apply the proposition above to find a real smooth function $F \in C^\infty(M)$ such that:

$$\rho = \rho^H + 2i\partial\bar{\partial}F. \tag{2.15}$$

The function F is called the *Ricci potential* and it is determined up to an additive constant, if the manifold M is compact. If we require that its integral over M is zero, then the Ricci potential is uniquely determined.

2.10 Differential operators

Definition 2.20. Let E and F be vector bundles over a M. A *differential operator* from smooth sections of E to smooth sections of F is a linear map:

$$C^\infty(M; E) \to C^\infty(M; F),$$

which with a choice of local coordinates can be written as a polynomial in the partial derivatives with coefficients in smooth functions. A differential operator is *of order at most n* if it has a local representation with no terms of degree higher than n in the partial derivatives.

We say that a differential operator is of *finite order* if it is of order at most n for some $n \in \mathbb{N}$, and the space of the differential operators of finite order from smooth sections of E to smooth sections of F is denoted $\mathcal{D}(M, E, F)$, or just $\mathcal{D}(M, E)$, if

$E = F$. The definition immediately specializes to the case in which the vector bundles both coincide with the trivial line bundle $\mathbb{R}$, in which case we just talk about differential operators on M and denote their space by $\mathcal{D}(M)$.

In this thesis we will be mostly interested in the study of the differential operators for which the two vector bundles coincide with a power of a certain line bundle over M that we shall define precisely in the next chapter, namely the prequantum line bundle $\mathcal{L}^k$.

Example 2.21. On the power of the prequantum line bundle $\mathcal{L}^k$ we have the Levi-Civita connection ∇. If $X_1, \ldots, X_n$ are smooth vector fields, then we can define the differential operator inductively on n by:

$$\nabla^n_{X_1,\ldots,X_n} = \nabla_{X_1}\nabla^{n-1}_{X_2,\ldots,X_n} s - \sum_{j=2}^{n} \nabla_{X_2,\ldots,\nabla_{X_1}X_j,\ldots,X_n} s, \tag{2.16}$$

for $n \geq 2$ and $\nabla^1_{X_1} s = \nabla_{X_1} s$. It amounts to an easy calculation to check that these expressions are tensorial in the vector fields, and therefore we get a map

$$\nabla^n \colon C^\infty(M, TM^m) \to \mathcal{D}(M, \mathcal{L}^n).$$

2.10.1 Symbols of differential operators

To any differential operator $D \in \mathcal{D}(M, \mathcal{L}^k)$ of order at most n, we can assign the *principal symbol* $\sigma_P(D) \in C^\infty(M, S^n(TM))$, which is a symmetric section of the n-th tensor power of the tangent bundle. If the principal symbol vanishes, then D is of order at most $n-1$.

To define symbols of all orders we need to use additional structure, namely the covariant derivative in our situation. Given differential operator $D \in \mathcal{D}(M, \mathcal{L}^k)$ of order at most n and with principal symbol $\sigma_P(D) \in C^\infty(M, S^n(TM))$, we look at the differential operator $D - \nabla^n_{S_n}$, where S_n is just another notation for the principal symbol. This differential operator is of order at most $n-1$, since its principal symbol vanishes. If we regard $D - \nabla^n_{S_n}$ as a differential operator of order at most $n-1$, we have its principal symbol in $C\infty(M, S^{n-1}(TM))$, which we denote by S_{n-1}. By iterating this process we can write uniquely D in the form:

$$D = \nabla^n_{S_n} + \nabla^{n-1}_{S_{n-1}} + \cdots + \nabla_{S_1} + S_0,$$

and $S_k \in C^\infty(M, S^k(TM))$ is the *symbol of order k* of D, and it gives rise to a *symbol map*

$$\sigma_k \colon \mathcal{D}(M, \mathcal{L}^k) \to C^\infty(M, S^k(TM)).$$

Note that the values of all the symbol maps determine completely a differential operator of finite order.

2.10.2 Inner product

We shall later consider the adjoints of differential operators with respect to the inner product defined here. If M is a compact manifold, then we can define an inner product on smooth sections of $\mathcal{L}^k$ by

$$\langle s_1, s_2 \rangle := \int_M h(s_1, s_2) \frac{\omega^m}{m!}, \tag{2.17}$$

for any smooth smooth sections s_1 and s_2. For any complex vector field X on M we have then that the adjoint of ∇_X is:

$$(\nabla_X)^* = -\nabla_{\bar{X}} - \delta \bar{X}. \tag{2.18}$$

2.11 Hochschild cohomology

We shall later on consider the Hochschild cohomology of the algebra of smooth functions on our manifold with a star product on it. Hence let us recall here the basic construction of this cohomology theory. For further details we refer the reader to Weinstein and Xu [WX98].

Let N be an associative algebra (not necessarily commutative) over $\mathbb{R}$ or $\mathbb{C}$. To construct the Hochschild cohomology of N with values in N we shall first define what the cochains are: for $p \in \mathbb{N}$, a *p-Hochschild cochain* is a p-linear map

$$N^p = N \times \cdots \times N \to N.$$

These cochains form a complex $C^{\cdot}(N, N)$ with the *Hochschild coboundary operator* $\partial \colon C^p(N, N) \to C^{p+1}(N, N)$, defined by:

$$\begin{aligned}(\partial C)(u_0, \ldots, u_p) &= u_0 C(u_1, \ldots, u_p) \\ &\quad + \sum_{r=1}^{p} (-1)^r C(u_0, \ldots, u_{r-1} u_r, \ldots, u_p) + (-1)^{p-1} C(u_0, \ldots, u_{p-1}) u_p,\end{aligned}$$

for any cochain $C \in C^p(N, N)$.

Example 2.22. We can see that on a 1-cochain F the coboundary operator gives:

$$(\partial F)(u, v) = uF(v) - F(uv) + F(u)v,$$

and on a 2-cochain C it gives:

$$(\partial C)(u, v, w) = uC(v, w) - C(uv, w) + C(u, vw) - C(u, v)w.$$

As usual, a *cocycle* is a p-cochain C such that $\partial C = 0$, and we say that C is a *coboundary* if there exists a $(p-1)$-cochain Q such that $C = \partial Q$.

In our study of Hochschild cohomology we shall limit ourselves to the case in which the algebra N is $C^\infty(M)[[h]]$ for a manifold M, i.e. formal power series of smooth functions on M. We are going to consider a star product on the algebra N, which is a particular non-commutative product that we introduce in Chapter 4. For the time being it is not important how such a product is defined, as here we are only interested in the algebraic aspect of the construction. Let us make these assumptions for the rest of the section.

We say that a p-cochain C is *differential* if it is given by a differential operator in each argument, and *k-differential* if it is differential and each argument is a differential operator of order at most k. A cochain *vanishes on constant* if it gives zero when at least one of the arguments is a constant function on M. Note that the coboundary of a differential cochain is also differential.

We are now ready to define Hochschild cohomology.

Definition 2.23. The p-th *differential Hochschild cohomology* of N is the following space:

$$H^p_{\text{diff}}(N,N) = \frac{\ker(\partial\colon C^p_{\text{diff}}(N,N) \to C^{p+1}_{\text{diff}}(N,N))}{\operatorname{Im}(\partial\colon C^{p-1}_{\text{diff}}(N,N) \to C^p_{\text{diff}}(N,N))},$$

where $C^p_{\text{diff}}(N,N)$ denotes the space of differential Hochschild p-cochains.

If C and D are respectively a p-cochain and a q-cochain, then we can define a $(p+q)$-cochain by:

$$C \otimes D(u_1,\dots,u_{p+q}) = C(u_1,\dots,u_p)D(u_{p+1},\dots,u_{p+q}),$$

and in the tensor product complex we have the coboundary map that is expressed by:

$$\partial(C \otimes D) = \partial C \otimes D + (-1)^p C \otimes \partial D.$$

Note that a k-differential operator D can be seen as a differential k-cochain. If $k=1$, i.e. D is a vector field, then $\partial D = 0$, while for $k \geq 2$ we can see by applying the Leibniz rule several time that the coboundary ∂D is a $(k-1)$-bi-differential operator.

3

Quantization

In this chapter we shall give some basic introductory ideas about what quantization is and what it aims to do. We will look at geometric quantization in some detail, introducing the prequantum condition, polarizations and quantum spaces. We will then mention Toeplitz operators and the Hitchin connection, which will be studied in further detail in the following chapters. Similarly we will briefly introduce deformation quantization, which is going to be the main object of study of the next chapter and indeed of most of the remaining part of this dissertation.

3.1 The idea of quantization

The theory of quantization aims at giving a precise formulation in mathematical terms of the correspondence between classical and quantum mechanics. In classical mechanics we can describe a physical system with a triple $(M, \{\cdot,\cdot\}, H)$ of an even-dimensional Poisson manifold, the *phase-space*, together with a smooth function H on M, the *Hamiltonian function*. These data describe completely a classical system: a point in the manifold corresponds to a physical state and an observable corresponds to a smooth function on M. The time evolution of such an observable is described by the equation:

$$\frac{df}{dt} = \{H, f\}.$$

A quantum mechanics system is described by a complex Hilbert space $\mathcal{H}$ and a Hamiltonian operator $\hat{H}$. Now a physical state is a vector in $\mathcal{H}$, and the physical observables are self-adjoint operators on $\mathcal{L}(\mathcal{H})$, the space of linear operators on $\mathcal{H}$. In this formalism, the equation describing the time evolution of the system is:

$$\frac{d\hat{f}}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{f}\right],$$

where the square bracket is the commutator.

The aim of quantization is to "translate" a classical system into a quantum one, in a well defined mathematical way, in the sense of associating a quantum observable to a classical one.

In mathematical terms, a *quantization scheme* has to satisfy the following: it associates a (projective) Hilbert space $\mathcal{Q}$ to any symplectic manifold (M, ω) and a self adjoint operator $Q(f)$ on $\mathcal{Q}$ to any smooth function f in M, such that the association $f \mapsto Q(f)$ is linear, and $Q(1) = \mathrm{Id}$. Moreover we require that

$$[Q(f), Q(g)] = i\hbar Q(\{f, g\}), \tag{3.1}$$

for any smooth functions f and g, and lastly it should reproduce the canonical quantization in the case M is $\mathbb{R}^{2m}$, with symplectic form

$$\omega = \sum dp_i \wedge dq_i.$$

It turns out that a quantization scheme satisfying all these requirements does not exist (see [AE05]), so we have to loosen some of them. One way of going around this problem is to restrict the set of observables we want to quantize, for instance only certain smooth functions (or observables). This has the side effect of limiting the number of quantizable functions in the quantization schemes that we shall consider here. Another approach is to replace the constant $\hbar$ in (3.1) with a variable h and interpret the requirement as something that should only hold asymptotically as h goes to zero, thereby replacing that equation with:

$$[Q(f), Q(g)] = ihQ(\{f, g\}) + O(h^2) \quad \text{as} \quad h \to 0. \tag{3.2}$$

In what follows we are going to discuss two different approaches to the quantization problem: *geometric quantization* and *deformation quantization*. The first approach is based on the idea of constructing the Hilbert space of quantum states as sections of a certain line bundle over the classical phase space. The latter is based on a different idea: introducing a new product on the space of classical observables, deforming it and reflecting the inherently non-commutative nature of the products in the quantum operators.

For a more extensive treatment of the general theory of quantization we refer the reader to [Woo97]. We shall now review the main points in the constructions of geometric quantization before we switch our focus to deformation quantization, which is the main object of study of this dissertation.

3.2 Geometric quantization

The Hilbert space in geometric quantization consists of the sections of a certain Hermitian line bundle over the phase space. The first step of the construction is called prequantization.

3.2.1 Prequantization

Let (M, ω) be the symplectic manifold of dimension $2m$ representing the classical phase space.

Definition 3.1. A *prequantum line bundle* over a symplectic manifold (M, ω) is the data of a complex line bundle $\mathcal{L}$ with a Hermitian metric h and a compatible connection ∇ whose curvature satisfies:

$$F_\nabla = -i\omega.$$

We say that a symplectic manifold is *prequantizable* if there exists a prequantum line bundle over it. This is not the case for every symplectic manifold; the following lemma gives a condition equivalent to being prequantizable.

Lemma 3.2. *A symplectic manifold (M, ω) is prequantizable if and only if it satisfies the* prequantum condition*, namely:*

$$\left[\frac{\omega}{2\pi}\right] \in \mathrm{Im}(H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{R})).$$

A proof of this lemma can be found in [Woo97], Proposition 8.3.1.

Note that the prequantum condition is clearly necessary, because the first Chern class of a prequantum line bundle is $\tilde{c}_1(\mathcal{L}) = \left[\frac{\omega}{2\pi}\right]$.

Let us now proceed to the construction of a Hilbert space: for any natural number k we define:

$$\mathcal{P}_k = C^\infty(M, \mathcal{L}^k)$$

to be the Hilbert space of quantum states, where we set the inner product:

$$\langle s_1, s_2 \rangle = \int_M h(s_1, s_2) \frac{\omega^m}{m!},$$

where h is the Hermitian metric on $\mathcal{L}$.

For a classical observable $f \in C^\infty(M)$ and $k \in \mathbb{N}$, we define the quantum observable to be the *prequantum operator*

$$\begin{aligned} P_k \colon C^\infty(M) &\to \mathcal{P}_k \\ f &\mapsto \frac{i}{k}\nabla_{X_f} + f, \end{aligned}$$

acting on $\mathcal{P}_k$. The prequantum operator is self-adjoint, as it can easily be checked, and it satisfies

$$[P_k(f), P_k(g)] = \frac{i}{k} P_k(\{f, g\}). \tag{3.3}$$

The natural number k is called the *level* of the quantization. It corresponds to a discrete version of the inverse h^{-1} of the formal parameter, as it can be seen by comparison the equations (3.1) and (3.3).

The previous considerations show that prequantization satisfies all the requirement for quantization, with one exception: prequantization fails to reproduce the canonical quantization of $\mathbb{R}^{2m}$. To realize why prequantization cannot satisfy this requirement, we can see that the Hilbert space that it produces depends on twice as many variables as it would be expected from the canonical quantization, namely $2m$. The way to solve this is to find a way of "choosing" half of the variables, and this is done by picking a *polarization* on M.

The term polarization is inspired by the physical phenomenon of polarization, where one selects only the waves that oscillate with a specific orientation from an electromagnetic radiation. In our mathematical formalism, we can choose polarizations in several ways, as explained in [Woo97]. In what follows we shall only concentrate on a specific type of them, namely complex polarizations.

3.2.2 Complex polarizations

Let (M, ω) denote our symplectic manifold that is prequantizable, and let us choose an almost complex structure I on it that is compatible and integrable, so that we obtain a Kähler manifold, which we denote by M_I. As pointed out earlier in Section 2.5, the Kähler form ω has type $(1,1)$, and therefore it gives a holomorphic structure to the prequantum line bundle $\mathcal{L}$ on M. Now we can use the complex structure to choose in a rather "natural" way a subset of half of the coordinates: let us define the space of quantum spaces to be the space of holomorphic sections of the tensor power of the line bundle:

$$\mathcal{Q}_k(I) = H^0(M_I; \mathcal{L}^k) = \{s \in \mathcal{P}_k \mid \nabla_X s = 0,\ \forall\ X \in T''M_I\}.$$

Note that this defines a subspace of the prequantum space $\mathcal{P}_k$, and moreover $\mathcal{Q}_k(I)$ is finite dimensional whenever the manifold M is compact, and its dimension is n, which is what we wanted to obtain. On the other hand, one sees that the prequantum operators do not generally preserve holomorphic sections, therefore when we apply them to an observable we might get a section which does not belong to our space.

One way of approaching this issue is to reduce our set of quantizable functions. In fact, the prequantum operator P_f preserves the space $\mathcal{Q}_k(I)$ if and only if f satisfies:

$$[X_f, T''M_I] \subseteq T''M_I.$$

If we decide only to quantize the functions that satisfy the previous relation, then the spaces $\mathcal{Q}_k(I)$ will be preserved. On the other hand, if we choose this approach we are limiting our set of quantizable functions strongly, because the requirement implies that the Hamiltonian vector field for a quantizable function f is a Killing vector field of the Kähler metric, and therefore the space of quantizable functions will be at most of finite dimension and often trivial (see [Woo97]).

We shall take a different approach to the issue: let us note that the space $\mathcal{Q}_k(I)$ is a closed subspace of $\mathcal{P}_k$, as proved for instance in [Woo97], and therefore the orthogonal projection

$$\pi_k(I)\colon \mathcal{P}_k \to \mathcal{Q}_k(I)$$

is well defined. Given a classical observable $f \in C^\infty(M)$ we can then define its quantization by the formula:

$$Q_{k,I}(f) = \pi_k(I) \circ P_k(f). \tag{3.4}$$

The quantum operators that we obtain this way will not generally form an algebra, but they satisfy (3.2) when M is compact, since:

$$\left\| [Q_{k,I}(f), Q_{k,I}(g)] - \frac{i}{k} Q_{k,I}(\{f,g\}) \right\| = O(k^{-2}) \qquad \text{as } k \to \infty, \tag{3.5}$$

where the norm is the operator norm on $\mathcal{Q}_k(I)$. The fact that the last equation is true is part of the theory of Toeplitz operators, which we shall look at in greater detail in Chapter 5.

The Hilbert space we obtain in this quantization scheme has the right dimension, but still one needs to apply a correction to get the right answers, when we compare with the results in examples from quantum mechanics. In the case of the one dimensional harmonic oscillator, this quantization scheme fails to produce the right answer by a shift, and to solve this issue one can apply what is called metaplectic correction, which we shall not treat in this dissertation. We refer the reader to [Gam10] for further details on this aspect.

3.2.3 Toeplitz operators

Definition 3.3. Let $f \in C^\infty(M)$. The *Toeplitz operator* $T_f^{(k)}\colon \mathcal{P}_k \to \mathcal{Q}_k$ is the map defined by

$$T_f^{(k)}(s) = \pi^{(k)}(fs),$$

mapping a smooth section s on $\mathcal{L}^k$ to its projection to the subspace of holomorphic sections.

Even if this is not evident from the notation, a Toeplitz operator depends on the complex structure on the manifold. Note that the definition of these operators does not require the manifold to be compact, but in case this is true, Bordemann, Meinrenken and Schlichenmaier [BMS94] proved interesting and strong results about the asymptotic properties of Toeplitz operator: therefore let us put ourselves in this assumption from now on.

The space $\mathcal{P}_k$ is equipped with the operator norm associated with the Hermitian inner product

$$\langle s, r\rangle = \int_M h(s,r)\frac{\omega^m}{m!},$$

and the Toeplitz operators restrict to endomorphisms of the finite-dimensional subspace $\mathcal{Q}_k$.

The following results says that the if we consider all the Toeplitz operators associated to a function we get a faithful representation of it.

Theorem 3.4. *Let $f \in C^\infty(M)$ be a smooth function on M. Then if we take the limit of the operator norm of its Toeplitz operators we get:*

$$\lim_{k\to\infty} \left\| T_f^{(k)} \right\| = \|f\|_\infty,$$

and this limit is approached from below.

This theorem has been proved by Bordemann, Meinrenken and Schlichenmaier [BMS94].

Note that the Toeplitz operators do not form an algebra, since the composition of two of them does not give a Toeplitz operator in general. Nevertheless, if we look at the product of two Toeplitz operators asymptotically, then we see that it can be approximated by operators of the same kind.

We shall say more about this when we discuss the Berezin-Toeplitz deformation quantization in Chapter 4, but for the time being we just state the following theorem about the commutator of two Toeplitz operators.

Theorem 3.5. *The Toeplitz operators satisfy*

$$\left\| \left[T_f^{(k)}, T_g^{(k)} \right] - \frac{i}{k} T_{\{f,g\}}^{(k)} \right\| = O(k^{-2}) \quad as \quad k \to \infty,$$

for any smooth functions $f, g \in C^\infty(M)$.

It follows that the Toeplitz operators satisfy (3.1) in the precise meaning of the theorem. The quantization scheme that uses the Toeplitz operators to quantize observables is called Berezin-Toeplitz quantization, and coincides with the quantum operators from geometric quantization that we defined in Chapter 3.

When M is a compact manifold, the Toeplitz operators are related to the quantum operators Q_k by the following theorem.

Theorem 3.6 (Tuynman)**.** *If M is compact, the quantum operators are Toeplitz operators and satisfy*

$$Q_k(f) = T_{\alpha(f,k)}^{(k)},$$

for

$$\alpha(f,k) = \frac{1}{2k}\Delta f + f$$

for any smooth function $f \in C^\infty(M)$ and $k \in \mathbb{N}$.

Note that it was shown in general in [BMS94] that the maps $T^{(k)}$ are surjective for any level k on a compact manifold. Tuynman's result gives us an explicit expression for the family of functions f_k whose Toeplitz operators are the quantum operators Q_k.

As an easy consequence of Tuynman's theorem and Theorem 3.5, the quantum operators also satisfy the condition (3.1).

Theorem 3.7. *The quantum operators satisfy*

$$\left\| [Q_k(f), Q_k(g)] - \frac{i}{k} Q_k(\{f,g\}) \right\| = O(k^{-2}) \quad as \quad k \to \infty,$$

for any smooth functions $f, g \in C^\infty(M)$.

Proposition 3.8. *If $X \in C^\infty(M, T'M)$ is a smooth section of the holomorphic tangent bundle on M, then we have*

$$\pi_k(\nabla_X s) = -T_{\delta(X)}^{(k)} s,$$

for any smooth section $s \in \mathcal{P}_k$.

Proof. The result follows by partial integration. If $s' \in \mathcal{Q}_k$ is any holomorphic section, then

$$X[h(s,s')] = h(\nabla_X s, s') + h(s, \nabla_{\bar{X}} s') = h(\nabla_X s, s'),$$

since s' is holomorphic and $\bar{X}$ is antiholomorphic. □

3.3 The Hitchin connection

The choice of a complex polarization involves the choice of the Kähler structure I, therefore the resulting quantum spaces $\mathcal{Q}_k(I)$ will be dependent on that choice as well. This is of course not justified from a physical perspective.

The Hitchin connection is a construction that aims to handle this apparent choice-dependence. For any Kähler structure I we see the spaces $\mathcal{Q}_k(I)$ as the fibres of a vector bundle over the space of Kähler structures. The strategy is to try to relate the fibres through the parallel transport of this connection. If the connection is flat, then the fibres can be identified in a canonical way.

We shall see in detail the construction of the Hitchin connection in Chapter 5.

3.4 Deformation quantization

Deformation quantization takes a different approach from that of geometric quantization: instead of trying to construct a Hilbert space of quantum states, in deformation quantization we construct a deformation of the algebra of the smooth functions, i.e. the classical observables. To do so, one looks at a family of *star products* $\{\star_h\}_{h\in\mathbb{R}}$, on the algebra $C^\infty(M)$. The family is parametrized by the real parameter h, in such a way that the star product corresponds with the pointwise product for $h = 0$. The product of two observables f and g can be written as a power series in h:

$$f \star_h g = \sum_{i\in\mathbb{N}} c_i(f,g)h^i, \tag{3.6}$$

with $c_0(f,g) = fg$.

While we shall postpone the precise definition of a star product to the next chapter, we can already see a correspondence between deformation quantization and geometric quantization: if we think in terms of quantum operators, the star product corresponds to the composition of them:

$$Q_f Q_g = Q_{f\star_h g},$$

therefore it is non-commutative. Looking at (3.2), we see that it gives a relation between the star-commutator and the Poisson bracket at first order in h, which is expressed by the equation:

$$c_1(f,g) - c_1(g,f) = i\{f,g\}.$$

When working with deformation quantization we can neglect completely the construction of the Hilbert space of quantum states, and only define observables using the star product, as done in [BFF+78].

The question of whether (3.6) is convergent is usually not answered in the general study of deformation quantization: there we take a different approach, and regard h solely as a *formal parameter*, thereby considering the former expression as a formal power series. Note that in this way we are enlarging the algebra of smooth functions on which the star product is defined to the algebra $C^\infty(M)[[h]]$ of the *formal* smooth functions.

Having in mind geometric quantization, we can construct a certain star product, the Berezin-Toeplitz one, by using the theory of Toeplitz operators, as showed by Schlichenmaier in [Sch11].

3.5 The formal Hitchin connection

Similarly to what we noted for geometric quantization, to construct star products like the Berezin-Toeplitz one we have to make a choice of a complex structure on the manifold, because the construction of this kind of star products relies on the presence of a Kähler structure. Again, the fact that we have to make a choice does not make sense physically, since the resulting quantization should not be dependent on any choice of extra structure.

Andersen [And12] proposed a way to approach this problem in a similar fashion to what one does when connecting the quantum spaces of geometric quantization with the Hitchin connection. In fact he introduced a *formal Hitchin connection* in the vector bundle with fibres $C^\infty(M)[[h]]$ over the base space parametrizing Kähler structures on M, which we shall define in a precise way in Chapter 7. The goal is to use the parallel transport of this connection to identify the different star products.

We shall look at the construction of the formal Hitchin connection and formal connections in general in Chapter 7.

4

Deformation quantization

The idea of deformation quantization, due to Bayen, Flato and others [BFF$^+$78], is to deform the algebra of functions $C^\infty(M)$ (where the product is the pointwise product of functions) into a non-commutative one. To do this one defines a new product, called star product. Star product is used as a synonym of deformation quantization.

