Demazure descent theory

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Contents

Abstract

This dissertation consists of several projects. In the first we introduce the notions of Demazure descent data (DDD) for a triangulated category. This is a collection of functors satisfying a categorical version of degenerate Hecke algebra relations. For such data we define the descent category which can be seen as a categorical version of taking invariants. We construct such data explicitly for the derived category of representations of a Borel subgroup B of an reductive algebraic group G and more generally for the derived category of B-equivariant quasi-coherent sheaves on a G-scheme X. We prove that the derived categories of representations of G and of G-equivariant quasi-coherent sheaves on X is equivalent to their respective descent categories. The result for quasi-coherent sheaves is a categorification of a result in K-theory by Harada, Landweber and Sjamaar.

The next project studies the absolute derived category of equivariant matrix factorizations, where the potential is induced by the moment map of the Hamiltonian action of G on the cotangent bundle of a smooth complex G-variety. Combining results of Isik and Polishchuk-Vaintrob one obtains that in the non-equivariant setting this category is equivalent to the derived category of coherent sheaves on the zero scheme of the moment map. We prove that this result extends to the equivariant setting. This provides an equivalence between the equivariant absolute derived category of matrix factorizations and the derived category of coherent sheaves on the Hamiltonian reduction.

The last project is inspired a by a construction by Bezrukavnikov and Riche of a categorical action of the affine braid group on the equivariant derived category of coherent sheaves on both the Grothendieck and the Springer variety. We use this result to construct such an action on the equivariant absolute derived category of a slightly modified version of the matrix factorizations studied in the previous project.

Résumé

Denne afhandling indeholder flere projekter. I det første introducerer vi begrebet Demazure descent data (DDD). Dette er en samling af funktorer som opfylder en kategorisk udgave af degenerate Hecke algebra relationer. For denne data definerer vi descent kategorien der kan ses som kategorisk måde at tage invarianter på. Vi konstruerer denne data eksplicit for den afledte kategori af repræsentationer af en Borel undergruppe B af en reduktiv algebraisk gruppe G, og mere generelt for den afledte kategori af B-ækvivariante kvasi-koherente knipper på et G-skema X. Vi viser at de afledte kategorier af repræsentationer af G og af G-ækvivariante kvasi-koherente knipper på X er ækvivalente til deres respektive descent kategorier. Resultatet for kvasi-koherente knipper er en kategorifisering af et resultat i K-theori af Harada, Landweber og Sjamaar.

Det næste projekt studerer den absolute afledte kategori af af ækvivariante matrix faktorisationer, hvor potentialet er induceret af moment afbildningen for den Hamiltoniske virkning af G på kotangent bundtet af en glat kompleks G-varietet. Ved at kombinerer resultater af Isik og Polishchuk-Vaintrob får man at i det ikke-ækvivariante tilfælde er denne kategori ækvivalent til den afledte kategori af koherente knipper på nul-skemaet af moment afbildningen. Vi viser at dette resultat kan generaliseres til den ækvivariantte situation. Denne giver en ækvivalens mellem den ækvivariante absolute afledte kategori af matrix faktorisationer og den afledte kategori af koherente knipper på den Hamiltoniske reduktion.

Det sidste projekt er inspireret af en konstruktion af Bezrukavnikov og Riche af en kategorisk virkning af den affine braid gruppe på den ækvivariante afledte kategori af koherente knipper på både Grothendieck og Springer varieteten. Vi anvender dette resultat til at konstruere en sådan virkning på den ækvivariante afledte kategori af koherente knipper på en let modificeret udgave af de matrix factorisationer vi studerede i det foregående projekt.

CHAPTER 1

Introduction

1.1. Introduction

There are many trends in modern representation theory. One is to realize algebraic objects as homological invariants (Borel-Moore homology, topological and algebraic K-theory) of spaces of different geometric nature (topological spaces, manifolds, algebraic varieties). Another one is categorification, in which algebraic objects are replaced by categories and morphisms by functors.

1.1.1. Geometric realizations of algebraic objects. One advantage of geometric realizations in terms of homological invariants is that such invariants often come with a tool box containing e.g. pull-backs, push-forwards, long exact sequences. These can be used to construct new structures, which are natural in this setting, but would have been very hard to come up with just looking at the algebraic description. One example of a triumph of this approach is the well known result of Kazhdan-Lusztig and Ginzburg realizing the group algebra of the Weyl group as the top Borel-Moore homology of the Steinberg variety, where the algebra structure on the later is given by a convolution. Using this description they were able to classify all irreducible representations. By varying the homology theory (e.g. replacing Borel-Moore homology by topological or algebraic K-theory) one can get deformations of the group representation.

1.1.2. Idea of categorification. The idea of categorification is to replace an algebraic object with a category from which the original object can be recovered by passing to the Grothendieck group. A vector space V is replaced by a (Abelian, differential graded or triangulated) category $\mathcal{C}$ whose Grothendieck group is isomorphic to the vector space. Additional structure should be encoded categorically in such a way that the original structure is recovered in the Grothendieck group. When passing to the Grothendieck group functors become morphisms so actions of various kinds should be given by a collection of functors satisfying some relations. The action of a group is encoded in the following way.

Definition 1.1. A categorical action of a group Γ on a category $\mathcal{C}$ is a collection of endofunctors $\{F_\gamma : \mathcal{C} \to \mathcal{C} \mid \gamma \in \Gamma\}$ satisfying $F_{\gamma_1} \circ F_{\gamma_2} \simeq F_{\gamma_1\gamma_2}$ for all γ_1, γ_2 in $\mathcal{C}$.

Remark 1.2. Notice that we do not impose any assumptions on the isomorphisms. This is sometimes called a weak group action as opposed to a strong group action in which the isomorphisms are required to satisfy some commutativity conditions.

One benefit of this approach is that categories offer higher flexibility and they often come with a natural set of distinguished objects (e.g. indecomposable projective objects or simple objects). When passing to the Grothendieck group these objects defines a canonical basis for V and such bases often have interesting combinatorial properties. Another advantage is that categories sometimes comes equipped with a natural set of functors, which can be used to create new structures on the vector space or study the existing ones. This is especially common when the category is of a geometric nature such as sheaves (coherent, quasi-coherent, constructible, D-modules) on a space (variety, scheme). These categories usually have tensor products and pull-back and push-forward along nice morphisms, which can be used to define a convolution product on the category. A convolution on the category can induce a ring structure on the Grothendieck group. This is the way most of the ring structures arises on the geometric realizations of algebraic objects mentioned above.

One of the most famous examples of a categorical approach to a result in pure algebra is the Kazhdan-Lusztig conjecture from 1979 [**KL1**, Conj. 1.5]. The conjecture is about the relation between the standard basis for the Hecke algebra and a canonical basis invariant with respect to an involution. Since the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for a semi-simple Lie algebra is a categorification of the Hecke algebra the conjecture can also be translated into a question about the relationship in the Grothendieck group between the Verma modules and the simple modules. The conjecture was proved in 1981 independently by Beilinson and Bernstein [**BB**] and by Brylinski and Kashiwara [**BK**] using geometrically defined categories equivalent to category $\mathcal{O}$ such as perverse sheaves and D-modules. An algebraic proof using the category of Soergel bimodules has recently been given by Elias and Williamson [**EW**].

1.1.3. Categorifications of results in K-theory. In the opposite direction, known constructions on the Grothendieck group can serve as inspiration for new constructions on the categorical level. K-theory is the Grothendieck group of the (derived) category of coherent sheaves so it is natural to try to lift K-theoretic constructions to the level of (derived) categories. The first part of this thesis construct such a lifting of a result by Harada, Landweber and Sjamaar [**HLS**]. They consider a algebraic variety X with an action of a split reductive algebraic group G with a maximal torus T. The authors show that the natural Weyl group action on the T-equivariant algebraic K-theory of X extends to an action of the degenerate Hecke ring generated by divided difference operators. These operators were originally introduced by Demazure and the corresponding operators on K-theory are called Demazure operators. The main result is that G-equivariant classes are exactly those T-equivariant classes, which are killed by the Demazure operators. Our goal is to replace the Demazure operators by Demazure functors on the derived category of coherent sheaves on X equivariant with respect to a Borel subgroup B (on the level of K-theory T-equivariance and B-equivariance is the same, but this is not the case on the categorical level). We aim to lift their main result to a categorical level showing that G-equivariant complexes are those B-equivariant complexes on which the constructed Demazure functors act as identity.

1.1.4. Matrix factorizations coming from Hamiltonian actions. Matrix factorizations are pairs of coherent sheaves on a scheme X with morphisms in each direction satisfying that both compositions are multiplication by a fixed function on X, called the potential. They where originally introduced by Eisenbud in [**Eis**] as a tool to study maximal Cohen-Macaulay modules. They have since become a standard tool and an object of study in commutative algebra (see e.g [**BGS**], [**Yos**]) and has also spread to other fields including topology and algebraic geometry.

Since the composition is not zero the ordinary definition of the derived category does not work. However, another notion of derived category for matrix factorizations has been defined and studied extensively by Positselski (see e.g. [**Pos**]). This kind of derived category is called the absolute derived category. Following a suggestion by Kontsevich it has been used in theoretical physics to describe D-branes of type B in Landau-Ginzburg models (see [**KaLi1**] and [**KaLi2**]). They've also found applications in various approaches to mirror symmetric and in the study of sigma model/Landau-Ginzburg correspondence (see [**KKP**], [**Efi2**], [**Sei**], [**BHLW**] and [**HHP**]). The mathematical foundations of these ideas is due to Orlov (see [**Orl1**], [**Orl2**] and [**Orl3**]).

There are several known generalizations of matrix factorizations. One of them is the notion of curved differential graded modules, which is also due to Positselski. In this context they have been studied extensively by Positselski and Efimov in the papers [**Pos**] and [**EP**]. It turns out that a lot of the homological algebra machinery known from ordinary derived categories can be made to work on the absolute derived category. Another generalization is to replace the category of coherent sheaves by any Abelian category. In this setting the potential becomes the application of a natural transformation $w : \mathrm{Id} \to \mathrm{Id}$. This approach has been developed in [**Efi1**].

We are mainly interested in the kind of matrix factorizations which comes from a Hamiltonian action. More precisely, we are interested in matrix factorizations on $T^*X \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of a complex reductive algebraic group G and X is a smooth complex G-variety. The potential W is the composition of the moment map $\mu : T^*X \to \mathfrak{g}^*$ times identity and the natural pairing $\mathfrak{g} \times \mathfrak{g}^* \to \mathbb{C}$. However, most of our results work in greater generality. It has been proved by Polishchuk and Vaintrob [**PV**] that in this case the absolute derived category of matrix factorizations is equivalent to the singularity category of $W^{-1}(0)$. Furthermore, Isik proved that this singularity category is equivalent to the derived category of coherent sheaves on the zero scheme of μ [**Isik**]. Our goal is to upgrade this result to the equivariant setting. One reason such a result is interesting is that it gives an equivalence between the equivariant absolute derived category coming from a Hamiltonian action and the derived category of coherent sheaves on the Hamiltonian reduction.

1.1.5. Braid group actions in representation theory. Categorical braid group actions appear in many places in representation theory. For example it turns out that many functors, which have now become standard tools in the study of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for a semisimple Lie algebra satisfy braid relations. These include shuffling functors, Joseph's completion functors and Arkhipov's twisting functors (see [**MS**]). Among other things they play an important role in the study of Kazhdan-Lusztig combinatorics of the category $\mathcal{O}$ (see [**KL1**]) together with other geometrically defined functors on perverse sheaves and D-modules also satisfying braid relations.

Another case in which an affine braid group action plays an important role is related to noncommutative resolutions of singularities of $\mathcal{N}$ (for a discussing see [**Bez3**]). A noncommutative resolution of singularities is a sheaf of associative $\mathcal{O}_{\mathcal{N}}$-algebras A satisfying $D(A-\mathrm{mod}) \simeq D(\mathrm{Coh}(\tilde{\mathcal{N}}))$. It is determined up to Morita equivalence by a t-structure on $D(\mathrm{Coh}(\tilde{\mathcal{N}}))$ called the exotic t-structure. The heart of this t-structure is called exotic sheaves and these can be described in terms of a certain action of the affine braid group for the Langlands dual group on $D(\mathrm{Coh}(\tilde{\mathcal{N}}))$. Exotic sheaves are connected both to perverse sheaves on the affine flag variety for the Langlands dual group, and to modular Lie algebra representations. This connection has been used to approach Lusztig's conjecture [**Lus**] relating the classes of irreducible $\mathfrak{g}$-modules to elements of the canonical basis in the Borel-Moore homology of a Springer fiber. The appearance of the affine braid group suggests a connection with homological mirror symmetry. This was the motivation for Seidel and Thomas [**ST**] to study braid group actions on derived categories of coherent sheaves.

In [**BR**] Bezrukavnikov and Riche constructed a categorical geometric action of the affine braid on the bounded derived categories of $G \times \mathbb{G}_m$-equivariant coherent sheaves on both the Grothendieck and the Springer variety as well as their DG-category analogs. Their construction works both in characteristic zero and in positive characteristic with some mild restrictions on the characteristic. The action has also been used by Bezrukavnikov and Mirković [**BM**] to prove several conjectures by Lusztig about some numerical properties of representations of semisimple Lie algebras in positive characteristic. It also plays a technical role in the study of Koszul duality for modular representations of semi-simple Lie algebras (see [**Ric2**]). In section 7.1 we describe the construction of the categorical affine braid group action in [**BR**] in detail, since we will use it to construct a categorical action of the affine braid group on matrix factorizations similar to the ones studied in the previous project.

1.2. Chapter overview

In chapter 2 we recall the algebro-geometric realizations in terms of homology or K-theory of some classical objects in representation theory. These realizations are by now standard results in geometric representation theory and most of them can be found in the book of Criss and Ginzburg [**CG**]. The realizations in this section will not be used directly, but the schemes involved will be the main players in this thesis, so this chapter can be seen as a motivation for why these particular schemes are natural to study from a representation theory point of view.

Chapter 3 introduces the notion of Demazure descent data and descent category for a triangulated category. These notions were first introduced by Arkhipov and the author in the paper [**AK1**]. They can be seen as a categorification of a result in K-theory by Harada, Landweber and Sjamaar [**HLS**], which we recall in the first part of the chapter. For the purpose of this thesis it only serves as motivation and is not used anywhere.

The content of chapter 4 is essentially the paper [**AK1**] by Arkhipov and the author. Here we provide the first example of Demazure descent for the derived category of representations of a reductive algebraic group G and a Borel subgroup B. The main results are the construction of demazure descent data on $D^b(\mathsf{Rep}(B))$, and a theorem stating that the descent category corresponding to this data is equivalent to $D^b(\mathsf{Rep}(G))$. Since the schemes involved are affine the proof is largely of an algebraic nature.

Chapter 5 is the paper [**AK2**] by Arkhipov and the author. In this paper we extend the results of [**AK1**] to the derived category of equivariant quasi-coherent sheaves on a G-scheme X. We define a convolution product on the quasi-coherent Hecke category $D(\mathrm{QCoh}^G(G/B\times G/B))$ and a monoidal action of this category on $D(\mathrm{QCoh}^B(X))$. Demazure descent data on $D(\mathrm{QCoh}^B(X))$ is constructed as convolution with certain elements. We prove that the corresponding descent category is equivalent to $D(\mathrm{QCoh}^G(X))$. This result is a categorification of the main result in [**HLS**]. Although the results in the previous chapter is the special case $X =$ point of the results in this chapter, the proofs are not just direct generalizations of the proofs in that chapter, since the algebraic setting is no longer sufficient when X is not assumed to be affine. Instead the proofs in this chapter are entirely of an algebro-geometric nature.

In chapter 6 we study equivariant matrix factorizations on the cotangent bundle of a smooth complex G-variety X whose potential W comes from the moment map of the Hamiltonian action of G on T^*X. The main result in this chapter is an extension of the result mentioned in the introduction to the equivariant setting. The result of Polishchuk-Vaintrob already works in this generality so the main content of this chapter is an extension of the result by Isik to the equivariant setting. In Isik's proof the main step uses linear Kozsul duality as developed by Mirković and Riche in [**MR1**]. Mirković and Riche has later extended linear Koszul duality to the equivariant setting in [**MR3**]. Our method of proof is to follow the approach of Isik with the linear Koszul duality from [**MR1**] replaced by the one from [**MR3**].

In chapter 7 we work with matrix factorizations on the product of the Grothendieck variety and cotangent bundle to a smooth complex G-variety X with potential induced by the moment map. The main result is the construction of a categorical action of the affine braid group on the equivariant absolute derived category of matrix factorizations of this kind. The main source of inspiration for this project is an affine braid group action constructed by Bezrukavnikov and Riche [**BR**]. In their proof they construct a monoidal category with a monoidal action on these categories. They then identify objects in the monoidal category whose convolution with each other satisfy affine braid group relations. In our proof we also construct a monoidal category with a monoidal action on the category we are interested in. Using the direct analogy of the convolution in [**BR**] would result in a functor mapping into matrix factorizations with the wrong potential so some modifications are needed. The generators of the affine braid group action our category are identified by constructing a monoidal functor from a full subcategory of the monoidal category in [**BR**], containing the generators, to our monoidal category.

Chapter 8 contains suggestions for further projects which are natural extensions of the projects presented in this dissertation.

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CHAPTER 2

Geometric realizations of objects in representation theory

2.1. Notations

We always work over an algebraically closed field of characteristic 0. Let G be a reductive algebraic group, B a fixed Borel subgroup and T a maximal torus contained in B. In this thesis all subgroups are closed. The weight lattice $\mathrm{Hom}_{\mathrm{alg}}(T, \mathbb{C}^*)$ is denoted by $\mathbb{X}$, and the coweight lattice $\mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}^*, T)$ by $\mathbb{X}^\vee$. The root and coroot system is denoted by Φ and $\Phi^\vee$ respectively. Let W be the Weyl group. The generator of W corresponding to the root $\alpha \in \Phi$ is denoted by s_α. The set of simple roots is denoted by $\Pi = \{\alpha_1, \dots, \alpha_n\}$ and the corresponding set of simple reflections by S. Recall that W can be written as the group with generators simple reflections and relations

$$s_\alpha s_\beta s_\alpha \cdots = s_\beta s_\alpha s_\beta \cdots \qquad m(\alpha, \beta) \text{ factors},$$
$$s_\alpha^2 = 1$$

The first type of relations are called braid relations. The minimal number of factors needed to write $w \in W$ as a product of simple reflections is denoted by $\ell(w)$. The function taking w to $\ell(x)$ is called the length function. An expression $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_m}}$ with all $s_{\alpha_{i_k}} \in S$ is a reduced expression if $m = \ell(w)$.

The (extended) affine Weyl group, W_{aff}, is the semi-direct product $W \ltimes \mathbb{X}$. It acts on the complexification of the Lie algebra of the torus $\mathfrak{t}_\mathbb{C}$ by affine linear transformations and is generated by reflections in the hyperplanes

$$H_{\alpha,n} := \{v \mid (\alpha, v) = n\} \qquad \alpha \text{ root}, n \in \mathbb{Z}$$

The connected components of the complement of all these hyperplanes are called alcoves. The alcove

$$\{v \mid 0 < (\alpha, v) < 1 \text{ for all positive roots}\}$$

is the fundamental alcove.

Remark 2.1. The group W_{aff} is not a Coxeter group. Classically, the affine Weyl group $W_{\mathrm{aff}}^{\mathrm{Cox}}$ has been defined as $W \ltimes \mathbb{Z}\Phi$. This is a Coxeter group associated to the affine root system so it comes with a length function ℓ. There is an isomorphism $W_{\mathrm{aff}} \simeq W_{\mathrm{aff}}^{\mathrm{Cox}} \rtimes \Omega$, where $\Omega \subset W_{\mathrm{aff}}$ is the stabilizer of the fundamental alcove under the standard action. Setting $\ell(w) = 0$ for $w \in \Omega$ one can extend ℓ to W_{aff}.

The flag variety is the quotient $\mathcal{B} := G/B$. The following spaces will play an important role throughout the dissertation.

DEFINITION 2.2. (1) The Grothendieck variety is the variety

$$\tilde{\mathfrak{g}} := \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}\}.$$

(2) The projection $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is called the Grothendieck-Springer resolution.
(3) An element $x \in \mathfrak{g}$ is nilpotent if the map $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ is nilpotent. The set of all nilpotent elements in $\mathfrak{g}$ is called the nilpotent cone and is denoted by $\mathcal{N}$.
(4) The Springer variety is the preimage

$$\tilde{\mathcal{N}} := \mu^{-1}(\mathcal{N}) = \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}\}$$

and the restriction $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is called the Springer resolution.
(5) The fiber product over the Springer resolution

$$Z := \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} = \{(x, \mathfrak{b}), (x', \mathfrak{b}') \in \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \mid x = x'\}$$

is the Steinberg variety.

REMARK 2.3. (1) Clearly, $\mathcal{N}$ is a closed $\mathrm{Ad}(G)$-stable subvariety of $\mathfrak{g}$. It is stable under k^*-dilations so it is a cone variety.
(2) There is an isomorphism

$$G \times_B \mathfrak{b} \xrightarrow{\sim} \tilde{\mathfrak{g}}, \qquad (g, x) \mapsto (g \cdot x \cdot g^{-1}, g \cdot B/B).$$

In particular, $\tilde{\mathfrak{g}}$ is smooth and have a natural G-action. Similarly $\tilde{\mathcal{N}} \simeq G \times_B \mathfrak{n}$, where $\mathfrak{n}$ is the Lie algebra of the unipotent radical.
(3) The Grothendieck-Springer resolution is proper since it factors through $\mathfrak{g} \times \mathcal{B}$ and $\mathcal{B}$ is projective.
(4) There is a natural G-equivariant vector bundle isomorphism $\tilde{\mathcal{N}} \simeq T^*\mathcal{B}$ (see [**CG**, Lemma 3.2.2]). In particular, $\tilde{\mathcal{N}}$ is smooth and $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is a resolution of singularities.

2.2. Weyl groups

2.2.1. Finite Weyl group. The group algebra of the Weyl group can be constructed geometrically in terms of Borel-Moore homology. The results in this section will not be needed anywhere but together with subsequent sections it provides a beautiful example of the general principal in geometric representation theory of passing from homology to K-theory and finally to categories. For details we refer to section 2.6 and 2.7 in [**CG**].

DEFINITION 2.4. Let X be a complex variety with one-point compactification $\hat{X} = X \cup \{\infty\}$. The Borel-Moore homology of X is defined as

$$H_*^{\mathrm{BM}}(X) := H_*(\hat{X}, \infty),$$

where H_* is the ordinary relative homology with complex coefficients of the pair $(\hat{X}, \infty)$.

The one advantage of this type of homology is that any (not necessarily smooth or compact) complex algebraic variety X has a fundamental class $[X] \in H^{\mathrm{BM}}_{\dim_{\mathbb{R}} X}(X)$. Many constructions in ordinary homology induces constructions in Borel-Moore homology. For example Borel-Moore homology has push-forward along proper maps and a Künneth formula.

$$\boxtimes : H^{\mathrm{BM}}_*(X_1) \otimes H^{\mathrm{BM}}_*(X_2) \xrightarrow{\sim} H^{\mathrm{BM}}_*(X_1 \times X_2).$$

For two closed subspaces X_1, X_2 inside X one can define a cap product

$$\cap : H^{\mathrm{BM}}_i(X_1) \times H^{\mathrm{BM}}_j(X_2) \to H^{\mathrm{BM}}_{i+j-\dim_{\mathbb{R}} X}(X_1 \cap X_2).$$

This is done by transporting the cap product in cohomology to Borel-Moore homology using Poincaré duality (for details see section 2.6.15 in [**CG**]). Notice that the top degree $H^{\mathrm{BM}}_{\dim_R X}$ is conserved under this operation. Set $d := \dim_{\mathbb{R}} \tilde{\mathcal{N}}$ and let $p_{13} : \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \to \tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$ be the projection to the first and last factor. We define a convolution product on $H^{\mathrm{BM}}_d(Z)$ in the following way

$$\begin{gathered} * : H^{\mathrm{BM}}_d(Z) \times H^{\mathrm{BM}}_d(Z) \to H^{\mathrm{BM}}_d(Z), \\ a * b := p_{13*}((a \boxtimes [\tilde{\mathcal{N}}]) \cap [\tilde{\mathcal{N}}] \boxtimes b)). \end{gathered}$$

The convolution products on the level of K-theory and categories we are going to work with later are similar in spirit.

THEOREM 2.5. *There is a canonical algebra isomorphism*

$$H^{BM}_d(Z) \simeq \mathbb{Q}[W]$$

PROOF. This is theorem 3.4.1 in [**CG**]. □

One of the main applications of this geometrical realization is a classification of isomorphism classes of simple W-modules. The Springer fiber for an element $n \in \mathcal{N}$ is $\mathcal{B}_x := \mu^{-1}(x)$. The centralizer $C(x)$ is finite and acts on $H^{\mathrm{BM}}_{\dim_{\mathbb{R}} \mathcal{B}_x}(\mathcal{B}_x)$. Write $C(x)^\wedge$ for the set of irreducible representations of $C(x)$ that occur in $H^{\mathrm{BM}}_{\dim_{\mathbb{R}} \mathcal{B}_x}(\mathcal{B}_x)$. For $\psi \in C(x)^\wedge$ let $H^{\mathrm{BM}}_{\dim_{\mathbb{R}} \mathcal{B}_x}(\mathcal{B}_x)_\psi$ denote the ψ-isotypic component.

THEOREM 2.6. *Assume that G is semisimple. Then the set*

$$\{H^{BM}_{\dim_{\mathbb{R}} \mathcal{B}_x}(\mathcal{B}_x)_\psi \mid G - \textit{conjugacy classes of pairs } (x \in \mathcal{N}, \psi \in C(x)^\wedge)\}$$

is the complete collection of isomorphism classes of simple W-modules.

PROOF. This is theorem 3.6.9 in [**CG**]. □

2.2.2. Affine Weyl group. A similar construction to the one above with Borel-Moore homology replaced by T-equivariant K-theory gives a geometric realization of the affine Weyl group. The analog of the convolution in Borel-Moore homology is the tensor product in K-theory. Let X be a G-variety. The K-group $K_G(X)$ is the Grothendieck group of $\mathrm{Coh}^G(X)$ and $D^b(\mathrm{Coh}^G(X))$. Let Y be another G-variety and p_X (resp. p_Y) be the projection from $X \times Y$ to X (resp. Y). The functor

$$\boxtimes : \mathrm{Coh}^G(X) \times \mathrm{Coh}^G(Y) \to \mathrm{Coh}^G(X \times Y)$$
$$(\mathcal{F}, \mathcal{G}) \mapsto p_X^* \mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} p_Y^* \mathcal{G}$$

is exact so it induces a function on equivariant K-theory. Assume that X is smooth and let $\Delta : X \to X \times X$ be the diagonal embedding. Equivariant K-theory also has pull-back along G-equivariant closed embeddings so for any two closed subsets Y_1, Y_2 one can define a tensor product

$$\otimes : K_G(Y_1) \times K_G(Y_2) \to K_G(Y_1 \cap Y_2),$$
$$(\mathcal{F}, \mathcal{G}) \mapsto \Delta^*(\mathcal{F} \boxtimes \mathcal{G}).$$

Consider the $G \times \mathbb{C}^*$-action (we will need the $\mathbb{C}^*$-action in the next section) on $\tilde{\mathcal{N}}$ given by

$$\tilde{\mathcal{N}} \simeq \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}\}$$
$$(g, z) \cdot (x, \mathfrak{b}) := (z^{-1} g x g^{-1}, g \mathfrak{b} g^{-1}).$$

The diagonal $G \times \mathbb{C}^*$-action on $\tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$ induces an action on Z, since the morphism $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is equivariant when the action of $G \times \mathbb{C}^*$ on $\mathcal{N}$ is given by $(g, z) \cdot x = z^{-1} g x g^{-1}$.

THEOREM 2.7. *There is a natural algebra isomorphism* $K_G(Z) \simeq \mathbb{Z}[W_{\mathrm{aff}}]$.

PROOF. This is theorem 7.2.2 in [**CG**]. □

2.3. Hecke algebras

Hecke algebras can be thought of as q-deformations of the group algebra of the corresponding Weyl group.

DEFINITION 2.8. The (finite) Hecke algebra, $\mathcal{H}_q$, of the Coxeter group (W, S) is the $\mathbb{Z}[q, q^{-1}]$ algebra with generators $\{T_s \mid s \in S\}$ and relations

(i) $T_{s_\alpha} T_{s_\beta} T_{s_\alpha} \cdots = T_{s_\beta} T_{s_\alpha} T_{s_\beta} \cdots$ with $m(\alpha, \beta)$ factors on both sides.
(ii) $(T_{s_\alpha} + 1)(T_{s_\alpha} - q) = 0$.

It can be rewritten in the following form.

PROPOSITION 2.9. *The Hecke algebra is isomorphic to the algebra with a free* $\mathbb{Z}[q, q^{-1}]$ *basis* $\{T_w \mid w \in W\}$ *satisfying the following multiplication relations*

(i) $T_y T_w = T_{yw}$ *if* $\ell(y) + \ell(w) = \ell(yw)$.

(ii) $(T_s+1)(T_s-q)=0$ *if* $s \in S$.

PROOF. This is [**CG**, Prop. 7.1.2]. □

Recall the Bruhat decomposition of G into a disjoint union

$$G = \sqcup_{w \in W} BwB.$$

The Hecke algebra can be realized as an algebra of functions on a double coset.

PROPOSITION 2.10. *Let p be a prime number and set $q = p^n$ for some $n \in \mathbb{N}$. Then*

$$\mathcal{H}_q \otimes_{q \to p^n} \mathbb{C} \simeq \mathbb{C}\big(B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q)\big), \qquad T_w \leftrightarrow \delta_{BwB},$$

where δ_{BwB} is the function which is 1 on BwB and 0 elsewhere.

Notice that the group algebra $\mathbb{Z}[\mathbb{X}]$ is isomorphic to the ring of representations of T denoted by $R(T)$. Write e^λ for the element in $\mathbb{Z}[\mathbb{X}]$ corresponding to the weight $\lambda \in \mathbb{X}$.

DEFINITION 2.11. The affine Hecke algebra, $\mathcal{H}_{\text{aff}}$, is a free $\mathbb{Z}[q,q^{-1}]$-module with basis $\{e^\lambda \cdot T_w \mid w \in W, \lambda \in \mathbb{X}\}$, such that

(1) The $\{T_w\}$ span a subalgebra of $\mathcal{H}_{\text{aff}}$ isomorphic to $\mathcal{H}_q$.
(2) The $\{e^\lambda\}$ span a $\mathbb{Z}[q,q^{-1}]$-subalgebra of $\mathcal{H}_{\text{aff}}$ isomorphic to $R(T)[q,q^{-1}]$.
(3) For $s_\alpha \in S$ with $\langle \lambda, \alpha^\vee \rangle = 0$ we have $T_{s_\alpha} e^\lambda = e^\lambda T_{s_\alpha}$.
(4) For $s_\alpha \in S$ with $\langle \lambda, \alpha^\vee \rangle = 1$ we have $T_{s_\alpha} e^{s_\alpha(\lambda)} T_{s_\alpha} = q e^\lambda$.

Using infinite dimensional spaces one can get a description of the affine Hecke algebra analogous to the one for the finite Hecke algebra. Set $F := \mathbb{F}_{q^n}((t))$ and $O := \mathbb{F}_{q^n}[[t]]$. The inclusion $O \hookrightarrow F$ and the morphism $O \to \mathbb{F}_q$ given by evaluation at 0 induce morphisms of algebraic groups

$$\begin{array}{ccc} G(F) & \hookleftarrow & G(O) \\ & & \downarrow \\ & & G(\mathbb{F}_{q^n}) \end{array}$$

The analog of the Borel subgroup is the Iwahori subgroup $I \subset G(O)$ which is the preimage of $B(\mathbb{F}_{q^n})$.

PROPOSITION 2.12. *There is an isomorphism*

$$H_{\mathit{aff}} \otimes_{q \to p^n} \mathbb{C} \xrightarrow{\sim} \mathbb{C}\big(I\backslash G(F)/I\big), \qquad T_w \leftrightarrow \delta_{IwI}.$$

The affine Hecke algebra also have a geometric realization similar to the one for $\mathbb{Z}[W_{\text{aff}}]$. Every irreducible representation of the group $\mathbb{C}^*$ is of the form $z \mapsto z^n$ for some $n \in \mathbb{Z}$ so there are natural ring isomorphisms

$$R(\mathbb{C}^*) \simeq \mathbb{Z}[q,q^{-1}], \qquad R(T \times \mathbb{C}^*) \simeq R(T)[q,q^{-1}].$$

Thus, the extra $\mathbb{Z}[q,q^{-1}]$-module structure corresponds to introducing a $\mathbb{C}^*$-action. Let $Z_\Delta \subset \tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$ be the diagonal.

THEOREM 2.13. *There is a natural algebra isomorphism* $K^{G\times\mathbb{C}^*}(Z) \simeq \mathcal{H}_{\text{aff}}$ *making the following diagram commutative*

$$\begin{array}{ccc} K^{G\times\mathbb{C}^*}(Z_\Delta) & \hookrightarrow & K^{G\times\mathbb{C}^*}(Z) \\ \downarrow\wr & & \downarrow\wr \\ R(T)[q,q^{-1}] & \hookrightarrow & \mathcal{H}_{\text{aff}} \end{array}$$

PROOF. This is theorem 7.2.5 in [**CG**]. □

REMARK 2.14. A classification of simple modules over $\mathcal{H}_{\text{aff}}$ along the same lines as the one for $\mathbb{Z}[W]$ has been obtain by Kazhdan and Lusztig in [**KL2**, Thm 7.1.2].

2.4. Braid groups

We will consider the braid monoid when defining Demazure descent in the next chapter.

DEFINITION 2.15. The monoid Br^+ with generators $\{T_w,\ w \in W\}$ and relations

$$T_{w_1}T_{w_2} = T_{w_1w_2} \text{ if } \ell(w_1) + \ell(w_2) = \ell(w_1w_2) \text{ in } W$$

is called the braid monoid of G.

In chapter 7 we deal with categorical actions of the affine braid group defined below. Note that in some papers this is called the extended affine braid group and the name affine braid group is used for a different group.

DEFINITION 2.16. The (extended) affine braid group B_{aff} is the group with generators $\{T_w,\ w \in W_{\text{aff}}\}$ and relations

$$T_{w_1}T_{w_2} = T_{w_1w_2} \text{ if } \ell(w_1) + \ell(w_2) = \ell(w_1w_2) \text{ in } W_{\text{aff}}$$

The affine braid group also has another presentation, which is the one used by Bezrukavnikov and Riche.

THEOREM 2.17. [**BR**, Thm 1.1.3] *The affine braid group* B_{aff} *admits a presentation with generators* $\{T_\alpha \mid \alpha \in \Pi\} \cup \{\theta_x \mid x \in \mathbb{X}\}$ *and relations:*

(1) $T_\alpha T_\beta T_\alpha \cdots = T_\beta T_\alpha T_\beta \cdots$ *with* $m(\alpha,\beta)$ *factors on each side.*
(2) $\theta_x\theta_y = \theta_{x+y}$.
(3) $T_\alpha\theta_x = \theta_x T_\alpha$ *if* $\langle x,\alpha\rangle = 0$, *i.e.* $s_\alpha(x) = x$.
(4) $\theta_x = T_\alpha\theta_{x-\alpha}T_\alpha$ *if* $\langle x,\alpha\rangle = 1$, *i.e.* $s_\alpha(x) = x - \alpha$.

If G is simply-connected the affine braid group can be realized geometrically as the fundamental group

$$B_{\text{aff}} = \pi_1\left(\frac{\mathfrak{t}_{\mathbb{C}}\setminus\bigcup_{\alpha,n} H_{\alpha,n}}{W_{\text{aff}}^{\text{Cox}}}\right).$$

Categorifications of braid groups has been studied by Rouquier in [**Rou**].

2.5. Demazure operators

2.5.1. Definitions. Demazure operators were introduced by Demazure in [**Dem1**]. The original application was to calculate the characters of simple modules for a semi-simple Lie algebra. The Demazure operators provide a factorization of the Weyl character formula. Recall the identification $R(T) = \mathbb{Z}[\mathbb{X}]$.

DEFINITION 2.18. Let $\alpha_i \in \Pi$. The Demazure operator δ_i is the endomorphism of $R(T)$ given by

$$\delta_i(u) := \frac{u - e^{-\alpha_i} s_{\alpha_i}(u)}{1 - e^{-\alpha_i}}.$$

Note that $1 - e^{-\alpha_i}$ divides $u - e^{-\alpha_i} s_{\alpha_i}(u)$ so $\delta_i(u)$ is a finite sum. Demazure observed that

$$\delta_i^2 = \delta_i.$$

He also proved the following proposition

PROPOSITION 2.19. *Let $w \in W$ with a reduced expression $w = s_{i_1} \cdots s_{i_n}$. Then*

$$\delta_w := \delta_{\alpha_{i_1}} \cdots \delta_{\alpha_{i_n}}$$

is independent of the choice of reduced expression.

DEFINITION 2.20. The affine 0-Hecke algebra, $\mathcal{H}^0_{\mathrm{aff}}$ is the $R(T)$-algebra generated by $\{\delta_i \mid \alpha_i \in \Pi\}$.

REMARK 2.21. We will often call $\mathcal{H}^0_{\mathrm{aff}}$ the degenerate Hecke algebra. We warn the reader that some sources use this name for the algebra one gets by setting $q = 0$ in the definition of the affine Hecke algebra.

