Spectral Analysis of Large Particle Systems

PhD Thesis

Jonas Dahlbæk

Main supervisor: Jacob Schach Møller
Co-supervisor: Oliver Matte

Department of Mathematics
Aarhus University

June 7, 2017

Preface

This thesis has three main chapters, each describing research that has been carried out in relation to my PhD studies. The three chapters consitute working drafts of papers that are intended to be submitted for publication before the thesis defense. Minor changes have been made to the drafts in order to make the thesis more coherent. The manuscript contained in the first chapter is joint work with David Hasler and Jacob Schach Møller. The two manuscripts contained in the second and third chapters are joint work with Oliver Matte.

Acknowledgements

I would like to thank Jacob Schach Møller and Oliver Matte for sharing with me many insights into the deeper aspects of non-relativistic quantum field theory, and in general for their excellent guidance throughout my time at Aarhus University as a PhD student.

My sincere thanks to David Hasler for his many fruitful remarks and observations, and to the University of Jena for their generous hospitality during my stay there.

A special thanks to Ida and Nayeli, and to all my family and friends for all their support!

This work has been supported by the Villum Foundation.

Abstract

The Fröhlich polaron model is defined as a quadratic form, and its discrete spectrum is studied for each fixed total momentum $\xi \in \mathbb{R}^d$ in the weak coupling regime. Criteria are determined by means of which the number of discrete eigenvalues may be deduced. The analysis is based on relating the spectral analysis of the Fröhlich polaron model to an equivalent problem in terms of a family of generalized Friedrichs models. This is possible by employing a combination of the Birman-Schwinger principle and the Haynsworth inertia additivity formula. The number of discrete eigenvalues of a generalized Friedrichs model is analyzed explicitly. In order to determine the family generalized Friedrichs models induced by the Fröhlich polaron model, it is necessary to compute a certain Feshbach operator.

A method for computing Feshbach operators in bosonic Fock space is developed. The focus is on obtaining a framework which unifies and generalizes frameworks that have appeared previously in the literature. The end result is a calculus for creation/annihilation symbols, where Wick's theorem provides a formula for the product of finitely many symbols. The framework is then applied to the Fröhlich polaron model. The framework is also applied to the spin boson model. The application to the spin boson model is based on the spectral renormalization group.

It is shown that the spectral renormalization group scheme can be naturally posed as an iterated Grushin problem. While it is already known that Schur complements, Feshbach maps and Grushin problems are three sides of the same coin, it seems to be a new observation that the smooth Feshbach method can also be formulated as a Grushin problem. Based on this, an abstract account of the spectral renormalization group is given.

Resumé

Fröhlich polaronen defineres som en kvadratisk form, og dens diskrete spektrum studeres for hvert fastholdt totalt momentum $\xi \in \mathbb{R}^d$ i svag-kobling regimet. Der bestemmes kriterier som kan benyttes til at udlede antallet af diskrete egenværdier. Analysen er baseret på at relatere den spektrale analyse af Fröhlich polaron modellen til et ækvivalent problem for en familie af generaliserede Friedrichs modeller. Dette er muligt ved at benytte en kombination af Birman-Schwinger princippet of Haynsworth inertiadditivitetsformlen. Antallet af diskrete egenværdier for en generaliseret Friedrichs model analyseres explicit. For at bestemme den familie af generaliserede Friedrichs modeller som Fröhlich polaron modellen inducerer, er det nødvendigt at udregne en bestemt Feshbach operator.

En metode til at udregne Feshbach operatorer i bosonisk Fock rum udvikles. Fokus er på at udvikle et maskineri som forener of generaliserer maskinerier der tidligere har optrådt i literaturen. Slutresultatet er en calculus for kreations/annihilations-symboler, hvor Wick's sætning forsyner en med formlen for produktet af endeligt mange symboler. Maskineriet anvendes efterfølgende på Fröhlich polaron modellen. Maskineriet anvendes også på spin boson modellen. Anvendelsen på spin boson modellen er baseret på den spektrale renormaliseringsgruppe.

Det vises at metoden bag den spektrale renormaliseringsgruppe naturligt kan formuleres som et gentagent Grushin problem. Altimens det allerede er kendt at Schur komplementer, Feshbach afbildninger of Grushin problemer er tre sider af samme sag, lader det til at være en ny observation at den glatte Feshbach metode også kan formuleres som et Grushin problem. Baseret på denne observation, gives en abstrakt gennemgang af den spektrale renormaliseringsgruppe.

Contents

Introduction

In this thesis, the spectral analysis of certain models of bosons in non-relativistic quantum field theory will be discussed. The purpose of this introductory chapter is to provide an introduction to the topics that occur in the later chapters. Most definitions and arguments will be heuristic and formal (or absent).

Creation/Annihilation Operators

Since this thesis deals with bosonic non-relativistic quantum field theory, it seems in its place to start out by introducing creation/annihilation operators in bosonic Fock space. Let $\mathcal{M} := \mathbb{R}^d$, denote by $\mathcal{F}^{(n)}_{\text{sym}} := \bigotimes^n_{\text{sym}} L^2(\mathcal{M})$ the space of symmetric, square integrable functions on $\mathcal{M}^n$, and denote bosonic Fock space by $\mathcal{F}_{\text{sym}} := \bigoplus_{n=0}^\infty \mathcal{F}^{(n)}_{\text{sym}}$. Consider a measurable function

$$\omega^{(r)}_{m,n} : \mathcal{M}^m \times \mathcal{M}^n \times \mathcal{M}^r \to \mathbb{C} \tag{1}$$

which is separately symmetric. By separately symmetric is meant that if S_l denotes the set of permutations of the l-point set $\{1, 2, \ldots, l\}$, and $\sigma \in S_m, \tau \in S_n, \pi \in S_r$, then

$$\omega^{(r)}_{m,n}(k_{\bar{\mathscr{M}}}, k_{\bar{\mathscr{N}}}, k_{\bar{\mathscr{R}}}) = \omega^{(r)}_{m,n}(k_{\sigma\bar{\mathscr{M}}}, k_{\tau\bar{\mathscr{N}}}, k_{\pi\bar{\mathscr{R}}}), \tag{2}$$

where

$$\begin{aligned} k_{\bar{\mathscr{M}}} &= (k_1, \ldots, k_m), & k_{\sigma\bar{\mathscr{M}}} &= (k_{\sigma(1)}, \ldots, k_{\sigma(m)}), \\ k_{\bar{\mathscr{N}}} &= (q_1, \ldots, q_n), & k_{\tau\bar{\mathscr{N}}} &= (q_{\tau(1)}, \ldots, q_{\tau(n)}), \\ k_{\bar{\mathscr{R}}} &= (t_1, \ldots, t_r), & k_{\pi\bar{\mathscr{R}}} &= (t_{\pi(1)}, \ldots, t_{\pi(r)}). \end{aligned}$$

Due to symmetry of $\omega^{(r)}_{m,n}$, the ordering of, say, the m-tuple $k_{\bar{\mathscr{M}}}$ does not matter. To emphasize this, introduce the notation $k_{\mathscr{M}}$, where the ordering is not specified. One may consider $k_{\mathscr{M}}$ to be the equivalence class of $k_{\bar{\mathscr{M}}} \in \mathcal{M}^m$ with respect to permutation of the variables. Denote, for each fixed $(k_{\mathscr{M}}, k_{\mathscr{N}})$, by $\omega^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}})$ the operator of multiplication by the function $k_{\mathscr{R}} \mapsto \omega^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}})$ in $\mathcal{F}^{(r)}_{\text{sym}}$. Similarly, put

$$\omega_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}) := \bigoplus_{r=0}^\infty \omega^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}).$$

With these conventions, for each $(k_{\mathscr{M}}, k_{\mathscr{N}})$, $\omega_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}})$ is an operator in $\mathcal{F}_{\text{sym}}$, which acts in each summand $\mathcal{F}^{(r)}_{\text{sym}}$ as a multiplication operator.

Suppose for each $m, n, r \in \mathbb{N}_0$ that $\omega_{m,n}^{(r)}$ is as in (1) and satisfies (2). If

$$W_{m,n} := \int dk_{\mathscr{M}} dk_{\mathscr{N}}\, a^*(k_{\mathscr{M}}) v^{\otimes m}(k_{\mathscr{M}}) \omega_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}) \bar{v}^{\otimes n}(k_{\mathscr{N}}) a(k_{\mathscr{N}}), \tag{3}$$

then $W = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} W_{m,n}$ will be referred to as a creation/annihilation operator. Here, $v : \mathcal{M} \to \mathbb{C}$ is some measurable function. Formula (3) should be interpreted as a quadratic form by means of the formula, for $f \in \mathcal{F}_{\mathrm{sym}}$,

$$(a(k_{\mathscr{N}})f)^{(r)}(k_{\mathscr{R}}) = \sqrt{\frac{(n+r)!}{r!}} f^{(n+r)}(k_{\mathscr{N}}, k_{\mathscr{R}}). \tag{4}$$

so that

$$\langle \psi, W_{m,n}\phi \rangle = \int dk_{\mathscr{M}} dk_{\mathscr{N}} \langle \bar{v}^{\otimes m}(k_{\mathscr{M}}) a(k_{\mathscr{M}})\psi, \omega_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}) \bar{v}^{\otimes n}(k_{\mathscr{N}}) a(k_{\mathscr{N}})\phi \rangle.$$

The triple sequence $(\omega_{m,n}^{(r)})_{m,n,r \in \mathbb{N}_0}$ of functions will be referred to as the symbol of the creation/annihilation operator W.

The Fröhlich Polaron Model

The Fröhlich polaron model [12], which describes an idealized system of a single electron interacting with a polar crystal, is defined in dimension d by the Hamiltonian

$$H_{\text{Fröh}} := -\frac{1}{2}\Delta + N - \sqrt{\alpha} \int_{x \in \mathbb{R}^d}^{\oplus} dx\, \Phi(v_x), \tag{5}$$

acting in the direct integral space

$$\int_{x \in \mathbb{R}^d}^{\oplus} dx\, \mathcal{F}_{\mathrm{sym}} := L^2(\mathbb{R}^d, \mathcal{F}_{\mathrm{sym}}) \cong L^2(\mathbb{R}^d) \otimes \mathcal{F}_{\mathrm{sym}}.$$

The parameter $\alpha \geqslant 0$ is a coupling constant whose value depends on the particular type of crystal under consideration, $v_x(k) := v(k) \cdot e^{-ik \cdot x}$, with $v(k) := |k|^{-\frac{d-1}{2}}$ denoting the coupling function in dimension d [33], and

$$\Phi(f) := \int_{k \in \mathbb{R}^d} dk \left[\bar{f}(k) a(k) + a^*(k) f(k) \right].$$

The free electron is modelled by the Laplacian $-\Delta/2$ acting in $L^2(\mathbb{R}^d)$, while the number operator $N := \int_{k \in \mathbb{R}^d} dk\, a^*(k) a(k)$ acting in bosonic Fock space $\mathcal{F}_{\mathrm{sym}}$ models the free polar crystal. Modelling the polar crystal in this way corresponds physically to considering only a single type of excitation of the crystal: Longitudinal optical phonons with fixed frequency $\omega = 1$ [31, 33].

The interaction between the electron and the polar crystal is modelled by the third term on the right hand side of equation (5),

$$\sqrt{\alpha}\int_{x\in\mathbb{R}^d}^{\oplus} dx \int_{k\in\mathbb{R}^d} dk \left[\bar{v}(k)e^{ik\cdot x}a(k) + a^*(k)v(k)e^{-ik\cdot x}\right],$$

which 'mixes' the electron position x with the single-phonon momentum k.

An important property of the Fröhlich polaron model is that it is translation invariant [15, 31], in the sense that the Hamiltonian $H_{\text{Fröh}}$ commutes strongly with the total momentum operator, $P_{\text{tot}} := -i\nabla + d\Gamma(p)$, where $-i\nabla$ is the momentum operator of the electron and $d\Gamma(p) := \int_{\mathbb{R}^d} dk\, a^*(k)ka(k)$ is the momentum operator of the phonon field. This can be verified by means of the canonical commutation relations

$$[a(k), a(k')] = 0, \qquad [a^*(k), a^*(k')] = 0, \qquad [a(k), a^*(k')] = \delta(k - k').$$

As a consequence, it is possible to simultaneously diagonalize the total momentum operator and the Fröhlich polaron Hamiltonian. Such a simultaneous diagonalization can be implemented by the Lee-Low-Pines transform [26], defined by

$$(S\psi)(x) := (2\pi)^{-d/2}\int_{\xi\in\mathbb{R}^d} d\xi\, e^{ix\cdot\xi}e^{-id\Gamma(p)\cdot\xi}\psi(\xi).$$

Computation reveals that $S^*P_{\text{tot}}S = \int_{\xi\in\mathbb{R}^d}^{\oplus} d\xi\,\xi$, i.e. the Lee-Low-Pines transform indeed diagonalizes the total momentum operator. Furthermore, if one recalls the pull-through formula, which ensures that we have

$$a(k)e^{-id\Gamma(p)\cdot\xi} = e^{-id\Gamma(p)\cdot\xi - ik\cdot\xi}a(k), \quad e^{id\Gamma(p)\cdot\xi}a^*(k) = a^*(k)e^{id\Gamma(p)\cdot\xi + ik\cdot\xi},$$

one may also deduce that

$$S^*H_{\text{Fröh}}S = \int_{\xi\in\mathbb{R}^d}^{\oplus} d\xi\, H_{\text{Fröh}}(\xi),$$

with

$$H_{\text{Fröh}}(\xi) := \frac{1}{2}|\xi - d\Gamma(p)|^2 + N - \sqrt{\alpha}\int_{\mathbb{R}^d} dk\,[v(k)a(k) + a^*(k)\bar{v}(k)]\,. \tag{6}$$

In this sense, in order to understand the spectral properties of the Fröhlich polaron model, it suffices to consider the family of so-called fiber Hamiltonians $H_{\text{Fröh}}(\xi)$ for total momentum $\xi \in \mathbb{R}^d$, acting in bosonic Fock space $\mathcal{F}_{\text{sym}}$.

Irrespective of whether one takes formula (5) or (6) as the starting point of the analysis of the Fröhlich polaron model, the question of well-definedness of the model must be adressed. The problem one is confronted with is that the coupling function $v(k) := |k|^{-\frac{d-1}{2}}$ is not square integrable in $\mathbb{R}^d$. One way to remedy this flaw is to introduce an ultraviolet cutoff $\Lambda > 0$ in the coupling function, thereby rendering the coupling function square integrable and thus yielding well-defined Hamiltonians $H^\Lambda_{\text{Fröh}}$ and $H^\Lambda_{\text{Fröh}}(\xi)$. Employing a unitary transform due to Gross [19], it is then possible to show that there are self-adjoint operators $H_{\text{Fröh}}$ and $H_{\text{Fröh}}(\xi)$, such that, as $\Lambda \to \infty$, one has $H^\Lambda_{\text{Fröh}} \to H_{\text{Fröh}}$ as well as $H^\Lambda_{\text{Fröh}}(\xi) \to H_{\text{Fröh}}(\xi)$ in norm resolvent sense [15, 18]. There is also an alternative route, by means of quadratic forms. Starting from formula (5), it has been observed by Griesemer and Wünsch [18] that a simple commutator estimate due to Lieb and Thomas [27] is sufficient to show that the interaction term of $H_{\text{Fröh}}$ is infinitesimally relatively form bounded with respect to the non-interacting Hamiltonian. Thus, $H_{\text{Fröh}}$ can be uniquely defined by means of the KLMN theorem.

Another remark is in order concerning the removal of the ultraviolet cutoff. It has been shown by Møller [31] that the essential spectrum of $H^\Lambda_{\text{Fröh}}(\xi)$ fulfills the identity $\sigma_{\text{ess}}(H^\Lambda_{\text{Fröh}}(\xi)) = [\Sigma_{\text{ess}}(H^\Lambda_{\text{Fröh}}(\xi)), \infty)$, where the quantity $\Sigma_{\text{ess}}(H^\Lambda_{\text{Fröh}}(\xi)) := \inf \sigma_{\text{ess}}(H^\Lambda_{\text{Fröh}}(\xi))$ denotes the bottom of the essential spectrum of $H^\Lambda_{\text{Fröh}}(\xi)$. Since $H^\Lambda_{\text{Fröh}}(\xi) \to H_{\text{Fröh}}(\xi)$ in norm resolvent sense, it follows that one also has $\sigma_{\text{ess}}(H_{\text{Fröh}}(\xi)) = [\Sigma_{\text{ess}}(H_{\text{Fröh}}(\xi)), \infty)$ without the ultraviolet cutoff. Therefore, the eigenvalues of $H_{\text{Fröh}}(\xi)$ below the bottom of the essential spectrum are precisely the discrete eigenvalues of $H_{\text{Fröh}}(\xi)$.

The topic of the first chapter of this thesis is the energy-momentum spectrum of the Fröhlich polaron model, by which is meant the set

$$\left\{(\xi, E) \in \mathbb{R}^d \times \mathbb{R} \,\middle|\, E \in \sigma(H_{\text{Fröh}}(\xi))\right\}.$$

More specifically, the discrete part of the energy-momentum spectrum is studied, with the aim of counting the number of discrete eigenvalues in the weak coupling regime at each total momentum $\xi \in \mathbb{R}^d$. We take formula (6) as the starting point of our analysis. The analysis has 4 essential ingredients:

1. A relative form bound, which allows a simple definition of the model as a quadratic form by means of the KLMN theorem.

2. A version of the Birman-Schwinger principle, by means of which the spectral analysis of the Fröhlich polaron Hamiltonian is transformed into an equivalent problem in terms of a Birman-Schwinger operator.

3. An extension of the Haynsworth inertia additivity formula to the setting of unbounded, self-adjoint operators, by means of which the study

of the Birman-Schwinger operator is further transformed into an equivalent problem in terms of a Feshbach operator, which turns out to have the form of a generalized Friedrichs model.

4. A direct analysis of the spectrum of the induced generalized Friedrichs model.

The relative form bound of part 1 is closely related to the commutator estimate of Lieb and Thomas [27]. Instead of a commutator estimate, our proof is based on a pointwise estimate for the symbol of a particular creation/annihilation operator. More precisely, and with a slight change of notation, if we put $g := \sqrt{\alpha}$ and define

$$T_{E,\xi} := \frac{1}{2}|\xi - d\Gamma(p)|^2 + N - E, \tag{7}$$

$$H_{g,E,\xi} := T_{E,\xi} - g\Phi(v), \tag{8}$$

then showing that $\Phi(v)$ is relatively form bounded with respect to $T_{E,\xi}$ is equivalent to deriving a norm bound for the creation/annihilation operator

$$B_{g,E,\xi} := g|T_{E,\xi}|^{-1/2}\Phi(v)|T_{E,\xi}|^{-1/2} \tag{9}$$

for an appropriate choice of E. Note that if $v \in L^2(\mathbb{R}^d)$, then boundedness of $B_{g,E,\xi}$ follows from standard estimates, but the case $v \notin L^2(\mathbb{R}^d)$ is more subtle.

The version of the Birman-Schwinger principle needed in part 2 is very similar to the version obtained by Birman [5] (an English translation exists [6]).

As for part 3, the procedure we employ is closely related to a procedure employed by Minlos [28]. An essential difference is that we assume much less regularity of v, i.e. we do not introduce infrared or ultraviolet cutoffs, nor do we assume that v is differentiable. The generalized Friedrichs model is defined by an operator A_g in $\mathcal{H} := \mathbb{C} \oplus L^2(\mathbb{R}^d)$ which has the form

$$(A_g\psi)_0 := e_0\psi_0 - g\int dk\, \bar{v}(k)\psi_1(k)$$

$$(A_g\psi)_1(k) := -gv(k)\psi_0 + M(k)\psi_1(k) - g^2\int dq\, v(k)C(k,q)\bar{v}(q)\psi_1(q),$$

with a real parameter $e_0 \in \mathbb{R}$, a real coupling constant $g \in \mathbb{R}$, a coupling function $v(k) = |k|^{-\frac{d-1}{2}}$, and functions M, C corresponding to a multiplication and an integral operator [25]. The spectral properties of A_g depend on the specific form of the functions M and C. A main technical ingredient of our analysis is a method for computing the functions M and C of the generalized Friedrichs model induced by the Fröhlich polaron model, without relying on regularity of v. This technical ingredient is presented in the second chapter of the thesis. As for the Haynsworth inertia additivity formula [23], our observation that it remains true in the setting of unbounded, self-adjoint operators seems to be new.

The analysis given in part 4 is closely related to an analysis of the Friedrichs model on the torus given by Ikromov and Sharipov [24]. Again, we have to deal with additional technical challenges due to the lack of regularity of v.

Feshbach Maps in Bosonic Fock Space

Given a closed operator H in a Hilbert space $\mathcal{H}$ along with a pair of orthogonal projections $P, \bar{P} := 1 - P$, suppose that the restriction of $H_E = H - E$ to the range of $\bar{P}$, $\bar{H}_E = \bar{P} H_E \restriction_{\mathrm{dom}(H) \cap \mathrm{ran}(\bar{P})}$, is invertible for some $E \in \mathbb{C}$. Then the Feshbach reduction method [7] allows one to reduce the spectral study of H_E to the spectral study of the Feshbach operator

$$F_E := P H_E P - P H \bar{P} \bar{H}_E^{-1} \bar{P} H P.$$

The advantage of analyzing the Feshbach operator as compared to analyzing H_E directly lies in the fact that F_E acts in a reduced subspace $\mathrm{ran}(P)$. On the other hand, the disadvantage is that the dependence on the spectral parameter E is more complicated.

The Feshbach reduction method was applied by Minlos [28] in order to reduce the Fröhlich polaron Hamiltonian $H_{\text{Fröh}}(\xi)$, which acts in bosonic Fock space $\mathcal{F}_{\mathrm{sym}} = \bigoplus_{n=0}^{\infty} \mathcal{F}_{\mathrm{sym}}^{(n)}$, to a generalized Friedrichs operator acting in $\mathcal{F}_{\mathrm{sym}}^{(\leqslant 1)} = \mathcal{F}_{\mathrm{sym}}^{(0)} \oplus \mathcal{F}_{\mathrm{sym}}^{(1)} = \mathbb{C} \oplus L^2(\mathbb{R}^d)$. With notation as in equations (7), (8) and (9), one must compute the Feshbach operator

$$\begin{aligned} F_{g,E,\xi} &:= P H_{g,E,\xi} P - g^2 P \Phi(v) \bar{P} (\bar{H}_{g,E,\xi})^{-1} \bar{P} \Phi(v) P \\ &= P H_{g,E,\xi} P - \sum_{n=0}^{\infty} g^2 P \Phi(v) \bar{P} \bar{T}_{E,\xi}^{-1/2} (\bar{B}_{g,E,\xi})^n \bar{T}_{E,\xi}^{-1/2} \bar{P} \Phi(v) P, \end{aligned} \tag{10}$$

where P denotes the orthogonal projection onto $\mathcal{F}_{\mathrm{sym}}^{(\leqslant 1)}$, $\bar{P} = 1 - P$, and $\bar{A}$ denotes the restriction of A to $\mathrm{ran}(\bar{P})$ for any operator A in $\mathcal{F}_{\mathrm{sym}}$. This computation was carried out by Minlos [28] for the case $v \in L^2(\mathbb{R}^d)$ (see also the related work by Angelescu et al. [2]), but his derivation is not directly applicable to the case $v \notin L^2(\mathbb{R}^d)$.

The Feshbach reduction method also forms the basis of the spectral renormalization group introduced by Bach et al. [4]. A crucial improvement to the technique was made by Bach et al. [3] with the introduction of the smooth Feshbach map, which was further developed by Griesemer and Hasler [17]. Here, one assumes that $H_{g,E} := T_E - gW$, with $T_E := T - E$ for some closed operators H, T and a coupling constant $g \in \mathbb{C}$. The orthogonal projections $P, \bar{P}$ are replaced by a smooth partition $\chi, \bar{\chi}$ satisfying $\chi^2 + \bar{\chi}^2 = 1$. Under appropriate conditions, the spectral study of $H_{g,E}$ is then reduced to the spectral study of the smooth Feshbach operator

$$\begin{aligned} F_{g,E} &:= T_E - g\chi W\chi - g^2\chi W\bar{\chi}(\bar{T}_E - g\bar{\chi}W\bar{\chi})^{-1}\bar{\chi}W\chi, \\ &= T_E - g\chi W\chi - \sum_{n=0}^{\infty} g^2\chi W\bar{\chi}\bar{T}_E^{-1/2}(g\bar{T}_E^{-1/2}\bar{\chi}W\bar{\chi}\bar{T}_E^{-1/2})^n\bar{T}_E^{-1/2}\bar{\chi}W\chi, \end{aligned} \tag{11}$$

acting in the reduced subspace $\overline{\mathrm{ran}(\chi)}$. The map $H_{g,E} \mapsto F_{g,E}$ is called the smooth Feshbach map, and one may say that the smooth Feshbach map decimates the degrees of freedom of the system [3]. The hope is that the smooth Feshbach operator has a simpler structure than the original operator, in some appropriate sense. For applications to bosonic non-relativistic quantum field theories, the operator W is usually a creation/annihilation operator. In proofs that rely on the spectral renormalization group, it is usually a main technical step to compute the sum (11), and to show that the result is again a creation/annihilation operator in a certain class, such that one is able to iterate the smooth Feshbach map.

In the second chapter of the thesis, a framework is developed for the computation of operators such as (10) and (11). The main result is a version of Wick's theorem, which generalizes a result due to Bach et al. [4]. It is a main focus that the framework should be applicable to

1. The study of the spin boson model by means of the spectral renormalization group.
2. The case $v \notin L^2(\mathbb{R}^d)$ of the Fröhlich polaron model.

