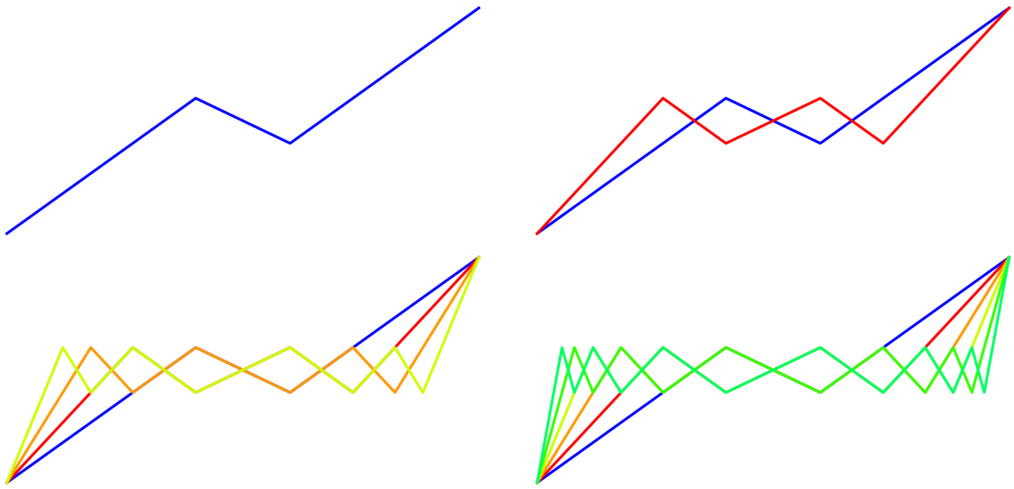
Circle Maps and C^*-algebras



Thomas Lundsgaard Schmidt

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Supervisor: Klaus Thomsen

Department of Mathematics
Faculty of Science and Technology, Aarhus University

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Resumé

Lad $\mathbb{T}$ være enhedscirklen i det komplekse plan, og lad $\varphi : \mathbb{T} \to \mathbb{T}$ være en afbildning som er kontinuert, surjektiv og stykvist monoton. Vi tillader φ at have kritiske punkter. Ved at generalisere de transformationsgrupoider for lokale homeomorfier, som først blev introduceret af Renault i [30], konstruerer vi to étale grupoider, Γ_φ og Γ_φ^+, fra en sådan afbildning. Afhandlingen omhandler forholdet mellem de dynamiske egenskaber ved φ, grupoid-egenskaberne ved Γ_φ og Γ_φ^+, strukturteorien for disse grupoiders reducerede C^*-algebraer, og – for visse klasser af cirkelafbildninger – disse algebraers K-teori. Vi viser, at hvis afbildningen φ er transitiv, er grupoid-C^*-algebraerne rent uendelige og opfylder den universelle koefficient-sætning. Ydermere finder vi nødvendige og tilstrækkelige betingelser for at disse C^*-algebraer er simple, og formulerer disse betingelser i termer af en bestemt type fixpunkter for φ. I det tilfælde, hvor algebraerne ikke er simple, bestemmer vi det primitive ideal-spektrum. Vi viser, at enhver irreducibel repræsentation faktoriserer gennem C^*-algebraen for reduktionen af grupoiden til banen af et punkt på cirklen, og at de tilhørende idealer falder i to typer, afhængigt af isotropien over det tilknyttede punkt på cirklen. Herefter retter vi opmærksomheden mod kritisk endelige afbildninger – afbildninger hvor fremad-banen for ethvert kritisk punkt er endelig – og udleder en algoritme, der gør det muligt med simple midler at bestemme K-teorien for de tilhørende C^*-algebraer. Til slut gør vi det samme for cirkelafbildninger uden periodiske punkter ved at kombinerere resultater fra tidligere kapitler med tidligere arbejde af Putnam, Schmidt og Skau i [28]. For en mere detaljeret kapiteloversigt henvises til den engelske introduktion.

Introduction

The relationship between dynamical systems and operator algebras is by now at least half a century old, and, as far as fifty-year relationships go, it seems to be a happy and fruitful one. Many interesting C^*-algebras can be realised as the C^*-algebra of a dynamical system, and knowledge of the dynamics often translate into results about the structure of the C^*-algebra. Conversely, the C^*-algebras associated to a class of dynamical systems often provide strong, computable invariants for these systems.

By now, there are many concrete ways of associating a C^*-algebra to a dynamical system – one, first suggested by Jean Renault in [30], involves creating a *topological groupoid* $\mathcal{G}$ from the dynamical system, and then constructing a C^*-algebra $C_r^*(\mathcal{G})$ from the groupoid by completing the $*$-algebra of continuous, compactly supported functions $C_c(\mathcal{G})$ on $\mathcal{G}$ in a suitable norm. As many mathematical constructs – e.g. equivalence relations, groups, and infinite paths on graphs – can be formulated in terms of groupoids, Renault's construction paved the way for associating C^*-algebras to these. In particular, in the case of an abelian group G acting by homeomorphisms α_g on a compact Hausdorff space X, the associated *reduced transformation groupoid C^*-algebra* $C_r^*(G, X, \alpha)$ is isomorphic to the well-know reduced crossed product $C(X) \rtimes_\alpha G$. However, as realised by Renault and many others (see e.g. [30], [2], [10]), the transformation groupoid construction is flexible enough to work even when the dynamics are non-invertible, i.e. when the map is not a homeomorphism. This has led to a wealth of C^*-algebras constructed from non-invertible dynamical systems. The goal of this dissertation is to take this construction one step further, and construct topological groupoids (Hausdorff, locally compact, second countable and étale) from dynamical systems that are not even *locally invertible* – more precisely, from maps with critical points.

The dynamical systems we consider are all *one-dimensional*. More precisely, they are all continuous, surjective and piecewise monotone self-maps of the unit circle. While one-dimensional dynamics only constitute a tiny corner in the wide world of dynamical systems, they often serve as very interesting test cases: On one hand, they are tractable enough to admit a general theory, while on the other, they are complicated enough to show chaotic, weird and wonderful behaviour. The study of circle dynamics goes back at least to Poincaré, who classified circle diffeomorphisms of to conjugacy in terms of their *rotation number*. Since then, the subject has grown in a multitude of directions (see [37] for a recent survey), and many connections with the field of C^*-algebras have appeared. This thesis presents yet another.

Let's give a brief overview of the contents of this dissertation:

Chapter 1 contains a number of preliminaries. We give a short overview of the theory of topological groupoids and their C^*-algebras, and prove a number of standard results. This chapter can be skipped upon first reading – note, however, that the chapter concludes with number of elementary examples of groupoids, and that these examples will feature prominently through the rest of the thesis.

Chapter 2 is where the real work begins. We introduce a construction of locally compact, second countable, Hausdorff étale groupoids Γ_φ and Γ_φ^+ associated to a self-map φ of the circle. Central to this construction is the idea of a *local transfer*: A locally defined homeomorphism η of the circle satisfying $\varphi^n = \varphi^m \circ \eta$ for some $m, n \in \mathbb{N}$. Using these local transfers in the construction of the groupoid reflects the fact that φ may have critical points, and that these points are somehow 'special'. In the construction of one groupoid, Γ_φ, we allow these transfers to revert the standard orientation of the circle, while for Γ_φ^+, only orientation-preserving transfers make the cut. We then prove a number of structural results about the C^*-algebras of the gropoids, and show that transitivity of the map implies that the algebras are purely infinite and satisfy the Universal Coefficient Theorem.

Chapter 3 focuses on core algebras $C_r^*(R_\varphi)$ and $C_r^*(R_\varphi^+)$, which arise as C^*-algebras of groupoids related to the equivalence relation on T given by $x \sim y$ if $\varphi^k(x) = \varphi^k(y)$. We realise these algebras as direct limits of so-called *building block*-algebras (as in [44]). This allows us to show that the algebras $C_r^*(\Gamma_\varphi)$ and $C_r^*(\Gamma_\varphi^+)$ are nuclear and satisfy the Universal Coefficient Theorem.

Chapter 4 determines when $C_r^*(\Gamma_\varphi)$ and $C_r^*(\Gamma_\varphi^+)$ are simple, and characterise the primitive ideal spectrum in the non-simple case. We show that one algebra is simple if and only if the other is, and then that simplicity is equivalent to exactness of the map φ and non-existence of *exceptional fixed points*; that is, fixed points e whose pre-image is contained in the critical points of φ (and $\{e\}$ itself). When $C_r^*(\Gamma_\varphi)$ and $C_r^*(\Gamma_\varphi)$ are non-simple, we investigate a close connection between primitive ideals and reductions $\Gamma_\varphi|_{\overline{[x]}}$ and $\Gamma_\varphi^+|_{\overline{[x]}}$ to the closure of the groupoid orbit of a single point x. We show that the primitive ideals come in two distinct families, depending on whether the set $[x]$ contains an isolated pre-periodic or pre-critical point or not. We end by finding the maximal ideals among the primitive ones.

Chapter 5 concerns the class of *critically finite* maps – maps with the property that the forward orbit of every critical point is a finite set. We use some heavy machinery, not least the theory of C^*-correspondences, to connect $C_r^*(\Gamma_\varphi)$ and $C_r^*(\Gamma_\varphi^+)$ to the building block algebras of Chapter 3, and exploit these connections to develop a simple algorithm for calculating the K-theory of $C_r^*(\Gamma_\varphi)$ and $C_r^*(\Gamma_\varphi^+)$.

Chapter 6 concerns circle maps without periodic points. After a historical detour through results of Poincaré and Denjoy on homeomorphisms with no periodic points, we show how the so-called Denjoy homeomorphisms can be modified to obtain any piecewise monotone circle map without periodic points. Using the results from Chapter 4, we then determine the primitive ideals of the groupoid C^*-algebras associated to such a map, and use results of Putnam, Skau and Schmidt to determine the K-theory groups of these algebras.

Chapter 7 takes another look at the core algebras from Chapter 3, but with the added assumption that the map in question is critically finite. The main result is that when the core algebras $C_r^*(R_\varphi)$ and $C_r^*(R_\varphi^+)$ are simple, the first is an AF-algebra, while the second is not.

A note on being self-referential: The beginnings of this project can be traced back to the year 2012, shortly after I was admitted to the PhD programme at Aarhus University. Rather quickly, the project converged to a joint paper with my supervisor Klaus Thomsen ([38]), published in Ergodic Theory and Dynamical Systems. Since publication, I have generalised the many results in [38] in a number of directions, and the contents of the paper have in many ways been subsumed by later results. Hence, instead of attaching the paper as part of the dissertation, I have opted to integrate its directly into the main body of the thesis. Some parts – especially in Chapter 4 – have been copied more or less verbatim from this paper, while others, such as in Chapter 5, appear here in more general versions. The same is true for my (unpublished) Qualification Exam report, which essentially contains all the material of Chapter 5. The rest of the material is new, and some, especially most of Chapter 6, will hopefully appear in publication at some point in the near future.

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Chapter 1

Preliminaries

This chapter contains a brief introduction to topological groupoids and their C^*-algebras. We fix some necessary notation, introduce some constructions which will be used extensively in the chapters to come, and develop a catalogue of examples of groupoid C^*-algebras which will feature prominently later. For more on the theory of groupoids, see e.g. [30] or [21].

1.1 Groupoids

We begin with a definition:

Definition 1.1. A *groupoid* $\mathcal{G}$ is a small category in which every morphism is an isomorphism.

While this definition is an excellent one-liner, one could argue that it needs some unwrapping to be useful in calculations. Starting from the definition of a category, a groupoid $\mathcal{G}$ consists of a set of objects $\mathcal{G}^0$, a set of morphisms $\mathcal{G}^1$, and range and source maps $r, s : \mathcal{G}^1 \to \mathcal{G}^0$, satisfying a number of category-theoretic axioms. One usually think of $\mathcal{G}^1$ as the 'elements' of $\mathcal{G}$, and refer to $\mathcal{G}^0$ as the 'unit space' of $\mathcal{G}$. Expanding this point of view yields a more 'algebraic' definition of a groupoid, which highlights how one may think of groupoids as 'generalised groups':

Definition 1.2. A *groupoid* is a set G with an inversion $^{-1} : G \to G$, a set of *composable elements* $G^{(2)} \subseteq G \times G$ and a composition $G^{(2)} \to G$ satisfying the following axioms for all $\gamma_1, \gamma_2, \gamma_3 \in G$:

- $(\gamma_1^{-1})^{-1} = \gamma_1$
- If $(\gamma_1, \gamma_2), (\gamma_2, \gamma_3) \in G^{(2)}$, then $(\gamma_1, \gamma_2\gamma_3), (\gamma_1\gamma_2, \gamma_3) \in G^{(2)}$ and $(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3)$.
- $(\gamma_1, \gamma_1^{-1}) \in G^{(2)}$, and if $(\gamma_1, \gamma_2) \in G^{(2)}$, then $\gamma_1^{-1}(\gamma_1\gamma_2) = \gamma_2$.
- $(\gamma_1^{-1}, \gamma_1) \in G^{(2)}$, and if $(\gamma_1, \gamma_2) \in G^{(2)}$, then $(\gamma_2\gamma_1)\gamma_1^{-1} = \gamma_2$.

Definition 1.3. Let G be a groupoid. Define *range* and *source* maps $r, s : G \to G$ by $r(\gamma) = \gamma\gamma^{-1}$ and $s(\gamma) = \gamma^{-1}\gamma$, and define the *unit space* G^0 of G as $r(G)$.

Note that $r(\gamma^{-1}) = s(\gamma)$, so we could equally define G^0 as $s(G)$. Going back and forth between the categorical and the algebraic definitions of a groupoid is straightforward, as are the next few lemmas. The first lemma explains the name *unit space* – a typical groupoid has no neutral element (unless it is a group!), but the elements of the unit space act 'locally' as units.

Lemma 1.4. *Let $\gamma \in G$. Then $(r(\gamma), \gamma), (\gamma, s(\gamma)) \in G^{(2)}$ and $r(\gamma)\gamma = \gamma s(\gamma) = \gamma$.*

Proof. If $(\gamma, \eta) \in G^{(2)}$, the axioms in Definition 1.2 imply that

$$s(\gamma) = \gamma^{-1}\gamma = \gamma^{-1}\gamma\eta\eta^{-1} = \eta\eta^{-1} = r(\eta).$$

On the other hand, if $s(\gamma) = \gamma^{-1}\gamma = \eta\eta^{-1} = r(\eta)$, it follows that $(\gamma^{-1}\gamma, \eta)$ is in $G^{(2)}$, and since $(\gamma, \gamma^{-1}\gamma) \in G^{(2)}$, we get that (γ, η) is in $G^{(2)}$. □

Lemma 1.5. *Let G be a groupoid with unit space G^0 and range and source maps $r, s : G \to G^0$. Then $(\gamma, \eta) \in G^{(2)}$ if and only if $s(\gamma) = r(\eta)$. If $(\gamma, \eta) \in G^{(2)}$, we have $r(\gamma\eta) = r(\gamma)$ and $s(\gamma\eta) = s(\eta)$.*

Proof. This follows directly from the axioms in Definition 1.2. □

Subgroupoids are defined just as one would expect:

Definition 1.6. Let G be a groupoid, and $H \subset G$ a subset. H is called a *subgroupoid* if it is closed under inversion and composition – that is, if $(\gamma, \eta) \in (H \times H) \cap G^{(2)}$, we have $\gamma\eta \in H$.

A typical way of creating subgroupoids is by taking reductions to invariant subsets of the unit space:

Definition 1.7. Let G be a groupoid with unit space G^0, range and source maps $r, s : G \to G^0$, and let $x \in G^0$. Define $[x]_G$, the *G-orbit* of x as $r(s^{-1}(x))$. We say that a subset $A \subseteq G^0$ is *G-invariant* if $x \in A$ implies $[x]_G \subseteq A$.

When the groupoid G is clear from the context, we will write $[x]$ for $[x]_G$. We remark that the word 'invariant' will appear with any number of prefixes (G-, forward, backward, totally...) in the chapters to come, and that a significant part of the thesis will be dedicated to untangling the various relations between different types of invariance.

Definition 1.8. Let G be a groupoid and $A \subset G^0$ a G-invariant set. We define $G|_A$, the *reduction* of G to A, as the set

$$G|_A = \{\gamma \in G \mid r(\gamma) \in A\}$$

Using the definition of G-invariance and Lemma 1.5, it is easy to see that $G|_A$ is a subgroupoid of G. Finally, we define isotropy groups:

Definition 1.9. Let G be a groupoid and $x \in G^0$. Define $\mathrm{Iso}(x)$, the *isotropy group* at x, as the set

$$\mathrm{Iso}(x) = \{\gamma \in G \mid r(\gamma) = s(\gamma) = x\}$$

It is easy to check that $\mathrm{Iso}(x)$ is indeed a group, with x as neutral element.

1.1.1 Topological groupoids

Having defined groupoids algebraically, we now add some topology:

Definition 1.10. Let G be a groupoid, give $G \times G$ the product topology and $G^{(2)}$ the induced topology from $G \times G$. We say that G is a *topological groupoid* if the inversion and composition maps are continuous.

To do any serious work – in particular, to make the construction of the reduced groupoid C^*-algebra work –, we need some assumptions on the topology of a given groupoid. In most cases, we will require the topology to be Hausdorff, locally compact and second countable. Furthermore, we would like the topology to be étale:

Definition 1.11. Let G be a topological groupoid. An open subset $S \subseteq G$ is called a *bisection* if the maps $r|_S : S \to r(S)$ and $s|_S : S \to s(S)$ are homeomorphisms onto their image. The groupoid G is *étale* if the topology on G has a basis of bisections.

Lemma 1.12. *Let G be an étale groupoid. Then the unit space G^0 is open in G.*

Proof. Let $x \in G^0$, and choose a bisection U containing x. Since $r(x) = x$, we may choose an open neighbourhood V of x such that $V \subset U \cap r^{-1}(U)$. If there is a $y \in V \cap (G \setminus G^0)$, we have $y, r(y) \in U$ with $y \neq r(y)$ (since $y \notin G^0$), but $r(y) = r(r(y))$, contradicting the injectivity of r on U. □

Renault ([30]) refers to étale groupoids as *r-discrete*, which is explained by the next lemma:

Lemma 1.13. *Let G be an etale groupoid with range and source maps $r, s : G \to G^0$, and let $x \in G^0$. Then the fibers $r^{-1}(x)$ and $s^{-1}(x)$ are discrete subspaces of G.*

Proof. For $\gamma \in r^{-1}(x)$, simply choose a bisection U containing γ. Then $r^{-1}(x) \cap U = \{\gamma\}$, showing that $r^{-1}(x)$ is discrete. The same goes for $s^{-1}(x)$. □

To a certain extent, one may think of étale groupoids as an analouge of discrete groups. For our purposes, the salient feature of étale groupoids is that we can forget about the technical difficulties of working with a Haar system, and simply equip each fiber over G^0 with the counting measure. As we shall see, this makes forming the convolution algebra – and hence, the groupoid C^*-algebra – over G much easier.

Lemma 1.14. *Let G be an étale groupoid, and A an open (resp. closed) G-invariant subset of G^0. Then $G|_A$ is open (resp. closed) in G.*

Proof. The range map is continuous, and $r^{-1}(A) = G|_A$. □

1.2 The reduced C^*-algebra of an étale groupoid

Given an second countable, locally compact Hausdorff étale groupoid G, we proceed to construct its reduced C^*-algebra $C^*_r(G)$. The process is similar to the construction of a group C^*-algebra: Define a convolution product on some suitable set of continuous functions on G, and complete this $*$-algebra in an appropriate norm to get a C^*-algebra. Denote by $C_c(G)$ the continuous, compactly supported functions on G, and note that for any $f \in C_c(G)$ and $x \in G^0$, the set $\mathrm{supp}(f) \cap r^{-1}(x)$ is finite by Lemma 1.13. Hence, we may define a convolution product $*$ by

$$f * g(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2), \qquad f, g \in C_c(G). \tag{1.1}$$

If we note that $\gamma_1\gamma_2 = \gamma$ is equivalent to $\gamma_2 = \gamma_1^{-1}\gamma$ and $r(\gamma_1) = r(\gamma)$, we can rewrite Equation 1.1 as

$$f * g(\gamma) = \sum_{\gamma_1 \in r^{-1}(r(\gamma))} f(\gamma_1)g(\gamma_1^{-1}\gamma), \qquad f, g \in C_c(G)$$

It follows that $f * g$ is again compactly supported and continuous. To define an involution, define f^* by

$$f^*(\gamma) = \overline{f(\gamma^{-1})}, \ \ f \in C_c(G).$$

Checking that the product and involution turns $C_c(G)$ into a $*$-algebra is straightforward. To obtain a C^*-algebra, we represent $C_c(G)$ as multiplication operators – we just need to consider many representations at the same time: For $x \in G^0$, define a representation $\pi_x : C_c(G) \to B(l^2(s^{-1}(x)))$ by

$$[\pi_x(f)\psi](\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)\psi(\gamma_2) \tag{1.2}$$

and define a norm on $C_c(G)$ by

$$\|f\| = \sup_{x\in G^0} \|\pi_x(f)\| \tag{1.3}$$

Definition 1.15. Let G be an étale, locally compact, second countable Hausdorff groupoid. The *reduced groupoid C^*-algebra $C^*_r(G)$* of G is the completion of the $*$-algebra $C_c(G)$ in the norm given by Equation 1.3.

Checking that Equation (1.3) actually defines a C^*-norm takes a bit of work – for a proof, see e.g. Chapter II.1 of [30]. If H is an open subgroupoid – for instance, the reduction of G to an open invariant set of the unit space –, there is an inclusion map $i_H : C_c(H) \to C_c(G)$ simply given by

$$i_H(f)(\gamma) = f(\gamma)$$

for $\gamma \in F, f \in C_c(H)$.

Lemma 1.16. *Let G be a locally compact, second countable Hausdorff étale groupoid, and H an open subgroupoid. Then the inclusion $i : C_c(H) \to C_c(G)$ extends to an injective $*$-homomorphism $i_H : C^*_r(H) \to C^*_r(G)$. If $H = G|_U$ for some open G-invariant set U, $C^*_r(H)$ is an ideal in $C^*_r(G)$.*

Proof. The first statement is Proposition 1.9 of [23]. For the second statement, let $f \in C_c(G|_U)$ and $h \in C_c(G)$, and let $\gamma \in G \setminus G|_U$. If we then write $\gamma = \gamma_1\gamma_2$, we have $r(\gamma_1) = r(\gamma) \notin U$, so $\gamma_1 \notin G|_U$. It follows that

$$f * h(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)h(\gamma_2) = 0,$$

so $f * h \in C_c(G|_U)$. Then claim then follows by continuity. □

In a similar way, if $F \subseteq G^0$ is closed and invariant, there is a restriction map $\pi_F : C_c(G) \to C_c(G|_F)$ given by

$$\pi_F(f)(\gamma) = f(\gamma),$$

for $\gamma \in F, f \in C_c(G)$. This is surjective by Tietzes extension theorem. It is straightforward to check that π_F is also a $*$-homomorphism.

Lemma 1.17. *Let G be a locally compact, second countable Hausdorff étale groupoid, and F a closed, invariant subset of the unit space. Then the restriction map $\pi_F : C_c(G) \to C_c(G|_F)$ extends to a surjective $*$-homomorphism $\pi_F : C^*_r(G) \to C^*_r(G|_F)$. In particular, $C^*_r(G|_F)$ is a quotient of $C^*_r(G)$.*

Proof. Let $x \in \mathbb{T}$, and let π_x be one of the representations from Equation 1.2. Then $\pi_x \circ \pi_F = \pi_x$ when $x \in F$, and $\pi_x \circ \pi_F = 0$ otherwise – in particular, π_F is a contraction, so it extends to a $*$-homomorphism from $C^*_r(G)$ to $C^*_r(G|_F)$. Surjectivity of π_F follows by continuity. The final statement is the isomorphism theorem. □

In the best of all possible worlds, one might concieve that the kernel of the map π_F is exactly $C_r^*(G|_{G^0 \setminus F})$. This is not always the case – see [31] for an example. It is, however, true for the groupoids we will focus on, and we prove this in Proposition 3.21.

Finally, a handy lemma:

Lemma 1.18. *Let G be a locally compact, second countable Hausdorff étale groupoid, and $f \in C_c(G)$. Then f may be decomposed as a finite sum*

$$f = \sum_{i=1}^{n} f_i$$

where each f_i is supported in a bisection.

Proof. Let $K = \mathrm{supp}(f)$. For each $\gamma \in K$, choose a bisection W_γ containing γ. Then the W_γ's cover K, and we may exhaust to a finite cover $\{W_{\gamma_i}\}_{i=1}^n$. Choose a partition of unity $\{\rho_i\}$ with respect to this cover, and put $f_i = \rho f$. □

1.3 Examples

In this section, we consider a number of elementary groupoids and calculate their C^*-algebras. These groupoids, or variations of them, will feature prominently in the chapters to come, usually arising as subgroupoids of the groupoid of a circle map.

Definition 1.19. Let X be a set, and $\sim$ an equivalence relation on X. Define G_R, the groupoid of the equivalence relation, as

$$G_R = \{(x, y) \in X \times X \mid x \sim y\}$$

with composition $(x, y)(y, z) = (x, z)$, inversion $(x, y)^{-1} = (y, x)$, and range and source maps given by the first and second coordinate projection, respectively. Particular examples are the full equivalence relation on X, where $x \sim y$ for any $x, y \in X$, and the trivial equivalence relation, where $x \sim y$ if and only if $x = y$.

Here's a very specific example:

Example 1.20. Let X be a countable set, and G_X the groupoid of the full equivalence relation on X, equipped with the discrete topology. We claim that

$$C_r^*(G_X) \simeq \mathbb{K}(l^2(X)),$$

where $\mathbb{K}(l^2(X))$ denotes the compact operators on $l^2(X)$. For this, note that $C_r^*(G_X)$ contains a set of matrix units, namely the characteristic functions $\{1_{x,y} | x, y \in X\}$. Indeed, one checks easily that

$$1_{x,y} * 1_{z,w} = \delta_{y,z} 1_{x,w}, \qquad 1_{x,y}^{-1} = 1_{y,x}$$

Since $\mathbb{K}$ (identified with $\mathbb{K}(l^2(X))$) is the universal C^*-algebra generated by such matrix units, we obtain a surjective $*$-homomorphism from $\mathbb{K}(l^2(X))$ to $C_r^*(G_X)$. But the compact operators is a simple C^*-algebra, so this map is also injective. ▲

In particular, when X is a finite set, the groupoid C^*-algebra of G_X is simply isomorphic to $M_n(\mathbb{C})$ with $n = |X|$.

Remark 1.21. The above example generalises easily to other equivalence relations – indeed, let X be countable, $\sim$ an equivalence relation on X and G_R the groupoid of the equivalence relation. Assume that $\sim$ has finitely many equivalence classes $X_1, \dots, X_n$. Then each set X_n is G_R-invariant and $G_R = \bigsqcup_{i=1}^n G_R|_{X_i}$. It follows that

$$C_r^*(G_R) \simeq \bigoplus_{i=1}^n C_r^*(G_R|_{X_i}),$$

and each of the $C_r^*(G_R|_{X_i})$ are determined by Example 1.20. ♦

Example 1.22. Consider the following situation: Let $\{X_i\}_{i\in\Lambda}$ be a collection of metric spaces, indexed by an at most countable set Λ, and let $\theta_{ij} : X_i \to X_j$ be a collection of homeomorphisms satisfying $\theta_{ii} = \mathrm{id}_{X_i}$, $\theta_{ij}^{-1} = \theta_{ji}$ and $\theta_{kj} \circ \theta_{ik} = \theta_{ij}$ for all i, j, k. Put $X = \sqcup_{i\in\Lambda} X_i$, the disjoint union of the sets $\{X_i\}_{i\in\Lambda}$, and define a groupoid G_θ by

$$G_\theta = \{(x, y) \in X \times X \mid \exists i, j : x \in X_i, y \in X_j, \theta_{ij}(x) = y\}$$

Equip G_θ with the topology generated by the sets $\{U \times \theta_{ij}(U)\}$, with i, j ranging over Λ and U over the open sets of X_i. Essentially, this is example 1.20 with each element in X replaced with a topological space. It is easy to see that $C_r^*(G_\theta)$ is Hausdorff, locally compact, second countable and étale. Fix a $\lambda \in \Lambda$, and put $X_0 = X_\lambda$. We claim that

$$C_r^*(G_\theta) \simeq C_0(X_0) \otimes \mathbb{K}(l^2(\Lambda)).$$

To see this, let $f \in C_c(G_\theta)$, and define $f_{ij} \in C_c(X_0)$ by

$$f_{ij}(x) = f(\theta_{0i}(x), \theta_{0j}(x)), \qquad x \in X_0$$

Since Λ is discrete, $\mathbb{K}(l^2(\Lambda))$ is generated by matrix units e_{ij}, $i, j \in \Lambda$. Consider the map $\Theta : C_c(G_\theta) \to C_c(X_0) \otimes \mathbb{K}(l^2(\Lambda))$ given by

$$f \mapsto \sum_{i,j\in\Lambda} f_{ij} \otimes e_{ij}$$

Θ is evidently linear, multiplicative, isometric and $*$-preserving. That f is compactly supported entails that only finitely many f_{ij}'s are non-zero, so the image of $C_c(G_\theta)$ under Θ is exactly $C_c(X_0)$ tensored with the finite rank operators in $\mathbb{K}(l^2(\Lambda))$. Since Θ is an isometry, we get the desired isomorphism

$$C_r^*(G_\theta) = C_0(X_0) \otimes \mathbb{K}(l^2(\Lambda))$$ ▲

A prominent feature of the groupoids in the example above – and indeed, of any groupoid generated by an equivalence relation – is that the isotropy groups $\mathrm{Iso}(x)$ are all trivial. The next proposition gives a tool for determining the C^*-algebra of a groupoid where the isotropy groups are identical over any element in the unit space.

Proposition 1.23. *Let G be a discrete groupoid with unit space G^0, and assume that there is a group H such that $\mathrm{Iso}(x) \simeq H$ for any $x \in G^0$. Then there is an isomorphism*

$$C_r^*(G) \simeq C_r^*(H) \otimes \mathbb{K}(l^2(G^0))$$

with $C_r^*(H)$ denoting the reduced group C^*-algebra of H.

Proof. See Lemma 4.11 of [47]. □

Example 1.24. An example that will occur a few times in the chapters to come is the following: Let G_X be the groupoid from Example 1.20, with X either at most countable, let $\mathbb{Z}_2$ be the group with two elements, and let K be the groupoid

$$K = G \times \mathbb{Z}_2 = \{(\gamma, p) \mid \gamma \in G, p \in \mathbb{Z}_2\}$$

with unit space G^0 and range and source maps inherited from G. Then, K is discrete and $\mathrm{Iso}(k) \simeq \mathbb{Z}_2$ for any $k \in K$. Since $C^*(\mathbb{Z}_2) \simeq \mathbb{C}^2$, Lemma 1.23 and 1.20 yields an isomorphism

$$C_r^*(K) \simeq C^*(\mathbb{Z}_2) \otimes \mathbb{K}(l^2(X)) \simeq \mathbb{C}^2 \otimes C_r^*(G) \simeq C_r^*(G) \oplus C_r^*(G)$$ ▲

1.3.1 K-theory

For the basic K-theory of C^*-algebras, we refer to [34].

Example 1.25. Let X be a countable set, and let G_X be the groupoid from Example 1.20. Since $C_r^*(G_X) \simeq \mathbb{K}(l^2(X))$, it follows that $K_0(C_r^*(G_X)) \simeq \mathbb{Z}$ and $K_1(G_X) \simeq 0$. Any rank-one projection in $C_r^*(G_X)$ will do as generator of $K_0(C_r^*(G_X))$; the natural choice is the characteristic function $1_{(x,x)}$ for some $x \in X$. ▲

Example 1.26. Continuing Example 1.24, let $L = G_X \times \mathbb{Z}_2$. It follows immediately that $K_0(L) \simeq \mathbb{Z}^2$. A situation that will occur often in the chapters to come is the following: Let $\mathcal{A}$ be some C^*-algebra and $\chi : C_r^*(L) \to \mathcal{A}$ a $*$-homomorphism, inducing a group homomorphism $\chi_* : K_0(C_r^*(L)) \to K_0(\mathcal{A})$. Describing χ_* requires explicit generators of $K_0(C_r^*(L))$. To find these, write $\mathbb{Z}_2 = \{+, -\}$, and let $\mathbb{K}(l^2(X))$ be generated by matrix units $e_{x,y}$ with $x, y \in X$. Define a map $\Psi : C_r^*(L) \to M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ by

$$\Psi(1_{(x,y,p)}) = \begin{cases} (e_{x,y}, e_{x,y}), & \text{if } p = + \\ (e_{x,y}, -e_{x,y}), & \text{if } p = - \end{cases}$$

It is straightforward to check that this map is a $*$-isomorphism. Now, fix an $x \in X$ and define elements p_+, p_- in $C_r^*(L)$ by

$$p_+ = \tfrac{1}{2}(1_{(x,x,+)} + 1_{(x,x,-)}), \quad , p_- = \tfrac{1}{2}(1_{(x,x,+)} - 1_{(x,x,-)}).$$

We note that

$$\Psi(p_+) = (e_{x,x}, 0), \;\; \Psi(p_-) = (0, e_{x,x}),$$

and that the elements p_+, p_- are projections in $C_r^*(L)$. The trace map $\mathrm{Tr} : M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \to \mathbb{C}^2$ induces an isomorphism $\mathrm{Tr}_* : K_0(M_n(\mathbb{C}) \oplus M_n(\mathbb{C})) \to \mathbb{Z}^2$. It follows that the composition $\mathrm{Tr}_* \circ \Psi_* : K_0(C_r^*(G_X \times \mathbb{Z}_2)) \to \mathbb{Z}^2$ is an isomorphism taking the K_0-classes of p_+ and p_- to $(1,0)$ and $(0,1)$, respectively. ▲

Finally, a general result: Let $\mathbb{A}$ and $\mathbb{B}$ be C^*-algebras, and $I, U : \mathbb{A} \to \mathbb{B}$ $*$-homomorphisms. Let I_0 and U_0 denote the induced maps between $K_0(\mathbb{A})$ and $K_0(\mathbb{B})$. Define $\mathbb{D}$, the *double mapping cylinder* of $\mathbb{A}$ and $\mathbb{B}$, as the C^*-algebra

$$\mathbb{D} = \{(a, f) \in \mathbb{A} \oplus C([0,1], \mathbb{B}) \mid I(a) = f(0), U(a) = f(1)\}$$

Then the sequence

$$0 \longrightarrow S\mathbb{B} \xrightarrow{\iota} \mathbb{D} \xrightarrow{\pi} \mathbb{A} \longrightarrow 0 \tag{1.4}$$

is exact, where $S\mathbb{B}$ denotes the suspension

$$S\mathbb{B} = \{f : [0,1] \to \mathbb{B} \mid f(0) = f(1) = 0\},$$

ι the inclusion $\iota(f) = (0, f)$, and π the projection onto the first coordinate. It follows that we have a six-term exact sequence on K-theory:

$$\begin{array}{ccccc} K_0(S\mathbb{B}) & \xrightarrow{\iota_0} & K_0(\mathbb{D}) & \xrightarrow{\pi_0} & K_0(\mathbb{A}) \\ \uparrow & & & & \downarrow \delta_1 \\ K_1(\mathbb{A}) & \xleftarrow{\pi_1} & K_1(\mathbb{D}) & \xleftarrow{\iota_1} & K_1(S\mathbb{B}) \end{array} \tag{1.5}$$

Lemma 1.27. *Let $\mathbb{D}$ be the double mapping cylinder of $\mathbb{A}$ and $\mathbb{B}$ via maps I and U as above. Let $\beta : K_0(\mathbb{B}) \to K_1(S\mathbb{B})$ be the Bott map, and $\delta_1 : K_0(\mathbb{A}) \to K_1(S\mathbb{B})$ the exponential map in (1.5). Then $\delta_1 = \beta \circ (I_0 - U_0)$.*

Proof. Assume first that $\mathbb{D}$, and hence also $\mathbb{A}$, are unital, with 1 denoting the unit of $\mathbb{A}$, and $(1, \underline{1})$ the unit in $\mathbb{D}$, $\underline{1}$ being the constant function 1. We use the setup from Proposition 12.2.2 in [34], and identify the unitisation $\widetilde{S\mathbb{B}}$ with the set $\{f : [0,1] \to \mathbb{B} \mid f(0) = f(1) \in \mathbb{C}\}$. Writing an element $f \in \widetilde{S\mathbb{B}}$ as $(f - f(0)) + f(0)$ and noting that $f - f(0) \in S\mathbb{B}$, we notice that the map $\bar{\iota} : \widetilde{S\mathbb{B}} \to \mathbb{D}$ is given by

$$\bar{\iota}(f) = (f - f(0), 0) + f(0)(1, 1) = (f, f(0)).$$

Now, let $g = [p]_0 \in K_0(\mathbb{A})$ for some projection $p \in M_n(\mathbb{A})$. Define $h : [0,1] \to M_n(\mathbb{B})$ by

$$h(t) = tU(p) + (1-t)I(p),$$

and note that $h(0) = I(p)$, $h(1) = U(p)$ and $h(t)$ is self-adjoint for any $t \in [0,1]$. With $d = (h, p)$ we certainly have $d \in \mathbb{D}$ and $\pi(d) = p$. Next, note that $\exp(2\pi i p) = 1$ (by spectral calculus), so $\exp(2\pi i d) = (\exp(2\pi i h), 1)$. If we define $u : [0,1] \to M_n(\mathbb{B})$ by $u(t) = \exp(2\pi i h(t))$, we have $u \in \mathcal{U}_n(\widetilde{S\mathbb{B}})$ and

$$\bar{\iota}(u) = (u, u(0)) = (\exp(2\pi i h), 1) = \exp(2\pi i d)$$

By Proposition 12.2.2, this implies that $\delta_0(g) = -[u]_1$.

On the other hand, let $\beta : K_0(\mathbb{B} \to K_1(S\mathbb{B})$ be the Bott map given by

$$\beta([q]) = [f_q], \;\; f_q(z) = zq + (1_n - q)$$

for a projection q in $M_n(\mathbb{B})$. We then have the following calculation:

$$\beta((I_* - U_*)([p]_0)) = \beta(I_*([p]_0)) - \beta(U_*([p]_0)) = [f_{I(p)}]_1 - [f_{U(p)}]_1 = [f_{I(p)} f^*_{U(p)}]_1$$

To finish the proof, we need to show that $[f^*_{I(p)} f_{U(p)}]_1 = -[f_{I(p)} f^*_{U(p)}]_1 = [u]_1$ in $K_1(S\mathbb{B})$. Now, again by spectral calculus, we have

$$f^*_{I(p)} f_{U(p)}(e^{2\pi i t}) = \left(e^{-2\pi i t} I(p) + (1_n - I(p))\right)\left(e^{2\pi i t} U(p) + (1_n - U(p))\right) = e^{-2\pi i t I(p)} e^{2\pi i t U(p)},$$

so the class of $f^*_{I(p)} f_{U(p)}$ in K_1 is (by the Whitehead lemma) represented by the element

$$\begin{pmatrix} e^{2\pi i t U(p)} & 0 \\ 0 & e^{-2\pi i t I(p)} \end{pmatrix} = \exp\left(2\pi i t \begin{pmatrix} U(p) & 0 \\ 0 & 0 \end{pmatrix}\right) \exp\left(2\pi i (1-t) \begin{pmatrix} 0 & 0 \\ 0 & I(p) \end{pmatrix}\right)$$

Similarly, the class of u is represented by

$$\exp\begin{pmatrix} 2\pi i \left[t U(p) + (1-t) I(p) \right] & 0 \\ 0 & 1 \end{pmatrix}$$

To establish a homotopy between these two maps, let R_θ, for $\theta \in [0, \pi/2]$, be the rotation matrix $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$, and put define $G : [0, \pi/2] \times [0,1] \to S\tilde{\mathbb{B}}$ by

$$G(\theta, t) = \exp\left(2\pi i \left[\begin{pmatrix} tU(p) & 0 \\ 0 & 0 \end{pmatrix} + R_\theta \begin{pmatrix} 0 & 0 \\ 0 & (1-t) I(p) \end{pmatrix} R_\theta^{-1} \right]\right)$$

Then an easy computation shows that $G(0, -) = f^*_{I(p)} f_{U(p)}$, $G(\pi/2, -) = u$ and that $G(\theta, 0) = G(\theta, 1) = 1$ for any fixed θ (Note that $R_\theta \begin{pmatrix} 0 & 0 \\ 0 & I(p) \end{pmatrix} R_\theta^{-1}$ is a projection). The existence of such a homotopy implies that $[f^*_{I(p)} f_{U(p)}]_1 = [u]_1$, hence $\delta_0([p]_0) = \beta_B((I_* - U_*)([p]_0))$.

When $\mathbb{D}$ and $\mathbb{A}$ are not unital, the result follows from a diagram chase, using naturality of the exponential map. □

Corollary 1.28. *Let $\mathbb{A}$ and $\mathbb{B}$ be C^*-algebras with $K_1(\mathbb{A}) = K_1(\mathbb{B}) = 0$. Let $\mathbb{D}$ be a double mapping cylinder of $\mathbb{A}$ and $\mathbb{B}$ via maps $I, U : \mathbb{A} \to \mathbb{B}$ as in Lemma 1.27. Then*

$$K_0(\mathbb{D}) \simeq \ker(I_0 - U_0), \qquad K_1(\mathbb{D}) \simeq \operatorname{coker}(I_0 - U_0)$$

Proof. Since $K_1(\mathbb{A})$ and $K_0(S\mathbb{B}) \simeq K_1(\mathbb{B})$ are trivial, the six-term exact sequence 1.5 simiplifies to

$$0 \longrightarrow K_0(\mathbb{D}_k) \xrightarrow{\pi_*} K_0(\mathbb{A}_k) \xrightarrow{\delta_1} K_1(S\mathbb{B}_k) \xrightarrow{\iota^*} K_1(\mathbb{D}_k) \longrightarrow 0 \tag{1.6}$$

Using Lemma 1.27, the map δ_1 is equal to $\beta_B \circ (I_0 - U_0)$, and since the Bott map is an isomorphism, the result follows. □

Chapter 2

The C^*-algebra of an amended transformation groupoid

The idea of associating a topological groupoid G_ψ to a dynamical system $\psi : X \to X$ has a long history, beginning with the work of Renault in [30] and developed further by Deaconu, Anatharaman-Delaroche and many others (see e.g. [10], [2]). The construction generalises the well-known reduced crossed product-C^*-algebra in the following way: Assume that $\psi : X \to X$ is a homeomorphism. Then ψ induces an $\mathbb{Z}$-action $\alpha : \mathbb{Z} \to \mathrm{Aut}(C(X))$ defined as

$$\alpha(p)(f)(x) = f(\psi^{-p}(x)), \qquad p \in \mathbb{Z}, \quad x \in X, \quad f \in C(X)$$

and we may construct the reduced crossed product $C(X) \rtimes_{\psi,r} \mathbb{Z}$ (see e.g. [51]). On the other hand, we may form the *transformation groupoid* G_ψ defined as the set

$$G_\psi = \{(x,p,y) \in X \times \mathbb{Z} \times X \mid \psi^p(x) = y\}$$

with composable elements

$$G_\psi^{(2)} = \{(x,p,\psi^p(x)),(y,q,\psi^q(y)) \in G_\psi \mid \psi^p(x) = y\}$$

and composition $(x,p,z)(z,q,y) = (x,p+q,y)$ and inversion $(x,p,y)^{-1} = (y,-p,x)$. Connecting the two constructions, there is a map $\Phi : C_c(G_\psi) \to C(X) \rtimes_{\psi,r} \mathbb{Z}$ given by

$$\Phi(f)(p)(x) = f(\varphi^{-p}(x),p,x), \qquad f \in C_c(G_\psi), \quad p \in \mathbb{Z}, \quad x \in X$$

which extends to an isomorphism $\Phi : C_r^*(G_\psi) \to C(X) \rtimes_{\psi,r} \mathbb{Z}$ (for a proof, see Proposition 1.8 of [23]). The groupoid construction, however, makes sense for a larger class of maps than just homeomorphisms – see e.g. [10], which studies the transformation groupoid of a local homeomorphism of a compact space, or [7] for the groupoid of a locally injective surjection on a metric space.

We now aim to take the construction of transformation groupoids (and their C^*-algebras) even further, and adapt the construction to the case where the dynamics ψ is no longer locally injective. To make things work – more precisely, to equip the groupoid with a sufficiently nice topology –, we restrict to the situation where the underlying space is the unit circle $\mathbb{T}$ and the map ψ has critical points. Before constructing the groupoids, we need some general results on dynamical systems.

2.1 Dynamical systems on the circle

This section contains a number of basic results on dynamical systems, starting in a very general setup, and later specialising to the case where the underlying space is the unit circle $\mathbb{T}$.

Definition 2.1. A *dynamical system* (X, φ) is a metric space X with a continuous map $\varphi : X \to X$.

As always, choices must be made: One could impose fewer restrictions on X (i.e. by letting X be a topological space), require X to be compact, or put more restrictions on φ (bijectivity is perhaps the most common). For now, we stick with this definition and specialise as necessary.

Definition 2.2. Let (X, φ) be a dynamical system, and $x \in X$. Define the *forward orbit* $\mathcal{O}^+_\varphi(x)$, the *backward orbit* $\mathcal{O}^-_\varphi(x)$ and the *full orbit* $\mathcal{O}_\varphi(x)$ of x as follows:

$$\mathcal{O}^+_\varphi(x) = \{\varphi^n(x) \mid n = 0, 1, 2, \dots\}$$
$$\mathcal{O}^-_\varphi(x) = \bigcup_{n=1}^{\infty} \varphi^{-n}(\{x\})$$
$$\mathcal{O}_\varphi(x) = \mathcal{O}^+_\varphi(x) \cup \mathcal{O}^-_\varphi(x)$$

Note that x is always an element of its own forward orbit, but not (necessarily) of its own backward orbit.

Definition 2.3. Let (X, φ) be a dynamical system. We say that x is *(n-)periodic* if there is an $n > 0$ such that

$$\varphi^n(x) = x \tag{2.1}$$

The *minimal period* of x is the smallest n satisfying Equation 2.1. A point x is *pre-periodic* (or eventually periodic) if there is an $k \geq 0$ such that $\varphi^k(x)$ is periodic.

Note that a periodic point is pre-periodic by definition. We now specialise to the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, given the usual metric topology, and the usual (counter-clockwise) orientation. We note that for any continuous map $\varphi : \mathbb{T} \to \mathbb{T}$, there is a unique map $f : [0, 1] \to \mathbb{R}$ satisfying

$$f(t) = \varphi(e^{2\pi i t}), \qquad t \in [0, 1]$$

and $f(0) \in [0, 1[$. f is called a lift of φ, and the integer $f(1) - f(0)$ the *degree* of φ.

Definition 2.4. Let $\varphi : \mathbb{T} \to \mathbb{T}$ be continuous with lift f. We say that φ is *piecewise monotone* if there are points $0 = c_0 < c_1 < \cdots < c_n = 1$ such that f is strictly increasing or decreasing at each interval $]c_{i-1}, c_i[$.

This condition is sometimes called *piecewise strictly monotone*, since we do not allow the map to be locally constant.

Definition 2.5. Let $l : \mathbb{T} \to [0, 1[$ be the inverse of the map $t \mapsto e^{2\pi i t}$, let $\varphi : \mathbb{T} \to \mathbb{T}$ be piecewise monotone, and let f be a lift of φ (i.e. $f(l(z)) = \varphi(z)$ for all $z \in \mathbb{T}$). We say that $z \in \mathbb{T}$ is *critical* for φ if $l(z)$ is critical for f, and call z a maximum/minimum if $l(z)$ is a maximum/minimum for f. A point $d \in \mathbb{T}$ is *pre-critical* if there is an $n \geq 0$ such that $\varphi^n(d)$ is critical, and *post-critical* if there is an $n \in \mathbb{N}$ and some element $c \in \varphi^{-n}(d)$ such that c is critical.

Note that the Definition 2.4 ensures that a piecewise monotone map has only finitely many critical points, and that the pre-image $\varphi^{-1}(x)$ of a point x is always finite. We can now define the class of maps that will concern us for the next many pages:

Definition 2.6. A *circle map* is a map $\varphi : \mathbb{T} \to \mathbb{T}$ which is continuous, surjective and piecewise monotone.

When nothing else is specified, the term 'circle map' will be used in the precise sense defined above.

2.2 Local transfers

Traditionally, the construction of algebraic objects from a dynamical system (X, φ) has been based on studying points $x, y \in X$ such that

$$\varphi^k(x) = \varphi^l(y) \text{ for some numbers } k, l \in \mathbb{N}. \tag{2.2}$$

This is also the approach taken here – but with an extra requirement stemming from the fact that a circle map may have critical points. In short, we want to amend Equation 2.2 to remove cases where x is critical for φ^k, while y is not critical for φ^l. Or put as a one-liner: φ^k around x should 'look like' φ^l around y. We encode this condition using *local transfers*, which we'll turn to now.

Definition 2.7. Let ψ be a circle map, let $U \subseteq \mathbb{T}$ be open, and let $\eta : U \to \eta(U)$ be a homeomorphism onto its image. We say that η is a *local transfer* (of degree (n, m)) if $\psi^n(\eta(x)) = \psi^m(x)$ for any $x \in U$. We write $\mathcal{T}(m, n)$ for the local transfers of degree (n, m), and put

$$\mathcal{T}(k) = \bigcup_{n-m=k} \mathcal{T}(n, m)$$

for $k \in \mathbb{Z}$. Finally, define $\mathcal{T} = \cup_{k \in \mathbb{Z}} \mathcal{T}(k)$.

Remark 2.8. The set $\mathcal{T}$ of local transfers constitute a *pseudogroup* – it has an identity element $\mathrm{id}_{\mathbb{T}}$, all elements have inverses, and given two transfers $\eta : U \to \eta(U)$ and $\rho : V \to \rho(V)$, we may form the composition $\rho \circ \eta : U \cap \eta^{-1}(V) \to \rho \circ \eta(U \cap \eta^{-1}(V))$, provided $\eta(U) \cap V \neq \emptyset$. Local transfers implements the idea that ψ^n at a point $x \in \mathbb{T}$ 'looks like' ψ^m at a point y. Indeed, assume that ψ^m has a critical point at $y \in \mathbb{T}$, and suppose there is a local transfer $\eta \in \mathcal{T}(n, m)$ such that $\eta(y) = x$. Then $\psi^n(x) = \psi^m(y)$, and x must be a critical point for ψ^n – indeed, if ψ^n was monotone at x, $\psi^m = \psi^n \circ \eta$ would be monotone at $\eta^{-1}(x) = y$. Furthermore, y is a local minimum for ψ^m if and only if x is a local minimum for ψ^n, and similarly for local maxima. ♦

The next lemma adresses the following question: Given two points $x, y \in \mathbb{T}$ and a $k \in \mathbb{N}$, when does there exist a transfer $\eta \in \mathcal{T}(k)$ with $\eta(y) = x$, and when is it unique?

Lemma 2.9. *Let $x, y \in \mathbb{T}$, and let $k \in \mathbb{N}$. Assume that there exist numbers n, m with $n - m = k$ such that $\psi^n(x) = \psi^m(y)$. Then:*

- *If both x and y are local maxima (or both local minima) for some iterate of ψ, there are two (germs of) local transfers $\eta_1, \eta_2 \in \mathcal{T}(k)$ with $\eta_i(y) = x$.*
- *If neither x nor y are pre-critical, there is exactly one (germ of a) local transfer $\eta \in \mathcal{T}(k)$ with $\eta(y) = x$.*

Proof. Assume first that x is not precritical for any $n \in \mathbb{N}$. In particular, for any n, there is an interval around x where ψ^n is monotone. Hence, for any n, we may consider the inverse map ψ^{-n} around some small neighbourhood of $\psi^n(x)$. Now, if η satisfies $\psi^n \circ \eta = \psi^m$ for some n, m with $n - m = k$, it follows that $\eta = \psi^{-n} \circ \psi^m$ locally (and indeed, this choice of η is a homeomorphism on a suitably small open set). Had we chosen other numbers n', m' with $n' - m' = k$, we would have had $\psi^{-n} \circ \psi^m = \psi^{-n'} \circ \psi^{m'}$, which shows uniqueness.

Note next that if η is a transfer taking y to x, and $\psi^n \circ \eta = \psi^m$, ψ^n can have a mininum (resp. maximum) at x if and only if ψ^m has a minimum (resp. maximum) at y. Now, assume that $\eta \in \mathcal{T}(n, m)$ for some n, m with $n - m = k$, and that $\eta(y) = x$. Also, assume that x and y are minima for ψ^n and ψ^m, respectively. Choose small intervals I_l and I_r to the left and right of x such that $\psi^n(I_l) = \psi^n(I_r)$, and let ψ_l^{-n} and ψ_r^{-n} be inverses to ψ^n (such that $\psi^n \circ \psi_l^{-n} = \mathrm{id}_{I_l}$ and $\psi^n \circ \psi_r^{-n} = \mathrm{id}_{I_r}$. Then two choices of η are

$$\eta_1(t) = \begin{cases} \psi_l^{-n} \circ \psi^m(t), & t < y \\ x, & t = y \\ \psi_r^{-n} \circ \psi^m(t), & t > y \end{cases}, \quad \eta_2(t) = \begin{cases} \psi_r^{-n} \circ \psi^m(t), & t < y \\ x, & t = y \\ \psi_l^{-n} \circ \psi^m(t), & t > y \end{cases}$$

As before, these two choices of η are the only ones possible, independently of choice of n and m. □

Another way of interpreting the above lemma is by introducing the notion of valency of a map:

Definition 2.10. Let $\psi : \mathbb{T} \to \mathbb{T}$ be continuous and piecewise strictly monotone. The *valency* $\mathrm{val}(\psi, x)$ of ψ at a point $x \in \mathbb{T}$ is an element of the set $\mathcal{V} = \{(-,+), (+,-), (+,+), (-,-)\}$ defined as follows:

- $\mathrm{val}(\psi, x) = (-,+)$ if x is a local minimum of ψ.
- $\mathrm{val}(\psi, x) = (+,-)$ if x is a local maximum of ψ.
- $\mathrm{val}(\psi, x) = (+,+)$ if x is increasing at x.
- $\mathrm{val}(\psi, x) = (-,-)$ if x is decreasing at x.

Using the following composition table, we may turn $\mathcal{V}$ into a semigroup with $(+,+)$ as neutral element:

$x \bullet y$	$y = (+,+)$	$y = (+,-)$	$y = (-,+)$	$y = (-,-)$
$x = (+,+)$	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
$x = (+,-)$	$(+,-)$	$(+,-)$	$(+,-)$	$(+,-)$
$x = (-,+)$	$(-,+)$	$(-,+)$	$(-,+)$	$(-,+)$
$x = (-,-)$	$(-,-)$	$(-,+)$	$(+,-)$	$(+,+)$

Table 2.1: The composition table for $\bullet$.

This composition respects composition of maps:

Lemma 2.11. *Let ψ, φ be continuous, piecewise strictly monotone circle maps, and $x \in \mathbb{T}$. Then*

$$\mathrm{val}(\varphi \circ \psi, x) = \mathrm{val}(\varphi, \psi(x)) \bullet \mathrm{val}(\psi, x)$$

Proof. Straightforward. □

We note that x is critical for ψ^n if $\text{val}(\psi^n, x) \in \{(+,-),(-,+)\}$. In particular, the above lemma implies that

$$\text{val}(\psi^n, x) = \text{val}(\psi, \psi^{n-1}(x)) \bullet \cdots \bullet \text{val}(\psi, \psi(x)) \bullet \text{val}(\psi, x)$$

a formula that will become useful time and time again. Together with Table 2.2, it implies for instance that if $\text{val}(\psi^n, x) \in \{(+,-),(-,+)\}$, it is also the case that $\text{val}(\psi^{n+k}, x) \in \{(+,-),(-,+)\}$ for all $k > 0$.

