Fusion rings and fusion ideals

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Abstract

This dissertation investigates fusion rings, which are Grothendieck groups of rigid, monoidal, semisimple, abelian categories. Special interest is in rational fusion rings, i.e., fusion rings which admit a finite basis, for as commutative rings they may be presented as quotients of polynomial rings by the so-called fusion ideals.

The fusion rings of Wess-Zumino-Witten models have been widely studied and are well understood in terms of precise combinatorial descriptions and explicit generating sets of the fusion ideals. They also appear in another, more general, setting via tilting modules for quantum groups at complex roots of unity. The main goal of this dissertation is to generalize previous results to this setting.

Resumé

Denne afhandling undersøger fusionsringe, som er Grothendieck-grupperne for rigide, monoidale, semisimple, abelske kategorier. Særlig interesse lægges i rationale fusionsringe, dvs. fusionsringe for hvilke der findes en endelig basis, for som kommutative ringe kan de præsenteres som kvotienter af polynomiumsringe med de såkalde fusionsidealer.

Fusionsringene der hører til Wess-Zumino-Witten-modeller har været et udbredt forskningsemne, og er blevet præcist beskrevet med hensyn til kombinatorikken og eksplicitte frembringere af fusionsidealerne. De forekommer også i en anden, mere generel, opsætning via tiltingmoduler for kvantegrupper hvis parametre er komplekse enhedsrødder. Hovedformålet med denne afhandling er at generalisere de tidligere resultater til denne opsætning.

Contents

Introduction

Consider for some field k an abelian category $\mathcal{F}$, which is k-linear and semisimple, i.e., the Hom-spaces are finite-dimensional k-vector spaces, composition of morphisms is k-linear and there is a countable collection of simple objects $\{L_i \mid i \in I\}$, such that $\dim_k \operatorname{Hom}_{\mathcal{F}}(L_i, L_j) = \delta_{i,j}$ and any object in $\mathcal{F}$ is isomorphic to a finite direct sum of the L_i. Assume furthermore that the category is rigid, monoidal and that the unit object is simple. We call $\mathcal{F}$ a fusion category, and its Grothendieck group $F = \mathrm{K}_0(\mathcal{F})$ is called a fusion ring. If there are only finitely many isomorphism classes of simple objects, then we call F a rational fusion ring.

We are mainly interested in fusion rings associated to a semisimple Lie algebra $\mathfrak{g}$. Examples of settings where they arise include:

(i) The category of finite-dimensional $\mathfrak{g}$-modules.

(ii) The category of fixed-level representations of the affine Kac-Moody algebra $\widetilde{\mathfrak{g}}$ associated to $\mathfrak{g}$.

(iii) The category of tilting modules of the associated quantum group at a complex root of unity.

(iv) The category of rational modules of the corresponding semisimple, simply connected algebraic group over a field of positive characteristic.

The first three examples will be examined in this dissertation, with the main focus on the case (iii). An examination of (iv) can be found in [AP95, Section 2]. Of these four examples the last three lead to rational fusion rings.

Let us deduce some properties of fusion rings with the aim to make the notion of a fusion ring independent of the category theory. Let $\{[i] \mid i \in I\}$ denote the $\mathbb{Z}$-basis for F corresponding to the isomorphism classes of the simple objects in $\mathcal{F}$. The ring structure on F relative to this basis is given by

$$[i][j] = \sum_{l \in I} N_{i,j}^{l} [l],$$

where $N_{i,j}^l = \dim_k \operatorname{Hom}_{\mathcal{F}}(L_i \otimes L_j, L_l) \geq 0$ is the multiplicity with which L_l occurs in a decomposition of $L_i \otimes L_j$ into simple summands. The duality functor $L \mapsto L^*$ on $\mathcal{F}$ has $L \simeq (L^*)^*$, so it maps simple objects to simple objects, giving an involution $i \mapsto i^*$ of I, which induces an antiautomorphism of F. If we let $i_0 \in I$ correspond to the unit object in $\mathcal{F}$, this means that $N_{i,j}^{i_0} = \delta_{i,j^*}$ together with $N_{i^*,j^*}^{l^*} = N_{i,j}^l$. We end up with the axiomatization of a fusion ring given in Definition 1.1.

Among other things, you may ask the following questions about fusion rings:

Given a fusion ring F, in the sense of a countable set I and a set of integers $\{N_{i,j}^l \in \mathbb{N} \mid i,j,l \in I\}$ satisfying the above properties, can we classify the non-equivalent fusion categories, whose Grothendieck group is isomorphic to F? By Ocneanu rigity, cf. [ENO05, Theorem 2.28], over a field of characteristic 0 the number of such is finite. An approach to this problem is to classify fusion categories with a small number of isomorphism classes of simple objects, and here [Ost03], [Ost08] and [Lar14] take care of the cases of 2, 3 and 4 classes.

Given a fusion category, give an effective method for calculating the structure constants $N_{i,j}^l$ in the fusion ring. We ask for either an algorithm that terminates in polynomial time or an identification of the structure constants with known numbers.

If we consider the basis elements $\{[i] \mid i \in I\}$ as formal variables we may present a rational fusion ring F as the quotient of the free polynomial ring in $|I|$ variables by some ideal, namely

$$F \simeq \mathbb{Z}\left[[i]\right]/J,$$

where J is the ideal generated by relations $[i][j] - \sum_{l \in I} N_{i,j}^l [l]$ for all $i,j \in I$. Some of these variables can be eliminated, e.g. we may identify $[i_0] = 1$. Fix a subset $\{i_1, \ldots, i_r\} \subseteq I$ such that $F \simeq \mathbb{Z}\left[[i_1], \ldots, [i_r]\right]/J'$ and no such elimination may occur. What is a minimal generating set of J'?

Of these questions we will focus on the last two.

Summary

The dissertation is structured as follows. Chapter 0 introduces notation and terminology used throughout the dissertation.

In Chapter 1 three main examples of fusion rings are examined. The case (i) above is used as a prototypical example to describe the combinatorial structure of a fusion ring in terms of fusion rules. From (ii) we define a rational fusion ring, which is realized as a quotient of the first example.

Finally (iii) gives rise to a class of fusion rings, which is seen to encompass the previous examples.

Chapter 2 is a historical overview of the development of the theory on fusion rings. Focus is on presenting the results used in this dissertation according to when they were first introduced in the literature.

In Chapter 3 we address the problem of finding a minimal generating set of the defining ideal in a presentation of the fusion ring as a quotient of a polynomial ring. A combinatorial approach is used for Lie algebras of classical type, giving explicit expressions for generating sets in these cases. For Lie algebras of low rank, general algebraic arguments give a non-constructive result on an upper bound for the number of elements in a minimal generating set.

In Chapter 4 we do a treatment of the fusion rings associated to simple Lie algebras of rank 2. Case by case, explicit generating sets of the fusion ideal are presented and shown by calculations to work. The findings are compared to the results in the previous chapter.

In Chapter 5 we relax the axioms of a fusion ring to get similar structures for which the previously studied methods and results apply. In particular we study tensor ideals in the category of tilting modules of a quantum group. We give a general result on finding a generating set of the tensor ideal, together with an explicit analysis in a case related to a Lie algebra of rank 2.

In Chapter 6 we present our unfinished work and open projects. In [Dou13] a canonical generating set of the fusion ideal is presented for each fusion ring in the setting of (ii). We present our progress in generalizing this result to fusion rings for quantum groups, and outline the obstructions that arise in this setting.

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Chapter 0

Notation and terminology

We fix once and for all notation for well known objects and constructions. General references are [Hum72] and [Jan96].

$\mathbb{N} \subseteq \mathbb{Z}$	The set of non-negative numbers sitting inside the ring of integers.
$\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$	The fields of rational numbers, real numbers and complex numbers.
$\mathfrak{g} \supseteq \mathfrak{b} \supseteq \mathfrak{h}$	A simple complex Lie algebra of rank r, a Borel and a Cartan subalgebra.
$\Phi \supseteq \Phi^+$	The corresponding root system and set of positive roots.
$\alpha_1, \ldots, \alpha_r$	The simple roots of Φ^+.
$\alpha_0, \beta_0 \in \Phi^+$	The highest short root and the highest long root.
Q	The root lattice $\mathbb{Z}\Phi$.
E	Euclidean vector space $Q \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by the root system.
$(\cdot, \cdot)$	Inner product on E such that $(\alpha, \alpha) = 2$ for all short roots α.
d_i	The numbers $\frac{(\alpha_i, \alpha_i)}{2}$ for each simple root which are either 1, 2 or 3 depending on $\mathfrak{g}$.
$\omega_1, \ldots, \omega_r$	The fundamental weights, dual in E to the basis $\frac{\alpha_1}{d_1}, \ldots, \frac{\alpha_r}{d_r}$.
P	The integral weight lattice $\sum_{i=1}^r \mathbb{Z}\omega_i$.
P^+	The dominant weights $\sum_{i=1}^r \mathbb{N}\omega_i$.
$\leq$	The order on P given by $\lambda \leq \mu \Leftrightarrow \mu - \lambda \in Q \cap P^+$.
ρ	Half the sum of the positive roots $\frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^r \omega_i$.
$\langle \lambda, \alpha^\vee \rangle$	The integers $\frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$ for $\lambda \in P, \alpha \in \Phi$.
s_α	The reflection $\lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ on P with regards to $\alpha \in \Phi$.
W	The Weyl group generated by all reflections $s_\alpha, \alpha \in \Phi$.
w_0	The longest element of the Weyl group.

$(a_{ij})_{i,j=1}^r$	The Cartan matrix, $a_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$.
A	The Laurent polynomial ring $\mathbb{Z}[v, v^{-1}]$ over the integers.
$\mathbb{Q}(v)$	The quotient field of A.
v_i	The number v^{d_i}.
$[n]_v$	The nth quantum number $\frac{v^n - v^{-n}}{v - v^{-1}}$.
$\begin{bmatrix} n \\ m \end{bmatrix}_v$	The Gaussian binomial coefficient $\frac{[n]_v \cdots [n-m+1]_v}{[m]_v \cdots [1]_v}$.
U_v	The generic quantum group over $\mathbb{Q}(v)$ associated to $\mathfrak{g}$ with generators E_i, F_i, K_i and $K_i^{-1}, i = 1, \dots, r$ and certain relations, cf. [Jan96, 4.3].
$E_i^{(n)}, F_i^{(n)}$	The divided powers $\frac{E_i^n}{[n]_{v_i}}, \frac{F_i^n}{[n]_{v_i}}, n \geq 1$.
$\begin{bmatrix} K_i \\ t \end{bmatrix}_{v_i}$	The element $\prod_{s=1}^t \frac{K_i v_i^{-s+1} - K_i^{-1} v_i^{s-1}}{v_i^s - v_i^{-s}}, t \in \mathbb{N}$.
U_A	The Lusztig A-form, the A-subalgebra of U_v spanned by all $E_i^{(n)}, F_i^{(n)}$ and $K_i^{\pm 1}, i = 1, \dots, r, n \geq 1$.
U_q	The quantum group $U_A \otimes_A \mathbb{C}$ over the A-algebra $\mathbb{C}$ by specializing v to a non-zero element $q \in \mathbb{C}$.
U_q^-, U_q^0, U_q^+	The subalgebras of U_q generated by $\{F_i^{(n)} \mid i = 1, \dots, r, n \geq 1\}, \{K_i^{\pm 1}, \begin{bmatrix} K_i^{\pm 1} \\ t \end{bmatrix}_{q_i} \mid i = 1, \dots, r, t \in \mathbb{N}\}$ resp. $\{E_i^{(n)} \mid i = 1, \dots, r, n \geq q\}$. Here $q_i = q^{d_1}$.

Unless otherwise stated tensor products are always over the complex numbers, i.e., we write $\otimes = \otimes_{\mathbb{C}}$. We also write dim short for $\dim_{\mathbb{C}}$.

We assume for simplicity that the Lie algebra $\mathfrak{g}$ is simple as the generalization to the case of a semisimple Lie algebra is immediate.

0.1 Monoidal categories

The main subject in this dissertation is on rings in different shades. The purpose of this section is to set up the language necessary to formalize categorification of rings. As a general reference one may use [ML98].

A category is *additive* if all Hom-spaces have a structure of abelian groups such that composition of morphisms is bilinear and every finite set of objects have a biproduct. Necessarily the empty biproduct is a zero object in the category. We use $\oplus$ for notation of the biproduct.

An additive category is *abelian* if all morphisms have a kernel and cokernel and if every monomorphism is the kernel of some morphism and every epimorphism is the cokernel of some morphism.

A *semisimple* category is an abelian category where every short exact sequence splits.

A *(weak) monoidal* category $\mathcal{A}$ consists of a bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ written $A \otimes B$, functorial isomorphisms $\alpha_{A,B,C} : (A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$ for any $A, B, C \in \mathcal{A}$, a unit object $I \in \mathcal{A}$, functorial isomorphisms $\lambda_A : I \otimes A \simeq A$ and $\rho_A : A \otimes I \simeq A$ for any $A \in \mathcal{A}$ subject to the following axioms:

(i) The *pentagon axiom*: For any $A, B, C, D \in \mathcal{A}$ the diagram

$$\begin{tikzcd} & ((A \otimes B) \otimes C) \otimes D \arrow[dl, "\alpha_{A,B,C}\otimes \mathrm{id}"'] \arrow[dr, "\alpha_{A\otimes B,C,D}"] & \\ (A \otimes (B \otimes C)) \otimes D \arrow[d, "\alpha_{A,B\otimes C,D}"] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C\otimes D}"] \\ A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\mathrm{id}\otimes\alpha_{B,C,D}"] & & A \otimes (B \otimes (C \otimes D)) \end{tikzcd}$$

commutes.

(ii) The *triangle axiom*: For any $A, B \in \mathcal{A}$ the diagram

$$\begin{tikzcd} (A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A\otimes\mathrm{id}"'] & & A \otimes (I \otimes B) \arrow[dl, "\mathrm{id}\otimes\lambda_B"] \\ & A \otimes B & \end{tikzcd}$$

commutes. $\mathcal{A}$ is called *strict* monoidal if the isomorphisms $\alpha_{A,B,C}$, λ_A, ρ_A are actually equalities.

By *Mac Lane's strictness theorem* every weak monoidal category is equivalent to a strict monoidal category. We will therefore often omit the associativity isomorphisms $\alpha_{A,B,C}$ and the unit isomorphisms λ_a, ρ_A.

Let $\mathcal{A}, \mathcal{A}'$ be monoidal categories. A *monoidal functor* consists of a functor $F : \mathcal{A} \to \mathcal{A}'$, functorial isomorphisms $\beta_{A,B} : F(A \otimes B) \to F(A) \otimes' F(B)$ for all $A, B \in \mathcal{A}$, and an isomorphism $\iota : F(I) \to I'$ satisfying the natural compatibilities: For any $A, B, C \in \mathcal{A}$ the diagrams

$$\begin{tikzcd} F((A \otimes B) \otimes C) \arrow[r, "\beta_{A\otimes B,C}"] \arrow[d, "F\alpha_{A,B,C}"'] & F(A \otimes B) \otimes' F(C) \arrow[r, "\beta_{A,B}\otimes'\mathrm{id}"] & (F(A) \otimes' F(B)) \otimes' F(C) \arrow[d, "\alpha'_{F(A),F(B),F(C)}"] \\ F(A \otimes (B \otimes C)) \arrow[r, "\beta_{A,B\otimes C}"] & F(A) \otimes' F(B \otimes C) \arrow[r, "\mathrm{id}\otimes'\beta_{B,C}"] & F(A) \otimes' (F(B) \otimes' F(C)) \end{tikzcd}$$

and

$$\begin{tikzcd}
F(I \otimes A) \arrow[r, "\beta_{I,A}"] \arrow[d, "F(\lambda_A)"'] & F(I) \otimes' F(A) \arrow[d, "\iota \otimes' \mathrm{id}"] \\
F(A) & I' \otimes' F(A) \arrow[l, "\lambda'_{F(A)}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A \otimes I) \arrow[r, "\beta_{A,I}"] \arrow[d, "F(\rho_A)"'] & F(A) \otimes' F(I) \arrow[d, "\mathrm{id} \otimes' \iota"] \\
F(A) & F(A) \otimes' I' \arrow[l, "\rho'_{F(A)}"]
\end{tikzcd}$$

are commutative.

Let $\mathcal{A}$ be a monoidal category and $A \in \mathcal{A}$ be a given object. A *right dual* to A is an object A^* together with two morphisms

$$\eta_A : A^* \otimes A \to I, \iota_A : I \to A \otimes A^*$$

such that the compositions

$$A \xrightarrow{\iota_A \otimes \mathrm{id}} A \otimes A^* \otimes A \xrightarrow{\mathrm{id} \otimes \eta_A} A$$

$$A^* \xrightarrow{\mathrm{id} \otimes \iota_A} A^* \otimes A \otimes A^* \xrightarrow{\eta_A \otimes \mathrm{id}} A^*$$

are equal to the identity morphisms. There is a similar notion of a *left dual* ${}^*A$ to A. The category $\mathcal{A}$ is called *rigid* if every object in $\mathcal{A}$ has a right and a left dual. If $\mathcal{A}$ is a semisimple, rigid, monoidal category, then for any object $A \in \mathcal{A}$ there is an isomorphism $A^* \simeq {}^*A$.

Let still $\mathcal{A}$ be a monoidal category. A *module category* $\mathcal{M}$ over $\mathcal{A}$ consists of an exact bifunctor $\otimes : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$, functorial isomorphisms $\mu_{A,B,M}$: $(A \otimes B) \otimes M \to A \otimes (B \otimes M)$ for any $A, B \in \mathcal{A}, M \in \mathcal{M}$ and $\varepsilon_M : I \otimes M \to M$ for any $M \in \mathcal{M}$ such that the diagrams

$$\begin{tikzcd}
& ((A \otimes B) \otimes C) \otimes M \arrow[dl, "\alpha_{A,B,C} \otimes \mathrm{id}"'] \arrow[dr, "\mu_{A \otimes B, C, M}"] & \\
(A \otimes (B \otimes C)) \otimes M \arrow[d, "\mu_{A, B \otimes C, M}"] & & (A \otimes B) \otimes (C \otimes M) \arrow[d, "\mu_{A,B,C \otimes M}"] \\
A \otimes ((B \otimes C) \otimes M) \arrow[rr, "\mathrm{id} \otimes \mu_{B,C,M}"] & & A \otimes (B \otimes (C \otimes M))
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A \otimes I) \otimes M \arrow[rr, "\mu_{A,I,M}"] \arrow[dr, "\rho_A \otimes \mathrm{id}"'] & & A \otimes (I \otimes M) \arrow[dl, "\mathrm{id} \otimes \varepsilon_M"] \\
& A \otimes M &
\end{tikzcd}$$

commute for any $A, B, C \in \mathcal{A}, M \in \mathcal{M}$.

If $\mathcal{A}$ is a rigid monoidal category and $\mathcal{M}$ a module category of $\mathcal{A}$ then we have canonical isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}}(A \otimes M, N) &\simeq \mathrm{Hom}_{\mathcal{A}}(M, {}^*A \otimes N),\\ \mathrm{Hom}_{\mathcal{A}}(M, A \otimes N) &\simeq \mathrm{Hom}_{\mathcal{A}}(A^* \otimes M, N)\end{aligned}$$

for any $A \in \mathcal{A}, M, N \in \mathcal{M}$.

Let $\mathcal{M}, \mathcal{M}'$ be module categories over a monoidal category $\mathcal{A}$. A *module functor* consists of a functor $F : \mathcal{M} \to \mathcal{M}'$, and functorial morphisms $\gamma_{A,M} : F(A \otimes M) \to A \otimes' F(M)$ for any $A \in \mathcal{A}, M \in \mathcal{M}$ such that the diagrams

$$\begin{tikzcd} & F((A \otimes B) \otimes M) \arrow[dl, "F\mu_{A,B,M}"'] \arrow[dr, "\gamma_{A\otimes B,M}"] & \\ F(A \otimes (B \otimes M)) \arrow[d, "\gamma_{A,B\otimes M}"] & & (A \otimes B) \otimes' F(M) \arrow[d, "\mu'_{A,B,F(M)}"] \\ A \otimes' F(B \otimes M) \arrow[rr, "\mathrm{id}\otimes'\gamma_{B,M}"] & & A \otimes' (B \otimes' F(M)) \end{tikzcd}$$

and

$$\begin{tikzcd} F(I \otimes M) \arrow[rr, "F\varepsilon_M"] \arrow[dr, "\gamma_{I,M}"'] & & F(M) \\ & I \otimes' F(M) \arrow[ur, "\varepsilon'_{F(M)}"'] & \end{tikzcd}$$

commute for any $A, B \in \mathcal{A}, M \in \mathcal{M}$. The module functor is called *strict* if all the morphisms $\gamma_{A,M}$ are isomorphisms. If $\mathcal{A}$ is rigid, any module functor is automatically strict.

Let $\mathcal{A}$ be any category. A *congruence relation* R on $\mathcal{A}$ consists of an equivalence relation $R_{A,B}$ on $\mathrm{Hom}_{\mathcal{A}}(A, B)$ for any $A, B \in \mathcal{A}$ such that the equivalence relations respects composition of morphisms. I.e., if f_1, f_2 are related in $\mathrm{Hom}_{\mathcal{A}}(A, B)$ and g_1, g_2 are related in $\mathrm{Hom}_{\mathcal{A}}(B, C)$ then $g_1 f_1, g_1 f_2$, $g_2 f_1, g_2 f_2$ are related in $\mathrm{Hom}_{\mathcal{A}}(A, C)$. We define the *quotient category* $\mathcal{A}/R$ as the category whose objects are those of $\mathcal{A}$ and morphisms are equivalence classes of morphisms in $\mathcal{A}$, i.e., $\mathrm{Hom}_{\mathcal{A}/R}(A, B) = \mathrm{Hom}_{\mathcal{A}}(A, B)/R_{A,B}$.

Let $\mathcal{A}$ be an additive, monoidal category. We say a full subcategory $\mathcal{I}$ of $\mathcal{A}$ is a *tensor ideal* if $K \oplus L \in \mathcal{I}$ if and only if $K, L \in \mathcal{I}$ and if $A \in \mathcal{A}, J \in \mathcal{I}$ implies $A \otimes J \in \mathcal{I}$. Let R be the congruence relation on $\mathcal{A}$ where $R_{A,B}(f, g)$ if $f - g : A \to B$ factors through an element of $\mathcal{I}$, and define $\mathcal{A}/\mathcal{I}$ to be the associated quotient category. This becomes an additive, monoidal category. Furthermore $\mathcal{A}/\mathcal{I}$ is a module category over $\mathcal{A}$.

0.2 The Grothendieck group

Let $\mathcal{A}$ be an additive category. The *split Grothendieck group* $\mathrm{K}_0^{\oplus}(\mathcal{A})$ of $\mathcal{A}$ is the quotient of the free abelian group generated by isomorphism classes $[A]$ of objects $A \in \mathcal{A}$ modulo relations $[A] = [B] + [C]$ whenever there is an isomorphism $A \simeq B \oplus C$.

If $\mathcal{A}$ is an abelian category then, as it is an additive category as well, we could consider its split Grothendieck group. This group however is in some ways too big. For instance two objects with the same composition factors may not be identified in $\mathrm{K}_0^{\oplus}(\mathcal{A})$. Instead we define the *(non-split) Grothendieck group* $\mathrm{K}_0(\mathcal{A})$ to be the quotient of the free abelian group generated by isomorphism classes $[A], A \in \mathcal{A}$ modulo relations $[A] = [B] + [C]$ whenever there is a short exact sequence $0 \to B \to A \to C \to 0$.

If $\mathcal{A}$ is additive, but not abelian, then we write $\mathrm{K}_0(\mathcal{A})$ instead of $\mathrm{K}_0^{\oplus}(\mathcal{A})$. Whether $\mathcal{A}$ is additive or abelian we call $\mathrm{K}_0(\mathcal{A})$ the *decategorification* of $\mathcal{A}$. For any abelian group G isomophic to $\mathrm{K}_0(\mathcal{A})$ we say that $\mathcal{A}$ *categorifies* G.

If $\mathcal{A}$ is an additive/abelian and monoidal category, then the (split/non-split) Grothendieck group has a ring structure induced by $\otimes$: $[A][B] = [A \otimes B]$. If an additive/abelian category $\mathcal{M}$ is a module category over $\mathcal{A}$ then its Grothendieck group $\mathrm{K}_0(\mathcal{M})$ has the structure of a $\mathrm{K}_0(\mathcal{A})$-module.

If $\mathcal{I} \subseteq \mathcal{A}$ is a full subcategory of an additive category such that $K \oplus L \in \mathcal{I}$ if and only if $K, L \in \mathcal{I}$ then there is an injective embedding of the Grothendieck group $\mathrm{K}_0(\mathcal{I})$ in $\mathrm{K}_0(\mathcal{A})$. If $\mathcal{A}$ is a monoidal category and $\mathcal{I}$ is a tensor ideal in $\mathcal{A}$ then $\mathrm{K}_0(\mathcal{I})$ furthermore has the structure of a $\mathrm{K}_0(\mathcal{A})$-ideal and we may form the quotient $\mathrm{K}_0(\mathcal{A})/\mathrm{K}_0(\mathcal{I})$.

Given a commutative, unital ring R and $\mathcal{A}$ the *R-decategorification* of an additive/abelian category $\mathcal{A}$ is $\mathrm{K}_0(\mathcal{A}) \otimes_{\mathbb{Z}} R$. If M is an R-module then an *R-categorification* of M is an additive/abelian category $\mathcal{A}$ and an isomorphism $M \simeq \mathrm{K}_0(\mathcal{A}) \otimes_{\mathbb{Z}} R$.

0.3 The character ring and character formulas

Define $\mathbb{Z}[P]$ to be the group algebra over $\mathbb{Z}$ of the weight lattice P. It is free with $\mathbb{Z}$-basis $\{e^\lambda \mid \lambda \in P\}$ and its multiplicative structure is $e^\lambda e^\mu = e^{\lambda+\mu}, \lambda, \mu \in P$. The Weyl group action on P induces an action on $\mathbb{Z}[P]$ by $we^\lambda = e^{w\lambda}, w \in W$ and we define the *character ring* $\mathbb{Z}[P]^W$ to consist of the invariant elements under this action. Then $\mathbb{Z}[P]^W$ is a subalgebra of $\mathbb{Z}[P]$.

Each finite-dimensional $\mathfrak{g}$-module L splits into a direct sum of its weight spaces $L = \oplus_{\lambda \in P} L_\lambda$, obeying $\dim L_\lambda = \dim L_{w(\lambda)}$ for all $w \in W$, so L

defines an element in the character ring: $\mathrm{ch}\, L = \sum_{\lambda \in P} \dim L_\lambda e^\lambda \in \mathbb{Z}[P]^W$. It is well-known that the character $\mathrm{ch}\, L(\lambda)$ of the simple finite-dimensional highest weight module $L(\lambda), \lambda \in P^+$ is given by the Weyl characters

$$\chi_\lambda = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{l(w)} e^{w\rho}} \tag{1}$$

and the set $\{\chi_\lambda \mid \lambda \in P^+\}$ constitutes a $\mathbb{Z}$-basis for $\mathbb{Z}[P]^W$. (1) makes sense for all $\lambda \in P$ and we note that if the stabilizer $\mathrm{Stab}_W(\lambda) = \{w \in W \mid w.\lambda = \lambda\}$ of λ in W under the shifted action $w.\lambda = w(\lambda + \rho) - \rho$ is non-trivial then $\chi_\lambda = 0$. If the stabilizer is trivial then there is a unique $w \in W$ such that $w.\lambda \in P^+$ and $\chi_\lambda = (-1)^{l(w)} \chi_{w.\lambda}$.

If $\sum_{\mu \in P} a_\mu e^\mu \in \mathbb{Z}[P]^W$ is a W-invariant element then multiplication with a Weyl character is explicitly given by

$$(\sum_{\mu \in P} a_\mu e^\mu) \chi_\lambda = \sum_{\mu \in P} a_\mu \chi_{\lambda+\mu}.$$

As a $\mathbb{Z}$-algebra $\mathbb{Z}[P]^W$ is generated freely by $\chi_{\omega_i}, i = 1, ..., r$, i.e., we have an isomorphism $\mathbb{Z}[X_1, \ldots, X_r] \simeq \mathbb{Z}[P]^W$ mapping X_i to χ_{ω_i}. This means that any W-invariant element $\sum_{\mu \in P} a_\mu e^\mu \in \mathbb{Z}[P]^W$ can be written as a polynomial in the fundamental characters.

Chapter 1

Fusion rings

Let $\mathcal{F}$ denote the category of finite-dimensional $\mathfrak{g}$-modules. For a dominant weight $\lambda \in P^+$ denote by $L(\lambda)$ the simple module of highest weight λ. These constitute all finite-dimensional simple $\mathfrak{g}$-modules, and any module $M \in \mathcal{F}$ splits as a direct sum of simple components: $M = \bigoplus_{\lambda \in P^+} L(\lambda)^{\oplus a_\lambda(M)}$, $a_\lambda(M) \in \mathbb{N}$.

For $\lambda, \mu \in P^+$ the module $L(\lambda) \otimes L(\mu)$ is again finite-dimensional and its decomposition into simple components are given by numbers $M^\nu_{\lambda,\mu} = a_\nu(L(\lambda) \otimes L(\mu))$. We recall a few well-known properties of these tensor products.

Direct sum $\oplus$ and tensor product $\otimes$ give $\mathcal{F}$ the structure of an abelian, monoidal category. The trivial highest weight module $L(0) = \mathbb{C}$ is a unit element, i.e., $M^\nu_{\lambda,0} = \delta_{\lambda,\nu}$.

The structure constants $M^\nu_{\lambda,\mu}$ are non-negative integers and given $\lambda, \mu \in P^+$ only for finitely many $\nu \in P^+$ are they non-zero.

For any $L(\lambda)$ the dual module $L(\lambda)^*$ is again a simple module of highest weight $\lambda^* = -w_0(\lambda) \in P^+$, where $w_0(\lambda)$ is the lowest weight of $L(\lambda)$. We have $(L(\lambda)^*)^* = L(\lambda)$, $L(0)$ appears exactly once in the tensor product $L(\lambda) \otimes L(\lambda^*)$ and in fact $M^0_{\lambda,\mu} = \delta_{\mu,\lambda^*}$. This means $\mathcal{F}$ is a rigid monoidal category.

The tensor product of dual modules is the dual of the tensor product of the modules themselves, in the sense that $M^{\nu^*}_{\lambda^*,\mu^*} = M^\nu_{\lambda,\mu}$. We also see explicitly that the structure constants are given by $M^\nu_{\lambda,\mu} = \dim \operatorname{Hom}_{\mathfrak{g}}(L(\lambda) \otimes L(\mu) \otimes L(\nu^*), \mathbb{C})$.

If we forget the categorical structure on $\mathcal{F}$ and consider only the structure constants $\{M^\nu_{\lambda,\mu} \mid \lambda, \mu, \nu \in P^+\}$, what can we say about them? We consider formal sums of elements indexed by P^+ and give them a product defined by the structure constants $M^\nu_{\lambda,\mu}$. This is the ring structure on the Grothendieck group $\mathrm{K}_0(\mathcal{F})$ with various additional properties corresponding to the properties deduced above. In the following section we formalize these properties.

1.1 Fusion ring axioms

Definition 1.1. *Let I be a countable set. A fusion rule on I is a set of non-negative integers $N = \{N_{\lambda,\mu}^{\nu} \in \mathbb{N} \mid \lambda, \mu, \nu \in I\}$ such that*

(i) $N_{\lambda,\mu}^{\nu} = N_{\mu,\lambda}^{\nu}$ *for all* $\lambda, \mu, \nu \in I$,

(ii) $\sum_{\eta \in I} N_{\lambda,\mu}^{\eta} N_{\eta,\nu}^{\zeta} = \sum_{\eta \in I} N_{\mu,\nu}^{\eta} N_{\lambda,\eta}^{\zeta}$ *for all* $\lambda, \mu, \nu, \zeta \in I$,

(iii) there is an element $\lambda_0 \in I$ *and an associated map* $\lambda \mapsto \lambda^*$ *given by* $N_{\lambda,\mu}^{\lambda_0} = \delta_{\mu,\lambda^*}$ *such that* $(\lambda^*)^* = \lambda$ *and* $N_{\lambda^*,\mu^*}^{\nu^*} = N_{\lambda,\mu}^{\nu}$ *for all* $\lambda, \mu, \nu \in I$.