For part 1, the immediate goal is to generalize a formula for the product of finitely many creation/annihilation symbols due to Bach et al. [4]. In order to write down the formula, it is necessary to introduce some more notation. For a creation/annihilation symbol $\omega^{(r)}_{m,n}$, define

$$\mathfrak{w}^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}})^{(w)}_{u,v}(k_{\mathscr{U}}, k_{\mathscr{V}}, k_{\mathscr{W}}) := \omega^{(r+w)}_{m+u,n+v}(k_{\mathscr{M}\sqcup\mathscr{U}}, k_{\mathscr{N}\sqcup\mathscr{V}}, k_{\mathscr{R}\sqcup\mathscr{W}}),$$

where $k_{\mathscr{M}\sqcup\mathscr{U}} := (k_{\mathscr{M}}, k_{\mathscr{U}})$ etc. One may consider $\mathfrak{w}$ to be a symbol valued symbol, and thus define the creation/annihilation operator valued functions

$$\begin{aligned} &\tilde{W}^{(r)}_{m,u,n,v}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) \\ &\quad := \int dk_{\mathscr{U}}\, dk_{\mathscr{V}}\, a^*(k_{\mathscr{U}})v^{\otimes u}(k_{\mathscr{U}})\mathfrak{w}^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}})_{u,v}(k_{\mathscr{U}}, k_{\mathscr{V}})\bar{v}^{\otimes v}(k_{\mathscr{V}})a(k_{\mathscr{V}}). \end{aligned}$$

Let $W^1, \ldots, W^J$ be creation/annihilation operators. The formula of Bach et al. [4, Theorem A.4] may be written in the notation introduced here as follows: $W^J \cdots W^1 = W$, where the symbol of W is given by the symmetrization

$$\omega^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) := \frac{1}{m!n!} \sum_{\sigma \in S_m, \tau \in S_n} \hat{\omega}^{(r)}_{m,n}(k_{\sigma \mathscr{M}}, k_{\tau \mathscr{N}}, k_{\mathscr{R}})$$

of the function

$$\hat{\omega}^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) := \sum_{\substack{m_1+\ldots+m_J=m \\ n_1+\ldots+n_J=n}} \sum_{\substack{p_1,\ldots,p_J \in \mathbb{N}_0 \\ q_1,\ldots,q_J \in \mathbb{N}_0}} \prod_{j=1}^{J} \binom{m_j+p_j}{p_j} \binom{n_j+q_j}{q_j} \cdot \langle \Omega, \prod_{j=1}^{J} \tilde{W}^{j(r_j)}_{m_j,p_j,n_j,q_j}(k_{\mathscr{M}_j}, k_{\mathscr{N}_j}, k_{\mathscr{R}_j}) \Omega \rangle,$$

where $\prod_{j=1}^{J} a^j = a^J \cdots a^1$, $k_{\mathscr{M}} = (k_{\mathscr{M}_J}, \ldots, k_{\mathscr{M}_1})$, $k_{\mathscr{N}} = (k_{\mathscr{N}_J}, \ldots, k_{\mathscr{N}_1})$ and

$$k_{\mathscr{R}_j} = (k_{\mathscr{N}_J}, \ldots, k_{\mathscr{N}_{j+1}}, k_{\mathscr{M}_{j-1}}, k_{\mathscr{M}_1}, k_{\mathscr{R}}).$$

A version of this formula will be derived which is completely explicit, in the sense that the vacuum expectation values

$$\langle \Omega, \prod_{j=1}^{J} \tilde{W}^{j(r_j)}_{m_j,p_j,n_j,q_j}(k_{\mathscr{M}_j}, k_{\mathscr{N}_j}, k_{\mathscr{R}_j}) \Omega \rangle$$

are also evaluated. The result is a formula in which the product symbol ω is expressed directly in terms of the factor symbols $\omega^J, \ldots, \omega^1$. Based on this formula, it is possible to introduce a calculus for creation/annihilation symbols, by defining a product # on the space of symbols such that one has $\omega = \omega^J \# \cdots \# \omega^1$. Furthermore, a formula is derived for the symbol valued symbol $\mathfrak{w}$ in terms of the symbol valued symbols $\mathfrak{w}^J, \ldots, \mathfrak{w}^1$. By means of this formula, it is possible to show for instance that differentiability in the variables $(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}})$ is preserved by the product #.

Having obtained such a formula for the product of finitely many symbols, a treatment is given of the existence and uniqueness problem for the ground state of the spin boson model in the weak coupling regime. The treatment is an adaptation of a proof given by Hasler and Herbst [20], based on the spectral renormalization group. We are able to treat a slightly more infrared singular coupling function with the use of our symbol calculus. The spectral renormalization group is discussed in the third chapter.

As for part 2, i.e. being able to treat the case $v \notin L^2(\mathbb{R}^d)$ of the Fröhlich polaron, the problem largely boils down to choosing an appropriate norm on the space of creation/annihilation symbols. The aim is to pick a norm which controls the operator norm of the corresponding creation/annihilation operator, while also ensuring that the symbols have certain differentiability properties. In the case of the spin boson model, it suffices to consider creation/annihilation operators that act in a reduced subspace $\mathcal{H}_{\mathrm{red}} := \mathrm{ran}(1_{[0,1]}(d\Gamma(|p|)))$, where $d\Gamma(|p|) := \int dk\, a^*(k)|k|a(k)$. For the treatment of the Fröhlich polaron, however, the creation/annihilation operator $\bar{B}_{g,E,\xi}$ which appears in (10) acts in $\mathcal{F}_{\mathrm{sym}}^{(\geqslant 2)} = \bigoplus_{n=2}^{\infty} \mathcal{F}_{\mathrm{sym}}^{(n)}$. The norms considered by Hasler and Herbst [20] are therefore not appropriate. On the other hand, the norm considered by Minlos [28] is only useful if $v \in L^2(\mathbb{R}^d)$. An appropriate choice of norm is made, and it is demonstrated that the framework applies to the Fröhlich polaron model.

Spectral Renormalization

The Feshbach reduction method is a variation on the Schur complement method, which goes back at least to Schur [36]. It is based on a block version of the Gaussian elimination algorithm,

$$M := \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ CA^{-1} & 1 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} 1 & A^{-1}B \\ 0 & 1 \end{bmatrix},$$

where A is assumed invertible, and $S := D - CA^{-1}B$ denotes the Schur complement. From this formula, it follows that M is invertible if and only if S is invertible, and one finds an isomorphism $\ker(M) \cong \ker(S)$ as well as an isomorphism $\mathrm{coker}(M) \cong \mathrm{coker}(S)$.

A related notion is that of a Grushin problem. Here, one has invertible operator block matrices $M, \mathcal{E} := M^{-1}$, with

$$M := \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad \mathcal{E} := \begin{bmatrix} E & E_2 \\ E_1 & E_{12} \end{bmatrix}. \tag{12}$$

In this setup, one finds that A is invertible if and only if E_{12} is invertible, along with isomorphisms $\ker(A) \cong \ker(E_{12})$ and $\mathrm{coker}(A) \cong \mathrm{coker}(E_{12})$.

Another related notion is that of the smooth Feshbach method. In the treatment given by Griesemer and Hasler [17], one considers two closed operators H, T on common domain $\mathrm{dom}(H) = \mathrm{dom}(T)$ along with $W := H - T$. Furthermore, one fixes two bounded operators $\chi, \bar{\chi}$ such that

$$\chi^2 + \bar{\chi}^2 = 1, \qquad \chi T \subseteq T\chi, \qquad \bar{\chi} T \subseteq T\bar{\chi}.$$

One assumes that $T, \bar{H} := T + \bar{\chi} W \bar{\chi}$ are bijective maps from $\mathrm{dom}(T) \cap \mathrm{ran}(\bar{\chi})$ to $\mathrm{ran}(\bar{\chi})$. Finally, one assumes that $\bar{\chi}\bar{H}^{-1}\bar{\chi}W\chi$ is bounded. Then, with

$$F = T + \chi W \chi - \chi W \bar{\chi} \bar{H}^{-1} \bar{\chi} W \chi,$$

it is found that H is invertible if and only if F is invertible, along with an isomorphism $\ker(H) \cong \ker(F)$.

The third chapter of this thesis starts out with a treatment of the Schur complement theorem and its relation to the Grushin problem, based on a work by Sjöstrand and Zworski [39]. It is then shown how to pose the smooth Feshbach method as a Grushin problem as well. Having two mutual inverses $M, \mathcal{E}$ as in formula (12) corresponds precisely to having 8 componentwise identities, corresponding to the two matrix identites $M\mathcal{E} = 1$ and $\mathcal{E}M = 1$. In this relation, it is worth noting that 6 of these identites are contained in the work by Griesemer and Hasler [17], while the remaining two identities needed to form a Grushin problem seem to be new.

The spectral renormalization group is based on iterated applications of the (smooth) Feshbach map. In this relation, we note that while there is a simple formula for the compsition of two Feshbach maps, no such formula exists for the smooth Feshbach map. On the other hand, iterated Grushin problems are well understood [39]. We provide a treatment of iterated Grushin problems. We then provide an abstract treatment of the spectral renormalization group, formulated as an iterated Grushin problem.

1 On the Discrete Energy-momentum Spectrum of the Fröhlich Polaron Model

Jonas Dahlbæk, David Hasler and Jacob Schach Møller

Abstract

The number of discrete eigenvalues of the Fröhlich polaron Hamiltonian $H_{g,0,\xi}$ is counted. The method applies in any dimension $d \in \mathbb{N}$ and for any total momentum $\xi \in \mathbb{R}^d$, but relies on a weak coupling assumption $|g| \ll 1$. The method does not require infrared or ultraviolet cut-offs.

In any dimension, there is at most 2 discrete eigenvalues, and outside a region of small total momentum there is at most 1 discrete eigenvalue. In dimension $d = 1, 2$, there is at least one discrete eigenvalue for all ξ. In dimension $d \geqslant 3$, there is bounded region of total momenta outside which there are no discrete eigenvalues.

The analysis consists in an application of the Birman-Schwinger principle, coupled with an application of the Feshbach method, reducing the model to a generalized Friedrichs model. In order to keep track of the number of eigenvalues below the bottom of the essential spectrum, the Haynsworth inertia additivity formula is employed.

1.1 Introduction

In this paper, we study the lower part of the spectrum of the Fröhlich polaron model [12], which describes an electron in an ionic crystal. Although this model has been extensively studied in the literature, there are still open questions regarding its spectral properties, even in the weak coupling regime.

The Fröhlich polaron Hamiltonian commutes strongly with the operator of total momentum, and using a unitary transformation due to Lee et al. [26], it may be fiber diagonalized with respect to total momentum, so that the study of its spectral properties is reduced to the study of the family of fiber Hamiltonians

$$H_{g,E,\xi} = \frac{1}{2}(\xi - d\Gamma(p))^2 + N - g\Phi(v) - E, \tag{13}$$

acting in the bosonic Fock space $\mathcal{F}_{\mathrm{sym}}$ [15, 31]. Here, $g \in \mathbb{R}$ is a coupling constant, $E \in \mathbb{R}$ is a spectral parameter, $\xi \in \mathbb{R}^d$ labels the total momentum, N is the number operator, $v(k) = |k|^{-\frac{d-1}{2}}$ is the coupling function in dimension d [33], and we put $p(k) = k$. Further definitions are given in Subsection 1.2.

Analysis of the Fröhlich polaron model is made complicated by the fact that $v \notin L^2(\mathbb{R}^d)$. In fact, for the full model $H_g = \int_{\xi\in\mathbb{R}^d}^{\oplus} H_{g,0,\xi} d\xi$, it has been demonstrated by Griesemer and Wünsch [18], based on a bound due to Frank and Schlein [9], that the domain of the Fröhlich polaron model satisfies the formula $\mathrm{dom}(H_g) \cap \mathrm{dom}(H_0) = \{0\}$ whenever $g \neq 0$. On the other hand, they also show that the quadratic form domain satisfies $\mathrm{qfd}(H_g) = \mathrm{qfd}(H_0)$. In other words, the perturbation is singular on the level of operator domains, but well-behaved on the level of quadratic form domains. It is further observed by Griesemer and Wünsch that a simple commutator estimate due to Lieb and Thomas [27] suffices to define H_g as a quadratic form.

We will define $H_{g,E,\xi}$ as a quadratic form by means of an estimate that is closely related to the commutator estimate by [27]. Our proof of the estimate is different, and based on a simple pointwise estimate of the Birman-Schwinger kernel. It is well known that if one introduces an ultraviolet cutoff $\Lambda > 0$ in the coupling function, then $H^{\Lambda}_{g,E,\xi} \to H_{g,E,\xi}$ in the norm resolvent sense as $\Lambda \to \infty$ [13, 15, 32]. It has been shown by Møller [31] that $\sigma_{\mathrm{ess}}(H^{\Lambda}_{g,E,\xi}) = [\Sigma_{\mathrm{ess}}(H^{\Lambda}_{g,E,\xi}), \infty)$, where $\Sigma_{\mathrm{ess}}(H^{\Lambda}_{g,E,\xi}) := \inf \sigma_{\mathrm{ess}}(H^{\Lambda}_{g,E,\xi})$ denotes the bottom of the essential spectrum of $H^{\Lambda}_{g,E,\xi}$. It follows from norm resolvent convergence that $\sigma_{\mathrm{ess}}(H_{g,E,\xi}) = [\Sigma_{\mathrm{ess}}(H_{g,E,\xi}), \infty)$. Therefore, the eigenvalues of $H_{g,E,\xi}$ below the bottom of the essential spectrum are precisely the discrete eigenvalues of $H_{g,E,\xi}$.

The spectrum of the free model is given by $\sigma(H_{0,0,\xi}) = \{\frac{1}{2}|\xi|^2\} \cup [1, \infty)$, so it was suprising when Spohn [40], using techniques from stochastic integration, proved that, in dimension $d = 1, 2$, the bottom of the spectrum $\Sigma_0(H_{g,0,\xi}) = \inf \sigma(H_{g,0,\xi})$ is in fact a discrete eigenvalue for all $\xi \in \mathbb{R}^d$, whenever $g \in \mathbb{R} \setminus \{0\}$.

Minlos [28] studied a class of operators similar to (13), but imposing the ultraviolet condition $v \in L^2(\mathbb{R}^d)$ as well as some amount of smoothness of v. For the class of operators he studied, he was able to verify and improve the analogue of the result of Spohn in the weak coupling regime. Specifically, he managed to give a condition in terms of which the number of eigenvalues below the bottom of the essential spectrum is characterized in any dimension. In dimension $d \geqslant 3$, Minlos was in particular able to decide from his condition that there can be at most 1 eigenvalue below the bottom of the essential spectrum, counted with multiplicity. Surprisingly, the result of Minlos suggests that, in dimension $d = 1, 2$, there may be a second eigenvalue below the bottom of the essential spectrum for small total momentum in the weak coupling regime.

The method of Spohn is effective for arbitrary values of the coupling constant, and applies to a class of linearly coupled models containing the Fröhlich polaron model. The method of Minlos is also effective for a class of linearly coupled models, but this class does not contain the Fröhlich polaron model, and the analysis reveals only information in the weak coupling regime.

Our analysis is based on the approach of Minlos, but modified in such a way that it applies to a class of models containing the Fröhlich polaron model. Our method is only applicable in the weak coupling regime. We will now give a short account of the similarities and differences between the method of Minlos and our method.

Minlos imposes the ultraviolet regularity condition $v \in L^2(\mathbb{R}^d)$ and shows that when the number $|g|(3 + \|v\|_{L^2(\mathbb{R}^d)})$ is suitably small, one may use the Feshbach method to reduce the Fröhlich polaron model to a generalized Friedrichs model [25]. He then carries out a direct analysis of the discrete spectrum in terms of Fredholm determinants. For this analysis, sufficient smoothness of the coupling function v is demanded. Due to the condition that the number $|g|(3 + \|v\|_{L^2(\mathbb{R}^d)})$ be suitably small, one cannot directly remove the ultraviolet regularity condition without taking $g \to 0$, i.e. the method does not directly apply to the Fröhlich polaron model.

In our approach, since we treat a more singular case, we first apply the well known Birman-Schwinger principle [5, 37], which allows us to relate the polaron Hamiltonian to a bounded self-adjoint operator acting in a weighted Fock space. We then apply a Feshbach map similar to the one applied by Minlos, which yields a generalized Friedrichs model. Unlike the analysis of Minlos, our analysis of the generalized Friedrichs model is based on a second application of a Feshbach map, which reduces the model to a Friedrichs model [10]. We count the number of discrete eigenvalues of the Friedrichs model by means of the Birman-Schwinger principle. In order to relate the number of discrete eigenvalues of the Friedrichs model to the number of discrete eigenvalues of the Fröhlich polaron model, we make use of the Haynsworth inertia additivity formula [23], which does not seem to be well known in our context.

The Birman-Schwinger principle has been extensively studied and generalized in the literature, see e.g. [35, 16, 8] for recent developments. For our purpose, we need a version of it which is very similar to the one obtained by Birman [5], whose work has been translated into English [6].

The Haynsworth inertia additivity formula is not so well known in our context. The formula was obtained for matrices by Haynsworth [23], but does not seem to have appeared previously for unbounded, self-adjoint operators. We demonstrate that, with suitable definitions, it applies equally well to the setting of unbounded, self-adjoint operators.

The Friedrichs model, introduced by Friedrichs [10, 11], has also been the object of extensive study. We remark that much of the work on the Friedrichs model has been carried out under relatively strong regularity assumptions. An exception is the work by Ikromov and Sharipov [24], where a method related to the technique we employ is used. Contrary to the situation we are concerned with, Ikromov and Sharipov consider the case where the underlying space is a torus rather than $\mathbb{R}^d$, and they assume that the coupling function is square integrable.

The generalized Friedrichs model was introduced and studied by Lakaev [25]. The discrete spectrum has been further studied by Minlos [28], and the essential spectrum has been further studied in dimension $d \geqslant 3$ by Angelescu et al. [2]. For a recent work on the generalized Friedrichs model in dimension $d = 3$, see the work by Akchurin [1]. In these works, it is assumed that the coupling function v is square integrable, typically along with some amount of smoothness. In a work by Miyao [29], a generalized Friedrichs model with non-integrable coupling function is studied. This generalized Friedrichs model, for which Miyao coins the term '$0, 1$-phonon polaron model', is obtained from the Fröhlich polaron model by neglecting terms with 2 or more phonons. In our analysis, we do not neglect these terms.

In dimension $d = 1, 2$, we recover the result of Spohn (note, however, that Spohn considers the general coupling situation) that the ground state energy is a discrete eigenvalue for any total momentum $\xi \in \mathbb{R}^d$, and we improve this result by showing, except for a region of small total momentum, that the ground state energy is in fact the unique discrete eigenvalue. Similarly, in dimension $d \geqslant 3$, we recover the result that there is a bounded region outside of which there are no discrete eigenvalues, and we improve the result be showing that there can be at most 2 discrete eigenvalues. Furthermore, the second eigenvalue, if it appears, can only appear in a region of small total momentum.

Similarly to Minlos, whose result we improve by treating a class of models containing the Fröhlich polaron model, we are able to give a condition which characterizes the number of eigenvalues below the bottom of the essential spectrum. In our singular setting, we are not able to rule out the existence of a second discrete eigenvalue for small total momentum even in dimension $d \geqslant 3$.

1.2 Fröhlich Polaron Model

In the following definitions, we collect basic notation for self-adjoint operators and quadratic forms in Hilbert spaces.

Definition 1.1. Let Q be a quadratic form on a vector space $\mathrm{qfd}(Q)$. If $\psi \in \mathrm{qfd}(Q)$, we will denote the value of Q at ψ by $Q[\psi] \in \mathbb{C}$.

Definition 1.2. Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$. We will denote by $\ker(A), \mathrm{ran}(A), \mathrm{dom}(A), \mathrm{qfd}(A) := \mathrm{dom}(|A|^{1/2})$ the kernel, range, operator domain and quadratic form domain of A, respectively. We denote by $\sigma(A)$ and $\sigma_{\text{ess}}(A)$ the spectrum and essential spectrum of A, respectively. We denote by $\Sigma_0(A) := \inf \sigma(A)$ and $\Sigma_{\text{ess}}(A) := \inf \sigma_{\text{ess}}(A)$ the bottom of the spectrum and the bottom of the essential spectrum of A, respectively. Finally, we will write

$$A[\psi] = \langle |A|^{1/2}\psi, \operatorname{sgn}(A)|A|^{1/2}\psi\rangle = \|A_+^{1/2}\psi\|^2 - \|A_-^{1/2}\psi\|^2, \quad \psi \in \operatorname{qfd}(A),$$

for the quadratic form corresponding to the self-adjoint operator A. Here, $A_\pm =$ denotes the positive/negative part of A, defined by functional calculus.

Having settled on basic notation, we will now introduce notation which is more specific to the Fröhlich polaron model.

Let $\mathcal{F}_{\text{sym}}^{(n)} := L^2_{\text{sym}}((\mathbb{R}^d)^n)$ denote the Hilbert space of complex valued functions $\psi^{(n)} : (\mathbb{R}^d)^n \to \mathbb{C}$ which are symmetric, i.e $\psi^{(n)}(k_{\sigma(1)}, \ldots, k_{\sigma(n)}) = \psi^{(n)}(k_1, \ldots, k_n)$ whenever $\sigma \in S_n$ is a permutation, and additionally satisfy boundedness of the norm

$$\|\psi^{(n)}\|_{\mathcal{F}_{\text{sym}}^{(n)}} := \left(\int dk_1 \cdots dk_n |\psi^{(n)}(k_1, \ldots, k_n)|^2\right)^{1/2}, \quad n \geqslant 1.$$

This notation is also consistent for $n = 0$, if we let $(\mathbb{R}^d)^0 = \{\star\}$ be the one-point measure space with counting measure. Then we denote bosonic Fock space by $\mathcal{F}_{\text{sym}} := \bigoplus_{n=0}^{\infty} \mathcal{F}_{\text{sym}}^{(n)}$. The inner product on $\mathcal{F}_{\text{sym}}$ is given by $\|\psi\|_{\mathcal{F}_{\text{sym}}} := (\sum_{n=0}^{\infty} \|\psi^{(n)}\|^2_{\mathcal{F}_{\text{sym}}^{(n)}})^{1/2}$. When there can be no confusion, we will omit the subscripts $\mathcal{F}_{\text{sym}}^{(n)}$ and $\mathcal{F}_{\text{sym}}$ from our notation, and we will likewise supress the variable $\star$, putting $\psi^{(0)} := \psi^{(0)}(\star)$.

For $E \in \mathbb{R}, \xi \in \mathbb{R}^d$, define a self-adjoint, lower semi-bounded operator $T_{E,\xi}$ in $\mathcal{F}_{\text{sym}}$ by

$$T_{E,\xi} := \frac{1}{2}|\xi - d\Gamma(p)|^2 + N - E := \bigoplus_{n=0}^{\infty} T_{E,\xi}^{(n)},$$

where $p(k) := k$ for all $k \in \mathbb{R}^d$, $d\Gamma(p)$ denotes the second quantization of the maximal multiplication operator corresponding to p, N denotes the number operator, and we define $T_{E,\xi}^{(n)}$ to be the maximal operator of multiplication by the function $T_{E,\xi}^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}$, given by $T_{E,\xi}^{(0)} = \frac{1}{2}|\xi|^2 - E$,

$$T_{E,\xi}^{(n)}(k_1, \ldots, k_n) := \frac{1}{2}|\xi - k_1 - \ldots - k_n|^2 + n - E, \quad n \geqslant 1.$$

The non-interacting fiber Fröhlich polaron model at spectral parameter $E \in \mathbb{R}$ and total momentum $\xi \in \mathbb{R}^d$ is defined by the Hamiltonian $H_{0,E,\xi} := T_{E,\xi}$.

Suppose that $v : \mathbb{R}^d \to \mathbb{C}$ is a measurable function, define, for $E < 1$,

$$L_E := \sqrt{\int \frac{dk\,|v(k)|^2}{\frac{1}{2}|k|^2 + 1 - E}}, \tag{14}$$

and assume that v is rotation invariant along with $0 < L_0 < \infty$. Note that the condition $L_0 > 0$ ensures that v is not identically zero, and rotational invariance means that $v(k) = v(\mathcal{O}k)$ for any orthogonal $d \times d$ matrix $\mathcal{O}$. Furthermore, $L_0 < \infty$ implies $L_E < \infty$ for all $E < 1$. It will be shown in section 1.5 that we have the relative form bound

$$|a(v)[\psi]| \leqslant \epsilon \|T_{0,\xi}^{1/2}\psi\|_{\mathcal{F}_{\text{sym}}}^2 + C_\epsilon \|\psi\|_{\mathcal{F}_{\text{sym}}}^2, \quad \psi \in \text{qfd}(T_{0,\xi}),$$

where the constants C_ϵ do not depend on ξ, and the form $a(v)$ defined by

$$a(v)[\psi] := \sum_{n=1}^{\infty} n^{\frac{1}{2}} \int dk_1 \cdots dk_n\, \bar{\psi}^{(n-1)}(k_1, \ldots, k_{n-1}) \bar{v}(k_n) \psi^{(n)}(k_1, \ldots, k_n)$$

is the annihilation form corresponding to v. In particular, the quadratic form

$$H_{g,E,\xi}[\psi] := T_{E,\xi}[\psi] - 2g \operatorname{re} a(v)[\psi], \quad \psi \in \text{qfd}(T_{0,\xi}), \tag{15}$$

is well defined, and by the KLMN theorem defines a unique self-adjoint Hamiltonian $H_{g,E,\xi}$ with form domain $\text{qfd}(H_{g,E,\xi}) = \text{qfd}(T_{0,\xi})$. The choice $v(k) = |k|^{-\frac{d-1}{2}}$ yields the fiber Fröhlich polaron Hamiltonian.

Since $U^* H_{g,E,\xi} U = H_{-g,E,\xi}$, where U denotes the unitary transformation defined by $(U\psi)^{(n)} = (-1)^n \psi^{(n)}$, one may assume $g \geqslant 0$ whenever convenient.

The set $\left\{(\xi, E) \in \mathbb{R}^d \times \mathbb{R} \,\middle|\, 0 \in \sigma(H_{g,E,\xi})\right\}$ is usually referred to as the energy-momentum spectrum of $H_g := \int_{\xi \in \mathbb{R}^d}^{\oplus} H_{g,0,\xi} d\xi$. Since v is rotationally invariant, we have $U_{\mathcal{O}}^* H_{g,E,\xi} U_{\mathcal{O}} = H_{g,E,\mathcal{O}\xi}$, where $(U_{\mathcal{O}}\psi)(k) = \psi(\mathcal{O}k)$, and we therefore have $\sigma(H_{g,E,\xi}) = \sigma(H_{g,E,\xi'})$ whenever $|\xi| = |\xi'|$. Thus, the energy momentum spectrum may be recovered from the set

$$\left\{(|\xi|, E) \in [0, \infty) \times \mathbb{R} \,\middle|\, \xi \in \mathbb{R}^d, 0 \in \sigma(H_{g,E,\xi})\right\}.$$

1.3 Statement of the Main Result

Before stating our main result, we recall that for the non-interacting model in any dimension $d \in \mathbb{N}$, the bottom of the essential spectrum is given by $\Sigma_{\text{ess}}(H_{0,0,\xi}) = 1$, and there is precisely one eigenvalue, counted with multiplicity, namely $\frac{1}{2}|\xi|^2$. This information about the discrete energy-momentum spectrum in the non-interacting case may thus be summarized in a simple

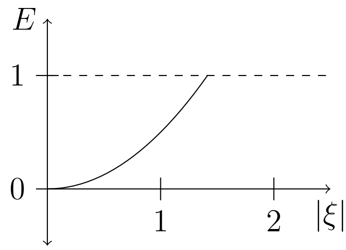

Figure 1: $g = 0$.

graph, plotting the discrete spectrum against the norm of the total momentum:

For the case $d = 1, 2$ of our main result, we impose the extra condition

$$\int \frac{dk\,|v(k)|^2}{\frac{1}{2}|\xi - k|^2} = \infty \tag{16}$$

for all $\xi \in \mathbb{R}$. This is satisfied if, say, for any compact set $C \subseteq \mathbb{R}^d$, there is a constant $M_C > 0$ such that $|v(k)| \geqslant M_C$ for $k \in M_C$. In particular, the coupling function $v(k) = |k|^{-\frac{d-1}{2}}$ of the Fröhlich polaron model satisfies this condition.