2.5.2. Geometric realization. Assume that G is a finite-dimensional, semi-simple, connected, simply-connected, complex algebraic group. Then $\mathcal{H}^0_{\mathrm{aff}}$ has the following geometric realization:

THEOREM 2.22 ([**KK**]). *There is an isomorphism $\mathcal{H}^0_{\mathrm{aff}} \simeq K_T(\mathcal{B})$.*

PROOF. Using [**KK**, Thm 3.13] one gets an action of $\mathcal{H}^0_{\mathrm{aff}}$ on $K_T(\mathcal{B})$ (note that δ_w is y_w in their notation). The Schubert variety X_w is the closure of BwB/B in G/B. Let $\mathcal{O}_w$ denote the structure sheaf $\mathcal{O}_{X_w}$ extended by zero to $\mathcal{B}$. By [**KK**, Lemma 4.9 and Prop. 4.10] the elements $\{[\mathcal{O}_w]\}_{w \in W}$ form a $R(T)$-basis for the $R(T)$-module $K_T(\mathcal{B})$. Let $* : K_T(\mathcal{B}) \to K_T(\mathcal{B})$ be the involution taking a vector bundle to its dual. The $\mathcal{H}^0_{\mathrm{aff}}$-action in this basis is given by [**KK**, Lemma 4.12]:

$$\delta_i(*[\mathcal{O}_w]) = \begin{cases} *[\mathcal{O}_w] & \text{if } ws_i < w \\ *[\mathcal{O}_{ws_i}] & \text{if } ws_i > w \end{cases}$$

Thus, $K_T(\mathcal{B})$ is generated by $*[\mathcal{O}_e]$ as a $\mathcal{H}^0_{\mathrm{aff}}$ module. □

CHAPTER 3

Demazure descent

3.1. Demazure operators on K-theory

The main motivation for the definition of Demazure descent is to try to categorify the following result by Harada, Landweber and Sjamaar:

THEOREM 3.1. [**HLS**, Prop. 6.5 and 6.6] *Let X be a quasi-projective k-scheme with a G action*

(1) There is a natural action of $\mathcal{H}^0_{\text{aff}}$ on $K_T(X)$.
(2) There is an isomorphism

$$K_G(X) \xrightarrow{\sim} K_T(X)^{I(\mathcal{H}^0_{\text{aff}})},$$

where $I(\mathcal{H}^0_{\text{aff}}) := \{\Delta \in \mathcal{H}^0_{\text{aff}} \mid \Delta(1) = 0\}$ and 1 is the identity element in $R(T)$.

We will not give the complete proof, but we will give their construction the $\mathcal{H}^0_{\text{aff}}$-action on $K_T(X)$. Their construction works for $K^*_G(X) := K_G(X) \oplus K^{-1}_G(X)$, but we are only interested in $K_G(X)$. The group $K_T(X)$ has a natural $K_T(\text{pt}) \simeq R(T)$-module structure so we only need to define the action of the δ_i's.

The closed embeddings $j : T \hookrightarrow G$ and $k : T \hookrightarrow B$ induces pull-back morphisms on the K-groups

$$j^* : K_G(X) \to K_T(X),$$
$$k^* : K_B(X) \to K_T(X).$$

Since B is contractible to T the map k^* is an isomorphism. The projection $\text{pr} : X \times \mathcal{B} \to X$ is a projective G-morphism since $\mathcal{B}$ is projective. Hence, it induces pull-back and push-forward morphisms on the K-groups

$$\text{pr}^* : K_G(X) \to K_G(X \times \mathcal{B}),$$
$$\text{pr}_* : K_G(X \times \mathcal{B}) \to K_G(X).$$

Finally, the closed embedding $i : X \simeq X \times [B] \hookrightarrow X \times \mathcal{B}$ induces an isomorphism

$$i^* : K_G(X \times \mathcal{B}) \xrightarrow{\sim} K_B(X).$$

Notice that j^* can be written as $j^* = k^* i^* \text{pr}^*$. It has a left inverse given by

$$j_* = \text{pr}_*(i^*)^{-1}(k^*)^{-1}$$

The composition $\delta := j^* j_*$ is a Demazure operator corresponding to the whole group.

For a simple root α_l we define $G_l := Z(\ker \alpha_l)$ with embedding $j_l : T \hookrightarrow G_i$. Let B_l be a Borel in G_l containing T and $k_l : T \hookrightarrow B_l$ the embedding. Replacing G by G_l and B by B_l in the above definition we get

$$j_l^* = k_l^* i_l^* \mathrm{pr}_l^*, \qquad j_{l*} = \mathrm{pr}_{l*}(i_l^*)^{-1}(k_l^*)^{-1}.$$
$$\delta_l := j_l^* j_{l*}.$$

The δ_l together with the $R(T)$-action generate the $\mathcal{H}^0_{\text{aff}}$-action in (1). The isomorphism in (2) is given by j^*.

3.2. Demazure descent

In this section we introduce the notion of Demazure descent on a triangulated category $\mathcal{C}$. Demazure descent data is supposed to be a categorical version of the Demazure operators part of the $\mathcal{H}^0_{\text{aff}}$-action in [**HLS**]. To categorify the action of the Demazure operators we require our functors to satisfy the same kind of relations. The first step is to require it to satisfy the braid relations from proposition 2.19.

Definition 3.2. A weak braid monoid action on the category $\mathcal{C}$ is a collection of triangulated functors

$$D_w : \mathcal{C} \to \mathcal{C}, \qquad w \in W$$

satisfying braid monoid relations, i.e. for all $w_1, w_2 \in W$ there exist isomorphisms of functors

$$D_{w_1} \circ D_{w_2} \simeq D_{w_1 w_2}, \qquad \text{if } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2).$$

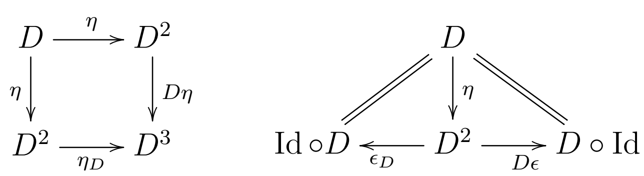
Notice that we neither fix the braid relations isomorphisms nor impose any additional relations on them.

Now, we would like to impose the $\delta_i^2 = \delta_i$ condition. In [**HLS**] the Demazure operators were constructed as a composition of two maps, which on the categorical level, come from a pair of adjoint functors. The composition of a functor with its left adjoint produces a comonad [**Mac**, section VI.1].

Definition 3.3. A comonad in a category, $\mathcal{C}$, consists of a functor $D : \mathcal{C} \to \mathcal{C}$ and natural transformations

$$\eta : D \to D^2, \qquad \epsilon : D \to \mathrm{Id},$$

for which the following diagrams are commutative



This gives a natural way to impose the idempotentcy condition.

DEFINITION 3.4. Demazure descent data on a triangulated category $\mathcal{C}$ is a weak braid monoid action $\{D_w\}$ such that for each simple root s_i the corresponding functor D_{s_i} is a comonad for which the comonad map $D_{s_i} \to D_{s_i}^2$ is an isomorphism.

Given a fixed Demazure descent data $\{D_w, w \in W\}$ the categorical analog of the invariance from theorem 3.1 is the descent category.

DEFINITION 3.5. The descent category $\mathfrak{Desc}(\mathcal{C}, D_w, w \in W)$ is the full subcategory in $\mathcal{C}$ consisting of objects M such that for all i the cones of the counit maps $D_{s_i}(M) \stackrel{\epsilon}{\to} M$ are isomorphic to 0.

Notice that the requirement that the cone of a map is isomorphic to 0 is equivalent to the map being an isomorphism.

REMARK 3.6. Suppose that $\mathcal{C}$ has functorial cones. Then $\mathfrak{Desc}(\mathcal{C}, D_w, w \in W)$ a full triangulated subcategory in $\mathcal{C}$ being the intersection of kernels of $\mathrm{Cone}(D_{s_i} \to \mathrm{Id})$. However, one can prove this statement not using functoriality of cones.

DEFINITION 3.7. An object X in $\mathcal{C}$ is a comodule over the comonad D if there exists a morphism $c : X \to D(X)$ such that the following diagrams are commutative.

$$\begin{array}{ccc} X & \xrightarrow{c} & D(X) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \eta_X} \\ D(X) & \xrightarrow[Dc]{} & D^2(X) \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{c} & D(X) \\ & \searrow\!\!\!\!\searrow & \downarrow{\scriptstyle \epsilon_X} \\ & & X \end{array}$$

LEMMA 3.8. *An object $M \in \mathfrak{Desc}(\mathcal{C}, D_w, w \in W)$ is naturally a comodule over each D_{s_i}.*

PROOF. By definition the comonad maps

$$\eta : D_{s_i} \to D_{s_i}^2, \qquad \epsilon : D_{s_i} \to \mathrm{Id}$$

makes the following diagram commutative

$$\begin{array}{ccc} & & D_{s_i} \\ & \swarrow\!\!\!\!\swarrow & \downarrow{\scriptstyle \eta} \\ D_{s_i} \circ \mathrm{Id} & \xleftarrow[D_{s_i}\epsilon]{} & D_{s_i}^2 \end{array}$$

For Demazure descent data we require that η is an isomorphism, so $D_{s_i}\epsilon$ is also an isomorphism. Let $M \in \mathfrak{Desc}(\mathcal{C}, D_w, w \in W)$. By assumption $\epsilon_M : D_{s_i}(M) \to M$

is an isomorphism so we have a commutative diagram.

$$\begin{array}{ccc} M & \xrightarrow{\epsilon_M^{-1}} & D_{s_i}(M) \\ {\scriptstyle \epsilon_M^{-1}}\big\downarrow & \diagup\!\!\diagup & \big\downarrow{\scriptstyle \eta_M} \\ D_{s_i}(M) & \xrightarrow[D_{s_i}\epsilon_M{}^{-1}]{} & D_{s_i}^2(M) \end{array}$$

The second diagram in the definition is clearly commutative so ϵ_M^{-1} defines a comodule structure on M. ☐

Remark 3.9. In the usual descent setting either in Algebraic Geometry or in abstract Category Theory (Barr-Beck theorem) descent data includes a pair of adjoint functors and their composition which is a comonad. By definition, the descent category for such data is the category of comodules over this comonad. Our definition of $\mathfrak{Desc}(C, D_w, w \in W)$ for Demazure descent data formally is not about comodules, yet the previous lemma demonstrates that every object of $\mathfrak{Desc}(C, D_w, w \in W)$ is naturally equipped with structures of a comodule over each D_{s_i}, and any morphism in $\mathfrak{Desc}(C, D_w, w \in W)$ is a morphism of D_{s_i}-comodules.

CHAPTER 4

Demazure descent for representations

4.1. Categories of representations.

For a linear algebraic group G, we denote the Hopf algebra of polynomial functions on G by $\mathcal{O}(G)$. Let $\mathsf{Rep}(G)$ be the category of $\mathcal{O}(G)$-comodules. This is an Abelian tensor category. For a closed subgroup H of G the H invariant part of a $M \in \mathsf{Rep}(G)$ is defined as

$$M^H := \mathrm{Hom}_{\mathsf{Rep}(H)}(k, M).$$

Fix a Borel subgroup B. The minimal parabolic subgroup P_i corresponding to the simple root α_i is the parabolic subgroup containing B such that its Levi subgroup has the root system $\{\pm\alpha_i\}$. For minimal parabolics the quotient map $G/B \to G/P_i$ is a locally trivial fibration with fiber $\mathbb{P}^1$.

Using the natural Hopf algebra maps $\mathcal{O}(G) \to \mathcal{O}(B)$ and $\mathcal{O}(P_i) \to \mathcal{O}(B)$ we get restriction functors

$$\mathrm{Res}_i : \ \mathsf{Rep}(P_i) \to \mathsf{Rep}(B), \text{ and } \mathrm{Res} : \ \mathsf{Rep}(G) \to \mathsf{Rep}(B).$$

The restriction functors are exact and naturally commute with taking tensor product of representations. Consider the induction functors

$$\begin{aligned} \mathrm{Ind}_i : \mathsf{Rep}(B) \to \mathsf{Rep}(P_i), \qquad & M \mapsto (\mathcal{O}(P_i) \otimes M)^B, \\ \mathrm{Ind} : \mathsf{Rep}(B) \to \mathsf{Rep}(G), \qquad & M \mapsto (\mathcal{O}(G) \otimes M)^B. \end{aligned}$$

This can also be reformulated in terms of cotensor products.

Definition 4.1. Let R be a ring and C a coalgebra over R. Consider a right C-comodule M with coaction map $\rho_M : M \to M \otimes_R C$ and a left C-comodule N with coaction map $\rho_N : N \to C \otimes_R N$. The cotensor product consists of a ring $N \otimes_R^C M$ together with a morphism $\phi : N \otimes_R^C M \to N \otimes_R M$ such that

$$(\rho_M \otimes \mathrm{Id}) \circ \phi = (\mathrm{Id} \otimes \rho_N) \circ \phi$$

and satisfying the following universal property: for any ring Y and morphism $h : Y \to M \otimes_R N$ with $(\rho_M \otimes \mathrm{Id}) \circ h = (\mathrm{Id} \otimes \rho_N) \circ h$ there exist a unique morphism

$u : Y \to N \otimes_R^C M$ making the following diagram commutative

$$\begin{array}{ccccc} M \otimes_R^C N & \xrightarrow{\phi} & M \otimes_R N & \underset{\mathrm{Id} \otimes \rho_N}{\overset{\rho_M \otimes \mathrm{Id}}{\rightrightarrows}} & M \otimes_R C \otimes_R N \\ \uparrow{\scriptstyle u} & \nearrow{\scriptstyle h} & & & \\ Y & & & & \end{array}$$

LEMMA 4.2. *Let* $M \in \mathrm{comod} - \mathcal{O}(B)$ *and* $N \in \mathcal{O}(B) - \mathrm{comod}$. *Then*

$$M \otimes^{\mathcal{O}(B)} N \simeq (M \otimes N)^B.$$

PROOF. $M \otimes^{\mathcal{O}(B)} N$ is the subring of $M \otimes N$ on which $\mathrm{coac}_M \otimes \mathrm{Id} = \mathrm{Id} \otimes \mathrm{coac}_N$ so $\sum m_i \otimes n_i \in M \otimes^{\mathcal{O}(B)} N$ if and only if $\sum \mathrm{coac}_M(m_i) \otimes n_i - \sum m_i \otimes \mathrm{coac}_N(n_i) = 0$. By Nullstellensatz this is equivalent to

$$ev_b \left(\sum \mathrm{coac}_M(m_i) \otimes n_i - \sum m_i \otimes \mathrm{coac}_N(n_i) \right) = 0 \qquad \text{for all } b \in B,$$

where ev_b is the evaluation at b. By definition of the B module structure (See [**Jant**, Section 2.8])

$$ev_b \left(\sum \mathrm{coac}_M(m_i) \otimes n_i - \sum m_i \otimes \mathrm{coac}_N(n_i) \right) = \sum m_i b \otimes n_i - \sum m_i \otimes b n_i$$

This is exactly the requirement $\sum m_i \otimes n_i \in (M \otimes N)^B$. □

Using the lemma we can rewrite the induction functors as

$$\begin{aligned} \mathrm{Ind}_i : \mathsf{Rep}(B) \to \mathsf{Rep}(P_i), \qquad & M \mapsto \mathcal{O}(P_i) \otimes^{\mathcal{O}(B)} M, \\ \mathrm{Ind} : \mathsf{Rep}(B) \to \mathsf{Rep}(G), \qquad & M \mapsto \mathcal{O}(G) \otimes^{\mathcal{O}(B)} M. \end{aligned}$$

Set $\Delta_i := \mathrm{Res}_i \circ \mathrm{Ind}_i \circlearrowright \mathsf{Rep}(B)$ and $\Delta := \mathrm{Res} \circ \mathrm{Ind} \circlearrowright \mathsf{Rep}(B)$. Notice that Δ_i and Δ are left exact, since the induction functors are left exact.

4.1.1. The derived categories. For an algebraic group H, the regular comodule $\mathcal{O}(H)$ is injective in $\mathsf{Rep}(H)$, moreover for any $M \in \mathsf{Rep}(H)$ the coaction map $M \to \mathcal{O}(H) \otimes M$ provides an embedding of M into an injective object. In particular, $\mathsf{Rep}(H)$ has enough injectives. The algebraic De Rham complex $\Omega^\bullet(H)$ provides an injective resolution for the trivial comodule, of the length equal to the dimension of H. For any $M \in \mathsf{Rep}(H)$ the complex $\Omega^\bullet(H) \otimes M$ provides an injective resolution for M of the same length. Thus, $\mathsf{Rep}(H)$ has finite homological dimension. In particular, we have a right derived functor for any left exact functor in the bounded derived categories $D^b(\mathsf{Rep}(B)), D^b(\mathsf{Rep}(P_i))$ and $D^b(\mathsf{Rep}(G))$. Let L_i and L be the derived functors of Res_i and Res respectively. Denote the right derived functors of Ind_i and Ind by I_i and I respectively. Set $D_i = L_i \circ I_i$ and $D = L \circ I$.

4.2. Main theorem

The main theorem in this chapter is the following.

THEOREM 4.3. *Let* $w \in W$ *and let* $w = s_{i_1} \cdots s_{i_n}$ *be a reduced expression. Then* $D_w := D_{i_1} \circ \cdots \circ D_{i_n}$ *is independent of the choice of reduced expression and the* $\{D_w\}_{w \in W}$ *form Demazure descent data on* $D^b(\mathsf{Rep}(B))$.

Schubert and Bott-Samuelson schemes play an important role in the proof of the main theorem in this chapter and also of the main theorem in the next, so we will take a moment to introduce them before starting the proof.

4.2.1. Schubert schemes. For $w \in W$ we define the Schubert scheme X_w to be the closure of BwB/B in G/B. It is known that

$$X_w = \sqcup_{w' \leq w} Bw'B/B,$$

where $\leq$ is the Bruhat order. In other words the union is over element in W which can be obtained from an expression for w by deleting a number of simple reflections. In particular, if w_0 denotes the unique longest element in W then $X_{w_0} = G/B$.

LEMMA 4.4. [**Dem2**, Coro. 5.1] *Schubert schemes are normal.*

Schubert schemes are generally singular, but they have a resolution of singularites given by so-called Bott-Samelson schemes: Consider the $B \times \cdots \times B$-action on $P_{i_1} \times \cdots \times P_{i_n}$ given by

$$(b_1, \ldots, b_n) \cdot (p_{i_1}, \ldots, p_{i_n}) = (p_{i_1} b_1, b_1^{-1} p_{i_2} b_2^{-1}, \ldots, b^{n-1} p_{i_n} b_n)$$

The quotient by this action is called the Bott-Samelson scheme and is denoted by

$$X_{i_1 \ldots i_n} := P_{i_1} \times^B \cdots \times^B P_{i_n}/B.$$

The multiplication $m : P_{i_1} \times \cdots P_{i_n} \to P_{i_1} \cdots P_{i_n}$ factions though $X_{i_1 \ldots i_n}$ giving a morphism

$$\bar{m} : X_{i_1 \ldots i_n} \to P_{i_1} \cdots P_{i_n}/B.$$

Notice that $X_{i_1 \ldots i_n}$ is complete since $P_{i_k}/B \simeq \mathbb{P}^1$ is complete. This implies that $\bar{m}$ is proper.

LEMMA 4.5. *Let* $w = s_{i_1} \cdots s_{i_n}$ *be a reduced expression. Then*

$$P_{i_1} \cdots P_{i_n} = \sqcup_{w' \leq w} Bw'B,$$

where the union is over all $w' \in W$ *which is* $\leq w$ *in the Bruhat order.*

PROOF. The proof goes by induction on $n = \ell(w)$. It is true for $n = 1$ by definition of P_i. Set $v = s_{i_1} \cdots s_{i_{n-1}}$. Using the hypotheses we get

$$P_{i_1} \cdots P_{i_{n-1}} P_{i_n} = \Big(\bigcup_{w' \leq v} Bw'B \Big) (B \cup Bs_{i_n} B) = \bigcup_{w' \leq v} Bw'B \cup \bigcup_{w' \leq v} (Bw'B)(Bs_{i_n} B)$$

Let w' be any element in W and s a simple reflection. Then by [**Hum2**, Cor. 28.3] we have $(Bw'B)(BsB) \subseteq Bw'sB \cup Bw'B$. Thus, if $w's_{i_n} \leq w' \leq v$ then $(Bw'B)(Bs_{i_n}B)$ is contained in the first union. If $w' \leq w's_{i_n}$ then we have $(Bw'B)(Bs_{i_n}B) = Bw's_{i_n}B$ by [**Hum2**, Lemma 29.3A and section 29.1]. Thus, the product can be written as

$$\begin{aligned} P_{i_1}\cdots P_{i_n} &= \bigcup_{w' \leq v} Bw'B \ \cup \bigcup_{\substack{w' \leq v \\ w' \leq w's_{i_n}}} Bw's_{i_n}B \\ &= \bigcup_{w' \leq v} Bw'B \ \cup \bigcup_{\substack{w''s_{i_n} \leq v, \\ w''s_{i_n} \leq w''}} Bw''B \end{aligned}$$

Claim 4.6. The conditions $w''s_{i_n} \leq v$ and $w''s_{i_n} \leq w''$ are equivalent to the conditions $w'' \leq w$ and $w''s_{i_n} \leq w''$.

Proof of the claim. Assume that $w''s_{i_n} \leq v$. By [**Hum3**, Prop. 5.9] this implies that $w'' \leq v$ or $w'' \leq vs_{i_n} = w$. In both cases we get $w'' \leq w$ since $v \leq w$. Assume now that $w'' \leq w$ and $w''s_{i_n} \leq w''$. w'' has a reduced expression of the form

$$w'' = s_{i_1} \cdots \hat{s}_{i_{j_1}} \cdots \hat{s}_{i_{j_2}} \cdots \hat{s}_{i_{j_k}} \cdots s_{i_n},$$

where the ˆ indicates that the term has been removed from the product. If $j_k \neq n$ then

$$w''s_{i_n} = s_{i_1} \cdots \hat{s}_{i_{j_1}} \cdots \hat{s}_{i_{j_2}} \cdots \hat{s}_{i_{j_k}} \cdots s_{i_{n-1}} \leq s_{i_1} \cdots s_{i_{n-1}} = v.$$

If $j_k = n$ then $w'' \leq v$. Since $w''s_{i_n} \leq w''$ by assumption we get $w''s_{i_n} \leq v$. ☐

If $w' \leq v$ in the first union satisfies that $w's_{i_n} \leq w'$ then it is also contained in the second union. Using the claim we get

$$P_{i_1}\cdots P_{i_n} = \bigcup_{\substack{w' \leq v \\ w' \leq w's_{i_n}}} Bw'B \ \cup \bigcup_{\substack{w'' \leq w, \\ w''s_{i_n} \leq w''}} Bw''B$$

Assume that $w' \leq w$ and $w' \leq w's_{i_n}$. Then w' has a reduced expression of the form

$$w' = s_{i_1} \cdots \hat{s}_{i_{j_1}} \cdots \hat{s}_{i_{j_2}} \cdots \hat{s}_{i_{j_k}} \cdots s_{i_n}.$$

If $j_k = n$ then $w' \leq v$. If $j_k \neq n$ then $w's_{i_n} \leq v$, but since $w' \leq w's_{i_n}$ we get $w' \leq v$. Hence, the conditions $w' \leq v$ and $w' \leq w's_{i_n}$ can be replaced by $w' \leq w$ and $w' \leq w's_{i_n}$. Thus,

$$P_{i_1}\cdots P_{i_n} = \bigcup_{\substack{w' \leq w \\ w' \leq w's_{i_n}}} Bw'B \ \cup \bigcup_{\substack{w'' \leq w, \\ w''s_{i_n} \leq w''}} Bw''B = \bigcup_{w' \leq w} Bw'B.$$

This finishes the induction step. ☐

By the lemma, when $s_{i_1}\cdots s_{i_n}$ is a reduced expression, then we have a morphism

$$\bar{m}: X_{i_1\ldots i_n}\to X_w.$$

This morphism is called the Bott-Samelson resolution and it is a resolution of singularities.

4.2.2. Comonad structure of functors associated to simple roots. In this section we prove that the D_{s_i}'s are comonads for which $D_{s_i}\to D_{s_i}^2$ is an isomorphism. The braid relations are proved in the next section. The proof of the first part of the following technical lemma is essentially some extra observations added to the proof of [**CPS**, Lemma 2.2].

LEMMA 4.7. *(1)* $\mathcal{O}(P_{i_1}\times^B\cdots\times^B P_{i_k})\simeq\mathcal{O}(P_{i_1}\cdots P_{i_k})$
(2) $H^q\left(P_{i_1}\times\cdots\times P_{i_n},\left(\mathcal{O}_{P_{i_1}}\otimes\cdots\otimes\mathcal{O}_{P_{i_n}}\right)^{B\times\cdots\times B}\right)=0$ *for* $q>0$.

For the proof we need the lemma.

LEMMA 4.8. *Let $f: X\to Y$ be a surjective proper morphism and suppose further that f is a birational isomorphism and Y is a normal variety. Then $f^{\#}: \mathcal{O}(Y)\to\mathcal{O}(X)$ is an isomorphism.*

PROOF. See [**EGA**, III, 4.3.12] □

PROOF OF LEMMA 4.7. The multiplication morphism factors as

$$P_{i_1}\times\cdots\times P_{i_n}\xrightarrow{\pi}P_{i_1}\times^B\cdots\times^B P_{i_n}\xrightarrow{m'}P_{i_1}\cdots P_{i_n}$$

Consider the functor

$$F:\mathrm{QCoh}^{B\times\cdots\times B}(P_{i_1}\times\cdots\times P_{i_n})\to\mathrm{QCoh}\left(P_{i_1}\times^B\cdots\times^B P_{i_n}\right),$$
$$F(M)(U)=(\pi_*(M)(U))^{B\times\cdots\times B}$$

For global sections we have

$$\begin{aligned}F(M)&=F(M)(P_{i_1}\times^B\cdots\times^B P_{i_n})\\&=(M(\pi^{-1}(P_{i_1}\times^B\cdots\times^B P_{i_n}))^{B\times\cdots\times B}\\&=(M(P_{i_1}\times\cdots\times P_{i_n}))^{B\times\cdots\times B}\\&=M^{B\times\cdots\times B}\end{aligned}$$

On the other hand

$$F(M)=F(M)(P_{i_1}\times^B\cdots\times^B P_{i_n})\simeq m'_*(F(M))(P_{i_1}\cdots P_{i_n})=m'_*(F(M))$$

so $M^{B\times\cdots\times B}\simeq m_*(F(M))$. This also gives an equivalence of the derived functors $R(\ )^{B\times\cdots\times B}\simeq R(m'_*\circ F)$. For $M=\mathcal{O}_{P_{i_1}\times\cdots\times P_{i_n}}$ we have $F(M)=\mathcal{O}_{P_{i_1}\times^B\cdots\times^B P_{i_n}}$ so

$$H^q\left(P_{i_1}\times\cdots\times P_{i_n},\left(\mathcal{O}_{P_{i_1}}\otimes\cdots\otimes\mathcal{O}_{P_{i_n}}\right)^{B\times\cdots\times B}\right)\simeq R^q m'_*\left(\mathcal{O}_{P_{i_1}\times^B\cdots\times^B P_{i_n}}\right)$$

To prove that this vanishes we will construct a cartesian diagram of locally trivial principal B-bundles inspired by the proof of [**CPS**, Lemma 2.2]. Recall the Bott-Samelson resolution for a reduced expression $w = s_{i_1} \cdots s_{i_n}$ from section 4.2.1

$$\bar{m} : X_{i_1 \ldots i_n} \to X_w.$$

As previously mentioned the Bott-Samelson resolution is proper. By [**Dem2**, Proposition 3.2]

$$m' : Y_{i_1 \cdots i_n} := P_{i_1} \times^B \cdots \times^B P_{i_n} \to P_{i_1} \cdots P_{i_n}$$

is a birational isomorphism. The quotient maps $\pi : Y_{i_1 \ldots i_n} \to X_{i_1 \ldots i_n}$ and $\pi' : P_{i_1} \cdots P_{i_n} \to X_w$ are locally trivial principal B-bundles. Since P_{i_n}/B is complete π is proper. We have a commutative diagram

$$\begin{array}{ccc} Y_{i_1 \ldots i_n} & \xrightarrow{m'} & P_{i_1} \ldots P_{i_n} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ X_{i_1 \ldots i_n} & \xrightarrow[\bar{m}]{} & X_w \end{array}$$

By [**Ser**][(3.1)] any locally trivial principal B-bundle F which fits into the diagram (with $F \to P_{i_1} \cdots P_{i_n}$ equivariant) is isomorphic to $X_{i_1 \ldots i_n} \times_{X_w} P_{i_1} \cdots P_{i_n}$. Hence, the diagram is cartesian so we have the formula

$$L\pi'^{*} \circ R\bar{m}_* \simeq Rm'_* \circ L\pi^*.$$

The Schubert varieties are normal, so we can apply lemma 4.8 to $\bar{m}$ and get

$$\mathcal{O}(X_{i_1 \ldots i_n}) \simeq \mathcal{O}(P_{i_1} \cdots P_{i_n}/B).$$

Properness is conserved under base change so m' is also proper. Applying lemma 4.8 to m' we get

$$\mathcal{O}(P_{i_1} \times^B \cdots \times^B P_{i_n}) \simeq \mathcal{O}(P_{i_1} \ldots P_{i_n}).$$

Proposition 5.2 and 5.3 in [**Dem2**] gives that

$$R^q \bar{m}_*(\mathcal{O}_{X_{i_1 \ldots i_n}}) = 0 \qquad q > 0.$$

Notice that $L\pi^*(\mathcal{O}_{X_{i_1 \ldots i_n}}) = \mathcal{O}_{Y_{i_1 \ldots i_n}}$ ($\mathcal{O}_{Y_{i_1 \ldots i_n}}$ is the only non-zero term in the complex and is siting in degree 0). Inserting this we get

$$R^q m'_*(\mathcal{O}_{P_{i_1} \times^B \cdots \times^B P_{i_n}}) = R^q \bar{m}_*(\mathcal{O}_{X_{i_1 \ldots i_n}}) = 0 \qquad q > 0. \qquad \square$$

Notice that the lemma applied to only one $\mathcal{O}(P_i)$ shows that I_i takes the trivial $\mathcal{O}(B)$-comodule to the trivial $\mathcal{O}(P_i)$-comodule. Using this and a couple of other nice properties for Ind_i and Res_i we can prove that D_i are comonads (the proofs are identical for Ind, Res and D).

PROPOSITION 4.9.

(a) The functor L_i is left adjoint to I_i.

(b) For $M \in D^b(\mathsf{Rep}(B))$ and $N \in D^b(\mathsf{Rep}(P_i))$ we have the tensor identitiy:

$$I_i(M \otimes L_i(N)) \simeq I_i(M) \otimes N.$$

(c) The D_i's are comonads for which the comonad map $D_i \to D_i^2$ are isomorphisms.

The same is true for L and I.

PROOF. (a) By proposition 3.4 in [**Jant**] Ind_i is right adjoint to Res_i. The derived functors of a pair of adjoint functors are also adjoint [**Stacks**, Lemma 13.28.4.]. (b) The tensor identity

$$\mathrm{Ind}_i(M \otimes \mathrm{Res}_i(N)) \simeq \mathrm{Ind}_i(M) \otimes N$$

is proposition 3.6 in [**Jant**]. Consider the functor $I_i(M \otimes L_i(N))$. Let J denote the operation of taking resolutions. Then the derived functor can be written as

$$\mathrm{Ind}_i \big(J(N) \otimes J(\mathrm{Res}_i(J(M)))\big)$$

For any X we have that $X \otimes I$ is injective if I is injective. Therefore, $J(N) \otimes \mathrm{Res}_i(J(M))$ is another injective resolution. Hence,

$$\begin{aligned}\mathrm{Ind}_i\Big(J(N) \otimes J\big(\mathrm{Res}_i(J(M))\big)\Big) &\simeq \mathrm{Ind}_i\Big(J(N) \otimes \mathrm{Res}_i(J(M))\Big)\\ &\simeq \mathrm{Ind}_i(J(N)) \otimes J(M).\end{aligned}$$

This is exactly $I_i(N) \otimes M$ so we get the derived tensor identity. (c) Part (a) together with [**Mac**, section VI.1] shows that the D_i's are comonads. Since I_i take the trivial $\mathcal{O}(B)$ comodule to the trivial $\mathcal{O}(P_i)$ comodule inserting $M = k$ into (b) we get $I_i \circ L_i(N) \simeq N$ for $N \in D^b(\mathsf{Rep}(P_i))$. Thus, $\mathrm{Id} \xrightarrow{\sim} I_i \circ L_i$. From this we get the desired isomorphism

$$D_i = L_i \circ I_i = L_i \circ \mathrm{Id} \circ I_i \xrightarrow{\sim} L_i \circ I_i \circ L_i \circ I_i = D_i^2. \qquad \square$$

REMARK 4.10. It follows that the restriction functors L_i and L are fully faithful.

4.3. Braid relations

For the braid relations we will need to consider compositions of induction functors. First we will consider a slightly more general setting.

Let C be a coalgebra and let $Q \in C$-mod-C. Assume that Q has an injective resolution $J(Q)$ as a $C \boxtimes C$ module. Assume also that $\mathrm{Res} : C \boxtimes C \to 1 \boxtimes C$ takes injectives to injectives. We have the functor

$$F : C\text{-comod} \to C\text{-comod}, \qquad M \mapsto Q \otimes^C M.$$

We want to study its right derived functor. Let $\otimes_R^C$ denote the right derived functor of $\otimes^C$.

LEMMA 4.11. *Suppose we have $Q_1, Q_2 \in C$-comod-C with finite injective resolutions $J(Q_1)$ and $J(Q_2)$ with $H^{\neq 0}(J(Q_2) \otimes^C J(Q_1)) = 0$ and functors F_1 and F_2 on C-comod given by*

$$F_1(X) = Q_1 \otimes^C_R X, \qquad F_2(X) = Q_2 \otimes^C_R X.$$

Then $F_2 \circ F_1 = (Q_2 \otimes^C Q_1) \otimes^C_R$.

PROOF. Let I be injective. We have

$$F_2 \circ F_1(I) = J\Big(J(Q_2) \otimes^C J\big(J(Q_1) \otimes^C I\big)\Big)$$

The derived cotensor product can be calculated by resolving any of the two factors or both [**EM**, Section 5] so we have

$$\begin{aligned} J\Big(J(Q_2) \otimes^C J\big(J(Q_1) \otimes^C I\big)\Big) &= J\Big(J(Q_2) \otimes^C \big(J(Q_1) \otimes^C I\big)\Big) \\ &= J\Big(\big(J(Q_2) \otimes^C J(Q_1)\big) \otimes^C I\Big). \end{aligned}$$

Since $H^{\neq 0}(J(Q_2) \otimes^C J(Q_1)) = 0$ we have a quasi-isomorphism $J(Q_2) \otimes^C J(Q_1) \simeq Q_2 \otimes^C Q_1$. Finite injective complexes are co-flat so $(Q_2 \otimes^C Q_1) \otimes^C I$ is quasi-isomorphic to $\big(J(Q_2) \otimes^C J(Q_1)\big) \otimes^C I$. We have proved that

$$\begin{aligned} F_2 \circ F_1(I) &= J\Big(\big(J(Q_2) \otimes^C J(Q_1)\big) \otimes^C I\Big) \\ &= (Q_2 \otimes^C Q_1) \otimes^C_R I \end{aligned}$$

□

Lemma 4.7 shows that the above assumptions are satisfied in the case $C = \mathcal{O}(B), Q_1 = \mathcal{O}(P_{i_1})$ and $Q_2 = \mathcal{O}(P_{i_2}) \otimes^{\mathcal{O}(B)} \cdots \otimes^{\mathcal{O}(B)} \mathcal{O}(P_{i_k})$. By induction

$$D_{i_1} \circ \cdots \circ D_{i_n} \simeq \big(\mathcal{O}(P_{i_1}) \otimes^{\mathcal{O}(B)} \cdots \otimes^{\mathcal{O}(B)} \mathcal{O}(P_{i_n})\big) \otimes^{\mathcal{O}(B)}_R$$

Thus, the study of composition of the D_i's reduces to the study of cotensor products of $\mathcal{O}(P_i)$'s. Since $B \subseteq P_i$ we can restrict the multiplication map from P_i to B. The category of affine schemes is the opposite category of the category of commutative rings so

$$\operatorname{Spec}\Big(\mathcal{O}(P_i) \otimes^{\mathcal{O}(B)} \mathcal{O}(P_j)\Big) \simeq \operatorname{Coeq}\Big(P_i \times B \times P_j \underset{\operatorname{Id} \times m}{\overset{m \times \operatorname{Id}}{\rightrightarrows}} P_i \times P_j\Big).$$

Let Y be a scheme and h a map $P_i \times P_j \to Y$ such that $h \circ (m \times \operatorname{Id}) = h \circ (\operatorname{Id} \times m)$.

$$m \times \operatorname{Id}(p_i, b, p_j) = (p_i b, p_j), \qquad \operatorname{Id} \times m(p_i, b, p_j) = (p_i, b p_j,)$$

Hence, h factors through $P_i \times^B P_j$ uniquely. Thus,

$$\mathcal{O}(P_i) \otimes^{\mathcal{O}(B)} \mathcal{O}(P_j) \simeq \mathcal{O}(P_i \times^B P_j).$$

Reviewing the proof we see that it extends to any number of factors

$$\mathcal{O}(P_{i_1}) \otimes^{\mathcal{O}(B)} \cdots \otimes^{\mathcal{O}(B)} \mathcal{O}(P_{i_n}) \simeq \mathcal{O}(P_{i_1} \times^B P_{i_2} \times^B \cdots \times^B P_{i_n}).$$

Using lemma 4.7 this proves that

$$D_{i_1} \circ \cdots \circ D_{i_n} \simeq \mathcal{O}(P_{i_1} \cdots P_{i_n}) \otimes_R^{\mathcal{O}(B)}$$

By lemma 4.5 the product $P_{i_1} \cdots P_{i_n}$ is independent of the choice of reduced expression. This finishes the proof of theorem 4.3.

4.4. The descent category

THEOREM 4.12. *$\mathfrak{Desc}(D^b(\mathsf{Rep}(B)), D_w, w \in W)$ is equivalent to $D^b(\mathsf{Rep}(G))$.*

PROOF. Let $M \in D^b(\mathsf{Rep}(B)$. It is also in $D^b(\mathsf{Rep}(G)$ if it is in the essential image of L. If $M \simeq D(M) = L \circ I(M)$ then M is in the essential image. If $M = L(N)$ then

$$D(M) = L \circ I \circ L(N) \xrightarrow{\sim} L(N) = M.$$

Thus, being in the essential image is equivalent to $D(M) \simeq M$. Let $w_0 = s_{i_1} \cdots s_{i_n}$ be a reduced expression for the longest element in the Weyl group. Then $P_{i_1} \cdots P_{i_n} = G$ by lemma 4.5 so

$$D(M) \simeq D_{w_0}(M) \simeq D_{i_1} \circ \cdots \circ D_{i_n}(M).$$

Thus, the descent category $\mathfrak{Desc}(D^b(\mathsf{Rep}(B)), D_w, w \in W)$ is a full subcategory in the essential image of L. The restriction factor as

$$\mathrm{Res}_B^G = \mathrm{Res}_B^{P_{i_k}} \circ \mathrm{Res}_{P_{i_k}}^G .$$

Passing to derived functors we get $L \simeq L_{i_k} \circ L\,\mathrm{Res}_{P_{i_k}}^G$. Hence, objects in the essential image of L are in the essential image of L_{i_k} for $k = 1, \ldots, n$. By the above argument this is equivalent to $D_{i_k}(M) \simeq M$ so $D^b(\mathsf{Rep}(G))$ is a full subcategory of $\mathfrak{Desc}(D^b(\mathsf{Rep}(B)), D_w, w \in W)$. $\square$

CHAPTER 5

Demazure descent for equivariant quasi-coherent sheaves

5.1. Equivariant quasi-coherent sheaves on a scheme

Below we collect the main facts about equivariant quasi-coherent sheaves to be used later. In this section, K denotes a not necessarily reductive algebraic group.