The associated fusion ring $F = F(N)$ *is the free* $\mathbb{Z}$*-module with basis* $\{[\lambda] \mid \lambda \in I\}$ *equipped with multiplication* $[\lambda][\mu] = \sum_{\nu \in I} N_{\lambda,\mu}^{\nu}[\nu]$*. If the indexing set* I *is finite the fusion ring has finite rank and we say it is rational.*

Remark. The requirements in the definition has the following interpretation. The condition (i) is equivalent to the multiplication being commutative

$$[\lambda][\mu] = \sum_{\nu \in I} N_{\lambda,\mu}^{\nu}[\nu] = \sum_{\nu \in I} N_{\mu,\lambda}^{\nu}[\nu] = [\mu][\lambda]$$

and the condition (ii) is equivalent to it being associative

$$\begin{aligned}([\lambda][\mu])[\nu] &= \sum_{\eta \in I} N_{\lambda,\mu}^{\eta}[\eta][\nu] = \sum_{\eta,\zeta \in I} N_{\lambda,\mu}^{\eta} N_{\eta,\nu}^{\zeta}[\zeta] = \sum_{\eta,\zeta \in I} N_{\mu,\nu}^{\eta} N_{\lambda,\eta}^{\zeta}[\zeta] \\ &= \sum_{\eta \in I} N_{\mu,\nu}^{\eta}[\lambda][\eta] = [\lambda]([\mu][\nu]).\end{aligned}$$

Define a map $[\lambda] \mapsto [\lambda]^* = [\lambda^*]$ and extend by linearity to all of F. Then (iii) is equivalent to this map being an involution $([\lambda]^*)^* = [\lambda]$ and a homomorphism $([\lambda][\mu])^* = [\lambda]^*[\mu]^*$. □

We say that two fusion rings $F^{(1)}$ and $F^{(2)}$ are isomorphic if there is a bijection of their underlying index sets $\sigma : I^{(1)} \to I^{(2)}$ which respects (i)-(iii) of Definition 1.1, i.e., such that $(N^{(1)})_{\lambda,\mu}^{\nu} = (N^{(2)})_{\sigma(\lambda),\sigma(\mu)}^{\sigma(\nu)}$, $\sigma(\lambda_0^{(1)}) = \lambda_0^{(2)}$, $\sigma(\lambda^*) = \sigma(\lambda)^*$.

Properties of fusion rings

We deduce some results from the axioms of a fusion ring $F = F(N)$ for a fusion rule N on a set I.

We have $N^{\lambda_0}_{\lambda,\mu} = \delta_{\mu,\lambda^*} = \delta_{\lambda^*,(\mu^*)^*} = N^{\lambda_0}_{\mu^*,\lambda^*}$ so $[\nu] = [\nu^*]^* = \sum_{\eta\in I} N^{\lambda_0}_{\nu^*,\eta^*}[\eta^*]$ $= \sum_{\eta\in I} N^{\lambda_0}_{\nu,\eta}[\eta^*]$ which lets us rewrite the multiplication

$$[\lambda][\mu] = \sum_{\nu\in I} N_{\lambda,\mu,\nu}[\nu^*]$$

where $N_{\lambda,\mu,\nu} = N^{\nu^*}_{\lambda,\mu} = \sum_{\eta\in I} N^{\nu}_{\lambda,\mu}N^{\lambda_0}_{\nu,\eta}$. The set of structure constants $\{N_{\lambda,\mu,\nu} \mid \lambda,\mu,\nu \in I\}$ are completely symmetric in the $\lambda,\mu,\nu \in I$, for instance $N_{\lambda,\mu,\nu} = \sum_{\eta\in I} N^{\eta}_{\lambda,\mu}N^{\lambda_0}_{\eta,\nu} = \sum_{\eta\in I} N^{\eta}_{\lambda,\nu}N^{\lambda_0}_{\eta,\mu} = N_{\lambda,\nu,\mu}$. As a consequence we see that

$$N^{\nu}_{\lambda_0,\mu} = N_{\lambda_0,\mu,\nu^*} = N_{\mu,\nu^*,\lambda_0} = N^{\lambda_0^*}_{\mu,\nu^*} = N^{\lambda_0}_{\mu^*,\nu} = \delta_{\mu,\nu}$$

so the special element $[\lambda_0]$ acts as the identity on F, denote it $1 = [\lambda_0]$. Moreover by the same formula $N^{\lambda_0}_{\lambda_0,\mu} = \delta_{\lambda_0,\mu}$ we see that $[\lambda_0]^* = [\lambda_0]$.

Given an $x \in F$ write it in the basis given by I as $x = \sum_{\lambda\in I} n_\lambda[\lambda]$ and define a $\mathbb{Z}$-linear form $t(x) = n_{\lambda_0}$. By assumption it satisfies $t([\lambda][\mu]^*) = N^{\lambda_0}_{\lambda,\mu^*} = \delta_{\lambda,\mu}$ and furthermore $t(x[\mu^*]) = \sum_{\lambda\in I} n_\lambda t([\lambda][\mu^*]) = n_\mu$. For $x, y \in F$ define a $\mathbb{Z}$-bilinear form $(x, y) = t(xy^*)$. It is positive-definite since $([\lambda],[\mu]) = \delta_{\lambda,\mu}$.

Assume now that the indexing set I is finite. Then the bilinear form defines an isomorphism of $\mathbb{Z}$-modules of F with $\mathrm{Hom}_{\mathbb{Z}}(F,\mathbb{Z})$ by $x \mapsto [f_x : y \mapsto (x, y)]$. If we give $\mathrm{Hom}_{\mathbb{Z}}(F,\mathbb{Z})$ an F-action by $zf(y) = f(z^*y)$, then this isomorphism is actually an isomorphism of F-modules since $f_{zx}(y) = (zx, y) = (x, z^*y) = zf_x(y)$ for all $z \in F$.

The exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ induces to

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(F,\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(F,\mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(F,\mathbb{Q}/\mathbb{Z}) \to 0.$$

Since $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules $\mathrm{Hom}_{\mathbb{Z}}(F,\mathbb{Q})$ and $\mathrm{Hom}_{\mathbb{Z}}(F,\mathbb{Q}/\mathbb{Z})$ are injective F-modules. This means that $\mathrm{Hom}_{\mathbb{Z}}(F,\mathbb{Z})$ has finite injective dimension, and by the module isomorphism $F \simeq \mathrm{Hom}_{\mathbb{Z}}(F,\mathbb{Z})$ constructed above, we conclude that so has F, i.e., rational fusion rings are Gorenstein.

1.2 The Wess-Zumino-Witten fusion rings

We resume in this section the work with representations of the Lie algebra $\mathfrak{g}$ initated in the beginning of the chapter. The representation ring

$R = \mathrm{K}_0(\mathcal{F})$ has as elements formal differences of isomorphism classes of finite-dimensional representations of $\mathfrak{g}$ with addition given by direct sums of representations and multiplication given by tensor product. The structure constants $M = \{M^{\nu}_{\lambda,\mu} \mid \lambda, \mu, \nu \in P^+\}$ define a fusion rule on P^+, and the representation ring with $\mathbb{Z}$-basis $\{[L(\lambda)] \mid \lambda \in P^+\}$ and product structure

$$[L(\lambda)][L(\mu)] = \sum_{\nu \in P_+} M^{\nu}_{\lambda,\mu}[L(\nu)]$$

is the associated fusion ring $F(M)$. The goal is now to show how to define a rational fusion ring in a natural way from this construction. The following is mainly based on [Bea96].

Associated affine Lie algebra

Consider the ring $\mathbb{C}[[z]]$ of formal power series $\sum_{n=0}^{\infty} c_n z^n$. Its quotient field $\mathbb{C}((z))$ consists of formal Laurent series $\sum_{n\in\mathbb{Z}} c_n z^n$ with $c_n \neq 0$ for only finitely many negative n, and thus we have the polynomial ring $\mathbb{C}[z^{-1}]$ embedded in $\mathbb{C}((z))$.

Let $\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((z)) \oplus \mathbb{C}K$ denote the affine Lie algebra associated to $\mathfrak{g}$. This is the central extension of the infinite-dimensional Lie algebra $\mathfrak{g} \otimes \mathbb{C}((z))$ with one-dimensional center $\mathbb{C}K$. The Lie bracket on $\mathfrak{g} \otimes \mathbb{C}((z))$ of two elements $x \otimes f$ and $y \otimes g$ is

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg + \kappa\,(x, y) \operatorname{Res}_{z=0}(gdf)K.$$

Here $\kappa\,(\cdot, \cdot)$ is the Killing form on $\mathfrak{g}$ normalized such that coroots $\beta^\vee = \frac{2\beta}{(\beta,\beta)} \in \mathfrak{h}$ corresponding to long roots $\beta \in \Phi$ has $\kappa\,(\beta^\vee, \beta^\vee) = 2$.

We have a decomposition

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}^- \oplus \mathfrak{g} \oplus \mathbb{C}K \oplus \widetilde{\mathfrak{g}}^+,$$

where $\widetilde{\mathfrak{g}}^- = \mathfrak{g} \otimes z^{-1}\mathbb{C}[z^{-1}]$, $\widetilde{\mathfrak{g}}^+ = \mathfrak{g} \otimes z\mathbb{C}[[z]]$ and $\mathfrak{g} = \mathfrak{g} \otimes z^0$ are subalgebras of $\widetilde{\mathfrak{g}}$. Let also $\mathfrak{p} = \mathfrak{g} \oplus \mathbb{C}K \oplus \widetilde{\mathfrak{g}}^+$ denote a parabolic subalgebra of $\widetilde{\mathfrak{g}}$.

For a dominant weight $\lambda \in P^+$ let $\Delta(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ denote the universal highest weight module of highest weight λ. Then $L(\lambda)$ occurs as the unique simple quotient of $\Delta(\lambda)$.

Denote by $\mathfrak{s}$ the subalgebra of $\widetilde{\mathfrak{g}}$ spanned by $x_{-\beta_0} \otimes z, K - \beta_0^\vee \otimes 1, x_{\beta_0} \otimes z^{-1}$, where $x_{\pm\beta_0} \in \mathfrak{g}_{\pm\beta_0}$ are root vectors chosen such that $[x_{\beta_0}, x_{-\beta_0}] = -\beta_0^\vee$. As a Lie algebra $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}_2$.

Given an integer $k \geq 0$ we extend the action of $\mathfrak{g}$ on $\Delta(\lambda)$ to $\mathfrak{p}$ by letting K act by multiplication with k and $\widetilde{\mathfrak{g}}^+$ act trivially. We define the induced highest weight modules $\widetilde{\Delta}(\lambda) = U(\widetilde{\mathfrak{g}}) \otimes_{U(\mathfrak{p})} \Delta(\lambda)$. $\widetilde{\Delta}(\lambda)$ has a unique simple quotient $\widetilde{L}(\lambda)$.

Recall the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ of the Lie algebra $\mathfrak{g}$. We call a $\mathfrak{g}$-module integrable if it is locally $\mathfrak{n}^-$-finite and locally $\mathfrak{n}$-finite. We say a $\widetilde{\mathfrak{g}}$-module L is integrable if L is integrable as a $\mathfrak{g}$-module and also as an $\mathfrak{s}$-module. Then the simple module $\widetilde{L}(\lambda)$ is integrable if and only if $\lambda \in P_k = \{\lambda \in P_+ \mid \langle \lambda, \beta_0^\vee \rangle \leq k\}$.

Rational conformal field theory

Consider the Lie algebra $\mathfrak{g} \otimes \mathcal{O}[U]$ where $U = \mathbb{P}^1 \setminus \{x_1, x_2, x_3\}$ is the projective line minus 3 distinct points and $\mathcal{O}[U]$ is the ring of regular functions on U. For an $f \in \mathcal{O}[U]$ let $(f)_{x_i} \in \mathbb{C}((z))$ denote the Laurent series expansion of f at x_i. For an integer $k \geq 0$ and 3 weights $\lambda, \mu, \nu \in P_k$ there is a $\mathfrak{g} \otimes \mathcal{O}[U]$-action on $\widetilde{L}(\lambda) \otimes \widetilde{L}(\mu) \otimes \widetilde{L}(\nu^*)$ given by

$$\begin{aligned}(x \otimes f)(v_1 \otimes v_2 \otimes v_3) =& (x \otimes (f)_{x_1} v_1) \otimes v_2 \otimes v_3 + v_1 \otimes (x \otimes (f)_{x_2} v_2) \otimes v_3 \\ &+ v_1 \otimes v_2 \otimes (x \otimes (f)_{x_3} v_3).\end{aligned}$$

Then we define the vector space of conformal blocks on $\mathbb{P}^1$ with 3 marked points

$$V^\dagger_{\mathbb{P}^1}(\lambda, \mu, \nu^*) = \mathrm{Hom}_{\mathfrak{g}\otimes\mathcal{O}[U]}(\widetilde{L}(\lambda) \otimes \widetilde{L}(\mu) \otimes \widetilde{L}(\nu^*), \mathbb{C}).$$

It is a fact that up to isomorphism these spaces do not depend on the choice of local coordinates x_1, x_2, x_3.

Definition 1.2. *The fusion ring $F = F(\mathfrak{g}, k)$ for the Lie algebra $\mathfrak{g}$ at level k is the free $\mathbb{Z}$-module with basis $\{[\lambda] \mid \lambda \in P_k\}$ and product structure $[\lambda][\mu] = \bigoplus_{\nu \in P_k} N^\nu_{\lambda,\mu}[\nu]$, where $N^\nu_{\lambda,\mu} = \dim_{\mathbb{C}} V^\dagger_{\mathbb{P}^1}(\lambda, \mu, \nu^*)$.*

Remark. This defines a fusion rule on P_k. It is clear that the proposed product structure is commutative. That it is associative follows from a result in [TUY89].

The involution $\lambda^* = -w_0(\lambda)$ preserves the positive part of the root system, therefore fixes the longest root β_0 and consequently $\lambda \in P_k \Leftrightarrow \lambda^* \in P_k$, i.e., it restricts to an involution on P_k.

An analogue of [Bea96, Proposition 2.8] shows that $V^\dagger_{\mathbb{P}^1}(\lambda, \mu, \nu^*) \simeq V^\dagger_{\mathbb{P}^1}(\lambda^*, \mu^*, \nu)$ so $N^\nu_{\lambda,\mu} = N^{\nu^*}_{\lambda^*,\mu^*}$ and thus the conditions in Definition 1.1 are satisfied.

Note that the indexing set P_k is finite so F is a rational fusion ring. $\square$

Combinatorial description of WZW fusion rings

We will identify the representation ring R and the fusion ring F defined above with certain anti-invariants in the free $\mathbb{Z}$-module over the weight lattice and describe their mutual relationship.

Let $H_\alpha \subseteq E$ denote the hyperplane fixed by the reflection s_α. For an integer $n \geq 0$ we let $s_{\alpha,n}$ denote reflection in the affine hyperplane $H_{\alpha,n} = n\alpha/2 + H_\alpha$, i.e., $s_{\alpha,n}(\lambda) = \lambda + (n - \langle \lambda, \alpha^\vee \rangle)\alpha$. In this section we define the affine Weyl group $W_{k+h^\vee}$ to be the group generated by W and $s_{\beta_0,k+h^\vee}$, where $h^\vee = \langle \rho, \beta_0^\vee \rangle + 1$ is the dual Coxeter number.

Consider the group algebra $\mathbb{Z}[P]$ of P with $\mathbb{Z}$-basis $\{e^\lambda \mid \lambda \in P\}$. The action of W and $W_{k+h^\vee}$ on P extends by linearity to $\mathbb{Z}[P]$. Let $\mathbb{Z}[P]_W$ resp. $\mathbb{Z}[P]_{W_{k+h^\vee}}$ denote quotients of $\mathbb{Z}[P]$ by the ideals generated by all $e^\lambda - \det(w)e^{w\lambda}$ for $\lambda \in P$ and w in W resp. $W_{k+h^\vee}$ (these rings are not to be confused with the character ring $\mathbb{Z}[P]^W$). Then $\mathbb{Z}[P]_{W_{k+h^\vee}}$ is a quotient of $\mathbb{Z}[P]_W$ and we denote the quotient map p.

By [Bea96, Lemma 8.2] we have the following

Lemma 1.3. *The linear maps*

$$\varphi : R \to \mathbb{Z}[P]_W, \quad \varphi_k : F \to \mathbb{Z}[P]_{W_{k+h^\vee}}$$

which maps $[L(\lambda)]$ to the resp. classes of $e^{\lambda+\rho}$ are bijections.

By the lemma there is a map $\pi : R \to F$ making the diagram

$$\begin{tikzcd} R \arrow[r, "\pi"] \arrow[d, "\varphi"] & F \arrow[d, "\varphi_k"] \\ \mathbb{Z}[P]_W \arrow[r, "p"] & \mathbb{Z}[P]_{W_{k+h^\vee}} \end{tikzcd} \tag{1.1}$$

commutative. Explicitly the map π is as follows. If a weight $\lambda \in P^+$ lies on an affine wall, i.e., $\lambda + \rho$ is fixed by an element of $W_{k+h^\vee}$ then $\pi([L(\lambda)]) = 0$. Otherwise there is a unique $w \in W_{k+h^\vee}$ such that $w.\lambda = w(\lambda+\rho) - \rho \in P_k$ and $\pi([L(\lambda)]) = \det(w)[L(w.\lambda)]$. In particular $\pi([L(\lambda)]) = [L(\lambda)]$ for $\lambda \in P_k$.

It is a fact that π is a ring homomorphism for all semisimple Lie algebras, cf. [BK09, Theorem 3.7]. If $\lambda, \mu \in P_k$ this means

$$\sum_{\xi \in P_+} M^\xi_{\lambda,\mu} \pi([L(\xi)]) = \sum_{\nu \in P_k} N^\nu_{\lambda,\mu}[L(\nu)],$$

i.e.

$$N^\nu_{\lambda,\mu} = \sum_{w \in W_{k+h^\vee}} \det(w) M^{w.\nu}_{\lambda,\mu}. \tag{1.2}$$

As a commutative, associative $\mathbb{Z}$-algebra the representation ring R is freely generated by the fundamental representations $X_i = [L(\omega_i)], i = 1, \ldots, r$, i.e., it may be presented as the polynomial ring $R \simeq \mathbb{Z}[X_1, \ldots, X_r]$. Since the map π in (1.1) is surjective and a ring homomorphism we may identify the fusion ring F as a quotient $\mathbb{Z}[X_1, \ldots, X_r]/I_k$ where the ideal $I_k \subseteq \mathbb{Z}[X_1, \ldots, X_r]$ corresponds to the kernel $\mathrm{Ker}(\pi) \subseteq R$.

1.3 Fusion rings for quantum groups

Let $q \in \mathbb{C}$ be a root of unity and let l denote the order of q^2. If l is even this means q is a primitive $2l$th root of unity, and if l is odd then q is a primitive lth or $2l$th root of unity.

We work with U_q, the quantum group at q obtained from the Lusztig A-form U_A of the generic quantum group U_v by specializing v to q. This is a Hopf algebra and it has a triangular decomposition $U_q = U_q^- U_q^0 U_q^+$. We denote by $B_q = U_q^- U_q^0$ the quantized version of the Borel subalgebra.

Each weight $\lambda = \sum_{i=1}^r \lambda_i \omega_i \in P$ defines a homomorphism $\lambda : U_q^0 \to \mathbb{C}$ by

$$\lambda(K_i) = q_i^{\lambda_i}, \quad \lambda(\begin{bmatrix} K_i \\ t \end{bmatrix}_{q_i}) = \begin{bmatrix} \lambda_i \\ t \end{bmatrix}_{q_i},$$

$i = 1, \ldots, r, t \in \mathbb{N}$. In this way λ defines a 1-dimensional U_q^0-module $\mathbb{C}_\lambda$ which becomes a B_q-module with trivial U_q^--action.

For a U_q^0-module M and a weight $\lambda \in P$ we define the λ-weight space

$$M_\lambda = \{m \in M \mid um = \lambda(u)m \ \forall u \in U_q^0\}.$$

We say λ is a weight of M if $M_\lambda \neq 0$. The different weight spaces of M form a direct sum, and if M is a U_q-module the subspace $\bigoplus_{\lambda \in P} M_\lambda$ is a U_q-submodule of M.

We say a U_q-module M is integrable if M is the sum of its weight spaces and if, for each $m \in M$ and each $i = 1, \ldots, r$, $E_i^{(n)} m = 0 = F_i^{(n)} m$ for large enough n. Let $\mathcal{C}_q$ denote the category of integrable U_q-modules (of type 1). Similarly we denote by $\mathcal{C}_q^-$ the category of integrable B_q-modules.

Induction modules

We have a functor F from the category of U_q-modules to $\mathcal{C}_q$ which takes a module M to its maximal integrable submodule $F(M) = \{m \in M \mid \forall i : E_i^{(n)} m = F_i^{(n)} m = 0 \text{ for } n \gg 0\}$.

Define the induction functor $H_q^0 : \mathcal{C}_q^- \to \mathcal{C}_q$ by

$$H_q^0(N) = F(\mathrm{Hom}_{B_q}(U_q, N))$$

for $N \in \mathcal{C}_q^-$. Here U_q is a B_q-module by left multiplication and $\mathrm{Hom}_{B_q}(U_q, N)$ is a U_q-module by $uf(x) = f(xu), u, x \in U_q, f \in \mathrm{Hom}_{B_q}(U_q, N)$. The induction functor is left exact, and we denote its ith right derived functor by $H_q^i = \mathcal{R}^i H_q^0$.

Recall that each $\lambda \in P$ defines a one-dimensional B_q-module $\mathbb{C}_\lambda$. We define the induction modules

$$\nabla_q(\lambda) = H_q^0(\mathbb{C}_\lambda).$$

Then $\nabla_q(\lambda) \neq 0$ if and only if $\lambda \in P^+$, and for such λ it has a unique simple submodule $L_q(\lambda)$. In this case it is finite-dimensional and its character $\mathrm{ch}\, \nabla_q(\lambda) = \sum_{\mu \in P} \dim \nabla_q(\lambda)_\mu e^\mu \in \mathbb{Z}[P]$ is equal to the Weyl character χ_λ. For any $M \in \mathcal{C}_q$ we have Frobenius reciprocity

$$\mathrm{Hom}_{U_q}(M, \nabla_q(\lambda)) \simeq \mathrm{Hom}_{B_q}(M, \mathbb{C}_\lambda). \tag{1.3}$$

Given an $M \in \mathcal{C}_q$ and a weight λ of M the projection $M \to M_\lambda$ is a non-zero U_q^0-homomorphism. If λ is a highest weight of M then $um \notin M_\lambda$ for any $u \in U_q^-, m \in M$, meaning the projection is a B_q-homomorphism. Taking a 1-dimensional summand of M_λ and identifying it with $\mathbb{C}_\lambda$ gives us an element of the right hand side of (1.3), which induces to a non-zero U_q-homomorphism $M \to \nabla_q(\lambda)$.

It follows that the modules $L_q(\lambda), \lambda \in P^+$ up to isomorphism form a complete set of nonisomorphic simple finite-dimensional modules in $\mathcal{C}_q$. For given a simple finite-dimensional U_q-module M, let λ be a highest weight of M and take a non-zero homomorphism $M \to \nabla_q(\lambda)$ as above. This must be an injection identifying M with the unique simple submodule $L_q(\lambda)$.

The antipode S on U_q gives the dual $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ a U_q-module structure by $uf(m) = f(S(u)m), u \in U_q, m \in M, f \in \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$. We define

$$\Delta_q(\lambda) = \nabla_q(\lambda^*)^*$$

where still $\lambda^* = -w_0\lambda$. Then $\Delta_q(\lambda)$ and $\nabla_q(\lambda)$ have the same characters, and $L_q(\lambda)$ occurs as the unique simple quotient of $\Delta_q(\lambda)$.

Lemma 1.4. *For all $\lambda, \mu \in P^+$ we have* $\mathrm{Ext}_{U_q}(\Delta_q(\lambda), \nabla_q(\mu)) = 0$.

Proof. Suppose that

$$0 \to \nabla_q(\mu) \to E \to \Delta_q(\lambda) \to 0$$

is a short exact sequence in $\mathcal{C}_q$. If $\lambda \not\geq \mu$ then μ is a maximal weight for E, and we have seen above that Frobenius reciprocity (1.3) gives us a non-zero homomorphism $E \to \nabla_q(\mu)$ and the sequence is split. If $\lambda \geq \mu$ we dualize the sequence and get

$$0 \to \nabla_q(\lambda^*) \to E^* \to \Delta_q(\mu^*) \to 0$$

which by the same argument is split, and consequently the first sequence is too. □

The Grothendieck group of $\mathcal{C}_q$

Let $\mathrm{K}_0(\mathcal{C}_q)$ denote the Grothendieck group generated by isomorphism classes $[M]$ of integrable modules $M \in \mathcal{C}_q$ modulo relations $[M] = [L] + [N]$ when we have a short exact sequence $0 \to L \to M \to N \to 0$ of modules in $\mathcal{C}_q$. The sets $\{[\nabla_q(\lambda)] \mid \lambda \in P^+\}$ and $\{[L_q(\lambda)] \mid \lambda \in P^+\}$ are two $\mathbb{Z}$-bases of $\mathrm{K}_0(\mathcal{C}_q)$. For $M \in \mathcal{C}_q$ we write $[M : \nabla_q(\lambda)]$ for the coefficient of $[\nabla_q(\lambda)]$ when $[M]$ is written in the former basis.

For a finite-dimensional B_q-module N we define the Euler character χ_N in $\mathrm{K}_0(\mathcal{C}_q)$ by

$$\chi_N = \sum_{i \geq 0} (-1)^i [H^i_q(N)].$$

For $N = \mathbb{C}_\lambda, \lambda \in P$ simply write χ_λ. By the quantum version of Kempf's vanishing theorem (cf. [APK91, Corollary 5.7], proved in general in [RH03]) all higher induction modules for $\mathbb{C}_\lambda, \lambda \in P^+$ vanish such that $\chi_\lambda = [\nabla_q(\lambda)]$ justifying the abuse of notation as $\mathrm{ch}\,\nabla_q(\lambda)$ is equal to the Weyl character. Likewise we have $\chi_\lambda = (-1)^{l(w)}\chi_{w.\lambda}$ for general $\lambda \in P$ and $w \in W$, cf. [And03, Corollary 3.8].

An integrable module $N \in \mathcal{C}_q^-$ splits into a direct sum of its weight spaces $N = \oplus_{\lambda \in P} N_\lambda$, which leads to a B_q-filtration of N, and we have $\chi_N = \sum_{\lambda \in P} (\dim N_\lambda)\chi_\lambda$. The comultiplication of U_q makes $\mathcal{C}_q$ into a monoidal category and gives $\mathrm{K}_0(\mathcal{C}_q)$ a ring structure. The tensor identity [APK91, Proposition 2.16] $H^i_q(M \otimes N) \simeq M \otimes H^i_q(N)$ for $M \in \mathcal{C}_q$ finite-dimensional, together with the fact that χ is additive with respect to short exact sequences, gives us for $\lambda \in P^+$

$$\begin{aligned}
[M][\nabla_q(\lambda)] &= \chi(M \otimes \mathbb{C}_\lambda) = \sum_{\nu \in P} (\dim M_\nu)\chi_{\lambda+\nu} \\
&= \sum_{\nu \in P^+} \left(\sum_{w \in W} (-1)^{l(w)} \dim M_{w.\nu - \lambda} \right) [\nabla_q(\nu)].
\end{aligned}$$

This can be rewritten as

$$[M \otimes \nabla_q(\lambda) : \nabla_q(\nu)] = \sum_{w\in W} (-1)^{l(w)} \dim M_{w.\nu-\lambda} \tag{1.4}$$

for $M \in \mathcal{C}_q$ finite-dimensional and $\lambda, \nu \in P^+$.

The linkage and translation principles

Following [And03] we define the strong linkage relation for U_q. Recall that the Weyl group W is generated by the reflections $s_\alpha, \alpha \in \Phi$ of P given by $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$, and that P^+ is a fundamental domain for this action. We also have the shifted action of W on P fixing $-\rho$ instead of 0, the shifted reflections being $s_\alpha.\lambda = s_\alpha(\lambda+\rho) - \rho = \lambda - \langle \lambda+\rho, \alpha^\vee \rangle \alpha$.

For $i = 1, \dots, r$ let $l_i = l/\gcd(l, d_i)$ and for any root $\alpha \in \Phi$ with $\alpha = w(\alpha_i)$, let $l_\alpha = l_i$. Note that this is independent of the choice of $i = 1, \dots, r$ and $w \in W$.

For $\alpha \in \Phi$ and $m \in \mathbb{Z}$ define the affine reflection $s_{\alpha,m}(\lambda) = s_\alpha(\lambda) + ml_\alpha\alpha$ and in the same way the shifted affine reflection. The affine Weyl group W_l is the group generated by all shifted affine reflections $s_{\alpha,m}, \alpha \in \Phi, m \in \mathbb{Z}$. Define the fundamental alcove

$$A_l = \{\lambda \in P \mid 0 < \langle \lambda+\rho, \alpha^\vee \rangle < l_\alpha \ \forall \alpha \in \Phi^+\}$$

The closure of A_l is

$$\bar{A}_l = \{\lambda \in P \mid 0 \le \langle \lambda+\rho, \alpha^\vee \rangle \le l_\alpha \ \forall \alpha \in \Phi^+\}$$

and this set is a fundamental domain for the affine Weyl group.

Remark. This notion of affine Weyl group is a generalization of the one in Section 1.2. There will be more details on this at the end of the chapter. □

The shape of A_l depends on whether $l = l_\alpha$ for all $\alpha \in \Phi$ or not. In the first case it is easy to see that $A_l = \{\lambda \in P^+ \mid \langle \lambda+\rho, \alpha_0^\vee \rangle < l\}$ and likewise in the second case $A_l = \{\lambda \in P^+ \mid \langle \lambda+\rho, \beta_0^\vee \rangle < l/d_{\beta_0}\}$. For $\mathfrak{g}$ of type B_r, C_r or F_4 this is a question of whether 2 divides l and for type G_2 of whether 3 divides l. If l is prime to all d_i, then the affine Weyl group is actually the affine Weyl group for the dual root system in the Bourbakian convention.

Let $s_i = s_{\alpha_i}, i = 1, \dots, r$ denote the simple reflections in W. The affine Weyl group is generated by the reflections in the walls of $\bar{A}_l$. Let s_0 denote the reflection in the upper wall of $\bar{A}_l$, i.e., if $l = l_\alpha$ for all $\alpha \in \Phi$ then

$s_0 = s_{\alpha_0,1}$ and otherwise $s_0 = s_{\beta_0,1}$. Then $\{s_0, s_1, \dots, s_r\}$ with the shifted action on P is a simple generating set of W_l.

Write $\lambda \uparrow_\beta \mu$ if $\lambda, \mu \in P$ are related by $\mu = s_{\beta,m}.\lambda$ for $\beta \in \Phi^+$ and $m \in \mathbb{N}$ with $\langle \lambda + \rho, \beta^\vee \rangle \leq ml_\beta$, i.e., $\lambda \leq s_{\beta,m}.\lambda$. We say that λ is strongly linked to μ and write $\lambda \uparrow_\Phi \mu$, or just $\lambda \uparrow \mu$, if there are $\lambda = \lambda_1, \lambda_2, \dots, \lambda_r = \mu$ such that $\lambda_i \uparrow_{\beta_i} \lambda_{i+1}$ for some $\beta_i \in \Phi^+$.

We record the linkage principle for which a proof can be found in [And03, Theorem 3.13].

Theorem 1.5. *Let $\lambda, \mu + \rho \in P^+$. Assume that $L_q(\lambda)$ is a composition factor of $H^i_q(w.\mu)$ for some $w \in W$ and $i \geq 0$. Then $\lambda \uparrow \mu$.*

Corollary 1.6. *Let $V \in \mathcal{C}_q$ be an indecomposable module. If $\lambda, \mu \in P^+$ such that both $L_q(\lambda)$ and $L_q(\mu)$ are composition factors of V then $\mu \in W_l.\lambda$.*

Proof. Suppose that $\mu \notin W_l.\lambda$. We will show that any extension

$$0 \to L_q(\lambda) \to V \to L_q(\mu) \to 0$$

splits. Assume that $\lambda \not\leq \mu$ such that λ is a maximal weight of V. We have seen that Frobenius reciprocity (1.3) gives us a non-zero U_q-homomorphism $V \to \nabla_q(\lambda)$. By the theorem $L_q(\mu)$ is not a composition factor of $\nabla_q(\lambda)$ so the map $V \to \nabla_q(\lambda)$ has image $L_q(\lambda)$ and the sequence is split. In case $\lambda \leq \mu$ we dualize the sequence, and the above shows that

$$0 \to L_q(\mu^*) \to V^* \to L_q(\lambda^*) \to 0$$

is split, which in turn tells us that the original sequence is split. □

For $\mu \in \bar{A}_l \cap P^+$ there are no dominant $\lambda \in P^+$ strongly linked to μ other than $\lambda = \mu$, so the module $\nabla_q(\mu) = L_q(\mu)$ is simple.

For $\lambda \in \bar{A}_l$ let $\mathcal{C}_q(\lambda)$ be the full subcategory of $\mathcal{C}_q$ consisting of modules whose composition factors have highest weights in $W_l.\lambda$. By the corollary the category $\mathcal{C}_q$ is a direct sum of these subcategories

$$\mathcal{C}_q = \bigoplus_{\lambda \in \bar{A}_l} \mathcal{C}_q(\lambda)$$

and we let $p_\lambda : \mathcal{C}_q \to \mathcal{C}_q(\lambda)$ denote projection functors.

For $\lambda, \mu \in \bar{A}_l$ let $w \in W$ be such that $w(\mu - \lambda) \in P^+$. Define the translation functor $T^\mu_\lambda : \mathcal{C}_q(\lambda) \to \mathcal{C}_q(\mu)$ by $T^\mu_\lambda(M) = p_\mu(M \otimes L_q(w(\mu - \lambda)))$.

We record the translation principle for which a proof can be found in [APK91, Theorem 8.3].