Theorem 1.3. *Consider the Fröhlich polaron operator $H_{g,0,\xi}$ introduced in Subsection 1.2. There is $g_0 > 0$ such that, for all $\xi \in \mathbb{R}^d$, and all $g \in \mathbb{R}$ satisfying $|g| \leqslant g_0$, $H_{g,0,\xi}$ has at most two discrete eigenvalues. Furthermore,*

1. *In dimension $d = 1, 2$, there is $g_0 > 0, C_1 > 0$ such that, for all $g \in \mathbb{R}$ satisfying $0 < |g| \leqslant g_0$, we have:*
   - *(a) If $\frac{1}{2}|\xi|^2 \leqslant C_1 g^2$, $H_{g,0,\xi}$ has at least one discrete eigenvalue.*
   - *(b) If $C_1 g^2 < \frac{1}{2}|\xi|^2$, $H_{g,0,\xi}$ has precisely one discrete eigenvalue.*
2. *In dimension $d \geqslant 3$, there is $g_0 > 0$ and $C_1, C_2, C_3 > 0$ such that, for all $g \in \mathbb{R}$ satisfying $|g| \leqslant g_0$, we have:*
   - *(a) If $\frac{1}{2}|\xi|^2 \leqslant C_1 g^2$, $H_{g,0,\xi}$ has at least one discrete eigenvalue.*
   - *(b) If $C_1 g^2 < \frac{1}{2}|\xi|^2 \leqslant \Sigma_{\mathrm{ess}}(H_{g,0,\xi}) + C_2 g^2$, $H_{g,0,\xi}$ has precisely one discrete eigenvalue.*
   - *(c) If $\Sigma_{\mathrm{ess}}(H_{g,0,\xi}) + C_2 g^2 < \frac{1}{2}|\xi|^2 \leqslant \Sigma_{\mathrm{ess}}(H_{g,0,\xi}) + C_3 g^2$, $H_{g,0,\xi}$ has at most one discrete eigenvalue.*
   - *(d) If $\Sigma_{\mathrm{ess}}(H_{g,0,\xi}) + C_3 g^2 < \frac{1}{2}|\xi|^2$, $H_{g,0,\xi}$ has no discrete eigenvalues.*

The result may be summarized by the following graphs, where the discrete spectrum is plotted against the absolute value of the total momentum. The dashed line represents the bottom of the essential spectrum, the grey boxes represent areas that contain at most one eigenvalue, and the dotted lines represent the picture in the non-interacting case, as seen in Figure 1.

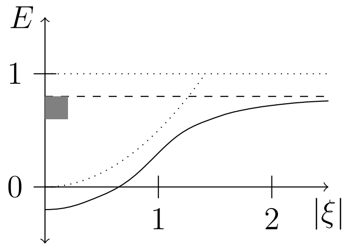

Figure 2: $g \neq 0$ and $d = 1, 2$.

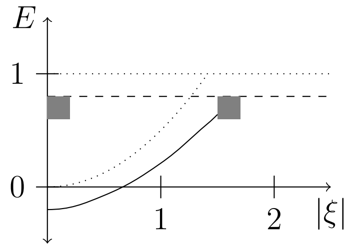

Figure 3: $g \neq 0$ and $d \geqslant 3$.

In the weak coupling regime of the Fröhlich polaron model, this result clearly extends the result of Spohn [40] by ruling out the existence of excited states below the bottom of the essential spectrum, except for a small total momentum regime.

Our main result should be compared to the result of Minlos [28], which we improve by treating a class of models containing the Fröhlich polaron model. Note that in our singular setting, contrary to the setting of Minlos, the analysis does not rule out the existence of an additional eigenvalue for small total momentum in dimension $d \geqslant 3$.

1.3.1 Outline of Proof

In this subsection, we give an outline of our proof of the main result.

In Subsection 1.4, we cover the eigenvalue counting arguments that will be essential for our proof. More precisely, we recall the Birman-Schwinger principle and we give a treatment of the well known Feshbach method with an emphasis on the less well known Haynsworth inertia additivity formula.

In Subsection 1.5, we cover the self-adjointedness of the model by means of relative form estimates. The results of this section do not rely on a weak coupling assumption.

In Subsection 1.6, we apply a Feshbach map, which reduces the Fröhlich polaron model to a generalized Friedrichs model. Furthermore, we give a precise analysis of the discrete spectrum of the generalized Friedrichs model in the weak coupling regime.

Finally, in Subsection 1.7, we derive the main result.

1.4 Eigenvalue Counting Arguments

Our analysis of the Fröhlich polaron model consists in a twofold reduction, first to a Birman-Schwinger operator, next to a Feshbach operator. In Subsubsection 1.4.1, we recall the well known Birman-Schwinger principle in a form which is useful for us. It is not so well known that a similar principle is true of the Feshbach operator. In Subsubsection 1.4.2, we will recall this less well known result, which is due to Haynsworth [23], along with some properties of the Feshbach map.

First, we recall a version of the min-max principle known as the Glazman lemma. It is usually stated for semi-bounded operators, though the proof applies equally well in the generality stated here. In relation to our application to the Haynsworth inertia additivity formula, it is useful to drop the condition of lower semi-boundedness. For the convenience of the reader, we provide a proof.

Definition 1.4. For a self-adjoint operator $A : \mathrm{dom}(A) \to \mathcal{H}$ in a Hilbert space $\mathcal{H}$, we let

$$\begin{aligned} N(\lambda, A) &:= \mathrm{tr}[1_{(-\infty,\lambda)}(A)] := \dim \mathrm{ran}(1_{(-\infty,\lambda)}(A)) \in \mathbb{N}_0 \cup \{\infty\}, \\ n(\lambda, A) &:= \mathrm{tr}[1_{(\lambda,\infty)}(A)] := \dim \mathrm{ran}(1_{(\lambda,\infty)}(A)) \in \mathbb{N}_0 \cup \{\infty\}. \end{aligned}$$

If $\lambda \leqslant \Sigma_{\mathrm{ess}}(A)$, then $N(\lambda, A)$ is the number of eigenvalues of A strictly below λ.

Lemma 1.5. *Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Let $\mathcal{E} \subseteq \mathrm{qfd}(A)$ be a subspace which is dense in $\mathrm{qfd}(A)$ with respect to the norm $\|\psi\|_{\mathrm{qfd}(A)} := (\|A_+^{1/2}\psi\|^2 + \|A_-^{1/2}\psi\|^2 + \|\psi\|^2)^{1/2}$. Then*

$$\begin{aligned} N(\lambda, A) &= \sup\{\dim \mathcal{F} \,|\, \mathcal{F} \subseteq \mathcal{E} \text{ finite dimensional subspace}, A < \lambda \text{ on } \mathcal{F}\}, \\ n(\lambda, A) &= \sup\{\dim \mathcal{F} \,|\, \mathcal{F} \subseteq \mathcal{E} \text{ finite dimensional subspace}, A > \lambda \text{ on } \mathcal{F}\}. \end{aligned}$$

Proof. Since $n(\lambda, A) = N(-\lambda, -A)$, the second identity follows from the first. Furthermore, since $N(\lambda, A) = N(0, A - \lambda)$ and $A[\phi] < \lambda\|\psi\|^2$ if and only if $(A - \lambda)[\phi] < 0$, we may without loss of generality assume $\lambda = 0$. We note also that the norm induced on $\mathrm{qfd}(A)$ by A coincides with the norm induced on $\mathrm{qfd}(A)$ by $A - \lambda$, so $\mathcal{E}$ is dense with respect to the first norm if and only if it is dense with respect to the second norm.

From $A_-^{1/2} = 1_{(-\infty,0)}(A)A_-^{1/2}$, we see that $\mathrm{ran}(A_-^{1/2}) \subseteq \mathrm{ran}(1_{(-\infty,0)}(A))$. Suppose then that $\mathcal{F} \subseteq \mathcal{E}$ is a finite dimensional subspace such that $A[\phi] < 0$ whenever $0 \neq \phi \in \mathcal{F}$. Since $\|A_+^{1/2}\phi\|^2 - \|A_-^{1/2}\phi\|^2 = A[\phi] < 0$, we find that $\phi \notin \ker(A_-^{1/2})$ if $0 \neq \phi \in \mathcal{F}$. Thus, $A_-^{1/2}$ is injective on $\mathcal{F}$, and therefore $\dim \mathcal{F} = \dim A_-^{1/2}(\mathcal{F}) \leqslant \dim \mathrm{ran}(A_-^{1/2}) \leqslant N(0, A)$.

Next, assuming $N(0, A) \geqslant k \in \mathbb{N}$, we have $\dim \operatorname{ran}(1_{(K,0)}(A)) \geqslant k$ for some $K < 0$, and thus find a subspace $\tilde{\mathcal{F}} \subseteq \operatorname{ran}(1_{(K,0)}(A)) \subseteq \operatorname{qfd}(A)$ of dimension k such that $A[\tilde{\phi}] = -\|A_-^{1/2}\tilde{\phi}\|^2 < 0$ if $0 \neq \tilde{\phi} \in \tilde{\mathcal{F}}$. In fact, since $\tilde{\mathcal{F}}$ is finite dimenional, there is $0 < \epsilon < 1$ such that $\|A_-^{1/2}\phi\|^2 \geqslant \epsilon\|\phi\|^2$ for $\phi \in \tilde{\mathcal{F}}$. Let $\tilde{\phi}_1, \ldots, \tilde{\phi}_k$ be an orthonormal basis of $\tilde{\mathcal{F}}$ and pick approximating vectors $\phi_1, \ldots, \phi_k \in \mathcal{E}$ with $\|\phi_j - \tilde{\phi}_j\|_{\operatorname{qfd}(A)} \leqslant \delta$. If $\phi = \sum_{j=1}^k c_j\phi_j$ and $\tilde{\phi} = \sum_{j=1}^k c_j\tilde{\phi}_j$, then we have

$$\big|\|\phi\| - \|\tilde{\phi}\|\big| \leqslant \|\phi - \tilde{\phi}\|_{\operatorname{qfd}(A)} \leqslant \delta \sum_{j=1}^k |c_j| \leqslant \delta\sqrt{k}\|\tilde{\phi}\|.$$

Thus, if $\phi = 0$ and $\delta < 1/\sqrt{k}$, we find $\tilde{\phi} = 0$, and therefore $c_1 = \ldots = c_k = 0$. We conclude that $\mathcal{F} = \operatorname{span}(\phi_1, \ldots, \phi_k) \subseteq \mathcal{E}$ has dimension k. Additionally, if $\delta \leqslant 1/(2\sqrt{k})$, then we have $\|\phi\|/2 \leqslant \|\tilde{\phi}\| \leqslant 3\|\phi\|/2$. Thus,

$$\begin{aligned} \|A_-^{1/2}\phi\| &\geqslant \epsilon\|\tilde{\phi}\| - \|\phi - \tilde{\phi}\|_{\operatorname{qfd}(A)} \geqslant (\epsilon - 3\delta\sqrt{k})\|\phi\|/2, \\ \|A_+^{1/2}\phi\| &= \|A_+^{1/2}(\phi - \tilde{\phi})\| \leqslant 3\delta\sqrt{k}\|\phi\|/2, \end{aligned}$$

and we therefore have, when $\delta < \epsilon/(6\sqrt{k})$ and $\phi \neq 0$,

$$A[\phi] = \|A_+^{1/2}\phi\|^2 - \|A_-^{1/2}\phi\|^2 < 0.$$

□

1.4.1 Birman-Schwinger Principle

In this subsection, we recall the well known Birman-Schwinger principle, which we need in a form very close to the one obtained by Birman [6]. First, we introduce some useful notation. Let $\mathcal{H}$ be a fixed Hilbert space.

Definition 1.6. Let T be a self-adjoint operator in $\mathcal{H}$. Define $\mathcal{H}^{1/2}[T]$ to be the topological vector space with underlying vector space $\operatorname{qfd}(T)$ and topology induced by the norm $\|\psi\|_{\mathcal{H}^{1/2}[T]} := \|(T^2+1)^{1/4}\psi\|$. Define $\mathcal{H}^{-1/2}[T]$ to be the topological vector space obtained by completing $\mathcal{H}$ with respect to the norm $\|\psi\|_{\mathcal{H}^{-1/2}[T]} := \|(T^2+1)^{-1/4}\psi\|$.

This definition introduces $\mathcal{H}^{\pm 1/2}[T]$ as topological vector spaces. For our treatment of the Birman-Schwinger principle, it is convenient to consider certain families of norms on $\mathcal{H}^{\pm 1/2}[T]$, to be introduced in the following two definitions.

Definition 1.7. Let T be a self-adjoint operator in $\mathcal{H}$. For $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(T)$, let P_{Ω_λ} denote the orthogonal projection in $\mathcal{H}$ onto the finite-dimensional subspace $\Omega_\lambda := \ker(T-\lambda)$, and define a self-adjoint, positive operator in $\mathcal{H}$ by

$$U_\lambda := |T-\lambda| + P_{\Omega_\lambda}.$$

Write $U_\lambda^{1/2} : \mathcal{H}^{1/2}[T] \to \mathcal{H}$, and let $\bar{U}_\lambda^{1/2} : \mathcal{H} \to \mathcal{H}^{-1/2}[T]$ be the unique continuous extension of $U_\lambda^{1/2}$ to $\mathcal{H}$.

Definition 1.8. Let $\lambda, U_\lambda^{1/2}$ and $\bar{U}_\lambda^{1/2}$ be as in Definition 1.7. Define $\mathcal{H}_\lambda^{\pm 1/2}[T]$ to be the Hilbert space with underlying topological vector space $\mathcal{H}^{\pm 1/2}[T]$ and norm

$$\|\psi\|_{\mathcal{H}_\lambda^{1/2}[T]} = \|U_\lambda^{1/2}\psi\|, \qquad \|\psi\|_{\mathcal{H}_\lambda^{-1/2}[T]} = \|\bar{U}_\lambda^{-1/2}\psi\|.$$

For notational simplicity, define also $\|\psi\|_\lambda = \|\psi\|_{\mathcal{H}_\lambda^{1/2}[T]}$ for $\psi \in \mathrm{qfd}(T)$.

Remark 1.9. The topologies of $\mathcal{H}_i^{\pm 1/2}[T]$, respectively, manifestly coincide with the topologies of $\mathcal{H}^{\pm 1/2}[T]$, respectively. Therefore, the inequalities

$$\begin{aligned} \|\psi\|_{\mathcal{H}_\lambda^{1/2}[T]} &= \|U_\lambda^{1/2} U_{\lambda'}^{-1/2} U_{\lambda'}^{1/2}\psi\| \leqslant \|U_\lambda^{1/2} U_{\lambda'}^{-1/2}\| \|\psi\|_{\mathcal{H}_{\lambda'}^{1/2}[T]}, \\ \|\psi\|_{\mathcal{H}_\lambda^{-1/2}[T]} &= \|\bar{U}_\lambda^{-1/2} \bar{U}_{\lambda'}^{1/2} \bar{U}_{\lambda'}^{-1/2}\psi\| \leqslant \|\bar{U}_\lambda^{-1/2} \bar{U}_{\lambda'}^{1/2}\| \|\psi\|_{\mathcal{H}_{\lambda'}^{-1/2}[T]}, \end{aligned} \tag{17}$$

which hold true for all $\lambda, \lambda' \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(T)$, ensure that the topologies of $\mathcal{H}_\lambda^{\pm 1/2}[T]$, respectively, coincide with the topologies of $\mathcal{H}^{\pm 1/2}[T]$, respectively, for all $\lambda, \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(T)$. Note that

$$\|\psi\|_i^2 = \|U_i^{1/2}\psi\|^2 = \||T-i|^{1/2}\psi\|^2, \qquad \psi \in \mathrm{qfd}(T), \tag{18}$$

Combinging the elementary inequality $|x-i| \leqslant |x| + 1 \leqslant 2|x-i|$ for $x \in \mathbb{R}$ with formulas (17) and (18), we find that there are constants $L_\lambda, L'_\lambda > 0$ such that

$$L_\lambda \|\psi\|_\lambda^2 \leqslant \||T|^{1/2}\psi\|^2 + \|\psi\|^2 \leqslant L'_\lambda \|\psi\|_\lambda^2, \quad \psi \in \mathrm{qfd}(T), \lambda \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(T). \tag{19}$$

Finally, there is a natural pairing of $\phi \in \mathcal{H}^{1/2}[T]$ and $\psi \in \mathcal{H}^{-1/2}[T]$, given by

$$\langle \phi, \psi \rangle' := \langle U_\lambda^{1/2}\phi, \bar{U}_\lambda^{-1/2}\psi \rangle, \tag{20}$$

where $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(T)$ is arbitrary.

Theorem 1.10. *Let T, H be self-adjoint operators in $\mathcal{H}$ with* $\mathrm{qfd}(H) = \mathrm{qfd}(T)$*, define the quadratic form $W[\psi] := T[\psi] - H[\psi]$ on* $\mathrm{qfd}(T)$*, and suppose that there is a positive constant $L > 0$ such that*

$$|W[\psi]| \leqslant L(\||T|^{1/2}\psi\|^2 + \|\psi\|^2), \quad \psi \in \mathrm{qfd}(T).$$

Then there is a unique continuous linear operator $W : \mathcal{H}^{1/2}[T] \to \mathcal{H}^{-1/2}[T]$ such that

$$\langle \psi, W\psi \rangle' = W[\psi], \quad \psi \in \mathrm{qfd}(T). \tag{21}$$

For $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(T)$, define the bounded operator $B_\lambda := \bar{U}_\lambda^{-1/2} W U_\lambda^{-1/2}$ on $\mathcal{H}$. If B_i is compact, then $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(T)$. Let J_λ denote the partial isometry from the polar decomposition $T - \lambda = J_\lambda |T - \lambda|$. If $\lambda \in \mathbb{R} \setminus \sigma_{\mathrm{ess}}(T)$, then $N(\lambda, H) = N(0, J_\lambda - B_\lambda)$.

Theorem 1.10 is essentially due to Birman [6]. Birman, however, only considers $\lambda < \Sigma_0(T)$, in which case $J_\lambda = 1$, and therefore $N(\lambda, H) = n(1, B_\lambda)$. We will refer to the operator B_λ as the Birman-Schwinger operator. Note that if, say, $W \geqslant 0$, then it is most common in the literature to refer to the 'sandwiched resolvent' $K_\lambda = W^{1/2}(T - \lambda)^{-1} W^{1/2}$ (considered by both Birman [6] and Schwinger [37]) as the Birman-Schwinger operator.

For the convenience of the reader, we give a proof of the result, although it is very similar to the proof given by Birman [6].

Proof. From inquality (19), we find $L'_\lambda > 0$ such that $|W[\psi]| \leqslant L'_\lambda \|\psi\|_\lambda$ when $\psi \in \mathcal{H}_\lambda^{1/2}[T]$. Thus, there is a unique bounded operator $\tilde{B}_\lambda$ on $\mathcal{H}_\lambda^{1/2}[T]$ with

$$\langle \psi, \tilde{B}_\lambda \psi \rangle_\lambda = W[\psi], \quad \psi \in \mathcal{H}_\lambda^{1/2}[T].$$

Define $W_\lambda : \mathcal{H}_\lambda^{1/2}[T] \to \mathcal{H}_\lambda^{-1/2}[T]$ by $W_\lambda = \bar{U}_\lambda^{1/2} U_\lambda^{1/2} \tilde{B}_\lambda$. Note, for all $\psi \in \mathrm{qfd}(T)$, that

$$\langle \psi, W_\lambda \psi \rangle' = \langle U_\lambda^{1/2} \psi, \bar{U}_\lambda^{-1/2} W_\lambda \psi \rangle = \langle U_\lambda^{1/2} \psi, U_\lambda^{1/2} \tilde{B}_\lambda \psi \rangle = \langle \psi, \tilde{B}_\lambda \psi \rangle_\lambda = W[\psi],$$

where we made use of formula (20). In particular, $W_\lambda \psi$ does not depend on λ, and it is therefore well-defined to put $W := W_\lambda$. Furthermore, we see that $B_\lambda := \bar{U}_\lambda^{-1/2} W U_\lambda^{-1/2} = U_\lambda^{1/2} \tilde{B}_\lambda U_\lambda^{-1/2}$ is unitarily equivalent to $\tilde{B}_\lambda$, so that B_i is compact if and only if $\tilde{B}_i$ is compact. Additionally, if we define the operator $\tilde{J}_\lambda := U_\lambda^{-1/2} J_\lambda U_\lambda^{1/2} = J_\lambda \restriction_{\mathrm{qfd}(T)}$, we have $N(0, J_\lambda - B_\lambda) = N(0, \tilde{J}_\lambda - \tilde{B}_\lambda)$.

By the Weyl criterion, if $C_i = (T - i)^{-1} - (H - i)^{-1}$ is compact, then $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(T)$. But we have, for $\psi \in \mathrm{dom}(H)$,

$$\langle \psi, |T - i|^{-1}(H - i)\psi \rangle_i = \langle \psi, (H - i)\psi \rangle = \langle \psi, (\tilde{J}_i - \tilde{B}_i)\psi \rangle_i,$$

which implies $(T-i)^{-1}(H-i) = 1 - \tilde{J}_i^{-1}\tilde{B}_i$ on $\mathrm{dom}(H)$. It follows that $C_i = -\tilde{J}_i^{-1}\tilde{B}_i(H-i)^{-1}$. Since $(H-i)^{-1}$ is bounded as an operator from $\mathcal{H}$ into $\mathcal{H}_i^{1/2}[T]$, and $\tilde{J}_i^{-1}$ is bounded as an operator from $\mathcal{H}_i^{1/2}[T]$ into $\mathcal{H}$, we see that C_i is compact if $\tilde{B}_i$ is compact.

Finally, since

$$\langle\psi, (\tilde{J}_\lambda - \tilde{B}_\lambda)\psi\rangle_\lambda = T[\psi] - \lambda\|\psi\|^2 - W[\psi] = H[\psi] - \lambda\|\psi\|^2, \quad \psi \in \mathrm{qfd}(T),$$

the Glazman lemma, Lemma 1.5, ensures that $N(0, \tilde{J}_\lambda - \tilde{B}_\lambda) = N(\lambda, H)$. □

1.4.2 Haynsworth Inertia Additivity Formula

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and assume that $\mathcal{D}_1 \subseteq \mathcal{H}_1, \mathcal{D}_2 \subseteq \mathcal{H}_2$ are dense subspaces. Suppose that we are given a block operator matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : \mathcal{D}_1 \oplus \mathcal{D}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2. \tag{22}$$

Let P denote the orthogonal projection in $\mathcal{H}_1 \oplus \mathcal{H}_2$ onto $\mathcal{H}_1 \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2$ and suppose that $A_{22} : \mathcal{D}_2 \to \mathcal{H}_2$ is invertible. The Feshbach map with respect to P maps A to the Feshbach operator $F := A_{11} - A_{12}A_{22}^{-1}A_{21} : \mathcal{D}_1 \to \mathcal{H}_1$. The Feshbach operator F is also called the Feshbach map of A with respect to P.

It is sometimes convenient to employ a notation where the matrix decomposition (22) is implicit. In that case, we introduce also the orthogonal projection $\bar{P}$ in $\mathcal{H}_1 \oplus \mathcal{H}_2$ onto $\mathcal{H}_2$, and note that $A_{22} = PA\restriction_{\mathrm{ran}(\bar{P})\cap\mathcal{D}_2}$.

Theorem 1.11. *Let A and P be as described above. Suppose that A is self-adjoint and that $A_{12}A_{22}^{-1}$ is bounded. Then the Feshbach operator F is self-adjoint, and we have $0 \in \sigma(A) \Leftrightarrow 0 \in \sigma(F)$. Furthermore, we have*

$$\begin{aligned} \dim\ker(A) &= \dim\ker(F), \\ N(0, A) &= N(0, F) + N(0, A_{22}), \\ n(0, A) &= n(0, F) + n(0, A_{22}). \end{aligned}$$

For matrices, this result goes by the name of the Haynsworth inertia additivity formula, and our proof below is essentially the same as the one given by Haynsworth [23], except for technical complications associated with unbounded operators. For unbounded, closed (but not necessarily self-adjoint) operators, the closedness of F and the relations $0 \in \sigma(A) \Leftrightarrow 0 \in \sigma(F)$ and $\dim\ker(A) = \dim\ker(F)$ are due to Bach et al. [4], albeit with a slightly different proof and imposing slightly different conditions. The two final relations of the theorem do not seem to have appeared previously in the literature for the case of unbounded, self-adjoint operators.

Proof. We will first argue that the Aitken block diagonalization formula,

$$\begin{bmatrix} 1 & -A_{12}A_{22}^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -A_{22}^{-1}A_{21} & 1 \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & A_{22} \end{bmatrix} \tag{23}$$

holds true in the sense of unbounded operators. With obvious choice of notation, we write this formula as $S(A_{12}A_{22}^{-1})AS'(A_{22}^{-1}A_{21}) = \operatorname{diag}(F, A_{22})$. It is clear that $\operatorname{diag}(F, A_{22}) \subseteq S(A_{12}A_{22}^{-1})AS'(A_{22}^{-1}A_{21})$, so we must prove that $\operatorname{dom}(AS'(A_{22}^{-1}A_{21})) = \operatorname{dom}(\operatorname{diag}(F, A_{22})) = \mathcal{D}_1 \oplus \mathcal{D}_2$. By inspection, we find that $S'(A_{22}^{-1}A_{21}) : \mathcal{D}_1 \oplus \mathcal{H}_2 \to \mathcal{D}_1 \oplus \mathcal{H}_2$ is a bijection, with inverse $S'(-A_{22}^{-1}A_{21})$. It follows that $\operatorname{dom}(AS'(A_{22}^{-1}A_{21})) = S'(-A_{22}^{-1}A_{21})(\mathcal{D}_1 \oplus \mathcal{D}_2)$. Furthermore, $S'(A_{22}^{-1}A_{21})$ and $S'(-A_{22}^{-1}A_{21})$ both map $\mathcal{D}_1 \oplus \mathcal{D}_2$ into $\mathcal{D}_1 \oplus \mathcal{D}_2$, and their restrictions to that subspace are therefore also mutual inverses. The Aitken block diagonalization formula (23) follows.