4.1 Star products

Let M be a Poisson manifold. Recall that $\mathbb{C}[[h]]$ denotes the ring of formal power series with complex coefficients, and similarly $C_h^\infty(M) = C^\infty(M)[[h]]$ is the algebra of *formal functions* on M, which are formal power series with coefficients in $C^\infty(M)$. $C_h^\infty(M)$ is then an algebra over $\mathbb{C}[[h]]$, and we can extend the Poisson bracket linearly to make it into a Poisson algebra. This allows us to formulate the following definition.

Definition 4.1. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold. A (formal) *star product* (or *deformation quantization*) on M is $\mathbb{C}[[h]]$-bilinear map $\star\colon C_h^\infty(M) \times C_h^\infty(M) \to C_h^\infty(M)$ written as

$$f \star g = \sum_{k=0}^{\infty} c^k(f,g)h^k,$$

where $c^k\colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, for $k \in \mathbb{N}$, are bilinear maps called *coefficients*. A star product is required to satisfy the following conditions:

1. associativity: $(f_1 \star f_2) \star f_3 = f_1 \star (f_2 \star f_3)$,
2. $c^0(f_1, f_2) = f_1 f_2$,
3. $c^1(f_1, f_2) - c^1(f_2, f_1) = i\{f_1, f_2\}$,

for all $f_1, f_2, f_3 \in C^\infty(M)$.

As remarked earlier, deformation quantization takes the formal point of view, meaning that the star product is defined on the algebra of formal functions and we do not look at convergence matters. Nevertheless we shall still call the product a star product, omitting the word “formal” for simplicity.

Remark 4.2. There are several different conventions in literature about the multiplicative constant in front of the Poisson bracket in (3). For instance Schlichenmaier [Sch11] uses the opposite constant $-i$. In this thesis we follow the convention of the definition above.

We see that the failure of the star product to be commutative is measured by the Poisson bracket. It is often useful to impose some extra conditions on star products.

Definition 4.3. A star product is said to be *differential* if the coefficients c^k are bidifferential operators, in the sense that, for a fixed $f \in C^\infty(M)$, both $c^k(f, \cdot)$ and $c^k(\cdot, f)$ are differential operators for all $k \in \mathbb{N}$. It is *null on constants* if $f \star 1 = 1 \star f = f$ for all $f \in C^\infty(M)$ (or, equivalently, if $c^k(1, f) = c^k(f, 1) = 0$ for all $k \geq 1$).

In this thesis we shall always assume that star products are null on constants, and this corresponds to the fact that the constant function 1 is the unit in the algebra of formal functions on M with the star product.

Remark 4.4. A star product is said to be *local*, if

$$\operatorname{supp} c^k(f, g) \subseteq \operatorname{supp} f \cap \operatorname{supp} g,$$

for any $f, g \in C^\infty(M)$, and in this case the global star product defines a star product on the algebra $C^\infty(U)$ for any open subset $U \subseteq M$.

It can be proven that a local star product is differential, which is not a trivial fact (see [CGDW80]).

Definition 4.5. A differential star product is said to be *with separation of variables (of anti-Wick type)* if $f \star h = fh$ and $h \star g = hg$ for any locally defined functions f and g with f holomorphic and g anti-holomorphic, and any function h.

Remark. Several papers in literature (e.g. [KS01]) consider the opposite notion to the definition of separation of variables we have defined above, namely star products where $f \star h = fh$ and $h \star g = hg$ for any locally defined functions f and g with f anti-holomorphic and g holomorphic, and any function h. These products are usually called *of Wick type*, and consequently, our products with separation of variables are also called *of anti-Wick type*.

The following proposition is an immediate consequence of the definition.

Proposition 4.6. *A differential star product $\star$ is with separation of variables if and only if the bidifferential operators c^k, for $k \geq 1$ only differentiate in anti-holomorphic derivatives in the first argument and only holomorphic derivatives in the second argument.*

4.2 Equivalence

Definition 4.7. Two star products $\star, \star'$ on M are said to be *equivalent* if there is a formal power series of linear maps

$$T = \sum_{k=0}^{\infty} T_k \qquad T_k \colon C_h^\infty(M) \to C_h^\infty(M), \quad k \in \mathbb{N},$$

such that $T_0 = \mathrm{Id}$ and $T(f_1) \star' T(f_2) = T(f_1 \star f_2)$, for f_1, f_2 smooth functions on M.

If the operators T_k are differential for every k, then the equivalence is called *differential*. An equivalence of two differential star product needs to be a differential equivalence, as showed in [GR99]. On the other hand, a star product equivalent to a product with separation of variables need not have the same property.

4.3 Existence of deformation quantizations

The problem of the existence of a star product is far from being trivial. In the symplectic case we have a classification of a family of star products, namely those that are differential and have separation of variables. These products can be written down explicitly, as we shall see in Chapter 6.

Originally the problem of existence of deformation quantizations in the symplectic case was solved by De Wilde and Lecomte in [DWL83] with a cohomological approach. Fedosov [Fed94] gave a more geometrical construction of star products, and a classification, which we shall look at further on, when we will study formal connections in Chapter 7.

In the general Poisson case, the question of existence was settled by Kontsevich, and is the Formality theorem [Kon03].

4.4 Classification of natural star products

We shall now introduce a particular type of differential star products that will be used later in Chapter 7.

Definition 4.8. A *natural star product* $\star = \sum_{k \in \mathbb{N}} c^k$ on a Poisson manifold M is a differential star product null on constants such that the k-th coefficient is a bidifferential operator of order at most k in each argument.

Remark 4.9. All the star products that we have considered so far that are constructed explicitly are natural, including:

- star products with separation of variables on a Kähler manifold (e.g. the Berezin-Toeplitz star product,
- star products arising from the Fedosov construction (see Chapter 7 for details),
- star products constructed by Kontsevich's formality [Kon03].

Gutt and Rawnsley in [GR03] gave a classification of natural star products on a symplectic manifold. The following theorem is the main classification result they obtained.

Theorem 4.10. *A natural star product $\star$ on a symplectic manifold (M, ω) determines uniquely:*

- *a symplectic connection ∇,*
- *a formal series of closed 2-forms $\alpha \in h\Lambda^2(M)[[h]]$,*

- *a formal series*
$$E = \sum_{k\geq 1} E_r h^r,$$
of differential operators such that
$$E_r u = \sum_{k=2}^{r+1} \left(E_r^{(k)}\right)^{i_1\dots i_k} \nabla^k_{i_1\dots i_k} u,$$
where, for any r, the $E_r^{(k)}$ are symmetric k-tensors.

Moreover the star product satisfies:
$$f \star g = \exp(-E)((\exp Ef) \star_{\nabla,\alpha} (\exp Eg)),$$
where $\star_{\nabla,\alpha}$ denotes the Fedosov star product corresponding to ∇ and α, which we shall construct later in this chapter.

4.5 Characteristic class

In this section we review Fedosov's construction of the characteristic class for a star product, following the presentation by Waldmann in [Wal07].

Let (M, ω) be a symplectic manifold of dimension $m = 2n$. To any tangent space T_xM, for $x \in M$, we have the associated Weyl algebra.

Recall that a *multi-index* is a tuple
$$\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m.$$
The *length* of a multi-index is $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and we can extend the factorial to multi-indices by: $\alpha! = \alpha_1! \cdots \alpha_m!$. We use the notation $y^\alpha = (y^1)^{\alpha_1} \cdots (y^m)^{\alpha_m}$.

Definition 4.11. The *formal Weyl algebra* W_x associated to T_xM, for $x \in M$, is an associative $\mathbb{C}$-algebra with unit whose elements are formal power series in h with formal polynomials on T_xM as coefficients. This means that an element in the algebra has the form:
$$a(y, h) = \sum_{k\in\mathbb{N}} h^k a_{k,\alpha} y^\alpha,$$
where $(y^1, \dots, y^m)$ are local coordinates on T_xM, $\alpha = (\alpha_1, \dots, \alpha_m)$ is a multi-index.

The formal Weyl algebra is equipped with the following *Moyal-Weyl star product*:
$$a \circ_{MW} b = \sum_{k=0}^{\infty} \left(\frac{ih}{2}\right)^k \frac{1}{k!} \pi^{i_1 j_1} \dots \pi^{i_k j_k} \frac{\partial^k a}{\partial y^{i_1} \cdots \partial y^{i_k}} \frac{\partial^k b}{\partial y^{i_1} \cdots \partial y^{i_k}}. \tag{4.1}$$

Let $W = \cup_{x\in M} W_x$. This defines a bundle of algebras over M, which is called the *Weyl bundle*. The space of smooth sections of this bundle, ΓW, gives an associative algebra with fibre-wise multiplication. This space of section can be thought of as a "quantized tangent bundle" of M. One can check that the fibrewise product that we have defined gives a deformation quantization with respect to the Poisson structure given by fibrewise Poisson bracket, with constant symplectic structure in each tangent space.

Note that the centre of ΓW is formed by the elements that do not contain any y^i, and therefore is naturally identified with $C^\infty(M)[[h]]$.

We can use the Weyl algebra to define a filtration. Let us assign degrees to the y^i's by $\deg y^i = 1$ for any i, and $\deg h = 2$.

$$C^\infty(M)[[h]] \subset \Gamma(W_1) \subset \cdots \subset \Gamma(W_i) \subset \cdots \subset \Gamma(W).$$

A differential form on M with values in W is a section of $W \otimes \Lambda^q T^*M$, and can be expressed as:

$$a(x,y,h,dx) = \sum h^k a_{k,i_1,\dots,i_p,j_1,\dots,j_q} y^{i_1} \dots y^{i_p} dx^{j_1} \wedge \cdots \wedge dx^{j_q},$$

in local coordinates, where the coefficients $a_{k,i_1,\dots,i_p,j_1,\dots,j_q}$ are symmetric in the i's and anti-symmetric in the j's.

Note that the fibrewise commutator in ΓW extends to $W \otimes \Lambda^q T^*M$. We can use the exterior derivative to define an operator δ on W-valued differential forms in the following way:

$$\delta(a) = \sum_i dx^i \wedge \frac{\partial a}{\partial y^i}, \qquad \text{for all } a \in W \otimes \Lambda^q T^*M.$$

This operator can be also written as

$$\delta(a) = -\left[\frac{i}{h}\omega_{ij} dy^i dx^j, a\right],$$

where the commutator is with respect to the Moyal-Weyl star product, and we can note that the operator δ does not involve derivatives with respect to x.

The operator δ can also be seen in a different way: if we consider the algebra of differential forms $(\Omega(M, W(TM))$ with values in $W(TM)$, we can define two commuting derivations δ and δ^* by considering the identity morphism $TM \to TM$ as a section of $TM \otimes T^*M$. We can now insert the TM-part into either ΛT^*M or $W(TM)$ and multiply the T^*M-part with the other factor, using the pointwise product on $W(TM)$. The operator δ corresponds to the latter case, while δ^* corresponds to the former. The symplectic form $\omega \in \Omega^2(M)$ on M can be seen as a section of

$$T^*M \otimes T^*M \subset \Lambda T^*M \otimes W(TM),$$

which we denote by $\tilde{\omega}$. The derivation δ can then be also written as

$$\delta = -\frac{i}{h}\,\mathrm{ad}(\tilde{\omega}),$$

where ad refers to the adjoint action with respect to the Moyal-Weyl product $\circ_{MW}$.

Definition 4.12. A *symplectic connection* on a symplectic manifold (M, ω) is a linear connection ∇ that is torsion-free and such that ω is parallel with respect to it, meaning that $\nabla\omega = 0$.

Let us fix a symplectic connection ∇ on M. Its curvature tensor is contracted with ω to yield an element $R \in \Omega^2(M, W(TM))$. Moreover, this connection yields a covariant derivative d_∇ on $\Omega(M, TM)$, which we dualize and extend – as a derivation with respect to $\circ_{MW}$ – to $\Omega(M, W(TM))$. It turns out that d_∇ commutes with δ and squares to $-\frac{i}{h}\,\mathrm{ad}(R)$.

Fedosov's idea is to find an appropriate $r \in \Omega^1(M, W(TM))$ such that the total operator

$$D_r := -\delta + d_\nabla + \frac{i}{h}\,\mathrm{ad}(r) \tag{4.2}$$

squares to zero. Note that D_r defines a connection on W. This connection is clearly a derivation of the Moyal-Weyl product defined above, meaning that the following relation holds:

$$D_r(a \circ_{MW} b) = D_r(a) \circ_{MW} b + a \circ_{MW} D_r(b).$$

We can now compute the square to D_r, which gives:

$$D_r^2 = \frac{i}{h}\,\mathrm{ad}\left(-\omega - \delta r + R + d_\nabla r + \frac{i}{h} r \circ_{MW} r\right),$$

where the tensor R is defined by

$$R = \frac{1}{4} R_{ijkl} y^i y^j dx^k \wedge dx^l,$$

where R_{ijkl} is the curvature tensor of the symplectic connection.

Hence the flatness of D_r is equivalent to the fact that

$$\alpha = -\omega - \delta r + R + d_\nabla r + \frac{i}{h} r \circ_{MW} r \tag{4.3}$$

is central. In fact a connection written in the form (4.2) is called *abelian* if α is a scalar 2-form, meaning that $\alpha \in \Omega^2(M)[[h]]$. If this is the case for a connection D_r, then by the Bianchi identity we have that $d\alpha = D_r\alpha = 0$, as shown in Lemma 6.4.12 of [Wal07], and so α is closed (i.e. $\alpha \in Z^2(M)[[h]]$) and is called the *Weyl curvature*.

The following theorem shows how to construct the appropriate r to plug into D_r.

Theorem 4.13. *(Fedosov) Let ∇ be a symplectic connection on M and*

$$\alpha = \omega + h\alpha_1 + h^2\alpha_2 + \cdots \in Z^2(M)[[h]]$$

be a closed formal 2-form that is a perturbation of the symplectic form. Then there exists a unique $r \in W_2 \otimes \Lambda T^*M$, such that the D_r given by (4.2) *is an abelian connection with Weyl curvature α and $\delta^* r = 0$.*

The connection given by the theorem is called the *Fedosov connection*.

Definition 4.14. The *characteristic class* of a star product $\star$ is the class

$$\mathrm{cl}(\star) = [\alpha] \in H^2(M; \mathbb{R})[[h]],$$

where α is the Weyl curvature of the corresponding Fedosov connection D_r.

We denote with $H^2(M;\mathbb{R})$ the de Rham second cohomology, and $H^2(M)[[h]]$ is formal power series with de Rham classes as coefficients, called *formal de Rham classes*.

Note that if M is contractible, then $H^2(M)$ is trivial and there is only one differential star product up to equivalence. A consequence of this is that locally all differential star products are equivalent to the Moyal–Weyl product. For a more extended treatment of classification results, we refer the reader to [Del95], [GR99], [Fed94].

Theorem 4.13 is proved in [Wal07, 6.4.14], and here we shall briefly describe the main ideas that go into the proof.

Proof. Recall that we have defined the total degree of an element of

$$\Omega(M, W(TM))[[h]]$$

to be the polynomial degree in $W(TM)$ plus two times the degree in h. One checks that δ decreases this degree, while d_∇ and $\circ_{MW}$ preserve it. We denote the homogeneous part of total degree k of an element X by $X_{(k)}$.

One can then decompose the equation for δr into the homogeneous pieces with respect to the total degree. By induction, one shows that

$$R_{(k)} + (d_\nabla r)_{(k)} + \frac{i}{h}(r \circ_{MW} r)_{(k)} + \alpha_{(k)}$$

is closed with respect to δ. Since the cohomology of δ vanishes in all relevant ranges, the equation

$$\delta X = R_{(k)} + (d_\nabla r)_{(k)} + \frac{i}{h}(r \circ_{MW} r)_{(k)} + \alpha_{(k)}$$

has a solution, which is moreover unique once we impose $\delta^* r = 0$. So we can iteratively construct r by adding the solution to these equations as its homogeneous part of total degree k. $\square$

4.6 Fedosov star product

To obtain the Fedosov star product, we build up on the construction seen in the previous section. The next step is to show that every element $f \in C^\infty(M)[[h]]$ can be uniquely extended to an element

$$\tau(f) \in \Omega^0(M, W(TM))[[h]]$$

which is constant with respect to the operator D_r. Again, the proof amounts to breaking up the equation $D_r\tau(f) = 0$ into its homogeneous pieces with respect to the total degree. We can rewrite the equation as

$$\delta(\tau(f)) = d_\nabla(\tau(f)) + \frac{i}{h}[\mathrm{ad}(r), f],$$

and check inductively that the right hand side is closed with respect to δ, hence we can solve the equation. The requirement for $\tau(f)$ is that:

$$\tau(f) \in \Omega^0(M, W(TM))[[h]]$$

determines the equation in each step uniquely. The element $\tau(f)$ is also uniquely characterised by $D_r\tau(f) = 0$ and $\sigma(\tau(f)) = f$, where σ is the natural projection

$$\sigma : \Omega(M, W(TM))[[h]] \to C^\infty(M)[[h]].$$

We are now ready to define the *Fedosov star product* $\star_{\nabla,\alpha}$ associated to the closed formal 2-form α and to the symplectic connection ∇ on M, which is given by the following formula:

$$f \star_{\nabla,\alpha} g := \sigma(\tau(f) \circ_{MW} \tau(g)). \tag{4.4}$$

Since D_r is a derivation with respect to $\circ_{MW}$, this defines an associative product and one checks inductively that one actually obtains a natural star product, i.e. $\star_{\nabla,\alpha}$ is given by bidifferential operators that at order h^k have order at most k in each argument.

4.7 Karabegov's classification

Karabegov provides us with a classification of formal star products with separation of variables (of Wick type) on Kähler manifolds. The classification is based on a deformation of the structure of the manifold, i.e. the Kähler form, and it is important to note that it gives a bijection with the actual star products not up to equivalence, as it will be explained in this section.

Let M be a Kähler manifold with Kähler metric g, and let us denote the Kähler form by ω_{-1}.

Definition 4.15. A *formal deformation* of the Kähler form ω_{-1} is a formal 2-form

$$\omega = \omega_{-1}\frac{1}{h} + \omega_0 + \omega_1 h + \omega_2 h^2 + \ldots, \tag{4.5}$$

where each ω_i is a closed 2-form on M of type $(1,1)$.

Karabegov's classification [Kar96] assigns to every star product $\star$ with separation of variables (of Wick type) on M a formal deformation of the Kähler form in the following way. Let $U \subset M$ be a contractible open subset of M, with local holomorphic coordinates $z^1, \ldots, z^m$. Then there exists a set of formal functions $\Psi^1, \ldots, \Psi^m$ on U, written as:

$$\Psi^k = \Psi^k_{-1}\frac{1}{h} + \Psi^k_0 + \Psi^k_1 h + \ldots,$$

satisfying the equations:

$$\Psi^k \star z^j - z^l \star \Psi^k = \delta^{kl}.$$

Then the formal 2-form ω is defined by:

$$\omega|_U = -i\bar{\partial}\left(\sum_{k=1}^m \Psi^k dz^k\right). \tag{4.6}$$

As showed by Karabegov [Kar96], the resulting 2-form is independent of the choice of the local coordinates and of the formal functions Ψ^k. The local expressions for the 2-forms can be pasted together giving a globally defined 2-form ω which takes the name of *Karabegov form* of the star product $\star$, denoted $\mathrm{Kar}(\star)$.

Theorem 4.16 (Karabegov)**.** *A differential star product null on constants and with separation of variables (of Wick type) on a Kähler manifold is totally determined by its Karabegov form.*

Let us note that, even though Karabegov's result is stated and proved in [Kar96] for star products of Wick type, one can get the corresponding result for star products of anti-Wick type by considering the opposite star product. We can define the *opposite* star product by $f \star_{op} g = g \star f$, for any star product, and this gives a star product on the same manifold with opposite symplectic form.

As mentioned, the Karabegov form identifies the product completely, not up to equivalence. Given a formal deformation ω of the Kähler form, then there is a unique star product with separation of variables satisfying:

$$\mathrm{Kar}(\star) = \omega.$$

We call any product that is classified this way a *Karabegov star product*. Karabegov proved in [Kar98] that the Karabegov form of a product is related to the characteristic class. The relation originally obtained by Karabegov contained a sign error, which was later corrected in [KS01], giving the following relation[1]

$$\mathrm{cl}(\star) = \frac{[\mathrm{Kar}(\star)]}{2\pi} - \frac{\tilde{c_1}(M)}{2},$$

where $\tilde{c_1}(M)$ is the real first Chern class of the manifold. It is then clear that two star products with separation of variables are equivalent if and only if their Karabegov classes are the same in cohomology.

4.8 The Berezin-Toeplitz deformation quantization

We conclude this chapter by introducing the Berezin-Toeplitz star product, which is related with geometric quantization and Toeplitz operators.

The statement of Theorem 3.5 can be generalised to describe asymptotically the product of two Toeplitz operators, and it turns out that this produces functions that satisfy the properties of the coefficients of a star product, which is called the *Berezin-Toeplitz star product*. On a compact Kähler manifold M that satisfies the hypotheses of Theorem 5.18, the star product is characterized by the following result, proved by Schlichenmaier [Sch00].

Theorem 4.17 (Schlichenmaier)**.** *There exists a unique star product $\star^{BT}$ for M, called the* Berezin-Toeplitz star product*, and expressed by:*

$$f_1 \star^{BT} f_2 = \sum_{k=0}^{\infty} c^{(k)}(f_1, f_2) h^k,$$

with $c^{(k)}(f_1, f_2) \in C^\infty(M)$ in such a way that, for all $f_1, f_2 \in C^\infty(M)$ and for any positive integer L the following holds:

$$\left\| T^{(k)}_{f_1,\sigma} T^{(k)}_{f_2,\sigma} - \sum_{l=0}^{L} T^{(k)}_{c^{(l)}_{\sigma}(f_1,f_2),\sigma} k^{-l} \right\| = O(k^{-(L+1)}).$$

[1] Recall that in our convention (2.4) the first Chern class of a symplectic manifold is the opposite of the first Chern class of the canonical line bundle.

Karabegov and Schlichenmaier proved furthermore [KS01] that this star product is differential and null on constants. Hawkins proved in [Haw00] that the characteristic class of this star product is:

$$\mathrm{cl}(\star^{BT}) = \frac{[\omega]}{2\pi h} - \frac{\tilde{c}_1(M)}{2}$$

Remark 4.18. Karabegov and Schlichenmaier [KS01] proved that this star product is of Wick type, namely it has the property of separation of variables, with the role of the holomorphic and anti-holomorphic functions switched with respect to Definition 4.5. If consider the opposite star product to the Berezin-Toeplitz star product, we obtain a star product, which is of anti-Wick type and has the Karabegov class, as it was proved in [KS01]:

$$\mathrm{Kar}(\star_{op}^{BT}) = -\frac{\omega}{h} + \rho,$$

where ρ is the Ricci form. We shall see in the next chapter how this star product can be represented combinatorially with graphs.

5

The Hitchin connection

We review here the construction of the Hitchin connection, in the differential geometric version of Andersen in [And12]. For more details the reader is referred also to [Woo97].

The Hitchin connection was introduced by Hitchin in [Hit90], as a connection over the Teichmüller space in the bundle one obtains by applying geometric quantization to the moduli spaces of flat $SU(n)$ connections. Furthermore Hitchin proved that this connection is projectively flat. Hitchin's construction was motivated by Witten's study of quantum Chern-Simons theory in $2+1$ dimensions in [Wit89]. In [And12] Andersen proposed a differential geometric construction of the Hitchin connection which works for a more general class of manifolds and showed its existence under certain assumptions on the manifolds.

We will begin the construction by introducing all the tools we need to define the Hitchin connection, which will be finally constructed in Section 5.7.

Let (M, ω) be a symplectic manifold that is prequantizable, with prequantum line bundle: $(\mathcal{L}, (\cdot, \cdot), \nabla)$. Recall that the connection is compatible with the Hermitian structure in the sense that, for any vector field X on M, and any two section s_1, s_2 of $\mathcal{L}$, we have

$$X(s_1, s_2) = (\nabla_X(s_1), s_2) + (s_1, \nabla_X(s_2)).$$

In general the curvature is a 2-form in the endomorphism bundle $\operatorname{End}(\mathcal{L})$ and, since we are considering a line bundle, the endomorphism bundle is $M \times \mathbb{C}$, therefore the curvature can be seen as a 2-form on M with values in $\mathbb{C}$.

For the time being we consider a fixed Kähler structure I on M, and define:

$$\mathcal{P}_k = C^\infty(M, \mathcal{L}^k),$$

where $\mathcal{L}^k$ denotes the k-th tensor power of the prequantum line bundle.

The Levi-Civita connection ∇ on M splits into $\nabla = \nabla^{1,0} + \nabla^{0,1}$, where

$$\nabla_X^{1,0} = \nabla_{X'}, \qquad \nabla_X^{0,1} = \nabla_{X''}.$$

The splitting means that, for example, if s is a smooth section of $\mathcal{L}^k$, then $\nabla^{0,1}s$ is the derivative of s in the anti-holomorphic direction. We define then

$$\mathcal{Q}_k = \{s \in \mathcal{P}_k \mid \nabla^{0,1}s = 0\},$$

which is the space of the holomorphic sections of $\mathcal{L}^k$.