Using this idea, we may reformulate Lemma 2.9:

Lemma 2.12. *Let $x, y \in \mathbb{T}$, and let $k \in \mathbb{N}$. Assume that there exist numbers n, m with $n - m = k$ such that $\psi^n(x) = \psi^m(y)$. Then:*

- *If $\text{val}(\psi^{n'}, x) = \text{val}(\psi^{m'}, y) \in \{(+,-),(-,+)\}$ for some n', m' with $n' - m' = k$, there are two germs $\eta_1, \eta_2 \in \mathcal{T}_k(\psi)$ such that $\eta_i(y) = x$.*
- *If $\text{val}(\psi^{n'}, x), \text{val}(\psi^{m'}, y) \in \{(+,+),(-,-)\}$ for all n', m', there is a unique germ $\eta \in \mathcal{T}_k(\psi)$ with $\eta(y) = x$.*

Proof. Immediate from Lemma 2.9 and the definition of valency. See also figure 2.4 at the end of this chapter. □

Remark 2.13. For future reference, we note a simple fact regarding the orientation of the local transfers in Lemma 2.12: In the first case, with two possible germs η_1 and η_2 taking y to x, observe that one germ preserves the standard orientation on $\mathbb{T}$, while the other reverses it. In the other case, the unique germ preserves orientation if and only if $\text{val}(\varphi^n, x) = \text{val}(\varphi^m, y)$, and reverses orientation if this is not the case. Thus, the lemmas become simpler if we restrict ourselves to only orientation-preserving local transfers: Such (germs of) transfers exist if only if $\text{val}(\varphi^n, x) = \text{val}(\varphi^m, y)$, and if they exist, they are also unique. ♦

2.3 The amended transformation groupoid of a circle map

Consider a circle map φ with an associated set of local transfers $\mathcal{T}$. Let Σ be a sub-pseudogroup of $\mathcal{T}$ – i.e. a set containing the identity morphisms and closed under inversion and compositions. We will refer to Σ as a *pseudogroup of local transfers*. The aim of this section is to take this data and construct a topological groupoid $\mathcal{G}_\varphi(\Sigma)$ that somehow encodes the dynamics of φ.

Remark 2.14. All these pseudogroups and sub-pseudogroups might seem rather abstract – to simplify matters, lets note that there are really only four particular psedogroups that we care about. The first is $\mathcal{T}$ itself, the set of *all* local transfers for φ. The next, which we will denote by $\mathcal{T}^+$, is the subset consisting of all local transfers that preserve orientation, i.e. $x < y$ implies $\eta(x) < \eta(y)$. The final two are the sets $\mathcal{T}(0)$ and $\mathcal{T}(0) \cap \mathcal{T}^+$, that is, the elements η satisfying $\varphi^k \circ \eta = \varphi^k$ for some k – i.e. η has to intertwine some iterate of φ with itself, and not just another iterate. The groupoids associated to the final two pseudogroups will be the focus of Chapter 3.♦

We define the groupoids in a number of steps:

Definition 2.15. Let ψ be a circle map, and Σ a pseudogroup of local transfers. Put

$$\mathcal{G}_\psi(\Sigma) = \{(x, k, \eta, y) \in \mathbb{T} \times \mathbb{Z} \times \Sigma \times \mathbb{T} \mid \eta \in \mathcal{T}_\Sigma(k), \eta(x) = y\}$$

One may turn $\mathcal{G} = \mathcal{G}_\psi(\Sigma)$ into a groupoid with unit space $\mathbb{T}$ and rules for composition and inversion inspired by those for regular transformation groupoids. However, $\mathcal{G}$ is not quite the groupoid we're looking for – if (x, k, η, y) is an element of $\mathcal{G}_\psi$, we can restrict η to any subset of its domain (containing x), obtain a new local transfer $\bar{\eta}$ and hence another element $(x, k, \bar{\eta}, y) \in \mathcal{G}_\psi$, a priori different from (x, k, η, y). We want such two elements to be identified, so we need to 'mod out' by an appropriate equivalence relation.

Definition 2.16. Let $U, V \subseteq \mathbb{T}$ be open sets, $x \in U$, and $\eta : U \to V$ a map. Define $[\eta]_x$, the germ of η at x, as the set of all homeomorphisms $\rho : U' \to V'$ such that ρ is equal to η on a sufficiently small neighbourhood of x.

Define an equivalence relation $\sim$ on $\mathcal{G}_\psi$ by $(x, k, \eta, y) \sim (x', k', \eta', y')$ if $(x', k', y') = (x, k, y)$ and $[\eta]_x = [\eta']_x$.

Definition 2.17. Let ψ be a circle map, Σ a pseudogroup of local transfers on $\mathbb{T}$, and $\mathcal{G}_\psi(\Sigma)$ the set from Definition 2.15. We define the *amended transformation groupoid* $G_\psi(\Sigma)$ of ψ and Σ as the set $\mathcal{G}_\psi(\Sigma)/\sim$. The unit space of $G_\psi(\Sigma)$ is $\mathbb{T}$, with range and source maps given by projection onto first and last component. The composable elements $G^{(2)}\Sigma)_\psi$ are given by

$$G^{(2)}(\Sigma)_\psi = \{(x, k, [\eta]_x, y), (x', k', [\eta']'_x, y') \in G_\psi \mid y = x'\}$$

and composition and inversion given by

$$(x, k, [\eta]_x, y)(y, k', [\eta']'_x, y') = (x, k + k', [\eta' \circ \eta]_x, y'), \quad (x, k, [\eta]_x, y)^{-1} = (y, -k, [\eta^{-1}]_y, x)$$

If φ is a circle map, we will write Γ_φ as a short-hand for $G_\varphi(\mathcal{T})$ and Γ_φ^+ for $G_\varphi(\mathcal{T}^+)$. Briefly put, the remainder of this dissertation studies the structure of Γ_φ, Γ_φ^+ and their C^*-algebras, as well as the relationship and the differences between these two algebras. Many theorems will be of the type 'Let G be either Γ_φ or $\Gamma_\varphi^+ \ldots$'. In some cases, we might be able to prove some result about both groupoids in one fell swoop; in other cases, we treat the two groupoids separately, or prove the result first for one groupoid and then use it to prove it for the other. As a useful intuition Γ_φ^+ can tell the valencies $(+,+)$ and $(-,-)$ apart, while Γ_φ cannot.

Remark 2.18. Dealing with the local transfers directly is troublesome. Here's another 'picture' of the two grupoids Γ_φ and Γ_φ^+, using Lemma 2.12. First, by this lemma and the subsequent remark, we have $(x, k, [\eta], y) \in \Gamma_\varphi^+$ if and only if there are $n, m \in \mathbb{N}$ with $n - m = k$, $\varphi^n(x) = \varphi^m(y)$ and $\mathrm{val}(\varphi^n, x) = \mathrm{val}(\varphi^m, y)$. On the other hand, if there are there are $n, m \in \mathbb{N}$ with $n - m = k$, $\varphi^n(x) = \varphi^m(y)$ and $\mathrm{val}(\varphi^n, x) = \mathrm{val}(\varphi^m, y)$, there is a unique local transfer η with $\eta(x) = y$. It follows that we can forget about the local transfers and write

$$\Gamma_\varphi^+ = \{(x, k, y) \in \mathbb{T} \times \mathbb{Z} \times \mathbb{T} \mid \exists n, m : k = n - m, \varphi^n(x) = \varphi^m(y), \mathrm{val}(\varphi^n, x) = \mathrm{val}(\varphi^m, y)\}$$

For Γ_φ, the situation is a bit more murky: Write the group $\mathbb{Z}_2$ as $\{+, -\}$ with the obvious composition, and define a map $V : \mathcal{P} \to \mathbb{Z}_2$ by $V(\eta) = +$ if η preserves orientation and $V(\eta) = -$ if η reverses orientation. V respects the composition in $\mathcal{P}$, and by Lemma 2.12, we have

$$\begin{aligned}\Gamma_\varphi = \{(x, k, p, y) \in \mathbb{T} \times \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{T} \exists n, m \in \mathbb{N}, \eta \in \mathcal{T}_\varphi(k) : & n - m = k, \varphi^n(x) = \varphi^m(y), \\ & \eta(x) = y, V(\eta) = p\}\end{aligned} \tag{2.3}$$

Essentially, for each triple (x, k, y), there are at most two transfers such that $(x, k, [\eta]_x, y)$ is in Γ_φ, and the transfer is determined uniquely by its orientation, represented as an element of $\mathbb{Z}_2$. ♦

We proceed to equip $G_\varphi(\Sigma)$ with a topology, and show that this topology is sufficiently nice.

Definition 2.19. Let φ be a circle map, Σ a pseudogroup of local transfers, $U \subseteq \mathbb{T}, k \in \mathbb{Z}$ and $\eta \in \Sigma(k)$ with U inside the domain of η. Put

$$U(\eta) = \{[z, k, \eta, \eta(z)] \mid z \in U\} \subseteq G_\varphi(\Sigma)$$

Let Ω_Σ denote the collection of sets $\{U(\eta)\}$ with U ranging over the open subsets of $\mathbb{T}$, k over $\mathbb{Z}$ and η over Σ. We equip $G_\varphi(\Sigma)$ with the topology generated by these sets.

Proposition 2.20. *In the topology defined above, the groupoids Γ_φ and Γ_φ^+ are locally compact, second countable, Hausdorff and étale.*

Proof. Let G denote either of the groupoids Γ_φ and Γ_φ^+, and let $U(\eta)$ be an open set in G. Both range and source maps restrict to local homeomorphisms on $U(\eta)$, so G is étale and locally compact. To show that G is Hausdorff, let $\gamma = [x, k, \eta, y]$ and $\gamma' = [x', k', \eta', y'] \in G$ with $\gamma \neq \gamma'$. If $x \neq x'$, $y \neq y'$ or $k \neq k'$, it is straightforward to choose disjoint neighbourhoods of γ and γ'. If $[\eta]_x \neq [\eta']_x$, it follows from Lemma 2.9 that that there is a neighbourhood U of x such that $\eta(z) = \eta'(z)$ implies $z = x$ for all $y \in U$. Hence, $[\eta]_z \neq [\eta']_z$ for all $z \in U$, so $U(\eta)$ and $U(\eta')$ are disjoint neighbourhoods of γ and γ'.

To show that G is second countable, we need to work a bit more. The approach is completely similar to that of Proposition 4.3 in [43]: Fix m and n, and note that the set of points critical for either φ^n or φ^m is finite. Choose a countable basis $\{U_i\}_{i=1}^\infty$ for the topology of $\mathbb{T}$ such that each U_i is connected and contains at most one critical of φ^n or φ^m, and such that φ^m is injective on the U_i's not containing critical points of φ^m. Let $i, j \in \mathbb{N}$. If U_i and U_j contains no critical points of φ^n and φ^m, respectively, there is only one transfer η with domain U_i such that

$$\eta(U_i) \subseteq U_j \text{ and } \varphi^n(z) = \varphi^m(\eta(z)) \text{ for all } z \in U_i \tag{2.4}$$

If U_i contains a critical point of φ^n and U_j a critical point of φ^m, there can be at most two transfers satisfying Equation 2.4 (the exact number depends on the given valencies of φ^n and φ^m and on the groupoid). In either case, the collection $A(n, m, i, j)$ of local transfers satisfying Equation 2.4 is finite for all n, m, i, j. Now, if η is a local transfer of order k and $U \subseteq \mathbb{T}$ is open, $U(\eta)$ is a countable union of sets

$$\{[z, k, \mu, \mu(z)] \mid z \in U_i\}$$

for some (n, m, i, j) with $n - m = k$ and $\mu \in A(n, m, i, j)$. It follows that these sets form a countable basis for the topology on G. □

2.4 On the structure of $C^*(\Gamma_\varphi)$ and $C_r^*(\Gamma_\varphi^+)$

Having equipped Γ_φ and Γ_φ^+ with sufficiently nice topologies, we can form their groupoid C^*-algebras $C_r^*(\Gamma_\varphi)$ and $C_r^*(\Gamma_\varphi^+)$. We now take the approach laid out in the introduction – assume some dynamical property of φ, and see how it reflects on the C^*-algebraic properties of the corresponding C^*-algebras. Or the other way around: Require $C_r^*(\Gamma_\varphi)$ or $C_r^*(\Gamma_\varphi^+)$ to have a certain property, and investigate how that affects the dynamics.

Definition 2.21. Let $\varphi : \mathbb{T} \to \mathbb{T}$ be a continuous map. We say that φ is *transitive* if for any pair of open subsets U and V of $\mathbb{T}$, there is an $n \in \mathbb{N}$ such that $\varphi^n(U) \cap V \neq \emptyset$.

Proposition 2.22. *Assume that φ is transitive circle map and not locally injective. It follows that there is an orientation-preserving homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that $h \circ \varphi \circ h^{-1}$ is uniformly piecewise linear with slope $s > 1$.*

Proof. We will show how the theorem follows from the work of Shultz in [39] on discontinuous piecewise monotone maps of the interval.

After conjugation by a rotation of the circle we can assume that $\varphi(1) \neq 1$ and $1 \notin \varphi(\mathcal{C}_1)$. (Indeed, since φ is piecewise monotone and transitive there are λ's in $\mathbb{T}$ arbitrary close to 1 such that $\varphi(\lambda) \neq \lambda$. Choose one of them such that $\lambda \notin \varphi(\mathcal{C}_1)$. Then $\varphi_1(t) = \lambda^{-1}\varphi(\lambda t)$ is conjugate to φ, does not fix 1 and all its critical values are different from 1.) Let $\mu : \mathbb{T} \to [0,1[$ be the inverse map of $[0,1[\ni t \mapsto e^{2\pi i t}$. Then

$$\tau(t) = \mu \circ \varphi(e^{2\pi i t})$$

is piecewise monotone in the sense of Shultz [39]. Since φ is surjective and $1 \notin \varphi\left(\mathcal{C}_1 \cup \{1\}\right)$, it follows that τ is discontinuous at a point in $]0,1[$ and $\tau\left([0,1]\right) = [0,1[$. We claim that τ is transitive in the sense of Definition 2.6 in [39]; that is, we claim that for every open non-empty subset $U \subseteq [0,1]$ there is an $n \in \mathbb{N}$ such that

$$\bigcup_{i=0}^{n} \widehat{\tau}^k(U) = [0,1] \tag{2.5}$$

Here $\widehat{\tau}$ is the possibly multivalued map on $[0,1]$ which associates to each $x \in [0,1]$ the left and right hand limits of τ at x. By construction this union is either $\{\tau(x)\}$ or $\{1,0\}$. In the latter case $0 = \tau(x)$. It follows therefore that $\widehat{\tau}(A)\backslash\{1\} = \tau(A)$ for every subset $A \subseteq [0,1]$. Thus

$$\widehat{\tau}^k(U) \supseteq \tau^k(U)$$

for all k. The strong transitivity of φ implies that $\bigcup_{i=0}^{n-1} \tau^k(U) = [0,1[$ for some $n \in \mathbb{N}$. As observed above τ is discontinuous at a point in $]0,1[$. It follows therefore that $1 \in \widehat{\tau}\left([0,1[\right)$ and hence that (2.5) holds since

$$\bigcup_{i=0}^{n} \widehat{\tau}^i(U) \supseteq \widehat{\tau}\Big(\bigcup_{i=0}^{n-1} \widehat{\tau}^i(U)\Big) \supseteq \widehat{\tau}\Big(\bigcup_{i=0}^{n-1} \tau^i(U)\Big) = \widehat{\tau}([0,1[) = [0,1].$$

It follows now from Propositions 4.3 and 3.6 in [39] that there is a homeomorphism $h : [0,1] \to [0,1]$ such that $f = h \circ \tau \circ h^{-1}$ is uniformly piecewise linear. From the proof of Proposition 3.6 in [39] we see that h is increasing. Since φ is not locally injective there are non-empty open intervals $I, I' \subseteq \mathbb{T}\backslash\{1\}$ such that $I \cap I' = \emptyset$ and $\varphi(I) = \varphi(I')$. Then $J = \mu(I)$ and $J' = \mu(I')$ are non-empty open intervals in $[0,1[$ such that $J \cap J' = \emptyset$ and $\tau(J) = \tau(J')$, i.e. τ is not essentially injective in the sense of Definition 4.1 of [39]. Hence the slope s of the linear pieces of f is > 1 by Proposition 4.3 of [39].

Since $h(0) = 0, h(1) = 1$ and $\tau(0) = \tau(1)$ we find that $f(0) = f(1)$ and we can therefore define $\varphi : \mathbb{T} \to \mathbb{T}$ such that $\varphi\left(e^{2\pi i t}\right) = e^{2\pi i f(t)}, t \in [0,1]$. Then $\varphi = g \circ \varphi \circ g^{-1}$ where $g = \mu^{-1} \circ h \circ \mu$. Then $g(1) = 1 = \lim_{\lambda \to 1} g(\lambda)$. Hence g is continuous and an orientation preserving homeomorphism on $\mathbb{T}$. It follows that φ is a continuous map on $\mathbb{T}$ and conjugate to φ. By construction φ is uniformly piecewise linear with slope $s > 1$. □

Remark 2.23. Let φ be transitive, and choose an orientation preserving homeomorphisms $h : \mathbb{T} \to \mathbb{T}$ such that $\psi = h \circ \varphi \circ h^{-1}$ is uniformly piecewise linear with slope $s > 1$. Then the map $H : \Gamma_\varphi \to \Gamma_\psi$ given by

$$H(x,k,p,y) = (h^{-1}(x), k, p, h^{-1}(y))$$

is an isomorphism of groupoids. Hence, whenever it is convenient, we may assume without loss of generality that our map is uniformly piecewise linear (assuming it is transitive). ♦

We say that a p-periodic point $x \in \mathbb{T}$ is *repelling* when there is an open interval I in $\mathbb{T}$ and a $r > 1$ such that $x \in I$ and $|\varphi^p(y) - x| \geq r\,|y - x|$ for all $y \in I$.

Lemma 2.24. *Assume that φ is transitive and uniformly piecewise linear with slope $s > 1$. Then the periodic points of φ are dense in $\mathbb{T}$ and they are all repelling.*

Proof. Since φ is transitive there is a point in $\mathbb{T}$ with dense forward orbit, cf. Theorem 5.9 in [50]. It follows therefore from Corollary 2 in [4] that φ has periodic points, and then by Corollary 3.4 in [9] that the periodic points are dense. For each $n \in \mathbb{N}$ the map φ^n is uniformly piecewise linear with slope $s^n > 1$. Therefore all periodic points of φ are repelling. □

Lemma 2.25. *Assume that φ is transitive. Then Γ_φ and Γ^+_φ are locally contractive; that is, for every open set $U \subseteq \mathbb{T}$, there is an open set $V \subseteq U$ and an open bisection S such that $\overline{V} \subseteq s(S)$ and $s(VS^{-1}) \subsetneq V$.*

Proof. Let $U \subseteq \mathbb{T}$ be open. By Lemma 2.24, there is a periodic and repelling point in U. Since there are only finitely many critical points, the union of orbits of periodic points which contain a critical point is a finite set – hence we may choose a $z_0 \in U$ that is periodic (of period n), repelling, and whose orbit contains no critical points. Now, $\mathrm{val}(\varphi^{2n}, z_0) = (+,+)$, so there is a neighbourhood $W \subseteq U$ of z_0 and a $\kappa > 1$ such that $|\varphi^{2n}(y) - z| \geq \kappa |y - z|$ for all $y \in W$. From here, one may follow the proof of Proposition 4.1 in [46], replacing the set $J(f) \setminus \mathcal{E}(f)$ by $\mathbb{T}$. □

Lemma 2.26. *Assume that φ is transitive. Then Γ_φ and Γ^+_φ are topologically principal, i.e. the set of points with trivial isotropy is dense in $\mathbb{T}$.*

Proof. We first consider Γ_φ: Let $x \in \mathbb{T}$, and note that the isotropy group Is_x is given by

$$\mathrm{Is}_x = \{\gamma \in \Gamma_\varphi \,|\, r(\gamma) = s(\gamma) = x\} = \{(x, k, p, x) \in \Gamma_\varphi\}$$

Assume that x has non-trivial isotropy. Then x is either pre-periodic (so $(x, k, p, x) \in \Gamma_\varphi$ for some $k > 0$) or pre-critical (so $(x, 0, -, x) \in \Gamma_\varphi$). The set $\bigcup_{j \in \mathbb{N}} \varphi^{-j}(\mathcal{C})$ is countable, so it has empty interior. We claim that the same is true for the set of pre-periodic points. Since φ is transitive, Lemma 2.22 implies that there is a homeomorphism $h : \mathbb{T} \to \mathbb{T}$ and a map $g : \mathbb{T} \to \mathbb{T}$ which is uniformly piecewise linear with slope $s > 1$ such that $g = h \circ \varphi \circ h^{-1}$. The set of periodic points for g is clearly a countable set, so the same is true for the set of pre-periodic points. Since h maps the pre-periodic points of g bijectively to the pre-periodic points of φ, this set is countable, too; in particular, it has empty interior.

The same proof works for Γ_φ, just delete the part about pre-critical points. □

For the statement of the next theorem, recall that a C^*-algebra is *purely infinite* if all proper hereditary subalgebras contain an infinite projection, see [2].

Theorem 2.27. *Assume that φ is transitive. Then $C^*_r(\Gamma_\varphi)$ and $C^*(\Gamma^+_\varphi)$ are purely infinite.*

Proof. The Lemmas 2.26 and 2.25, combined with Proposition 2.4 of [2], yields the desired conclusion immediately. □

In the next chapter, we shall see that transitivity of φ also has a great deal to say in deciding simplicity of $C_r^*(\Gamma_\varphi)$ and $C_r^*(\Gamma_\varphi^+)$.

Finally, let us establish a connection between the algebras $C_r^*(\Gamma_\varphi)$ and $C_r^*(\Gamma_\varphi^+)$: First, note that by Remark 2.18, there is an inclusion $i : \Gamma_\varphi^+ \to \Gamma_\varphi$ given by

$$i((x,k,y)) = (x,k,+,y), \qquad (x,k,y) \in \Gamma_\varphi^+$$

It follows that Γ_φ^+ sits inside Γ_φ as a subgroupoid, and by comparing the topologies, we see immediately that this subgroupoid is clopen. This yields an inclusion of C^*-algebras: For $f \in C_c(\Gamma_\varphi^+)$, put

$$i(f)(x,k,p,y) = \begin{cases} f(x,k,y) & \text{if } p = +, \\ 0 & \text{if } p = - \end{cases} \tag{2.6}$$

The map $i : C_c(\Gamma_\varphi^+) \to C_c(\Gamma_\varphi^+)$ is isometric, injective and respects composition and inversion, and therefore extends to a C^*-embedding $i : C_r^*(\Gamma_\varphi^+) \to C_r^*(\Gamma_\varphi)$, realising $C_r^*(\Gamma_\varphi^+)$ as a subalgebra of $C_r^*(\Gamma_\varphi)$. It turns out that something stronger holds: $C_r^*(\Gamma_\varphi)$ is the fixed point-algebra of an order-two automorphism Λ of $C_r^*(\Gamma_\varphi)$. To see this, let $f \in C_c(\Gamma_\varphi)$, and define

$$\Lambda(f)(x,k,p,y) = (-1)^p f(x,k,p,y), \;\; (x,k,p,y) \in \Gamma_\varphi \tag{2.7}$$

with the convention that $(-1)^+ = 1$ and $(-1)^- = -1$.

Proposition 2.28. *The map Λ from Equation 2.7 extends to an order-two $*$-automorphism of $C_r^*(\Gamma_\varphi)$, with fixed-point algebra $C_r^*(\Gamma_\varphi)^\Lambda$ isomorphic to $C_r^*(\Gamma_\varphi^+)$.*

Proof. Most of this is completely straightforward – for instance, multiplicativity amounts to seeing that

$$\begin{aligned}
\Lambda(f * g)(x,k,p,y) &= (-1)^p (f * g)(x,k,p,y) \\
&= (-1)^p \sum_{z,k_1+k_2=k,p_1p_2=p} f(x,k_1,p_1,z) g(z,k_2,p_2,y) \\
&= \sum_{z,k_1+k_2=k,p_1p_2=p} (-1)^{p_1} f(x,k_1,p_1,z)(-1)^{p_2} g(z,k_2,p_2,y) \\
&= \sum_{z,k_1+k_2=k,p_1p_2=p} \Lambda(f)(x,k_1,p_1,z)\Lambda(g)(z,k_2,p_2,y) \\
&= \Lambda(f) * \Lambda(g)(x,k,p,y)
\end{aligned}$$

for $f, g \in C_c(\Gamma_\varphi)$. Similarly, $\Lambda^2 = \mathrm{id}$ follows from the fact that $(-1)^p(-1)^p = 1$. Since $|\Lambda(f)(\gamma)| = |f(\gamma)|$ for any $f \in C_c(\Gamma_\varphi)$ and $\gamma \in \Gamma_\varphi$, Λ extends to an isometry $\Lambda : C_r^*(\Gamma_\varphi) \to C_r^*(\Gamma_\varphi)$, and since $\Lambda^2 = \mathrm{id}$ on the dense subset $C_c(\Gamma_\varphi)$, this also holds on $C_r^*(\Gamma_\varphi)$. Finally,

$$(-1)^p f(x,k,p,y) = f(x,k,p,y)$$

for $f \in C_c(\Gamma_\varphi)$ and all $(x,k,p,y) \in \mathrm{supp}(f)$ if and only if $f \in C_c(\Gamma_\varphi^+)$. Hence, $C_c(\Gamma_\varphi^+) \subseteq C_r^*(\Gamma_\varphi)^\Lambda$, so by continuity $C_r^*(\Gamma_\varphi^+) \subseteq C_r^*(\Gamma_\varphi)^\Lambda$. On the other hand, if $a \in C_r^*(\Gamma_\varphi)^\Lambda$, we can approximate a with elements $\{a_n\} \subseteq C_c(\Gamma_\varphi)$. But then $a_n' = 1/2(a_n + \Lambda(a_n))$ is an element of $C_c(\Gamma_\varphi)^\Lambda = C_c(\Gamma_\varphi^+)$ for all n, and a_n' converges to a. It follows that a is in $C_r^*(\Gamma_\varphi^+)$. ☐

The map Λ is sometimes known as the *flip automorphism*. We shall use it several times in the following chapters.

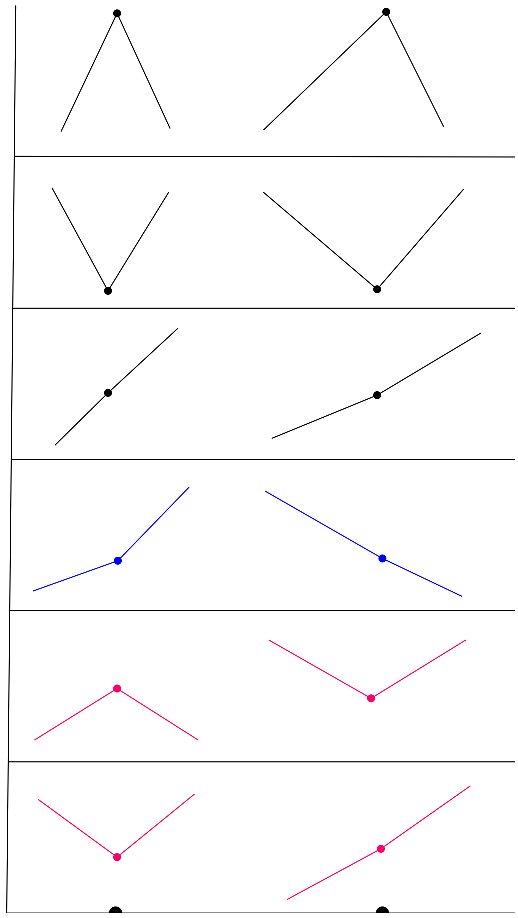


Figure 2.1: This figure shows various combinations of valencies. The graphs on the left shows the graph of some iterate φ^l around some point x, and the graphs on the right show an iterate φ^k around y. In the first three cases, we have $\text{val}(\varphi^k, x) = \text{val}(\varphi^k, y)$, so there is at least one local transfer intertwining φ^k and φ^l around x and y. In the fourth case, $\text{val}(\varphi^k, x)$ is $(+,+)$, while $\text{val}(\varphi^l, y) = (-,-)$ – so there is a local transfer intertwining φ^k and φ^l, but it reverses orientation. In the fifth and sixth case, no transfers exist..

Chapter 3

The core algebras

Let φ be a circle map (continuous, surjective and piecewise strictly monotone), and consider the groupoids Γ_φ and Γ_φ^+ introduced in Chapter 2. Each of these have a distinguished subgroupoid:

Definition 3.1. Let $\varphi : \mathbb{T} \to \mathbb{T}$ be a circle map, and let Γ_φ and Γ_φ^+ the amended transformation groupoids associated to φ as defined in Definition 2.17. Define groupoids R_φ and R_φ^+ by

$$R_\varphi = \{(x,k,[\eta]_x,y) \in \Gamma_\varphi | k = 0\}, \;\; R_\varphi^+ = \{(x,k,[\eta]_x,y) \in \Gamma_\varphi^+ | k = 0\}$$

Give R_φ and R_φ^+ the topology inherited from Γ_φ and Γ_φ^+, respectively.

It is immediate that R_φ (resp. R_φ^+) is clopen in Γ_φ (resp. Γ_φ^+). It follows that these groupoids are locally compact, second countable, Hausdorff étale groupoids in their own right. To ease notation, we will drop the k when writing elements of the groupoids – combining this with Remark 2.18, we have

$$R_\varphi^+ = \{(x,y) \in \mathbb{T} \times \mathbb{T} \mid \exists n \in \mathbb{N} : \varphi^n(x) = \varphi^n(y), \mathrm{val}(\varphi^n, x) = \mathrm{val}(\varphi^n, y)\}$$

and

$$R_\varphi = \{(x,p,y) \in \mathbb{T} \times \mathbb{Z}_2 \times \mathbb{T} \mid \exists n \in \mathbb{N}, \eta \in \mathcal{T}(0) : \varphi^n(x) = \varphi^n(y), \eta(y) = x, V(\eta) = p\}$$

The C^*-algebras of these two groupoids are sometimes referred to as the 'core algebras' of φ. In many ways, these algebras are more tractable – not least due to the fact that the groupoids have very little isotropy (R_φ^+ is the groupoid of an equivalence relation, while R_φ has $\mathbb{Z}_2$-isotropy over the countable set of pre-critical points). Apart from being interesting in themselves, studying these algebras serve a twofold purpose: First, it paves the way for calculating the K-theory of the algebras $C_r^*(\Gamma_\varphi)$ and $C_r^*(\Gamma_\varphi^+)$ and obtaining some crucial structural results about these. Second, in some cases, it gives a naturally occuring example of an AF-algebra with an order-two-automorphism whose fixed-point algebra is not AF.

We note that each core algebra arise as the fixed-point algebra of a *gauge action*:

Definition 3.2. Let $\mu \in \mathbb{T}$, and $f \in C_c(\Gamma_\varphi)$. Define $\beta_\mu(f) \in C_c(\Gamma_\varphi)$ by

$$\beta_\mu(f)(x,k,p,y) = \mu^k f(x,k,p,y), \qquad (x,k,p,y) \in \Gamma_\varphi$$

The map $f \mapsto \beta_\mu(f)$ is isometric, so β_μ extends to a $*$-automorphism of $C_r^*(\Gamma_\varphi)$. The $\mu \mapsto \beta_\mu$ from $\mathbb{T}$ to $\mathrm{Aut}(C_r^*(\Gamma_\varphi))$ called the *gauge action*.

It is easy to check that $\beta_\mu \circ \beta_\rho = \beta_{\mu\rho}$ and $\beta_\mu^{-1} = \beta_{\overline{\mu}}$ for $\mu, \rho \in \mathbb{T}$. It follows that β is a group homomorphism.

Proposition 3.3. *Let β be the gauge action on $C_r^*(\Gamma_\varphi)$, and*

$$C_r^*(\Gamma_\varphi)^\beta = \{a \in C_r^*(\Gamma_\varphi) \mid \beta_\mu(a) = a \text{ for all } \mu \in \mathbb{T}\}$$

the fixed-point algebra of β. Then $C_r^*(\Gamma_\varphi)^\beta$ is isomorphic to $C_r^*(R_\varphi)$.

Proof. As in [47], Lemma 3.1. ☐

In a completely similar way, there is a gauge action β on $C_r^*(\Gamma_\varphi^+)$, given by

$$\beta_\mu(f)(x,k,y) = \mu^k f(x,k,y)$$

for $f \in C_c(\Gamma_\varphi^+)$, and the fixed point algebra of this action is $C_r^*(R_\varphi^+)$.

To understand the algebras $C_r^*(R_\varphi)$ and $C_r^*(R_\varphi^+)$ better, we begin by realising them as inductive limits of well-understood algebras.

3.1 The finite-dimensional building blocks

Fix a $k \in \mathbb{N}$, and put

$$R_\varphi(k) = \left\{(x,p,y) \in R_\varphi \,\middle|\, \varphi^k(x) = \varphi^k(y)\right\}$$

and

$$R_\varphi^+(k) = \left\{(x,y) \in R_\varphi^+ \,\middle|\, \varphi^k(x) = \varphi^k(y)\right\}$$

These groupoids can almost be visualised – one simply has do draw the graph of φ^k and keep track of points with same image under φ^k, subject to the appropriate valency condition. The ultimate goal of the section is the following result:

Theorem 3.4. *Let $k \in \mathbb{N}$ and let R be either $R_\varphi(k)$ or $R_\varphi^+(k)$. Then there are finite-dimensional algebras $\mathbb{A}_k$ and $\mathbb{B}_k$ and $*$-homomorphisms $I_k, U_k : \mathbb{A}_k \to \mathbb{B}_k$ such that*

$$C_r^*(R) \simeq \{(a,f) \in \mathbb{A}_k \oplus C([0,1],\mathbb{B}_k) \mid I_k(a) = f(0), U_k(a) = f(1)\}$$

Combined with Lemma 1.27, this determines the K-theory of the algebras. The strategy of the proof is roughly the same for $C_r^*(R_\varphi(k))$ and $C_r^*(R_\varphi^+(k))$, with a few crucial differences – we give all details for the case $C_r^*(R_\varphi(k))$ and then outline the differences when considering $C_r^*(R_\varphi^+(k))$ instead.

Let $\mathcal{C}_k$ denote the critical points of φ^k, and fix $\mathcal{D}$ be a finite set with $\varphi^k(\mathcal{C}_k) \subseteq \mathcal{D}$. The set $\mathcal{E} = \varphi^{-k}(\mathcal{D})$ is finite, $R_\varphi(k)$-invariant and contains all the critical points of φ^k. It follows that we have reductions

$$R_\varphi(k)|_{\mathcal{E}} = \left\{(x,p,y) \in R_\varphi(k) \,\middle|\, \varphi^k(x), \varphi^k(y) \in \mathcal{D}\right\}$$

and

$$R_\varphi(k)|_{\mathbb{T}\setminus\mathcal{E}} = \left\{(x,p,y) \in R_\varphi(k) \,\middle|\, \varphi^k(x), \varphi^k(y) \notin \mathcal{D}\right\}$$

The philosophy is now as follows: Given a function $f \in C_c(R_\varphi(k))$, we get, by restriction, functions on the two reductions above. On the other hand, given maps on the two reductions above, we may – given that the maps satisfy certain compatibility relations – piece them together and get an element of $C_c(R_\varphi(k))$. Giving a precise description of these compatibility relations will occupy us for much of this section.

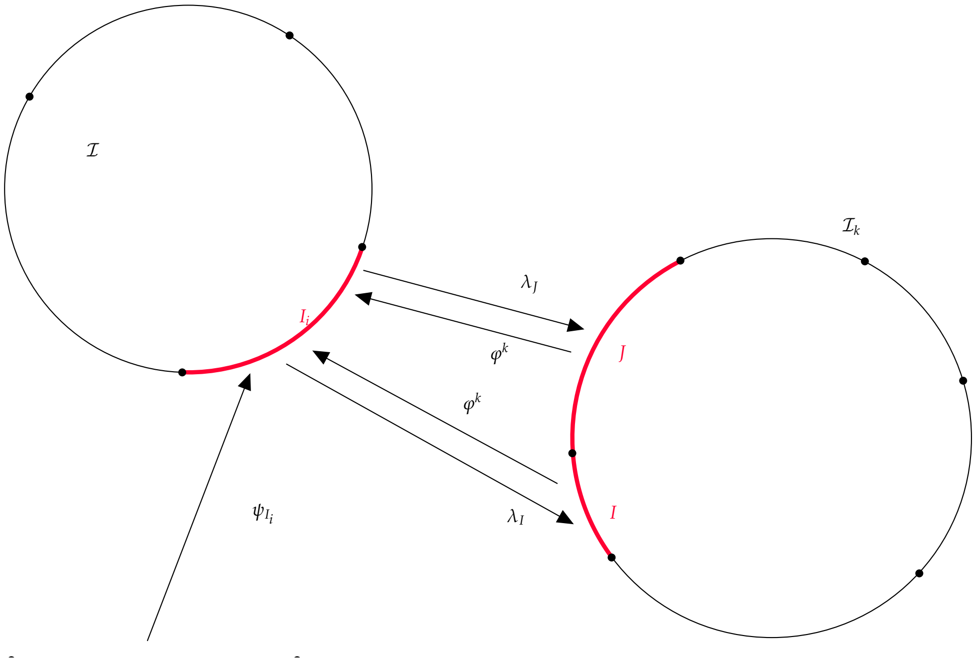


Figure 3.1: A picture of the sets $\mathcal{I}$ and $\mathcal{I}_k$ and the maps ψ_{I_i}, λ_I and λ_J. The leftmost circle is divided into intervals $\mathcal{I}$ by the points of $\mathcal{D}$, while the rightmost circle is subdivided into intervals $\mathcal{I}_k$ by the set $\mathcal{E}$. Each interval I of $\mathcal{I}_k$ is mapped to some interval in $\mathcal{I}$ by φ^k, with inverse λ_I. The map ψ_{I_i} parametrises the interval I_i along the unit interval $(0,1)$..

Remark 3.5. Note that the obvious choice of $\mathcal{D}$ would simply be $\varphi^k(\mathcal{C}_k)$. However, we will later need the extra flexibility of being able to choose any finite set containing $\varphi^k(\mathcal{C}_k)$. ♦

Write $\mathcal{D} = \{c_1, c_2, \dots, c_N\}$, and put $c_0 = c_N$. Let $\mathcal{I} = \{I_1, I_2, \dots, I_N\}$ denote the set of connected components of $\mathbb{T} \setminus \mathcal{D}$ such that $c_{i-1} < I_i < c_i$, and let $\mathcal{I}_k$ be the connected components of $\mathbb{T} \setminus \mathcal{E}$. Note that by construction, φ^k maps each element of $\mathcal{I}_k$ bijectively onto an element of $\mathcal{I}$ – hence for each $I \in \mathcal{I}_k$, φ^k has an inverse λ_I. Furthermore, for each $I_i \in \mathcal{I}$, we can define an orientation-preserving homeomorphism $\psi_{I_i} : (0,1) \to I_i$ such that $\lim_{t\to 0} \psi_{I_i}(t) = c_{i-1}$ and $\lim_{t\to 1} \psi_{I_i}(t) = c_i$ (see Figure 3.1).

Lemma 3.6. *For $i = 1, \dots, N$, let $m_i = \#\{I \in \mathcal{I}_k | \varphi^k(I) = I_i\}$. Put*

$$\mathbb{B}_k = \bigoplus_{i=1}^{N} M_{m_i}(\mathbb{C})$$

Then there is an isomorphism

$$C_r^*\left(R_\varphi(k)|_{\mathbb{T}\setminus\mathcal{E}}\right) \simeq S\mathbb{B}_k$$

where $S\mathbb{B}_k$ denotes the suspension of $\mathbb{B}_k$.

Proof. This is mainly an application of Example 1.22: Fix an $I_i \in \mathcal{I}$, and consider the set

$$A_i = \left\{ I \in \mathcal{I}_k \,\middle|\, \varphi^k(I) = I_i \right\}.$$

For $I, J \in A_i$, the map $\lambda_J \circ \varphi^k$ is a homeomorphism from I to J. Since φ^k is monotone on each I in A_i, we have $(x, p, y) \in R_\varphi(k)|_{\mathbb{T}\setminus\mathcal{E}}$ if and only if $\varphi^k(x) = \varphi^k(y)$, or, equivalently, if $y = \lambda_I(\varphi^k(x))$ for an appropriate I. It follows from Example 1.22 that

$$C_r^*(R_\varphi(k)|_{A_i}) \simeq C_0((0,1)) \otimes M_{m_i}(\mathbb{C}).$$

This works for each $I_i \in \mathcal{I}$, and the result follows. □

Put

$$\mathcal{I}_k^{(2)} = \left\{ (I, J) \in \mathcal{I}_k \times \mathcal{I}_k \,\middle|\, \varphi^k(I) = \varphi^k(J) \right\}$$

We recall from Example 1.22 that the algebra $\mathbb{B}_k$ is generated by matrix units $e_{I,J}$, $(I, J) \in \mathcal{I}_k^{(2)}$. Let $f \in C_c(R_\varphi(k))$, $(I, J) \in \mathcal{I}_k^{(2)}$, with $\varphi^k(I) = \varphi^k(J) = I_i$, and consider the map $f_{I,J} \in C((0,1))$ given by

$$f_{I,J}(t) = f(\lambda_I \circ \psi_{I_i}(t), p, \lambda_J \circ \psi_{I_i}(t)), \qquad t \in (0,1)$$

where $p = p_{I,J}$ is $+$ if $\mathrm{val}(\varphi^k, I) = \mathrm{val}(\varphi^k, J)$, and $-$ otherwise. This map has a continuous extension to a map – also denoted $f_{I,J}$ – in $C([0,1])$. Indeed, let

$$\overline{\lambda_I(c_{i-1})} = \lim_{x \to c_{i-1}} \lambda_I(x), \quad \overline{\lambda_I(c_i)} = \lim_{x \to c_i} \lambda_I(x)$$

If $(\overline{\lambda_I(c_{i-1})}), p_{I,J}, \overline{\lambda_J(c_{i-1})}) \in R_\varphi(k)$, then

$$\lim_{t \downarrow 0} f(\lambda_I \circ \psi_{I_i}(t), p_{I,J}, \lambda_J \circ \psi_{I_i}(t)) = f(\overline{\lambda_I(c_{i-1})}), p_{I,J}, \overline{\lambda_J(c_{i-1})})$$

On the other hand, if $(\overline{\lambda_I(c_{i-1})}), p_{I,J}, \overline{\lambda_J(c_{i-1})}) \notin R_\varphi(k)$, we must have that

$$\lim_{t \downarrow 0} f(\lambda_I \circ \psi_{I_i}(t), p_{I,J}, \lambda_J \circ \psi_{I_i}(t)) = 0$$

since f is compactly supported. In either case, we define $f_{I,J}(0)$ to be this limit. The same considerations hold for the limit $t \uparrow 1$, and we thus get a map $f_{I,J} : [0,1] \to \mathbb{C}$. Doing this for each pair of intervals $(I, J) \in \mathcal{I}_k^{(2)}$ yields a $*$-homomorphism $b : C_c(R_\varphi(k)) \to C([0,1], \mathbb{B}_k)$ given by

$$b(f) = \sum_{(I,J) \in \mathcal{I}_k^{(2)}} f_{I,J} e_{I,J}$$

which extends to a $*$-homomorphism $b : C_r^*(R_\varphi(k)) \to C([0,1], \mathbb{B}_k)$.

Next, we do a similar analysis for the reduction $R_\varphi(k)|_{\mathcal{E}}$. Unlike the reduction to the complement, $\mathcal{E}$ contains critical points, so we have to deal with non-trivial isotropy groups. However, Example 1.24 has paved the way:

Lemma 3.7. *Put $\mathbb{A}_k = C_r^*(R_\varphi(k)|_{\mathcal{E}})$. For $i = 1, \ldots, N$, let $n_i = |\varphi^{-k}(\{c_i\})|$. Write $n_i = n_i^{\min} + n_i^{\max} + n_i^r$ with*

$$\begin{aligned} n_i^{\min} &= \left|\left\{ x \in \varphi^{-k}(\{c_i\}) \,\middle|\, \mathrm{val}(\varphi^k, x) = (-,+) \right\}\right| \\ n_i^{\max} &= \left|\left\{ x \in \varphi^{-k}(\{c_i\}) \,\middle|\, \mathrm{val}(\varphi^k, x) = (+,-) \right\}\right| \\ n_i^{r} &= \left|\left\{ x \in \varphi^{-k}(\{c_i\}) \,\middle|\, \mathrm{val}(\varphi^k, x) \in \{(+,+),(-,-)\} \right\}\right| \end{aligned}$$

Then

$$\mathbb{A}_k \simeq \bigoplus_{i=1}^{N} \left(M_{n_i^{\min}}(\mathbb{C})^2 \oplus M_{n_i^{\max}}(\mathbb{C})^2 \oplus M_{n_i^r}(\mathbb{C}) \right)$$

Proof. Fix a $c_i \in \mathcal{D}$, and let $A = \left\{ x \in \mathcal{E} \,\middle|\, \varphi^k(x) = c_i, \mathrm{val}(\varphi^k, x) = (-,+) \right\}$. The set A is $R_\varphi(k)$-invariant and has $n_i^{\min}$ elements. If $x, y \in A$, both points are critical for φ^k, and it follows that there are two (germs of) local transfers in $\mathcal{T}(k,k)$ taking x to y. Hence,

$$R_\varphi(k)|_A \simeq A \times A \times \mathbb{Z}_2$$

Appealing to Examples 1.20 and 1.24, we get

$$C_r^*(R_\varphi(k)|_A) \simeq M_{n_i^{\min}}(\mathbb{C}) \oplus M_{n_i^{\min}}(\mathbb{C})$$

Putting

$$B = \left\{ x \in \mathcal{E} \,\middle|\, \varphi^k(x) = c_i, \mathrm{val}(\varphi^k, x) = (+,-) \right\}, \qquad C = \left\{ x \in \mathcal{E} \,\middle|\, \varphi^k(x) = c_i, \mathrm{val}(\varphi^k, x) \in \{(+,-),(-,+)\} \right\}$$

we get by analogous calculations that

$$C_r^*(R_\varphi(k)|_B) \simeq M_{n_i^{\max}}(\mathbb{C}) \oplus M_{n_i^{\max}}(\mathbb{C})$$

and

$$C_r^*(R_\varphi(k)|_C) \simeq M_{n_i^r}(\mathbb{C})$$

since the isotropy over C is trivial. Doing this for each $i = 1, \ldots, N$ yields the result. □

Note that since the set $\mathcal{E}$ is finite, the groupoid $R_\varphi(k)|_{\mathcal{E}}$ is discrete. It follows that the algebra $\mathbb{A}_k$ is generated by the characteristic functions $1_{(x,p,y)}$, $(x,p,y) \in R_\varphi(k)|_{\mathcal{E}}$. For brevity, we will sometimes write these characteristic functions as 'matrix units' $e_{x,p,y}$ for $(x,p,y) \in R_\varphi(k)|_{\mathcal{E}}$. However, while they obey the same composition rules as regular matrix units – i.e. $e_{(x,p,y)} e_{(y,q,z)} = e_{(x,pq,z)}$, it is important to note that the elements $e_{(x,-,x)}$ are *not* projections (indeed, $e_{(x,-,x)}^2 = e_{(x,+,x)}$).

As above, we get a $*$-homomorphism $a : C_r^*(R_\varphi(k)) \to \mathbb{A}_k$ by setting

$$a(f) = \sum_{(x,p,y) \in R_\varphi(k)|_{\mathcal{E}}} f(x,p,y) e_{x,p,y}.$$

We have now seen how elements in $C_c(R_\varphi(k))$ give rise to elements in the algebras $\mathbb{A}_k$ and $C([0,1], \mathbb{B}_k)$. Now, we go the other way and analyse when an $a \in \mathbb{A}_k$ and an $f \in C([0,1], \mathbb{B}_k)$ match up to form a function $g \in C_c(R_\varphi(k))$. We encode these compatibility relations in two $*$-homomorphisms $I_k, U_k : \mathbb{A}_k \to \mathbb{B}_k$. Define the maps I_k and U_k first on the generating matrix units $e_{x,p,y} \in \mathbb{A}_k$: For each $x \in \mathcal{E}$, let I_l^x and I_r^x be the intervals of $\mathcal{I}_k$ immediately to the left and right of x. Abbreviate $\mathrm{val}(\varphi^k, x)$ by v_x, and define maps I_k and U_k by the following table:

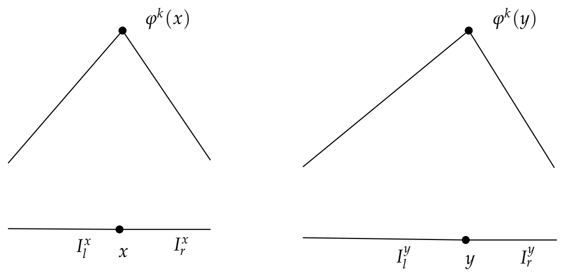


Figure 3.2: The intervals I_l^x and I_r^x to the left and right of x, and I_l^y and I_r^y to the left and right and y..

$e_{x,p,y}$	$I_k(e_{x,p,y})$	$U_k(e_{x,p,y})$
$v_x = v_y = (-,+)$, $p = +$	$e_{I_l^x,I_l^y} + e_{I_r^x,I_r^y}$	0
$v_x = v_y = (-,+)$, $p = -$	$e_{I_l^x,I_r^y} + e_{I_r^x,I_l^y}$	0
$v_x = v_y = (+,-)$, $p = +$	0	$e_{I_l^x,I_l^y} + e_{I_r^x,I_r^y}$
$v_x = v_y = (+,-)$, $p = -$	0	$e_{I_l^x,I_r^y} + e_{I_r^x,I_l^y}$
$v_x = v_y = (+,+)$	$e_{I_r^x,I_r^y}$	$e_{I_l^x,I_l^y}$
$v_x = v_y = (-,-)$	$e_{I_l^x,I_l^y}$	$e_{I_r^x,I_r^y}$
$v_x = (+,+), v_y = (-,-)$	$e_{I_r^x,I_l^y}$	$e_{I_l^x,I_r^y}$
$v_x = (-,-), v_y = (+,+)$	$e_{I_l^x,I_r^y}$	$e_{I_r^x,I_l^y}$

Checking that I_k and U_k respect multiplication and involution is now a straightforward, albeit tedious, task. The following lemma is crucial:

Lemma 3.8. *Let $f \in C_c(R_\varphi(k))$, and $a : C_c(R_\varphi(k)) \to \mathbb{A}_k$, $b : C_c(R_\varphi(k)) \to C([0,1], \mathbb{B}_k)$ and $I_k, U_k : \mathbb{A}_k \to \mathbb{B}_k$ be as above. Then*

$$I_k(a(f)) = b(f)(0), \quad U_k(a(f)) = b(f)(1).$$

Proof. Let $f \in C_c(\Gamma_\varphi)$. It is enough to consider the case where f is a bump function around a point $(x,p,y) \in R_\varphi(k)|_\mathcal{E}$. Assume, for instance, that $(x,+,y)$ is the only element in $\mathrm{supp}(f) \cap \Gamma_\varphi|_\mathcal{E}$, and that $v_x = v_y = (-,+)$. Then $a(f) = f((x,+,y))e_{x,+,y}$, hence

$$I_k(a(f)) = f((x,+,y))\left(e_{I_l^x,I_l^y} + e_{I_r^x,I_r^y}\right), \qquad U_k(a(f)) = 0$$

by Table 3.1. Meanwhile, $b(f)$ is given by

$$b(f) = f_{I_l^x,I_l^y} e_{I_l^x,I_l^y} + f_{I_r^x,I_r^y} e_{I_r^x,I_r^y}.$$

Since $\mathrm{val}(\varphi^k, x) = (-,+)$, there is an $I \in \mathcal{I}$ such that

$$\varphi^k(I_l^x) = \varphi^k(I_l^y) = \varphi^k(I_r^x) = \varphi^k(I_r^y) = I.$$

Then $f_{I_l^x,I_l^y}$ is given by

$$f_{I_l^x,I_l^y}(t) = f\left(\lambda_{I_l^x} \circ \psi_I(t), +, \lambda_{I_l^y} \circ \psi_I(t)\right) = f(x,+,y)$$

since

$$\lim_{t\to 0} \lambda_{I_l^x} \circ \psi_I(t) = x, \quad \lim_{t\to 0} \lambda_{I_l^y} \circ \psi_I(t) = y.$$

The same thing holds for the map $f_{I_r^x, I_r^y}(t)$, which shows that $b(f)(0) = I_k(a(f))$. Similarly, we have $U_k(a(f)) = 0$, and

$$b(f)(0) = \lim_{t\to 1} b(f)(t) = \lim_{t\to 1} f_{I_l^x, I_l^y}(t) e_{I_l^x, I_l^y} + f_{I_r^x, I_r^y}(t) e_{I_r^x, I_r^y} = 0$$

by assumption on f. The other cases follow using analogous arguments. □

Put

$$\mathbb{D}_k = \{(a, f) \in \mathbb{A}_k \oplus C([0,1], \mathbb{B}_k) \mid I_k(a) = f(0), U_k(a) = f(1)\}.$$

The lemma above then gives a map $\mu_k : C_c(R_\varphi(k)) \to \mathbb{D}_k$ given by $\mu_k(f) = (a(f), b(f))$.

Lemma 3.9. *The map $\mu_k : C_c(R_\varphi(k)) \to \mathbb{D}_k$ is injective, isometric, and extends to an isomorphism between $C_r^*(R_\varphi(k))$ and $\mathbb{D}_k$.*

Proof. Showing that μ_k is an isometry is a little tricky: Recall that

$$\|f\| = \sup_{x\in\mathbb{T}} \|\pi_x(f)\|_{B(l^2(s^{-1}(x)))}, \qquad f \in C_c(R_\varphi(k))$$

with

$$\pi_x(f) g(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1) g(\gamma_2).$$

Assume first that $x \in \varphi^{-k}(\mathcal{D})$ with $\mathrm{val}(\varphi^k, x) = (-,+)$ and $\varphi^k(x) = c_i$ for some $c_i \in \mathcal{D}$. Write

$$s^{-1}(x) = \{(y_1, 0, +, x), (y_2, 0, +, x), \ldots, (y_n, 0, +, x), (y_1, 0, -, x), \ldots, (y_n, 0, -, x)\}$$

with $n = n_i^{\min}$. We identify $l^2(s^{-1}(x))$ with $\mathbb{C}^{2n}$. Now, one calculates that

$$\pi_x(f) 1_{(y_i,0,+,x)} = \sum_{\substack{j=1,\ldots,n \\ p=+,-}} f(y_j, 0, p, y_i) 1_{y_i,0,p,x}, \quad \pi_x(f) 1_{(y_i,0,-,x)} = \sum_{\substack{i=j,\ldots,n \\ p=+,-}} f(y_j, 0, p, y_i) 1_{y_i,0,\bar{p},x}$$

with $\bar{p}$ denoting the opposite element of p of $\mathbb{Z}_2$. Hence, a matrix representation of $\pi_x(f)$ in $M_{2n}(\mathbb{C})$ has the form

$$\pi_x(f) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

with $A = (f(y_j, 0, +, y_i))_{i,j=1}^n$ and $B = (f(y_j, 0, -, y_i))_{i,j=1}^n$. We now compare this with

$$a(f) \in \mathbb{A}_k = \bigoplus_i \left(M_{n_i^{\min}}(\mathbb{C})^2 \oplus M_{n_i^{\max}}(\mathbb{C})^2 \oplus M_{n_i^r}(\mathbb{C}) \right).$$

By definition,

$$a(f) = \sum_{(x,0,p,y) \in R_\varphi(k)|_{\varphi^{-k}(\mathcal{D})}} f(x, 0, p, y) e_{x,p,y}.$$

Under the isomorphism from Lemma 3.7, the component of $a(f)$ in $M_{n_i^{\min}}(\mathbb{C})^2$ is represented by $(A+B, A-B) \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$, with A and B as above. The unitary map $\Psi : \mathbb{C}^n \oplus \mathbb{C}^n \to \mathbb{C}^n \oplus \mathbb{C}^n$ given by $\Psi(a, b) = \frac{1}{\sqrt{2}}(a + b, a - b)$ implements a unitary equivalence between $\pi_x(f)$

and $a(f)|_{M_{n_i^{\min}}(\mathbb{C})^2}$. Doing this for all $x \in \varphi^{-k}(\mathcal{D})$ yields $\|\pi_x(f)\| = \|a(f)\|$. When $x \notin \varphi^{-k}(\mathcal{D})$ a similar argument yields that $\|\pi_x(f)\| = \|b(f)\|$, which proves that μ_k is an isometry. Hence, it extends to an injective map $\mu_k : C_r^*(R_\varphi(k)) \to \mathbb{D}_k$.