Next, we observe that, since A is self-adjoint, we have $A_{22}^{-1} \subseteq (A_{22}^{-1})^*$ and $A_{12} \subseteq A_{21}^*$ and therefore also $A_{22}^{-1}A_{21} \subseteq (A_{12}A_{22}^{-1})^*$. We conclude that $S'(A_{22}^{-1}A_{21}) \subseteq S(A_{12}A_{22}^{-1})^*$, and therefore $\operatorname{diag}(F, A_{22}) \subseteq SAS^*$, where we put $S = S(A_{12}A_{22}^{-1})$. But $S(A_{12}A_{22}^{-1})$, and therefore also $S(A_{12}A_{22}^{-1})^*$, is bounded with a bounded inverse on $\mathcal{H}_1 \oplus \mathcal{H}_2$. It therefore follows as before that $\operatorname{dom}(SAS^*) = (S^*)^{-1}(\mathcal{D}_1 \oplus \mathcal{D}_2) = \mathcal{D}_1 \oplus \mathcal{D}_2 = \operatorname{dom}(\operatorname{diag}(F, A_{22}))$, and therefore that we have $SAS^* = \operatorname{diag}(F, A_{22})$ in the sense of unbounded operators.

We now argue that $T = \operatorname{diag}(F, A_{22}) = SAS^*$ is self-adjoint, from which it follows that F and A_{22} are self-adjoint. It is clear that T is symmetric, so it suffices to show that $\operatorname{dom}(T^*) \subseteq \operatorname{dom}(T)$. If $\psi \in \operatorname{dom}(T^*)$, then the linear functional Λ defined by

$$(S^*)^{-1}(\operatorname{dom}(A)) \ni \phi \mapsto \Lambda(\phi) = \langle \psi, SAS^*\phi \rangle$$

is bounded. It follows that the linear functional

$$\operatorname{dom}(A) \ni \phi \mapsto \langle S^*\psi, A\phi \rangle = \Lambda((S^*)^{-1}\phi)$$

is also bounded, and therefore that $S^*\psi \in \operatorname{dom} A^* = \operatorname{dom} A$. In conclusion, $\psi \in \operatorname{dom}(AS^*) = \operatorname{dom}(T)$, and this was what we wanted. Note that self-adjointedness of A_{22} implies closedness, which in turn implies closedness of A_{22}^{-1}. The closed graph theorem therefore ensures boundedness of A_{22}^{-1}.

The assertions $0 \in \sigma(A) \Leftrightarrow 0 \in \sigma(F)$ and $\dim\ker(A) = \dim\ker(F)$ now follow directly from the formula $SAS^* = \operatorname{diag}(F, A_{22})$, as do the assertions $N(0, A) = N(0, F) + N(0, A_{22})$ and $n(0, A) = n(0, F) + n(0, A_{22})$ according to the Glazman lemma, Lemma 1.5, with $\mathcal{E} = \mathcal{D}_1 \oplus \mathcal{D}_2$. □

1.5 General Coupling Considerations

In this section we will make some considerations for general coupling $g \in \mathbb{R}$, in particular regarding the definition of the model we consider. The starting point of our analysis is the following bound, which is related to a commutator bound due to Lieb and Thomas [27] (see also [18, Lemma 2.1]). Recall from formula (14) the definition of the constants L_E for $E < 1$.

Lemma 1.12. *Let $v : \mathbb{R}^d \to \mathbb{C}$ be a measurable function satisfying $L_0 < \infty$. For $\xi \in \mathbb{R}^d$, we have $\mathrm{qfd}(T_{0,\xi}) \subseteq \mathrm{qfd}(a(v))$, and for any $E \leqslant -1$, we have the inequality*

$$|a(v)[\psi]| \leqslant 2L_E \|T_{E,\xi}^{1/2}\psi\|^2_{\mathcal{F}_{\mathrm{sym}}}, \quad \psi \in \mathrm{qfd}(T_{0,\xi}). \tag{24}$$

Proof. Define, for $n \geqslant 0$, $\tilde{\psi}^{(n)}(k^{(n)}) := \sqrt{T_{E,\xi}^{(n)}(k^{(n)})}|\psi^{(n)}(k^{(n)})|$ and

$$\tilde{\phi}^{(n)}(k^{(n)}) := \left(\int dk_{n+1}\, T_{E,\xi}^{(n+1)}(k^{(n)}, k_{n+1})|\psi^{(n+1)}(k^{(n)}, k_{n+1})|^2\right)^{1/2},$$

where $k^{(n)} = (k_1, \ldots, k_n)$. Then we have, for $E < 0, \xi \in \mathbb{R}^d$, $n \geqslant 0$,

$$\begin{aligned} A_{n+1}^{\psi} &:= \sqrt{n+1}\int dk^{(n)} \int dk_{n+1}\, |\psi^{(n)}(k^{(n)}) v(k_{n+1})\psi^{(n+1)}(k^{(n)}, k_{n+1})| \\ &\leqslant \int dk^{(n)} \left(\int \frac{dk_{n+1}\,(n+1)|v(k_{n+1})|^2}{T_{E,\xi}^{(n)}(k^{(n)}) T_{E,\xi}^{(n+1)}(k^{(n)}, k_{n+1})}\right)^{1/2} \cdot \tilde{\psi}^{(n)}(k^{(n)})\tilde{\phi}^{(n)}(k^{(n)}). \end{aligned}$$

With $\xi^{(n)} := \xi - k_1 - \ldots - k_n$ and $E \leqslant -1$, the pointwise estimate

$$\frac{(n+1)|v(k_{n+1})|^2}{(\frac{1}{2}|\xi^{(n)}|^2 + n - E)(\frac{1}{2}|\xi^{(n)} - k_{n+1}|^2 + n + 1 - E)} \leqslant \frac{4|v(k_{n+1})|^2}{\frac{1}{2}|k_{n+1}|^2 + 1 - E}$$

follows by noting that we must have $|k_{n+1}| \leqslant 2|\xi^{(n)} - k_{n+1}|$ or $|k_{n+1}| \leqslant 2|\xi^{(n)}|$. Thus, after an application of the Cauchy-Schwarz inequality, we deduce that

$$\sum_{n=1}^{\infty} A_n^{\psi} \leqslant 2L_E \sum_{n=1}^{\infty} \|[T_{E,\xi}^{(n-1)}]^{1/2}\psi^{(n-1)}\|_{\mathcal{F}_{\mathrm{sym}}^{(n-1)}} \|[T_{E,\xi}^{(n)}]^{1/2}\psi^{(n)}\|_{\mathcal{F}_{\mathrm{sym}}^{(n)}}.$$

Another application of the Cauchy-Schwarz inequality finishes the proof. □

Corollary 1.13. *Suppose v satisfies $L_0 < \infty$. Then we find, for any $\epsilon > 0$, a positive number $C_\epsilon > 0$ independent from $\xi \in \mathbb{R}^d$ such that*

$$|a(v)[\psi]| \leqslant \epsilon\|T_{0,\xi}^{1/2}\psi\|^2_{\mathcal{F}_{\mathrm{sym}}} + C_\epsilon\|\psi\|^2_{\mathcal{F}_{\mathrm{sym}}}, \quad \psi \in \mathrm{qfd}(T_{0,\xi}).$$

In particular, $a(v)$ is infinitesimally relatively $T_{0,\xi}$-form-bounded.

Proof. Recall that $\|T_{E,\xi}^{1/2}\psi\|^2 = \|T_{0,\xi}^{1/2}\psi\|^2 - E\|\psi\|^2$ for $\psi \in \mathrm{qfd}(T_{0,\xi})$. This corollary therefore follows from inequality (24) by picking E sufficiently negative that $2L_E \leqslant \epsilon$, in which case we may take $C_\epsilon = 2|E|L_E$. □

It now follows from the KLMN Theorem that there is a unique self-adjoint operator $H_{g,E,\xi}$ in $\mathcal{F}_{\mathrm{sym}}$ such that formula (15) holds true. We may therefore apply the Birman-Schwinger principle, Theorem 1.10, with $H := H_{g,0,\xi}, T := T_{0,\xi}$. For the convience of the reader, we write down the conclusion of Theorem 1.10 as applied to this setting, including the explicit form of the corresponding Birman-Schwinger operator, denoted $B_{g,E,\xi}$.

Theorem 1.14. *For $E < 1$, we have $N(E, H_{g,0,\xi}) = N(0, J_{E,\xi} - B_{g,E,\xi})$. Here, $J_{E,\xi} := \bigoplus_{n=0}^\infty J_{E,\xi}^{(n)}$, with $J_{E,\xi}^{(n)} := 1$ if $n \geqslant 1$, while $J_{E,\xi}^{(0)} := \mathrm{sgn}(\frac{1}{2}|\xi|^2 - E)$. The Birman-Schwinger operator $B_{g,E,\xi}$ has the structure $B_{g,E,\xi} = g(b_{E,\xi} + b_{E,\xi}^*)$, where, for $\psi \in \mathcal{F}_{\mathrm{sym}}$,*

$$(b_{E,\xi}\psi)^{(n)}(k_1, \ldots, k_n) := \int \frac{dk_{n+1}\,(n+1)^{1/2}\bar{v}(k_{n+1})\psi^{(n+1)}(k_1, \ldots, k_{n+1})}{U_{E,\xi}^{(n)}(k_1, \ldots, k_n)U_{E,\xi}^{(n+1)}(k_1, \ldots, k_{n+1})},$$
$$(b_{E,\xi}^*\psi)^{(n)}(k_1, \ldots, k_n) := \sum_{j=1}^n \frac{n^{-1/2}v(k_j)\psi^{(n-1)}(k_1, \ldots, \hat{k}_j, \ldots, k_n)}{U_{E,\xi}^{(n)}(k_1, \ldots, k_n)U_{E,\xi}^{(n-1)}(k_1, \ldots, \hat{k}_j, \ldots, k_n)},$$

and the notation $\hat{k}_j$ indicates that the variable k_j is omitted. Here, we denote by $U_{E,\xi}^{(n)}$ the function

$$U_{E,\xi}^{(n)}(k_1, \ldots, k_n) = (\frac{1}{2}|\xi - k_1 - \ldots - k_n|^2 + n - E)^{1/2}, \quad \text{for } n \geqslant 1,$$

while $U_{E,\xi}^{(0)} = |\frac{1}{2}|\xi|^2 - E|^{1/2}$ if $\frac{1}{2}|\xi|^2 \neq E$, and $U_{E,\xi}^{(0)} = 1$ if $\frac{1}{2}|\xi|^2 = E$.

It is also convenient to introduce notation which mirrors the notation introduced in Subsection 1.4.2

Definition 1.15. In $\mathcal{F}_{\mathrm{sym}}$, define orthogonal projections $P_{\leqslant 1}$ onto $\mathcal{F}_{\mathrm{sym}}^{(\leqslant 1)} := \mathcal{F}_{\mathrm{sym}}^{(0)} \oplus \mathcal{F}_{\mathrm{sym}}^{(1)} \subseteq \mathcal{F}_{\mathrm{sym}}$, and $P_{\geqslant 2} = 1 - P_{\leqslant 1}$ onto $\mathcal{F}_{\mathrm{sym}}^{(\geqslant 2)} := \bigoplus_{n=2}^\infty \mathcal{F}_{\mathrm{sym}}^{(n)} \subseteq \mathcal{F}_{\mathrm{sym}}$. Given an operator A in $\mathcal{F}_{\mathrm{sym}}$, denote by $\bar{A} := P_{\geqslant 2}A\restriction_{\mathcal{F}_{\mathrm{sym}}^{(\geqslant 2)} \cap \mathrm{dom}(A)}$ the restriction of A to $\mathcal{F}_{\mathrm{sym}}^{(\geqslant 2)}$.

Corollary 1.16. *If $\psi \in \mathcal{F}_{\mathrm{sym}}^{(\geqslant 2)}$ and $E \leqslant 1$, then*

$$|a(v)[\psi]| \leqslant 6L_{-1}\|T_{E,\xi}^{1/2}\psi\|_{\mathcal{F}_{\mathrm{sym}}}^2.$$

Proof. Considering inequality (24), note simply that in $\mathcal{F}_{\mathrm{sym}}^{(\geqslant 2)}$, we have the inequality $N+1 \leqslant 3(N-E)$ whenever $E \leqslant 1$, and thus also $T_{-1,\xi} \leqslant 3T_{E,\xi}$. □

Corollary 1.17. *In $\mathcal{F}_{\mathrm{sym}}^{(\geqslant 2)}$, we have $\|\bar{B}_{g,E,\xi}\|_{\mathcal{F}_{\mathrm{sym}}^{(\geqslant 2)}} \leqslant 6gL_{-1}$.*

1.6 Generalized Friedrichs Model

In the following definition, we introduce the generalized Friedrichs model (with a singular interaction). The definition is justified by Proposition 1.19. The remainder of the section is devoted to clarifying the precise relation between the generalized Friedrichs model and the Fröhlich polaron model.

Definition 1.18. Fix $k_{\min} \in \mathbb{R}^d$ and a measurable function $M : \mathbb{R}^d \to [0, \infty)$ satisfying $\frac{1}{4}|k - k_{\min}|^2 \leqslant M(k) \leqslant |k - k_{\min}|^2$. Let $C \in C_b^2(\mathbb{R}^d \times \mathbb{R}^d)$ be a function with two continuous, bounded derivates satisfying $\bar{C}(k_1, k_2) = C(k_2, k_1)$, and let $v : \mathbb{R}^d \to \mathbb{C}$ be a measurable function which is not identically zero and satisfies $L_0 < \infty$, with L_0 given by formula (14). We denote also by M the maximal operator of multiplication by the function M in $L^2(\mathbb{R}^d)$ and by C the quadratic form on $\operatorname{qfd}(M) \subseteq L^2(\mathbb{R}^d)$ defined by

$$(M\psi)(k) = M(k)\psi(k), \; C[\psi] := \int dk_1 dk_2 \, \bar{\psi}(k_1) v(k_1) C(k_1, k_2) \bar{v}(k_2) \psi(k_2). \quad (25)$$

The non-interacting generalized Friedrichs model is defined by the Hamiltonian $A_0 := \operatorname{diag}(e_0, M)$, acting in the direct sum space $\mathcal{H} := \mathbb{C} \oplus L^2(\mathbb{R}^d)$. Here, $e_0 \in \mathbb{R}$ is a real parameter. The generalized Friedrichs model (with a singular interaction) at coupling strength $g \in \mathbb{R}$ is defined by the quadratic form, for $\psi = (\psi^{(0)}, \psi^{(1)}) \in \operatorname{qfd}(A_0) = \mathbb{C} \oplus \operatorname{qfd}(M)$,

$$A_g[\psi] := e_0 |\psi^{(0)}|^2 + M[\psi^{(1)}] - 2g \operatorname{re} \int dk \, \bar{\psi}^{(0)} \bar{v}(k) \psi^{(1)}(k) - g^2 C[\psi^{(1)}]. \quad (26)$$

Proposition 1.19. *There is a unique self-adjoint operator A_g in the Hilbert space $\mathcal{H} = \mathbb{C} \oplus L^2(\mathbb{R}^d)$ with quadratic form domain $\operatorname{qfd}(A_g) = \operatorname{qfd}(A_0)$ defined by the quadratic form (26). We have $\sigma_{\mathrm{ess}}(A_g) = [0, \infty)$.*

Proof. The bounds $\frac{1}{4}|k - k_{\min}|^2 \leqslant M(k) \leqslant |k - k_{\min}|^2$ and $L_0 < \infty$ ensure that $\phi_\lambda \in L^2(\mathbb{R}^d)$, with $\phi_\lambda(k) := v(k)|M(k) - \lambda|^{-1/2}$. Putting $\psi = (\psi^{(0)}, \psi^{(1)}) \in \mathcal{H}$ and defining the interaction term

$$A_{\mathrm{int}}^{(g)}[\psi] := A_0[\psi] - A_g[\psi] = 2g \operatorname{re} \int dk \, \bar{\psi}^{(0)} \bar{v}(k) \psi^{(1)}(k) + g^2 C[\psi^{(1)}],$$

we have, whenever $\lambda < \min(e_0, 0)$, the inequality

$$|A_{\mathrm{int}}^{(g)}[\psi]| \leqslant g \|\phi_\lambda\|_{L^2(\mathbb{R}^d)} (2(e_0 - \lambda)^{-1/2} + g \|\phi_\lambda\|_{L^2} \|C\|_{L^\infty}) \|(A_0 - \lambda)^{1/2} \psi\|_{\mathcal{H}}^2.$$

The monotone convergence theorem ensures $\|\phi_\lambda\|_{L^2(\mathbb{R}^d)} \to 0$ as $\lambda \to -\infty$, so this shows that $A_{\mathrm{int}}^{(g)}$ is infinitesimally relatively form bounded with respect to the non-interacting term A_0. By the KLMN theorem, the quadratic form

A_g induces a unique self-adjoint operator in $\mathbb{C} \oplus L^2(\mathbb{R}^d)$ with form domain $\mathrm{qfd}(A_g) = \mathrm{qfd}(A_0)$.

We will now show that $\sigma_{\mathrm{ess}}(A_g) = \sigma_{\mathrm{ess}}(A_0) = [0, \infty)$. According to Theorem 1.10, it suffices to show that the Birman-Schwinger operator is compact for $\lambda = i$. We have

$$B_\lambda := \begin{bmatrix} 0 & -g|e_0 - \lambda|^{-1/2}\langle\phi_\lambda| \\ -g|\phi_\lambda\rangle|e_0 - \lambda|^{-1/2} & -g^2 C_\lambda \end{bmatrix},$$

where, $\langle\phi_\lambda| : L^2(\mathbb{R}^d) \to \mathbb{C}$ denotes the bounded linear functional defined by $\langle\phi_\lambda|(f) := \langle\phi_\lambda, f\rangle$ and $|\phi_\lambda\rangle : \mathbb{C} \to L^2(\mathbb{R}^d)$ denotes the bounded rank 1 operator $|\phi_\lambda\rangle(z) := z\phi_\lambda$. Finally, $C_\lambda : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the integral operator with integral kernel $\phi_\lambda(k_1)\phi_\lambda(k_2)C(k_1, k_2)$, which is seen to be Hilbert-Schmidt. It is now clear that B_i is compact. □

We now turn to the relation between the generalized Friedrichs model and the Fröhlich polaron model. Recalling Definition 1.15 and Corollary 1.17, we find that $1 - \bar{B}_{g,E,\xi}$ is invertible when g is sufficiently small. We may therefore consider the Feshbach map of $J_{E,\xi} - B_{g,E,\xi}$ with respect to $P_{\leqslant 1}$. The Feshbach operator, $F_{g,E,\xi}$, acts in the direct sum space $\mathcal{F}_{\mathrm{sym}}^{(\leqslant 1)} = \mathcal{F}_{\mathrm{sym}}^{(0)} \oplus \mathcal{F}_{\mathrm{sym}}^{(1)}$,

$$F_{g,E,\xi} := \begin{bmatrix} \mathrm{sgn}(\frac{1}{2}|\xi|^2 - E) & -g b_{E,\xi} \\ -g b_{E,\xi}^* & 1 - g^2 b_{E,\xi}(1 - \bar{B}_{g,E,\xi})^{-1} b_{E,\xi}^* \end{bmatrix}. \tag{27}$$

Theorem 1.20. *There is $g_0 > 0$ such that, for $|g| \leqslant g_0$, $E \leqslant 1$, $\xi \in \mathbb{R}^d$, there is a unique choice of $k_{\min}(g, E, \xi), M_{g,E,\xi}, C_{g,E,\xi}$ and $\lambda_{g,E,\xi} \in \mathbb{R}$ such that*

$$k_{\min} := k_{\min}(g, E, \xi), \quad M := M_{g,E,\xi}, \quad C := C_{g,E,\xi},$$
$$\lambda := \lambda_{g,E,\xi}, \quad e_0 := e_{0,g,E,\xi} := \frac{1}{2}|\xi|^2 - E + \lambda_{g,E,\xi},$$

fulfill the conditions of Definition 1.18 and $F_{g,E,\xi}$ is the Birman-Schwinger operator corresponding to the choice of operators $T := \mathrm{diag}(\frac{1}{2}|\xi|^2 - E, T_{E,\xi}^{(1)})$, $H := A_g$ at $\lambda := \lambda_{g,E,\xi}$.

A detailed account of this result will be given in Subsection 2.5 based on a creation/annihilation calculus developed in that chapter. Here, we provide instead a sketch, based on well established methods in the field.

Sketch of Proof. We use the notation $P_f := d\Gamma(p)$. We define

$$t_{E,\xi}(k, n) := \frac{1}{2}(k - \xi)^2 + n - E, \quad \bar{t}_{E,\xi}(k, n) := t_{E,\xi}(k, n) 1_{n \geq 2}.$$

Then, using a generalized version of Wick's theorem [4, 21], one obtains the following formulae:

$$\lambda_{g,E,\xi}(k) := -\min_{k\in\mathbb{R}^d}[t_{E,\xi}(k,1) - g^2\tilde{M}_{g,E,\xi}(k)],$$
$$M_{g,E,\xi}(k) := t_{E,\xi}(k,1) - g^2\tilde{M}_{g,E,\xi}(k) + \lambda_{g,E,\xi},$$

$$\tilde{M}_{g,E,\xi}(k) := \sum_{L=0}^{\infty} g^L\langle\Omega, a(v)\bar{t}_{E,\xi}(P_f+k,N+1)^{-1} \cdot \prod_{l=2}^{L+1}\{\phi(v)\bar{t}_{E,\xi}(P_f+k,N+1)^{-1}\}a^*(v)\Omega\rangle$$

$$C_{g,E,\xi}(k,q) := \sum_{\substack{L\in\mathbb{N}_0\\ \underline{m},\underline{n},\underline{p},\underline{q}\in\mathbb{N}_0^{L+2}\\ n_l+q_l+m_l+p_l=1\\ m_1+p_1=0\\ n_{L+2}+q_{L+2}=0\\ \sum_{l=1}^{L+2} m_l=1\\ \sum_{l=1}^{L+2} n_l=1}} g^L\langle\Omega, a(v)^{q_1}\bar{t}_{E,\xi}(P_f+\pi_1(k,q),N+\nu_1)^{-1} \cdot \prod_{l=2}^{L+1}\{a^*(v)^{p_l}a(v)^{q_l}\bar{t}_{E,\xi}(P_f+\pi_l(k,q),N+\nu_l)^{-1}\} \cdot a^*(v)^{p_{L+2}}\Omega\rangle,$$

where we defined

$$\pi_l(k,q) := k1_{(\exists l'\geq l+1: m_{l'}=1)} + q1_{(\exists l'\leq l: n_{l'}=1)},$$
$$\nu_l := 1_{(\exists l'\geq l+1: m_{l'}=1)} + 1_{(\exists l'\leq l: n_{l'}=1)}.$$

Using these formulae, it is possible to show that $C_{g,E,\xi}$ and $M_{g,E,\xi}$ are well defined for small g and have the desired properties. To estimate the derivatives of $\tilde{M}_{g,E,\xi}$ and $C_{g,E,\xi}$, we use the Leibniz rule and we write differentiated factors as

$$\nabla_k t_{E,\xi}(x+k,n) = t_{E,\xi}(x+k,n)^{-1/2}\frac{x+k-\xi}{t_{E,\xi}(x+k,n)}t_{E,\xi}(x+k,n)^{-1/2},$$

where we note that the middle factor on the right hand side is bounded. Analogously, we can show that also second derivatives (as well as higher derivatives) are bounded. Factors which are twice differentiated can be written as

$$\partial_{k_j}\partial_{k_i}t_E(x+k,n) = t_{E,\xi}(x+k,n)^{-1/2} \cdot \left(\frac{\delta_{ij}}{t_{E,\xi}(x+k,n)} + \frac{(x+k-\xi)_i(x+k-\xi)_j}{t_{E,\xi}(x+k,n)^2}\right) \cdot t_{E,\xi}(x+k,n)^{-1/2},$$

where we note that the term in the bracket on the right hand side is bounded. □

Definition 1.21. In the setting of Theorem 1.20, the generalized Friedrichs model at coupling strenght $g \in (-g_0, g_0)$, denoted $A_g := A_{g,E,\xi}$, will be referred to as the generalized Friedrichs model induced by the Fröhlich polaron model $H_{g,E,\xi}$.

Theorem 1.22. *For $|g| \leqslant g_0, E \leqslant 1, \xi \in \mathbb{R}^d$, the generalized Friedrichs model induced by the Fröhlich polaron model satisfies*

$$N(\lambda_{g,E,\xi}, A_{g,E,\xi}) = N(E, H_{g,0,\xi}). \tag{28}$$

Proof. For the case $E < 1$, we have by combining the Glazman lemma, Lemma 1.5, and the Haynsworth inertia additivity formula, Theorem 1.11,

$$N(\lambda_{g,E,\xi}, A_{g,E,\xi}) = N(0, F_{g,E,\xi}) = N(0, J_{E,\xi} - B_{g,E,\xi}) = N(E, H_{g,0,\xi}).$$

For $E = 1$ and $g = 0$, we have $\lambda_{0,1,\xi} = 0$ and $e_{0,0,1,\xi} = \frac{1}{2}|\xi|^2 - 1$, and it therefore follows from direct observation that

$$N(0, A_{0,1,\xi}) = N(1, H_{0,0,\xi}) = \begin{cases} 1 & \frac{1}{2}|\xi|^2 < 1, \\ 0 & \frac{1}{2}|\xi|^2 \geqslant 1. \end{cases}$$

Similarly, for $E = 1$ and $g \neq 0$, we have $1 > E_{\mathrm{ess}} = \Sigma_{\mathrm{ess}}(H_{g,0,\xi})$ along with $\lambda_{g,1,\xi} > 0 = \Sigma_{\mathrm{ess}}(A_{g,1,\xi})$, and thus $N(\lambda_{g,1,\xi}, A_{g,1,\xi}) = N(1, H_{g,0,\xi}) = \infty$. □

Remark 1.23. The Fröhlich polaron model in our regime has thus succesfully been reduced to a family of generalized Friedrichs models. As is typical of the Feshbach method, we have obtained a simpler effective model at the price of a more complicated dependence on the relevant parameters, as evidenced by formula (28). If one neglects the term $g^2 b_{E,\xi}(1 - \bar{B}_{g,E,\xi})^{-1} b^*_{E,\xi}$ in formula (27), then one obtains the quadratic form considered by Miyao [29].

The following lemma can be proven by using the same tools as were used in the sketch of the proof of Theorem 1.20. A detailed proof is given in Subsection 2.5.