Let X be a K-scheme. Denote the action (resp., the projection) map $K \times X \to X$ by ac (resp., by p). Consider further the multiplication map and two projections $m, p_0, p_1 : K \times K \times X \to K \times X$ and the coordinate embedding $s : X \to K \times X, x \mapsto (1, x)$.

Definition 5.1. A K-equivariant quasi-coherent sheaf on a K-scheme X is a pair (M, θ), where $M \in \mathrm{QCoh}(X)$ and θ is an isomorphism $\mathrm{ac}^* M \tilde{\to} p^* M$ satisfying $m^*\theta = p_0^*\theta \circ p_1^*\theta$ and $s^*\theta = \mathrm{Id}_M$. The category of K-equivariant quasi-coherent sheaves on X is denoted by $\mathrm{QCoh}^K(X)$.

Let X be a K-scheme. The forgetful functor $\mathrm{For} : \mathrm{QCoh}^K(X) \to \mathrm{QCoh}(X)$ is exact and has an exact right adjoint functor $\mathrm{Av}^K = \mathrm{ac}_* p^*$ called the averaging functor. Being right adjoint to an exact functor implies that Av^K takes injectives to injectives. For any $M \in \mathrm{QCoh}(X)$ the natural map $M \to \mathrm{For} \circ \mathrm{Av}^K(M)$ is an embedding. It follows that the category $\mathrm{QCoh}^K(X)$ has enough injective objects since $\mathrm{QCoh}(X)$ has enough injectives (see [**Bez1**], Section 2).

Let $f : X \to Y$ be a K-equivariant map of K-schemes. The functors of push-forward and pull-back are extended naturally to the categories of equivariant sheaves:

$$f^* : \mathrm{QCoh}^K(Y) \to \mathrm{QCoh}^K(X), \quad (M, \theta) \mapsto (f^*M, f^*\theta \circ \text{canonical isomorphisms}),$$

$$f_* : \mathrm{QCoh}^K(X) \to \mathrm{QCoh}^K(Y), \quad (M, \theta) \mapsto (f_*M, (\mathrm{Id} \times f)_*\theta \circ \text{canonical isomorphisms}).$$

Notice that both f^* and f_* commute with the forgetful functor $\mathrm{For} : \mathrm{QCoh}^K(X) \to \mathrm{QCoh}(X)$.

Let K, H be algebraic groups acting on a scheme X so that the actions commute. Assume that X admits an H-equivariant quotient $q : X \to X/K$ which is a locally trivial principal K-bundle. Denote the quotient scheme X/K by Y.

Lemma 5.2. *The inverse image functor provides an equivalence of Abelian categories* $\mathrm{QCoh}^H(Y) \to \mathrm{QCoh}^{H \times K}(X)$.

PROOF. See [**Bri**], discussion in Section 2. □

In order for all derived functors to exist and to satisfy the same relations as in the non-equivariant case we assume that all K-schemes are Noetherian, normal and quasi-projective. To avoid further restrictions on X, we work in the unbounded derived category $D\,\mathrm{QCoh}^K(X)$. Since it is exact the functor For extends to the functor $D\,\mathrm{QCoh}^K(X) \to D\,\mathrm{QCoh}(X)$. Under these assumptions the following derived functors exist.

PROPOSITION 5.3. *Assume that all schemes are Noetherian, normal and quasi-projective. Then we have the following properties*

(a) Any unbounded complex in $\mathrm{QCoh}^K(X)$ *has a K-injective resolution and a K-flat resolution.*

(b) The tensor product has a left derived functor

$$\otimes^L_X : D\,\mathrm{QCoh}^K(X) \times D\,\mathrm{QCoh}^K(X) \to D\,\mathrm{QCoh}^K(X).$$

(c) Let $f : X \to Y$ *be a* K*-morphism. Then the derived functors exist*

$$Lf^* : \ D\,\mathrm{QCoh}^K(Y) \to D\,\mathrm{QCoh}^K(X)$$
$$Rf_* : \ D\,\mathrm{QCoh}^K(X) \to D\,\mathrm{QCoh}^K(Y).$$

PROOF. This is proposition 1.5.6 and 1.5.7 in [**VV**]. □

The equivariant derived functors satisfy many of the same relations as the non-equivariant derived functors.

PROPOSITION 5.4. *Assume that all schemes are Noetherian, normal and quasi-projective. Then the derived functors satisfy the following relations*

1. *The functors* Rf_* *and* Lf^* *commute with the forgetful functor.*
2. *The functor* Lf^* *is left adjoint to* Rf_**.*
3. *The functors satisfy the projection formula, i.e. for* $N \in D\,\mathrm{QCoh}^K(Y)$ *and* $M \in D\,\mathrm{QCoh}^K(X)$ *we have a canonical isomorphism*

$$Rf_*N \overset{L}{\otimes}_Y M \simeq Rf_*(N \overset{L}{\otimes}_X Lf^*M).$$

4. *The flat base change theorem also works in the equivariant setting: Let* $g : Z \to Y$ *be a flat* K*-morphism. Consider the Cartesian square*

$$\begin{array}{ccc} Z \times_Y X & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

The standard adjunction map provides an isomorphism of functors

$$Lg^*Rf_* \simeq Rf'_*Lg'^*.$$

Proof. This is section 1.5.8 and the discussion immediately after lemma 1.5.9 in [**VV**]. □

Remark 5.5. In [**VV**] proposition 5.3 this is proved under weaker assumptions, but the stronger assumptions are needed for proposition 5.4.

5.2. Convolution and the quasi-coherent Hecke category

5.2.1. Convolution monoidal structure. Let Z, Y and X be Noetherian, normal, quasi-projective K-schemes. Consider the projections

$$\begin{array}{ccccc} & & Z \times Y \times X & & \\ & \swarrow^{\mathrm{pr}_{12}} & \downarrow{\scriptstyle \mathrm{pr}_{13}} & \searrow^{\mathrm{pr}_{23}} & \\ Z \times Y & & Z \times X & & Y \times X \end{array}$$

The group K acts on each of the four schemes in the diagram diagonally, and the projections are K-equivariant.

The convolution product $*$ is defined as follows:

$$* : D(\mathrm{QCoh}^K(Z \times Y)) \times D(\mathrm{QCoh}^K(Y \times X)) \to D(\mathrm{QCoh}^K(Z \times X)),$$

$$M_1 * M_2 := R\mathrm{pr}_{13*}(L\mathrm{pr}_{12}^* M_1 \overset{L}{\otimes}_{Z \times Y \times X} L\mathrm{pr}_{23}^* M_2).$$

Proposition 5.6. *The convolution product defines a monoidal structure*

$$*: \ D(\mathrm{QCoh}^K(X \times X)) \times D(\mathrm{QCoh}^K(X \times X)) \to D(\mathrm{QCoh}^K(X \times X)).$$

It also produces a monoidal action of $D(\mathrm{QCoh}^K(X \times X))$ *on* $D(\mathrm{QCoh}^K(X \times Y))$.

Proof. Let $M_1, M_2 \in D(\mathrm{QCoh}^K(X \times X))$ and $M_3 \in D(\mathrm{QCoh}^K(X \times Y))$. We need to show that $(M_1 * M_2) * M_3 \simeq M_1 * (M_2 * M_3)$. First we calculate the LHS.

$$\begin{aligned} &(M_1 * M_2) * M_3 \\ &\qquad = Rp_{13*}\Big(Lp_{12}^*\big(Rp_{13*}(Lp_{12}^* M_1 \otimes^L_{X \times X \times X} Lp_{23}^* M_2)\big) \otimes^L_{X \times X \times Y} Lp_{23}^* M_3\Big). \end{aligned}$$

Consider the cartesian diagram

$$\begin{array}{ccc} X \times X \times X \times Y & \xrightarrow{p_{123}} & X \times X \times X \\ \downarrow{\scriptstyle p_{134}} & & \downarrow{\scriptstyle p_{13}} \\ X \times X \times Y & \xrightarrow[p_{12}]{} & X \times X \end{array}$$

By flat base change we have $Lp_{12}^* Rp_{13*} \simeq Rp_{134*} Lp_{123}^*$. Inserting this we get

$$\begin{aligned} &(M_1 * M_2) * M_3 \\ &\qquad \simeq Rp_{13*}\big(Rp_{134*} Lp_{123}^*(Lp_{12}^* M_1 \otimes^L_{X \times X \times X} Lp_{23}^* M_2) \otimes^L_{X \times X \times Y} Lp_{23}^* M_3\big). \end{aligned}$$

Using the projection formula we get

$$(M_1 * M_2) * M_3$$
$$\simeq Rp_{13*}Rp_{134*}\big(Lp_{123}^*(Lp_{12}^*M_1 \otimes^L_{X\times X\times X} Lp_{23}^*M_2) \otimes^L_{X\times X\times X\times Y} Lp_{134}^*Lp_{23}^*M_3\big).$$

Pull-back commutes with tensor product so

$$(M_1 * M_2) * M_3$$

$$\simeq Rp_{13*}Rp_{134*}\big(Lp_{123}^*Lp_{12}^*M_1 \otimes^L_{X\times X\times X\times Y} Lp_{123}^*Lp_{23}^*M_2 \otimes^L_{X\times X\times X\times Y} Lp_{134}^*Lp_{23}^*M_3\big)$$

$$\simeq Rp_{14*}\big(Lp_{12}^*M_1 \otimes^L_{X\times X\times X\times Y} Lp_{23}^*M_2 \otimes^L_{X\times X\times X\times Y} Lp_{34}^*M_3\big).$$

Now we make a similar calculation for the RHS.

$$M_1*(M_2*M_3) = Rp_{13*}\Big(Lp_{12}^*M_1 \otimes^L_{X\times X\times Y} Lp_{23}^*\big(Rp_{13*}(Lp_{12}^*M_2 \otimes^L_{X\times X\times Y} Lp_{23}^*M_3)\big)\Big).$$

Consider the cartesian diagram

$$\begin{array}{ccc} X\times X\times X\times Y & \xrightarrow{p_{234}} & X\times X\times Y \\ \downarrow{\scriptstyle p_{124}} & & \downarrow{\scriptstyle p_{13}} \\ X\times X\times Y & \xrightarrow[p_{23}]{} & X\times Y \end{array}$$

Using flat base change we get

$$M_1 * (M_2 * M_3) \simeq Rp_{13*}\big(Lp_{12}^*M_1 \otimes^L_{X\times X\times Y} Rp_{124*}Lp_{234}^*(Lp_{12}^*M_2 \otimes^L_{X\times X\times Y} Lp_{23}^*M_3)\big).$$

Applying the projection formula we get

$$M_1 * (M_2 * M_3)$$

$$\simeq Rp_{13*}Rp_{124*}\big(Lp_{124}^*Lp_{12}^*M_1 \otimes^L_{X\times X\times X\times Y} Lp_{234}^*(Lp_{12}^*M_2 \otimes^L_{X\times X\times Y} Lp_{23}^*M_3)\big)$$

$$\simeq Rp_{13*}Rp_{124*}\big(Lp_{124}^*Lp_{12}^*M_1 \otimes^L_{X\times X\times X\times Y} Lp_{234}^*Lp_{12}^*M_2 \otimes^L_{X\times X\times X\times Y} Lp_{234}^*Lp_{23}^*M_3)\big)$$

$$\simeq Rp_{14*}(Lp_{12}^*M_1 \otimes^L_{X\times X\times X\times Y} Lp_{23}^*M_2 \otimes^L_{X\times X\times X\times Y} Lp_{34}^*M_3).$$

This finishes the proof. □

REMARK 5.7. In the proposition everything is in a weak sense: the associativity constraint $(M_1 * M_2) * M_3 \xrightarrow{\sim} M_1 * (M_2 * M_3)$ is not specified.

One often need to compare convolutions on two different spaces. This requires the following setup. Suppose we have Noetherian, normal K-schemes X_1, X_2 and X_3, Z_{12}, Z'_{12} and Z_{23}. We are given the K-equivariant flat maps

$$p_1 :\ Z_{12} \to X_1,\ p_2 :\ Z_{12} \to X_2,\ q_2 :\ Z_{23} \to X_2,\ q_3 :\ Z_{23} \to X_3,$$

and $\alpha : Z'_{12} \to Z_{12}$ such that $p'_1 = p_1 \circ \alpha$ and $p'_2 = p_2 \circ \alpha$ are also flat. Consider the projections

$$\begin{aligned} \mathrm{pr}_{12} :\ & Z_{12} \times_{X_2} Z_{23} \to Z_{12}, & \mathrm{pr}_{23} :\ & Z_{12} \times_{X_2} Z_{23} \to Z_{23},\\ \mathrm{pr}'_{12} :\ & Z'_{12} \times_{X_2} Z_{23} \to Z'_{12}, & \mathrm{pr}'_{23} :\ & Z'_{12} \times_{X_2} Z_{23} \to Z'_{23},\\ \mathrm{pr}_{13} :\ & Z_{12} \times_{X_2} Z_{23} \to X_1 \times X_3, & \mathrm{pr}'_{13} :\ & Z'_{12} \times_{X_2} Z_{23} \to X_1 \times X_3 \end{aligned}$$

We introduce the convolution products

$$\begin{aligned} *' :\ & D(\mathrm{QCoh}^K(Z'_{12})) \times D(\mathrm{QCoh}^K(Z_{23})) \to D(\mathrm{QCoh}^K(X_1 \times X_3)),\\ & M_1 * M_2 := R\mathrm{pr}_{13*}(L\mathrm{pr}^*_{12}(R\alpha_* M_1) \otimes^L_{Z_{12}\times_{X_2} Z_{23}} L\mathrm{pr}^*_{23} M_2) \end{aligned}$$

and

$$\begin{aligned} *'' :\ & D(\mathrm{QCoh}^K(Z'_{12})) \times D(\mathrm{QCoh}^K(Z_{23})) \to D(\mathrm{QCoh}^K(X_1 \times X_3)),\\ & M_1 *'' M_2 := R\mathrm{pr}'_{13*}(L\mathrm{pr}'^*_{12} M_1 \otimes^L_{Z'_{12}\times_{X_2} Z_{23}} L\mathrm{pr}'^*_{23} M_2). \end{aligned}$$

Lemma 5.8. *The convolutions $*$ and $*'$ are canonically isomorphic.*

Proof. Let $M_1 \in D(\mathrm{QCoh}^K(Z'_{12}))$ and $M_2 \in D(\mathrm{QCoh}^K(Z_{23}))$. Consider the cartesian diagram

$$\begin{array}{ccc} Z'_{12} \times_{X_2} Z_{23} & \xrightarrow{\mathrm{pr}'_{12}} & Z'_{12} \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} \\ Z_{12} \times_{X_2} Z_{23} & \xrightarrow[\mathrm{pr}_{12}]{} & Z_{12} \end{array}$$

By flat base change we get

$$M_1 *' M_2 \simeq Rp_{13*}\big(R\beta_* L\mathrm{pr}'^*_{12} M_1 \otimes^L_{Z_{12}\times_{X_2} Z_{23}} L\mathrm{pr}^*_{23} M_2\big).$$

Using the projection formula we get

$$\begin{aligned} M_1 *' M_2 &\simeq Rp_{13*} R\beta_*\big(L\mathrm{pr}'^*_{12} M_1 \otimes^L_{Z'_{12}\times_{X_2} Z_{23}} L\beta^* L\mathrm{pr}^*_{23} M_2\big)\\ &\simeq R\mathrm{pr}'_{13*}(L\mathrm{pr}'^*_{12} M_1 \otimes^L_{Z'_{12}\times_{X_2} Z_{23}} L\mathrm{pr}^*_{23} M_2)\\ &= M_1 *'' M_2. \end{aligned}$$

□

Remark 5.9. A typical special case in which Lemma 5.8 is applied is as follows. Take $X_1 = X_2 = X_3 = X$. For a flat surjective K-equivariant map $X \to Y$ consider

$$Z'_{12} = X \times_Y X,\ Z_{12} = Z_{23} = Z_{13} = X \times X.$$

Lemma 5.8 implies that convolution operations defined via $X \times_Y X \times X$ and push-forward from $X \times X \times X$ coincide.

Remark 5.10. In particular the unit object in the monoidal category $D(\mathrm{QCoh}^K(X\times X))$ is given by the structure sheaf of the diagonal in $X \times X$ denoted by $\mathcal{O}_{X_\Delta}$.

5.2.2. Convolution and correspondences. Let $X, Y_1, \dots Y_n$ be regular K-schemes. Suppose we are given flat surjective maps $\phi_i : \ X \to Y_i$, $i = 1, \dots, n$. Denote the fiber product $X \times_{Y_i} X \subset X \times X$ by X_I and let $\alpha_i : \ X_i \to X \times X$ be the inclusion. Consider the iterated fibered product

$$Z_{i_1,\dots,i_k} := X_{i_1} \times_X \dots \times_X X_{i_k} = X \times_{Y_{i_1}} X \dots \times_{Y_{i_k}} X \subset X^{k+1}.$$

We have the map provided by the projections to the first and last factors

$$\alpha_{i_1,\dots,i_k} : \ Z_{i_1,\dots,i_k} \to X \times X.$$

Denote the image of the map by $X_{i_1,\dots,i_k} \subset X \times X$. All the defined schemes are acted on naturally by K and all the defined maps are K-equivariant.

Consider the sheaves $M_i := R\alpha_{i\,*}(\mathcal{O}_{X_i})$. The category $D(\mathrm{QCoh}^K(X\times X))$ acts on the category $D(\mathrm{QCoh}^K(X))$ by convolution. Denote the functor of convolution with M_i by D_i.

Lemma 5.11. *The functor* $D_i : \ D(\mathrm{QCoh}^K(X)) \to D(\mathrm{QCoh}^K(X))$ *is isomorphic to the functor* $L\phi_i^* \circ R\phi_{i*}$.

Proof. Denote the two projections $X_i \to X$ by $q_{1,i}$ and $q_{2,i}$. Let $N \in D(\mathrm{QCoh}^K(X))$. We apply lemma 5.8 as suggested in remark 5.9 with $Z_{12} = X\times X$, $Z'_{12} = X_i$ and $Z_{23} = X$. In the notation of the lemma $\mathrm{pr}'_{12} = \mathrm{Id}$, $\mathrm{pr}'_{23} = q_2$ and $\mathrm{pr}'_{13} = q_1$.

$$\begin{aligned} D_i(N) &= \mathcal{O}_{X_i} *' N \\ &\simeq Rq_{1*}(\mathcal{O}_{X_i} \otimes^L_{X_i} Lq_2^* N) \\ &\simeq Rq_{1*} Lq_2^* N. \end{aligned}$$

Consider the cartesian diagram

$$\begin{array}{ccc} X \times_{Y_i} X & \xrightarrow{q_2} & X \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle \phi_i} \\ X & \xrightarrow[\phi_i]{} & Y_i \end{array}$$

Applying flat base change we obtain the statement of the lemma. □

Corollary 5.12. Each functor D_i is isomorphic to a comonad. Suppose additionally that the maps $\phi_i : \ X \to Y_i$ are locally trivial fibrations with fiber F such that $H^i(F) = 0$ for $i > 0$ and $H^0(F) = k$. Then the comonads D_i are coprojectors, i.e. the coproduct maps $D_i \to D_i \circ D_i$ are isomorphisms of functors.

PROOF. Since $R\phi_{i*}$ is right adjoint to $L\phi_i^*$ the first statement follows from [**Mac**, section VI.1]. Let $\{U_j\}$ be a trivialization of ϕ_i. Then

$$\begin{aligned}(R^k\phi_{i*}\mathcal{O}_X)|_{U_j} &\simeq R^k\phi_{i*}(\mathcal{O}_X|_{\phi_i^{-1}(U_j)})\\ &\simeq R^k(\mathrm{Id}\times\Gamma)_*(\mathcal{O}_{U_j\times F})\\ &\simeq \mathcal{O}_{U_j}\otimes H^k(F,\mathcal{O}_F)\\ &\simeq \begin{cases}\mathcal{O}_{U_j} & k=0\\ 0 & k>0\end{cases}\end{aligned}$$

Thus, $R\phi_{i*}\mathcal{O}_X \simeq \mathcal{O}_{Y_i}$. Inserting this into the projection formula for $M \in D\,\mathrm{QCoh}^G(Y_i)$ we get

$$\begin{aligned}R\phi_{i*}L\phi_i^*M &\simeq R\phi_{i*}(\mathcal{O}_X\otimes^L_X L\phi_i^*M)\\ &\simeq R\phi_{i*}\mathcal{O}_X\otimes^L_{Y_i} M\\ &\simeq \mathcal{O}_{Y_i}\otimes^L_{Y_i} M \simeq M.\end{aligned}$$

Thus, the map

$$\mathrm{Hom}(N,M)\to\mathrm{Hom}(N,R\phi_{i*}L\phi_i^*M)=\mathrm{Hom}(N,M)$$

induced by the adjunction map $M\to R\phi_{i*}L\phi_i^*M$ is a bijection for all N, M. This implies that the adjunction map is an isomorphism. The coproduct map

$$D_i\simeq L\phi_i^*\circ\mathrm{Id}\circ R\phi_{i*}\to L\phi_i^*\circ R\phi_{i*}\circ L\phi_i^*\circ R\phi_{i*}\simeq D_i\circ D_i$$

is defined via the adjunction map so it is also an isomorphism. □

Our goal is to describe the composition of the functors $D_{i_1}\circ\ldots\circ D_{i_n}$ explicitly. Denote $R\alpha_{i_1,\ldots,i_k*}(\mathcal{O}_{Z_{i_1,\ldots,i_k}})\in D(\mathrm{QCoh}^K(X\times X))$ by $M_{i_1,\ldots,i_k}$.

LEMMA 5.13. *We have a natural isomorphism of objects in* $D(\mathrm{QCoh}^K(X\times X))$

$$M_{i_1}*\ldots*M_{i_k}\xrightarrow{\sim}M_{i_1,\ldots,i_k}.$$

PROOF. We proceed by induction. As in the proof of lemma 5.11 we consider the two projections $q_{1,i}$ and $q_{2,i}: X\times_{Y_i}X\times X\to X\times X$ and notice that

$$M_{i_1}*\ldots*M_{i_k}\xrightarrow{\sim}M_{i_1}*M_{i_2,\ldots,i_k}\xrightarrow{\sim}Rq_{1,i*}Lq_{2,i}^*(M_{i_2,\ldots,i_k}).$$

Applying base change to the diagram

$$\begin{array}{ccc} Z_{i_1,\ldots,i_k}=X\times_{Y_i}Z_{i_2,\ldots,i_k} & \xrightarrow{\pi_2} & Z_{i_2,\ldots,i_k}\\ \downarrow{\scriptstyle p_{1,2,k+1}} & & \downarrow{\scriptstyle \alpha_{i_2,\ldots,i_k}}\\ X\times_{Y_i}X\times X & \xrightarrow[q_{2,i}]{} & X\times X\end{array}$$

we get

$$\begin{aligned} M_{i_1} * \ldots * M_{i_k} &\simeq Rq_{1,i*} Lq_{2,i}^* R\alpha_{i_2,\ldots,i_k}(\mathcal{O}_{Z_{i_2,\ldots i_k}}) \\ &\simeq Rq_{1,i*} Rp_{1,2,k+1*} L\pi_2^*(\mathcal{O}_{Z_{i_2,\ldots,i_k}}) \\ &\simeq R(q_{1,i} \circ p_{1,2,k+1})_*(\mathcal{O}_{Z_{i_1,\ldots,i_k}}) \\ &\simeq R\alpha_{i_1,\ldots,i_k*}(\mathcal{O}_{Z_{i_1,\ldots,i_k}}) \\ &= M_{i_1,\ldots,i_k}. \end{aligned}$$

This finishes the proof. □

5.2.3. Quasi-coherent Hecke category. Fix a reductive algebraic group K with an algebraic subgroup H. Consider the K-scheme $Y = K/H$.

DEFINITION 5.14. The monoidal category

$$(D(\mathrm{QCoh}^K(K/H \times K/H)), *)$$

is called the quasi-coherent Hecke category and it is denoted by $\mathrm{QCHecke}(K, H)$.

Notice that for a K-scheme X we have

$$D(\mathrm{QCoh}^H(X)) \simeq D(\mathrm{QCoh}^K(K/H \times X)).$$

Taking $Z = Y = K/H$ in the setting 5.2.1 we get the monoidal action

$$\mathrm{QCHecke}(K, H) \times D(\mathrm{QCoh}^H(X)) \to D(\mathrm{QCoh}^H(X)).$$

5.3. Demazure Descent for $D(\mathrm{QCoh}^B(X))$

Let X be a scheme equipped with an action of a reductive algebraic group G. For every element of the Weyl group $w \in W$ we construct a functor D_w acting on the category $D(\mathrm{QCoh}^G(G/B \times X))$. The functor is defined in terms of the monoidal action of $\mathrm{QCHecke}(G, B)$.

The elements in the Weyl group is in bijection with the orbits for the the diagonal action of G on $G/B \times G/B$ via the map $w \mapsto G \cdot (BwB, B)$ [**CG**, Theorem 3.1.9]. We denote the closure of the orbit corresponding to w by $\overline{\mathbb{O}}_w$. To simplify the notations below we write $\overline{\mathbb{O}}_i$ for $\overline{\mathbb{O}}_{s_{\alpha_i}}$ when s_{α_i} is a simple reflection. It can be expressed in terms of minimal parabolics

$$\begin{aligned} \overline{\mathbb{O}}_i &= (G/B)_\Delta \cup G \cdot \{(B, p_iB) \mid p_i \in P_i \backslash B\} \\ &= G/B \times_{G/P_i} G/B. \end{aligned}$$

Since $G/B \to G/P_i$ is a locally trivial fibration with fiber $\mathbb{P}^1$ so is the projections to the factors $p_1, p_2 : \overline{\mathbb{O}}_i \to G/B$. In particular, they are flat. The structure sheaves of the orbit closures $\mathcal{O}_{\overline{\mathbb{O}}_w}, w \in W$, are objects of the category $\mathrm{QCHecke}(G, B)$. Consider the functor

$$\begin{aligned} D_w : D(\mathrm{QCoh}^G(G/B \times X)) &\to D(\mathrm{QCoh}^G(G/B \times X)), \\ D_w(M) &:= \mathcal{O}_{\overline{\mathbb{O}}_w} * M. \end{aligned}$$

Below we prove that the functors $D_w, w \in W$, form Demazure descent data on the category $D(\mathrm{QCoh}^G(G/B \times X))$. Consider the projection $p_i : G/B \times X \to G/P_i \times X$ and the corresponding inverse and direct image functors

$$Lp_i^*,\ Rp_{i*} :\ D(\mathrm{QCoh}^G(G/B \times X)) \overset{\longleftarrow}{\longrightarrow} D(\mathrm{QCoh}^G(G/P_i \times X)).$$

PROPOSITION 5.15. *The Demazure functor D_{s_i} is isomorphic to $Lp_i^* Rp_{i*}$. This gives a comonad structure with $D_{s_i} \xrightarrow{\sim} D_{s_i} D_{s_i}$.*

PROOF. The projections p_i are locally trivial fibrations with fiber $\mathbb{P}^1$ so the result follows from lemma 5.11 and corollary 5.12. □

Notice that the associativity up to isomorphism for the monoidal action of $\mathrm{QCHecke}(G, B)$ on the category $D(\mathrm{QCoh}^G(G/B \times X))$ implies that all relations up to a non-specified isomorphism can be checked in the Hecke category.

PROPOSITION 5.16. *Let $w = s_{k_1} \cdots s_{k_n}$ be a reduced expression. Then*

$$\mathcal{O}_{\overline{\mathbb{O}}_{k_1}} * \cdots * \mathcal{O}_{\overline{\mathbb{O}}_{k_n}} \simeq \mathcal{O}_{\overline{\mathbb{O}}_w}$$

PROOF. For $w \in W$ choose a a reduced expression $w = s_{k_1} \cdots s_{k_n}$. Recall the Bott-Samelson resolution $\phi_w : X_{k_1 \cdots k_n} := P_{k_1} \times^B P_{k_2} \times^B \cdots \times^B P_{k_n}/B \to X_w$ from section 4.2.1. With the projections $\phi_{i_k} : G/B \to G/P_{i_k}$ we are in the setting of 5.2.2. Set

$$\begin{aligned}
\overline{\mathbb{O}}_{k_1 \cdots k_n} &:= (G/B \times_{G/P_{k_1}} G/B) \times_{G/B} \cdots \times_{G/B} (G/B \times_{G/P_{k_n}} G/B) \\
&\simeq G/B \times_{G/P_{k_1}} G/B \times_{G/P_{k_2}} \cdots \times_{G/P_{k_n}} G/B \\
&\simeq G \times^B P_{k_1} \times^B P_{k_2} \times^B \cdots \times^B P_{k_n}/B \\
&\simeq \frac{G \times X_{k_1 \cdots k_n}}{B}.
\end{aligned}$$

Thus, we have the following equivalences of categories

$$\mathrm{QCoh}^G(\overline{\mathbb{O}}_{k_1 \ldots k_n}) \simeq \mathrm{QCoh}^G\left(\frac{G \times X_{k_1 \cdots k_n}}{B}\right) \simeq \mathrm{QCoh}^B(X_{k_1 \cdots k_n}).$$

As a set

$$\begin{aligned}
G/B \times_{G/P_{k_1}} G/B \times_{G/P_{k_2}} \cdots \times_{G/P_{k_n}} G/B& \\
= G \cdot \{(p_{i_1} \cdots p_{i_n} B, \ldots, p_{i_{n-1}} p_{i_n} B, p_{i_n} B, B) \mid p_{i_k} \in P_{i_k}\}&.
\end{aligned}$$

Hence, $\alpha_{i_1,\ldots,i_n}(\overline{\mathbb{O}}_{k_1 \cdots k_n}) \simeq \frac{X_w \times G}{B} \simeq \overline{\mathbb{O}}_w$ and the following diagram is commutative

$$\begin{array}{ccc}
\frac{G \times X_{k_1 \cdots k_n}}{B} & \xrightarrow{\bar{\phi}_w} & \frac{G \times X_w}{B} \\
\wr\Big| & & \Big|\wr \\
\overline{\mathbb{O}}_{k_1 \ldots k_n} & \xrightarrow[\alpha_{i_1,\ldots,i_n}]{} & \overline{\mathbb{O}}_w
\end{array}$$

Thus, on the level of categories we have

$$\begin{array}{ccc} D(\mathrm{QCoh}^B(X_{k_1\cdots k_n})) & \xrightarrow{R\phi_{w*}} & D(\mathrm{QCoh}^B(X_w)) \\ \psi_1 \downarrow \wr & & \psi_2 \downarrow \wr \\ D(\mathrm{QCoh}^G(\overline{\mathbb{O}}_{k_1\ldots k_n})) & \xrightarrow[R\alpha_{i_1,\ldots,i_n*}]{} & D(\mathrm{QCoh}^G(\overline{\mathbb{O}}_w)) \end{array}$$

It is known (see e.g. [**And**][Theorem 3.1]) that in the non-equivariant setting $R\phi_{w*}\mathcal{O}_{X_{k_1\cdots k_n}} \simeq \mathcal{O}_{X_w}$. Since $R\phi_{w*}$ commutes with the forgetful functor the same is true in the equivariant setting. Inserting this we get

$$\begin{aligned} R\alpha_{i_1,\ldots,i_n*}(\mathcal{O}_{\overline{\mathbb{O}}_{k_1\ldots k_n}}) &\simeq R\alpha_{i_1,\ldots,i_n*}\psi_1(\mathcal{O}_{X_{k_1\cdots k_n}}) \simeq \psi_2 R\phi_{w*}(\mathcal{O}_{X_{k_1\cdots k_n}}) \\ &\simeq \psi_2(\mathcal{O}_{X_w}) \simeq \mathcal{O}_{\overline{\mathbb{O}}_w}. \end{aligned}$$

By lemma 5.13 $\mathcal{O}_{\overline{\mathbb{O}}_{k_1}} * \cdots * \mathcal{O}_{\overline{\mathbb{O}}_{k_n}} \simeq R\alpha_{i_1,\ldots,i_n*}(\mathcal{O}_{\overline{\mathbb{O}}_{k_1\ldots k_n}})$ so this finishes the proof. □

Now we are prepared to prove the central result.

THEOREM 5.17. *The functors $\{D_w, w \in W\}$ form Demazure Descent Data on the category $D(\mathrm{QCoh}^G(G/B \times X))$.*

PROOF OF THEOREM 5.17. We have proved that each of the functors D_{s_i} is a comonad and the coproduct maps are isomorphisms of functors. It remains to show that for all $w, w_2 \in W$ with $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ we have

$$D_{w_1} \circ D_{w_2} \simeq D_{w_1w_2}.$$

Fix reduced expressions for the Weyl group elements $w_1 = s_{k_1} \cdots s_{k_n}$ and $w_2 = s_{j_1} \cdots s_{j_m}$. Since $\ell(w_1w_2) = \ell(w_1)+\ell(w_2)$ the expression $w_1w_2 = s_{k_1} \cdots s_{k_n} s_{j_1} \cdots s_{j_m}$ is reduced. For any $M \in D(\mathrm{QCoh}^G(G/B \times X))$ we obtain

$$\begin{aligned} D_{w_1} \circ D_{w_2}(M) &\simeq \mathcal{O}_{\overline{\mathbb{O}}_{w_1}} * \mathcal{O}_{\overline{\mathbb{O}}_{w_2}} * M \\ &\simeq \mathcal{O}_{\overline{\mathbb{O}}_{k_1}} * \cdots * \mathcal{O}_{\overline{\mathbb{O}}_{k_n}} * \mathcal{O}_{\overline{\mathbb{O}}_{j_1}} * \cdots * \mathcal{O}_{\overline{\mathbb{O}}_{j_m}} * M \\ &\simeq \mathcal{O}_{\overline{\mathbb{O}}_{w_1w_2}} * M \\ &\simeq D_{w_1w_2}(M). \end{aligned}$$

This finishes the proof. □

5.4. Descent category

Consider the descent category for the constructed Demazure descent data on the category $D(\mathrm{QCoh}^G(G/B \times X))$.

THEOREM 5.18. *The descent category $\mathfrak{Desc}(D(\mathrm{QCoh}^B(X)), D_w, w \in W)$ is equivalent to $D(\mathrm{QCoh}^G(X))$.*

The projection $p_i : G/B \times X \to G/P_i \times X$ (resp. $p : G/B \times X \to X$) is a locally trivial fibration with fiber $\mathbb{P}^1$ (resp. G/B). Notice that $H^k(\mathbb{P}^1) = 0 = H^k(G/B)$ for $k > 0$ and $H^0(\mathbb{P}^1) \simeq k \simeq H^0(G/B)$. In the proof of corollary 5.12 we proved that this implies that the adjunction map $\mathrm{Id} \to Rp_{i*}Lp_i^*$ (resp. $\mathrm{Id} \to Rp_*Lp^*$) is an isomorphism. Hence, Lp_i^* (resp. Lp^*) is fully faithful since

$$\mathrm{Hom}(Lp_i^*N, Lp_i^*M) = \mathrm{Hom}(Rp_{i*}Lp_i^*N, M) = \mathrm{Hom}(N, M).$$

We identify $D(\mathrm{QCoh}^G(X))$ with the full subcategory of $D(\mathrm{QCoh}^B(X))$ given by the essential image of Lp^*.

LEMMA 5.19. *An object M in $D(\mathrm{QCoh}^G(G/B \times X))$ belongs to the essential image of Lp_i^* if and only if the coaction map $M \to D_{s_i}(M)$is an isomorphism.*

PROOF. We identified the functor D_{s_i} with the composition $Lp_i^*Rp_{i*}$. Thus, $M \simeq D_{s_i}(M)$ implies that M belongs to the essential image of Lp_i^*. Assume that $M = Lp_i^*(N)$ for some $N \in D(\mathrm{QCoh}^G(G/P_i \times X))$. Since the adjunction map is an isomorphism we have

$$D_{s_i}(M) \simeq Lp_i^*Rp_{i*}Lp_i^*(N) \simeq Lp_i^*(N) = M. \qquad \square$$

REMARK 5.20. Since $\overline{\mathbb{O}}_{w_0} = G/B \times G/B$ the same argument shows that an object M in $D(\mathrm{QCoh}^G(G/B \times X))$ belongs to the essential image of Lp^* if and only if $D_{w_0}(M)$ is isomorphic to M.

PROOF OF THEOREM 5.18. Let $M \in \mathfrak{Desc}(D(\mathrm{QCoh}^B(X)), D_w, w \in W)$. For every simple root α_i the object $D_{s_i}(M)$ is isomorphic to M. Choose a reduced expression $s_{i_1} \cdots s_{i_n}$ for w_0. We have

$$D_{s_{i_1}} \circ \cdots \circ D_{s_{i_n}}(M) \simeq D_{w_0}(M).$$

It follows that M belongs to the essential image of Lp^*. In particular, the descent category $\mathfrak{Desc}(D(\mathrm{QCoh}^B(X)), D_w, w \in W)$ is a full subcategory in the essential image of the functor Lp^*.