To prove surjectivity, let $(a,b) \in \mathbb{D}_k$. Write

$$a = \sum_{(x,p,y)\in R_\varphi(k)|_{\mathcal{E}}} \lambda_{x,p,y} e_{x,p,y}$$

with each $\lambda_{x,p,y} \in \mathbb{C}$. Since $R_\varphi(k)|_{\mathcal{E}}$ is finite, there is a function $g \in C_c(R_\varphi(k))$ with $g(x,p,y) = \lambda_{x,p,y}$. It follows that $\mu_k(g) = (a,b')$ for some $b' \in C_c(R_\varphi(k))$, hence

$$(a,b) = \mu_k(g) + (0, b - b')$$

Since $b(0) = b'(0)$ and $b(1) = b'(1)$, it follows that $f = b - b' \in C_0((0,1), \mathbb{B}_k)$. Write

$$f = \sum_{(I,J)\in\mathcal{I}_k^{(2)}} f_{I,J} e_{I,J}$$

with each $f_{I,J} \in C((0,1))$. Fix a pair (I,J) and put

$$\mathcal{U} = \{(\lambda_I \circ \psi_{I_i}(t), p, \lambda_J \circ \psi_{I_i}(t) | t \in (0,1)\}$$

$\mathcal{U}$ is open in $R_\varphi(k)$, and we can define a map $h \in C_0(\mathcal{U})$ such that

$$h(\lambda_I \circ \psi_{I_i}(t), p, \lambda_J \circ \psi_{I_i}(t)) = f_{I,J}(t).$$

Choose a sequence $h_n \in C_c(\mathcal{U})$ such that h_n converges uniformly to h. Then $\mu_k(h_n)$ converges to $(0, f_{I,J} e_{I,J})$, and by extending linearly, we are done. □

As an immediate corollary, we get:

Corollary 3.10. *The sequence*

$$0 \longrightarrow C_r^*(R_\varphi(k)|_{\mathbb{T}\setminus\mathcal{E}}) \xrightarrow{\ i\ } C_r^*(R_\varphi(k)) \xrightarrow{\ a\ } C_r^*(R_\varphi(k)|_{\mathcal{E}}) \longrightarrow 0 \tag{3.1}$$

is exact, and we have

$$K_0(C_r^*(R_\varphi(k))) \simeq \ker((I_k)_0 - (U_k)_0), \qquad K_1(C_r^*(R_\varphi(k))) \simeq \operatorname{coker}((I_k)_0 - (U_k)_0)$$

where $(I_k)_0$ and $(U_k)_0$ denote the induced maps between $K_0(\mathbb{A}_k)$ and $K_0(\mathbb{B}_k)$.

Proof. The first statement follows from the isomorphisms of Lemmas 3.6, 3.7 and 3.9, and the fact that the sequence

$$0 \longrightarrow S\mathbb{B}_k \xrightarrow{\ i\ } \mathbb{D}_k \xrightarrow{\ a\ } \mathbb{A}_k \longrightarrow 0 \tag{3.2}$$

is exact. The second statement follows from Corollary 1.28. □

Remark 3.11. So far, we have dealt only with the groupoids $R_\varphi(k)$ and the algebras $C_r^*(R_\varphi(k))$. With only minor modifications, we can decompose $C_r^*(R_\varphi^+(k))$ in the same way – and some definitions become simpler, due to the absence of $\mathbb{Z}_2$-isotropy. We have

$$R_\varphi^+(k)|_{\mathcal{E}} \simeq \left\{(x,y) \in \mathcal{E}\times\mathcal{E} \,\middle|\, \varphi^k(x) = \varphi^k(y), \operatorname{val}(\varphi^k, x) = \operatorname{val}(\varphi^k, y)\right\},$$

so, writing $\mathbb{A}_k^+$ for $C_r^*(R_\varphi^+(k)|_\mathcal{E})$, we get

$$\mathbb{A}_k^+ = \bigoplus_{i=1}^{N} M_{n_i^{(-,+)}}(\mathbb{C}) \oplus M_{n_i^{(+,-)}}(\mathbb{C}) \oplus M_{n_i^{(+,+)}}(\mathbb{C}) \oplus M_{n_i^{(-,-)}}(\mathbb{C}) \tag{3.3}$$

with

$$n_i^v = |\Big\{x \in \mathcal{E} \;\Big|\; \varphi^k(x) = c_i, \mathrm{val}(\varphi^k, x) = v\Big\}|,$$

for $v \in \mathcal{V}$. We also have $C_r^*(R_\varphi(k)|_{\mathbb{T}\setminus\mathcal{E}}) \simeq C_0((0,1), \mathbb{B}_k^+)$, with

$$\mathbb{B}_k^+ = \bigoplus_{i=1}^{N} M_{m_i^+}(\mathbb{C}) \oplus M_{m_i^-}(\mathbb{C}), \tag{3.4}$$

and $m_i^v = |\Big\{I \in \mathcal{I}_k \;\Big|\; \varphi^k(I) = I_i, \mathrm{val}(\varphi^k, I) = v\Big\}|$ for $v \in \{(+,+),(-,-)\}$. The maps I_k^+ and U_k^+ are given by

$e_{x,y}$	$I^+(e_{x,y})$	$U^+(e_{x,y})$
$v_x = v_y = (-,+)$	$e_{I_l^x,I_l^y} + e_{I_r^x,I_r^y}$	0
$v_x = v_y = (+,-)$	0	$e_{I_l^x,I_l^y} + e_{I_r^x,I_r^y}$
$v_x = v_y = (+,+)$	$e_{I_r^x,I_r^y}$	$e_{I_l^x,I_l^y}$
$v_x = v_y = (+,+)$	$e_{I_l^x,I_l^y}$	$e_{I_r^x,I_r^y}$

As above, there are $*$-homomorphisms $a : C_r^*(R_\varphi^+(k)) \to \mathbb{A}_k^+$ and $b : C_r^*(R_\varphi^+(k)) \to C([0,1], \mathbb{B}_k^+)$ such that

$$C_r^*(R_\varphi^+(k)) \simeq \{(a,f) \in \mathbb{A}_k^+ \oplus C([0,1], \mathbb{B}_k^+) \mid I_k^+(a) = f(0), U_k^+(a) = f(1)\}.$$

This yields a short exact sequence like (5.10), and shows that

$$K_0(C_r^*(R_\varphi^+(k))) \simeq \ker((I_k^+)_0 - (U_k^+)_0), \;\; K_1(C_r^*(R_\varphi^+(k))) \simeq \mathrm{coker}((I_k^+)_0 - (U_k^+)_0).$$

♦

3.1.1 Calculations

Let's provide some more details on how to determine the K-theory of these algebras. We begin with $C_r^*(R_\varphi^+(k))$, as this is a bit simpler. It follows from Equation 3.3 that $K_0(\mathbb{A}_k^+)$ is a free abelian group with one summand for each element in the set

$$\mathcal{D}(\pm) = \Big\{[d,v] \in \mathcal{D} \times \mathcal{V} \;\Big|\; \exists x \in \varphi^{-k}(d) : \mathrm{val}(\varphi^k, x) = v\Big\}.$$

For each such pair $[d,v]$, the corresponding summand in $K_0(\mathbb{A}_k^+)$ is generated by the K_0-class of the matrix unit $e_{x,x}$ for any $x \in \varphi^{-k}(d)$ with $\mathrm{val}(\varphi^k, x) = v$.

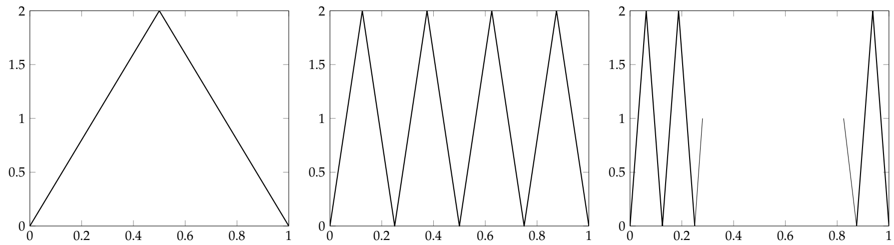
Similarly, from Equation 3.4 it follows that $K_0(\mathbb{B}_k^+)$ is a free abelian group, with one summand for each element in the set

$$\mathcal{I}(\pm) = \Big\{[I_i, v] \in \mathcal{I} \times \{(+,+),(-,-)\} \;\Big|\; \exists I \in \mathcal{I}_k : \varphi^k(I) = I_i, \mathrm{val}(\varphi^k, I) = v\Big\}.$$

For each such pair $[I_i, v]$, any interval I with $\varphi^k(I) = I_i$ and $\mathrm{val}(\varphi^k, I) = v$ gives rise to a rank-one projection $e_{I,I}$ generating the corresponding summand of $K_0(\mathbb{B}_k^+)$.

This gives the recipe for determining the induced map $(I_k^+)_0 - (U_k^+)_0$: Take an element $[d,v] \in \mathcal{D}(\pm)$, choose a corresponding x with $x \in \varphi^{-k}(d)$ with $\mathrm{val}(\varphi^k, x) = v$, write $I_k^+(e_{x,x})$ and $U_k^+(e_{x,x})$ as a sum of matrix units $e_{I,I}$, and determine, for each term in this sum, the corresponding element in $\mathcal{I}(\pm)$. We illustrate with an example:

Example 3.12. Let $\tau : [0,1] \to \mathbb{R}$ denote map given by $\tau(t) = 4t$ for $t \in [0,1/2]$ and $\tau(t) = 4-4t$ for $t \in [1/2,1]$. τ induces a map (which we will also refer to as τ) of the unit circle onto itself. This map is continuous, surjective, piecewise linear and non-injective. Iterating τ is straightforward, as this picture of the graphs of τ, τ^2 and τ^3 shows:



To simplify matters, let us consider the case $k = 2$. The critical points $\mathcal{C}_2$ are $\{0, 1/8, 1/4, \dots, 7/8\}$, so we may put $\mathcal{D} = \tau^2(\mathcal{C}_2) = \{0\}$. Then

$$\mathcal{D}(\pm) = \{[0,(-,+)],[0,(+,-)],[0,(+,+)],[0,(-,-)]\}$$

so $K_0(\mathbb{A}_2^+) \simeq \mathbb{Z}^4$, with the elements above (in that order) as a basis. Since $\mathbb{T} \setminus \mathcal{D}$ has only one connected component I, the set $\mathcal{I}(\pm)$ is just $\{[I,(+,+)],[I,(-,-)]\}$. It follows that $K_0(\mathbb{B}_2^+) \simeq \mathbb{Z}^2$. To calculate $(I_2^+)_0$ and $(U_2^+)_0$, we proceed as sketched above: Consider, for instance, $[0,(+,-)] \in \mathcal{D}(\pm)$. Since $\tau^2(1/8) = 0$ and $\mathrm{val}(\tau^2, 1/8) = (+,-)$, we should determine $I_2^+(e_{1/8,1/8})$ and $U_2^+(e_{1/8,1/8})$. We observe that

$$\tau^{-2}(0) = \{0, 1/16, 1/8, 3/16, 1/4, \dots, 7/8, 15/16\},$$

so the connected components of $\mathbb{T} \setminus \tau^{-2}(\mathcal{D})$ immediately to the left and right of $1/8$ is $J = (1/16, 1/8)$ and $K = (1/8, 3/16)$. Using Table 3.11, we get

$$I_2^+(e_{1/8,1/8}) = 0, \qquad U_2^+(e_{1/8,1/8}) = e_{J,J} + e_{K,K}$$

Since $\tau^2(J) = \tau^2(K) = I$, $\mathrm{val}(\tau^2, J) = (+,+)$ and $\mathrm{val}(\tau^2, K) = (-,-)$, the K_0-classes of $e_{J,J}$ and $e_{K,K}$ correspond to $[I,(+,+)]$ and $[I,(-,-)]$, respectively. It follows that

$$\left((I_2^+)_0 - (U_2^+)_0\right)([0,(+,-)]) = -[(I,(+,+))] - [(I,(-,-))]$$

Doing similar calculations for the other elements of $\mathcal{D}(\pm)$ yields a matrix representation of $\left((I_2^+)_0 - (U_2^+)_0\right)$ as a map from $\mathbb{Z}^4$ to $\mathbb{Z}^2$, ordering the bases of $K_0(\mathbb{A}_2^+)$ and $K_0(\mathbb{B}_2^+)$ as above:

$$(I_2^+)_0 - (U_2^+)_0 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

so

$$K_0(C_r^*(R_\tau^+(2))) \simeq \ker((I_2^+)_0 - (U_2^+)_0) \simeq \mathbb{Z}^3$$

and

$$K_1(C_r^*(R_\tau^+(2))) \simeq \mathrm{coker}((I_2^+)_0 - (U_2^+)_0 \simeq \mathbb{Z}$$

Considering other iterates of τ and not just τ^2 yield the same results: For any k, the only critical value of τ^k is 0, so we choose $\mathcal{D} = \{0\}$ and get $K_0(\mathbb{A}_k^+) \simeq \mathbb{Z}^4$ and $K_0(\mathbb{B}_k^+) = \mathbb{Z}^2$, with the induced map $(I_k^+)_0 - (U_k^+)_0$ given as above. Hence, the K-theory groups of $C_r^*(R_\tau^+(k))$ are $\mathbb{Z}^3$ and $\mathbb{Z}$ for any k. ▲

Example 3.13. We now turn to the other family of groupoids, $R_\tau(k)$. The presence of $\mathbb{Z}_2$-isotropy makes the calculations slightly more complicated. By Lemma 3.6, $\mathbb{B}_k$ has a full matrix summand for each element of $\mathcal{I}$ – in comparison to $\mathbb{B}_k^+$, $\mathbb{B}_k$ has fewer summands, but these summands typically have higher dimension. More precisely, $K_0(\mathbb{B}_k) \simeq \mathbb{Z}^N$, where N is the number of elements in $\mathcal{I}$. $\mathbb{A}_k$, on the other hand, tends to have more summands: Given a $d \in \mathcal{D}$, a $v \in \{(-,+),(+,-)\}$ and an $x \in \varphi^{-k}(d)$ with $\mathrm{val}(\varphi^k, x) = v$, we obtain two elements $(x,+,x)$ and $(x,-,x)$ in $R_\varphi(k)|_{\mathcal{E}}$, each yielding the two linearly independent functions $1_{(x,+,x)}$ and $1_{(x,-,x)}$ in $C_r^*(R_\varphi(k)|_{\mathcal{E}}) = \mathbb{A}_k$. As in Example 1.26, one checks that

$$p_{+,x} = \tfrac{1}{2}\big(1_{(x,+,x)} + 1_{(x,-,x)}\big), \quad p_{-,x} = \tfrac{1}{2}\big(1_{(x,+,x)} - 1_{(x,-,x)}\big)$$

are projections in $\mathbb{A}_k$, each generating a summand of $K_0(\mathbb{A}_k)$. On the other hand, if $\varphi^k(x) = d$ with $\mathrm{val}(\varphi^k, x) \in \{(+,+),(-,-)\}$, the element $1_{(x,+,x)}$ is a projection in $\mathbb{A}_k$, generating a summand of $K_0(\mathbb{A}_k)$.

Again, as an example, we consider the map τ and start by looking at τ^2. Using Lemmas 3.6 and 3.7, it follows that $K_0(\mathbb{A}_2) \simeq \mathbb{Z}^5$ and $K_0(\mathbb{B}_2) \simeq \mathbb{Z}$. It is still the case that $\tau^2(1/8) = 0$ and $\mathrm{val}(\tau^2, 1/8) = (+,-)$ – but we now have two elements $(1/8,+,1/8)$ and $(1/8,-,1/8)$ in $R_\tau(2)|_{\mathcal{E}}$, each giving rise to characteristic functions $1_{(1/8,+,1/8)}$ and $1_{(1/8,-,1/8)}$. Following Example 1.26, the K_0-classes of each of the projections

$$p_+ = \tfrac{1}{2}\big(1_{(1/8,+,1/8)} + 1_{(1/8,-,1/8)}\big), \quad p_- = \tfrac{1}{2}\big((1_{(1/8,+,1/8)} - 1_{(1/8,-,1/8)}\big)$$

generate a summand in $K_0(\mathbb{A}_2)$. Appealing to Table 3.1, we see that

$$I_2(p_+) = I_2(p_-) = 0$$

and

$$U_2(p_+) = \tfrac{1}{2}\left(e_{J,J} + e_{K,K} + e_{J,K} + e_{K,J}\right)$$

Since $\tau^2(J) = \tau^2(K) = I$, it follows that

$$((I_2)_0 - (U_2)_0)([p_+]) = ((I_2)_0 - (U_2)_0([p_-])) = -[I]$$

Doing similar calculations for the other generators of $K_0(\mathbb{A}_2)$ shows that $(I_2)_0 - (U_2)_0$ as a map from $\mathbb{Z}^5$ to $\mathbb{Z}$ is given by the 1-by-5-matrix $(11(-1)(-1)0)$, in particular

$$K_0(C_r^*(R_\tau(2))) \simeq \ker((I_2)_0 - (U_2)_0) \simeq \mathbb{Z}^4$$

and

$$K_1(C_r^*(R_\tau^+(2))) \simeq \mathrm{coker}((I_2^+)_0 - (U_2^+)_0) \simeq 0$$

Again, one may check that these K-theory groups are independent of k. ▲

We note that $K_1(C_r^*(R_\tau(k)))$ is trivial. This turns out to be a general fact:

Lemma 3.14. *For $k \in \mathbb{N}$, the induced map $(I_k)_0 - (U_k)_0 : K_0(\mathbb{A}_k) \to K_0(\mathbb{B}_k)$ is surjective. In particular, $K_1(C_r^*(R_\varphi(k))) = 0$.*

Proof. Let $k \in \mathbb{N}$, let $\mathcal{D} = \varphi^k(\mathcal{C}_k)$ and $\mathcal{E} = \varphi^{-k}(\mathcal{D})$. Assume that $\mathcal{D}$ has N elements, and write $\mathbb{T} \setminus \mathcal{D} = \{I_1, \ldots, I_N\}$. By Lemma 3.6, we have

$$\mathbb{B}_k = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C})$$

with the i'th summand generated by matrix units $e_{I,J}$ where I and J are intervals such that $\varphi^k(I) = \varphi^k(J) = I_i$. Hence

$$K_0(\mathbb{B}_k) \simeq \mathbb{Z}^N = \mathbb{Z}[I_1] \oplus \ldots \mathbb{Z}[I_N].$$

Fix an I_i, and let $d, d' \in \mathcal{D}$ such that $d < I_i < d'$. We divide into three cases:

- Assume that there is an $x \in \varphi^{-k}(d)$ such that $\text{val}(\varphi^k, x) = (-,+)$. x gives rise to elements $e_{x,+,x}$ and $e_{x,-,x}$ in $\mathbb{A}_k$. Let I and J be the intervals of $\mathbb{T} \setminus \mathcal{E}$ immediately to the left and right of x. One then checks that $p = 1/2(e_{x,+,x} + e_{x,-,x})$ is a projection in $\mathbb{A}_k$, and by using Table 3.1, we see that

$$I_k(p) = \tfrac{1}{2}\left(e_{I,I} + e_{J,J} + e_{I,J} + e_{J,I}\right), \qquad U_k(p) = 0$$

  Since $\varphi^k(I) = \varphi^k(J) = I_i$, it follows that

$$((I_k)_0 - (U_k)_0)([p]) = [I_i].$$

- If there is an $x \in \varphi^k(d')$ with $\text{val}(\varphi^k, x) = (+,-)$, we may define p as above and get that

$$I_k(p) = 0, \qquad U_k(p) = \tfrac{1}{2}\left(e_{I,I} + e_{J,J} + e_{I,J} + e_{J,I}\right).$$

  Hence, $((I_k)_0 - (U_k)_0)(-[p]) = [I_i]$.

- Finally, assume that neither of the above cases occur. Then there is a x such that $\varphi^k(x) = d$, $\text{val}(\varphi^k, x) = (+,-)$. Since φ^k is surjective, there is also a point $y \in \mathbb{T}$ such that $\varphi^k(y) = d$, $\text{val}(\varphi^k, y) = v \in \{(+,+),(-,-)\}$. Then the element

$$p = \tfrac{1}{2}\left(e_{x,+,x} + e_{x,-,x}\right) - e_{y,+,y}$$

  is a projection, and one checks that $((I_k)_0 - (U_k)_0)(p) = [I_i]$.

This shows surjectivity of $(I_k)_0 - (U_k)_0$. □

This, on the other hand, is *not* the case for $C_r^*(R_\varphi^+(k))$:

Lemma 3.15. *Let $k \in \mathbb{N}$. The map $(I_k^+)_0 - (U_k^+)_0 : K_0(\mathbb{A}_k^+) \to K_0(\mathbb{B}_k^+)$ is not surjective. In particular, $K_1(C_r^*(R_\varphi^+(k)))$ is non-trivial.*

Proof. We have $K_0(\mathbb{B}_k^+) \simeq \mathbb{Z}^{\mathcal{I}(\pm)}$ with a basis given by elements $\{[I,(+,+)],[I,(-,-)]\}_{I \in \mathcal{I}}$. We show that $(I_k)_0 - (U_k)_0$ maps $\mathbb{A}_k^+$ into the subspace V of $\mathbb{Z}^L$ given by

$$V = \left\{ \sum_{I \in \mathcal{I}} c_{I,+}[I,(+,+)] + c_{I,-}[I,(-,-)] \;\middle|\; \sum_{I \in \mathcal{I}} c_{I,+} - c_{I,-} = 0 \right\}$$

To do this, let $d \in \mathcal{D}$, $v \in \mathcal{V}$, and let $[d, v]$ be a basis element of $K_0(\mathbb{A}_k^+)$. Assume, for instance, that $v = (-, +)$, and let $J \in \mathcal{I}$ be the element of $\mathcal{I}$ immediately above d. Then

$$(I_k^+)_0 - (U_k^+)_0([d, v]) = [J, (+, +)] + [J, (-, -)] \in V.$$

Similarly, if $v = (+, +)$, let I and J be intervals of $\mathcal{I}$ such that $J < d < I$. Then

$$(I_k^+)_0 - (U_k^+)_0([d, v]) = [I, (+, +)] - [J, (+, +)] \in V.$$

The other two cases are similar. It follows that $\mathrm{Im}((I_k^+)_0 - (U_k^+)_0) \subseteq V$, and V is a subspace of codimension 1 in $\mathbb{Z}^{\mathcal{I}(\pm)}$. In particular, $(I_k^+)_0 - (U_k^+)_0$ is not surjective, so the cokernel is non-zero. From this it follows that

$$K_1(C_r^*(R_\varphi^+(k))) \simeq \mathrm{coker}((I_k^+)_0 - (U_k^+)_0)$$

is non-trivial. □

3.2 On the structure of $C_r^*(\Gamma_\varphi)$ and $C_r^*(\Gamma_\varphi^+)$

We observe that for each k, the groupoid $R_\varphi(k)$ is an open subgroupoid of $R_\varphi(k+1)$, and that $R_\varphi = \bigcup_{k \in \mathbb{N}} R_\varphi(k)$. It follows that

$$C_r^*(R_\varphi) = \overline{\bigcup_{k \in \mathbb{N}} C_r^*(R_\varphi(k))},$$

and in similar fashion that $C_r^*(R_\varphi^+) = \overline{\bigcup_{k \in \mathbb{N}} C_r^*(R_\varphi^+(k))}$.

We will need a short discussion on amenability of groupoids: For groupoids, the word 'amenable' can have any number of prefixes ('strongly', 'topological', 'measure-wise', etc.) In our discussion, we will only need amenability as a tool to prove other properties, so we forego any discussion of how the various definitions of amenability are related. For a thorough discussion, see [3], in which it is proved (see Remark 3.3.9) that for étale groupoids, topological and measure-wise amenability coincide, and that is all we will need.

Proposition 3.16. *Let G be a locally compact, étale groupoid and assume that* $\mathrm{Iso}(x)$ *is discrete for all $x \in G^0$. Then G is (measure-wise) amenable if and only if $C_r^*(G)$ is nuclear.*

Proof. This is Corollary 6.2.14 in [3]. □

For the definition of nuclearity, and a number of results about nuclear C^*-algebras, see [35]. In particular, we note that quotients of nuclear C^*-algebras are nuclear (see [6], Corollary 9.4.4).

Secondly, we need a brief discussion of the full C^*-algebra of a (locally compact, Hausdorff étale) groupoid G. Recall that to define $C_r^*(G)$, we put a norm on $C_c(G)$ given by

$$\|f\| = \sup_{x \in \mathbb{T}} \|\pi_x(f)\|, \quad f \in C_c(G)$$

with π_x the left-regular representation on $l^2(s^{-1}(x))$. $C_r^*(G)$ was then the completion of $C_c(G)$ in this norm. Had we instead defined

$$\|f\| = \sup \|\pi(f)\|, \quad f \in C_c(G)$$

with π ranging over *all* representations of $C_c(G)$, and completed $C_c(G)$ in this norm, we would have gotten the *full* groupoid C^*-algebra $C^*(G)$. When the groupoid is amenable, these constructions coincide:

Proposition 3.17. *Let G be locally compact groupoid which is measure-wise amenable. Then $C_r^*(G)$ is isomorphic to $C^*(G)$.*

Proof. This is Proposition 6.1.8 of [3]. □

Proposition 3.18. *The algebra $C_r^*(R_\varphi^+)$ is nuclear and satisfies the Universal Coefficient Theorem.*

Proof. Fix a k and consider the groupoid $R_\varphi^+(k)$. The C^*-algebra $C_r^*(R_\varphi^+(k))$ is nuclear, as it is the extension of the nuclear C^*-algebras $\mathbb{A}_k^+$ and $S\mathbb{B}_k^+$ by Equation 5.10 and Remark 3.11. Since $C_r^*(R_\varphi^+)$ is an inductive limit of nuclear C^*-algebras, it is also nuclear (see e.g. Proposition 2.1.2 of [35]). It follows by Proposition 3.16 that R_φ^+ is amenable. By a result of Tu in [49], it follows then that $C_r^*(R_\varphi^+)$ satisfies the UCT. □

To obtain similar results about $C_r^*(\Gamma_\varphi^+)$, recall the gauge action from Definition 3.2, given by

$$\beta_\mu(f)(x,k,y) = \mu^k f(x,k,y)$$

for $\mu \in \mathbb{T}$, $(x,k,y) \in \Gamma_\varphi^+$ and $f \in C_c(\Gamma_\varphi^+)$.

Proposition 3.19. *Γ_φ^+ is amenable, and the algebra $C_r^*(\Gamma_\varphi^+)$ is nuclear and satisfies the UCT.*

Proof. Let β be the gauge action. Let $c : \Gamma_\varphi^+ \to \mathbb{Z}$ be the map given by $c((x,k,y)) = k$. This is a continuous homomorphism, and $c^{-1}(0) = R_\varphi^+$. By (the proof of) Proposition 3.18, R_φ^+ is amenable, so by Proposition 9.3 of [41], Γ_φ^+ is amenable, and $C_r^*(\Gamma_\varphi^+) \simeq C^*(\Gamma_\varphi^+)$ is nuclear. By the previously mentioned result of Tu, $C_r^*(\Gamma_\varphi^+)$ satisfies the UCT. □

To obtain the same results for $C_r^*(\Gamma_\varphi)$ and $C_r^*(R_\varphi)$, define $d : \Gamma_\varphi \to \mathbb{Z}_2 = \{+,-\}$ by

$$d((x,k,p,y)) = p.$$

Then $d^{-1}(+) = \Gamma_\varphi^+$, which is amenable, and now another application of the result of Spielberg in [41] shows that Γ_φ is amenable and that $C_r^*(\Gamma_\varphi)$ is nuclear and satisfies the UCT. Restricting d to R_φ shows that the same holds for $C_r^*(R_\varphi)$.

Remark 3.20. From the Remarks to Theorem 1.12 in [30], it follows that $C_r^*(G)$ is separable when the groupoid G is second countable. It follows that when φ is transitive, the C^*-algebras $C_r^*(\Gamma_\varphi)$ and $C_r^*(\Gamma_\varphi)$ are unital, separable, purely infinite (by Theorem 2.27), nuclear and satisfies the UCT. Hence, whenever the algebras are simple, the Kirchberg-Phillips Classification Theorem (see [26]) implies that the algebras are classified by their K-theory. ♦

One consequence of the lemmas above is the following result, which determines the kernel of a restriction map π_F:

Proposition 3.21. *Let G be an amenable locally compact second countable Hausdorff étale groupoid with unit space G^0, and assume that G has discrete isotropy groups. Let F be a closed invariant subset of G^0, and put $U = G^0 \setminus F$. Let $i : C^*(G|_U) \to C^*(G)$, and $\pi_F : C^*(G) \to C^*(G|_F)$ be the inclusion and restriction maps defined in 1.16 and 1.17, respectively. Then the sequence*

$$0 \longrightarrow C_r^*(G|_U) \xrightarrow{\ i\ } C_r^*(G) \xrightarrow{\ \pi_F\ } C_r^*(G|_F) \longrightarrow 0 \tag{3.5}$$

is exact.

Proof. The statement for the full groupoid C^*-algebra is true by Lemma 2.10 of [19]. Since G is amenable, $C^*(G) \simeq C_r^*(G)$ by Proposition 3.17, and this algebra is nuclear by Proposition 3.16. We then know that $C^*(G|_F)$ is a quotient of $C^*(G)$, so $C_r^*(G|_F)$ is nuclear and $C_r^*(G|_F) \simeq C^*(G|_F)$. In then follows (from e.g. the five-lemma) that $C_r^*(G|_U)$ is isomorphic to $C^*(G|_U)$, and we have our result. □

We can generalise this lemma slightly: If A is closed in $\mathbb{T}$ and $B \subseteq A$ is closed in A and G-invariant, we have a surjective restriction map $\pi_{A,B} : C_r^*(G|_A) \to C_r^*(G|_B)$. Arguing exactly as above, it follows that $\ker(\pi_{A,B}) \simeq C_r^*(G|_{A\setminus B})$. Finally, if both B and C are closed and invariant in $\mathbb{T}$, the same is true for $B \cup C$, and there is a restriction map $\pi_{B\cup C}$ from $C_r^*(G)$ to $C_r^*(G|_{B\cup C})$. The next lemma compares the kernel of this map with the kernels of π_B and π_C:

Lemma 3.22. *Let G be an amenable locally compact second countable Hausdorff étale groupoid with unit space G^0, and assume that G has discrete isotropy groups. Let B and C be closed, G-invariant subsets of G^0. Then*

$$\ker(\pi_{B\cup C}) = \ker(\pi_B) \cap \ker(\pi_C).$$

Proof. Since $C_r^*(G|_{B^c})$ is a subalgebra of $C_r^*(G)$, the map $\pi_C : C_r^*(G) \to C_r^*(G|_C)$ restricts to a map $\pi_C|_{B^c} : C_r^*(G|_{B^c}) \to C_r^*(G|_C)$. $\pi_C|_{B^c}$ is equal to the map $\pi_{B^c, B^c\cap C} : C_r^*(G|_{B^c}) \to C_r^*(G|_{B^c\cap C})$, since they agree on the dense subset $C_c(G|_{B^c})$. Combining this with the above lemma yields

$$\ker(\pi_B) \cap \ker(\pi_C) = C_r^*(G|_{B^c}) \cap \ker(\pi_C) = \ker(\pi_C|_{B^c}) = \ker(\pi_{B^c, B^c\cap C}).$$

Using the lemma on $\pi_{B^c, B^c\cap C}$ thus shows that

$$\ker(\pi_{B^c, B^c\cap C}) = C_r^*(G|_{B^c\setminus(B^c\cap C)}) = C_r^*(G|_{(B\cup C)^c}) = \ker(\pi_{B\cup C}),$$

which is the desired conclusion. □

Chapter **4**

Simplicity and primitive ideals

4.1 Orbits and invariant sets

This section contains a number of general results. We determine the possible isotropy groups of Γ_φ and Γ_φ^+ and investigate the topology of orbits $[x]$ under Γ_φ and Γ_φ^+.

4.1.1 Prime subsets

We begin with some general results on topological groupoids.

Definition 4.1. Let G be a topological groupoid, and A a closed and G-invariant subset of the unit space. Assume that $A \subseteq A_1 \cup A_2$ for some other closed G-invariant subsets A_1 and A_2. If either $A \subseteq A_1$ or $A \subseteq A_2$ for any such decomposition, we say that A is *prime*.

In the following, we determine the prime subsets of the unit space of a second countable, locally compact Hausdorff étale groupoid. The discussion follows [13] and [40] closely, but we include full proofs here for completeness. Note that the results hold for a rather large class of groupoids, and not just the particular ones under scrutiny in this paper.

We need some terminology from general topology: Let X be a topological space. A subset $A \subseteq X$ is *locally closed* if A is the intersection of an open and a closed set. X is a *Baire space* if the intersection of a countable collection of open dense sets is again dense, and X is *totally Baire* if any locally closed subset of X is a Baire space. A closed set $F \subseteq X$ is *irreducible* if it is not the union of two closed proper subsets.

Lemma 4.2. *Let X and Y be topological spaces with X second countable and totally Baire, and let $\psi : X \to Y$ be an open, continuous surjection. Then Y is also second countable and totally Baire.*

Proof. Let $\{U_i\}_{i\in\mathbb{N}}$ be a countable basis for the topology on X, and put $V_i = \psi(U_i)$. Then each V_i is open. We claim that $\{V_i\}_{i\in\mathbb{N}}$ is a countable basis for the topology on Y: If $A \subseteq Y$ is open, $\psi^{-1}(A)$ is open in X, so $\psi^{-1}(A) = \cup_k U_{i_k}$ for some sequence of indices i_k. But since ψ is surjective,

$$A = \psi(\psi^{-1}(A)) = \psi(\bigcup_k U_{i_k}) = \bigcup_k \psi(U_{i_k}) = \bigcup_k V_{i_k}$$

and the claim follows.

Next, let F be locally closed in Y, and let $\{D_i\}$ be a countable collection of dense sets that are open in F. Write $F = V \cap C$ with V open and C closed. Then $\psi^{-1}(F) = \psi^{-1}(V) \cap \psi^{-1}(C)$, so $\psi^{-1}(F)$ is locally closed in X, hence a Baire space. The sets $\psi^{-1}(D_i)$ are open in $\psi^{-1}(F)$, and since ψ is open and surjective, each set is also dense in $\psi^{-1}(F)$. By the Baire property of $\psi^{-1}(F)$, the intersection $\cap_i \psi^{-1}(D_i)$ is dense in $\psi^{-1}(F)$. By continuity and surjectivity of ψ, the intersection of the D_i's is then dense in F. □

Lemma 4.3. *Let X be a topological space which is second countable and totally Baire, and let $F \subseteq X$ be non-empty and closed. Then F is irreducible if and only if it is the closure of a single point.*

Proof. Asumme that F is irreducible, and note that F is second countable and totally Baire since X is. Any open non-empty set $U \subseteq F$ must be dense – indeed, writing $F = \overline{U} \cup (F \setminus U)$, we get by irreducibility that $F = \overline{U}$. Let $\{U_i\}_{i \in \mathbb{N}}$ be a countable base for the topology of F. Since each U_i is dense, the Baire property implies that $\cap_i U_i$ is dense as well. In particular, it is non-empty, so we may pick an $x \in \cap_i U_i$. If $y \in F$, and V is an open neighbourhood of y, there is an i such that $y \in U_i \subseteq V$. Since $x \in U_i$, it follows that $y \in \overline{\{x\}}$, hence $F = \overline{\{x\}}$. For the other direction, it is immediate that *any* set which is the closure of a single point is irreducible. □

To apply the above lemma to our situation, assume that G is locally compact, second countable, Hausdorff and étale. Define an equivalence relation $\sim$ on $G^{(0)}$ by $x \sim y$ if $[x]_G = [y]_G$. The *orbit space* $G^{(0)}/G$ is the quotient space of $G^{(0)}$ under this equivalence relation, equipped with the quotient topology. The quotient map $q : G^{(0)} \to G^{(0)}/G$ is continuous and surjective by definition, and also open: We have $q^{-1}(q(A)) = s(r^{-1}(A))$ for any $A \subseteq G^{(0)}$, and if A is open, $s(r^{-1}(A))$ is open since G is étale. By Lemma 4.2, the properties of being second countable and totally Baire is preserved under continuous, open surjective maps, so since $G^{(0)}$ is second countable and totally Baire, the same holds for the orbit space $G^{(0)}/G$.

Lemma 4.4. *Let $F \subseteq G^{(0)}$ be closed and G-invariant, and let $\tilde{q} : F \to q(F)$ be the restriction of the quotient map $q : G^{(0)} \to G^{(0)}/G$ to F. Equip F and $q(F)$ with the induced topologies. Then $\tilde{q}$ is an open map.*

Proof. Let $U \subseteq F$ be open, and choose $V \subseteq G^{(0)}$ open such that $U = F \cap V$. The inclusion $q(F \cap V) \subseteq q(F) \cap q(V)$ is clearly true. On the other hand, let $x \in q(F) \cap q(V)$ and choose $f \in F$ and $v \in V$ such that $x = q(f) = q(v)$. By G-invariance of F, we see that $v \in F$, hence $x = q(v) \in q(F \cap V)$. It follows that $q(U) = q(F) \cap q(V)$, and since $q(V)$ is open in $G^{(0)}/G$, we conclude that $q(U)$ is open in $q(F)$. □

For the next lemma, we note that any locally compact Hausdorff space is totally Baire (since locally compact spaces are totally Baire, and locally closed subsets of locally compact spaces again are locally compact). Hence, if G is a topological groupoid which is second countable, locally compact and Hausdorff, the unit space $G^{(0)} \subseteq G$ has the same properties, and is in particular totally Baire.

Proposition 4.5. *(Cf. Lemma 2.1 of [40]) Let G be a second countable, locally compact Hausdorff étale groupoid, and let $F \subseteq G^{(0)}$ be closed and G-invariant. Then F is prime if and only if there is an $x \in G^{(0)}$ such that $F = \overline{[x]_G}$.*

Proof. Let $F \subseteq G^{(0)}$ be prime. Note that $q(F)$ is closed, and that the assumptions on G and $G^{(0)}$ imply that $G^{(0)}/G$ is second countable and totally Baire. Now, if $q(F) = F_1 \cup F_2$ with F_1 and F_2 closed, we have $F \subseteq q^{-1}(F_1) \cup q^{-1}(F_2)$, so since F is prime, we may assume that $F \subseteq q^{-1}(F_1)$. It follows that $q(F) = F_1$, so $q(F)$ is irreducible. But then Lemma 4.3 implies that $q(F)$ is the closure of a single point, say $q(F) = \overline{\{\xi\}}$ for some $\xi \in G^{(0)}/G$. Choose $x \in F$ such that $q(x) = \xi$. Assume towards a contradiction that $H = F \setminus \overline{[x]}$ is non-empty. Then H is open in F, so $q(H)$ is open in $q(F)$ by Lemma 4.4. Since $\{\xi\}$ is dense in $q(F)$, we have $\xi \in q(H)$, so we may choose a $y \in H$ with $q(y) = \xi = q(x)$. But then y is both in $[x]$ and in $F \setminus \overline{[x]}$, which is absurd. It follows that $F = \overline{[x]}$.

For the other direction, simply observe that if $\overline{[x]} = F_1 \cup F_2$ for some closed invariant sets F_1 and F_2, there is an $i = 1, 2$ such that $x \in F_i$. But then $\overline{[x]} \subseteq F_i$, so $\overline{[x]}$ is irreducible. □

4.1.2 Orbit closures and isotropy

Recall that a primitive ideal of a C^*-algebra $\mathcal{A}$ is the kernel of an irreducible representation of $\mathcal{A}$. We note the following result, which is a special case of Lemma 2.4 of [40]:

Lemma 4.6. *Let G be an amenable second countable, locally compact Hausdorff étale groupoid, and π an irreducible representation of $C_r^*(G)$. Then π factors through $C_r^*(G_{\overline{[x]}})$ for some $x \in G^{(0)}$.*

Proof. This is Lemma 2.4 of [40], with amenability ensuring that $C^*(G) \simeq C_r^*(G)$. □

Lemma 4.6 implies that reductions $G|_{\overline{[x]}}$ hold the key to determining the irreducible representations of $C_r^*(G)$. In this section, we investigate to topology of the sets $\overline{[x]}$ with respect to the groupoids Γ_φ and Γ_φ^+. Along the way, we calculate the possible isotropy groups for each of the groupoids. Recall that a point $x \in \mathbb{T}$ is *periodic* (for φ) if there is an $n \in \mathbb{N}$ such that $\varphi^n(x) = x$, and that the *minimal* period of x is the smallest n satisfying this. We denote by Per_n the set of points with minimal period n. Furthermore, x is *preperiodic* if there is a $k \in \mathbb{N}$ such that $\varphi^k(x)$ is periodic. Similarly, we say that $x \in \mathbb{T}$ is pre-critical if there is a $k \in \mathbb{N}$ such that $\varphi^k(x)$ is critical for φ (in particular, critical points are precritical). Finally, x is *G-isotropic* if the isotropy group $\mathrm{Iso}(x)$ (with respect to G) is non-trivial.

Proposition 4.7. *Let $G = \Gamma_\varphi^+$ and $x \in \mathbb{T}$. Then x is G-isotropic if and only if it is preperiodic, in which case* $\mathrm{Iso}(x) = \mathbb{Z}$.

Proof. It is clear from the definition that it is necessary for x to be preperiodic for $\mathrm{Iso}(x)$ to be non-trivial. To show that it is also sufficient, assume that x is preperiodic, that is, $\varphi^j(x)$ is periodic with period p, and both numbers are chosen minimal. Let $\mathcal{O}$ denote the periodic orbit $\{\varphi^j(x), \varphi^{j+1}(x), \ldots, \varphi^{j+p-1}(x)\}$. There are now two cases: If $\mathcal{O}$ contains no critical points of φ, we have

$$\mathrm{val}(\varphi^{j+2p}, x) = \mathrm{val}(\varphi^p, \varphi^j(x)) \bullet \mathrm{val}(\varphi^p, \varphi^j(x)) \bullet \mathrm{val}(\varphi^j, x),$$

using that $\varphi^{j+p}(x) = \varphi^j(x)$. Since $\mathrm{val}(\varphi^p, \varphi^j(x)) \in \{(+,+), (-,-)\}$, we get

$$\mathrm{val}(\varphi^p, \varphi^j(x)) \bullet \mathrm{val}(\varphi^p, \varphi^j(x)) = (+,+)$$

and it follows that $\mathrm{val}(\varphi^{j+2p}, x) = \mathrm{val}(\varphi^j, x)$, so $(x, 2p, x) \in \Gamma_\varphi^+$, and the isotropy at x is non-trivial.

If there are critical points in $\mathcal{O}$, we have $\mathrm{val}(\varphi^p, \varphi^j(x)) \in \{(+,-), (-,+)\}$, so

$$\mathrm{val}(\varphi^{j+p}, x) = \mathrm{val}(\varphi^p, \varphi^j(x)) \bullet \mathrm{val}(\varphi^j, x) = \mathrm{val}(\varphi^p, \varphi^j(x)),$$

which means that $(x, p, x) \in \Gamma_\varphi^+$, and the isotropy at x is non-trivial.

In both cases, there is a $j \in \mathbb{N}$ such that

$$\mathrm{Iso}(x) = \{(x, kj, x) \mid k \in \mathbb{Z}\} \simeq \mathbb{Z}$$ □

Proposition 4.8. *Let $x \in \mathbb{T}$. Then x is isotropic for Γ_φ if and only if*

- *x is pre-periodic but not pre-critical, in which case* $\mathrm{Iso}(x) \simeq \mathbb{Z}$, *or*
- *x is pre-critical but not pre-periodic, in which case* $\mathrm{Iso}(x) \simeq \mathbb{Z}_2$, *or*
- *x is pre-critical and pre-periodic, in which case* $\mathrm{Iso}(x) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$.

Proof. Assume that x is isotropic. Then there is a $k \in \mathbb{Z}$ or a local transfer η such that $[x, k, \eta, x] \in \Gamma_\varphi$, and $k \neq 0$ or $[\eta]_x \neq [\mathrm{id}]_x$. Note that $\eta(x) = x$. If $k \neq 0$, there are $n, m \in \mathbb{N}$ with $n - m = k$ and $\varphi^n(x) = \varphi^n(\eta(x)) = \varphi^m(x)$, hence x is pre-periodic. If $k = 0$, there is an $n \in \mathbb{N}$ with $\varphi^n \circ \eta = \varphi^n$ around x. Since η reverses orientation, x must be critical for φ^n, and hence x is pre-critical.

Assume now that x is pre-critical for φ, i.e. that $\varphi^n(x) \in \mathcal{C}$ for some $n \in \mathbb{N}$, and that x is not pre-periodic. Then x is critical for φ^{n+1}, so there is a unique (germ of) η with $\eta(x) = x$, $\varphi^{n+1} \circ \eta = \varphi^{n+1}$ and $[\eta]_x \neq [\mathrm{id}]_x$. Then $[x, 0, \eta, x] \in \mathrm{Iso}(x)$. Note that this η is unique (up to germ-wise identification) and satisfies $[\eta \circ \eta]_x = [\mathrm{id}]_x$. If furthermore x is *not* preperiodic, we have

$$\mathrm{Iso}(x) = \{[x, 0, \mathrm{id}, x], [x, 0, \eta, x]\} \simeq \mathbb{Z}_2.$$

If x is pre-periodic, but not pre-critical, we get in a similar way that $\mathrm{Iso}(x) \simeq \mathbb{Z}$. If finally x is both pre-periodic and pre-critical, choose n such that $\varphi^n(x)$ is critical, and $k, p \in \mathbb{N}$ such that $\varphi^{k+p}(x) = \varphi^k(x)$, with p the minimal period of $\varphi^k(x)$. We may assume without loss of generality that $k > n$. There is a unique (germ of a) local transfer η such that $\varphi^n \circ \eta = \varphi^n$, $\eta(x) = x$ and $[\eta]_x \neq [\mathrm{id}]_x$. It follows that the elements $(x, jp, [\eta]_x, x)$ (for $j \in \mathbb{Z}$) are all in $\mathrm{Iso}(x)$, as are the elements $(x, jp, [\mathrm{id}]_x, x)$. Then

$$\mathrm{Iso}(x) = \{[x, jp, \mathrm{id}, x], [x, jp, \eta, x] | j \in \mathbb{Z}\} \simeq \mathbb{Z} \oplus \mathbb{Z}_2$$

as we wanted. □

The key result in this section is the following:

Lemma 4.9. *Let $G = \Gamma_\varphi^+$ or Γ_φ, and $x \in \mathbb{T}$. Then either $\overline{[x]}_G$ contains an isolated, isotropic point, or the non-isotropic points are dense in $\overline{[x]}_G$.*

The proof comes in two parts, depending on what groupoid we consider:

Proof (if $G = \Gamma_\varphi^+$:). Assume first that there is an open set $V \subset \mathbb{T}$ such that $\varphi^n|_V = \mathrm{id}_V$ for some n, and that $[x] \cap V$ is not empty. We show that $[x] \cap V$ is finite. If $y \in [x] \cap V$, there are k and l such that $\varphi^k(x) = \varphi^l(y)$. Since y is n-periodic, we may assume that $\varphi^l(y) = x$ and $0 \leq l \leq n$. But then $y \in \bigcup_{0 \leq l \leq n} \varphi^{-l}(x)$, which is a finite set. In particular, y is isolated and isotropic in $\overline{[x]}$.

Assume, on the other hand, that no iterate φ^n restricts to the identity on any open subset of $\mathbb{T}$. If the non-preperiodic points are *not* dense in $\overline{[x]}$, there is an open set U of $\overline{[x]}$ such that

$$U \subseteq \bigcup_{n,j} \varphi^{-j}(\mathrm{Per}_n)$$

By the Baire category theorem, there are then numbers n and j and an open set W of $\overline{[x]}$ such that

$$W \subseteq \varphi^{-j}(\mathrm{Per}_n)$$

Let $y \in W$, and choose $\varepsilon > 0$ such that $B(y, \varepsilon) \cap \overline{[x]} \subseteq \varphi^{-j}(\mathrm{Per}_n)$. The set $V = B(y, \varepsilon) \cap [x]$ is non-empty. We claim that it is also finite. Indeed, pick a $z \in V$, and note that $\varphi^j(z) = \varphi^{n+j}(z)$. Since $z \in [x]$, there is a k such that $\varphi^k(x) = \varphi^j(z)$, so the forward orbit of x is finite. For any $z' \in V$, the forward orbit of z' meets the forward orbit of x after at most $j + n$ iterations. Hence, $\varphi^{j+n}(V)$ is finite, which implies that V is finite since φ is piecewise monotone. Finiteness of V in turn means that any element of V is isolated in $\overline{[x]}$, and the inclusion $V \subseteq \varphi^{-j}(\mathrm{Per}_n)$ shows that V consists of isotropic points. □

Proof (Proof if $G = \Gamma_\varphi$). If there is a $n \in \mathbb{N}$ and an open set V such that $\varphi^n|_V$ is the identity, the argument from the proof above works. Assume not, and assume again that the set of non-isotropic points is *not* dense in $\overline{[x]}$. Bear in mind that isotropic points in Γ_φ^+ can be either preperiodic or precritical, so we know that there is a set U, open in $\overline{[x]}$ such that

$$U \subseteq \bigcup_{n,j} \varphi^{-j}(\mathrm{Per}_n \cup \mathcal{C})$$

The Baire category theorem yields numbers n and j and an open set W of $\overline{[x]}$ such that

$$W \subseteq \varphi^{-j}(\mathrm{Per}_n \cup \mathcal{C}).$$

Let $y \in W$ and choose $\varepsilon > 0$ such that

$$V = B(y, \varepsilon) \cap \overline{[x]} \subseteq \varphi^{-j}(\mathrm{Per}_n \cup \mathcal{C}) = \varphi^{-j}(\mathrm{Per}_n) \cup \varphi^{-j}(\mathcal{C})$$

Write $V_1 = V \cap \varphi^{-j}(\mathrm{Per}_n)$ and $V_2 = V \cap \varphi^{-j}(\mathcal{C})$. Now, V_1 is finite by the same argument as in the proof above, and V_2 is finite since $\varphi^{-j}(\mathcal{C})$ is finite. It follows that V is finite, so any element of V is isolated in $\overline{[x]}$ and isotropic. □

4.2 Simplicity

Let G denote either Γ_φ or Γ_φ^+. The goal of this section is to obtain a 'dynamical' criterion for simplicity of the algebras $C_r^*(G)$ The strategy consists of three steps: First, we show that $C_r^*(G)$ is simple if and only if $[x]_G$ is dense in $\mathbb{T}$ for all x (this is easy). Then, we establish necessary and sufficient conditions for this to be true for Γ_φ^+, expressed in terms of transitivity of φ and the existence of a certain class of fixed points. This part is subdivided into several cases depending on the degree of φ. Finally, we exploit a connection between $C_r^*(\Gamma_\varphi)$ and $C_r^*(\Gamma_\varphi)$ and show that one algebra is simple if and only if the other is.

As we have seen in Lemma 3.21, open (or closed) invariant subsets of $\mathbb{T}$ give rise to ideals in $C_r^*(G)$. In particular, given an $x \in \mathbb{T}$ such that $[x]_G$ is not dense, we can put $A = \overline{[x]_G}$ and obtain an ideal $C_r^*(\Gamma_\varphi|_{A^c}) = \ker(\pi_A)$. We now aim to show a converse result – if $[x]_G$ is dense for all $x \in \mathbb{T}$, $C_r^*(G)$ is simple.

Lemma 4.10. *$C_r^*(G)$ is simple if and only if $[x]_G$ is dense for all $x \in \mathbb{T}$.*

Proof. One direction is clear already. For the other direction, assume that $[x]_G$ is dense in $\mathbb{T}$ for all $x \in G$. Then certainly no $[x]_G$ contains an isolated, isotropic point, so by Lemma 4.9, the non-isotropic points are dense in $\mathbb{T}$. But then Corollary 2.18 of [45] gives the desired conclusion. □

As a standing assumption for the remainder of this chapter, we will assume that φ is not locally injective, i.e. has at least one critical point. However, if φ is locally injective, it is a local homeomorphism, and by Proposition 4.3 of [11], simplicity of the algebra is then equivalent to φ being strongly transitive (since $\mathbb{T}$ is an infinite set).

4.2.1 Exceptional fixed points

In this section, we focus only on Γ_φ^+ – meaning, in particular, that we will write $[x]$ for $[x]_{\Gamma_\varphi^+}$. First, we expand on the definition of transitivity from 2.21:

Definition 4.11. Let $\varphi : \mathbb{T} \to \mathbb{T}$ be a map. We say that

- *totally transitive* if $\varphi^n : T \to \mathbb{T}$ is transitive for all n,
- *strongly transitive* if for any open set $U \subseteq \mathbb{T}$ there is an N such that $\bigcup_{i=1}^N \varphi^i(U) = \mathbb{T}$,
- *exact* if for any open set $U \subseteq \mathbb{T}$, there is an N such that $\varphi^N(U) = \mathbb{T}$.

Remark 4.12. It is clear that a totally (or strongly) transitive map is transitive, and that an exact map is strongly transitive. Furthermore, by Corollary 4.2 of [52], a transitive, piecewise monotone circle map is strongly transitive. Furthermore, φ has a periodic point by Lemma 2.24, and then Theorem C of [9] shows that exactness and total transitivity are equivalent conditions.♦

Finally, given an $e \in \mathbb{T}$, we say that e an *exceptional fixed point* if $\varphi^{-1}(e) \setminus \mathcal{C} = \{e\}$. The main theorem, which will take several pages to prove, is the following:

Theorem 4.13. *Let φ be a circle map. The following are equivalent:*

1. *$C_r^*(\Gamma_\varphi^+)$ is simple.*
2. *φ is totally transitive and has no exceptional fixed points.*
3. *φ is exact and has no exceptional fixed points.*

The proof requires a deep plunge into the dynamics of circle maps. We begin by showing that transitivity of φ is a necessary condition for simplicity:

Lemma 4.14. *Assume that $C_r^*(\Gamma_\varphi^+)$ is simple. Then φ is transitive.*

Proof. Let $E \subseteq \mathbb{T}$ be closed with non-empty interior and φ-invariant in the sense that $\varphi(E) \subseteq E$. By Theorem 5.9 of [50] it suffices to show that $E = \mathbb{T}$. For each $n, m \geq 1$ set

$$U_{n,m} = \{x \in \mathbb{T} \mid \varphi^n(x) = \varphi^m(y), \mathrm{val}(\varphi^n, x) = \mathrm{val}(\varphi^m, y) \text{ for some } y \in \operatorname{Int} E\}$$

where $\operatorname{Int} E$ is the interior of E. Note that $U_{n,m}$ is open and non-empty and that $\bigcup_{n,m} U_{n,m}$ is Γ_φ^+-invariant. It follows therefore from Lemma 1.16 that $\bigcup_{n,m} U_{n,m} = \mathbb{T}$. By compactness there is an $N \in \mathbb{N}$ such that $\mathbb{T} = \bigcup_{n,m=1}^N U_{n,m}$. Since $U_{n,m} \subseteq \varphi^{-n}(E)$ we find then that

$$\mathbb{T} = \varphi^N(\mathbb{T}) \subseteq \varphi^N\Big(\bigcup_{n=1}^N \varphi^{-n}(E)\Big) \subseteq E.$$

□

The converse of Lemma 4.14 is not true in general; transitivity of φ does not imply that $C_r^*(\Gamma_\varphi^+)$ is simple.

For the next lemma, let $\mathcal{C}$ denote the critical points of φ, recall that the critical values of φ is the set $\varphi(\mathcal{C})$, and that the post-critical points is the set $\bigcup_{i=1}^\infty \varphi^i(\mathcal{C})$. Note that critical points are pre-critical, but not post-critical.

Lemma 4.15. *Assume that φ is transitive. Let $A \subseteq \mathbb{T}$ be a non-empty Γ_φ^+-invariant subset which is not dense in $\mathbb{T}$. Then A is finite and consists of points that are post-critical and not pre-critical.*

Proof. By assumption there is an open non-empty interval $J \subseteq \mathbb{T}$ such that

$$A \cap J = \emptyset. \tag{4.1}$$

By Remark 4.12, φ is not only transitive, but also strongly transitive. There is therefore an $N \in \mathbb{N}$ such that

$$\bigcup_{i=0}^{N} \varphi^i(J) = \mathbb{T}. \tag{4.2}$$

If $x \in A$ and $\operatorname{val}(\varphi^j, x) \in \{(+,-),(-,+)\}$ for some $j \geq 1$ we can choose $y \in J$ such that $\varphi^k(y) = x$ for some $k \in \{1,2,\dots,N\}$. It follows from the composition table for $\bullet$ that

$$\operatorname{val}(\varphi^{k+j}, y) = \operatorname{val}(\varphi^j, x) \bullet \operatorname{val}(\varphi^k, y) = \operatorname{val}(\varphi^j, x).$$

Hence $y \in [x] \subseteq A$, contradicting (4.1). It follows that $\operatorname{val}(\varphi^j, x) \in \{(+,+),(-,-)\}$ for all $j \in \mathbb{N}$ when $x \in A$; i.e A consists of points that are not pre-critical.

Since φ is not locally injective there is a $z \in \mathbb{T}$ such that $\operatorname{val}(\varphi, z) \in \{(+,-),(-,+)\}$. Choose $z_0 \in J$ and $k \in \{1,2,\dots,N\}$ such that $\varphi^k(z_0) = z$ and note that $\operatorname{val}(\varphi^{k+1}, z_0) \in \{(+,-),(-,+)\}$. There are therefore subintervals J_+, J_- of J such that $\operatorname{val}(\varphi^{k+1}, y) = (+,+)$ when $y \in J_+$, $\operatorname{val}(\varphi^{k+1}, y) = (-,-)$ when $y \in J_-$, and $\varphi^{k+1}(J_+) = \varphi^{k+1}(J_-) =: I$. Since φ is strongly transitive there is a $K \in \mathbb{N}$ such that $\bigcup_{i=1}^{K} \varphi^i(I) = \mathbb{T}$. Set $M_i = I \cap \mathcal{C}_i$. Let $a \in \mathbb{T}$ be a non-critical element, i.e. $\operatorname{val}(\varphi, a) \in \{(+,+),(-,-)\}$. Assume that $a \notin \bigcup_{i=1}^{K} \varphi^i(M_i)$. We claim that $[a] \cap J \neq \emptyset$. To see this note that there is an $i \in \{1,2,\dots,K\}$ and a $y' \in I \backslash M_i$ such that $\varphi^i(y') = a$. Then $\operatorname{val}(\varphi^i, y') \in \{(+,+),(-,-)\}$ and there is also an element $y \in J_+ \cup J_-$ such that $\varphi^{k+1}(y) = y'$ and

$$\operatorname{val}(\varphi^{i+k+2}, y) = \operatorname{val}(\varphi^{i+1}, y') \bullet \operatorname{val}(\varphi^{k+1}, y) = \operatorname{val}(\varphi, a)$$

It follows that $y \in [a] \cap J$, proving the claim.