Lemma 1.24. *With notation as in Definition 1.21, we have, for fixed $|g| \leqslant g_0, \xi \in \mathbb{R}^d$, that the function $E \mapsto \lambda_{g,E,\xi}$ is strictly increasing and continuous. There is a unique number $E_{\mathrm{ess}} := E_{\mathrm{ess}}(g, \xi) \leqslant 1$ such that $\lambda_{g,E_{\mathrm{ess}},\xi} = 0$. In particular, $E_{\mathrm{ess}} = \Sigma_{\mathrm{ess}}(H_{g,0,\xi})$. If $0 < |g| \leqslant g_0$, then $E_{\mathrm{ess}} < 1$. Finally, we* *have the expansions in g*

$$k_{\min} = \xi + g^4 k^{(4)}_{\min}(g, E, \xi),$$

$$E_{\mathrm{ess}} = 1 - \int \frac{dq\, g^2|v(q)|^2}{\frac{1}{2}|q|^2 + 1} + g^4 E^{(4)}_{\mathrm{ess}}(g, \xi),$$

$$\begin{aligned} C(k_1, k_2) = {} & \frac{1}{T^{(2)}_{E,\xi}(k_1, k_2)} + \int \frac{dq\, g^2|v(q)|^2}{T^{(2)}_{E,\xi}(k_1, k_2) T^{(3)}_{E,\xi}(k_1, k_2, q) T^{(2)}_{E,\xi}(k_1, k_2)} \\ & + \int \frac{dq\, g^2|v(q)|^2}{T^{(2)}_{E,\xi}(k_1, k_2) T^{(3)}_{E,\xi}(k_1, k_2, q) T^{(2)}_{E,\xi}(q, k_2)} \\ & + \int \frac{dq\, g^2|v(q)|^2}{T^{(2)}_{E,\xi}(k_1, q) T^{(3)}_{E,\xi}(k_1, k_2, q) T^{(2)}_{E,\xi}(k_1, k_2)} \\ & + \int \frac{dq\, g^2|v(q)|^2}{T^{(2)}_{E,\xi}(k_1, q) T^{(3)}_{E,\xi}(k_1, k_2, q) T^{(2)}_{E,\xi}(q, k_2)} + g^4 C^{(4)}_{g,E,\xi}(k_1, k_2). \end{aligned}$$

where the remainder terms satisfy

$$\sup_{|g| \leqslant g_0, E \leqslant 1, \xi \in \mathbb{R}^d} \Big(|\lambda^{(4)}_{g,E,\xi}| + |e^{(4)}_{0,g,E,\xi}| + |E^{(4)}_{\mathrm{ess}}(g, \xi)| + |k^{(4)}_{\min}(g, E, \xi)| \Big) < \infty,$$

$$\sup_{|g| \leqslant g_0, E \leqslant 1, \xi \in \mathbb{R}^d} \left(\sum_{|\alpha| \leqslant 2} \|\partial^\alpha M^{(2)}_{g,E,\xi}\|_{L^\infty(\mathbb{R}^d)} + \sum_{|\alpha| \leqslant 2} \|\partial^\alpha C^{(4)}_{g,E,\xi}\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} \right) < \infty.$$

1.6.1 Discrete Spectrum Analysis

In this subsection, we give an analysis of the discrete spectrum of the generalized Friedrichs model, see Definition 1.18. The number $c_0 := C(k_{\min}, k_{\min})$ will play an important role in our analysis, and it is useful to introduce the notation $\tilde{C}(k_1, k_2) := C(k_1, k_2) - c_0$.

Before giving the main result of this section, we derive a simple consequence of the fundamental theorem of calculus.

Lemma 1.25. *Let $C \in C^2_b(\mathbb{R}^d \times \mathbb{R}^d)$ and $M : \mathbb{R}^d \to [0, \infty)$ be as in Definition 1.18. Then we have*

$$\sqrt{\frac{\frac{1}{2}|k - k_{\min}|^2 + 1}{M(k)}} |\tilde{C}(k, k_{\min})| \leqslant 2\sqrt{2} \sum_{|\alpha| \leqslant 1} \|\partial^\alpha \tilde{C}\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)}, \tag{29}$$

and we have $C(k_1, k_2) = c_0 + \tilde{C}(k_1, k_{\min}) + \tilde{C}(k_{\min}, k_2) + R(k_1, k_2)$, where the remainder term $R : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ satisfies

$$\sqrt{\frac{(\frac{1}{2}|k_1|^2 + 1)(\frac{1}{2}|k_2|^2 + 1)}{M(k_1) M(k_2)}} |R(k_1, k_2)| \leqslant 16 \sum_{|\alpha| \leqslant 2} \|\partial^\alpha \tilde{C}\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)}. \tag{30}$$

Proof. Assume without loss of generality that $k_{\min} = 0$. When $|k| \geqslant \sqrt{2}$,

$$\sqrt{\frac{\frac{1}{2}|k|^2+1}{M(k)}}|\tilde{C}(k,0)| \leqslant \sqrt{\frac{\frac{1}{2}|k|^2+1}{\frac{1}{4}|k|^2}}|\tilde{C}(k,0)| \leqslant 2\|\tilde{C}\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}.$$

If $|k| \leqslant \sqrt{2}$, then the identity $\tilde{C}(k,0) = \int_0^1 \frac{d}{dt}\tilde{C}(tk,0)dt$ implies

$$\sqrt{\frac{\frac{1}{2}|k|^2+1}{M(k)}}|\tilde{C}(k,0)| \leqslant \sqrt{\frac{\frac{1}{2}|k|^2+1}{\frac{1}{4}|k|^2}}|\tilde{C}(k,0)| \leqslant 2\sqrt{2}\sum_{|\alpha|=1}\|\partial^\alpha\tilde{C}\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}.$$

Combining these two bounds, we obtain inequality (29).

For the second relation, we have

$$\begin{aligned}
R(k_1,k_2) &= \tilde{C}(k_1,k_2) - \tilde{C}(k_1,0) - \tilde{C}(0,k_2)\\
&= \int_0^1 \frac{d}{dt_1}\tilde{C}(t_1k_1,k_2)dt_1 - \int_0^1 \frac{d}{dt_1}\tilde{C}(t_1k_1,0)dt_1\\
&= \int_0^1 \frac{d}{dt_2}\tilde{C}(k_1,t_2k_2)dt_2 - \int_0^1 \frac{d}{dt_2}\tilde{C}(0,t_2k_2)dt_2\\
&= \int_0^1\int_0^1 \frac{d}{dt_2}\frac{d}{dt_1}\tilde{C}(t_1k_1,t_2k_2)dt_1dt_2,
\end{aligned}$$

which leads us to the four bounds

$$|R(k_1,k_2)| \leqslant 3\|\tilde{C}\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}, \qquad \frac{|R(k_1,k_2)|}{\frac{1}{2}|k_1|} \leqslant 4\sum_{|\alpha|=1}\|\partial^\alpha\tilde{C}\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)},$$

$$\frac{|R(k_1,k_2)|}{\frac{1}{2}|k_2|} \leqslant 4\sum_{|\alpha|=1}\|\partial^\alpha\tilde{C}\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}, \quad \frac{|R(k_1,k_2)|}{\frac{1}{2}|k_1|\frac{1}{2}|k_2|} \leqslant 4\sum_{|\alpha|=2}\|\partial^\alpha\tilde{C}\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}.$$

After a small amount of patchwork as above, this implies the bound (30). □

Lemma 1.26. *Consider the generalized Friedrichs model A_g defined by formula* (26)*. Define, for $\lambda \leqslant 0$,*

$$\phi_\lambda := \frac{v}{\sqrt{M-\lambda}}, \quad \tilde{\phi}_\lambda(k) := \phi_\lambda(k)\tilde{C}(k,k_{\min}), \quad \zeta_\lambda := \bar{P}_\lambda\tilde{\phi}_\lambda = \tilde{\phi}_\lambda - \frac{\langle\phi_\lambda,\tilde{\phi}_\lambda\rangle}{\|\phi_\lambda^2\|}\phi_\lambda,$$

where $\bar{P}_\lambda = 1 - P_\lambda$, P_λ denotes the orthogonal projection onto the subspace spanned by ϕ_λ, and we put $\zeta_0 := \tilde{\phi}_0$ if $\|\phi_0\| = \infty$. Furthermore, define, for $e_0 < \lambda < 0$ or $\lambda < 0 \leqslant e_0$,

$$\psi_\lambda := \tilde{\phi}_\lambda + \frac{1}{2}((e_0-\lambda)^{-1} + c_0)\phi_\lambda, \tag{31}$$

$$E_\lambda^\pm := \mathrm{re}\langle\phi_\lambda,\psi_\lambda\rangle \pm \sqrt{\|\phi_\lambda\|^2\|\zeta_\lambda\|^2 + |\mathrm{re}\langle\phi_\lambda,\psi_\lambda\rangle^2|}. \tag{32}$$

Then the limit $E^+ := \lim_{\lambda\to 0} E_\lambda^+ \in [0,\infty]$ exists. In fact, we find:

1. *If* $e_0 = 0$*, then* $E^+ = \infty$.

2. *If* $e_0 \neq 0$ *and* $\|\phi_0\| < \infty$*, then formulas* (31) *and* (32) *extend to* $\lambda = 0$ *by continuity.*

3. *If* $e_0 \neq 0$*,* $\|\phi_0\| = \infty$*, and* $e_0^{-1} + c_0 > 0$*, then* $E^+ = \infty$.

4. *If* $e_0 \neq 0$*,* $\|\phi_0\| = \infty$*,* $\|\tilde{\phi}_0\| \neq 0$ *and* $e_0^{-1} + c_0 = 0$*, then* $E^+ = \infty$.

5. *If* $e_0 \neq 0$*,* $\|\phi_0\| = \infty$*,* $\|\tilde{\phi}_0\| = 0$ *and* $e_0^{-1} + c_0 = 0$*, then* $E^+ = 0$.

6. *If* $e_0 \neq 0$*,* $\|\phi_0\| = \infty$ *and* $e_0^{-1} + c_0 < 0$*, then* $E^+ = \frac{\|\tilde{\phi}_0\|^2}{|e_0^{-1}+c_0|}$.

Proof. Assume without loss of generality that $k_{\min} = 0$. We will give a case-by-case derivation of the limiting behavour of E_λ^+ as $\lambda \to 0$, thereby also proving existence of the limit E^+.

1.: If $e_0 = 0$, we have $2\langle \phi_\lambda, \psi_\lambda \rangle = (c_0 - \lambda^{-1})\|\phi_\lambda\|^2 + 2\langle \phi_\lambda, \tilde{\phi}_\lambda \rangle$. For $|\lambda|$ sufficiently small, $c_0 - \lambda^{-1} \geqslant -\lambda^{-1}/2$ and

$$-\lambda^{-1}\|\phi_\lambda\|/2 - 2\|\tilde{\phi}_\lambda\| \geqslant -\lambda^{-1}\|\phi_\lambda\|/4$$

so $E_\lambda^+ \geqslant 2\langle \phi_\lambda, \psi_\lambda \rangle \geqslant -\lambda^{-1}\|\phi_\lambda\|/4 \to \infty$ as $\lambda \nearrow 0$.

2.: This is clear from the formula for E_λ^+.

Before considering 3.−6., we observe that $\|\phi_\lambda\|^{-1}\phi_\lambda \to 0$ weakly as $\lambda \nearrow 0$, due to the condition $\|\phi_0\| = \infty$. In fact, if $f \in L^2(\mathbb{R}^d)$, then

$$|\langle \phi_\lambda, f \rangle| \leqslant 2\delta^{-1/2} \int_{|k| \geqslant \delta} |v(k)f(k)|\, dk + \|\phi_\lambda\| \sqrt{\int_{|k| \leqslant \delta} |f(k)|^2\, dk}$$

for any $\delta > 0$, which suffices to prove the claim. In particular,

$$\left|\frac{\langle \phi_\lambda, \tilde{\phi}_\lambda \rangle}{\|\phi_\lambda\|}\right| \leqslant \left|\frac{\langle \phi_\lambda, \tilde{\phi}_0 \rangle}{\|\phi_\lambda\|}\right| + \|\tilde{\phi}_\lambda - \tilde{\phi}_0\| \to 0$$

as $\lambda \nearrow 0$, and therefore also $\zeta_\lambda \to \tilde{\phi}_0$ as $\lambda \nearrow 0$.

3.: If $e_0 \neq 0$, $\|\phi_0\| = \infty$ and $e_0^{-1} + c_0 > 0$, then $2\frac{\langle \phi_\lambda, \psi_\lambda \rangle}{\|\phi_\lambda\|^2} \to e_0^{-1} + c_0 > 0$, and therefore $\frac{E_\lambda^+}{\|\phi_\lambda\|^2} \to (e_0^{-1} + c_0)$. Thus, $E_\lambda^+ \to \infty$ as $\lambda \nearrow 0$.

4.: Since $e_0^{-1} + c_0 = 0$ and $\|\tilde{\phi}_0\| \neq 0$, we have $\frac{E_\lambda^+}{\|\phi_\lambda\|} \to \|\tilde{\phi}_0\| > 0$ as $\lambda \nearrow 0$.

5.: Since $\|\tilde{\phi}_0\| = 0$, we have

$$E_\lambda^+ \leqslant 2|\mathrm{re}\langle \phi_\lambda, \psi_\lambda \rangle| = \|\phi_\lambda\|^2 |(e_0 - \lambda)^{-1} + c_0|.$$

We note that if $e_0^{-1} + c_0 = 0$, then we have

$$|(e_0 - \lambda)^{-1} + c_0| = |(e_0 - \lambda)^{-1} + c_0 - e_0^{-1} - c_0| \leqslant |\lambda||e_0|^{-2},$$

and therefore

$$\|\phi_\lambda\|^2 |(e_0 - \lambda)^{-1} + c_0| \leqslant \int \frac{dk\, |v(k)|^2 |\lambda|}{(M(k) - \lambda)|e_0|^2}.$$

Since the function $\lambda \mapsto \frac{-\lambda}{m(k)-\lambda}$ is decreasing for every $k \neq k_{\min}$, the dominated convergence theorem ensures that $E_\lambda^+ \leqslant \|\phi_\lambda\|^2 |(e_0 - \lambda)^{-1} + c_0| \to 0$ as $\lambda \nearrow 0$.

6.: As in 3., $\frac{E_\lambda^-}{\|\phi_\lambda\|^2} \to e_0^{-1} + c_0 < 0$. On the other hand, we have the identity $E_\lambda^+ E_\lambda^- = -\|\phi_\lambda\|^2 \|\zeta_\lambda\|^2$, so $E_\lambda^+ \to \frac{\|\tilde{\phi}_0\|^2}{|e_0^{-1} + c_0|}$. □

The proof of the following result is closely related to a proof by Ikromov and Sharipov [24].

Theorem 1.27. *Consider the setting of Lemma 1.26, put*

$$L := 16 \int \frac{dk\, |v(k)|^2}{\frac{1}{2}|k_{\min} - k|^2 + 1} \sum_{|\alpha| \leqslant 2} \|\partial^\alpha \tilde{C}\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)},$$

and assume $g^2 L < 1$. In terms of the number $E^+ \in [0, \infty]$, we find:

(i) If $e_0 \geqslant 0$, then $N(0, A_g) \leqslant 1$. If $e_0 \geqslant 0$ and $g^2 E^+ > 1 + g^2 L$, then $N(0, A_g) = 1$. If $e_0 \geqslant 0$ and $g^2 E^+ \leqslant 1 - g^2 L$, then $N(0, A_g) = 0$.

(ii) If $e_0 < 0$, then $1 \leqslant N(0, A_g) \leqslant 2$. If $e_0 < 0$ and $g^2 E^+ > 1 + g^2 L$, then $N(0, A_g) = 2$. If $e_0 < 0$ and $g^2 E^+ \leqslant 1 - g^2 L$, then $N(0, A_g) = 1$.

Proof. Assume without loss of generality that $k_{\min} = 0$. Theorem 1.10 allows us to reduce the problem to a study of the Birman-Schwinger operator B_λ for $\lambda < 0$. Putting $J_\lambda := \operatorname{sgn}(A_0 - \lambda)$, we have the relation $N(\lambda, A_g) = N(0, J_\lambda - B_\lambda)$. With notation as in Proposition 1.19,

$$J_\lambda - B_\lambda = \begin{bmatrix} \operatorname{sgn}(e_0 - \lambda) & -g|e_0 - \lambda|^{-1/2}\langle\phi_\lambda| \\ -g|\phi_\lambda\rangle|e_0 - \lambda|^{-1/2} & 1 - g^2 C_\lambda \end{bmatrix}.$$

Since $\lambda < 0 \leqslant e_0$ or $e_0 < \lambda < 0$, we have $\operatorname{sgn}(e_0 - \lambda) \neq 0$, and we may therefore consider the Feshbach map with respect to the orthogonal projection P_2 in $\mathcal{H}$ onto $L^2(\mathbb{R}^d)$. The Feshbach operator is given by

$$F_{g,\lambda} := 1 - g^2 \left((e_0 - \lambda)^{-1} |\phi_\lambda\rangle\langle\phi_\lambda| - C_\lambda \right)$$

Noting that $N(0, F_{g,\lambda}) = n(1, ((e_0 - \lambda)^{-1}|\phi_\lambda\rangle\langle\phi_\lambda| - C_\lambda))$, it follows from the Haynsworth inertia additivity formula, Theorem 1.11, that we have

$$N(\lambda, A_g) = n(1, g^2((e_0 - \lambda)^{-1}|\phi_\lambda\rangle\langle\phi_\lambda| - C_\lambda)), \qquad e_0 \geqslant 0, \tag{33}$$
$$N(\lambda, A_g) = n(1, g^2((e_0 - \lambda)^{-1}|\phi_\lambda\rangle\langle\phi_\lambda| - C_\lambda)) + 1, \qquad e_0 < 0. \tag{34}$$

Defining $D_\lambda = |\phi_\lambda\rangle\langle\psi_\lambda| + |\psi_\lambda\rangle\langle\phi_\lambda|$, we find from Lemma 1.25 that

$$(e_0 - \lambda)^{-1}|\phi_\lambda\rangle\langle\phi_\lambda| + C_\lambda = D_\lambda + R_\lambda,$$

where R_λ is the integral operator with integral kernel $\phi_\lambda(k_1)\phi_\lambda(k_2)R(k_1, k_2)$. Thus, $\|(e_0-\lambda)^{-1}|\phi_\lambda\rangle\langle\phi_\lambda|+C_\lambda-D_\lambda\|_{\mathrm{HS}} \leqslant L$. Due to the the Glazman lemma (Lemma 1.5), equations (33) and (34) therefore imply

$$n(1 + g^2L, g^2D_\lambda) \leqslant N(\lambda, A_g) \leqslant n(1 - g^2L, g^2D_\lambda), \qquad e_0 \geqslant 0, \tag{35}$$
$$n(1 + g^2L, g^2D_\lambda) + 1 \leqslant N(\lambda, A_g) \leqslant n(1 - g^2L, g^2D_\lambda) + 1, \quad e_0 < 0. \tag{36}$$

Parts (i) and (ii) now follow, if we can show that E_λ^+ given by formula (32) is the unique non-negative eigenvalue of D_λ.

We first observe that $\zeta_\lambda = \psi_\lambda - \frac{\langle\phi_\lambda,\psi_\lambda\rangle}{\|\phi_\lambda^2\|}\phi_\lambda$. Thus, if ψ_λ and ϕ_λ are linearly dependent, then $\zeta_\lambda = 0$, and therefore $D_\lambda = 2\,\mathrm{re}\langle\phi_\lambda, \psi_\lambda\rangle\frac{|\phi_\lambda\rangle\langle\phi_\lambda|}{\|\phi_\lambda^2\|}$ is a rank 1 operator with spectrum $\sigma(D_\lambda) = \{E_\lambda^-, E_\lambda^+\}$ (note that either $E_\lambda^- = 0$ or $E_\lambda^+ = 0$). In particular, E_λ^+ is the unique non-negative eigenvalue of D_λ.

Next, assume that ψ_λ and ϕ_λ are linearly independent, so the operator D_λ is a rank 2 operator

$$D_\lambda = 2\,\mathrm{re}\langle\psi_\lambda, \phi_\lambda\rangle\frac{|\phi_\lambda\rangle\langle\phi_\lambda|}{\|\phi_\lambda\|^2} + |\zeta_\lambda\rangle\langle\phi_\lambda| + |\phi_\lambda\rangle\langle\zeta_\lambda|,$$

with eigenvalues $E_\lambda^- \leqslant 0 \leqslant E_\lambda^+$ given by (32). Again, it follows that E_λ^+ is the unique non-negative eigenvalue of D_λ, which finishes the proof. □

1.7 Proof of the Main Result

In this section we present the proof of our main result, Theorem 1.3. We first show that there is $\tilde{g}_0 > 0$ such that, when $|g| \leqslant \tilde{g}_0$, $E = E_{\mathrm{ess}}(g, \xi)$, $\xi \in \mathbb{R}^d$, $e_0 \neq 0$ then $e_0^{-1} + c_0$ has the same sign as e_0.

Lemma 1.28. *Consider the situation of Definition 1.21, and fix $E = E_{\mathrm{ess}}$. If $e_0 \neq 0$, then we have*

$$e_0^{-1} + c_0 = \frac{1}{e_0}\left[\frac{|\xi|^2}{T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi, \xi)} + g^2\int\frac{|v(q)|^2N_{g,\xi}(q)dq}{D_{g,\xi}(q)} + g^4K^{(4)}_{g,\xi}\right] + g^4\tilde{K}^{(4)}_{g,\xi}, \tag{37}$$

where $\sup_{|g|\leqslant g_0, \xi\in\mathbb{R}^d}|K^{(4)}_{g,\xi}| + |\tilde{K}^{(4)}_{g,\xi}| < \infty$, *and*

$$D_{g,\xi}(q) := \left[T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,\xi)\right]^2 \left[T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,q)\right]^2 T^{(3)}_{E_{\mathrm{ess}},\xi}(\xi,\xi,q), \tag{38}$$

$$\begin{aligned} N_{g,\xi}(q) := &\left[\frac{1}{2}|2\xi+q|^2 + 2(1-E_{\mathrm{ess}})\right] T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,\xi) T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,q) \\ &+ \frac{3}{4}|\xi|^2 \left[T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,\xi)\right]^2 + \left[\frac{1}{4}|\xi|^2 + \frac{1}{2}|q|^2\right]\left[T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,\xi)\right]^2 \\ &+ \frac{1}{4}|\xi|^2\left[\frac{1}{2}|\xi|^2 - \frac{1}{2}|q|^2\right]^2. \end{aligned} \tag{39}$$

Proof. According to Lemma 1.24, we have

$$e_0 = \frac{1}{2}|\xi|^2 - E_{\mathrm{ess}}, \quad k_{\min} = \xi + O(g^4), \quad E_{\mathrm{ess}} = 1 - g^2 \int \frac{|v(q)|^2 dq}{\frac{1}{2}|q|^2+1} + O(g^4),$$

and if we define $\tilde{K}^{(4)}_{g,\xi} := C^{(4)}_{g,E_{\mathrm{ess}},\xi}(k_{\min},k_{\min})$, we have

$$\begin{aligned} &\frac{1}{\frac{1}{2}|\xi|^2 - E_{\mathrm{ess}}} + C_{g,E_{\mathrm{ess}},\xi}(k_{\min},k_{\min}) \\ &\quad = \frac{1}{\frac{1}{2}|\xi|^2 - E_{\mathrm{ess}}} + \frac{1}{T^{(2)}_{E_{\mathrm{ess}},\xi}(k_{\min},k_{\min})} \\ &\quad + g^2 \int \frac{dq\,|v(q)|^2}{T^{(2)}_{E_{\mathrm{ess}},\xi}(k_{\min},k_{\min}) T^{(3)}_{E_{\mathrm{ess}},\xi}(k_{\min},k_{\min},q) T^{(2)}_{E_{\mathrm{ess}},\xi}(k_{\min},k_{\min})} \\ &\quad + 2g^2 \int \frac{dq\,|v(q)|^2}{T^{(2)}_{E_{\mathrm{ess}},\xi}(k_{\min},k_{\min}) T^{(3)}_{E_{\mathrm{ess}},\xi}(k_{\min},k_{\min},q) T^{(2)}_{E_{\mathrm{ess}},\xi}(k_{\min},q)} \\ &\quad + g^2 \int \frac{dq\,|v(q)|^2}{T^{(2)}_{E_{\mathrm{ess}},\xi}(k_{\min},q) T^{(3)}_{E_{\mathrm{ess}},\xi}(k_{\min},k_{\min},q) T^{(2)}_{E_{\mathrm{ess}},\xi}(k_{\min},q)} + g^4 \tilde{K}^{(4)}_{g,\xi}. \end{aligned}$$

We see that there is $K^{(4)}_{g,\xi}$ with $\sup_{|g|\leqslant g_0,\xi\in\mathbb{R}^d}|K^{(4)}_{g,\xi}| < \infty$ such that

$$\begin{aligned} &\frac{1}{\frac{1}{2}|\xi|^2 - E_{\mathrm{ess}}} + \frac{1}{T^{(2)}_{E_{\mathrm{ess}},\xi}(k_{\min},k_{\min})} = \frac{1}{\frac{1}{2}|\xi|^2 - E_{\mathrm{ess}}} + \frac{1}{\frac{1}{2}|\xi - 2k_{\min}|^2 + 2 - E_{\mathrm{ess}}} \\ &= \frac{1}{e_0}\left[\frac{|k_{\min}|^2 + |\xi - k_{\min}|^2 + 2(1-E_{\mathrm{ess}})}{\frac{1}{2}|\xi - 2k_{\min}|^2 + 2 - E_{\mathrm{ess}}}\right] \\ &= \frac{1}{e_0}\left[\frac{|\xi|^2}{\frac{1}{2}|\xi|^2 + 2 - E_{\mathrm{ess}}} + g^2 \frac{\int \frac{|v(q)|^2 dq}{\frac{1}{2}|q|^2+1}}{\frac{1}{2}|\xi|^2 + 2 - E_{\mathrm{ess}}} + g^4 K^{(4)}_{g,\xi}\right] \end{aligned}$$

Defining $D_{g,\xi}$ by formula (38) and $\tilde{N}_{g,\xi}$ by

$$\tilde{N}_{g,\xi}(q) := 2T^{(3)}_{E_{\mathrm{ess}},\xi}(\xi,\xi,q)T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,\xi)T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,q) + e_0\left[T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,q)\right]^2 \\ + 2e_0 T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,\xi)T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,q) + e_0\left[T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,\xi)\right]^2,$$

we find formula (37) with $N_{g,\xi}$ replaced by $\tilde{N}_{g,\xi}$. We finish the proof by showing that $\tilde{N}_{g,\xi} = N_{g,\xi}$. Noting that

$$2T^{(3)}_{E_{\mathrm{ess}},\xi}(\xi,\xi,q)T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,\xi)T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,q) \\ = \left[\frac{1}{2}|\xi+q|^2 + 2 + \xi\cdot q\right]T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,\xi)T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,q) \\ + T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,q)\left[T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,\xi)\right]^2 + T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,\xi)\left[T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,q)\right]^2,$$

we see that

$$\tilde{N}_{g\xi}(q) = \left[\frac{3}{2}|\xi|^2 + \frac{1}{2}|q|^2 + 2\xi\cdot q + 2(1-E_{\mathrm{ess}})\right]T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,\xi)T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,q) \\ + \frac{|\xi|^2+|q|^2}{2}\left[T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,\xi)\right]^2 + |\xi|^2\left[T^{(2)}_{E_{\mathrm{ess}},\xi}(\xi,q)\right]^2.$$

Since $\frac{3}{2}|\xi|^2 + \frac{1}{2}|q|^2 + 2\xi\cdot q = -\frac{1}{2}|\xi|^2 + \frac{1}{2}|2\xi+q|^2$, this finishes the proof. $\square$

We come to the proof of the main theorem. Let us first consider the case $d \leqslant 2$, where we note that condition (16) ensures in the setting of Lemma 1.26 that $\|\phi_0\| = \infty$, since $1/4 \leqslant M \leqslant 1$.