5.1 Families of Kähler structures

Now, instead of choosing a fixed Kähler structure, we will consider a family of them parametrized by a manifold. Let $\mathcal{T}$ be a smooth manifold that parametrizes smoothly a family of Kähler structures on (M, ω). This means that we have a smooth map

$$I\colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$$

which associates to each $\sigma \in \mathcal{T}$ an integrable and compatible almost complex structure on M. The requirement that the map I is smooth means that it defines a smooth section of the pullback bundle

$$\pi_M^*(\mathrm{End}(TM)) \to \mathcal{T} \times M,$$

where $\pi_M\colon \mathcal{T} \times M \to M$ denotes the canonical projection map.

If we have a symmetry group Γ acting on M, we require that is acts on $\mathcal{T}$ too, and that the map I is equivariant with respect to the action.

We denote with M_σ the manifold M with Kähler structure given by ω and $I_\sigma := I(\sigma)$. Note that the constructions seen previously in this section for a manifold with a fixed Kähler structure in particular apply to M_σ for all $\sigma \in \mathcal{T}$. In what follows we will suppress the σ in the notation when this does not cause ambiguity.

As seen before, for each $\sigma \in \mathcal{T}$, we use I_σ to split the complexified tangent bundle $TM_\mathbb{C}$ into the holomorphic and the anti-holomorphic parts, denoted respectively $T'M_\sigma$ and $T''M_\sigma$.

Let V be a vector field on $\mathcal{T}$: we can differentiate the family of Kähler structures in the direction of V:

$$V[I]\colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM)).$$

By definition, I_σ defines an almost complex structure for any $\sigma \in \mathcal{T}$, and thus it satisfies the identity $I_\sigma^2 = -\mathrm{Id}$, which gives the following equation when differentiated along a vector field V on $\mathcal{T}$:

$$V[I]_\sigma I_\sigma + I_\sigma V[I]_\sigma = 0,$$

which shows that $V[I]$ anti-commutes with I, and therefore $V[I]_\sigma$ switches types of vectors on M_σ. So we can decompose the tangent space as:

$$V[I]_\sigma = V[I]'_\sigma + V[I]''_\sigma \tag{5.1}$$

where $V[I]'_\sigma \in C^\infty(M, T''M_\sigma^* \otimes T'M_\sigma)$ and $V[I]''_\sigma \in C^\infty(M, T'M_\sigma^* \otimes T''M_\sigma)$.

5.1.1 The canonical bundle for a family

When given a family of complex structures, we can consider the vector bundle $\hat{T}'M$ over $\mathcal{T} \times M$, where the fibre over a point (σ, p) is the holomorphic part of the tangent space T'_pM_σ. Let $\hat{d}$ denote the external differential on the product $\mathcal{T} \times M$.

We have a Hermitian structure $\hat{h}$ on the bundle $\hat{T}'M$ induced by the Kähler metric, and the Levi-Civita connection gives a compatible partial connection along M. Now we wish to extend this partial connection to a connection in the bundle: for a section $Z \in C^\infty(\mathcal{T} \times M, \hat{T}'M)$ and a vector field V on $\mathcal{T}$, we define:

$$\hat{\nabla}_V Z = \pi^{1,0}(V[Z]),$$

which means to take the holomorphic projection of the derivative of Z along V, considering Z as a smooth family of sections of the complexified tangent bundle.

This way we have defined a connection $\hat{\nabla}$, and we can see that it preserves the Hermitian structure in the direction of M, being induced by the Levi-Civita connection. Moreover, if V is a vector field on $\mathcal{T}$ and X, Y are sections of the bundle, we have:

$$\begin{aligned} V[h(X,Y)] = V[g(X,\bar{Y})] &= V[g](X,\bar{Y}) + g(V[X],\bar{Y}) + g(X,V[\bar{Y}]) \\ &= h(\hat{\nabla}_V X, Y) + h(X, \hat{\nabla}_V Y), \end{aligned}$$

because the $(1,1)$-part of $V[g]$ vanishes. Therefore the connection $\hat{\nabla}$ preserves the Hermitian structure on $\hat{T}'M$.

Let us now define the line bundle

$$\hat{K} = \Lambda^m \hat{T}'M^*$$

over $\mathcal{T} \times M$, which we call the *canonical line bundle of the family of complex structures*. The previous discussion shows that we have a Hermitian structure and a compatible connection on $\hat{K}$ induced by those on $\hat{T}'M$, which we shall denote again respectively $\hat{h}$ and $\hat{\nabla}$.

The curvature of the connection $\hat{\nabla}$ in $\hat{K}$ can be computed via a rather technical calculation, which we shall omit. The resulting expression is given in the following formula.

$$\begin{aligned} F_{\hat{\nabla}}(X,Y) &= i\rho(X,Y) \\ F_{\hat{\nabla}}(V,X) &= \frac{i}{2}\delta(V[I])X \\ F_{\hat{\nabla}}(V,W) &= i\vartheta(V,W), \end{aligned} \tag{5.2}$$

for vector fields X, Y on M and V, W on $\mathcal{T}$, where ϑ is a two-form in $\Omega^2(\mathcal{T}, C^\infty(M))$ expressed by:

$$\vartheta(V,W) = -\frac{i}{4}\operatorname{Tr}\pi^{1,0}\left[V[I], W[I]\right].$$

We refer the reader to [Gam10] for a proof of these expressions. Using these expressions for the curvature and the Bianchi identity, we can obtain a useful result about the variation of the Ricci form.

Proposition 5.1. *For a vector field V on $\mathcal{T}$, the variation of the Ricci form is given by:*

$$V[\rho] = \frac{1}{2} d\delta(V[I]).$$

Proof. Let X and Y be commuting vector fields on M. By the Bianchi identity for $\hat{\nabla}$ we have:

$$V[F_{\hat{\nabla}}(X,Y)] + X[F_{\hat{\nabla}}(Y,V)] + Y[F_{\hat{\nabla}}(V,X)] = 0.$$

We can now substitute the expressions for the curvature from (5.2) and obtain:

$$V[i\rho(X,Y)] = X[\frac{i}{2}\delta(V[I])Y] - Y[\frac{i}{2}\delta(V[I])X].$$

Since X and Y commute, this implies that

$$2V[\rho](X,Y) = d\delta(V[I])(X,Y),$$

which concludes the proof. □

The following proposition is a consequence of the Bianchi identity for two vector fields on $\mathcal{T}$ and one vector field on M.

Proposition 5.2. *Let V and W be vector fields on $\mathcal{T}$. Then we have that:*

$$d\vartheta(V,W) = \frac{1}{2}W[\delta](V[I]) - \frac{1}{2}V[\delta](W[I]),$$

where $\vartheta \in \Omega^2(\mathcal{T}, C^\infty(M))$ is the 2-form:

$$\vartheta(V,W) = -\frac{i}{4}\operatorname{Tr}\pi^{1,0}\left[V[I], W[I]\right].$$

The proof follows the same argument as the previous proposition, using the expressions for the curvature obtained in (5.2).

5.2 The bivector field $\tilde{G}(V)$

The symplectic form ω is non-degenerate, therefore we get an isomorphism from it by contracting in the first entry:

$$i_\omega\colon TM_{\mathbb{C}} \to TM_{\mathbb{C}}^*.$$

Moreover i_ω interchanges types on M_σ, since ω is I_σ invariant. Similarly we get a type-interchanging isomorphism

$$i_{g_\sigma}\colon TM_{\mathbb{C}} \to TM_{\mathbb{C}}^*$$

induced by the Kähler metric on M_σ. The two isomorphisms are related by the equation: $i_{g_\sigma} = I_\sigma i_\omega$.

Now we can define $\tilde{G}(V) \in C^\infty(M, TM_{\mathbb{C}} \otimes TM_{\mathbb{C}})$ by

$$V[I] = (\mathrm{Id} \otimes i_\omega)(\tilde{G}(V)), \tag{5.3}$$

for all vector fields V, and define $G(V) \in C^\infty(M, T'M_\sigma \otimes T'M_\sigma)$ such that we get a decomposition

$$\tilde{G}(V) = G(V) + \bar{G}(V),$$

where $\bar{G}(V) \in C^\infty(M, T''M_\sigma \otimes T''M_\sigma)$.

Recalling the identity $g = \omega \cdot I$, we can obtain the variation of the Kähler metric by differentiating it and thereby obtain:

$$V[g] = \omega \cdot V[I] = \omega \cdot \tilde{G}(V) \cdot \omega,$$

and the $(1,1)$-part of $V[g]$ vanishes because of the types of ω and $\tilde{G}(V)$.

Similarly we can compute the variation of the Levi-Civita connection. It is a tensor field:

$$V[\nabla] \in C^\infty(M, S^2(TM^*) \otimes TM),$$

and we have the following expression for it:

$$2g(V[\nabla]_X Y, Z) = \nabla_X(V[g])(Y, Z) + \nabla_Y(V[g])(X, Z) - \nabla_Z(V[g])(X, Y), \quad (5.4)$$

for any X, Y, Z vector fields on M. The proof of this formula can be found in Besse [Bes87] Theorem 1.174.

5.3 Laplacian-like operators

Before defining the Hitchin connection, we need to define a certain differential operator associated to a bivector field.

From a symmetric holomorphic bivector field $Z \in C^\infty(M, S^2(T'M_\sigma))$ we can obtain a holomorphic bundle map $Z\colon T'M_\sigma^* \to T'M_\sigma$ by contraction, and we can define an operator Δ_Z by the composition:

$$\begin{aligned} C^\infty(M, \mathcal{L}^k) &\xrightarrow{\nabla_\sigma^{(1,0)}} C^\infty(M, T'M_\sigma^* \otimes \mathcal{L}) \xrightarrow{Z \otimes \mathrm{Id}} C^\infty(M, T'M_\sigma \otimes \mathcal{L}) \\ &\xrightarrow{\tilde{\nabla}_\sigma^{(1,0)} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla_\sigma^{(1,0)}} C^\infty(M, T'M_\sigma^* \otimes T'M_\sigma \otimes \mathcal{L}^k) \to C^\infty(M, \mathcal{L}^k), \end{aligned}$$

where $\tilde{\nabla}_\sigma^{(1,0)}$ is the holomorphic part of the Levi-Civita connection, and the last arrow is the trace.

The operator Δ_Z can also be expressed as:

$$\Delta_Z = \nabla_Z^2 + \nabla_{\delta Z}, \quad (5.5)$$

using the power of the covariant derivative defined in 2.16, and its adjoint can be expressed using the following lemma.

Lemma 5.3. *Let Z be a complex, symmetric bivector field on M. The adjoint of the operator Δ_Z has the expression:*

$$(\Delta_Z)^* = \Delta_{\bar{Z}}.$$

Proof. We can write the bivector field as: $Z = \sum_j (X_j \otimes Y_j)$, and so the operator is written as

$$\Delta_Z = \sum_j \left(\nabla_{X_j} \nabla_{Y_j} + \nabla_{\delta(X_j) Y_j} \right).$$

Since X_j and Y_j are complex vector fields, we have by (2.18) that

$$(\nabla_{X_j}\nabla_{Y_j})^* = (\nabla_{Y_j})^*(\nabla_{X_j})^* = (\nabla_{\bar{Y}_j} + \delta(\bar{Y}_j))(\nabla_{\bar{X}_j} + \delta(\bar{X}_j))$$
$$= \nabla_{\bar{Y}_j}\nabla_{\bar{X}_j} + \nabla_{\bar{Y}_j}\delta(\bar{X}_j) + \delta(\bar{Y}_j)\nabla_{\bar{X}_j} + \delta(\bar{Y}_j)\delta(\bar{X}_j).$$

Similarly we have:

$$(\nabla_{\delta(X_j)Y_j})^* = -\nabla_{\bar{Y}_j}\delta(\bar{X}_j) - \delta(\bar{X}_j)\delta(\bar{Y}_j),$$

and so we get:

$$(\Delta_Z)^* = \sum_j \nabla_{\bar{Y}_j}\delta(\bar{X}_j) + \delta(\bar{Y}_j)\nabla_{\bar{X}_j} = \Delta_{\bar{Z}}. \qquad \square$$

We shall use the operator $\Delta_{G(V)}$ several times in the coming sections to construct the Hitchin connection, where it will play an important role.

5.4 Holomorphic families of Kähler structures

If the manifold $\mathcal{T}$ has a complex structure, then it makes sense to require the family I to be a holomorphic map from $\mathcal{T}$ to the space of complex structures. Keeping in mind the splitting of $V[I]$ given in (5.1), we can make the following definition to make precise this requirement:

Definition 5.4. Let $\mathcal{T}$ be a complex manifold, and I a family of Kähler structures on M that is parametrized $\mathcal{T}$. We say that I is *holomorphic* if:

$$V'[I] = V[I]' \quad \text{and} \quad V''[I] = V[I]'',$$

for any vector field V on $\mathcal{T}$.

Let $\mathcal{T}$ be a complex manifold, parametrizing a family I of Kähler structures on M. Since $\mathcal{T}$ is complex, we can use the associated integrable almost complex structure J on it to define an almost complex structure $\hat{I}$ on the product $\mathcal{T} \times M$ in the following way:

$$\hat{I}(V \oplus X) = JV \oplus I_\sigma X,$$

for $V \otimes X \in T_{(\sigma,p)}(\mathcal{T} \times M)$.

The property of being holomorphic for a family of Kähler structure is related to integrability in the way made precise by the following proposition.

Proposition 5.5. *The family I is holomorphic if and only if and only if the almost complex structure $\hat{I}$ on $\mathcal{T} \times M$ is integrable.*

Proof. By Theorem 2.8 it is sufficient to show that the Nijenhuis tensor for $\hat{I}$ vanishes. It is clear that, if we evaluate the tensor on vectors tangent to $\mathcal{T}$ then it vanishes, since the almost complex structure on $\mathcal{T}$ is integrable. If the family is holomorphic, then the Nijenhuis tensor vanishes also when applied only to vectors tangent to M, since each structure in the family is integrable. Let us consider the only case that is left, namely when the tensor is applied to mixed vectors.

Let X and V be vector fields respectively on M and $\mathcal{T}$. Then we have that

$$[V, IX] = V[I]X,$$

and

$$\begin{aligned} N_{\hat{I}}(V', X) &= [JV', IX] - [V', X] - \hat{I}[JV', X] - \hat{I}[V', IX] \\ &= i[V', IX] - \hat{I}[V', IX] = iV'[I]X - IV'[I]X \\ &= 2i\pi^{0,1}V'[I]X. \end{aligned}$$

One can do a similar computation to obtain: $N_{\hat{I}}(V'', X) = -2i\pi^{1,0}V''[I]X$, and therefore we see that the Nijenhuis tensor vanishes if and only if

$$\pi^{0,1}V'[I]X = 0 \quad \text{and} \quad 2i\pi^{1,0}V''[I]X = 0,$$

completing the proof. □

In the case where the family of Kähler structures is holomorphic we have the following lemma about the variation of the family at second order.

Lemma 5.6. *If I is a holomorphic family of Kähler structures, then*

$$W''V'[I] = \frac{i}{2}\left[V'[I], W''[I]\right], \tag{5.6}$$

for any vector fields V and W on $\mathcal{T}$ such that V' and W'' commute.

Proof. Since I is holomorphic we have that

$$V'[I]\pi^{1,0} = V[I]'\pi^{1,0} = 0,$$

and therefore we obtain, by differentiating along W'':

$$W''V'[I]\pi^{1,0} = \frac{i}{2}V'[I]W''[I].$$

Similarly as above, we have the equation: $W''[I]\pi^{0,1} = 0$, which we can differentiate along V' to obtain:

$$V'W''[I]\pi^{0,1} = -\frac{i}{2}W''[I]V'[I].$$

If we add the two relations and use the fact that V' and W'' commute, we obtain the relation we wanted. □

When we assume that the family I is holomorphic we have also the following useful fact about the bivector field $\tilde{G}(V)$ defined in (5.3) for any vector field V:

$$\tilde{G}(V') = V'[I]\cdot\tilde{\omega} = V[I]'\cdot\tilde{\omega} = G(V).$$

Similarly we can obtain:

$$\tilde{G}(V'') = V''[I]\cdot\tilde{\omega} = V[I]''\cdot\tilde{\omega} = \bar{G}(V).$$

5.5 The rigidity condition

In this section we should study the notion of *rigidity* for a family of Kähler structures on a manifold. This condition is a rather strong requirement, and it will turn out to be very important in order to construct the Hitchin connection.

Definition 5.7. We say that the family I of Kähler structures on M is *rigid* if

$$\nabla_{X''} G(V) = 0, \tag{5.7}$$

for all vector fields V on $\mathcal{T}$ and X on M.

In other words, the family I is rigid if $G(V)$ is a holomorphic section of $S^2(T'M)$, for any vector field V on $\mathcal{T}$.

Let us see an example of a rigid family of holomorphic Kähler structures.

Example 5.8. Let (M, ω) be $\mathbb{R}^2$ with the standard symplectic form $\omega = dx \wedge dy$, and let $\mathcal{T} = \mathbb{R}^n$. Consider the following family of complex structures, for $\sigma \in \mathcal{T}$:

$$I_\sigma \left(\frac{\partial}{\partial x} \right) = A(\sigma, x, y) \frac{\partial}{\partial x} + B(\sigma, x, y) \frac{\partial}{\partial y},$$

for smooth functions $A, B \in \mathbb{C}^\infty(\mathcal{T} \times M)$. By the identity $I^2 = -\operatorname{Id}$ we have:

$$I_\sigma \left(\frac{\partial}{\partial y} \right) = - \left(\frac{1 + A^2(\sigma, x, y)}{B(\sigma, x, y)} \right) \frac{\partial}{\partial x} - A(\sigma, x, y) \frac{\partial}{\partial y},$$

and one can verify that ω is I-invariant, and that $g = \omega \cdot I$ is a positive definite bilinear form if B is positive.

Let us assume that B is constant with respect to $\sigma \in \mathcal{T}$. Given a vector field V on $\mathcal{T}$ we have:

$$V[I] = V[A] \frac{\partial}{\partial x} - \left(\frac{2AV[A]}{B} \frac{\partial}{\partial x} + V[A] \frac{\partial}{\partial y} \right) dy.$$

Recall the relation defining $\tilde{G}(V)$:

$$V[I] = \tilde{G}(V) \cdot \omega,$$

which gives:

$$\tilde{G}(V) = -2V[A] \frac{\partial}{\partial x} \frac{\partial}{\partial y} - 2 \frac{AV[A]}{B} \frac{\partial^2}{\partial x^2}.$$

We can then compute

$$G(V) = -2i \frac{V[A]}{B} \frac{\partial^2}{\partial z^2},$$

for $z = x + iy$, from which we see that our family of Kähler structures is rigid if and only if the following two equations are satisfied:

$$0 = -V[A] \frac{\partial B}{\partial y} + B \frac{\partial V[A]}{\partial y}$$

$$0 = V[A] \frac{\partial B}{\partial x} - B \frac{\partial V[A]}{\partial x}.$$

We note that the following expressions for A and B give a solution:

$$B(x,y) = B_0(x,y)$$
$$A(\sigma,x,y) = A_0(x,y) + \sum_{i=1}^{n} \sigma_i B_0(x,y),$$

where A_0 and B_0 are any functions on M. This means that for any initial Kähler structure

$$I_0\left(\frac{\partial}{\partial x}\right) = A_0(x,y)\frac{\partial}{\partial x} + B_0(x,y)\frac{\partial}{\partial y}$$

we get a rigid family of Kähler structures on $M = \mathbb{R}^2$, parametrised by $\mathbb{R}^n$.

If we differentiate the rigidity condition (5.7) along $\mathcal{T}$, we get the following result.

Proposition 5.9. *Any rigid and holomorphic family of Kähler structures satisfies*

$$\mathcal{S}(G(V)\cdot\nabla G(W)) = \mathcal{S}(G(W)\cdot\nabla G(V)), \tag{5.8}$$

for any vector fields V and W on $\mathcal{T}$, where $\mathcal{S}$ denotes the symmetrization of the tensors.

Before proving the proposition, let us check the following lemma about the variation of the Levi-Civita connection.

Lemma 5.10. *Let M be a manifold with a family of rigid and holomorphic family of Kähler structures, and let V and Z be vector fields respectively on $\mathcal{T}$ and M. For $\sigma \in \mathcal{T}$, let us take a vector field X of type $(1,0)$ on M_σ. Then we have the following expression for the variation of the Levi-Civita connection:*

$$V'[\nabla]_{Z''}X = \frac{i}{2} Z\cdot\omega\cdot\nabla_X(G(V)). \tag{5.9}$$

Proof. We observe that equation (5.4) implies the following relation:

$$2g(V'[\nabla]_{Z''}X,U) = \nabla_{Z''}(V'[g])(X,U) + \nabla_X(V'[g])(Z'',U) - \nabla_U(V'[g])(Z'',X).$$

In our setting the first added on the right hand side vanishes, because the family is holomorphic and rigid, and the last one is also zero because $V'[g]$ has no $(1,1)$ part. Hence we get:

$$V'[\nabla]_{Z''}X\cdot g\cdot U = \frac{1}{2}\nabla_X(V'[g])(Z'',U) = \frac{i}{2}Z\cdot\omega\cdot\nabla_X(G(V))\cdot g\cdot U,$$

and from this we obtain the relation we wanted. □

Proof of Proposition 5.9. Let us consider two vector fields V and W on $\mathcal{T}$ such that $[V',W'] = 0$, and let U and Z be vector fields on M. Let us take $\sigma \in \mathcal{T}$ and choose two vector fields X and Y of type $(1,0)$ on M_σ, seen as a complex manifold with the structure coming from the rigid family I.

By the previous lemma we have that:

$$V'[\nabla]_{Z''}X = \frac{i}{2} Z\cdot\omega\cdot\nabla_X(G(V)),$$

and if we write locally $G(W)$ as $\sum_j X_j \otimes Y_j$, then we get:

$$\begin{aligned} V'[\nabla]_{Z''}(G(W)) &= \sum_j V'[\nabla]_{Z''} X_j \otimes Y_j + \sum_j X_j \otimes V'[\nabla]_{Z''} Y_j \\ &= \frac{i}{2} \sum_j Z \cdot \omega \cdot \nabla_{X_j}(G(V)) \otimes Y_j + \frac{i}{2} \sum_j X_j \otimes Z \cdot \omega \cdot \nabla_{Y_j}(G(V)) \\ &= \frac{3i}{2} Z \cdot \omega \cdot \mathcal{S}(G(W) \cdot \nabla G(V)) - \frac{i}{2} Z \cdot \omega \cdot G(W) \cdot \nabla G(V), \end{aligned} \tag{5.10}$$

where we can check the last equality by writing the symmetrization and using the symmetry of $G(W)$.

Remembering that the family is rigid we have that $\nabla_{Z''} G(W) = 0$, and we can differentiate this along V' to obtain:

$$0 = V'[\nabla]_{Z''}(G(W)) - \frac{i}{2} Z \cdot \omega \cdot G(V) \nabla G(W) + \nabla_{Z''}(V'[G(W)]).$$

We can replace in (5.10) and get:

$$\begin{aligned} &3Z \cdot \omega \cdot \mathcal{S}(G(W) \cdot \nabla G(V)) \\ &\quad = Z \cdot \omega \cdot G(V) \nabla G(W) + Z \cdot \omega \cdot G(W) \nabla G(V) + 2i \nabla_{Z''}(V'[G(W)]), \end{aligned} \tag{5.11}$$

which is symmetric in V and W because:

$$W'[G(V)] = -W'V'[\tilde{g}] = -V'W'[\tilde{g}] = V'[G(W)].$$

This shows that the left hand side of (5.11) is also symmetric in V and W, which concludes the proof. □

The following results is a consequence of the proposition.

Proposition 5.11. *If the manifold M has a rigid and holomorphic family of Kähler structures, then the following equality holds:*

$$\begin{aligned} &\nabla^2_{G(V)} G(W) + \nabla_{\delta G(V)} G(W) + 2\mathcal{S}(G(V) \cdot \nabla \delta G(W)) \\ &\qquad = \nabla^2_{G(W)} G(V) + \nabla_{\delta G(W)} G(V) + 2\mathcal{S}(G(W) \cdot \nabla \delta G(V)), \end{aligned}$$

for any smooth vector fields V and W on $\mathcal{T}$.

This is proved by taking the divergence of (5.8).

We shall now put aside the notion of rigidity and explore how the Ricci potential behaves in presence of a family of Kähler structures.

5.6 Families of Ricci potentials

In this section we shall consider a compact manifold M with a family of Kähler structures I parametrized by a complex manifold $\mathcal{T}$.

Let us introduce the notation:

$$C_0^\infty(M,\mathbb{R}) = \left\{ f \in C^\infty(M,\mathbb{R}) \mid \int_M f\omega^m = 0 \right\},$$

for the set of smooth real functions on M with zero average. We denote with F the smooth function $\mathcal{T} \to C^\infty(M,\mathbb{R})$, such that F_σ is the Ricci potential of the manifold M_σ, which means that is satisfies:

$$\mathrm{Ric}_\sigma = \mathrm{Ric}_\sigma^H + 2i\partial_\sigma\bar{\partial}_\sigma F_\sigma, \tag{5.12}$$

where $\mathrm{Ric}_\sigma \in \Omega^{1,1}(M_\sigma)$ and Ric_σ^H is its harmonic part. The existence of such a potential is guaranteed by Hodge theory as seen in Section 2.9. A smooth map $F\colon \mathcal{T} \to C_0^\infty(M,\mathbb{R})$ satisfying (5.12) for all $\sigma \in \mathcal{T}$ is called a *smooth family of Ricci potentials*.