To prove the other embedding, notice that the map p factors as

$$G/B \times X \to G/P_i \times X \to X.$$

It follows that the essential image of Lp^* is a full subcategory in the essential image of Lp_i^* for all i. This completes the proof of the theorem. $\square$

CHAPTER 6

Equivariant matrix factorizations and Hamiltonian reduction

6.1. Context

6.1.1. Known results in the non-equivariant setting. Let X be a smooth variety over an algebraically closed field of characteristic zero and let $\pi : E \to X$ be a vector bundle. Fix a regular global section s of that vector bundle. The dual vector bundle is denoted by $\pi^\vee : E^\vee \to X$. This defines a pull-back section via the cartesian diagram

$$\begin{array}{ccc} E^\vee \times_X E & \longrightarrow & E \\ {\scriptstyle (\pi^\vee)^* s}\uparrow & & \uparrow{\scriptstyle s} \\ E^\vee & \xrightarrow{\pi^\vee} & X \end{array}$$

Define the function W to be the composition with the natural pairing

$$W : E^\vee \to E^\vee \times_X E \xrightarrow{\langle\,,\rangle} k,$$
$$a_x \mapsto \langle ((\pi^\vee)^* s)(a_x), a_x \rangle.$$

Consider W as a function on $E^\vee$. Acting by this element defines a potential for matrix factorizations on $E^\vee$. Set $X_0 := W^{-1}(0)$. Let $\mathrm{Perf}(X_0)$ denote the full subcategory of $D^b \operatorname{Coh}(X_0)$ of perfect complexes, i.e. complexes quasi-isomorphic to a bounded complex of locally free sheaves of finite rank. The singularity category denoted by $D_{\mathrm{sg}}(X_0)$ is the quotient category $D^b \operatorname{Coh}(X_0)/\mathrm{Perf}(X_0)$. Introducing an additional k^*-equivariance, equivalent to an additional grading, Isik proved [**Isik**, Theorem 3.6] using linear Koszul duality that $D^{k^*}_{\mathrm{sg}}(X_0)$ is equivalent to $D^b \operatorname{Coh}(Y)$, where Y is the zero scheme of s.

Polishchuk and Vaintrob proved in [**PV**, Theorem 3.14] (generalizing a result of Orlov) that assuming W is not a zero-divisor the absolute derived category $\mathrm{DMF}(E^\vee, W)$ is isomorphic to $D_{\mathrm{sg}}(X_0)$. The precise definition of the absolute derived category is given in section 6.5.1.

6.1.2. Moment maps and Hamiltonian reduction. Symplectic differential geometry is the language of classical mechanics. The definitions naturally carries over to the algebro-geometric setting. Let X be a smooth algebraic variety. The definition of a symplectic form on X is the same as in differential geometry only we require it to be algebraic.

DEFINITION 6.1. A symplectic form ω on X is a 2-form satisfying

- ω is closed, i.e. $d\omega = 0$ where d is the exterior derivative.
- ω is non-degenerate, i.e. for each $x \in X$ the map $\omega_x : T_xX \times T_xX \to k$ satisfies that if we have $Y \in T_xX$ such that $\omega_x(Y, Z) = 0$ for all $Z \in T_xX$ then $Y = 0$.

A symplectic algebraic variety is pair (X, ω) of a smooth algebraic variety X and a symplectic form ω.

Our main example of interest is the following

EXAMPLE 6.2. Let X be a smooth algebraic variety. Then T^*X is a symplectic algebraic variety with a symplectic form defined in the following way. Let $\pi : T^*X \to X$ be the projection. For $x \in X$ and $\beta \in T^*_xX$ we define the 1-form

$$\alpha_{(x,\beta)} : T_{(x,\beta)}(T^*X) \to k, \qquad \alpha_{(x,\beta)}(\xi) := \langle \beta, d_{(x,\beta)}\pi(\xi)\rangle.$$

Then we get a symplectic form $\omega := -d\alpha$.

The non-degeneracy condition gives a canonical isomorphism $TX \simeq T^*X$ and it defines a unique k-linear map

$$\mathcal{O}(X) \to \mathrm{Vect}(X), \qquad f \mapsto \xi_f$$

defined by the requirement

$$\omega(\cdot, \xi_f) = df.$$

DEFINITION 6.3. A Poisson algebra is a commutative algebra with a Lie bracket satisfying the Leibniz rule.

LEMMA 6.4. *The algebra $\mathcal{O}(X)$ of global functions has a Poisson algebra structure with the bracket defined by*

$$\{f, g\} := \omega(\xi_f, \xi_g).$$

The map $f \mapsto \xi_f$ intertwines the bracket on $\mathcal{O}(X)$ with the Lie bracket on $\mathrm{Vect}(X)$.

PROOF. See section 1.2 in [**CG**]. □

The vector field $v(f) := \{f, \cdot\}$ is called the Hamiltonian vector field for f. Let G be a reductive algebraic group acting on X. The action is called symplectic if it preserves the symplectic form, i.e. $\omega(z, y) = \omega(gz, gy)$ for all $x \in X$, $z, y \in T_xX$ and $g \in G$. Such an action induces a Lie algebra homomorphism

$$\mathfrak{g} \to \mathrm{Vect}(X), \qquad \xi \mapsto \xi_X.$$

Defined in the following way. For $x \in X$ consider the action $\sigma_x : G \to G \cdot x \subset X$. Then for $\xi \in \mathfrak{g}$ we define $\xi_X(x) := d\sigma_x(\xi) \in T_xX$.

Definition 6.5. A symplectic G-action on a smooth symplectic algebraic variety X is called Hamiltonian if a G-equivariant (with respect to the adjoint action of G on $\mathfrak{g}$) Lie algebra homomorphism

$$\mu^* : \mathfrak{g} \to \mathcal{O}(X)$$

is given, which makes the following diagram commutative

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mu^*} & \mathcal{O}(X) \\ & {\scriptstyle \xi \mapsto \xi_X} \searrow \quad \swarrow {\scriptstyle f \mapsto \xi_f} & \\ & \mathrm{Vect}(X) & \end{array}$$

Such μ^* is called a comoment map.

Remark 6.6. If it exists a comoment map μ^* is unique up to an element of $(\mathfrak{g}^*)^G$. For $\lambda \in (\mathfrak{g}^*)^G$ the map $\mu^*(\xi) + \langle \lambda, \xi \rangle$ is also a comoment map.

Given a comoment map one can define the dual map $\mu : X \to \mathfrak{g}^*$ by the requirement

$$\langle \mu(x), \xi \rangle = \mu^*(\xi)(x).$$

This is called a moment map.

Example 6.7. An action of a reductive algebraic group G on a smooth algebraic variety X lifts to a symplectic action of G on T^*X. A comoment map for this action is given by

$$\mathfrak{g} \ni \xi \mapsto \xi_X \in \mathrm{Vect}(X) \hookrightarrow \mathcal{O}(T^*X) = S_{\mathcal{O}(X)}\mathrm{Vect}(X).$$

Here S is the symmetric algebra.

One of the main applications is the following construction.

Definition 6.8. Let G and X as above and let $\lambda \in (\mathfrak{g}^*)^G$. The Hamiltonian reduction is the GIT quotient

$$\mu^{-1}(\lambda)//G = \mathrm{Spec}(\mathcal{O}(\mu^{-1}(\lambda))^G)$$

When the action of G on $\mu^{-1}(\lambda)$ is free the Hamiltonian reduction is a smooth symplectic algebraic variety. Thus, this is an important way of constructing new symplectic varieties from old ones. In physics this kind of construction is often used to eliminate degrees of freedom in a system in classical mechanics using a symmetry of the system. A classical object of study in geometric representation theory are Nakajima quiver varieties and some of these are of this form. In our setup $\mu^{-1}(0)$ will be a derived scheme.

6.1.3. Matrix factorizations coming from Hamiltonian reduction. Let X be a smooth algebraic variety and G a linear algebraic group acting on X. The moment map $\mu : T^*X \to \mathfrak{g}^*$ is a section of the trivial vector bundle $\pi : T^*X \times \mathfrak{g}^* \to T^*X$ so it defines a potential

$$W : T^*X \times \mathfrak{g} \xrightarrow{\mu\times \mathrm{Id}} \mathfrak{g}^* \times \mathfrak{g} \xrightarrow{\langle\, ,\rangle} k.$$

Inserting this into the theorems in section 6.1.1 we get $\mathrm{DMF}(T^*X \times \mathfrak{g}, W)$ on one side and $D^b\,\mathrm{Coh}(\mu^{-1}(0))$ on the other. If one could extend these results to the G-equivariant setting one would get an equivalence between the equivariant derived category of matrix factorizations and the derived category of coherent sheaves on the Hamiltonian reduction. This is the goal of this chapter.

6.2. The setting

We recall the basic definitions of equivariant quasi-coherent sheaves of differential graded modules.

Definition 6.9. Let $\mathcal{A} = \bigoplus_{p\in\mathbb{Z}} \mathcal{A}^p$ be a sheaf of $\mathbb{Z}$-graded $\mathcal{O}_X$-algebras on a complex algebraic variety X. Denote the multiplication map by $\mu_{\mathcal{A}} : \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{A}$.

(1) The sheaf $\mathcal{A}$ is a sheaf of dg-algebras if it is provided with an endomorphism of $\mathcal{O}_X$-modules $d_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ of degree 1, such that $d_{\mathcal{A}} \circ d_{\mathcal{A}} = 0$, satisfying the following formula on $\mathcal{A}_i \otimes \mathcal{A}$ for any $i \in \mathbb{Z}$:
$$d_{\mathcal{A}} \circ \mu_{\mathcal{A}} = \mu_{\mathcal{A}} \circ (d_{\mathcal{A}} \otimes \mathrm{Id}_{\mathcal{A}}) + (-1)^i \mu_{\mathcal{A}} \circ (\mathrm{Id}_{\mathcal{A}^p} \otimes d_{\mathcal{A}}).$$
(2) A morphism of sheaves of dg-algebras on the same scheme is a morphism of sheaves of graded algebras commuting with the differentials.
(3) A morphism of dg-algebras on different schemes $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is the data of a morphism of schemes $f_0 : X \to Y$, and a morphism of sheaves of dg-algebras $f_0^*\mathcal{B} \to \mathcal{A}$.
(4) Define the opposite dg-algebra $\mathcal{A}^{op}$ to have the same elements and differential as $\mathcal{A}$ but a new multiplication $a \circ b := (-1)^{\deg(a)\deg(b)} ba$. The sheaf of dg-algebras $\mathcal{A}$ is called graded-commutative if the identity map $\mathrm{Id} : \mathcal{A} \to \mathcal{A}^{op}$ is an isomorphism of sheaves of dg-algebras.
(5) A $\mathcal{A}$-dg-module is a sheaf of $\mathbb{Z}$-graded left $\mathcal{A}$-modules $\mathcal{F}$ on X together with an endomorphism of $\mathcal{O}_X$-modules $d_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$ of degree 1, such that $d_{\mathcal{F}} \circ d_{\mathcal{F}} = 0$ and satisfying the following formula on $\mathcal{A}^i \otimes_{\mathcal{O}_X} \mathcal{F}$ for $i \in \mathbb{Z}$, where $\alpha_{\mathcal{F}} : \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$ is the action map:
$$d_{\mathcal{F}} \circ \alpha_{\mathcal{F}} = \alpha_{\mathcal{F}} \circ (d_{\mathcal{A}} \otimes \mathrm{Id}_{\mathcal{F}}) + (-1)^i \alpha_{\mathcal{F}} \circ (\mathrm{Id}_{\mathcal{A}^p} \otimes d_{\mathcal{F}}).$$
(6) A morphism of $\mathcal{A}$-dg-modules is a morphism of sheaves of graded $\mathcal{A}$-modules commuting with differentials.
(7) A quasi-coherent dg-sheaf $\mathcal{F}$ on $(X, \mathcal{A})$ is an $\mathcal{A}$-dg-module such that $\mathcal{F}^i$ is a quasi-coherent $\mathcal{O}_X$-module for all $i \in \mathbb{Z}$.

Definition 6.10. Let G be a complex reductive algebraic group acting on a complex algebraic variety X. Let $\mathcal{A}$ be a sheaf of dg-algebras on X and assume that $\mathcal{A}_i$ is G-equivariant for all $i \in \mathbb{Z}$ and that the multiplication and differential are G-equivariant. A $\mathcal{A}$-dg-module $\mathcal{F}$ is G-equivariant if $\mathcal{F}_i$ is G-equivariant for all $i \in \mathbb{Z}$ and the differential and action morphisms are G-equivariant.

The category of G-equivariant quasi-coherent left dg-modules over the dg-algebra $\mathcal{A}$ is denoted by $\mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A})$. The definition of the equivariant derived category is analogous to the non-equivariant derived category as defined in [**BL**]. We recall the definitions.

Definition 6.11.

(1) The translation functor

$$[1] : \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A}) \to \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A})$$

is given by

$$(\mathcal{M}[1])^i = \mathcal{M}^{i+1}, \qquad d_{\mathcal{M}[1]} = -d_{\mathcal{M}}, \qquad a \cdot m := (-1)^{\deg(a)} am, \;\; a \in \mathcal{A}.$$

(2) Two morphisms $f, g : \mathcal{M} \to \mathcal{N}$ in $\mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A})$ are homotopic if there exists a morphism of modules over the graded ring $\mathcal{A}$ (but not necessarily a morphism of $\mathcal{A}$-modules) $s : \mathcal{M} \to \mathcal{N}[-1]$ s.t.

$$f - g = s d_{\mathcal{M}} + d_{\mathcal{N}} s.$$

We write $f \sim g$.

(3) The homotopy category $H^0(\mathrm{QCoh}^G(\mathcal{A}))$ has the same objects as $\mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A})$ and morphisms

$$\mathrm{Hom}_{H^0(\mathrm{QCoh}^G(\mathcal{A}))}(\mathcal{M}, \mathcal{N}) := \mathrm{Hom}_{\mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A})}(\mathcal{M}, \mathcal{N}) / \{\text{morphisms } \sim 0\}.$$

(4) Let $u : \mathcal{M} \to \mathcal{N}$ be a morphism. The cone of u, $C(u)$, is defined as

$$C(u) = \mathcal{N} \oplus \mathcal{M}[1], \qquad d_{C(u)} = (d_{\mathcal{N}} + u, -d_{\mathcal{M}}).$$

(5) An exact triangle in $H^0(\mathrm{QCoh}^G(\mathcal{A}))$ is a sequence isomorphic to a sequence of the form

$$\mathcal{M} \xrightarrow{u} \mathcal{N} \to C(u) \to \mathcal{M}[1].$$

(6) The cohomology of $\mathcal{M} \in \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A})$ is the graded sheaf of $\mathcal{O}_X$-modules $\mathcal{H}(\mathcal{M}) = \ker(d_{\mathcal{M}})/\mathrm{im}(d_{\mathcal{M}})$. $\mathcal{M}$ is acyclic if $\mathcal{H}(\mathcal{M}) = 0$.

(7) A morphism is a quasi-isomorphism if it induces an isomorphism on cohomology. The derived category $\mathcal{D}\,\mathrm{QCoh}^G(\mathcal{A})$ is the localization of $H^0(\mathrm{QCoh}^G(\mathcal{A}))$ with respect to quasi-isomorphisms.

(8) A coherent dg-module $\mathcal{M}$ over $\mathcal{A}$ is a quasi-coherent dg-sheaf whose cohomology sheaf $\mathcal{H}(\mathcal{M})$ is coherent over $\mathcal{H}(\mathcal{A})$. The full subcategory of $\mathcal{D}\,\mathrm{QCoh}^G(\mathcal{A})$ whose objects are coherent is denoted by $\mathcal{D}\,\mathrm{Coh}^G(\mathcal{A})$.

(9) The full subcategory of $\mathcal{D}\,\mathrm{QCoh}^G(\mathcal{A})$ consisting of objects whose cohomology is bounded and coherent as a $\mathcal{O}_X$-module is denoted by $\mathcal{D}^{bc}\,\mathrm{QCoh}^G(\mathcal{A})$.

LEMMA 6.12. *The derived categories $\mathcal{D}\,\mathrm{QCoh}^G(\mathcal{A})$ and $\mathcal{D}\,\mathrm{Coh}^G(\mathcal{A})$ are triangulated.*

PROOF. See [**BL**, Cor. 10.4.3]. □

REMARK 6.13. Consider $\mathcal{O}_X$ as a dg-algebra with $\mathcal{O}_X$ in degree zero and 0 elsewhere. Then

$$\mathcal{D}\,\mathrm{QCoh}^G(\mathcal{O}_X) \simeq D^b(\mathrm{QCoh}^G(X)), \qquad \mathcal{D}\,\mathrm{Coh}^G(\mathcal{O}_X) \simeq D^b(\mathrm{Coh}^G(X)).$$

6.2.1. Functors. Let G be a reductive algebraic group acting on a complex algebraic variety X. To be able to define the derived functors we will assume that the following property hold:

$$\begin{aligned} &\text{For any } \mathcal{F} \in \mathrm{Coh}^G(X), \text{ there exists } \mathcal{P} \in \mathrm{Coh}^G(X) \\ &\text{which is flat over } \mathcal{O}_X \text{ and a surjection } \mathcal{P} \twoheadrightarrow \mathcal{F} \text{ in } \mathrm{Coh}^G(X). \end{aligned} \tag{6.2.1}$$

REMARK 6.14. Property 6.2.1 is satisfied e.g. when X admits an ample family of line bundles in the sense of [**VV**, Definition 1.5.3] or when X is normal and quasi-projective (see [**CG**, Proposition 5.1.26]).

DEFINITION 6.15. Let $\mathcal{A}$ be an equivariant sheaf of dg-algebras on X. If $\mathcal{A}$ is quasi-coherent, non-positively graded and graded-commutative then the pair $(X, \mathcal{A})$ is called a dg-scheme.

From now on we will always assume that we are working with a dg-scheme. In particular, the category of left $\mathcal{A}$-dg-modules is equivalent to the category of right $\mathcal{A}$-dg-modules (see [**BL**, 10.6.3]). Furthermore, we will assume that $\mathcal{A}$ is locally finitely generated over $\mathcal{A}^0$, that $\mathcal{A}^0$ is locally finitely generated as an $\mathcal{O}_X$-algebra, and that $\mathcal{A}$ is K-flat as a $\mathbb{G}_m$-equivariant $\mathcal{A}^0$-dg-module.

The last assumption is justified by the following observation in [**MR3**, Section 2.2]: If A is the G-equivariant affine scheme over X such that the push-forward of $\mathcal{O}_A$ to X is $\mathcal{A}^0$, then there exists a $\mathbb{G}_m \times G$-equivariant quasi-coherent $\mathcal{O}_A$-dg-algebra $\mathcal{A}'$ whose direct image to X is $\mathcal{A}$ and there is an equivalence of categories $\mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A}') \simeq \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A})$. Using this trick one can always reduce to the situation in which $\mathcal{A}$ is $\mathcal{O}_X$-coherent and K-flat as an $\mathcal{O}_X$-dg-module.

Let $\mathcal{D}^{bc}(\mathrm{QCoh}^G(\mathcal{A}))$ denote the full subcategory of $\mathcal{D}\,\mathrm{QCoh}^G(\mathcal{A})$ whose objects are the dg-modules $\mathcal{M}$ for which the complex $\mathcal{M}_j$ has bounded and coherent cohomology for any $j \in \mathbb{Z}$.

LEMMA 6.16. [**MR3**, Lemma 2.5] *Let $\mathcal{A}$ be as above. For any $\mathcal{M}, \mathcal{N} \in \mathcal{D}^{bc}(\mathrm{QCoh}^{G\times\mathbb{C}^*}(\mathcal{A}))$ the $\mathbb{C}$-vector space $\mathrm{Hom}_{\mathcal{D}^{bc}(\mathrm{QCoh}^{\mathbb{C}^*}(\mathcal{A}))}(\mathcal{M}, \mathcal{N}))$ has a natural structure of an algebraic G-module. Moreover, the natural morphism*

$$\mathrm{Hom}_{\mathcal{D}^{bc}(\mathrm{QCoh}^{G\times\mathbb{C}^*}(\mathcal{A}))}(\mathcal{M}, \mathcal{N}) \to (\mathrm{Hom}_{\mathcal{D}^{bc}(\mathrm{QCoh}^{\mathbb{C}^*}(\mathcal{A}))}(\mathcal{M}, \mathcal{N}))^G$$

induced by the forgetful functor is an isomorphism.

One can define the usual functors on $\mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A})$. We define the internal $\mathcal{H}om$ functor

$$\mathcal{H}om^G_{\mathcal{A}}(-,-) : \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A}) \times \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A}) \to \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{O}_X)$$

For $\mathcal{M}, \mathcal{N} \in \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A})$ the sheaf of $\mathcal{O}_X$-dg-modules $\mathcal{H}om^G_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ is the graded sheaf of $\mathcal{O}_X$-modules with the i-th component being local equivariant homomorphisms of graded $\mathcal{A}$-modules $\mathcal{M} \to \mathcal{N}[i]$ (not necessarily commuting with the differentials). For $\phi \in \mathcal{H}om^G_{\mathcal{A}}(\mathcal{M}, \mathcal{N})^i$ the differential is given by

$$d(\phi) = d_{\mathcal{M}} \circ \phi - (-1)^i \phi \circ d_{\mathcal{N}}.$$

We also have a tensor product

$$- \otimes_{\mathcal{A}} - : \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A}) \times \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A}) \to \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{O}_X)$$

The sheaf of $\mathcal{O}_X$-dg-modules $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ is graded in the natural way and on local sections of $\mathcal{F}^i \otimes_{\mathcal{A}} \mathcal{G}$ the differential is given by

$$d(f \otimes g) = d(f) \otimes g + (-1)^i f \otimes d(g).$$

It is equivariant with respect to the diagonal G-action.

Let $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a G-equivariant morphism of dg-schemes. This defines a the morphism of sheaves of dg-algebras since, by adjunction, the morphism $f^*\mathcal{B} \to \mathcal{A}$ corresponds to a morphism $\mathcal{B} \to f_*\mathcal{A}$. We define the direct image functor to be restriction of scalars using this map.

$$f_* : \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A}) \to \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{B}).$$

We can also define an inverse image functor using the tensor product

$$f^* : \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{B}) \to \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A}),$$
$$\mathcal{F} \to \mathcal{A} \otimes_{f^*\mathcal{B}} f^*\mathcal{F}.$$

Lemma 6.17. [**MR3**, Lemma 2.7 and Prop. 2.8] *Assume that (X, G) satisfies the above assumptions and let $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a $G \times \mathbb{G}_m$-equivariant morphism of dg-schemes. Then*

1. *For any object $\mathcal{M} \in \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{B})$, there exits an object $\mathcal{P} \in \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{B})$, which is K-flat as a $\mathcal{B}$-dg-module and a quasi-isomorphism of $G \times \mathbb{G}_m$-equivariant $\mathcal{B}$-dg-modules $\mathcal{P} \to \mathcal{M}$.*
2. *The functor of pull-back admits a derived functor*

$$Lf^* : \mathcal{D}\,\mathrm{QCoh}^G(\mathcal{B}) \to \mathcal{D}\,\mathrm{QCoh}^G(\mathcal{A})$$

and the following diagram is commutative

$$\begin{CD} \mathcal{D}\,\mathrm{QCoh}^G(\mathcal{B}) @>{Lf^*}>> \mathcal{D}\,\mathrm{QCoh}^G(\mathcal{A}) \\ @V{For}VV @VV{For}V \\ \mathcal{D}\,\mathrm{QCoh}^G(\mathcal{B}) @>{Lf^*}>> \mathcal{D}\,\mathrm{QCoh}^G(\mathcal{A}) \end{CD}$$

(3) *For any* $\mathcal{N} \in \mathcal{C}^+\,\mathrm{QCoh}^G(\mathcal{A})$*, there exists an object* $\mathcal{I} \in \mathcal{C}^+\,\mathrm{QCoh}^G(\mathcal{A})$ *which is* K*-injective in* $\mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A})$ *and a quasi-isomorphism* $\mathcal{N} \to \mathcal{I}$*.*

(4) *The functor of push-forward admits a derived functor*

$$Rf_* : \mathcal{D}^+\,\mathrm{QCoh}^G(\mathcal{A}) \to \mathcal{D}\,\mathrm{QCoh}^G(\mathcal{B})$$

and the following diagram is commutative up to isomorphism

$$\begin{CD} \mathcal{D}^+\,\mathrm{QCoh}^G(\mathcal{A}) @>{Rf_*}>> \mathcal{D}\,\mathrm{QCoh}^G(\mathcal{B}) \\ @V{For}VV @VV{For}V \\ \mathcal{D}^+\,\mathrm{QCoh}(\mathcal{A}) @>{Rf_*}>> \mathcal{D}\,\mathrm{QCoh}(\mathcal{B}) \end{CD}$$

LEMMA 6.18. [**BR**, Prop. 5.2.1] *Let* H *be an algebraic group (not necessarily reductive) and* $f : (X, \mathcal{A}) \to (X, \mathcal{B})$ *a* H*-equivariant quasi-isomorphism of complex algebraic* H*-varieties satisfying the conditions above. Then the pull-back and push-forward functors induce equivalences of categories*

$$\mathcal{D}\,\mathrm{QCoh}^H(\mathcal{A}) \simeq \mathcal{D}\,\mathrm{QCoh}^H(\mathcal{B}).$$

The equivalence restricts to an equivalence

$$\mathcal{D}\,\mathrm{Coh}^H(\mathcal{A}) \simeq \mathcal{D}\,\mathrm{Coh}^H(\mathcal{B}).$$

6.3. Equivariant linear Koszul duality

In the paper [**MR3**] Mirković and Riche extend the linear Koszul duality from [**MR1**] and [**MR2**] to the equivariant setting. In this section we recall their construction. Consider a complex algebraic variety X with an action of a reductive algebraic group G. Again we assume that property (6.2.1) is satisfied. Consider a two term complex of locally free G-equivariant $\mathcal{O}_X$-modules of finite rank.

$$\mathcal{X} := (\cdots 0 \to \mathcal{V} \xrightarrow{f} \mathcal{W} \to 0 \cdots).$$

Here $\mathcal{V}$ sits in homological degree -1 and $\mathcal{W}$ is in homological degree 0. We consider it as a complex of graded $\mathcal{O}_X$-modules with both $\mathcal{V}$ and $\mathcal{W}$ siting in internal degree 2. We define the graded symmetric algebra $\mathcal{S}\mathrm{ym}_{\mathcal{O}_X}(\mathcal{X})$ to be the sheaf tensor algebra of $\mathcal{X}$ modulo the graded commutation relations $a \otimes b = (-1)^{\deg_h(a)\deg_h(b)} b \otimes$

a, where $\deg_h$ is the homological degree. More explicitly $\mathcal{S}\mathrm{ym}_{\mathcal{O}_X}(\mathcal{X})$ is the bi-graded complex for which the term in homological degree k and internal degree $2k+2n$ is

$$\mathcal{S}\mathrm{ym}_{\mathcal{O}_X}(\mathcal{X})^k_{2k+2n} = \Lambda^k\mathcal{V} \otimes_{\mathcal{O}_X} \mathrm{Sym}^n(\mathcal{W}).$$

The differential is given by

$$d(v_1 \wedge \cdots \wedge v_n \otimes w) = \sum_{i=1}^{n} (-1)^i v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_n \otimes f(v_i)w.$$

For a bi-graded sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ we denote by $\mathcal{M}^\vee$ the bi-graded $\mathcal{O}_X$-module with $(\mathcal{M}^\vee)^i_j = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}^{-i}_{-j}, \mathcal{O}_X)$. The dual complex is defined as

$$\mathcal{Y} := (\cdots 0 \to \mathcal{W}^\vee \xrightarrow{-f^\vee} \mathcal{V}^\vee \to 0 \cdots),$$

where $\mathcal{W}^\vee$ sits in bi-degree (-1,-2) and $\mathcal{V}^\vee$ sits in bi-degree (0,-2). A shift in homological degree is denoted by [ ] and shift in internal degree is denoted by ( ). We introduce the following notation

$$\mathcal{T} := \mathcal{S}\mathrm{ym}_{\mathcal{O}_X}(\mathcal{X}), \qquad \mathcal{R} := \mathcal{S}\mathrm{ym}_{\mathcal{O}_X}(\mathcal{Y}), \qquad \mathcal{S} := \mathcal{S}\mathrm{ym}_{\mathcal{O}_X}(\mathcal{Y}[-2]).$$

Mirković and Riche proved the following theorem known as equivariant linear Koszul duality.

THEOREM 6.19. [**MR3**, Theorem 3.1] *There is an equivalence of triangulated categories*

$$\kappa : \mathcal{D}\,\mathrm{Coh}^{G\times\mathbb{C}^*}(\mathcal{T}) \xrightarrow{\sim} \mathcal{D}\,\mathrm{Coh}^{G\times\mathbb{C}^*}(\mathcal{R})^{op},$$

satisfying $\kappa(\mathcal{M}[n](m)) = \kappa(\mathcal{M})[-n+m](-m)$.

REMARK 6.20. In [**MR3**] the theorem is stated in less generality. However, the corresponding statement in the non-equivariant setting [**MR2**, Thm 1.7.1 and section 1.8] is stated in this generality and the proof in [**MR3**] shows that this non-equivariant equivalence can be lifted to the equivariant setting.

We now recall their construction of the functor κ. For a dg-algebra $\mathcal{A}$ let $\mathcal{C}_-\,\mathrm{QCoh}^{G\times\mathbb{C}^*}(\mathcal{A})$ (resp. $\mathcal{C}_+\,\mathrm{QCoh}^{G\times\mathbb{C}^*}(\mathcal{A})$) denote the full subcategory of $\mathcal{C}\,\mathrm{QCoh}^{G\times\mathbb{C}^*}(\mathcal{A})$ consisting of objects whose internal degree is bounded above (resp. below) uniformly in the homological degree. The associated derived category is denoted by $\mathcal{D}_-\,\mathrm{QCoh}^{G\times\mathbb{C}^*}(\mathcal{A})$ (resp. $\mathcal{D}_+\,\mathrm{QCoh}^{G\times\mathbb{C}^*}(\mathcal{A})$). The functor κ is a restriction of a functor

$$\kappa : \mathcal{D}^{bc}_+\,\mathrm{QCoh}^{G\times\mathbb{C}^*}(\mathcal{T}) \xrightarrow{\sim} \mathcal{D}^{bc}_-\,\mathrm{QCoh}^{G\times\mathbb{C}^*}(\mathcal{R})^{op}.$$

This functor is the composition of three functors. The first is the functor

$$\mathscr{A} : \mathcal{C}\,\mathrm{QCoh}^{G\times\mathbb{C}^*}(\mathcal{T}) \to \mathcal{C}\,\mathrm{QCoh}^{G\times\mathbb{C}^*}(\mathcal{S}).$$

As a bi-graded equivariant $\mathcal{O}_X$-module $\mathscr{A}(\mathcal{M}) = \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{M}$. The $\mathcal{S}$-action is induced by the left multiplication of $\mathcal{S}$ on itself. The differential is the sum of two terms d_1 and d_2. The term d_1 is the natural differential on the tensor product

$$d_1(s \otimes m) = d_{\mathcal{S}}(s) \otimes m + (-1)^{|s|} s \otimes d_{\mathcal{M}}(m).$$

The term d_2 is the composition of the following morphisms. First the morphism

$$\rho : \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{M}, \qquad s \otimes m \mapsto (-1)^{|s|} s \otimes m.$$

The second morphism

$$\psi : \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{S} \otimes \mathcal{X}^\vee \otimes \mathcal{X} \otimes \mathcal{M}$$

is induced by the natural morphism $i : \mathcal{O}_X \to \mathcal{E}nd(\mathcal{X}) \simeq \mathcal{X}^\vee \otimes \mathcal{X}$. The last map is the morphism

$$\Psi : \mathcal{S} \otimes \mathcal{X}^\vee \otimes \mathcal{X} \otimes \mathcal{M} \to \mathcal{S} \otimes \mathcal{M}$$

induced by the right multiplication $\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{X}^\vee \to \mathcal{S}$ and the action $\mathcal{X} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M}$. The term d_2 is defined as $d_2 = \Psi \circ \psi \circ \rho$. Locally, choosing a basis $\{x_\alpha\}$ of $\mathcal{X}$ and the dual basis $\{x_\alpha^*\}$ of $\mathcal{X}^\vee$ it can be written as

$$d_2(s \otimes m) = (-1)^{|s|} \sum_\alpha s x_\alpha^* \otimes x_\alpha \cdot m.$$

This data defines a $\mathcal{S}$-dg-module structure on $\mathscr{A}(\mathcal{M})$. Part of the proof of [**MR3**, Thm. 3.1] is showing that $\mathscr{A}$ induces an equivalence of categories

$$\bar{\mathscr{A}} : \mathcal{D}^{bc}_- \operatorname{QCoh}^{G\times\mathbb{C}^*}(\mathcal{T}) \xrightarrow{\sim} \mathcal{D}^{bc}_- \operatorname{QCoh}^{G\times\mathbb{C}^*}(\mathcal{S}).$$

By [**Bez1**, Example 2.16] under the assumption 6.2.1 there exists an object $\Omega \in \mathcal{D}^b \operatorname{Coh}^G(X)$ whose image under the forgetful functor For: $\mathcal{D}^b \operatorname{Coh}^G(X) \to \mathcal{D}^b \operatorname{Coh}(X)$ is a dualizing object in $\mathcal{D}^b \operatorname{Coh}(X)$. Let $\mathcal{I}_\Omega$ be a bounded below complex of injective objects of $\operatorname{QCoh}^G(X)$ whose image in the derived category $\mathcal{D}^+ \operatorname{QCoh}(X)$ is Ω. It defines a functor on the category of complexes of all equivariant sheaves on X.

$$\mathcal{RH}om_{\mathcal{O}_X}(-, \mathcal{I}_\Omega) : \mathcal{C}(\operatorname{Sh}^G(X)) \to \mathcal{C}(\operatorname{Sh}^G(X))^{op}.$$

In [**MR3**, Lemma 2.3] it is proved that this functor is exact and that the induced functor on derived categories restricts to a functor

$$D^X_\Omega : D^b \operatorname{Coh}^G(X) \to D^b \operatorname{Coh}^G(X)^{op}.$$

Let $\tilde{\mathcal{C}}(\mathcal{T} - \operatorname{mod}^{G\times\mathbb{C}^*})$ denote the category of all sheaves of $G \times \mathbb{C}^*$-equivariant $\mathcal{T}$ dg-modules on X. It's derived category is denoted by $\tilde{\mathcal{D}}(\mathcal{T} - \operatorname{mod}^{G\times\mathbb{C}^*})$. Consider the functor

$$\tilde{D}_\Omega : \tilde{\mathcal{C}}(\mathcal{T} - \operatorname{mod}^{G\times\mathbb{C}^*}) \to \tilde{\mathcal{C}}(\mathcal{T} - \operatorname{mod}^{G\times\mathbb{C}^*})^{op},$$

which sends $\mathcal{M} \in \tilde{\mathcal{C}}(\mathcal{T} - \operatorname{mod}^{G\times\mathbb{C}^*})$ to the dg-module whose underlying $G \times \mathbb{G}_m$-equivariant $\mathcal{O}_X$-dg-module is $\mathcal{H}om(\mathcal{M}, \mathcal{I}_\Omega)$, with $\mathcal{T}$-action defined by

$$(t \cdot \phi)(m) = (-1)^{|t|\cdot|\phi|} \phi(t \cdot m), \qquad t \in \mathcal{T}, m \in \mathcal{M}.$$

In proposition 2.6 Mirković and Riche prove that the induced functor restricts to an equivalence

$$D_\Omega : \mathcal{D}^{bc}\,\mathrm{QCoh}^{G\times\mathbb{C}^*}(\mathcal{T}) \xrightarrow{\sim} \mathcal{D}^{bc}\,\mathrm{QCoh}^{G\times\mathbb{C}^*}(\mathcal{T})^{op}.$$

The last functor is the regrading functor

$$\xi : \mathcal{C}\,\mathrm{QCoh}^{G\times\mathbb{C}^*}(\mathcal{S}) \to \mathcal{C}\,\mathrm{QCoh}^{G\times\mathbb{C}^*}(\mathcal{R})$$

sending $\mathcal{M} \in \mathcal{C}\,\mathrm{QCoh}^{G\times\mathbb{C}^*}(\mathcal{S})$ to the $\mathcal{R}$-dg-module with (i,j) component $\xi(\mathcal{M}) = \mathcal{M}_j^{i-j}$. If one forgets the grading then $\mathcal{S}$ and $\mathcal{R}$ coincides and so does $\mathcal{M}$ and $\xi(\mathcal{M})$. The $\mathcal{R}$-action on the differential on $\xi(\mathcal{M})$ is the same as the $\mathcal{S}$-action on the differential of $\mathcal{M}$. This is an equivalence of categories. The functor κ from theorem 6.19 is the restriction of the composition $\xi \circ \bar{\mathscr{A}} \circ D_\Omega$.