The last two paragraphs show that $A \subseteq \bigcup_{i=1}^{K} \varphi^i(M_i)$. This completes the proof because $\bigcup_{i=1}^{K} \varphi^i(M_i)$ is finite and consists of post-critical points. $\square$

We will refer to points x with $[x]$ finite as *exposed* points. By the lemma above, exposed points are the only obstruction for simplicity of $C_r^*(\Gamma_\varphi^+)$. To figure out when a map has exposed points, we divide into cases depending on the degree of the map:

4.2.2 $|\deg(\varphi)| \geq 2$

Lemma 4.16. *Assume that φ is transitive and that $|\deg \varphi| \geq 2$. It follows that $[x]$ is dense in $\mathbb{T}$ for all $x \in \mathbb{T}$.*

Proof. Let $n \in \mathbb{N}$. By looking at the graph of a lift $f : [0,1] \to \mathbb{R}$ of φ^{2n} one sees that for any $x \in \mathbb{T}$, the set

$$A_n = \left\{ y \in \mathbb{T} \;\middle|\; \varphi^{2n}(y) = x, \operatorname{val}(\varphi^{2n}, y) = (+,+) \right\}$$

contains at least $\deg \varphi^{2n}$ elements. Since $A_n \subseteq [x]$ we conclude that $[x]$ is infinite for all $x \in \mathbb{T}$. It follows then from Lemma 4.15 that $[x]$ is dense for all x. $\square$

Corollary 4.17. *Assume that φ is transitive and that $|\deg \varphi| \geq 2$. Then $C_r^*(\Gamma_\varphi^+)$ is simple.*

4.2.3 $|\deg \varphi| = 1$

We next assume that the degree of φ is 1 or -1. If φ is not totally transitive, we have the following decomposition theorem:

Lemma 4.18. *Assume that φ is transitive, but not totally transitive. It follows that there is a $p > 1$ and closed intervals $I_i, i = 0, 1, 2, \dots, p-1$, such that*

1. $\varphi(I_i) = I_{i+1}$ *(addition mod p),*
2. $I_i \cap \operatorname{Int} I_j = \emptyset,\ i \neq j,$
3. $\bigcup_{i=0}^{p-1} I_i = \mathbb{T},$
4. $\varphi^p|_{I_i}$ *is totally transitive for each i.*

Proof. This is a special case of Corollary 2.7 in [1]. □

Note that the number p and the collection $\{I_0, I_1, \dots, I_{p-1}\}$ of intervals in Lemma 4.18 are unique. We will refer to p as *the global period of φ*, and say that it is 1 when φ is totally transitive. In the following we denote the set of endpoints of the intervals I_i from Lemma 4.18 by $\mathcal{E}$.

Lemma 4.19. *Assume that φ is transitive but not totally transitive. Then*

$$\varphi^{-1}(\mathcal{E}) \backslash \mathcal{C} = \mathcal{E}. \tag{4.3}$$

Proof. Assume for a contradiction that $e \in \mathcal{E}$, but $\varphi(e) \notin \mathcal{E}$. There are then intervals $I_i, I_{i'}, I_j$ as in Lemma 4.18 such that $i \neq i'$, $e \in I_i \cap I_{i'}$ and $\varphi(e) \in \operatorname{Int} I_j$. By continuity of φ and condition 1) from Lemma 4.18 it follows that $I_{i+1} \cap \operatorname{Int} I_j \neq \emptyset$ and $I_{i'+1} \cap \operatorname{Int} I_j \neq \emptyset$. Since $i+1 \neq i'+1$ this violates condition 2). Thus

$$\varphi(\mathcal{E}) \subseteq \mathcal{E}. \tag{4.4}$$

If $e \in \mathcal{E}$ is a critical point the images $I_{i+1} = \varphi(I_i)$ and $I_{i'+1} = \varphi(I_{i'})$ of the two intervals $I_i, I_{i'}$ containing e will both have non-trivial intersection with the same interval I_j containing $\varphi(e)$; contradicting 2) again. Hence

$$\mathcal{E} \cap \mathcal{C} = \emptyset. \tag{4.5}$$

Consider then an element $x \in \varphi^{-1}(\mathcal{E})$ and assume that $x \notin \mathcal{C}_1$. Let I_i and $I_{i'}$ be the two intervals among the intervals from Lemma 4.18 which contain $\varphi(x)$. If $x \notin \mathcal{E}$ there is a third interval I_j which contains x in its interior. Since x is not critical it follows that $\varphi(I_j) = I_{j+1}$ has non-trivial intersection with both $\operatorname{Int} I_i$ and $\operatorname{Int} I_{i'}$, contradicting 2) once more. Hence

$$\varphi^{-1}(\mathcal{E}) \backslash \mathcal{C}_1 \subseteq \mathcal{E}. \tag{4.6}$$

□

This completes the proof since (4.3) is equivalent to (4.4), (4.5) and (4.6).

Lemma 4.20. *Assume that φ is transitive but not totally transitive. When $\deg \varphi = 1$ the set $\mathcal{E}$ is a p-periodic orbit where p is the global period of φ, and $\operatorname{val}(\varphi, x) = (+,+)$ for all $x \in \mathcal{E}$. When $\deg \varphi = -1$ the global period of φ is 2 and $\mathcal{E}$ consists of two distinct fixed points of valency $(-,-)$.*

Proof. Let $I_i, i = 0, 1, \cdots, p-1$, be the intervals from Lemma 4.18, and let e_i^-, be the left endpoint of I_i, defined using the orientation of $\mathbb{T}$. When $\deg \varphi = 1$ we see by looking at the graph of a lift of φ that $\operatorname{val}(\varphi, e_i^-) = (+,+)$ and $\varphi(e_i^-) = e_{i+1}^-$ (addition mod p). It follows that $\mathcal{E} = [e_0^-]$, and that this is also the (forward) orbit of e_0^-. When $\deg \varphi = -1$ observe first φ has a fixed point x. This fixed point lies in one of the intervals I_i. Since x also lies in I_{i+1} and I_{i+2} it follows that two of the intervals I_i, I_{i+1} and I_{i+2} must be the same, i.e. $p = 2$. By looking at the graph of a lift of φ we see that $\mathcal{E}$ consists of two fixed points of valency $(-,-)$. □

Lemma 4.21. *If* $\deg \varphi = 1$ *there is for all* $x \in \mathbb{T}$ *an element* $y \in \varphi^{-1}(x)$ *such that* $\operatorname{val}(\varphi, y) = (+,+)$. *If* $\deg \varphi = -1$ *there is for all* $x \in \mathbb{T}$ *an element* $y \in \varphi^{-1}(x)$ *such that* $\operatorname{val}(\varphi, y) = (-,-)$.

Proof. Look at the graph of a lift of φ. □

Lemma 4.22. *Assume that* $\deg \varphi \in \{1, -1\}$. *Then* $[x]$ *is infinite for all* $x \in \mathbb{T}$ *that are not periodic under* φ.

Proof. Let $x \in \mathbb{T}$. It follows from Lemma 4.21 that there are sequences $\{n_i\}$ in $\mathbb{N}$ and $\{x_i\}$ in $\mathbb{T}$ such that $\varphi^{n_1}(x_1) = x$, $\varphi^{n_i}(x_i) = x_{i-1}$, $i \geq 2$, and $\operatorname{val}(\varphi^{n_i}, x_i) = (+,+)$ for all i. Then $x_i \in [x]$ for all i. The set $\{x_i : i \in \mathbb{N}\}$ is infinite when x is not periodic. □

Lemma 4.23. *Assume that* $\deg \varphi \in \{-1, 1\}$ *and* φ *is transitive but not totally transitive. Then* $\mathcal{E}$ *is the set of exposed points for* φ.

Proof. Let $e \in \mathcal{E}$ and $y \in [e]$. There are natural numbers $i, j \in \mathbb{N}$ such that $\varphi^i(e) = \varphi^j(y)$ and $\operatorname{val}(\varphi^i, e) = \operatorname{val}(\varphi^j, y)$. It follows from (4.3) that $\varphi^j(y) = \varphi^i(x) \in \mathcal{E}$ and that $\operatorname{val}(\varphi^i, e) \in (\pm, \pm)$ since $e \in \mathcal{E}$. This implies first that $\operatorname{val}(\varphi, \varphi^k(y)) \in (\pm, \pm)$ for all $k \leq j-1$ and then that $\varphi^{j-1}(y) \in \varphi^{-1}(\mathcal{E}) \backslash \mathcal{C}_1 = \mathcal{E}$. But then $\varphi^{j-2}(y) \in \varphi^{-1}(\mathcal{E}) \backslash \mathcal{C}_1 = \mathcal{E}$, and so on. After j steps we conclude that $y \in \mathcal{E}$. This shows that $\mathcal{E}$ is Γ_φ^+-invariant.

It remains to show that $\mathcal{E}$ contains all exposed points. Assume therefore that y_0 is an exposed point. It follows from Lemma 4.22 that all exposed points are periodic. Since they are also post-critical by Lemma 4.15 and there are only finitely many critical points it follows that there are only finitely many exposed points. Let $m \in \mathbb{N}$ be an even number divisible by the global period p and by all the periods of exposed points. Then $\varphi^m(y_0) = y_0$. Furthermore, if $z \in \varphi^{-m}(y_0)$ and $\operatorname{val}(\varphi^m, z) = (+,+)$ we see that $z \in [y_0]$ and hence z is exposed. By definition of m this implies that $\varphi^m(z) = z$, i.e. $z = y_0$. To see that there can not be any $z \in \varphi^{-m}(y_0)$ with $\operatorname{val}(\varphi^m, z) = (-,-)$ observe by looking at the graph of the lift of φ^m, that since $\deg \varphi^m = 1$ the existence of such a z would imply the existence of a $z' \in \varphi^{-m}(y_0) \backslash \{y_0\}$ with $\operatorname{val}(\varphi^m, z') = (+,+)$ which is impossible as we have just seen. Now assume for a contradiction that $y_0 \notin \mathcal{E}$. Then y_0 lies in the interior of one of the intervals from Lemma 4.18, say I_i. We can then write I_i as the union $I_i = J_1 \cup J_2$ of two closed non-degenerate intervals such that $J_1 \cap J_2 = \{y_0\}$. As we have just seen an element of $I_i \cap (\varphi^{-m}(y_0) \backslash \{y_0\})$ must be critical for φ^m and it follows therefore that $\varphi^m(J_1) = J_1$. This contradicts the total transitivity of $\varphi^p|_{I_i}$. □

Lemma 4.24. *Assume that* φ *is totally transitive and that* $\deg \varphi \in \{-1, 1\}$. *It follows that there is at most a single exposed point, and it must be a fixed point* e *such that* $\varphi^{-1}(e) \backslash \mathcal{C}_1 = \{e\}$.

Proof. Let m be the same number as in the proof of Lemma 4.23. In that proof it was shown that

$$\varphi^{-m}(y) \backslash \{y\} \subseteq \mathcal{C}_m \tag{4.7}$$

for every exposed point y. It follows that if there are two exposed points, say e_1 and e_2, we could write $\mathbb{T} = J_1 \cup J_2$ where J_1 and J_2 are non-degenerate closed intervals such that $J_1 \cap J_2 = \{e_1, e_2\}$

and $\varphi^{2m}(J_i) = J_i, i = 1,2$. This contradicts the assumed total transitivity of φ. Therefore there is at most at single exposed point e, and it is fixed by φ^m. When $\deg\varphi = 1$ it follows from Lemma 4.21 that there is an element $z \in \varphi^{-1}(e)$ such that $\mathrm{val}(\varphi, z) = (+,+)$. Then z is exposed (since $z \in [e]$) and the uniqueness of e implies that $z = e$, proving that e is a fixed point for φ.

To reach the same conclusion when $\deg\varphi = -1$ it suffices to consider the case where $\mathrm{val}(\varphi, e) = (-,-)$. By Lemma 4.21 there are elements $z_1, z \in \mathbb{T}$ such that $\varphi(z_1) = z$, $\varphi(z) = e$ and $\mathrm{val}(\varphi, z_1) = \mathrm{val}(\varphi, z) = (-,-)$. Then $z_1 \in [e]$ and hence $z_1 = e$ because e is the only exposed point. It follows that $z = \varphi(e)$, i.e. $\varphi^2(e) = e$. Note that $\mathrm{val}(\varphi, \varphi(e)) = (-,-)$. We claim that

$$\varphi^{-1}(\{e, \varphi(e)\}) \backslash \mathcal{C}_1 = \{e, \varphi(e)\}. \tag{4.8}$$

□

To show this let $x \in \varphi^{-1}(e)\backslash\mathcal{C}_1$. If $\mathrm{val}(\varphi, x) = (+,+)$ we find that $x = e$ since e is the only exposed point. If $\mathrm{val}(\varphi, x) = (-,-)$ an application of Lemma 4.21 shows that $x = \varphi(e)$. Consider then an element $y \in \varphi^{-1}(\varphi(e))\backslash\mathcal{C}_1$. If $\mathrm{val}(\varphi, y) = (-,-)$ it follows that $y \in [e]$ and hence $y = e$ by uniqueness of e. If instead $\mathrm{val}(\varphi, y) = (+,+)$ an application of Lemma 4.21 shows that that $y = \varphi(e)$. Having established (4.8) note that it implies that $\varphi(e)$ is exposed, whence equal to e.

To show that $\varphi^{-1}(e)\backslash\mathcal{C}_1 = \{e\}$ we may assume that $\deg\varphi = 1$, since the other case follows from (4.8). Furthermore, it suffices to show that $\varphi^{-1}(e)\backslash\mathcal{C}_1 \subseteq \{e\}$ since exposed points are not critical by Lemma 4.15. Consider therefore an element $x \in \varphi^{-1}(e)\backslash\mathcal{C}_1$. If $\mathrm{val}(\varphi, x) = (+,+)$ it follows that $x \in [e]$ and hence x is exposed. Since e is the only exposed points this shows that $x = e$. Assume then that $\mathrm{val}(\varphi, x) = (-,-)$. If $x \neq e$ a look at the graph for a lift of φ shows that there is then also a point $y \in \varphi^{-1}(e)\backslash\{e\}$ with $\mathrm{val}(\varphi, y) = (+,+)$ which we have just seen is not possible. Hence $x = e$.

Proposition 4.25. *Assume that φ is transitive and that $\deg\varphi \in \{-1, 1\}$. Then $C_r^*(\Gamma_\varphi^+)$ is simple unless either*

1. *φ is not totally transitive, or*
2. *φ is totally transitive and there is an exceptional fixed point.*

In case 2) there is an extension

$$0 \longrightarrow B \longrightarrow C_r^*\left(\Gamma_\varphi^+\right) \longrightarrow C(\mathbb{T}) \longrightarrow 0 \tag{4.9}$$

where B is simple and purely infinite. When φ is not totally transitive and $\deg\varphi = 1$ there is an extension

$$0 \longrightarrow B \longrightarrow C_r^*\left(\Gamma_\varphi^+\right) \longrightarrow C(\mathbb{T}) \otimes M_p(\mathbb{C}) \longrightarrow 0, \tag{4.10}$$

where p is the global period of φ and B is simple and purely infinite. When φ is not totally transitive and $\deg\varphi = -1$ there is an extension

$$0 \longrightarrow B \longrightarrow C_r^*\left(\Gamma_\varphi^+\right) \longrightarrow C(\mathbb{T}) \oplus C(\mathbb{T}) \longrightarrow 0, \tag{4.11}$$

where B is simple and purely infinite.

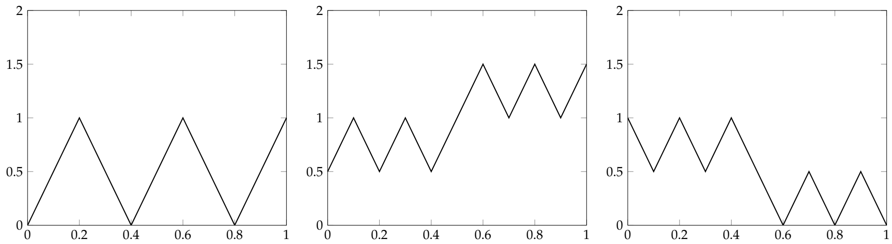


Figure 4.1: Graphs of (lifts of) the maps φ_1, φ_2 and φ_3..

Proof. Assume that none of the two cases 1) or 2) occur. It follows from Lemma 4.24 that there are no exposed points, and then from Corollary 4.15 that $C_r^*(\Gamma_\varphi^+)$ is simple.

In case 2) it follows from Lemma 4.24 that there is exactly one exposed point, e, which is a fixed point. From Lemma 3.21 we get then the extension

$$0 \longrightarrow B \longrightarrow C_r^*\left(\Gamma_\varphi^+\right) \longrightarrow C_r^*\left(\Gamma_\varphi^+|_{\{e\}}\right) \longrightarrow 0 \tag{4.12}$$

where $B = C_r^*(\Gamma_\varphi^+|_{\mathbb{T}\backslash\{e\}})$. By Proposition 1.23, we see that $C_r^*(\Gamma_\varphi^+|_{\{e\}}) \simeq C^*(\mathbb{Z}) \simeq C(\mathbb{T})$. Furthermore, B is purely infinite because B is an ideal in $C_r^*(\Gamma_\varphi^+)$ which is purely infinite by Theorem 2.27. To conclude that B is simple we argue as in the proof of Proposition 4.10 in [47]: The elements of $\mathbb{T}\backslash\{e\}$ with non-trivial isotropy in $C_r^*(\Gamma_\varphi^+|_{\mathbb{T}\backslash\{e\}})$ are pre-periodic. It follows from Lemma 2.26 that the pre-periodic points are countable, whence $\mathbb{T}\backslash\{e\}$ must contain a point with trivial isotropy. By Corollary 2.18 of [45] it suffices therefore to show that $\mathbb{T}\backslash\{e\}$ does not contain any non-trivial (relatively) closed Γ_φ^+-invariant subsets. Let therefore L be such a set. Then $L \cup \{e\}$ is closed and Γ_φ^+-invariant in $\mathbb{T}$ and hence either equal to $\mathbb{T}$ or contained in $\{e\}$ by Lemma 4.15 and Lemma 4.24. It follows that $L = \emptyset$ or $L = \mathbb{T}\backslash\{e\}$. This completes the proof in case 2).

In case 1) we argue as above, except that we use Lemma 4.23 to replace Lemma 4.24, and Lemma 4.20 to determine $C_r^*(\Gamma_\varphi^+|_{\mathcal{E}})$. □

Example 4.26. Let's give examples of the various cases in the Proposition above. Consider the three maps below. The first map φ_1 is exact, with 0 as an exceptional fixed point, so we obtain an extension of the form in (4.9). The second map φ_2 is of degree 1 and transitive but not totally transitive (to see this, observe that φ_2^2 leaves the intervals $(0, 1/2)$ and $(1/2, 1)$ invariant). It follows that we obtain an extension like the one in (4.10), with $p = 2$. The third map φ_3 shows a map of degree -1 which is not totally transitive, giving rise to an extension like (4.11). ▲

4.2.4 $\deg \varphi = 0$

A point $z \in \mathbb{T}$ will be called *an exceptional critical value* when $\varphi^{-1}(z) \subseteq \mathcal{C}$.

Lemma 4.27. *Assume that* $\deg \varphi = 0$ *and that* φ *is surjective. There is at most one exceptional critical value, and for all other elements* $x \in \mathbb{T}$ *there are points* $y_\pm \in \varphi^{-1}(x)$ *such that* $\operatorname{val}(\varphi, y_\pm) = (\pm, \pm)$.

Proof. Look at the graph of a lift of φ. □

Lemma 4.28. *Assume that φ is transitive and that* $\deg \varphi = 0$*. If $y \in \mathbb{T}$ is an exposed point there is an exceptional critical value $e \in \mathbb{T}$ such that $\varphi^2(e) = \varphi(e) \neq e$, $[y] = \{e, \varphi(e)\}$ and $\varphi^{-1}(\varphi(e)) \setminus \mathcal{C} = \{e, \varphi(e)\}$.*

Proof. The main part of the proof will be to show that there is an exceptional critical value e such that one of the following holds:

1. $\varphi^2(e) = \varphi(e) \neq e$, $\text{val}(\varphi, e) = (-,-)$ and $[y] = \{\varphi(e)\}$,
2. $\varphi^2(e) = \varphi(e) \neq e$, $\text{val}(\varphi, \varphi(e)) = (+,+)$ and $[y] = \{e\}$,
3. $\varphi^2(e) = \varphi(e) \neq e$, $[y] = \{e, \varphi(e)\}$.

Assume first that $[y]$ does not contain an exceptional critical value. Let $z \in [y]$. By using Lemma 4.27 we can construct $y_k, k = 0,1,2,3,\ldots$ such that $y_0 = z$, $\varphi(y_k) = y_{k-1}$ and $\text{val}(\varphi, y_k) = (+,+)$, $k \geq 1$. Then $y_k \in [y]$ for all k so there are $k \neq k'$ such that $y_k = y_{k'}$. It follows that z is periodic and that $\text{val}(\varphi, u) = (+,+)$ for all u in the orbit $\mathcal{O}(z)$ of z. Hence $\mathcal{O}(z) \subseteq [y]$. Since this conclusion holds for all $z \in [y]$ and since the forward orbits of elements from $[y]$ must intersect we conclude that $[y] = \mathcal{O}(y)$ and $\text{val}(\varphi, \varphi^k(y)) = (+,+)$ for all $k \in \mathbb{N}$. Let $z \in [y]$. Using Lemma 4.27 again we find $u_1, v_1 \in \varphi^{-1}(z)$ such that $\text{val}(\varphi, u_1)) = (+,+)$ and $\text{val}(\varphi, v_1) = (-,-)$. Then $u_1 \in [y]$ and u_1 is therefore an element of the orbit of y. Since $v_1 \neq u_1$ (or since $\text{val}(\varphi, v_1) = (-,-)$), it follows that v_1 is not in the orbit of y. If v_1 is not an exceptional critical value we can find $v_2 \in \varphi^{-1}(v_1)$ such that $\text{val}(\varphi, v_2) = (-,-)$. It follows that $v_2 \in [y]$ and v_2 must therefore be an element of $\mathcal{O}(y)$. This contradicts that v_1 is not, and we conclude that v_1 must be an exceptional critical value e, which by Lemma 4.27 is unique. This shows that $z = \varphi(e)$ and we conclude therefore that case 1 occurs.

We consider then the case where $[y]$ contains an exceptional critical value e. By looking at the graph of a lift of φ we see that a non-critical exceptional critical value e can not be fixed since the degree is 0. Thus $\varphi(e) \neq e$ since exposed points are not critical. To see that $\varphi(e)$ is a fixed point assume that it is not. Consider first the case where $\varphi(e)$ is periodic, say of period $p > 1$. Since $\varphi(e) \neq e$ it follows from Lemma 4.27 that there is a point $b_1 \in \varphi^{-1}(\varphi(e))$ such that $\text{val}(\varphi, b_1) \neq \text{val}(\varphi, e)$. Then $b_1 \notin \{e, \varphi(e)\}$ and we use Lemma 4.27 again to find $b_2 \in \varphi^{-1}(b_1)$ such that $\text{val}(\varphi, b_2) \neq \text{val}(\varphi, \varphi(e))$. It follows that $b_2 \notin \{e, \varphi(e), b_1\}$. By requiring in each step that $\text{val}(\varphi, b_i) \neq \text{val}(\varphi, \varphi(e))$ we obtain through repeated application of Lemma 4.27 elements $b_i, i = 1,2,\ldots,p+1$, such that $\varphi(b_{k+1}) = b_k$ and $b_{k+1} \notin \{e, \varphi(e), b_1, b_2, \ldots, b_k\}$ for all $k = 1,2,\ldots,p$. Then, for $j > p+1$ we require in each step instead that $\text{val}(\varphi^j, b_j) = \text{val}(\varphi, e)$. It is then still automatic that $b_{k+1} \notin \{e, b_1, b_2, \ldots, b_k\}$ for all k, while the fact that $b_j \neq \varphi(e)$ follows for $j \geq p+1$ because j is larger than the period of $\varphi(e)$. Since $b_j \in [e] = [y]$ when $j > p+1$, we have contradicted the assume finiteness of $[y]$. To get the same contradiction when $\varphi(e)$ is not assumed to be periodic we proceed in the same way, except that the steps between b_1 and b_{p+1} can be bypassed. In any case we conclude that $\varphi^2(e) = \varphi(e)$. We next argue, in a similar way, that $\varphi^{-1}(\varphi(e)) \setminus \mathcal{C}_1 \subseteq \{e, \varphi(e)\}$. Indeed, if $b_1 \in \varphi^{-1}(\varphi(e)) \setminus (\{e, \varphi(e)\} \cup \mathcal{C}_1)$ we use Lemma 4.27 to get a sequence b_i such that $\varphi(b_{i+1}) = b_i, i \geq 1$, and $\text{val}(\varphi, b_i) = (-,-), i \geq 2$. Then $i \neq i' \Rightarrow b_i \neq b_{i'}$, and $b_i \in [e]$ for infinitely many i; again contradicting the infiniteness of $[e]$. Since e is not pre-critical by Lemma 4.15 we have shown that $\varphi^{-1}(\varphi(e)) \setminus \mathcal{C}_1 = \{e, \varphi(e)\}$. If $\text{val}(\varphi, e) = (-,-)$ and $\text{val}(\varphi, \varphi(e)) = (+,+)$ we find now easily that $[y] = [e] = \{e\}$, which is case ii), and in all other cases that $[y] = [e] = \{e, \varphi(e)\}$, which is case 3.

Finally we argue that the cases 1 and 2 are impossible. Indeed, in both cases we must have that $\text{val}(\varphi, e) = (-,-)$ and $\text{val}(\varphi, \varphi(e)) = (+,+)$ since otherwise $e \in [\varphi(e)]$. But then the two closed intervals J_1 and J_2 defined such that $J_1 \cap J_2 = \{e, \varphi(e)\}$ and $J_1 \cup J_2 = \mathbb{T}$ are both φ-invariant, which contradicts the transitivity of φ. It follows that only case 3 can occur. □

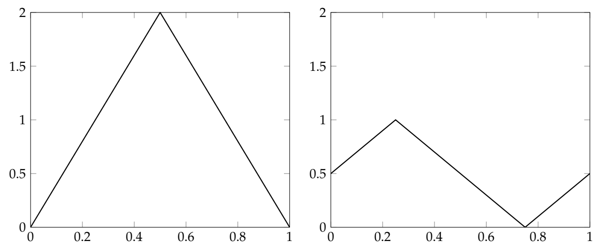


Figure 4.2: Two circle maps – the C^*-algebra of the first map is simple, while the second is not..

Lemma 4.29. *Assume that φ is transitive and $\deg \varphi = 0$. Then there are exposed points if and only if φ is not totally transitive.*

Proof. If φ is not totally transitive there are exposed points by (the proof of) Lemma 4.23. Conversely, if there are exposed points it follows from Lemma 4.28 that there is an exceptional critical value e such that $e \neq \varphi(e) = \varphi^2(e)$ and $\{e, \varphi(e)\}$ is the set of exposed points. Furthermore, $\varphi^{-1}(\varphi(e)) \backslash \mathcal{C}_1 = \{e, \varphi(e)\}$. The points e and $\varphi(e)$ define closed intervals J_1 and J_2 such that $\mathbb{T} = J_1 \cup J_2$, $J_1 \cap J_2 = \{e, \varphi(e)\}$ and $\varphi(J_i) = J_i$, $i = 1, 2$, or $\varphi(J_1) = J_2$ and $\varphi(J_2) = J_1$. The first case is ruled out by transitivity, and the second implies that φ is not totally transitive. □

Proposition 4.30. *Assume that φ is transitive and that $\deg \varphi = 0$. Then $C_r^*\left(\Gamma_\varphi\right)$ is simple if and only if φ is totally transitive. When φ is not totally transitive there is an extension*

$$0 \longrightarrow B \longrightarrow C_r^*\left(\Gamma_\varphi^+\right) \longrightarrow C(\mathbb{T}) \otimes M_2(\mathbb{C}) \longrightarrow 0 \tag{4.13}$$

where B is simple and purely infinite.

Proof. With Lemma 4.29 and Lemma 4.28 at hand all the necessary arguments can be found in the proof of Proposition 4.25. □

Example 4.31. The picture below shows two degree-zero circle maps. The first map is the tent map from Example 3.12, which is totally transitive. From the theorem above, it follows that the corresponding C^*-algebra is simple. The second map is transitive, but not totally transitive (again, the second iterate of the map leaves $(0, 1/2)$ and $(1/2, 1)$ invariant). It follows that the set $\{0, 1/2\}$ is invariant, and that the C^*-algebra of the map sits in a extension like 5.5.

We are ready to give the

Proof. (Of Theorem 4.13) Note first that by Remark 4.12, exactness and total transitivity are equivalent, which shows that 2) and 3) are equivalent. If $C_r^*(\Gamma_\varphi)$ is simple, it follows by Lemma 4.14 that φ is transitive. But then φ is also totally transitive – otherwise the set $\mathcal{E}$ of Lemma 4.18 would be non-empty, finite, and Γ_φ-invariant (by the proof of Lemma 4.22). φ has no exceptional fixed points, since an exceptional fixed point is its own Γ_φ^+-orbit. Assume, on the other hand that φ is totally transitive and without exceptional fixed points. Simplicity of $C_r^*(\Gamma_\varphi^+)$ then follows from Propositions 4.30, 4.25 and 4.17. □

4.2.5 An order two-automorphism

We're now ready to deal with simplicity of $C_r^*(\Gamma_\varphi)$. Our main result is the following:

Theorem 4.32. *Let φ be a circle map. Then $C_r^*(\Gamma_\varphi)$ is simple if and only if $C_r^*(\Gamma_\varphi^+)$ is.*

There is at least two ways to show this. One involves jumping through the same hoops as in the previous section – showing that non-dense Γ_φ-orbits are finite, and then considering several cases depending on the degree of φ, ending up with a copy of Theorem 4.13 for Γ_φ. This gets a little repetitive, however, so we take a shortcut and exploit the relationship between the groupoids Γ_φ and Γ_φ^+ instead. Recall the flip automorphism from Proposition 2.28 given by

$$\Lambda(f)(x,k,p,y) = (-1)^p f(x,k,p,y)$$

for $f \in C_c(\Gamma_\varphi)$, and that the fixed-point algebra of Λ is equal to $C_r^*(\Gamma_\varphi^+)$.

Lemma 4.33. *The automorphism $\Lambda : C_r^*(\Gamma_\varphi) \to C_r^*(\Gamma_\varphi)$ is not implemented by conjugation with a unitary element of $C_r^*(\Gamma_\varphi)$.*

The proof of this is surpringsingly convoluted. Here's the main idea: Take an $x \in \mathbb{T}$ critical for φ, and consider the characteristic function $f = 1_{(x,0,-,x)}$. An easy calculation shows that $\Lambda(f)(x,0,-,x) = -1$, while $(u^*fu)(x,0,-,x) = \sum_k |u(x,k,p,x)|^2 \geq 0$ for any unitary $u \in C_c(\Gamma_\varphi)$. Of course, f is not an element of $C_r^*(\Gamma_\varphi)$, and we cannot be sure that a conjugating unitary is in $C_c(\Gamma_\varphi)$ – so to make the approach work, we need to do our calculations in the space $l^2(s^{-1}(x))$ and do a number of approximations.

Proof. Fix an x critical for φ, and note that $(y,k,+,x)$ is in Γ_φ if and only if $(y,k,-,x)$ is. As a shorthand, write 1_+ and 1_- for the characteristic functions of the elements $(x,0,+,x)$ and $(x,0,-,x)$, respectively. Choose a sequence g_n in $C_c(\Gamma_\varphi)$ converging pointwise to 1_-. Using the representation $\pi_x : C_r^*(\Gamma_\varphi) \to B(l^2(s^{-1}(x)))$, we observe that

$$\begin{aligned}
\langle \pi_x(g_n)1_\gamma, 1_{\gamma'}\rangle &= \sum_{\rho \in s^{-1}(x)} (\pi_x(g_n)1_\gamma)(\rho)\overline{1_{\gamma'}(\rho)} \\
&= (\pi_x(g_n)1_\gamma)(\gamma') \\
&= \sum_{\gamma_1\gamma_2=\gamma'} g_n(\gamma_1)1_\gamma(\gamma_2) \\
&= g_n(\gamma'\gamma^{-1}) \to_{n\to\infty} 1_-(\gamma'\gamma^{-1})
\end{aligned}$$

for $\gamma, \gamma' \in s^{-1}(x)$. Writing $\gamma = (y_1,k_1,p_1,x)$ and $\gamma' = (y_2,k_2,p_2,x)$ (and using that $p = p^{-1}$ in $\mathbb{Z}_2$), we have $\gamma'\gamma^{-1} = (y_2, k_2 - k_1, p_2p_1, y_1)$. Hence, $1_{(x,0,-,x)}(\gamma'\gamma^{-1}) = 1$ if $y_1 = y_2 = x$, $k_1 = k_2$ and $p_1 \neq p_2$, and 0 otherwise. Define an operator P on $l^2(s^{-1}(x))$ by

$$(Pf)(y,k,p,x) = \begin{cases} (y,k,\bar{p},x) & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}, \qquad f \in l^2(s^{-1}(x))$$

where $\bar{p}$ denotes 'flipping' the sign in $\mathbb{Z}_2$, i.e. $\overline{+} = -$ and $\overline{-} = +$. One checks that P is a partial isometry, and that the sequence $\pi_x(g_n)$ converges weakly to P in $B(l^2(s^{-1}(x)))$. If we furthermore define an operator V on $l^2(s^{-1}(x))$ by

$$(Vf)(x,k,p,y) = (-1)^p f(x,k,p,y),$$

calculations similar to those above show V implements Λ on $l^2(s^{-1}(x))$, i.e. that $\pi_x(\Lambda(f)) = V^*\pi_x(f)V$ for any $f \in l^2(s^{-1}(x))$. Hence,

$$\begin{aligned}\langle \pi_x(\Lambda(g_n))1_+, 1_-\rangle &= \langle \pi_x(g_n)V1_+, V1_-\rangle \\ &\xrightarrow[n\to\infty]{\langle} PV1_+, V1_-\rangle \\ &= -PV1_+((x,0,-,x)) \\ &= -V1_+(x,0,+,x) = -1.\end{aligned}$$

Now, assume that Λ is implemented by a unitary $u \in C_r^*(\Gamma_\varphi)$, i.e. that there is a unitary in $C_r^*(\Gamma_\varphi)$ such that $\Lambda(f) = u^*fu$ for any f in $C_r^*(\Gamma_\varphi)$. Then

$$\begin{aligned}-1 &= \lim_n \langle \pi_x(\Lambda(g_n))1_+, 1_-\rangle = \lim_n \langle \pi_x(u^*g_nu)1_+, 1_-\rangle && (4.14)\\ &= \lim_n \langle \pi_x(g_n)\pi_x(u)1_+, \pi_x(u)1_-\rangle = \langle P\pi_x(u)1_+, \pi(u)1_-\rangle. && (4.15)\end{aligned}$$

Choose a sequence $\{u_n\} \subseteq C_c(\Gamma_\varphi)$ converging to u in $C_r^*(\Gamma_\varphi)$. Then, by definition of the norm on $C_r^*(\Gamma_\varphi)$, $\pi_x(u_n)$ converges to $\pi_x(u)$ in norm on $B(l^2(s^{-1}(x)))$, so

$$\lim_{n\to\infty} \langle P\pi_x(u_n)1_+, \pi_x(u_n)1_-\rangle = -1.$$

Observe that $\pi_x(u_n)1_{\gamma_1}(\gamma_2) = u_n(\gamma_2\gamma_1^{-1})$, and

$$Pu_n1_{\gamma_1}(\gamma_2) = u_n1_{\gamma_1}(\bar{\gamma}_2) = u_n(\bar{\gamma}_2\gamma_1^{-1})$$

when $r(\gamma_2) = x$, with $\bar{\gamma}_2$ denoting γ_2 with the sign 'flipped'. Hence

$$\begin{aligned}\langle P\pi_x(u_n)1_+, \pi_x(u_n)1_-\rangle &= \sum_{\gamma_1 \in s^{-1}(x)} P\pi_x(u_n)1_+(\gamma_1)\pi_x(u_n)1_-(\gamma_1) \\ &= \sum_{\gamma_1 \in s^{-1}(x), r(\gamma_1)=x} u_n(\gamma_1(x,0,+,x))\overline{u_n(\bar{\gamma}_1(x,0,-,x))} \\ &= \sum_{\gamma_1 \in s^{-1}(x), r(\gamma_1)=x} |u_n(\gamma_1)|^2 \geq 0.\end{aligned}$$

This contradicts equation 4.14, so Λ is not implemented by a unitary. □

Proof. (Of Theorem 4.32): Assume first that $C_r^*(\Gamma_\varphi^+)$ is simple. Then $[x]_{\Gamma_\varphi^+}$ is dense in $\mathbb{T}$ for all $x \in \mathbb{T}$ by Lemma 4.10. But $[x]_{\Gamma_\varphi^+}$ is a subset of $[x]_{\Gamma_\varphi}$, so $[x]_{\Gamma_\varphi}$ is dense for any $x \in \mathbb{T}$, and the other direction in Lemma 4.10 then shows that $C_r^*(\Gamma_\varphi)$ is simple.

Assume on the other hand that $C_r^*(\Gamma_\varphi)$ is simple. Then, by Theorem 3.1 in [16], the crossed product $C_r^*(\Gamma_\varphi) \rtimes_\Lambda \mathbb{Z}_2$ is also simple. But then, by the Corollary in [36], the fixed point algebra $C_r^*(\Gamma_\varphi)^\Lambda = C_r^*(\Gamma_\varphi^+)$ is simple, too. □

4.3 Primitive ideals

Section 4.2 gave a criterion for simplicity of $C_r^*(\Gamma_\varphi)$ and $C_r^*(\Gamma_\varphi^+)$: The algebra is simple if and only if φ is totally transitive and without exceptional fixed points. In this chapter, we investigate what happens when this is not the case. By digging deep into the connections between primitive ideals, isotropy groups and invariant subsets of the unit space, we obtain a fairly concrete description of the primitive ideals of the two algebras. As a corollary, we also determine the maximal ideals. The approach is similar to that of [47].

Throughout this section, G denotes either of the groupoids Γ_φ or Γ_φ^+. Let I be an ideal in $C_r^*(G)$ (I is, by definition, closed and two-sided). We say that I is *prime* if it has the property that $I_1 I_2 \subseteq I$ implies $I_1 \subset I$ or $I_2 \subset I$ for any pair of ideals I_1, I_2. We note that for any seperable C^*-algebra – in particular, for $C_r^*(G)$ – an ideal is primitive if and only if it is prime (see e.g. Theorem 4.3.6 of [22]).

Definition 4.34. Let I be an ideal of $C_r^*(G)$. The *co-support* $\rho(I)$ of I is the set

$$\rho(I) = \{y \in \mathbb{T} \mid f(y) = 0 \text{ for all } f \in C(\mathbb{T}) \cap I\}$$

We note that ρ reverses inclusions: If $I \subset J$, it follows that $\rho(J) \subset \rho(I)$.

Lemma 4.35. *For any ideal I, $\rho(I)$ is a closed, G-invariant subset of $\mathbb{T}$.*

Proof. If $\{y_n\}$ is a sequence in $\rho(I)$ converging to $y \in \mathbb{T}$, and $f \in C(\mathbb{T}) \cap I$, we have $f(y_n) = 0$ for all n, hence $f(y) = 0$ by continuity. This shows that $\rho(I)$ is closed. For G-invariance, we show that the set $\mathbb{T} \setminus \rho(I)$ is G-invariant. Let $\gamma = [x, k, \eta, y] \in G$ with $x \in \mathbb{T} \setminus \rho(I)$. Choose $f \in C(T) \cap I$ with $f(x) \neq 0$ and an open bisection W around γ. Choose a map $h \in C_c(G)$ with $h(\gamma) = 1$ and $\mathrm{supp}(h) \subseteq W$. It is straightforward to check that $h^* f h$ is in $C(\mathbb{T}) \cap I$ and that $(h^* f h)(y) = f(x) \neq 0$, so $y \in \mathbb{T} \cap \rho(I)$ and $\rho(I)$ is G-invariant. □

Lemma 4.36. *Let I be an ideal in $C_r^*(G)$, and A a closed, G-invariant set with $\rho(I) \subseteq A$. Then $\ker(\pi_A) \subseteq I$.*

Proof. Let $S = C_0(\mathbb{T} \setminus A) \cap I$. Then S separates points in $\mathbb{T} \setminus A$: Let x and y be different points of $\mathbb{T} \setminus A$, and choose a function $f \in C(\mathbb{T}) \cap I$ with $f(x) \neq 0$ (this is possible since $\mathbb{T} \setminus A \subseteq \mathbb{T} \setminus \rho(I)$). Let U be a small neighbourhood of x not containing y, and $h \in C(\mathbb{T} \setminus A)$ a function with $h(x) = 1$ and vanishing outside U. Then fh is in S with $fh(x) \neq 0$ and $fh(y) = 0$. Furthermore, S vanishes nowhere on $\mathbb{T} \setminus A$: Given $x \in \mathbb{T} \setminus A$, the map fh constructed above satisfies $fh(x) \neq 0$. From the Stone-Weierstrass theorem, we get that S is dense in $C_0(\mathbb{T} \setminus A)$, from which it follows that $C_0(\mathbb{T} \setminus A)$ is a subset of $C(\mathbb{T}) \cap I$. Take an approximate unit $\{i_n\}$ in $C_0(\mathbb{T} \setminus A)$. We claim that this is also an approximate unit in $\ker(\pi_A)$: By Lemma 3.21, we have $\ker(\pi_A) \simeq C_r^*(\Gamma_\varphi|_{\mathbb{T} \setminus A})$. Let $f \in C_c(\Gamma_\varphi|_{\mathbb{T} \setminus A})$. Then $r(\mathrm{supp}(f))$ is a compact subset of $\mathbb{T} \setminus A$, so we may choose a $h \in C_0(\mathbb{T} \setminus A)$ such that $hf = f$. Then $i_n f = i_n h f$ converges to $hf = f$, so i_n is an approximate unit in $C_c(G|_{\mathbb{T} \setminus A})$, and by continuity also in $\ker(\pi_A)$. It follows that $\ker(\pi_A) \subseteq I$. □

The next lemma gives a crucial connection between primitive ideals of $C_r^*(G)$ and prime subsets of $G^{(0)}$.

Lemma 4.37. *Let $I \subseteq C_r^*(G)$ be a primitive ideal. Then $\rho(I)$ is prime.*

Proof. First, note that for any G-invariant closed set A, we have

$$\ker(\pi_A) \cap C(\mathbb{T}) = C_0(\mathbb{T} \setminus A).$$

From this, it follows that $\rho(\ker(\pi_A)) = A$. Now, let B and C be closed and G-invariant with $\rho(I) \subseteq B \cup C$. By Lemma 4.36 and 3.22 and we have

$$\ker(\pi_B) \cap \ker(\pi_C) = \ker(\pi_{B \cup C}) \subseteq I$$

Since I is primitive, and therefore a prime ideal, we must have $\ker(\pi_B) \subseteq I$ or $\ker(\pi_C) \subseteq I$. In the first case, we have

$$B = \rho(\ker(\pi_B)) \supset \rho(I).$$

and in the other case, we have $\rho(I) \subseteq C$. This shows that $\rho(I)$ is prime. □

Define the *quasi-orbit space* $\mathcal{Q}(G)$ of G as the set

$$\mathcal{Q}(G) = \left\{ \overline{[x]} \;\middle|\; x \in \mathbb{T} \right\}.$$

By the lemma above and Proposition 4.5, we have a map $\rho : \mathrm{Prim}(C_r^*(G)) \to \mathcal{Q}(G)$ taking a primitive ideal I to the co-support $\rho(I)$. Our next step is to determine, for each $x \in \mathbb{T}$, the set $\left\{ I \in \mathrm{Prim}(C_r^*(G)) \;\middle|\; \rho(I) = \overline{[x]} \right\}$. We split our analysis into two cases according to the dichotomy of Proposition 4.9. Put

$$\mathcal{Q}(G)_{ex} = \{A \in \mathcal{Q}(G) \mid \ A \text{ contains an isolated isotropic point}\}.$$

and $\mathcal{Q}(G)_{fr} = \mathcal{Q}(G) \setminus \mathcal{Q}(G)_{ex}$ (*ex* and *fr* for exceptional and free, respectively).

Lemma 4.38. *Let $A \in \mathcal{Q}(G)_{fr}$. Then $\ker(\pi_A)$ is a primitive ideal, and the unique ideal with $\rho(\ker(\pi_A)) = A$.*

Proof. To show that $\ker(\pi_A)$ is primitive, we show that $C_r^*(G)/\ker(\pi_A) \simeq C_r^*(G|_A)$ is a prime C^*-algebra. Let I_1 and I_2 be ideals in $C_r^*(G|_A)$ with $I_1 I_2 = \{0\}$, and let $y \in A$. Write A as $A = A_1 \cup A_2$ with

$$A_i = \{y \in A \mid f(y) = 0 \text{ for all } f \in I_i \cap C(A)\}, \qquad i = 1, 2.$$

We can do this – otherwise, we would have a $f_1 \in I_1 \cap C(A)$ with $f_1(y) \neq 0$ and a $f_2 \in I_2 \cap C(A)$ with $f_2(y) \neq 0$, so $f_1 f_2$ would be a non-zero element in $I_1 I_2$. Now, choose $x \in \mathbb{T}$ such that $\overline{[x]} = A$. Assume without loss of generality that $x \in A_1$. Arguing as in Lemma 4.35, we see that A_1 is closed and $\Gamma_\varphi|_A$-invariant, so $A = A_1$. This means that $I_1 \cap C(A) = 0$, and as above, we conclude that $I_1 = 0$, so $C_r^*(G|_A)$ is prime. From this, it follows that $\rho(\ker(\pi_A))$ is a prime subset of $\mathbb{T}$, and since A is invariant and $A \subseteq \rho(\ker(\pi_A))$, Lemma 4.37 shows that $A = \rho(\ker(\pi_A))$.

For uniqueness, let I be an ideal with $\rho(I) = A$. Then $\ker(\pi_A) \subseteq I$ by Lemma 4.36. We must show that $I \subseteq \ker(\pi_A)$, or, equivalently, that $\pi_A(f) = 0$ for any $f \in I$. Begin by letting $f \in \pi_A(I) \cap C(A)$. Choose a function $h \in C(\mathbb{T})$ with $h|_A = f$ and an $a \in I$ with $\pi_A(a) = f$. Then $\pi_A(a - h) = 0$, so $a - h \in \ker(\pi_A) \subseteq I$. Then $h = a - (a - h) \in I \cap C(\mathbb{T})$, hence $h(x) = 0$ for all $x \in \rho(I) = A$. This means that $f = h|_A = 0$, so $\pi_A(I) \cap C(A) = 0$. Now, we use that points with trivial isotropy group are dense in A. Let $P : C_r^*(G) \to C(\mathbb{T})$ denote the conditional expectation, and apply Lemma 2.15 of [45] to show that $P(f)(x) = 0$ for all $f \in \pi_A(I)$ and $x \in A$. Faithfulness of P then implies that $\pi_A(I) = 0$, as we wanted. □

Next, we look at primitive ideals I with $\rho(I) \in \mathcal{Q}(G)_{ex}$. Given a set $A \in \mathcal{Q}(G)_{ex}$, we know by Proposition 4.5 that $A = \overline{[x]}$ for some $x \in \mathbb{T}$, and that A contains an isolated, isotropic point y. Note that $\overline{[y]} \subseteq A$ since A is G-invariant. On the other hand, since y is isolated in $A = \overline{[x]}$, we must have $y \in [x]$, so $x \in [y]$, and it follows that $A = \overline{[x]} \subseteq \overline{[y]}$. Now, for such $A = \overline{[y]} \in \mathcal{Q}(G)_{ex}$ with y isolated and isotropic, let $\gamma \in \mathrm{Iso}(y)$. Then γ is isolated in $G|_A$, so the characteristic function 1_γ is in $C_r^*(G|_A)$. Let $\widehat{\mathrm{Iso}(y)}$ denote the Pontryagin dual group of $\mathrm{Iso}(y)$, and let $\omega \in \widehat{\mathrm{Iso}(y)}$. Denote by $I_0(y, \omega)$ the ideal of $C_r^*(G|_A)$ generated by the elements

$$\left\{ 1_{[y,0,\mathrm{id},y]} - \overline{\omega(\gamma)} 1_\gamma \;\middle|\; \gamma \in \mathrm{Iso}(y) \right\}.$$

Finally note that given two such isolated isotropic points $x, y \in A$ with $\overline{[x]} = \overline{[y]}$, the groups $\mathrm{Iso}(x)$ and $\mathrm{Iso}(y)$, as well as their dual groups, are isomorphic, and will be identified from here on.

Lemma 4.39. *Let $A \in \mathcal{Q}(G)_{ex}$, and let $x, y \in A$ be isolated, isotropic points with $\overline{[x]} = \overline{[y]} = A$. For any $\omega \in \widehat{\mathrm{Iso}(y)} = \widehat{\mathrm{Iso}(x)}$, we have $I_0(y, \omega) = I_0(x, \omega)$.*

Proof. It suffices to show that $I_0(y, \omega) \subseteq I_0(x, \omega)$. As in the discussion above, since y is isolated in $\overline{[x]}$, we must have $y \in [x]$, and hence there are $i \in \mathbb{Z}$ and a local transfer η such that $\gamma = [x, i, \eta, y] \in G|_A$. Since both x and y are isolated in A, 1_γ is an element of $C_r^*(G|_A)$. One checks easily that

$$1_\gamma^* 1_{[x,0,\mathrm{id},x]} 1_\gamma = 1_{[y,0,\mathrm{id},y]}$$

Similarly, let $[y, l, \rho, y] \in \mathrm{Iso}(y)$. Then $\lambda = [x, l, \eta \circ \rho \circ \eta^{-1}, x] \in \mathrm{Iso}(x)$, and

$$1_\lambda^* 1_\gamma 1_\lambda = 1_{[y,l,\rho,y]}.$$

It follows that $I_0(y, \omega) \subseteq I_0(x, \omega)$. □

The above lemma shows that the ideal $I_0(x, \omega)$ is independent of our choice of isotropic point $x \in A$, so we may as well denote it by $I_0(A, \omega)$. Similarly, we let $\mathrm{Iso}(A)$ denote the group $\mathrm{Iso}(x)$ for some arbitrary isotropic isolated $x \in A$. Let $I(A, \omega)$ denote its preimage $\pi_A^{-1}(I_0(A, \omega))$ in $C_r^*(G)$.

Proposition 4.40. *Let $A \in \mathcal{Q}(G)_{ex}$. The map $\omega \mapsto I(A, \omega)$ is a bijection from $\widehat{\mathrm{Iso}(A)}$ to $\{I \in \mathrm{Prim}(C_r^*(G)) \mid \rho(I) = A\}$*

Proof. First, note that the map π_A gives a bijection between the sets

$$\{I \in \mathrm{Prim}(C_r^*(G)) \mid \ker(\pi_A) \subseteq I\} \xrightarrow[\pi_A]{\mathrm{Prim}} (C_r^*(G|_A))$$

This follows from Theorem 4.1.11 (ii) of [22]. If $I \in \mathrm{Prim}(C_r^*(G))$ with $\rho(I) = A$, it follows from Lemma 4.36 that $\ker(\pi_A) \subseteq I$, so $\pi_A(I)$ is a primitive ideal in $C_r^*(G|_A)$. Furthermore, one checks that $I \mapsto \pi_A(I)$ maps primitive ideals I in $C_r^*(G)$ with $\rho(I) = A$ bijectively to ideals $\pi_A(I)$ in $C_r^*(G|_A)$ with $\rho(\pi_A(J)) = A$. Now, let Q_y denote the ideal in $C_r^*(G|_A)$ generated by $p_y = 1_{[y,0,\mathrm{id},y]}$. Since

$$p_y(p_y - \overline{\omega(\gamma)} 1_\gamma) = p_y - \overline{\omega(\gamma)} 1_\gamma$$

we have $I_0(y, \omega) \subseteq Q_y$. Another application of Theorem 4.1.11 (ii) of [22] gives a bijection

$$\{I \in \mathrm{Prim}(C_r^*(G|_A) \mid Q_y \nsubseteq I\} \xrightarrow{I \mapsto I \cap Q_y} \mathrm{Prim}(Q_y)$$

Let J be an ideal in $C_r^*(G|_A)$. We claim that $Q_y \nsubseteq J$ if and only if $\rho(J) = A$. If $p_y \in J$, we have $y \notin \rho(J)$, so $\rho(J) \neq A$. On the other hand, if $\rho(J) \neq A$, we have $y \notin \rho(J)$ since $\rho(J)$ is closed and G-invariant with $A = \overline{[y]}$. It follows that there is an $f \in J \cap C(A)$ with $f(y) \neq 0$, so $p_y = f p_y \in J$. Hence, there is a bijection between primitive ideals in Q_y and primitive ideals J in $C_r^*(G_A)$ with $\rho(J) = A$. Now, observe that $p_y C_r^*(G|_A)$ is a $p_y C_r^*(G|_A) p_y - Q_y$-imprimitivity bimodule, so the map $J \mapsto p_y J p_y$ takes primitive ideals in Q_y bijectively to primitive ideals in $p_y C_r^*(G|_A) p_y$. Since

$$\{\gamma \in G|_A \mid r(\gamma) = s(\gamma) = y\} = \mathrm{Iso}(y),$$

we obtain an isomorphism

$$p_y C_r^*(G|_A) p_y \simeq C^*(\mathrm{Iso}(y)) \simeq C(\widehat{\mathrm{Iso}(y)}).$$

It follows that the primitive ideals of Q_y are in one-to-one-correspondence with $\widehat{\mathrm{Iso}(y)}$, and we are done. □

Combining Lemmas 4.40 and 4.38, we get the following classification of the primitive ideal spectrum of $C_r^*(G)$:

Theorem 4.41. *The set of primitive ideals in $C_r^*(G)$ is the disjoint union of the sets*

$$\left\{ \ker(\pi_A) \,\middle|\, A \in \mathcal{Q}(G)_{fr} \right\} \sqcup \left\{ I(A,\omega) \,\middle|\, A \in \mathcal{Q}(G)_{ex}, \omega \in \widehat{\mathrm{Iso}(A)} \right\}$$

4.3.1 The maximal ideals

The next step is to determine the maximal ideals among the primitive ones.

Lemma 4.42. *Let $F \subseteq T$, assume that not all points of F are pre-periodic, and that $C_r^*(G|_F)$ contains a non-trivial ideal. Then it contains a non-trivial, gauge-invariant ideal J with $J \cap C(F) \neq \{0\}$.*

Proof. As in [7], Lemma 4.17. □

We say that a closed, G-invariant set $F \subseteq \mathbb{T}$ is *minimal* if it does not contain any proper, closed G-invariant subset.