Theorem 1.7. *Let $\lambda \in A_l, \mu \in \bar{A}_l$. Then*

(i) $T_\lambda^\mu H_q^i(w.\lambda) \simeq H_q^i(w.\mu)$ *for all* $i \geq 0, w \in W_l$,

(ii) if $w \in W_l$ *with* $w.\lambda \in P^+$ *then*

$$T_\lambda^\mu L_q(w.\lambda) = \begin{cases} L_q(w.\mu) & \textit{if } w.\mu \textit{ is in the upper closure of } w.A_l \\ 0 & \textit{otherwise} \end{cases},$$

(iii) if $\mathrm{Stab}_{W_l}(\mu) = \{1, s\}$ *and* $w \in W_l$ *satisfies* $w.\lambda < ws.\lambda$*, then there is an exact sequence*

$$\begin{aligned} &0 \to H_q^0(w.\lambda) \to T_\mu^\lambda H_q^0(w.\mu) \to H_q^0(ws.\lambda) \to \\ &\dots \\ &\to H_q^i(w.\lambda) \to T_\mu^\lambda H_q^i(w.\mu) \to H_q^i(ws.\lambda) \to \\ &\dots \end{aligned}$$

Tilting modules

A finite-dimensional module $T \in \mathcal{C}_q$ is a tilting module if it has both a ∇_q- and a Δ_q-filtration. Let $\mathcal{T}_q$ denote the subcategory of $\mathcal{C}_q$ consisting of all tilting modules. It is obvious that direct sums and direct summands of tilting modules are again tilting modules. We also note the non-trivial fact that the tensor product of two tilting modules is also a tilting module, cf. [Par92, Theorem 3.3].

We immediately see that for all $\lambda \in \bar{A}_l \cap P^+$ the module $\nabla_q(\lambda) \simeq L_q(\lambda) \simeq \Delta_q(\lambda)$ is tilting, and in fact for general λ the induction module $\nabla_q(\lambda)$ is tilting if and only if it is simple.

The tensor product of a tilting module with a finite-dimensional module is again a tilting module, and therefore the category of tilting modules is closed under translation functors.

Proposition 1.8. *Let $\lambda \in P^+$. Then there exists an indecomposable tilting module $T_q(\lambda) \in \mathcal{T}_q$ such that*

(i) all weights μ of $T_q(\lambda)$ have $\mu \leq \lambda$,

(ii) $\dim T_q(\lambda)_\lambda = 1$ *and*

(iii) $T_q(\lambda)$ *is unique up to isomorphism.*

The proof is an analogue of [AP95, Proposition 1.7], which is originally due to Ringel [Rin91] and Donkin [Don93].

Proof. We have already noted that if $\lambda \in \bar{A}_l \cap P^+$ then $T_q(\lambda) = \nabla_q(\lambda)$ is a tilting module that meets the requirements. We will use induction on the length $l(w)$ of elements $w \in W_l$ for which $w.A_l \subseteq P^+$ to show existence of $T_q(w.\lambda)$ where $\lambda \in \bar{A}_l$.

Let $w \in W_l$ be such that $w.A_l \subseteq P^+$ and pick a simple reflection s for which $l(w) < l(ws)$. If $ws.\lambda = w.\lambda$ there is nothing to prove. Otherwise pick $\mu \in \bar{A}_l$ with $\mathrm{Stab}_{W_l}(\mu) = \{1, s\}$. Then $T_q(w.\mu)$ exists by the induction hypothesis. Consider $T_\mu^\lambda T_q(w.\mu)$ which is a tilting module. By Theorem 1.7 we have a short exact sequence

$$0 \to \nabla_q(w.\lambda) \to T_\mu^\lambda \nabla_q(w.\mu) \to \nabla_q(ws.\lambda) \to 0$$

showing that $\nabla_q(ws.\lambda)$ will occur with multiplicity 1 in a ∇_q-filtration of $T_\mu^\lambda T_q(w.\mu)$. Therefore we take $T_q(ws.\lambda)$ to be the unique indecomposable summand of this module such that $T_q(ws.\lambda)_{ws.\lambda} \neq 0$.

To prove uniqueness, suppose that $T_q(\lambda)$ and $T'_q(\lambda)$ are two indecomposable tilting modules with the properties (i) and (ii). Then $\Delta_q(\lambda)$ occurs as a submodule in the Δ_q-filtrations of both $T_q(\lambda)$ and $T'_q(\lambda)$. This fits into a diagram

$$\begin{tikzcd}
0 \arrow[r] & \Delta_q(\lambda) \arrow[r] \arrow[d, equal] & T_q(\lambda) \arrow[r] & T_q(\lambda)/\Delta_q(\lambda) \arrow[r] & 0 \\
0 \arrow[r] & \Delta_q(\lambda) \arrow[r] & T'_q(\lambda) \arrow[r] & T'_q(\lambda)/\Delta_q(\lambda) \arrow[r] & 0
\end{tikzcd}$$

As $T_q(\lambda)/\Delta_q(\lambda)$ has a Δ_q-filtration and $T'_q(\lambda)$ has a ∇_q-filtration induction on the lengths of the filtrations together with Lemma 1.4 gives us $\mathrm{Ext}_{U_q}(T_q(\lambda)/\Delta_q(\lambda), T'_q(\lambda)) = 0$. Under the surjection $\mathrm{Hom}_{U_q}(T_q(\lambda), T'_q(\lambda)) \twoheadrightarrow \mathrm{Hom}_{U_q}(\Delta_q(\lambda), T'_q(\lambda))$ let $\varphi : T_q(\lambda) \to T'_q(\lambda)$ be a homomorphism which restricts to the identity on $\Delta_q(\lambda)$. Similarly we get a homomorphism $\varphi' : T'_q(\lambda) \to T_q(\lambda)$ which restricts to the identity on $\Delta_q(\lambda)$. The composition $\varphi' \circ \varphi$ is an endomorphism of $T_q(\lambda)$ which is the identity on $T_q(\lambda)_\lambda$. Since $T_q(\lambda)$ is indecomposable and of finite length, it is a standard result that $\varphi' \circ \varphi$ is either bijective or nilpotent, and we see the latter is impossible. As the same applies to $\varphi \circ \varphi'$ we conclude that $T_q(\lambda) \simeq T'_q(\lambda)$. □

In the split Grothendieck group $\mathrm{K}_0(\mathcal{T}_q)$ we have a $\mathbb{Z}$-basis $\{[T_q(\lambda)] \mid \lambda \in P^+\}$ and let $(T : T_q(\lambda))$ denote the multiplicity of $T_q(\lambda)$ as a direct summand of $T \in \mathcal{T}_q$. As a ring $\mathrm{K}_0(\mathcal{T}_q)$ is freely generated by the isomorphism classes of the indecomposable tilting modules belonging to the fundamental weights, i.e., we have a presentation $\mathrm{K}_0(\mathcal{T}_q) \simeq \mathbb{Z}[X_1, \ldots, X_r]$ with $X_i = [T(\omega_i)], i = 1, \ldots, r$.

Proposition 1.9. *Let $T \in \mathcal{T}_q$ be a tilting module. Then for all $\lambda \in A_l$ we have*

$$(T : T_q(\lambda)) = \sum_{\substack{w \in W_l \\ w.\lambda \in P^+}} (-1)^{l(w)} \left[T : \nabla_q(w.\lambda)\right].$$

We repeat the proof from [AP95, Proposition 3.20].

Proof. Since both sides are additive in T, it is enough to show the formula for $T = T_q(\nu), \nu \in P^+$, for which the left hand side is $(T_q(\nu) : T_q(\lambda)) = \delta_{\nu,\lambda}$. If $\nu = \lambda \in A_l$ then $T_q(\nu) = \nabla_q(\nu)$ by the linkage principle, and the right hand side is also 1. We are left to show that the right hand side is 0 for all other values of ν.

By the linkage principle Corollary 1.6 the right hand side is 0 unless $\nu \in W_l.\lambda$. So suppose that $\nu = w.\lambda$ for some non-trivial element $w \in W_l$. Consider the construction of $T_q(w.\lambda)$ as a summand of $T_\mu^\lambda T_q(w.\mu)$ where $\mu \in \bar{A}_l$ is a weight fixed by a single non-trivial reflection s such that $w.\mu$ is in the lower closure of the alcove containing ν. Since translation is exact, for each $\nabla_q(y.\mu), y \in W_l$ occuring as a sub-quotient in a ∇_q-filtration of $T_q(w.\mu)$, we will see $T_\mu^\lambda \nabla_q(y.\mu)$ as a subquotient in a filtration of $T_\mu^\lambda T_q(w.\mu)$. If $y.\lambda < ys.\lambda$ the modules fit into an exact sequence

$$0 \to \nabla_q(y.\lambda) \to T_\mu^\lambda \nabla_q(y.\mu) \to \nabla_q(ys.\lambda) \to 0.$$

If $y.\lambda > ys.\lambda$ we have a similar result. Since the sequence is non-split we see, that if $\nabla_q(y.\lambda)$ occurs in a ∇_q-filtration of $T_q(w.\lambda)$, then $\nabla_q(ys.\lambda)$ occurs too and contributes with opposite sign to the formula. $\square$

The above proposition together with (1.4) applied to $T = T_q(\lambda) \otimes T_q(\mu), \lambda, \mu \in \bar{A}_l \cap P^+$, and $\nu \in A_l$, gives us the following multiplicity formula

$$(T_q(\lambda) \otimes T_q(\mu) : T_q(\nu)) = \sum_{w \in W_l} (-1)^{l(w)} \dim \nabla_q(\lambda)_{w.\nu - \mu}. \tag{1.5}$$

The negligible tilting modules

Let $\mathcal{N}_q$ denote the subcategory of $\mathcal{T}_q$ consisting of those tilting modules which have no summands $T_q(\lambda)$ with $\lambda \in A_l$, these are called the negligible tilting modules. It is shown in [AP95, Thm 3.21] that $\mathcal{N}_q$ is a tensor ideal in $\mathcal{T}_q$ and that the quotient $\mathcal{T}_q/\mathcal{N}_q$ is semisimple.

Definition 1.10. *Define the fusion ring $F_q = F_q(\mathfrak{g}, l)$ for the quantum group U_q to be the quotient of the Grothendieck group $\mathrm{K}_0(\mathcal{T}_q)$ by $\mathrm{K}_0(\mathcal{N}_q)$.*

Remark. The product structure on $\mathrm{K}_0(\mathcal{T}_q)$ induces a product structure on the fusion ring F_q, which satisfies Definition 1.1: It is commutative because the multiplication on $\mathrm{K}_0(\mathcal{T}_q)$ is.

Let $[\lambda]$ denote the image of $[T_q(\lambda)]$ in F_q for $\lambda \in A_l$. As a $\mathbb{Z}$-module F_q is free with finite basis $[\lambda], \lambda \in A_l$. We extend the notation $[\lambda] \in F_q$ to all $\lambda \in P$ by saying $[\lambda] = 0$ if λ lies on a wall for W_l, and otherwise $[\lambda] = (-1)^{l(w_\lambda)}[w_\lambda.\lambda]$ for the unique $w_\lambda \in W_l$ with $w_\lambda.\lambda \in A_l$.

From (1.5) the multiplication in F_q is then given by

$$\begin{aligned}
[\lambda][\mu] &= \sum_{\nu \in A_l} (T_q(\lambda) \otimes T_q(\mu) : T_q(\nu))\,[\nu] \\
&= \sum_{\nu \in A_l} (\sum_{w \in W_l} (-1)^{l(w)} \dim \nabla_q(\lambda)_{w.\nu-\mu})[\nu] \\
&= \sum_{\nu \in P} \dim \nabla_q(\lambda)_\nu [\mu + \nu]
\end{aligned}$$

for $\lambda, \mu \in A_l$. In this notation we show associativity of the product structure:

$$\begin{aligned}
([\lambda][\mu])[\eta] = [\eta]([\lambda][\mu]) &= \sum_{\nu \in P} \dim \nabla_q(\lambda)_\nu [\eta][\mu + \nu] \\
&= \sum_{\nu, \zeta \in P} \dim \nabla_q(\lambda)_\nu \dim \nabla_q(\eta)_\zeta [\mu + \nu + \zeta] \\
&= \sum_{\zeta \in P} \dim \nabla_q(\eta)_\zeta [\lambda][\mu + \zeta] \\
&= [\lambda]([\eta][\mu]) = [\lambda]([\mu][\eta]).
\end{aligned}$$

The notation above satisfies $[\lambda]^* = [\lambda^*]$ for all $\lambda \in P$: Since the involution $-w_0$ preserves both the highest short root and the highest long root it preserves A_l. Then it also permutes the walls $\bar{A}_l \setminus A_l$ and therefore the walls of W_l, i.e., $[\lambda] = 0$ if and only if $[\lambda^*] = 0$.

On the other hand, if λ lies off the walls, then $w_0 w_{\lambda^*} w_0$ is an element of W_l with $w_0 w_{\lambda^*} w_0.\lambda = -w_0(w_{\lambda^*}.(-w_0\lambda)) = (w_{\lambda^*}.\lambda^*)^* \in A_l$, so by uniqueness it equals w_λ, i.e., $[\lambda] = [\lambda^*]^*$.

Using this we see

$$\begin{aligned}
([\lambda][\mu])^* &= \sum_{\nu \in P} \dim \nabla_q(\lambda)_\nu [\mu + \nu]^* \\
&= \sum_{\nu \in P} \dim \nabla_q(\lambda)_{\nu^*} [\mu^* + \nu] \\
&= \sum_{\nu \in P} \dim \nabla_q(\lambda^*)_\nu [\mu^* + \nu] = [\lambda^*][\mu^*]
\end{aligned}$$

showing that the involution on F_q is a homomorphism. □

As a ring F_q is generated by the images of $X_i = [L_q(\omega_i)], i = 1, \dots, r$, when l is big enough that $A_l \neq \emptyset$. In the presentation of the Grothendieck group $\mathrm{K}_0(\mathcal{T}_q)$ as a polynomial ring $\mathbb{Z}[X_1, \dots, X_r]$ let I_l be the ideal corresponding to the Grothendieck group $\mathrm{K}_0(\mathcal{N}_q)$. Then $F_q \simeq \mathbb{Z}[X_1, \dots, X_r]/I_l$.

Identification with the WZW fusion rings

Given a Lie algebra $\mathfrak{g}$ the fundamental alcove A_l matches the region P_k for certain integers $k, l \in \mathbb{N}$, and it is reasonable to expect the fusion rings $F(\mathfrak{g}, k)$ and $F_q(\mathfrak{g}, l)$ to be isomorphic. To be precise, we require that l is divisible by all $d_i, i = 1, \dots, r$, (if $\mathfrak{g}$ is simply laced this is always the case) and then that $k = l_{\beta_0} - h^\vee = l_{\beta_0} - (\langle \rho, \beta_0^\vee \rangle + 1)$. In this case the definition of $W_{k+h^\vee}$ from Section 1.2 and the definition of W_l of this section match up, since they are both generated by reflections of P in the walls of $P_k = A_l$.

Under the above assumptions we show equality of the fusion rule $N = \{N_{\lambda,\mu}^\nu \mid \lambda, \mu, \nu \in P_k\}$ from (1.2) and the fusion rule $N' = \{N'^\nu_{\lambda,\mu} \mid \lambda, \mu, \nu \in A_l\}$ from (1.5). To do this we recall Klimyk's formula [Hum72, Exercise 24.9] for $\lambda, \mu \in P^+$

$$[L(\lambda) \otimes L(\mu)] = \sum_{\eta \in P} t_{\mu+\eta+\rho} \dim L(\lambda)_\eta [L((\mu + \eta + \rho)^+ - \rho)]$$

where $t_\zeta = 0$ if $\zeta \in P$ is fixed by a non-trivial element of W, and otherwise $t_\zeta = (-1)^{l(w)}$ for $w \in W$ the unique element of the Weyl group such that $w\zeta \in P^+$ is dominant. In both cases ζ^+ is the unique element in P^+ conjugate to ζ by W. Then the coefficient of $[L(\xi)], \xi \in P^+$ appearing in this sum is

$$M_{\lambda,\mu}^\xi = \sum_{\substack{w \in W \\ w.(\mu+\eta)=\xi}} t_{\mu+\eta+\rho} \dim L(\lambda)_\xi = \sum_{\substack{w \in W \\ w.\xi=\mu+\eta}} (-1)^{l(w)} \dim L(\lambda)_{w.\xi-\mu}.$$

Finally we sum over all $w' \in W_{k+h^\vee}, w'.\nu = \xi$ and get

$$N_{\lambda,\mu}^\nu = \sum_{\substack{w' \in W_{k+h^\vee} \\ w'.\nu=\xi}} (-1)^{l(w')} M_{\lambda,\mu}^\xi = \sum_{w \in W_{k+h^\vee}} (-1)^{l(w)} \dim L(\lambda)_{w.\nu-\mu}.$$

Now this number matches exactly $N'^\nu_{\lambda,\mu} = \sum_{w \in W_l} (-1)^{l(w)} \dim \nabla_q(\lambda)_{w.\nu-\mu}$, since they are both given by the Weyl character χ_λ. Thus the fusion rules N and N' on $P_k = A_l$ are the same, and the associated fusion rings $F(\mathfrak{g}, k)$ and

$F_q(\mathfrak{g}, l)$ are isomorphic. We will therefore mostly work with the quantum version of the fusion ring in the rest of the dissertation.

We note that all results and properties regarding the combinatorics of the fusion ring $F(\mathfrak{g}, k)$ in the Lie algebra setup are valid for the corresponding fusion ring $F_q(\mathfrak{g}, l)$ for the quantum group where the order l of q^2 is divisible by all $d_i, i = 1, \ldots, r$. Much of the work in this dissertation is focused on giving self-contained proofs in the quantum group setup and generalizing them to arbitrary orders l.

The fusion ring for $U_q(\mathfrak{sl}_2)$

As an example we explore the structure of the fusion rings associated to the quantum group $U_q(\mathfrak{sl}_2)$. We let $q \in \mathbb{C}$ be a root of unity such that, for l the order of q^2, we have $k = l - 2 \geq 0$.

Identify the dominant weights P^+ with the natural numbers $\mathbb{N}$. For each dominant weight $n \in P^+$ the induction module $\nabla_q(n)$ is $(n+1)$-dimensional with a basis $e_0, \ldots, e_n$ such that e_i has weight $n-2i$. The simple submodule $L_q(n) \subseteq \nabla_q(n)$ is the span of those e_i for which the quantum binomial coefficient $\begin{bmatrix} n \\ i \end{bmatrix}_q \neq 0$. When $n < l = \mathrm{ord}(q^2)$ all $[\mu]_q = \frac{q^\mu - q^{-\mu}}{q - q^{-1}} \neq 0$ for $0 \leq \mu \leq n$ and the induction module $\nabla_q(n) = L_q(n)$ is simple.

For $n, i \in \mathbb{N}$ write $n = n^{(0)} + ln^{(1)}, i = i^{(0)} + li^{(1)}$ with $0 \leq n^{(0)}, i^{(0)} < l$. By [Lus10, Lemma 34.1.2 (c)] we have

$$\begin{bmatrix} n \\ i \end{bmatrix}_q = q^{(n^{(0)}i^{(1)} - n^{(1)}i^{(0)})l + (n^{(1)}+1)i^{(1)}l^2} \binom{n^{(1)}}{i^{(1)}} \begin{bmatrix} n^{(0)} \\ i^{(0)} \end{bmatrix}_q .$$

This shows that

(i) If $0 \leq n < l$ then $\nabla_q(n) = L_q(n)$ is simple.

(ii) If $n \equiv -1 \mod l$ then $\nabla_q(n) = L_q(n)$ is simple.

(iii) If $n = n^{(0)} + ln^{(1)}$ with $0 \leq n^{(0)} < l - 1$ and $n^{(1)} \geq 1$ then we have a short exact sequence

$$0 \to L_q(n) \to \nabla_q(n) \to L_q(n - 2n^{(0)} - 2) \to 0.$$

We conclude that $\nabla_q(n) = T_q(n)$ is a tilting module if and only if $0 \leq n < l$ or $n \equiv -1 \mod l$.

Consider as in case (iii) an $n = n^{(0)} + ln^{(1)}$ with $0 \leq n^{(0)} < l-1$ and $n^{(1)} \geq 1$, i.e., n is a regular weight. For $m = ln^{(1)} - 1$ we know that $T_q(m) = \nabla_q(m)$, and that $T_q(n)$ is the direct summand of $\nabla_q(m) \otimes L_q(n^{(0)}+1)$ involving the weight n. In conclusion we have a short exact sequence

$$0 \to \nabla_q(n - 2n^{(0)} - 2) \to T_q(n) \to \nabla_q(n) \to 0. \tag{1.6}$$

The ideal $\mathcal{N}_q$ has an $\mathbb{N}$-basis in the indecomposable tilting modules $T_q(n)$ with highest weight outside the fundamental alcove, i.e., in $P^+ \setminus A_l = \{n \in \mathbb{N} \mid n \geq l-1\}$. We claim that the tilting module $T_q(l-1) = \nabla_q(l-1)$ generates $\mathcal{N}_q$ over $\mathcal{T}_q$. To see this either use the general Proposition 3.1 or check directly for low values of $n \geq l-1$

$$\begin{aligned}
&T_q(l-1) \otimes \nabla_q(1) \simeq T_q(l) \\
&T_q(l-1) \otimes \nabla_q(2) \simeq T_q(l+1) \oplus T_q(l-1) \\
&T_q(l-1) \otimes \nabla_q(3) \simeq T_q(l+2) \oplus T_q(l) \\
&T_q(l-1) \otimes \nabla_q(4) \simeq T_q(l+3) \oplus T_q(l+1) \oplus T_q(l-1) \\
&\qquad\qquad\qquad \vdots
\end{aligned}$$

This is seen by looking at which induction modules occur in a ∇_q-filtration of $\nabla_q(l-1) \otimes L$ for L finite-dimensional, and compare it with (1.6).

In the Grothendieck group of $\mathcal{T}_q$ we set $X = [\nabla_q(1)]$, and identify $\mathrm{K}_0(\mathcal{T}_q) \simeq \mathbb{Z}[X]$. The character of $\nabla_q(1)$ is $e + e^{-1}$ meaning that for any $n \geq 1$

$$X[\nabla_q(n)] = [\nabla_q(n+1)] + [\nabla_q(n-1)]. \tag{1.7}$$

For low values of n we calculate

$$\begin{aligned}
[\nabla_q(2)] &= X[\nabla_q(1)] - [\nabla_q(0)] = X^2 - 1 \\
[\nabla_q(3)] &= X[\nabla_q(2)] - [\nabla_q(1)] = X^3 - 2X \\
[\nabla_q(4)] &= X[\nabla_q(3)] - [\nabla_q(2)] = X^4 - 3X^2 + 1 \\
[\nabla_q(5)] &= X[\nabla_q(4)] - [\nabla_q(3)] = X^5 - 4X^3 + 3X \\
&\vdots
\end{aligned}$$

Inspired by these calculations we set up the hypothesis

$$[\nabla_q(n)] = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-1}{i} X^{n-2i}$$

and check that it respects (1.7). Then the presentation of the fusion ring $F_q = F_q(\mathfrak{sl}_2, l)$, as a quotient of a polynomial ring, is given by

$$\mathbb{Z}[X] / \left\langle \sum_{i=0}^{\lfloor \frac{l-1}{2} \rfloor} (-1)^i \binom{l-2}{i} X^{l-2i-1} \right\rangle.$$

Chapter 2

Known results and conjectures

In this chapter we go through the development of the theory of fusion rings associated to semisimple Lie algebras. Our focus is on getting precise expressions of the fusion rules and on presenting the fusion ring as a quotient of a polynomial ring. We present key points in a seletion of papers contributing to these areas in the notation introduced in Chapter 1.

2.1 Gepner 1991

In [Gep91] the fusion ring $F = F(\mathfrak{g}, k)$ at level $k \in \mathbb{N}$ for a simple complex Lie algebra $\mathfrak{g}$ of type $A_r, r \geq 1$, is studied. The study exploits heavily the symmetry of the Weyl group $W = S_r$ and the character ring $\mathbb{Z}[P]^W$.

A simple finite-dimensional representation $L(\lambda)$ is identified with the Young diagram corresponding to $\lambda \in P^+$. For $\lambda = \sum_{i=1}^r m_i\omega_i \in P^+$ set $\lambda_i = m_i + \cdots + m_r, i = 1, \ldots, r$. Then $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$ and the level of λ is simply $\langle \lambda, \beta_0^\vee \rangle = \lambda_1$, i.e., $\lambda \in P_k$ if and only if $\lambda_1 \leq k$. The Young diagram of λ is then the non-increasing rows of boxes with λ_i boxes in the ith row. Use the notation $[a_1, \ldots, a_n]$ where $n = \lambda_1$ is the number of columns in the Young diagram, and a_i is the number of boxes in the ith column. Then $r \geq a_1 \geq a_2 \geq \cdots \geq a_n > 0$.

There are two types of special representations of $\mathfrak{g}$: the fundamental representations $L(\omega_i), i = 1, \ldots, r$, whose Young diagram $[i]$ consist of one column with i boxes, and $L(n\omega_1), n \in \mathbb{N}$, whose Young diagram $[1, \ldots, 1]$ consist of one row with n boxes.

Gepner proves a (Pieri) formula for the product of the isomorphism class $X_i = [L(\omega_i)]$ of a fundamental representation with any other element of R represented in Young diagram by

$$[i][a_1, \ldots, a_n] = \sum [b_1, \ldots, b_m] \tag{2.1}$$

where the sum is over all those $r+1 \geq b_1 \geq \cdots \geq b_m > 0$ with $\sum b_j = \sum a_j + i$, $a_j + 1 \geq b_j \geq a_j$, and with the convention that $b_j = r+1$ is the same as $b_j = 0$. This corresponds to adding i boxes to the Young diagram without adding two boxes in the same row, and if a column has $r+1$ boxes in it we remove it. As a consequence of this, he proves a (Giambelli) formula which expresses any element in R as a polynomial in the X_i:

$$[a_1, \dots, a_n] = \det([a_i + i - j])_{1 \leq i,j \leq n} \tag{2.2}$$

where det is the determinant of this matrix with the convention that $[0] = [r+1] = 1$ and $[i] = 0$ for $i > r+1$ or $i < 0$.

For a weight $\lambda \in P_k$ we denote by $[\lambda]$ the corresponding basis element in F. The associated Young diagram has at most k columns, and we write it $[a_1, \dots, a_k]$ allowing some of the a_i to be 0. The corresponding (Pieri) formula for multiplication of $[\omega_i]$ with another element of F is a truncated version of (2.1):

$$[i][a_1, \dots, a_k] = \sum [b_1, \dots, b_k] \tag{2.3}$$

where the sum is over all those $r+1 \geq b_1 \geq \cdots \geq b_k \geq 0$ with $\sum b_j = \sum a_j + i$, $a_j + 1 \geq b_j \geq a_j$, and with the convention that $b_j = r+1$ is the same as $b_j = 0$. This corresponds to adding i boxes to the Young diagram without adding two boxes in the same row, and we do not allow more than a total of k columns. If a column ends up with $r+1$ boxes in it, we remove it.

As the representation ring R is generated freely by the classes of the fundamental representations $X_i, i = 1, \dots, r$, there is an identification $R \simeq \mathbb{Z}[X_1, \dots, X_r]$ with the polynomial ring in r variables. Gepner shows that there is a surjective algebra homomorphism $\varphi : R \to F$ whose kernel $I = \operatorname{Ker}(\varphi)$ is generated by the classes $[L((k+1)\omega_1)], \dots, [L((k+r)\omega_1)]$, i.e., we have an isomorphism

$$F \simeq R / \langle [L((k+1)\omega_1)], \dots, [L((k+r)\omega_1)] \rangle .$$

This is done by showing that all elements in R of level $k+1$ is contained in the ideal using the determinantal formula (2.2). Then the two formulas for multiplication (2.1) and (2.3) agree.

The next step is to show that the relations $[L((k+i)\omega_1)] \in R$ can be integrated to a single polynomial $V(X_1, \dots, X_r)$, i.e. the partial derivatives of V w.r.t. the variables X_i gives the relations $[L((k+i)\omega_1)]$. Such a polynomial is called a potential function for the ideal I.

Denote by $\varepsilon_1 = \omega_1, \varepsilon_i = \omega_i - \omega_{i-1}, i = 2, \dots, r$ and $\varepsilon_{r+1} = -\omega_r = -(\varepsilon_1 + \cdots + \varepsilon_r)$ the weights of the standard representation $L(\omega_1)$ and set $q_i = e^{\varepsilon_i}$, i.e., $\operatorname{ch} L(\omega_1) = \sum_{i=1,\dots,r+1} q_i \in \mathbb{Z}[P]^W$.

Written in these variables the character of the fundamental representation $L(\omega_i)$ is

$$\chi_{\omega_i} = \sum_{1\le j_1<\cdots<j_i\le r+1} q_{j_1}\cdots q_{j_i},$$

and the character of the representation $L(n\omega_1)$ is

$$\chi_{n\omega_1} = \sum_{1\le j_1\le\cdots\le j_n\le r+1} q_{j_1}\cdots q_{j_n},$$

which can be seen, for instance, by the Weyl character formula (1).

For each positive number $m \geq 1$ consider the element

$$\frac{1}{m}\sum_{i=1}^{r+1} q_i^m \tag{2.4}$$

living in $\mathbb{Z}[P]\otimes_{\mathbb{Z}}\mathbb{Q}$. The set of weights $\{m\varepsilon_i \mid i = 1,\ldots,r+1\}$ is W-invariant, so the element $\sum_{i=1}^{r+1} e^{m\varepsilon_i}$ defines an element of the character ring $\mathbb{Z}[P]^W$. Consequently the element in (2.4) can be written as a polynomial with rational coefficients in the fundamental characters $\chi_{\omega_i}, i = 1,\ldots,r$. Let $V_m(X_1,\ldots,X_r)\in R\otimes_{\mathbb{Z}}\mathbb{Q}$ be the same polynomial in the isomorphism classes of the fundamental representations $X_i, i = 1,\ldots,r$. Gepner proves by calculations on the generating function $V(t) = \sum_{m=1}^{\infty}(-1)^{m-1}V_m t^m$ that the derivatives of V_m w.r.t. the X_i are elements of R satisfying

$$\frac{\partial V_m}{\partial X_i} = (-1)^{i-1}[L((m-i)\omega_1)]$$

for $m > i$. We therefore set $V = V_{k+r+1}$, and we have the desired formula

$$F \simeq R/\left\langle \frac{\partial V}{\partial X_1},\ldots,\frac{\partial V}{\partial X_r}\right\rangle.$$

Gepner ends the work with the type A_r case stating a conjecture on fusion rings for Lie algebras of other types.

Conjecture 2.1. *All fusion rings defined in the setting of rational conformal field theory are presentable as the quotient of a polynomial ring by an ideal generated by the partial derivatives of a potential function V, which is a polynomial with integral coefficients.*

2.2 Gepner and Schwimmer 1992

In [GS92] much of the work from [Gep91] is generalized to a Lie algebra $\mathfrak{g}$ of type $C_r, r \geq 2$. Again the focus is on describing the fusion ring $F = F(\mathfrak{g}, k), k \in \mathbb{N}$, as a quotient of a polynomial ring modulo an ideal generated by partial derivatives of a potential function.

The methods in this article are less combinatorial than in the previous one, and characterizes the fusion ideal as an ideal of polynomials vanishing on a specific set of points. The result, however, is phrased in the same language as before.

Denote by $\varepsilon_1 = \omega_1$, $\varepsilon_i = \omega_i - \omega_{i-1}, i = 2, \dots, r$, some of the weights of the standard representation $L(\omega_1)$ and set $q_i = e^{\varepsilon_i}$, i.e., $\operatorname{ch} L(\lambda) = \sum_{i=1}^r (q_i + q_i^{-1}) \in \mathbb{Z}[P]^W$. For $m \geq 1$ the set of weights $\{\pm m\varepsilon_i \mid i = 1, \dots, r\}$ is invariant under the Weyl group action, so the element

$$\frac{1}{m} \sum_{i=1}^{r} (q_i^m + q_i^{-m}) \tag{2.5}$$

defines an element of $\mathbb{Z}[P]^W \otimes_{\mathbb{Z}} \mathbb{Q}$ and can therefore be written as a polynomial in in the fundamental characters $\chi_{\omega_i}, i = 1, \dots, r$ with rational coefficients. Let $X_i = [L(\omega_i)], i = 1, \dots, r$, denote the isomorphism classes of the fundamental representations. Let $V_m(X_1, \dots, X_r) \in R \otimes_{\mathbb{Z}} \mathbb{Q}$ be the polynomials obtained from (2.5). Then the partial derivatives of $V = V_{r+k+1}$ are elements of R that generate the fusion ideal:

$$F \simeq R / \left\langle \frac{\partial V}{\partial X_1}, \dots, \frac{\partial V}{\partial X_r} \right\rangle.$$

In conclusion, Gepner and Schwimmer confirms Conjecture 2.1 in the case of a Lie algebra of type C_r.

2.3 Bouwknegt and Ridout 2006

In [BR06] the method from [Gep91] and [GS92] for presenting the fusion ring $F = F(\mathfrak{g}, k), k \in \mathbb{N}$, for a Lie algebra $\mathfrak{g}$ of type A_r or C_r in terms of a potential function is revisited. It is shown that analogous potential functions cannot describe the fusion ring for Lie algebras of other types, which explains why no progress has since been made on Conjecture 2.1.