6.3.1. Extension to an arbitrary linear algebraic group. We want to be able to work with equivariance with respect to a Borel subgroup. Thus, we need to extend linear Koszul duality to work with a not necessarily reductive linear algebraic group H sitting inside a reductive group G. Let X be a H-variety. Consider the variety $\tilde{X} := \mathrm{Ind}_H^G(X) = \frac{G\times X}{H}$. The projection and quotient morphisms

$$X \xleftarrow{\mathrm{pr}} G\times X \xrightarrow{\pi} \tilde{X}.$$

induce equivalences of categories

$$A := \mathrm{pr}^* : \mathrm{QCoh}^H(X) \xrightarrow{\sim} \mathrm{QCoh}^{G\times H}(G\times X),$$
$$B := \pi^* : \mathrm{QCoh}^G(\tilde{X}) \xrightarrow{\sim} \mathrm{QCoh}^{G\times H}(G\times X).$$

Lemma 6.21. *The functors A and B are monoidal.*

Proof. Let $\Delta : X \to X\times X$ be the diagonal embedding. By definition the monoidal action on $\mathrm{QCoh}^H(X)$ is given by

$$M \otimes_{\mathcal{O}_X} N := \Delta^*\,\mathrm{Res}_{H_\Delta}^{H\times H}(M \boxtimes N).$$

Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X\times X \\ {\scriptstyle\mathrm{pr}}\uparrow & & \uparrow{\scriptstyle\mathrm{pr}_2} \\ G\times X & \xrightarrow{\Delta_G} & G\times X\times G\times X \end{array}$$

Using this we calculate

$$\begin{aligned} A(M) \otimes_{\mathcal{O}_{G\times X}} A(N) &= \Delta_G^* \operatorname{Res}^{G\times H\times G\times H}_{(G\times H)_\Delta}(\mathrm{pr}^* M \boxtimes \mathrm{pr}^* N) \\ &\simeq \Delta_G^* \operatorname{Res}^{G\times H\times G\times H}_{(G\times H)_\Delta} \mathrm{pr}_2^*(M \boxtimes N) \\ &\simeq \Delta_G^* \mathrm{pr}_2^* \operatorname{Res}^{H\times H}_{H_\Delta}(M \boxtimes N) \\ &\simeq \mathrm{pr}^* \Delta^* \operatorname{Res}^{H\times H}_{H_\Delta}(M \boxtimes N) \\ &= A(M \otimes_{\mathcal{O}_X} N). \end{aligned}$$

Thus, A is monoidal. For B we have the following diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\Delta}} & \tilde{X} \times \tilde{X} \\ \uparrow{\scriptstyle \pi} & & \uparrow{\scriptstyle \pi_2} \\ G \times X & \xrightarrow{\Delta_G} & G \times X \times G \times X \end{array}$$

This gives

$$\begin{aligned} B(M) \otimes_{\mathcal{O}_{G\times X}} B(N) &= \Delta_G^* \operatorname{Res}^{G\times H\times G\times H}_{(G\times H)_\Delta}(B(M) \boxtimes B(N)) \\ &\simeq \Delta_G^* \pi_2^* \operatorname{Res}^{G\times G}_{G_\Delta}(M \boxtimes N) \\ &\simeq \pi^* \tilde{\Delta}^* \operatorname{Res}^{G\times G}_{G_\Delta}(M \boxtimes N) \\ &= B(M \otimes_{\mathcal{O}_{\tilde{X}}} N). \end{aligned}$$

Hence, both functors are monoidal. □

By the lemma we have a monoidal equivalence of categories

$$B^{-1}A : \mathrm{QCoh}^H(X) \xrightarrow{\sim} \mathrm{QCoh}^G(\tilde{X}).$$

Consider a complex of H-equivariant vector bundles

$$\mathcal{X} := (\cdots \to 0 \to \mathcal{V} \to \mathcal{W} \to 0 \to \cdots).$$

Applying $B^{-1}A$ we get a new complex

$$\tilde{\mathcal{X}} := B^{-1}A(\mathcal{X}) = (\cdots \to 0 \to B^{-1}A(\mathcal{V}) \to B^{-1}A(\mathcal{W}) \to 0 \to \cdots).$$

Notice that

$$\begin{aligned} \mathcal{H}om_{\mathcal{O}_{\tilde{X}}}(B^{-1}A(\mathcal{M}), \mathcal{O}_{\tilde{X}}) &\simeq \mathcal{H}om_{\mathcal{O}_{\tilde{X}}}(B^{-1}A(\mathcal{M}), B^{-1}A(\mathcal{O}_X)) \\ &\simeq B^{-1}A(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)). \end{aligned}$$

Thus, $\widetilde{\mathcal{X}^\vee} \simeq (\tilde{\mathcal{X}})^\vee$.

In the construction of $\mathcal{S}\mathrm{ym}_{\mathcal{O}_X}(\mathcal{X})$ we only used the monoidal structure on $\mathrm{QCoh}^H(X)$. Since $B^{-1}A$ is monoidal we get

$$B^{-1}A(\mathcal{S}\mathrm{ym}_{\mathcal{O}_X}(\mathcal{X})) \simeq \mathcal{S}\mathrm{ym}_{\mathcal{O}_{\tilde{X}}}(\tilde{\mathcal{X}}).$$

Let $\mathcal{M}$ be a dg-module over $\mathcal{S}\mathrm{ym}_{\mathcal{O}_X}(\mathcal{X})$. I.e. there is a collection of linear maps $\mathcal{S}\mathrm{ym}^n_{\mathcal{O}_X}(\mathcal{X}) \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M}$ respecting the differentials. $B^{-1}A$ respects these maps so $B^{-1}A(\mathcal{M})$ is a dg-module over $\mathcal{S}\mathrm{ym}_{\mathcal{O}_{\tilde{X}}}(\tilde{\mathcal{X}})$. Thus, we have proved

PROPOSITION 6.22. *There is a natural equivalence of dg-categories*

$$\mathcal{C}\,\mathrm{QCoh}^H(\mathcal{S}\mathrm{ym}_{\mathcal{O}_X}(\mathcal{X})) \simeq \mathcal{C}\,\mathrm{QCoh}^G(\mathcal{S}\mathrm{ym}_{\mathcal{O}_{\tilde{X}}}(\tilde{\mathcal{X}})).$$

The functor sends quasi-isomorphisms to quasi-isomorphisms so it descends to the derived category. Using this equivalence we obtain the desired version of linear Koszul duality.

THEOREM 6.23. *Let G be a complex reductive group acting on a variety X satisfying condition 6.2.1. Let H be a closed subgroup of G and define $\mathcal{T}$ and $\mathcal{R}$ as in the previous section. Then there is an equivalence of triangulated categories*

$$\kappa : \mathcal{D}\,\mathrm{Coh}^{H\times\mathbb{C}^*}(\mathcal{T}) \xrightarrow{\sim} \mathcal{D}\,\mathrm{Coh}^{H\times\mathbb{C}^*}(\mathcal{R})^{op},$$

satisfying $\kappa(\mathcal{M}[n](m)) = \kappa(\mathcal{M})[-n+m](-m)$.

6.4. Derived category of equivariant DG-modules for G-schemes

In this section we extend the construction in [**Isik**] to the equivariant setting. Let X be a smooth complex algebraic variety with an action of a reductive algebraic group G. In particular, X is Noetherian, separated and regular. Then X has an ample family of G-equivariant line bundles and property (6.2.1) is satisfied (see [**VV**, Remark 1.5.4]). Let $\pi : E \to X$ be a G-equivariant vector bundle of rank n. Denote the sheaf of G-equivariant sections of the bundle by $\mathcal{E}$ and let $s \in H^0(X, \mathcal{E})$ be a G-equivariant regular section. The zero scheme of s is denoted by Y. In order to use linear Koszul duality we need to introduce an additional $\mathbb{Z}$-grading or equivalently a $\mathbb{C}^*$-action. Consider $\mathcal{O}_Y[t, t^{-1}]$ as a bi-graded dg-algebra sitting in homological degree 0 with zero differential and t a formal variable sitting in internal degree -2.

PROPOSITION 6.24. *There is an equivalence of categories*

$$\mathcal{D}\,\mathrm{Coh}^{G\times\mathbb{C}^*}(\mathcal{O}_Y[t,t^{-1}]) \simeq D^b\,\mathrm{Coh}^G(Y)$$

PROOF. The pull-back along the projection $Y \times \mathbb{C}^* \to Y$ is an equivalence of categories $\mathrm{Coh}^G(Y) \simeq \mathrm{Coh}^{G\times\mathbb{C}^*}(Y \times \mathbb{C}^*)$. By remark 6.13

$$D^b\,\mathrm{Coh}^{G\times\mathbb{C}^*}(Y\times\mathbb{C}^*) \simeq \mathcal{D}\,\mathrm{Coh}^{G\times\mathbb{C}^*}(\mathcal{O}_{Y\times\mathbb{C}^*}).$$

Notice that $\mathcal{O}_{Y\times\mathbb{C}^*} \simeq \mathcal{O}_Y[t,t^{-1}]$. □

By lemma 6.18 we may replace $\mathcal{O}_Y[t,t^{-1}]$ by a quasi-isomorphic dg-algebra sitting in non-positive homological degrees which fits into the setting of linear Koszul duality. When Y is the zero locus of a regular section $s \in H^0(X,\mathcal{E})$ the sheaf $\mathcal{O}_Y$ has an equivariant Koszul resolution

$$0 \to \Lambda^n\mathcal{E}^\vee \to \cdots \to \Lambda^2\mathcal{E}^\vee \to \mathcal{E}^\vee \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

with differential given by $d(f) = f(s)$ and extended by Leibnitz rule. Using shifted copies of this resolution in each internal degree we get a bi-complex with is a resolution of $\mathcal{O}_Y[t, t^{-1}]$.

$$\cdots \longrightarrow \Lambda^3\mathcal{E}^\vee t^{-1} \longrightarrow \Lambda^2\mathcal{E}^\vee \longrightarrow \mathcal{E}^\vee t \longrightarrow \mathcal{O}_X t^2 \qquad i=-4$$

$$\cdots \longrightarrow \Lambda^3\mathcal{E}^\vee t^{-2} \longrightarrow \Lambda^2\mathcal{E}^\vee t^{-1} \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{O}_X t \qquad i=-2$$

$$\cdots \longrightarrow \Lambda^3\mathcal{E}^\vee t^{-3} \longrightarrow \Lambda^2\mathcal{E}^\vee t^{-2} \longrightarrow \mathcal{E}^\vee t^{-1} \longrightarrow \mathcal{O}_X \qquad i=0$$

$$\cdots \longrightarrow \Lambda^3\mathcal{E}^\vee t^{-4} \longrightarrow \Lambda^2\mathcal{E}^\vee t^{-3} \longrightarrow \mathcal{E}^\vee t^{-2} \longrightarrow \mathcal{O}_X t^{-1} \qquad i=2$$

$$\cdots \longrightarrow \Lambda^3\mathcal{E}^\vee t^{-5} \longrightarrow \Lambda^2\mathcal{E}^\vee t^{-4} \longrightarrow \mathcal{E}^\vee t^{-3} \longrightarrow \mathcal{O}_X t^{-2} \qquad i=4$$

We denote this bi-complex by $\mathcal{A}_{X\times\mathbb{C}^*}$. By construction $H(\mathcal{A}_{X\times\mathbb{C}^*}) = \mathcal{O}_Y[t, t^{-1}]$ and the morphism

$$\psi : \mathcal{A}_{X\times\mathbb{C}^*} \to \mathcal{O}_Y[t, t^{-1}],$$

which takes $t^k f$ to $t^k f|_Y$ for $f \in \mathcal{O}_X$ and everything else to zero, is a quasi-isomorphism. Thus, we have shown that

PROPOSITION 6.25. *There is an equivalence of categories*

$$\mathcal{D}\operatorname{Coh}^{G\times\mathbb{C}^*}(\mathcal{A}_{X\times\mathbb{C}^*}) \simeq \mathcal{D}\operatorname{Coh}^{G\times\mathbb{C}^*}(\mathcal{O}_Y[t, t^{-1}]).$$

Consider the following bi-graded complex with $\mathcal{E}^\vee$ in degree (-1,-2), $\mathcal{O}_X$ in degree (0,0) and t in degree $(0, -2)$.

$$\mathcal{A}_{X\times\mathbb{C}} := \bigwedge \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{O}_X[t]$$

with differential $d(f) = tf(s)$ and extended by Leibnitz. Observe that

$$\mathcal{A}_{X\times\mathbb{C}^*} = \mathcal{A}_{X\times\mathbb{C}} \otimes_{\mathcal{O}_X} \mathcal{O}_X[t^{-1}].$$

The bi-complex $\mathcal{A}_{X\times\mathbb{C}}$ fits into the setting of linear Koszul duality as it can be written in the form

$$\mathcal{A}_{X\times\mathbb{C}} = \mathcal{S}\mathrm{ym}_{\mathcal{O}_X}\big(0 \to \mathcal{E}^\vee \xrightarrow{-s^\vee} t\mathcal{O}_X \to 0\big)$$

DEFINITION 6.26. Let $(X, \mathcal{A})$ be a $G \times \mathbb{C}^*$-equivariant dg-scheme.

(1) The full subcategory of $\mathcal{D}\operatorname{Coh}^{G\times\mathbb{C}^*}(\mathcal{A})$ whose objects are locally in $\langle\mathcal{A}(i)\rangle_{i\in\mathbb{Z}}$, i.e. is quasi-isomorphic to a bounded complex of free $\mathcal{A}$-modules of finite rank, is denoted by $\operatorname{Perf}(\mathcal{A})$. Such complexes are called perfect.

(2) The full subcategory of $\mathcal{D}\operatorname{Coh}^{G\times\mathbb{C}^*}(\mathcal{A})$ whose objects are locally in $\langle\mathcal{O}_X(i)\rangle_{i\in\mathbb{Z}}$ is denoted by $\mathcal{D}_X\operatorname{Coh}^{G\times\mathbb{C}^*}(\mathcal{A})$. We say that these modules are supported on X.

LEMMA 6.27. *Let G be a complex reductive algebraic group acting on X such that assumption 6.2.1 is satisfied. Let $\mathcal{R}$ be a sheaf of dg-algebras sitting in non-positive homological and non-positive internal degrees with $\mathcal{R}_0^0 = \mathcal{O}_X$ and $H^0(\mathcal{R})_0 = \mathcal{O}_X$. If $\mathcal{M}$ is a coherent module over $\mathcal{R}$ and $H(\mathcal{M})$ is coherent when considered as a module over $\mathcal{O}_X$, then $\mathcal{M}$ is locally in $\langle\mathcal{O}_X(i)\rangle_{i\in\mathbb{Z}}$.*

PROOF. In the non-equivariant setting this is [**Isik**, Lemma 3.3]. Under the assumption that (6.2.1) is satisfied the proof extends to the equivariant setting. We recall it here. The property is local so we may assume that X is affine. Assume that $H(\mathcal{M})$ is coherent as a $\mathcal{O}_X$-module. In particular the cohomology of $\mathcal{M}$ is bounded above and below and there are only finitely many pairs (i,j) such that $H^i(\mathcal{M})_j \neq 0$. The proof is by induction on the number of such pairs. Acyclic modules are in $\langle\mathcal{O}_X(i)\rangle_{i\in\mathbb{Z}}$ so the start is clear.

Let n be the lowest degree such that $H^n(\mathcal{M}) \neq 0$. Then $\mathcal{M}$ is quasi-isomorphic to the truncated complex

$$\tau_{\geq n}\mathcal{M} = \quad \cdots \to 0 \to \operatorname{coker} d_{\mathcal{M}}^{n-1} \to \mathcal{M}^{n+1} \to \mathcal{M}^{n+2} \to \cdots$$

The complex $\tau_{\geq n}\mathcal{M}$ is also a $\mathcal{R}$-module since $\mathcal{R}$ sits in non-positive homological degrees. Let $\mathcal{F}$ be the kernel of the morphism

$$d^n : \operatorname{coker} d_{\mathcal{M}}^{n-1} \to \mathcal{M}^{n+1}.$$

The assumption on $\mathcal{R}$ means that $\mathcal{F}$ is a $\mathcal{R}$-submodule of $\tau_{\geq n}\mathcal{M}$. Notice that $\mathcal{F} \simeq H^n(\mathcal{M})$, so $\mathcal{F}$ is coherent as an $\mathcal{O}_X$-module.

Let m be the lowest internal degree such that $\mathcal{F}_m \simeq H^n(\mathcal{M})_m \neq 0$. Since $\mathcal{R}$ sits in non-positive both homological and internal degree $\mathcal{R}_j^i$ acts by zero on the coherent $\mathcal{O}_X$-module $\mathcal{F}_m$ for $(i,j) \neq (0,0)$. Thus, $\mathcal{F}_m$ which is concentrated in degree (n,m), is also an $\mathcal{R}$-module.

Since X is smooth there exist a finite G-equivariant free resolution of $\mathcal{F}_m$. Hence, $\mathcal{F}_m$ is quasi-isomorphic to a complex of $\mathcal{O}_X$-modules of the form

$$0 \to \mathcal{O}_X^{\oplus r_k}(m) \to \cdots \to \mathcal{O}_X^{\oplus r_2}(m) \to \mathcal{O}_X^{\oplus r_1}(m) \to 0.$$

Since $\mathcal{R}_j^i$ acts by zero on $\mathcal{F}_m$ for $(i,j) \neq (0,0)$ and $\mathcal{R}_0^0 = \mathcal{O}_X$ this complex is also quasi-isomorphic to $\mathcal{F}_m$ as a $\mathcal{R}$-module with $\mathcal{R}$ acting trivially except for the $(0,0)$ piece. Hence, $\mathcal{F}_m$ represents an object in $\langle\mathcal{O}_X(i)\rangle_{i\in\mathbb{Z}}$. The cone of the inclusion $\mathcal{F}_m \hookrightarrow \tau_{\geq n}\mathcal{M}$ has the same cohomology as $\mathcal{M}$ except for the piece in degree (n,m) which is zero. By induction the cone is in $\langle\mathcal{O}_X(i)\rangle_{i\in\mathbb{Z}}$ so $\tau_{\geq n}\mathcal{M}$ and consequently $\mathcal{M}$ are in $\langle\mathcal{O}_X(i)\rangle_{i\in\mathbb{Z}}$. □

PROPOSITION 6.28. *There is an equivalence of categories*

$$\frac{\mathcal{D}\operatorname{Coh}^{G\times\mathbb{C}^*}(\mathcal{A}_{X\times\mathbb{C}})}{\mathcal{D}_X\operatorname{Coh}^{G\times\mathbb{C}^*}(\mathcal{A}_{X\times\mathbb{C}})} \simeq \mathcal{D}\operatorname{Coh}^{G\times\mathbb{C}^*}(\mathcal{A}_{X\times\mathbb{C}^*}).$$

PROOF. In the non-equivariant setting this is [**Isik**, Prop. 3.2]. In the proof Isik uses the inclusion morphism $\phi : \mathcal{A}_{X\times\mathbb{C}} \to \mathcal{A}_{X\times\mathbb{C}}[t^{-1}] = \mathcal{A}_{X\times\mathbb{C}^*}$ to construct two functors

$$\begin{gathered}\phi_* : \mathcal{D}\operatorname{QCoh}(\mathcal{A}_{X\times\mathbb{C}}) \to \mathcal{D}\operatorname{QCoh}(\mathcal{A}_{X\times\mathbb{C}^*}),\\ \mathcal{M} \mapsto \mathcal{A}_{X\times\mathbb{C}^*} \otimes_{\mathcal{A}_{X\times\mathbb{C}}} \mathcal{M} \simeq \mathcal{O}_X[t,t^{-1}] \otimes_{\mathcal{O}_X[t]} \mathcal{M},\\ \phi^* : \mathcal{D}\operatorname{QCoh}(\mathcal{A}_{X\times\mathbb{C}^*}) \to \mathcal{D}\operatorname{QCoh}(\mathcal{A}_{X\times\mathbb{C}}),\\ \mathcal{N} \mapsto \mathcal{N}_{\leq 0}.\end{gathered}$$

He proves that ϕ_* factors through $\mathcal{D}_X\operatorname{Coh}(\mathcal{A}_{X\times\mathbb{C}})$ and that the functors induce mutually inverse equivalences of categories

$$\frac{\mathcal{D}\operatorname{Coh}(\mathcal{A}_{X\times\mathbb{C}})}{\mathcal{D}_X\operatorname{Coh}(\mathcal{A}_{X\times\mathbb{C}})} \underset{\phi^*}{\overset{\phi_*}{\underset{\sim}{\rightleftarrows}}} \mathcal{D}\operatorname{Coh}(\mathcal{A}_{X\times\mathbb{C}^*})$$

Both functors naturally extend to the equivariant setting so we get functors

$$\frac{\mathcal{D}\operatorname{Coh}^{G\times\mathbb{C}^*}(\mathcal{A}_{X\times\mathbb{C}})}{\mathcal{D}_X\operatorname{Coh}^{G\times\mathbb{C}^*}(\mathcal{A}_{X\times\mathbb{C}})} \underset{\phi^*}{\overset{\phi_*}{\rightleftarrows}} \mathcal{D}\operatorname{Coh}^{G\times\mathbb{C}^*}(\mathcal{A}_{X\times\mathbb{C}^*})$$

As in the non-equivariant setting we need to prove that the natural transformations

$$\phi_* \circ \phi^* \to \operatorname{Id}$$

given by

$$\mathcal{A}_{X\times\mathbb{C}}[t^{-1}] \otimes_{\mathcal{A}_{X\times\mathbb{C}}} (\mathcal{N})_{\leq 0} \to \mathcal{N}, \qquad a\otimes n \mapsto an.$$

and

$$\operatorname{Id} \to \phi^* \circ \phi_*$$

given by

$$\begin{gathered}\mathcal{M} \to (\mathcal{A}_{X\times\mathbb{C}}[t^{-1}] \otimes_{\mathcal{A}_{X\times\mathbb{C}}} \mathcal{M})_{\leq 0} \simeq (k[t^{-1}] \otimes_k \mathcal{M})_{\leq 0},\\ m \mapsto \begin{cases} 1\otimes m & \text{internal degree non-positive}\\ 0 & \text{else}\end{cases}\end{gathered}$$

are both isomorphisms.

Then first natural transformation $\phi_* \circ \phi^* \to \operatorname{Id}$ is clearly surjective. Assume that we have a section $t^k \otimes n$ whose image is 0. Then $n = t^{-k}t^k n = 0$, so the morphism is also injective. Thus, this natural transformation is an isomorphism

Let $\mathcal{J}$ be the cone of the morphism $\mathcal{M} \to \phi^*\phi_*\mathcal{M}$ from the second natural transformation. We want to show that $\mathcal{J}$ is supported on X. By the lemma it is enough to show that $H(\mathcal{J})$ is coherent over $\mathcal{O}_X$. Consider the long exact sequence of sheaves of $\mathcal{O}_X$-modules in cohomology

$$\cdots \to H^i(\mathcal{M}) \to H^i(\phi^*\phi_*\mathcal{M}) \to H^i(\mathcal{J}) \to H^{i+1}(\mathcal{M}) \to H^{i+1}(\phi^*\phi_*\mathcal{M}) \to \cdots$$

So we get short exact sequences

$$0 \to \operatorname{coker}(\alpha_i) \to H^i(\mathcal{J}) \to \ker(\alpha_{i+1}) \to 0,$$

where

$$\alpha_i : H^i(\mathcal{M}) \to H^i((\mathcal{O}_X[t,t^{-1}] \otimes_{\mathcal{O}_X[t]} \mathcal{M})_{\leq 0})$$

is the induced map on cohomology. From the short exact sequence it follows that $H(\mathcal{J})$ is coherent over $\mathcal{O}_X$ if $\operatorname{coker}(\alpha_i)$ and $\ker(\alpha_{i+1})$ are. Recall what the terms in $\mathcal{A}_{X\times\mathbb{C}}$ look like

$$\cdots \longrightarrow 0 \longrightarrow \Lambda^2\mathcal{E}^\vee \longrightarrow \mathcal{E}^\vee t \longrightarrow \mathcal{O}_X t^2 \qquad i=-4$$

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{O}_X t \qquad i=-2$$

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathcal{O}_X \qquad i=0$$

In each degree we have a truncation of the resolution of $\mathcal{O}_Y$. In degrees lower than $-n$ we have the full resolution so the only non-zero cohomology in lower degrees are of the form $t^k\mathcal{O}_Y$. All individual terms in $H(\mathcal{A}_{X\times\mathbb{C}})$ are coherent over $\mathcal{O}_X$ so the only way $\operatorname{coker}(\alpha_i)$ and $\ker(\alpha_{i+1})$ could fail to be coherent is if infinitely many powers of t are required to generate them. However, everything in $\ker(\alpha_{i+1})$ sits in strictly positive internal degree and since the internal degree of t is -2 this is not the case. Isik shows that $H(\mathcal{O}_X[t,t^{-1}] \otimes_{\mathcal{O}_X[t]} \mathcal{M}) \simeq \mathcal{O}_X[t,t^{-1}] \otimes_{\mathcal{O}_X[t]} H(\mathcal{M})$ so elements in $\operatorname{coker}(\alpha_i)$ are represented by $t^{-k} \otimes m$ with $2k + \deg_i(m) \leq 0$, where $\deg_i(m)$ is the internal degree of m. Thus, $\operatorname{coker}(\alpha_i)$ is also coherent over $\mathcal{O}_X$ and we get the result. $\square$

The Koszul dual to $\mathcal{A}_{X\times\mathbb{C}}$ is the dg-algbera

$$\mathcal{B} := \mathcal{S}\mathrm{ym}(0 \to \epsilon\mathcal{O}_X \xrightarrow{s} \mathcal{E} \to 0).$$

Here ϵ is a formal variable with $\epsilon^2 = 0$ sitting in homological degree -1 and internal degree 2. This is just a convenient notation expressing the fact that $\Lambda^n\mathcal{O}_X = 0$ for $n \geq 2$. $\mathcal{E}$ is in homological degree 0 and internal degree 2. As a complex $\mathcal{B}$ is $\epsilon \operatorname{Sym}\mathcal{E} \to \operatorname{Sym}\mathcal{E}$ with differential $d_{\mathcal{B}}(\epsilon f) = sf$.

The functor κ from theorem 6.19 gives an equivalence of categories

$$\kappa : \mathcal{D}\operatorname{Coh}^{G\times\mathbb{C}^*}(\mathcal{B}) \xrightarrow{\sim} \mathcal{D}\operatorname{Coh}^{G\times\mathbb{C}^*}(\mathcal{A}_{X\times\mathbb{C}})^{op}.$$

LEMMA 6.29. *The functor κ restricts to an equivalence*

$$\operatorname{Perf}(\mathcal{B}) \simeq \mathcal{D}_X\operatorname{Coh}^{G\times\mathbb{C}^*}(\mathcal{A}_{X\times\mathbb{C}})^{op}.$$

PROOF. In the non-equivariant setting this is similar to proposition 3.1 in [**Isik**]. However, Isik uses the Koszul duality from [**MR1**], which is slightly different from the linear Koszul duality from [**MR3**] that we are using, so some additional arguments are needed. The functor κ is defined locally so for any open subset $i : U \hookrightarrow X$ the following diagram is commutative

$$\begin{array}{ccc} \mathcal{D}\,\mathrm{Coh}^{G\times\mathbb{C}^*}(\mathcal{B}) & \xrightarrow{\kappa} & \mathcal{D}\,\mathrm{Coh}^{G\times\mathbb{C}^*}(\mathcal{A}_{X\times\mathbb{C}}) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ \mathcal{D}\,\mathrm{Coh}^{G\times\mathbb{C}^*}(\mathcal{B}|_U) & \xrightarrow{\kappa|_U} & \mathcal{D}\,\mathrm{Coh}^{G\times\mathbb{C}^*}(\mathcal{A}_{X\times\mathbb{C}}|_U) \end{array}$$

It suffices to show that for any open affine U the functor $\kappa|_U$ takes modules of the form $\mathcal{B}(i)$ to objects in $\langle \mathcal{O}_X(i)\rangle_{i\in\mathbb{Z}}$. Theorem 6.19 states that $\kappa(\mathcal{M}[j](i)) = \kappa(\mathcal{M})[-j+i](-i)$, so it is enough to prove the statement for $i = 0$.

Recall that κ is the composition of three functors ξ, $\bar{\mathscr{A}}$ and D_Ω. Since X is smooth the dualizing sheaf Ω is just top forms shifted by dimension. In particular, it is locally free of rank 1 so the functor $\mathcal{H}om(-,\Omega)$ is exact and there is no need to take the injective resolution I_Ω. That Ω is a line bundle implies that locally $D_\Omega(\mathcal{B}) \simeq \mathcal{H}om(\mathcal{B},\mathcal{O}_X)$. Applying $\bar{\mathcal{A}}$ on the opposite category corresponds to reversing the grading for all dg-modules. Thus, $\bar{\mathcal{A}} \circ D_\Omega(\mathcal{B})$ corresponds to $\bar{\mathcal{A}}(\mathcal{B}^\vee)$, where $(\mathcal{B}^\vee)^i_j = \mathcal{H}om(\mathcal{B}^{-i}_{-j},\mathcal{O}_X)$. By [**MR1**, Lemma 2.6.1] the projection on the $(0,0)$-component $\bar{\mathcal{A}}(\mathcal{B}^\vee) \to \mathcal{O}_X$ is a quasi-isomorphism. Clearly, $\xi(\mathcal{O}_X) = \mathcal{O}_X$. This finishes the proof. □

DEFINITION 6.30. The singularity category of a dg-algebra $\mathcal{A}$ is the Verdier quotient

$$\mathcal{D}^{G\times\mathbb{C}^*}_{sg}(\mathcal{A}) := \frac{\mathcal{D}\,\mathrm{Coh}^{G\times\mathbb{C}^*}(\mathcal{A})}{\mathrm{Perf}(\mathcal{A})}.$$

COROLLARY 6.31. There is an equivalence of categories

$$\mathcal{D}^{G\times\mathbb{C}^*}_{sg}(\mathcal{B}) \simeq \frac{\mathcal{D}\,\mathrm{Coh}^{G\times\mathbb{C}^*}(\mathcal{A}_{X\times\mathbb{C}})^{op}}{\mathcal{D}_X\,\mathrm{Coh}^{G\times\mathbb{C}^*}(\mathcal{A}_{X\times\mathbb{C}})^{op}}.$$

Recall the notation from the beginning of this section.

Let $\pi^\vee : E^\vee \to X$ denote the dual vector bundle. This defines a pull-back section via the cartesian diagram

$$\begin{array}{ccc} E^\vee \times_X E & \longrightarrow & E \\ {\scriptstyle (\pi^\vee)^*s}\uparrow & & \uparrow{\scriptstyle s} \\ E^\vee & \xrightarrow{\pi^\vee} & X \end{array}$$

Define the function W to be the composition with the natural pairing

$$W : E^\vee \to E^\vee \times_X E \xrightarrow{\langle\,,\rangle} \mathbb{C}, \qquad a_x \mapsto \langle ((\pi^\vee)^*s)(a_x), a_x\rangle.$$

Set $Z := W^{-1}(0)$. There is a short exact sequence

$$0 \to \epsilon\mathcal{S}\mathrm{ym}\,\mathcal{E} \xrightarrow{s} \mathcal{S}\mathrm{ym}\,\mathcal{E} \longrightarrow \pi_*\mathcal{O}_Z \to 0.$$

Thus, the map $\phi : \mathcal{B} \to \pi_*\mathcal{O}_Z$ sending $\mathcal{S}\mathrm{ym}\,\mathcal{E}$ to $\pi_*\mathcal{O}_Z$ and ϵ to 0 is a quasi-isomorphism of sheaves of equivariant graded dg-algebras. Proposition 6.18 gives an equivalence

$$\mathcal{D}\,\mathrm{Coh}^{G\times\mathbb{C}^*}(\mathcal{B}) \simeq \mathcal{D}\,\mathrm{Coh}^{G\times\mathbb{C}^*}(\pi_*\mathcal{O}_Z).$$

The equivalence takes $\mathcal{B}$ to $\pi_*\mathcal{O}_Z$ so it descends to the singularity categories.

LEMMA 6.32. *There is an equivalence of categories* $\mathcal{D}_{sg}^{G\times\mathbb{C}^*}(\mathcal{B}) \simeq \mathcal{D}_{sg}^{G\times\mathbb{C}^*}(\pi_*\mathcal{O}_Z)$.

LEMMA 6.33. *Let* $f : X \to Y$ *be a* G*-equivariant affine morphism and set* $\mathcal{C} := f_*\mathcal{O}_X$*. Then* f_* *induces an equivalence of categories*

$$f_* : \mathrm{QCoh}^G(X) \xrightarrow{\sim} \mathrm{QCoh}^G(\mathcal{C}).$$

Here $\mathrm{QCoh}^G(\mathcal{C})$ *denotes* G*-equivariant quasi-coherent* $\mathcal{O}_Y$*-modules with a* $\mathcal{C}$*-action.*

PROOF. In the non-equivariant setting this is [**Har1**, Exercise II.5.17]. The functor f_* lifts to the equivariant setting

$$\begin{array}{ccc} \mathrm{QCoh}^G(X) & \xrightarrow{f_*^G} & \mathrm{QCoh}^G(\mathcal{C}) \\ \downarrow{\scriptstyle \mathrm{For}} & & \downarrow{\scriptstyle \mathrm{For}} \\ \mathrm{QCoh}(X) & \underset{\sim}{\xrightarrow{f_*}} & \mathrm{QCoh}(\mathcal{C}) \end{array}$$

Since f_* is fully-faithful we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}^G(f_*^G(\mathcal{N}), f_*^G(\mathcal{M})) &= (\mathrm{Hom}_{\mathcal{C}}(f_*(\mathcal{N}), f_*(\mathcal{M})))^G \\ &= (\mathrm{Hom}_X(\mathcal{N},\mathcal{M}))^G \\ &= \mathrm{Hom}_X^G(\mathcal{N},\mathcal{M}). \end{aligned}$$

Thus, f_*^G is fully-faithful. We need to show that f_*^G is essentially surjective. Let $(\mathcal{M},\phi) \in \mathrm{QCoh}^G(\mathcal{C})$ where $\mathcal{M} \in \mathrm{QCoh}(\mathcal{C})$ and $\phi : ac_Y^*(\mathcal{M}) \xrightarrow{\sim} p_Y^*(\mathcal{M})$. Here $ac_Y, p_Y : G \times Y \to Y$ is the action and projection morphism, respectively. Use similar notation for X. Since f_* is essentially surjective forgetting the equivariance $\mathcal{M} \simeq f_*(\mathcal{N})$ for some $\mathcal{N} \in \mathrm{QCoh}(X)$. Using base change we get isomorphisms

$$(1\times f_*)ac_X^*(\mathcal{N}) \simeq ac_Y^* f_*(\mathcal{N}) \underset{\sim}{\xrightarrow{\phi}} p_Y^* f_*(\mathcal{N}) \simeq (1\times f_*)p_X^*(\mathcal{N})$$

Since $1\times f_*$ is an equivalence of categories this induces an isomorphism $\psi := (1\times f_*)^{-1}\phi : ac_X^*(\mathcal{N}) \xrightarrow{\sim} p_X^*(\mathcal{N})$. Thus, $\mathcal{N}$ lifts to $\mathrm{QCoh}^G(X)$ so f_*^G is essentially surjective. □

The morphism $\pi : Z \to X$ is affine so the lemma gives an equivalence

$$D^b \operatorname{Coh}^{G\times\mathbb{C}^*}(Z) \simeq \mathcal{D}\operatorname{Coh}^{G\times\mathbb{C}^*}(\pi_*\mathcal{O}_Z).$$

The equivalence descends to the singularity categories

$$\mathcal{D}_{sg}^{G\times\mathbb{C}^*}(\pi_*\mathcal{O}_Z) \simeq D_{sg}^{G\times\mathbb{C}^*}(Z).$$

Combining all these equivalences of categories we can now prove the equivariant version of the main theorem in [**Isik**].

THEOREM 6.34. *There is an equivalence of categories* $D_{sg}^{G\times\mathbb{C}^*}(Z) \simeq D^b(\operatorname{Coh}^G(Y))$.

PROOF. We already proved the following series of equivalences

$$\begin{aligned} D_{sg}^{G\times\mathbb{C}^*}(Z) &\simeq \mathcal{D}_{sg}^{G\times\mathbb{C}^*}(\pi_*\mathcal{O}_Z) \\ &\simeq \mathcal{D}_{sg}^{G\times\mathbb{C}^*}(\mathcal{B}) \\ &\simeq \frac{\mathcal{D}\operatorname{Coh}^{G\times\mathbb{C}^*}(\mathcal{A}_{X\times\mathbb{C}})^{op}}{\mathcal{D}_X\operatorname{Coh}^{G\times\mathbb{C}^*}(\mathcal{A}_{X\times\mathbb{C}})^{op}} \\ &\simeq \mathcal{D}\operatorname{Coh}^{G\times\mathbb{C}^*}(\mathcal{A}_{X\times\mathbb{C}^*})^{op} \\ &\simeq \mathcal{D}\operatorname{Coh}^{G\times\mathbb{C}^*}((\mathcal{O}_Y[t,t^{-1}])^{op} \\ &\simeq D^b(\operatorname{Coh}^G(Y))^{op}. \end{aligned}$$

Since X has an ample family of G-equivariant line bundles so does the closed subvariety Y by restriction. Thus, Y satisfies property (6.2.1) so a functor similar to D_Ω^X from section 6.3 can also be constructed for Y. This gives an equivalence $D^b(\operatorname{Coh}^G(Y))^{op} \simeq D^b(\operatorname{Coh}^G(Y))$. □

6.5. Equivariant matrix factorizations and singularity category

6.5.1. Definitions. Let X be an algebraic stack and $W \in H^0(X, \mathbb{C})$ a section. The section W is called the potential.

DEFINITION 6.35. (1) A matrix factorization $\bar{E} = (E_\bullet, \delta_\bullet)$ of W on X consists of a pair of vector bundles, i.e. locally free sheaves of finite rank, E_0, E_1 on X together with homomorphisms

$$\delta_1 : E_1 \to E_0 \quad \text{and} \quad \delta_0 : E_0 \to E_1$$

such that $\delta_1\delta_0 = W \cdot \operatorname{Id} = \delta_0\delta_1$.

(2) The dg-category of matrix factorizations is defined in the following way. Let $\bar{E}, \bar{F}$ be matrix factorizations. Then $\mathcal{H}om_{\mathrm{MF}}(\bar{E}, \bar{F})$ is the $\mathbb{Z}$-graded complex

$$\mathcal{H}om_{\mathrm{MF}}(\bar{E}, \bar{F})^{2n} := \operatorname{Hom}(E_0, F_0) \oplus \operatorname{Hom}(E_1, F_1),$$
$$\mathcal{H}om_{\mathrm{MF}}(\bar{E}, \bar{F})^{2n+1} := \operatorname{Hom}(E_0, F_1) \oplus \operatorname{Hom}(E_1, F_0).$$

with differential

$$df := \delta_F \circ f - (-1)^{|f|} f \circ \delta_E.$$

REMARK 6.36. Matrix factorizations can be defined for a general line bundle over X (see [**PV**]) but for the application we have in mind we only need $L = \mathbb{C}$.

Let G be a linear algebraic group acting on X and assume that W is invariant with respect to the action. Then we define G-equivariant matrix factorizations in the following way.

DEFINITION 6.37. A matrix factorization $\bar{E} = (E_\bullet, \delta_\bullet)$ of W on X is G-equivariant if (E_0, E_1) are G-equivariant vector bundles and (δ_0, δ_1) are G-invariant. We define $\mathcal{H}om_{\mathrm{MF}_G}(\bar{E}, \bar{F})$ to be the complex

$$\mathcal{H}om_{\mathrm{MF}_G}(\bar{E}, \bar{F})^{2n} := \mathrm{Hom}(E_0, F_0)^G \oplus \mathrm{Hom}(E_1, F_1)^G,$$
$$\mathcal{H}om_{\mathrm{MF}_G}(\bar{E}, \bar{F})^{2n+1} := \mathrm{Hom}(E_0, F_1)^G \oplus \mathrm{Hom}(E_1, F_0)^G.$$

The differential is the same as for non-equivariant matrix factorizations.

Denote the corresponding homotopy categories by $\mathrm{HMF}(X, W) := H^0(\mathrm{MF}(X, W))$ and $\mathrm{HMF}_G(X, W) := H^0(\mathrm{MF}_G(X, W))$.