Clearly we can take a normalized smooth family of Ricci potential $\hat{F}$ by imposing $\hat{F}_\sigma \in C_0^\infty(M,\mathbb{R})$ for every $\sigma \in \mathcal{T}$. This family has the property of being equivariant with respect to the action of a symmetry group acting by symplectomorphisms on M, since the action will preserve the symplectic (Kähler) form.

Let us now assume that there exists $n \in \mathbb{Z}$ such that the first real Chern class of (M,ω) has the form:

$$\tilde{c}_1(M,\omega) = n\left[\frac{\omega}{2\pi}\right].$$

Note that $c_1(M,\omega) = -c_1(K_\sigma)$, and therefore the first real Chern class is also represented by $\frac{\rho}{2\pi}$. We can then rewrite (5.12) as:

$$\rho = n\omega + 2i\partial\bar{\partial}F, \tag{5.13}$$

since the Kähler form is harmonic.

Proposition 5.12. *Let M be a compact symplectic manifold such that $H^1(M;\mathbb{R}) = 0$, whose first real Chern class satisfies: $\tilde{c}_1(M,\omega) = n\left[\frac{\omega}{2\pi}\right]$. Let I be a holomorphic family of Kähler structures on M. Let F be any smooth family of Ricci potentials on $\mathcal{T}$. Then the variation of the potentials satisfies:*

$$-\bar{\partial}V'[F] = \frac{i}{4}\delta(V'[I]) + \frac{i}{2}dF \cdot V'[I], \tag{5.14}$$

for any vector field V on $\mathcal{T}$.

Proof. Let us differentiate (5.13) with respect to V':

$$V'[\rho] = -d(dF \cdot V'[I]) + 2i\partial\bar{\partial}V'[F].$$

If we apply Proposition 5.1 on the left side of the equality we get:

$$d\delta(V'[I]) + 2d(dF \cdot V'[I]) - 4id\bar{\partial}V'[F] = 0.$$

If we now look at the form

$$\delta(V'[I]) + 2dF \cdot V'[I] - 4i\bar{\partial}V'[F],$$

we have that it is closed by the previous equation, and hence exact because we assume that the first cohomology is trivial. We also see that it is a form of type $(0,1)$, since I is holomorphic, and therefore it has to be zero, being exact, which concludes the proof. ☐

The next result about the divergence of $V[I]$ follows from Proposition 5.12.

Proposition 5.13. *Let M be a compact symplectic manifold such that $H^1(M;\mathbb{R}) = 0$, whose first real Chern class satisfies: $\tilde{c}_1(M,\omega) = n\left[\frac{\omega}{2\pi}\right]$. Let I be a holomorphic family of Kähler structures on M. Let F be any smooth family of Ricci potentials on $\mathcal{T}$. Then*

$$\delta(V[I])X = 4iV'X''[F] - 4iV''X'[F],$$

for any vector fields V on $\mathcal{T}$ and X on M.

Proof. Let us pair the one-form examined earlier in the proof of Proposition 5.12 with the vector field X, and use (5.14). This way we can write:

$$\delta(V'[I])X = 4iX''V'[F] - 2(V'[I]X)F = 4iV'X''[F].$$

By conjugating the first and the last term of the equality we get:

$$\delta(V''[I])X = -4iV''X'[F].$$

Therefore we can compute:

$$\delta(V[I])X = \delta(V'[I])X + \delta(V''[I])X = 4iV'X''[F] - 4iV''X'[F],$$

which is what we wanted to obtain. □

5.7 Construction of the Hitchin connection

We are now ready to define the Hitchin connection. We are going to do this in the framework of geometric quantization, and we follow the differential geometric construction proposed by Andersen in [And12]. The construction is done under the following assumptions.

Consider a compact, symplectic manifold (M,ω), equipped with a prequantum line bundle $\mathcal{L}$, and assume that $H^1(M,\mathbb{R}) = 0$, and that the real first Chern class of (M,ω) is given by

$$\tilde{c}_1(M,\omega) = n\left[\frac{\omega}{2\pi}\right], \tag{5.15}$$

for some $n \in \mathbb{Z}$. Further, assume that I is a rigid and holomorphic family of Kähler structures on (M,ω) parametrized by a complex manifold $\mathcal{T}$.

The prequantum space $\mathcal{P}_k = C^\infty(M,\mathcal{L}^k)$ forms the fibre of a trivial, infinite-rank vector bundle over $\mathcal{T}$,

$$\hat{\mathcal{P}}_k = \mathcal{T} \times \mathcal{P}_k. \tag{5.16}$$

Let ∇^t denote the trivial connection on this bundle.

Definition 5.14. A *Hitchin connection* in the bundle $\hat{\mathcal{P}}_k$ is a connection of the form

$$\nabla = \nabla^t + a, \tag{5.17}$$

where $a \in \Omega^1(\mathcal{T},\mathcal{D}(M,\mathcal{L}^k))$ is a one-form on $\mathcal{T}$ with values in the space of differential operators on sections of $\mathcal{L}^k$, such that ∇ preserves the quantum subspaces

$$\mathcal{Q}_k(\sigma) = H^0(M_\sigma,\mathcal{L}^k)$$

of holomorphic sections of the k-th power of the prequantum line bundle, inside each fibre of $\hat{\mathcal{P}}_k$.

The existence of a Hitchin connection in the bundle $\hat{\mathcal{P}}_k$ implies that the subspaces $\mathcal{Q}_k(\sigma)$ form a subbundle, because it can be trivialized locally through parallel transport by ∇.

To prove the existence of such a connection, we first look at the bundle $\hat{\mathcal{P}}_k$ from a slightly different perspective: consider the pullback

$$\hat{\mathcal{L}} = \pi_M^*(\mathcal{L})$$

of the line bundle $\mathcal{L}$ along the projection

$$\pi_M \colon \mathcal{T} \times M \to M.$$

Sections of $\hat{\mathcal{P}}_k$ are in one-to-one correspondence with sections of $\hat{\mathcal{L}}^k$, and the connection on $\mathcal{L}^k$ defines a partial connection on $\hat{\mathcal{L}}$ along the directions of M. We can easily extend this to a full connection $\hat{\nabla}^k$ on $\hat{\mathcal{L}}^k$: for any section s of $\hat{\mathcal{L}}^k$ and any vector field V on $\mathcal{T}$, we define

$$\hat{\nabla}^k_V(s) = \nabla^t_V(s) = V[s],$$

simply expressing differentiation in the direction of V, in which $\hat{\mathcal{L}}^k$ is trivial.

The bundle $\hat{\mathcal{L}}^k$ has a Hermitian structure, which is induced from $\mathcal{L}$, and $\hat{\nabla}^k$ easily seen to be compatible with this. It is easy to check that the curvature of this connection has the expression given in the proposition below.

Proposition 5.15. *The curvature of the connection* $\hat{\nabla}^k$ *is*

$$F_{\hat{\nabla}^k} = -ik\pi_M^*(\omega),$$

where π_M *is the projection* $\mathcal{T} \times M \to M$.

Then we see that the curvature of $\hat{\nabla}^k$ has type $(1,1)$ on $\mathcal{T} \times M$, and so it defines a holomorphic structure on $\hat{\mathcal{L}}^k$. Since $\hat{\nabla}^k$ only has curvature in directions along M we can prove the following result.

Proposition 5.16. *The connection* $\hat{\nabla}^k$ *preserves the fibrewise subspaces* $\mathcal{Q}_k(\sigma)$ *of* $\hat{\mathcal{P}}_k$ *if and only if the one-form* a *satisfies*

$$\nabla^{(0,1)} a(V)s + \frac{i}{2}\omega \cdot G(V) \cdot \nabla s = 0, \tag{5.18}$$

for any vector field V *on* $\mathcal{T}$*, any point* $\sigma \in \mathcal{T}$ *and any section* $s \in \mathcal{Q}_k(\sigma)$.

Proof. Let V and X be vector fields on $\mathcal{T}$ and M, respectively. It is easily calculated that

$$[X'', V] = -\frac{i}{2}V[I]X = \frac{i}{2}X \cdot \omega \cdot \tilde{G}(V). \tag{5.19}$$

Consider a point $\sigma \in \mathcal{T}$, and suppose that $s \in \mathcal{Q}_k(\sigma)$. Let $\hat{s}$ be any extension of s to a smooth section of the bundle $\hat{\mathcal{P}}_k$ over $\mathcal{T}$. Then we have that $\hat{\nabla}_{X''}\hat{s} = 0$ at the point σ, and we obtain:

$$\begin{aligned} \hat{\nabla}^k_{X''}\nabla_V \hat{s} &= \hat{\nabla}_{X''}\hat{\nabla}^k_V \hat{s} + \hat{\nabla}_{X''} a(V)\hat{s} \\ &= R_{\hat{\nabla}^k}(X'', V)\hat{s} + \hat{\nabla}^k_V \hat{\nabla}_{X''}\hat{s} + \hat{\nabla}_{[X'',V]}\hat{s} + \hat{\nabla}_{X''} a(V)\hat{s} \\ &= \frac{i}{2}X \cdot \omega \cdot G(V) \cdot \hat{\nabla}\hat{s} + \hat{\nabla}_{X''} a(V)\hat{s}, \end{aligned} \tag{5.20}$$

at the point σ, where the curvature term vanishes by Proposition 5.16. Finally, it is clear that $(\nabla_V \hat{s})_\sigma \in \mathcal{Q}_k(\sigma)$ if and only if the left-hand side of (5.20) vanishes at σ, and the proposition follows. □

If we can find a one-form a satisfying (5.18), then it follows that ∇ preserves the fibrewise subspaces $\mathcal{Q}_k$ of the bundle $\hat{\mathcal{P}}_k$. In this case, these subspaces must form a smooth subbunble $\hat{\mathcal{Q}}_k$ as we have observed earlier, which is trivialized by the parallel transport of the connection ∇, and furthermore, ∇ induces a connection in this subbundle.

We shall now prove a proposition about $G(V)$ which we can use to produce a form a satisfying the condition of (5.18).

Proposition 5.17. *The operator $\Delta_{G(V)}$ satisfies:*

$$\nabla^{0,1}\Delta_{G(V)}s = -2ik\omega \cdot G(V) \cdot \nabla s - i\rho \cdot G(V) \cdot \nabla s - ik\omega \cdot \delta(G(V))s, \tag{5.21}$$

for and any (local) holomorphic section s of $\mathcal{L}^k$.

Proof. Fix a vector field V and a point $\sigma \in \mathcal{T}$. The statement of the proposition is local on M, and since the family of complex structures is rigid, the bivector field $G(V)$ is holomorphic and can therefore be expressed locally as

$$G(V) = \sum_j X_j \otimes Y_j,$$

for X_j and Y_j local holomorphic vector fields on M_σ. With this notation, the operator $\Delta_{G(V)}$ has the expression

$$\Delta_{G(V)} = \sum_j \nabla_{X_j}\nabla_{Y_j} + \nabla_{\delta(X_j)Y_j}.$$

For an local holomorphic section s of $\mathcal{L}^k$ and any local anti-holomorphic vector field $\bar{Z}$ we get that:

$$\nabla_{\bar{Z}}\nabla_{X_j}\nabla_{Y_j}s = -ik\omega(\bar{Z}, X_j)\nabla_{Y_j}s - ik\omega(\bar{Z}, Y_j)\nabla_{X_j}s - ik\omega(\bar{Z}, \nabla_{X_j}Y_j)s.$$

On the other hand, we have

$$\bar{Z}[\delta(X_j)] = \bar{Z}[\mathrm{Tr}(\nabla X_j)] = \mathrm{Tr}\,\nabla_{\bar{Z}}\nabla X_j = \mathrm{Tr}\,R(\bar{Z}, \cdot)X_j = -i\rho(\bar{Z}, X_j),$$

and therefore:

$$\nabla_{\bar{Z}}\nabla_{\delta(X_j)Y_j}s = -ik\omega(\bar{Z}, \delta(X_j)Y_j)s - i\rho(\bar{Z}, X_j)\nabla_{Y_j}s.$$

We can then combine the relations above and finally obtain:

$$\nabla_{\bar{Z}}\Delta_{G(V)} = -\sum_j -ik\omega(\bar{Z}, X_j)\nabla_{Y_j}s + i\rho(\bar{Z}, X_j)\nabla_{Y_j}s + ik\omega(\bar{Z}, \delta(X_j \otimes Y_j))s,$$

which is what we wanted to show. □

We emphasize the importance of the rigidity condition on the family of Kähler structures in this proposition.

If not for the last two terms of (5.21), we could use $\frac{1}{4k}\Delta_{G(V)}$ as our $a(V)$: we shall now see how one can try to get rid of those terms.

Let us recall that the assumption (5.15) on the first Chern class implies that $\rho^H = n\omega$, since the Kähler form is harmonic. In particular, for any smooth family of Ricci potentials F_σ, we have the identity

$$\rho = n\omega + 2i\partial\bar{\partial}F. \tag{5.22}$$

Inserting this in (5.18), and using the identity

$$\nabla^{0,1}\nabla_{G(V)\cdot dF}s = -\partial\bar{\partial}F \cdot G(V) \cdot \nabla s - ik\omega \cdot G(V) \cdot dFs,$$

which holds for any $\sigma \in \mathcal{T}$ and for any holomorphic section s of $\mathcal{L}^k$, we can obtain the following equation:

$$\begin{aligned}\nabla^{0,1} &\left(\Delta_{G(V)}s + 2\nabla_{G(V)\cdot dF}s\right) \\ &= -(2k+n)i\omega \cdot G(V) \cdot \nabla s - ik\omega \cdot \delta(G(V))s - 2ik\omega \cdot G(V) \cdot dFs. \end{aligned}\tag{5.23}$$

We can easily convince ourselves that this is an improvement with respect to (5.21) because we have replaced the second summand on the right side, which was a first-order term, with a zero-order term.

Moreover, we can use Proposition 5.12 to get rid of the last two terms: if we now define a one-form $a \in \Omega^1(\mathcal{T}, \mathcal{D}(M, \mathcal{L}^k))$ by:

$$a(V) = \frac{1}{4k+2n}\left(\Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} + 4kV'[F]\right), \tag{5.24}$$

it will satisfy our requirement (5.18).

For later use let us introduce now the following manner of writing $a(V)$, using the following operator $b(V)$, with the purpose of splitting $a(V)$ into orders.

$$b(V) = \frac{1}{4}\left(\Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} - 2nV'[F]\right)$$

$$a(V) = \frac{1}{k+n/2}b(V) + V'[F].$$

The results that we have gone through in this section allow us to prove the following theorem, which establish the existence of the Hitchin connection in the setting of geometric quantization.

Theorem 5.18 (Andersen)**.** *Let (M, ω) be a compact, prequantizable, symplectic manifold which satisfies that there exists an $n \in \mathbb{Z}$ such that the first Chern class of (M, ω) is $n\left[\frac{\omega}{2\pi}\right] \in H^2(M;\mathbb{Z})$ and $H^1(M;\mathbb{R}) = 0$. Moreover suppose that I is a rigid holomorphic family of Kähler structures on M, parametrized by a complex manifold $\mathcal{T}$. Then there exists a Hitchin connection in the bundle $\hat{\mathcal{Q}}_k$ over $\mathcal{T}$, given by the following expression:*

$$\hat{\nabla}_V = \nabla^t_V + \frac{1}{4k+2n}\{\Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} + 4kV'[F]\},$$

where ∇^t_V is the trivial connection in $\hat{\mathcal{P}}_k$, and V is any smooth vector field on $\mathcal{T}$.

Note that, if the smooth family of Ricci potential is normalized to be equivariant, then the Hitchin connection defined in this theorem is also equivariant with respect to the action of the symmetry group Γ.

6

Graph language

This chapter is concerned with reviewing Gammelgaard's construction that allows to express Karabegov's star products in a combinatorial manner in local coordinates, using differential operators associated to certain graphs. We shall afterwards extend this combinatorial language to make use of this formalism to study the problem of trivialization of formal connections in Chapter 8.

In this chapter we consider a Kähler manifold M of dimension m equipped with a differential star product $\star$ with separation of variables that is null on constants. Such a star product is classified by Karabegov as seen in Chapter 4, and therefore it has a Karabegov form $\mathrm{Kar}(\star)$.

6.1 A family of graphs

Let us consider finite graphs consisting of vertices connected by directed edges.

Definition 6.1. A *finite directed graph G* is the data of a finite set of vertices V_G, a finite set of edges E_G and two maps $t_G, h_G \colon E_G \to V_G$ named respectively *tail* and *head* that specify respectively the vertex where the directed edge originates and the vertex where it ends.

We can consider *paths* in a graph G, which are obtained following subsequent edges along the given direction. A path which starts and ends at the same vertex is called a *cycle*. Two edges are called *parallel* if they connect the same pair of vertices and have the same orientation. A graph without any cycle is called *acyclic*.

From now on, with graph we will mean finite directed acyclic graphs. Note that we allow parallel edges.

Definition 6.2. A *half-edge* in a graph is one of the two ends of an edge.

If v is a vertex in a graph G, we denote with $\deg^+(v)$ the number of edges incoming to v and call this number *in-degree*, and with $\deg^-(v)$ the number of edges going out from v, the *out-degree*. Their sum gives the total degree of the vertex v, denoted $\deg(v)$. The vertex is a *source* if $\deg^+(v) = 0$, and a *sink* if $\deg^-(v) = 0$.

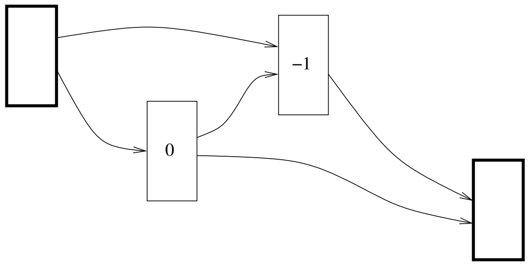


Figure 6.1: A weighted graph with 2 external vertices and total weight 4.

Among the vertices of a graph G we distinguish an ordered subset $\text{Ext}(G)$ of the *external vertices*. The remaining vertices are called *internal* and form the set $\text{Int}(G) = V_G \setminus \text{Ext}(G)$. The first external vertex is required to be a source, and the last one a sink, while there are no requirements on the degree of the remaining external vertices.

A graph is *weighted* if we assign an integer $w(v) \in \mathbb{N} \cup \{-1\}$ called *weight* to all internal vertices $v \in \text{Int}(G)$. The weight $w(v)$ is only allowed to be -1 if $\deg(w) \geq 3$. The total weight of a graph G is defined by:

$$W(G) = |E_G| + \sum_{v \in \text{Int}(G)} w(v).$$

Definition 6.3. An isomorphism of graphs $G_1 \to G_2$ is the data of two bijections

$$V_{G_1} \to V_{G_2} \qquad E_{G_1} \to E_{G_2}$$

preserving the way vertices are connected with edges, mapping external vertices to external vertices preserving the ordering. If the graphs are weighted, an isomorphism should preserve the weight of vertices.

Definition 6.4. We denote with $\mathcal{A}_n(k)$ the set of the isomorphism classes of finite acyclic weighted graphs with n external vertices and weight k. We denote the union of all these graphs by:

$$\mathcal{A}_n = \bigcup_k \mathcal{A}_n(k).$$

6.2 The partition function of a graph

Let us consider a formal deformation of the Kähler form of M, denoted by ω as in (4.5). We want to write the Karabegov star product with class ω graph-theoretically using the graphs defined here. To do this we define certain partition functions associated to graphs in our family and using them to define a product on functions on M. Then one checks that this product satisfies the requirements to be a star product and checks that it in fact coincides with the Karabegov star product we wished to express.

This is done locally, on a contractible neighbourhood U of M, where we have local holomorphic coordinates $z^1, \ldots, z^m$.

Let $G \in \mathcal{A}_n$ be a graph and let $f_1, \ldots, f_n \in C^\infty(M)$ be smooth functions. If $f \in C^\infty(M)$ is a smooth function and $(p, q) \in \mathbb{Z} \times \mathbb{Z}$, we define a covariant tensor $f^{(p,q)}$ of type (p, q) by:

$$f^{(p,q)}\left(\frac{\partial}{\partial z^{i_1}}, \ldots, \frac{\partial}{\partial z^{i_p}}, \frac{\partial}{\partial \bar{z}^{j_1}}, \ldots, \frac{\partial}{\partial \bar{z}^{j_q}}\right) = \frac{\partial^{p+q} f}{\partial z^{i_1} \cdots \partial z^{i_p}, \partial \bar{z}^{j_1}, \cdots, \partial \bar{z}^{j_q}}.$$

For each vertex v in the graph G, we assign to it the tensor $f^{(p,q)}$, where p is the in-degree and q the out-degree of v, and

$$f = \begin{cases} f_i & \text{if } v \text{ is the } i\text{-th external vertex} \\ -\Phi_w & \text{if } v \text{ is an internal vertex and } w(v) = v, \end{cases}$$

where Φ is a formal potential on U of the Karabegov form ω defined in (4.5), which means that

$$\Phi = \frac{1}{h}\Phi_{-1} + \Phi_0 + \Phi_1 h + \Phi_2 h^2 + \ldots,$$

and $\omega|_U = i\partial\bar{\partial}\Phi$. Note that such a potential exists, since ω is a closed form of type $(1, 1)$ and we work on a contractible neighbourhood.

Definition 6.5. The *partition function* $\Gamma_G(f_1, \ldots, f_n)$ associated to the graph $G \in \mathcal{A}_n$ is the smooth function on M obtained by contracting the tensors we have associated to each vertex using the Kälher metric, along the edges of the graph.

Note that, since the tensors are completely symmetric, the contraction is well-defined.

6.3 Coefficients of the product

We can now use these partition functions to define the coefficients for a product that we shall see is the star product in which we are interested. Let us define:

$$c^k(f_1, \ldots, f_n) = \sum_{G \in \mathcal{A}_n(k)} \frac{1}{|\mathrm{Aut}(G)|} \Gamma_G(f_1, \ldots, f_n), \tag{6.1}$$

Then the star product will be expressed by the formula:

$$f_1 \star f_2 = \sum_{k=0}^{\infty} c^k(f_1, f_2) h^k. \tag{6.2}$$

Equivalently, if we define the operator:

$$c(f_1, \ldots, f_n) = \sum_{G \in \mathcal{A}_n} \frac{1}{|\mathrm{Aut}(G)|} \Gamma_G(f_1, \ldots, f_n) h^{w(G)}, \tag{6.3}$$

then the statement that we shall prove is the following:

$$f_1 \star f_2 = c(f_1, f_2). \tag{6.4}$$

Remark 6.6. It is immediate to verify that $c^0(f_1, f_2) = f_1 f_2$ for all smooth functions f_1, f_2, since the only graph with two external vertices and weight zero is the graph with no internal vertices and no edges. Similarly, we have only one graph with two external vertices and total weight one, namely the graph with no internal vertices and one edge connecting the two external vertices. The definition above associates to this graph the partition function:

$$c^1(f_1, f_2) = \sum_{p,q=0}^{m} g^{qp} \frac{\partial f_1}{\partial \bar{z}^q} \frac{\partial f_2}{\partial z^p}, \tag{6.5}$$

and so we obtain the relation:

$$c^1(f_1, f_2) - c^1(f_2, f_1) = i\{f_1, f_2\},$$

that is satisfied by any $f_1, f_2 \in C^\infty(M)$.

The product that we have defined is clearly $\mathbb{C}[[h]]$-bilinear and it satisfies the second and third requirement of Definition 4.1. We only need to show its associativity to prove that it is in fact a star product.

6.4 Another type of graph expressions

To show that the product defined here combinatorially associative, we shall write it in a slightly different form, which makes it easier to control how the functions are differentiated by the operators. We shall also build up on the new type of graphs here defined to extend the combinatorial language and define notions such as differentiation and composition of graphs in Chapter 8.

Definition 6.7. A *labelling* l of a graph G is the assignment of an integer in the set $\{1, \ldots, m\}$ to each half-edge of the graph.

A *labelled* graph is a graph together with a labelling l.

A *circuit structure* of a graph is the data of an ordering of the incoming edges and of the outgoing edges at each vertex of the graph.

We say that the two structures just defined above are *compatible* if, at each vertex of the graph, the labels of the incoming edges and of the outgoing edges increase with the orderings given by the circuit structure.

Let G be a graph with a labelling l. We want to count the number of circuit structures compatible with l. The labelling can be seen as the data of two functions

$$\alpha_l, \beta_l \colon V_g \to \mathbb{Z}^m_{\geq 0},$$

assigning to each vertex two multi-indices, where $\alpha_l(v)_i$ counts the number of occurrences of the label $i \in \{1, \ldots, m\}$ among the incoming edges at v, and similarly $\beta_l(v)_i$ is the number of occurrences of the label $i \in \{1, \ldots, m\}$ among the outgoing edges. The following lemma, whose proof is a straightforward calculation, expresses the wanted quantity.

Lemma 6.8. *Let G be a graph with a labelling l. The number of circuit structures compatible with l is expressed by:*

$$C(G, l) = \prod_{v \in V_G} \alpha_l(v)! \beta_l(v)!.$$

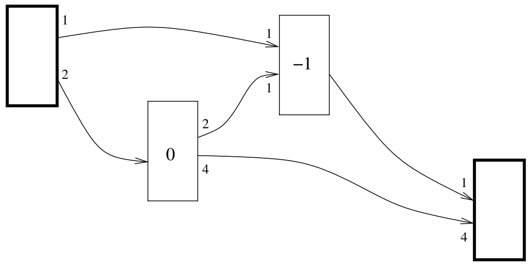


Figure 6.2: A labelled graph with 2 external vertices with a compatible circuit structure.