More precisely, for a simple complex Lie algebra $\mathfrak{g}$ let P_λ denote the set of weights with multiplicities of the representation $L(\lambda)$, i.e. $\operatorname{ch} L(\lambda) =$

$\sum_{\mu\in P_\lambda} e^{\mu}$, and consider the element

$$\frac{1}{k+h^\vee}\sum_{\mu\in P_\lambda} e^{(k+h^\vee)\mu} \in \mathbb{Z}[P]\otimes_{\mathbb{Z}}\mathbb{Q}. \tag{2.6}$$

For $\lambda = \omega_1$ this is formula (2.4) if $\mathfrak{g}$ has type A_r and formula (2.5) if $\mathfrak{g}$ has type C_r. The set of weights $\{(k+h^\vee)\mu \mid \mu \in P_\lambda\}$ is W-invariant, so $\sum_{\mu\in P_\lambda} e^{(k+h^\vee)\mu}$ is an element of the character ring $\mathbb{Z}[P]^W$, and (2.6) can be written as a polynomial in the fundamental characters $\chi_{\omega_i}, i = 1,\dots,r$, with rational coefficients. Let $V^\lambda \in \mathbb{Z}[X_1,\dots,X_r]\otimes_{\mathbb{Z}}\mathbb{Q}$ denote the same polynomial in the isomorphism classes $X_i = [L(\omega_i)], i = 1,\dots,r$, of the fundamental representations.

Recall from Section 1.2 the presentation of the fusion ring F as a quotient of R by an ideal I_k. It is shown by Bouwknegt and Ridout that the ideal $\left\langle \frac{\partial V^\lambda}{\partial X_1},\dots,\frac{\partial V^\lambda}{\partial X_r}\right\rangle \subseteq R$ generated by the derivatives of V^λ does not equal the fusion ideal I_k for any $\lambda \in P^+$, unless $\mathfrak{g}$ has type A_r or C_r. The crucial difference is, that for type A_r or C_r all integers $\langle \omega_i, \beta_0^\vee\rangle = 1, i = 1,\dots,r$, but for all other types there is an i with $\langle \omega_i, \beta_0^\vee\rangle > 1$. Thus another method is required in order to prove Conjecture 2.1.

Having established this negative result Bouwknegt and Ridout turn to a different method for describing a generating set of the fusion ideal I_k. Let $\lambda = \sum_{i=1}^r \lambda_i\omega_i$ be a dominant weight. In a decomposition of $L(\omega_1)^{\lambda_1}\otimes\dots\otimes L(\omega_r)^{\lambda_r}$ into irreducible components the module $L(\lambda)$ occurs exactly once, and all other occuring components $L(\mu)$ are of lower height $\langle\mu,\rho^\vee\rangle < \langle\lambda,\rho^\vee\rangle$. Then in the presentation of the representation ring as a polynomial ring we have

$$[L(\lambda)] = X_1^{\lambda_1}\dots X_r^{\lambda_r} - \text{"lower terms"}.$$

They proceed to define a monomial ordering on R making this precise, such that the leading term of $[L(\lambda)]$ is indeed $X_1^{\lambda_1}\dots X_r^{\lambda_r}$.

Define a monomial ordering $\prec$ on R by saying $X_1^{\lambda_1}\dots X_r^{\lambda_r} \prec X_1^{\mu_1}\dots X_r^{\mu_r}$ if and only if

$$\begin{aligned}
&\langle\lambda,\beta_0^\vee\rangle < \langle\mu,\beta_0^\vee\rangle \text{ or}\\
&\langle\lambda,\beta_0^\vee\rangle = \langle\mu,\beta_0^\vee\rangle \text{ and } \langle\lambda,\rho^\vee\rangle < \langle\mu,\rho^\vee\rangle \text{ or}\\
&\langle\lambda,\beta_0^\vee\rangle = \langle\mu,\beta_0^\vee\rangle \text{ and } \langle\lambda,\rho^\vee\rangle = \langle\mu,\rho^\vee\rangle \text{ and } X_1^{\lambda_1}\dots X_r^{\lambda_r} \prec' X_1^{\mu_1}\dots X_r^{\mu_r}
\end{aligned}$$

where $\prec'$ is any other monomial ordering on R. This order indeed picks out the leading term $\mathrm{LT}_\prec([L(\lambda)]) = X_1^{\lambda_1}\dots X_r^{\lambda_r}$.

Note that the fusion ring has as $\mathbb{Z}$-basis the elements $\{[L(\lambda)] \mid \lambda \in P^+, \langle\lambda,\beta_0^\vee\rangle \le k\}$, which are distinguished by $\prec$ from those elements $[L(\mu)]$ with weights $\mu \in P^+\setminus P_k$ outside the alcove $P_k = \{\lambda \in P^+ \mid \langle\lambda,\beta_0^\vee\rangle \le k\}$.

Consider the ideal $\langle \mathrm{LT}_{\prec}(I_k)\rangle$ in R generated by the leading terms w.r.t $\prec$ of polynomials in I_k. Bouwknegt and Ridout show, that as an abelian group $\langle \mathrm{LT}_{\prec}(I_k)\rangle$ is generated freely by the set of monomials $\mathcal{M} = \{X_1^{\lambda_1} \dots X_r^{\lambda_r} \mid \langle \lambda, \beta_0^\vee \rangle > k\}$, and as an ideal it is generated by the atomic monomials in $\mathcal{M}$, i.e., those monomials which cannot be expressed as the product of a monomial in $\mathcal{M}$ and an X_i.

Note that the atomic monomials in $\mathcal{M}$ include all monomials $X_1^{\lambda_1} \dots X_r^{\lambda_r}$ associated to weights on the boundary $\langle \lambda, \beta_0^\vee \rangle = k+1$. For $\mathfrak{g}$ of type A_r or C_r these account for all atomic monomials, since $\langle \omega_i, \beta_0^\vee \rangle = 1$ for all $i = 1, \dots, r$, so if a monomial $X_1^{\mu_1} \dots, X_r^{\mu_r}$ has $\langle \mu, \beta_0^\vee \rangle > k+1$ then we can find an i such that $\mu - \omega_i \in P^+$ and $\langle \mu - \omega_i, \beta_0^\vee \rangle \geq k+1$. For the remaining types there will generally be other atomic monomials.

For all monomials $X_1^{\lambda_1} \dots X_r^{\lambda_r} \in \mathcal{M}$ define a polynomial $p_\lambda \in I_k \subseteq R$ whose leading term w.r.t. $\prec$ is $X_1^{\lambda_1} \dots X_r^{\lambda_r}$: If λ is on a shifted affine alcove boundary, then $[L(\lambda)]$ is in I_k, so set $p_\lambda = [L(\lambda)]$. Otherwise, find a $w \in W_k$ such that $w.\lambda \in P_k$ and set $p_\lambda = [L(\lambda)] - \det(w)[L(w.\lambda)] \in I_k$.

The following result is [BR06, Proposition 3]:

Proposition 2.2. *The polynomials p_λ associated to atomic monomials form a Gröbner basis for the fusion ideal w.r.t. the monomial ordering $\prec$, i.e.*

$$I_k = \big\langle p_\lambda \in R \mid \lambda \in P^+ \setminus P_k \text{ with } \lambda - \omega_i \notin P^+ \setminus P_k \text{ for all } i \big\rangle$$

2.4 Boysal and Kumar 2009

In [BK09] a number of conjectures regarding specific generators of the fusion ideal is proposed. Again the isomorphism class of a simple representation $[L(\lambda)]$ is presented as an element of the polynomial ring $\mathbb{Z}[X_1, \dots, X_r]$ with variables the isomorphism classes of the fundamental representations $X_i = [L(\omega_i)], i = 1, \dots, r$.

For a natural number $k \in \mathbb{N}$ they work with the fusion ideal $I_k \subseteq R$ and seek to conjecturally describe it in terms of a generating set whose size is independent of the level k. They give 3 equivalent definitions of the fusion ring where the first one is repeated in Section 1.2. Therefore we may use the formula in Proposition 2.2 as a definition of I_k.

Below we state the main theorem of the paper and the subsequent conjecture.

Theorem 2.3. *Let k be any positive integer. We have the following inclusions of ideals* *(a) For $\mathfrak{g}$ of type B_r, D_r or E_6*

$$I_k \supseteq \sqrt{\langle [L((k+1)\omega_1)], [L((k+2)\omega_1)], \ldots, [L((k+h^\vee-1)\omega_1)] \rangle}.$$

(b) For $\mathfrak{g}$ of type G_2

$$I_k \supseteq \begin{cases} \sqrt{\langle [L((k+1)\omega_1)], [L((k+2)\omega_1)], [L(((k+1)/2)\omega_2)] \rangle}, & \text{if } k \text{ is odd} \\ \sqrt{\langle [L((k+1)\omega_1)], [L((k+2)\omega_1)], [L(\omega_1+(k/2)\omega_2)] \rangle}, & \text{if } k \text{ is even.} \end{cases}$$

(c) For $\mathfrak{g}$ of type F_4

$$I_k \supseteq \sqrt{\langle [L((k+1)\omega_4)], [L((k+2)\omega_4)], \ldots, [L((k+6)\omega_4)] \rangle}.$$

(d) For $\mathfrak{g}$ of type E_7

$$I_k \supseteq \sqrt{\langle [L((k+1)\omega_7)], [L((k+2)\omega_7)], \ldots, [L((k+h^\vee-1)\omega_7)] \rangle}.$$

(e) For $\mathfrak{g}$ of type E_8

$$I_k \supseteq \begin{cases} \sqrt{\langle [L((k+2)\omega_8)], [L((k+3)\omega_8)], \ldots, [L((k+29)\omega_8)] \rangle}, & \text{if } k \text{ is even} \\ \sqrt{\langle [L((k+2)\omega_8)], \ldots, [L((k+29)\omega_8)], [L(((k+1)/2)\omega_8)] \rangle}, & \text{if } k \text{ is odd.} \end{cases}$$

Conjecture 2.4. *(a) All the inclusions in* (a)-(b) *in the above theorem are equalities for $\mathfrak{g}$ of type B_r, D_r or G_2.*

(b) In addition, for $\mathfrak{g}$ of type B_r

$$\begin{aligned} &\sqrt{\langle [L((k+1)\omega_1)], [L((k+2)\omega_1)], \ldots, [L((k+h^\vee-1)\omega_1)] \rangle} \\ &= \langle [L((k+1)\omega_1)], [L((k+2)\omega_1)], \ldots, [L((k+h^\vee-1)\omega_1)], [L(k\omega_1+\omega_r)] \rangle, \end{aligned}$$

and for $\mathfrak{g}$ of type D_r

$$\begin{aligned} &\sqrt{\langle [L((k+1)\omega_1)], [L((k+2)\omega_1)], \ldots, [L((k+h^\vee-1)\omega_1)] \rangle} \\ &= \langle [L((k+1)\omega_1)], \ldots, [L((k+h^\vee-1)\omega_1)], [L(k\omega_1+\omega_{r-1})], [L(k\omega_1+\omega_r)] \rangle. \end{aligned}$$

Verifying the conjectures

Computer aided we verify several of the conjectures for low levels k. For each weight $\lambda \in P^+$ we first calculate an expression of $[L(\lambda)]$ in terms of the fundamental classes X_i in the polynomial presentation of R.

Given an ideal $I \subseteq R$ we use a version of Buchberger's algorithm to calculate a Gröbner basis for I, cf. for example [KRK88]. To test whether a given element $p \in R$ belongs to I, we check whether p reduces to zero by the Gröbner basis.

To test for membership of p in the radical $\sqrt{I}$ of an ideal we add an extra variable t to the polynomial presentation of the representation ring and embed both p and I in $R[t]$. By the following lemma $p \in \sqrt{I}$ if and only if the ideal $\langle I, tp-1\rangle$ is the whole ring $R[t]$.

Lemma 2.5. *Let R be a noetherian ring and let $I \subseteq R$ be an ideal. Then $p \in \sqrt{I}$ if and only if the ideal localized at p contains a unit: $I_p = R_p$.*

Proof. Assume $p \in \sqrt{I}$, i.e., $p^n \in I$ for some $n \geq 1$. Then $1 = p^n/p^n \in I_p$ so $I_p = R_p$. On the other hand, let $I \subseteq \mathfrak{p}$ be an arbitrary prime ideal. If $I_p = R_p$ then $\mathfrak{p}_p = R_p$, so write $1 = a/f^n$ for some $a \in \mathfrak{p}$. Then $p^{n+k} = ap^k$ in $\mathfrak{p}$ for some $k \geq 0$ and consequently $p \in \mathfrak{p}$. So $p \in \bigcap_{I\subseteq\mathfrak{p}} \mathfrak{p} = \sqrt{I}$. □

These techniques and algorithms have been implemented in a java program[1], and results of an attempted verification of the conjectures can be seen in Table 2.3. Calculations for higher levels were initiated but aborted, as the algorithm didn't terminate after 24 hours of calculating. We see that Conjecture 2.4 (a) for $\mathfrak{g}$ of type G_2 is refuted for certain levels.

Let us make this concrete in the case where $\mathfrak{g}$ has type G_2 and the level is $k = 7$. The polynomials in $\mathbb{Z}[X_1, X_2]$ corresponding to the generators of the radical ideal are

$$\begin{aligned}
[L(8\omega_1)]: \quad & p_1 = X_1^8 - X_1^7 - 7X_1^6X_2 - 5X_1^6 + 4X_1^5X_2 + 6X_1^5 + 15X_1^4X_2^2 + \\
& 21X_1^4X_2 + 5X_1^4 + 2X_1^3X_2^2 - 6X_1^3X_2 - 8X_1^3 - 10X_1^2X_2^3 - 27X_1^2X_2^2 - \\
& 15X_1^2X_2 - 8X_1X_2^3 - 12X_1X_2^2 + 3X_1 + X_2^4 + X_2^3 \\
[L(9\omega_1)]: \quad & p_2 = X_1^9 - X_1^8 - 8X_1^7X_2 - 6X_1^7 + 5X_1^6X_2 + 8X_1^6 + 21X_1^5X_2^2 + \\
& 30X_1^5X_2 + 6X_1^5 - 13X_1^4X_2 - 13X_1^4 - 20X_1^3X_2^3 - 50X_1^3X_2^2 - \\
& 30X_1^3X_2 + 2X_1^3 - 14X_1^2X_2^3 - 12X_1^2X_2^2 + 9X_1^2X_2 + 7X_1^2 + 5X_1X_2^4 + \\
& 16X_1X_2^3 + 21X_1X_2^2 + 8X_1X_2 - 2X_1 + 4X_2^4 + 11X_2^3 + 6X_2^2 - 2X_2 - 1 \\
[L(4\omega_2)]: \quad & p_3 = X_1^6 - X_1^5 - 2X_1^4X_2 - 3X_1^4 - 3X_1^3X_2^2 - 2X_1^3X_2 + 2X_1^3 + \\
& 6X_1^2X_2 + 3X_1^2 + 6X_1X_2^3 + 9X_1X_2^2 + 2X_1X_2 - X_1 + X_2^4 + 5X_2^3 + \\
& 4X_2^2 - 2X_2 - 1
\end{aligned}$$

[1] Available at `http://home.imf.au.dk/troels/Del-B/ReAlGriDPCv1.2.zip`

The weight $4\omega_1 + 2\omega_2$ lies on the affine wall so the element

$$[L(4\omega_1+2\omega_2)] : \quad q = -X_1^7 + 2X_1^6 + 4X_1^5X_2 + X_1^5 + X_1^4X_2^2 - 2X_1^4X_2 - 4X_1^4 - 5X_1^3X_2^2 - 10X_1^3X_2 + 2X_1^3 - 3X_1^2X_2^3 - 3X_1^2X_2^2 + 3X_1^2X_2 + 2X_1^2 + 6X_1X_2^2 + 6X_1X_2 - 2X_1 + X_2^4 + 2X_2^3 + X_2^2$$

belongs to the fusion ideal. However if we consider the ideal $\langle p_1, p_2, p_3, qT-1\rangle \subseteq \mathbb{Z}[X_1, X_2, T]$ we do not get the whole ring.

Type of $\mathfrak{g}$	Conjecture 2.4 (a)	Conjecture 2.4 (b)
B_3	verified for levels $1-7$	verified for levels $1-7$
B_4	verified for levels $1-4$	verified for levels $1-4$
D_4	verified for levels $1-4$	verified for levels $1-5$
G_2	verified for levels $1-6, 8, 10, 12$ refuted for levels $7, 9, 11, 13$	

Table 2.3

2.5 Korff and Stroppel 2010

In [KS10] the combinatorial structures of two geometric constructions related to $\mathfrak{g} = \mathfrak{sl}_{r+1}$ and an integer $k \geq 0$ are extrinsically defined in an analogues way.

One is the fusion ring $F = F(\mathfrak{sl}_{r+1}, k)$ emerging in the theory of rational conformal field theory. The natural $\mathbb{Z}$-basis is indexed by partitions λ whose Young diagram fits into a bounding box of size $r \times k$, to which the multiplicative structure constants are given as dimensions of conformal blocks on $\mathbb{P}^1$ with 3 marked points.

The other is the quantum cohomology ring $qH^\bullet(\mathrm{K}_{0k,r+k+1})$, which is a deformation of the ordinary cohomology ring of $\mathrm{K}_{0k,r+k+1}$. It has a $\mathbb{Z}[q]$-basis in Shubert classes $[\Omega_\lambda]$ indexed by partitions λ, whose Young diagram fits into a bounding box of size $k \times (r+1)$, and the structure constants are given by certain Gromov-Witten invariants.

In both cases the multiplicative structures are defined in terms of symmetric polynomials in pairwise non-commuting variables. The main theorem of the paper is a realization of F as a quotient of $qH^\bullet(\mathrm{K}_{0k,r+k+1})$, which comes as a consequence of the analogy of the combinatorial descriptions.

An interpretation of these variables are given as particle hopping operators on the extended Dynkin diagram of $\mathfrak{sl}_{r+1}$: A partition $\lambda = \sum_{i=1}^r m_i\omega_i$

gives a basis element of F if and only if $m = \sum_{i=1}^{r} m_i \leq k$. In this case we set $m_0 = k - m$ and we interpret the basis element as a configuration of k particles on a circle with $r+1$ marked points with m_i particles at place i. Let $a_i, i = 0, \dots, r$ be the operator that moves a particle from the ith place to the $i+1$st place in clockwise direction. If there are no particles at place i the operator kills the configuration instead.

For a subset $I \subseteq \{0, 1, \dots, r\}$, let $\underline{a}_I = a_{i_1} \dots a_{i_n}$ be the product of the operators a_{i_s} with i_s running through I in anticlockwise cyclical order, i.e., such that if $i_t = i_s + 1 \mod (r+1)$ then a_{i_s} occurs to the right of a_{i_t}. Define the elementary symmetric polynomial $e_n(\underline{a}) = \sum_{|I|=n} \underline{a}_I$ for $1 \leq n \leq r$, set $e_0(\underline{a})$ and $e_{r+1}(\underline{a})$ to be the identity and $e_m(\underline{a}) = 0$ if $m < 0$ or $m > r+1$. Then by [KS10, Corollary 5.14] the operators $e_n(\underline{a})$ commute and therefore

$$s_\lambda(\underline{a}) = \det(e_{\lambda_i^t - i + j}(\underline{a})) \tag{2.7}$$

is well-defined. Here λ_i^t is the number of boxes in the ith column of the dynkin diagram of λ. Now by [KS10, Theorem 6.18] multiplication in F is given on the basis elements λ by

$$\lambda \star \mu = s_\lambda(\underline{a})\mu. \tag{2.8}$$

2.6 Douglas 2013

In his paper [Dou09] Douglas gives an abstract presentation of the fusion ring $F = F(\mathfrak{g}, k)$ for a simple complex Lie algebra $\mathfrak{g}$ and integer $k \in \mathbb{N}$ as a quotient of the representation ring R by an ideal I_k generated by a set $p_1(k), \dots, p_{n_\mathfrak{g}}(k) \in R$, where the $p_i(k), i = 1, \dots, n_\mathfrak{g}$, are isomorphism classes of representations of $\mathfrak{g}$ depending on the level k, such that the number $n_\mathfrak{g}$ is independent of the level. An upper bound on the number of generators $n_\mathfrak{g}$ of the fusion ideal is $\sum_{i=1}^{r} |W| / \left|W(\hat{i})\right|$ where $W(\hat{i})$ is the subgroup of W generated by all simple reflections $s_1, \dots, \hat{s}_i, \dots, s_r$ except the ith.

In his sequential paper [Dou13] Douglas improves drastically on the upper bound of $n_\mathfrak{g}$ and extended the method to give a complete computation of explicit generators in each case. For $\mathfrak{g}$ a simple complex Lie algebra of classical type or of type G_2 we repeat in Table 2.4 his results on specific generators for the fusion ideal I_k for level $k \in \mathbb{N}$.

We also refer to Chapter 6 where we have sought to generalize part of the work in the two papers to the setting of quantum groups.

$A_r, r \geq 1$	$I_k = \langle [L((k+1)\omega_1)], [L(k\omega_1+\omega_2)], \dots, [L(k\omega_1+\omega_r)] \rangle$
$B_r, r \geq 3$	$I_k = \langle [L((k-1)\omega_1+\omega_2)], \dots, [L((k-1)\omega_1+\omega_{r-1})],$ $[L((k-1)\omega_1+2\omega_r)], [L(k\omega_1+\omega_r)], \dots, [L(k\omega_1+\omega_3)],$ $[L(k\omega_1+\omega_2)]+[L(k\omega_1)], [L((k+1)\omega_1)] \rangle$
$C_r, r \geq 2$	$I_k = \langle [L((k+1)\omega_1)], [L(k\omega_1+\omega_2)], \dots, [L(k\omega_1+\omega_r)] \rangle$
$D_r, r \geq 4$	$I_k = \langle [L((k-1)\omega_1+\omega_2)], \dots, [L((k-1)\omega_1+\omega_{r-2})],$ $[L((k-1)\omega_1+\omega_{r-1}+\omega_r)], [L(k\omega_1+\omega_r)], [L(k\omega_1+\omega_{r-1})],$ $[L(k\omega_1+\omega_{r-1}+\omega_r)], [L(k\omega_1+\omega_{r-2})], \dots, [L(k\omega_1+\omega_3)],$ $[L(k\omega_1+\omega_2)]+[L(k\omega_1)], [L((k+1)\omega_1)] \rangle$
G_2	$I_k = \begin{cases} \langle [L((\frac{k}{2}-1)\omega_2)]+[L(\frac{k}{2}\omega_2)], [L(\omega_1+\frac{k}{2}\omega_2)], \\ \quad [L(3\omega_1+(\frac{k}{2}-1)\omega_2)] \rangle, & k \text{ even} \\ \langle [L(\frac{k+1}{2}\omega_2)], [L(2\omega_1+\frac{k-1}{2}\omega_2)], \\ \quad [L(3\omega_1+\frac{k-1}{2}\omega_2)]+[L(3\omega_1+\frac{k-3}{2}\omega_2)] \rangle, & k \text{ odd} \end{cases}$

Table 2.4: Part of the table in [Dou13, Theorem 1.1]

2.7 Andersen and Stroppel 2014

In [AS14] an overview of the theory of fusion rings in the setting of tilting modules for quantum groups, as presented in our Section 1.3, is provided, and several of the techniques from the above-mentioned papers are applied.

In particular a generating set $\mathcal{G}$ for the category of negligible tilting modules $\mathcal{N}_q$, as a tensor ideal in $\mathcal{T}_q$, is presented analogous to the Gröbner basis analysis that lies behind Proposition 2.2.

Similarly they realize the fusion ring as a quotient of a polynomial ring. The defining ideal in this commutative presentation is generated by the polynomials corresponding to the elements in $\mathcal{G}$, and in many cases they are able to produce alternative generating sets of much smaller sizes via classical determinantal identities.

The story here deviates from the one in the previous papers, as the type of the affine Weyl group W_l depends on whether l is divisible by all off-diagonal entries in the Cartan matrix or not: In the first case the type matches that of the associated affine Lie algebra $\widetilde{\mathfrak{g}}$, but in the second case it is the dual type in Bourbakian convention. The fusion rings that are constructed in this latter case have not been studied much before. Thus the fusion rings that arise in this setting encompass all the ones studied in the previous papers and as well as everal others.

In the case where $\mathfrak{g}$ has type A_r the multiplicative structure in the fusion ring is compared to the explicit expression given in (2.8). Once it is shown that the two multiplicative structures acts identical relative to chosen bases, it follows as a consequence of commutativity of the tensor product in $\mathcal{T}_q$, that the determinant in (2.7) is well-defined and (2.8) is commutative.

An analogues description of the multiplicative structure in the fusion ring in terms of symmetric polynomials in non-commutative variables is made in the case where $\mathfrak{g}$ has type C_r. An interpretation of the variables are again given as particle hopping on the extended Dynkin diagram associated to W_l. The type of the diagram and the number of particles depends on whether l is odd or even.

Chapter 3

Generators of the fusion ideal

In this and the following chapters we return to working with the fusion ring $F_q = F_q(\mathfrak{g}, l)$ for the quantum group U_q associated to a simple Lie algebra $\mathfrak{g}$, with l the order of $q^2 \in \mathbb{C}$. We consider the polynomial presentation of the (split) Grothendieck group $\mathrm{K}_0(\mathcal{T}_q) \simeq R = \mathbb{Z}[X_1, \ldots, X_r]$ and will explore the ideal $I_l \subseteq R$ for which $F_q \simeq R/I_l$.

A priori the ideal is generated by infinitely many relations, and the focus in this chapter is to reduce the number of generators. F_q is a finitely generated free $\mathbb{Z}$-module, so it has Krull dimension 1, and being a quotient of the $r+1$-dimensional ring R the ideal I_l is generated by at least r elements. In case I_l is generated by r elements, these generators will constitute a regular R-sequence, and since R is a regular ring the fusion ring is a complete intersection ring.

Assuming we have proven I_l is a complete intersection we may be able to find explicit generators $p_1, \ldots, p_r \in R$ of I_l with $\frac{\partial}{\partial X_j} p_i$ symmetric in i and j. Then $\alpha = \sum_{i=1}^r p_i dX_i$ is a closed 1-form, $d\alpha = \sum_{i,j=1}^r \frac{\partial}{\partial X_j} p_i dX_j \wedge dX_i = 0$, hence it integrates to a potential function $V \in R$, $\alpha = dV = \sum_{i=1}^r \frac{\partial}{\partial X_i} V dX_i$, and the generators $p_i = \frac{\partial}{\partial X_i} V$ are described by a single polynomial V.

We end up with the following set of problems on the defining ideal I_l of the fusion ring F_q.

(i) Find a finite generating set of I_l for each value of l.

(ii) Find a level-independent upper bound on the number of elements necessary to generate I_l, preferably together with a uniform explicit generating set.

(iii) Decide whether I_l can be generated by r elements, preferably together with r explicit generators.

(iv) Describe these r generators as the partial derivatives of an explicit potential function.

The plan for this chapter is to explore these problems for each Lie algebra $\mathfrak{g}$ of classical type or of type G_2. To sum it up (i) has been achieved in all cases, (ii) has been achieved achieved in all cases but type C_r for odd l, (iii) has been achieved for type A_l, type B_r for odd l, type C_r for even l and for type G_2 and finally (iv) has been achieved for type A_l, type B_r for odd l and type C_r for even l.

If $l = l_\alpha$ for all $\alpha \in \Phi$ then set $k = l - (\langle \rho, \alpha_0^\vee \rangle + 1)$ and otherwise $k = l/d_{\beta_0} - (\langle \rho, \beta_0^\vee \rangle + 1)$. In the first case we also say that the level of a weight $\lambda \in P$ is $\langle \lambda, \alpha_0^\vee \rangle$ and otherwise $\langle \lambda, \beta_0^\vee \rangle$. The fundamental alcove A_l consists of all dominant weights of level at most k. We will assume that $A_l \neq \emptyset$ which is equivalent to $k \geq 0$.

3.1 A finite generating set

We describe in this section a method for solving problem (i) listed above for any simple complex Lie algebra $\mathfrak{g}$. Next we go through this method in detail for $\mathfrak{g}$ of specific types.

The following is inspired by the method in Section 2.3 originally developed in [BR06]. Let $\preceq$ denote the ordering on P defined by $\lambda \preceq \mu$ if $\mu - \lambda = \sum_{i=1}^r a_i \omega_i$ with $a_1, \ldots, a_r \geq 0$. We have the following

Proposition 3.1. *The tensor ideal $\mathcal{N}_q \subseteq \mathcal{T}_q$ is generated by the set*

$$\{T_q(\mu) \mid \mu \text{ minimal in } P^+ \setminus A_l \text{ with respect to } \preceq\}. \tag{3.1}$$

For a proof see [AS14, Proposition 2.4]. To find a finite generating set of I_l we take the isomorphism class $[T_q(\lambda)]$ of a tilting module belonging to a minimal highest weight $\lambda \in P^+ \setminus A_l$ and describe it as a polynomial in the fundamental classes $X_i = [L_q(\omega_i)], i = 1, \ldots, r$.

Note that the size of this set is finite but dependent on l.

Type A_r

Let $\mathfrak{g}$ be of type $A_r, r \geq 1$. We have $k = l - (r+1)$. A weight $\lambda = \sum_{i=1}^r m_i \omega_i \in P$ has level $\langle \lambda, \alpha_0^\vee \rangle = \sum_{i=0}^r m_i$, thus the minimal weights in $P^+ \setminus A_l$ certainly includes all

$$\{\sum_{i=1}^r m_i \omega_i \in P^+ \mid \sum_{i=1}^r m_i = k+1\}. \tag{3.2}$$

If a weight $\lambda = \sum_{i=1}^r m_i\omega_i \in P^+ \setminus A_l$ has level $\sum_{i=1}^r m_i > k+1$, then it is not minimal. For say $m_i > 0$, then $\lambda - \omega_i \in P^+$ and has level $\geq k+1$, so is not in A_l. Thus the above set includes all the minimal weights in $P^+ \setminus A_l$.

The cardinality of the set above is a polynomial in k of degree $r-1$, in fact it is equal to the binomial coefficient $\binom{k+r}{r-1}$: When $0 \leq m_1 \leq k+1$ is fixed, there is by induction on r $\binom{(k+1-m_1)+(r-2)}{r-2}$ choices of $m_2, \ldots, m_r \geq 0$ summing up to $k+1-m_1$, making the total number of choices equal to $\sum_{m_1=0}^{k+1}\binom{k+r-m_1-1}{r-2} = \sum_{m_1=0}^{k+1}(\binom{k+r-m_1}{r-1} - \binom{k+r-m_1-1}{r-1}) = \binom{k+r}{r-1} - \binom{r-2}{r-1}$.

Type C_r

Let $\mathfrak{g}$ be of type $C_r, r \geq 2$. If l is even, then $k = l/2 - (r+1)$. A weight $\lambda = \sum_{i=1}^r m_i\omega_i$ has level $\langle \lambda, \beta_0^\vee \rangle = \sum_{i=1}^r m_i$, so the same argument as before shows that the minimal weights are

$$\{\sum_{i=1}^r m_i\omega_i \in P^+ \mid \sum_{i=1}^r m_i = k+1\}, \tag{3.3}$$

and that this set contains $\binom{k+r}{r-1}$ elements.

If l is odd, then $k = l - 2r$. A weight $\lambda = \sum_{i=1}^r m_i\omega_i$ has level $\langle \lambda, \alpha_0^\vee \rangle = m_1 + \sum_{i=2}^r 2m_i$. The set of minimal weights in $P^+ \setminus A_l$ contains all the elements

$$\{\sum_{i=1}^r m_i\omega_i \in P^+ \mid m_1 + \sum_{i=2}^r 2m_i = k+1\} \tag{3.4}$$

and a priori it also contains the elements $\{\sum_{i=2}^r m_i\omega_i \in P^+ \mid \sum_{i=2}^r 2m_i = k+2\}$, but since k is odd, $k+2$ is also odd, and this set is empty. Therefore the above set exhausts all the minimal weights.

In (3.4) $m_1 = 2m$ must be even, and when $0 \leq m \leq \frac{k+1}{2}$ is fixed, the number of choices of elements $m_2, \ldots, m_r \geq 0$ for which $2\sum_{i=2}^r m_i = k+1-2m$ has already been determined to be $\binom{\frac{k+1}{2}-m+r-2}{r-2}$, making the total number of elements in (3.4) equal to $\sum_{m=0}^{\frac{k+1}{2}}\binom{\frac{k+1}{2}-m+r-2}{r-2} = \sum_{m=0}^{\frac{k+1}{2}}(\binom{\frac{k+1}{2}-m+r-1}{r-1} - \binom{\frac{k+1}{2}-m+r-2}{r-1}) = \binom{\frac{k+1}{2}+r-1}{r-1}$, which is also a polynomial of degree $r-1$ in k.

Type B_r

Let $\mathfrak{g}$ be of type $B_r, r \geq 3$. If l is even, then $k = l/2 - (2r-1)$. A weight $\lambda = \sum_{i=1}^r m_i\omega_i$ has level $\langle \lambda, \beta_0^\vee \rangle = m_1 + \sum_{i=2}^{r-1} 2m_i + m_r$. Thus besides the weights of level $k+1$, the minimal weights in $P^+ \setminus A_l$ include those

$\sum_{i=1}^{r} m_i\omega_i$ of level $k+2$ with $m_1 = m_r = 0$, i.e., the set is

$$\begin{aligned}&\{\sum_{i=1}^{r} m_i\omega_i \in P^+ \mid m_1 + \sum_{i=2}^{r-1} 2m_i + m_r = k+1\}\\ &\quad\cup\{\sum_{i=2}^{r-1} m_i\omega_i \in P^+ \mid \sum_{i=2}^{r-1} 2m_i = k+2\}.\end{aligned} \tag{3.5}$$

If k is even, the first set in the union contains $2\binom{\frac{k}{2}+r-1}{r-1}$ elements and the second set $\binom{\frac{k}{2}+r-2}{r-3}$ elements. If k is odd, the first set contains $\binom{\frac{k+1}{2}+r-1}{r-1} + \binom{\frac{k-1}{2}+r-1}{r-1}$ elements and the second set is empty. This is verified by doing manipulations similar to those in the previous cases.