Proof of part 1 of Theorem 1.3. From Theorem 1.22, we have the identity $N(E_{\mathrm{ess}}, H_{g,0,\xi}) = N(0, A_g)$, so we will proceed by computing $N(0, A_g)$.

(a): This follows from part (ii) of Theorem 1.27.

(b): Consider first $\frac{1}{2}|\xi|^2 = E_{\mathrm{ess}}$, i.e. $e_0 = 0$. From part 3 of Lemma 1.26 and part (i) of Theorem 1.27, we have $N(0, A_g) = 1$. Next, find $C > 0$ such that if $Cg^2 \leqslant \frac{1}{2}|\xi|^2 < E_{\mathrm{ess}}$, then there is precisely one eigenvalue. Using Lemma 1.28, pick $C > 0$ (independent from ξ) sufficiently large that $e_0^{-1} + c_0 < -Cg^2/2$ for all $|g| < g_0$, and also sufficiently large that $g^2E^+ = \frac{g^2\|\tilde{\phi}_0\|^2}{|e_0^{-1}+c_0|} < 1/2$. Thus, combining part 6 of Lemma 1.26 with part (ii) of Theorem 1.27, we find $\tilde{g}_0$ such that when $|g| \leqslant \tilde{g}_0$, have $N(0, A_g) = 1$. Finally, consider the case $\frac{1}{2}|\xi|^2 > E_{\mathrm{ess}}$. According to Lemma 1.28, we have $e_0^{-1} + c_0 > 0$. Combining par 3 of Lemma 1.26 with part (i) of Theorem 1.27, we therefore conclude that $N(0, A_g) = 1$. $\square$

Proof of part 2 of Theorem 1.3. As in the previous proof, we proceed by computing $N(0, A_g)$.

(a): This follow from part (ii) of Theorem 1.27.

(b): We will first find C_1 such that when $C_1 g^2 \leqslant \frac{1}{2}|\xi|^2 < E_{\text{ess}}$, there is precisely one eigenvalue. According to Lemma 1.28, we have a bound of the form $e_0^{-1} + c_0 < -C_1 g^2/2$, assuming C_1 is sufficiently large. Assuming $|e_0^{-1} + c_0|\|\tilde{\phi}_0\| \leqslant 2\|\tilde{\phi}_0\|$, we find $E^+ \leqslant C'$ for an absolute constant. On the other hand, if $|e_0^{-1}+c_0|\|\tilde{\phi}_0\| \geqslant 2\|\tilde{\phi}_0\|$, then we find $E_0^- \leqslant -|e_0^{-1}+c_0|\|\phi_0\|^2/2$, and therefore $E^+ = \|\phi_0\|^2\|\zeta_0\|^2/|E_0^-| \leqslant 4\|\zeta_0\|^2/C_1 g^2$. Combining part 2 of Lemma 1.26 with part (ii) of Theorem 1.27, we then have $N(0, A_g) = 1$ if C_1 is sufficiently large. Next, consider $\frac{1}{2}|\xi|^2 = E_{\text{ess}}$: Combining part 1 of Lemma 1.26 with part (i) of Theorem 1.27, we have $N(0, A_g) = 1$. Finally, we pick C_2 such that if $E_{\text{ess}} + g^2 C_2 \geqslant \frac{1}{2}|\xi|^2 > E_{\text{ess}}$, then there is precisely one eigenvalue. According to Lemma 1.28, we have $e_0^{-1} + c_0 > 0$, and we also have $e_0^{-1} + c_0 \geqslant e_0^{-1} \geqslant g^{-2}C_2^{-1}$. This ensures that $E^+ > g^{-2}C_2^{-1}/2$ if C_2 is sufficiently small. Thus, combining part 2 of Lemma 1.26 with part (i) of Theorem 1.27, we have $N(0, A_g) = 1$ if we pick C_2 sufficiently small.

(c): This follows from part (i) of Theorem 1.27.

(d): We will find C_3 such that if $\frac{1}{2}|\xi|^2 > E_{\text{ess}} + g^2 C_3$, then there is no eigenvalue. According to Lemma 1.28, we have $e_0^{-1} + c_0 > 0$, and there is an absolute constant $c > 0$ such that $e_0^{-1} + c_0 \leqslant c(g^{-2}C_3^{-1} + g^4) \leqslant 2cg^{-2}C_3^{-1}$ assuming g_0 is sufficiently small. Note that $|k_{\text{ess}}| \geqslant 1$, so we have a ξ-uniform bound $\|\phi_0\| < \infty$, $\|\psi_0\| < c' g^{-2} C_3^{-1}\|\phi_0\|$ for an absolute constant $c' > 0$. This implies $g^2 E^+ \leqslant c'' C_3^{-1}$ for an absolute constant c'', and thus, combining Part 2 of Lemma 1.26 with Part (i) of Theorem 1.27, we have $N(0, A_g) = 0$ if we pick C_3 sufficiently large. □

1.8 Concluding Remarks

By going to the next order in g^2, we hope in the future to be able to decide whether or not the Fröhlich polaron Hamiltonian has a second discrete eigenvalue at $\xi = 0$ in the weak coupling regime.

In dimension $d = 1, 2$, it would be interesting to obtain an expansion of the ground state energy in the weak coupling regime for large total momentum. It has been shown by Møller [31] that $\lim_{|\xi|\to\infty}[\Sigma_{\text{ess}}(H^{\Lambda}_{g,0,\xi}) - \Sigma_0(H^{\Lambda}_{g,0,\xi})] = 0$, where $\Lambda > 0$ is a suitably implemented ultraviolet cutoff. We conjecture that $\lim_{|\xi|\to\infty}[\Sigma_{\text{ess}}(H_{g,0,\xi}) - \Sigma_0(H_{g,0,\xi})] = 0$ also without an ultraviolet cutoff.

In this paper, we only obtain information about the discrete part of the energy-momentum spectrum. It would be natural to attempt to probe the essential spectrum. This has been done by Angelescu et al. [2] in extension of the work by Minlos [28]. Unfortunately, their work does not apply to the

Fröhlich polaron model for technical reasons very similar to why the work of Minlos does not apply to the Fröhlich polaron model. A natural first question to ask would be whether $\Sigma_0(H_{g,0,\xi})$ is an eigenvalue if $\Sigma_0(H_{g,0,\xi}) = \Sigma_{\text{ess}}(H_{g,0,\xi})$. Note that Møller [31] has shown, in dimension $d = 3, 4$, that $\Sigma_0(H^{\Lambda}_{g,0,\xi})$ is not an eigenvalue if $\Sigma_0(H^{\Lambda}_{g,0,\xi}) = \Sigma_{\text{ess}}(H^{\Lambda}_{g,0,\xi})$.

A natural generalization of the Fröhlich polaron model is the bipolaron model. Miyao and Spohn [30] have given an analysis of the strong coupling regime of this model, and it would be interesting to examine whether our analysis of the weak coupling regime of the Fröhlich polaron model can be applied to the bipolaron model.

2 Creation/Annihilation Operators and Feshbach Maps in Bosonic Fock Space

Jonas Dahlbæk and Oliver Matte

Abstract

A symbol calculus for creation/annihilation operators is developed. Wick's theorem provides the formula for the product of two symbols. It is demonstrated that the framework can be used to compute Feshbach maps in models of non-relativistic quantum field theory. Specifically, the Fröhlich polaron model is reduced to a generalized Friedrichs model, and a main technical step in the study of the spin boson model by means of the spectral renormalization group is carried out.

2.1 Notation

Let $(\mathcal{M}, \Sigma, \mu)$ be a measure space with measure $\mu : \Sigma \to [0, \infty)$. We assume that $\mathcal{M}$ is σ-finite, and in our notation for integration, we suppress the μ, i.e. if $f : \mathcal{M} \to \mathbb{C}$ is integrable, then we write

$$\int dk\, f(k) := \int \mu(dk)\, f(k).$$

Denote by $\mathcal{L}^0(\mathcal{M}), \mathcal{L}^+(\mathcal{M})$, respectively, the spaces of measurable functions $f : \mathcal{M} \to \mathbb{C}, f : \mathcal{M} \to [0, \infty]$, respectively. If $f, g \in \mathcal{L}^0(\mathcal{M}) \cup \mathcal{L}^+(\mathcal{M})$, we put

$$\|f\|_{\mathcal{L}^2(\mathcal{M})} := \sqrt{\langle f, f \rangle_{\mathcal{L}^+(\mathcal{M})}}, \quad \langle f, g \rangle_{\mathcal{L}^+(\mathcal{M})} := \int dk\, |f(k_{\mathscr{M}}) g(k_{\mathscr{M}})|. \tag{40}$$

We denote by $\mathcal{L}^2(\mathcal{M}) \subseteq \mathcal{L}^0(\mathcal{M})$ the subspace of functions $f \in \mathcal{L}^0(\mathcal{M})$ that satisfy $\|f\|_{\mathcal{L}^2(\mathcal{M})} < \infty$. We denote by $L^0(\mathcal{M}), L^+(\mathcal{M})$, respectively, the spaces of equivalence classes of functions from $\mathcal{L}^0(\mathcal{M}), \mathcal{L}^+(\mathcal{M})$, respectively, that agree almost everywhere, and we denote by $L^2(\mathcal{M}) \subseteq L^0(\mathcal{M})$ the Hilbert space of equivalence classes of function from $\mathcal{L}^2(\mathcal{M})$ with respect to the semi-norm (40). As is standard, we will sometimes pass from equivalence classes to representatives without mention, and vice versa.

When working with explicit expressions for products of creation and annihilation operators, one quickly ends up with expressions involving large quantities of variables being integrated, permuted or otherwise nontrivially manipulated. It is therefore useful to introduce a notation suited for the purpose of working with creation and annihilation operators. Such a notation will be introduced presently.

Consider the space $\mathcal{M}^m$, where $m \in \mathbb{N}$. For elements of this space, we employ the notation $x_{\bar{\mathscr{M}}} \in \mathcal{M}^m$, where $\bar{\mathscr{M}} = (j_1, \ldots, j_m)$ is an ordered set of m elements, and $x_{\bar{\mathscr{M}}} = (x_{j_1}, \ldots, x_{j_m})$. Suppose $k_1, \ldots, k_n$ are n distinct elements of the set $\{j_1, \ldots, j_m\}$, and put $\bar{\mathscr{N}} = (k_1, \ldots, k_n)$. Then we define $x_{\bar{\mathscr{N}}} = (x_{k_1}, \ldots, x_{k_n})$. Letting S_m denote the set of permutations of the m-point set $\{1, \ldots, m\}$, we define $\sigma\bar{\mathscr{M}} = (j_{\sigma(1)}, \ldots, j_{\sigma(m)})$ whenever $\sigma \in S_m$ is a permutation. If $m = 0$, we let $\mathcal{M}^m = \{\star\}$ be the 1-point set, and we put $\bar{\mathscr{M}} = \emptyset$ and $x_{\bar{\mathscr{M}}} = \star$. If $m \in \mathbb{N}$, we consider in the space $\mathcal{M}^m$ the product measure $\mu^{\otimes m}$, and if $m = 0$, we let $\mu^{\otimes 0}$ denote counting measure on the one-point set.

Our notation introduced thus far has a bar on $\bar{\mathscr{M}}$, which is there to remind us that $\bar{\mathscr{M}}$ is an m-tuple. The lowercase letter m denotes the corresponding number of elements of the m-tuple $\bar{\mathscr{M}}$. It is important to emphasize that the m-tuple $\bar{\mathscr{M}}$ consists simply of m distinct 'elements', and we do not identify $\bar{\mathscr{M}}$ with the m-tuple of integers $(1, \ldots, m)$. When we introduce different such tuples, say $\bar{\mathscr{M}}$ and $\bar{\mathscr{N}}$, then it is always assumed that these tuples have no common elements, unless otherwise explicitly stated. With this convention, when we write $x_{\bar{\mathscr{M}}} \in \mathcal{M}^m$ and $x_{\bar{\mathscr{N}}} \in \mathcal{M}^n$, there is no a priori relation between the vectors $x_{\bar{\mathscr{M}}}$ and $x_{\bar{\mathscr{N}}}$. Had we instead made the identification $\bar{\mathscr{M}} = \{1, \ldots, m\}$ and $\bar{\mathscr{N}} = \{1, \ldots, n\}$, then this would imply that the first $\min(m, n)$ components of $x_{\bar{\mathscr{M}}}$ and $x_{\bar{\mathscr{N}}}$ would coincide.

Suppose $f : \mathcal{M}^{m_1} \times \cdots \times \mathcal{M}^{m_n} \to \mathbb{C}$ is a function. Then we say that f is separately symmetric if, for all $\sigma_1 \in S_{m_1}, \ldots, \sigma_n \in S_{m_n}$, we have

$$f(x_{\sigma_1\bar{\mathscr{M}}_1}, \ldots, x_{\sigma_n\bar{\mathscr{M}}_n}) = f(x_{\bar{\mathscr{M}}_1}, \ldots, x_{\bar{\mathscr{M}}_n}).$$

It is then clear that, for a function such as this, we need not keep track of the ordering of, say, $\bar{\mathscr{M}}_1 = (j_1, \ldots, j_m)$, and so we introduce instead the notation $\mathscr{M}_1$ for the (unordered) set $\mathscr{M}_1 = \{j_1, \ldots, j_m\}$. Thus, we may write formulas such as, say $(x_{\mathscr{M}_1}, \ldots, x_{\mathscr{M}_n}) \mapsto f(x_{\mathscr{M}_1}, \ldots, x_{\mathscr{M}_n})$. We will make use of both the notation with the bars, $\bar{\mathscr{M}}$, as well as the un-barred notation, $\mathscr{M}$, depending on whether the functions under consideration are separately symmetric or not. If $n = 1$ and if $f : \mathcal{M}^m \to \mathbb{C}$ is separately symmetric, then we say that f is totally symmetric.

Denote by $L^0_{\text{sym}}(\mathcal{M}^m), L^+_{\text{sym}}(\mathcal{M}^m), L^2_{\text{sym}}(\mathcal{M}^m)$, respectively, the sets of functions of $L^0(\mathcal{M}^m), L^+(\mathcal{M}^m), L^2(\mathcal{M}^m)$, respectively, that have a totally symmetric representative.

2.2 Preliminaries

We put $(\mathcal{F}^0)^{(m)} := L^0(\mathcal{M}^m)$ and $(\mathcal{F}^+)^{(m)} := L^+(\mathcal{M}^m)$ and define the spaces $\mathcal{F}^0 := \bigoplus_{m=0}^{\infty}(\mathcal{F}^0)^{(m)}$ and $\mathcal{F}^+ := \bigoplus_{m=0}^{\infty}(\mathcal{F}^+)^{(m)}$. If $\psi, \phi \in \mathcal{F}^0$, we put

$$\|\psi\|_{\mathcal{F}} := \sqrt{\langle \psi, \phi \rangle_{\mathcal{F}^+}}, \langle \psi, \phi \rangle_{\mathcal{F}^+} := \left(\sum_{m=0}^{\infty} \int dk_{\mathscr{M}} \, |\psi^{(m)}(k_{\mathscr{M}}) \phi^{(m)}(k_{\mathscr{M}})| \right)^{\frac{1}{2}}. \quad (41)$$

For $\psi \in \mathcal{F}^0 \cup \mathcal{F}^+$, we do not demand that the number $\|\psi\|$ be finite, i.e. ψ is simply a formal sequence $\psi = (\psi^{(0)}, \psi^{(1)}, \ldots)$. We put $\mathcal{F}^{(m)} := L^2(\mathcal{M}^m)$ and denote by $\mathcal{F} := \bigoplus_{m=0}^{\infty} \mathcal{F}^{(m)} \subseteq \mathcal{F}^0$ the un-symmetrized Fock space, consisting of those $\psi \in \mathcal{F}^0$ which satisfy $\|\psi\|_{\mathcal{F}} < \infty$.

Similarly, put $(\mathcal{F}^+_{\text{sym}})^{(m)} := L^+_{\text{sym}}(\mathcal{M}^m)$ and let $\mathcal{F}^+_{\text{sym}} := \bigoplus_{m=0}^{\infty}(\mathcal{F}^+_{\text{sym}})^{(m)}$ without demanding finiteness of the number $\|\psi\|_{\mathcal{F}}$ if $\psi \in \mathcal{F}^+$. Finally, put $\mathcal{F}^{(m)}_{\text{sym}} := L^2_{\text{sym}}(\mathcal{M}^m)$ and denote by $\mathcal{F}_{\text{sym}} := \bigoplus_{m=0}^{\infty} \mathcal{F}^{(m)}_{\text{sym}}$ the bosonic (symmetrized) Fock space, where we do impose finiteness of the norm $\|\psi\|_{\mathcal{F}_{\text{sym}}} = \|\psi\|_{\mathcal{F}}$ if $\psi \in \mathcal{F}_{\text{sym}}$.

It is important to emphasize that we distinguish between the inner product in $\mathcal{F}$, given by $\langle \psi, \phi \rangle_{\mathcal{F}} = \sum_{m=0}^{\infty} \int dk_{\mathscr{M}} \, \bar{\psi}^{(m)}(k_{\mathscr{M}}) \phi^{(m)}(k_{\mathscr{M}})$, and the positive number $\langle \psi, \phi \rangle_{\mathcal{F}_+}$, given by formula (41). Of course, if $\psi, \phi \geqslant 0$, then no confusion can occur, since the two numbers coincide, and in this case we shall often write simply $\langle \psi, \phi \rangle$.

Definition 2.1. Let $f : \mathcal{M} \to \mathbb{C}$ be a measurable function and let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable functions $f_n : \mathcal{M} \to \mathbb{C}$. We say that $f_n \to f$ locally in measure provided that $\lim_{n \to \infty} \mu(F \cap \{|f - f_n| \geqslant \epsilon\}) = 0$, whenever $F \subseteq \mathcal{M}$ is measurable with $\mu(F) < \infty$. We say that the sequence $(f_n)_{n \in \mathbb{N}}$ is Cauchy locally in measure provided that

$$\forall \epsilon, \delta > 0 : \exists N \in \mathbb{N} : \forall n, m \geqslant N : \mu(F \cap \{|f_m - f_n| \geqslant \epsilon\}) \leqslant \delta.$$

We regard the following results as well known and therefore omit the proofs. The statements are given for the convenience of the reader, because we will refer to them later.

Proposition 2.2. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable functions. If f_n is Cauchy locally in measure, then there is a measurable $f : \mathcal{M} \to \mathbb{C}$ such that $f_n \to f$ locally in measure. Furthermore, there is a subsequence $(f_{n_j})_{j \in \mathbb{N}}$ such that $f_{n_j} \to f$ almost everywhere.*

Proposition 2.3. *Let f be a measurable function and let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable functions. Then $f_n \to f$ locally in measure if and only if every subsequence $(f_{n_j})_{j \in \mathbb{N}}$ has a subsubsequence $(f_{n_{j_l}})_{l \in \mathbb{N}}$ such that $f_{n_{j_l}} \to f$ almost everywhere.*

Proposition 2.4. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable non-negative functions $f_n : \mathcal{M} \to [0,\infty)$ and let $f : \mathcal{M} \to \mathbb{C}$ be a measurable function. If $f_n \to f$ locally in measure, then $\int f d\mu \leqslant \liminf_{n\to\infty} \int f_n d\mu$.*

2.3 Standard Creation/Annihilation Symbols

For ease of later reference, we pick a measurable function $v : \mathcal{M} \to \mathbb{C}$. In applications, v is typically chosen to be a coupling function. There is no essential loss of generality in assuming $v = 1$; see Remark 2.7. We first have a long definition. Recall for the following definition our convention $\mathcal{M}^0 = \{\star\}$ from Subsection 2.1.

Definition 2.5. Whenever given $m, n, r \in \mathbb{N}_0$, we will denote by $\mathcal{W}^{(r)0}_{m,n}$ and $\mathcal{W}^{(r)+}_{m,n}$, respectively, the spaces of separately symmetric, measurable functions $\omega^{(r)}_{m,n} : \mathcal{M}^m \times \mathcal{M}^n \times \mathcal{M}^r \to \mathbb{C}$ and $\omega^{(r)}_{m,n} : \mathcal{M}^m \times \mathcal{M}^n \times \mathcal{M}^r \to [0,\infty]$, respectively. Let

$$\mathcal{W}^0 := \bigoplus_{m,n,r\in\mathbb{N}_0} \mathcal{W}^{(r)0}_{m,n}, \qquad \mathcal{W}^+ := \bigoplus_{m,n,r\in\mathbb{N}_0} \mathcal{W}^{(r)+}_{m,n},$$

denote the spaces of (formal) triple sequences of the respective spaces.

If $\omega \in \mathcal{W}^0 \cup \mathcal{W}^+$ and $f : \mathcal{M} \to \mathbb{C}$ is measurable, we define

$$\begin{aligned} f_m(k_{\bar{\mathscr{M}}}) &:= f^{\otimes m}(k_{\bar{\mathscr{M}}}), \\ f_{m,n}(k_{\bar{\mathscr{M}}}, k_{\bar{\mathscr{N}}}) &:= f^{\otimes m}(k_{\bar{\mathscr{M}}})\overline{f^{\otimes n}(k_{\bar{\mathscr{N}}})}, \\ (f_{m,n}\omega^{(r)}_{m,n})(k_{\bar{\mathscr{M}}}, k_{\bar{\mathscr{N}}}, k_{\bar{\mathscr{R}}}) &:= f_{m,n}(k_{\bar{\mathscr{M}}}, k_{\bar{\mathscr{N}}})\omega^{(r)}_{m,n}(k_{\bar{\mathscr{M}}}, k_{\bar{\mathscr{N}}}, k_{\bar{\mathscr{R}}}), \\ (f\cdot\omega)^{(r)}_{m,n} &:= f_{m,n}\omega^{(r)}_{m,n}. \end{aligned}$$

If $\omega \in \mathcal{W}^0$, we define $|\omega| \in \mathcal{W}^+$ by $|\omega|^{(r)}_{m,n} := |\omega^{(r)}_{m,n}|$, and we say that $\omega \geqslant 0$ if $\omega \in \mathcal{W}^+$. If $(\omega_l)_{l\in\mathbb{N}}$ is a sequence in $\mathcal{W}^0$ and $\omega \in \mathcal{W}^0$, then we say that $\omega_l \to \omega$ locally in measure if, for all $m, n, r \in \mathbb{N}_0$, $(\omega_l)^{(r)}_{m,n} \to \omega^{(r)}_{m,n}$ locally in measure.

Given $\omega \in \mathcal{W}^{(r)0}_{m,n} \cup \mathcal{W}^{(r)+}_{m,n}$, define $\mathrm{Op}'_+(\omega^{(r)}_{m,n}) : (\mathcal{F}^+)^{(n+r)} \to (\mathcal{F}^+)^{(m+r)}$ in terms of representatives by

$$\begin{aligned} &(\mathrm{Op}'_+(\omega^{(r)}_{m,n})\phi^{(n+r)})^{(a)}(k_{\bar{\mathscr{M}}}, k_{\bar{\mathscr{R}}}) \\ &\quad := \int dk_{\bar{\mathscr{N}}}\, |(v_{m,n}\omega^{(r)}_{m,n})(k_{\bar{\mathscr{M}}}, k_{\bar{\mathscr{N}}}, k_{\bar{\mathscr{R}}})|\phi^{(n+r)}(k_{\bar{\mathscr{N}}\sqcup\bar{\mathscr{R}}}). \end{aligned}$$

Let $\mathcal{W}^{(r)}_{m,n} \subseteq \mathcal{W}^{(r)0}_{m,n}$ consist of those elements $\omega^{(r)}_{m,n} \in \mathcal{W}^{(r)0}_{m,n}$ satisfying finiteness of the number

$$\|\omega^{(r)}_{m,n}\|_{\mathcal{W}^{(r)}_{m,n}} := \sup_{\substack{\psi^{(m+r)}\in(\mathcal{F}^+)^{(m+r)} \\ \phi^{(n+r)}\in(\mathcal{F}^+)^{(n+r)} \\ \|\psi^{(m+r)}\|=\|\phi^{(n+r)}\|=1}} \langle\psi^{(m+r)}, \mathrm{Op}'_+(\omega^{(r)}_{m,n})\phi^{(n+r)}\rangle. \tag{42}$$

Remark 2.6. Note that if $\psi^{(m+r)} \in \mathcal{L}^2(\mathcal{M}^{m+r})$, then we have the pointwise inequality $|\psi^{(m+r)}| \leqslant (m+r)!\phi^{(m+r)}$, where $\phi^{(m+r)}$ denotes the symmetrization of $|\psi^{(m+r)}|$. Since we have $\|\phi^{(m+r)}\|_{\mathcal{L}^2(\mathcal{M}^{m+r})} \leqslant \|\psi^{(m+r)}\|_{\mathcal{L}^2(\mathcal{M}^{m+r})}$, one obtains the same space $\mathcal{W}^{(r)}_{m,n}$, with an equivalent norm, if one only considers symmetric functions $\psi^{(m+r)}, \phi^{(n+r)}$ in the supremum (42).

Remark 2.7. We include the function $v : \mathcal{M} \to \mathbb{C}$ because it will make it easier for us to refer back to results of this subsection later on, where v will be an explicitly fixed function. However, from the abstract point of view, it is sometimes useful to be able to change the function v at will. Let us therefore for a moment consider the dependence of the space $\mathcal{W}^{(r)}_{m,n} = \mathcal{W}^{(r)}_{m,n}(v)$ on the function v. Suppose $u : \mathcal{M} \to \mathbb{C}$ is measurable and satisfies

$$\mu(\{u = 0\} \setminus \{v = 0\}) = 0.$$

Letting $\mathcal{W}^{(r)}_{m,n}(u)$ denote the similarly defined space but replacing everywhere v by u, we see that the mapping $B : \mathcal{W}^{(r)}_{m,n}(v) \to \mathcal{W}^{(r)}_{m,n}(u)$ given by

$$B(\omega^{(r)}_{m,n})(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) = \frac{v^{\otimes(m+n)}(k_{\mathscr{M} \sqcup \mathscr{N}})}{u^{\otimes(m+n)}(k_{\mathscr{M} \sqcup \mathscr{N}})} \omega^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}})$$

is an isometry. This isometry is surjective if and only if

$$\mu((\{u = 0\} \setminus \{v = 0\}) \cup (\{v = 0\} \setminus \{u = 0\})) = 0.$$

Furthermore, since any isometry is injective, we may always recover results concerning $\mathcal{W}^{(r)}_{m,n}(v)$ from corresponding results concerning $\mathcal{W}^{(r)}_{m,n}(u)$ with $u = 1$. At times, it is also convenient to take, $u \in \mathcal{L}^2(\mathcal{M})$, $\|u\|_{\mathcal{L}^2(\mathcal{M})} = 1$ and $u > 0$ or $u \geqslant 0$. Note that such a function $u > 0$ exists, since we assume that $\mathcal{M}$ is σ-finite.