Lemma 4.43. *Assume that $F \subseteq \mathbb{T}$ is closed, G-invariant, minimal and non-empty. Then one of the following two cases hold:*

- *$F \in \mathcal{Q}(G)_{fr}$, and $\ker(\pi_F)$ is a maximal ideal.*
- *F is finite, and there is an isotropic point $x \in \mathbb{T}$ such that $F = [x]$.*

Proof. Note first that minimality ensures that $F = \overline{[x]}$ for any $x \in F$ – otherwise, $\overline{[x]}$ would be a proper, closed G-invariant subset of F. Assume first that the non-isotropic points are dense in F, and that there is a proper ideal I containing $\ker(\pi_F)$. Then $\pi_F(I)$ is a proper ideal in $C_r^*(G|_F)$, so we may appeal to Lemma 4.42 and pick a proper gauge-invariant ideal J in $C_r^*(G|_F)$. Then $\pi_F^{-1}(J)$ is a proper gauge-invariant ideal in $C_r^*(G)$. Since $\ker(\pi_F) \subseteq \pi_F^{-1}(J)$, we have

$$F = \rho(\ker(\pi_F)) \supset \rho(\pi_F^{-1}(J))$$

contradicting the minimality of F. It follows that $\ker(\pi_F)$ is maximal. Assume on the other hand, that F contains an isolated, isotropic point x. It follows that x is isolated in F, and hence that $F = [x]$ – otherwise, there would be a $y \in F \setminus [x]$, but then $x \notin \overline{[y]} = F$, which is absurd. Since F is compact, $[x]$ must be finite. □

Proposition 4.44. *Let I be a maximal ideal in $C_r^*(G)$. Then either $I = \ker(\pi_F)$ for some minimal, closed, G-invariant set $F \in \mathcal{Q}(G)_{fr}$, or $I = I([x], \omega)$ with $[x]$ a finite set and $\omega \in \widehat{\mathrm{Iso}(x)}$.*

Proof. Since I is maximal, it is also primitive, so Proposition 4.41 yields that either $I = \ker(\pi_F)$ for some closed invariant set $F \in \mathcal{Q}(G)_{fr}$ or $I = I(A, \omega)$, where $A = \overline{[x]}$ with x isolated and isotropic in A and $\omega \in \widehat{\mathrm{Iso}(x)}$. In the first case, maximality of I implies minimality of F. In the second case, recall from the proof of Lemma 4.40 that $I_0(A, \omega) \subsetneq Q_x$. The inclusion is strict, since Q_x is gauge-invariant while $I_0(A, \omega)$ is not. Since $I_0(A, \omega)$ is maximal in $C_r^*(G_A)$, we conclude that $Q_x = C_r^*(G_A)$. Since any point of $[x]$ is isolated, $[x]$ is an open G-invariant subset of A, and since $p_x \in C_r^*(G_{[x]})$, it follows that $C_r^*(G_{[x]}) = C_r^*(G_A)$. From this it follows that

$$C_0([x]) = C(A) \cap C_r^*(G_{[x]}) = C(A),$$

so $A = [x]$. Since A is compact, $[x]$ is finite. □

Having determined the maximal ideals of $C_r^*(G)$, we proceed to determine the simple quotients. Like the maximal ideals, these come in two flavours:

Lemma 4.45. *Let I be a maximal ideal in $C_r^*(G)$ such that $I = \ker(\pi_F)$ for some minimal, closed, G-invariant set $F \in \mathcal{Q}(G) \setminus \mathcal{Q}(G)_{ex}$. Then the quotient $C_r^*(G)/I$ is isomorphic to $C^*(G|_F)$.*

Proof. This is a direct consequence of Lemma 3.21. □

Lemma 4.46. *Let I be a maximal ideal in $C^*(G)$ such that $I = I([x], \omega)$ for $[x]$ finite and $\omega \in \widehat{\mathrm{Iso}(x)}$. Then the quotient $C_r^*(G)/I$ is finite dimensional.*

Proof. First, note that $\ker(\pi_{[x]}) \subseteq I$ by Lemma 4.36. It follows that the quotient map $q_I : C_r^*(G) \to C_r^*(G)/I$ factors through $C_r^*(G_{[x]})$ as in the following diagram:

$$\begin{array}{ccc} C_r^*(G) & \xrightarrow{\pi_{[x]}} & C_r^*(G_{[x]}) \\ & \searrow^{q_I} & \downarrow \tilde{q}_I \\ & & C_r^*(G)/I \end{array}$$

Since $[x]$ is finite, Lemma 4.11 of [47] implies that

$$C_r^*(G_{[x]}) \simeq C^*(\mathrm{Iso}(x)) \otimes \mathbb{K}(l^2([x])) = C(\widehat{\mathrm{Iso}(x)}) \otimes M_n(\mathbb{C})$$

where $n = |[x]|$. It follows that $C_r^*(G)/I$ is a simple quotient of $C(\widehat{\mathrm{Iso}(x)}) \otimes M_n(\mathbb{C})$, hence finite dimensional. □

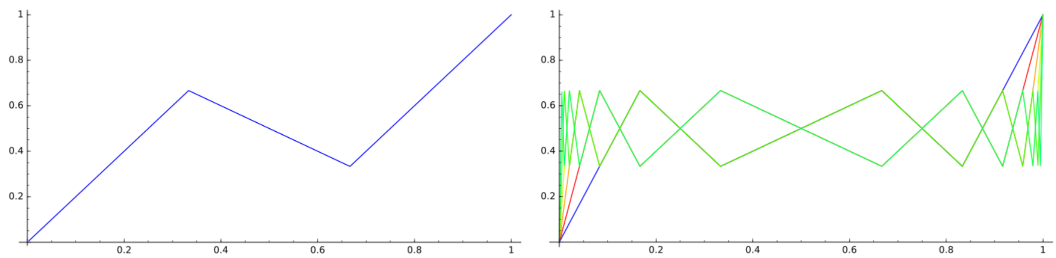
Proposition 4.47. *Let Q be a simple quotient of $C_r^*(G)$. Then either Q is either finite-dimensional or isomorphic to $C_r^*(G_F)$ for some $F \subseteq \mathbb{T}$ such that F is minimal, closed and G-invariant.*

Proof. Combine Lemmas 4.45 and 4.46. □

Example 4.48. We give a few examples of non-transitive circle maps and the primitive ideals of their groupoid C^*-algebras. The first map, ψ_1, is the piecewise linear function through the points $(0,0), (1/3, 2/3), (2/3, 1/3)$ and $(1,1)$. This map is not transitive – in particular, the interval $(1/3, 2/3)$ is forward invariant. The figure below shows ψ_1 and the first few iterates. From these pictures, determining the set $\mathcal{Q}(\Gamma_{\psi_1})$, is straightforward: Any point $z \in \mathbb{T} \setminus \{0\}$ is eventually mapped to a point in the interval $[1/3, 1/2]$, and two points have the same groupoid orbit if and only if they map to the same element of $[1/3, 1/2]$. Finally, 0 is a fixed point, and the groupoid orbit of 0 is just $\{0\}$. It follows that there is bijection from $\mathcal{Q}(\Gamma_{\psi_1})$ to the set $\{0\} \sqcup [1/3, 1/2]$. In particular, any point in $\mathbb{T}$ is pre-periodic and has an isolated, isotropic point in its groupoid-orbit. It follows that there are *many* primitive ideals in $C_r^*(\Gamma_{\psi_1})$: For each point in $x \in \{0\} \sqcup [1/3, 1/2]$, Theorem 4.41 give uncountable many primitive ideals $I(x, \omega)$, parametrised by the group $\widehat{\mathrm{Iso}(x)}$. The isotropy groups are

$$\mathrm{Iso}(x) \simeq \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & \text{when z=1/3 so } z \text{ is critical,} \\ \mathbb{Z}, & \text{when } z \neq 1/3, \end{cases}$$

so the dual groups are either $\mathbb{T}$ or $\mathbb{T} \oplus \mathbb{Z}_2$.



To distinguish the maximal ideals among the primitive ones, note that for any $z \in \mathbb{T}$, 0 is a limit point for $\mathcal{O}^-(z)$, so $0 \in \overline{[z]}$ for any $z \in \mathbb{T}$. It follows that $\{0\}$ is the only Γ_{ψ_1}-minimal subset of $\mathbb{T}$, so the maximal ideals are $I(0, \omega)$ with $\omega \in \widehat{\mathrm{Iso}(0)} \simeq \mathbb{T}$. Note that

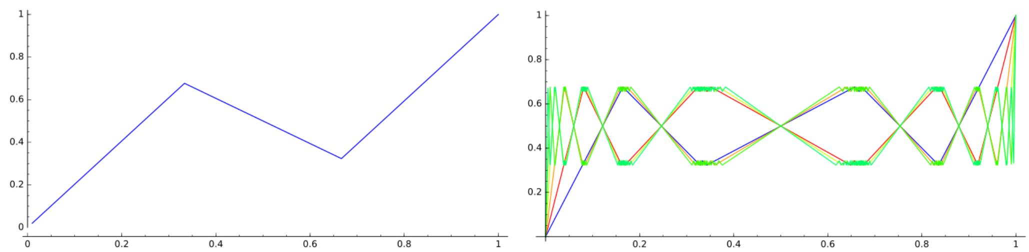
$$\Gamma_{\psi_1}|_{\{0\}} = \{(0, k, +, 0) | k \in \mathbb{Z}\},$$

so $\Gamma_{\psi_1}|_{\{0\}}$ is discrete and $C_r^*(\Gamma_{\psi_1}|_{\{0\}}) \simeq C(\mathbb{T})$ via the isomorphism taking a characteristic function $1_{(0,k,+,0)}$ to the map $e_k \in C(\mathbb{T})$ given by $e_k(z) = z^k$. The isomorphism $\widehat{\mathbb{Z}} \simeq \mathbb{T}$ is implemented by the map $z \mapsto \omega_z$, where $\omega_z(n) = z^n$. It follows that the ideal $I_0(0, \omega_z)$, as an ideal in $C(\mathbb{T})$, is generated by the functions $u_{z,k} = 1 - \overline{\omega_z(k)} e_k$ for $k \in \mathbb{Z}$. But

$$u_{z,k}(z) = 1 - \overline{z}^k z^k = 0,$$

so $I_0(0, \omega_z)$ is simply the maximal ideal of $C(\mathbb{T})$ given by

$$I_0(0, \omega_z) = \{f \in C(\mathbb{T}) \mid f(z) = 0\}$$



▲

Now consider the map ψ_2, which can be obtained by modifying ψ_1 very slightly: Instead of mapping $1/3$ to $2/3$ and $2/3$ to $1/3$, ψ_2 maps $1/3$ to $2/3 + \varepsilon$ and $2/3$ to $1/3 - \varepsilon$ for some small

$\varepsilon > 0$. As the figure shows, this indicates the structure of the groupoid orbits significantly: For any $x \in \mathbb{T} \setminus \{0, 1/2\}$, the orbit $[x]$ is now dense in $\mathbb{T}$. $1/2$ is still a fixed point which is isolated in its Γ_{φ_2}-orbit, giving rise to uncountably many primitive ideals $I(1/2, \omega_z)$ parametrised by $z \in \mathbb{T}$. Similarly, 0 is a fixed point whose Γ_{ψ_2}-orbit is just 0 itself, giving rise to another family of primitive ideals parametrised by $\mathbb{T}$. The picture reveals that 0 is a limit point for any backward orbit $\mathcal{O}^-(z)$, so as before, $\{0\}$ is the only Γ_{ψ_2}-minimal subset, and the corresponding ideals $I(0, \omega)$ the only maximal ideals.

Remark 4.49. It is easy to continue experimenting with various classes of circle maps – any piece of mathematics software makes it possible to visualise iterates of a given circle map and the structure of the corresponding groupoid orbits. There seem to be a general pattern – if the map has a periodic point x, it attracts either the forward or backward orbit of any point in some neighbourhood of x. In particular, the minimal sets are orbits of periodic points, and the maximal ideals sit above the algebra of the reduction to these points. In Chapter 6, we will investigate a wholly different situation, namely the one where the map has no periodic points at all. ♦

CHAPTER 5

Critically finite maps

Since the algebras $C_r^*(\Gamma_\varphi)$ and $C_r^*(\Gamma_\varphi^+)$ are classfied by their K-theory, it seems reasonable to develop a method of determining these groups. This seems unreasonably hard for general circle maps – the crucial requirement for getting anywhere seems to be some degree of control over the orbits of the critical points of the map. In this chapter, we focus on *critically finite maps*, i.e. maps where the forward orbit of any critical point is finite. For such maps, we develop an algorithm to determine the K-theory of the corresponding algebras. The road to this algorithm is long and winding, and we will need to overcome a great deal of technicalities along the way. Let's give a rough sketch of our approach: In Chapter 3, we considered the building block algebras $C_r^*(R_\varphi(k))$ for some number k. In Theorem 5.17 below, we show that $C_r^*(\Gamma_\varphi)$ arises as the C^*-algebra of a C^*-correspondence over $C_r^*(R_\varphi(k))$ for some k. Using results of Katsura, this yields a six-term exact sequence connecting the K-theory groups of $C_r^*(\Gamma_\varphi)$ and $C_r^*(R_\varphi(k))$. Then, in section 5.3, we use the machinery of linking algebras to reduce the problem to determining a certain map between the building block-algebras at level k and level $k+1$. Finally, we reduce the problem even further, showing that all we need to do is to determine the induced *inclusion* map between level k and $k+1$ on K-theory, and connect it with the K-theory calculations from Chapter 3. We end up with an algorithm that involves nothing more than looking at the graph of φ^k and doing some linear algebra.

5.1 C^*-correspondences

We will need some results from the theory of C^*-correspondences. For a detailed introduction to C^*-correspondences, see [15] – here, we will just mention some key definitions and results. We recall that a (right) Hilbert A-module over a C^*-algebra A is a Banach space X with a (right) action of A and a A-valued inner product $\langle , \rangle_A$ satisfying certain conditions (see e.g. [17]). We denote by $\mathbb{L}(X)$ the adjointable operators on X, and by $\mathbb{K}(X)$ the compact operators, i.e. the closed span of the 'rank-one projections' $\theta_{x,y}$, $x, y \in X$ defined by $\theta_{x,y}(z) = x\langle y, z\rangle_A$.

Definition 5.1. Let A be a C^*-algebra and X a (right) Hilbert A-module. If $\varphi_X : A \to \mathbb{L}(X)$ is a $*$-homomorphism, we say that (X, φ_X) is a C^*-*correspondence* over A.

We call φ_X the *action* of the C^*-correspondence.

Definition 5.2. Let (X, φ_X) be a C^*-correspondence over A, and B a C^*-algebra. A *representation* of the correspondence is a $*$-homomorphism $\pi : A \to B$ and a linear map $t : X \to B$ satisfying

- $t(x)^*t(y) = \pi(\langle x,y\rangle_A)$ for $x,y \in X$.
- $\pi(a)t(x) = t(\varphi_X(a)x)$ for $a \in A, x \in X$.

The representation is *injective* if π is injective.

The images of π and t form a sub-C^*-algebra $C^*(\pi,t)$ of B. Furthermore, we may define a $*$-homomorphism $\psi_t : \mathbb{K}(X) \to B$ given by

$$\psi_t(\theta_{x,y}) = t(x)t(y)^*$$

on rank-one projections. We say that a representation (π,t) of (X,φ_X) on B is *covariant* if $\pi(a) = \psi_t(\varphi_X(a))$ for any $a \in \varphi_X^{-1}(\mathbb{K}(X)) \cap (\ker(\varphi_X))^\perp$.

While the theory of C^*-correspondences is applicable in a variety of circumstances, we will only need it in a very particular case:

Example 5.3. Let B be a C^*-algebra, A a sub-C^*-algebra, and $E \subset B$ a closed subspace such that $EA \subset E$, $AE \subset E$, and $E^*E \subset A$. If we let A act on E by multiplication, E is a Hilbert A-module with inner product $\langle x,y\rangle_A = x^*y$. Furthermore, defining $\varphi_E(a)x = ax$ for $x \in E$ and $a \in A$, (E,φ_E) becomes a C^*-correspondence over A. Let $\pi : A \to B$ and $t : E \to B$ be the inclusions of A and E into B. Then (π,t) is a injective representation of (E,φ_E). If we assume that the closed span of E^*E is *equal* to A (i.e. that E is a *full* module), then φ_E is an isomorphism from A to $\mathbb{K}(E)$, so we have $\varphi_E^{-1}(\mathbb{K}(E)) \cap (\ker(\varphi_E))^\perp = A$. We also see that $\pi(a) = \psi_t(\varphi_E(a))$ – indeed, if we write an $a \in A$ as x^*y for $x,y \in E$, we have

$$\varphi_E(a)b = ab = x^*yb = x^*\langle y^*,b\rangle_A = \theta_{x^*,y^*}(b),$$

so $\varphi_E(a) = \theta_{x^*,y^*}$, hence

$$\psi_t(\varphi_E(a)) = \psi_t(\theta_{x^*,y^*}) = t(x^*)t(y^*)^* = x^*y = a = \pi(a).$$

For general a, the same claim follows by linearity and continuity. In short, fullness of E implies that the representation is covariant. ▲

Definition 5.4. Let (X,φ_X) be a C^*-correspondence over A, and (π,t) a representation on B. We say that the representation *admits a gauge action* if, for any $z \in \mathbb{T}$, there is a $*$-homomorphism $\beta_z : C^*(\pi,t) \to C^*(\pi,t)$ such that $\beta_z(\pi(a)) = \pi(a)$ and $\beta_z(t(x)) = zt(x)$ for any $a \in A$ and $x \in X$.

For a general C^*-correspondence (X,φ_X), denote by (π_X,t_X) the universal covariant representation, and let $\mathcal{O}_X = C^*(\pi_X,t_X)$. By universality, the is a canonical surjection $s : \mathcal{O}_X \to C^*(\pi,t)$ onto the C^*-algebra of any other covariant representation. Now, one may ask when this surjection is an isomorphism. This is answered by Theorem 6.4 in [15].

Theorem 5.5. *Let (π,t) be a representation of (X,φ_X). The surjection $s : \mathcal{O}_X \to C^*(\pi,t)$ is an isomorphism if and only if (π,t) is covariant and admits a gauge action.*

If the conditions of the theorem above are fulfilled, we get a six-term exact sequence on K-theory relating $\mathcal{O}_X$ and A, cf. Theorem 8.6 in [15]. In the context of our example, it might be possible to prove that the hypotheses of the theorem are fulfilled, and that $C^*(\pi,t) = B$. Then, the six-term sequence yields a connection between the K-theory of B and the K-theory of the subalgebra A, which we may exploit for various purposes.

5.2 The groupoid of a critically finite map

Let $\varphi : \mathbb{T} \to \mathbb{T}$ be a circle map satisfying our standard assumptions, i.e. φ is continuous, surjective and piecewise monotone. Assume furthermore that φ is critically finite. This entails that

$$\mathcal{D}_0 = \bigcup_{c\in\mathcal{C}} \mathcal{O}^+(c), \tag{5.1}$$

is a finite set. For each critical point c, define the number $k_c = \min\left\{k \in \mathbb{N} \,\middle|\, \exists p \in \mathbb{N} : \varphi^{k-p}(c) = \varphi^k(c)\right\}$. Define k_1 as the maximum of the k_c's. Finally, we need to assume that the algebras $C_r^*(\Gamma_\varphi)$ and $C_r^*(\Gamma_\varphi^+)$ are simple – by Theorems 4.13 and 4.32, this is equivalent to assuming that φ is exact and without exceptional fixed points. Finally, we assume that φ has at least one critical point – if not, φ is conjugate to an irrational rotation, and this case is well studied already. We need some notation:

Definition 5.6. Let $k \in \mathbb{Z}$, $n \in \mathbb{N}$, and assume that $n + k \geq 1$. Define subsets

$$\Gamma_\varphi(k,n) = \left\{(x,k,p,y) \in \Gamma_\varphi \,\middle|\, \varphi^{k+n}(x) = \varphi^n(y)\right\}$$

and

$$\Gamma_\varphi^+(k,n) = \left\{(x,k,y) \in \Gamma_\varphi^+ \,\middle|\, \varphi^{k+n}(x) = \varphi^n(y)\right\}$$

of Γ_φ and Γ_φ^+, respectively. For $l \in \mathbb{Z}$, put

$$\Gamma_\varphi(l) = \bigcup_{i\in\mathbb{N}} \Gamma_\varphi(l,i), \qquad \Gamma_\varphi^+(l) = \bigcup_{i\in\mathbb{N}} \Gamma_\varphi^+(l,i)$$

We note that $\Gamma_\varphi = \bigcup_l \Gamma_\varphi(l)$ and $\Gamma_\varphi^+ = \bigcup_l \Gamma_\varphi^+(l)$. The following lemmas analyze the structure of the sets defined above – in particular, we ask the question: Given a $\gamma \in \Gamma_\varphi^+(k,n)$, how can γ be written as a product of elements from other sets $\Gamma_\varphi^+(k',n')$? We prove the lemmas first for Γ_φ^+, and then use these results as shortcuts to prove the same statements for Γ_φ. The statements might seem technical, and the proofs are often case-by-case-arguments – the reader short on time can skip to Remark 5.14 for the conclusion.

Lemma 5.7. *Assume that φ is exact. Then there is a number k_2 such that for any $x \in \mathbb{T}$ and for any $j \geq k_2$, there are points z_+ and z_- in $\varphi^{-j}(x)$ such that* either $\operatorname{val}(\varphi^j, z_+) = (+,+)$ *and* $\operatorname{val}(\varphi^j, z_-) = (-,-)$ or *z_+ and z_- are critical for φ^j.*

Proof. Choose intervals I_+, I_- and I with $\varphi(I_+) = \varphi(I_-) = I$ and $\operatorname{val}(\varphi, I_\pm) = (\pm,\pm)$. By exactness of φ, there is an k_2 such that $\varphi^{k_2-1}(I) = \mathbb{T}$. Note that for any $j \geq k_2$, there is a $z \in I$ with $\varphi^{j-1}(z) = x$. Choose points $z_\pm \in I_\pm$ with $\varphi(z_\pm) = z$, and note that $z_+, z_- \in \varphi^{-j}(x)$ and that one z is critical for φ^j only if the other one is. Assume that neither z_+ nor z_- is critical for φ^j. Then $\operatorname{val}(\varphi^j, z_+)$ is either $(+,+)$ or $(-,-)$, and $\operatorname{val}(\varphi^j, z_-)$ is the opposite. If necessary, we interchange z_+ and z_-, and we are done. □

Note that the choice of the number k_2 depends on the interval I chosen in the proof above. Other choices of intervals would lead to other values of k_2 – the important thing is that such a number exists. Put $k = 2k_1 + k_2$. This number is fixed for the rest of the chapter.

Lemma 5.8. *Assume that φ is exact and has no exceptional fixed points. Let $l \in \mathbb{Z}$, $n \in \mathbb{N}$, and assume that $l + n \geq k + 1$. Let $(x,l,y) \in \Gamma_\varphi^+(l,n)$. Then there exists a $z \in \mathbb{T}$ such that $(x,1,z) \in \Gamma_\varphi^+(1,k)$ and $(z, l-1, y) \in \Gamma_\varphi^+(l-1,n)$.*

Proof. To prove the lemma, we need to find a $z \in \mathbb{T}$ such that

$$\varphi^{k+1}(x) = \varphi^k(z) \qquad , \mathrm{val}(\varphi^{k+1}, x) = \mathrm{val}(\varphi^k, z),$$
$$\varphi^n(y) = \varphi^{n+l-1}(z) \qquad , \mathrm{val}(\varphi^n, y) = \mathrm{val}(\varphi^{n+l-1}, z).$$

Choose elements z_+ and z_- as in the lemma above, i.e. such that $\varphi^{k_2}(z_\pm) = \varphi^{k_2+1}(x)$. Consider the sets

$$\left\{\varphi^j(x) \,\middle|\, j = 0, \dots, k\right\} \text{ and } \left\{\varphi^j(z_\pm) \,\middle|\, j = 0, \dots, k-1\right\}.$$

We divide the proof into four cases:

Case 1: Assume that neither of the sets above contain a critical point. Then there is a $z \in \{z_+, z_-\}$ such that $\varphi^{k_2}(z) = \varphi^{k_2+1}(x)$ and such that $\mathrm{val}(\varphi^{k_2}, z) = \mathrm{val}(\varphi^{k_2+1}, x)$. Then $\varphi^k(z) = \varphi^{k+1}(x)$ and (using that $k_2 + 1 = k + 1 - 2k_1$)

$$\begin{aligned}\mathrm{val}(\varphi^k, z) &= \mathrm{val}(\varphi^{2k_1}, \varphi^{k_2}(z)) \bullet \mathrm{val}(\varphi^{k_2}, z) \\ &= \mathrm{val}(\varphi^{2k_1}, \varphi^{k+1-2k_1}(x)) \bullet \mathrm{val}(\varphi^{k+1-2k_1}, x) = \mathrm{val}(\varphi^{k+1}, x),\end{aligned}$$

so $(x, 1, z) \in \Gamma^+_\varphi(1, k)$. Similarly we get $(z, l-1, y) \in \Gamma^+_\varphi(l-1, n)$ since

$$\begin{aligned}\mathrm{val}(\varphi^n, y) = \mathrm{val}(\varphi^{n+l}, x) &= \mathrm{val}(\varphi^{n+l-m-1}, x) \bullet \mathrm{val}(\varphi^{k+1}, x) \\ &= \mathrm{val}(\varphi^{n+l-k-1}, x) \bullet \mathrm{val}(\varphi^k, z) = \mathrm{val}(\varphi^{n+l-1}, z)\end{aligned}$$

when $l + n \geq k + 2$ and

$$\mathrm{val}(\varphi^{n+l-1}, z) = \mathrm{val}(\varphi^k, z) = \mathrm{val}(\varphi^{k+1}, x) = \mathrm{val}(\varphi^{l+n}, x) = \mathrm{val}(\varphi^n, y)$$

when $l + n = k + 1$.

Case 2: Assume next that $\varphi^j(x)$ is critical for some j between 1 and k. Put $z = \varphi(x)$. By the composition table for valency, we get

$$\begin{aligned}\mathrm{val}(\varphi^{k+1}, x) &= \mathrm{val}(\varphi^{k-j}, \varphi^{j+1}(x)) \bullet \mathrm{val}(\varphi, \varphi^j(x)) \bullet \mathrm{val}(\varphi^j, x) \\ &= \mathrm{val}(\varphi^{k-j}, \varphi^j(z)) \bullet \mathrm{val}(\varphi, \varphi^{j-1}(z)) \bullet \mathrm{val}(\varphi^{j-1}, z) = \mathrm{val}(\varphi^k, z)\end{aligned}$$

so $(x, 1, z) \in \Gamma^+_\varphi(1, k)$. Next, one observes that

$$\begin{aligned}\mathrm{val}(\varphi^n, y) = \mathrm{val}(\varphi^{n+l}, x) &= \mathrm{val}(\varphi^{n+l-k-1}, \varphi^{k+1}(x)) \bullet \mathrm{val}(\varphi^k, x) \\ &= \mathrm{val}(\varphi^{n+l-k-1}, \varphi^k(z)) \bullet \mathrm{val}(\varphi^k, z) = \mathrm{val}(\varphi^{n+l-1}, z),\end{aligned}$$

so $(z, l-1, y) \in \Gamma^+_\varphi(l-1, n)$.

Case 3: Assume that x is critical for φ, but that $\varphi^j(x)$ is non-critical for any j between 1 and k. By choice of k, we know that $\varphi^k(x)$ is a periodic point with some period p, and that $\varphi^{k-2p}(x) = \varphi^k(x)$. Choose a $z \in \mathbb{T}$ such that $\varphi^{2p-1}(z) = x$, and note that this entails that $\varphi^{k+1}(x) = \varphi^k(z)$. Using that x is critical for φ^{k+1} yields

$$\begin{aligned}\mathrm{val}(\varphi^{k+1}, x) &= \mathrm{val}(\varphi^{k+1}, x) \bullet \mathrm{val}(\varphi^{2p-1}, z) \\ &= \mathrm{val}(\varphi^{k+1}, \varphi^{2p-1}(z)) \bullet \mathrm{val}(\varphi^{2p-1}, z) = \mathrm{val}(\varphi^{k+2p}, z) \\ &= \mathrm{val}(\varphi^p, \varphi^{m+p}(z)) \bullet \mathrm{val}(\varphi^p, \varphi^m(z)) \bullet \mathrm{val}(\varphi^m, z) \\ &= (\mathrm{val}(\varphi^p, \varphi^m(z)))^2 \bullet \mathrm{val}(\varphi^m, z)\end{aligned}$$

Now, $\varphi^k(z) = \varphi^{k-1}(x)$ is not critical for any φ^j, hence $\mathrm{val}(\varphi^p, \varphi^k(z))$ is either $(+,+)$ or $(-,-)$. The composition table then gives that $(\mathrm{val}(\varphi^p, \varphi^k(z)))^2 = (+,+)$, hence $\mathrm{val}(\varphi^{k+1}, x) = \mathrm{val}(\varphi^k, z)$, so $(x, 1, z) \in \Gamma_\varphi^+(1, k)$. Calculations as in case 1 shows that $(z, l-1, y) \in \Gamma_\varphi^+(l-1, n)$.

Case 4: Finally, assume that $\{\varphi^j(x) | j = 0, \ldots, k\}$ contains no critical points, but that the set $\{\varphi^j(z_\pm) | j = 0, \ldots, k_1 - 1\}$ does. Then $\varphi^{k_2}(x) = \varphi^{k_2-1}(z_\pm)$ is in $\mathcal{D}_0$, so $\{\varphi^j(x) | j = 0, \ldots, k\}$ contains a periodic orbit $\mathcal{O}$ consisting only of non-critical points. By simplicity of Γ_φ^+, $[\varphi^{k-1}(x)]_{\Gamma_\varphi^+}$ is infinite, so we may choose a z_0 in $[\varphi^{k-1}(x)]_{\Gamma_\varphi^+} \cap (\mathbb{T} \setminus \mathcal{D}_0)$, and even arrange it such that $\varphi^{2k_1}(z_0) = \varphi^{k-1}(x)$. Finally choose a z in $\varphi^{-k_2}(z_0)$ such that

$$\mathrm{val}(\varphi^k, z) = \mathrm{val}(\varphi^{2k_1}, z_0) \bullet \mathrm{val}(\varphi^{k_2}, z) = \mathrm{val}(\varphi^{k-1}, x).$$

Calculations similar to those above show that $(z, l-1, y) \in \Gamma_\varphi^+(l-1, n)$, which proves the lemma. □

Corollary 5.9. *Assume that $n \geq k+1$, and let $(x, 0, y) \in \Gamma_\varphi^+(0, n)$. Then there are elements $z_1, z_2 \in \mathbb{T}$ such that $(z_1, 0, z_2) \in \Gamma_\varphi^+(0, n-1)$ and $(x, 1, z_1), (y, 1, z_2) \in \Gamma_\varphi^+(1, k)$.*

Proof. Use Lemma 5.8 to get a $z_1 \in \mathbb{T}$ such that $(x, 1, z_1) \in \Gamma_\varphi^+(1, k)$ and $(z_1, -1, y) \in \Gamma_\varphi^+(-1, n)$. Then $(y, 1, z_1) \in \Gamma_\varphi^+(1, n-1)$, and a second application of Lemma 5.8 gives a $z_2 \in \mathbb{T}$ such that $(y, 1, z_2) \in \Gamma_\varphi^+(1, k)$ and $(z_1, 0, z_2) \in \Gamma_\varphi^+(0, n-1)$. We note that

$$(x, 0, y) = (x, 1, z_1)(z_1, 0, z_2)(z_2, -1, y)$$ □

Lemma 5.10. *Assume that $C_r^*(\Gamma_\varphi^+)$ is simple, let $j \geq k$, and let $(x, 0, y) \in R_\varphi^+(j)$. Then there is a $z \in \mathbb{T}$ such that*

$$(z, 1, x), (z, 1, y) \in \Gamma_\varphi^+(1, j)$$

and $(x, 0, y) = (x, -1, z)(z, 1, y) \in \Gamma_\varphi^+$.

Proof. Choose points z_+ and z_- in $\varphi^{-k_2}(\varphi^{k_2-1}(x))$ as in Lemma 5.7. Now, there are several cases:

Case 1: If neither of the sets $\{\varphi^i(x) | i = 0, \ldots, k_2 - 2\}$ and $\{\varphi^i(z_\pm) | i = 0, \ldots, k_2 - 1\}$ contain a critical point, let $z \in \{z_+, z_-\}$ be such that $\mathrm{val}(\varphi^{k_2}, z) = \mathrm{val}(\varphi^{k_2-1}, x)$. One now easily sees that $(z, 1, x)$ and $(z, 1, y)$ is in $\Gamma_\varphi^+(1, j)$.

Case 2: If the set $\{\varphi^i(x) | i = 0, \ldots, j-1\}$ contains a critical point, any z in $\varphi^{-1}(x)$ will do, since the composition table for $\bullet$ gives

$$\mathrm{val}(\varphi^{j+1}, z) = \mathrm{val}(\varphi^j, x) \bullet \mathrm{val}(\varphi, z) = \mathrm{val}(\varphi^j, x)$$

whenever $\mathrm{val}(\varphi^j, x) \in \{(+,-), (-,+)\}$. Hence, for this z we have $(z, 1, x), (z, 1, y) \in \Gamma_\varphi^+(1, j)$.

Case 3: Assume finally that there is a critical point in the set $\{\varphi^j(z_\pm) | j = 0, \ldots, k_2 - 1\}$, but none in $\{\varphi^i(x) | i = 0, \ldots, j-1\}$. Then, by choice of $j \geq k$, the set $\{\varphi^l(x) | l = 0, \ldots, j\}$ contains a periodic orbit $\mathcal{O}$ without any critical points. Since $[\varphi^j(x)]_{\Gamma_\varphi^+}$ is infinite and contains no critical points, we may choose a $z_0 \in [\varphi^j(x)]_{\Gamma_\varphi^+} \cap (\mathbb{T} \setminus \mathcal{D}_0)$ and a $l \leq 2k_1$ such that $\varphi^l(z_0) = \varphi^j(x)$. Finally, note that $j - l + 1 \geq k_2$, so we may choose a $z \in \mathbb{T}$ such that $\varphi^{j-l-1}(z) = z_0$ and such that

$$\mathrm{val}(\varphi^l, z_0) \bullet \mathrm{val}(\varphi^{j-l+1}, z) = \mathrm{val}(\varphi^{j+1}, z) = \mathrm{val}(\varphi^j, x)$$

Hence $(z, 1, x)$, and also $(z, 1, y)$, are in $\Gamma_\varphi^+(1, j)$. □

The lemmas above also holds if we replace Γ_φ^+ with Γ_φ, but the proofs require a little notation: Define an equivalence relation $\equiv$ on $\mathcal{V}$, the set of valencies, by putting $(+,+) \equiv (-,-)$ and $(+,-)$ and $(-,+)$ only equivalent to themselves. Then, if $\varphi^i(x) = \varphi^j(y)$ and $\mathrm{val}(\varphi^i, x) \equiv \mathrm{val}(\varphi^j, y)$, there is a p – maybe two – such that $(x, i-j, p, y) \in \Gamma_\varphi$.

Lemma 5.11. *Assume that φ is exact. Let $l \in \mathbb{Z}$, $n \in \mathbb{N}$, and assume that $l + n \geq k+1$. Let $(x, l, p, y) \in \Gamma_\varphi(l, n)$. Then there exists a $z \in \mathbb{T}$ and $p_1, p_2 \in \mathbb{Z}_2$ such that $(x, 1, p_1, z) \in \Gamma_\varphi(1, k)$, $(z, l-1, p_2, y) \in \Gamma_\varphi(l-1, n)$ and $(x, l, p, y) = (x, 1, p_1, z)(z, l-1, p_2, y)$.*

Proof. Consider first the case where $\mathrm{val}(\varphi, x) = (\pm, \pm)$. Put $z = \varphi(x)$, and note that $\varphi^k(z) = \varphi^{k+1}(x)$ and by the composition table for $\bullet$,

$$\mathrm{val}(\varphi^{k+1}, x) = \mathrm{val}(\varphi^k, \varphi(x)) \bullet \mathrm{val}(\varphi, x) = \mathrm{val}(\varphi^k, z) \bullet \mathrm{val}(\varphi, x) \equiv \mathrm{val}(\varphi^k, z).$$

Hence, there is an $\eta_1 \in \mathcal{P}$ with $p_1 = V(\eta_1)$ $(x, 1, p_1, z) \in \Gamma_\varphi(1, k)$. Next, note that

$$\begin{aligned} \mathrm{val}(\varphi^{l+n-1}, z) &= \mathrm{val}(\varphi^{l+n-1-k}, \varphi^k(z)) \bullet \mathrm{val}(\varphi^k, z) \\ &\equiv \mathrm{val}(\varphi^{l+n-1-k}, \varphi^{k+1}(x)) \bullet \mathrm{val}(\varphi^{k+1}, x) \\ &= \mathrm{val}(\varphi^{l+n}, x) \equiv \mathrm{val}(\varphi^n, y) \end{aligned}$$

and that $\varphi^n(y) = \varphi^{l+n-1}(z)$, so there is an p_2 with $(z, l-1, p_2, y) \in \Gamma_\varphi(l-1, n)$. Now

$$(x, 1, p_1, z)(z, l-1, p_2, y) = (x, 1, p_2 p_1, y),$$

so we just need to check that $p_2 p_1 = p$. If φ^{n+l} is monotone at x, the choice of local transfer is unique, and we're done. If x is critical for φ^{n+l}, we must have that y is critical for φ^n, and there are two possible choices of p_2. In this case, we simply choose the one making $p_2 p_1$ equal to p.

Next, assume that x is critical for φ. Then it is also critical for φ^{n+l}, so we must have $\mathrm{val}(\varphi^{n+l}, x) = \mathrm{val}(\varphi^n, y)$. Note next that either $\varphi^j(x) \notin \mathcal{C}$ for any $j = 1, \ldots, k$ (corresponding to case 2 of the lemma above), or there is a j between 1 and k with $\varphi^j(x) \in \mathcal{C}$ (corresponding to case 3 above). In both cases, we found a $z \in \mathbb{T}$ with $\mathrm{val}(\varphi^k, z) = \mathrm{val}(\varphi^{k+1}, x)$ and $\varphi^{k+1}(x) = \varphi^k(x)$ as well as $\mathrm{val}(\varphi^{l+n-1}, z) = \mathrm{val}(\varphi^n, y)$ and $\varphi^{l+n-1}(z) = \varphi^n(y)$. This implies first that there is an p_1 such that $(x, 1, p_1, z) \in \Gamma_\varphi(1, k)$, and secondly that there are *two* possible p_2's such that $(z, l-1, p_2, y) \in \Gamma_\varphi(l-1, n)$. Choosing the p_2 such that $p_2 p_1 = p$ yields the desired result. □

Corollary 5.12. *Assume that $n \geq k+1$, and let $(x, 0, p, y) \in \Gamma_\varphi^+(0, n)$. Then there are elements $z_1, z_2 \in \mathbb{T}$ and $p_1, p_2, q \in \mathbb{Z}_2$ such that $(z_1, 0, q, z_2) \in \Gamma_\varphi^+(0, n-1)$ and $(x, 1, p_1, z_1), (y, 1, p_2, z_2) \in \Gamma_\varphi^+(1, k)$ and*

$$(x, 0, p, y) = (x, 1, p_1, z_1)(z_1, 0, q, z_2)(z_2, -1, p_2, y).$$

Proof. Exactly as the proof of Corollary 5.9. □

Lemma 5.13. *Let $(x, 0, p, y) \in R_\varphi(k)$. Then there is a $z \in \mathbb{T}$ and $p_1, p_2 \in \mathbb{Z}_2$ such that*

$$(z, 1, p_1, x), (z, 1, p_2, y) \in \Gamma_\varphi(1, k)$$

and $(x, 0, p, y) = (x, -1, p_1, z)(z, 1, p_2, y) \in \Gamma_\varphi$.

Proof. Assume first that x is critical for φ^k. Then $\operatorname{val}(\varphi^k, x) = \operatorname{val}(\varphi^k, y)$. By Lemma 5.10, there is a $z \in \mathbb{T}$ with $\varphi^{k+1}(z) = \varphi^k(x) = \varphi^k(y)$ and $\operatorname{val}(\varphi^{k+1}, z) = \operatorname{val}(\varphi^k, x) = \operatorname{val}(\varphi^k, y)$. This means that

$$(z, -1, +, x), (z, -1, -, x), (z, 1, +, y), (z, 1, -, y) \in \Gamma_\varphi(1, k)$$

Choosing the right p_1 and p_2 yields the desired result.

Next, assume that x is not critical for φ^k. Then the sign p in $(x, 0, p, y)$ is uniquely determined. We may use Lemma 5.10 to find a $z \in \mathbb{T}$ with $\varphi^k(x) = \varphi^{k+1}(z)$ and $\operatorname{val}(\varphi^k, x) = \operatorname{val}(\varphi^{k+1}, z)$. Choose the unique p_1 such that $(z, 1, p_1, x) \in \Gamma_\varphi(1, k)$. Since $\varphi^{k+1}(z) = \varphi^k(y)$ and y is not critical for φ^k, there is a unique p_2 such that $(z, 1, p_2, y) \in \Gamma_\varphi(1, k)$, and we get $(x, -1, p_1, z)(z, 1, p_2, y) = (x, 0, p, y)$. □

Remark 5.14. Here's the short version of the results of this section: Let G be either of the groupoids Γ_φ or Γ_φ^+, and $G(n, m)$ the corresponding subsets introduced in Definition 5.6. Corollaries 5.9 and 5.12 shows that given a $\gamma \in G(0, n)$, for $n \geq k+1$, there are elements $\gamma_1 \in G(1, k)$, $\gamma_2 \in G(0, n-1)$ and $\gamma_3 \in G(-1, k+1)$ such that $\gamma = \gamma_1\gamma_2\gamma_3$. The Lemmas 5.10 and 5.13, on the other hand, shows that any $\gamma \in G(0, k)$ may be written as $\gamma = \gamma_1\gamma_2$ with $\gamma_1 \in G(-1, k+1)$ and $\gamma_2 \in G(1, k)$.

Many results in the next sections depend not on the particular groupoid, but only on the factorisation results that we proved above. Hence, in what's to come we will let G denote either groupoid Γ_φ or Γ_φ^+. Similarly, we will let R denote either R_φ or R_φ^+, and $R(k)$ the sub-groupoids $R_\varphi(k)$ or $R_\varphi^+(k)$. ♦

We can now take the first step of the program laid out in the introduction to this chapter and realise $C_r^*(G)$ as the algebra of a C^*-correspondence over $C_r^*(R(k))$. Let E be the closure of $C_c(G(1, k))$ inside $C_r^*(G)$. It is straightforward to check that

$$E^*E \subseteq C_r^*(R(k)),\ C_r^*(R(k+1))E \subseteq E,\ EC_r^*(R(k)) \subseteq E \tag{5.2}$$

It follows that we are in the situation from Example 5.3 (here, $A = C_r^*(R(k))$ and $B = C_r^*(R(k+1))$) where E is a C^*-correspondence over $C_r^*(R(k))$. We now aim to show that the associated C^*-algebra $\mathcal{O}_E$ is isomorphic to $C_r^*(G)$. We need some properties of E:

Lemma 5.15. *The following holds:*

1. $\overline{\operatorname{Span} EE^*} = C_r^*(R(k+1))$
2. $C_r^*(R(n)) \subseteq EC_r^*(R(n+1))E^*$ *for* $n \geq k+1$.
3. $C_r^*(G)$ *is generated by* E.

Proof. The inclusion $EE^* \subset C_r^*(R(k+1))$ is straightforward, so $\overline{\operatorname{Span} EE^*} \subseteq C_r^*(R(k+1))$ by continuity. For the other inclusion, take a $h \in C_c(R(k+1))$, and assume without loss of generality that h is supported in a bisection U. Put $K = r(\operatorname{supp} h)$. If $x \in K$, there is a unique $\gamma \in U$ such that $r(\gamma) = x$. From Lemma 5.11 (or 5.8) we get elements $\gamma_1 \in G(1, k), \gamma_2 \in G(-1, k+1)$ such that $\gamma = \gamma_1\gamma_2$. Let $W_1 \subseteq G(1, k)$ and $W_2 \subseteq G(-1, k+1)$ be open bisections containing γ_1 and γ_2 respectively, chosen such that $W_1 W_2 \subseteq U$. For each ρ in a neighbourhood of γ_1 in W_1, there is a unique ρ' in U with $r(\rho) = r(\rho')$, so we can define an $f \in C_c(W_1)$ by $f(\rho) = h(\rho')$. Defining $g \equiv 1$ in a neighbourhood of γ_2, we get a neighbourhood V_x of x such that $fg = h$ on $r^{-1}(V)$. The sets $\{V_x\}$ cover K, so by compactness, we may exhaust to a finite cover $\{V_{x_i}\}_{i=1}^n$. Let $\{\psi_i\}_i = 1^n \subseteq C(\mathbb{T})$ be a partition of unity with respect to this cover, and $f_i, g_i \in C_c(G(1, k))$ maps defined from the x_i's as above. Then $h = \sum_i \psi_i f_i g_i^*$. Since $\psi_i f_i, g_i \in E$, it follows that

h is in the span of EE^*. The second claim follows by very similar arguments. Finally, since $C_c(R_\varphi(l)) \subseteq C_c(R_\varphi(l+1))$ for all $l \in \mathbb{N}$, it follows from 1) and 2) that $C_r^*(R)$ is contained in the C^*-algebra generated by E. But with arguments similar to those above, it follows from Lemmas 5.11 (or 5.8) that $C_c(G(1))$ is contained in $\operatorname{Span} EC_c(R)$, and then that $C_c(G(2))$ is in the span of $EC_c(G(1))$, and so on. Since $G(-l) = G(l)^*$, it follows that all $C_c(G(l))$ are contained in the C^*-algebra generated by E. Since $\bigcup_l C_c(G(l))$ is dense in $C_r^*(G)$, we are done. □

From Equation 5.2, it follows that $C_r^*(R(k+1))$ acts on E by right multiplication, so there is a map $\pi : C_r^*(R(k+1)) \to \mathbb{L}(E)$ given by $\pi(a)b = ab$.

Lemma 5.16. *The map $\pi : C_r^*(R(k+1)) \to \mathbb{L}(E)$ is injective, and the image of π is equal to $\mathbb{K}(E)$.*

Proof. This follows immediately from the equality $\overline{\operatorname{Span} EE^*} = C_r^*(R(k+1))$. □

Theorem 5.17. *Let φ be critically finite of order k, and let E be the closure of $C_c(G(1,k))$. Then $C_r^*(G) \simeq \mathcal{O}_E$, and there is a six-terms exact sequence*

$$\begin{array}{ccccc} K_0\left(C_r^*\left(R(k)\right)\right) & \xrightarrow{\mathrm{id}-[E]_0} & K_0\left(C_r^*\left(R(k)\right)\right) & \xrightarrow{\iota_0} & K_0\left(C_r^*\left(G\right)\right) \\ \uparrow & & & & \downarrow \\ K_1\left(C_r^*\left(G\right)\right) & \xleftarrow[\iota_1]{} & K_1\left(C_r^*\left(R(k)\right)\right) & \xleftarrow[\mathrm{id}-[E]_1]{} & K_1\left(C_r^*\left(R(k)\right)\right) \end{array} \tag{5.3}$$

where $\iota : C_r^*\left(R(k)\right) \to C_r^*\left(G\right)$ is the inclusion map and $[E]$ is the KK-theory element defined from $\pi : C_r^*\left(R(k)\right) \to \mathbb{K}(E)$.

Proof. The isomorphism $\mathcal{O}_E \simeq C_r^*(G)$ follows from point 3 of Lemma 5.15, Example 5.3 and Theorem 5.5. The six-terms exact sequence is then a consequence of the lemma above and Theorem 8.6 of [15]. □

Corollary 5.18. *Let E be as above. Then there are extensions*

$$0 \longrightarrow \operatorname{coker}(\mathrm{id}-[E]_0) \longrightarrow K_0(C_r^*(G)) \longrightarrow \ker(\mathrm{id}-[E]_1) \longrightarrow 0 \tag{5.4}$$

and

$$0 \longrightarrow \operatorname{coker}(\mathrm{id}-[E]_1) \longrightarrow K_1(C_r^*(G)) \longrightarrow \ker(\mathrm{id}-[E]_0) \longrightarrow 0. \tag{5.5}$$

5.3 The linking algebra

From Corollary 5.18, it follows that computing the maps $[E]_0$ and $[E]_1$ is the next step towards computing the K-theory of $C_r^*(G)$. To do that, we need some bits and pieces from the theory of groupoid equivalences. This was introduced by Muhly, Renault and Williams in [18]. Here, we follow the approach taken in [40].

Definition 5.19. Let G be a locally compact groupoid with unit space $G^{(0)}$ and source map s_G, and Z a locally compact space. Z is a *(left) G-space* if there is a continuous, open map $r_Z : Z \to G^{(0)}$ (the *structure map*) and a continuous action

$$\{(\gamma, z) \in G \times Z \mid s_G(\gamma) = r_Z(z)\} =: G * Z \ni (\gamma, z) \mapsto \gamma \cdot z \in Z$$

such that $r_Z(z) \cdot z = z$ for all $z \in Z$ and $(\gamma_1\gamma_2)z = \gamma_1(\gamma_2 z)$ for all $(\gamma_1, \gamma_2) \in G^{(2)}$ with $s_G(\gamma_2) = r_Z(z)$. The action is *free* if $\gamma \cdot z = z$ implies $\gamma = r_Z(z)$ and *proper* if $(\gamma, z) \mapsto (\gamma \cdot z, z)$ is a proper map from $G * Z \to Z \times Z$.

Right G-spaces are defined analogously, except that the structure map is denoted s_Z.

Definition 5.20. Let G and H be locally compact groupoids. Z is a (G,H)-equivalence if

- Z is a free and proper left G-space.
- Z is a free and proper right H-space.
- The actions of G and H on Z commute.
- r_Z and s_Z induces homeomorphisms $Z/H \simeq G^{(0)}$ and $G\backslash Z \simeq H^{(0)}$.

Lemma 5.21. *$G(1,k)$ is a $(R(k+1),R(k))$-equivalence, with multiplication inherited from G and r_Z and s_Z the range- and source maps on G.*

Proof. Checking that $G(1,k)$ is a free and proper left $R(k+1)$-space and right $R(k)$-space is straightforward, as is commutativity of the right and left actions. The set $G(1,k)/R(k)$ consists of equivalence classes of elements $\gamma \in G(1,k)$, where $\gamma \sim \rho$ if $r(\gamma) = r(\rho)$. To show that the induced structure map $r_Z : G(1,k)/R(k) \to \mathbb{T}$ is injective, let $\gamma, \rho \in G(1,k)$ with $r(\gamma) = r(\rho)$. Then $\rho^{-1}\gamma \in R(k)$, and $\gamma = \rho(\rho^{-1}\gamma)$, so $\gamma \sim \rho$. For surjectivity, let $x \in \mathbb{T}$, and think of x as an element of $R(k+1)$. Using Corollaries 5.9 and 5.12 yields an element $\gamma \in G(1,k)$ such that $r(\gamma) = x$, which gives surjectivity. The proof that $R(k+1)\backslash G(1,k) \simeq \mathbb{T}$ is similar, using Lemmas 5.10 and 5.13 instead. □

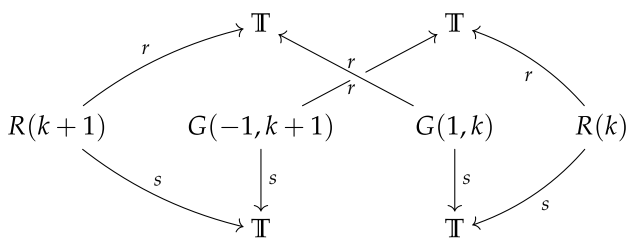
Using the lemma above, Theorem 13 of [40] now yields the following conclusion:

Lemma 5.22. *The completion E of $C_c(G(1,k))$ is a $C_r^*(R(k+1)) - C_r^*(R(k))$-imprimitivity bimodule. Hence, $C_r^*(R(k))$ and $C_r^*(R(k+1))$ are Morita equivalent.*

To get any further, we need a description of the linking groupoid $\mathcal{L}$ of $R(k)$ and $R(k+1)$. Following the description in [40], $\mathcal{L}$ as a topological space is the disjoint union

$$\mathcal{L} = R(k+1) \sqcup R(k) \sqcup G(1,k) \sqcup G(-1,k+1)$$

with each of the four sets clopen in $\mathcal{L}$. We may regard $\mathcal{L}$ as a groupoid with unit space $\mathbb{T} \sqcup \mathbb{T}$, and range and source maps inherited from G, but with target spaces as in the picture below:



(5.6)

Similarly, $\mathcal{L}^{(2)} = \{(\gamma,\gamma') \in \mathcal{L}^2 \mid s(\gamma) = r(\gamma')\}$ with multiplication inherited from G. One checks that $\mathcal{L}$ is a second countable étale locally compact Hausdorff groupoid. The reduction of $\mathcal{L}$ to the first (resp. second) copy of $\mathbb{T}$ is naturally identified with $R(k+1)$ (resp. $R(k)$), giving embeddings

$$a : C_r^*(R(k+1)) \to C_r^*(\mathcal{L}), \qquad b : C_r^*(R(k)) \to C_r^*(\mathcal{L})$$

Given a function $f \in C_c(\mathcal{L})$, write $f = f_{11} + f_{12} + f_{21} + f_{22}$ with $f_{11} \in C_c(R(k+1))$, $f_{12} \in C_c(G(1,k))$, $f_{21} \in C_c(G(-1,k+1))$ and $f_{22} \in C_c(R(k))$. Define $\Psi(f) \in \mathbb{L}(E \oplus R(k))$ by

$$\Psi(f)(e,g) = (f_{11}e + f_{12}g, f_{21}e + f_{22}g).$$

Theorem 13 in [40] and Corollary 3.21 in [29] implies that Ψ extends to a $*$-isomorphism

$$\Psi : C_r^*(\mathcal{L}) \to \mathbb{K}\left(E \oplus C_r^*(R(k))\right).$$

There are canonical embeddings $\mathbb{K}(E) \to \mathbb{K}(E \oplus C_r^*(R(k)))$ and

$$C_r^*(R(k)) \simeq \mathbb{K}(C_r^*(R(k))) \to \mathbb{K}(E \oplus C_r^*(R(k)))$$

making the diagram

$$\begin{array}{ccccccc} C_r^*(R(k)) & \xrightarrow{\rho} & C_r^*(R(k+1)) & \xrightarrow{a} & C_r^*(\mathcal{L}) & & \\ & \searrow^{\pi} & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \Psi} & \nwarrow^{b} & \\ & & \mathbb{K}(E) & \longrightarrow & \mathbb{K}(E \oplus C_r^*(R(k))) & \longleftarrow & C_r^*(R(k)) \end{array} \tag{5.7}$$

commute. Here, π is the isomorphism from Lemma 5.16, and $\rho : C_r^*(R(k)) \to C_r^*(R(k+1))$ the inclusion. On K-theory, the map going diagonally down from $C_r^*(R(k))$ to $\mathbb{K}(E)$ and then horizontally to $C_r^*(R(k))$ is – cf. [15], Theorem 8.3 and Appendix B – equal to the map $[E]_*$, i.e.

$$[E]_* = b_*^{-1} \circ a_* \circ \rho_* \tag{5.8}$$

where the $*$ denotes the induced maps on either K_0 or K_1. The path forward is now clear: We need to determine the three maps in the equation above. The strategies for the two groupoids Γ_φ and Γ_φ^+ are roughly the same – however, there are enough differences in the proofs to justify first going through the results in great detail for Γ_φ^+, and then doing the same for Γ_φ, highlighting the places where the proofs are different.

5.4 On $K_*(C_r^*(\Gamma_\varphi^+))$

We begin by recalling the setup of Chapter 3: Fix a $k \in \mathbb{N}$, and consider the groupoid $R_\varphi^+(k)$ from Remark 3.11. Let $\mathcal{C}$ denote the critical points of φ, and put

$$\mathcal{D}_0 = \bigcup_{k=0}^{\infty} \varphi^k(\mathcal{C}).$$

Put $\mathcal{D} = \varphi(\mathcal{D}_0)$. Then $\mathcal{E}_k = \varphi^{-k}(\mathcal{D})$ is finite and $R_\varphi^+(k)$-invariant, and we obtain from Lemma 3.21 a short exact sequence

$$0 \longrightarrow C_r^*(R_\varphi^+(k)|_{\mathbb{T}\setminus\mathcal{E}_k}) \xrightarrow{i} C_r^*(R_\varphi^+(k)) \xrightarrow{\pi} C_r^*(R_\varphi^+(k)|_{\mathcal{E}_k}) \longrightarrow 0 \tag{5.9}$$

of C^*-algebras. It also follows from Section 3.1 that there are finite dimensional C^*-algebras $\mathbb{A}_k^+$ and $\mathbb{B}_k^+$ such that

$$C_r^*(R_\varphi^+(k)|_{\mathbb{T}\setminus\mathcal{E}_k}) \simeq S\mathbb{B}_k^+, \qquad C_r^*(R_\varphi^+(k)|_{\mathcal{E}_k}) \simeq \mathbb{A}_k^+.$$

Briefly put, $\mathbb{B}_k^+$ is generated by matrix units $e_{I,J}$, where I and J are connected components of $\mathbb{T} \setminus \mathcal{E}_k$ satisfying $\varphi^k(I) = \varphi^k(J)$ and $\mathrm{val}(\varphi^k, I) = \mathrm{val}(\varphi^k, J)$, and $\mathbb{A}_k^+$ is generated by matrix units $e_{x,y}$, with $x, y \in \mathcal{E}_k$ satisfying $\varphi^k(x) = \varphi^k(y)$ and $\mathrm{val}(\varphi^k, x) = \mathrm{val}(\varphi^k, y)$. Now, we may do the exact same thing for $R_\varphi^+(k+1)$, using $\mathcal{E}_{k+1} = \varphi^{-k-1}(\mathcal{D})$ as invariant set. We obtain two extensions

$$0 \longrightarrow S\mathbb{B}_j^+ \xrightarrow{i} C_r^*(R_\varphi^+(k)) \xrightarrow{a} \mathbb{A}_j^+ \longrightarrow 0 \tag{5.10}$$

for $j = k, k+1$. Finally, we have seen that the summands in $\mathbb{A}_j^+$ is in one-to-one-correspondence with the set

$$\mathcal{D}_j(\pm) = \left\{(d,v) \in \mathcal{D} \times \mathcal{V} \,\middle|\, \exists x \in \mathcal{E}_k : \varphi^j(x) = d, \mathrm{val}(\varphi^j, x) = v\right\}$$

and the summands in $\mathbb{B}_j$ is in one-to-one-correspondence with the set

$$\mathcal{I}_j(\pm) = \left\{(I_i, v) \in \mathcal{I} \times \{(+,+),(-,-)\} \,\middle|\, \exists I \in \mathcal{I}_j : \varphi^j(I) = I_i, \mathrm{val}(\varphi^j, I) = v\right\}$$

where $\mathcal{I}$ denotes the connected components of $\mathbb{T} \setminus \mathcal{D}$ and $\mathcal{I}_j$ the connected components of $\mathbb{T} \setminus \mathcal{E}_j$.