If l is odd, then $k = l - 2r$. A weight $\lambda = \sum_{i=1}^{r} m_i\omega_i$ has level $\langle\lambda, \alpha_0^\vee\rangle = \sum_{i=1}^{r-1} 2m_i + m_r$, so the minimal weights are

$$\{\sum_{i=1}^{r} m_i\omega_i \in P^+ \mid \sum_{i=1}^{r-1} 2m_i + m_r = k+1\}. \tag{3.6}$$

Again the set of minimal weights a priori contains the elements $\{\sum_{i=1}^{r-1} m_i\omega_i \in P^+ \mid \sum_{i=1}^{r-1} 2m_i = k+2\}$, but since k is odd this set is empty. It is easy to see that (3.6) contains $\binom{\frac{k+1}{2}+r-1}{r-1}$ elements.

Type D_r

Let $\mathfrak{g}$ be of type $D_r, r \geq 4$. We have $k = l-(2r-3)$. A weight $\lambda = \sum_{i=1}^{r} m_i\omega_i$ has level $\langle\lambda, \alpha_0^\vee\rangle = m_1 + \sum_{i=2}^{r-2} 2m_i + m_{r+1} + m_r$, so the minimal weights are

$$\begin{aligned}&\{\sum_{i=1}^{r} m_i\omega_i \in P^+ \mid m_1 + \sum_{i=2}^{r-2} 2m_i + m_{r+1} + m_r = k+1\}\\ &\quad\cup\{\sum_{i=2}^{r-2} m_i\omega_i \in P^+ \mid \sum_{i=2}^{r-2} 2m_i = k+2\}.\end{aligned} \tag{3.7}$$

If k is even, the first set in the union contains $3\binom{\frac{k}{2}+r-1}{r-1} + \binom{\frac{k}{2}+r-2}{r-1}$ elements and the second set $\binom{\frac{k}{2}+r-3}{r-4}$ elements. If k is odd, the first set contains $\binom{\frac{k+1}{2}+r-1}{r-1} + 3\binom{\frac{k+1}{2}+r-2}{r-1}$ elements and the second set is empty.

Type G_2

Let $\mathfrak{g}$ be of type G_2. If 3 divides l, then $k = l/3 - 4$. A weight $\lambda = m_1\omega_1 + m_2\omega_2$ has level $\langle \lambda, \beta_0^\vee \rangle = m_1 + 2m_2$, so the minimal weights are

$$\begin{gathered}\{m_1\omega_1 + m_2\omega_2 \in P^+ \mid m_1 + 2m_2 = k+1\} \\ \cup \{m_2\omega_2 \in P^+ \mid 2m_2 = k+2\}.\end{gathered} \tag{3.8}$$

If k is even, the first set in the union contains $\frac{k}{2}$ elements and the second 1 element. If k is odd, the first set contains $\frac{k+1}{2}$ elements and the second set is empty.

If l is prime to 3, then $k = l - 6$ is also prime to 3. A weight $\lambda = m_1\omega_1 + m_2\omega_2$ has level $\langle \lambda, \alpha_0^\vee \rangle = 2m_1 + 3m_2$, so the set of minimal weights contains all weights of level $k+1$ and $k+2$. If a weight $m_1\omega_1 + m_2\omega_2$ has level $k+3$, then it must have $m_1 > 0$, and it is not minimal in $P^+ \setminus A_l$. The minimal weights are therefore

$$\{m_1\omega_1 + m_2\omega_2 \in P^+ \mid k+1 \leq 2m_1 + 3m_2 \leq k+2\}. \tag{3.9}$$

If $k \equiv 1 \mod 6$ this set contains $\frac{k-1}{3}$ elements, if $k \equiv 2 \mod 6$ it contains $\frac{k-2}{3}$ elements, if $k \equiv 4 \mod 6$ it contains $\frac{k-1}{3}$ elements and if $k \equiv 5 \mod 6$ it contains $\frac{k-2}{3}$ elements.

3.2 Finding canonical generators

We now review the method of expressing irreducible characters of the classical Lie algebras as determinants of matrices whose entries are given by symmetric polynomials, following [BR06, Appendix A]. The calculations start with the Weyl character formula

$$\chi_\lambda = \frac{A_{\lambda+\rho}}{A_\rho} \in \mathbb{Z}[P]^W$$

for a dominant $\lambda \in P^+$, where $A_\lambda = \sum_{w\in W}(-1)^{l(w)}e^{w(\lambda)} \in \mathbb{Z}[P]$ is an antisymmetric element with maximal weight λ. Case by case we will rewrite this as a polynomial in variables $q = e^\varepsilon$ for weights ε of the standard representation $L(\omega_1)$. By careful manipulations we may be able to show that the characters of the generators of Proposition 3.1 can be expressed by a canonical set characters of elements belonging to $\mathcal{N}_q$.

Another strategy is to use some of the results presented in Chapter 2 when possible. If l is divisible by all $d_i, i = 1, \ldots, r$, we have an isomorphism of the fusion ring $F_q(\mathfrak{g}, l)$ with the WZW fusion ring $F(\mathfrak{g}, k)$ for $k = l/d_{\beta_0} - (\langle \rho, \beta_0^\vee \rangle + 1)$. In these cases the presentations in Table 2.4 applies to give us canonical sets of generators of the fusion ideal I_l.

Type A_r

Assume $\mathfrak{g}$ has type $A_r, r \geq 1$. The weights of the standard representation are $\varepsilon_1 = \omega_1$, $\varepsilon_i = \omega_i - \omega_{i-1}$ for $1 < i \leq r$ and $\varepsilon_{r+1} = -\omega_r = -(\varepsilon_1 + \cdots + \varepsilon_r)$. The Weyl group $W = S_{r+1}$ permutes these weights. Write $q_i = e^{\varepsilon_i}$ for $i = 1, \ldots, r$ as formal exponentials.

Given a weight $\lambda = \sum_{i=1}^r m_i\omega_i \in P^+$ we write it as $\sum_{i=1}^r \lambda_i\varepsilon_i$ with $\lambda_i = m_i + \cdots + m_r, 1 \leq i \leq r$, and $\lambda_{r+1} = 0$. Then $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$. Specifically $\rho = \sum_{i=1}^r \rho_i\varepsilon_i$ has $\rho_i = r + 1 - i$. We calculate using the Leibniz formula for determinants

$$A_\lambda = \sum_{w\in W} \det(w)e^{\sum_{i=1}^{r+1} \lambda_i w(\varepsilon_i)} = \sum_{\sigma\in S_{r+1}} \operatorname{sgn}(\sigma) \prod_{i=1}^{r+1} q_{\sigma(i)}^{\lambda_i} = \left|q_i^{\lambda_j}\right|_{i,j=1}^{r+1}.$$

We now identify the character $\chi_\lambda = A_{\lambda+\rho}/A_\rho$ with the Schur polynomial

$$S_\lambda = S_\lambda(q_1, \ldots, q_{r+1}) = \frac{\left|q_j^{\lambda_i+r+1-i}\right|}{\left|q_j^{r+1-i}\right|}.$$

We let $H_m = H_m(q_1, \ldots, q_{r+1}), m \geq 0$, denote the complete symmetric polynomial in $r + 1$ variables, i.e., the sum of all distinct monomials of degree m with the constraint that $q_1 \ldots q_{r+1} = 1$. Formally, let $H_m = 0$ for $m < 0$ and note also that $H_0 = 1$. A relationship between the Schur polynomials and the complete symmetric polynomials is given by a Jacobi-Trudi identity, cf. [FH91, (24.10)]:

Lemma 3.2. *Let λ be a partition of a positive integer into r parts. Then the Schur polynomial is given by*

$$S_\lambda = \left|H_{\lambda_j+i-j}\right|_{i,j=1}^{r+1}$$

As $\lambda_{r+1} = 0$ the last column is just $(0, \ldots, 0, 1)^T$ and this determinant equals

$$\left|H_{\lambda_j+i-j}\right|_{i,j=1}^{r} = \begin{vmatrix} H_{\lambda_1} & H_{\lambda_2-1} & \ldots & H_{\lambda_r-r+1} \\ H_{\lambda_1+1} & H_{\lambda_2} & \ldots & H_{\lambda_r-r+2} \\ \vdots & \vdots & & \vdots \\ H_{\lambda_1+r-1} & H_{\lambda_2+r-2} & \ldots & H_{\lambda_r} \end{vmatrix}. \tag{3.10}$$

The formula applied to $\lambda = m\omega_1 = m\varepsilon_1, m \geq 0$, shows us that $\chi_{m\omega_1} = H_m$.

The following result is [Gep91, (2.20)], and the proof is from [BR06, 3.2].

Proposition 3.3. *The fusion ideal I_l for the fusion ring $F_q(\mathfrak{g}, l)$, where $\mathfrak{g}$ has type A_r and $k = l - (r+1) \geq 0$, has the following sets of r generators*

$$\begin{aligned} I_l &= \langle [\nabla_q((k+1)\omega_1)], [\nabla_q((k+2)\omega_2)], \ldots, [\nabla_q((k+r)\omega_1)] \rangle \\ &= \langle [\nabla_q((k+1)\omega_1)], [\nabla_q(k\omega_1 + \omega_2)], \ldots, [\nabla_q(k\omega_1 + \omega_r)] \rangle \,. \end{aligned}$$

Proof. We know from the previous section that the ideal I_l is generated by all $[T_q(\lambda)]$ with $\lambda = \sum_{i=1}^r m_i\omega_i$ the minimal weights in (3.2). All of these weights lie in the closure of the alcove $\bar{A}_l$, so $[T_q(\lambda)] = [\nabla_q(\lambda)]$. Since the character of $\nabla_q(\lambda)$ is the Weyl character χ_λ, we use the lemma to get expressions of these generators.

When we write such a λ in the basis $\varepsilon_1, \ldots, \varepsilon_r$ it satisfies $k+1 = \lambda_1 \geq \cdots \geq \lambda_r \geq 0$. Expanding the determinant (3.10) along the first column, we get $[\nabla_q(\lambda)]$ written as a linear combination of $[\nabla_q((k+1)\omega_1)], [\nabla_q((k+2)\omega_1)], \ldots, [\nabla_q((k+r)\omega_1)]$, showing that I_l is included in the first ideal.

Next we show that the first ideal is contained in the second ideal. Consider the weight $k\omega_1 + \omega_i = (k+1)\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i$, for which (3.10) is

$$\chi_{k\omega_1+\omega_i} = \begin{vmatrix} H_{k+1} & H_0 & & & & & & \\ H_{k+2} & H_1 & H_0 & & & & & \\ \vdots & & & \ddots & & & 0 & \\ H_{k+i-1} & & & & H_0 & & & \\ H_{k+i} & H_{i-1} & H_{i-2} & \ldots & H_1 & & & \\ H_{k+i+1} & & & & H_2 & H_0 & & \\ \vdots & & & & \vdots & & \ddots & \\ H_{k+r} & H_{r-1} & H_{r-2} & \ldots & H_{r-i+1} & & & H_0 \end{vmatrix}$$

showing that $\chi_{k\omega_1+\omega_i} + (-1)^i H_{k+i} \in \langle H_{k+1}, \ldots, H_{k+i-1} \rangle$. By induction all $\chi_{(k+i)\omega_1} = H_{k+i} \in \left\langle \chi_{(k+1)\omega_1}, \chi_{k\omega_1+\omega_2}, \ldots, \chi_{k\omega_1+\omega_i} \right\rangle$.

Lastly we notice that all the weights $k\omega_1 + \omega_i$ belong to (3.2) so the last ideal is contained in I_l. $\square$

Type C_r

Assume that $\mathfrak{g}$ has type $C_r, r \geq 2$. The weights of the standard representation are $\pm\varepsilon_i, i = 1, \ldots, r$, where $\varepsilon_1 = \omega_1$ and $\varepsilon_i = \omega_i - \omega_{i-1}$ for $2 \leq i \leq r$. The Weyl group is $W = S_r \ltimes \mathbb{Z}_2^r$, where S_r permutes the indices and each $\mathbb{Z}_2$ changes sign on one ε_i and leaves the others invariant. Write $q_i = e^{\varepsilon_i}$.

Given a weight $\lambda = \sum_{i=1}^r m_i\omega_i \in P^+$ we again write it $\lambda = \sum_{i=1}^r \lambda_i\varepsilon_i$ with $\lambda_i = m_i + \cdots + m_r, i = 1, \ldots, r$. Then again $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$ and

$\rho_i = r+1-i$. We calculate

$$
\begin{aligned}
A_\lambda &= \sum_{w \in S_r \ltimes \mathbb{Z}_2^r} \det(w) e^{\sum_{i=1}^r \lambda_i w(\varepsilon_i)} \\
&= \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) \prod_{i=1}^r (q_{\sigma(i)}^{\lambda_i} - q_{\sigma(i)}^{-\lambda_i}) = \left| q_i^{\lambda_j} - q_i^{-\lambda_j} \right|_{i,j=1}^r ,
\end{aligned}
$$

giving a formula for the irreducible character of highest weight λ:

$$
\chi_\lambda = \frac{\left| q_i^{\lambda_j+r+1-j} - q_i^{-(\lambda_j+r+1-j)} \right|}{\left| q_i^{r+1-j} - q_i^{-(r+1-j)} \right|}.
$$

A version of the Jacobi-Trudi identity, [FH91, Proposition 24.22], calculates the character in terms of complete symmetric polynomials $J_m = H_m(q_1, \dots, q_r, q_1^{-1}, \dots, q_r^{-1})$ in the variables q_i and their inverses q_i^{-1}. Again $J_0 = 1$ and $J_m = 0$ for $m < 0$.

Lemma 3.4. *Let $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$ and consider the character ring $\mathbb{Z}[P]^W$ for $\mathfrak{g}$ of type C_r. Then the irreducible character of highest weight $\lambda = \sum_{i=1}^r \lambda_i \varepsilon_i$ equals*

$$
\chi_\lambda = \begin{vmatrix} J_{\lambda_1} & \dots & J_{\lambda_r - r+1} \\ J_{\lambda_1+1} + J_{\lambda_1-1} & \dots & J_{\lambda_r-r+2} + J_{\lambda_r - r} \\ \vdots & & \vdots \\ J_{\lambda_1-1+r} + J_{\lambda_1-r+1} & \dots & J_{\lambda_r} + J_{\lambda_r-2r+2} \end{vmatrix}. \tag{3.11}
$$

Again the formula applied to $\lambda = m\omega_1 = m\varepsilon_1, m \geq 0$, shows us $\chi_{m\omega_1} = J_m$.

The following result is [BMRS92, (2.9)], and the proof is from [BR06, 3.2].

Proposition 3.5. *Let $\mathfrak{g}$ have type C_r and let l be a positive integer. If l is even, we assume that $k = l/2 - (r+1) \geq 0$. The fusion ideal I_l for the fusion ring $F_q(\mathfrak{g}, l)$ has the following sets of r generators*

$$
\begin{aligned}
I_l = &\langle [\nabla_q((k+1)\omega_1)], [\nabla_q((k+2)\omega_1)] + [\nabla_q(k\omega_1)], \dots, \\
&[\nabla_q((k+r)\omega_1)] + [\nabla_q((k+2-r)\omega_1)] \rangle \\
= &\langle [\nabla_q((k+1)\omega_1)], [\nabla_q(k\omega_1 + \omega_2)], \dots, [\nabla_q(k\omega_1 + \omega_r)] \rangle .
\end{aligned}
$$

If l is odd, we assume $k = l - 2r \geq 0$. The fusion ideal I_l has the following set of generators

$$
I_l = \langle [\nabla_q(\lambda)] \mid \lambda_1 + \lambda_2 = k+1 \rangle .
$$

Proof. When l is even, the fusion ideal is generated by the classes $[T_q(\lambda)] = [\nabla_q(\lambda)]$ with $\lambda = \sum_{i=1}^r m_i\omega_i \in P^+$ ranging over the minimal weights in (3.3) of level $\sum_{i=1} m_i = k+1$. Written in the basis $\varepsilon_1, \dots, \varepsilon_r$ such a weight satisfies $k+1 = \lambda_1 \geq \cdots \geq \lambda_r \geq 0$, and an expansion of the determinantal expression in (3.11) along the first column gives us $[\nabla_q(\lambda)]$ written as a linear combination of $[\nabla_q((k+1)\omega_1)], [\nabla_q((k+2)\omega_1)] + [\nabla_q(k\omega_1)], \dots, [\nabla_q((k+r)\omega_1)] + [\nabla_q((k+2-r)\omega_1)]$, i.e., I_l is included in the first ideal.

Now consider the weight $k\omega_1 + \omega_i = (k+1)\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i$ for which (3.11) looks like

$$\chi_{k\omega_1+\omega_i} = \begin{vmatrix} J_{k+1} & J_0 & & & & \\ J_{k+2}+J_k & J_1 & & & & \\ \vdots & & \ddots & J_0 & & \text{\Large 0} \\ J_{k+i}+J_{k+2-i} & & \dots & J_1 & & \\ J_{k+i+1}+J_{k+1-i} & & & J_2 & J_0 & \\ \vdots & & & \vdots & & \ddots \\ J_{k+r}+J_{k+2-r} & & \dots & J_{r-i+1} & & & J_0 \end{vmatrix},$$

so $\chi_{k\omega_1+\omega_i} + (-1)^i(J_{k+i} + J_{k+2-i}) \in \langle J_{k+1}, J_{k+2} + J_k, \dots, J_{k+i-1} + J_{k+3-i}\rangle$. By induction $J_{k+i} + J_{k+2-i} \in \left\langle \chi_{(k+1)\omega_1}, \chi_{k\omega_1+\omega_2}, \dots, \chi_{k\omega_1+\omega_i}\right\rangle$, showing that the first ideal is contained in the second. Finally all weights $k\omega_1 + \omega_i$ has weight $k+1$ so belong to (3.3).

When l is odd, a $\lambda = \sum_{i=1}^r m_i\omega_i$ is an element of (3.4) if and only if $m_1 + \sum_{i=2}^r 2m_i = k+1$. Written in the basis $\varepsilon_1, \dots, \varepsilon_r$ this means $\lambda_1 + \lambda_2 = k+1$. □

We note that for odd l the determinant method does not give us a canonical set of generators of the fusion ideal. The culprit is that the level of λ is $\langle \lambda, \alpha_0^\vee \rangle = \lambda_1 + \lambda_2$, which means that the elements in a single column of (3.11) will not belong to I_l. We refer to [BR06, Section 4] for a method of expanding the determinant down the first two columns simultaniously. The result is a much smaller set of generators compared to the set in the proposition, but one whose size is not level-independent.

Type B_r

Assume that $\mathfrak{g}$ has type $B_r, r \geq 3$. The weights of the standard representation are $\pm\varepsilon_i, i = 1, \dots, r$, and 0, where $\varepsilon_1 = \omega_1$, $\varepsilon_i = \omega_i - \omega_{i-1}, i = 2, \dots, r-1$ and $\varepsilon_r = 2\omega_r - \omega_{r-1}$. The Weyl group is again $W = S_r \ltimes \mathbb{Z}_2^r$ acting on the non-zero weights $\pm\varepsilon_i$ by permutation and sign change. Write $q_i = e^{\varepsilon_i}$.

Given a weight $\lambda = \sum_{i=1}^{r} m_i\omega_i \in P^+$ write it as $\sum_{i=1}^{r} \lambda_i\varepsilon_i$ with $\lambda_i = m_i + \cdots + m_{r-1} + m_r/2, 1 \leq i \leq r-1$, and $\lambda_r = m_r/2$, i.e., the λ_i are all either integers or half-integers. In any case $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$ and $\rho_i = r + 1/2 - i$. Since the Weyl group only permutes the non-zero weights of the standard representation, we get the same calculation as in the C_r case:

$$A_\lambda = \sum_{\sigma\in S_r} \operatorname{sgn}(\sigma) \prod_{i=1}^{r} (q_{\sigma(i)}^{\lambda_i} - q_{\sigma(i)}^{-\lambda_i}) = \left| q_i^{\lambda_j} - q_i^{-\lambda_j} \right|_{i,j=1}^{r},$$

Therefore we have a formula for the irreducible character of highest weight λ:

$$\chi_\lambda = \frac{\left| z_i^{\lambda_j+r-j+1/2} - z_i^{-(\lambda_j+r-j+1/2)} \right|}{\left| z_i^{r-j+1/2} - z_i^{-(r-j+1/2)} \right|}.$$

Then [FH91, Proposition 24.33] expresses χ_λ as a determinant in the complete symmetric polynomials $H_m(q_1, \ldots, q_r, 1, q_1^{-1}, \ldots, q_r^{-1})$ in the variables q_i, their inverses q_i^{-1} and 1. Use the following notation:

$$\begin{aligned} K_n &= H_n(q_1, \ldots, q_r, 1, q_1^{-1}, \ldots, q_r^{-1}), n \leq 1 \\ K_m &= H_m(q_1, \ldots, q_r, 1, q_1^{-1}, \ldots, q_r^{-1}) \\ &\quad - H_{m-2}(q_1, \ldots, q_r, 1, q_1^{-1}, \ldots, q_r^{-1}), m \geq 2. \end{aligned}$$

Lemma 3.6. *Let $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$ and consider the character ring $\mathbb{Z}[P]^W$ for $\mathfrak{g}$ of type B_r. Then the irreducible character of highest weight $\lambda = \sum_{i=1}^{r} \lambda_i\varepsilon_i$ equals*

$$\chi_\lambda = \begin{vmatrix} K_{\lambda_1} & \ldots & K_{\lambda_r-r+1} \\ K_{\lambda_1+1} + K_{\lambda_1-1} & \ldots & K_{\lambda_r-r+2} + K_{\lambda_r-r} \\ \vdots & & \vdots \\ K_{\lambda_1-1+r} + K_{\lambda_q-r+1} & \ldots & K_{\lambda_r} + K_{\lambda_r-2r+2} \end{vmatrix} \tag{3.12}$$

The formula tells us that for $\lambda = m\omega_1 = m\varepsilon_1, m \geq 0$, we have $\chi_{m\omega_1} = K_m$.

The following is [AS14, Theorem 6.1].

Proposition 3.7. *Let $\mathfrak{g}$ have type B_r and let l be a positive integer. If l is even, we assume $k = l/2 - (2r-1) \geq 0$. The fusion ideal I_l has the following set of generators*

$$I_l = \langle [\nabla_q(\lambda)], [T_q(\mu)] \mid \lambda_1 + \lambda_2 = k+1, 2\mu_1 = 2\mu_2 = k+2, \mu_r = 0 \rangle .$$

If l is odd, we assume $k = l - 2r \geq 0$. Then I_l has the following set of r generators

$$\begin{aligned}I_l =&\langle[\nabla_q(\frac{k+1}{2}\omega_1)], [\nabla_q((\frac{k-1}{2}+i)\omega_1)] + \nabla_q[((\frac{k-1}{2}-i)\omega_1)] \mid i = 2, \dots, r\rangle \\ =&\langle[\nabla_q(\frac{k+1}{2}\omega_1)], [\nabla_q(\frac{k-1}{2}\omega_1+\omega_2)], \dots, [\nabla_q(\frac{k-1}{2}\omega_1+\omega_{r-1})], \\ &[\nabla_q(\frac{k-1}{2}\omega_1+2\omega_r)]\rangle.\end{aligned}$$

Proof. When l is even, a minimal weight from (3.5) is either a $\lambda = \sum_{i=1}^{r} m_i\omega_i \in P^+$ of level $m_1 + \sum_{i=2}^{r-1} 2m_2 + m_r = k+1$ or a $\mu = \sum_{i=2}^{r-1} n_i\omega_i \in P^+$ of level $\sum_{i=2}^{r-1} 2n_i = k+2$. Written in the basis $\varepsilon_1, \dots, \varepsilon_r$ this means $\lambda_1 + \lambda_2 = k+1$ or $2\mu_1 = 2\mu_2 = k+2, \mu_r = 0$.

When l is odd, we know that the fusion ideal is generated by $T_q(\lambda) = \nabla_q(\lambda)$ with $\lambda = \sum_{i=1}^{r} \lambda_i\varepsilon_i$ the level $k+1$ weights in (3.6), i.e., $2\lambda_1 = k+1$. An expansion of the determinantal formula (3.12) for the character of these weights along the first column show that χ_λ is in the ideal generated by $K_{\frac{k+1}{2}}$ and the elements $K_{\frac{k-1}{2}+i} - K_{\frac{k-1}{2}-i}, i = 2, \dots, r$, i.e., I_l is included in the first ideal.

Now consider the weights $\lambda^{(1)} = \frac{k+1}{2}\omega_1, \lambda^{(i)} = \frac{k-1}{2}\omega_1 + \omega_i, i = 2, \dots, r-1, \lambda^{(r)} = \frac{k-1}{2}\omega_1 + 2\omega_r$ for which $\lambda^{(i)} = \frac{k+1}{2}\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_i$. The character of $\lambda^{(i)}$ is given by (3.12)

$$\chi_{\lambda^{(i)}} = \begin{vmatrix} K_{\frac{k+1}{2}} & K_0 & & & & \\ K_{\frac{k-1}{2}+2} + K_{\frac{k-1}{2}-2} & K_1 & & & & \\ \vdots & & \ddots & K_0 & & 0 \\ K_{\frac{k-1}{2}+i} + K_{\frac{k-1}{2}-i} & & \dots & K_1 & & \\ K_{\frac{k-1}{2}+i+1} + K_{\frac{k-1}{2}-i-1} & & & K_2 & K_0 & \\ \vdots & & & \vdots & & \ddots \\ K_{\frac{k-1}{2}+r} + K_{\frac{k-1}{2}-r} & & \dots & K_{r-i+1} & & K_0 \end{vmatrix}$$

showing that $\chi_{\lambda^{(j)}} + (-1)^i(K_{\frac{k-1}{2}+i} + K_{\frac{k-1}{2}-i})$ belongs to the ideal generated by $K_{\frac{k+1}{2}}$ and the $K_{\frac{k-1}{2}+i} - K_{\frac{k-1}{2}-i}, i = 2, \dots, i-1$. By induction $K_{\frac{k-1}{2}+i} + K_{\frac{k-1}{2}-i} \in \langle\chi_{\lambda^{(1)}}, \dots, \chi_{\lambda^{(i)}}\rangle$ and the first ideal is contained in the second. Finally all weights $\lambda^{(j)}$ have level $k+1$, so the second ideal is contained in I_l. □

Note that for l even the level of a weight λ is $\langle\lambda, \beta_0^\vee\rangle = \lambda_1 + \lambda_2$ so again the determinant method does not give a canonical set of generators.

This time however we may use the identification of $F_q(\mathfrak{g}, l)$ with the WZW fusion ring $F(\mathfrak{g}, l/2 - (2r-1))$ for which we have a canonical set of $2r-1$ generators given in Table 2.4:

$$\begin{aligned} I_l =&\langle[\nabla_q((k-1)\omega_1 + \omega_2)], \ldots, [\nabla_q((k-1)\omega_1 + \omega_{r-1})], \\ &[\nabla_q((k-1)\omega_1 + 2\omega_r)], [\nabla_q(k\omega_1 + \omega_r)], \ldots, [\nabla_q(k\omega_1 + \omega_3)], \\ &[\nabla_q(k\omega_1 + \omega_2)] + [\nabla_q(k\omega_1)], [\nabla_q((k+1)\omega_1)]\rangle \end{aligned}$$

As the root system B_r is dual to C_r you might expect the associated fusion rings to be related. More precisely, let $\mathfrak{g}'$ have type C_r and let l' be a positive integer. When the parities of l and l' are different the affine Weyl groups W_l and $W'_{l'}$ have the same type. Assume that the fusion rings $F_q(\mathfrak{g}, l)$ and $F_q(\mathfrak{g}', l')$ isomorphic. Then in specific we have a bijection of the underlying index sets I and I'.

Assume first that l is odd, and let $l' = 2(l - r - 1)$. When we look at the fundamental alcoves A_l for $\mathfrak{g}$ and $A_{l'}$ for $\mathfrak{g}'$, we see that the index sets are given by

$$\begin{aligned} I &= \{(m_1, \ldots, m_r) \in \mathbb{N}^r \mid \sum_{i=1}^{r-1} 2m_i + m_r \leq k\} \\ I' &= \{(m_1, \ldots, m_r) \in \mathbb{N}^r \mid \sum_{i=1}^{r} m_i \leq k\} \end{aligned}$$

where $k = l - 2r = l'/2 - (r+1)$ and these sets are clearly of different size.

If we assume that 2 divides l exactly once we may similarly let $\mathfrak{g}'$ have type C_r with $l' = \frac{l}{2} + 1$, and the index sets are now given by

$$\begin{aligned} I &= \{(m_1, \ldots, m_r) \in \mathbb{N}^r \mid m_1 + \sum_{i=2}^{r-1} 2m_i + m_r \leq k\} \\ I' &= \{(m_1, \ldots, m_r) \in \mathbb{N}^r \mid m_1 + \sum_{i=2}^{r} 2m_i \leq k\} \end{aligned}$$

where $k = l/2 + (2r - 1) = l' - 2r$.

Though we do not have an isomorphism of the fusion rings they do share some properties. We note from Propositions 3.5 and 3.7, that when l is even and l' is odd, the fusion ideals I_l and $I_{l'}$ can be generated by r elements but, for l odd and l' even we have not been able to prove this.

Type D_r

Assume that $\mathfrak{g}$ has type $D_r, r \geq 4$. The weights of the standard representation are $\pm\varepsilon_i, i = 1, \dots, r$, where $\varepsilon_1 = \omega_1, \varepsilon_i = \omega_i - \omega_{i-1}, i = 2, \dots, r - 2, \varepsilon_{r-1} = \omega_n + \omega_{n-1} - \omega_{n-2}$ and $\varepsilon_r = \omega_n - \omega_{n-1}$. The Weyl group is $W = S_r \ltimes \mathbb{Z}_2^{r-1}$, where S_r permutes the indices and $\mathbb{Z}_2^{r-1}$ acts by changing the sign on an even number of the weights $\pm\varepsilon_i$ and leaving the rest invariant. Write $q_i = e^{\varepsilon_i}$.

Given a weight $\lambda = \sum_{i=1}^r m_i\omega_i \in P^+$, write it as $\sum_{i=1}^r \lambda_i\varepsilon_i$ with $\lambda_i = m_i + \cdots + m_{r-2} + (m_{r-1} + m_r)/2, 1 \leq i \leq r-2$, $\lambda_{r-2} = (m_{r-1} + m_r)/2$ and $\lambda_r = (-m_{r-1} + m_r)/2$. Again, all λ_i are either integers or half-integers. Now $\lambda_1 \geq \cdots \geq |\lambda_r|$ and $\rho_i = r - i$. We have

$$2A_\lambda = \left|q_i^{\lambda_j} + q_i^{-\lambda_j}\right|_{i,j=1}^r + \left|q_i^{\lambda_j} - q_i^{-\lambda_j}\right|_{i,j=1}^r .$$

If any $\lambda_j = 0$ then the second determinant vanishes. In particular $A_\rho = \frac{1}{2}\left|q_i^{r-j} + q_i^{-(r-j)}\right|_{i,j=1}^r$. Therefore we have a formula for the irreducible character of highest weight $\lambda \in P^+$:

$$\chi_\lambda = \frac{\left|z_i^{\lambda_j+r-j} + z_i^{-(\lambda_j+r-j)}\right| + \left|z_i^{\lambda_j+r-j} - z_i^{-(\lambda_j+r-j)}\right|}{\left|z_i^{r-j} - z_i^{-(r-j)}\right|}.$$

Then [FH91, Proposition 24.44] expresses χ_λ as a determinant in the complete symmetric polynomials $H_m(q_1, \dots, q_r, q_1^{-1}, \dots, q_r^{-1})$ in the variables q_i and their inverses q_i^{-1}. Use the following notation:

$$L_n = H_n(q_1, \dots, q_r, q_1^{-1}, \dots, q_r^{-1}), n \leq 1$$
$$L_m = H_m(q_1, \dots, q_r, q_1^{-1}, \dots, q_r^{-1}) - H_{m-2}(q_1, \dots, q_r, q_1^{-1}, \dots, q_r^{-1}), m \geq 2.$$

Lemma 3.8. *Let $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$ and consider the character ring $\mathbb{Z}[P]^W$ for $\mathfrak{g}$ of type D_r. Then the irreducible character of highest weight $\lambda = \sum_{i=1}^r \lambda_i\varepsilon_i$ equals*

$$\chi_\lambda = \begin{vmatrix} L_{\lambda_1} & \dots & L_{\lambda_r-r+1} \\ L_{\lambda_1+1} + L_{\lambda_1-1} & \dots & L_{\lambda_r-r+2} + L_{\lambda_r-r} \\ \vdots & & \vdots \\ L_{\lambda_1-1+r} + L_{\lambda_q-r+1} & \dots & L_{\lambda_r} + L_{\lambda_r-2r+2} \end{vmatrix} \tag{3.13}$$

The formula tells us that for $\lambda = m\omega_1 = m\varepsilon_1, m \geq 0$ we have $\chi_{m\omega_1} = L_m$.