REMARK 6.38. Let $\bar{W}$ denote the induced potential $X/G \to \mathbb{C}$. Then the dg-categories $\mathrm{MF}_G(X, W)$ and $\mathrm{MF}(X/G, \bar{W})$ are equivalent.

The category $\mathrm{HMF}(X, W)$ is a triangulated category. Consider the triangulated subcategory $\mathrm{LHZ}(X, W)$ consisting of matrix factorizations $\bar{E}$ that are locally contractible (i.e. there exists an open covering U_i of X in the smooth topology such that $\bar{E}|_{U_i} = 0$ in $\mathrm{HMF}(U_i, W|_{U_i})$).

DEFINITION 6.39. For a stack X we define the derived category of matrix factorizations by

$$\mathrm{DMF}(X, W) := \mathrm{HMF}(X, W) / \mathrm{LHZ}(X, W).$$

6.5.2. Connection with singularity categories. In this subsection we recall the connection between matrix factorizations and singularity categories as stated by Polishchuk and Vaintrob in [**PV**]. Let X be an algebraic stack and $W \in H^0(X, \mathbb{C})$ a potential. Assume that W is not a zero divisor (i.e. the morphism $W : \mathcal{O}_X \to \mathbb{C}$ is injective). Set $X_0 := W^{-1}(0)$. Then there is a natural functor (see [**PV**, Section 3]).

$$\mathfrak{C} : \mathrm{HMF}(X, W) \to D_{sg}(X_0),$$
$$(E_\bullet, \delta_\bullet) \mapsto \mathrm{coker}(\delta_1 : E_1 \to E_0).$$

DEFINITION 6.40. (i) X has the resolution property (RP) if for every coherent sheaf $\mathcal{F}$ on X there exists a vector bundle V on X and a surjection $V \to \mathcal{F}$.

(ii) X has finite cohomological dimension (FCD) if there exists an integer N such that for every quasi-coherent sheaf $\mathcal{F}$ on X one has $H^i(X, \mathcal{F}) = 0$ for $i > N$.
(iii) X is called a FCDRP-stack if it has property (i) and (ii).

THEOREM 6.41. [**PV**, Theorem 3.14] *Let X be a smooth FCDRP-stack and W a potential which is not a zero-divisor. Then the functor*

$$\bar{\mathfrak{C}} : \mathrm{DMF}(X, W) \to D_{sg}(X_0)$$

induced by $\mathfrak{C}$ is an equivalence of triangulated categories.

PROPOSITION 6.42. *Let U be a Noetherian scheme and G a reductive algebraic group acting on it. Assume that U has an ample family of G-equivariant line bundles. Then the quotient stack U/G is a FCDRP-stack.*

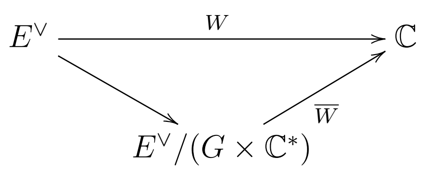
PROOF. See [**PV**, Section 3]. □

6.6. Application to the setting from Isik

Recall the notation from section 6.4: X is a smooth complex algebraic variety with an action of a reductive algebraic group G. We also have a G-equivariant vector bundle $\pi : E \to X$ of rank n. Its sheaf of G-equivariant sections is denoted by $\mathcal{E}$. Let $s \in H^0(X, \mathcal{E})$ be a regular G-equivariant section. The zero scheme of s is denoted by Y. We also defined the function.

$$W : E^\vee \to \mathbb{C}, \qquad a_x \mapsto \langle ((\pi^\vee)^* s)(a_x), a_x \rangle.$$

We assume that W is not a zero divisor. The section s is G-equivariant by assumption and the pairing is G-invariant so W is G-invariant. The $\mathbb{C}^*$-action is given by dilation of the fibers and W is also invariant with respect to this action. Thus, W factors through the quotient



For any linear algebraic group K acting on a scheme V there is an equivalence $\mathrm{Coh}^K(V) \simeq \mathrm{Coh}(V/K)$. This induces equivalences

$$D_{sg}^{G\times\mathbb{C}^*}(W^{-1}(0)) \simeq D_{sg}(W^{-1}(0)/(G \times \mathbb{C}^*)) \simeq D_{sg}(\overline{W}^{-1}(0)).$$

PROPOSITION 6.43. *With the above assumptions there is an equivalence of categories*

$$\mathrm{DMF}(E^\vee/(G \times \mathbb{C}^*), \bar{W}) \simeq D_{sg}(\overline{W}^{-1}(0)).$$

PROOF. We need to show that the assumptions of theorem 6.41 are satisfied. By assumption X is smooth, hence $E^\vee$ is smooth so the stack $E^\vee/(G \times \mathbb{C}^*)$ is smooth. The variety X has an ample family of $G\times\mathbb{C}^*$-equivariant line bundles. The pull-back of such a family along the $G \times \mathbb{C}^*$-equivariant bundle map $\pi^\vee : E^\vee \to X$ is an ample family of $G \times \mathbb{C}^*$-equivariant line bundles on $E^\vee$. It then follows from proposition 6.42 that $E^\vee/(G \times \mathbb{C}^*)$ is a FCDRP-stack. □

By remark 6.38 we have

$$\mathrm{DMF}(E^\vee/(G \times \mathbb{C}^*), \overline{W}) \simeq \mathrm{DMF}_{G\times\mathbb{C}^*}(E,^\vee W).$$

Combining these equivalences with corollary 6.34 we obtain

THEOREM 6.44. *There is an equivalence of categories*

$$\mathrm{DMF}_{G\times\mathbb{C}^*}(E^\vee, W) \simeq D^b(\mathrm{Coh}^G(Y)).$$

6.6.1. Extension to an arbitrary linear algebraic group. In this section we extend theorem 6.44 to arbitrary linear algebraic groups. Let X be a smooth complex algebraic variety with an action of a linear algebraic group H. Let $\pi : E \to X$ be a H-equivariant vector bundle and s a H-equivariant regular section. Let Y denote the zero section of s and let W be the function

$$W : E^\vee \to \mathbb{C}, \qquad a_x \mapsto \langle ((\pi^\vee)^* s)(a_x), a_x \rangle.$$

THEOREM 6.45. *With the above assumptions there is an equivalence of categories*

$$\mathrm{DMF}_{\mathbb{C}^*}(E^\vee/H, \bar{W}) \simeq D^b(\mathrm{Coh}^H(Y)).$$

PROOF. Embed H into a reductive algebraic group G. Then H acts freely on $G \times X$ by $h \cdot (g, x) := (gh^{-1}, hx)$. Consider the quotient by this action. The morphism

$$\pi_G : \tfrac{G\times E}{H} \to \tfrac{G\times X}{H}, \qquad (g, e) \mapsto (g, \pi(e)).$$

is a G-equivariant vector bundle. Consider the section

$$s_G : \tfrac{G\times X}{H} \to \tfrac{G\times E}{H}, \qquad (g, x) \mapsto (g, s(x)).$$

The zero scheme for this section is $\frac{G\times Y}{H} =: Y_G$. We have the corresponding function.

$$W_G : \left(\tfrac{G\times E}{H}\right)^\vee = \tfrac{G\times E^\vee}{H} \xrightarrow{\mathrm{Id}\times s_G} \tfrac{G\times E^\vee}{H} \times_{\frac{G\times X}{H}} \tfrac{G\times E}{H} \xrightarrow{\langle\, ,\, \rangle} \mathbb{C}.$$

Notice that $W_G(g, e) = W(e)$ for all $(g, e) \in \frac{G\times E}{H}$. Inserting this into theorem 6.44 gives an equivalence

$$\mathrm{DMF}_{G\times\mathbb{C}^*}(\tfrac{G\times E^\vee}{H}, W_G) \simeq D^b(\mathrm{Coh}^G(Y_G)).$$

Notice that $D^b(\mathrm{Coh}^G(Y_G)) \simeq D^b(\mathrm{Coh}^H(Y))$ and that

$$\begin{aligned}\mathrm{DMF}_{G\times\mathbb{C}^*}(\tfrac{G\times E^\vee}{H}, W_G) &\simeq \mathrm{DMF}_{\mathbb{C}^*}(\tfrac{G\times E^\vee}{H}/G, \bar{W}_G)\\ &\simeq \mathrm{DMF}_{\mathbb{C}^*}(E^\vee/H, \bar{W}).\end{aligned}$$

This finishes the proof. □

6.6.2. Application to Hamiltonian reduction. Let X be a smooth complex algebraic variety with a free action of a linear algebraic group G such that the quotient $Y := X/G$ is a scheme. The moment map $\mu : T^*X \to \mathfrak{g}^*$ provides a G-equivariant section of the trivial vector bundle $\pi : T^*X \times \mathfrak{g}^* \to T^*X$. The potential W is the composition

$$W : T^*X \times \mathfrak{g} \xrightarrow{\mu \times \mathrm{Id}} \mathfrak{g}^* \times \mathfrak{g} \xrightarrow{\langle\,,\rangle} \mathbb{C}.$$

Theorem 6.45 gives an equivalence of categories between matrix factorizations and the Hamiltonian reduction

$$\mathrm{DMF}_{\mathbb{C}^*}((T^*X \times \mathfrak{g})/G, \bar{W}) \simeq D^b(\mathrm{Coh}(\mu^{-1}(0)/G)).$$

Write $\mu^{-1}(0)$ as $T^*X \times_{\mathfrak{g}^*} 0$. Let $p : X \to X/G$ be the quotient morphism and $\{U_i\}$ a trivialization. Then

$$T^*(p^{-1}(U_i)) \times_{\mathfrak{g}^*} 0 \simeq (G \times \mathfrak{g}^* \times T^*U_i) \times_{\mathfrak{g}^*} 0 \simeq G \times T^*U_i.$$

This shows that $\mu^{-1}(0)/G \simeq T^*(X/G)$. In particular, we proved that

THEOREM 6.46. *There is an equivalence of categories*

$$\mathrm{DMF}_{\mathbb{C}^*}((T^*X \times \mathfrak{g})/G, \bar{W}) \simeq D^b(\mathrm{Coh}(T^*(X/G))).$$

REMARK 6.47. When X is a reductive algebraic group G and the linear algebraic group from the theorem is a Borel subgroup B then we get the Springer resolution $\tilde{\mathcal{N}} \simeq T^*(G/B)$. Thus, we get $\mathrm{DMF}_{\mathbb{C}^*}((T^*G \times \mathfrak{b})/B, \bar{W}) \simeq D^b(\mathrm{Coh}^G(\tilde{\mathcal{N}}))$.

CHAPTER 7

Braid group actions on equivariant matrix factorizations

7.1. The braid group action of Bezrukavnikov and Riche

In this section we recall the construction of an action of the (extended) affine braid group by Bezrukavnikov and Riche in [**BR**] and [**Ric**]. Let G be a reductive algebraic group over an algebraically closed field k. Their proof works both in characteristic zero and when the characteristic is bigger than the Coxeter number. Fix a maximal torus T and Borel subgroup B containing it. Recall that the extended affine braid group has the following presentation.

THEOREM 7.1. [**Ric**, Thm 1.1.3] *The extended affine braid group, B_{aff}, admits a presentation with generators $\{T_\alpha \mid \alpha \in \Pi\} \cup \{\theta_x \mid x \in \mathbb{X}\}$ and relations:*

(1) $T_\alpha T_\beta T_\alpha \cdots = T_\beta T_\alpha T_\beta \cdots$ with $m(\alpha, \beta)$ factors on each side.
(2) $\theta_x \theta_y = \theta_{x+y}$.
(3) $T_\alpha \theta_x = \theta_x T_\alpha$ if $\langle x, \alpha \rangle = 0$, i.e. $s_\alpha(x) = x$.
(4) $\theta_x = T_\alpha \theta_{x-\alpha} T_\alpha$ if $\langle x, \alpha \rangle = 1$, i.e. $s_\alpha(x) = x - \alpha$.

We sketch their construction. Let X and Y be G-varieties. Denote the projections $X \times Y \to X$ and $X \times Y \to Y$ by p_X and p_Y respectively. These projections are not assumed to be proper, so push-forward might not take coherent sheaves to coherent sheaves. This problem is fixed by introducing the following full subcategory

$$D^b_{\mathrm{prop}}(\mathrm{Coh}(X \times Y)) \subset D^b(\mathrm{Coh}(X \times Y))$$

in the following way. An object in $D^b(\mathrm{Coh}(X \times Y))$ belongs to $D^b_{\mathrm{prop}}(\mathrm{Coh}(X \times Y))$ if its cohomology sheaves are topologically supported on a closed subscheme $Z \subset X \times Y$ such that the restrictions of p_X and p_Y to Z are proper. They define a convolution product

$$* : D^b_{\mathrm{prop}}(\mathrm{Coh}^G(Y \times Z)) \times D^b_{\mathrm{prop}}(\mathrm{Coh}^G(X \times Y)) \to D^b_{\mathrm{prop}}(\mathrm{Coh}^G(X \times Z)),$$
$$\mathcal{F} * \mathcal{G} := Rp_{X,Z*}(p^*_{X,Y}\mathcal{F} \otimes^L_{X \times Y \times Z} p^*_{Y,Z}\mathcal{G}),$$

where $p_{X,Y}, p_{Y,Z}$ and $p_{X,Z}$ are the projections from $X \times Y \times Z$ to the listed factors. Any $\mathcal{F} \in D^b_{\mathrm{prop}}(\mathrm{Coh}^G(X \times Y))$ defines a functor

$$F^{\mathcal{F}}_{X \to Y} : D^b(\mathrm{Coh}^G(X)) \to D^b(\mathrm{Coh}^G(X)),$$
$$\mathcal{M} \mapsto Rp_{Y*}(\mathcal{F} \otimes^L_{X \times Y} p^*_X \mathcal{M}).$$

LEMMA 7.2. [**Ric**, Lemma 1.2.1] *Let* $\mathcal{F} \in D^b_{prop}(\mathrm{Coh}^G(X \times Y))$ *and* $\mathcal{G} \in D^b_{prop}(\mathrm{Coh}^G(Y \times Z))$. *Then*

$$F^{\mathcal{G}}_{Y \to Z} \circ F^{\mathcal{F}}_{X \to Y} \simeq F^{\mathcal{G} * \mathcal{F}}_{X \to Z}.$$

The categorical action of B_{aff} to be constructed will be a weak action. Recall that a weak action of a group A on a category $\mathcal{C}$ is a group morphism from A to the group of isomorphism classes of auto-equivalences of the category $\mathcal{C}$. Note that no compatibility conditions are imposed on the morphisms. In particular, to construct an action of B_{aff} on $D^b(\mathrm{Coh}^G(X))$ it suffices to find objects in $(D^b_{\mathrm{prop}}(\mathrm{Coh}^G(X \times X)), *)$, whose convolution with each other satisfy affine braid group relations.

In this construction the variety is going to be the Grothendieck variety $\tilde{\mathfrak{g}}$. Recall that $\tilde{\mathfrak{g}}$ is smooth and that $\nu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is proper. It follows that the base change morphisms $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ are proper, so we have a full monoidal subcategory

$$D^b_{\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}}(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}})) \subset D^b_{\mathrm{prop}}(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}})),$$

whose objects are topologically supported on $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$. It is a monoidal category with $*$. We call this monoidal category the affine Hecke category

$$\mathrm{Hecke}_{\mathrm{af}}(G, B) := (D^b_{\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}}(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}})), *).$$

Consider the composition

$$\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \hookrightarrow \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \twoheadrightarrow \mathcal{B} \times \mathcal{B}.$$

For $w \in W$ we denote by Z_w the closure of the inverse image of the G diagonal orbit of $(B/B, w^{-1}B/B)$. For $x \in \mathbb{X}$ we have the canonical line bundle $\mathcal{O}_{\mathcal{B}}(x)$. It is the sheaf corresponding to the G-equivariant line bundle $G \times_B k_x \to \mathcal{B}$. Here k_x is the vector space k with a B-action given by

$$b \cdot z = x(\pi(b))z,$$

where π is the quotient map $B \to B/[B, B] \simeq T$. We define $\mathcal{O}_{\Delta(\tilde{\mathfrak{g}})}(x)$ to be the pull-back of $\mathcal{O}_{\mathcal{B}}(x)$ along the projection of the diagonal $\Delta(\tilde{\mathfrak{g}}) \to \mathcal{B}$. We can now state the main result of Bezrukavnikov and Riche, which can be seen as a categorification of the result about representations of the Weyl group in chapter 2.

THEOREM 7.3. [**BR**, Theorem 1.3.2] *There is a categorical* B_{aff}*-action on* $D^b(\mathrm{Coh}^G(\tilde{\mathfrak{g}}))$ *in which* T_{α_i} *acts by convolution with* $\mathcal{O}_{Z_{s_{\alpha_i}}}$ *and* θ_x *acts by convolution with* $\mathcal{O}_{\Delta(\tilde{\mathfrak{g}})}(x)$.

Moreover, they prove that for a reduced expression $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_n}}$

$$\mathcal{O}_{Z_{s_{\alpha_{i_1}}}} * \cdots * \mathcal{O}_{Z_{s_{\alpha_{i_n}}}} \simeq \mathcal{O}_{Z_w}.$$

In [**BR**, Section 4 and 5] Bezrukavnikov and Riche also lift this construction to the DG-category setting. Consider two $G \times \mathbb{G}_m$-equivariant dg-algebras $X = (X_0, \mathcal{A}_X)$ and $Y = (Y_0, \mathcal{A}_Y)$ and an equivariant functor of dg-algebras $f : X \to Y$.

By choosing resolutions (in the appropriate sense) one can define a derived fiber product $X \times_Y^R X$ up to quasi-isomorphism. This is sufficient to get a well-defined category

$$\mathcal{K}_{X,Y} := \mathcal{D}\,\mathrm{QCoh}^{G\times\mathbb{G}_m}(X \times_Y^R X).$$

REMARK 7.4. Assume that X and Y are ordinary schemes (i.e. $\mathcal{A}_X = \mathcal{O}_X$ and $\mathcal{A}_Y = \mathcal{O}_Y$) satisfying

$$\mathrm{Tor}_{\neq 0}^{f^{-1}\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) = 0.$$

Then the dg-scheme $X \times_Y^R X$ is just the ordinary fiber product $X_0 \times_{Y_0} X_0$ and

$$\mathcal{K}_{X,Y} \simeq \mathcal{D}\,\mathrm{QCoh}^{G\times\mathbb{G}_m}(X_0 \times_{Y_0} X_0).$$

This is the case when $X = \tilde{\mathfrak{g}}$, $Y = \mathfrak{g}$ and f is the Grothendieck-Springer resoluion (see [**Bez2**, Section 1.2.2]).

The category $\mathcal{K}_{X,Y}$ is monoidal. When f is smooth (any quasi-projective morphism can be replaced by a smooth morphism using a trick explained in [**BR**, Section 3.7]) the derived fiber product is the ordinary fiber product of dg-schemes $X \times_Y X$ and the convolution product is defined in the following way. Let $q_{ij} : X_0 \times_{Y_0} X_0 \times_{Y_0} X_0 \to X_0 \times_{Y_0} X_0$ be the projection to the (i,j)-th factor. Consider the dg-schme

$$Z_{ij} := (X_0 \times_{Y_0} X_0 \times_{Y_0} X_0, q_{ij}^* \mathcal{A}_{X\times_Y X}.)$$

It has a natural morphism of dg-schemes $p_{ij} : Z_{ij} \to X \times_Y X$. Let $q_2 : X_0 \times_{Y_0} X_0 \times_{Y_0} X_0 \to X_0$ be projection to the second factor, and consider the sheaf of dg-algebras $q_2^* \mathcal{A}_X$ on $X_0 \times_{Y_0} X_0 \times_{Y_0} X_0$,. Then there exist a derived tensor product

$$\otimes^L_{q_2^*\mathcal{A}_X} : \mathcal{D}\,\mathrm{QCoh}^{G\times\mathbb{G}_m}(Z_{12}) \times \mathcal{D}\,\mathrm{QCoh}^{G\times\mathbb{G}_m}(Z_{23}) \to \mathcal{D}\,\mathrm{QCoh}^{G\times\mathbb{G}_m}(Z_{13})$$

Using this they define the convolution product $* : \mathcal{K}_{X,Y} \times \mathcal{K}_{X,Y} \to \mathcal{K}_{X,Y}$

$$\mathcal{M} * \mathcal{N} := Rp_{13*}(Lp_{12}^* \mathcal{N} \otimes^L_{q^*\mathcal{A}_X} Lp_{23}^* \mathcal{M}).$$

Let $p_1, p_2 : X \times_Y X \to X$ be the two projections. There is a monoidal action of $\mathcal{K}_{X,Y}$ on $\mathcal{D}\,\mathrm{QCoh}^{G\times\mathbb{G}_m}(X)$.

$$\mathcal{K}_{X,Y} \times \mathcal{D}\,\mathrm{QCoh}^{G\times\mathbb{G}_m}(X) \to \mathcal{D}\,\mathrm{QCoh}^{G\times\mathbb{G}_m}(X),$$
$$\mathcal{M} * \mathcal{N} := Rp_{2*}(\mathcal{M} \otimes^L_{X\times_Y^R X} Lp_1^* \mathcal{N}).$$

DEFINITION 7.5. Let $\mathcal{K}_{X,Y}^{\mathrm{Coh}}$ be the full subcategory of $\mathcal{K}_{X,Y}$ whose objects are complexes with only finitely many non-zero cohomology sheaves, each of which is a coherent sheaf on $X \times_Y X$.

PROPOSITION 7.6. [**BR**, Prop. 4.2.1] *Assume that X and Y are ordinary schemes and that f is proper. Then $\mathcal{K}_{X,Y}^{\mathrm{Coh}}$ is a monoidal category with the restricted convolution product and the action of $\mathcal{K}_{X,Y}^{\mathrm{Coh}}$ on $\mathcal{D}\,\mathrm{QCoh}^{G\times\mathbb{G}_m}(X)$ preserves the full subcategory $\mathcal{D}^b\,\mathrm{Coh}^{G\times\mathbb{G}_m}(X)$.*

In the setting of the proposition there is a "direct image under closed embedding" functor

$$\mathcal{K}^{\mathrm{Coh}}_{X,Y} \to \mathcal{D}^b \operatorname{Coh}_{X\times_Y X}(X \times X)$$

This functor is monoidal as remarked in [**MR3**, Section 4.1]. The generators of the braid group action from theorem 7.3 are all schemes on $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ so they can naturally be considered as objects in $\mathcal{K}^{\mathrm{Coh}}_{\tilde{\mathfrak{g}},\mathfrak{g}}$.

THEOREM 7.7. *There is a (weak) geometric action of B_{aff} on $\mathcal{K}^{\mathrm{Coh}}_{\tilde{\mathfrak{g}},\mathfrak{g}}$.*

7.2. Braid group action on matrix factorizations coming from Hamiltonian actions

Let G be reductive algebraic group over an algebraically closed field of characteristic zero. We would like to construct a categorical affine braid group action on the equivariant absolute derived category of matrix factorizations coming from the Hamiltonian action of G on the cotangent bundle of a smooth complex G-variety X. However, in order to get such an action one should replace $\mathfrak{g}$ by the Grothendieck variety $\tilde{\mathfrak{g}}$ and instead consider matrix factorizations on $\tilde{\mathfrak{g}} \times T^*X$. Denote the moment map of the action by μ. One can use the Grothendieck-Springer resolution $\nu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ to modify the potential to work with $\tilde{\mathfrak{g}}$. The new potential is the function

$$w : \tilde{\mathfrak{g}} \times T^*X \xrightarrow{\nu\times\mu} \mathfrak{g} \times \mathfrak{g}^* \xrightarrow{\langle\,,\rangle} \mathbb{C}.$$

The theorem we are going to prove is the following.

THEOREM 7.8 (Main theorem). *There is a B_{aff}-action on $D^{\mathrm{abs}}(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times T^*X), w)$.*

The idea of the proof is to construct a monoidal category of matrix factorizations and a monoidal action of this category on $D^{\mathrm{abs}}(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times T^*X), w)$. Then we construct a monoidal functor from a subcategory of the category from Bezrukavnikov and Riche's theorem 7.7 below, containing the generators of the braid group action, to this monoidal category.

7.3. Matrix factorizations as CDG-modules

In this section we give an another but equivalent definition of matrix factorizations, which is more convenient for the purpose of this project. Here we will consider matrix factorizations as a special (zero differential, 2-periodic and with constant curvature) case of curved differential graded modules following the approach of Efimov and Positselski in [**EP**] and [**Pos**]. The previous chapter relied on the result [**PV**, Thm. 3.14]. In order to deal with the equivariance, it was important that the result works for quotient stacks. The same result formulated in the language of this chapter but only for schemes is [**EP**, Thm. 2.7]. It is possible that the proof of Efimov and Positselski extends to the equivariant setting, but we did not check this. The notion of matrix factorizations and their derived category can be extended to any Abelian category (see [**Efi1**]).

Our main reason for choosing the CDG-module setting is that all the homological algebra machinery we need has been developed in [**Pos**] and [**EP**]. Some machinery has also been developed in [**Efi1**] in the categorical generalization, but it is not sufficient for our purposes. An attempt to develop it to the extend we need was done by Ballard, Deliu, Favero, Isik and Katzarkov in the paper [**BDFIK**]. However, the paper contains many errors, so we decided to stick with the CDG-module setting. We are fairly convinced that the results we need also holds in the categorical setting, and that our construction would work well also in this formulation, but this would require more work.

7.3.1. Curved differential graded modules.

Definition 7.9. (1) A curved differential graded (CDG) ring is a triple $B = (B, d, h)$, where B is a $\mathbb{Z}$-graded ring with an odd derivation $d : B \to B$ of degree 1 and h is an element in B^2 such that $d^2(b) = [h, b]$ for all $b \in B$ and $d(h) = 0$. Here $[\ ,\ ]$ is the supercommutator $[a, b] = ab - (-1)^{|a||b|}ba$.

(2) A morphism of CDG-rings is a pair $(f, a) : B \to A$, where $f : B \to A$ is a morphism of graded rings and $a \in A^1$ an element satisfying $f(d_B b) = d_A f(b)+[a, f(b)]$ for all $b \in B$ and $f(h_B) = h_A+d_A a+a^2$. The composition of morphisms is defined as $(f, a) \circ (g, b) = (f \circ g, a + f(b))$.

(3) A left CDG-module over a CDG-ring B is a pair (M, d_M), where M is a graded B-module with an odd derivation $d_M : M \to M$ compatible with d_B and satisfying $d_B^2(m) = hm$ for all $m \in M$.

Remark 7.10. In the cases we are interested in all morphisms of CDG-rings will have the change of curvature element $a = 0 \in A^1$.

The category of left CDG-modules over a CDG- ring B has a DG-category structure with the following Hom complex

$$\mathrm{Hom}^n(M, N) := \{f : M \to N \text{ homogeneous} \mid f(bm) = (-1)^{n|b|} bf(m) \ \ \forall b \in B, m \in M\}$$
$$d(f)(m) := d_N(f(m)) - (-1)^{|f|} f(d_M(m)).$$

Since

$$\begin{aligned} d^2(f)(m) &= d_N^2 f(m) - (-1)^{|f|+1} d_N f d_M(m) - (-1)^{|f|} d_N f d_M(m) - f d_M^2(m) \\ &= hf(m) - f(hm) = hf(m) - (-1)^{n2} hf(m) \\ &= 0. \end{aligned}$$

From now on all schemes are separated Noetherian and we assume that they have enough vector bundles, i.e. every coherent sheaf on X is the quotient sheaf of a locally free sheaf of finite rank.

DEFINITION 7.11. Let X be a separated Noetherian scheme with enough vector bundles.

(1) A quasi-coherent CDG-algebra $\mathcal{B}$ over X is a graded quasi-coherent $\mathcal{O}_X$-algebra such that for every affine open subspace $U \subseteq X$ the graded ring $\mathcal{B}(U)$ is a CDG-ring (the differential is not required to be $\mathcal{O}_X$-linear). For each pair of embedded affine open subspaces $U \subseteq V \subseteq X$ we fix an element $a_{UV} \in \mathcal{B}^1(U)$ which together with the restriction morphism $\mathcal{B}(V) \to \mathcal{B}(U)$ form a morphism of CDG-rings. Moreover, we impose the usual compatibility conditions for triples of embedded affine open subsets.
(2) A quasi-coherent left CDG-module $\mathcal{M}$ over $\mathcal{B}$ is an $\mathcal{O}_X$ quasi-coherent sheaf endowed with a family of differentials $d : \mathcal{M}(U) \to \mathcal{M}(U)$ for each affine open subset U satisfying $d(s)|_U = d(s|_U)+a_{UV}s|_U$ for all $s \in \mathcal{M}(V)$.

DEFINITION 7.12. Let $f : Y \to X$ be a morphism of separated Noetherian schemes with enough vector bundles and $\mathcal{B}_X$, $\mathcal{B}_Y$ CDG-algebras on X and Y respectively. A morphism of CDG-algebras $\mathcal{B}_X \to \mathcal{B}_Y$ compatible with f is a CDG-ring morphism $\mathcal{B}_X(U) \to \mathcal{B}_Y(V)$ for each pair of affine open subsets $U \subseteq X$, $V \subseteq Y$ with $f(V) \subseteq U$ satisfying the following compatibility condition: for all affine open subsets $U' \subseteq U$, $V' \subseteq V$ with $f(V') \subseteq U'$ the following diagram is commutative

$$\begin{array}{ccc} \mathcal{B}_X(U) & \longrightarrow & \mathcal{B}_Y(V) \\ \downarrow & & \downarrow \\ \mathcal{B}_X(U') & \longrightarrow & \mathcal{B}_Y(V') \end{array}$$

For two quasi-coherent left CDG-modules $\mathcal{M}$ and $\mathcal{N}$ we define the complex $\mathrm{Hom}(\mathcal{M},\mathcal{N})$ to be the complex of morphisms such that $f|_U : \mathcal{M}(U) \to \mathcal{N}(U)$ is in the Hom complex of CDG-modules over the CDG-ring $\mathcal{B}(U)$ for every affine open subset $U \subset X$. The differential is defined locally as the supercommutator

$$d(f)(m) := d_N(f(m)) - (-1)^{|f|} f(d_M(m))$$

This gives a DG-category structure and we denote the DG-category by $\mathcal{B} - \mathrm{QCoh}$.

The CDG-algebra $\mathcal{B}$ is called Noetherian if $\mathcal{B}(U)$ is Noetherian for all affine open U. In this case we denote by $\mathcal{B} - \mathrm{Coh}$ the full DG-subcategory of quasi-coherent CDG-modules whose underlying graded $\mathcal{B}$-modules are finitely generated over $\mathcal{B}$.

These DG-categories admits shifts and cones so the homotopy categories $H^0(\mathcal{B}-\mathrm{QCoh})$ and $H^0(\mathcal{B} - \mathrm{Coh})$ are triangulated.

7.3.2. Exotic derived category. Since the differential does not square to zero the conventional definition of the derived category does not make sense. Instead one can form the exotic derived category introduced by Positselski.

DEFINITION 7.13. For a complex $\cdots \to X_i \xrightarrow{g_i} X_{i+1} \to \dots$ in one of these DG-categories we define the total object to be the object $\bigoplus_n X_n[n]$ with differential defined by $d = \sum_n d_{X_n} + (-1)^{\text{vertical degree}} g_n$

DEFINITION 7.14 (Exotic derived category).

(1) A quasi-coherent CDG-module over $\mathcal{B}$ is absolutely acyclic if it belongs to the minimal thick subcategory of $H^0(\mathcal{B} - \mathrm{QCoh})$ containing all total complexes of short exact sequences in $\mathcal{B} - \mathrm{QCoh}$.
(2) The absolute derived category $D^{\mathrm{abs}}(\mathcal{B} - \mathrm{QCoh})$ is the quotient category of $H^0(\mathcal{B} - \mathrm{QCoh})$ by the thick subcategory of absolutely acyclic CDG-modules.
(3) Assume that $\mathcal{B}$ is Noetherian. A coherent CDG-module is absolutely acyclic if it belongs to the minimal thick subcategory of $H^0(\mathcal{B} - \mathrm{Coh})$ containing all total complexes of short exact sequences in $\mathcal{B} - \mathrm{Coh}$.
(4) The absolute derived category $D^{\mathrm{abs}}(\mathcal{B} - \mathrm{Coh})$ is the quotient category of $H^0(\mathcal{B} - \mathrm{Coh})$ by the thick subcategory of coherent absolutely acyclic CDG-modules.
(5) Let $\mathcal{E}$ be an exact subcategory in the Abelian category of quasi-coherent graded left $\mathcal{B}$-modules. Denote the full subcategory of $\mathcal{B} - \mathrm{QCoh}$ with objects for which underlying graded $\mathcal{B}$-modules are in $\mathcal{E}$ by $\mathcal{B} - \mathrm{QCoh}_{\mathcal{E}}$. The relative absolute derived category $D^{\mathrm{abs}}(\mathcal{B} - \mathrm{QCoh}_{\mathcal{E}})$ is the quotient of $H^0(\mathcal{B} - \mathrm{QCoh}_{\mathcal{E}})$ by the minimal thick subcategory containing the total objects of all short exact sequences, whose underlying graded modules belong to $\mathcal{E}$. The category $D^{\mathrm{abs}}(\mathcal{B} - \mathrm{Coh}_{\mathcal{E}})$ is defined similarly.

LEMMA 7.15. [**EP**, Rem. 1.3] *Absolute acyclicity is a local notion, i.e. to check that $\mathcal{M}$ is acyclic it is enough to show that $\mathcal{M}$ restricted to each U_α is acyclic, where $\{U_\alpha\}$ is a finite affine open cover of X.*

PROPOSITION 7.16. [**EP**, Prop. 1.5(c)] *The functor $D^{\mathrm{abs}}(\mathcal{B}-\mathrm{Coh}) \to D^{\mathrm{abs}}(\mathcal{B}-\mathrm{QCoh})$ induced by the inclusion $\mathcal{B} - \mathrm{Coh} \hookrightarrow \mathcal{B} - \mathrm{QCoh}$ is fully faithful.*

7.3.3. Matrix factorizations. Let X be a separated Noetherian scheme with enough vector bundles and let $\mathcal{L}$ be a line bundle on X. Fix a section $w \in \mathcal{L}(X)$. This section is called the potential. We define the $\mathbb{Z}$-graded quasi-coherent CDG-algebra $(X, \mathcal{L}, w)$ as follows.

$$(X, \mathcal{L}, w)^n := \begin{cases} \mathcal{L}^{\otimes n/2} & n \in 2\mathbb{Z} \\ 0 & n \in 2\mathbb{Z}+1 \end{cases}$$

In particular, the differential is zero and the elements a_{UV} defining the restriction morphism $(X, \mathcal{L}, w)(U) \to (X, \mathcal{L}, w)(V)$ for affine open $V \subseteq U \subseteq X$ vanish. For U affine open the curvature element is $w|_U \in (X, \mathcal{L}, w)^2 = \mathcal{L}(X)$. The multiplication comes from the natural isomorphisms

$$\mathcal{L}^{\otimes n/2} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m/2} \to \mathcal{L}^{\otimes (n+m)/2}.$$

There is an equivalence of categories between quasi-coherent $\mathbb{Z}$-graded $(X, \mathcal{L}, w)$-modules and $\mathbb{Z}/2$-graded quasi-coherent $\mathcal{O}_X$-modules given by

$$\begin{array}{ccc} \mathrm{QCoh}_{\mathbb{Z}}(X, \mathcal{L}, w) - \mathrm{mod} & \longleftrightarrow & \mathrm{QCoh}_{\mathbb{Z}/2}(X) \\ \mathcal{M} & \longmapsto & (\mathcal{U}^0 = \mathcal{M}^0, \mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2} = \mathcal{M}^1) \\ \mathcal{M}^n = \mathcal{U}^{n \mod 2} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n/2} & \longleftarrow & (\mathcal{U}^0, \mathcal{U}^1) \end{array}$$

This equivalence preserves all the properties we are interested in, e.g. coherence, flatness ect. Hence, an object in $(X, \mathcal{L}, w) - \mathrm{QCoh}$ is a pair of quasi-coherent $\mathcal{O}_X$-modules $\mathcal{U}^0$ and $\mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2}$ with $\mathcal{O}_X$-linear morphisms

$$\mathcal{U}^0 \to \mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2}, \qquad \mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2} \to \mathcal{U}^0 \otimes_{\mathcal{O}_X} \mathcal{L}$$

such that both compositions

$$\mathcal{U}^0 \to \mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2} \to \mathcal{U}^0 \otimes_{\mathcal{O}_X} \mathcal{L}$$
$$\mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2} \to \mathcal{U}^0 \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{U}^1 \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes 3/2}$$

are equal to the multiplication by w.

Let $f : X \to Y$ be a morphism of separated Noetherian schemes with enough vector bundles. For any affine open $U \subseteq X$ and $V \subseteq Y$ with $f(U) \subseteq V$ we have a morphism compatible with restriction

$$f_* \mathcal{L}^{\otimes n/2}(V) = \mathcal{L}(f^{-1}(V))^{\otimes n/2} \to \mathcal{L}^{\otimes n/2}(U).$$

By adjunction we get a map $\mathcal{L}^{\otimes n/2}(U) \to f^* \mathcal{L}^{\otimes n/2}(V)$. This defines a morphism of CDG-algebras $(X, \mathcal{L}, w) \to (Y, f^*\mathcal{L}, f^*w)$, where $f^*w = f(w)$.

The absolute derived category of matrix factorizations is $D^{\mathrm{abs}}(\mathrm{Coh}(X), \mathcal{L}, w) := D^{\mathrm{abs}}((X, \mathcal{L}, w) - \mathrm{Coh})$. Likewise, we write $D^{\mathrm{abs}}(\mathrm{QCoh}(X), \mathcal{L}, w) := D^{\mathrm{abs}}((X, \mathcal{L}, w) - \mathrm{QCoh})$. When $\mathcal{L} = \mathcal{O}_X$ we leave it out from the notation.

7.3.4. Equivariant matrix factorizations. Let G be a reductive algebraic group acting on a smooth scheme X. In the case we are interested in $\mathcal{L} = \mathcal{O}_X$ and w is a G-invariant section. The notation in the equivariant setting is analogues to the non-equivariant setting.