Lemma 5.23. *For $j \geq k$, there are isomorphisms $K_0(\mathbb{A}_j^+) \simeq K_0(\mathbb{A}_{j+1}^+)$ and $K_0(\mathbb{B}_j^+) \simeq K_0(\mathbb{B}_{j+1}^+)$.*

Proof. By the discussion above, this amounts to showing that there is a k such that

$$\mathcal{I}_j(\pm) = \mathcal{I}_{j+1}(\pm), \;\; \mathcal{D}_j(\pm) = \mathcal{D}_{j+1}(\pm)$$

for $j \geq k$. We choose k to be the order of the map as defined in Section 5.2. We begin with $\mathbb{B}_j^+$: Let k be the order of the map, and let $j \geq k$. Let $I_i \in \mathcal{I}$, $v \in \{(+,+),(-,-)\}$, and assume that there is an interval $I \in \mathcal{I}_j$ such that $\varphi^j(I) = I_i$ with $\mathrm{val}(\varphi^k, I) = v$. We must show that there is an interval $J \in \mathcal{I}_{j+1}$ with the same properties. Choose an $x \in I_i$ such that $\varphi^{-j-1}(x)$ contains points non-critical for φ^{j+1}. Then Lemma 5.7 implies that $\varphi^{-j}(x)$ contains a point z such that $\mathrm{val}(\varphi^{j+1}, z) = v$. We may then choose J to be the connected component of z in $\mathcal{I}_{j+1}$. This shows that $\mathcal{I}_j(\pm) \subseteq \mathcal{I}_{j+1}(\pm)$, and the other way is completely similar.

For $\mathbb{A}_j^+$, let $(d,v) \in \mathcal{D}_j(\pm)$ and choose $x \in \varphi^{-j}(d)$ with $\mathrm{val}(\varphi^j, x) = v$. Then $(x, 0, x) \in R_\varphi(j)$, so by Lemma 5.10, there is a y in $\mathbb{T}$ such that $(y, 1, x) \in \Gamma_\varphi(1, j)$ – that is, $\varphi^{j+1}(y) = d$ and $\mathrm{val}(\varphi^{j+1}, y) = v$. On the other hand, if $\varphi^{j+1}(y) = d$ with $\mathrm{val}(\varphi^{j+1}, y) \equiv v$, we use Lemma 5.11 (with $l = 0, n = j+1$) instead to get an x such that $(x, -1, y) \in \Gamma_\varphi(-1, j+1)$; that is, $\varphi^j(x) = \varphi^{j+1}(y) = d$ and $\mathrm{val}(\varphi^j, x) = v$. □

Remark 5.24. Note that the number k might be fairly large, and it might very well be the cases that the K-theory groups above stabilise at an earlier level – for instance, for the tent map of Example 3.12, we saw that $\mathcal{D}(\pm)$ and $\mathcal{I}(\pm)$ were independent of the number k. It is also worth noting that the induced maps $(I_k)_0$ and $(U_k)_0$ between $K_0(\mathbb{A})_k$ and $K_0(\mathbb{B}_k)$ are independent of k – once the K-theory groups stabilise, so do the induced maps. ♦

From this point onwards, we fix the number k from the Lemma above. Finally, we can reveal what all this has to do with the linking groupoud $\mathcal{L}$. Consider the set $\mathcal{E}_{k+1} \sqcup \mathcal{E}_k$ as a subset of $\mathbb{T} \sqcup \mathbb{T}$, the unit space of $\mathcal{L}$. Then $\mathcal{E}_{k+1} \sqcup \mathcal{E}_k$ is finite and $\mathcal{L}$-invariant, and yields an extension

$$0 \longrightarrow C_r^*(\mathcal{L}|_{(\mathbb{T}\sqcup\mathbb{T})\setminus(\mathcal{E}_{k+1}\sqcup\mathcal{E}_k)}) \xrightarrow{\;i\;} C_r^*(\mathcal{L}) \xrightarrow{\;\pi\;} C_r^*(\mathcal{L}|_{\mathcal{E}_{k+1}\sqcup\mathcal{E}_k}) \longrightarrow 0 \tag{5.11}$$

Write $\mathbb{A}_\mathcal{L}$ for $C_r^*(\mathcal{L}|_{\mathcal{E}_{k+1}\sqcup\mathcal{E}_k})$. Then, like the algebras $\mathbb{A}_k^+$ and $\mathbb{A}_{k+1}^+$, $\mathbb{A}_\mathcal{L}$ is generated by matrix units $e_{x,y}$, $x, y \in \varphi^{-k-1}(\mathcal{D}) \sqcup \varphi^{-k}(\mathcal{D})$ subject to relations similar to those above:

- $\varphi^{k+1}(x) = \varphi^{k+1}(y)$ and $\mathrm{val}(\varphi^{k+1}, x) = \mathrm{val}(\varphi^{k+1}, y)$ when $x, y \in \varphi^{-k-1}(\mathcal{D})$.
- $\varphi^k(x) = \varphi^k(y)$ and $\mathrm{val}(\varphi^k, x) = \mathrm{val}(\varphi^k, y)$ when $x, y \in \varphi^{-k}(\mathcal{D})$.
- $\varphi^{k+1}(x) = \varphi^k(y)$ and $\mathrm{val}(\varphi^{k+1}, x) = \mathrm{val}(\varphi^k, y)$ when $x \in \varphi^{-k-1}(\mathcal{D}), y \in \varphi^{-k}(\mathcal{D})$.

- $\varphi^k(x) = \varphi^{k+1}(y)$ and $\mathrm{val}(\varphi^k, x) = \mathrm{val}(\varphi^{k+1}, y)$ when $x \in \varphi^{-k}(\mathcal{D}), y \in \varphi^{-k-1}(\mathcal{D})$.

In particular, $\mathbb{A}_{\mathcal{L}}$ is finite-dimensional with summands in one-to-on-correspondence with the set $\mathcal{D}_k(\pm)$. The inclusions from $C_r^*(R_\varphi^+(j)) \to C_r^*(\mathcal{L})$ (for $j = k, k+1$) restrict to maps $\mathbb{A}_j^+ \to \mathbb{A}_{\mathcal{L}}$, realising $\mathbb{A}_k^+$ and $\mathbb{A}_{k+1}^+$ as orthogonal subalgebras of $\mathbb{A}_{\mathcal{L}}$, as in this picture:

$$\mathbb{A}_{\mathcal{L}} = \begin{bmatrix} \mathbb{A}_k^+ & * \\ * & \mathbb{A}_{k+1}^+ \end{bmatrix} \tag{5.12}$$

Similarly, we have $C_r^*(\mathcal{L}|_{(\mathbb{T}\sqcup\mathbb{T})\setminus(\mathcal{E}_{k+1}\sqcup\mathcal{E}_k)}) \simeq S\mathbb{B}_{\mathcal{L}}$, where $\mathbb{B}_{\mathcal{L}}$ is a finite-dimensional C^*-algebra generated by matrix units $e_{I,J}$, $I, J \in \mathcal{I}_{k+1} \sqcup \mathcal{I}_k$, subject to relations

- $\varphi^{k+1}(I) = \varphi^{k+1}(J)$ and $\mathrm{val}(\varphi^{k+1}, I) = \mathrm{val}(\varphi^{k+1}, J)$ when $I, J \in \mathcal{I}_{k+1}$.
- $\varphi^k(I) = \varphi^k(J)$ and $\mathrm{val}(\varphi^k, I) = \mathrm{val}(\varphi^k, J)$ when $I, J \in \mathcal{I}_k$.
- $\varphi^{k+1}(I) = \varphi^k(J)$ and $\mathrm{val}(\varphi^{k+1}, I) = \mathrm{val}(\varphi^k, J)$ when $I \in \mathcal{I}_{k+1}, J \in \mathcal{I}_k$.
- $\varphi^k(I) = \varphi^{k+1}(J)$ and $\mathrm{val}(\varphi^k, I) = \mathrm{val}(\varphi^{k+1}, J)$ when $I \in \mathcal{I}_k, J \in \mathcal{I}_{k+1}$.

As above, any matrix unit $e_{I,J}$ in $\mathbb{B}_k^+$ or $\mathbb{B}_{k+1}^+$ gives rise to a matrix unit in $\mathbb{B}_{\mathcal{L}}$, and the full matrix summands in $\mathbb{B}_k^+$, $\mathbb{B}_{k+1}^+$ and $\mathbb{B}_{\mathcal{L}}$ are all indexed by the set $\mathcal{I}_k(\pm)$. Pictorially, we can think of the inclusions of $\mathbb{B}_k^+$ and $\mathbb{B}_{k+1}^+$ into $\mathbb{B}_{\mathcal{L}}$ as

$$\mathbb{B}_{\mathcal{L}} = \begin{bmatrix} \mathbb{B}_k^+ & * \\ * & \mathbb{B}_{k+1}^+ \end{bmatrix} \tag{5.13}$$

It is straightforward to see that we get a commutative diagram of extensions:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S\mathbb{B}_{k+1}^+ & \longrightarrow & C_r^*\left(R_\varphi^+(k+1)\right) & \longrightarrow & \mathbb{A}_{k+1}^+ & \longrightarrow & 0 \\ & & \downarrow & & \downarrow a & & \downarrow & & \\ 0 & \longrightarrow & S\mathbb{B}_{\mathcal{L}} & \longrightarrow & C_r^*(\mathcal{L}) & \longrightarrow & \mathbb{A}_{\mathcal{L}} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow b & & \uparrow & & \\ 0 & \longrightarrow & S\mathbb{B}_k^+ & \longrightarrow & C_r^*\left(R_\varphi^+(k)\right) & \longrightarrow & \mathbb{A}_k^+ & \longrightarrow & 0. \end{array} \tag{5.14}$$

where every square commutes. By Lemma 3.10 there are isomorphisms

$$K_0(C_r^*(R_\varphi^+(j))) \simeq \ker((I_j^+)_0 - (U_j^+)_0), \quad K_1(C_r^*(R_\varphi^+(j))) \simeq \mathrm{coker}((I_j^+)_0 - (U_j^+)_0)$$

for $j = k, k+1$. From the diagram above, it follows that $b_0^{-1} \circ a_0$ and $b_1^{-1} \circ a_1$ are realised as homomorphisms

$$b_0^{-1} \circ a_0 : \ker((I_{k+1}^+)_0 - (U_{k+1}^+)_0) \to \ker((I_k^+)_0 - (U_k^+)_0)$$

and

$$b_1^{-1} \circ a_1 : \mathrm{coker}((I_{k+1}^+)_1 - (U_{k+1}^+)_1) \to \mathrm{coker}((I_k^+)_1 - (U_k^+)_1)$$

From the discussion above the diagram, it follows that

$$K_0(\mathbb{B}_k) \simeq K_0(\mathbb{B}_{k+1}) \simeq K_0(\mathbb{B}_{\mathcal{L}}) \simeq \mathbb{Z}^{\mathcal{I}(\pm)}$$

and

$$K_0(\mathbb{A}_k) \simeq K_0(\mathbb{A}_{k+1}) \simeq K_0(\mathbb{A}_{\mathcal{L}}) \simeq \mathbb{Z}^{\mathcal{D}(\pm)}$$

and that under these isomorphisms, the maps $b_0^{-1} \circ a_0$ and $b_1^{-1} \circ a_1$ becomes identities. Recalling Formula 5.8, the final task in determining the maps $[E]_*$ is then to give a description of the induced inclusion maps $\rho_* : K_*(C_r^*(R_\varphi^+(k))) \to K_*(C_r^*(R_\varphi^+(k+1)))$.

Recall Lemma 3.9 from Chapter 3: For any j, the algebra $C_r^*(R_\varphi^+(j))$ is isomorphic to the algebra $\mathbb{D}_j$ given as

$$\mathbb{D}_j = \left\{(a,f) \in \mathbb{A}_j \oplus C([0,1],\mathbb{B}_j) \mid I_k^+(a) = f(0), U_k^+(a) = f(1)\right\}$$

via an isomorphism $\mu_j : C_r^*(R_\varphi^+(j)) \to \mathbb{D}_j$. It follows that there is a unique $*$-homomorphism $\Phi : \mathbb{D}_k \to \mathbb{D}_{k+1}$ making the diagram

$$\begin{array}{ccc} C_r^*(R_\varphi^+(k)) & \xrightarrow{\mu_k} & \mathbb{D}_k \\ \rho \downarrow & & \downarrow \Phi \\ C_r^*(R_\varphi^+(k+1)) & \xrightarrow{\mu_{k+1}} & \mathbb{D}_{k+1} \end{array} \tag{5.15}$$

commute. Write $\Phi = (\Phi_1, \Phi_2)$ with $\Phi_1 : \mathbb{D}_k \to \mathbb{A}_{k+1}$ and $\Phi_2 : \mathbb{D}_k \to C([0,1],\mathbb{B}_{k+1})$. We can write

$$\Phi_1(a,f) = \chi(a) + \mu(f)$$

where $\chi : \mathbb{A}_k \to \mathbb{A}_{k+1}$ and $\mu : C([0,1],\mathbb{B}_k) \to \mathbb{A}_{k+1}$ are $*$-homomorphisms. Let's describe these maps: Let $f \in C_c(R_\varphi^+(k))$, and let $a_j : C_r^*(R_\varphi^+(j)) \to \mathbb{A}_j$ and $b_j : C_r^*(R_\varphi^+(j)) \to C([0,1],\mathbb{B}_j)$ (for $j = k, k+1$) be the maps introduced in Chapter 3. For Diagram (5.15) to be commutative, we must have

$$a_{k+1}(\rho(f)) = \Phi_1((a_k(f), b_k(f)) = \chi(a_k(f)) + \mu(b_k(f))$$

We know that

$$a_{k+1}(\rho(f)) = \sum_{(x,y)\in A} f(x,y)e_{x,y}$$

with $A = \{(x,y) \in \varphi^{-k}(\mathcal{D}) | \varphi^{k+1}(x) = \varphi^{k+1}(y)\}$. There are two types of matrix units in the sum above: those arising from pairs (x,y) where $\varphi^k(x) = \varphi^k(y) \in \mathcal{D}$, and those arising from pairs where $\varphi^k(x) = \varphi^k(y) \notin \mathcal{D}$. The first kind can be obtained as the image of matrix units in $\mathbb{A}_k$, while the second kind comes from evaluating an element of $C([0,1],\mathbb{B}_k)$ at the right places. More precisely: First, define

$$\chi(e_{x,y}) = e_{x,y}$$

whenever $x, y \in \varphi^{-k}(\mathcal{D})$ with $\varphi^k(x) = \varphi^k(y)$. Second, let $(I,J) \in \mathcal{I}_{k+1}^{(2)}$ and let

$$N_{I,J} = \{(x,y) \in \varphi^{-k-1}(\mathcal{D})^2 | x \in I, y \in J, \varphi^k(x) = \varphi^k(y) \notin \mathcal{D}\}$$

Put

$$\mu(g \otimes e_{I,J}) = \sum_{(x,y)\in N_{I,J}} g\left(\psi_{\varphi^k(I)}^{-1}(\varphi^k(x))\right) e_{x,y} \tag{5.16}$$

for $g \in C([0,1])$. Then

$$\chi(a_k(f)) + \mu(b_k(f)) = \sum_{\varphi^k(x)=\varphi^k(y)\in\mathcal{D}} f(x,y)e_{x,y} + \sum_{(I,J)\in\mathcal{I}_k^{(2)}} \mu(f_{I,J} \otimes e_{I,J})$$

$$
\begin{aligned}
&= \sum_{\varphi^k(x)=\varphi^k(y)\in\mathcal{D}} f(x,y)e_{x,y} + \sum_{(I,J)\in\mathcal{I}_k^{(2)}} \left(\sum_{(x,y)\in N_{I,J}} f_{I,J}(\psi^{-1}_{\varphi^k(I)}(\varphi^k(x)))e_{x,y} \right) \\
&= \sum_{\varphi^k(x)=\varphi^k(y)\in\mathcal{D}} f(x,y)e_{x,y} + \sum_{(I,J)\in\mathcal{I}_k^{(2)}} \left(\sum_{(x,y)\in N_{I,J}} f(x,y)e_{x,y} \right) \\
&= a_{k+1}(\rho(f))
\end{aligned}
$$

as we wanted. We note a special case of the formula 5.16: When $I = J$ and $g \equiv 1$, we have

$$N_{I,I} = \{(x,x)|x \in I \cap \varphi^{-k-1}(\mathcal{D})\}.$$

since φ^k is one-to-one on I. Then

$$\mu(e_{I,I}) = \sum_{(x,x)\in N_{I,I}} e_{x,x}. \tag{5.17}$$

We also note that the ranges of χ and μ are orthogonal. Next, let $h_s : C([0,1],\mathbb{B}_k) \to C([0,1],\mathbb{B}_k)$ be the family of maps given by $h_s(f)(t) = h(st)$ for $s \in [0,1]$, and define $H_s : \mathbb{D}_k \to \mathbb{A}_{k+1}$ by

$$H_s((a,f)) = \chi(a) + \mu(h_s(f)).$$

Then

$$H_0((a,f)) = \chi(a) + \mu(f(0)) = (\chi + \mu \circ I_k)(a)$$

and

$$H_1((a,f)) = \chi(a) + \mu(f) = \Phi_1((a,f))$$

which shows that the diagram

$$
\begin{array}{ccc}
\mathbb{D}_k & \xrightarrow{p_k} & \mathbb{A}_k \\
\Phi \big\downarrow & & \big\downarrow \chi+\mu\circ I_k \\
\mathbb{D}_{k+1} & \xrightarrow{p_{k+1}} & \mathbb{A}_{k+1}
\end{array}
\tag{5.18}
$$

commutes op to homotopy. It follows that

$$\rho_0 = \chi_0 + \mu_0 \circ (I_k)_0 \tag{5.19}$$

on $K_0(\mathbb{D}_k) \simeq \ker((I_k)_0 - (U_k)_0)$. This map has a very concrete description, which we proceed to give here. Recall that $K_0(\mathbb{A}_k) \simeq \mathbb{Z}^{\mathcal{D}(\pm)}$, so a basis of $K_0(\mathbb{A}_k)$ is given by elements $[d,v]$ for $(d,v) \in \mathcal{D}(\pm)$. For each basis element, we should do the following: Choose an $x \in \varphi^{-k}(d)$ with $\mathrm{val}(\varphi^k, x) = v$, calculate $(\chi + \mu \circ I_k)(e_{x,x})$, and express the K_0-class of this element in terms of the basis elements. Consider, for instance, a $d \in \mathcal{D}$ and the corresponding basis element $[d,(-,+)]$. Let $x \in \mathbb{T}$ with $\varphi^k(x) = d$ and $\mathrm{val}(\varphi^k, x) = (-,+)$. Let I and J denote the elements of $\mathcal{I}_k$ immediately to the left and right of x, and note that $\varphi^k(I) = \varphi^k(J) = I_d$, where I_d is the element of $\mathcal{I}$ above d. Then

$$(\chi + \mu \circ I_k)(e_{x,x}) = e_{x,x} + \sum_{(y,y)\in N_{I,I}} e_{y,y} + \sum_{(y,y)\in N_{J,J}} e_{y,y}$$

Each of the matrix units in the sum above determines a basis element $[d', v']$ in $\mathbb{Z}^{\mathcal{D}}(\pm)$, determined by $d' = \varphi^{k+1}(y)$ and $v' = \mathrm{val}(\varphi^{k+1}, y)$. Since φ^k is injective on I and J and $\varphi^k(I) = \varphi^k(J) = I_i$ there is a bijection between the sets $N_{I,I}$, $N_{J,J}$ and the set

$$I_i \cap \varphi^{-1}(\mathcal{D}) = \left\{ z \in I_i \,\middle|\, \varphi^k(z) \in \mathcal{D} \right\}.$$

Hence, if $z = \varphi^k(y)$, we have $\mathrm{val}(\varphi^{k+1}, y) = \mathrm{val}(\varphi, z) \bullet \mathrm{val}(\varphi^k, y)$, and $\mathrm{val}(\varphi^k, y)$ is $(-,-)$ when $y \in I$ and $(+,+)$ when $y \in J$. It follows that we have

$$\begin{aligned}(\chi_0 + \mu_0 \circ (I_k)_0)([d,(-,+)]) =&[\varphi(d), \mathrm{val}(\varphi, d)] + \\ &\sum_{z \in I_d \cap \varphi^{-1}(\mathcal{D})} ([\varphi(z), \mathrm{val}(\varphi, z) \bullet (+,+)] + [\varphi(z), \mathrm{val}(\varphi, z) \bullet (-,-)])\end{aligned}$$

For brevity, denote the map $(\chi_0 + \mu_0 \circ (I_k)_0)$ by A. Then, by arguments analogous to those above, we see that A is given as follows:

$$A[d,v] = [\varphi(d), \mathrm{val}\,(\varphi, d)]$$

when $v = (+,-)$,

$$\begin{aligned}A[d,v] =& [\varphi(d), \mathrm{val}\,(\varphi, d)] + \\ &\sum_{z \in I_d^+ \cap \varphi^{-1}(\mathcal{D})} [\varphi(z), \mathrm{val}(\varphi, z) \bullet (+,+)] + [\varphi(z), \mathrm{val}(\varphi, z) \bullet (-,-)])\end{aligned}$$

when $v = (-,+)$,

$$\begin{aligned}A[d,v] =& [\varphi(d), \mathrm{val}\,(\varphi, d)] + \\ &\sum_{z \in I_d^+ \cap \varphi^{-1}(\mathcal{D})} [\varphi(z), \mathrm{val}\,(\varphi, z) \bullet (+,+)]\end{aligned}$$

when $v = (+,+)$ and finally

$$\begin{aligned}A[d,v] =& [\varphi(d), \mathrm{val}\,(\varphi, d)] + \\ &\sum_{z \in I_d^+ \cap \varphi^{-1}(\mathcal{D})} [\varphi(z), \mathrm{val}\,(\varphi, z) \bullet (-,-)]\end{aligned}$$

when $v = (-,-)$. Let $\tilde{A}$ denote the restriction of A to $\ker((I_k)_0 - (U_k)_0)$ – and don't worry, we'll do an example shortly. First, we need to determine the map $\rho_1 : K_1(C_r^*(R_\varphi^+(k))) \to K_1(C_r^*(R_\varphi^+(k+1)))$. Here's the general idea: Let $i_j : S\mathbb{B}_j \to C_r^*(R_\varphi^+(j))$ be the inclusion (for $j = k, k+1$) from Extension . The induced maps $(i_j)_1 : K_1(S\mathbb{B}_j) \to K_1(C_r^*(R_\varphi^+(j))$ are surjective since $K_1(\mathbb{A}_j) = 0$, and the groups $K_1(S\mathbb{B}_j)$ are free, so there is a homomorphism $B : K_1(S\mathbb{B}_k) \to K_1(S\mathbb{B}_{k+1})$ making the diagram

$$\begin{array}{ccc} K_1(S\mathbb{B}_k) & \xrightarrow{(i_k)_1} & K_1(C_r^*(R_\varphi^+(k))) \\ \Big\downarrow B & & \Big\downarrow \rho_1 \\ K_1(S\mathbb{B}_{k+1}) & \xrightarrow{(i_{k+1})_1} & K_1(C_r^*(R_\varphi^+(k+1))) \end{array} \tag{5.20}$$

commute. One way to describe B is as follows: We have $K_1(S\mathbb{B}_k) \simeq K_0(\mathbb{B}_k) \simeq \mathbb{Z}^{\mathcal{I}(\pm)}$, and the basis elements of $K_1(S\mathbb{B}_k)$ are in one-to-one-correspondence with the set $\mathcal{I}(\pm)$. Let $J \in \mathcal{I}$ and

$v \in \{(+,+),(-,-)\}$. Choose a $I \in \mathcal{I}_k$ with $\varphi^k(I) = J$ and $\mathrm{val}(\varphi^k, I) = v$, and let $u : I \to \mathbb{T} - 1$ be a continuous path of winding number 1. Then

$$(u \circ \lambda_I \circ \psi_J) \otimes e_{I,I}$$

is an element of $S\mathbb{B}_k$ representing $(-1)^v[J,v]$ in $K_1(S\mathbb{B}_k)$. Choose a subinterval I_1 of I such that $I_1 \in \mathcal{I}_{k+1}$. Deform u to a map $u' : I \to \mathbb{T} - 1$ which wraps once round $\mathbb{T} - 1$ while traversing I_1, and is zero elsewhere. Then

$$(u' \circ \lambda_I \circ \psi_J) \otimes e_{I,I}$$

represents the same element $(-1)^v[J,v]$ in $K_1(S\mathbb{B}_k)$ (by homotopy invariance). Let $J_1 = \varphi^{k+1}(I_1) \in \mathcal{D}$. Using the inclusion $\rho : C_r^*(R_\varphi(k)) \to C_r^*(R_\varphi(k+1))$ on u, we obtain an element

$$(u' \circ \lambda_{I_1} \circ \psi_{J_1}) \otimes e_{I_1,I_1} \in S\mathbb{B}_{k+1}$$

representing $(-1)^v(-1)^w[\varphi(J_1), v \bullet w]$ in $K_1(S\mathbb{B}_{k+1})$ (where $w = \mathrm{val}(\varphi, J)$). This yields an algorithm for determining B: For each $J \in \mathcal{I}$ and $v \in \{(+,+),(-,-)\}$ choose a subinterval $J' \subseteq J$ such that J' is a connected component of $\varphi^{-1}(\mathcal{D}) \cap J$. Then $\varphi(J) \in \mathcal{I}$ and

$$B[J,v] = (-1)^{\mathrm{val}(\varphi,J')}[\varphi(J'), \mathrm{val}(\varphi,J') \bullet v] \tag{5.21}$$

Then B is a homomorphism between $K_0(\mathbb{B}_k)$ and $K_0(\mathbb{B}_{k+1})$, descending to a homomorphism $\tilde{B}$ between $\mathrm{coker}((I_k)_0 - (U_k)_0)$ and $\mathrm{coker}((I_{k+1})_0 - (U_{k+1})_0)$ which is equal to the induced inclusion map ρ_1.

It's time to put all the pieces together: We have obtained two endomorphisms

$$\tilde{A} : \ker((I_k)_0 - (U_k)_0) \to \ker((I_{k+1})_0 - (U_{k+1})_0)$$
$$\tilde{B} : \mathrm{coker}((I_k)_0 - (U_k)_0) \to \mathrm{coker}((I_{k+1})_0 - (U_{k+1})_0)$$

determining the K-theory of $C_r^*(\Gamma_\varphi^+)$ in the sense that there are extensions

$$0 \longrightarrow \mathrm{coker}\,(1-\tilde{A}) \longrightarrow K_0(C_r^*(\Gamma_\varphi^+)) \longrightarrow \ker\,(1-\tilde{B}) \longrightarrow 0 \tag{5.22}$$

and

$$0 \longrightarrow \mathrm{coker}\,(1-\tilde{B}) \longrightarrow K_1(C_r^*(\Gamma_\varphi^+)) \longrightarrow \ker\,(1-\tilde{A}) \longrightarrow 0. \tag{5.23}$$

Note that the last extension is always split and hence

$$K_1(C_r^*(\Gamma_\varphi^+)) \simeq \mathrm{coker}\,(1-\tilde{B}) \oplus \ker\,(1-\tilde{A})\,.$$

To identify the C^*-algebra from its K-theory groups it is important to know which element of $K_0(C_r^*(\Gamma_\varphi^+))$ represents the unit 1 of $C_r^*(\Gamma_\varphi^+)$. Note therefore that $[1] \in K_0(C_r^*(\Gamma_\varphi^+))$ is the image of $[1] \in K_0(C_r^*(R_\varphi^+(k)))$ under the map ι_0 in (5.3). Under the identification $K_0(\mathbb{A}_k) = \mathbb{Z}^{\mathcal{D}(\pm)}$ we have that

$$[1] = \sum_{(d,v)\in\mathcal{D}(\pm)} m(d,v)[d,v]$$

in $K_0(\mathbb{A}_k)$, where $m(d,v) = \left\{x \in \varphi^{-k}(d) \,\middle|\, \mathrm{val}(\varphi^k, x) = v\right\}$. Since I_k and U_k are unital $*$-homomorphisms this element is always in $\ker((I_k)_0 - (U_k)_0)$ and gives therefore rise to an element of $\mathrm{coker}\,(1-\tilde{A})$ which under the embedding $\mathrm{coker}\,(1-\tilde{A}) \subseteq K_0(C_r^*(\Gamma_\varphi^+))$ from (5.22) gives us the element representing $[1] \in K_0(C_r^*(\Gamma_\varphi^+))$.

Example 5.25. We illustrate the algorithm above with a simple example. Let φ be the tent map introduced in Example , i.e. the map with lift $\varphi : [0,1] \to \mathbb{R}$ given by

$$\varphi(t) = \begin{cases} 2t \text{ for } t \in [0,1/2], \\ 2-2t \text{ for } t \in [1/2,1] \end{cases} \tag{5.24}$$

This map is critically finite, with critical points $\mathcal{C} = \{0, 1/2\}$. We may put $\mathcal{D} = \{0\}$ and $\mathcal{I} = \{(0,1)\}$. Since

$$\mathcal{D}_1(\pm) = \{[0,(-,+)],[0,(+,-)],[0,(+,+)],[0,(-,-)]\}$$

and

$$\mathcal{I}_1(\pm) = \{[I,(+,+)],[I,(-,-)]\}$$

the groups $K_0(\mathbb{A}_1^+) \simeq \mathbb{Z}^4$ and $K_0(\mathbb{B}_1^+) = \mathbb{Z}^2$ stabilise from the first step. The maps I and U are given by the table

$[d,v]$	$[0,(-,+)]$	$[0,(+,-)]$	$[0,(+,+)]$	$[0,(-,-)]$
$I[d,v]$	$[I,(+,+)]+[I,(-,-)]$	0	$[I,(+,+)]$	$[I,(-,-)]$
$U[d,v]$	0	$[I,(+,+)]+[I,(-,-)]$	$[I,(+,+)]$	$[I,(-,-)]$

from which it follows that the map $I - U : \mathbb{Z}^4 \to \mathbb{Z}^2$ has the matrix representation

$$I - U = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \tag{5.25}$$

In particular,

$$K_0(C_r^*(R_\varphi(1))) \simeq \ker(I-U) \simeq \left\{(x,x,y,z) \in \mathbb{Z}^4 \,\middle|\, x,y,z \in \mathbb{Z}\right\} \simeq \mathbb{Z}^3$$

and

$$K_1(C_r^*(R_\varphi(1))) \simeq \operatorname{coker}(I-U) \simeq \mathbb{Z}.$$

The map $A : \mathbb{Z}^{\mathcal{D}(\pm)} \to \mathbb{Z}^{\mathcal{D}(\pm)}$ is easy to calculate: There is only one interval $I = (0,1)$ in $\mathcal{I}$, $I \cap \varphi^{-1}(\mathcal{D}) = \{1/4, 1/2, 3/4\}$, and these points have valency $(+,+),(+,-)$ and $(-,-)$ respectively. Finally, we note that $\varphi(0) = 0$ and $\operatorname{val}(\varphi, 0) = (-,+)$. It follows that, for instance,

$$A[0,(+,+)] = [0,(-,+)] + [0,(+,+)] + [0,(+,-)] + [0,(-,-)].$$

Doing the same computations on the other basis elements yields a matrix representation of A and its restriction $\tilde{A}$ to $\ker(I-U)$:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}, \qquad \tilde{A} = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}, \qquad \tilde{A} - 1 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}. \tag{5.26}$$

A few row and column operations show that $\ker(\tilde{A}-1)$ is trivial and that $\operatorname{coker}(\tilde{A}-1) \simeq \mathbb{Z}_3$. Next, we determine the map $B : \mathbb{Z}^{\mathcal{I}(\pm)} \to \mathbb{Z}^{\mathcal{I}(\pm)}$. Following the description above, consider for instance the interval $J = (0,1/4)$. We have $\varphi(J) = I$ and $\operatorname{val}(\varphi, J) = (+,+)$. The subinterval $(0,1/16)$ of J is in $\mathcal{I}_2$ (since $\varphi^2((0,1/16) = I)$ and $\operatorname{val}(\varphi,(0,1/16)) = (+,+)$. Hence

$B[I,(+,+)] = [I,(+,+)]$. Similarly, $B[I,(-,-)] = B[I,(-,-)]$. It follows that B, hence also the quotient map $\tilde{B}$, is the identity, so $\tilde{B}-1$ as a map from $\mathbb{Z}$ to $\mathbb{Z}$ is the zero map. In particular

$$\ker(1-\tilde{B}) = \operatorname{coker}(1-\tilde{B}) \simeq \mathbb{Z}.$$

From 5.22 and 5.23, there are extensions

$$0 \longrightarrow \mathbb{Z}_3 \longrightarrow K_0\left(C_r^*\left(\Gamma_\varphi^+\right)\right) \longrightarrow \mathbb{Z} \longrightarrow 0, \tag{5.27}$$

and

$$0 \longrightarrow \mathbb{Z} \longrightarrow K_1\left(C_r^*\left(\Gamma_\varphi^+\right)\right) \longrightarrow 0 \longrightarrow 0. \tag{5.28}$$

We conclude that

$$K_0(C_r^*(\Gamma_\varphi^+)) \simeq \mathbb{Z}\oplus\mathbb{Z}_3, \;\; K_1(C_r^*(\Gamma_\varphi^+)) \simeq \mathbb{Z}.$$ ▲

In the appendix, we calculate the K-theory of a larger class of critically finite maps.

5.5 On $K_*(C_r^*(\Gamma_\varphi))$

The previous section gave an algorithm for calculating the K-theory of $C_r^*(\Gamma_\varphi^+)$ by looking at a graph of φ and doing some linear algebra. We can do the same for $C_r^*(\Gamma_\varphi)$, with a completely similar approach. Some aspects are easier – due to the fact that $K_1(C_r^*(R_\varphi(j))) = 0$ – while others are more technical (as always, due to the presence of $\mathbb{Z}_2$-isotropy in Γ_φ).

As above, we put

$$\mathcal{D}_0 = \bigcup_{k=0}^{\infty} \varphi^k(\mathcal{C}), \;\; \mathcal{D} = \varphi(\mathcal{D}_0)$$

and $\mathcal{E}_j = \varphi^{-j}(\mathcal{D})$. Then $\mathcal{E}_j$ is $R_\varphi(j)$-invariant (for $j = k, k+1$). Furthermore, if $\mathcal{L}$ is the linking groupoid of $R_\varphi(k+1)$ and $R_\varphi^+(k)$, the set $\mathcal{E}_{k+1}\sqcup\mathcal{E}_k$ is a $\mathcal{L}$-invariant subset of $\mathbb{T}\sqcup\mathbb{T}$. As in the previous section, we obtain an extension

$$0 \longrightarrow C_r^*(\mathcal{L}|_{(\mathbb{T}\sqcup\mathbb{T})\setminus(\mathcal{E}_{k+1}\sqcup\mathcal{E}_k)}) \xrightarrow{\;i\;} C_r^*(\mathcal{L}) \xrightarrow{\;a\;} C_r^*(\mathcal{L}|_{\mathcal{E}_{k+1}\sqcup\mathcal{E}_k}) \longrightarrow 0. \tag{5.29}$$

Let $\mathcal{I}$ be the connected components of $\mathbb{T}\setminus\mathcal{D}$, and $\mathcal{I}_j$ the connected components of $\mathbb{T}\setminus\mathcal{E}_j$. We let $\mathbb{B}_\mathcal{L}$ be the C^*-algebra generated by matrix units $e_{I,J}$ with $I,J\in\mathcal{I}_k\cup\mathcal{I}_{k+1}$, subject to the conditions that $\varphi^j(I) = \varphi^{j'}(J)$ for $j,j'\in\{k,k+1\}$. Then $\mathbb{B}_\mathcal{L}$ is finite-dimensional with summands in one-to-one-correspondence with set $\mathcal{I}$, and there is an isomorphism

$$S\mathbb{B}_\mathcal{L} \simeq C_r^*(\mathcal{L}|_{(\mathbb{T}\sqcup\mathbb{T})\setminus(\varphi^{-k-1}(\mathcal{D})\sqcup\varphi^{-k}(\mathcal{D})})$$

There are embeddings $\mathbb{B}_k\to\mathbb{B}_\mathcal{L}$ and $\mathbb{B}_{k+1}\to\mathbb{B}_\mathcal{L}$ into orthogonal corners:

$$\mathbb{B}_\mathcal{L} = \begin{bmatrix} \mathbb{B}_k & * \\ * & \mathbb{B}_{k+1}\end{bmatrix}. \tag{5.30}$$

Next, since $\mathcal{L}|_{\mathcal{E}_{k+1}\sqcup\mathcal{E}_k}$ is finite, the algebra $\mathbb{A}_\mathcal{L} = C_r^*(\mathcal{L}|_{\mathcal{E}_{k+1}\sqcup\mathcal{E}_k})$ is finite-dimensional, and the summands are in one-to-one-correspondence with those of $\mathbb{A}_k$ and $\mathbb{A}_{k+1}$. These algebras embed as orthogonal corners in $\mathbb{A}_\mathcal{L}$, and we have

$$K_0(\mathbb{A}_k) \simeq K_0(\mathbb{A}_{k+1}) \simeq K_0(\mathbb{A}_\mathbb{L})$$

We obtain a commutative diagram of extensions identical to (5.14), from which we see that the map

$$b_0^{-1} \circ a_0 : \ker((I_{k+1})_0 - (U_{k+1})_0) \to \ker((I_k)_0 - (U_k)_0)$$

is an identity. Furthermore, we recall from Lemma 3.14 that $K_1(C_r^*(R_\varphi(j)))$ is zero, which in particular means that the map

$$b_1^{-1} \circ a_1 : \operatorname{coker}((I_{k+1})_0 - (U_{k+1})_0) \to \operatorname{coker}((I_k)_0 - (U_k)_0)$$

is zero. From Corollary 5.18, it follows that

$$K_0(C_r^*(\Gamma_\varphi)) \simeq \operatorname{coker}(1-\rho_0), \quad K_1(C_r^*(\Gamma_\varphi)) \simeq \ker(1-\rho_0).$$

It follows that we need to determine the map

$$\rho_0 : K_0(C_r^*(R_\varphi(k))) \to K_0(C_r^*(R_\varphi(k+1)))$$

induced by the inclusion $\rho : C_r^*(R_\varphi(k)) \to C_r^*(R_\varphi(k+1))$. The setup is essentially the same as in the case of $C_r^*(R_\varphi^+(k))$: Since the maps μ_j, $j = k, k+1$ are isomorphisms, there is a $*$-homomorphism $\Phi : \mathbb{D}_k \to \mathbb{D}_{k+1}$ making the diagram

$$\begin{array}{ccc} C_r^*(R_\varphi^+(k)) & \xrightarrow{\mu_k} & \mathbb{D}_k \\ \rho \downarrow & & \downarrow \Phi \\ C_r^*(R_\varphi^+(k+1)) & \xrightarrow{\mu_{k+1}} & \mathbb{D}_{k+1} \end{array} \tag{5.31}$$

commute. Writing $\Phi = (\Phi_1, \Phi_2)$ with $\Phi_1 : \mathbb{D}_k \to \mathbb{A}_{k+1}$ and $\Phi_2 = \mathbb{D}_k \to C([0,1], \mathbb{B}_{k+1})$, we need to describe Φ_1 and Φ_2. And this is where things get nasty: In the previous section, we had a bijection between summands of $\mathbb{A}_k^+$ and elements of the set $\mathcal{D}_k(\pm)$, which in turn gave an isomorphism $K_0(\mathbb{A}_k^+) \simeq \mathbb{Z}^{\mathcal{D}(\pm)}$ making it fairly straightforward to describe the induced map of a $*$-homomorphism from $\mathbb{A}_k^+$ on K-theory. For $C_r^*(R_\varphi(k))$ and $\mathbb{A}_k$, things are less clear. Now, we may write $\Phi_1 : \mathbb{D}_k \to \mathbb{A}_{k+1}$ may be written as $\chi + \mu$, with $\chi : \mathbb{A}_k \to \mathbb{A}_{k+1}$ and $\mu : C([0,1], \mathbb{B}_k) \to \mathbb{A}_{k+1}$. To define χ, let $(x,p,y) \in R_\varphi(k)$, and put

$$\chi(1_{(x,p,y)}) = 1_{(x,p,y)}$$

This makes sense, since $(x,p,y) \in R_\varphi(k+1)$. Next, let $(I,J) \in \mathcal{I}_k^{(2)}$, and put $p = p(I,J) = +$ if $\operatorname{val}(\varphi^k, I) = \operatorname{val}(\varphi^k, J)$ and $p(I,J) = -$ otherwise. Define

$$N_{I,J} = \left\{ (x,p,y) \,\middle|\, x \in I, y \in J, \varphi^k(x) = \varphi^k(y) \notin \mathcal{D}, \varphi^{k+1}(x) \in \mathcal{D} \right\}$$

and

$$\mu(f \otimes e_{I,J}) = \sum_{(x,p,y) \in N_{I,J}} f(\psi_{\varphi^k(I)}^{-1}(\varphi^k(x))) 1_{(x,p,y)}$$

χ and μ are seen to have orthogonal ranges. We won't write down an explicit description of Φ_2 – as we shall see, we will not need it. Indeed, we note that diagram

$$\begin{array}{ccc} \mathbb{D}_k & \xrightarrow{p_k} & \mathbb{A}_k \\ \Phi \downarrow & & \downarrow \chi + \mu \circ I_k \\ \mathbb{D}_{k+1} & \xrightarrow{p_{k+1}} & \mathbb{A}_{k+1} \end{array} \tag{5.32}$$

commutes op to homotopy, hence $\rho_0 = \chi_0 + \mu_0 \circ (I_k)_0$ on $K_0(C^*_r(R_\varphi(k))) \simeq \ker((I_k)_0 - (U_k)_0)$. To give an explicit formula for ρ_0, we begin by giving a 'canonical' basis of $\mathbb{A}_k$:

Lemma 5.26. *Let $c_i \in \mathcal{D}$. As far as such elements exist, choose $x_i^{\min}$, $x_i^{\max}$ and $x_i^r \in \mathbb{T}$ such that $\varphi^k(x_i^{\min}) = \varphi^k(x_i^{\max}) = \varphi^k(x_i^r) = c_i$ and*

$$\mathrm{val}(\varphi^k, x_i^{\min}) = (-,+),\ \ \mathrm{val}(\varphi^k, x_i^{\max}) = (+,-),\ \ \mathrm{val}(\varphi^k, x_i^r) = (\pm,\pm)$$

Define projections in $\mathbb{A}_k$ by

$$\begin{aligned} E^{(-,+)}_{c_i,+} &= \tfrac{1}{2}\left(e_{x_i^{\min},+,x_i^{\min}} + e_{x_i^{\min},-,x_i^{\min}}\right), E^{(-,+)}_{c_i,-} = \tfrac{1}{2}\left(e_{x_i^{\min},+,x_i^{\min}} - e_{x_i^{\min},-,x_i^{\min}}\right) \\ E^{(+,-)}_{c_i,+} &= \tfrac{1}{2}\left(e_{x_i^{\max},+,x_i^{\max}} + e_{x_i^{\max},-,x_i^{\max}}\right), E^{(+,-)}_{c_i,-} = \tfrac{1}{2}\left(e_{x_i^{\max},+,x_i^{\max}} - e_{x_i^{\max},-,x_i^{\max}}\right) \\ E^r_{c_i} &= e_{x^r,+,x^r} \end{aligned}$$

Then $K_0(\mathbb{A}_k) \subset \mathbb{Z}^M$ for some $M \leq 5|\mathcal{D}|$, and the K-theory classes of the E_{c_i}'s correspond to the standard basis of $\mathbb{Z}^M$.

Proof. This is an immediate consequence of Examples 1.24 and 1.26. □

Note the identities

$$E^{(-,+)}_{c_i,+} + E^{(-,+)}_{c_i,-} = e_{x_i^{\min},+,x_i^{\min}},\ \ E^{(-,+)}_{c_i,+} - E^{(-,+)}_{c_i,-} = e_{x_i^{\min},-,x_i^{\min}}$$

and similarly for $e_{x_i^{\max},\pm,x_i^{\max}}$. To describe the action of ρ_0 on these basis elements, we introduce the following notation: For $c_i \in \mathcal{D}$, let I_i^+ denote the interval in $\mathcal{I}$ such that $c_i < I_i^+$, and I_i^- the interval in $\mathcal{I}$ such that $c_i > I_i^-$. Assume now, for instance, that $\varphi^k(x) = c_i$ and $\mathrm{val}(\varphi^k, x) = (-,+)$. Then both L_x and R_x, the intervals in $\mathcal{I}_k$ to the right and left of x, are mapped homeomorphically to I_i^+. Hence, the projections $e_{L_x,L_x}, e_{R_x,R_x} \in \mathbb{B}_k$ represent the element $[I_i^+] \in K_0(\mathbb{B}_k)$. Similarly, if $\mathrm{val}(\varphi^k, x) = (+,-)$, e_{L_x,L_x} and e_{R_x,R_x} are represented by $[I_i^-]$. Now, recall that Lemma 5.26 gave basis elements for $K_0(\mathbb{A}_k)$. Combining this with the Table 5.5 for the map I_k show that

$$I_k(E^{(-,+)}_{c_i,+}) = \tfrac{1}{2}(e_{L_{x_i^{\max}},L_{x_i^{\max}}} + e_{R_{x_i^{\max}},x_i^{\max}} + e_{L_{x_i^{\max}},R_{x_i^{\max}}} + e_{R_{x_i^{\max}},L_{x_i^{\max}}})$$

and that

$$I_k(E^{(-,+)}_{c_i,-}) = \tfrac{1}{2}(e_{L_{x_i^{\max}},L_{x_i^{\max}}} + e_{R_{x_i^{\max}},e_{R_i^{\max}}}} - e_{L_{x_i^{\max}},R_{x_i^{\max}}} - e_{R_{x_i^{\max}},L_{x_i^{\max}}})$$

Both these elements correspond to $[I_i^+]$ in $K_0(\mathbb{B}_k)$. Now,

$$I_k(E^{(+,-)}_{c_i,\pm}) = 0$$

and $I_k(E^r_{c_i})$ is either $e_{L_{x_i^r},L_{x_i^r}}$ (if φ^k is decreasing at x_i^r) or $e_{R_{x_i^r},R_{x_i^r}}$ (if φ^k is increasing at x_i^r). This, too, represents the same K-theory element as $I_k(E^{(-,+)}_{c_i,+})$. Carrying out the same considerations about U_k yields the following table:
Next, we describe how $\mu_0 : K_0(\mathbb{B}_k) \to K_0(\mathbb{A}_{k+1})$ works on K-theory. For each $I_i \in \mathcal{I}$, choose an $I \in \mathcal{I}_k$ with $\varphi^k(I) = I_i$. Then,

$$\mu_0([I_i]) = [\mu(e_{I,I})] = \sum_{x \in N_{I,I}} [e_{x,+,x}],\ \ N_{I,I} = \left\{ x \in I \,\middle|\, \varphi^k(x) \in \varphi^{-1}(\mathcal{D}) \setminus \mathcal{D} \right\}.$$

	$[E^{(-,+)}_{c_i,+}]$	$[E^{(-,+)}_{c_i,-}]$	$[E^{(+,-)}_{c_i,+}]$	$[E^{(+,-)}_{c_i,+}]$	$[E^{r}_{c_i}]$
$(I_k)_0$	$[I_i^+]$	$[I_i^+]$	0	0	$[I_i^+]$
$(U_k)_0$	0	0	$[I_i^-]$	$[I_i^-]$	$[I_i^-]$

Table 5.1: Table of $(I_k)_0, (U_k)_0 : K_0(\mathbb{A}_k) \to K_0(\mathbb{B}_k)$.

Note that φ^k yields a bijection between the set $N_{I,J}$ and the set $\{z \in I_i \mid \varphi(z) \in \mathcal{D}\}$, and that $\operatorname{val}(\varphi^{k+1}, x) = \operatorname{val}(\varphi, \varphi^k(x)) \operatorname{val}(\varphi^k, x) \equiv \operatorname{val}(\varphi, \varphi^k(x))$. Taking all this into account gives the following formula:

$$(\mu \circ I_k)_0([E^{(-,+)}_{c_i,+}]) = \sum_{z \in I^+_{c_i} \cap \mathcal{C}} \left([E^{\operatorname{val}(\varphi,z)}_{\varphi(z),+}] + [E^{\operatorname{val}(\varphi,z)}_{\varphi(z),-}] \right) + \sum_{z \in I^+_{c_i} \cap \varphi^{-1}(\mathcal{D}) \setminus \mathcal{C}} [E^r_{\varphi(z)}]$$

By the considerations above, this is also the value of $(\mu \circ I_k)_0$ on $[E^{(-,+)}_{c_i,-}]$ and $[E^r_{c_i}]$, and the value on $[E^{(+,-)}_{c_i,+}]$ and $[E^{(+,-)}_{c_i,-}]$ is just 0. Recall that ρ_0 was equal to $\chi_0 + (\mu \circ I_k)_0$ and that $\chi(e_{x,v,y}) = e_{x,v,y} \in \mathbb{A}_{k+1}$. Note, however, that – unless φ is Markov – a $c_i \in \mathcal{D}$ is not necessarily critical for φ. Hence, a 'critical' basis element can be mapped to a 'non-critical' basis element, and vice versa. Phrasing this in terms of basis elements, we get

$$\chi_0([E^v_{c_i,p}]) = \begin{cases} [E^{\operatorname{val}(\varphi,c_i)}_{\varphi(c_i),p}] & \text{if } c_i \in \mathcal{C} \\ [E^r_{\varphi(c_i)}] & \text{if } c_i \notin \mathcal{C} \end{cases}, \qquad \chi_0([E^r_{c_i}]) = \begin{cases} [E^{\operatorname{val}(\varphi,c_i)}_{\varphi(c_i),+}] + [E^{\operatorname{val}(\varphi,c_i)}_{\varphi(c_i),-}] & \text{if } c_i \in \mathcal{C} \\ [E^r_{\varphi(c_i)}] & \text{if } c_i \notin \mathcal{C} \end{cases} \tag{5.33}$$

for $v \in \{(+,-),(-,+)\}$, $p \in \{+,-\}$. Combining all of this yields a formula for ρ_0:

$$\begin{aligned}
\rho_0([E^{(-,+)}_{c_i,+}]) &= \chi_0([E^{(-,+)}_{c_i,+}]) + \sum_{z \in I^+_{c_i} \cap \mathcal{C}} \left([E^{\operatorname{val}(\varphi,z)}_{\varphi(z),+}] + [E^{\operatorname{val}(\varphi,z)}_{\varphi(z),-}] \right) + \sum_{z \in I^+_{c_i} \cap \varphi^{-1}(\mathcal{D}) \setminus \mathcal{C}} [E^r_{\varphi(z)}] \\
\rho_0([E^{(-,+)}_{c_i,-}]) &= \chi_0([E^{(-,+)}_{c_i,-}]) + \sum_{z \in I^+_{c_i} \cap \mathcal{C}} \left([E^{\operatorname{val}(\varphi,z)}_{\varphi(z),+}] + [E^{\operatorname{val}(\varphi,z)}_{\varphi(z),-}] \right) + \sum_{z \in I^+_{c_i} \cap \varphi^{-1}(\mathcal{D}) \setminus \mathcal{C}} [E^r_{\varphi(z)}] \\
\rho_0([E^{(+,-)}_{c_i,+}]) &= \chi_0([E^{(+,-)}_{c_i,+}]) \\
\rho_0([E^{(+,-)}_{c_i,-}]) &= \chi_0([E^{(+,-)}_{c_i,-}]) \\
\rho_0([E^r_{c_i}]) &= [\chi_0([E^r_{c_i}]) + \sum_{z \in I^+_{c_i} \cap \mathcal{C}} \left([E^{\operatorname{val}(\varphi,z)}_{\varphi(z),+}] + [E^{\operatorname{val}(\varphi,z)}_{\varphi(z),-}] \right) + \sum_{z \in I^+_{c_i} \cap \varphi^{-1}(\mathcal{D}) \setminus \mathcal{C}} [E^r_{\varphi(z)}]
\end{aligned} \tag{5.34}$$

While this formula looks frightening, apperances are decieving – calculating the $\mathbb{Z}$-matrix given by ρ_* is possible simply by considering the graph of φ, and is, at least for fairly simple maps, doable by hand. We illustrate with an example:

Example 5.27. Let φ be the tent map from Example and Example 5.25. We may then choose $\mathcal{D} = \{0\}$ and $\mathcal{I} = \{I\}$ with $I = (0,1)$. In the notation of Lemma 5.26, we have

$$K_0(\mathbb{A}_k) \simeq \mathbb{Z}[E^{(-,+)}_{0,+}] \oplus \mathbb{Z}[E^{(-,+)}_{0,-}] \oplus \mathbb{Z}[E^{(+,-)}_{0,+}] \oplus \mathbb{Z}[E^{(+,-)}_{0,-}] \oplus \mathbb{Z}[E^r_0] \simeq \mathbb{Z}^5$$

We have $\varphi^{-1}(0) \cap I = \{1/4, 1/2, 3/4\}$ with $\operatorname{val}(\varphi, 1/4) = (+,+)$, $\operatorname{val}(\varphi, 1/2) = (+,-)$ and $\operatorname{val}(\varphi, 3/4) = (-,-)$. In particular, $1/2$ is in $\mathcal{C} \cap \varphi^{-1}(\mathcal{D}$ while $1/4$ and $3/4$ are in $\varphi^{-1}(\mathcal{D}) \setminus \mathcal{C}$.

We may then use the formula for ρ_0 developed above – for instance,

$$\chi_0([E_{0,+}^{(-,+)}]) = E_{\varphi(0),+}^{\mathrm{val}(\varphi,0)} = E_{0,+}^{(-,+)}$$

so

$$\rho_0([E_{0,+}^{(-,+)}]) = E_{0,+}^{(-,+)} + E_{0,+}^{(+,-)} + E_{0,-}^{(+,-)} + 2E_0^r$$

Doing these calculations for all basis elements yields a matrix representation of ρ_0:

$$\rho_0 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 2 & 2 & 0 & 0 & 2 \end{pmatrix}.$$

By using Table 5.5, we observe that $(I_1)_0 - (U_1)_0$ as a map from $\mathbb{Z}^5$ to $\mathbb{Z}$ is given by the 1-by-5-matrix $(1, 1, -1, -1, 0)$, so

$$\ker((I_1)_0 - (U_1)_0) \simeq \{(x, y, z, v, w) \in \mathbb{Z}^5 | v = x + y - z\} \simeq \mathbb{Z}^4$$

and the restriction of ρ_0 to $\ker((I_1)_0 - (U_1)_0)$ is represented by the matrix

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 \end{pmatrix}.$$

We conclude that

$$K_0(C_r^*(\Gamma_\varphi)) \simeq \mathrm{coker}(1 - \tilde{A}) \simeq \mathbb{Z} \oplus \mathbb{Z}_3$$

and

$$K_1(C_r^*(\Gamma_\varphi)) \simeq \ker(1 - \tilde{A}) \simeq \mathbb{Z}.$$ ▲

See the Appendix for a more involved calculation of the K-theory of some of these algebras.

Chapter 6

Circle maps with no periodic points

In this chapter, we consider a rather special class of circle maps – in addition to our stading assumptions (continuous, surjective, piecewise strictly monotone with finitely many critical points), we also assume that our maps have no periodic points. This puts severe limitations on the possible isotropy groups of the corresponding groupoid, which in turn allows us to apply the general theory of Chapter 4 to determine the ideal structure of the groupoid C^*-algebras.

One can say quite a lot about continuous maps of the circle without periodic points – the theory goes back to Poincaré's work on circle homeomorphisms and Denjoy and Arnold's examples of circle homeomorphisms not conjugate to an irrational rotation. For the case where the map is not necessarily a homeomorphism, the paper [4] is the definitive reference. This chapter begins with a sketch of the classical theory, goes on to describe some of the main results of [4], and finally develops the theory of the groupoid C^*-algebras arising from these maps.