Proposition 3.9. *Let $\mathfrak{g}$ have type D_r and let l be a positive integer with $k = l - (2r-3) \geq 0$. If k is even the fusion ideal I_l for the fusion ring $F_q(\mathfrak{g}, l)$ has the following set of generators*

$$I_l = \langle [\nabla_q(\lambda)], [T_q(\mu)] \mid \lambda_1 + \lambda_2 = k+1, 2\mu_1 = 2\mu_2 = k+2, \mu_{r-1} = \mu_r = 0 \rangle .$$

If k is odd the fusion ideal I_l has the following set of generators

$$I_l = \langle [\nabla_q(\lambda)] \mid \lambda_1 + \lambda_2 = k+1 \rangle .$$

Proof. A minimal weight from (3.7) is either a $\lambda = \sum_{i=1}^{r} m_i\omega_i \in P^+$ of level $m_1 + \sum_{i=2}^{r-2} 2m_2 + m_{r-1} + m_r = k+1$ or a $\mu = \sum_{i=2}^{r-2} n_i\omega_i \in P^+$ of level $\sum_{i=2}^{r-2} 2n_i = k+2$. Written in the basis $\varepsilon_1, \dots, \varepsilon_r$ this means $\lambda_1 + \lambda_2 = k+1$ or $2\mu_1 = 2\mu_2 = k+2, \mu_{r-1} = \mu_r = 0$. If k is odd there are no weights $\mu \in P^+$ with $2\mu_1 = k+2$. □

Again we do not get a canonical set of generators of the fusion ideal by using the determinant method. We may however use the identification of $F_q(\mathfrak{g}, l)$ with the WZW fusion ring $F(\mathfrak{g}, l-(2r-3))$ to show that the fusion ideal is generated by $2r-1$ generators:

$$\begin{aligned} I_l = \langle &[\nabla_q((k-1)\omega_1 + \omega_2)], \dots, [\nabla_q((k-1)\omega_1 + \omega_{r-2})], \\ &[\nabla_q((k-1)\omega_1 + \omega_{r-1} + \omega_r)], [\nabla_q(k\omega_1 + \omega_r)], [\nabla_q(k\omega_1 + \omega_{r-1})], \\ &[\nabla_q(k\omega_1 + \omega_{r-1} + \omega_r)], [\nabla_q(k\omega_1 + \omega_{r-2})], \dots, [\nabla_q(k\omega_1 + \omega_3)], \\ &[\nabla_q(k\omega_1 + \omega_2)] + [\nabla_q(k\omega_1)], [\nabla_q((k+1)\omega_1)] \rangle \end{aligned}$$

3.3 Complete intersections and fusion potentials

For a Lie algebra of type A_r and l arbitrary, of type B_r and l odd or of type C_r and l even we have seen that the fusion ideal $I_l \subseteq R = \mathbb{Z}[X_1, \dots, X_r]$ is a complete intersection, and we have an explicit set of r generators $p_1, \dots, p_r$. Furthermore, for type A_r and type C_r these generators integrates to an explicit potential function $V(X_1, \dots, X_r)$, i.e., $p_i = \frac{\partial}{\partial X_i} V$.

If we do a change of basis to the complex numbers this is always the case. Let $I_l^{\mathbb{C}} = I_l \otimes_{\mathbb{Z}} \mathbb{C} \subseteq \mathbb{C}[X_1, \dots, X_r]$ and $F_q^{\mathbb{C}} = \mathbb{C}[X_1, \dots, X_r]/I_l^{\mathbb{C}}$. The algebraic variety $V(I_l^{\mathbb{C}}) \subseteq \mathbb{C}^r$ consists of a finite number of points, each corresponding to a basis element of $F_q^{\mathbb{C}}$, i.e., an element of A_l. As points in the variety corresponds to maximal ideals in the coordinate ring, the ideal $I_l^{\mathbb{C}}$ is locally a complete intersection. By [Kun85, Theorem 5.21] it is globally a complete intersection, i.e., $I_l^{\mathbb{C}}$ can be generated by r elements in the coordinate ring $\mathbb{C}[X_1, \dots, X_r]$.

We refer to [Fuc94, 6.2] for an exposition on how to do a change of variable $X_1 \mapsto \widetilde{X}_1$ and construct an abstract polynomial $V(\widetilde{X}_1, X_2, \dots, X_r) \in \mathbb{C}[\widetilde{X}_1, X_2, \dots, X_r]$, for which $F_q^{\mathbb{C}} \simeq \mathbb{C}[\widetilde{X}_1, X_2, \dots, X_r]/dV$, i.e., the complexified fusion ring $F_q \otimes_{\mathbb{Z}} \mathbb{C}$ can be presented as the quotient of a complex polynomial ring by an ideal generated by the partial derivatives of a potential function. It should be noted, that there is no reason to expect the coefficients of the potential function V to be even rational integers, so there is poor chance it will give a similar presentation of F_q as a quotient $\mathbb{Z}[\widetilde{X}_1, X_2, \dots, X_r]/dV$. Also, the polynomial V is far from unique, so it should not be expected to possess any independent meaning.

In the case where we have explicit generators $p_1', \dots, p_r' \in \mathbb{Z}[X_1, \dots, X_r]$ of the fusion ideal I_l, we may look for an invertible $r \times r$-matrix S over $\mathbb{Z}[X_1, \dots, X_r]$, such that the alternative generating set given by $(p_1 \dots p_r) = (p_1' \dots p_r')S$, satisfies $\frac{\partial}{\partial X_j} p_i$ is symmetric in i and j. As mentioned this would imply that the 1-form $\sum_{i=1}^r p_i dX_i$ integrates to a potential function V. However there is currently no method to produce such a matrix, and in the cases where it has been done, it seems that the starting point has been a concrete potential function, obtained by educated guessing, with a subsequent verification of its properties, cf. the discussion in the last section of [BR06].

3.4 Non-constructive methods

So far we have used combinatorial tools to find concrete generating sets of the fusion ideal. If we are only interested in setting an upper bound on the minimal number of elements necessary to generate the ideal, then several algebraic tools may help us.

Rank 2

We prove in this subsection that the fusion ideal for quantum groups associated to a Lie algebra of rank 2 can always be generated by 2 elements. The proof utilizes commutative algebra and works with the fusion ring from an abstract point of view. Therefore there is little chance to say anything explicit about actual generators in specific examples.

We work with a commutative presentation of the fusion ring F as a quotient of the polynomial ring $R = \mathbb{Z}[X, Y]$ in 2 variables by an ideal I. We know F is a Gorenstein ring, meaning that we have an isomorphism of R/I-modules $\mathrm{Hom}_{\mathbb{Z}}(R/I,\mathbb{Z}) \simeq R/I$, which lifts to an isomorphism of R-modules.

We first prove a technical lemma.

Lemma 3.10. *Let S be a noetherian ring, let T be a ring that is a finitely generated S-module and let $b \in T$. Consider T as an $S[X]$-module by the mapping $X \mapsto b$. Then there is an isomorphism of $S[X]$-modules*

$$\mathrm{Ext}^i_S(T,S) \simeq \mathrm{Ext}^{i+1}_{S[X]}(T,S[X])$$

for all $i \geq 0$.

Proof. Set $T[X] = S[X] \otimes_S T$ with trivial $S[X]$-action. Multiplication on $T[X]$ by $X-b$ fits in to a short exact sequence

$$0 \to T[X] \overset{X-b}{\to} T[X] \to T \to 0.$$

Apply the left exact functor $\mathrm{Hom}_{S[X]}(-,S[X])$

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{Ext}^i_{S[X]}(T[X],S[X]) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Ext}^i_{S[X]}(T[X],S[X]) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Ext}^{i+1}_{S[X]}(T,S[X]) \arrow[r] & \dots \\
0 \arrow[r] & S[X] \otimes_S \mathrm{Ext}^i_S(T,S) \arrow[r] & S[X] \otimes_S \mathrm{Ext}^i_S(T,S) \arrow[r] & \mathrm{Cok} \arrow[r] & 0
\end{tikzcd}$$

where the vertical isomorphisms come from the fact that $S \to S[X]$ is flat. The lower homomorphism makes the diagram commutative, so by naturality of the isomorphisms it is multiplication by $X-b$, identifying the cokernel with $\mathrm{Ext}^i_S(T,S)$. Diagram chasing gives us a map

$$\mathrm{Ext}^i_S(T,S) \to \mathrm{Ext}^{i+1}_{S[X]}(T,S[X])$$

which is an isomorphism by the Five Lemma. □

Proposition 3.11. *Let $I \subseteq R = \mathbb{Z}[X,Y]$ be an ideal such that we have an isomorphism of R-modules*

$$\mathrm{Hom}_{\mathbb{Z}}(R/I,\mathbb{Z}) \simeq R/I. \tag{3.14}$$

Then I is generated by an R-regular sequence of length 2.

Proof. Necessarily from the duality (3.14) R/I is a finitely generated $\mathbb{Z}$-module, so Lemma 3.10 with $S = \mathbb{Z}$, $T = R/I$ and $x = X + I \in R/I$ gives us

$$\mathrm{Ext}^i_{\mathbb{Z}}(R/I,\mathbb{Z}) \simeq \mathrm{Ext}^{i+1}_{\mathbb{Z}[X]}(R/I,\mathbb{Z}[X]).$$

Now R/I is still finitely generated as an $S' = \mathbb{Z}[X]$-module, so the lemma with $y = Y + I \in R/I$ gives us

$$\operatorname{Ext}^{i+1}_{\mathbb{Z}[X]}(R/I, \mathbb{Z}[X]) \simeq \operatorname{Ext}^{i+2}_{\mathbb{Z}[X,Y]}(R/I, \mathbb{Z}[X,Y]).$$

With $i = 0$ and the assumption, we get

$$\operatorname{Ext}^2_R(R/I, R) \simeq \operatorname{Hom}_{\mathbb{Z}}(R/I, \mathbb{Z}) \simeq R/I.$$

Pick a unit $e \in R/I$. With the identification $\operatorname{Ext}^2_R(R/I, R) \simeq \operatorname{Ext}^1_R(I, R)$ this element corresponds to a nonsplit short exact sequence

$$0 \to R \to M \to I \to 0. \tag{3.15}$$

The goal is to show that $M \simeq R^2$, for then the image of two generators under the surjection in (3.15) will generate I.

We prove first that $\operatorname{Ext}^i_R(M, R) = 0$ for all $i \geq 1$. Consider the long exact sequence associated to (3.15):

$$\operatorname{Hom}_R(R, R) \xrightarrow{p} \operatorname{Ext}^1_R(I, R) \to \operatorname{Ext}^1_R(M, R) \to \operatorname{Ext}^1_R(R, R) = 0.$$

By construction we have $p(\mathrm{id}) = e \in \operatorname{Ext}^1_R(I, R) \simeq R/I$. Choose an $f \in R$ with $e^{-1} = f + I$. Since p is an R-homomorphism $p(f) = fe = (f+I)e = 1$, so p is surjective and $\operatorname{Ext}^1_R(M, R) = 0$. Let now $i \geq 2$. Since R/I is Gorenstein it is Cohen-Macaulay, and localizing at a prime ideal $\mathfrak{p} \subseteq R$ containing I gives us $\operatorname{proj\,dim}_R(R/I) = 2$ by the Auslander-Buchsbaum formula. Then $0 = \operatorname{Ext}^{i+1}_R(R/I, R) \simeq \operatorname{Ext}^i_R(I, R) \simeq \operatorname{Ext}^i_R(M, R)$.

Now induction on the length of a projective resolution on a given module N gives $\operatorname{Ext}^i_R(M, N) = 0$ for all $i \geq 1$, i.e., M is projective.

Then [Qui76, Theorem 4] says that all projective modules over $\mathbb{Z}[X,Y]$ are actually free, so $M \simeq R^k$. Choose any prime ideal $\mathfrak{p} \subseteq R$ not containing I. Localizing (3.15) at $\mathfrak{p}$ we get

$$0 \to R_{\mathfrak{p}} \to M_{\mathfrak{p}} \to R_{\mathfrak{p}} \to 0$$

showing that $k = 2$. Here we used that $I \cap (R \setminus \mathfrak{p}) \neq \emptyset$ so $I_{\mathfrak{p}}$ contains a unit. □

Note that the proof does not give us an explicit set of generators. Our method for taking the local data of a projective module to a global setting using Quillen's theorem does not give us any information on the structure of the resulting free module.

Rank 3 and higher

Many of the techniques in the above proof generalize to higher ranks. When we consider a general ideal $I \subseteq R = \mathbb{Z}[X_1, \dots, X_r]$ such that $\mathrm{Hom}_{\mathbb{Z}}(R/I, \mathbb{Z}) \simeq R/I$, Lemma (3.10) applied r times still gives us

$$R/I \simeq \mathrm{Hom}_{\mathbb{Z}}(R/I, \mathbb{Z}) \simeq \mathrm{Ext}^1_{\mathbb{Z}[X_1]}(R/I, \mathbb{Z}[X_1]) \simeq \\ \cdots \simeq \mathrm{Ext}^r_{\mathbb{Z}[X_1,\dots,X_r]}(R/I, \mathbb{Z}[X_1, \dots, X_r]).$$

Locally, the Auslander-Buchsbaum formula still gives us $\operatorname{proj\,dim}_R(R/I) = r$, so $\mathrm{Ext}^i_R(R/I, R) = 0$ for $i > r$. We also have our tool to give us global information from local data: By [Qui76, Theorem 4] any projective $\mathbb{Z}[X_1, \dots, X_r]$-module is free. The case $r = 2$ is special though, since only then will a unit in $\mathrm{Ext}^r_R(R/I, R) \simeq \mathrm{Ext}^{r-1}_R(I, R)$ give us a short exact sequence as in (3.15), which is a paramount part of the proof.

In [Ser63] it was proved that a quotient of a regular local ring of codimension 2 is a complete intersection ring if and only if it is Gorenstein. After more than 50 years this result has not been generalized to higher codimensions, suggesting that a generalization of Theorem 3.11 to higher ranks is not possible without further assumptions on the ideal.

For a Lie algebra of rank 3 we may instead use a result from [BE74], stating that for a local noetherian ring and an ideal such that the quotient is Gorenstein the minimal number of generators of the ideal must be odd. Applied to our problem, we may consider the fusion ring for $\mathfrak{g}$ of type C_3 for odd l or type B_3 for even l. If we were able to find an explicit generating set of the fusion ideal consisting of 4 elements, then we would know, that locally the minimal number of generators of the ideal was actually 3, i.e., the fusion ideal is locally a complete intersection. Note that this does not say anything about the ideal itself being globally generated by 3 explicit generators.

Chapter 4

Fusion rings for $\mathfrak{g}$ of rank 2

In this chapter we do a treatment of the fusion ring $F_q(\mathfrak{g}, l)$ for each simple complex Lie algebra of rank 2. In Section 3.4 of the previous chapter it was proven, that in this case the fusion ideal I_l can always be generated by 2 elements. For each value of l we propose a generating set of the fusion ideal I_l and show explicitly that it works. For type A_2 or C_2 we find a generating set of 2 elements as predicted by Proposition 3.11, but for type G_2 we can only find an explicit generating set of 3 elements.

We identify a weight $\lambda = a\omega_1 + b\omega_2 \in P$ with $(a, b) \in \mathbb{Z}^2$.

4.1 Type A_2

We have two simple roots α_1, α_2 and positive roots $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_0\}$, where $\alpha_0 = \alpha_1 + \alpha_2$ is the highest root. The level is $k = l - 3$, so we assume $l \geq 3$. The fundamental weights are $\omega_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$ and $\omega_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$, so the fundamental alcove is

$$A_l = \{(a, b) \in \mathbb{N}^2 \mid a + b \leq k\}.$$

The characters of the fundamental representations are

$$\begin{aligned} \operatorname{ch} \nabla_q(\omega_1) &= e^{(1,0)} + e^{(-1,1)} + e^{(0,-1)} \\ \operatorname{ch} \nabla_q(\omega_2) &= e^{(0,1)} + e^{(1,-1)} + e^{(-1,0)}. \end{aligned}$$

Proposition 4.1. *Let $\mathfrak{g}$ have type A_2, let $l \geq 3$ and set $k = l - 3$. The fusion ideal I_l for the fusion ring $F_q(\mathfrak{g}, l)$ has the following set of 2 generators*

$$I_l = \langle [\nabla_q(k+1, 0)], [\nabla_q(k, 1)] \rangle .$$

Proof. We have already seen in Proposition 3.3 that the two proposed generators work, but let us show it by calculations in an inductive manner. We know from (3.2) that the fusion ideal is generated by all elements that have level $k+1$. We define elements

$$g_i = [\nabla_q(k+1-i,i)], 0 \le i \le k+1,$$
$$r_i = [T_q(k+1-i,1+i)] = [\nabla_q(k+1-i,1+i)] + [\nabla_q(k-i,i)], 0 \le i \le k.$$

The g_i's are all the elements in $\mathrm{K}_0(\mathcal{T}_q)$ of level $k+1$, and the r_i's are some of the elements of level $k+2$. Let J be the ideal generated by g_0 and g_1. We will show that all $g_i, i = 2, \dots, k+1$, belong to J.

We use the formula (1.4) to calculate

$$[\nabla_q(\omega_1)]g_i = g_{i+1} + r_{i-1}, k \ge i > 0$$
$$[\nabla_q(\omega_2)]g_i = \begin{cases} r_0, & i = 0 \\ g_{i-1} + r_i, & k \ge i > 0 \end{cases}$$

Then we have $r_0 = [L_q(\omega_2)]g_0 \in J$. Assume that $g_i, g_{i+1}, r_i \in J$ for some $k-1 > i \ge 0$. Then also $r_{i+1} = [L_q(\omega_2)]g_{i+1} - g_i \in J$ and $g_{i+2} = [L_q(\omega_1)]g_{i+1} - r_i \in J$ and we are done. □

Let us quickly compare with the calculations used in the proof of Proposition 3.3. The element g_i belonging to the weight $(k+1-i)\omega_1 + i\omega_2 = (k+1)\varepsilon_1 + i\varepsilon_2$ has the following determinantal description, cf. (3.10):

$$g_i = \begin{vmatrix} [\nabla_q(k+1,0)] & [\nabla_q(i-1,0)] \\ [\nabla_q(k+2,0)] & [\nabla_q(i,0)] \end{vmatrix}.$$

Together with the formula for g_1

$$[\nabla_q(k,1)] = \begin{vmatrix} [\nabla_q(k+1,0)] & [\nabla_q(0,0)] \\ [\nabla_q(k+2,0)] & [\nabla_q(1,0)] \end{vmatrix}$$

we get

$$\begin{aligned} g_i &= [\nabla_q(k+1,0)][\nabla_q(i,0)] - [\nabla_q(k+2,0)][\nabla_q(i-1,0)] \\ &= g_0\left([\nabla_q(i,0)] - [\nabla_q(i-1,0)][\nabla_q(1,0)]\right) + g_1[\nabla_q(i-1,0)]. \end{aligned}$$

4.2 Type C_2

We have the two simple roots α_1, α_2 with α_1 short, such that the set of positive roots are $\{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\}$. Then $\alpha_0 = \alpha_1 + \alpha_2$ and

$\beta_0 = 2\alpha_1 + \alpha_2$ are the highest short resp. long roots. If l is odd we have level $k = l - 4$ and for l even $k = l/2 - 3$. The two fundamental weights are $\omega_1 = \beta_0/2$ and $\omega_2 = \alpha_0$, so the fundamental alcove is

$$A_l = \begin{cases} \{(a,b) \in \mathbb{N}^2 \mid a + 2b \le k\}, & l \text{ odd} \\ \{(a,b) \in \mathbb{N}^2 \mid a + b \le k\}, & l \text{ even} \end{cases}$$

The characters of the fundamental representations are

$$\begin{aligned} \operatorname{ch} \nabla_q(\omega_1) &= e^{(1,0)} + e^{(-1,1)} + e^{(1,-1)} + e^{(-1,0)}, \\ \operatorname{ch} \nabla_q(\omega_2) &= e^{(0,1)} + e^{(2,-1)} + e^{(0,0)} + e^{(-2,1)} + e^{(0,-1)}. \end{aligned}$$

Proposition 4.2. *Let $\mathfrak{g}$ have type C_2 and let l be a positive integer. If l is even, we assume $k = l/2 - 3 \ge 0$. The fusion ideal I_l for the fusion ring $F_q(\mathfrak{g}, l)$ has the following set of 2 generators*

$$I_l = \langle [\nabla_q(k+1, 0)], [\nabla_q(k, 1)] \rangle .$$

If l is odd, we assume $k = l - 4 \ge 1$. The fusion ideal I_l has the following 2 generators

$$I_l = \left\langle [\nabla_q(0, \frac{k+1}{2})], [\nabla_q(2, \frac{k-1}{2})] \right\rangle .$$

Proof. We have already seen, in Proposition 3.5, that the two proposed generators work when l is even, so consider l odd. Set $n = \frac{k+1}{2}$ and define elements

$$\begin{aligned} g_i &= [\nabla_q(2i, n-i)], 0 \le i \le n, \\ r_i &= [T_q(1+2i, n-i)] \\ &= [\nabla_q(1+2i, n-i)] + [\nabla_q(1+2i, n-1-i)], 0 \le i < n, \\ s_i &= [T_q(2i, n+1-i)] \\ &= [\nabla_q(2i, n+1-i)] + [\nabla_q(2i, n-1-i)], 0 \le i < n. \end{aligned}$$

Then the g_i's, the r_i's resp. the s_i's are elements in $\mathrm{K}_0(\mathcal{T}_q)$ of level $k+1$, $k+2$ resp. $k+3$. We let $J \subseteq \mathrm{K}_0(\mathcal{T}_q)$ be the ideal generated by the elements g_0 and g_1. We know from (3.4) that I_l is generated by all the $g_0, \dots, g_n$, so we just need to show that they all belong to J.

Calculate by (1.4)

$$[\nabla_q(\omega_2)] g_i = \begin{cases} g_1 + s_0, & i = 0 \\ g_{i-1} + g_i + g_{i+1} + s_i, & n > i > 0 \end{cases}$$

$$[\nabla_q(\omega_1)] g_i = \begin{cases} r_0, & i = 0 \\ r_i + r_{i-1}, & n > i > 0 \end{cases}$$

$$[\nabla_q(\omega_1)] r_i = 2g_i + 2g_{i+1} + s_i + s_{i+1}, n - 1 > i \ge 0.$$

Then we have $r_0 = [\nabla_q(\omega_1)]g_0 \in J$ and $s_0 = [\nabla_q(\omega_2)]g_0 - g_0 \in J$. Assume that $g_i, g_{i+1}, r_i, s_i \in J$ for some $n-1 > i \geq 0$. Then also $r_{i+1} = [\nabla_q(\omega_1)]g_{i+1} - r_i \in J$, $s_{i+1} = [\nabla_q(\omega_1)]r_i - 2g_i - 2g_{i+1} - s_i \in J$ and $g_{i+2} = [\nabla_q(\omega_2)]g_{i+1} - g_i - s_{i+1} - g_{i+1} \in J$, and we are done. $\square$

Another case

When we look at the situation above, we can imagine a case that does not occur for any l, namely if the affine alcove $A_l = \{(a,b) \in \mathbb{N}^2 \mid a+2b \leq k\}$ is determined by the longest short root α_0, and k is even. If we consider the ideal I of $\mathrm{K}_0(\mathcal{T}_q)$ generated by all elements $[\nabla_q(\mu)], \mu \in P^+$, of level $k+1$ together with $[\nabla_q(0, \frac{k}{2}+1)]+[\nabla_q(0, \frac{k}{2})]$, these are the elements corresponding to minimal weights in $P^+ \setminus A_l$. We show that this ideal can be generated by the 2 elements $[\nabla_q(1, \frac{k}{2})]$ and $[\nabla_q(0, \frac{k}{2}+1)] + [\nabla_q(0, \frac{k}{2})]$. Let $n = \frac{k}{2}$ and define elements

$$\begin{aligned}
g_i &= [\nabla_q(1+2i, n-i)], 0 \leq i \leq n, \\
r_i &= [\nabla_q(2i, n+1-i)] + [\nabla_q(2i, n-i)], 0 \leq i \leq n \\
s_i &= [\nabla_q(1+2i, n+1-i)] + [\nabla_q(1+2i, n-1-i)], 0 \leq i < n.
\end{aligned}$$

Let $J \subseteq \mathrm{K}_0(\mathcal{T}_q)$ be the ideal generated by g_0 and r_0. We calculate

$$\begin{aligned}
[\nabla_q(\omega_2)]g_i &= \begin{cases} g_0 + g_1 + s_0, & i = 0 \\ g_{i-1} + g_i + g_{i+1} + s_i, & n > i > 0 \end{cases} \\
[\nabla_q(\omega_1)]g_i &= r_i + r_{i+1}, n > i \geq 0 \\
[\nabla_q(\omega_1)]r_i &= \begin{cases} 2g_0 + s_0, & i = 0 \\ 2g_{i-1} + 2g_i + s_{i-1} + s_i, & n > i > 0 \end{cases}
\end{aligned}$$

Then we have $s_0 = [\nabla_q(\omega_1)]r_0 - 2g_0 \in J$, $r_1 = [\nabla_q(\omega_1)]g_0 - r_0 \in J$ and $g_1 = [\nabla_q(\omega_2)]g_0 - g_0 - s_0 \in J$. Assume that $g_i, g_{i+1}, r_{i+1}, s_i \in J$ for some $n-1 > i \geq 0$. Then also $s_{i+1} = [\nabla_q(\omega_1)]r_{i+1} - 2g_i - 2g_{i+1} - s_i \in J$, $r_{i+2} = [\nabla_q(\omega_1)]g_{i+1} - r_{i+1} \in J$ and $g_{i+2} = [\nabla_q(\omega_2)]g_{i+1} - g_{i+1} - g_i - s_{i+1} \in J$, and we are done.

4.3 Type G_2

We have the two simple roots α_1, α_2, where α_1 is short and the set of positive roots is $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}$. Then $\alpha_0 = 2\alpha_1 + \alpha_2$ and $\beta_0 = 3\alpha_1 + 2\alpha_2$ are the highest short resp. long roots. If

l is divisible by 3, we have level $k = l/3 - 4$, and otherwise $l = k - 6$. The fundamental weights are $\omega_1 = \alpha_0$ and $\omega_2 = \beta_0$, and the fundamental alcove is then

$$A_l = \begin{cases} \{(a,b) \in \mathbb{N}^2 \mid 2a + 3b \leq k\}, & 3 \nmid l \\ \{(a,b) \in \mathbb{N}^2 \mid a + 2b \leq k\}, & 3 \mid l \end{cases}$$

The characters of the fundamental representations are

$$\begin{aligned} \operatorname{ch} \nabla_q(\omega_1) &= e^{(1,0)} + e^{(-1,1)} + e^{(2,-1)} + e^{(0,0)} + e^{(-2,1)} + e^{(1,-1)} + e^{(-1,0)} \\ \operatorname{ch} \nabla_q(\omega_2) &= e^{(0,1)} + e^{(3,-1)} + e^{(1,0)} + e^{(-1,1)} + e^{(2,-1)} + e^{(-3,2)} + 2e^{(0,0)} \\ &\quad + e^{(3,-2)} + e^{(-2,1)} + e^{(1,-1)} + e^{(-1,0)} + e^{(-3,1)} + e^{(0,-1)} \end{aligned}$$

Proposition 4.3. *Let $\mathfrak{g}$ have type G_2 and let l be a positive integer. If l is divisible by 3, we assume $k = l/3 - 4 \geq 0$. The fusion ideal I_l for the fusion ring $F_q(\mathfrak{g}, l)$ has the following generators*

$$\begin{cases} [T_q(0,1)], [\nabla_q(1,0)], & k = 0, \\ [\nabla_q(0,1)], [\nabla_q(2,0)], & k = 1, \\ [T_q(0, \frac{k+2}{2})], [\nabla_q(1, \frac{k}{2})], [\nabla_q(3, \frac{k-2}{2})], & k \text{ even}, k \geq 2 \\ [\nabla_q(0, \frac{k+1}{2})], [\nabla_q(2, \frac{k-1}{2})], [\nabla_q(4, \frac{k-3}{2})], & k \text{ odd}, k \geq 3. \end{cases}$$

If l is not divisible by 3, we assume $k = l - 6 \geq 0$. The fusion ideal I_l has the following generators

$$\begin{cases} [\nabla_q(1,0)], [\nabla_q(0,1)], & k = 1, \\ [\nabla_q(0,1)], [T_q(2,0)], & k = 2, \\ [\nabla_q(1,1)], [T_q(3,0)], [\nabla_q(0,2)], & k = 4, \\ [\nabla_q(\frac{k+1}{2}, 0)], [\nabla_q(\frac{k-5}{2}, 2)], [T_q(\frac{k-1}{2}, 1)], & k \text{ odd}, k \geq 5, \\ [\nabla_q(\frac{k-2}{2}, 1)], [T_q(\frac{k+2}{2}, 0)], [T_q(\frac{k-4}{2}, 2)], & k \text{ even}, k \geq 8. \end{cases}$$

Proof. We refer to [AS14, Theorem 7.1] for the explicit calculations in the case where 3 divides l, so assume that it does not. The claim for $k \in \{1, 2, 4\}$ follows immediately, so let $k \geq 5$.

Consider first k odd. If $k \equiv 1 \mod 3$, then $k \equiv 1 \mod 6$, and we set $n = \frac{k-1}{6}$. Otherwise $k \equiv 5 \mod 6$, and we set $n = \frac{k+1}{6}$. In both cases, define elements

$$\begin{aligned}
g_i &= [\nabla_q(\tfrac{k+1}{2} - 3i, 2i)], 0 \le i \le n,\\
r_i &= [T_q(\tfrac{k-1}{2} - 3i, 1+2i)]\\
&= [\nabla_q(\tfrac{k-1}{2} - 3i, 1+2i)] + [\nabla_q(\tfrac{k-3}{2} - 3i, 1+2i)], 0 \le i < n,\\
s_i &= [T_q(\tfrac{k+3}{2} - 3i, 2i)]\\
&= [\nabla_q(\tfrac{k+3}{2} - 3i, 2i)] + [\nabla_q(\tfrac{k-1}{2} - 3i, 1+2i)], 0 \le i < n,\\
t_i &= [T_q(\tfrac{k+1}{2} - 3i, 1+2i)]\\
&= [\nabla_q(\tfrac{k+1}{2} - 3i, 1+2i)] + [\nabla_q(\tfrac{k-5}{2} - 3i, 1+2i)], 0 \le i < n,\\
u_i &= [T_q(\tfrac{k+5}{2} - 3i, 2i)] = [\nabla_q(\tfrac{k+5}{2} - 3i, 2i)] + [\nabla_q(\tfrac{k-3}{2} - 3i, 2i)], 0 \le i < n,
\end{aligned}$$

and if $k \equiv 1 \mod 3$, we also define $r_n = [\nabla_q(0, \frac{k+2}{3})]$ and $s_n = [T_q(2, \frac{k-1}{3})] = [\nabla_q(2, \frac{k-1}{3})] + [\nabla_q(0, \frac{k-1}{3})]$. We let J be the ideal in $\mathrm{K}_0(\mathcal{T}_q)$ generated by g_0, g_1 and r_0. The generators from (3.9) are all the g_i and r_i's, and we show that they all belong to J.

First we calculate

$$\begin{aligned}
[\nabla_q(\omega_1)]g_i &= \begin{cases} g_0 + r_0 + s_0, & i = 0\\ g_i + r_{i-1} + r_i + s_i, & n > i > 0\end{cases}\\
[\nabla_q(\omega_2)]g_i &= \begin{cases} g_1 + r_0 + s_0 + t_0, & i = 0\\ g_{i-1} + g_i + g_{i+1} + r_{i-1} + r_i + s_i + t_{i-1} + t_i, & n > i > 0\end{cases}\\
[\nabla_q(\omega_1)]r_i &= 2g_i + 2g_{i+1} + 2r_i + s_i + s_{i+1} + t_i, \quad n-1 > i \ge 0\\
[\nabla_q(\omega_2)]r_i &= \begin{cases} 2g_0 + 2g_1 + 2r_0 + r_1 + s_0 + u_0 + u_1, & i = 0\\ 2g_i + 2g_{i+1} + r_{i-1} + 2r_i + r_{i+1} + s_i + u_i + u_{i+1}, & n-1 > i > 0\end{cases}\\
[\nabla_q(\omega_1)]s_i &= \begin{cases} 2g_0 + r_0 + t_0 + u_0, & i = 0\\ 2g_i + r_{i-1} + r_i + s_i + t_{i-1} + t_i + u_i, & n > i > 0.\end{cases}
\end{aligned}$$

Note that in the case $k \equiv 1 \mod 3$ we still have

$$\begin{aligned}
[\nabla_q(\omega_1)]r_{n-1} &= 2g_{n-1} + 2g_n + 2r_{n-1} + s_{n-1} + s_n + t_{n-1},\\
[\nabla_q(\omega_1)]g_n &= g_n + r_{n-1} + r_n + s_n.
\end{aligned}$$

We see that

$$\begin{aligned}
s_0 &= [\nabla_q(\omega_1)]g_0 - g_0 - r_0 \in J\\
t_0 &= [\nabla_q(\omega_2)]g_0 - g_1 - r_0 - s_0 \in J\\
s_1 &= [\nabla_q(\omega_1)]r_0 - 2g_0 - 2g_1 - 2r_0 - s_0 - t_0 \in J\\
r_1 &= [\nabla_q(\omega_1)]g_1 - g_1 - r_0 - s_1 \in J\\
u_0 &= [\nabla_q(\omega_1)]s_0 - 2g_0 - r_0 - t_0 \in J\\
u_1 &= [\nabla_q(\omega_2)]r_0 - 2g_0 - 2g_1 - 2r_0 - r_1 - s_0 - u_0 \in J\\
t_1 &= [\nabla_q(\omega_1)]s_1 - 2g_1 - r_0 - r_1 - s_1 - t_0 - u_1 \in J.
\end{aligned}$$

Now assume that all $g_j, r_j, s_j, t_j, u_j \in J$ for $j \leq i$, where $n-1 > i > 0$. Then also

$$\begin{aligned}
g_{i+1} &= [\nabla_q(\omega_2)]g_i - g_{i-1} - g_i - r_{i-1} - r_i - s_i - t_{i-1} - t_i \in J\\
s_{i+1} &= [\nabla_q(\omega_1)]r_i - 2g_i - 2g_{i+1} - 2r_i - s_i - t_i \in J\\
r_{i+1} &= [\nabla_q(\omega_1)]g_{i+1} - g_{i+1} - r_i - s_{i+1} \in J\\
u_{i+1} &= [\nabla_q(\omega_2)]r_i - 2g_i - 2g_{i+1} - r_{i-1} - 2r_i - r_{i+1} - s_i - u_i \in J\\
t_{i+1} &= [\nabla_q(\omega_1)]s_{i+1} - 2g_{i+1} - r_i - r_{i+1} - s_{i+1} - t_i - u_{i+1} \in J.
\end{aligned}$$

We end the inductive argument with $g_n = [\nabla_q(\omega_2)]g_{n-1} - g_{n-2} - g_{n-1} - r_{n-2} - r_{n-1} - s_{n-1} - t_{n-2} - t_{n-1} \in J$, and if $k \equiv 1 \mod 3$ also $s_n = [\nabla_q(\omega_1)]r_{n-1} - 2g_{n-1} - 2g_n - 2r_{n-1} - s_{n-1} - t_{n-1} \in J$ and $r_n = [\nabla_q(\omega_1)]g_n - g_n + r_{n-1} - s_n \in J$.