DEFINITION 7.17.
(1) The subcategory of $(X, \mathcal{O}_X, w) - \mathrm{QCoh}$ whose objects are the CDG-modules for which the underlying $\mathcal{O}_X$-modules are G-equivariant and morphisms are G-equivariant is denoted by $(\mathrm{QCoh}^G(X), w)$.
(2) Objects in $H^0(\mathrm{QCoh}^G(X), w)$ are called equivariant absolutely acyclic if they belong to the minimal thick subcategory of $H^0(\mathrm{QCoh}^G(X), w)$ containing all total complexes of short exact sequences in $(\mathrm{QCoh}^G(X), w)$.

(3) The equivariant absolute derived category $D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), w)$ is the quotient category of $H^0(\mathrm{QCoh}^G(X), w)$ by the thick subcategory of equivariant absolutely acyclic CDG-modules.

We use similar notation in the coherent case and in the relative case. The proof from proposition 7.16 generalizes to the equivariant setting.

PROPOSITION 7.18. *Let G be a reductive algebraic group acting on a smooth scheme X. Then the functor $D^{\mathrm{abs}}(\mathrm{Coh}^G(X), w) \to D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), w)$ induced by the inclusion $(\mathrm{Coh}^G(X), w) \hookrightarrow (\mathrm{QCoh}^G(X), w)$ is fully faithful.*

7.4. Derived functors

7.4.1. Derived functors on exotic derived categories. Let f be a morphism of CDG-algebras. Then we can define push-forward and pull-back as usual

$$f^* : \mathcal{B}_X - \mathrm{QCoh} \to \mathcal{B}_Y - \mathrm{QCoh},$$
$$f^*\mathcal{M} := \mathcal{B}_Y \otimes_{f^{-1}\mathcal{B}_X} f^{-1}\mathcal{M}.$$
$$f_* : \mathcal{B}_Y - \mathrm{QCoh} \to \mathcal{B}_X - \mathrm{QCoh},$$
$$f_*\mathcal{N} := \mathcal{N} \text{ as a } \mathcal{B}_X\text{-module}.$$

These functors descend to the homotopy categories and f_* is right adjoint to f^*.

The usual tensor product for graded complexes defines a quasi-coherent graded $\mathcal{O}_X$-module $\mathcal{N} \otimes_{\mathcal{B}_X} \mathcal{M}$ for $\mathcal{N}$ a quasi-coherent graded right $\mathcal{B}_X$-module $\mathcal{N}$ and $\mathcal{M}$ a quasi-coherent graded left $\mathcal{B}_X$-module. A module $\mathcal{M}$ is called flat if the functor $- \otimes_{\mathcal{B}_X} \mathcal{M}$ is exact on the abelian category of quasi-coherent graded right $\mathcal{B}_X$-modules. Equivalently, $\mathcal{M}$ is flat if the graded left $\mathcal{B}_X(U)$-module $\mathcal{M}(U)$ is flat for any affine open subscheme $U \subseteq X$. Denote the full subcategory in $\mathcal{B}_X - \mathrm{QCoh}$ of CDG-modules whose underlying graded $\mathcal{B}_X$-modules are flat by $\mathcal{B}_X - \mathrm{QCoh}_{\mathrm{fl}}$. Since $\otimes_{\mathcal{B}_Z}$ takes short exact sequences in $\mathcal{B}_X - \mathrm{QCoh}_{\mathrm{fl}}$ to short exact sequences it takes absolutely acyclic modules to absolutely acyclic modules. Hence, it induces a functor

$$\otimes^L_{\mathcal{B}_X} : D^{\mathrm{abs}}(\mathrm{QCoh} - \mathcal{B}_X) \times D^{\mathrm{abs}}(\mathcal{B}_X - \mathrm{QCoh}_{\mathrm{fl}}) \to D^{\mathrm{abs}}(\mathcal{O}_X - \mathrm{QCoh}).$$

The functor f^* restricts to a functor $H^0(\mathcal{B}_X - \mathrm{QCoh}_{\mathrm{fl}}) \to H^0(\mathcal{B}_Y - \mathrm{QCoh})$ preserving short exact sequences. Hence, it takes modules absolutely acyclic with respect to $\mathcal{B}_X - \mathrm{QCoh}_{\mathrm{fl}}$ to modules absolutely acyclic with respect to $\mathcal{B}_Y - \mathrm{QCoh}_{\mathrm{fl}}$ so it induces a functor

$$Lf^* : D^{\mathrm{abs}}(\mathcal{B}_X - \mathrm{QCoh}_{\mathrm{fl}}) \to D^{\mathrm{abs}}(\mathcal{B}_Y - \mathrm{QCoh}_{\mathrm{fl}}).$$

Notice that if $g : (X, \mathcal{B}_X) \to (Z, \mathcal{B}_Z)$ is another morphism of CDG-algebras then $L(g \circ f)^* \simeq Lf^* \circ Lg^*$ since pull-back takes flat to flat.

Similarly, we denote the full subcategory in $\mathcal{B}_X - \mathrm{QCoh}$ of CDG-modules whose underlying graded $\mathcal{B}_X$-modules are injective (respectively of finite injective dimension) by $\mathcal{B}_X - \mathrm{QCoh}_{\mathrm{inj}}$ (respectively $\mathcal{B}_X - \mathrm{QCoh}_{\mathrm{fid}}$).

LEMMA 7.19. [**EP**, Lemma 1.7(a)] *The natural functor* $H^0(\mathcal{B}_X - \mathrm{QCoh}_{\mathrm{inj}}) \to D^{\mathrm{abs}}(\mathcal{B}_X - \mathrm{QCoh}_{\mathrm{fid}})$ *is an equivalence of triangulated categories.*

By the lemma f_* induces a functor

$$Rf_* : D^{\mathrm{abs}}(\mathcal{B}_Y - \mathrm{QCoh}_{\mathrm{fid}}) \to D^{\mathrm{abs}}(\mathcal{B}_X - \mathrm{QCoh}).$$

Let $D^{\mathrm{abs}}(\mathrm{QCoh}^G(X)_{\mathrm{lf}}, w)$ denote the relative absolute derived category for the full subcategory in $\mathrm{QCoh}^G(X)$ whose underlying graded modules are locally free of finite rank.

7.4.2. Derived functors on matrix factorizations. Let $f : X \to Y$ be an equivariant morphism of separated Noetherian schemes G-schemes with enough G-equivariant vector bundles. As noted in section 7.3.3 this defines a morphism of CDG-algebras. Hence, we can define the absolute derived pull-back and push-forward as in the previous section

$$Lf^* : D^{\mathrm{abs}}(\mathrm{QCoh}^G(X)_{\mathrm{fl}}, w) \to D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y)_{\mathrm{fl}}, f^*w).$$
$$Rf_* : D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y)_{\mathrm{inj}}, f^*w) \to D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), w).$$

When X and Y are smooth these can be extended to functors

$$Lf^* : D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), w) \to D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y), f^*w).$$
$$Rf_* : D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y), f^*w) \to D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), w).$$

by the following proposition

PROPOSITION 7.20. *Let G be a reductive algebraic group acting on a smooth scheme X.*

(1) *The natural functor*

$$H^0(\mathrm{QCoh}^G(X)_{\mathrm{inj}}, h) \to D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), h)$$

is an equivalence of triangulated categories.

(2) *The natural functor*

$$D^{\mathrm{abs}}(\mathrm{QCoh}^G(X)_{\mathrm{lf}}, w) \to D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), w)$$

is an equivalence of triangulated categories.

PROOF. (1) The proof of lemma 7.19 extends to the equivariant setting so we get an equivalence of triangulated categories

$$H^0(\mathrm{QCoh}^G(X)_{\mathrm{inj}}, w) \simeq D^{\mathrm{abs}}(\mathrm{QCoh}^G(X)_{\mathrm{fid}}, w).$$

For smooth schemes all equivariant quasi-coherent sheaves have finite equivariant injective dimension so the result follows.

(2) [**EP**, Cor. 2.4(b)+rem.] states that when X is a regular separated Noetherian scheme of finite Krull dimension then the natural functor $D^{\mathrm{abs}}(\mathrm{QCoh}(X)_{\mathrm{lf}}, w) \to D^{\mathrm{abs}}(\mathrm{QCoh}(X), w)$ is an equivalence of triangulated categories. The proof extends

to the equivariant setting when X has finite G-equivariant locally free dimension. This is satisfied when X is smooth. □

PROPOSITION 7.21. *Let $f : X \to Y$ be an equivariant morphism of smooth G-schemes. Then Rf_* is right adjoint to Lf^*.*

PROOF. In the non-equivariant setting this follows from [**EP**] Prop. 1.9 and Cor. 2.3(b)+(f). The proof also works in the equivariant case. □

LEMMA 7.22. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of smooth schemes. Then*

$$L(g \circ f)^* \simeq Lf^* \circ Lg^* \qquad \text{and} \qquad R(g \circ f)_* \simeq Rg_* \circ Rf_*.$$

PROOF. The first part follows from the fact that pull-back take flat to flat. The second part follows from the first by adjunction. □

We also have an equivariant tensor product

$$\otimes^L_X : D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), w_1) \times D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), w_2) \to D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), w_1 + w_2).$$

LEMMA 7.23. *Let $f : X \to Y$ be an equivariant morphism of smooth schemes. There is an isomorphism of functors*

$$Lg^*(-) \otimes^L_X Lg^*(-) \simeq Lg^*(- \otimes^L_X -).$$

PROOF. The pull-back takes flat modules to flat modules and the tensor product is a functor $\otimes^L_Y : D^{\mathrm{abs}}(\mathrm{QCoh}(Y)_{\mathrm{fl}}, w_1) \times D^{\mathrm{abs}}(\mathrm{QCoh}(Y)_{\mathrm{fl}}, w_2) \to D^{\mathrm{abs}}(\mathrm{QCoh}(Y)_{\mathrm{fl}}, w_1 + w_2)$. Thus, on $D^{\mathrm{abs}}(\mathrm{QCoh}(Y)_{\mathrm{fl}}, w_1) \times D^{\mathrm{abs}}(\mathrm{QCoh}(Y)_{\mathrm{fl}}, w_2)$.

$$Lg^*(-) \otimes^L_X Lg^*(-) \simeq L(g^*(-) \otimes_X g^*(-))$$
$$Lg^*(- \otimes^L_X -) \simeq L(g^*(- \otimes_Y -)).$$

Thus, it is enough to show that $g^*(-) \otimes_X g^*(-) \simeq g^*(- \otimes_Y -)$

$$\begin{aligned} g^*(\mathcal{F}_1, \mathcal{F}_2) &\otimes_X g^*(\mathcal{G}_1, \mathcal{G}_2) \\ &= (g^*\mathcal{F}_1 \otimes_X g^*\mathcal{G}_2 \oplus g^*\mathcal{F}_2 \otimes_X g^*\mathcal{G}_1, g^*\mathcal{F}_1 \otimes_X g^*\mathcal{G}_1 \oplus g^*\mathcal{F}_2 \otimes_X g^*\mathcal{G}_2) \\ &\simeq (g^*(\mathcal{F}_1 \otimes_Y \mathcal{G}_2) \oplus g^*(\mathcal{F}_2 \otimes_Y \mathcal{G}_1), g^*(\mathcal{F}_1 \otimes_Y \mathcal{G}_1) \oplus g^*(\mathcal{F}_2 \otimes_Y \mathcal{G}_2)) \\ &= g^*((\mathcal{F}_1, \mathcal{F}_2) \otimes_Y (\mathcal{G}_1, \mathcal{G}_2)). \end{aligned}$$

Clearly, the differentials also match. The result now follows from proposition 7.20. □

PROPOSITION 7.24 (Projection formula). *Let $g : X \to Y$ be an equivariant morphism of smooth G-schemes. Then there are isomorphisms of functors*

$$Rg_*(Lg^*(-) \otimes^L_X -) \simeq - \otimes^L_Y Rg_*(-), \qquad Rg_*(- \otimes^L_X Lg^*(-)) \simeq Rg_*(-) \otimes^L_Y -.$$

PROOF. The proof is similar to the proof for the usual derived category of quasi-coherent sheaves (see for example [**Stacks**, Lemma 20.8.2 or 21.37.1]). We only prove the first formula since the other is similar. Like for quasi-coherent sheaves we can use the adjunction to construct a morphism.

$$- \otimes^L_Y Rg_*(-) \to Rg_*(Lg^*(-) \otimes^L_X -).$$

Indeed, the adjunction morphism $Lg^* Rg_* \to \mathrm{Id}$ induces a morphism

$$Lg^*(- \otimes^L_Y Rg_*(-)) \simeq Lg^*(-) \otimes^L_X Lg^* Rg_*(-) \to Lg^*(-) \otimes^L_X -.$$

We obtain the desired morphism from the above morphism by adjunction.

If $\mathcal{F}$ is flat and $\mathcal{I}$ is injective then $\mathcal{F} \otimes_X \mathcal{I}$ is injective by [**EP**, Lemma 2.5]. Thus, when restricting to $D^{\mathrm{abs}}(\mathrm{QCoh}(Y)_{\mathrm{fl}}, w_Y) \times D^{\mathrm{abs}}(\mathrm{QCoh}(X)_{\mathrm{inj}}, w_X)$ we have

$$\begin{aligned} Rg_*(Lg^*(-) \otimes^L_X -) &= g_*(g^*(-) \otimes_X -), \\ - \otimes^L_Y Rg_*(-) &= - \otimes_Y g_*(-). \end{aligned}$$

Thus, we only need to show that $g_*(g^*\mathcal{F} \otimes_X \mathcal{G}) \simeq \mathcal{F} \otimes_Y g_*(\mathcal{G})$. By [**EP**, Cor. 2.3(h)] the inclusion of $D^{\mathrm{abs}}(\mathrm{QCoh}(Y)_{\mathrm{lf}}, w_Y)$ into $D^{\mathrm{abs}}(\mathrm{QCoh}(Y)_{\mathrm{fl}}, w_Y)$ is an equivalence of categories, so we may assume that $\mathcal{F} \in D^{\mathrm{abs}}(\mathrm{QCoh}(Y)_{\mathrm{lf}}, w_Y)$. By Lemma 7.15 proving that we have an isomorphism can be done locally. Hence, we may assume that $\mathcal{F} = (\mathcal{O}_Y^{\otimes n}, \mathcal{O}_Y^{\otimes m})$. Write $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)$. Then

$$\begin{aligned} g_*(g^*(\mathcal{O}_Y^{\otimes n}, \mathcal{O}_Y^{\otimes m}) \otimes_X (\mathcal{G}_1, \mathcal{G}_2)) &= g_*((\mathcal{O}_X^{\otimes n}, \mathcal{O}_X^{\otimes m}) \otimes_X (\mathcal{G}_1, \mathcal{G}_2)) \\ &= g_*(\mathcal{G}_2^{\otimes n} \oplus \mathcal{G}_1^{\otimes m}, \mathcal{G}_1^{\otimes n} \oplus \mathcal{G}_2^{\otimes m}) \\ &= (g_*\mathcal{G}_2^{\otimes n} \oplus g_*\mathcal{G}_1^{\otimes m}, g_*\mathcal{G}_1^{\otimes n} \oplus g_*\mathcal{G}_2^{\otimes m}). \end{aligned}$$

On the other side we have

$$(\mathcal{O}_Y^{\otimes n}, \mathcal{O}_Y^{\otimes m}) \otimes_Y g_*(\mathcal{G}_1, \mathcal{G}_2)) = (g_*\mathcal{G}_2^{\otimes n} \oplus g_*\mathcal{G}_1^{\otimes m}, g_*\mathcal{G}_1^{\otimes n} \oplus g_*\mathcal{G}_2^{\otimes m}).$$

We check that the differentials match

$$\begin{aligned} g_*(g^* d_i \otimes_X d_i^{\mathcal{G}})(1 \otimes a) &= g_* g^* d_i(1) \otimes a + (-1)^{|1|} 1 \otimes g^* d_i^{\mathcal{G}}(a) \\ &= (d_i(1) \otimes 1) \otimes a + (-1)^{|1|} 1 \otimes 1 \otimes d_i^{\mathcal{G}}(a) \\ &\simeq d_i(1)a + (-1)^{|1|} d_i^{\mathcal{G}}(a) \end{aligned}$$

On the other side we have

$$\begin{aligned} (d_i \otimes_Y g_* d_i^{\mathcal{G}})(1 \otimes a) &= d_i(1) \otimes a + (-1)^{|1|} 1 \otimes g_* d_i^{\mathcal{G}}(a) \\ &\simeq d_i(1)a + (-1)^{|1|} d_i^{\mathcal{G}}(a). \end{aligned}$$

The result now follows from proposition 7.20. □

PROPOSITION 7.25. *Consider a Cartesian square of equivariant morphisms*

$$\begin{array}{ccc} Z & \xrightarrow{h} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[g]{} & W \end{array}$$

Assume that either u or v is flat. Then there is a natural isomorphism between the composition of derived functors

$$Lg^* \circ Rf_* \simeq Rf'_* \circ Lh^*.$$

PROOF. The proof is inspired by the proof of tor-independent base change for quasi-coherent sheaves (see for example [**Stacks**, Lemma 35.17.3]). By [**EP**, Proposition 1.9] derived push-forward for matrix factorizations is right adjoint to pull-back so we can use the same construction as for coherent sheaves to get a canonical base change morphism

$$Lg^*Rf_*\mathcal{M} \to Rf'_*Lh^*\mathcal{M}, \qquad \mathcal{M} \in D^{\mathrm{abs}}(\mathrm{QCoh}(Y, w)).$$

This morphism is the adjoint of the morphism

$$Lf'^*Lg^*Rf_*\mathcal{M} \simeq Lh^*Lf^*Rf_*\mathcal{M} \to Lh^*\mathcal{M},$$

which is induced by the adjunction morphism $Lf^*Rf_*\mathcal{M} \to \mathcal{M}$.

That the base change morphism is an isomorphism can be checked locally by lemma 7.15. Hence, we may assume that S' and S are affine so g_* is exact. We claim that it is enough to show that

$$Rg_*Lg^*Rf_*\mathcal{M} \to Rg_*Rf'_*Lh^*\mathcal{M} \simeq Rf_*Rh_*Lh^*\mathcal{M} \tag{7.4.1}$$

is an isomorphism. The reason is that a morphism α is an isomorphism if and only if $Rg_*\alpha$ is an isomorphism. Indeed, by exactness $\mathrm{Cone}(Rg_*\alpha) = g_*\mathrm{Cone}(\alpha)$ but g_* is just restriction so if $g_*\mathrm{Cone}(\alpha) \simeq 0$ it is because $\mathrm{Cone}(\alpha) \simeq 0$.

To simplify the notation we write $\overline{\mathcal{F}}$ for the matrix factorization $(0, \mathcal{F}, 0, 0)$. Notice that

$$(\mathcal{G}_0, \mathcal{G}_1, d_0, d_1) \otimes_X \overline{\mathcal{F}} = (\mathcal{G}_0 \otimes_X \mathcal{F}, \mathcal{G}_1 \otimes_X \mathcal{F}, d_0, d_1).$$

In particular, $\otimes_X \overline{\mathcal{O}_X}$ is the identity. Using the projection formula we get

$$Rg_*Lg^*\mathcal{N} \simeq Rg_*(Lg^*\mathcal{N} \otimes^L_X \overline{\mathcal{O}_X}) \simeq \mathcal{N} \otimes^L_W Rg_*\overline{\mathcal{O}_X}.$$

By base change we get that h is an affine morphism so it is exact and the same formula works for h. Thus, (7.4.1) can be rewritten as

$$Rf_*\mathcal{M} \otimes^L_W Rg_*\overline{\mathcal{O}_X} \to Rf_*(\mathcal{M} \otimes^L_Y h_*\overline{\mathcal{O}_Z}).$$

Notice that $h_*\overline{\mathcal{O}_Z} = h_*f'^*\overline{\mathcal{O}_X}$. Inserting this we get

$$Rf_*\mathcal{M} \otimes^L_W Rg_*\overline{\mathcal{O}_X} \to Rf_*(\mathcal{M} \otimes^L_Y f^*g_*\overline{\mathcal{O}_X})$$

If $f^*g_*\overline{\mathcal{O}_X} \simeq Lf^*g_*\overline{\mathcal{O}_X}$ this is the morphism in the projection formula and we are done. For f flat this is clear. In the case where g is flat we use the projection formula

$$\begin{aligned} Lf^*g_*\overline{\mathcal{O}_X} &= g_*\overline{\mathcal{O}_X} \otimes^L_W \overline{\mathcal{O}_Y} \\ &\simeq g_*(\overline{\mathcal{O}_X} \otimes^L_X Lg^*\overline{\mathcal{O}_X}) \simeq g_*Lg^*\overline{\mathcal{O}_X}. \end{aligned}$$

Hence, we also have $f^*g_*\overline{\mathcal{O}_X} \simeq Lf^*g_*\overline{\mathcal{O}_X}$ in the case where g is the flat morphism. □

7.5. Convolution

In this section we define a monoidal action on the category from section 7.2. As in [**BR**] the categorical action of the affine braid group will come from convolution with certain elements.

7.5.1. Definition. Let X and Y be smooth G-schemes and V a G-vector space. Let V^* denote the dual vector space with the following G-action

$$g \cdot f(x) = f(g^{-1}x) \qquad \forall g \in G, f \in \mathrm{Hom}(V, k), x \in V.$$

Assume that we have equivariant morphisms $\mu : X \to V^*$ and $\nu : Y \to V$. These determine a G-invariant section $w \in \mathcal{O}(Y \times X)$ by

$$w : Y \times X \to k, \qquad (y, x) \mapsto \mu(x)(\nu(y)).$$

Remark 7.26. In the case we are interested in $X = T^*X$, $Y = \tilde{\mathfrak{g}}$, $V = \mathfrak{g}$, μ is the moment map and ν is the Grothendieck-Springer resolution.

We would like construct an action on $D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times X), w)$ similar to the one for coherent sheaves in [**BR**]. However, if we use the exact same formula then the potentials will not match and we will land in the wrong category. To correct this, we introduce an additional factor of V^*

$$\begin{array}{ccccc} & & Y \times Y \times X & & \\ & \swarrow^{p} & \downarrow^{p_{13}} & \searrow^{p_{23}} & \\ Y \times Y \times V^* & & Y \times X & & Y \times X \end{array}$$

where $p := \mathrm{Id} \times \mathrm{Id} \times \mu$. Define the potential on $Y \times Y \times V^*$ to be the section given by

$$h : Y \times Y \times V^* \to k, \qquad (y_1, y_2, g) \mapsto g(\nu(y_1) - \nu(y_2)).$$

Then we have

$$\begin{aligned} (w \circ p_{23} + h \circ p)(y_1, y_2, x) &= \mu(x)(\nu(y_2)) + \mu(x)(\nu(y_1) - \nu(y_2)) \\ &= \mu(x)(\nu(y_1)) = w \circ p_{13}(y_1, y_2, x), \end{aligned}$$

Thus, we can define the action $*$ to be the composition.

$$\begin{array}{c}
D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times V^*), h) \times D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times X), w) \\
\Big\downarrow {\scriptstyle Lp^* \times p_{23}^*} \\
D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times X), h \circ p) \times D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times X), w \circ p_{23}) \\
\Big\downarrow {\scriptstyle \otimes^L_{Y \times Y \times X}} \\
D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times X), h \circ p + w \circ p_{23} = w \circ p_{13}) \\
\Big\downarrow {\scriptstyle Rp_{13*}} \\
D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times X), w)
\end{array}$$

Now we need to define a monoidal structure on $D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times V^*), h)$. Consider the projection maps

$$\begin{array}{ccccc}
 & & Y \times Y \times Y \times V^* & & \\
 & \swarrow^{p_{12}} & \Big\downarrow {\scriptstyle p_{13}} & \searrow^{p_{23}} & \\
Y \times Y \times V^* & & Y \times Y \times V^* & & Y \times Y \times V^*
\end{array}$$

Notice that

$$\begin{aligned}
(h \circ p_{12} + h \circ p_{23})(y_1, y_2, y_3, g) &= g(\nu(y_1) - \nu(y_2)) + g(\nu(y_2) - \nu(y_3)) \\
&= g(\nu(y_1) - \nu(y_3)) = h \circ p_{13}(y_1, y_2, y_3, g).
\end{aligned}$$

Thus, we can define the convolution product $*$ as the composition.

$$\begin{array}{c}
D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times V^*), h) \times D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times V^*), h) \\
\Big\downarrow {\scriptstyle p_{12}^* \times p_{23}^*} \\
D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times Y \times V^*), h \circ p_{12}) \times D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times Y \times V^*), h \circ p_{23}) \\
\Big\downarrow {\scriptstyle \otimes^L_{Y \times Y \times Y \times V^*}} \\
D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times Y \times V^*), h \circ p_{12} + h \circ p_{23} = h \circ p_{13}) \\
\Big\downarrow {\scriptstyle Rp_{13*}} \\
D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times V^*), h).
\end{array}$$

The proof that the convolution product is associative is similar to the proof of proposition 7.27 below and both are similar to the proof of associativity in chapter 5.

PROPOSITION 7.27. *Let* $\mathcal{M}_1, \mathcal{M}_2 \in D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times V^*), h)$ *and* $\mathcal{N} \in D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times X), w)$. *Then*

$$\mathcal{M}_1 * (\mathcal{M}_2 * \mathcal{N}) \simeq (\mathcal{M}_1 * \mathcal{M}_2) * \mathcal{N}.$$

PROOF. Consider the following cartesian diagram of projections

$$\begin{array}{ccc} Y \times Y \times Y \times X & \xrightarrow{p_{234}} & Y \times Y \times X \\ {\scriptstyle p_{124}}\downarrow & & \downarrow{\scriptstyle p_{13}} \\ Y \times Y \times X & \xrightarrow[p_{23}]{} & Y \times X \end{array}$$

Using the flat base change from proposition 7.25 and the projection formula from proposition 7.24 we get

$$\begin{aligned} &\mathcal{M}_1 * (\mathcal{M}_2 * \mathcal{N}) \\ &\qquad = Rp_{13*}(Lp^*\mathcal{M}_1 \otimes^L_{Y\times Y\times X} p_{23}^* Rp_{13*}(Lp^*\mathcal{M}_2 \otimes^L_{Y\times Y\times X} p_{23}^*\mathcal{N})) \\ &\qquad \simeq Rp_{13*}(Lp^*\mathcal{M}_1 \otimes^L_{Y\times Y\times X} Rp_{124*}p_{234}^*(Lp^*\mathcal{M}_2 \otimes^L_{Y\times Y\times X} p_{23}^*\mathcal{N})) \\ &\qquad \simeq Rp_{13*}Rp_{124*}(p_{124}^*Lp^*\mathcal{M}_1 \otimes^L_{Y\times Y\times Y\times X} p_{234}^*(Lp^*\mathcal{M}_2 \otimes^L_{Y\times Y\times X} p_{23}^*\mathcal{N})) \\ &\qquad \simeq Rp_{13*}Rp_{124*}(p_{124}^*Lp^*\mathcal{M}_1 \otimes^L_{Y\times Y\times Y\times X} p_{234}^*Lp^*\mathcal{M}_2 \otimes^L_{Y\times Y\times Y\times X} p_{234}^*p_{23}^*\mathcal{N}). \end{aligned}$$

Since $p_{13} \circ p_{124} = p_{13} \circ p_{134}$ and $p_{23} \circ p_{234} = p_{23} \circ p_{134}$ we get

$$\begin{aligned} &\mathcal{M}_1 * (\mathcal{M}_2 * \mathcal{N}) \\ &\qquad \simeq Rp_{13*}Rp_{134*}(p_{124}^*Lp^*\mathcal{M}_1 \otimes^L_{Y\times Y\times Y\times X} p_{234}^*Lp^*\mathcal{M}_2 \otimes^L_{Y\times Y\times Y\times X} p_{134}^*p_{23}^*\mathcal{N}) \\ &\qquad \simeq Rp_{13*}(Rp_{134*}(p_{124}^*Lp^*\mathcal{M}_1 \otimes^L_{Y\times Y\times Y\times X} p_{234}^*Lp^*\mathcal{M}_2) \otimes^L_{Y\times Y\times X} p_{23}^*\mathcal{N}). \end{aligned}$$

Set $p_4 := \mathrm{Id} \times \mathrm{Id} \times \mathrm{Id} \times \mu$. Notice that $p \circ p_{124} = \pi_{12} \circ p_4$ and $p \circ p_{234} = \pi_{23} \circ p_4$.

$$\begin{aligned} &\mathcal{M}_1 * (\mathcal{M}_2 * \mathcal{N}) \\ &\qquad \simeq Rp_{13*}(Rp_{134*}(Lp_4^*\pi_{12}^*\mathcal{M}_1 \otimes^L_{Y\times Y\times Y\times X} Lp_4^*\pi_{23}^*\mathcal{M}_2) \otimes^L_{Y\times Y\times X} p_{23}^*\mathcal{N}) \\ &\qquad \simeq Rp_{13*}(Rp_{134*}Lp_4^*(\pi_{12}^*\mathcal{M}_1 \otimes^L_{Y\times Y\times Y\times V^*} \pi_{23}^*\mathcal{M}_2) \otimes^L_{Y\times Y\times X} p_{23}^*\mathcal{N}) \end{aligned}$$

Consider the cartesian diagram

$$\begin{array}{ccc} Y \times Y \times Y \times X & \xrightarrow{p_4} & Y \times Y \times Y \times V^* \\ {\scriptstyle p_{134}}\downarrow & & \downarrow{\scriptstyle \pi_{13}} \\ Y \times Y \times X & \xrightarrow[p]{} & Y \times Y \times V^* \end{array}$$

Using flat base change we get the result

$$\begin{aligned} &\mathcal{M}_1 * (\mathcal{M}_2 * \mathcal{N}) \\ &\qquad \simeq Rp_{13*}(Lp^*R\pi_{13*}(\pi_{12}^*\mathcal{M}_1 \otimes^L_{Y\times Y\times Y\times V^*} \pi_{23}^*\mathcal{M}_2) \otimes^L_{Y\times Y\times X} p_{23}^*\mathcal{N}) \\ &\qquad = (\mathcal{M}_1 * \mathcal{M}_2) * \mathcal{N}. \end{aligned}$$

□

7.5.2. Restriction to an action on the coherent category. The category normally referred to as derived equivariant matrix factorizations is the category $D^{\mathrm{abs}}(\mathrm{Coh}^G(Y \times X), w)$ so we would like our action to restrict to this category. The derived pull-back and tensor products both restricts to the coherent category. However, this is not the case for the push-forward along a non-proper map. Bezrukavnikov and Riche solved the corresponding problem for coherent sheaves by introducing a support condition. A similar notion exists in our setting.

DEFINITION 7.28. (i) The category-theoretical support of an equivariant coherent sheaf $\mathcal{M}$ on X is the minimal closed subset $T \subset X$ such that $\mathcal{M}|_{X \setminus T}$ is absolutely acyclic in $D^{\mathrm{abs}}(\mathrm{Coh}^G(X), w)$.

(ii) For T a closed subset of a scheme X we denote by $D^{\mathrm{abs}}_T(\mathrm{Coh}^G(X), w)$ the quotient category of the homotopy category of coherent matrix factorizations category-theoretically supported inside T by the thick subcategory of matrix factorizations which are absolutely acyclic in $D^{\mathrm{abs}}(\mathrm{Coh}^G(X), w)$.

The category $D^{\mathrm{abs}}_T(\mathrm{Coh}^G(X), w)$ is a full subcategory in $D^{\mathrm{abs}}(\mathrm{Coh}^G(X), w)$. In the non-equivariant setting this is [**EP**, Prop. 1.10(d)] and the proof extends to the equivariant case.

LEMMA 7.29. *Let $\phi : X \to Y$ be a G-equivariant morphism of Noetherian separated G-schemes with enough G-equivariant vector bundles and T a G-invariant closed subset in X.*

(1) If $\phi|_T : T \to Y$ is proper of finite type and S is a closed subset in $\phi(T)$ then $R\phi_*$ restricts to

$$R\phi_* : D^{\mathrm{abs}}_T(\mathrm{Coh}^G(X), w \circ \phi) \to D^{\mathrm{abs}}_S(\mathrm{Coh}^G(Y), w).$$

(2) Let T_1, T_2 be closed subsets of X. Then the tensor product restricts to a functor

$$\otimes^L_X : D^{\mathrm{abs}}_{T_1}(\mathrm{Coh}^G(X), h_1) \times D^{\mathrm{abs}}_{T_2}(\mathrm{Coh}^G(X), h_2) \to D^{\mathrm{abs}}_{T_1 \cap T_2}(\mathrm{Coh}^G(X), h_1 + h_2).$$

(3) Let S be a closed subset of Y. Then the pull-back restricts to a functor

$$L\phi^* : D^{\mathrm{abs}}_S(\mathrm{Coh}^G(Y), h) \to D^{\mathrm{abs}}_{X \setminus \phi^{-1}(Y \setminus S)}(\mathrm{Coh}^G(X), h \circ \phi).$$

PROOF. 1) By [**EP**, lemma 3.5] $R\phi_*$ restricts to a functor

$$R\phi_* : D^{\mathrm{abs}}_T(\mathrm{Coh}^G(X), w \circ \phi) \to D^{\mathrm{abs}}(\mathrm{Coh}^G(Y), w)$$

so we only need to check the support. Since acyclicity is a local property we may assume that X and Y are affine so ϕ_* is exact. For $V \subseteq Y$ open $\phi_*\mathcal{M}(V) = \mathcal{M}(\phi^{-1}(V))$ so if $\mathcal{M}|_U$ is absolutely acyclic then $\phi_*\mathcal{M}|_V$ is absolutely acyclic for $V \subseteq \phi(U)$.

2) It is well-know that the derived tensor product of coherent sheaves on a Noetherian scheme is coherent. A tensor product is acyclic if one of the factors is acyclic and the other is flat. Thus, for an open set $\mathcal{V}$ the matrix factorization

$(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})|_{\mathcal{V}} = \mathcal{M}|_{\mathcal{V}} \otimes_{\mathcal{O}_X|_{\mathcal{V}}} \mathcal{N}|_{\mathcal{V}}$ is acyclic when $\mathcal{V} \subseteq X \backslash T_1$ or $\mathcal{V} \subseteq X \backslash T_2$. Hence if $\mathcal{V} \subseteq X \backslash T_1 \cup X \backslash T_2 = X \backslash (T_1 \cap T_2)$.

3) Let $\mathcal{M} \in D^{\text{abs}}_S(\text{Coh}^G(Y), h)$. Then $\phi^{-1}\mathcal{M}|_{\phi^{-1}(Y\backslash S)}$ is absolutely acyclic. By (2) this implies that $L\phi^*(\mathcal{M}) \in D^{\text{abs}}_{X\backslash \phi^{-1}(Y\backslash S)}(\text{Coh}^G(X), h \circ \phi)$. □

Corollary 7.30. (1) The convolution action restricts to

$$* : D^{\text{abs}}_{Y\times_V Y \times V^*}(\text{Coh}^G(Y \times Y \times V^*), h) \times D^{\text{abs}}(\text{Coh}^G(Y \times X), w) \to D^{\text{abs}}(\text{Coh}^G(Y \times X), w)$$

(2) The category $D^{\text{abs}}_{Y\times_V Y \times V^*}(\text{Coh}^G(Y \times Y \times V^*), h)$ is monoidal.

Proof. 1) By the lemma the functors in the convolution all restrict to the full subcategory of coherent matrix factorizations.

$$\begin{array}{c}
D^{\text{abs}}_{Y\times_V Y \times V^*}(\text{Coh}^G(Y \times Y \times V^*), h) \times D^{\text{abs}}(\text{Coh}^G(Y \times X), w) \\
\Big\downarrow {\scriptstyle p^* \times p_{23}^*} \\
D^{\text{abs}}_{Y\times_V Y \times X}(\text{Coh}^G(Y \times Y \times X), h \circ p) \times D^{\text{abs}}(\text{Coh}^G(Y \times Y \times X), w \circ p_{23}) \\
\Big\downarrow {\scriptstyle \otimes^L_{Y\times Y \times X}} \\
D^{\text{abs}}_{Y\times_V Y \times X}(\text{Coh}^G(Y \times Y \times X), w \circ p_{13}) \\
\Big\downarrow {\scriptstyle Rp_{13*}} \\
D^{\text{abs}}(\text{Coh}^G(Y \times X), w)
\end{array}$$

2) In the same way we get

$$\begin{array}{c}
D^{\text{abs}}_{Y\times_V Y \times V^*}(\text{Coh}^G(Y \times Y \times V^*), h) \times D^{\text{abs}}_{Y\times_V Y \times V^*}(\text{Coh}^G(Y \times Y \times V^*), h) \\
\Big\downarrow {\scriptstyle p_{12}^* \times p_{23}^*} \\
D^{\text{abs}}_{Y\times_V Y \times Y \times V^*}(\text{Coh}^G(Y \times Y \times Y \times V^*), h \circ p_{12}) \\
\times D^{\text{abs}}_{Y\times Y\times_V Y \times V^*}(\text{Coh}^G(Y \times Y \times Y \times V^*), h \circ p_{23}) \\
\Big\downarrow {\scriptstyle \otimes^L_{Y\times Y \times Y \times V^*}} \\
D^{\text{abs}}_{Y\times_V Y \times_V \times Y \times V^*}(\text{Coh}^G(Y \times Y \times Y \times V^*), h \circ p_{13}) \\
\Big\downarrow {\scriptstyle Rp_{13*}} \\
D^{\text{abs}}_{Y\times_V Y \times V^*}(\text{Coh}^G(Y \times Y \times V^*), h).
\end{array}$$

Since all categories are full subcategories of the ones involved in the convolution in the quasi-coherent setting the restricted convolutions are also associative. □

7.6. Koszul duality

The main theorem 7.8 would follow from [**BR**] if we can construct a monoidal functor

$$D(\mathrm{Coh}^G(Y \times_V^R Y)) \to D^{\mathrm{abs}}_{Y\times_V Y \times V^*}(\mathrm{Coh}^G(Y \times Y \times V^*), h),$$

This turns out to be a bit hard so we will only construct the functor on a full subcategory which contains the generators of the braid group action. This functor is called Koszul duality, but it is not the functor from the previous chapter and no result from that chapter will be used. We prove all the properties of this functor needed for our proof.