6.1 Denjoy homeomorphisms – the classical theory

Let $\psi : \mathbb{T} \to \mathbb{T}$ be a orientation-preserving *homeomorphism* of the circle, and let $\widetilde{\psi} : \mathbb{R} \to \mathbb{R}$ be a lift of ψ to a map of the real line. The starting point for studying circle homeomorphisms is the following classical theorem:

Theorem 6.1. *Define the* rotation number *of ψ as*

$$\rho(\psi) = \lim_{n\to\infty} \frac{\widetilde{\psi}^n(x) - x}{n}, \qquad x \in \mathbb{R} \tag{6.1}$$

Then $\rho(\psi)$ exists and is, as an element of $\mathbb{T}$, independent both of the choice of lift $\widetilde{\psi}$ and of the point $x \in \mathbb{R}$. Furthermore $\rho(\psi)$ is rational if and only if ψ has a periodic point.

For a proof, se e.g. [8], Chapter 3.3, Theorem 1. From this point on, we'll focus only on maps with irrational rotation number. We start by analysing the limit sets of forward orbits under ψ:

Lemma 6.2. *Let $\psi : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving homeomorphism with irrational rotation number, let $x \in \mathbb{T}$ and let $m, n \in \mathbb{Z}$ with $m \neq n$. Let Δ and Δ' be the two arcs of $\mathbb{T}$ connecting $\psi^n(x)$ and $\psi^m(x)$. Then for any $y \in \mathbb{T}$, each of these arcs has non-empty intersection with $\mathcal{O}^+(y)$ as well as $\mathcal{O}^-(y)$.*

Proof. Assume without loss of generality that $m > n$. Note first that all the arcs Δ, $\psi^{m-n}(\Delta)$, $\psi^{2(m-n)}(\Delta)$ are adjacent to each other. Furthermore, for some k large enough we have

$$\bigcup_{j=1}^{k} \psi^{j(m-n)}(\Delta) = \mathbb{T}$$

Indeed, if this was not the case, the sequence $\widetilde{\psi}^{j(m-n)}(x)$, where $\widetilde{\psi}$ is a lift of ψ, would be monotonic and bounded, and hence converge to some limit x_0. But then x_0 would be a periodic point of period k, contradicting our assumption. Now let $y \in \mathbb{T}$. Then $y \in \psi^{j(m-n)}(\Delta)$ for some j, which means that $\varphi^{-j(m-n)}(y) \in \Delta$. It follows that $\mathcal{O}^-(y) \cap \Delta \neq \emptyset$. Interchanging m and n shows that $\mathcal{O}^+(y) \cap \Delta \neq \emptyset$. □

Corollary 6.3. *Let $x \in \mathbb{T}$, and P be the set of limit points of $\mathcal{O}^+(x)$. Then P is independent of the point x, and P is also the set of limit points of $\mathcal{O}^-(x)$.*

Proof. Let $y \in \mathbb{T}$, and $z \in P$. For the first claim, it suffices to show that z is a limit point of $\mathcal{O}^+(y)$. By assumption, there is a sequence $\{n_l\}$ of positive integers such that $\psi^{n_l}(x)$ converges to z. For each l, the lemma above yields a number k_l such that $\psi^{k_l}(y) \in [\psi^{n_l}(x), \psi^{n_{l+1}}(x)]$ (i.e. the shortest of the two arcs joining $\psi^{n_l}(x)$ and $\psi^{n_{l+1}}(x)$). It follows that $\psi^{k_l}(y)$ converges to z. For the second statement, the lemma above yields a sequence $\{m_l\}$ of positive integers such that $\psi^{-m_l}(x) \in [\psi^{n_l}(x), \psi^{n_{l+1}}(x)]$, so $\psi^{-m_l}(x)$ converges to z. By symmetry, we see that a limit point of the forward orbit is also a limit point of the backward orbit. □

The next theorem is also classical – for the proof, see e.g. [28]:

Proposition 6.4. *Let ψ be an orientation preserving homeomorphism of the circle with irrational rotation number α. Then ψ is semi-conjugate to the irrational rotation R_α, i.e. there is a continuous, surjective, orientation-preserving map $h : \mathbb{T} \to \mathbb{T}$ such that*

$$h \circ \psi = R_\alpha \circ h.$$

If the map h in the proposition above is invertible, ψ is *conjugate* to an irrational rotation – in particular, any forward orbit under φ is dense and the map is transitive. There are some rigidity results stating sufficient conditions for h to be invertible, see e.g. Theorem 1, Chapter 3.4 of [8]. We, however, are interested in the opposite case:

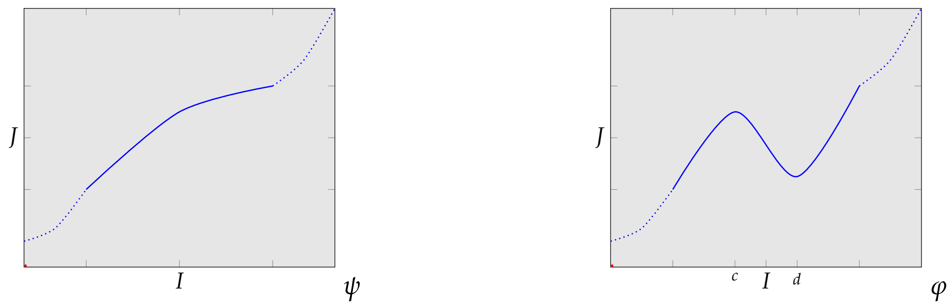
Definition 6.5. Let ψ and h be as in (6.4). If h is not invertible, ψ is called a *Denjoy homeomorphism*.

The following result, cf. Proposition 3.4 of [28], describes some properties of these maps. We say that a set A is totally invariant (under ψ) if $\psi(A) = \psi^{-1}(A) = A$:

Theorem 6.6. *Let ψ be a Denjoy homeomorphism. Then ψ has a unique minimal closed totally invariant set $\Sigma \subset \mathbb{T}$. Σ is a Cantor set (totally disconnected, compact and without isolated points), and for any $x \in \mathbb{T}$, Σ is the set of limit points of $\mathcal{O}(x)$.*

Let ψ be a Denjoy homeomorphism with invariant Cantor set Σ. The complement $Y = \mathbb{T} \setminus \Sigma$ is open, hence a countable union of disjoint open intervals $Y = \cup_{n \in \mathbb{Z}} Y_n$. Since ψ has no periodic points, it follows that the intervals Y_n are permuted acyclically by ψ, i.e. $\psi^k(Y_n) \cup Y_n = \emptyset$ for any $n, k \in \mathbb{Z}$. The action of ψ on these intervals may have one, several, or even countably infinite many orbits.

It is straightforward to modify a Denjoy homeomorphism ψ to a map with a critical point without altering the structural properties listed above: Let Y_n be a component of $\mathbb{T} \setminus \Sigma$, and note that ψ maps Y_n homeomorphically to another interval $Y_{n'} \subseteq \mathbb{T} \setminus \Sigma$. Now define φ as the map obtained from ψ by adding critical points in Y_n in a way such that $\varphi(Y_n) = \psi(Y_n) = Y_{n'}$. It is evident that φ is continuous, surjective, piecewise strictly monotone and without periodic points. One might add several critical points in other intervals in the same way. Since we only modify ψ away from Σ, and only on finitely many intervals, it is still the case that Σ is the set of limit points for any (forward or backward) orbit under φ. We will refer to these maps as *modified Denjoy maps* or just *Denjoy maps*.



6.2 Circle maps without periodic points

The previous section gave a construction of circle maps – i.e. piecewise linear, continuous, surjective maps with critical points – without periodic points, starting from a Denjoy homeomorphism and modifying it suitably on the complement of the invariant Cantor set. In this section we shall see that essentially *all* circle maps without periodic points arise in this way. The discussion follows [4] closely. We let $\varphi : \mathbb{T} \to \mathbb{T}$ be continuous and without periodic points. This implies surjectivity of φ. For each $x \in \mathbb{T}$, and let J_x be the largest interval containing x such that $J_x \cap \mathcal{O}^+(x)$ is empty. The intervals J_x have a number of interesting properties, cf. Theorem 1 of [4]:

Lemma 6.7. *Let φ, x and J_x be as above. Then the following hold:*

1. *If ξ and ξ' are the endpoints of J_x, then $\varphi(\xi)$ and $\varphi(\xi')$ are the endpoints of $\varphi(J_x)$.*
2. *$\varphi^n(J_x) = J_{\varphi^n(x)}$ for any $n \in \mathbb{N}$.*
3. *The sets $\{J_{\varphi^n(x)}\}_{n\in\mathbb{N}}$ are pairwise disjoint.*
4. *The sets J_x form a decomposition of $\mathbb{T}$ (i.e. for $x, y \in \mathbb{T}$, either $J_x = J_y$, or $J_x \cap J_y = \emptyset$).*
5. *At most countably many of the sets J_x are non-degenerate (i.e. $J_x \neq \{x\}$).*
6. *If $\varphi(x) = \varphi(y)$, then $J_x = J_y$.*

Recall that a subset Y of a dynamical system (X, φ) is *minimal* if the forward orbit of any point of Y is dense in Y. Let Σ be the set of endpoints of the intervals $\{J_x\}_{x\in\mathbb{T}}$. Theorem 2 of [4] states some properties of Σ:

Lemma 6.8. *The set Σ is the unique minimal subset of $\mathbb{T}$, and if φ is not a homeomorphism, Σ is nowhere dense in $\mathbb{T}$. For $x \in \Sigma$, $|\varphi^{-1}(x) \cap \Sigma|$ is 1 or 2, and there are at most countably many $x \in \Sigma$ with $|\varphi^{-1}(x) \cap \Sigma| = 2$.*

Lemma 6.9. *Σ is a Cantor set, i.e. compact, totally disconnected and perfect.*

Proof. Since Σ is closed in $\mathbb{T}$, it is compact, and since it is nowhere dense, it is totally disconnected. Given $x \in \Sigma$, we have by minimality that $\overline{\mathcal{O}^+(\varphi(x))} = \Sigma$, and $x \notin \mathcal{O}^+(\varphi(x))$ since φ has no periodic points. It follows that Σ is perfect. □

Putting these lemmas together gives a fairly clear picture of the dynamics of φ: $\mathbb{T}$ has a unique minimal set Σ, which is a Cantor set, and whose complement is a countable union of open intervals. Given an $x \in \mathbb{T}$ such that J_x is non-degenerate, it follows from Lemma (6.7) that $\varphi(J_x) = J_{\varphi(x)}$ is either another open interval or collapses to a single point $\{f(x)\}$.

Proposition 6.10. *Assume that φ, in addition to the other hypotheses in this section, is also piecewise strictly monotone – in particular, not locally constant anywhere. Then Σ is totally invariant, and φ restricted to Σ is a homeomorphism.*

Proof. We show first that Σ is backwards invariant: Let $x \in \Sigma$, and assume that $y \in \mathbb{T} \setminus \Sigma$ with $\varphi(y) = x$. Then $J_x = \{x\}$, and J_y is a non-degenerate interval. But since $\varphi(J_y) = J_{\varphi(y)} = J_x = \{x\}$, φ is locally constant on the open set J_y. This is against our assumptions, hence $\varphi^{-1}(\Sigma) = \Sigma$ and φ restricted to Σ is surjective. Finally, if $y_1, y_2 \in \Sigma$ with $\varphi(y_1) = \varphi(y_2)$, we have $\{y_1\} = J_{y_1} = J_{y_2} = \{y_2\}$ by Lemma 6.7, so φ restricted to Σ is injective, and hence a homeomorphism. □

Theorem 6.11. *Let $\varphi : \mathbb{T} \to \mathbb{T}$ be a continuous, surjective, piecewise monotone map with no periodic points, and assume that φ is not a homeomorphism. Then φ is a modified Denjoy map.*

Proof. From Lemma 6.9 and Proposition 6.10, it follows that there is a Cantor set Σ which is totally invariant, and such that $\varphi|_\Sigma$ is a homeomorphism. The complement $Y = \mathbb{T} \setminus \Sigma$ is an at most countable collection of intervals $Y = \sqcup Y_n$, and each interval Y_n contains at most finitely many critical points. Since Σ is totally invariant, the complement Y is also totally invariant – hence, by continuity, each interval Y_n is mapped to some other interval $Y_{n'}$ with $n \neq n'$. Since φ has no periodic points, $\varphi^k(Y_n) \cap Y_n = \emptyset$ for all k and n. Each restriction $\varphi|_{Y_n} : Y_n \to Y_{n'}$ can thus be homotoped to a homeomorphism $\psi_{Y_n} : Y_n \to Y_{n'}$. Doing this on each interval containing critical points yields a map $\psi : \mathbb{T} \to \mathbb{T}$ such that $\psi|_\Sigma = \varphi|_\Sigma$ and $\psi(Y_n) = \varphi(Y_n)$ for each n. It follows that ψ is a Denjoy homeomorphism, and that φ can be obtained from ψ be reverting the homotopies. □

6.3 The groupoid C^*-algebra of a Denjoy map

Having obtained a fairly complete description of circle maps without critical points, we now turn to their associated groupoids and groupoid C^*-algebras. Let φ be a map like those described above, with minimal Cantor set Σ, and let Γ_φ be its amended transformation groupoid introduced in Chapter 2. We restrict our attention to Γ_φ in this chapter, but the case of Γ_φ^+ should not be significantly different. Denote by $\mathcal{C}$ the critical points of φ. First, we apply the theory developed in Chapter 4 to determine the primitive ideal space of $C_r^*(\Gamma_\varphi)$. Recall that we write $[x]$ for the Γ_φ-orbit of $x \in \mathbb{T}$, i.e. the set $r(s^{-1}(x))$, that a set A is Γ_φ-invariant if $x \in A$ implies $[x] \subseteq A$, and that A is Γ_φ-minimal if it contains no proper closed Γ_φ-invariant subsets.

6.3.1 The primitive ideals

We begin by determining the primitive and maximal ideals of $C_r^*(\Gamma_\varphi)$.

Proposition 6.12. *The set Σ is Γ_φ-invariant and $\Sigma \subseteq \overline{[x]}$ for any $x \in \mathbb{T}$. In particular, it is the only non-trivial Γ_φ-minimal set.*

Proof. If $x \in \Sigma$, we have $\overline{\mathcal{O}^+(x)} \subseteq \Sigma$ by forward invariance, and then $\overline{[x]} \subseteq \Sigma$ since Σ is backwards invariant and closed. On the other hand, φ is increasing at any point x of Σ, so we have $\mathcal{O}(x) \subseteq [x]$, so $\Sigma \subseteq \overline{\mathcal{O}(x)} \subseteq \overline{[x]}$. This shows that $\overline{[x]} = \Sigma$ for any $x \in \Sigma$ and that Σ is Γ_φ-minimal.

Next, assume that $x \in \mathbb{T} \setminus \Sigma$. If the forward orbit of x intersects $\mathcal{C}$ trivially, φ is monotone at $\varphi^n(x)$ for all n, and we have $\mathcal{O}^+(x) \subseteq [x]$. Hence, $\Sigma \subseteq \overline{[x]}$. If, on the other hand, there is an n such that $\varphi^n(x) \in \mathcal{C}$, we must have $\mathcal{O}^-(x) \subseteq [x]$: if $\varphi^k(y) = x$, we have

$$\operatorname{val}(\varphi^{k+n+1}, y) = \operatorname{val}(\varphi, \varphi^n(x)) \bullet \operatorname{val}(\varphi^{k+n}, y) = \operatorname{val}(\varphi, \varphi^n(x)) = \operatorname{val}(\varphi^{n+1}, x).$$

Since Σ is also the set of limit points of $\mathcal{O}^-(x)$, it follows that $\Sigma \subseteq \overline{[x]}$. In particular, Σ is the only Γ_φ-minimal set. □

Theorem 6.13. *Let $\varphi : \mathbb{T} \to \mathbb{T}$ be a continuous, surjective, piecewise monotone map with no periodic points. Then $C_r^*(\Gamma_\varphi)$ has a unique maximal ideal I, and the corresponding simple quotient is isomorphic to the crossed product of a minimal homeomorphism on a Cantor set.*

Proof. Let Σ be the minimal Cantor set of φ. By combining Proposition 6.12 and Lemma 4.45, we see that the only maximal ideal in $C_r^*(\Gamma_\varphi)$ is $\ker(\pi_\Sigma)$, and that $C_r^*(\Gamma_\varphi)/I$ is isomorphic to $C_r^*(\Gamma_\varphi|_\Sigma)$. Since Σ is totally invariant and $\varphi|_\Sigma$ is a minimal homeomorphism, it follows from Proposition 1.8 of [23] that $C_r^*(\Gamma_\varphi|_\Sigma)$ is isomorphic to the reduced crossed product C^*-algebra $C(\Sigma) \rtimes_{\varphi,r} \mathbb{Z}$. □

To determine the primitive ideals, recall from Theorem 4.41 that these come in two types: Either the primitive ideal I is isomorphic to $\ker(\pi_F)$ with $F = \overline{[x]}$ for some $x \in \mathbb{T}$ and the non-isotropic points dense in F, or $I = I(\overline{[x]}, \omega)$ where x is isotropic and and isolated in $[x]$, and ω is a character on $\operatorname{Iso}(x)$.

Lemma 6.14. *Let $x \in \mathbb{T}$ and assume that x is not pre-critical. Put $F = \overline{[x]}$. Then $\ker(\pi_F)$ is a primitive ideal, and $\ker(\pi_F) \subseteq \ker(\pi_\Sigma)$.*

Proof. If x is not precritical, the orbit closure $\overline{[x]}$ contains no isotropic points, so $\ker(\pi_F)$ is a primitive ideal by Theorem 4.41. Since $\Sigma \subseteq F$, we have $\ker(\pi_F) \subseteq \ker(\pi_\Sigma)$. □

Next, assume that x is precritical. It follows that x is isolated in $F = \overline{[x]}$, and by Lemma 4.8, we have

$$\operatorname{Iso}(x) = \{(x, 0, +, x), (x, 0, -, x)\} \simeq \mathbb{Z}_2.$$

The dual group $\widehat{\operatorname{Iso}(x)}$ has two elements χ_+ and χ_-, where $\chi_+ \equiv 1$, and χ_- is given by $\chi_-((x, 0, +, x)) = 1$ and $\chi_-((x, 0, -, x)) = -1$. In $C_r^*(\Gamma_\varphi|_F)$, there are two ideals $I_0(F, \chi_+)$ and $I_0(F, \chi_-)$, generated by the elements

$$a_+ = 1_{(x,0,+,x)} - \overline{\chi_+((x,0,-,x))} 1_{(x,0,-,x)} = 1_{(x,0,+,x)} - 1_{(x,0,-,x)}$$

and

$$a_- = 1_{(x,0,+,x)} - \overline{\chi_-((x,0,-,x))} 1_{(x,0,-,x)} = 1_{(x,0,+,x)} + 1_{(x,0,-,x)}$$

respectively.

Lemma 6.15. *Let $x \in \mathbb{T}$ such that x is isotropic and isolated in $[x]$, and let $F = \overline{[x]}$. Let $\pi_F : C_r^*(\Gamma_\varphi) \to C_r^*(\Gamma_\varphi|_F)$, and let $I_0(F, \chi_+)$ and $I_0(F, \chi_-)$ be as above. Then the ideals*

$$I(F,+) = \pi_F^{-1}\left(I_0(F,\chi_+)\right), \quad I(F,-) = \pi_F^{-1}\left(I_0(F,\chi_-)\right)$$

are primitive ideals in $C_r^*(\Gamma_\varphi)$, with

$$I(F,+) \cap I(F,-) = \ker(\pi_F) \text{ and } I(F,+) + I(F,-) = \ker(\pi_\Sigma).$$

Proof. It follows directly from Lemma 4.40 that $I(F,+)$ and $I(F,-)$ are primitive ideals in $C_r^*(\Gamma_\varphi)$. To show that $I(F,+) \cap I(F,-) = \ker(\pi_F)$, we show that

$$(I_0(F,\chi_+)) \cap (I_0(F,\chi_-)) = \{0\}.$$

This, in turn, follows from showing that $a_+ C_r^*(\Gamma_\varphi|_F) a_- = \{0\}$, which again by continuity amounts to showing that $a_+ C_c(\Gamma_\varphi|_F) a_- = \{0\}$. So, let $f \in C_c(\Gamma_\varphi|_F)$ and $\gamma = (x,k,p,y) \in \Gamma_\varphi|_F$. Since a_+ and a_- are supported on $\text{Iso}(x) = \{(x,0,+,x),(x,0,-,x)\}$, the same holds for $a_+ f a_-$. Now we calculate:

$$\begin{aligned}(a_+ f a_-)(x,0,+,x) &= a_+(x,0,+,x) f(x,0,+,x) a_-(x,0,+,x) + a_+(x,0,-,x) f(x,0,+,x) a_-(x,0,-,x) \\ &\quad + a_+(x,0,+,x) f(x,0,-,x) a_-(x,0,-,x) + a_+(x,0,-,x) f(x,0,-,x) a_-(x,0,+,x) \\ &= f(x,0,+,x) - f(x,0,+,x) + f(x,0,-,x) - f(x,0,-,x) = 0.\end{aligned}$$

Similarly, $(a_+ f a_-)(x,0,-,x) = 0$, which shows what we want.

To conclude the final statement, we note a few things: First, the characteristic function $1_{(x,0,+,x)}$ is in $I_0(F,\chi_+) \oplus I_0(F,\chi_-)$. Second, F decomposes as the disjoint union $F = [x] \sqcup \Sigma$. Hence, if $\gamma \in \Gamma_\varphi|_F$ there is a $\rho \in \Gamma_\varphi|_F$ such that $\gamma = \rho(x,0,+,x)\rho^{-1}$ if and only if $\gamma \in \Gamma_\varphi|_{F\setminus\Sigma}$. For such a γ we have $1_\gamma = 1_\rho 1_{(x,0,+,x)} 1_{\rho^{-1}}$, and since each element of $[x]$ is isolated in $\overline{[x]}$, it follows that $C_c(\Gamma_\varphi|_{F\setminus\Sigma}) \subseteq I_0(F,\chi_+) \oplus I_0(F,\chi_-)$. By continuity, we have $C_r^*(\Gamma_\varphi|_{F\setminus\Sigma}) \subseteq I_0(F,\chi_+) \oplus I_0(F,\chi_-)$. But, in the notation of Lemma 3.21, $\pi_\Sigma = \pi_{F,\Sigma} \circ \pi_F$, so if $a \in \ker(\pi_\Sigma)$ we have

$$\pi_F(a) \in \ker(\pi_{F,\Sigma}) = C_r^*(\Gamma_\varphi|_{F\setminus\Sigma}) \subseteq I_0(F,\chi_+) \oplus I_0(F,\chi_-),$$

and hence $a \in I(F,+) + I(F,-)$. It follows that

$$\ker(\pi_\Sigma) \subseteq I(F,+) + I(F,-).$$

The other inclusion follows from the fact that $1_{(x,0,+,x)}$ and $1_{(x,0,-,x)}$ are in $\ker \pi_\Sigma$ since $x \notin \Sigma$. □

Theorem 6.16. *Let ψ be a Denjoy map. Then any primitive ideal I of $C_r^*(\Gamma_\varphi)$ is either*

- *$I = \ker(\pi_F)$, where $F = \overline{[x]}$ and x is not pre-critical, or*
- *$I = I(F,\pm)$, where $F = \overline{[x]}$ and x is a pre-critical point.*

Proof. This is immediate from Theorem 4.41. □

6.3.2 Two fundamental extensions

By Theorem 6.13, $C_r^*(\Gamma_\varphi)$ has a unique maximal ideal $\ker(\pi_\Sigma)$. Putting $Y = \mathbb{T} \setminus \Sigma$, we have (by Lemma 3.21) $\ker(\pi_\Sigma) \simeq C_r^*(\Gamma_\varphi|_Y)$, and an extension

$$0 \longrightarrow C_r^*(\Gamma_\varphi|_Y) \longrightarrow C_r^*(\Gamma_\varphi) \longrightarrow C_r^*(\Gamma_\varphi|_\Sigma) \longrightarrow 0. \tag{6.2}$$

Since Y is open in $\mathbb{T}$, we may write $Y = \sqcup_{i\in\mathbb{N}} Y_i$ as a countable disjoint union of open intervals. After relabeling, we may assume that ψ maps Y_i homeomorphically onto Y_{i+1}. The set of critical points $\mathcal{C}$ is finite and contained in Y. Since we require φ to have only finitely many critical points, there is a number $N \in \mathbb{N}$ such $\mathcal{C} \subseteq \sqcup_{i=-N}^{N} Y_i$.

Remark 6.17. Note that for a general Denjoy map ψ, the action of ψ on the intervals $\{Y_i\}$ might have not just one, but several orbits – that is, we may partion Y into (finitely or infinitely many) disjoint subsets $Y^1, Y^2, \ldots$ with each Y^i left totally invariant by ψ, and hence also by φ. Since each Y^i is Γ_φ-invariant and clopen in Y, we get a direct sum decomposition

$$C_r^*(\Gamma_\varphi|_Y) = \bigoplus_i C_r^*(\Gamma_\varphi|_{Y^i}).$$

Hence, in analysing $C_r^*(\Gamma_\varphi|_Y)$, we lose no generality assuming that the action of φ on the intervals Y_i has only one orbit. ♦

To understand $C_r^*(\Gamma_\varphi|_Y)$, we construct another extension like (6.2), but now with $C_r^*(\Gamma_\varphi|_Y)$ in the middle: For a point $x \in Y$, put

$$\mathcal{D}(x) = \{y \in Y \mid \exists m, n \in \mathbb{N} : \varphi^n(x) = \varphi^m(y)\} = \bigcup_{n\in\mathbb{N}} \mathcal{O}^-(\varphi^n(x)),$$

and put

$$\mathcal{D} = \bigcup_{c\in\mathcal{C}} \mathcal{D}(c) \tag{6.3}$$

The set $\mathcal{D}$ is illustrated in the first figure of 6.3.2, in the case where φ has two critical points c and d.

Lemma 6.18. *The set $\mathcal{D}$ is closed in Y, totally φ-invariant and $\Gamma_\varphi|_Y$-invariant.*

Proof. Since $\mathcal{D}$ is the finite union of the sets $\mathcal{D}(c)$, it suffices to show that each of these has the desired properties. So, fix a $c \in \mathcal{C}$. Note that for $n \geq N$, $\varphi|_{Y_n} : Y_n \to Y_{n+1}$ is one-to-one, in particular, $\varphi^{-1}(\varphi^{n+1}(c)) = \{\varphi^n(c)\}$. It follows that

$$\mathcal{D}(c) = \mathcal{O}^+(\varphi^{N+1}(c)) \cup \mathcal{O}^-(\varphi^N(c))$$

Now, if $\{x_i\}$ is a sequence in $\mathcal{D}(c)$ which is convergent in Y, it must have a subsequence x_{i_k} in either $\mathcal{O}^+(\varphi^N(c))$ or $\mathcal{O}^-(\varphi^N(c))$. But if $\{x_i\}$ is not eventually constant, it follows from 6.3 that the limit of $\{x_{i_k}\}$, and hence of $\{x_i\}$ is in Σ, which is against our assumption. It follows that $\mathcal{D}(c)$ is closed in Y. Total invariance is immediate from the definition, and since $[c] \subseteq \mathcal{D}(c)$, it follows that $\mathcal{D}(c)$ is $\Gamma_\varphi|_Y$-invariant. □

One way to think of the set $\mathcal{D}$ is as the smallest totally φ-invariant set that containts all critical points. Let $Z = Y \setminus \mathcal{D}$. Observe that each $Z \cap Y_i$ is a finite union of open intervals, and that each connected component of $Z \cap Y_i$ is mapped homeomorphically onto some connected component

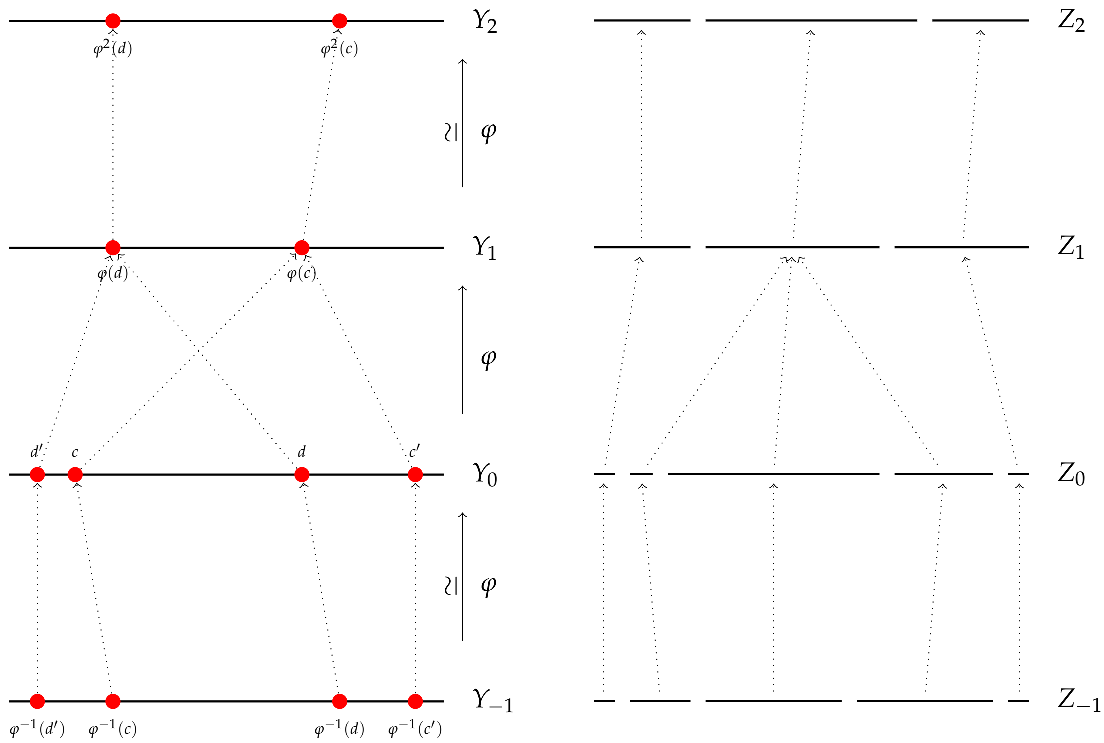


Figure 6.1: The first picture shows the set $\mathcal{D}$ in the case where φ has two critical points c and d, as in Figure 6.1. The red, marked points are the set $\mathcal{D}$. After removing $\mathcal{D}$ from Y, we obtain the set Z, which consists of a finite number of intervals at each level, each mapped homeomorphically onto an interval one level above..

of $Z \cap Y_{i+1}$ – this is basically the situation in the second picture in 6.3.2. Furthermore, since we have removed the total orbits of all critical points from Y, we have $\text{val}(\varphi^n, z) \in \{(+,+),(-,-)\}$ for any $z \in Z$ and $n \in \mathbb{N}$. Since Z is also $\Gamma_\varphi|_Y$-invariant, we have by Lemma 3.21 an extension

$$0 \longrightarrow C_r^*(\Gamma_\varphi|_Z) \longrightarrow C_r^*(\Gamma_\varphi|_Y) \longrightarrow C_r^*(\Gamma_\varphi|_{\mathcal{D}}) \longrightarrow 0 \tag{6.4}$$

We now have the ingredients for determining the K-theory of $C_r^*(\Gamma_\varphi)$: First, we analyse extension 6.4, obtain concrete descriptions of $C_r^*(\Gamma_\varphi|_Z)$ and $C_r^*(\Gamma_\varphi|_{\mathcal{D}})$ and realise $C_r^*(\Gamma_\varphi|_Y)$ as a double mapping cylinder of these two algebras. We then plug this information into Extension 6.2 and combine this with results from [28] to determine the K-theory of $C_r^*(\Gamma_\varphi)$.

6.3.3 On $\Gamma_\varphi|_Y$

In this section, we describe the structure of $C^*(\Gamma_\varphi|_Y)$. It is worth stressing that the situation in this section is very similar to the one in Section 3.1 of Chapter 3, in particular the extension in

Equation 5.10. Indeed, Equation 6.4 shows that $C_r^*(\Gamma_\varphi)$ is the extension of a C^*-algebra of the principal groupoid $\Gamma_\varphi|_Z$ by the C^*-algebra of the discrete groupoid $\Gamma_\varphi|_\mathcal{D}$.

We start, as before, with a Denjoy homeomorphism $\psi : \mathbb{T} \to \mathbb{T}$ with associated Cantor set Σ, and put $Y = \mathbb{T} \setminus \Sigma$. Write $Y = \sqcup_{i\in\mathbb{N}} Y_i$ as a countable disjoint union of open intervals. After relabeling, we may assume that ψ maps Y_i homeomorphically onto Y_{i+1}. We now modify ψ to get a map with critical points: At each interval Y_i, twist ψ to add a (finite) number of critical points $\{c_{i,1}, c_{i,2}, \ldots, c_{i,l_i}\}$, and denote the resulting map by φ. As long as $\varphi(Y_i) = Y_{i+1}$ and φ is piecewise strictly monotone, we make no particular demands on how we obtain φ from ψ. In particular, the forward orbits of critical points are allowed to meet. Let $\mathcal{C}$ denote the set of critical points. Note that we require φ to have a finite number of critical points, hence there is a number $N \in \mathbb{N}$ such $\mathcal{C} \subseteq \sqcup_{i=-N}^{N} Y_i$.

The structure of Y as a disjoint union of intervals has great influence on the structure of $\Gamma_\varphi|_Y$:

Lemma 6.19. *Let φ and Y be as above, and assume that $(x, k, p, y) \in \Gamma_\varphi|_Y$. Then the number k is uniquely determined by x and y.*

Proof. Choose numbers $i, j \in \mathbb{Z}$ such that $x \in Y_i$ and $y \in Y_j$, and $m, n \in \mathbb{N}$ such that $\varphi^n(x) = \varphi^m(y)$. It follows that $\varphi^n(x) \in Y_{i+n}$ and that $\varphi^m(y) \in Y_{j+m}$, so we have $i + n = j + m$. This means that $k = n - m = j - i$. $\square$

Lemma 6.20. *Let $x \in Y$. The sequence $\mathrm{val}(\varphi^n, x)$ is eventually constant.*

Proof. Choose N such that $\mathcal{C} \subseteq \sqcup_{i=-N}^{N} Y_n$. If x is in Y_k for some $k \in \mathbb{Z}$, we note that $\varphi^j(x)$ is non-critical for any $j \geq |k| + N$, indeed, $\mathrm{val}(\varphi, \varphi^j(x)) = (+,+)$. Since $(+,+)$ is a neutral element in the semigroup $\mathcal{V}$, it follows that

$$\mathrm{val}(\varphi^{|k|+N+i}, x) = \mathrm{val}(\varphi^i, \varphi^{|k|+N}(x)) \bullet \mathrm{val}(\varphi^{|k|+N}, x) = \mathrm{val}(\varphi^{|k|+N}, x)$$

as we wanted. $\square$

We denote this 'eventual valency' at x by v_x. Put an equivalence relation $\equiv$ on $\mathcal{V}$ by putting $(+,+) \equiv (-,-)$ and $(+,-)$ and $(-,+)$ only equivalent to itself. As we have seen, for points $x, y \in \mathbb{T}$ with $\varphi^n(x) = \varphi^m(y)$, there is a local transfer η with $\eta(x) = y$ and $\varphi^n = \varphi^m \circ \eta$ if and only if $\mathrm{val}(\varphi^n, x) \equiv \mathrm{val}(\varphi^m, y)$. If this valency is critical (i.e. $(+,-)$ or $(-,+)$) there are two choices of (germ of) local transfer, one reversing orientation and one preserving it. If the valencies at x and y are non-critical, there is one unique choice of (germ of) local transfers.

Lemma 6.21. *Let $x, y \in Y$ and assume that there are $n, m \in \mathbb{N}$ such that $\varphi^n(x) = \varphi^m(y)$. Put $k = n - m$. Then there is a local transfer η such that $(x, k, [\eta]_x, y) \in \Gamma_\varphi|_Y$ if and only if $v_x \equiv v_y$.*

Proof. If there is a transfer η such that $(x, k, [\eta]_x, y) \in \Gamma_\varphi|_Y$, there are n', m' with $\varphi^{n'}(x) = \varphi^{m'}(y)$, $\eta(y) = x$ and $\varphi^{n'}(x) = \varphi^{m'}(y)$. Then $\mathrm{val}(\varphi^{n'}, x) \equiv \mathrm{val}(\varphi^{m'}, y)$, so $v_x \equiv v_y$. On the other hand, if $v_x \equiv v_y$, there is an l such that $\mathrm{val}(\varphi^{n+l}, x) \equiv \mathrm{val}(\varphi^{m+l}, y)$. Since $\varphi^{n+l}(x) = \varphi^{m+l}(y)$, there is a transfer η such that $(x, k, [\eta]_x, y) \in \Gamma_\varphi|_Y$. $\square$

Putting all these lemmas together, we obtain a description of the groupoid $\Gamma_\varphi|_Y$ as

$$\Gamma_\varphi|_Y = \left\{(x, p, y) \in Y \times \mathbb{Z}_2 \times Y \,\middle|\, \exists n, m : \varphi^n(x) = \varphi^m(y), v_x \equiv v_y \right\}$$

with these possible choices of p determined by v_x and v_y as discussed above.

Lemma 6.22. *Let F be a topological space, assume that F is a countably infinite union of disjoint clopen sets $\{F_i\}_{i\in\mathbb{Z}}$, and that each F_i is a finite disjoint union of l_i sets $\{F_{i,l}\}_{l=1}^{l_i}$ such that each $F_{i,l}$ is homeomorphic to an open interval. Let $\theta : F \to F$ be a continuous, surjective map such that each set $F_{i,l}$ at level i is mapped homeomorphically onto some $F_{i+1,l'}$ at level $i+1$. Then θ is a local homeomorphism, the sequence $\{l_i\}$ is convergent to some number L, and we have an isomorphism of C^*-algebras*

$$C_r^*(\Gamma_\theta) \simeq (C_0((0,1)) \otimes \mathbb{K})^L \simeq C_0((0,1), \mathbb{K}^L)$$

Proof. Since θ is one-to-one on each interval $F_{i,l}$, it is immediate that θ is a local homeomorphism. Since θ is surjective and each interval at level F_i is mapped homeomorphically onto some interval at level F_{i+1}, we must have $l_i \geq l_{i+1}$ for all $i \in \mathbb{Z}$. It follows that $\{l_i\}$ converges to some integer $L \geq 1$.

The next part is an application of Example 1.22: Choose a number N such that F_i is a disjoint union of L sets for all $i \geq N$. Write $F_N = F_{N,1} \sqcup \cdots \sqcup F_{N,n_N}$. Define an equivalence relation $\sim$ on F by putting $F_{i,l} \sim F_{j,k}$ if $\varphi^n(F_{i,l}) = \varphi^m(F_{j,k})$ for some numbers n, m. Then $\sim$ partitions F into L equivalence classes. Furthermore, if $\varphi^n(F_{i,l}) = \varphi^m(F_{j,k})$, the map

$$\varphi^{-m} \circ \varphi^n : F_{i,l} \to F_{j,k}$$

is a homeomorphism. By Example 1.22, we have

$$C_r^*(\Gamma_\theta) \simeq C_0((0,1)) \otimes \mathbb{K}^L. \qquad \square$$

Corollary 6.23. *The algebra $C_r^*(\Gamma_\varphi|_Z)$ is isomorphic to $(C_0((0,1)) \otimes \mathbb{K})^L$ for some $L \in \mathbb{N}$.*

Proof. This is an immediate consequence of Lemma 6.22. $\square$

Note that Example 1.22 gives a little more info: Let $\mathcal{I}$ denote the set of connected components of Z. Then the relation $\sim$ from the proof of Lemma 6.22 partitions $\mathcal{I}$ into L equivalence classes. Let $\mathbb{B}$ be the algebra generated by matrix units $e_{I,J}$ for $I, J \in \mathcal{I}$ with $I \sim J$. Then $\mathbb{B} \simeq \mathbb{K}^L$, and $C_r^*(\Gamma_\varphi|_Z) \simeq C_0((0,1), \mathbb{K}^L)$.

We now turn to the algebra arising from the reduction of $\Gamma_\varphi|_Y$ to the set $\mathcal{D}$. Since $\mathcal{D}$ contains critical points, the Γ_φ-orbit structure of points in $\mathcal{D}$ is somewhat more complicated.

Lemma 6.24. *Let $\mathcal{D}$ be as in Equation 6.3. Then there is a finite set of points $\{x_1, \ldots, x_m\} \subseteq \mathcal{D}$ such that $\mathcal{D}$ decomposes as the disjoint union of the $\Gamma_\varphi|_Y$-orbits of these points, i.e.*

$$\mathcal{D} = [x_1] \sqcup \cdots \sqcup [x_m].$$

Furthermore, we have $C_r^*(\Gamma_\varphi|_{\mathcal{D}}) \simeq \mathbb{K}^M$ for some $M \in \mathbb{N}$.

Proof. As above, we may choose a number N such that $\mathcal{C} \subseteq \sqcup_{i=-N}^N Y_i$. Let K be the number of elements in $\mathcal{D} \cap Y_{N+1}$ (note that with L as in Corollary 6.23, we have $K = L - 1$). Write $\mathcal{D} \cap Y_{N+1} = \{y_1, \ldots, y_K\}$, and note that we may write $\mathcal{D}$ as the disjoint union of the sets $\mathcal{D}(y_i)$. It suffices to show that each $\mathcal{D}(y_i)$ may be written as a finite union of $\Gamma_\varphi|_Y$-orbits. We notice that each $\mathcal{D}(y_i)$ looks like an upside-down tree – the set $\varphi^{-1}(y_i)$ is finite, and for each $z \in \varphi^{-1}(y_i)$, the set $\varphi^{-1}(z)$ is finite, etc. Finally, if $z \in \varphi^{-2(N+1)}(y_i)$, we have $z \in Y_{-N-1}$, so z is not critical and $\varphi^{-1}(z)$ contains exactly one element. Now, observe the following:

- If $z \in \mathcal{O}^+(y_i)$, we have $[z] = [y_i]$.

- If $z \in \varphi^{-2(N+1)}(y_i)$, and $x \in \mathcal{O}^-(z)$, we have $[z] = [x]$.
- The set $\mathcal{D}(y_i) \cap (\sqcup_{i=-N}^{N} Y_i)$ is finite.

Taken together, these three facts imply that $\mathcal{D}(y_i)$ is the union of at most finitely many $\Gamma_\varphi|_Y$-orbits, and this shows the first part of the lemma.

For the second part, let $x \in \mathcal{D}$. Now one of two things happen: Either there is an n such that x is critical for φ^n, in which case the isotropy group $\mathrm{Iso}(x)$ at x is isomorphic to $\mathbb{Z}_2$, or no such n exists, in which case $\mathrm{Iso}(x)$ is trivial. In either case, the set $[x]$ is infinite: if x is pre-critical, any point in $\mathcal{O}^-(x)$ is contained in $[x]$, and if it isn't, we have $\mathcal{O}^+(x) \subseteq [x]$. We may now appeal to Examples 1.20 and 1.24 to conclude that

$$C_r^*(\Gamma_\varphi|_{[x]}) \simeq C^*(\mathrm{Iso}(x)) \otimes \mathbb{K}(l^2([x])) \simeq \begin{cases} \mathbb{K}^2, & \text{if } \mathrm{Iso}(x) \simeq \mathbb{Z}^2. \\ \mathbb{K}, & \text{if } \mathrm{Iso}(x) \simeq \{e\}. \end{cases} \tag{6.5}$$

Since $C_r^*(\Gamma_\varphi|_{\mathcal{D}})$ is the finite direct sum of the C^*-algebras of the Γ_φ-orbits contained in $\mathcal{D}$, the result follows. □

Proposition 6.25. *Let $\varphi : \mathbb{T} \to \mathbb{T}$ be a twisted Denjoy map with invariant Cantor set Σ, and put $Y = \mathbb{T} \setminus \Sigma$. Then there are extensions*

$$0 \longrightarrow C_r^*(\Gamma_\varphi|_Y) \longrightarrow C_r^*(\Gamma_\varphi) \longrightarrow C(\Sigma) \rtimes_{\varphi,r} \mathbb{Z} \longrightarrow 0 \tag{6.6}$$

and

$$0 \longrightarrow (C_0((0,1)) \otimes \mathbb{K})^L \longrightarrow C_r^*(\Gamma_\varphi|_Y) \longrightarrow \mathbb{K}^M \longrightarrow 0 \tag{6.7}$$

for some natural numbers L and M.

Proof. Since φ is a homeomorphism on Σ, it follows from Proposition 1.8 of [23] $C_r^*(\Gamma_\varphi|_\Sigma) \simeq C(\Sigma) \rtimes_\varphi \mathbb{Z}$. The second extension is a direct consequence of Lemmas 6.23 and 6.24. □

Remark 6.26. The numbers L and M can be determined directly from φ. L is given as in Lemma 6.22 and is simply the eventual number of connected components in $Y_N \setminus \mathcal{D}$. The number M is slightly more tricky – from Lemma 6.24 we may choose $x_1, \dots, x_m \in \mathcal{D}$ such that $\mathcal{D} = [x_1] \sqcup \dots \sqcup [x_m]$. Partition them according to their eventual valency, such that $v_{x_i} \in \{(+,-),(-,+)\}$ for $i = 1, \dots, m_1$ and $v_{x_i} \in \{(+,+),(-,-)\}$ for $i = m_1 + 1, \dots m$. From the lemma, it follows that each $C_r^*(\Gamma_\varphi|_{[x_i]}) \simeq \mathbb{K}^2$ for $i = 1, \dots, m'$ and $C_r^*(\Gamma_\varphi|_{[x_i]}) \simeq \mathbb{K}$ for $i = m' + 1, \dots m$. If we put $m_2 = m - m_1$, it follows that we have $M = 2m_1 + m_2$. ♦

6.3.4 K-theory

Lemma 6.25 gives a fairly clear picture of the C^*-algebra $C_r^*(\Gamma_\varphi)$ when φ is a modified Denjoy map. To calculate its K-theory, we use the same strategy as in Section 3.1 for determining the K-theory of $C_r^*(\Gamma_\varphi|_Y)$, and then combine these results with those of [28] to get hold of the K-theory of $C_r^*(\Gamma_\varphi)$. Start by taking the extension (6.7) and its associated six-term exact sequence on K-theory:

$$\begin{array}{ccccc} K_0((C_0((0,1)) \otimes \mathbb{K})^L) & \longrightarrow & K_0(C_r^*(\Gamma_\varphi|_Y)) & \longrightarrow & K_0(\mathbb{K}^M) \\ \uparrow & & & & \downarrow \\ K_1(\mathbb{K}^M) & \longleftarrow & K_1(C_r^*(\Gamma_\varphi|_Y)) & \longleftarrow & K_1((C_0((0,1)) \otimes \mathbb{K})^L) \end{array} \tag{6.8}$$

The K-theory of $\mathbb{K}$ and $C_0((0,1)) \otimes \mathbb{K}$ is well-known from e.g. [34]. Plugging this into (6.8) yields

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(C_r^*(\Gamma_\varphi|_Y)) & \longrightarrow & \mathbb{Z}^M \\ & & & & \downarrow \delta_0 \\ 0 & \longleftarrow & K_1(C_r^*(\Gamma_\varphi|_Y)) & \longleftarrow & \mathbb{Z}^L \end{array} \tag{6.9}$$

which in turn implies that

$$K_0(C_r^*(\Gamma_\varphi|_Y)) \simeq \ker(\delta_0), \qquad K_1(C_r^*(\Gamma_\varphi|_Y)) \simeq \operatorname{coker}(\delta_0).$$

Write $\mathbb{A}$ for the algebra $C_r^*(\Gamma_\varphi|_{\mathcal{D}})$, and $\mathbb{B}$ for the algebra of the equivalence relation on $\mathcal{I}$ as in the proof of Lemma 6.22. The extension in Lemma 6.7 then takes the form

$$0 \longrightarrow C_0((0,1),\mathbb{B}) \longrightarrow C_r^*(\Gamma_\varphi|_Y) \longrightarrow \mathbb{A} \longrightarrow 0. \tag{6.10}$$

We now aim to construct $*$-homomorphisms $I, U : \mathbb{A} \to \mathbb{B}$ such that

$$C_r^*(\Gamma_\varphi|_Y) \simeq \{(a,f) \in \mathbb{A} \oplus C([0,1],\mathbb{B}) \mid I(a) = f(0), U(a) = f(1)\}$$

since this by Lemma 1.27 would mean that the map δ_0 in (6.9) is equal to $I_* - U_*$, the difference of the two induced maps $I_*, U_* : K_0(\mathbb{A}) \to K_0(\mathbb{B})$. We define the maps on matrix units, extend them linearly, and then check that they give rise to honest $*$-homomorphisms. First, for a point $x \in \mathcal{D}$, choose i such that $x \in Y_i$, and denote by I_l^x and I_r^x the connected component of Z_i immediately to the left and right of x. Then, if $(x,p,y) \in \Gamma_\varphi|_{\mathcal{D}}$, define I and U on the matrix unit $e_{x,p,y}$ by the following – horrible – table:

$e_{x,p,y}$	$I(e_{x,p,y})$	$U(e_{x,p,y})$
$v_x = v_y = (-,+)$, $p = +$	$e_{I_l^x,I_l^y} + e_{I_r^x,I_r^y}$	0
$v_x = v_y = (-,+)$, $p = -$	$e_{I_l^x,I_r^y} + e_{I_r^x,I_l^y}$	0
$v_x = v_y = (+,-)$, $p = +$	0	$e_{I_l^x,I_l^y} + e_{I_r^x,I_r^y}$
$v_x = v_y = (+,-)$, $p = -$	0	$e_{I_l^x,I_r^y} + e_{I_r^x,I_l^y}$
$v_x = v_y = (+,+)$	$e_{I_r^x,I_r^y}$	$e_{I_l^x,I_l^y}$
$v_x = v_y = (-,-)$	$e_{I_l^x,I_l^y}$	$e_{I_r^x,I_r^y}$
$v_x = (+,+), v_y = (-,-)$	$e_{I_r^x,I_l^y}$	$e_{I_l^x,I_r^y}$
$v_x = (-,-), v_y = (+,+)$	$e_{I_l^x,I_r^y}$	$e_{I_r^x,I_l^y}$

Now extend I and U linearly to maps defined on all of $\mathbb{A}$. Checking that these maps respect the $*$-operation is a simple case by case-argument. So is checking multiplicativity, albeit with a lot more cases. For instance, let $(x,+,y),(y,-,z) \in \Gamma_\varphi|_{\mathcal{D}}$ with $v_x = v_y = (-,-)$ and $v_z = (+,+)$. Then

$$I(e_{x,+,y})I(e_{y,-,z}) = e_{I_l^x,I_l^y} \cdot e_{I_l^y,I_r^z} = e_{I_l^x,I_r^z} = I(e_{x,-,z}) = I(e_{x,+,y} \cdot e_{y,-,z}).$$

The other cases are similar. It follows that I and U are $*$-homomorphisms. Finally, construct maps $a : C_r^*(\Gamma_\varphi|_Y) \to \mathbb{A}$ and $b : C_r^*(\Gamma_\varphi|_Y) \to C((0,1),\mathbb{B})$ as follows: for $f \in C_c(\Gamma_\varphi|_Y)$, let $A = \operatorname{supp}(f) \cap \Gamma_\varphi|_{\mathcal{D}}$, and put

$$a(f) = \sum_{(x,p,y) \in A} f(x,p,y) e_{x,p,y}$$

and

$$b(f) = \sum_{(I,J)} f_{I,J} e_{I,J}$$

again with $f_{I,J}$ defined as in Lemma 6.22. It is straightforward to check that a and b extend to $*$-homomorphisms from $C_r^*(\Gamma_\varphi|_Y)$ into $\mathbb{A}$ and $C([0,1],\mathbb{B})$.

Lemma 6.27. *Let $f \in C_r^*(\Gamma_\varphi|_Y)$. Then $I(a(f)) = b(f)(0)$ and $U(a(f)) = b(f)(1)$.*

Proof. As the proof of Lemma 3.8. □

Consider the the C^*-algebra $\mathbb{D}$ given by

$$\mathbb{D} = \{(a,f) \in \mathbb{A} \oplus C([0,1],\mathbb{B}) \mid I(a) = f(0), U(a) = f(1)\}.$$

By Lemma 6.27, we have a map $\mu : C_r^*(\Gamma_\varphi|_Y) \to \mathbb{D}$ given by $\mu(f) = (a(f), b(f))$, which is seen to be injective and isometric.