Consider now k even. If $k \equiv 1 \mod 3$, then $k \equiv 4 \mod 6$, and we set $n = \frac{k-4}{6}$. Otherwise $k \equiv 2 \mod 6$, and we set $n = \frac{k-2}{6}$. We define new elements

$$\begin{aligned}
g_i &= [\nabla_q(\tfrac{k-2}{2} - 3i, 1+2i)], 0 \leq i \leq n,\\
r_i &= [T_q(\tfrac{k+2}{2} - 3i, 2i)] = [\nabla_q(\tfrac{k+2}{2} - 3i, 2i)] + [\nabla_q(\tfrac{k}{2} - 3i, 2i)], 0 \leq i \leq n,\\
s_i &= [T_q(\tfrac{k}{2} - 3i, 1+2i)]\\
&= [\nabla_q(\tfrac{k}{2} - 3i, 1+2i)] + [\nabla_q(\tfrac{k-4}{2} - 3i, 1+2i)], 0 \leq i < n,\\
t_i &= [T_q(\tfrac{k+4}{2} - 3i, 2i)] = [\nabla_q(\tfrac{k+4}{2} - 3i, 2i)] + [\nabla_q(\tfrac{k-2}{2} - 3i, 2i)], 0 \leq i \leq n,\\
u_i &= [T_q(\tfrac{k+2}{2} - 3i, 1+2i)]\\
&= [\nabla_q(\tfrac{k+2}{2} - 3i, 1+2i)] + [\nabla_q(\tfrac{k-6}{2} - 3i, 1+2i)], 0 \leq i < n,
\end{aligned}$$

and if $k \equiv 1 \mod 3$ then we also define $r_{n+1} = [\nabla_q(0, \frac{k+2}{3})]$ and $s_n = [T_q(2, \frac{k-1}{3})] = [\nabla_q(2, \frac{k-1}{3})] + [\nabla_q(0, \frac{k-1}{3})]$. This time, let J be the ideal generated by g_0, r_0 and r_1. Again we show that all the g_i's and r_i's belong to J.

We get similar calculations as before:

$$
\begin{aligned}
&[\nabla_q(\omega_1)]g_i = g_i + r_i + r_{i+1} + s_i, n > i \geq 0\\
&[\nabla_q(\omega_2)]g_i = \begin{cases} g_0 + g_1 + r_0 + r_1 + s_0 + t_0 + t_1, & i = 0\\ g_{i-1} + g_i + g_{i+1} + r_i + r_{i+1} + s_i + t_i + t_{i+1}, & n > i > 0 \end{cases}\\
&[\nabla_q(\omega_1)]r_i = \begin{cases} 2g_0 + 2r_0 + s_0 + t_0, & i = 0\\ 2g_{i-1} + 2g_i + 2r_i + s_{i-1} + s_i + t_i, & n > i \geq 0 \end{cases}\\
&[\nabla_q(\omega_2)]r_i = \begin{cases} 2g_0 + r_0 + r_1 + 2s_0 + t_0 + u_0, & i = 0\\ 2g_{i-1} + 2g_i + r_{i-1} + 2r_i + r_{i+1} + 2s_{i-1} & \\ +2s_i + t_i + u_{i-1} + u_i, & n > i > 0 \end{cases}\\
&[\nabla_q(\omega_1)]s_i = 2g_i + r_i + r_{i+1} + s_i + t_i + t_{i+1} + u_i, n > i \geq 0.
\end{aligned}
$$

If $k \equiv 1 \mod 3$, then still

$$
\begin{aligned}
[\nabla_q(\omega_1)]r_n &= 2g_{n-1} + 2g_n + 2r_n + s_{n-1} + s_n + t_n,\\
[\nabla_q(\omega_1)]g_{n-1} &= g_{n-1} + r_{n-1} + r_n + s_{n-1}.
\end{aligned}
$$

First we see that

$$
\begin{aligned}
s_0 &= [\nabla_q(\omega_1)]g_0 - g_0 - r_0 - r_1 \in J\\
t_0 &= [\nabla_q(\omega_1)]r_0 - 2g_0 - 2r_0 - s_0 \in J\\
u_0 &= [\nabla_q(\omega_2)]r_0 - 2g_0 - r_0 - r_1 - 2s_0 - t_0 \in J\\
t_1 &= [\nabla_q(\omega_1)]s_0 - 2g_0 - r_0 - r_1 - s_0 - t_0 - u_0 \in J\\
g_1 &= [\nabla_1(\omega_2)]g_0 - g_0 - r_0 - r_1 - s_0 - t_0 - t_1 \in J.
\end{aligned}
$$

Now assume that all $g_j, r_j, s_{j-1}, t_j, u_{j-1} \in J$ for $j \leq i$, where $n > i > 0$. Then also

$$
\begin{aligned}
s_i &= [\nabla_q(\omega_1)]r_i - 2g_{i-1} - 2g_i - 2r_i - s_{i-1} - t_i \in J\\
r_{i+1} &= [\nabla_q(\omega_1)]g_i - g_i - r_i - s_i \in J\\
u_i &= [\nabla_q(\omega_2)]r_i - 2g_{i-1} - 2g_i - r_{i-1} - 2r_i - r_{i+1} - 2s_{i-1} - 2s_i - t_i - u_{i-1} \in J\\
t_{i+1} &= [\nabla_q(\omega_1)]s_i - 2g_i - r_i - r_{i+1} - s_i - t_i - u_i \in J\\
g_{i+1} &= [\nabla_q(\omega_2)]g_i - g_{i-1} - g_i - r_i - r_{i+1} - s_i - t_i - t_{i+1} \in J.
\end{aligned}
$$

If $k \equiv 1 \mod 3$ we end the inductive argument with $s_n = [\nabla_q(\omega_1)]r_n - 2g_{n-1} - 2g_n - 2r_n - s_{n-1} - t_n$ and $r_{n+1} = [\nabla_q(\omega_1)]g_n - g_n - r_n - s_n$. $\square$

Chapter 5

Tensor ideals in $\mathcal{T}_q$

In this chapter we will generalize the constructions in Section 1.3 to get structures that might not satisfy all the axioms in Section 1.1, but for which the methods in Chapter 3 give similar results.

We will first explore suitable subcategories $\mathcal{I}$ of the category of tilting modules $\mathcal{T}_q$, that can replace the role of $\mathcal{N}_q$. We will require that the Grothendieck group $\mathrm{K}_0(\mathcal{I})$ is a subgroup of $\mathrm{K}_0(\mathcal{T}_q)$ which is also an ideal with regards to the ring structure. This leads us to the definition of a tensor ideal in an additive, monoidal category, cf. Section 0.1.

Definition 5.1. *A subcategory $\mathcal{I} \subset \mathcal{T}_q$ is called a tensor ideal if $V_1 \oplus V_2 \in \mathcal{I}$ implies $V_1, V_2 \in \mathcal{I}$, and $V \in \mathcal{I}, Q \in \mathcal{T}_q$ implies $V \otimes Q \in \mathcal{I}$.*

When $\mathcal{I} \subseteq \mathcal{T}_q$ is a tensor ideal, we may form the quotient category $\mathcal{T}_q/\mathcal{I}$. The story deviates here from the one in Section 1.3, as this quotient in general will not be semisimple and may have an infinite number of isomorphism classes of indecomposable objects.

5.1 Weight cells in the dominant chamber

Following [Ost01] we define a preorder on P^+.

Definition 5.2. *Let $\lambda, \mu \in P^+$. We say $\lambda \leq_{\mathcal{T}_q} \mu$ if there exists a $Q \in \mathcal{T}_q$ such that $T_q(\lambda)$ is a direct summand of $T_q(\mu) \otimes Q$. If both $\lambda \leq_{\mathcal{T}_q} \mu$ and $\mu \leq_{\mathcal{T}_q} \lambda$, then we say $\lambda \sim_{\mathcal{T}_q} \mu$, and we call the equivalence classes of $\sim_{\mathcal{T}_q}$ for the weight cells in P^+.*

For all $\lambda, \nu \in P^+$ we have $\lambda + \nu \leq_{\mathcal{T}_q} \lambda$, in fact $T_q(\lambda + \nu)$ is a direct summand of $T_q(\lambda) \otimes T_q(\nu)$.

The preorder $\leq_{\mathcal{T}_q}$ on P^+ induces a partial order on the set of weight cells, which we also denote by $\leq_{\mathcal{T}_q}$. Write $\underline{c}' <_{\mathcal{T}_q} \underline{c}$ if $\underline{c}' \leq_{\mathcal{T}_q} \underline{c}$ but $\underline{c}' \neq \underline{c}$. Furthermore, write $\lambda \leq_{\mathcal{T}_q} \underline{c}$ when λ belongs to a cell $\underline{c}'$ with $\underline{c}' \leq_{\mathcal{T}_q} \underline{c}$.

One may show directly that $(l-1)\rho + P^+$ is a single weight cell, cf. [And04, Proposition 6]. By the above observation, this is the unique minimal cell in the partial ordering on the weight cells. Furthermore, the fundamental alcove A_l is a single weight cell, cf. [And04, Proposition 9], and this is the unique maximal cell in the ordering.

The following is [And04, Proposition 8].

Proposition 5.3. *Each weight cell in P^+ is a union of lower closures of alcoves intersected with P^+.*

Characterization of tensor ideals

Given a weight cell $\underline{c} \subseteq P^+$, denote by $\mathcal{T}_q(\leq \underline{c})$ the subcategory of $\mathcal{T}_q$ whose objects are finite direct sums of $T_q(\lambda)$ with $\lambda \leq_{\mathcal{T}_q} \underline{c}$. Then $\mathcal{T}_q(\leq \underline{c})$ is by construction a tensor ideal in $\mathcal{T}_q$. It is clear that for a collection of weight cells $\underline{c}_i, i \in J$, the subcategory of $\mathcal{T}_q$ consisting of finite direct sums of $T_q(\lambda)$ with $\lambda \leq_{\mathcal{T}_q} \underline{c}_i$ for some $i \in J$ is also a tensor ideal in $\mathcal{T}_q$. On the other hand, all tensor ideals in $\mathcal{T}_q$ have this description. As a special case we denote by $\mathcal{T}_q(< \underline{c})$ the tensor ideal obtained from the collection of cells $\underline{c}' <_{\mathcal{T}_q} \underline{c}$.

There is an identification of W_l with the alcoves in P, obtained by matching $w \in W_l$ with $w.A_l$. Let W_l^+ denote the $w \in W_l$ for which $w.A_l \subseteq P^+$. By Proposition 5.3 we have a partition of W_l^+ under this identification. This partion is not new, as is shown in the following

Proposition 5.4. *The weight cells in P^+ correspond to the right Kazhdan-Lusztig cells in W_l^+.*

This is [Ost97, Theorem 5.5] and the following remark. Here a right Kazhdan-Lusztig cell is an equivalence class of the equivalence relation $\sim_R$ generated by the preorder $\leq_R$ on W_l defined in the next section.

The proposition gives us a characterization of all possible tensor ideals of $\mathcal{T}_q$ in terms of right Kazhdan-Lusztig cells. However, the definition of $\leq_R$ on W_l is quite intricate, and a complete decomposition of W_l into right cells has not been done in general. In the next section we review some of the progress on this problem.

We see that we have generalized the setting from Section 1.3, as $\mathcal{N}_q = \mathcal{T}_q(< A_l)$. We have the following generalization of Proposition 3.1:

Proposition 5.5. *Let $\underline{c} \subseteq P^+$ be a weight cell. Then the tensor ideal $\mathcal{T}_q(\leq \underline{c}) \subseteq \mathcal{T}_q$ is generated by the set*

$$\{T_q(\lambda) \mid \lambda \textit{ is minimal in } \cup_{\underline{c}' \leq_{\mathcal{T}_q} \underline{c}} \underline{c}' \textit{ with respect to } \preceq\}$$

The proof is formulated in the same way, but note that the generating set is not necessarily finite.

Comparison with fusion rings

Write $R = \mathrm{K}_0(\mathcal{T}_q)$, and for a weight cell $\underline{c}$, write $I_{\leq \underline{c}} = \mathrm{K}_0(\mathcal{T}_q(\leq \underline{c}))$. As a $\mathbb{Z}$-module $R/I_{\leq \underline{c}}$ has a basis in $\{[\lambda] \mid \lambda \in P^+, \lambda \not\leq_{\mathcal{T}_q} \underline{c}\}$, where $[\lambda] = [T_q(\lambda)] + I_{\leq \underline{c}}$ is the coset represented by the class of $T_q(\lambda)$. $R/I_{\leq \underline{c}}$ inherits a multiplicative structure from R, which we encode with regards to the basis as

$$[\lambda][\mu] = \sum_{\substack{\nu \in P^+ \\ \lambda \not\leq_{\mathcal{T}_q} \underline{c}}} N^{\nu}_{\lambda,\mu}[\nu].$$

We now ask ourselves what can be said about this structure.

Obviously the structure is commutative, and by construction it is associative. The involution $\lambda \mapsto \lambda^*$ may preserve the set $I = \{\lambda \in P^+ \mid \lambda \not\leq_{\mathcal{T}_q} \underline{c}\}$, in which case we have $N^{\nu^*}_{\lambda^*,\mu^*} = N^{\nu}_{\lambda,\mu}$ for all $\lambda, \mu, \nu \in I$. However, unless we are in the situation of Section 1.3 where $I = A_l$, it will not define a fusion structure on I: If $\lambda \in I \setminus A_l$ and $\nu \in A_l$, then $N^{\nu}_{\lambda,\mu} = 0$ for any $\mu \in I$, since we know that $\mathcal{T}_q(< A_l)$ is a tensor ideal. This means in particular that $N^{\lambda_0}_{\lambda,\lambda^*} = 0$, and the condition (iii) of Definition 1.1 is not satisfied.

Continue to assume that the involution preserves the set I. In all cases except $\mathfrak{g}$ of type A_r, D_r(r odd) or E_6 this is obvious, as the involution is just the identity, cf. [Bou02, Chapter VI, Plates I-IX]. We define a $\mathbb{Z}$-bilinear form on $R/I_{\leq \underline{c}}$: For the basis elements $\{[\lambda] \mid \lambda \in P^+, \lambda \not\leq_{\mathcal{T}_q} \underline{c}\}$ define

$$([\lambda], [\mu]) = \dim \mathrm{Hom}_{\mathcal{T}_q}(T_q(\lambda), T_q(\mu)) \tag{5.1}$$

and extend linearly. Since $\mathrm{Hom}_{\mathcal{T}_q}(L \otimes T, T') \simeq \mathrm{Hom}_{\mathcal{T}_q}(T, L^* \otimes T')$ for any $L, T, T' \in \mathcal{T}_q$ we have $(xz, y) = (x, z^*y)$ for all $x, y, z \in R/I_{\leq \underline{c}}$.

If I is a finite set, let $B = (b_{\lambda,\mu}), b_{\lambda,\mu} = ([\lambda], [\mu])$, be the matrix associated with the bilinear form. If the determinant of B is non-zero, then the bilinear form gives an injection of $R/I_{\leq \underline{c}}$ into $\mathrm{Hom}_{\mathbb{Z}}(R/I_{\leq \underline{c}}, \mathbb{Z})$, which is an isomorphism if the determinant is a unit. We calculate B using the formula

$$\dim \mathrm{Hom}_{\mathcal{T}_q}(T, T') = \sum_{\lambda \in P^+} [T : \Delta_q(\lambda)]\, [T' : \nabla_q(\lambda)] \tag{5.2}$$

which follows from the fact that $\mathrm{Ext}^n_{\mathcal{T}_q}(\Delta_q(\lambda), \nabla_q(\mu))$ is 1-dimensional if $n = 0$ and $\lambda = \mu$, and 0 otherwise.

5.2 Kazhdan-Lusztig cells in affine Weyl groups

We consider the affine Weyl group W_l generated by the set S of reflections in the walls of the fundamental alcove A_l. Let s_0 denote the reflection in the upper wall of A_l, i.e., if $l = l_\alpha$ for all $\alpha \in \Phi$ then $s_0 = s_{\alpha_0,1}$, and otherwise $s_0 = s_{\beta_0,1}$. Then the set of simple reflections is $S = \{s_1, \ldots, s_r, s_0\}$. As mentioned in the previous section, the partition of P^+ into weight cells corresponds to the partion of W_l^+ into right Kazhdan-Lusztig cells. We quickly go through the construction of the partitions of W_l into subsets of right, left and two-sided Kazhdan-Lusztig cells for the infinite Coxeter system (W_l, S) first defined in [KL79], but with the notation of [Soe97a].

Let $\mathcal{H}$ denote the Hecke algebra of (W_l, S) over $A = \mathbb{Z}[v, v^{-1}]$, i.e., $\mathcal{H} = \bigoplus_{w \in W} AH_w$ with multiplication defined by $H_s H_w = H_{sw}$ if $l(sw) > l(w)$ and $H_s H_w = H_{sw} + (v^{-l(w)} - v^{l(w)})H_w$ otherwise. There is an involution on $\mathcal{H}$ given by $\bar{v} = v^{-1}$ and $\bar{H}_w = (H_{w^{-1}})^{-1}$, and we call $H \in \mathcal{H}$ self-dual if $\bar{H} = H$.

We have the classical result, [KL79, Theorem 1.1]:

Theorem 5.6. *For all $x \in W_l$ there exists a unique self-dual element $\underline{H}_x \in \mathcal{H}$ such that $\underline{H}_x \in H_x + \sum_{y<x} v\mathbb{Z}[v]H_y$. The $\underline{H}_x$ form an A-basis of $\mathcal{H}$.*

Definition 5.7. *Let $\leq_L, \leq_R$ and $\leq_{LR}$ be the preorders on W_l generated by*

$$x \leq_L y \Leftrightarrow \underline{H}_x \text{ appears in } H\underline{H}_y \text{ for some } H \in \mathcal{H}$$
$$x \leq_R y \Leftrightarrow \underline{H}_x \text{ appears in } \underline{H}_y H \text{ for some } H \in \mathcal{H}$$
$$x \leq_{LR} y \Leftrightarrow \underline{H}_x \text{ appears in } H\underline{H}_y H' \text{ for some } H, H' \in \mathcal{H}.$$

We write $x \sim_L y$ if both $x \leq_L y$ and $y \leq_L x$. This is the equivalence relation generated by $\leq_L$, and we call its equivalence classes the left Kazhdan-Lusztig cells in W_l. Similarly, we have the equivalence relations $\sim_R$ and $\sim_{LR}$ on W_l generated by $\leq_R$ and $\leq_{LR}$ and we call their equivalence classes the right resp. the two-sided Kazhdan-Lusztig cells.

We notice that $x \leq_L y \Leftrightarrow x^{-1} \leq_R y^{-1}$, so there is a redundancy in the notation. As previously noted, the partitions of W_l into right and two-sided Kazhdan-Lusztig cells have not been computed in general. As a first general result, we note [Lus87, Theorem 2.2]:

Theorem 5.8. *The affine Weyl group W_l has only finitely many two-sided Kazhdan-Lusztig cells.*

Though some two-sided cells may consist of finitely many elements, this result tells us that in general they are infinite. As another result, we note [LX88, Theorem 1.2]:

Theorem 5.9. *Let $A \subseteq W_l$ be a two-sided Kazhdan-Lusztig cell. Then the intersection $A \cap W_l^+$ is a single right Kazhdan-Lusztig cell.*

Then, in order to characterize the weight cells in P^+, we may either characterize the two-sided Kazhdan-Lusztig cells in W_l or the right Kazhdan-Lusztig cells in W_l^+.

Some examples

Assume that $\mathfrak{g}$ has type $A_r, r \geq 2$. We know that the number of two-sided Kazhdan-Lusztig cells in W_l is finite, but more precisely it is equal to the number of partitions of $r+1$, as is shown in [Shi86, 17.5]. This is a corollary of a complete characterization of the two-sided cells in W_l through a technical map associating to each affine Weyl group element a partition of $r+1$. In the same book it is also shown that each non-trivial right Kazhdan-Lusztig cell in W_l is an infinite set of elements, cf. [Shi86, Proposition 19.1.5]. Therefore $\mathcal{N}_q(\mathfrak{sl}_{r+1})$ is the only tensor ideal in $\mathcal{T}_q(\mathfrak{sl}_{r+1})$ for which the quotient of the respective Grothendieck groups has finite rank.

Assume that $\mathfrak{g}$ is simple of rank 2. In his paper [Lus85] Lusztig describes explicitly the right and two-sided Kazhdan-Lusztig cells of W_l. The affine Weyl group is partitioned into unions of two-sided Kazhdan-Lusztig cells $W_l = A \cup B \cup C$ for $\mathfrak{g}$ of type A_2, $W_l = A \cup B \cup C \cup D$ for type C_2 and $W_l = A \cup B \cup C \cup D \cup E$ for type G_2. In all cases the last set is the trivial cell $\{1\}$, and in the first two cases the remaining sets A, B resp. A, B, C contain infinitely many elements. For type G_2 the sets A, B and C are infinite, while the set D is finite. Thus the set $D_1 = D \cap W_l^+$ is a finite right Kazhdan-Lusztig cell. The next section is dedicated to the study of it.

There are a number of results on partial and complete characterizations of the right and two-sided Kazhdan-Lusztig cells for affine Weyl groups of other types, but let us pick just one out. In [Lus83, 3.13] for each simple $\mathfrak{g}$ of rank $r \geq 2$ a number of graphs are constructed, each corresponding to a left Kazhdan-Lusztig cell in W_l. In case $\mathfrak{g}$ does not have type A_r or C_r, one of these graphs is finite and thus corresponds to a finite left Kazhdan-Lusztig cell.

5.3 The D_1 cell for type G_2

In this section we shall consider an example of a non-trivial finite weight cell in P^+ when $\mathfrak{g}$ has type G_2. This is the cell corresponding to the subset of W_l named D_1 in [Lus85, 11.2], consisting of the elements $s_0, s_0s_1, s_0s_1s_2, s_0s_1s_2s_1, s_0s_1s_2s_1s_0, s_0s_1s_2s_1s_2, s_0s_1s_2s_1s_2s_1, s_0s_1s_2s_1s_2s_1s_0$. Let D_1 denote this weight cell too, i.e., the union of the lower closures of $w.A_l$ intersected with P^+ for w one of the eight aforementioned elements of W_l. Thus the subcategory $\mathcal{T}_q(<D_1)$ of $\mathcal{T}_q$ is a tensor ideal, such that the quotient category has finitely many isomorphism classes of indecomposable objects.

Proposition 5.10. *Let $\mathfrak{g}$ have type G_2, and let $l \geq 6$ be an even integer not divisible by 3. Then the tensor ideal $\mathcal{T}_q(<D_1)$ in $\mathcal{T}_q$ is generated by the set $\{T_q((\frac{l}{2}-1)\rho), T_q((\frac{l}{2}-1)\rho - \omega_1 + \omega_2), T_q((\frac{l}{2}-1)\rho + 3\omega_1 - \omega_2)\}$.*

Proof. Define elements

$$\lambda_i = (\tfrac{l}{2}-1)\rho + i(\omega_2 - \omega_1), i = 0, \ldots, \tfrac{l}{2}-1,$$
$$\mu_j = (\tfrac{l}{2}-1)\rho + j(3\omega_1 - \omega_2), j = 1, \ldots, \tfrac{l}{2}-1.$$

Let $\mathcal{I}$ be the ideal in $\mathcal{T}_q$ generated by $T_q(\lambda_0), T_q(\lambda_1)$ and $T_q(\mu_1)$. The weights $\{\lambda_i, \mu_j\}$ are the minimal weights in the set $\cup_{\underline{c}<_{\mathcal{T}_q}D_1}\underline{c} = P^+ \setminus (D_1 \cup A_l)$ with respect to $\preceq$, so by Proposition 5.5 we just need to show that all the remaining $T_q(\lambda_i)$ and $T_q(\mu_j)$ belong to $\mathcal{I}$.

Under the identification of W_l with the set of alcoves in P, if $w = s_{i_n} \ldots s_{i_1}$ then denote $A_{i_n \ldots i_1} = w.A_l$. In the following we compute the characters for tilting modules belonging to lower closures of some alcoves in $P^+ \setminus (D_1 \cup A_l)$, specifically the alcoves $A_{012120}, A_{0121201}$ and $A_{01212010}$. For the deduction of these characters we use [Soe97b, Conjecture 7.1] (proven in [Soe98]) and the following remark.

Let $\lambda \in \bar{A}_l$. We consider first A_{012120}, so let $w = s_0s_1s_2s_1s_2s_0$. If $\lambda \in A_l$ is regular, then

$$\operatorname{ch} T_q(w.\lambda) = \chi_{w.\lambda} + \chi_{ws_0.\lambda} + \chi_{ws_2.\lambda} + \chi_{ws_0s_2.\lambda}.$$

If λ is subregular and fixed by s_2, then

$$\operatorname{ch} T_q(w.\lambda) = \chi_{w.\lambda} + \chi_{ws_0.\lambda}$$

and if λ is fixed by s_0, then

$$\operatorname{ch} T_q(w.\lambda) = \chi_{w.\lambda} + \chi_{ws_2.\lambda}.$$

If λ is fixed by both s_2 and s_0, then $T_q(w.\lambda)$ is simple.

Consider then $A_{0121201}$ and let $w = s_0s_1s_2s_1s_2s_0s_1$. If $\lambda \in A_l$ then

$$\operatorname{ch} T_q(w.\lambda) = \chi_{w.\lambda} + \chi_{ws_1.\lambda} + \chi_{ws_1s_0.\lambda} + \chi_{ws_1s_0s_1.\lambda} + \chi_{ws_1s_0s_2.\lambda} + \chi_{ws_1s_0s_2s_1.\lambda}.$$

If λ is fixed by s_1, then

$$\operatorname{ch} T_q(w.\lambda) = \chi_{w.\lambda} + \chi_{ws_1s_0.\lambda} + \chi_{ws_1s_0s_2.\lambda}.$$

Finally consider $A_{01212010}$ so let $w = s_0s_1s_2s_1s_2s_0s_1s_0$. If $\lambda \in A_l$ then

$$\operatorname{ch} T_q(w.\lambda) = \chi_{w.\lambda} + \chi_{ws_0.\lambda} + \chi_{ws_0s_1.\lambda} + \chi_{ws_0s_1s_0.\lambda} + \chi_{ws_1.\lambda} + \chi_{ws_1s_0.\lambda}.$$

If λ is fixed by s_0 then

$$\operatorname{ch} T_q(w.\lambda) = \chi_{w.\lambda} + \chi_{ws_0s_1.\lambda} + \chi_{ws_1.\lambda}$$

and if λ is fixed by s_1 then

$$\operatorname{ch} T_q(w.\lambda) = \chi_{w.\lambda} + \chi_{ws_0.\lambda} + \chi_{ws_1s_0.\lambda}.$$

If λ is fixed by s_0 and s_1 then $T_q(w.\lambda)$ is simple.

We show first that all $T_q(\lambda_i), i = 2, \ldots, \frac{l}{2} - 1$, belong to $\mathcal{I}$. Explicitly we see that

$$\begin{array}{ll}
T_q(\lambda_0 + \omega_1) \in \mathcal{I} & (T_q(\lambda_0) \otimes T_q(\omega_1) \simeq T_q(\lambda_0 + \omega_1) \oplus T_q(\lambda_1) \oplus T_q(\lambda_0)) \\
T_q(\lambda_0 + \omega_2) \in \mathcal{I} & (T_q(\lambda_0) \otimes T_q(\omega_2) \simeq T_q(\lambda_0 + \omega_2) \oplus T_q(\mu_1) \oplus T_q(\lambda_0 + \omega_1) \oplus \\
& T_q(\lambda_1) \oplus 2T_q(\lambda_0)) \\
T_q(\lambda_2) \in \mathcal{I} & (T_q(\lambda_1) \otimes T_q(\omega_1) \simeq T_q(\lambda_2) \oplus T_q(\lambda_0 + \omega_2) \oplus T_q(\lambda_0 + \omega_1) \oplus \\
& T_q(\lambda_1) \oplus 2T_q(\lambda_0))
\end{array}$$

and inductively

$$\begin{array}{ll}
T_q(\lambda_i + 2\omega_1) \in \mathcal{I} & (\text{direct summand of } T_q(\lambda_i + \omega_1) \otimes T_q(\omega_1) \text{ as above}) \\
T_q(\lambda_{i+1} + \omega_2) \in \mathcal{I} & (\text{direct summand of } T_q(\lambda_{i+1}) \otimes T_q(\omega_2)) \\
T_q(\lambda_{i+3}) \in \mathcal{I} & (\text{direct summand of } T_q(\lambda_{i+2}) \otimes T_q(\omega_1))
\end{array}$$

for $i = 0, \ldots, \frac{l}{2} - 4$. Note for $i = \frac{l}{2} - 4$ that $\lambda_i + 2\omega_1$ and $\lambda_{i+1} + \omega_2$ lie on the wall between A_{012120} and $A_{0121201}$.

In the same manner we show that all $T_q(\mu_i), i = 2, \ldots, \frac{l}{2} - 1$, belong to $\mathcal{I}$.

$$\begin{array}{ll}
T_q(\mu_i + 2\omega_1) \in \mathcal{I} & (\text{direct summand of } T_q(\mu_i + \omega_1) \otimes T_q(\omega_1)) \\
T_q(\mu_{i+1} + \omega_1) \in \mathcal{I} & (\text{direct summand of } T_q(\mu_{i+1}) \otimes T_q(\omega_1))
\end{array}$$

$$\begin{array}{ll}
T_q(\mu_i+\omega_1+\omega_2) \in \mathcal{I} & \text{(direct summand of } T_q(\mu_i+\omega_1)\otimes T_q(\omega_2))\\
T_q(\mu_{i+1}+\omega_2) \in \mathcal{I} & \text{(direct summand of } T_q(\mu_1+2\omega_1)\otimes T_q(\omega_1))\\
T_q(\mu_{i+2}) \in \mathcal{I} & \text{(direct summand of } T_q(\mu_{i+1})\otimes T_q(\omega_2))
\end{array}$$

for $i = 0, \dots, \frac{l}{2}-3$ (here $\mu_0 = \lambda_0$). The calculations become complicated when we hit the walls below $A_{0121201}$ and $A_{01212010}$, but in a decomposition of the tensor product in any of the 5 cases above, all but 1 summand has already been seen to belong to $\mathcal{I}$, and consequently the last one (the one in the left column) does too.

Note the similarity of the first inductive step with the calculations in the proof of [AS14, Theorem 7.1] and of the second with the proof of Proposition 4.3. They really are the same. □

The restrictions on l in the proposition are just for convenience. In case l is an odd integer not divisible by 3, or in case $l \geq 12$ is divisible by 3, we have a similar result on the tensor ideal $\mathcal{T}_q(< D_1)$ being generated by 3 elements.

Now consider $R = \mathrm{K}_0(\mathcal{T}_q)$ and $I_{<D_1} = \mathrm{K}_0(\mathcal{T}_q(< D_1))$. Let us calculate the determinant of the bilinear form (5.1) on $R/I_{<D_1}$. Since $\mathrm{Hom}_{\mathcal{T}_q}(T_q(\lambda), T_q(\mu)) = 0$ unless $\mu \in W_l.\lambda$, we may assume $\lambda \in \bar{A}_l$ and $\mu = w.\lambda$, where w is one of the elements $1, s_0, s_0s_1, s_0s_1s_2, s_0s_1s_2s_1, s_0s_1s_2s_1s_0, s_0s_1s_2s_1s_2, s_0s_1s_2s_1s_2s_1, s_0s_1s_2s_1s_2s_1s_0$.