We say that two graphs with a labelling (respectively, a circuit structure) are isomorphic if they are isomorphic as graphs via a map that preserves the labelling (respectively, the circuit structure).

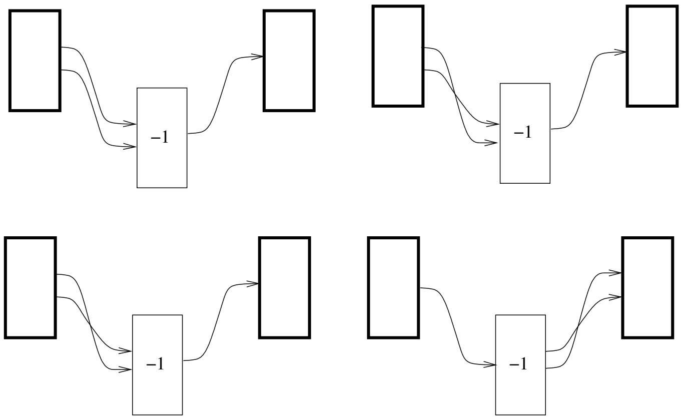
Definition 6.9. We denote with $\mathcal{L}^c_{n,k}$ the set of isomorphism classes of labelled graphs with n external vertices and weight k with a compatible circuit structure. For a unlabelled graph G, we denote with $\mathcal{L}(G)$ the set of isomorphism classes of labelled graphs with a compatible circuit structure whose underlying graph is G.

We represent labelled graphs with circuit structure as in the Figure 6.2. The vertices are represented as boxes, with the incoming edges on the left side and the outgoing ones on the right side. The half-edges on each of the sides of a vertex are ordered according to the circuit structure (increasing from top to bottom). This means that sequence of the labels on each side of a vertex is increasing going downwards. The n external vertices are drawn with a thicker line, and they are numbered from 1 to n in case of ambiguity. A number inside an internal vertex represents its weight.

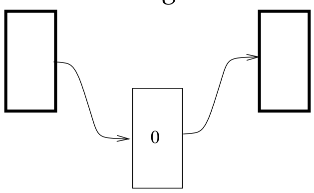
Example 6.10. Here we display with diagrams all the labelled graphs with compatible circuit structure, two external vertices and weight 2. The isomorphism classes of the graphs displayed here form the set $\mathcal{L}^c_{2,2}$. Following our convention, vertices are represented by boxes, with the incoming edges on the left side and the outgoing ones on the right. The ordering of the edges on the sides of the boxes (increasing from the top to the bottom) represents the circuit structure. The labellings are omitted from the diagrams: it is understood that every graph represented below can be given a labelling compatible with the circuit structure. Recall that we represent external vertices with a thicker line, and that number contained in internal vertices is their weight. The external vertices are ordered on the unique way given by the edge orientation.

- Graphs with no internal vertices.

- Graphs with one internal vertex of weight -1.



- Graphs with one internal vertex of weight 0.



For the rest of the chapter we shall consider labelled graphs with a compatible circuit structure. To express the product in the new graph language, we associate to a graph $G \in \mathcal{L}^c_{2,k}$ a new type of partition function: $\Lambda_G(\cdot,\cdot)$ defined as follows. Let us first consider a vertex v of the graph, with in-degree p and out-degree q, and $f_1, f_2 \in C^\infty(M)$. We associate to v an expression $F_{f_1,f_2}(v)$ defined by:

$$F_{f,g}(v) = -\frac{\partial^{p+q}\Phi_w}{\partial z^{i_1}\cdots\partial z^{i_p}\partial \bar{z}^{j_1},\cdots,\partial \bar{z}^{j_q}},$$

if $v \in \mathrm{Int}(G)$ and $w(v) = w$, and by:

$$F_{f,g}(v) = \frac{\partial^{p+q} f_i}{\partial z^{i_1}\cdots\partial z^{i_p}\partial \bar{z}^{j_1},\cdots,\partial \bar{z}^{j_q}},$$

if v is the i-th external vertex.

We associate also an expression to edges: if e is an edge with labels p and q (ordered along the edges orientation), then we define:

$$F_{f_1,f_2}(e) = g^{pq}.$$

Now we are ready to define the partition function:

$$\Lambda_G(f,g) = \prod_{v\in V_G} F_{f,g}(v) \prod_{e\in E_G} F_{f,g}(e),$$

for all $f, g \in C^\infty(M)$.

We can now rewrite the operator c defined in (6.3) using the new graph language just defined.

$$c(f_1,\ldots,f_n) = \sum_{G\in\mathcal{A}_n}\sum_{l\in\mathcal{L}(G)}\sum_{c\in\mathcal{C}(G,l)} \frac{1}{|\mathrm{Aut}(G)|\; C(G,l)} \Lambda^l_G(f_1,\ldots,f_n) h^{W(G)}. \qquad (6.6)$$

It is a matter of counting the graphs in each equivalence class and comparing the partition functions to realize that (6.6) defines the same operators that we had defined previously in (6.3). This allows us to re-write the coefficients of the product in the following way:

$$c^i(f,g) = \sum_{G \in \mathcal{L}^c_{2,i}} \frac{1}{C(G)} \Lambda_G(f,g). \tag{6.7}$$

6.5 The fusion of two graphs

Let us consider two graphs G_1 and G_2, with 2 external vertices each. Fusion is a process that puts the two graphs together to obtain a new graph with 3 external vertices. If we think at the partition functions associated to graphs, the partition function of the fusion is related to the composition of the partition functions of the two fused graphs. Let v_1, v_2 be the two external vertices of G_1, taken in their order, and v_3, v_4 be the ones of G_2. To make a fusion, we cut off v_2 and glue each of the obtained loose edges of G_1 to G_2 obeying to one of the following rules:

- we can attach a loose edge to a vertex of G_2, in any way that does not break the compatibility of the circuit structure. This is called fusion of *type A*.

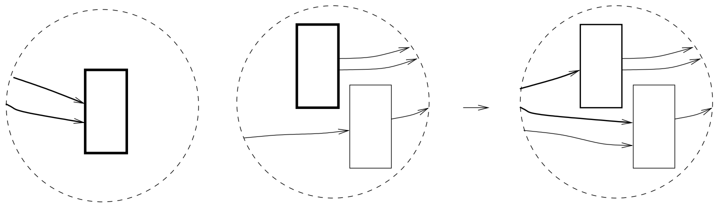


Figure 6.3: Local view of a fusion of type A.

- we can attach a loose edge to an edge of G_2. To do so, we insert a new internal vertex of weight -1 along this edge, and give new labels to the two half-edges we obtain. This is called fusion of *type B*.

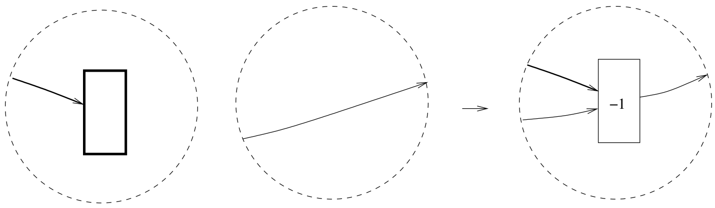


Figure 6.4: Local view of a fusion of type B.

We denote with $\mathcal{F}(G_1, G_2)$ the set of the isomorphism classes of fusions of two graphs G_1 and G_2.

Given a labelled graph $G \in \mathcal{L}^c_{3,k}$ with three external vertices, we can try to see if it can be obtained as the fusion of two graphs G_1, G_2 with two external vertices. It is not difficult to see that we can reconstruct from G the isomorphism class of G_2. The isomorphism class of G_1 cannot be reconstructed completely: we can delete all vertices coming after the second external vertex of G (in the ordering given by the orientation of edges), and connect the loose ends we get to an external vertex, which is then the second external vertex of G_1. But we cannot reconstruct the circuit structure at this vertex. The rest of the structure of G_1 can instead be reconstructed from G. To take in account for this, we can introduce the following equivalence relation: given two graphs G, G' with two external vertices, we say that they are equivalent ($G \sim G'$) if there is an isomorphism between the two graphs preserving the labelling at all vertices and preserving the circuit structure at all vertices except possibly at the second external vertex. We denote with $[G]$ the equivalence class of G, and it amounts to an immediate computation to show that $[G]$ consists of $\alpha_I(u)$ elements, if u is the second external vertex of G.

The alternative type of graph expressions that we have described in this section allow to prove the associativity of the product whose coefficients are expressed by (6.6). In particular we have that, if G_1, G_2 are graphs with two external vertices

$$\sum_{G\in[G_1]} \frac{1}{C(G)C(G_2)} \Lambda_G(f_1, \Lambda_{G_2}(f_2, f_3)) = \sum_{G\in\mathcal{F}(G_1,G_2)} \frac{1}{C(G)} \Lambda(f_1, f_2, f_3).$$

This is proved by Gammelgaard in [Gam10], by carefully examining how the fusion of two graphs works and comparing the coefficients. The conclusion is stated in the following theorem that establishes the associativity of the product.

Theorem 6.11 (Gammelgaard)**.** *For $n = 3$, the operator c defined in* (6.6) *is associative:*

$$c(f_1, c(f_2, f_3)) = c(f_1, f_2, f_3) = c(c(f_1, f_2), f_3),$$

for any smooth function f_1, f_2, f_3 on M.

Gammelgaard shows also that this product is coordinate invariant, and that the resulting star product on M has the Karabegov form that we started from at the beginning of the construction, ω. Therefore we have now a combinatorial way to express any Karabegov star product in local coordinates.

6.6 Extending the graph language

We shall now introduce a new feature on the graphs that we have considered in Section 6.4. As done there, all the graphs considered here are finite directed acyclic weighted graphs with 2 external vertices and with labelling and compatible circuit structures, and we shall reuse the same notation introduced in that chapter.

Our goal is to make sense of differentiating a graph, in the sense of writing in graph language the expressions we get when applying the vector field V to the partition functions associated to a graph.

Remark 6.12. As we shall discuss in the Chapter 8 to work on the trivialization problem, we need to consider also graphs that do not satisfy the constraints that we have set for them previously. In particular, we want to relax the condition that the first external vertex has to be a source and the last one has to be a sink, and that an internal vertex can only have weight -1 if its degree is at least 3.

We call *extended class* of labelled graphs with n external vertices and total weight k and with compatible circuit structure the set of the isomorphism classes of such graphs (as defined in Section 6.4) where any external vertex is allowed to have incoming and outgoing edges, and where an internal vertex of degree 2 can have weight -1. We denote it by $\mathcal{L}'_{n,k}$.

Now we add some extra structure to the graphs, which will be used to make sense of differentiation of a graph.

Definition 6.13. A *starred graph* is a graph with all the structure introduced in Chapter 6 plus one marking (star). The star can be placed on an edge, or on a half-edge of the graph.

To associate a partition function to a starred graph, we proceed as for the usual graphs (as described in Section 6.4) with the following modification:

- If the star is on the edge e, then the corresponding function is

$$F'_{f_1,f_2}(e) = V[F_{f_1,f_2}(e)] = V[g^{pq}],$$

  where p and q are the labels respectively at the tail and at the head of the edge.

- If the star is placed on a half-edge labelled $\bar{i}$ incoming to a vertex v, then the function associated to v is:

$$F'_{f_1,f_2}(v) = V\left[\frac{\partial}{\partial z^{\bar{i}}}\right] \frac{\partial^{p+q-1}\Phi_w}{\partial z^{i_1}\cdots\partial z^{\bar{i}-1}\partial z^{\bar{i}+1}\ldots\partial z^{i_p},\partial \bar{z}^{j_1},\cdots,\partial \bar{z}^{j_q}},$$

  if v is an internal vertex of weight w and

$$F'_{f_1,f_2}(v) = V\left[\frac{\partial}{\partial z^{\bar{i}}}\right] \frac{\partial^{p+q-1} f_i}{\partial z^{i_1}\cdots\partial z^{\bar{i}-1}\partial z^{\bar{i}+1}\ldots\partial z^{i_p},\partial \bar{z}^{j_1},\cdots,\partial \bar{z}^{j_q}},$$

  if v is the i-th external vertex. In the case where the star is on an out-going half-edge the definition is analogous, with V acting on the derivative $\frac{\partial}{\partial \bar{z}^{\bar{j}}}$ corresponding to the label of the starred half-edge.

The partition function associated to the starred graph is then defined as usual:

$$\Lambda_G(f_1, f_2) = \prod_{v\in V_G} F'_{f_1,f_2}(v) \prod_{e\in E_G} F'_{f_1,f_2}(e).$$

6.6.1 Differentiating graphs

Starred graphs can be used to make sense of what it means to "differentiate" a graph or, better, to differentiate the partition function we associate to a graph. As described in section 6.4, the partition function associated to a graph G is

$$\Lambda_G(f_1,\ldots,f_n) = \prod_{v\in V_G} F_{f_1,\ldots,f_n}(v) \prod_{e\in E_G} F_{f_1,\ldots,f_n}(e).$$

When we differentiate it in the direction of a vector field V, because of the Leibniz rule, we will get a sum of terms, in each of which V hits one of the factors in the product. Let us illustrate this with an example.

Example 6.14. We consider now the following graph G with two external vertices.

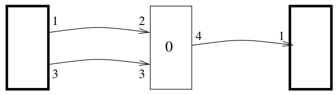


Figure 6.5: The graph G with 2 external vertices.

We can now compute the partition function associated to G to get:

$$\Lambda_G(f_1, f_2) = \frac{\partial^2}{\partial \bar{z}^1 \partial \bar{z}^3} f_1 \frac{\partial^3}{\partial z^2 \partial z^3 \partial \bar{z}^4} \Phi_0 \frac{\partial}{\partial z^1} f_2 g^{12} g^{33} g^{41}.$$

When we differentiate Λ_G in the direction of V, we get a sum of 9 terms, and in each of them V hits one of the factors in the expression. As described above, each of the 9 terms can be represented as a starred graph. More in detail we have the following possibilities:

- V hits one of the derivatives $\frac{\partial}{\partial z^i}$ or $\frac{\partial}{\partial \bar{z}^j}$. This corresponds to a star on a half-edge.
- V hits one of the metric components g^{pq}. This is the case of a star on the corresponding edge.

Remark 6.15. Let G be a graph. The partition function associated to sum of $3\,|V_G|$ copies of starred graphs with underlying graph G can be expressed as a total derivative if and only if the star is placed in all the possible $3\,|V_G|$ locations in the graph.

7

Formal connections

The idea of formal connections arises as a generalization of the formal Hitchin connection. This connection was defined by Andersen [And12] as an adaptation of the Hitchin connection to the language of deformation quantization. Andersen showed that the formal Hitchin connection that he studied is flat under certain hypotheses, and he studied the problem of the trivialization of this connection together with Gammelgaard [AG11].

In this chapter we shall study formal connections in general on a symplectic manifold with a family of Kähler structures on it. The main property of a formal connection is that it has to be a derivation with respect to the star product. We give a result about existence and describe the space of formal connections for such a manifold.

7.1 Formal connections

Let M be a symplectic manifold and let $\mathcal{T}$ be a complex manifold parametrizing Kähler structures on M. We denote with C_h the trivial bundle over $\mathcal{T}$ with fibres the formal power series of smooth functions on M with formal parameter h. Namely,

$$C_h = \mathcal{T} \times C^\infty(M)[[h]].$$

Definition 7.1. A *formal connection* D is a connection in the bundle C_h over $\mathcal{T}$ that can be written as

$$D_V f = V[f] + A(V)(f), \tag{7.1}$$

where A is a smooth 1-form on $\mathcal{T}$ with values in $\mathcal{D}(M)[[h]]$ such that $A = 0$ (mod h), f is a smooth section of C_h, V is any smooth vector field on $\mathcal{T}$, and $V[f]$ denotes the derivative of f along V.

The operator $A(V)(f)$ can be expressed as a series of differential operators as follows:

$$A(V)(f) = \sum_{k=0}^{\infty} A^k(V)(f)h^k,$$

where each A^k is a smooth 1-form on $\mathcal{T}$ with values in $\mathcal{D}(M)$.

Normally we are interested in looking at formal connections in presence of a family of star products on the manifold, and then we require that they are compatible in the following sense.

Definition 7.2. Let $\{\star_\sigma\}_{\sigma\in\mathcal{T}}$ be a family of star products. We say that D is a *formal connection compatible with the family of star products* if D_V is a derivation of $\star_\sigma$ for every V vector field and every $\sigma \in \mathcal{T}$, that is, the following equality holds:

$$D_V(f \star_\sigma g) = D_V(f) \star_\sigma g + f \star_\sigma D_V(g). \tag{7.2}$$

When a formal connection is compatible with a family of star product as defined above, then its parallel transport gives equivalences between the different products in the family.

In this chapter we are going to study the space of formal connections for a family of star products. In particular we will focus on the case where $H^1(M;\mathbb{R})$ vanishes, which allows us to get global results on M.

7.2 Formal Hitchin connections

Definition 7.3. Let M be a symplectic manifold with a family of compatible almost complex structures parametrized by a complex manifold $\mathcal{T}$, so that for any $\sigma \in \mathcal{T}$, the manifold M_σ is a Kähler manifold, and let $\{\star_\sigma^{BT}\}_{\sigma\in\mathcal{T}}$ be the associated family of Berezin-Toeplitz star products. A *formal Hitchin connection* on M is a formal connection that is compatible with this family of star products and that is flat.

The notion of formal Hitchin connection was initially introduced by Andersen in [And12]. Andersen considered one particular formal Hitchin connection, which we shall construct below, that is obtained from the Hitchin connection in geometric quantization, and he showed that it is flat if the Hitchin connection is projectively flat.

Note that we have a more general notion of a formal Hitchin connection here, and that the one studied by Andersen is still a formal Hitchin connection according to our Definition 7.3. We shall refer to the formal Hitchin connection studied by Andersen as formal Hitchin connection *associated to the Hitchin connection of geometric quantization*, and we shall denote its 1-form by $\tilde{A}$.

7.2.1 The formal Hitchin connection associated to geometric quantization

The construction relies on the existence of a Hitchin connection in geometric quantization, therefore we shall put ourselves in the hypotheses of Theorem 5.18. Let (M,ω) be a compact, prequantizable, symplectic manifold which satisfies that there exists an $n \in \mathbb{Z}$ such that the first Chern class of (M,ω) is $n\left[\frac{\omega}{2\pi}\right] \in H^2(M;\mathbb{Z})$ and $H^1(M;\mathbb{R}) = 0$. Moreover suppose that I is a rigid holomorphic family of Kähler structures on M, parametrized by a complex manifold $\mathcal{T}$. Let $\hat{\nabla}$ be the Hitchin connection in the bundle $\hat{\mathcal{Q}}_k$ whose existence is granted by the theorem.

Here we wish to construct a formal connection in the bundle C_h that is compatible with the Berezin-Toeplitz star products $\{\star_\sigma^{BT}\}_{\sigma\in\mathcal{T}}$ and therefore gives equivalences between the star products on different fibres.

The Hitchin connection $\hat{\nabla}$ in $\hat{\mathcal{Q}}_k$ induces a connection $\hat{\nabla}^e$ in the endomorphism bundle $\operatorname{End}(\hat{\mathcal{Q}}_k)$. The following result, proved in [And12], where we assume the same hypotheses of Theorem 5.18 establishes the existence of the *formal Hitchin connection*.

Theorem 7.4 (Andersen). *There is a unique formal connection D, written as $D_V = V + \tilde{A}(V)$ for any vector field V on $\mathcal{T}$ that satisfies*

$$\hat{\nabla}^e_V T_f^{(k)} \sim T^{(k)}_{(D_V f)(1/(2k+n))} \tag{7.3}$$

for all smooth section f of C_h and all smooth vector fields V on $\mathcal{T}$. Here the symbol $\sim$ has the following meaning: for any positive integer L we have that

$$\left\| \hat{\nabla}^e_V T_f^{(k)} - \left(T^{(k)}_{V[f]} + \sum_{l=1}^{L} T^{(k)}_{\tilde{A}_V^{(l)} f} \frac{1}{(2k+n)^l} \right) \right\| = O(k^{-(L+1)})$$

uniformly over compact subsets of $\mathcal{T}$ for all smooth maps $f \colon \mathcal{T} \to C^\infty(M)$.

Andersen derived also an explicit formula for $\tilde{A}$:

$$\tilde{A}(V)(f) = -V[F]f + V[F] \star^{BT} f + h(E(V)(f) - H(V) \star^{BT} f), \tag{7.4}$$

where E is a 1-form on $\mathcal{T}$ with values in $\mathcal{D}(M)$ and H is a 1-form with values in $C^\infty(M)$ such that $H(V) = E(V)(1)$. This result has been further refined by Andersen and Gammelgaard, who obtained an explicit formula for E in [AG11], which will prove useful for our computations in the next chapter:

$$E(V)(f) = -\frac{1}{4}(\Delta_{\tilde{G}(V)}(f) - 2\nabla_{\tilde{G}(V)dF}(f) - 2\Delta_{\tilde{G}(V)}(F)f - 2nV[F]f). \tag{7.5}$$

From this equation we immediately get an expression for H:

$$H(V) = E(V)(1) = \frac{1}{2}(\Delta_{\tilde{G}(V)}(F) + nV[F]).$$

We can summarize the previous results by writing the following formula for the formal Hitchin connection studied by Andersen:

$$\begin{aligned} D_V f =& V[f] - \frac{1}{4}h\Delta_{\tilde{G}(V)}(f) + \frac{1}{2}h\nabla_{\tilde{G}(V)dF}(f) + V[F] \star^{BT} f - V[F]f \\ &- \frac{1}{2}h(\Delta_{\tilde{G}(V)}(F) \star^{BT} f - nV[F] \star^{BT} f - \Delta_{\tilde{G}(V)}(F)f - nV[F]f). \end{aligned} \tag{7.6}$$

The following lemma, proved in [And12], shows that the formal Hitchin connection coming from geometric quantization is a derivation with respect to the Berezin-Toeplitz star product, and thereby that it is a formal connection compatible with this family of star products.

Proposition 7.5. *The formal operator D_V is a derivation with respect to the star product $\star_\sigma^{BT}$ for each $\sigma \in \mathcal{T}$, meaning that it satisfies the relation:*

$$D_V(f_1 \star^{BT} f_2) = D_V(f_1) \star^{BT} f_2 + f_1 \star^{BT} D_V(f_2) \tag{7.7}$$

for all $f_1, f_2 \in C^\infty(M)$.

The star product can be expressed as

$$f_1 \star^{BT} f_2 = \sum_{k\geq 0} c^{(k)}(f_1, f_2)h^k.$$

By applying this relation in the formula above we get another expression for $\tilde{A}$:

$$\tilde{A}_V(f) = hE(V)(f) - \sum_{k\geq 1} \left(c^{(k)}(V[F], f) + c^{(k-1)}(H(V), f)\right) h^k,$$

which in particular highlights that $\tilde{A}$ is zero modulo h.

We conclude this section by stating the result about the flatness of the formal Hitchin connection coming from geometric quantization that was proved by Andersen in [And12].

Proposition 7.6. *If the Hitchin connection $\hat{\nabla}$ in $\hat{\mathcal{Q}}_k$ is projectively flat, then the formal Hitchin connection associated to it $D_V = V + \tilde{A}(V)$ in C_h is flat.*

7.2.2 The formal Hitchin connection at low orders

The fact that the formal Hitchin connection obtained by Andersen is a derivation of the Berezin-Toeplitz star product, which we saw in Proposition 7.5, was proved by using the theory of geometric quantization and Toeplitz operators, translating those results to the setting of deformation quantization, and thereby its validity relies on the fact that there exists a Hitchin connection, which puts many requirements on the objects involved, and among the other things requires that the family of compatible Kähler structures is rigid and holomorphic.

But the explicit expression we have for the formal Hitchin connection studied by Andersen (7.6), in terms of differential operators, makes sense in a more general setting. Therefore we can ask ourselves to which extent that expression defines a formal connection of the Berezin-Toeplitz star product, or in other words, whether that expression defines a derivation of the star product.

Here we shall answer this question up to order one with a direct computation, for which we need not assume that a Hitchin connection exists.

We include here two preliminary lemmas about properties of the coefficients of the Berezin-Toeplitz star product. Recall that we use the notation $c^{(k)}$ for the coefficients, when we consider the Berezin-Toeplitz star product.