Remark 2.8. In applications, the function v will often not be differentiable. The symbols $\omega^{(r)}_{m,n}$ on the other hand will usually enjoy certain regularity properties. It is in order to capture these regularity properties that we include the function v in our framework. This is also the reason why we remain on the level of semi-norms and do not pass to the quotient space with respect to the semi-norm $\|\cdot\|_{\mathcal{W}^{(r)}_{m,n}}$.

It follows from our choice of semi-norm in $\mathcal{W}^{(r)}_{m,n}$ that $\omega^{(r)}_{m,n} \in \mathcal{W}^{(r)}_{m,n}$ defines a unique bounded operator in the following way.

Definition 2.9. If $\omega^{(r)}_{m,n} \in \mathcal{W}^{(r)}_{m,n}$, then $\mathrm{Op}'(\omega^{(r)}_{m,n}) : \mathcal{F}^{(n+r)} \to \mathcal{F}^{(m+r)}$ is the operator defined in terms of representatives by

$$\begin{aligned}&(\mathrm{Op}'(\omega^{(r)}_{m,n})\phi^{(n+r)})^{(a)}(k_{\bar{\mathscr{M}}}, k_{\bar{\mathscr{R}}}) \\ &\quad := \int dk_{\bar{\mathscr{N}}}\, (v_{m,n}\omega^{(r)}_{m,n})(k_{\bar{\mathscr{M}}}, k_{\bar{\mathscr{N}}}, k_{\bar{\mathscr{R}}})\phi^{(n+r)}(k_{\bar{\mathscr{N}} \sqcup \bar{\mathscr{R}}})\end{aligned}$$

Theorem 2.10. *Let $(\omega_l)_{l\in\mathbb{N}}$ be a sequence in $\mathcal{W}^{(r)0}_{m,n}$. If $v_{m,n}\omega_l \to v_{m,n}\omega \in \mathcal{W}^{(r)0}_{m,n}$ locally in measure, then $\|\omega\|_{\mathcal{W}^{(r)}_{m,n}} \leqslant \liminf_{l\to\infty}\|\omega_l\|_{\mathcal{W}^{(r)}_{m,n}}$.*

Proof. Due to Proposition 2.4, we have, whenever $\psi^{(m+r)} \in (\mathcal{F}^+)^{(m+r)}$, $\phi^{(n+r)} \in (\mathcal{F}^+)^{(n+r)}$ satisfy $\|\psi^{(m+r)}\| = \|\phi^{(n+r)}\| = 1$, the inequality

$$\langle\psi^{(m+r)}, \mathrm{Op}'_+(\omega)\phi^{(n+r)}\rangle \leqslant \liminf_{l\to\infty}\langle\psi^{(m+r)}, \mathrm{Op}'_+(\omega_l)\phi^{(n+r)}\rangle \leqslant \liminf_{l\to\infty}\|\omega_l\|_{\mathcal{W}^{(r)}_{m,n}}.$$

Here, we used that if $v_{m,n}\omega_l \to v_{m,n}\omega$ locally in measure, then we also have $|v_{m,n}\omega_l| \to |v_{m,n}\omega|$ locally in measure, as follows from Proposition 2.3. □

Theorem 2.11. *If $(\omega_l)_{l\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{W}^{(r)}_{m,n}$, then there is $\omega \in \mathcal{W}^{(r)}_{m,n}$ such that $\omega_l \to \omega$ in $\mathcal{W}^{(r)}_{m,n}$ and $v_{m,n}\omega_l \to v_{m,n}\omega$ locally in measure.*

Proof. Let $(\omega_j)_{j=1}^\infty$ be a Cauchy sequence in $\mathcal{W}^{(r)}_{m,n}$ and fix a measurable set F of finite measure. Since $\mathcal{M}$ is σ-finite, we may pick a sequence $E_j \subseteq \mathcal{M}$ of measurable sets of finite measure such that $E_j \subseteq E_{j+1}$ and $\bigcup_{j=1}^\infty E_j = \mathcal{M}$. Then the sets $F_j = E_j^m \times E_j^n \times E_j^r$ form a sequence of measurable sets of finite measure with $F_j \subseteq F_{j+1}$ and $\bigcup_{j=1}^\infty F_j = \mathcal{M}^m \times \mathcal{M}^n \times \mathcal{M}^r$. In particular,

$$\begin{aligned}
&\mu(F \cap \{|v_{m,n}\omega_l - v_{m,n}\omega_k| \geqslant \epsilon\}) \\
&\quad\leqslant \mu(F_j \cap \{|v_{m,n}\omega_l - v_{m,n}\omega_k| \geqslant \epsilon\}) + \mu(F \setminus F_j) \\
&\quad\leqslant \frac{1}{\epsilon}\left[\int dk_{\mathscr{M}}dk_{\mathscr{N}}dk_{\mathscr{R}}\, 1_{F_j}|v_{m,n}(\omega_l - \omega_k)|\right] + \mu(F \setminus F_j) \\
&\quad\leqslant \frac{\|1_{E_j^m\times E_j^r}\|\|1_{E_j^n\times E_j^r}\|}{\epsilon}\|\omega_l - \omega_k\|_{\mathcal{W}^{(r)}_{m,n}} + \mu(F \setminus F_j),
\end{aligned}$$

Thus, $(v_{m,n}\omega_l)_{l\in\mathbb{N}}$ is Cauchy locally in measure. Then Proposition 2.2 implies that there is $\omega \in \mathcal{W}^{(r)0}_{m,n}$ such that $v_{m,n}\omega_l \to v_{m,n}\omega$ locally in measure. Theorem 2.10 then ensures that $\|\omega\|_{\mathcal{W}^{(r)}_{m,n}} \leqslant \liminf_{l\to\infty}\|\omega_l\|_{\mathcal{W}^{(r)}_{m,n}}$, i.e. that $\omega \in \mathcal{W}^{(r)}_{m,n}$, and similarly that $\|\omega_l - \omega\|_{\mathcal{W}^{(r)}_{m,n}} \leqslant \liminf_{j\to\infty}\|\omega_l - \omega_j\|_{\mathcal{W}^{(r)}_{m,n}}$, i.e. that $\omega_l \to \omega$ in $\mathcal{W}^{(r)}_{m,n}$. □

We have now covered the preliminary notions, and wish to define proper creation and annihilation operators. These come with combinatorial prefactors. Consider the operator $\mathrm{Op}(\omega^{(r)}_{m,n}) : \mathcal{F}^{(n+r)}_{\mathrm{sym}} \to \mathcal{F}^{(m+r)}_{\mathrm{sym}}$, defined by

$$\mathrm{Op}(\omega^{(r)}_{m,n})\psi^{(n+r)} = \frac{\sqrt{(m+r)!(n+r)!}}{r!}\,\mathrm{Op}'(\omega^{(r)}_{m,n})\psi^{(n+r)}.$$

While the operator $\mathrm{Op}'(\omega^{(r)}_{m,n})$ acts between the full (un-symmetrized) spaces $\mathcal{F}^{(n+r)} \to \mathcal{F}^{(m+r)}$, we emphasize that the operator $\mathrm{Op}(\omega^{(r)}_{m,n})$ acts between the bosonic (symmetrized) spaces $\mathcal{F}^{(n+r)}_{\mathrm{sym}} \to \mathcal{F}^{(m+r)}_{\mathrm{sym}}$. Explicitly, we have

Definition 2.12. Let $\omega_{m,n}^{(r)} \in \mathcal{W}_{m,n}^{(r)}$. Then $\mathrm{Op}(\omega_{m,n}^{(r)}) : \mathcal{F}_{\mathrm{sym}}^{(n+r)} \to \mathcal{F}_{\mathrm{sym}}^{(m+r)}$ is the operator defined on representatives by

$$\begin{aligned}&(\mathrm{Op}(\omega_{m,n}^{(r)})\phi^{(n+r)})^{(a)}(k_{\mathscr{A}}) \\ &\quad := \sum_{\mathscr{M}\sqcup\mathscr{R}=\mathscr{A}} m!\sqrt{\frac{(n+r)!}{(m+r)!}} \int dk_{\mathscr{N}}\, (v_{m,n}\omega_{m,n}^{(r)})(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}})\phi^{(n+r)}(k_{\mathscr{N}\sqcup\mathscr{R}}),\end{aligned}$$

where $a = m + r$, and $\mathrm{Op}_+(\omega_{m,n}^{(r)}) : (\mathcal{F}_{\mathrm{sym}}^+)^{(n+r)} \to (\mathcal{F}_{\mathrm{sym}}^+)^{(m+r)}$ is defined similarly, but with $v_{m,n}\omega_{m,n}^{(r)}$ replaced by $|v_{m,n}\omega_{m,n}^{(r)}|$.

Remark 2.13. According to subsection 2.1, sets such as $\mathscr{A}, \mathscr{M}$ and $\mathscr{R}$ are disjoint unless otherwise explicitly specified. In the formula above, however, we explicitly specify $\mathscr{M} \sqcup \mathscr{R} = \mathscr{A}$. The sum ranges over all choices of partitions of $\mathscr{A}$ into two disjoint subsets $\mathscr{M}$ and $\mathscr{R}$ with $\#\mathscr{M} = m$ and $\#\mathscr{R} = r$. That is, it is a finite sum, with $\binom{a}{m} = \frac{a!}{m!r!}$ terms.

At this point we fix a measurable function $\varepsilon : \mathcal{M} \to (0, \infty)$. In applications, ε is typically a boson dispersion relation.

Definition 2.14. Let $\mathcal{W}_{m,n}^{st} = \bigoplus_{r=0}^{\infty} \mathcal{W}_{m,n}^{(r)}$ denote the space of standard creation/annihilation symbols with m creations and n annihilations, endowed with the semi-norm

$$\|\omega_{m,n}\|_{\mathcal{W}_{m,n}^{st}} := \sup_{\substack{\psi,\phi\in\mathcal{F}^+ \\ \|\psi\|=\|\phi\|=1}} \sum_{r=0}^{\infty} \langle \psi^{(m+r)}, \mathrm{Op}'_+(\gamma_{m,n}\omega_{m,n}^{(r)})\phi^{(n+r)}\rangle,$$

where $\gamma = 1 \vee \varepsilon^{-\frac{1}{2}}$. Let $\mathcal{W}^{st} = \bigcup_{h=0}^{\infty} \bigoplus_{m+n\leqslant h} \mathcal{W}_{m,n}^{st}$ denote the space of standard creation/annihilation symbols, with the semi-norm

$$\|\omega\|_{\mathcal{W}^{st}} := \sum_{m,n\in\mathbb{N}_0} \|\omega_{m,n}\|_{\mathcal{W}_{m,n}^{st}}.$$

Definition 2.15. Let $\omega_{m,n} \in \mathcal{W}_{m,n}^{st}$. Then $\mathrm{Op}(\omega_{m,n})$ denotes the possibly unbounded operator in $\mathcal{F}_{\mathrm{sym}}$ given by $\mathrm{Op}(\omega_{m,n}) := \sum_{r=0}^{\infty} \mathrm{Op}(\omega_{m,n}^{(r)})$ on the domain

$$\mathcal{D}_{fin} := \bigcup_{l=0}^{\infty} \bigoplus_{k=0}^{l} \mathcal{F}_{\mathrm{sym}}^{(k)}. \tag{43}$$

Remark 2.16. In formal notation, $\mathrm{Op}(\omega) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} W_{m,n}$, where

$$W_{m,n} := \int dk_{\mathscr{M}} dk_{\mathscr{N}}\, a^*(k_{\mathscr{M}}) v^{\otimes m}(k_{\mathscr{M}}) \omega_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}) \bar{v}^{\otimes n}(k_{\mathscr{N}}) a(k_{\mathscr{N}}).$$

Here, $\omega_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}) := \bigoplus_{r=0}^{\infty} \omega_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}})$, where $\omega_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}})$ denotes the operator of multiplication in $\mathcal{F}_{\text{sym}}^{(r)}$ by the function $k_{\mathscr{R}} \mapsto \omega_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}})$. Furthermore, for $f \in \mathcal{F}_{\text{sym}}$,

$$(a(k_{\mathscr{N}})f)^{(r)}(k_{\mathscr{R}}) = \sqrt{\frac{(n+r)!}{r!}} f^{(n+r)}(k_{\mathscr{N}}, k_{\mathscr{R}}),$$

so that

$$\langle \psi, W_{m,n}\phi \rangle = \int dk_{\mathscr{M}} dk_{\mathscr{N}} \langle \bar{v}^{\otimes m}(k_{\mathscr{M}}) a(k_{\mathscr{M}})\psi, \omega_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}) \bar{v}^{\otimes n}(k_{\mathscr{N}}) a(k_{\mathscr{N}})\phi \rangle.$$

Definition 2.17. Whenever $f : \mathcal{M} \to \mathbb{C}$ is measurable, we define

$$d\Gamma(f)^{(n)}(k_{\mathscr{N}}) := \sum_{j \in \mathscr{N}} f(k_j).$$

Theorem 2.18. *Let $\omega_{m,n} \in \mathcal{W}_{m,n}^{st}$. We have the sesquilinear form bound*

$$\begin{aligned} &|\langle \psi^{(m+r)}, \mathrm{Op}(\omega_{m,n}^{(r)})\phi^{(n+r)} \rangle| \\ &\quad \leqslant \|\omega_{m,n}^{(r)}\|_{\mathcal{W}_{m,n}^{st}} \|[d\Gamma(1 \wedge \varepsilon)^{(m+r)}]^{\frac{m}{2}} \psi^{(m+r)}\| \|[d\Gamma(1 \wedge \varepsilon)^{(m+r)}]^{\frac{n}{2}} \phi^{(n+r)}\|. \end{aligned}$$

In particular, we have for all $\phi \in \mathcal{D}(d\Gamma(1 \wedge \varepsilon)^{\frac{m+n}{2}})$ the bound

$$\|\mathrm{Op}(\omega_{m,n})\phi\| \leqslant \|\omega_{m,n}\|_{\mathcal{W}_{m,n}^{st}} \|(d\Gamma(1 \wedge \varepsilon) + m)^{\frac{m}{2}} d\Gamma(1 \wedge \varepsilon)^{\frac{n}{2}} \phi\|.$$

Thus, $\mathrm{Op}(\omega_{m,n})$ extends uniquely to a bounded operator in $\mathcal{D}(d\Gamma(1 \wedge \varepsilon)^{\frac{m+n}{2}})$ with respect to the graph norm.

Proof. Assume without loss of generality that $v = 1$. We have

$$\begin{aligned} &|\langle \psi^{(m+r)}, \mathrm{Op}(\omega_{m,n}^{(r)})\phi^{(n+r)} \rangle| \\ &\quad \leqslant \int dk_{\mathscr{M}} dk_{\mathscr{N}} dk_{\mathscr{R}} \Bigg[\sqrt{\frac{(m+r)!}{r!}} (1 \wedge \varepsilon)_m (k_{\mathscr{M}})^{\frac{1}{2}} |\psi^{(m+r)}(k_{\mathscr{M}}, k_{\mathscr{R}})| \\ &\qquad \cdot \frac{|\omega_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}})|}{(1 \wedge \varepsilon)_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}})^{\frac{1}{2}}} \\ &\qquad \cdot \sqrt{\frac{(n+r)!}{r!}} (1 \wedge \varepsilon)_n (k_{\mathscr{N}})^{\frac{1}{2}} |\phi^{(n+r)}(k_{\mathscr{N}}, k_{\mathscr{R}})| \Bigg] \\ &\quad \leqslant \|\omega_{m,n}^{(r)}\|_{\mathcal{W}_{m,n}^{st}} \|\tilde{\psi}\|_{\mathcal{F}^{(m+r)}} \|\tilde{\phi}\|_{\mathcal{F}^{(n+r)}}, \end{aligned}$$

where

$$\tilde{\psi}(k_{\bar{\mathscr{M}}}, k_{\bar{\mathscr{R}}}) = \sqrt{\frac{(m+r)!}{r!}} (1 \wedge \varepsilon)_m^{\frac{1}{2}} (k_{\bar{\mathscr{M}}}) \psi^{(m+r)}(k_{\bar{\mathscr{M}}}, k_{\bar{\mathscr{R}}}),$$

and similarly for $\tilde{\phi}$. Now, if $\psi \in \mathcal{F}_{\rm sym}$, symmetry of $\psi^{(m+r)}$ implies

$$\begin{aligned}\int dk_{\mathscr{M}} dk_{\mathscr{R}}\, (1\wedge\varepsilon)_m(k_{\mathscr{M}})|\psi^{(m+r)}(k_{\mathscr{M}},k_{\mathscr{R}})|^2 \\ \leqslant \frac{r!}{(m+r)!}\int dk_{\mathscr{M}} dk_{\mathscr{R}}\, \left[d\Gamma(1\wedge\varepsilon)^{(m+r)}(k_{\mathscr{M}},k_{\mathscr{R}})\right]^m |\psi^{(m+r)}(k_{\mathscr{M}},k_{\mathscr{R}})|^2.\end{aligned}$$

Performing a similar computation for ϕ, it follows that we have

$$\begin{aligned}&|\langle \psi^{(m+r)}, \mathrm{Op}(\omega^{(r)}_{m,n})\phi^{(n+r)}\rangle| \\ &\quad \leqslant \|\omega^{(r)}_{m,n}\|_{\mathcal{W}^{st}_{m,n}} \|[d\Gamma(1\wedge\varepsilon)^{(m+r)}]^{\frac{m}{2}}\psi^{(m+r)}\| \|[d\Gamma(1\wedge\varepsilon)^{(m+r)}]^{\frac{n}{2}}\phi^{(n+r)}\|.\end{aligned}$$

This takes care of the sesquilinear form inequality.

Next, we prove the second inequality. We have the pointwise inequality

$$\begin{aligned}d\Gamma(1\wedge\varepsilon)^{(m+r)}(k_{\mathscr{M}},k_{\mathscr{R}}) &= d\Gamma(1\wedge\varepsilon)^{(m)}(k_{\mathscr{M}}) + d\Gamma(1\wedge\varepsilon)^{(r)}(k_{\mathscr{R}}) \\ &\leqslant m + d\Gamma(1\wedge\varepsilon)^{(r)}(k_{\mathscr{R}}),\end{aligned}$$

which implies the pointwise bound

$$\begin{aligned}&d\Gamma(1\wedge\varepsilon)^{(m+r)}(k_{\mathscr{M}},k_{\mathscr{R}})^{\frac{m}{2}}|\omega^{(r)}_{m,n}(k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}})| \\ &\quad \leqslant |\omega^{(r)}_{m,n}(k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}})|(d\Gamma(1\wedge\varepsilon)^{(n+r)}(k_{\mathscr{N}},k_{\mathscr{R}})+m)^{\frac{m}{2}}.\end{aligned}$$

This allows us to conclude that

$$\begin{aligned}&|\langle \psi^{(m+r)}, \mathrm{Op}(\omega^{(r)}_{m,n})\phi^{(n+r)}\rangle| \\ &\quad = |\langle d\Gamma(1\wedge\varepsilon)^{-\frac{m}{2}}\psi^{(m+r)}, d\Gamma(1\wedge\varepsilon)^{\frac{m}{2}}\,\mathrm{Op}(\omega^{(r)}_{m,n})\phi^{(n+r)}\rangle| \\ &\quad \leqslant \langle d\Gamma(1\wedge\varepsilon)^{-\frac{m}{2}}|\psi^{(m+r)}|, d\Gamma(1\wedge\varepsilon)^{\frac{m}{2}}\,\mathrm{Op}(|\omega^{(r)}_{m,n}|)|\phi^{(n+r)}|\rangle \\ &\quad \leqslant \langle d\Gamma(1\wedge\varepsilon)^{-\frac{m}{2}}|\psi^{(m+r)}|, \mathrm{Op}(|\omega^{(r)}_{m,n}|)[d\Gamma(1\wedge\varepsilon)+m]^{\frac{m}{2}}|\phi^{(n+r)}|\rangle.\end{aligned}$$

After an application of the Cauchy-Schwarz inequality, the sesquilinear form inequality then implies

$$\|\mathrm{Op}(\omega_{m,n})\phi\| \leqslant \|\omega_{m,n}\|_{\mathcal{W}^{st}_{m,n}} \|(d\Gamma(1\wedge\varepsilon)+m)^{\frac{m}{2}} d\Gamma(1\wedge\varepsilon)^{\frac{n}{2}}\phi\|.$$

□

2.4 Bounded Creation/Annihilation Symbols

Definition 2.19. For $\omega \in \mathcal{W}^0$, we define

$$\|\omega\|_{\mathcal{W}} := \sup_{\substack{\psi,\phi\in\mathcal{F}^+_{\rm sym}\\ \|\psi\|=\|\phi\|=1}} \sum_{m,n,r\in\mathbb{N}_0} \langle \psi^{(m+r)}, \mathrm{Op}_+(\omega^{(r)}_{m,n})\phi^{(n+r)}\rangle,$$

and we define the space of bounded creation/annihilation symbols, $\mathcal{W}\subseteq\mathcal{W}^0$, by $\omega\in\mathcal{W}$ if and only if $\|\omega\|_{\mathcal{W}}<\infty$.

Remark 2.20. We emphasize that the supremum here is with respect to bosonic Fock space vectors only.

Lemma 2.21. *Suppose $(\omega_l)_{l\in\mathbb{N}}$ is a sequence in $\mathcal{W}^0$ and $v\cdot\omega_l \to v\cdot\omega$ locally in measure. Then we have $\|\omega\|_{\mathcal{W}} \leqslant \liminf_{l\to\infty}\|\omega_l\|_{\mathcal{W}}$.*

Proof. We have $v_{m,n}(\omega_l)^{(r)}_{m,n} \to v_{m,n}\omega^{(r)}_{m,n}$ locally in measure. Thus, if $\psi,\phi \in \mathcal{F}^+_{\mathrm{sym}}$ and $\|\psi\| = \|\phi\| = 1$, we have by first applying Proposition 2.4 and then applying Fatou's Lemma for sums,

$$\begin{aligned}
&\sum_{m,n,r\in\mathbb{N}_0} \langle \psi^{(m+r)}, \mathrm{Op}_+(\omega^{(r)}_{m,n})\phi^{(n+r)}\rangle \\
&\quad\leqslant \sum_{m,n,r\in\mathbb{N}_0} \liminf_{l\to\infty} \langle \psi^{(m+r)}, \mathrm{Op}_+((\omega_l)^{(r)}_{m,n})\phi^{(n+r)}\rangle \\
&\quad\leqslant \liminf_{l\to\infty} \sum_{m,n,r\in\mathbb{N}_0} \langle \psi^{(m+r)}, \mathrm{Op}_+((\omega_l)^{(r)}_{m,n})\phi^{(n+r)}\rangle \leqslant \liminf_{l\to\infty}\|\omega_l\|_{\mathcal{W}}.
\end{aligned}$$

□

Theorem 2.22. *Suppose $(\omega_l)_{l\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{W}$. Then there is $\omega\in\mathcal{W}$ such that $\omega_l\to\omega$ in $\mathcal{W}$ and $v\cdot\omega_l\to v\cdot\omega$ locally in measure.*

Proof. Suppose $(\omega_l)_{l\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{W}$. According to Remark 2.6, we have

$$\|(\omega_j-\omega_k)^{(r)}_{m,n}\|_{\mathcal{W}^{(r)}_{m,n}} \leqslant r!\sqrt{(m+r)!(n+r)!}\|\omega_j-\omega_k\|_{\mathcal{W}},$$

so we find that $((\omega_l)^{(r)}_{m,n})_{l\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{W}^{(r)}_{m,n}$. It follows from Theorem 2.11 that there is $\omega^{(r)}_{m,n}$ such that $(\omega_l)^{(r)}_{m,n} \to \omega^{(r)}_{m,n}$ in $\mathcal{W}^{(r)}_{m,n}$ and $v_{m,n}(\omega_l)^{(r)}_{m,n} \to v_{m,n}\omega^{(r)}_{m,n}$ locally in measure. This last convergence may also be written simply $v\cdot\omega_l\to v\cdot\omega$ locally in measure. But then Lemma 2.21 ensures that $\|\omega\|_{\mathcal{W}} \leqslant \liminf_{l\to\infty}\|\omega_l\|_{\mathcal{W}}$, so that $\omega\in\mathcal{W}$, and then also that $\|\omega_j-\omega\|_{\mathcal{W}} \leqslant \liminf_{l\to\infty}\|\omega_j-\omega_l\|_{\mathcal{W}}$, such that $\omega_l\to\omega$ in $\mathcal{W}$. □

Due to our choice of semi-norm in $\mathcal{W}$, any $\omega\in\mathcal{W}$ induces a unique bounded operator $\mathrm{Op}(\omega) : \mathcal{F}_{\mathrm{sym}} \to \mathcal{F}_{\mathrm{sym}}$ by the formula

$$\mathrm{Op}(\omega) = \sum_{m,n,r\in\mathbb{N}_0} \mathrm{Op}(\omega^{(r)}_{m,n}).$$

The operator $\mathrm{Op}(\omega)$ is called a bounded creation/annihilation operator, and explicitly, we have

Definition 2.23. Given $\omega \in \mathcal{W}$, we define $\operatorname{Op}(\omega) : \mathcal{F}_{\text{sym}} \to \mathcal{F}_{\text{sym}}$ in terms of representatives by

$$
\begin{aligned}
&(\operatorname{Op}(\omega)\phi)^{(a)}(k_{\mathscr{A}}) \\
&\quad := \sum_{\substack{m,n,r \in \mathbb{N}_0 \\ \mathscr{M} \sqcup \mathscr{R} = \mathscr{A}}} m! \sqrt{\frac{(n+r)!}{(m+r)!}} \int dk_{\mathscr{N}}\, (v_{m,n}\omega^{(r)}_{m,n})(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) \phi^{(n+r)}(k_{\mathscr{N} \sqcup \mathscr{R}}),
\end{aligned}
$$

and we define $\operatorname{Op}_+(\omega) : \mathcal{F}^+_{\text{sym}} \to \mathcal{F}^+_{\text{sym}}$ similarly, with $v_{m,n}\omega^{(r)}_{m,n}$ replaced by $|v_{m,n}\omega^{(r)}_{m,n}|$.

Remark 2.24. The sum here should be understood as described in Remark 2.13.

Remark 2.25. Note that we have the essential identity

$$
\|\omega\|_{\mathcal{W}} = \|\operatorname{Op}_+(\omega)\| = \sup_{\substack{\psi,\phi \in \mathcal{F}^+_{\text{sym}} \\ \|\psi\| = \|\phi\| = 1}} \langle \psi, \operatorname{Op}_+(\omega)\phi \rangle.
$$

2.4.1 Wick's Theorem

In this subsubsection, we compute the product of finitely many bounded creation/annihilation operators.