Assume first $\lambda \in A_l$ is regular. We again calculate the characters of the indecomposable tilting modules with highest weight in $D_1 \cup A_l$ linked to λ, using [Soe97b, Conjecture 7.1].

$$\begin{aligned}
\operatorname{ch} T_q(\lambda) &= \chi_\lambda\\
\operatorname{ch} T_q(s_0.\lambda) &= \chi_{s_0.\lambda} + \chi_\lambda\\
\operatorname{ch} T_q(s_0s_1.\lambda) &= \chi_{s_0s_1.\lambda} + \chi_{s_0.\lambda}\\
\operatorname{ch} T_q(s_0s_1s_2.\lambda) &= \chi_{s_0s_1s_2.\lambda} + \chi_{s_0s_1.\lambda}\\
\operatorname{ch} T_q(s_0s_1s_2s_1.\lambda) &= \chi_{s_0s_1s_2s_1.\lambda} + \chi_{s_0s_1s_2.\lambda}\\
\operatorname{ch} T_q(s_0s_1s_2s_1s_0.\lambda) &= \chi_{s_0s_1s_2s_1s_0.\lambda} + \chi_{s_0s_1s_2s_1.\lambda}\\
\operatorname{ch} T_q(s_0s_1s_2s_1s_2.\lambda) &= \chi_{s_0s_1s_2s_1s_2.\lambda} + \chi_{s_0s_1s_2s_1.\lambda}\\
\operatorname{ch} T_q(s_0s_1s_2s_1s_2s_1.\lambda) &= \chi_{s_0s_1s_2s_1s_2s_1.\lambda} + \chi_{s_0s_1s_2s_1s_2.\lambda}\\
\operatorname{ch} T_q(s_0s_1s_2s_1s_2s_1s_0.\lambda) &= \chi_{s_0s_1s_2s_1s_2s_1s_0.\lambda} + \chi_{s_0s_1s_2s_1s_2s_1.\lambda}\\
&\quad + \chi_{s_0s_1s_2s_1s_2s_0.\lambda} + \chi_{s_0s_1s_2s_1s_2.\lambda}
\end{aligned}$$

Using (5.2) the determinant of the principal minor of B corresponding to the linkage class of λ is seen to be:

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 \end{vmatrix} = 2.$$

Now let $\lambda \in \bar{A}_l$ be a subregular weight fixed by the affine reflection s_0. There are three distinct weights in $D_1 \cup A_l$ in the linkage class of λ, namely $\lambda, s_0s_1s_2s_1.\lambda, s_0s_1s_2s_1s_2s_1.\lambda$. We use [Soe97b, Remark 7.2] to calculate the characters of the related indecomposable tilting modules.

$$\begin{aligned} &\operatorname{ch} T_q(\lambda) = \chi_\lambda \\ &\operatorname{ch} T_q(s_0s_1s_2s_1s_0.\lambda) = \chi_{s_0s_1s_2s_1s_0.\lambda} \\ &\operatorname{ch} T_q(s_0s_1s_2s_1s_2s_1.\lambda) = \chi_{s_0s_1s_2s_1s_2s_1.\lambda} \end{aligned}$$

So the tilting modules are all simple, and the determinant of the principal minor of B corresponding to the linkage class of λ is 1.

If $\lambda \in \bar{A}_l$ is subregular and fixed by s_1, there are three weights in $D_1 \cup A_l$ in the linkage class of λ, namely $s_1.\lambda, s_0s_1s_2.\lambda, s_0s_1s_2s_1s_2.\lambda$. The indecomposable tilting modules related to these weights are all simple, so the corresponding subdeterminant is again 1. If λ is fixed by s_2, there are two weights in $D_1 \cup A_l$ in the linkage class of λ, the related tilting modules are also simple and the subdeterminant is 1.

Put together, the determinant of B is $d = 2^{|A_l|}$, and we conclude that the bilinear form (5.1) is non-degenerate but does not give an isomorphism of $R/I_{<D_1}$ with its $\mathbb{Z}$-dual.

However, if we make a change of the base ring to $\mathbb{Z}[\frac{1}{d}] = \{d^n, n \in \mathbb{N}\}^{-1}\mathbb{Z}$, the bilinear $\mathbb{Z}[\frac{1}{d}]$-form on $R/I_{<D_1}[\frac{1}{d}]$ gives an isomorphism of $R[\frac{1}{d}]$-modules

$$R/I_{<D_1}[\tfrac{1}{d}] \simeq \operatorname{Hom}_{\mathbb{Z}[\frac{1}{d}]}(R/I_{<D_1}[\tfrac{1}{d}], \mathbb{Z}[\tfrac{1}{d}]),$$

and then the proof of Proposition 3.11 carries over directly to give us the following

Proposition 5.11. *Let $I \subseteq R = \mathbb{Z}[X, Y]$ be an ideal such that R/I is finite free as a $\mathbb{Z}$-module and there is a non-degenerate bilinear $\mathbb{Z}$-form on R/I. Let $d \in \mathbb{Z}$ be the determinant of the matrix associated with the bilinear form for any basis of R/I. Then $I[\frac{1}{d}]$ is generated by an $R[\frac{1}{d}]$-regular sequence of length 2.*

In the proof it is important to note that the ring $\mathbb{Z}[\frac{1}{d}]$ is a principal ideal domain so the Quillen-Suslin theorem still holds over $R[\frac{1}{d}] \simeq \mathbb{Z}[\frac{1}{d}][X, Y]$.

Chapter 6

Future perspectives

We use in this chapter the notation of Section 1.3. In particular $q \in \mathbb{C}$ is a root of unity, and l denotes the order of q^2. Assume that l is big enough that $A_l \neq \emptyset$. The Weyl group W is generated by the simple reflections $s_i = s_{\alpha_i}, i = 1, \ldots, r$ while the affine Weyl group W_l is generated by W and the reflection s_0 in the upper wall of the fundamental alcove A_l, which is either $s_{\alpha_0,1}$ or $s_{\beta_0,1}$ depending on whether all $l_\alpha = l$ or not.

Let $D = \{0, 1, \ldots, r\}$ be a numbering of the nodes in the affine Dynkin diagram corresponding to the simple generating set $\{s_0, s_1, \ldots, s_r\}$ of W_l, and let $\bar{D} = \{1, \ldots, r\}$ denote the Dynkin diagram of W. For any nonempty subset $S \subseteq D$, let $W_S = \langle s_i \mid i \in S \rangle$ be the reflection group generated by the simple reflections corresponding to S. Let W_S act on E via the shifted action. This action preserves P, and we denote it by the usual dot-notation. Call an element of E S-singular if it is fixed by some non-trivial element of W_S, and let $H_S \subseteq E$ denote the set of S-singular elements. Let $C_S \subseteq E \setminus H_S$ be the unique connected component containing A_l. Then $\bar{C}_S$ is a fundamental domain for the action of W_S on E.

The subset of the extended Dynkin diagram corresponding to S is the Dynkin diagram of a (possibly reducible) root system $\Phi_S \subseteq \Phi$ whose Weyl group is W_S. Let Φ_S^+ denote the positive part of Φ_S corresponding to C_S, and let $\rho_S = \frac{1}{2}\sum_{\alpha \in \Phi_S} \alpha \in E$. Though ρ_S is not necessarily in P, the action $\lambda \mapsto w.(\lambda - \rho_S) + \rho_S$ preserves P for all $w \in W_S$. With abuse of notation denote this action by $w(\lambda), w \in W_S$. If $0 \notin S$ then W_S is a subgroup of W, and the actions of the two groups on P are the same.

Note that the shifted action of W_S fixes a face of $\bar{A}_l$, while the non-shifted action fixes a face of A_l.

The non-shifted action of W_S on P induces an action on $\mathbb{Z}[P]$. For $\lambda \in P$ the element $A_\lambda^{W_S} = \sum_{w \in W_S} (-1)^{l(w)} e^{w(\lambda)}$ is antisymmetric with regards to this action. Then the fraction

$$\frac{A^{W_S}_{\lambda+\rho_S}}{A^{W_S}_{\rho_S}} \tag{6.1}$$

is a W_S-invariant element of $\mathbb{Z}[P]$. If $S = \bar{D}$ then $W_{\bar{D}} = W$ and (6.1) is the Weyl character $\chi_\lambda \in \mathbb{Z}[P]^W$. When we identify $Z[P]^W$ with the split Grothendieck group $\mathrm{K}_0(\mathcal{T}_q)$ there is a quotient of $\mathbb{Z}[P]^W$ which is identified with the fusion ring $\mathrm{K}_0(\mathcal{T}_q)/\mathrm{K}_0(\mathcal{N}_q)$.

The following is [Dou09, Proposition 2.4].

Proposition 6.1. *Assume that l is divisible by all $d_i, i = 1, \dots, r$. There is a complex of $\mathbb{Z}[P]^W$-modules $\bigoplus_{S\subseteq D, |S|=r-i} \mathbb{Z}[P]^{W_S}$ whose differentials have components $d^{S,T} : \mathbb{Z}[P]^{W_S} \to \mathbb{Z}[P]^{W_T}$ for $T = S \cup \{i_s\}$, given by*

$$d^{S,T}\left(\left[\frac{A^{W_S}_{\lambda+\rho_S}}{A^{W_S}_{\rho_S}}\right]\right) = (-1)^s \left[\frac{A^{W_T}_{\lambda+\rho_T}}{A^{W_T}_{\rho_T}}\right].$$

Here $D \setminus S = \{i_0, \dots, i_p\}$ is ordered $i_0 < \cdots < i_p$ as in D. This complex is acyclic except in degree $i = 0$ for which it has homology identified with the fusion ring.

The complex is an interpretation of a twisted Mayer-Vietoris spectral sequence for the twisted K-homology of a simple, simply connected Lie group whose Lie algebra is $\mathfrak{g}$, cf. [Dou09, Section 2]. While the mechanics of this spectral sequence is deeply rooted in equivariant K-theory, the interpretation is formulated purely in terms of invariants in $\mathbb{Z}[P]$ of subgroups of the affine Weyl group. When we consider only the combinatorial description of the complex the assumptions on l seem artificial, which suggests they might be superfluous.

In this chapter we are interested in giving a representation theoretic realization of the complex (for any l) satisfying

(i) For each nonempty $S \subseteq D$ we want an additive, monoidal category $\mathcal{D}_{q,S}$ such that its split Grothendieck group $\mathrm{K}_0(\mathcal{D}_{q,S})$ is isomorphic to $\mathbb{Z}[P]^{W_S}$ as rings.

(ii) The categories $\mathcal{D}_{q,S}$ should be $\mathcal{D}_{q,\bar{D}}$-module categories, such that the $\mathbb{Z}[P]^{W_S}$ become finite free $\mathbb{Z}[P]^W$-modules.

(iii) For $S \subseteq D$ and $T = S \cup \{j\}$ we want a functor $\mathcal{D}_{q,S} \to \mathcal{D}_{q,T}$ categorifying the differentials $d^{S,T}$ above.

(iv) For $S = \bar{D}$ we want $\mathcal{D}_{q,\bar{D}} \simeq \mathcal{T}_q$ to be equivalent as additive, monoidal categories.

6.1 Candidate categories

We introduce parabolic subalgebras of the quantum group, following [Pul06]. As a reference to the constructions, we use [CP95]. Alternatively one could take the proofs in [Jan96, Chapter 8] and generalize them to the root of unity case.

Let $S \subseteq \bar{D} = \{1, \dots, r\}$ be a subset of the Dynkin diagram and let $\mathfrak{b}^- \subseteq \mathfrak{p}_S \subseteq \mathfrak{g}$ be the parabolic subalgebra associated to it. Let $\mathfrak{p}_S = \mathfrak{l}_S \oplus \mathfrak{u}_S$ be a Levi decomposition of $\mathfrak{p}_S$, with $\mathfrak{l}_S$ the Levi factor and $\mathfrak{u}_S$ the unipotent part. The group $W_S \subseteq W$ is the Weyl group of $\mathfrak{l}_S$, and we let $w_{0,S}$ be its longest element.

Choose a reduced expression $w_{0,S} = s_{i_n} \dots s_{i_1}, i_t \in S$, and extend it to a reduced expression $w_0 = s_{i_N} \dots s_{i_{n+1}} w_{0,S}$. Define elements

$$\beta_t = s_{i_1} \dots s_{i_{t-1}}(\alpha_{i_t}),\ t = 1, \dots, N.$$

Then $\Phi_S^+ = \{\beta_1, \dots, \beta_n\}$ and $\Phi^+ \setminus \Phi_S^+ = \{\beta_{n+1}, \dots, \beta_N\}$.

Let $T_i, i = 1, \dots, r$, denote the Braid group operators on the generic quantum group U_v, cf. [CP95, 8.1] or [Jan96, 8.14]. It is a direct calculation to check that the T_i preserve the A-form U_A. We use these to define elements of U_A associated to the roots above. Let

$$\begin{aligned} E_{\beta_t}^{(a)} &= T_{i_1} \dots T_{i_{t-1}}(E_{i_t}^{(a)}),\ t = 1, \dots, N, a \geq 1, \\ F_{\beta_t}^{(a)} &= T_{i_1} \dots T_{i_{t-1}}(F_{i_t}^{(a)}),\ t = 1, \dots, N, a \geq 1. \end{aligned}$$

Then [CP95, Proposition 9.3.3] or the proof of [Jan96, Theorem 8.24] shows that the set consisting of the elements

$$E_{\beta_N}^{(a_N)} \dots E_{\beta_1}^{(a_1)} K_1^{\sigma_1} \begin{bmatrix} K_1 \\ b_1 \end{bmatrix}_{v_1} \dots K_r^{\sigma_r} \begin{bmatrix} K_r \\ b_r \end{bmatrix}_{v_r} F_{\beta_1}^{(c_1)} \dots F_{\beta_N}^{(c_N)} \tag{6.2}$$

for $a_t, b_i, c_t \in \mathbb{N}, \sigma_i \in \{0, 1\}$, is an A-basis of U_A.

Definition 6.2. *The integral parabolic subalgebra $U_A(\mathfrak{p}_S)$ associated to $\mathfrak{p}_S$ is the subspace of U_A spanned by the elements*

$$E_{\beta_n}^{(a_n)} \dots E_{\beta_1}^{(a_1)} K_1^{\sigma_1} \begin{bmatrix} K_1 \\ b_1 \end{bmatrix}_{v_1} \dots K_r^{\sigma_r} \begin{bmatrix} K_r \\ b_r \end{bmatrix}_{v_r} F_{\beta_1}^{(c_1)} \dots F_{\beta_N}^{(c_N)}$$

for $a_t, b_i, c_s \in \mathbb{N}, \sigma_i \in \{0, 1\}$.

This is an algebra, which is a consequence of the following

Lemma 6.3. *For $1 \leq i < j \leq n$ and $a, b \in \mathbb{N}$ we have*

$$E_{\beta_i}^{(a)} E_{\beta_j}^{(b)} - v^{(b\beta_j, a\beta_i)} E_{\beta_j}^{(b)} E_{\beta_i}^{(a)} = \sum_{\underline{a} \in \mathbb{N}^{j-i+1}} c_{\underline{a}} E_{\beta_j}^{(a_j)} \dots E_{\beta_i}^{(a_i)} \tag{6.3}$$

where $c_{\underline{a}} \in A$ is nonzero only if $\underline{a} = (a_i, \dots, a_j)$ has $a_i < a$ and $a_j < b$.

Proof. When $a = b = 1$ this is the Levendorskii-Soibelman relation [LS91, Proposition 5.5.2]. The result is proven by induction, first over $j - i$ and then over a and b. □

Though the basis in (6.2) depends on the choices of the reduced expressions of $w_{0,S}$ and w_0, the definition of $U_A(\mathfrak{p}_S)$ does not.

We conclude that $U_q(\mathfrak{g})$ is free as a right $U_q(\mathfrak{p}_S)$-module, with an A-basis

$$E_{\beta_N}^{(a_N)} \dots E_{\beta_{n+1}}^{(a_{n+1})}$$

for $a_t \in \mathbb{N}$.

Definition 6.4. *The parabolic quantum group is the specialization $U_q(\mathfrak{p}_S) = U_A(\mathfrak{p}_S) \otimes_A \mathbb{C}$ of v to q.*

Parabolic modules

We define $\mathcal{C}_{q,S}$ to be the category of integrable $U_q(\mathfrak{p}_S)$-modules, i.e., it consists of modules $M = \bigoplus_{\lambda \in P} M_\lambda$ such that for all $m \in M$ and each $i \in S, j \in \bar{D}$ $E_i^{(a)} m = 0 = F_j^{(a)} m$ for big enough a. Let F_S be the functor from the category of $U_q(\mathfrak{p}_S)$-modules to $\mathcal{C}_{q,S}$ which sends a module to its maximal integrable submodule. We define an induction functor $H_q^0(S, -) : \mathcal{C}_q^- \to \mathcal{C}_{q,S}$ by

$$H_q^0(S, N) = F_S(\mathrm{Hom}_{B_q}(U_q(\mathfrak{p}_S), N))$$

for $N \in \mathcal{C}_q^-$. For each $\lambda \in P$ we define the induction module

$$\nabla_{q,S}(\lambda) = H_q^0(S, \mathbb{C}_\lambda).$$

Note that $\nabla_{q,S}(\lambda) \neq 0$ if and only if $\lambda \in P_S^+$, and for such λ it has a unique simple submodule $L_{q,S}(\lambda)$. $\nabla_{q,S}(\lambda)$ is finite-dimensional, and its character equals

$$\mathrm{ch}\, \nabla_{q,S}(\lambda) = \frac{\sum_{w \in W_S} (-1)^{l(w)} e^{w(\lambda + \rho_S)}}{\sum_{w \in W_S} (-1)^{l(w)} e^{w(\rho_S)}} \in \mathbb{Z}[P]^{W_S}.$$

The antipode on U_q restricts to $U_q(\mathfrak{p}_S)$ and gives a $U_q(\mathfrak{p}_S)$-structure on dual modules. We define

$$\Delta_{q,S}(\lambda) = \nabla_{q,S}(-w_{0,S}(\lambda))^*.$$

Then $\nabla_{q,S}(\lambda)$ and $\Delta_{q,S}(\lambda)$ have the same character, and $L_{q,S}(\lambda)$ occurs as the unique simple quotient of $\Delta_{q,S}(\lambda)$.

Let $H^i_q(S,-)$ be the ith right derived functor of $H^0_q(S,-)$. Then the linkage principle, Theorem 1.5, generalizes to show that for $\lambda \in P^+_S, \mu \in P \cap \bar{C}_S$, if $L_{q,S}(\lambda)$ is a composition factor of $H^i_{q,S}(w.\mu)$ for some $w \in W_S$ and $i \geq 0$, then $\lambda \uparrow_{\Phi_S} \mu$. Let $W_{l,S}$ be the subgroup of W_l generated by all affine reflections $s_{\alpha,m}, \alpha \in \Phi_S, m \in \mathbb{Z}$. Then Corollary 1.6 generalizes to $\mathcal{C}_{q,S}$, and we see that the module $\nabla_{q,S}(\mu)$ is simple when $\mu \in \bar{A}_{l,S} \cap P^+_S$, where $A_{l,S} = \{\lambda \in P \mid 0 < \langle \lambda + \rho_S, \alpha^\vee \rangle < l_\alpha \ \forall \alpha \in \Phi^+_S\}$. Note that $A_{l,S}$ is an infinite set unless $S = \bar{D}$.

As another consequence of the corollary, we get a decomposition of $\mathcal{C}_{q,S}$ into subcategories $\mathcal{C}_{q,S}(\lambda), \lambda \in \bar{A}_{l,S}$, consisting of modules whose composition factors have highest weights in $W_{l,S}.\lambda$. Similarly, we get translation functors between them.

We say that a finite-dimensional module $T \in \mathcal{C}_{q,S}$ is tilting if it has both a $\nabla_{q,S}$- and a $\Delta_{q,S}$-filtration. Denote by $\mathcal{T}_{q,S}$ the category of tilting modules. We have all the tools needed to generalize Proposition 1.8, i.e., for each $\lambda \in P^+_S$ we have a unique up to isomorphism indecomposable tilting module $T_{q,S}(\lambda) \in \mathcal{T}_{q,S}$ of highest weight λ.

Given another subset $S \subseteq T \subseteq \bar{D}$, all of the above applies with S replaced by T. Moreover, given a reduced expression $w_{0,S} = s_{i_n} \dots s_{i_1}$ we may extend it to a reduced expression $w_{0,T} = s_{i_m} \dots s_{i_{n+1}} w_{0,S}$ and again to a reduced expression $w_0 = s_{i_N} \dots s_{i_{m+1}} w_{0,T}$. We see that $U_q(\mathfrak{p}_T)$ is free as a right $U_q(\mathfrak{p}_S)$-module, with basis

$$E^{(a_m)}_{\beta_m} \dots E^{(a_{n+1})}_{\beta_{n+1}}$$

for $a_t \in \mathbb{N}$. We get a functor $H^0_q(T/S,-) : \mathcal{C}_{q,S} \to \mathcal{C}_{q,T}$ by

$$H^0_q(T/S, M) = F_T(\mathrm{Hom}_{U_q(\mathfrak{p}_S)}(U_q(\mathfrak{p}_T), M))$$

for $M \in \mathcal{C}_{q,S}$. Transitivity of induction gives us $\nabla_{q,T}(\lambda) \simeq H^0_q(T/S, \nabla_{q,S}(\lambda))$ for $\lambda \in P$.

We have a restriction functor $\mathcal{C}_q \to \mathcal{C}_{q,S}$ which makes $\mathcal{C}_{q,S}$ into a $\mathcal{C}_q$-module category. Furthermore, if $V \in \mathcal{C}_q$ is finite-dimensional, then by the tensor identity $H^0_q(T/S, V \otimes M) \simeq V \otimes H^0_q(T/S, M)$ for $M \in \mathcal{C}_{q,S}$, this module structure is compatible with induction.

Conclusions and problems

Consider the direct sums of Grothendieck groups $C^i = \bigoplus_{S\subseteq \bar{D},|S|=r-i} \mathrm{K}_0(\mathcal{C}_{q,S})$ and the maps $d : C^i \to C^{i-1}$ given by components $d^{S,T} : \mathrm{K}_0(\mathcal{C}_{q,S}) \to \mathrm{K}_0(\mathcal{C}_{q,T})$ for $T = S \cup \{i_s\}$, where $i_s \in \bar{D} \setminus S = \{i_0, \dots, i_p\}$ is $(-1)^s$ times the map induced by $H^0_q(T/S, -)$. This is an acyclic complex of $\mathrm{K}_0(\mathcal{C}_q)$-modules. $\mathrm{K}_0(\mathcal{C}_{q,S})$ has a $\mathbb{Z}$-basis in $\{[\nabla_{q,S}(\lambda)] \mid \lambda \in P^+_S\}$, and w.r.t. these bases the components $d^{S,T}$ are given by

$$d^{S,T}([\nabla_{q,S}(\lambda)]) = [\nabla_{q,T}(\lambda)].$$

The above satisfies part of our goal but does not completely realize the complex. We have not defined a parabolic subalgebra $U_q(\mathfrak{p}_S) \subseteq U_q$ for subsets $S \subseteq D$ containing 0, i.e., a subalgebra whose Levi component $U_q(\mathfrak{l}_S)$ has an associated root system Φ_S, whose simple roots are $\{-\theta, \alpha_{i_1}, \dots, \alpha_{i_p}\}$, where θ is either the highest short or long root. We could take a reduced expression $w_0 = s_{i_N} \dots s_{i_1}$ and look at the root vectors $E^{(r)}_{\beta_t}, F^{(r)}_{\beta_t}$ associated to $\beta_t = s_{i_1} \dots s_{i_{t-1}}(\alpha_{i_t})$. Let $U_q(\mathfrak{l}_S)$ be the span of the elements

$$E^{(a_m)}_{\beta_{t_m}} \dots E^{(a_1)}_{\beta_{t_1}} K_1^{\sigma_1} \begin{bmatrix} K_1 \\ b_1 \end{bmatrix}_{q_q} \dots K_r^{\sigma_r} \begin{bmatrix} K_r \\ b_r \end{bmatrix}_{q_r} F^{(c_1)}_{\beta_{t_1}} \dots F^{(c_m)}_{\beta_{t_m}}$$

where the β_{t_i} run over all roots $\Phi_S \cap \Phi^+$. Unfortunately, this is not necessarily an algebra. The problem is that we do not have an analogue of the Levendorskii-Soibelman relations for subsets of the ordered set $\{\beta_1, \dots, \beta_N\}$ like this.

Another detail is that the categories $\mathcal{C}_{q,S}$ are abelian, monoidal categories, while we asked for additive categories. In particular, for $S = \bar{D}$, $\mathcal{C}_{q,\bar{D}} = \mathcal{C}_q$ is the category of integrable U_q-modules and not the category $\mathcal{T}_q$ of tilting modules we wished for. As rings though, the Grothendieck group $\mathrm{K}_0(\mathcal{C}_q)$ and the split Grothendieck group $\mathrm{K}_0(\mathcal{T}_q)$ are both isomorphic to $\mathbb{Z}[P]^W$. We cannot use the tilting categories $\mathcal{T}_{q,S}$ for the part of $\mathcal{D}_{q,S}$, as there is no reason to expect that the induction functor $H^0_q(T/S, -) : \mathcal{C}_{q,S} \to \mathcal{C}_{q,T}$ restricts to a functor $\mathcal{T}_{q,S} \to \mathcal{T}_{q,T}$.

Bibliography

[And03] Henning Haahr Andersen. The strong linkage principle for quantum groups at roots of 1. *J. Algebra*, 260(1):2–15, 2003. Special issue celebrating the 80th birthday of Robert Steinberg.

[And04] Henning Haahr Andersen. Cells in affine Weyl groups and tilting modules. In *Representation theory of algebraic groups and quantum groups*, volume 40 of *Adv. Stud. Pure Math.*, pages 1–16. Math. Soc. Japan, Tokyo, 2004.

[AP95] Henning Haahr Andersen and Jan Paradowski. Fusion categories arising from semisimple Lie algebras. *Comm. Math. Phys.*, 169(3):563–588, 1995.

[APK91] Henning Haahr Andersen, Patrick Polo, and Wen Kexin. Representations of quantum algebras. *Inventiones mathematicae*, 104(1):1–59, 1991.

[AS14] Henning Haahr Andersen and Catharina Stroppel. Fusion rings for quantum groups, 2014.

[BE74] D.A. Buchsbaum and D. Eisenbud. *Algebra Structures for Finite Free Resolutions: And Some Structure Theorems for Ideals of Codimension 3*. Brandeis University, 1974.

[Bea96] Arnaud Beauville. Conformal blocks, fusion rules and the Verlinde formula. In *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993)*, volume 9 of *Israel Math. Conf. Proc.*, pages 75–96, Ramat Gan, 1996. Bar-Ilan Univ.

[BK09] Arzu Boysal and Shrawan Kumar. A conjectural presentation of fusion algebras. In *Algebraic analysis and around*, volume 54 of *Adv. Stud. Pure Math.*, pages 95–107. Math. Soc. Japan, Tokyo, 2009.

[BMRS92] Michelle Bourdeau, Eli J. Mlawer, Harold Riggs, and Howard J. Schnitzer. Topological Landau-Ginzburg matter from $\mathrm{Sp}(N)_K$ fusion rings. *Modern Phys. Lett. A*, 7(8):689–700, 1992.

[Bou02] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 4–6*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.

[BR06] Peter Bouwknegt and David Ridout. Presentations of Wess-Zumino-Witten fusion rings. *Rev. Math. Phys.*, 18(2):201–232, 2006.

[CP95] Vyjayanthi Chari and Andrew Pressley. *A guide to quantum groups*. Cambridge University Press, Cambridge, 1995. Corrected reprint of the 1994 original.

[Don93] Stephen Donkin. On tilting modules for algebraic groups. *Math. Z.*, 212(1):39–60, 1993.

[Dou09] Christopher L. Douglas. On the structure of the fusion ideal. *Comm. Math. Phys.*, 290(1):335–355, 2009.

[Dou13] Christopher L. Douglas. Fusion rings of loop group representations. *Comm. Math. Phys.*, 319(2):395–423, 2013.

[ENO05] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik. On fusion categories. *Ann. of Math. (2)*, 162(2):581–642, 2005.

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[Fin96] M. Finkelberg. An equivalence of fusion categories. *Geom. Funct. Anal.*, 6(2):249–267, 1996.

[Fuc94] Jürgen Fuchs. Fusion rules in conformal field theory. *Fortschr. Phys.*, 42(1):1–48, 1994.

[Gep91] Doron Gepner. Fusion rings and geometry. *Comm. Math. Phys.*, 141(2):381–411, 1991.

[GS92] Doron Gepner and Adam Schwimmer. Symplectic fusion rings and their metric. *Nucl. Phys. B*, 380:147–167, 1992.

[Hum72] James E. Humphreys. *Introduction to Lie algebras and representation theory*. Springer-Verlag, New York, 1972. Graduate Texts in Mathematics, Vol. 9.

[Jan96] Jens Carsten Jantzen. *Lectures on quantum groups*, volume 6 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1996.

[KL79] David Kazhdan and George Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53(2):165–184, 1979.

[KRK88] Abdelilah Kandri-Rody and Deepak Kapur. Computing a Gröbner basis of a polynomial ideal over a Euclidean domain. *J. Symbolic Comput.*, 6(1):37–57, 1988.

[KS10] Christian Korff and Catharina Stroppel. The $\widehat{\mathfrak{sl}}(n)_k$-WZNW fusion ring: a combinatorial construction and a realisation as quotient of quantum cohomology. *Adv. Math.*, 225(1):200–268, 2010.

[Kun85] Ernst Kunz. *Introduction to commutative algebra and algebraic geometry*. Birkhäuser Boston, Inc., Boston, MA, 1985. Translated from the German by Michael Ackerman, With a preface by David Mumford.

[Lar14] Hannah K. Larson. Classification of some fusion categories of rank 4, 2014.

[LS91] Serge Levendorskiĭ and Yan Soibelman. Algebras of functions on compact quantum groups, Schubert cells and quantum tori. *Comm. Math. Phys.*, 139(1):141–170, 1991.

[Lus83] George Lusztig. Some examples of square integrable representations of semisimple p-adic groups. *Trans. Amer. Math. Soc.*, 277(2):623–653, 1983.

[Lus85] George Lusztig. Cells in affine Weyl groups. In *Algebraic groups and related topics*, volume 6 of *Adv. Stud. Pure Math.*, pages 255–287. North-Holland, Amsterdam, 1985.

[Lus87] George Lusztig. Cells in affine Weyl groups. II. *J. Algebra*, 109(2):536–548, 1987.

[Lus10] George Lusztig. *Introduction to quantum groups.* Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2010. Reprint of the 1994 edition.

[LX88] George Lusztig and Nan Hua Xi. Canonical left cells in affine Weyl groups. *Adv. in Math.*, 72(2):284–288, 1988.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1998.

[Ost97] V. Ostrik. Tensor ideals in the category of tilting modules. *Transform. Groups*, 2(3):279–287, 1997.

[Ost01] Viktor Ostrik. Dimensions of quantized tilting modules. *Mosc. Math. J.*, 1(1):65–71, 2001.

[Ost03] Viktor Ostrik. Fusion categories of rank 2. *Math. Res. Lett.*, 10(2-3):177–183, 2003.

[Ost08] Victor Ostrik. Pre-modular categories of rank 3. *Mosc. Math. J.*, 8(1):111–118, 184, 2008.

[Par92] J. Paradowski. *Filtrations of Modules Over the Quantum Algebra.* Matematisk Institut Århus: Preprint series. Aarhus Universitet, 1992.

[Pul06] Riccardo Pulcini. Degree of parabolic quantum groups, 2006.

[Qui76] Daniel Quillen. Projective modules over polynomial rings. *Inventiones Mathematicae*, 36(1):167–171, 1976.

[RH03] Steen Ryom-Hansen. A q-analogue of Kempf's vanishing theorem. *Mosc. Math. J.*, 3(1):173–187, 260, 2003.

[Rin91] Claus Michael Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Math. Z.*, 208(2):209–223, 1991.

[Ser63] Jean-Pierre Serre. Sur les modules projectifs. Algebre Theorie Nombres, Sem. P. Dubreil, M.-L. Dubreil-Jacotin et C. Pisot 14 (1960/61), No.2, 16 p. (1963)., 1963.

[Shi86] Jian Yi Shi. *The Kazhdan-Lusztig cells in certain affine Weyl groups*, volume 1179 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1986.

[Soe97a] Wolfgang Soergel. Kazhdan-Lusztig-Polynome und eine Kombinatorik für Kipp-Moduln. *Represent. Theory*, 1:37–68 (electronic), 1997.

[Soe97b] Wolfgang Soergel. Kazhdan-Lusztig polynomials and a combinatoric[s] for tilting modules. *Represent. Theory*, 1:83–114 (electronic), 1997.

[Soe98] Wolfgang Soergel. Character formulas for tilting modules over Kac-Moody algebras. *Represent. Theory*, 2:432–448 (electronic), 1998.

[TUY89] Akihiro Tsuchiya, Kenji Ueno, and Yasuhiko Yamada. Conformal field theory on universal family of stable curves with gauge symmetries. In *Integrable systems in quantum field theory and statistical mechanics*, volume 19 of *Adv. Stud. Pure Math.*, pages 459–566. Academic Press, Boston, MA, 1989.