Lemma 7.7. *Let M be a symplectic manifold with a family of compatible Kähler structures parametrized by a manifold $\mathcal{T}$ on it, and let $c^{(k)}$ denote the coefficients of the Berezin-Toeplitz star product of Wick-type associated to the complex structure for a certain $\sigma \in \mathcal{T}$. Then we have:*

$$V[c^{(1)}](f_1, f_2) = \frac{1}{4}\left(\Delta_{\tilde{G}(V)}(f_1 f_2) - \Delta_{\tilde{G}(V)}(f_1)f_2 - \Delta_{\tilde{G}(V)}(f_2)f_1\right). \tag{7.8}$$

Proof. From a result of Karabegov [Kar96], we have that the degree 1 coefficient of the Berezin-Toeplitz star product can be written as:

$$c^{(1)}(f_1, f_2) = g(\partial f_1, \bar{\partial} f_2) = i\nabla_{X''_{f_2}}(f_1), \tag{7.9}$$

for any functions $f_1, f_2 \in C^\infty(M)$, where X_f is the Hamiltonian vector field associated to f. By differentiating equation (7.9), we get the following relation:

$$V[c^{(1)}](f_1, f_2) = \frac{1}{2} df_1 \tilde{G}(V) df_2. \tag{7.10}$$

The operator $\Delta_{\tilde{G}(V)}$ is written as $\Delta_{\tilde{G}(V)} = \nabla^2_{\tilde{G}(V)} + \nabla_{\delta\tilde{G}(V)}$ by (5.5), so the right hand side of (7.8) becomes:

$$\frac{1}{4}\left(\nabla^2_{\tilde{G}(V)}(f_1 f_2) - \nabla^2_{\tilde{G}(V)}(f_1) f_2 - \nabla^2_{\tilde{G}(V)}(f_2) f_1\right),$$

since the summand of order one vanishes. We can express the symmetric bivector field $\tilde{G}(V)$ as $\sum_j (X_j \otimes Y_j)$ for vector fields X_j and Y_j, and so we rewrite the previous expression as:

$$\begin{aligned}
&\frac{1}{4}\left(\sum_j \nabla_{X_j}\nabla_{Y_j}(f_1 f_2) - f_2 \sum_j \nabla_{X_j}(\nabla_{Y_j} f_1) - f_1 \sum_j \nabla_{X_j}(\nabla_{Y_j} f_2)\right) \\
&= \frac{1}{4}\sum_j \left(\nabla_{Y_j} f_1 \nabla_{X_j} f_2 + \nabla_{X_j} f_1 \nabla_{Y_j} f_2\right) = \frac{1}{2}\sum_j \left(\nabla_{X_j} f_1 \nabla_{Y_j} f_2\right) = \frac{1}{2} df_1 \tilde{G}(V) df_2,
\end{aligned}$$

where we use the symmetry of the bivector field, and this concludes the proof. □

Remark 7.8. Let us note that the expression we got for $V[c^{(1)}]$ also shows that it is symmetric in the two variables.

The first lemma is about a derivation property of $c^{(1)}$.

Lemma 7.9. *Let $f_1, f_2, f_3 \in C^\infty(M)$ be smooth functions. Then $c^{(1)}$ satisfies the following relation:*

$$c^{(1)}(f_1 f_2, f_3) = f_1 c^{(1)}(f_2, f_3) + f_2 c^{(1)}(f_1, f_3),$$

and the corresponding result holds when the product is at the second coordinate.

Proof. By (7.9) we have:

$$\begin{aligned}
c^{(1)}(f_1 f_2, f_3) &= g(\partial(f_1 f_2), \bar{\partial} f_3) \\
&= f_1 g(\partial(f_2), \bar{\partial} f_3) + f_2 g(\partial(f_1), \bar{\partial} f_3) \\
&= f_1 c^{(1)}(f_2, f_3) + f_2 c^{(1)}(f_1, f_3).
\end{aligned}$$

□

Note that the previous result holds for any Karabegov star product, as one can easily verify looking at the graph theoretical expression for c^1 obtained in Chapter 6.

We are now ready to show that the expression (7.6) always gives a derivation of the Berezin-Toeplitz star product, up to order 1 in h. The derivation relation (7.7) can be written as:

$$f_1 \; V[\star^{BT}] \; f_2 = \tilde{A}(V)(f_1) \star^{BT} f_2 + f_1 \star^{BT} \tilde{A}(V)(f_2) - \tilde{A}(V)(f_1 \star^{BT} f_2), \tag{7.11}$$

where $V[\star^{BT}]$ denotes the product with coefficients $V[c^{(k)}]$. We can write this equation modulo h^2, recalling that $\tilde{A}^0 = 0$, and this way we see that the condition that has to hold is the following, for all vector fields V on $\mathcal{T}$ and smooth functions f_1 and f_2 on M:

$$V[c^{(1)}](f_1, f_2) = -\tilde{A}^1(V)(f_1 f_2) + \tilde{A}^1(V)(f_1) f_2 + f_1 \tilde{A}^1(V)(f_2). \tag{7.12}$$

If we extract the expression for $\tilde{A}^1$ from (7.6) we obtain:

$$\tilde{A}^1(V)(f) = -\frac{1}{4}\Delta_{\tilde{G}(V)}(f) + c^{(1)}(V[F], f) + V[c^{(1)}](F, f),$$

which can be inserted into (7.12) to get:

$$V[c^{(1)}](f_1, f_2) = \frac{1}{4}\left(\Delta_{\tilde{G}(V)}(f_1 f_2) - f_2 \Delta_{\tilde{G}(V)}(f_1) - f_1 \Delta_{\tilde{G}(V)}(f_2)\right),$$

since the summands of order one vanish. But this is precisely the result we obtained in Proposition 7.7.

Moreover we can check that the expression (7.6) defines a formal connection that is flat up to order one in h. This amounts to showing that its curvature vanishes modulo h^2. Note that the only terms that we see when we compute the curvature modulo h^2 are those coming from $d_{\mathcal{T}}\tilde{A}^1$, since the commutator terms in the curvature are multiple of h^2. Therefore our expression defines a flat connection up to order one if and only if $\tilde{A}^1$ is closed. In our discussion about trivialization in the next chapter we will see that $\tilde{A}^1$ is actually exact, because we can define a 0-form $P_1 = \frac{1}{4}\Delta - c^{(1)}(F, f)$ such that $V[-P_1] = \tilde{A}^1(V)$ for any vector field on $\mathcal{T}$, as we show in Proposition 8.4. In particular we obtain that $\tilde{A}^1$ is closed.

The discussion above can be summed up in the following proposition.

Proposition 7.10. *Let M be a symplectic manifold with a family of compatible Kähler structures parametrized by a complex manifold $\mathcal{T}$. Then the expression* (7.6) *defines a formal connection that, up to order one in the formal parameter, is a derivation of the family of Berezin-Toeplitz star products on M, and that is flat up to order one in the formal parameter. Therefore it defines up to order one a formal Hitchin connection in the sense of Definition 7.3.*

Remark 7.11. Doing the same check at higher orders becomes much more difficult, because we do not have a simple expression for higher coefficients of the star product as we did at order one.

Instead we can take a different point of view, and start from the fact that the formal Hitchin connection is a derivation of the Berezin-Toeplitz star product. If we compute what that entails, we can get expressions for the variation of the higher coefficients of the star product. If we go through this process at order 2, we get the following expression:

$$
\begin{aligned}
&V[c^{(2)}](f_1, f_2) = \\
&\qquad - c^{(2)}(V[F], f_1 f_2) + f_1 c^{(2)}(V[F], f_2) + f_2 c^{(2)}(V[F], f_1) \\
&\qquad + \frac{1}{4}\left[\Delta_{\tilde{G}(V)} c^{(1)}(f_1, f_2) - c^{(1)}(\Delta_{\tilde{G}(V)}(f_1), f_2) - c^{(1)}(f_1, \Delta_{\tilde{G}(V)}(f_2))\right] \\
&\qquad - V[c^{(1)}](F, c^{(1)}(f_1, f_2)) + c^{(1)}(V[c^{(1)}](F, f_1), f_2) + c^{(1)}(f_1, V[c^{(1)}](F, f_2)) \\
&\qquad - c^{(1)}(V[F], c^{(1)}(f_1, f_2)) + c^{(1)}(c^{(1)}(V[F], f_1), f_2) + c^{(1)}(f_1, c^{(1)}(V[F], f_2)).
\end{aligned}
\tag{7.13}
$$

This will be used in Chapter 8 to calculate the equations that a trivialization of the formal Hitchin connection has to satisfy.

7.3 Derivations of star products

We shall now put aside the formal Hitchin connection and look at formal connections in general. We aim to give a result describing the space of formal connections on a symplectic manifold and to prove their existence under some assumptions.

We begin this study by studying the space of derivations of a star product. Let us recall the notion we are considering: we say that a map $B\colon C_h^\infty(M) \to C_h^\infty(M)$ is a derivation of the star product if it satisfies the relation:

$$B(f \star g) = B(f) \star g + f \star B(g), \tag{7.14}$$

for any f, g smooth (formal) functions on M. Let us now introduce some notation that will be useful in following results.

Definition 7.12. If $\star$ is a star product on (M, ω), we define the *star commutator* by:

$$[f, g]_\star = f \star g - g \star f,$$

for f, g smooth (formal) functions on M.

Let us note that this definition makes $C_h^\infty(M)$ into a Lie algebra with adjoint representation

$$\mathrm{ad}_\star f = [f, \cdot]_\star.$$

Remark 7.13. Since the term of order 0 in the star product is the pointwise commutative product, then $\mathrm{ad}_\star f$ is always divisible by the formal parameter h, and the first term in the expansion is the Poisson bracket[1], because of the properties of star product:

$$\mathrm{ad}_\star f(g) = h\{f, g\} + \dots.$$

Gutt and Rawnsley [GR99] give a characterization of self-equivalences of star products under a cohomological condition.

[1]we remove the i factor in front of it by renormalizing the formal parameter, to simplify the computations

Proposition 7.14 (Gutt, Rawnsley). *Let $\star$ be a differential star product on a symplectic manifold (M, ω), and suppose that $H^1(M;\mathbb{R}) = 0$. Then any self-equivalence $A = \mathrm{Id} + \sum_{k\geq 1} A_k h^k$ of $\star$ is* inner, *namely it can be written as $A = \exp \mathrm{ad}_\star(a)$ for a formal function $a \in C_h^\infty(M)$.*

The following proposition was proved in [GR99] and shows that any derivation of a star product on a symplectic manifold with trivial first cohomology is *essentially inner*, which means that is can be expressed as the adjoint representation of a formal smooth function. Here we include a proof by induction that shows how to construct the required element of $C_h^\infty(M)$.

Proposition 7.15. *Let (M, ω) be a symplectic manifold satisfying $H^1(M;\mathbb{R}) = 0$, and let $\star$ be a star product on M. If $B\colon C_h^\infty(M) \to C_h^\infty(M)$ is a derivation of the star product, then B there exists $b \in C_h^\infty(M)$ such that $B = h^{-1}\,\mathrm{ad}_\star(b)$.*

Proof. Let us write the map B as:

$$B = \sum_{k\in\mathbb{N}} B^k h^k$$

and prove the statement inductively. If we look at (7.14), it is immediate to see that B^0 satisfies:

$$B^0(fg) = B^0(f)g + fB^0(g),$$

for any smooth functions f and g. This means that B^0 is a differential operator of order 1 and so a vector field. If we look now at the terms of the same equation that are of order 1 in the formal parameter, then we obtain the relation:

$$B^0(c^1(f,g)) - c^1(B^0(f), g) - c^1(f, B^0(g)) = -B^1(fg) + B^1(f)g + fB^1(g).$$

Since the right hand side of the equation is symmetric in f and g, so is the left hand side and we can then symmetrize it and use property (3) from Definition 4.1 to get:

$$B^0(\{f,g\}) - \{B^0(f), g\} - \{f, B^0(g)\}.$$

This means that B^0 is a symplectic vector field. But we can now use our cohomological assumption to conclude that it is also Hamiltonian, and so we obtain that there exists $b_0 \in C^\infty(M)$ so that B^0 can be written as $B^0 = \{b_0, \cdot\}$. In other words we have proved that

$$hB = \mathrm{ad}_\star(b_0) + O(h^2).$$

Let us now assume that

$$hB = \mathrm{ad}_\star(b^{(k-1)}) + O(h^{k+1})$$

holds for $k \in \mathbb{N}$ where $b^{(k-1)} = b_0 + b_1 h + \cdots + b_{k-1}h^{k-1}$ and b_i is a smooth function on M for each i. This can be rewritten as

$$hB = \mathrm{ad}_\star(b^{(k-1)}) + \beta_k h^{k+1} + O(h^{k+2}).$$

Let us denote

$$B' = hB - \mathrm{ad}_\star(b^{(k-1)}) = \beta_k h^{k+1} + O(h^{k+2}). \tag{7.15}$$

It is immediate to see that B' is a derivation of the star product, since it is the sum of two derivations. If we look at the right hand side of (7.15) at the order $k+1$ of the formal parameter and use the derivation relation, then we obtain that the following holds:

$$\beta_k(fg) = \beta_k(f)g + f\beta_k(g)$$

for any smooth function f and g on M, showing that β_k is a differential operator of order 1, or a vector field. We can look now at the same equation at the order $k+2$ of the formal parameter, and we obtain that the expression

$$\beta_k(c^1(f,g)) - c^1(\beta_k(f),g) - c^1(f,\beta_k(g))$$

is equal to terms that are all symmetric in f and g, therefore we can symmetrize as done earlier and conclude that:

$$\beta_k(\{f,g\}) - \{\beta_k(f),g\} - \{f,\beta_k(g)\},$$

therefore we can write β_k as $\beta_k = \{b_k,\cdot\}$, for a $b_k \in C^\infty(M)$. Let us define $b^{(k)} = b^{(k-1)} + b_k h^k$. Then we have:

$$[hB - \mathrm{ad}_\star(b^{(k)})](f) = hB(f) - (b^{(k-1)} + b_k h^k)\star f + f \star (b^{(k-1)} + b_k h^k),$$

and it is an immediate computation to check that all the terms of order less or equal to $k+2$ vanish. Therefore we can write

$$hB = \mathrm{ad}_\star(b^{(k)}) + O(h^{k+2}),$$

concluding the inductive step and our proof. $\square$

Note that, since the star products we consider are null on constants, if $b \in C^\infty_h(M)$ is a constant formal functions, the associated essentially inner derivation is the null function. We shall use the notation $\tilde{C}^\infty_h(M)$ to indicate the space of formal functions on M modulo the constant ones.

Let us denote with $\mathrm{Der}(M,\star)$ the space of derivations of the star product $\star$ on M, and let $\mathrm{Der}_0(M,\star)$ denote the subset of the derivations that are zero modulo h.

Proposition 7.16. *Let (M,ω) be a symplectic manifold satisfying $H^1(M;\mathbb{R}) = 0$, and let $\star$ be a star product on M. The space of formal functions on M modulo the constants is isomorphic to the space of derivations of $\star$ that are zero modulo h via the map:*

$$\begin{aligned} \tilde{C}^\infty_h(M) &\to \mathrm{Der}_0(M,\star) \\ b &\mapsto \mathrm{ad}_*(b). \end{aligned}$$

Proof. It is immediate to check that, for any formal smooth function $b \in C^\infty_h(M)$, the star commutator $\mathrm{ad}_*(b)$ is a derivation of $\star$, and that it is trivial modulo h. If we add to b any constant $b_0 \in \mathbb{C}[[h]]$, then clearly $\mathrm{ad}_*(b+b_0) = \mathrm{ad}_*(b)$, since the star product is null on constants, so the map is well defined on the quotient $\tilde{C}^\infty_h(M)$ and it is linear. To check injectivity we assume that $\mathrm{ad}_*(b)$ is trivial for a formal smooth function b. In correspondence of h^1 this means that $\{b,f\} = 0$ for all $f \in C^\infty_h(M)$, and this implies that b is a constant. Surjectivity follows from Proposition 7.15: if A is any derivation in $\mathrm{Der}_0(M,\star)$, then it is divisible by h, and A is again a derivation. So by the previous proposition, we can find $b \in C^\infty_h(M)$ such that $h^{-1}A = h^{-1}\,\mathrm{ad}_*(b)$, and therefore $A = \mathrm{ad}_*(b)$. $\square$

7.4 The affine space of formal connections

Let D and D' be two formal connections on M for the same family of star products parametrized by $\mathcal{T}$. It is immediate to see that

$$D'_V - D_V = A'(V) - A(V) = (A' - A)(V).$$

Hence their difference is a 1-form on $\mathcal{T}$ with values in the derivations of the star products of the family, and it is zero modulo h.

If M is a symplectic manifold equipped with a family of star products $\{\star_\sigma\}_{\sigma\in\mathcal{T}}$ parametrized by $\mathcal{T}$, let us denote by $\mathcal{F}(M, \star_\sigma)$ the space of the formal connections that are compatible with the family.

We see that $\mathcal{F}(M, \star_\sigma)$ is an affine space over the space of 1-forms on $\mathcal{T}$ with values in the derivations of the star product that are zero modulo h, and thus it can be written as:

$$\mathcal{F}(M, \star_\sigma) = D_0 + \Omega^1(\mathcal{T}, \mathrm{Der}_0(M, \star_\sigma)),$$

for a fixed formal connection D_0.

If we assume that $H^1(M; \mathbb{R})$ vanishes, then Proposition 7.16 tells us that all derivations of $\star$ are essentially inner, and therefore they are parametrized by an element in $\tilde{C}_h^\infty(M)$, the space of formal functions on M modulo the constants. Therefore the compatible formal connections form an affine space modelled on the 1-forms on $\mathcal{T}$ with values in $\tilde{C}_h^\infty(M)$.

$$\mathcal{F}(M, \star_\sigma) \cong D_0 + \Omega^1(\mathcal{T}, \tilde{C}_h^\infty(M)),$$

also in this case for a fixed formal connection D_0.

7.5 Gauge transformations of formal connections

We shall study gauge transformations in the space of formal connections $\mathcal{F}(M, \star_\sigma)$. The transformations we look at are self-equivalences of the family of star products, since the connections should still act as derivations when we transform them. This means that we look at $P \in C^\infty(\mathcal{T}, \mathcal{D}_h(M))$ such that

$$P_\sigma(f \star_\sigma g) = P_\sigma(f) \star_\sigma P_\sigma(g), \tag{7.16}$$

for any $\sigma \in \mathcal{T}$ and any smooth function f and g and such that P_σ is invertible for any $\sigma \in \mathcal{T}$.

Theorem 7.17. *Let M be a symplectic manifold with a family of star products $\{\star_\sigma\}_{\sigma\in\mathcal{T}}$ parametrized by a smooth manifold $\mathcal{T}$ such that $H^1(\mathcal{T}, \mathbb{R}) = 0$. Let $D, D' \in \mathcal{F}(M, \star_\sigma)$ be formal connections for the family and let us assume that they are flat. Then they are gauge equivalent via a self-equivalence of the family of star products $P \in C^\infty(\mathcal{T}, \mathcal{D}_h(M))$, meaning that*

$$D'_V = P^{-1} D_V P, \tag{7.17}$$

for any vector field V on $\mathcal{T}$.

Proof. As usual we can write the formal connections in the form:

$$D_V = V + A(V)$$
$$D_V = V + A'(V),$$

for two 1-forms $A, A' \in \Omega(\mathcal{T}, \mathcal{D}_h(M))$ with values in formal differential operators on M, and any vector field V on $\mathcal{T}$. Then we can rewrite (7.17) plugging in a section f of the bundle as:

$$\begin{aligned} V[f] + A'(V)(f) &= P^{-1}(V + A(V))P(f) \\ &= P^{-1}\left(V[P](f) + P(V[f]) + A(V)(P(f))\right). \end{aligned} \tag{7.18}$$

Therefore if we apply P on both sides we get the following equation:

$$V[P] = PA'(V) - A(V)P. \tag{7.19}$$

If we can find a $P = \sum_{k\in\mathbb{N}} P_k h^k$ that solves the equation, then we get the wanted gauge transformation. To do so we proceed inductively: if we look at (7.19) modulo h, we see that it is enough to choose a P_0 such that $V[P_0] = 0$.

Let us now assume that we have determined $P^{(l)} = \sum_{k\leq l} P_k h^k$ such that

$$V[P^{(l)}] = P^{(l)}A'(V) - A(V)P^{(l)} + O(h^{l+1}).$$

This can be written as:

$$B_{l+1}(V)h^{l+1} = V[P^{(l)}] - (P^{(l)}A'(V) - A(V)P^{(l)}) + O(h^{l+2}), \tag{7.20}$$

where B_{l+1} is a 1-form on $\mathcal{T}$ with values in differential operators on M. Let us define a 1-form α_l on $\mathcal{T}$ with values in formal differential operators on M by $\alpha_l(V) = (P^{(l)}A'(V) - A(V)P^{(l)})$. We want to show that α_l is closed modulo h^2. At any point on $\mathcal{T}$ we can choose two vector fields V and W that commute. Then we have that:

$$\begin{aligned} &d_{\mathcal{T}}\alpha_l(V, W) \\ &= V[\alpha(W)] - W[\alpha(V)] \\ &= V[P^{(l)}A'(W) - A(W)P^{(l)}] - W[P^{(l)}A'(V) - A(V)P^{(l)}] \\ &= P^{(l)}\left(A'(V)A'(W) - A'(W)A'(V) + V[A'(W)] - W[A'(V)]\right) \\ &\quad - \left(A(V)A(W) - A(W)A(V) + V[A(W)] - W[A(V)]\right)P^{(l)} \\ &\quad + h^{l+1}(B_{l+1}(V)A'(W) - A(W)B_{l+1}(V) - B_{l+1}(V)A'(V) + A(V)B_{l+1}(V)), \end{aligned} \tag{7.21}$$

where to obtain the last equality we substitute again the expression for $V[P^{(l)}]$ and $W[P^{(l)}]$. Note also that the following expression, which appears in the last line of the equation,

$$B_{l+1}(V)A'(W) - A(W)B_{l+1}(V) - B_{l+1}(V)A'(V) + A(V)B_{l+1}(V)$$

is multiple of h. Let us now compute the expressions for the curvature of D, which we are assuming is flat:

$$0 = F_D(V, W) = D_V D_W - D_W D_V - D_{[V,W]}.$$

The last summand vanishes because we chose commuting vector fields, hence we get, for any section:

$$\begin{aligned} 0 &= A(V)A(W)(f) - A(W)A(V)(f) - W[A(V)(f)] \\ &\quad + A(V)W[f] + V[A(W)(f)] - A(W)V[f] \\ &= A(V)A(W)(f) - A(W)A(V)(f) - W[A(V)](f) + V[A(W)](f), \end{aligned} \tag{7.22}$$

which is the same as:

$$0 = F_D(V, W) = A(V)A(W) - A(W)A(V) + V[A(W)] - W[A(V)].$$

By computing the curvature in the same way for D' we obtain:

$$0 = F_{D'}(V, W) = A'(V)A'(W) - A'(W)A'(V) + V[A'(W)] - W[A'(V)].$$

By comparing with (7.21), we see that $d_{\mathcal{T}}\alpha_l = 0 \pmod{h^{l+2}}$, and therefore, by (7.20), we also have that $d_{\mathcal{T}}B_{l+1} = 0 \pmod{h^{l+2}}$. Since $H^1(\mathcal{T}; \mathbb{R})$ is trivial, we have that the 1-form B_{l+1} is exact modulo h^{l+2}, meaning that there exists a smooth function $P_{l+1} \colon \mathcal{T} \to \mathcal{D}(M)$ such that $V[P_{l+1}h^{l+1}] = -B_{l+1}h^{l+1} + O(h^{l+2})$. So we can set $P^{(l+1)} = P^{(l)} + P_{l+1}h^{l+1}$ as the notation suggests. To conclude the inductive step and the proof it is enough to show that:

$$V[P^{(l+1)}] - (P^{(l+1)}A'(V) - A(V)P^{(l+1)}) = 0 \pmod{h^{l+2}}.$$

By expanding we get:

$$V[P^{(l)}] - (P^{(l)}A'(V) - A(V)P^{(l)}) - h^{l+1}B_{l+1} + h^{l+1}(P_{l+1}A'(V) - A(V)P_{l+1}),$$

which is a multiple of h^{l+2} because of how we have chosen P_{l+1} and because the expression $P_{l+1}A'(V) - A(V)P_{l+1}$ is a multiple of h. $\square$

As seen earlier in Proposition 7.6, Andersen showed [And12] that the formal Hitchin connection associated to the Hitchin connection of geometric quantization is flat if the Hitchin connection is projectively flat. Therefore we have the following corollary.

Corollary 7.18. *Let $\mathcal{T}$ be a smooth manifold such that $H^1(\mathcal{T}; \mathbb{R}) = 0$. If there exists a formal Hitchin connection D in the bundle C_h on $\mathcal{T}$, then it is unique up to gauge equivalence.*

7.6 Existence of formal connections

In this section we study the problem of the existence of formal connections on a given symplectic manifold (M, ω) equipped with a smooth family of natural star products parametrized by $\mathcal{T}$. As we will see, this problem can be re-conducted to a cohomological condition via Hochschild cohomology.

In fact, if we look at Definition 7.2, we can immediately obtain that, given a family of natural star products $\{\star_\sigma\}_{\sigma\in\mathcal{T}}$, there exists a formal connection $D_V = V + A(V)$ compatible with the family if and only if A satisfies the following relation:

$$A(V)(f \star g) - A(V)(f) \star g - f \star A(V)(g) = fV[\star]g, \tag{7.23}$$

for any $f, g\colon \mathcal{T} \to C_h^\infty(M)$, where we write $V[\star]$ to indicate the product whose i-th coefficient is $V[c^i]$.

It is immediate to observe that the trivial solution, corresponding to $A = 0$ is not a candidate for a formal connection, because the equation above is clearly not satisfied in that case. It is on the contrary required some sort of interplay between $A(V)$ and the star product for the relation to have a chance of being true.