7.6.1. Definition of a Koszul duality functor. Recall that $D(\mathrm{Coh}^G(Y \times_V^R Y))$ can be expressed as the normal derived category of a DG-category in the following way. Consider the function

$$\rho : Y \times Y \to V, \qquad (y_1, y_2) \mapsto \nu(y_2) - \nu(y_1).$$

This induces a map $\rho^\sharp : V^* \to \mathcal{O}_{Y\times Y}$. Since $Y \times_V Y = \rho^{-1}(0)$ we have a resolution

$$\cdots \longrightarrow \mathcal{O}_{Y\times Y} \otimes \bigwedge^2 V^* \longrightarrow \mathcal{O}_{Y\times Y} \otimes V^* \xrightarrow{\theta} \mathcal{O}_{Y\times Y} \to \mathcal{O}_{Y\times_V Y} \to 0,$$

where $\theta(f \otimes s) = f\rho^\sharp(s)$ and the differential is extended by Leibniz rule. Thus,

$$D(\mathrm{Coh}^G(Y \times_V^R Y)) \simeq D(DG(\mathcal{O}_{Y\times Y} \otimes \Lambda(V^*) - \mathrm{mod}^G))$$

The first step in the construction is to define a functor

$$DG(\mathcal{O}_{Y\times Y} \otimes \Lambda(V^*) - \mathrm{mod}^G) \to CDG(\mathcal{O}_{Y\times Y} \otimes \mathrm{Sym}(V), h) - \mathrm{mod}^G.$$

One way to construct such a functor is to tensor with a $\mathcal{O}_{Y\times Y} \otimes \Lambda(V^*) \otimes \mathrm{Sym}(V)$-bimodule with differential d satisfying $d^2 = h$.

DEFINITION 7.31. Pick a basis $(t_1, \ldots, t_n)$ for V and a dual basis $(\xi_1, \ldots, \xi_n)$ for V^*. We define a grading with $\mathcal{O}_{Y\times Y}$ in degree 0, ξ_i in degree -1 and t_i in degree 2. Consider the complex K with terms

$$K^m := \bigoplus_{m=2i-j} \mathcal{O}_{Y\times Y} \otimes \Lambda^j(\xi_1, \ldots, \xi_n) \otimes \mathrm{Sym}^i(t_1, \ldots, t_n)$$

and differential

$$d(f \otimes x \otimes y) := f \otimes d_\Lambda(x) \otimes y + \sum_{k=1}^n f \otimes \xi_i x \otimes t_i y,$$

where d_Λ is the usual differential on Λ. We call K the Koszul complex.

LEMMA 7.32. *The Koszul complex is in* $CDG(\mathcal{O}_{Y\times Y} \otimes \mathrm{Sym}(V), h) - \mathrm{mod}^G$

PROOF. The only thing to check is that $d^2 = h$.

$$\begin{aligned} d^2(f\otimes x\otimes y) &= f\otimes d_\Lambda^2 x\otimes y + \sum_{k=1}^n f\otimes d_\Lambda(\xi_k x)\otimes t_k y + \sum_{k=1}^n f\otimes \xi_k(d_\Lambda x)\otimes t_k y \\ &\quad + \sum_{k,\ell=1}^n f\otimes \xi_k\xi_\ell x\otimes t_k t_\ell y \\ &= \sum_{k=1}^n f\otimes (d_\Lambda(\xi_k x)+\xi_k d_\Lambda x)\otimes t_k y \\ &= \sum_{k=1}^n f\otimes (\rho^\sharp(\xi_k)x-\xi_k d_\Lambda x+\xi_k d_\Lambda x)\otimes t_k y \\ &= \sum_{k=1}^n f\otimes \rho^\sharp(\xi_k)x\otimes t_k y. \end{aligned}$$

By definition we have $h : Y\times Y\times V^* \xrightarrow{\rho\times \mathrm{Id}} V\times V^* \xrightarrow{\langle\,,\rangle} k$. Hence,

$$h^\sharp = (\rho^\sharp\otimes \mathrm{Id})\circ\langle\,,\,\rangle^\sharp = \sum_{k=1}^n \rho^\sharp(\xi_k)\otimes t_k.$$

So $d^2 = h$. □

Using the lemma we can define the functor

$$\kappa : DG(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*)-\mathrm{mod}^G)\to CDG(\mathcal{O}_{Y\times Y}\otimes\mathrm{Sym}(V),h)-\mathrm{mod}^G,$$
$$\mathcal{M}\mapsto \mathcal{M}\otimes_{\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*)} K\simeq \mathcal{M}\otimes\mathrm{Sym}(V).$$

To make it $\mathbb{Z}/2$-graded we take the direct sum of the all the odd terms and all the even terms. Notice that

$$(\mathcal{M}\otimes_{\Lambda(V^*)} K)^m = \bigoplus_{m=i+2s-r}\mathcal{M}^i\otimes_{\Lambda(V^*)}\Lambda^r(V^*)\otimes\mathrm{Sym}^s(V)$$

So $(\mathcal{M}\otimes_{\Lambda(V^*)} K)^{\mathrm{odd}}\simeq\mathcal{M}^{\mathrm{odd}}\otimes\mathrm{Sym}(V)$ and likewise for the even part.

LEMMA 7.33. *κ descends to the homotopy categories.*

PROOF. Let $f:\mathcal{M}\to\mathcal{N}$ be a homotopy equivalence in $DG(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*)-\mathrm{mod}^G)$. We want to show that $\prod f\otimes\mathrm{Id}:\kappa(\mathcal{M})\to\kappa(\mathcal{N})$ is a homotopy equivalence in $H^0(\mathrm{QCoh}^G(Y\times Y\times V^*),h)$, i.e. the diagram

$$\begin{array}{ccc} \mathcal{M}^{\mathrm{odd}}\otimes\mathrm{Sym}(V) & \xrightarrow{\prod f\otimes\mathrm{Id}} & \mathcal{N}^{\mathrm{odd}}\otimes\mathrm{Sym}(V) \\ {\scriptstyle d_{\kappa(\mathcal{M})}}\downarrow\uparrow{\scriptstyle d_{\kappa(\mathcal{M})}} & & {\scriptstyle d_{\kappa(\mathcal{N})}}\downarrow\uparrow{\scriptstyle d_{\kappa(\mathcal{N})}} \\ \mathcal{M}^{\mathrm{even}}\otimes\mathrm{Sym}(V) & \xrightarrow{\prod f\otimes\mathrm{Id}} & \mathcal{N}^{\mathrm{even}}\otimes\mathrm{Sym}(V) \end{array}$$

is commutative.

$$\begin{aligned}(f\otimes \mathrm{Id})\circ d_{\kappa(\mathcal{M})}(m\otimes s) &= f(d_{\mathcal{M}}m)\otimes s+\sum_{k=1}^{n} f(\xi_k m)\otimes t_k s\\ &= d_{\mathcal{N}}f(m)\otimes s+\sum_{k=1}^{n}\xi_k f(m)\otimes t_k s\\ &= d_{\kappa(\mathcal{N})}\circ (f\otimes \mathrm{Id})(m\otimes s).\end{aligned}$$ □

We want a functor into the absolute derived category of coherent sheaves so we cannot take infinite direct sums of coherent modules. To avoid this we restrict our functor to the following category.

DEFINITION 7.34. Let $\mathcal{A}$ be a dg-scheme. The category $\mathrm{Perf}^G(\mathcal{A})$ is the full subcategory of the dg-category of G-equivariant $\mathcal{A}$-dg-modules whose objects are finite complexes of locally free modules of finite rank.

By the lemma we get a functor

$$\begin{gathered}\kappa : \mathrm{H}^0(\mathrm{Perf}^G(\mathcal{O}_{Y\times Y}\otimes \Lambda(V^*)))\to D^{\mathrm{abs}}(\mathrm{Coh}^G(Y\times Y\times V^*),h)\\ \mathcal{M}\mapsto (\mathcal{M}^{\mathrm{odd}}\otimes \mathrm{Sym}(V),\mathcal{M}^{\mathrm{even}}\otimes \mathrm{Sym}(V),d,d),\\ d(m\otimes s):=d_{\mathcal{M}}m\otimes s+\sum_{k=1}^{n}\xi_k m\otimes t_k s.\end{gathered}$$

We want the functor to descend to the derived category $D_{\mathrm{Perf}}(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*)-\mathrm{mod}^G)$. By lemma 7.15 checking that something is acyclic can be done locally so we may assume that Y is affine.

LEMMA 7.35. *Assume that Y is affine. Then there is an equivalence of triangulated categories*

$$H^0(\mathrm{Perf}^G(\mathcal{O}_{Y\times Y}\otimes \Lambda(V^*)))\simeq D_{\mathrm{Perf}}(\mathcal{O}_{Y\times Y}\otimes \Lambda(V^*)-\mathrm{mod}^G).$$

PROOF. When Y is affine the categories reduce to categories of modules over a DG-algebra. Set $A:=\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*)$. The result would follow from showing that objects in $\mathrm{Perf}^G(A)$ are projective in $A-\mathrm{mod}^G$. Equivalently,

$$\mathrm{Ext}^1_{A-\mathrm{mod}^G}(P,M)=0$$

for all $P\in\mathrm{Perf}^G(A)$ and $M\in A-\mathrm{mod}^G$. Recall that

$$\mathrm{Ext}^1_{A-\mathrm{mod}^G}(P,M)=(\mathrm{Ext}^1_{A-\mathrm{mod}}(P,M))^G.$$

The result now follows from the fact that perfect complexes are projective in $A-\mathrm{mod}$. □

Thus, we have constructed a functor

$$\kappa : D_{\mathrm{Perf}}(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*)-\mathrm{mod}^G)\to D^{\mathrm{abs}}(\mathrm{Coh}^G(Y\times Y\times V^*),h).$$

However, the full subcategory $D_{\mathrm{Perf}}(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*)-\mathrm{mod}^G)$ is not preserved by convolution since the derived push-forward along a non-proper maps does not send Perf to Perf. To fix this, we restrict to the full subcategory whose cohomology over $Y\times Y$ is set-theoretically supported on $Y\times_V Y$, so that the final projection is proper on the support.

$$\kappa : D_{\mathrm{Perf},Y\times_V Y}(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*)-\mathrm{mod}^G)\to D^{\mathrm{abs}}(\mathrm{Coh}^G(Y\times Y\times V^*),h).$$

7.6.2. Compatibility with convolution. To prove that κ commutes with convolution we need some preparatory lemmas.

LEMMA 7.36. *Let* $\mathcal{M},\mathcal{N}\in D_{\mathrm{Perf}}(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*)-\mathrm{mod}^G)$. *Then* $\kappa(\mathcal{M})\boxtimes\kappa(\mathcal{N})\simeq\kappa(\mathcal{M}\boxtimes\mathcal{N})$.

PROOF. The functor $\boxtimes$ is clearly exact and takes Perf $\times$ Perf to Perf. First we check that the matrix factorizations agree on terms

$$\begin{aligned}
&\kappa(\mathcal{M})\boxtimes\kappa(\mathcal{N})\\
&=\left(\begin{array}{c}\mathcal{M}^{\mathrm{odd}}\otimes\mathrm{Sym}(V)\boxtimes\mathcal{N}^{\mathrm{even}}\otimes\mathrm{Sym}(V)\oplus\mathcal{M}^{\mathrm{even}}\otimes\mathrm{Sym}(V)\boxtimes\mathcal{N}^{\mathrm{odd}}\otimes\mathrm{Sym}(V)\\ \uparrow\quad\downarrow\\ \mathcal{M}^{\mathrm{odd}}\otimes\mathrm{Sym}(V)\boxtimes\mathcal{N}^{\mathrm{odd}}\otimes\mathrm{Sym}(V)\oplus\mathcal{M}^{\mathrm{even}}\otimes\mathrm{Sym}(V)\boxtimes\mathcal{N}^{\mathrm{even}}\otimes\mathrm{Sym}(V)\end{array}\right)\\
&=\left(\begin{array}{c}(\mathcal{M}^{\mathrm{odd}}\boxtimes\mathcal{N}^{\mathrm{even}}\oplus\mathcal{M}^{\mathrm{even}}\boxtimes\mathcal{N}^{\mathrm{odd}})\otimes\mathrm{Sym}(V)\otimes\mathrm{Sym}(V)\\ \uparrow\quad\downarrow\\ (\mathcal{M}^{\mathrm{odd}}\boxtimes\mathcal{N}^{\mathrm{odd}}\oplus\mathcal{M}^{\mathrm{even}}\boxtimes\mathcal{N}^{\mathrm{even}})\otimes\mathrm{Sym}(V)\otimes\mathrm{Sym}(V)\end{array}\right)\\
&=\left(\begin{array}{c}(\mathcal{M}\boxtimes\mathcal{N})^{\mathrm{odd}}\otimes\mathrm{Sym}(V)\otimes\mathrm{Sym}(V)\\ \uparrow\quad\downarrow\\ (\mathcal{M}\boxtimes\mathcal{N})^{\mathrm{even}}\otimes\mathrm{Sym}(V)\otimes\mathrm{Sym}(V)\end{array}\right)\\
&=\kappa(\mathcal{M}\boxtimes\mathcal{N}).
\end{aligned}$$

Now we check the differentials

$$d_{\kappa(\mathcal{M})\boxtimes\kappa(\mathcal{N})}(a\otimes r\boxtimes b\otimes s)=d(a\otimes r)\boxtimes b\otimes s+(-1)^{|a|}a\otimes r\boxtimes d(b\otimes s)$$

$$=(d_{\mathcal{M}}a\otimes r+\sum_k\xi_k a\otimes t_k r)\boxtimes b\otimes s+(-1)^{|a|}a\otimes r\boxtimes(d_{\mathcal{N}}b\otimes s+\sum_k\xi_k b\otimes t_k s)$$

$$d_{\kappa(\mathcal{M}\boxtimes\mathcal{N})}(a\boxtimes b\otimes r\otimes s)$$

$$=d(a\boxtimes b)\otimes r\otimes s+\sum_k(\xi_k,0)\cdot(a\boxtimes b)\otimes(t_k,0)\cdot(r\otimes t)$$

$$+\sum_k (0,\xi_k)\cdot(a\boxtimes b)\otimes(0,t_k)\cdot(r\otimes s)$$

$$= d_{\mathcal{M}}a\boxtimes b\otimes r\otimes s + (-1)^{|a|}a\boxtimes d_{\mathcal{N}}b\otimes r\otimes s$$

$$+\sum_k(-1)^{|a||0|}\xi_k a\boxtimes b\otimes t_k r\otimes s+\sum_k(-1)^{|a||\xi_k|}a\boxtimes \xi_k b\otimes r\otimes t_k s$$

$$= d_{\mathcal{M}}a\boxtimes b\otimes r\otimes s + (-1)^{|a|}a\boxtimes d_{\mathcal{N}}b\otimes r\otimes s$$
$$+\sum_k(-1)^{|a|}\xi_k a\boxtimes b\otimes t_k r\otimes s+\sum_k a\boxtimes \xi_k b\otimes r\otimes t_k s \qquad \square$$

LEMMA 7.37. *Let $\theta : Z_1 \to Z_2$ be a morphism of schemes.*

(1) *If Z_3 is a closed subscheme of Z_1 and θ restricted to Z_3 is proper then the following diagram is commutative*

$$\begin{array}{ccc} D_{\mathrm{Perf}}(\mathcal{O}_{Z_2}\otimes\Lambda(V)-\mathrm{mod}^G) & \xleftarrow{(\theta\times\mathrm{Id})^\sharp_*} & D_{\mathrm{Perf},Z_3}(\mathcal{O}_{Z_1}\otimes\Lambda(V)-\mathrm{mod}^G) \\ \downarrow{\scriptstyle\kappa_1} & & \downarrow{\scriptstyle\kappa_2} \\ D^{\mathrm{abs}}(\mathrm{QCoh}^G(Z_2\times V),h_2) & \xleftarrow{R(\theta\times\mathrm{Id})^f_*} & D^{\mathrm{abs}}(\mathrm{QCoh}^G(Z_1\times V),h_1) \end{array}$$

(2) *If θ is flat then the following diagram is commutative*

$$\begin{array}{ccc} D_{\mathrm{Perf}}(\mathcal{O}_{Z_2}\otimes\Lambda(V)-\mathrm{mod}^G) & \xrightarrow{(\theta\times\mathrm{Id})^{\sharp *}} & D_{\mathrm{Perf}}(\mathcal{O}_{Z_1}\otimes\Lambda(V)-\mathrm{mod}^G) \\ \downarrow{\scriptstyle\kappa_1} & & \downarrow{\scriptstyle\kappa_2} \\ D^{\mathrm{abs}}(\mathrm{QCoh}^G(Z_2\times V),h_2) & \xrightarrow{(\theta\times\mathrm{Id})^*} & D^{\mathrm{abs}}(\mathrm{QCoh}^G(Z_1\times V),h_1) \end{array}$$

PROOF. 1) Since θ is proper on the support the functor $(\theta\times\mathrm{Id})^\sharp_*$ sends Perf to Perf so the composition is well-defined. Proving that $\kappa_1(\theta\times\mathrm{Id})^\sharp_*(\mathcal{M})\simeq(\theta\times\mathrm{Id})_*\kappa_2(\mathcal{M})$ can be done locally so we may assume that all schemes are affine in which case the push-forward is exact.

$$\kappa_2(\theta\times\mathrm{Id})^\sharp_*(\mathcal{M}) = \begin{pmatrix} ((\theta\times\mathrm{Id})^\sharp_*\mathcal{M})^{\mathrm{odd}}\otimes\mathrm{Sym}(V^*) \\ \uparrow\quad\downarrow \\ ((\theta\times\mathrm{Id})^\sharp_*\mathcal{M})^{\mathrm{even}}\otimes\mathrm{Sym}(V^*) \end{pmatrix}$$
$$= \begin{pmatrix} (\theta\times\mathrm{Id})_*(\mathcal{M}^{\mathrm{odd}}\otimes\mathrm{Sym}(V^*)) \\ \uparrow\quad\downarrow \\ (\theta\times\mathrm{Id})_*(\mathcal{M}^{\mathrm{even}}\otimes\mathrm{Sym}(V^*)) \end{pmatrix}$$

$$\simeq (\theta \times \mathrm{Id})_* \kappa_1(\mathcal{M}).$$

The functor $(\theta \times \mathrm{Id})^\sharp_*$ does not change the action of $\Lambda(V)$ so

$$\begin{aligned} d_{\kappa_2(\theta\times \mathrm{Id})^\sharp_*(\mathcal{M})}(m\otimes s) &= dm\otimes s + \sum_{k=1}^n \xi_k m \otimes t_k s \\ &= d_{(\theta\times \mathrm{Id})_*\kappa_1(\mathcal{M})}(m\otimes s). \end{aligned}$$

2) Pull-back preserve Perf and $(\theta \times \mathrm{Id})^*$ is exact so we have.

$$\begin{aligned} \kappa_1(\theta\times \mathrm{Id})^{\sharp *}(\mathcal{M}) &= \begin{pmatrix} (\mathcal{M}\otimes_{Z_2}\mathcal{O}_{Z_1})^{\mathrm{odd}}\otimes \mathrm{Sym}(V^*) \\ d\uparrow \quad \downarrow d \\ (\mathcal{M}\otimes_{Z_2}\mathcal{O}_{Z_1})^{\mathrm{even}}\otimes \mathrm{Sym}(V^*) \end{pmatrix} \\ &\simeq \begin{pmatrix} \mathcal{M}^{\mathrm{odd}}\otimes_{Z_2}\mathcal{O}_{Z_1}\otimes \mathrm{Sym}(V^*) \\ d\uparrow \quad \downarrow d \\ \mathcal{M}^{\mathrm{even}}\otimes_{Z_2}\mathcal{O}_{Z_1}\otimes \mathrm{Sym}(V^*) \end{pmatrix} \\ &\simeq (\theta\times \mathrm{Id})^*\kappa_2(\mathcal{M}). \end{aligned}$$

For differentials we have

$$\begin{aligned} d_{\kappa_1(\theta\times \mathrm{Id})^{\sharp *}(\mathcal{M})}(m\otimes f\otimes s) &= d_{\mathcal{M}}(m)\otimes f\otimes s + \sum_{k=1}^n \xi_k m\otimes f\otimes t_k s \\ &= d_{(\theta\times \mathrm{Id})^*\kappa_2(\mathcal{M})}(m\otimes f\otimes s). \end{aligned}$$

This finishes the proof. □

LEMMA 7.38. *Let $f: V \hookrightarrow V\oplus W$ be an inclusion of vector spaces. Then the following diagram is commutative*

$$\begin{array}{ccc} D_{\mathrm{Perf}}(\mathcal{O}_Z\otimes\Lambda(V)\otimes\Lambda(W)-\mathrm{mod}^G) & \xrightarrow{(\mathrm{Id}\times f)^\sharp_*} & D_{\mathrm{Perf}}(\mathcal{O}_Z\otimes\Lambda(V)-\mathrm{mod}^G) \\ \downarrow \kappa_1 & & \downarrow \kappa_2 \\ D^{\mathrm{abs}}(\mathrm{QCoh}^G(Z\times V\times W),h_1) & \xrightarrow{(\mathrm{Id}\times f)^*} & D^{\mathrm{abs}}(\mathrm{QCoh}^G(Z\times V),h_2) \end{array}$$

PROOF. Since f is injective and $\Lambda(W)$ is a finite complex with a finite dimensional vector space in each degree the functor $(\mathrm{Id}\times f)^\sharp_*$ preserves Perf. On the level of components we have

$$L(\mathrm{Id}\times f)^*\kappa_1(\mathcal{M}) = \begin{pmatrix} \mathcal{M}^{\mathrm{odd}}\otimes \mathrm{Sym}(V^*)\otimes \mathrm{Sym}(W^*)\otimes^L_{\mathrm{Sym}(W^*)\otimes \mathrm{Sym}(V^*)}\mathrm{Sym}(V^*) \\ \uparrow \quad \downarrow \\ \mathcal{M}^{\mathrm{even}}\otimes \mathrm{Sym}(V^*)\otimes \mathrm{Sym}(W^*)\otimes^L_{\mathrm{Sym}(W^*)\otimes \mathrm{Sym}(V^*)}\mathrm{Sym}(V^*) \end{pmatrix}$$

$$\simeq \begin{pmatrix} \mathcal{M}^{\mathrm{odd}} \otimes \mathrm{Sym}(V^*) \\ \uparrow \quad \downarrow \\ \mathcal{M}^{\mathrm{even}} \otimes \mathrm{Sym}(V^*) \end{pmatrix}$$
$$= \kappa_2(\mathrm{Id} \times f)^{\sharp}_*(\mathcal{M}).$$

Let $(\xi_k^V, t_k^V)_{k=1}^n$ be a pair of a basis for V and its dual basis and $(\xi_l^W, t_l^W)_{l=1}^m$ a pair of a basis for W and its dual basis. Then we have

$$\begin{aligned} & d_{L(\mathrm{Id} \times f)^* \kappa_1(\mathcal{M})}(m \otimes 1 \otimes 1 \otimes s) \\ &= d_{\mathcal{M}} m \otimes 1 \otimes 1 \otimes s + \sum_{k=1}^n \xi_k^V m \otimes 1 \otimes 1 \otimes t_k^V s + \sum_{l=1}^m \xi_l^W m \otimes 1 \otimes t_l^W \otimes s \\ &= d_{\mathcal{M}} m \otimes 1 \otimes 1 \otimes s + \sum_{k=1}^n \xi_k^V m \otimes 1 \otimes 1 \otimes t_k^V s + \sum_{l=1}^m \xi_l^W m \otimes 1 \otimes 1 \otimes f^{\sharp}(t_l^W) s \\ &= d_{\mathcal{M}} m \otimes 1 \otimes 1 \otimes s + \sum_{k=1}^n \xi_k^V m \otimes 1 \otimes 1 \otimes t_k^V s \\ &\simeq d_{\mathcal{M}} m \otimes s + \sum_{k=1}^n \xi_k^V m \otimes t_k^V s \\ &= d_{\kappa_2(\mathrm{Id} \times f)^{\sharp}_*(\mathcal{M})}(m \otimes s). \end{aligned}$$

This finishes the proof. □

PROPOSITION 7.39. *The functor κ is monoidal.*

PROOF. Consider the derived projections

$$\begin{array}{ccccc} & & Y \times_V^R Y \times_V^R Y & & \\ & \swarrow^{\bar{p}_{12}} & \downarrow{\scriptstyle \bar{p}_{13}} & \searrow^{\bar{p}_{23}} & \\ Y \times_V^R Y & & Y \times_V^R Y & & Y \times_V^R Y \end{array}$$

Recall that in [**BR**] the convolution product for $\mathcal{M}, \mathcal{N} \in D_{\mathrm{Perf}, Y \times_V Y}(\mathcal{O}_{Y \times Y} \otimes \Lambda(V^*) - \mathrm{mod}^G)$ is defined as.

$$\mathcal{M} * \mathcal{N} := R\bar{p}_{13*}(L\bar{p}_{12}^* \otimes_{Y \times_V^R Y \times_V^R Y} L\bar{p}_{23}^*)$$

On the level of DG-schemes this translates into the following picture

$$
\begin{array}{ccccc}
 & & \mathcal{O}_{Y\times Y\times Y}\otimes\Lambda(V^*)\otimes\Lambda(V^*) & & \\
 & \nearrow^{q_{12}} & \uparrow{\scriptstyle q_{13}} & \nwarrow^{q_{23}} & \\
\mathcal{O}_{Y\times Y\times Y}\otimes\Lambda(V^*) & & \mathcal{O}_{Y\times Y\times Y}\otimes\Lambda(V^*) & & \mathcal{O}_{Y\times Y\times Y}\otimes\Lambda(V^*) \\
\uparrow{\scriptstyle p_{12}^*} & & \uparrow{\scriptstyle p_{13}^*} & & \uparrow{\scriptstyle p_{23}^*} \\
\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*) & & \mathcal{O}_{Y\times Y}\otimes\Lambda(V^*) & & \mathcal{O}_{Y\times Y}\otimes\Lambda(V^*)
\end{array}
$$

Explicitly, the maps are given by

$$
\begin{gathered}
q_{12}(f\otimes v) = f\otimes v\otimes 1, \qquad q_{23}(f\otimes v) = f\otimes 1\otimes v, \\
q_{13}(f\otimes v) = f\otimes v\otimes 1 + f\otimes 1\otimes v.
\end{gathered}
$$

The map p_{ij}^* is the pull-back corresponding to the projection to the (i,j)'th factor of Y and identity on V^*. The formula becomes

$$
\begin{aligned}
\mathcal{M}*\mathcal{N} &= Rp_{13*}Rq_{13*}(q_{12}^*p_{12}^*\mathcal{M}\otimes^L_{\mathcal{O}_{Y\times Y\times Y}\otimes\Lambda(V^*)\otimes\Lambda(V^*)} q_{23}^*p_{23}^*\mathcal{N}) \\
&\simeq Rp_{13*}Rq_{13*}((\Lambda(V^*)\otimes p_{12}^*\mathcal{M})\otimes^L_{\mathcal{O}_{Y\times Y\times Y}\otimes\Lambda(V^*)\otimes\Lambda(V^*)} (p_{23}^*\mathcal{N}\otimes\Lambda(V^*))) \\
&\simeq Rp_{13*}Rq_{13*}(p_{12}^*\mathcal{M}\otimes^L_{\mathcal{O}_{Y\times Y\times Y}} p_{23}^*\mathcal{N}) \\
&\simeq Rp_{13*}Rq_{13*}L\Delta_Y^*(p_{12}^*\mathcal{M}\boxtimes p_{23}^*\mathcal{N}),
\end{aligned}
$$

where Δ_Y is the diagonal embedding.

The corresponding diagram on the Koszul dual side is the following

$$
\begin{array}{ccccc}
 & & Y\times Y\times Y\times V^*\times V^* & & \\
 & \nearrow^{\psi_{12}} & \uparrow{\scriptstyle \psi_{13}} & \nwarrow^{\psi_{23}} & \\
Y\times Y\times Y\times V^* & & Y\times Y\times Y\times V^* & & Y\times Y\times Y\times V^* \\
\downarrow{\scriptstyle \pi_{12}} & & \downarrow{\scriptstyle \pi_{13}} & & \downarrow{\scriptstyle \pi_{23}} \\
Y\times Y\times V^* & & Y\times Y\times V^* & & Y\times Y\times V^*
\end{array}
$$

The morphisms in the Koszul dual picture are

$$
\begin{gathered}
\psi_{12}(y_1,y_2,y_3,v) = (y_1,y_2,y_3,v,0), \qquad \psi_{23}(y_1,y_2,y_3,v) = (y_1,y_2,y_3,0,v) \\
\psi_{13}(y_1,y_2,y_3,v) = (y_1,y_2,y_3,v,v).
\end{gathered}
$$

The π_{ij} is projection to the (i,j)'th factor times identity. Using the above lemmas we get

$$\begin{aligned}
\kappa(\mathcal{M} * \mathcal{N}) &\simeq R\pi_{13*}\kappa(Rq_{13*}L\Delta_Y^*(p_{12}^*\mathcal{M} \boxtimes p_{23}^*\mathcal{N})) \\
&\simeq R\pi_{13*}L\psi_{13}^*\kappa(L\Delta_Y^*(p_{12}^*\mathcal{M} \boxtimes p_{23}^*\mathcal{N})) \\
&\simeq R\pi_{13*}L\psi_{13}^*L\Delta_Y^*\kappa(p_{12}^*\mathcal{M} \boxtimes p_{23}^*\mathcal{N}) \\
&\simeq R\pi_{13*}L(\Delta_Y \circ \psi_{13})^*(\pi_{12}^*\kappa(\mathcal{M}) \boxtimes \pi_{23}^*\kappa(\mathcal{N}))
\end{aligned}$$

Notice that $\Delta_Y \circ \psi_{13} : Y \times Y \times Y \times V^* \to Y \times Y \times Y \times Y \times Y \times Y \times V^* \times V^*$ is the diagonal embedding Δ so

$$\begin{aligned}
\kappa(\mathcal{M} * \mathcal{N}) &\simeq R\pi_{13*}L\Delta^*(\pi_{12}^*\kappa(\mathcal{M}) \boxtimes \pi_{23}^*\kappa(\mathcal{N})) \\
&\simeq R\pi_{13*}(\pi_{12}^*\kappa(\mathcal{M}) \otimes^L_{Y \times Y \times Y \times V^*} \pi_{23}^*\kappa(\mathcal{N})) \\
&= \kappa(\mathcal{M}) * \kappa(\mathcal{N}).
\end{aligned}$$

This finishes the proof. $\square$

PROPOSITION 7.40. *The functor κ takes the unit to the unit.*

PROOF. The unit in $\mathcal{K}_{Y,V}$ is the structure sheaf of the diagonal $\mathcal{O}_{\Delta Y}$ sitting in degree 0.

$$\kappa(\mathcal{O}_{\Delta Y}) = \left(\begin{array}{c} 0 \\ \uparrow \downarrow \\ \mathcal{O}_{\Delta Y} \otimes \mathrm{Sym}(V) \end{array} \right)$$

This is a push-forward along the inclusion

$$i : Y \times_Y Y \times V^* \hookrightarrow Y \times Y \times V^*.$$

Let $\mathcal{M} \in D^{\mathrm{abs}}(\mathrm{Coh}^G(Y \times Y \times V^*), h)$. We need to check that $\kappa(\mathcal{O}_{\Delta Y}) * \mathcal{M} \simeq \mathcal{M} \simeq \mathcal{M} * \kappa(\mathcal{O}_{\Delta Y})$. Consider the following commutative diagram

$$\begin{array}{ccccc}
Y \times Y \times V^* & \xrightarrow{\sim} & Y \times_Y Y \times Y \times V^* & \xrightarrow{p_1} & Y \times_Y Y \times V^* \simeq Y \times V^* \\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle i_4} & & \downarrow{\scriptstyle i} \\
Y \times Y \times V^* & \underset{p_{13}}{\overset{p_{23}}{\longleftarrow}} & Y \times Y \times Y \times V^* & \xrightarrow[p_{12}]{} & Y \times Y \times V^*
\end{array}$$

Using flat base change and the projection formula we get

$$\begin{aligned}
\kappa(\mathcal{O}_{\Delta Y}) * \mathcal{M} &= Rp_{13*}(p_{12}^* Ri_* \kappa(\mathcal{O}_{\Delta Y}) \otimes^L_{Y\times Y\times Y\times V^*} p_{23}^*\mathcal{M}) \\
&\simeq Rp_{13*}(Ri_{4*} p_1^* \kappa(\mathcal{O}_{\Delta Y}) \otimes^L_{Y\times Y\times Y\times V^*} p_{23}^*\mathcal{M}) \\
&\simeq Rp_{13*} Ri_{4*}(p_1^* \kappa(\mathcal{O}_{\Delta Y}) \otimes^L_{Y\times_Y Y\times Y\times V^*} Li_4^* p_{23}^*\mathcal{M}) \\
&\simeq \mathrm{Id}_*(p_1^*\kappa(\mathcal{O}_Y) \otimes^L_{Y\times Y\times V*} \mathrm{Id}^*\mathcal{M}) \\
&\simeq \left(\begin{pmatrix} 0 \\ \uparrow\downarrow \\ \mathcal{O}_{Y\times Y\times V^*} \end{pmatrix} \otimes_{Y\times V^*} \begin{pmatrix} 0 \\ \uparrow\downarrow \\ \mathcal{O}_Y \otimes \mathrm{Sym}(V) \end{pmatrix} \right) \otimes^L_{Y\times Y\times V^*} \mathcal{M} \\
&\simeq \begin{pmatrix} 0 \\ \uparrow\downarrow \\ \mathcal{O}_{Y\times Y\times V^*} \end{pmatrix} \otimes^L_{Y\times Y\times V^*} \mathcal{M} \\
&\simeq \mathcal{M}.
\end{aligned}$$

Similarly, we get $\mathcal{M} * \kappa(\mathcal{O}_{\Delta Y}) \simeq \mathcal{M}$. □

We can now finish the proof of the main theorem.

PROOF OF MAIN THEOREM 7.8. From theorem 7.7 we have explicit generators of a braid group action in $\mathcal{K}^{\mathrm{Coh}}_{\tilde{\mathfrak{g}},\mathfrak{g}}$. These are sheaves on closed subschemes sitting in one degree so they lie in the full subcategory $D_{\mathrm{Perf}}(\mathcal{O}_{\tilde{\mathfrak{g}}\times\tilde{\mathfrak{g}}} \otimes \Lambda(\mathfrak{g}^*) - \mathrm{mod}^G)$. Since all of them are supported on $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ the images under the monoidal functor κ land in the full subcategory $D^{\mathrm{abs}}_{\tilde{\mathfrak{g}}\times_{\mathfrak{g}}\tilde{\mathfrak{g}}\times\mathfrak{g}^*}(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \times \mathfrak{g}^*), h)$. Thus, they act on $D^{\mathrm{abs}}(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times T^*X), w)$ and generate the desired geometric braid group action. □

CHAPTER 8

Further directions

In this chapter we suggest some further projects which are natural continuations of the projects in this dissertation.

8.1. Affine Demazure descent

Having studied Demazure descent and categorical actions of the affine braid group it is natural to upgrade Demazure descent to the affine setting by replacing the Weyl group by the affine Weyl group.

DEFINITION 8.1. Affine Demazure descent data on the triangulated category $\mathcal{C}$ is a collection of triangulated functors $\{D_w : \mathcal{C} \to \mathcal{C}, w \in W_{\text{aff}}\}$ satisfying weak affine braid monoid relations:

$$D_{w_1} \circ D_{w_2} \simeq D_{w_1 w_2} \text{ for all } w_1, w_2 \in W_{\text{aff}} \text{ with } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2).$$

For each simple reflection $s_k \in W$ the corresponding functor D_{s_k} should be a comonad with the coproduct map being an isomorphism.

A natural goal is to try to upgrade the Demazure descent data from chapter 5 to affine Demazure descent data. In [**BR**] the action of $x \in X$ in the affine braid group is given by convolution with $\mathcal{O}_{\Delta\tilde{\mathfrak{g}}}(x) \in D^b_{\text{prop}} \operatorname{Coh}(\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}})$ so a natural candidate for the Demazure functor $D_x \circlearrowright D(\operatorname{QCoh}^B(X))$ is convolution by the analogous line bundle $\mathcal{O}_{\Delta G/B}(x)$ in $\operatorname{QCHecke}(G, B)$.

CONJECTURE 8.2. *The* $\{D_w \mid w \in W_{\mathit{aff}}\}$ *is affine Demazure descent data on* $D(\operatorname{QCoh}^B(X))$*. In particular,* $\operatorname{QCHecke}(G, B)$ *categorifies the degenerate affine Hecke algebra.*

8.2. Degenerate double affine Hecke algebra

Let G be a simple, connected and simply-connected complex algebraic group. The double affine Hecke algebra (DAHA) is a version of the affine Hecke algebra with an additional lattice. It has been constructed geometrically by Varagnolo and Vasserot in terms of K-theory (see [**VV**]) using the algebraic loop group $LG = G((t))$ and an Iwahori subgroup I. They proved that as a ring the DAHA is isomorphic to the I-equivariant K-theory on an affine analogue of the Steinberg variety $\mathfrak{N}$, with the ring structure on $K^I(\mathfrak{N})$ coming from a convolution on $D(\operatorname{Coh}^I(\mathfrak{N}))$. The isomorphism is defined on generators and each generator is mapped either to the class of the structure sheaf of a subspace in $\mathfrak{N}$ or to the class of a cone. Thus, their construction naturally suggests a categorification.

Assuming that the previous project is successful a natural next step is to attempt to categorify the degenerate double affine Hecke algebra by constructing a double affine version of Demazure descent data on an affine analog of $\mathrm{QCHecke}(G, B) \simeq D(\mathrm{QCoh}^B(G/B))$. Pursuing the analogy a natural candidate is

$$\mathrm{QCHecke}(LG, I) := D(\mathrm{QCoh}^I(LG/I))$$

with the monoidal structure defined in the same way as for $\mathrm{QCHecke}(G, B)$.

CONJECTURE 8.3. *The category* $\mathrm{QCHecke}(LG, I)$ *categorifies the degenerate DAHA.*

8.3. Categorification of the DAHA

If conjecture 8.3 holds it is natural to try to use the same setting to construct an affine version of Bezrukavnikov and Riche, i.e. a categorical action of the DAHA. In [**BR**] the B_{aff}-action is on $D^b(\mathrm{Coh}^G(\tilde{\mathfrak{g}})) \simeq D^b(\mathrm{Coh}^B(\mathfrak{b}))$. Following the analogy we hope that a analogous construction will produce an action of B_{aff} on $D(\mathrm{Coh}^I(\mathrm{Lie}(I)))$. For the action of the second lattice we would use a categorical version of [**VV**].

CONJECTURE 8.4. *There exist an action of the DAHA on* $D(\mathrm{Coh}^I(\mathrm{Lie}(I)))$.

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