Lemma 6.28. *The map $\mu : C_r^*(\Gamma_\varphi|_Y) \to \mathbb{D}$ is a $*$-isomorphism.*

Proof. The proof goes as the proof of Lemma 3.9, with an approximation argument towards the end as an added bonus. First, let $(a,f) \in \mathbb{D}$ and assume that the rank of a (as a compact operator) is finite. It follows from the definition of I that $I(a)$ is finite-rank in $C([0,1],\mathbb{B})$, and since $f(0) = I(a)$ and the rank of $f(t)$ is locally constant in t, we have that $f(t) \in \mathbb{B}$ is finite rank for any t. Proceeding as in the proof of Lemma 3.9, we may find a $g \in C_r^*(\Gamma_\varphi|_Y)$ such that $\mu(g) = (a,f)$. Next, for arbitrary $(a,f) \in \mathbb{D}$, let $n \in \mathbb{N}$ and define (a_n, f_n) by restricting a and f to points in the set $\sqcup_{i=-n}^n Y_n$. Then a_n and f_n are of finite rank, and $(a_n, f_n) \in \mathbb{D}$ with $(a_n, f_n) \to (a,f)$ as $n \to \infty$. Choosing elemens $g_n \in C_r^*(\Gamma_\varphi|_Y)$ with $\mu(g_n) = (a_n, f_n)$ gives a sequence $\{g_n\}$ which is Cauchy (since μ is an isometry), hence the limit point g is mapped to (a,f) by μ. □

Corollary 6.29. *With $I, U : C_r^*(\Gamma_\varphi|_{\mathcal{D}}) \to C_r^*(\Gamma_\varphi|_Z)$ defined as above, and I_*, U_* denoting the induced maps on K_0, we have $K_0(C_r^*(\Gamma_\varphi|_Y)) = \ker(I_* - U_*)$ and $K_0(C_r^*(\Gamma_\varphi|_Y)) = \operatorname{coker}(I_* - U_*)$.*

Proof. Immediate from Lemmas 3.9 and 1.27. □

These computations can be done for more complicated maps, but determining a matrix representation of $I_* - U_*$ quickly becomes rather involved. However, as we only need to determine the kernel and the cokernel of the map, the next lemma is all we need:

Proposition 6.30. *The map $I_* - U_* : K_0(C_r^*(\Gamma_\varphi|_{\mathcal{D}})) \to K_0(C_r^*(\Gamma_\varphi|_Z))$ is surjective.*

Proof. First, we choose a basis for $K_0(C_r^*(\Gamma_\varphi|_Z))$: Choose $N \in \mathbb{N}$ such that $\mathcal{C} \subseteq \sqcup_{i=-N}^N Y_i$, and let $I_1, \ldots, I_L$ be the connected components of $Y_N \setminus \mathcal{D}$. Each of these intervals Z_i yield a matrix unit $e_{Z_i,Z_i} \in C_r^*(\Gamma_\varphi|_Z)$, and with slight abuse of notation, we have $Z = [I_1] \sqcup \cdots \sqcup [I_L]$ – more precisely, each connected component of Z lies in exactly one of the sets $\bigcup_{n,m} \varphi^{-n}(\varphi^m(I_i))$, and these total orbits are disjoint sets. It follows that the K_0-classes of the matrix units $e_{I_1,I_1}, \ldots, e_{I_L,I_L}$ generate $\mathbb{Z}^L = K_0(C_r^*(\Gamma_\varphi|_Z))$. Now, fix an i between 1 and L, and denote by x and y the endpoints of I_i. Assume first that $i \neq L$, i.e. that I_i is not the rightmost interval of Y_N and y is not the right endpoint of Y_N. Since $y \in \mathcal{D} \cap Y_N$, there is a $y' \in \mathcal{C} \cap \varphi^{-j}(y)$ for some j. Let J be the connected component of Z with y' as the right endpoint, and J' the connected component of Z with y' as left endpoint. Since y' is critical, we have that the eventual valency $v_{y'}$ is either $(+,-)$ or $(-,+)$. If $v_{y'} = (+,-)$, we have $I(e_{y',+,y'}) = I(e_{y',-,y'}) = 0$ and

$$U\left(\tfrac{1}{2}(e_{y',+,y'} + e_{y',-,y'})\right) = \tfrac{1}{2}\left(e_{J,J} + e_{J',J'} + e_{J,J'} + e_{J',J}\right)$$

Since both J and J' are mapped to I_i by φ^j, it follows that the projection $\frac{1}{2}\left(e_{J,J} + e_{J',J'} + e_{J,J'} + e_{J',J}\right)$ is Murray-von Neumann equivalent to e_{I_i,I_i} via the partial isometry

$$v = \frac{1}{\sqrt{2}}\left(e_{J,I_i} + e_{J',I_i}\right).$$

It follows that on K_0, we have

$$I_* - U_*([\tfrac{1}{2}(e_{y',+,y'} + e_{y',-,y'}]) = [\tfrac{1}{2}([J,J + e_{J',J'} + e_{J,J'} + e_{J',J})] = [e_{I_i,I_i}]$$

If, on the other hand, $v_{y'} = (-,+)$, it follows that φ^j maps J and J' not to I_i, but I_{i+1}, and hence by the same calculations as above, we have

$$I_* - U_*([\tfrac{1}{2}(e_{y',+,y'} + e_{y',-,y'}]) = \left[\tfrac{1}{2}\left(e_{J,J} + e_{J',J'} + e_{J,J'} + e_{J',J}\right)\right] = [e_{I_{i+1},I_{i+1}}]$$

However, we note that $v_y = (+,+)$, so

$$I(e_{y,y}) = e_{I_{i+1},I_{i+1}}, \qquad U(e_{y,y}) = e_{I_i,I_i}$$

and hence

$$I_* - U_*([e_{y,y}]) = [e_{I_{i+1},I_{i+1}}] - [e_{I_i,I_i}]$$

Applying $I_* - U_*$ to the difference between $[\frac{1}{2}(e_{y',+,y'} + e_{y',-,y'})]$ and $[e_{y,y}]$ thus yields the element $[e_{I_i,I_i}]$. As noted, this argument works for all intervals I_i, except for the rightmost interval I_L. But here, applying the same argument to the left endpoint, with the roles of $(+,-)$ and $(-,+)$ interchanged, yields the desired result. □

This lemma allows to calculate the K-theory of $C_r^*(\Gamma_\varphi|_Y)$:

Proposition 6.31. *Let L and M be as in Remark 6.26. Then*

$$K_0(C_r^*(\Gamma_\varphi|_Y)) \simeq \mathbb{Z}^{M-L}, \qquad K_1(C_r^*(\Gamma_\varphi|_Y)) = 0.$$

Proof. Recall that $I_* - U_*$ is a linear map between $K_0(C_r^*(\Gamma_\varphi|_{\mathcal{D}})) \simeq \mathbb{Z}^M$ and $K_0(C_r^*(\Gamma_\varphi|_Z)) \simeq \mathbb{Z}^L$. By Proposition 6.30, it is surjective, so $\mathrm{Im}(I_* - U_*) \simeq \mathbb{Z}^L$ and hence

$$K_1(C_r^*(\Gamma_\varphi|_Y)) \simeq \mathrm{coker}(I_* - U_*) = 0.$$

Furthermore, the rank-nullity theorem tells us that

$$\dim(\ker(I_* - U_*)) + \dim(\mathrm{Im}(I_* - U_*)) = M,$$

so $K_0(C_r^*(\Gamma_\varphi|_Y)) \simeq \mathbb{Z}^{M-L}$. □

Using Bott periodicity on Equation (6.6) gives a six-term exact sequence on K-theory:

$$\begin{array}{ccccc} K_0(C_r^*(\Gamma_\varphi|_Y)) & \longrightarrow & K_0(C_r^*(\Gamma_\varphi)) & \longrightarrow & K_0(C(\Sigma)\rtimes_{\varphi,r}\mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(C(\Sigma)\rtimes_{\varphi,r}\mathbb{Z}) & \longleftarrow & K_1(C_r^*(\Gamma_\varphi)) & \longleftarrow & K_1(C_r^*(\Gamma_\varphi|_Y)) \end{array} \tag{6.11}$$

The K-theory of $C(\Sigma)\rtimes_{\varphi,r}\mathbb{Z}$ is calculated in Theorem 5.3 of [28], and we have, under the assumption of Remark 6.17, that

$$K_0(C(\Sigma)\rtimes_{\varphi,r}\mathbb{Z}) = \mathbb{Z}^2, \;\; K_1(C(\Sigma)\rtimes_{\varphi,r}\mathbb{Z}) = \mathbb{Z}.$$

Using this and Proposition 6.31 on (6.11), we get the following sequence:

$$\begin{array}{ccccc} \mathbb{Z}^{M-L} & \longrightarrow & K_0(C_r^*(\Gamma_\varphi)) & \longrightarrow & \mathbb{Z}^2 \\ \uparrow{\scriptstyle\delta} & & & & \downarrow \\ \mathbb{Z} & \xleftarrow{r_0} & K_1(C_r^*(\Gamma_\varphi)) & \longleftarrow & 0 \end{array} \tag{6.12}$$

where r_0 is the induced map from the restriction $r : C_r^*(\Gamma_\varphi) \to C_r^*(\Gamma_\varphi|_\Sigma)$, and £$\delta$ the index map. We observe that the map r_* from $K_1(C_r^*(\Gamma_\varphi))$ to $K_1(C_r^*(\Gamma_\varphi|_\Sigma)) \simeq \mathbb{Z}$ is injective, so $K_1(C_r^*(\Gamma_\varphi))$ is either $\mathbb{Z}$ or 0. We claim the following:

Proposition 6.32. *The map $r_0 : K_1(C_r^*(\Gamma_\varphi)) \to K_1(C_r^*(\Gamma_\varphi|_\Sigma))$ is surjective. In particular, $K_1(C_r^*(\Gamma_\varphi)) \simeq \mathbb{Z}$.*

We write $[a]_0$ and $[a]_1$ for the K_0- and K_1-classes of elements a of a C^*-algebra. Since $C_r^*(\Gamma_\varphi|_\Sigma) \simeq C(\Sigma)\rtimes_r\mathbb{Z}$, its K_1-group is generated by the unitary element $u = 1_{\{(x,1,\varphi(x)|x\in\Sigma\}}$. Showing that r_0 is surjective is equivalent to showing that the index map δ is zero, which in turn follows if $\delta([u]_1) = 0$.

Proof. Note that $C_r^*(\Gamma_\varphi)$ is unital. We use the picture of the index map given in Theorem 9.2.3 of [34]: Let $\widetilde{C_r^*(\Gamma_\varphi|_Y)}$ be the unitization of $C_r^*(\Gamma_\varphi|_Y)$ and $\bar{\imath} : \widetilde{C_r^*(\Gamma_\varphi|_Y)} \to C_r^*(\Gamma_\varphi)$ the map induced by the inclusion. Let $s : \widetilde{C_r^*(\Gamma_\varphi|_Y)} \to \widetilde{C_r^*(\Gamma_\varphi|_Y)}$ be the scalar mapping (i.e. $s(x+\alpha 1) = \alpha 1$ for $x \in C_r^*(\Gamma_\varphi|_Y)$). The map δ is then defined as follows: Given a unitary lift $V \in M_2(C_r^*(\Gamma_\varphi))$ of $\operatorname{diag}(u,u^*)$ and a projection $p \in M_2(\widetilde{C_r^*(\Gamma_\varphi|_Y)})$ with $\bar{\imath}(p) = V\operatorname{diag}(1,0)V^*$, we have

$$\delta([u]_1) = [p]_0 - [s(p)]_0.$$

To construct V, let $v \in C_c(\Gamma_\varphi)$ be a real-valued function such that

$$\operatorname{supp}(v) \subseteq \{(x,k,p,y) \in \Gamma_\varphi | k = 1, p = +\},$$

such that $v(x,1,+,\varphi(x)) = 1$, such that $v(x,k,p,y) = 0$ for $(x,k,p,y) \in \Gamma_\varphi|_{\mathcal{D}}$, and finallysuch that φ is one-to-one at x if $(x,1,+,\varphi(x)) \in \operatorname{supp}(v)$. Essentially, v is a bump function for the set $\{(x,1,\varphi(x))|x\in\Sigma\}$. It follows that $r(v) = u$, and that

$$v^*v(x,k,y) = v(y,0,y)^2$$

if $x = y$ and $k = 0$, and zero elsewhere. Define $V \in M_2(\widetilde{C_r^*(\Gamma_\varphi)})$ by

$$V = \begin{bmatrix} v & (1-v^*v)^{1/2} \\ -(1-v^*v)^{1/2} & v^* \end{bmatrix}.$$

Then V is unitary in $M_2(C_r^*(\Gamma_\varphi))$ and a lift of $\mathrm{diag}(u, u^*)$. It follows that

$$V \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} V^* = \begin{bmatrix} vv^* & -(1-v^*v)^{1/2}v^* \\ -(1-v^*v)^{1/2}v & 1-v^*v \end{bmatrix}$$
$$= \begin{bmatrix} vv^*-1 & -(1-v^*v)^{1/2}v^* \\ -(1-v^*v)^{1/2}v & 1-v^*v \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} =: V' + \mathrm{diag}(1,0).$$

The matrix V' is in $M_2(C_r^*(\Gamma_\varphi|_Y))$, so $p = V' + \mathrm{diag}(1,0) \in \widetilde{C_r^*(\Gamma_\varphi|_Y)}$ and $\bar{i}(p) = V\,\mathrm{diag}(1,0)V^*$. We see that $s(p) = \mathrm{diag}(1,0)$. As stated above, we want to show that $[p]_0 = [s(p)]_0$. To do this, recall that we have another restriction map in play – namely the map $r_{|\mathcal{D}} : C_r^*(\Gamma_\varphi|_Y) \to C_r^*(\Gamma_\varphi|_\mathcal{D})$. By looking at Equation (6.9), we see that the induced map $(r_\mathcal{D})_0$ on K_0 is injective, so it suffices to show that

$$(r_\mathcal{D})_0([p]_0) = (r_\mathcal{D})_0([s(p)]_0)$$

in $K_0(\widetilde{C_r^*(\Gamma_\varphi|_\mathcal{D})})$. By construction, we have $r_\mathcal{D}(v) = 0$, so since $\mathcal{D}$ is totally invariant, we also have

$$r_\mathcal{D}(vv^*) = r_\mathcal{D}((1-v^*v)^{1/2}v) = r_\mathcal{D}(v^*(1-v^*v)^{1/2}) = 0$$

It follows that

$$r_\mathcal{D}(p) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = r_\mathcal{D}(s(p))$$

with $\sim$ denoting Murray-von Neumann equivalence in $M_2(\widetilde{C_r^*(\Gamma_\varphi|_\mathcal{D})})$. This is what we wanted, so we conclude that $\delta([u]_1) = 0$. □

Theorem 6.33. *Let φ be a Denjoy map with constants L and M as in Remark 6.26. Then $K_0(C_r^*(\Gamma_\varphi)) \simeq \mathbb{Z}^{M-L+2}$ and $K_1(C_r^*(\Gamma_\varphi)) \simeq \mathbb{Z}$.*

Proof. We know already that $K_1(C_r^*(\Gamma_\varphi)) \simeq \mathbb{Z}$ and that the index map δ in Equation 6.12 is zero. It follows then that the sequence

$$0 \longrightarrow \mathbb{Z}^{M-L} \longrightarrow K_0(C_r^*(\Gamma_\varphi)) \longrightarrow \mathbb{Z}^2 \longrightarrow 0 \tag{6.13}$$

is exact, hence $K_0(C_r^*(\Gamma_\varphi)) \simeq \mathbb{Z}^{M-L+2}$. □

Chapter 7

Return of the core algebras

In this chapter, we return to the core algebras $C_r^*(R_\varphi)$ and $C_r^*(R_\varphi^+)$ from Chapter 3. The ultimate goal of this chapter is to show that in a number of situations, the algebra $C_r^*(R_\varphi)$ is an AF-algebra, while the algebra $C_r^*(R_\varphi^+)$ is not. We do this by proving a number of structural results about the algebras, and then appealing to some deep classificiation results. These results typically assume simplicity of the algebras, so we begin by obtaining some dynamical consequences of this assumption. I would like to stress that everything in this chapter is joint work with Benjamin Johannesen, and some of the results (Lemmas 7.1, 7.6, 7.7 and 7.9) appeared first in his Qualification Exam report [14].

Proposition 7.1. *Let φ be a circle map, and assume that $C_r^*(R_\varphi)$ is simple. Then φ is critically finite and transitive.*

Proof. Let $C_r^*(R_\varphi)$ be simple. This means that the orbit $[x] = [x]_{R_\varphi}$ of any point $x \in \mathbb{T}$ is dense in $\mathbb{T}$ – otherwise, the reduction $R_\varphi|_{\mathbb{T}\setminus\overline{[x]}}$ would give rise to an ideal $C_r^*(R_\varphi|_{\mathbb{T}\setminus\overline{[x]}})$ of $C_r^*(R_\varphi)$ by Lemma 1.16. Now, assume towards a contradiction that φ has a critical point c whose forward orbit $\mathcal{O}^+(c)$ is infinite. $\mathcal{O}^+(c)$ may contain other critical points, but since the number of critical points of φ is finite, there is a number $j \in \mathbb{N}$ such that $d = \varphi^j(c)$ is critical for φ, and such that $\mathcal{O}^+(d)$ contains no critical points. Note that $\mathrm{val}(\varphi^n, d) \in \{(+,-),(-,+)\}$ for all $n \in \mathbb{N}$. We have

$$[d]_{R_\varphi} = \{x \in \mathbb{T} \mid \exists n : \varphi^n(x) = \varphi^n(d), \mathrm{val}(\varphi^n, x) = \mathrm{val}(\varphi^n, d)\}$$

Let $\{e_1, \ldots, e_N\}$ be the set of critical points of φ whose forward orbit meets the forward orbit of d. For each e_i, we may choose $n_i \in \mathbb{N}$ minimal such that $\varphi^{n_i}(e_i) = \varphi^{m_i}(d)$ for some $m_i \in \mathbb{N}$. The number m_i is unique – otherwise, d would be pre-periodic. If $x \in [d]_{R_\varphi}$, x has to be pre-critical, and it then follows that $x \in \varphi^{m_i - n_i}(e_i)$ for some i. Hence

$$[d]_{R_\varphi} \subseteq \bigcup_{i=1}^{N} \varphi^{m_i - n_i}(e_i)$$

which is a finite set. But this is a contradiction, so φ must be critically finite.

For the second statement, note that simpicity of $C_r^*(R_\varphi)$ entails that $[x]_{R_\varphi}$ is dense for any $x \in \mathbb{T}$ by Lemma 1.16. But $[x]_{R_\varphi} \subset [x]_{\Gamma_\varphi}$, so any Γ_φ-orbit is dense. But then $C_r^*(\Gamma_\varphi)$ is simple Lemma 4.10, and then φ is transitive by Lemma 4.14. □

Note that the same proof works for $C_r^*(R_\varphi^+)$. For the rest of this chapter, we assume that the map φ is critically finite.

7.1 Recursive subhomogeneous algebras

We begin by showing that $C_r^*(R_\varphi)$ and $C_r^*(R_\varphi^+)$ are inductive limits of recursive subhomogeneous algebras, as defined by Phillips in [25]:

Definition 7.2. Let M_n denote the complex n-by-n-matrices. The class $\mathcal{R}$ of *recursive subhomogeneous C^*-algebras* (or RSH-algebras) is the smallest class of C^*-algebras such that

- If X is a compact Hausdorff space and $n \geq 1$, then $C(X, M_n) \in \mathcal{R}$, and
- If $A \in \mathcal{R}$, X is a compact Hausdorff space, $X^{(0)} \subseteq X$ is closed, $\varphi : A \to C(X^{(0)}, M_n)$ is a unital $*$-homomorphism and $\rho : C(X, M_n) \to C(X^{(0)}, M_n)$ is the restriction map, then the pullback
$$A \oplus_{C(X^{(0)}, M_n)} C(X, M_n) = \{(a, f) \in A \oplus C(X, M_n) \mid \varphi(a) = \rho(f)\}$$
is in $\mathcal{R}$.

Note that $X^{(0)} = \emptyset$ is allowed, in which case the pullback above is simply a direct sum.

Lemma 7.3. *Let $k \in \mathbb{N}$. Then the algebras $C_r^*(R_\varphi(k))$ and $C_r^*(R_\varphi^+(k))$ are RSH-algebras.*

Proof. Recall from Lemma 3.9 that the algebra $C_r^*(R_\varphi(k))$ is isomorphic to the algebra $\mathbb{D}_k$ given by
$$\mathbb{D}_k = \{\mathbb{A}_k \oplus C([0,1], \mathbb{B}_k) \mid I_k(a) = f(0), U_k(a) = f(1)\}$$
for some unital $*$-homomormphisms $I_k, U_k : \mathbb{A}_k \to \mathbb{B}_k$ with $\mathbb{A}_k$ and $\mathbb{B}_k$ finite dimensional. Now, $\mathbb{A}_k$ is an RSH-algebra, and, writing $\mathbb{B}_k = \oplus_{i=1}^n M_{n_i}$, each algebra $C([0,1], M_{n_i})$ is RSH. Let I_k^i and U_k^i denote the partial maps between $\mathbb{A}_k$ and M_{n_i}, and put $X = [0,1]$ and $X^{(0)} = \{0,1\}$. Then $(I_k^1, U_k^1) : \mathbb{A}_k \to C(X^{(0)}, M_{n_1})$ is a unital $*$-homomorphism, so
$$\mathbb{D}_k^1 = \left\{(a, f) \in \mathbb{A}_k \oplus C([0,1], M_{n_1}) \,\middle|\, I_k^1(a) = f(0), U_k^1(a) = f(1)\right\}$$
is an RSH-algebra. But then
$$\mathbb{D}_k^2 = \left\{(a, f, f') \in \mathbb{D}_k^2 \oplus C([0,1], M_{n_2} \,\middle|\, I_k^2(a) = f'(0), U_k^2(a) = f'(1)\right\}$$
is also an RSH-algebra, and, inductively, we get that $\mathbb{D}_k$ is RSH. The same proof goes for $C_r^*(R_\varphi^+)$. □

As in Chapter 5, we put
$$\mathcal{D} = \varphi\Big(\bigcup_{c\in\mathcal{C}} \bigcup_{i=0}^{\infty} \varphi^i(c)\Big)$$
When φ is critically finite, $\mathcal{D}$ is a finite set. As we have seen, for any $k \in \mathbb{N}$, the number of summands in $\mathbb{A}_k$, $\mathbb{A}_k^+$, $\mathbb{B}_k$, $\mathbb{B}_k^+$ are all bounded by some constant times the number of elements of $\mathcal{D}$. In the language of RSH-algebras (see definition 1.2 in [25]), this means that the *length* of the decompositions of $\mathbb{D}_k$ and $\mathbb{D}_k^+$ given above is uniformly bounded. Furthermore, the base space for both $\mathbb{D}_k$ and $\mathbb{D}_k^+$ is a finite disjoint union of spaces homeomorphic to the unit interval $[0,1]$; in particular, the *dimension functions* for all the algebras $\mathbb{D}_k$ and $\mathbb{D}_k^+$ are bounded by the constant 1. It follows that the inductive systems
$$\{C_r^*(R_\varphi(k)), \rho_k\}_{k\in\mathbb{N}}, \quad \{C_r^*(R_\varphi^+(k)), \rho_k^+\}_{k\in\mathbb{N}}$$

where ρ_k and ρ_k^+ are the inclusion maps, have *no dimension growth*. The limits of both systems, $C_r^*(R_\varphi)$ and $C_r^*(R_\varphi^+)$, are infinite-dimensional (since they each contain a copy of $C(\mathbb{T})$). It follows from Corollary 1.9 of [24] that both systems have *strict slow dimension growth*. We will need this fact in a bit.

7.2 Traces on $C_r^*(R_\varphi)$ and $C_r^*(R_\varphi^+)$

To understand the core algebras better, we now determine their tracial states. As we shall see, simplicity of the algebras implies that they each have a unique tracial state. We prove this first for $C_r^*(R_\varphi^+)$, and then use results of Nesvehyev to conclude the same for $C_r^*(R_\varphi)$. The lemma below reveals a close connection between tracial states on the core algebras and invariant measures on the circle. Generally, given a groupoid G with unit space G^0 and range and source maps $r, s : G \to G^0$, a measure μ on G^0 is G-invariant if $\mu(r(W)) = \mu(s(W))$ for any bisection $W \subseteq G$.

Lemma 7.4. *If μ is a regular R_φ^+-invariant Borel probability measure on $\mathbb{T}$, the state given by*

$$\omega(f) = \int_{\mathbb{T}} f(x,x)\mu(dx), f \in C_c(R_\varphi^+)$$

is a tracial state on $C_r^*(R_\varphi^+)$. Conversely, given a tracial state on $C_r^*(R_\varphi^+)$, it is given by integration against a regular R_φ-invariant Borel probability measure as above.

Proof. Since R_φ^+ is principal, this result follows from Lemmas 3.4.4 and 3.4.5 of [27]. □

Lemma 7.5. *Let $\varphi : \mathbb{T} \to \mathbb{T}$ be a uniformly piecewise linear transitive circle map. Then there exists a R_φ^+-invariant Borel probability measure on $\mathbb{T}$.*

Proof. Let λ be the normalised Lebesgue measure on $\mathbb{T}$. Let W be a bisection. Then we may assume that W has the form

$$W = \left\{ (x,y) \in \mathbb{T} \times \mathbb{T} \,\middle|\, x \in I, y \in J, \varphi^k(x) = \varphi^k(y), \mathrm{val}(\varphi^k, x) = \mathrm{val}(\varphi^k, y) \right\}$$

where I and J are open intervals. By the assumption on φ, it follows that

$$\lambda(r(W)) = \lambda(I) = \lambda(J) = \lambda(s(W))$$

It follows that λ is R_φ^+-invariant. □

Lemma 7.6. *Let μ be a Borel probability measure on $\mathbb{T}$ that is R_φ^+-invariant, and assume that $C_r^*(R_\varphi)$ is simple. Then μ is non-atomic.*

Proof. Assume towards a contradiction that there is an $x_0 \in \mathbb{T}$ such that $\mu(\{x_0\}) > 0$. Since $C_r^*(R_\varphi^+)$ is simple, the orbit $[x_0]_{R_\varphi^+}$ of x_0 is infinite. Clearly, we are done if we can show that $\mu(\{x\}) = \mu(\{x_0\})$ for any $x \in [x_0]_{R_\varphi^+}$, so choose such an x and a $\gamma \in R_\varphi^+$ with $r(\gamma) = x$ and $s(\gamma) = x_0$. Choose a decreasing sequence of open bisections W_n with $\bigcap_n W_n = \{\gamma\}$. Then

$$\mu(\{x\}) = \mu\Big(\bigcap_n r(W_n)\Big) = \lim_{n\to\infty} \mu(r(W_n)) = \lim_{n\to\infty} \mu(s(W_n)) = \mu\Big(\bigcap_n s(W_n)\Big) = \mu(\{x_0\}).$$

□

Lemma 7.7. *Assume that $C_r^*(R_\varphi^+)$ is simple. Then there is a unique tracial state on $C_r^*(R_\varphi^+)$.*

Proof. Since $C_r^*(R_\varphi^+)$ is simple, φ is transitive, hence conjugate to a uniformly piecewise monotone map ψ. Since $\Gamma_\psi^+ \simeq \Gamma_\varphi^+$ by Remark 2.23, their groupoid C^*-algebras are isomorphic, so we lose no generality assuming that our map is uniformly piecewise monotone. Then existence follows from the existence of a R_φ^+-invariant measure in Lemma 7.5. For uniqueness, note that Lemma 3.2 of [42] says that $C_r^*(R_\varphi^+)$ has at most one non-atomic tracial state, and Lemmas 7.4 7.6 says that any tracial state is non-atomic. □

Proposition 7.8. *Any trace on $C_r^*(R_\varphi)$ factors through $C_r^*(R_\varphi^+)$. In particular, $C_r^*(R_\varphi)$ has a unique tracial state.*

Proof. Let $Q : C_r^*(R_\varphi) \to C_r^*(R_\varphi^+)$ denote the restriction map, let $f \in C_c(R_\varphi)$, and write $f = f_+ + f_-$, with $f_+ \in C_c(R_\varphi^+)$ and $f_- \in C_c(\{(x,p,y) \in R_\varphi | p = -\}$. Since $f_+ = Q(f)$, we are done if we can show that $\tau(f_-) = 0$. Choose a probability measure μ on $\mathbb{T}$ and a field of states $\{\varphi_x\}_x$ representing τ, cf. [20], i.e. such that

$$\tau(h) = \int_{\mathbb{T}} \sum_{g \in \mathrm{Iso}(x)} h(g)\varphi_x(u_g)d\mu(x)$$

for any $h \in C_c(R_\varphi)$. Consider the function F given by

$$F(x) = \sum_{g \in \mathrm{Iso}(x)} f_-(g)\varphi_x(u_g), \quad x \in \mathbb{T}.$$

Given $x \in \mathbb{T}$ such that x is non-critical for any φ^k, the isotropy group $\mathrm{Iso}(x)$ consists of the single element $(x,+,x)$. Since $f_-(x,+,x) = 0$, we have $F(x) = 0$. The set of $x \in \mathbb{T}$ critical for some φ^k is countable, so the function F is zero μ-almost everywhere. By Lemma 7.6, μ is non-atomic, so we have $\tau(f_-) = \int_{\mathbb{T}} F(x)d\mu(x) = 0$. □

Lemma 7.9. *Let φ be uniformly piecewise linear, and let τ be the unique tracial state on $C_r^*(R_\varphi^+)$. Then for any $\varepsilon > 0$, there is a projection p in $C_r^*(R_\varphi^+)$ with $\tau(p) < \varepsilon$.*

Proof. Recall that $C_r^*(R_\varphi^+)$ is the infinite-dimensional limit of the inductive system $\{C_r^*(R_\varphi^+(k)), \rho_k\}$, that all the inclusion maps ρ_k are unital and injective, and that each $C_r^*(R_\varphi^+(k))$ is isomorphic to the algebra $\mathbb{D}_k$, as discussed in Lemma 7.3. Since $C_r^*(R_\varphi^+)$ is the infinite-dimensional inductive limit of a system of recursive subhomogeneous algebras with unital and injective inclusion maps, Lemma 1.8 of [25] shows the following: Let ρ_{ij} be the composition $\rho_{j-1} \circ \cdots \circ \rho_{i+1} \circ \rho_i : \mathbb{D}_i \to \mathbb{D}_j$. Then, given an $n \in \mathbb{N}$, for any non-zero $d \in \mathbb{D}_i$ (for any i), there is a j_0 such that for $j \geq j_0$,

$$\mathrm{rank}(\mathrm{ev}_x(\rho_{ij}(d))) \geq n$$

which in our notation simply boils down to the fact the the matrix dimension of any summand of $\mathbb{B}_k$ goes to infinity as k goes to infinity. In particular it follows that there is a k and a pair of intervals $(I,J) \in \mathcal{I}_k^{(2)}$ such that the set $v(I,J)$, consisting of I, J and the shortest of the two arcs on $\mathbb{T}$ between I and J, has length strictly less than ε. To prove the lemma, we proceed to construct a projection $p \in C_c(R_\varphi^+)$ such that p restricted to $\mathbb{T}$ is supported in $v(I,J)$, and such that $|p(x,y)| \leq 1$ for all $(x,y) \in R_\varphi^+$. If we can do this, it follows

$$\tau(p) = \int_{\mathbb{T}} pdt < \varepsilon$$

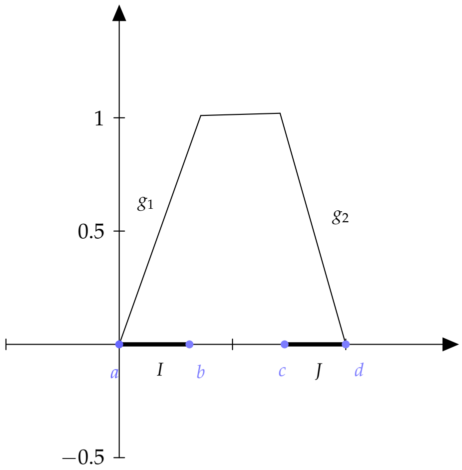


Figure 7.1: The map g defined on the set $v(I,J)$..

and we are done. p is essentially constructed as a Rieffel projection, introduced in [32]. Write $I = (a,b)$ and $J = (c,d)$, and assume that $v(I,J)$ is the set (a,d). Let $g_1 \in C(I)$ be any function such that $0 \leq g_1(t) \leq 1$, such that

$$\lim_{t \to a^+} g_1(t) = 0, \qquad \lim_{t \to b^-} g_1(t) = 1.$$

Let $\gamma : I \to J$ be the map $\lambda_J \circ \varphi^k$, which is a homeomorphism with inverse $\gamma^{-1} = \lambda_I \circ \varphi^k$. Define $g_2 \in C(J)$ by $g_2(t) = 1 - g_1(\gamma(t))$. Then

$$\lim_{t \to c^-} g_2(t) = 1 \qquad \lim_{t \to d^+} g_2(t) = 0$$

Finally, define g on $v(I,J)$ by putting $g \equiv g_1$ on I, $g \equiv g_2$ on J and $g \equiv 1$ on (b,c). Put

$$p(z,w) = \begin{cases} g(z) & \text{if } z = w \in v(I,J) \subseteq T \\ \sqrt{g(z) - g(z)^2} & \text{if } z \in I, w \in J, (z,w) \in R_\varphi^+(k), \\ \sqrt{g(w) - g(w)^2} & \text{if } z \in J, w \in I, (z,w) \in R_\varphi^+(k) \end{cases} \tag{7.1}$$

and 0 elsewhere. Checking that $p^* = p$ is straightforward, and checking that $p^2 = p$ is a simple calculation: Then, if $x \in I$, we have $s(r^{-1}(x) \cap \operatorname{supp}(p)) = \{x, \gamma(x)\}$, so

$$p^2(x,x) = p(x,\gamma(x))p(\gamma(x),x) + p(x,x)^2 = g(x) - g(x)^2 + g(x)^2 = g(x) = p(x,x).$$

The same calculation shows that $p^2(x,x) = p(x,x)$ when $x \in J$, and when $x \in v(I,J)$, but not in I or J, have $s(r^{-1} \cap \operatorname{supp}(p)) = \{x\}$, so

$$p^2(x,x) = p(x,x)p(x,x) = 1 = p(x,x).$$

Likewise, the calculation

$$\begin{aligned} p^2(x,\gamma(x)) &= p(x,x)p(x,\gamma(x)) + p(x,\gamma(x))p(\gamma(x),\gamma(x)) \\ &= g(x)\sqrt{g(x) - g(x)^2} + g(\gamma(x))\sqrt{g(x) - g(x)^2} = \sqrt{g(x) - g(x)^2} = p(x,\gamma(x)) \end{aligned} \tag{7.2}$$

shows that $p^2(x, \gamma(x)) = p(x, \gamma(x))$ when $x \in I$, and a similar calculation does the trick when $x \in J$. In all other cases, $p^2(x, y) = 0 = p(x, y)$, which shows that p is a projection. □

The use of the Rieffel projection in the proof above was communicated to me by Benjamin Johannesen, who in turn got the idea from conversations with Ian Putnam.

Corollary 7.10. *Let τ be the unique tracial state on $C_r^*(R_\varphi^+)$, and $\tau_* : K_0(C_r^*(R_\varphi^+)) \to \mathbb{R}$ the induced map. Then the image $\tau_*(K_0(C_r^*(R_\varphi^+)))$ is dense in $\mathbb{R}$.*

Since the unique tracial state on $C_r^*(R_\varphi)$ factors through $C_r^*(R_\varphi^+)$, the result above also holds for $C_r^*(R_\varphi)$.

7.3 A classification result

Let us recapitulate what we have shown so far: Assuming that the algebras $C_r^*(R_\varphi)$ and $C_r^*(R_\varphi^+)$ are simple, they each have a unique tracial state whose image in $\mathbb{R}$ is dense. Furthermore, $K_1(C_r^*(R_\varphi)) = 0$ by Lemma 3.14 and continuity of K_1. Finally, $C_r^*(R_\varphi^+)$ is the fixed point-algebra of the order-two automorphism Λ restricted to $C_r^*(R_\varphi)$. We now boldy claim the following:

Theorem 7.11. *Assume that $C_r^*(R_\varphi)$ is simple. Then it is an AF-algebra.*

Proving this will occupy the rest of the section, and requires some deep results. The proof relies heavily on the K-theory of AF-algebras, which we outline briefly (for more, see e.g. [34]): Assume that $\mathcal{A}$ is AF. Then $(K_0(\mathcal{A}), K_0(\mathcal{A})^+)$ is a *dimension group*, i.e. the limit of a system of ordered, free abelian groups:

$$\mathbb{Z}^{n_1} \xrightarrow{\alpha_1} \mathbb{Z}^{n_2} \xrightarrow{\alpha_2} \mathbb{Z}^{n_3} \longrightarrow \dots \tag{7.3}$$

with each $\mathbb{Z}^n$ given the usual ordering and the α's being positive group homomorphisms. Conversely, given a dimension group, there is an AF-algebra with K_0 isomorphic to this group. The Effros-Handelman-Shen Theorem (see [12]) characterises dimension groups intrinsically:

Theorem 7.12. *An ordered Abelian group (G, G^+) is a dimension group if and only if it is*

- unperforated*: For any $x \in G$, $nx \geq 0$ for some $n \in \mathbb{N}$ implies $x \geq 0$, and*
- *satisfies the* Riesz interpolation property*: if $x_1, x_2, y_1, y_2 \in G$ with $x_i \leq y_j$ for $i, j = 1, 2$, there is a $z \in G$ with $x_i \leq z \leq y_j$ for $i, j = 1, 2$.*

Finally, any AF-algebra has trivial K_1-group. The general strategy for proving Theorem 7.11 is then as follows: First, we prove that $C_r^*(R_\varphi)$, when simple, has the K-theory of an AF-algebra. Then we appeal to some general classification results for limits of RSH-algebras to show that if it quacks like an AF-algebra and walks like an AF-algebra, then...

We need a general result. We say that a C^*-algebra $\mathcal{A}$ satisfies *Blackadars Second Comparability Condition* if the partial order on projections in $M_\infty(\mathcal{A})$ is determined by traces: If $p, q \in M_\infty(\mathcal{A})$ with $\tau(p) < \tau(q)$ for all normalised traces on $\mathcal{A}$, we have $p \leq q$.

Lemma 7.13. *Let $\mathcal{A}$ be a simple C^*-algebra satisfying Blackadars Second Comparability Condition, and assume that $\mathcal{A}$ has a unique tracial state τ such that $\tau(\mathcal{P}_\infty(\mathcal{A}))$ is dense in $\mathbb{R}$. Then $K_0(\mathcal{A})$ has the Riesz interpolation property.*

Proof. Let $x_1, x_2, y_1, y_2 \in K_0(\mathcal{A})^+$ with $x_i \leq y_j$ for $i = 1, 2$, and choose projections $X_i, Y_j \in M_\infty(\mathcal{A})$ representing these. By assumption, $\tau(X_i) \leq \tau(Y_j)$ for $i, j = 1, 2$. Assume first that the traces of the X_i's are strictly less than those of the Y_j's, say $\tau(X_1) \leq \tau(X_2) < \tau(Y_1) \leq \tau(Y_2)$. Since the image of τ is dense in $\mathbb{R}$, there is a projection Z in some $M_n(\mathcal{A})$ with $\tau(X_2) < \tau(Z) < \tau(Y_1)$. By the Comparability Condition, the K_0-class of Z is an interpolating element between the x_i's and y_j's. On the other hand, assume that we have equality of some of the traces, say $\tau(X_2) = \tau(Y_1)$. Since $x_2 \leq y_1$, there are projections Z, R in some $M_n(\mathcal{A})$ such that $Y_1 \oplus R \sim X_2 \oplus Z \oplus R$. We get that $\tau(R) = 0$, and since the set $\{A \in M_n(\mathcal{A}) | \tau(A^*A) = 0\}$ is an ideal (and hence equal to $\{0\}$), we must have $R = 0$. Thus, $X_2 \sim Y_1$, so $x_2 = y_1$ works as an interpolating element. □

With these lemmas in place, we can show that $C_r^*(R_\varphi)$ has the K-theory of an AF-algebra:

Proposition 7.14. *Assume that $C_r^*(R_\varphi)$ is simple. Then $K_0(C_r^*(R_\varphi))$ is a dimension group.*

Proof. We know that $C_r^*(R_\varphi)$ is the unital direct limit of the system $(C_r^*(R_\varphi(k)), i_k)$ of recursive subhomogeneous algebras. Since the system has slow dimension growth, it follows from Theorem 0.1 of [24] that $K_0(C_r^*(R_\varphi))$ is unperforated and that $C_r^*(R_\varphi)$ satisfies Blackadars Second Comparability Condition. By Lemmas 7.9 and 7.13, $K_0(C_r^*(R_\varphi))$ has the Riesz interpolation property. By the Effros-Handelman-Shen Theorem, it is a dimension group. □

Proof (Of Theorem 7.11). By Proposition 7.14, we know that $K_0(C_r^*(R_\varphi))$ is a dimension group. Since $C_r^*(R_\varphi)$ is assumed simple and admits a trace by 7.8, it is stably finite by Theorem 2.2 of [33]. But then $K_0(C_r^*(R_\varphi))$ is a simple ordered group by Theorem 6.3.5 of [5]. It follows that there is an AF-algebra $\mathcal{A}$ whose K-theory is isomorphic to the K-theory of $C_r^*(R_\varphi)$, and then that $\mathcal{A}$ is simple since $K_0(\mathcal{A})$ is. But by [48], Corollary 1.4, ordered K-theory is a complete invariant in the class of unital simple separable limits of recursive subhomogeneous C^*-algebras with slow dimension growth whose projections separate traces (phew!), in particular for those algebras with a unique tracial state. It follows that $C_r^*(R_\varphi) \simeq \mathcal{A}$. □

To finish the programme laid out in the beginning of this chapter, we need to show that $C_r^*(R_\varphi)$ has non-trivial K_1 whenever it is simple. Recall that the K_1-groups of the building blocks $C_r^*(R_\varphi^+(k))$ all were non-trivial – if we were able to show the same for $C_r^*(R_\varphi^+)$, we would have an example of an AF-algebra (i.e. $C_r^*(R_\varphi)$ whenever it is simple) with an order two-automorphism whose fixed-point algebra is not AF. We know, by continuity of K_1, that

$$K_1(C_r^*(R_\varphi^+)) = \varinjlim(K_1(C_r^*(R_\varphi^+(k))), (\rho_k)_1)$$

where ρ_k is the inclusion from $C_r^*(R_\varphi^+(k))$ to $C_r^*(R_\varphi^+(k+1))$. But in the critically finite case, Equation 5.21 gives a formula for this map. Let's put that to use:

Proposition 7.15. *Let φ be a transitive, critically finite circle map. Then $K_1(C_r^*(R_\varphi^+))$ is non-trivial.*

Proof. Let's recall the setup from Lemma 3.15: For each k, there is a map $(I_k)_0 - (U_k)_0 : K_0(\mathbb{A}_k) \to K_0(\mathbb{B}_k)$ such that

$$K_1(C_r^*(R_\varphi^+(k))) \simeq \operatorname{coker}((I_k)_0 - (U_k)_0).$$

As we saw in Lemma 3.15, this map is not surjective, and its image is contained in the subspace V of $K_0(\mathbb{B}_k)$ defined by

$$V = \left\{ \sum_{I \in \mathcal{I}} c_{I,+}[I,(+,+)] + c_{I,-}[I,(-,-)] \;\middle|\; \sum_{I \in \mathcal{I}} c_{I,+} - c_{I,-} = 0 \right\}$$

Recall furthermore that $C_r^*(R_\varphi)$ is the limit of the building block algebras $C_r^*(R_\varphi^+(k))$, so by continuity of K_1, it follows that

$$K_1(C_r^*(R_\varphi^+)) = \varinjlim_k (K_1(C_r^*(R_\varphi(k))), (\rho_k)_1) \tag{7.4}$$

with ρ_k denoting the inclusion from $C_r^*(R_\varphi(k))$ to $C_r^*(R_\varphi(k+1))$. In Equation 5.21, we found a formula for a map $B : K_0(\mathbb{B}_k) \to K_0(\mathbb{B}_{k+1})$ making the diagram

$$\begin{array}{ccc} K_0(\mathbb{B}_k) & \xrightarrow{\pi} & K_1(C_r^*(R_\varphi^+(k))) \\ \downarrow B & & \downarrow \rho_1 \\ K_0(\mathbb{B}_{k+1}) & \xrightarrow{\pi} & K_1(C_r^*(R_\varphi^+(k+1))) \end{array} \tag{7.5}$$

commute, with π denoting the quotient map. To prove the proposition, we find an element v in the complement of V such that $B^n(v) \notin V$ for all n. It then follows that v gives rise to a non-zero element in the inductive limit $\varinjlim_k (K_1(C_r^*(R_\varphi(k))), (\rho_k)_1)$. First, lets describe the map B in some detail: Let $J \in \mathcal{I}$, $v \in \{(+,+), (-,-)\}$, and let $[J, v]$ denote the corresponding basis element of $K_0(\mathbb{B}_k)$. By Equation 5.21, we have

$$B[J, v] = (-1)^{\mathrm{val}(\varphi, J')}[\varphi(J'), v \bullet \mathrm{val}(\varphi, J')]$$

where $J' \subseteq J$ is an interval such that $\varphi(J') \in \mathcal{I}$ and $J' \cap \mathcal{C} = \emptyset$. (Recall that for each $J \in \mathcal{I}$, there might be many subintervals J' satisfying the conditions above, giving rise to many possible maps B – we just fix some particular choice of intervals). It follows that B is completely determined by the map $\Psi : \mathcal{I} \to \mathcal{I}$ taking an interval J to $\varphi(J')$ along with the set of valencies $\mathrm{val}(\varphi, J')$. Fix an $I \in \mathcal{I}$, and note that $v = [I, (+,+)] - [I, (-,-)]$ is in the complement of V, so $\pi(v)$ is a non-zero element of $K_1(C_r^*(R_\varphi^+(1)))$. Now, we calculate that

$$B[I, (+,+)] = \begin{cases} [\Psi(I), (+,+)] & \text{if } \mathrm{val}(\varphi, I') = (+,+) \\ -[\Psi(I), (-,-)] & \text{if } \mathrm{val}(\varphi, I') = (-,-) \end{cases}$$

and

$$B[I, (-,-)] = \begin{cases} [\Psi(I), (-,-)] & \text{if } \mathrm{val}(\varphi, I') = (+,+) \\ -[\Psi(I), (+,+)] & \text{if } \mathrm{val}(\varphi, I') = (-,-) \end{cases}$$

In both cases, it follows that

$$B\left([I, (+,+)] - [I, (-,-)]\right) = [\Psi(I), (+,+)] - [\Psi(I), (-,-)],$$

so $B(v)$ is also in the complement of V. But this calculation can be repeated to show that $B^2(v)$, and, in general, $B^n(v)$ is in the complement of V. It follows $\pi(B^k(v))$ is non-trivial in $K_1(C_r^*(R_\varphi^+(k)))$ for any k, and that

$$\rho_1(\pi(B^k(v)) = \pi(B^{k+1}(v)).$$

But then the direct limit

$$K_1(C_r^*(R_\varphi^+)) = \varinjlim_k (K_1(C_r^*(R_\varphi(k))), (\rho_k)_1)$$

is non-zero, as we wanted. □

Appendix A

Computations

This appendix contains a number of K-theory computations for critically finite maps. The setup is the following: Let $\varphi : \mathbb{T} \to \mathbb{T}$ be a circle map, satisfying the following assumptions: 0 is a critical point of valency $(-,+)$, and for any $c \in \mathcal{C}$, we have $\varphi(c) = 0$. This ensures that the map is critically finite – indeed, the map is a *Markov* map, i.e. all critical points are mapped to critical points. In the notation of Chapter 3, we may choose $\mathcal{D} = \{0\}$ and $\mathcal{I} = \{I\}$, with $I = (0,1)$.

Example A.1. We begin by considering $C_r^*(\Gamma_\varphi)$. Put

$$n_1 = \#\{z \in \varphi^{-1}(0) \cap (0,1) \,|\, \mathrm{val}(\varphi,z) = (+,-)\}, \quad n_2 = \#\{z \in \varphi^{-1}(0) \cap (0,1) \,|\, \mathrm{val}(\varphi,z) = (\pm,\pm)\},$$

and note that it is possible to construct a Markov map φ with arbitrary values of n_1 and n_2. Note first that $\#\{z \in \varphi^{-1}(0) \cap (0,1) \,|\, \mathrm{val}(\varphi,z) = (-,+)\} = n_1 - 1$. Using 5.34, we calculate:

$$\begin{aligned}\rho_0(E_{0,+}^{(-,+)}) &= E_{0,+}^{\mathrm{val}(\varphi,0)} + \sum_{z \in \varphi^{-1}(0) \cap \mathcal{C}} \left(E_{0,+}^{\mathrm{val}(\varphi,z)} + E_{0,-}^{\mathrm{val}(\varphi,z)}\right) + \sum_{z \in \varphi^{-1}(0) \setminus \mathcal{C}} E_0^r \\ &= n_1 E_{0,+}^{(-,+)} + (n_1 - 1)E_{0,-}^{(-,+)} + n_1 E_{0,+}^{(+,-)} + n_1 E_{0,-}^{(+,-)} + n_2 E_0^r.\end{aligned}$$

Similar computations reveal that the matrix A is given by

$$A = \begin{pmatrix} n_1 & n_1 - 1 & 1 & 0 & n_1 \\ n_1 - 1 & n_1 & 0 & 1 & n_1 \\ n_1 & n_1 & 0 & 0 & n_1 \\ n_1 & n_1 & 0 & 0 & n_1 \\ n_2 & n_2 & 0 & 0 & n_2 \end{pmatrix}.$$

Using 6.3.4 shows that $(I_1)_0 - (U_1)_0$ as a map from $\mathbb{Z}^5$ to $\mathbb{Z}$ is given by $(1,1,-1,-1,0)$. Hence,

$$\ker((I_1)_0 - (U_1)_0) = \{(x,y,z,x+y-z,w) \,|\, x,y,z,w \in \mathbb{Z}\} \simeq \mathbb{Z}^4.$$

A simple calculation shows that

$$\tilde{A} - 1 = \begin{pmatrix} n_1 - 1 & n_1 - 1 & 1 & n_1 \\ n_1 & n_1 & -1 & n_1 \\ n_1 & n_1 & -1 & n_1 \\ n_2 & n_2 0 & n_2 - 1 & \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 & n_1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n_2 - 1 \end{pmatrix},$$

from which it is easily seen that $K_1(C_r^*(\Gamma_\varphi)) = \ker(\tilde{A}-1) \simeq \mathbb{Z}$ if $n_2 \neq 1$, and $K_1(C_r^*(\Gamma_\varphi)) = \ker(\tilde{A}-1) \simeq \mathbb{Z}^2$ if $n_2 = 1$. Similarly, some coloumn operations transforms $\tilde{A}-1$ into the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2n_1 - 1 + n_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Putting $N = 2n_1 + n_2 - 1$, we get that the Smith normal normal form of $\tilde{A}-1$ is the diagonal matrix $(1,1,N,0)$, hence $K_0(C_r^*(\Gamma_\varphi)) = \operatorname{coker}(\tilde{A}-1) \simeq \mathbb{Z} \oplus \mathbb{Z}_N$. In other words, the possible K-theory groups for $C_r^*(\Gamma_\varphi)$, where $|\mathcal{D}| = 1$, is

$$(K_0(C_r^*(\Gamma_\varphi)), K_1(C_r^*(\Gamma_\varphi)) = (\mathbb{Z} \oplus \mathbb{Z}_N, \mathbb{Z})$$

with N arbitrary, and

$$(K_0(C_r^*(\Gamma_\varphi)), K_1(C_r^*(\Gamma_\varphi)) = (\mathbb{Z} \oplus \mathbb{Z}_N, \mathbb{Z}^2)$$ ▲

for N even (since choosing $n_2 = 1$ forces N to be even).

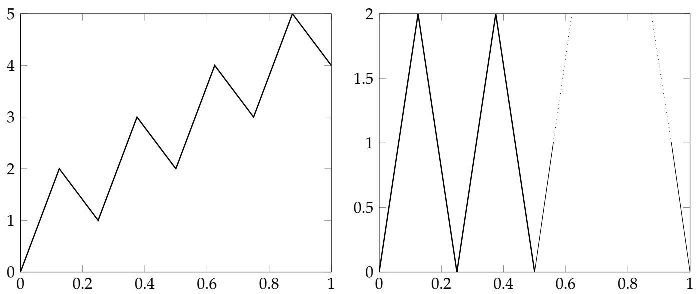


Figure A.1: Here are two examples of maps in the class we considered. For the first map φ_1, we have $n_1 = n_2 = 4$, so $N = 2n_1 + n_2 - 1 = 11$, and it follows that $(K_0(C_r^*(\Gamma_{\varphi_1})), K_1(C_r^*(\Gamma_{\varphi_1})) \simeq (\mathbb{Z} \oplus \mathbb{Z}_{11}, \mathbb{Z})$. The second map φ_2 is supposed to have n_1 local maxima (for some arbitrary $n_1 \in \mathbb{N}$). It then follows that $n_2 = 2n_1$, so $N = 3n_1 - 1$, so $K_0(C_r^*(\Gamma_{\varphi_2})), K_1(C_r^*(\Gamma_{\varphi_2})) \simeq (\mathbb{Z} \oplus \mathbb{Z}_{3n_1-1}, \simeq \mathbb{Z})$..

Example A.2. We now consider $C_r^*(\Gamma_\varphi^+)$. This is a bit more complicated: We are now able to distinguish between $(+,+)$ and $(-,-)$, so we need a parameter more compared to the sitation above. We also need to determine the map B between $K_0(\mathbb{B}_k)$ and $K_0(\mathbb{B}_{k+1})$. First, note that the assumptions on φ implies that $\mathcal{D}(\pm) = \{0\} \times \mathcal{V}$, so $K_0(\mathbb{A}_k) \simeq \mathbb{Z}^4$, and $K_0(\mathbb{B}_k) \simeq \mathbb{Z}^2$ since $\mathcal{I}(\pm) = (0,1) \times \{(+,+),(-,-)\}$. By the same arguments as in 5.25, $(I_k)_0 - (U_k)_0 : Z^4 \to \mathbb{Z}^2$ is given by the matrix

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

In particular, we have

$$\ker((I_k)_0 - (U_k)_0) = \left\{ (x,y,z,w) \in \mathbb{Z}^4 \,\middle|\, x = y \right\} \simeq \mathbb{Z}^3, \quad \operatorname{coker}((I_k)_0 - (U_k)_0) \simeq \mathbb{Z}$$

Next, note that since φ is assumed surjective, and since any critical point maps to 0, there is a subinterval J of $(0,1)$ such that $\varphi(J) = (0,1)$, J contains no critical points, and $\mathrm{val}(\varphi, J) = (+,+)$. It follows that the map B given in Equation 5.21, and hence also the restriction $\tilde{B}$, is simply the identity. Hence

$$\ker(1-\tilde{B}) \simeq \mathrm{coker}(1-\tilde{B}) \simeq \mathbb{Z}$$

Define numbers n_1, n_2 and n_3 as follows:

$$\begin{aligned} n_1 &= \#\Big\{z \in \varphi^{-1}(0) \cap (0,1) \,\Big|\, \mathrm{val}(\varphi, z) = (+,-)\Big\} \\ n_2 &= \#\Big\{z \in \varphi^{-1}(0) \cap (0,1) \,\Big|\, \mathrm{val}(\varphi, z) = (+,+)\Big\} \\ n_3 &= \#\Big\{z \in \varphi^{-1}(0) \cap (0,1) \,\Big|\, \mathrm{val}(\varphi, z) = (-,-)\Big\} \end{aligned}$$

Observe that $\#\{z \in \varphi^{-1}(0) \cap (0,1) | \mathrm{val}(\varphi, z) = (-,+)\} = n_1 - 1$. Using the formula for A given above shows that

$$A = \begin{pmatrix} 2n_1 - 1 & 1 & n_1 & n_1 \\ 2n_1 & 0 & n_1 & n_1 \\ n_2 + n_3 & 0 & n_2 & n_3 \\ n_2 + n_3 & 0 & n_3 & n_2 \end{pmatrix}.$$

Since Using the vectors $(1,1,0,0), (0,0,1,0), (0,0,0,1)$ as a basis for $\ker((I_k)_0 - (U_k)_0)$, a calculation reveals that

$$\tilde{A} - 1 = \begin{pmatrix} 2n_1 - 1 & n_1 & n_1 \\ n_2 + n_3 & n_2 - 1 & n_3 \\ n_2 + n_3 & n_3 & n_2 - 1 \end{pmatrix}$$

which, after some invertible row and coloumn operations, turn out to be equivalent to

$$\tilde{A} - 1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & n_1 + n_2 - 1 & n_1 + n_3 \\ 0 & n_1 + n_3 & n_1 + n_2 - 1 \end{pmatrix}$$

from which we see that $\ker(\tilde{A} - 1) = 0$ if

$$\det(\tilde{A} - 1) = (n_1 + n_2 - 1)^2 - (n_1 + n_3)^2 = n_2^2 + n_3^3 - 2n_1 n_2 - 2n_1 - 2n_2 + 1 \neq 0$$

and $\ker(\tilde{A} - 1) = \mathbb{Z}$ otherwise. It follows from 5.23 that $K_1(C_r^*(\Gamma_\varphi)) \simeq \mathbb{Z}$ in the first case, and $K_1(C_r^*(\Gamma_\varphi)) \simeq \mathbb{Z}^2$ in the second. Putting $u = n_1 + n_2 - 1$ and $v = n_1 + n_3$, a calculation shows that the Smith normal form of $\tilde{A} - 1$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \gcd(u,v) & 0 \\ 0 & 0 & \frac{u^2 - v^2}{\gcd(u,v)} \end{pmatrix}.$$

Hence $\mathrm{coker}(\tilde{A} - 1) = \mathbb{Z}_{\gcd(u,v)} \oplus \mathbb{Z}_{\frac{|u^2 - v^2|}{\gcd(u,v)}}$, so

$$K_0(C_r^*(\Gamma_\varphi^+)) \simeq \mathbb{Z} \oplus \mathbb{Z}_{\gcd(u,v)} \oplus \mathbb{Z}_{\frac{|u^2 - v^2|}{\gcd(u,v)}} \tag{A.1}$$

by 5.22. For any triple (n_1, n_2, n_3) with $n_1 > 0$ and $n_2, n_3 \geq 0$, we can find a critically finite map having the given constants. Indeed, fixing $u \geq 0$ and $v \geq 1$, and choosing a map

φ with $n_1 = 1$, $n_3 = v - 1$ and $u = n_2$, we get $n_1 + n_2 - 1 = u$ and $n_1 + n_3 = v$. To determine the possible K-theory groups, we simply need to calculate the range of the map $f(u, v) = (\gcd(u, v), |u^2 - v^2| / \gcd(u, v))$.

To do this, note first that $\gcd(u, v)$ divides $|u^2 - v^2| / \gcd(u, v)$, and that

$$\frac{(cu)^2 - (cv)^2}{\gcd(cu, cv)} = c \frac{u^2 - v^2}{\gcd(u, v)}$$

In other words, if $f(u, v) = (1, p)$, then $f(cu, cv) = (c, cp)$, so we may reduce to the case where $\gcd(u, v) = 1$. Since $\gcd(u, u + 1) = 1$ and $|u^2 - (u + 1)^2| = 2u + 1$, any pair $(1, 2u + 1)$ is in the range of f. Furthermore, with $u = 2k + 1$ odd and $v = 2k + 3$, we have $\gcd(u, v) = 1$ and $u^2 - v^2 = 8k + 8$, so any pair $(1, 8k)$ is in the range. Conversely, if $u^2 - v^2 = (u - v)(u + v)$ is even with u, v coprime, both u and v are necessarily odd, so both $u + v$ and $u - v$ are even. Furthermore, 4 divides at least one of them (if, say $u - v = 4k + 2$, then $u + v = u - v + 2v = 4k + 2 + 2(2j + 1) = 4(k + j + 1)$), so $u^2 - v^2$ is divisible by eight. Combining everything and using 5.22, we get that the possible groups are

$$K_0(C_r^*(\Gamma_\varphi^+)) = \mathbb{Z} \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{de}$$

where $d \in \mathbb{N}$ and e is either odd or a multiple of 8, or $d = 1$ and $e = 0$ (if $u = v$). ▲

We return to the concrete maps φ_1 and φ_2 given in the figure above. For the first map φ_1, we have $n_1 = n_2 = 4$ and $n_3 = 0$. This means that $u = 7$ and $n = 4$, so $\gcd(u, v) = 1$ and $|u^2 - v^2| / \gcd(u, v) = 33$. It follows that

$$K_0(C_r^*(\Gamma_{\varphi_1}^+)) \simeq \mathbb{Z} \oplus \mathbb{Z}_{33}, \quad K_1(C_r^*(\Gamma_{\varphi_1}^+)) \simeq \mathbb{Z}.$$

The second map φ_2 has n_1 local maxima in $(0, 1)$, and $n_1 = n_2 = n_3$. In the notation of the example, $u = 2n_1 - 1$ and $v = 2n_1$, so $\gcd(u, v) = 1$ and $|u^2 - v^2| = 2n_1 - 1$. It follows that the K-theory groups are $(\mathbb{Z} \oplus \mathbb{Z}_{2n_1 - 1}, \mathbb{Z})$.

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