We note that $B(f, g) = fV[\star]g$, for f and g depending on $\sigma \in \mathcal{T}$, can be seen as a cochain in the Hochschild cochain complex, which we introduced in Section 2.11. Hence we can interpret (7.23) as in the following proposition.

Proposition 7.19. *Let (M, ω) be a symplectic manifold and let $\{\star_\sigma\}$ be a family of natural star products on M parametrized by $\mathcal{T}$. The family admits a compatible formal connection if and only if the Hochschild 2-cochain $B(\sigma, V) = V[\star_\sigma]$ is exact for any $\sigma \in \mathcal{T}$ and any vector field V on $\mathcal{T}$, namely if and only if there exists a 1-cochain $A(\sigma, V)$ such that $A(\sigma, V) = 0 \pmod h$ that satisfies the equation*

$$dA(\sigma, V) = B(\sigma, V),$$

where d denotes the coboundary operator with respect to the star product $\star_\sigma$.

We aim to show that $B(\sigma, V)$ is exact, and thereby a formal connection compatible with the family of natural star products exists, if and only if the characteristic class across the family is constant. To prove this result we do a modified version of Fedosov's construction of a star product that takes into account the fact that we have a family of products, and not only one. This way we can construct A and show that B is exact. In the next sections we shall see in detail the steps of our construction that leads to the proof of our main result, stated in Theorem 7.21.

Remark 7.20. If we consider the following part of the Hochschild cochain complex

$$B_1 \xrightarrow{d} B_2 \xrightarrow{d} B_3,$$

we can note that the condition $d^2 = 0$ is in fact equivalent to the associativity of the star product, as one can check easily specializing the definition of the differential to the relevant degrees of the complex:

$$d(C)(f, g) = c(f \star g) - c(f) \star g - f \star c(g), \quad \text{for } C \in \mathcal{B}_1,$$
$$d(B)(f, g, h) = B(f, g \star h) - B(f, g) \star h + f \star B(g, h) - B(f \star g, h), \quad \text{for } B \in \mathcal{B}_2.$$

Theorem 7.21. *Let ∇ be a symplectic connection for (M, ω), and let $\{\star_\sigma\}$ be a smooth family of star products parametrized by $\mathcal{T}$. Let α_σ denote the smooth family of formal 2-forms on M representing the characteristic class of the family of star products,*

$$\alpha_\sigma \in h\Omega^2_{\mathcal{T}}(M)[[h]].$$

Then the following statements are equivalent:

1. *The cohomology class of α is constant in $\mathcal{T}$.*
2. *There is a* 1*-form $A \in \Omega^1(\mathcal{T}, h\mathcal{D}(M)[[h]])$ with values in formal differential operators on M such that for any vector field V on $\mathcal{T}$, and any smooth functions f and g on M the identity*

$$fV[\star]g = A(V)(f \star g) - A(V)(f) \star g - f \star A(V)(g)$$

holds.
3. *The family of star products admits a formal connection.*

In the theorem, $\star$ denotes the smooth family of star products on M, which is parametrized by $\mathcal{T}$ and that is given by performing Fedosov's construction fibrewise. The second equivalent condition in the theorem says in particular that the variation of $\star$ is exact in the second Hochschild cohomology of the family, which is the condition for the existence of a formal connection that we have identified in Proposition 7.19. Therefore we know already that the second and the third conditions are equivalent. It remains to prove that the first two conditions are equivalent, which will be the topic of the most of the remaining part of this chapter.

Remark 7.22. By the result of Gutt and Rawnsley seen in Theorem 4.10, one can extend the result in a straight-forward manner to all natural star products, since they are all equivalent to Fedosov star products.

7.7 Smooth families of differential forms

Suppose M and $\mathcal{T}$ are smooth manifolds. We will consider smooth families of objects on M, parametrized by $\mathcal{T}$. Let

$$\Omega_{\mathcal{T}}(M) = C^\infty(\mathcal{T} \times M, (\pi_2)^*(\Lambda T^*M)),$$

where π_2 denotes the projection $\mathcal{T} \times M \to M$, be the space of differential forms on M parametrized by $\mathcal{T}$. We denote the subspaces of differential forms of degree k by $\Omega^k_{\mathcal{T}}(M)$, and we have:

$$\bigoplus_k \Omega^k_{\mathcal{T}}(M) = \Omega_{\mathcal{T}}(M)$$

A family $\alpha \in \Omega_{\mathcal{T}}(M)$ associates to each $\sigma \in \mathcal{T}$ a differential form α_σ on M.

Proposition 7.23. *Assume that α is a family of closed differential forms in $\Omega_{\mathcal{T}}(M)$ and let k be a natural number for which the map*

$$\begin{aligned} [\alpha_\cdot] \colon \mathcal{T} &\to H^k(M) \\ b &\mapsto [\alpha_b] \end{aligned}$$

is constant. Let $b_0 \in \mathcal{T}$ be a fixed basepoint. Then there exists a smooth family of $(k-1)$-forms $\beta \in \Omega^{k-1}_{\mathcal{T}}(M)$ such that

$$\alpha_b - \alpha_{b_0} = d_M(\beta_b).$$

Proof. This follows from the existence of a continuous linear map

$$d^* : d(\Omega^{k-1}(M)) \to \Omega^{k-1}(M),$$

called the anti-differential, such that $dd^*\gamma = \gamma$ holds, and which maps smooth families to smooth families. For M compact, d^* might be constructed using Hodge-theory. For arbitrary M one can use the Čech-de Rham double complex to construct such an operator, as done in [AG14]. Apply this to the family $\alpha - \alpha|_{b_0}$, which is exact, to obtain β. □

Equivalently, one can phrase the result as follows:

Proposition 7.24. *Assume that the differential forms in $\Omega_{\mathcal{T}}(M)$ are closed and let k be a natural number for which the map*

$$\begin{aligned} [\alpha.]\colon \mathcal{T} &\to H^k(M) \\ b &\mapsto [\alpha_b] \end{aligned}$$

is constant. Then there is $\beta \in \Omega^1(\mathcal{T}, \Omega^{k-1}(M))$ such that $d_M\alpha = 0$ and $d_{\mathcal{T}}\alpha = d_M\beta$.

7.8 Fibrewise Fedosov products

Having in mind the construction of the Fedosov star product seen in Chapter 4, we do here a similar construction for the situation of a manifold with a smooth family of star products.

Let (M, ω) be a symplectic manifold of dimension $m = 2n$. We consider now the case of a family of closed 2-forms α_σ on M parametrized by a manifold $\mathcal{T}$ and describe a fibrewise construction of a family of Fedosov star products having these forms as their characteristic class.

We shall complete the construction of the formal connection in the special case where the parameter space $\mathcal{T}$ is just the interval $[0, 1]$ for simplicity. Let t be the global coordinate on the interval.

Our aim is to understand the dependence of Fedosov's star product on the choice of the closed 2-form. We know that star products are classified up to equivalence by an element of $H^2(M)[[h]]$, namely their characteristic class, and as we have seen earlier in Chapter 4, the characteristic class of $\star_{\nabla,\alpha_t}$ is represented by α_t.

In other words, two Fedosov star products $\star_{\nabla,\alpha}$ and $\star_{\nabla,\tilde{\alpha}}$ are equivalent if and only if the classes of α and $\tilde{\alpha}$ coincide in cohomology.

We assume now that $\alpha_t \in \Omega^2(M)[[h]]$ is a 1-parameter family of closed forms with constant cohomology class.

We also choose a primitive $\beta_t \in \Omega^1(M)[[h]]$, such that

$$\frac{d}{dt}\alpha_t = d_M\beta_t$$

holds for all $t \in [0, 1]$, by Proposition 7.23.

Let us consider Fedosov's construction, tensored with the de Rham algebra $(\Omega([0,1]), dt\frac{d}{dt})$ of the interval. This means that we first repeat Fedosov's construction for fixed t to obtain $r_t \in \Omega^1(M, W(TM))[[h]]$, and hence an abelian connection

$$D_{r_t} := -\delta + d_\nabla + \frac{i}{h}\operatorname{ad}(r_t),$$

as done in Chapter 4.

Our next aim is to understand the variation of D_{r_t} with respect to t. One way to achieve this is to look for an operator

$$\hat{D}_{s_t} = dt\frac{\partial}{\partial t} + \frac{i}{h}dt\operatorname{ad}(s_t),$$

for appropriate $s_t \in \Omega^0_T(M, W(TM))[[h]]$, such that

$$[\hat{D}_{s_t}, D_{r_t}] = 0$$

holds. Computing this yields:

$$\frac{i}{h}dt\operatorname{ad}\left(\left(-\delta + d_\nabla + \frac{i}{h}\operatorname{ad}(r_t)\right)(s_t) + \frac{i}{h}\frac{\partial r_t}{\partial t}\right).$$

Hence we need that the element

$$\gamma_t = \left(-\delta + d_\nabla + \frac{i}{h}\operatorname{ad}(r_t)\right)(s_t) + \frac{i}{h}\frac{\partial r_t}{\partial t}$$

is central, or in other words that γ_t of $\Omega_{[0,1]}(M)[[h]]$. To impose this we consider the equation:

$$\delta(s_t) = d_\nabla s_t + \frac{i}{h}\left([r_t, s_t] + \frac{\partial r_t}{\partial t}\right) + \gamma_t.$$

The following lemma gives a necessary condition for finding such an element.

Lemma 7.25. *Suppose $s_t \in \Omega^0(M, W(TM))[[h]]$ is such that*

$$\gamma_t = \delta(s_t) - \left(\frac{i}{h}\left([r_t, s_t] + \frac{\partial r_t}{\partial t}\right)\right)$$

is central. Then

$$d\gamma_t - \frac{\partial \alpha_t}{\partial t} = 0$$

holds, i.e. γ_t is a primitive for the family α_t

This is proved by applying D_r to both sides. On the left hand side we get $d\gamma_t$. On the right hand side we have several addends, but the only component in $\Omega(M)[[h]]$ is $\delta(\frac{\partial r_t}{\partial t}) = \frac{\partial \alpha_t}{\partial t}$.

The following proposition can be proved with the same argument used to determine r in Fedosov's construction.

Proposition 7.26. *Let β_t be a primitive for α_t. There is a unique*

$$s \in \Omega^0(M, W(TM))[[h]]$$

such that

$$\begin{aligned}\delta(s_t) =& d_\nabla s_t + \frac{i}{h}\left([r_t, s_t] + \frac{\partial r_t}{\partial t}\right) + \beta_t,\\ \delta^* s_t =& 0\end{aligned}$$

hold. Consequently, the operator $\hat{D}_{s_t}$ commutes with D_{r_t}.

Note that the equations above are simpler than the original ones for r_t since they are affine in s_t.

One can then construct an explicit operator $D_{r_t}^{-1}$ such that

$$s_t = D_{r_t}^{-1}\left(\frac{\partial r_t}{\partial t} + \beta_t\right).$$

We have to understand the variation of $\tau_t(f)$. Recall that this element is the (t-dependent) element of

$$\Omega^0(M, W(TM))[[h]]$$

that satisfies $D_{r_t}\tau_t(f) = 0$ and $\sigma(\tau_t(f)) = f$. Let us consider X such that

$$Xdt = \hat{D}_{s_t}\tau_t(f) = dt\left(\frac{d}{dt}\tau_t(f) + \frac{i}{h}[s_t, \tau_t(f)]\right).$$

Since $D_{r_t}X = \hat{D}_{s_t}D_{r_t}\tau_t(f) = 0$, X is closed with respect to D_{r_t}. Moreover

$$\sigma(X) = \sigma\left(\frac{d}{dt}\tau_t(f)\right) + \sigma\left(\frac{i}{h}[s_t, \tau_t(f)]\right) = \sigma\left(\frac{i}{h}[s_t, \tau_t(f)]\right),$$

where the first summand vanishes since σ and $\frac{d}{dt}$ commute.

Since X is closed with respect to D_{r_t}, as a consequence of [Wal07, 6.4.19], we obtain that $X = \tau_t(\sigma(X))$, and therefore

$$\frac{d}{dt}\tau_t(f) + \frac{i}{h}[s_t, \tau_t(f)] = X = \tau_t(\sigma(X)) = \tau_t\sigma\left(\frac{i}{h}[s_t, \tau_t(f)]\right).$$

If we rewrite this equality, we obtain

$$\frac{d}{dt}\tau_t(f) = \tau_t\sigma\left(\frac{i}{h}[s_t, \tau_t(f)]\right) - \frac{i}{h}[s_t, \tau_t(f)]. \tag{7.24}$$

We are now ready to complete the proof of our theorem.

Proof. (of Theorem 7.21)

To complete the proof it is enough to show that the variation of the star product admits a primitive in the Hochschild cochain complex if and only if the characteristic class of the products is constant in cohomology.

If the variation of the star product admits a primitive in the Hochschild cochain complex, then the family admits a formal connection by Proposition 7.19, and therefore we can use parallel transport along a path on $\mathcal{T}$ to construct an equivalence between any two star products in the family, which then have the same characteristic class in cohomology, by Fedosov's result that we described in Theorem 4.13.

Let us now assume that the characteristic class of the products of the family is constant in cohomology. By the discussion above, we have a 1-parameter family of Fedosov star products $\star_t := \star_{\nabla,\alpha_t}$ parametrized by the unit interval, and the computations above allow us to understand the variation of the family.

In fact we can compute:

$$\frac{d}{dt}(f \star_t g) = \sigma(\frac{d}{dt}\tau_t(f) \circ_{MW} \tau_t(g)) + \sigma(\frac{d}{dt}\tau_t(f) \circ_{MW} \tau_t(g)).$$

Plugging in the expression that we obtained for the variation of τ_t in (7.24), we obtain:

$$\begin{aligned}\frac{d}{dt}(f \star_t g) = \sigma(\tau_t\left(\frac{i}{h}[s_t, \tau_t(f)]\right) \circ_{MW} \tau_t(g)) + \sigma(\tau_t(f) \circ_{MW} \tau_t\left(\frac{i}{h}[s_t, \tau_t(g)]\right))\\ - \sigma(-\frac{i}{h}[s_t, \tau_t(f) \circ_{MW} \tau_t(g)]).\end{aligned}$$

It follows that if we set

$$A_t(f) := -\frac{i}{h}\sigma([s_t, \tau_t(f)]), \tag{7.25}$$

the previous equation reads as

$$\frac{d}{dt}(f \star_t g) = -A_t(f) \star_t g - f \star_t A_t(g) + A_t(f \star_t g),$$

which is exactly saying that A_t is a primitive for the variation of $\star_t$. One has to check that A_t is a differential operator, which follows by the same arguments that show that the Fedosov star product is differential, which are explained in detail in [Wal07, 6.4.22]. The idea is that $\tau_t(f)$ can be written as a Taylor expansion of f and $[s_t, \tau_t(f)]$ and after projecting with σ, one obtains differential operators on M.

It remains to check that $A_t = 0 \pmod h$, which is one of the requirements in the definition of a formal connection. To do so, we can look at the low orders of s_t and $\tau_t(f)$. If $\beta_t = \sum_k \beta_t^k h^k$ denotes the primitive for the family of differential forms α_t, then we have that

$$s_t = h\delta^* \beta_t^1 + O(h^2)$$

and

$$\tau_t(f) = f + df + O(h).$$

When we compute the Moyal-Weyl star product commutator of these two elements in (7.25), we see that its lowest orders terms have degree 2 in h, because the star product commutator is always multiple of h and s_t is multiple of h too. The factor $-\frac{i}{h}$ in (7.25) makes so that the lowest order terms in A_t have degree 1 in h. This shows that the variation of the family of star products admits a primitive $A \in \Omega^1(\mathcal{T}, h\mathcal{D}(M)[[h]])$ and concludes the proof. $\square$

7.9 Families of Karabegov products

The construction specializes to the case of a compact symplectic manifold M with a family of compatible Kähler structures parametrized by $\mathcal{T}$. Then one can choose for any $\sigma \in \mathcal{T}$ the anti-differential d_σ^* coming from Hodge theory with respect to the Kähler form for σ. We can choose a family of Karabegov star products with a characteristic class that in cohomology is independent of $\sigma \in \mathcal{T}$. For instance we can choose the unique Karabegov product with the trivial Karabegov form ωh^{-1} for every $\sigma \in \mathcal{T}$, which is of Wick type. Then the Fedosov classes of the family $\star$ is constant in cohomology, and we obtain the following result.

Theorem 7.27. *Let (M, ω) be a compact, symplectic manifold equipped with a family of compatible Kähler structures parametrized by a manifold $\mathcal{T}$. Let us consider a family of Karabegov star products $\{\star_\sigma\}_{\sigma\in\mathcal{T}}$ in which the Karabegov form is independent of σ in cohomology. Then the family admits a formal connection.*

In particular, if we choose the family of Berezin-Toeplitz star products, which are Karabegov star products as seen before, we obtain that this family admits a formal connection.

7.10 Formal connections in the graph language

On a Kähler manifold M equipped with a smooth family of Karabegov star products $\{\star_\sigma\}_{\sigma\in\mathcal{T}}$ parametrized by $\mathcal{T}$ such that the characteristic class of the family is constant, we can use the graph language of Chapter 6 to write a set of recursive equations that the connection has to satisfy. As usual let us denote with A the formal 1-form associated to the formal connection, with $A = 0 \pmod h$. Then A gives a formal connection if and only if satisfies (7.23), which means that it is a derivation for the family of star products. We can use our combinatorial formalism to express that equation.

Proposition 7.28. *The formal 1-form A with values in formal differential operators on M and such that $A = 0 \pmod h$ defines a formal connection if and only if it satisfies the following equations for any $k \geq 1$, for any vector field V on $\mathcal{T}$ and any two smooth functions f and g:*

$$\begin{aligned}
&A^k(V)(fg) - A^k(V)(f)g - fA^k(V)(g) = \sum_{G\in\mathcal{L}^c_{2,k}} \frac{1}{C(G)} V[\Lambda_G](f,g) \\
&- \sum_{i=1}^{k} \Bigg(A^{k-i}(V)\big(\sum_{G\in\mathcal{L}^c_{2,i}} \frac{1}{C(G)} \Lambda_G(f,g)\big) \\
&- \sum_{G\in\mathcal{L}^c_{2,i}} \frac{1}{C(G)} \Lambda_G(A^{k-i}(V)(f), g) - \sum_{G\in\mathcal{L}^c_{2,i}} \frac{1}{C(G)} \Lambda_G(f, A^{k-i}(V)(g)) \Bigg).
\end{aligned}$$

8

The trivialization problem

8.1 Formal trivializations

Definition 8.1. A *formal trivialization* of a formal connection D is a smooth map $P \colon \mathcal{T} \to \mathcal{D}_h(M)$ which modulo h is the identity, for all $\sigma \in \mathcal{T}$, and which satisfies

$$D_V(P(f)) = 0,$$

for all vector fields V on $\mathcal{T}$ and all $f \in C_h^\infty(M)$.

In this chapter we shall specifically look at trivializations of the formal Hitchin connection D that is associated to the Hitchin connection of geometric quantization and that has the explicit expression (7.6). Recall that D is a connection in the bundle $\mathcal{T} \times C_h^\infty(M) \to \mathcal{T}$.

Note that the fact that D is flat is a necessary condition for the existence of a trivialization, which otherwise cannot exist even locally. However we have the following result, proved in [And12], which assures the existence locally on $\mathcal{T}$ when the connection is flat. As mentioned, Andersen showed in [AG11] that this is the case if the Hitchin connection from geometric quantization is projectively flat.

Proposition 8.2. *Assume that D is flat and that $A = 0$ mod h. Then locally around any point in $\mathcal{T}$, there exists a formal trivialization. If $\mathcal{T}$ is contractible then there exists a formal trivialization defined globally on $\mathcal{T}$.*

If we use this result on the formal Hitchin connection obtained in Theorem 7.4, we can define a new star product:

$$f_1 \star f_2 = P_\sigma^{-1}(P_\sigma(f_1) \star_\sigma^{BT} P_\sigma(f_2)), \tag{8.1}$$

where f_1, f_2 are sections of C_h evaluated at $\sigma \in \mathcal{T}$. One differentiates the expression along a vector field on $\mathcal{T}$ and checks that the right side of the equality is actually independent of $\sigma \in \mathcal{T}$. Thus this defines a star product on M which does not depend on σ.

8.2 Trivialization at first order

In this section we shall obtain explicitly a global trivialization of the formal Hitchin connection (7.6) up to order one, following the construction of Andersen and Gammelgaard [AG11]. To do so we shall use the following preliminary result that relates the variation of the Laplace operator and the operators $\Delta_{\tilde{G}(V)}$.

Proposition 8.3. *We have the following relation between the variation of the Laplace-Beltrami operator and $\Delta_{\tilde{G}(V)}$:*

$$\Delta_{\tilde{G}(V)} = V[\Delta].$$

Proof. Let δ denote the divergence of a vector field, which can be expressed in terms of the Levi-Civita connection as

$$\delta(X) = \operatorname{Tr} \nabla X.$$

We can observe that

$$\delta(\tilde{G}(V)df) = \Delta_{\tilde{G}(V)}(f),$$

for any function $f \in C^\infty(M)$. Then we have that:

$$V[\Delta]f = V[\delta(-g^{-1}df)] = -\delta(V[g^{-1}]df),$$

and, recalling that $\tilde{G}(V) = -V[g^{-1}]$, we can write:

$$-\delta(V[g^{-1}]df) = \delta(\tilde{G}(V)df) = \Delta_{\tilde{G}(V)}(f). \qquad \square$$

Let us denote with P_i the i-th order part of the trivialization. Then we can write a global expression for P_0 and P_1, as shown in the proposition below.

Proposition 8.4 (Andersen-Gammelgaard)**.** *The family of operators*

$$P = \operatorname{Id} + h(\frac{1}{4}\Delta - c^{(1)}(F, f)) + O(h^2),$$

gives a formal trivialization of the formal Hitchin connection (7.6)*.*

Proof. Looking at the condition that $D_V(P(f)) = 0$ modulo h, one sees immediately that $P_0 = \operatorname{Id}$. Now we can consider the corresponding equation modulo h^2, denoting $P_1(f) = \tilde{f}_1$:

$$V[\tilde{f}_1] = -\tilde{A}^1(V)(f) = +\frac{1}{4}\Delta_{\tilde{G}(V)}(f) - \frac{1}{2}\nabla_{\tilde{G}(V)dF}(f) - c^{(1)}(V[F], f).$$

We can differentiate the expression (7.9) for the degree 1 coefficient of the Berezin-Toeplitz star product obtained by Karabegov and obtain:

$$V[c^{(1)}](f, g) = \frac{1}{2}\nabla_{\tilde{G}(V)df}(g).$$

If we apply this observation and Proposition 8.3 to the equation above we get:

$$V[\tilde{f}_1] = +\frac{1}{4}V[\Delta](f) - V[c^{(1)}](F, f) - c^{(1)}(V[F], f).$$

The usual chain rule give us the following:

$$V[c^{(1)}(F,f)] = V[c^{(1)}](F,f) + c^{(1)}(V[F],f),$$

where we note that the term with $V[f]$ vanishes, because f does not depend on $\sigma \in \mathcal{T}$. Thus our equation can be written as

$$V[\tilde{f}_1] = \frac{1}{4}V[\Delta](f) - V[c^{(1)}(F,f)] = V[\frac{1}{4}\Delta f - c^{(1)}(F,f)],$$

which is what we wanted to show. □

8.3 Trivialization at second order

The trivialization problem at second order requires solving the equation

$$D_V(P(f)) = 0 \mod h^3.$$

With an immediate computation we can write this equation as follows:

$$V[\tilde{f}_2] = -\tilde{A}^1(V)(\tilde{f}_1) - \tilde{A}^2(V)(f), \tag{8.2}$$

where $\tilde{f}_i = P_i(f)$.

Since we know that the formal connection is flat, a trivialization exists, at least locally, and can be written as an integral in a neighbourhood around any point.

To study this problem we can use two different approaches based on different techniques, and we shall give account for them in this chapter. The first approach is to try to find a solution through a direct calculation, in a similar fashion to what done for the problem at order one: this approach is described in Section 8.6. The second approach used the graph formalism introduced in Chapter 6, and is the topic of the next section.

8.3.1 Finding P_2 with graphs

We are trying to find an operator P_2 satisfying the following equation:

$$V[P_2] = -\tilde{A}^1(V)P_1 - \tilde{A}^2(V)P_0,$$

where $P_0 = \mathrm{Id}$, and

$$P_1(f) = +\frac{1}{4}\Delta(f) - c^{(1)}(F,f),$$

for all $f \in C^\infty(M)$. To approach this problem with graphs, we try to express all the terms on the right hand side of the equation in graph language, and then try to transgress what we obtain, or to read it as the derivative of P_2. Before doing this, let us observe that, from the case of P_1, we have the relation:

$$V[P_1] = -\tilde{A}^1(V),$$

and therefore we can easily check that our equation can be re-written as:

$$V[P_2] = -V[P_1]P_1 - \tilde{A}^2(V). \tag{8.3}$$

We already have a graph-theoretical expression for the last summand in P_1, namely $c^{(1)}$, in the graph language developed in Chapter 6. So we continue in the same fashion, trying to express the other terms in a compatible language.

Let us begin with Δ, the Laplace-Beltrami operator. This can then be expressed as the partition function associated to the following graph G_0, with one external vertex, and one internal of weight -1. Note that this graph breaks the requirements we had in Chapter 6, since it has a vertex of weight -1 with only 2 edges.

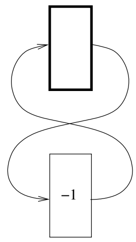


Figure 8.1: The graph G_0 representing Δ.

Lemma 8.5. *The Beltrami-Laplace operator Δ is expressed graph theoretically as follows:*

$$\Delta(f) = 2i \sum_{l \in \mathcal{L}(G_0)} \frac{1}{C(G_0)} \Lambda^l_{G_0}(f), \tag{8.4}$$

for any $f \in C^\infty(M)$.

Proof. By [Bes87], we can express the operator as

$$\Delta(f) = \Lambda_\omega(2i\bar{\partial}\partial f),$$

where $\Lambda_\omega(\alpha) = \langle \alpha, \omega \rangle$. Let us write down the partition function associated to G_0 following the definition:

$$\sum_{i,j,k,l} \frac{\partial^2 f}{\partial \bar{z}_j \partial z_i} g^{ik} g^{lj} \frac{\partial^2 \Phi_{-1}}{\partial z_k \partial \bar{z}_l}.$$

Note that the Kähler metric is given by a matrix with entries:

$$g_{kl} = \frac{\partial^2 \Phi_{-1}}{\partial z_k \partial \bar{z}_l},$$

and so, since $\sum_k g^{ik} g_{kl} = \delta^l_i$, the partition function looks like:

$$\sum_{i,j} \frac{\partial^2 f}{\partial \bar{z}_j \partial z_i} g^{ij} = \Lambda_\omega(\bar{\partial}\partial f) = \frac{\Delta(f)}{2i}. \qquad \square$$

We have now all the ingredients to re-write P_1 in graph language:

$$P_1(f) = +\frac{2i}{4} \sum_{l \in \mathcal{L}(G_0)} \Lambda^l_{G_0}(f) - \sum_{l \in \mathcal{L}(G_1)} \frac{1}{C(G)} \Lambda^l_{G_1}(V[F], f). \tag{8.5}$$

Recall that $\tilde{A}^1(V)$ has the following expression:

$$\tilde{A}^2(V)(f) = c^{(2)}(V[F], f) - \frac{n}{2} c^{(1)}(V[F], f) - \frac{1}{2} c^{(1)}(\Delta_{\tilde{G}(V)}(F), f).$$

Hence we can easily express it in graph language, since we can write the star product coefficients:

$$\begin{aligned}\tilde{A}^2(V)(f) = &\sum_{G\in\mathcal{L}^c_{2,2}} \frac{1}{C(G)}\Lambda_G(V[F],f) - \frac{n}{2}\sum_{\mathcal{L}(G_1)}\Lambda_{G_1}(V[F],f)\\ &-\frac{1}{2}\sum_{\mathcal{L}(G_1)}\Lambda_{G_1}(\Delta_{\tilde{G}(V)}(F),f).\end{aligned} \tag{8.6}$$

It is now immediate to combine the previous expressions and re-write (8.3) in graph language:

$$\begin{aligned}V[\tilde{f}_2] = &- V\left[\frac{2i}{4}\sum_{l\in\mathcal{L}(G_0)}\Lambda^l_{G_0}(\cdot) - \sum_{l\in\mathcal{L}(G_1)}\frac{1}{C(G)}\Lambda^l_{G_1}(V[F],\cdot)\right]\\ &\left(\frac{2i}{4}\sum_{l\in\mathcal{L}(G_0)}\Lambda^l_{G_0}(\cdot) - \sum_{l\in\mathcal{L}(G_1)}\frac{1}{C(G)}\Lambda^l_{G_1}(V[F],\cdot)\right)(f)\\ &-\sum_{G\in\mathcal{L}^c_{2,2}}\frac{1}{C(G)}\Lambda_G(V[F],f) + \frac{n}{2}\sum_{\mathcal{L}(G_1)}\Lambda_{G_1}(V[F],f)\\ &+\frac{1}{2}\sum_{\mathcal{L}(G_1)}\Lambda_{G_1}(\Delta_{\tilde{G}(V)}(F),f).\end{aligned} \tag{8.7}$$

We can now examine the equation written in this form to try to understand what kinds of graphs can appear in the expression of P_2. It is useful to study what happens to the weight of graphs under fusion.

Lemma 8.6. *Let G_1, G_2 be graphs in the extended family, with weight respectively w_1 and w_2. If G is a graph obtained by fusing them, then G has weight $w_1 + w_2$.*

Proof. Let us consider the two types of fusion at a vertex separately. In fusion of type A, we cut out the last external vertex of G_1 and glue the loose ends to some vertices of G_2. Therefore the number of edges $|E_G|$ in the fused graph will be the sum $|E_{G_1}| + |E_{G_2}|$. The internal vertices are not altered by this kind of fusion, therefore the result holds.

If we consider a fusion of type B, where a loose end is glued to an edge of G_2, by inserting a new internal vertex of weight -1 along that edge, we see that the total number of edges in G is $|E_{G_1}| + |E_{G_2}| + 1$, and the $+1$ is compensated by the weight of the new internal vertex, so that we have also in this case $w(G) = w_1 + w_2$. □

Let us now study what happens to the weight of a graph under differentiation: as mentioned, differentiating a graph G gives a sum of $3|E_G|$ starred graph that are isomorphic to G, modulo the location of the star. Then we can make immediately the following observation.

Lemma 8.7. *The weight of a graph does not change under differentiation.*

Now that we have some control of the weight as a graph invariant under fusion and differentiation, we can get some result restricting the structures of graphs that can appear in the expression of P_2. By examining (8.7), we see that the right hand side of the equality only presents graphs of weights 1 and 2.

Graph	Weight
G_0	1
G_1	1
$G \in \mathcal{L}^l_{2,2}$	2

In conclusion we have proved the following proposition.

Proposition 8.8. *The graphs that appear in the expression of P_2 can only have weight* 1 *or* 2.

8.4 The formal Hitchin connection in graph language

The graph language that we have developed in this chapter allows us to express the formal Hitchin connection (7.6) in terms of graphs. Let D be the formal Hitchin connection expressed by:

$$D_V(f) = V[f] + \tilde{A}(V)(f),$$

for any vector field V on $\mathcal{T}$, where the 1-form $\tilde{A}$ is determined explicitly in Andersen and Gammelgaard's work [AG11] and has the form that we recall here:

$$\begin{aligned}\tilde{A}(V)(f) = &-\frac{1}{4}h\Delta_{\tilde{G}(V)}(f) + \frac{1}{2}h\nabla_{\tilde{G}(V)dF}(f) + V[F] \star^{BT} f - V[F]f \\ &- \frac{1}{2}h(\Delta_{\tilde{G}(V)}(F) \star^{BT} f - nV[F] \star^{BT} f - \Delta_{\tilde{G}(V)}(F)f - nV[F]f).\end{aligned}$$

Then $\tilde{A}$ can be written as $\tilde{A} = \sum_{i\geq 1} \tilde{A}^i$, where

$$\begin{aligned}\tilde{A}^1(V)(f) &= -\frac{1}{4}\Delta_{\tilde{G}(V)}(f) + c^{(1)}(V[F], f) + V[c^{(1)}](F, f),\\ \tilde{A}^k(V)(f) &= c^{(k)}(V[F], f) - \frac{1}{2}c^{(k-1)}(\Delta_{\tilde{G}(V)}(F), f) + \frac{n}{2}c^{(k-1)}(V[F], f), \qquad \text{for } k \geq 2.\end{aligned}$$

We can see that $\tilde{A}^k$ for $k \geq 2$ is completely given in terms of coefficients of the star product, therefore we can easily express it in graph form. As to $\tilde{A}^1$, we can see that the only term that is not written as a coefficient of the star product or a derivative of that is the differential operator $\Delta_{\tilde{G}(V)}$, which can be written as $V[\Delta]$ by Proposition 8.3, and therefore can also be written in graph language by using Lemma 8.5. Summing up, we can write $\tilde{A}$ as follows:

$$\begin{aligned}\tilde{A}^1(V)(f) &= \sum_{G\in\mathcal{L}^c_{2,1}} \frac{1}{C(G)}\Lambda_G(V[F], f) - \frac{i}{2}V[\Lambda_{G_0}](f) + \sum_{G\in\mathcal{L}^c_{2,1}} V[\Lambda_G](F, f),\\ \tilde{A}^k(V)(f) &= \sum_{G\in\mathcal{L}^c_{2,k}} \frac{1}{C(G)}\Lambda_G(V[F], f)\\ &\quad + \sum_{G\in\mathcal{L}^c_{2,k-1}} \frac{1}{C(G)}\Lambda_G(-\frac{1}{2}\Delta_{\tilde{G}(V)}(F) + \frac{n}{2}V[F], f), \qquad \text{for } k \geq 2.\end{aligned}$$

By combining these expression we obtain a formula for the formal Hitchin connection (7.6) in which the 1-form $\tilde{A}$ is completely written in graph language.

8.5 The trivialization problem at all degrees

We can now generalize this approach to the trivialization problem to all degrees, and write the equations determining the formal trivialization P in terms of graphs. Recall that solving the trivialization problem means to find a smooth map

$$P = \sum_{k \in \mathbb{N}} P_k h^k \colon \mathcal{T} \to \mathcal{D}_h(M), \tag{8.8}$$

such that $P_0 = \mathrm{Id}$ and $D_V(P_\sigma(f)) = 0$ for any $\sigma \in \mathcal{T}$ and any vector field V on $\mathcal{T}$. This can be rewritten in the following form by using the general expression for a formal connection (7.1). We omit the σ dependency for notational reasons.

$$\begin{aligned} P_0 &= \mathrm{Id} \\ V[P_k] &= -A^1(V)(P_{k-1}) - A^2(P_{k-2}) - \cdots - A^k(V)(Id), \qquad \text{for } k \geq 2, \end{aligned} \tag{8.9}$$

for any vector field V on $\mathcal{T}$.

Note that we can use the graph theoretical expression for $\tilde{A}^k$ that we have obtained in the previous section to formulate the trivialization problem at all degrees in graph language. The following proposition summarizes the discussion of this section and gives a set of recursive equations completely written in graph terms which determine a formal trivialization for a given formal connection. Note that each of the recursive equations involves transgressing a linear combination of extended graphs, as done for P_2 in Section 8.3.1, in the sense of finding a graph whose derivative (as defined in Section 6.6.1 is the given linear combination.

Proposition 8.9. *Let D_V be the formal Hitchin connection* (7.6) *in the bundle $C_h(M) \times \mathcal{T}$ over $\mathcal{T}$ expressed in the form $D_V = V + \tilde{A}(V)$. A smooth map $P\colon \mathcal{T} \to \mathcal{D}_h(M)$ is a formal trivialization of D if it satisfies, for any $k \in \mathbb{N}$, the following recursive sequence of relations expressed in graph theoretical language:*

$$\begin{aligned} P_0 &= \textit{empty graph} \\ V[P_k] &= -\tilde{A}^1(V)(P_{k-1}) - \tilde{A}^2(V)(P_{k-2}) - \cdots - \tilde{A}^k(V)(P_0) \\ &= -\sum_{G \in \mathcal{L}^c_{2,1}} \frac{1}{C(G)} \Lambda_G(V[F], P_{k-1}) + \frac{i}{2} V[\Lambda_{G_0}](P_{k-1}) - \frac{1}{2} \sum_{G \in \mathcal{L}^c_{2,1}} V[\Lambda_G](F, P_{k-1}) \\ &\quad - \sum_{G \in \mathcal{L}^c_{2,2}} \frac{1}{C(G)} \Lambda_G(V[F], P_{k-2}) - \sum_{G \in \mathcal{L}^c_{2,1}} \frac{1}{C(G)} \Lambda_G(-\frac{1}{2}\Delta_{\tilde{G}(V)}(F) + \frac{n}{2} V[F], P_{k-2}) \\ &\quad - \dots \\ &\quad - \sum_{G \in \mathcal{L}^c_{2,k}} \frac{1}{C(G)} \Lambda_G(V[F], \mathrm{Id}) - \sum_{G \in \mathcal{L}^c_{2,k-1}} \frac{1}{C(G)} \Lambda_G(-\frac{1}{2}\Delta_{\tilde{G}(V)}(F) + \frac{n}{2} V[F], \mathrm{Id}), \end{aligned} \tag{8.10}$$

for $k \geq 1$.

8.6 Another approach to the trivialization problem

In this section we shall look at a different approach to the trivialization problem, which mimics more closely the approach used to get the global result in order 1 that we have treated in previously in this chapter, namely trying to solve it in a more calculative way.

To illustrate the idea we shall present an example of a computation first, and then look at the general procedure.

We start from the trivialization equation (8.3) at order 2. If we insert the expression we have for P_1, $\tilde{A}^1$ and $\tilde{A}^2$, then the equation reads as:

$$\begin{aligned} V[\tilde{f}_2] = &+ \frac{1}{16}\Delta_{\tilde{G}(V)}(\Delta f) - \frac{1}{4}\Delta_{\tilde{G}(V)}(c^{(1)}(F,f)) - \frac{1}{4}V[c^{(1)}](F,\Delta f) \\ &+ V[c^{(1)}](F,c^{(1)}(F,f)) - \frac{1}{4}c^{(1)}(V[F],\Delta f) + c^{(1)}(V[F],c^{(1)}(F,f)) \\ &- c^{(2)}(V[F],f) - \frac{n}{2}c^{(1)}(V[F],f) + \frac{1}{2}c^{(1)}(\Delta_{\tilde{G}(V)}(F),f). \end{aligned} \tag{8.11}$$

If we look at the expression we have obtained for $V[c^{(2)}]$ in (7.13), we see there some addends that also appear in this equation. Then we can substitute the addend

$$-c^{(2)}(V[F],f) = V[c^{(2)}](F,f) - V[c^{(2)}(F,f)]$$

using the expression from (7.13), and we note that we obtain the term $V[c^{(2)}(F,f)]$, which is already transgressed. Of course we get also several extra addends from the cited expansion, some of which cancel out, while others remain. What we wish for is that we can find expansions for other addends appearing in (8.11) so that every term can be transgressed.

To streamline this process we can proceed this way: we look at the right side of (8.11) and from it we get inspiration to choose a set of functions with the idea that $\tilde{f}_2 = P_2(f)$ can be written as a linear combination of them. If we compare this with the previous case of P_1, where we already have a solution, the functions involved there are $\Delta(f)$ and $c^{(1)}(F,f)$, with coefficients respectively $-\frac{1}{4}$ and -1. Then we differentiate in direction of V all the functions we have chosen, and impose that the obtained coefficients match the ones we see in (8.11). If the linear system for the coefficients admits a solution, then we have found an expression for P_2. In our calculation we have considered the following operators, applied to a smooth function f on M:

$\Delta^2 f$	$\Delta^2 F$	$c^{(1)}(F,\Delta f)$	$c^{(1)}(\Delta F,f)$
$\Delta f \Delta F$	$c^{(2)}(F,f)$	$c^{(1)}(c^{(1)}(F,f),f)$	$c^{(1)}(c^{(1)}(F,f),F)$
$c^{(1)}(c^{(1)}(f,F),F)$	$c^{(1)}(c^{(1)}(f,F),F)$	$c^{(1)}(f,c^{(1)}(F,f))$	$c^{(1)}(F,c^{(1)}(F,f))$
$c^{(1)}(f,c^{(1)}(f,F))$	$c^{(1)}(F,c^{(1)}(f,F))$	$\Delta c^{(1)}(F,f)$	$\Delta c^{(1)}(f,F)$
Δf	ΔF	$\Delta(f)F$	$\Delta(F)f$
$c^{(1)}(F,f)$	$c^{(1)}(f,F)$	$\Delta(f)c^{(1)}(F,f)$	$\Delta(f)c^{(1)}(F,F)$

We have then tried to express P_2 as a linear combination of these functions. After differentiating each of them along V, we have imposed that the coefficients match the ones found on the right side of (8.11). Differentiating the functions in the table we have of course obtained several extra terms, which unfortunately do not seem obvious to transgress. In other words, this approach did not give the wanted result yet, and the partial conclusion we can make is that, if P_2 can be expressed with a global expression, then it contains terms that are not listed in the table above.

Remark 8.10. As an aside, while studying the problem of transgressing the terms in the expansion for $V[c^{(2)}]$ obtained in (7.13), we have studied the sum of the three terms:

$$V[F]c^{(2)}(f_1, f_2) - c^{(2)}(V[F]f_1, f_2) + f_1 c^{(2)}(V[F], f_2).$$

To approach this in generality, let us define:

$$X(a,b,c) = c^{(2)}(ab,c) - ac^{(2)}(b,c) - bc^{(2)}(a,c),$$

where a, b, c are smooth functions. One immediately notices that the expression is symmetric in a and b, and also that, if we took the same expression with $c^{(1)}$ in place of $c^{(2)}$, we would get 0, by Lemma 7.9. We can now use graphs to write down an expression for $X(a,b,c)$ in local coordinates, following (6.3):

$$\begin{aligned} X(a,b,c) = &- \sum_{i,\bar{\imath},j,\bar{\jmath},k,\bar{k}} g^{i\bar{\imath}} g^{j\bar{\jmath}} g^{k\bar{k}} \frac{\partial a}{\partial \bar{z}^i} \frac{\partial b}{\partial \bar{z}^j} \frac{\partial^3 \Phi_{-1}}{\partial z^{\bar{\imath}} \partial z^{\bar{\jmath}} \partial \bar{z}^k} \frac{\partial c}{\partial z^{\bar{k}}} \\ &+ \sum_{i,\bar{\imath},j,\bar{\jmath}} g^{i\bar{\imath}} g^{j\bar{\jmath}} \frac{\partial a}{\partial \bar{z}^i} \frac{\partial b}{\partial \bar{z}^j} \frac{\partial^2 c}{\partial z^{\bar{\imath}} \partial z^{\bar{\jmath}}}. \end{aligned} \tag{8.12}$$

If we examine the expression, we recognize that the two sums are partition function associated to two graphs with 3 external vertices and weight 2 in the set $\mathcal{A}_3(2)$, as defined in Chapter 6.

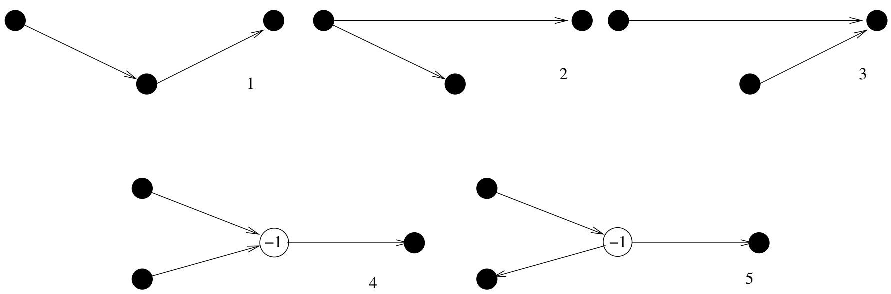


Figure 8.2: Graphs in $\mathcal{A}_3(2)$ (unlabelled).

Namely, we get the partition functions corresponding to the graphs 3 and 4 in the picture, where we insert the function a, b, c at the 3 external vertices (drawn as filled dots). Similarly, if we define the quantity:

$$Y(a,b,c) = c^{(2)}(a,bc) - bc^{(2)}(a,c) - cc^{(2)}(a,b),$$

we get the partition functions corresponding to the graphs 2 and 5 in the picture.

If we look at the equation we are trying to solve, we note that terms in the sum we are looking at here are the only ones where $c^{(2)}$ appears: in fact they are $X(V[F], F, f)$. We see also that we have second derivatives only for f, while $V[F]$ and F only appear as first derivatives. It could be therefore possible to re-write them, possibly in terms of $c^{(1)}$ and simplify the expression.

8.7 Trivialization on abelian varieties

In this section we shall consider a case of the trivialization problem where we can find a solution at all degrees, namely that of abelian varieties.

We follow the work of Andersen in [And05] and in his paper joint with Blaavand [AB11], which give us an explicit formula for a formal Hitchin connection in this setting.

Let V be a real vector space with a symplectic form ω, and let Λ be a discrete lattice of maximal rank. We consider the quotient

$$M = V/\Lambda,$$

and we require that M is *principally polarised*, meaning that the symplectic form ω must be integral and unimodular when restricted to Λ. Then M is called an *abelian variety*.

As usual $\mathcal{T}$ is a complex manifold parametrizing the Kähler structures I_σ on M that are compatible with ω for $\sigma \in \mathcal{T}$. Andersen derives an expression for a formal Hitchin connection[1] on such an abelian variety, which simplifies the formula that we got in 7.6, since in this setting the Ricci potential F is constantly zero.

Proposition 8.11. *Let M be a principally polarised abelian variety and $\mathcal{T}$ a complex manifold parametrizing the Kähler structures I_σ on M that are compatible with ω for $\sigma \in \mathcal{T}$. Then the expression below defines a formal Hitchin connection:*

$$D_V f = V[f] - \frac{h}{4}\Delta_{\tilde{G}(V)}(f), \tag{8.13}$$

for any vector field V on $\mathcal{T}$ and any smooth function f on M.

We note that in this setting we have that $[\Delta, \Delta_{\tilde{G}(V)}] = 0$, for any vector field V on $\mathcal{T}$. One can see this by writing the operators as matrices for a coordinate chart and diagonalizing them. The coefficients are constant, and therefore they commute. By Proposition 8.3, the relation above is equivalent to $[\Delta, V[\Delta]] = 0$. The following lemma is a consequence of this fact.

Lemma 8.12. *For any positive integer k we have:*

$$V[\Delta^k] = k\Delta^{k-1}V[\Delta].$$

Proof. We have that

$$V[\Delta^2] = V[\Delta]\Delta + \Delta V[\Delta] = 2\Delta V[\Delta].$$

If the claim holds for k, then we can write:

$$V[\Delta^{k+1}] = V[\Delta]\Delta^k + \Delta V[\Delta^k] = \Delta^k V[\Delta] + k\Delta\Delta^{k-1}V[\Delta] = (k+1)\Delta^k V[\Delta],$$

and so we can conclude by induction. □

[1] Here we use a different sign convention with respect to [AB11], therefore we get opposite signs for the 1-form in the formal Hitchin connection and for the trivialization.

If we write the equations for the trivialization problem (8.9) in the current case, they read as:

$$V[P_k(f)] = V[\tilde{f}_k] = -\tilde{A}^1(V)(\tilde{f}_{k-1}) = \frac{1}{4}\Delta_{\tilde{G}(V)}(\tilde{f}_{k-1}), \tag{8.14}$$

for any positive integer k, any vector field V on $\mathcal{T}$ and any smooth function f on M.

For a positive integer k, let

$$\tilde{f}_k = \alpha_k \Delta^k(f),$$

and let us substitute this expression in (8.14), we get:

$$\alpha_k V[\Delta^k]f = \frac{1}{4}\alpha_{k-1} V[\Delta]\Delta^{k-1} f.$$

If we apply Lemma 8.12 to the equation above, we see that our expression for $\tilde{f}_k$ provides a solution for all k by choosing:

$$\alpha_k = \frac{1}{4^k k!}.$$

In other words, the trivialization of the formal Hitchin connection in the abelian case has the form:

$$P = \sum_{k\in\mathbb{N}} h^k \frac{\Delta^k}{4^k k!} = \exp\left(h\frac{\Delta}{4}\right). \tag{8.15}$$

We can note that this expression matches the one obtained by Andersen and Gammelgaard [AG11] at order one if we let the Ricci potential vanish in their formula that we saw in Proposition 8.4.

The same expression that we got here for the trivialization was obtained by Andersen in [And05], where it was seen as a transformation of the Berezin-Toeplitz deformation quantization, and later in [AB11], where Andersen and Blaavand with different methods obtained the same trivialization of the formal Hitchin connection in the case of abelian varieties.

8.8 Remarks about the trivialization problem

Let us recapitulate the content of this chapter: we obtain partial results that are based on the graph theoretical language that we have introduced extending the one used by Gammelgaard [Gam14] to express the coefficients for the Berezin-Toeplitz star product. By encoding composition and differentiation in this graph language, we get some constraints on how the expression of the trivialization at degree 2 can be, by limiting the class of graphs that can appear in it.

We note anyhow that this graph language is based on the choice of a set of local coordinates, therefore it will not immediately give a global expression for the trivialization. It is anyhow interesting to investigate further in this direction, trying to better understand the trivialization at order 2 and higher.

Even though our results do not give a solution of the trivialization problem, they build a framework which will be possible to use to get further results in that direction. The problem of finding a trivialization is now completely encoded in graph terms. It is not evident to direct inspection which graphs one should consider, but the combinatorial nature of this approach allows to treat this question on a computer: implementing the problem in a mathematical software is a clear possibility for future work on this question.

It would also be interesting to search for a global expression for the trivialization. Section 8.6 gives account for our attempts in this direction. The calculations we carried out show a rather high degree of symmetry in the formulae we obtain. It could be for instance interesting to better understand the $c^{(2)}$ terms appearing in the expansion of $V[c^{(2)}]$ (described in Remark 8.10).

The results about classification of formal connections from Chapter 7 could also be used to progress on the trivialization problem: the formal Hitchin connection coming from geometric quantization is an example of such a connection, but possibly it is not the easiest to trivialize: one might be able to get a formal connection easier to trivialize by transforming the one obtained by Andersen from the Hitchin connection in geometric quantization.

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