Theorem 2.26. *Given $\omega^2, \omega^1 \in \mathcal{W}$, we may define $\omega^2 \# \omega^1 \in \mathcal{W}$ by*

$$
\begin{aligned}
&(\omega^2 \# \omega^1)^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) \\
&\quad := \sum_{l=0}^{\infty} \sum_{\substack{\mathscr{M}_2 \sqcup \mathscr{M}_1 = \mathscr{M} \\ \mathscr{N}_2 \sqcup \mathscr{N}_1 = \mathscr{N}}} \frac{m_2! m_1!}{m!} \frac{n_2! n_1!}{n!} \binom{n_2 + l}{l} \binom{m_1 + l}{l} l! \\
&\qquad \cdot \int dk_{\mathscr{L}}\, |v^{\otimes l}(k_{\mathscr{L}})|^2 \omega^{2(m_1+r)}_{m_2, n_2+l}(k_{\mathscr{M}_2}, k_{\mathscr{N}_2 \sqcup \mathscr{L}}, k_{\mathscr{M}_1 \sqcup \mathscr{R}}) \\
&\qquad \cdot \omega^{1(n_2+r)}_{m_1+l, n_1}(k_{\mathscr{M}_1 \sqcup \mathscr{L}}, k_{\mathscr{N}_1}, k_{\mathscr{N}_2 \sqcup \mathscr{R}}).
\end{aligned} \tag{44}
$$

The mapping $\# : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ is an associative product, and we have

$$
\operatorname{Op}(\omega^2 \# \omega^1) = \operatorname{Op}(\omega^2) \operatorname{Op}(\omega^1), \qquad \|\omega^2 \# \omega^1\|_{\mathcal{W}} \leqslant \|\omega^2\|_{\mathcal{W}} \|\omega^1\|_{\mathcal{W}}.
$$

If $\omega^J, \dots, \omega^1 \in \mathcal{W}$, then we have the formula

$$
\begin{aligned}
&(\omega^J \# \cdots \# \omega^1)^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) \\
&= \sum_{\substack{l\in\mathbb{N}_0 \\ \mathscr{M}_J\sqcup\ldots\sqcup\mathscr{M}_1=\mathscr{M} \\ \mathscr{N}_J\sqcup\ldots\sqcup\mathscr{N}_1=\mathscr{N} \\ \bigsqcup_{J\geqslant k>i\geqslant 1}\mathscr{L}_{ki}=\mathscr{L}}} \left[\frac{1}{m!}\prod_{j=1}^J m_j!\right]\left[\frac{1}{n!}\prod_{j=1}^J n_j!\right]\left[\frac{1}{l!}\prod_{J\geqslant k>i\geqslant 1} l_{ki}!\right] \\
&\qquad \cdot \left[\prod_{j=1}^J \binom{m_j+p_j}{p_j}\binom{n_j+q_j}{q_j}\right]\left[\frac{\prod_{j=1}^J p_j!q_j!}{\prod_{J\geqslant k>i\geqslant 1} l_{ki}!}\right] \\
&\qquad \cdot \int dk_{\mathscr{L}}\, |v^{\otimes l}(k_{\mathscr{L}})|^2 \prod_{j=1}^J \omega^{j(r_j+s_j)}_{m_j+p_j,n_j+q_j}(k_{\mathscr{M}_j\sqcup\mathscr{P}_j}, k_{\mathscr{N}_j\sqcup\mathscr{Q}_j}, k_{\mathscr{R}_j\sqcup\mathscr{S}_j}),
\end{aligned}
$$

with

$$\mathscr{P}_j = \bigsqcup_{J\geqslant k>j} \mathscr{L}_{kj}, \qquad \mathscr{Q}_j = \bigsqcup_{j>i\geqslant 1} \mathscr{L}_{ji}, \tag{I}$$

$$\mathscr{R}_j = \left(\bigsqcup_{J\geqslant k>j} \mathscr{N}_k\right) \sqcup \left(\bigsqcup_{j>i\geqslant 1} \mathscr{M}_i\right) \sqcup \mathscr{R}, \tag{II}$$

$$\mathscr{S}_j = \bigsqcup_{J\geqslant k>j>i\geqslant 1} \mathscr{L}_{ki}. \tag{III}$$

Remark 2.27. As will be evident from the proof, the theorem remains true if one replaces everywhere $\mathcal{W}$ by $\mathcal{W}^+$ and Op by Op_+.

Remark 2.28. The formulas in the statement of the theorem should be understood in the sense that, for each fixed value of $l \in \mathbb{N}_0$, we fix a set $\mathscr{L}$ with $\#\mathscr{L} = l$. Other than this, the formulas should be understood in a fashion similar to that described in Remark 2.13.

Remark 2.29. The formula for the iterated product may equivalently be written

$$(\omega^J \# \cdots \# \omega^1) = \sum_{\bar{m},\bar{n},\bar{p},\bar{q}\in\mathbb{N}_0^J} \mathop{\#}^{J}_{\bar{m},\bar{p},\bar{n},\bar{q}} [\omega^J, \dots, \omega^1], \tag{45}$$

where $\#^J_{\bar{m},\bar{p},\bar{n},\bar{q}} : \mathcal{W}^J \to \mathcal{W}$ is the multilinear form defined by

$$
\begin{aligned}
&\mathop{\#}^{J}_{\bar{m},\bar{p},\bar{n},\bar{q}} [\omega^J, \dots, \omega^1]^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) \\
&:= \sum_{\substack{\mathscr{M}_J\sqcup\ldots\sqcup\mathscr{M}_1=\mathscr{M} \\ \mathscr{N}_J\sqcup\ldots\sqcup\mathscr{N}_1=\mathscr{N} \\ \mathscr{P}_J\sqcup\ldots\sqcup\mathscr{P}_1=\mathscr{L} \\ \mathscr{Q}_J\sqcup\ldots\sqcup\mathscr{Q}_1=\mathscr{L} \\ \mathscr{P}_k\cap\mathscr{Q}_i=\emptyset, k\geqslant i}} \frac{\prod_{j=1}^J m_j! n_j!}{m!n!}\left[\prod_{j=1}^J \binom{m_j+p_j}{p_j}\binom{n_j+q_j}{q_j}\right]\frac{\prod_{j=1}^J p_j!q_j!}{l!} \\
&\qquad \cdot \int dk_{\mathscr{L}} |v^{\otimes l}(k_{\mathscr{L}})|^2 \prod_{j=1}^J \omega^{j(r_j+s_j)}_{m_j+p_j,n_j+q_j}(k_{\mathscr{M}_j\sqcup\mathscr{P}_j}, k_{\mathscr{N}_j\sqcup\mathscr{Q}_j}, k_{\mathscr{R}_j\sqcup\mathscr{S}_j}),
\end{aligned}
$$

with $\mathscr{R}_j$ given by (II) and

$$\mathscr{S}_j = \left(\bigsqcup_{J \geqslant k > j} \mathscr{Q}_k \right) \cap \left(\bigsqcup_{j > i \geqslant 1} \mathscr{P}_i \right). \tag{IV}$$

This formula should be understood in the sense that if

$$l := p_1 + \ldots + p_J = q_1 + \ldots + q_J,\ m_1 + \ldots + m_J = m,\ n_1 + \ldots + n_J = n, \tag{46}$$

then we fix a set $\mathscr{L}$ with $\#\mathscr{L} = l$ and interpret the formula in accordance with Remark 2.13. On the other hand, if the relations (46) are not satisfied, then we put $\#^J_{\bar{m},\bar{p},\bar{n},\bar{q}} \left[\omega^J, \ldots, \omega^1\right]^{(r)}_{m,n} = 0$. We observe that

$$\| \underset{\bar{m},\bar{p},\bar{n},\bar{q}}{\overset{J}{\#}} \left[\omega^J, \ldots, \omega^1\right] \|_{\mathcal{W}} \leqslant \|\omega^J_{m_J+p_J,n_J+q_J}\|_{\mathcal{W}} \cdots \|\omega^1_{m_1+p_1,n_1+q_1}\|_{\mathcal{W}}. \tag{47}$$

In order to see that formula (45) indeed coincides with the one given in the statement of the theorem, note that decomposing $\mathscr{L}$ into $J(J-1)/2$ sets $\mathscr{L}_{ki}$, with $J \geqslant k > i \geqslant 1$, corresponds precisely to decomposing $\mathscr{L}$ into J sets $\mathscr{P}_J, \ldots, \mathscr{P}_1$ and J sets $\mathscr{Q}_J, \ldots, \mathscr{Q}_1$ satisfying $\mathscr{P}_k \cap \mathscr{Q}_i = \emptyset$ if $k \geqslant i$. Explicitly, a correspondence is given by putting $\mathscr{L}_{ki} = \mathscr{Q}_k \cap \mathscr{P}_i$ in one direction, and by defining $\mathscr{P}_j, \mathscr{Q}_j$ by equation (I) in the other direction.

Proof. Making use of Remark 2.7, we observe that the statement of the theorem for general $v \geqslant 0$ follows from the statement of the theorem for $v = 1$. We may therefore without loss of generality assume $v = 1$.

We will first show that we may define $\omega \in \mathcal{W}^0$ by formula 44, i.e. that the formula yields a well defined measurable function $\omega^{(r)}_{m,n}$ in the sense that, for any choice $\omega^2, \omega^1 \in \mathcal{W}$, the integrand is integrable for almost every $(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}})$. In order to show this, it suffices to consider $\omega^2, \omega^1 \geqslant 0$, since the integrand in the formula for $|\omega^2|\#|\omega^1|$ dominates the integrand in the formula for $\omega = \omega^2 \# \omega^1$. But then we realise that it suffices to show that $\|\omega\|_{\mathcal{W}} < \infty$. Namely, if we have $\|\omega\|_{\mathcal{W}} < \infty$ and we fix $v \in \mathcal{L}^2(\mathcal{M})$ such that $v > 0$ and $\|v\| = 1$ (as we may, because $\mathcal{M}$ is σ-finite), then we may define $\psi, \phi \in \mathcal{F}_{\text{sym}}$ with $\psi, \phi > 0$ and $\|\psi\| = \|\phi\| = 1$ by $\psi^{(n)} = \phi^{(n)} = 2^{-n-1} v^{\otimes n}$. But if $\|\omega\|_{\mathcal{W}} < \infty$, then

$$\int dk_{\mathscr{M}} dk_{\mathscr{N}} dk_{\mathscr{R}}\, \psi^{(m+r)}(k_{\mathscr{M}}, k_{\mathscr{R}}) \omega^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) \phi^{(n+r)}(k_{\mathscr{N}}, k_{\mathscr{R}})$$
$$\leqslant \frac{r!}{\sqrt{(m+r)!(n+r)!}} \|\omega\|_{\mathcal{W}} < \infty,$$

which of course implies that $\omega^{(r)}_{m,n}$ is finite almost surely. We will therefore proceed by showing that $\|\omega\|_{\mathcal{W}} \leqslant \|\omega^2\|_{\mathcal{W}} \|\omega^1\|_{\mathcal{W}}$ when $\omega^2, \omega^1 \geqslant 0$.

Unraveling our definitions reveals, for any $\phi \in \mathcal{F}_{\text{sym}}$ with $\phi \geqslant 0$, that

$$
\begin{aligned}
&(\operatorname{Op}(\omega^2)\operatorname{Op}(\omega^1)\phi)^{(a)}(k_{\mathscr{A}}) \\
&= \sum_{\substack{m_2,l_2,r_2\in\mathbb{N}_0\\ \mathscr{M}_2\sqcup\mathscr{R}_2=\mathscr{A}}} m_2! \sqrt{\frac{(l_2+r_2)!}{(m_2+r_2)!}} \int dk_{\mathscr{L}_2}\, \omega^{2(r_2)}_{m_2,l_2}(k_{\mathscr{M}_2}, k_{\mathscr{L}_2}, k_{\mathscr{R}_2}) \\
&\qquad\qquad \cdot (\operatorname{Op}(\omega^1)\phi)^{(l_2+r_2)}(k_{\mathscr{L}_2}, k_{\mathscr{R}_2}) \\
&= \sum_{\substack{m_2,l_2,r_2\in\mathbb{N}_0\\ l_1,n_1,r_1\in\mathbb{N}_0\\ \mathscr{M}_2\sqcup\mathscr{R}_2=\mathscr{A}\\ \mathscr{L}_1\sqcup\mathscr{R}_1=\mathscr{L}_2\sqcup\mathscr{R}_2}} m_2! l_1! \sqrt{\frac{(n_1+r_1)!}{(m_2+r_2)!}} \int dk_{\mathscr{L}_2} dk_{\mathscr{N}_1}\, \omega^{2(r_2)}_{m_2,l_2}(k_{\mathscr{M}_2}, k_{\mathscr{L}_2}, k_{\mathscr{R}_2}) \\
&\qquad\qquad \cdot \omega^{1(r_1)}_{l_1,n_1}(k_{\mathscr{L}_1}, k_{\mathscr{N}_1}, k_{\mathscr{R}_1})\phi^{(n_1+r_1)}(k_{\mathscr{N}_1}, k_{\mathscr{R}_1}).
\end{aligned}
$$

Decomposing a set $\mathscr{L}_2 \sqcup \mathscr{R}_2$ into two sets $\mathscr{L}_1, \mathscr{R}_1$ corresponds precisely to decomposing $\mathscr{L}_2$ into two sets $\mathscr{N}_2, \mathscr{P}_{21}$ and decomposing $\mathscr{R}_2$ into two sets $\mathscr{R}, \mathscr{M}_1$, so we continue our computation with an application of the Tonelli-Fubini Theorem for positive functions,

$$
\begin{aligned}
&= \sum_{\substack{m_2,l_2,r_2\in\mathbb{N}_0\\ n_2,m_1,p_{21},r,n_1\in\mathbb{N}_0\\ \mathscr{M}_2\sqcup\mathscr{R}_2=\mathscr{A}\\ \mathscr{N}_2\sqcup\mathscr{P}_{21}=\mathscr{L}_2\\ \mathscr{R}\sqcup\mathscr{M}_1=\mathscr{R}_2}} m_2!(m_1+p_{21})! \sqrt{\frac{(n_1+n_2+r)!}{(m_1+m_2+r)!}} \\
&\qquad \cdot \int dk_{\mathscr{N}_2} dk_{\mathscr{N}_1} dk_{\mathscr{P}_{21}}\, \omega^{2(m_1+r)}_{m_2,n_2+p_{21}}(k_{\mathscr{M}_2}, k_{\mathscr{N}_2\sqcup\mathscr{P}_{21}}, k_{\mathscr{M}_1\sqcup\mathscr{R}}) \\
&\qquad\qquad \cdot \omega^{1(n_2+r)}_{m_1+p_{21},n_1}(k_{\mathscr{M}_1\sqcup\mathscr{P}_{21}}, k_{\mathscr{N}_1}, k_{\mathscr{N}_2\sqcup\mathscr{R}}) \\
&\qquad\qquad \cdot \phi^{(n_1+n_2+r)}(k_{\mathscr{N}_2\sqcup\mathscr{N}_1\sqcup\mathscr{R}}),
\end{aligned}
$$

and here we note that first decomposing $\mathscr{A}$ into two sets $\mathscr{M}_2, \mathscr{R}_2$ followed by decomposing $\mathscr{R}_2$ into two sets $\mathscr{R}, \mathscr{M}_1$ corresponds precisely to directly decomposing $\mathscr{A}$ into three sets $\mathscr{M}_2, \mathscr{M}_1, \mathscr{R}$,

$$
\begin{aligned}
&= \sum_{\substack{m_2,l_2\in\mathbb{N}_0\\ n_2,m_1,p_{21},r,n_1\in\mathbb{N}_0\\ \mathscr{M}_2\sqcup\mathscr{M}_1\sqcup\mathscr{R}=\mathscr{A}\\ \mathscr{N}_2\sqcup\mathscr{P}_{21}=\mathscr{L}_2}} m_2!(m_1+p_{21})! \sqrt{\frac{(n_1+n_2+r)!}{(m_1+m_2+r)!}} \\
&\qquad \cdot \int dk_{\mathscr{N}_2} dk_{\mathscr{N}_1} dk_{\mathscr{P}_{21}}\, \omega^{2(m_1+r)}_{m_2,n_2+p_{21}}(k_{\mathscr{M}_2}, k_{\mathscr{N}_2\sqcup\mathscr{P}_{21}}, k_{\mathscr{M}_1\sqcup\mathscr{R}}) \\
&\qquad\qquad \cdot \omega^{1(n_2+r)}_{m_1+p_{21},n_1}(k_{\mathscr{M}_1\sqcup\mathscr{P}_{21}}, k_{\mathscr{N}_1}, k_{\mathscr{N}_2\sqcup\mathscr{R}}) \\
&\qquad\qquad \cdot \phi^{(n_1+n_2+r)}(k_{\mathscr{N}_2\sqcup\mathscr{N}_1\sqcup\mathscr{R}}).
\end{aligned}
$$

Here, with l_2, n_2, p_{21} fixed, all terms in the sum

$$\sum_{\mathscr{N}_2 \sqcup \mathscr{P}_{21} = \mathscr{L}_2} m_2!(m_1+p_{21})! \sqrt{\frac{(n_1+n_2+r)!}{(m_1+m_2+r)!}} \cdot \int dk_{\mathscr{N}_2} dk_{\mathscr{N}_1} dk_{\mathscr{P}_{21}}\, \omega^{2(m_1+r)}_{m_2,n_2+p_{21}}(k_{\mathscr{M}_2}, k_{\mathscr{N}_2\sqcup\mathscr{P}_{21}}, k_{\mathscr{M}_1\sqcup\mathscr{R}}) \cdot \omega^{1(n_2+r)}_{m_1+p_{21},n_1}(k_{\mathscr{M}_1\sqcup\mathscr{P}_{21}}, k_{\mathscr{N}_1}, k_{\mathscr{N}_2\sqcup\mathscr{R}}) \cdot \phi^{(n_1+n_2+r)}(k_{\mathscr{N}_2\sqcup\mathscr{N}_1\sqcup\mathscr{R}}).$$

coincide, and there are $\binom{n_2+p_{21}}{p_{21}}$ terms, allowing us to simplify our expression further,

$$= \sum_{\substack{m_2,m_1,r\in\mathbb{N}_0 \\ n_2,p_{21},n_1\in\mathbb{N}_0 \\ \mathscr{M}_2\sqcup\mathscr{M}_1\sqcup\mathscr{R}=\mathscr{A}}} \binom{n_2+p_{21}}{p_{21}} m_2!(m_1+p_{21})! \sqrt{\frac{(n_1+n_2+r)!}{(m_1+m_2+r)!}} \cdot \int dk_{\mathscr{N}_2} dk_{\mathscr{N}_1} dk_{\mathscr{P}_{21}}\, \omega^{2(m_1+r)}_{m_2,n_2+p_{21}}(k_{\mathscr{M}_2}, k_{\mathscr{N}_2\sqcup\mathscr{P}_{21}}, k_{\mathscr{M}_1\sqcup\mathscr{R}}) \cdot \omega^{1(n_2+r)}_{m_1+p_{21},n_1}(k_{\mathscr{M}_1\sqcup\mathscr{P}_{21}}, k_{\mathscr{N}_1}, k_{\mathscr{N}_2\sqcup\mathscr{R}}) \cdot \phi^{(n_1+n_2+r)}(k_{\mathscr{N}_2\sqcup\mathscr{N}_1\sqcup\mathscr{R}}),$$

which finally allows us to conclude, after another application of the Tonelli-Fubini Theorem for positive functions, for any $\psi \in \mathcal{F}_{\mathrm{sym}}$ with $\psi \geqslant 0$, that

$$\langle \psi, \mathrm{Op}(\omega^2)\,\mathrm{Op}(\omega^1)\phi\rangle = \sum_{m,n,r\in\mathbb{N}_0} \frac{\sqrt{(m+r)!(n+r)!}}{r!} \int dk_{\mathscr{M}} dk_{\mathscr{N}} dk_{\mathscr{R}}\, \psi^{(m+r)}(k_{\mathscr{M}}, k_{\mathscr{R}})\phi^{(n+r)}(k_{\mathscr{N}\sqcup\mathscr{R}}) \cdot \omega^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}})$$

with ω defined by the formula (44). But then we have

$$\|\omega\|_{\mathcal{W}} = \sup_{\substack{\psi,\phi\in\mathcal{F}^+_{\mathrm{sym}} \\ \|\psi\|=\|\phi\|=1}} |\langle \psi, \mathrm{Op}(\omega^2)\,\mathrm{Op}(\omega^1)\phi\rangle| \leqslant \|\mathrm{Op}(\omega^2)\|_{\mathcal{F}_{\mathrm{sym}}\to\mathcal{F}_{\mathrm{sym}}} \|\mathrm{Op}(\omega^1)\|_{\mathcal{F}_{\mathrm{sym}}\to\mathcal{F}_{\mathrm{sym}}} = \|\omega^2\|_{\mathcal{W}} \|\omega^1\|_{\mathcal{W}},$$

which was what we wanted.

We have therefore shown, for each $\omega^2, \omega^1 \in \mathcal{W}$, that $\omega^2 \# \omega^1 \in \mathcal{W}$ is well defined and satisfies $\|\omega^2\#\omega^1\|_{\mathcal{W}} \leqslant \|\omega^2\|_{\mathcal{W}} \|\omega^1\|_{\mathcal{W}}$. But if we now repeat the argument, this time for arbitrary $\psi, \phi \in \mathcal{F}_{\mathrm{sym}}$ and without imposing $\omega^2, \omega^1 \geqslant 0$, then we also find $\mathrm{Op}(\omega^2\#\omega^1) = \mathrm{Op}(\omega^2)\,\mathrm{Op}(\omega^1)$. The only change one has to make to the argument is that one must appeal to the Tonelli-Fubini

Theorem for integrable functions instead of the Tonelli-Fubini Theorem for positive functions.

Now, we compute the symbol $(\omega^3\#\omega^2)\#\omega^1$. We have

$$\begin{aligned}
&((\omega^3\#\omega^2)\#\omega^1)^{(r)}_{m,n}(k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}}) \\
&\quad= \sum_{p_1=0}^{\infty}\sum_{\substack{\mathscr{M}'\sqcup\mathscr{M}_1=\mathscr{M}\\ \mathscr{A}\sqcup\mathscr{N}_1=\mathscr{N}}} \frac{m'!m_1!}{m!}\frac{a!n_1!}{n!}\binom{a+p_1}{p_1}\binom{m_1+p_1}{p_1}p_1! \\
&\qquad\qquad\cdot\int dk_{\mathscr{P}_1}\,(\omega^3\#\omega^2)^{(m_1+r)}_{m',a+p_1}(k_{\mathscr{M}'},k_{\mathscr{A}\sqcup\mathscr{P}_1},k_{\mathscr{M}_1\sqcup\mathscr{R}}) \\
&\qquad\qquad\qquad\cdot\omega^{1(a+r)}_{m_1+p_1,n_1}(k_{\mathscr{M}_1\sqcup\mathscr{P}_1},k_{\mathscr{N}_1},k_{\mathscr{A}\sqcup\mathscr{R}}),
\end{aligned}$$

where

$$\begin{aligned}
&(\omega^3\#\omega^2)^{(r')}_{m',n'}(k_{\mathscr{M}'},k_{\mathscr{N}'},k_{\mathscr{R}'}) \\
&\quad= \sum_{l_{32}=0}^{\infty}\sum_{\substack{\mathscr{M}_3\sqcup\mathscr{M}_2=\mathscr{M}'\\ \mathscr{N}_3'\sqcup\mathscr{N}_2'=\mathscr{N}'}} \frac{m_3!m_2!}{m'!}\frac{n_3'!n_2'!}{n'!}\binom{n_3'+l_{32}}{l_{32}}\binom{m_2+l_{32}}{l_{32}}l_{32}! \\
&\qquad\qquad\cdot\int dk_{\mathscr{L}_{32}}\,\omega^{3(m_2+r')}_{m_3,n_3'+l_{32}}(k_{\mathscr{M}_3},k_{\mathscr{N}_3'\sqcup\mathscr{L}_{32}},k_{\mathscr{M}_2\sqcup\mathscr{R}'}) \\
&\qquad\qquad\qquad\cdot\omega^{2(n_3'+r')}_{m_2+l_{32},n_2'}(k_{\mathscr{M}_2\sqcup\mathscr{L}_{32}},k_{\mathscr{N}_2'},k_{\mathscr{N}_3'\sqcup\mathscr{R}'}).
\end{aligned}$$

Combined, this yields

$$\begin{aligned}
&((\omega^3\#\omega^2)\#\omega^1)^{(r)}_{m,n}(k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}}) \\
&\quad= \sum_{\substack{p_1,l_{32}\in\mathbb{N}_0\\ \mathscr{M}'\sqcup\mathscr{M}_1=\mathscr{M}\\ \mathscr{A}\sqcup\mathscr{N}_1=\mathscr{N}\\ \mathscr{M}_3\sqcup\mathscr{M}_2=\mathscr{M}'\\ \mathscr{N}_3'\sqcup\mathscr{N}_2'=\mathscr{A}\sqcup\mathscr{P}_1}} \frac{m'!(a+p_1)!(m_1+p_1)!n_1!}{m!n!p_1!}\frac{m_3!n_2'!(n_3'+l_{32})!(m_2+l_{32})!}{m'!(n_3'+n_2')!l_{32}!} \\
&\qquad\cdot\int dk_{\mathscr{P}_1}dk_{\mathscr{L}_{32}}\,\omega^{3(m_2+m_1+r)}_{m_3,n_3'+l_{32}}(k_{\mathscr{M}_3},k_{\mathscr{N}_3'\sqcup\mathscr{L}_{32}},k_{\mathscr{M}_2\sqcup\mathscr{M}_1\sqcup\mathscr{R}}) \\
&\qquad\qquad\cdot\omega^{2(n_3'+m_1+r)}_{m_2+l_{32},n_2'}(k_{\mathscr{M}_2\sqcup\mathscr{L}_{32}},k_{\mathscr{N}_2'},k_{\mathscr{N}_3'\sqcup\mathscr{M}_1\sqcup\mathscr{R}}) \\
&\qquad\qquad\cdot\omega^{1(a+r)}_{m_1+p_1,n_1}(k_{\mathscr{M}_1\sqcup\mathscr{P}_1},k_{\mathscr{N}_1},k_{\mathscr{A}\sqcup\mathscr{R}}).
\end{aligned}$$

Before proceeding, we point out that the integrand in this formula is integrable for almost every $(k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}})$. Indeed, if we pick $v\in\mathcal{F}^{(1)}_{\mathrm{sym}}$ with $v>0,\|v\|=1$ and define functions $\psi^{(n)}=\phi^{(n)}=2^{-n-1}v^{\otimes(n)}>0$, then the integrand of $(\omega^3\#\omega^2)\#\omega^1$ is dominated by the integrand of $(|\omega^3|\#|\omega^2|)\#|\omega^1|$, and we have

$$\begin{aligned}
&\sum_{m,n,r\in\mathbb{N}_0}\int dk_{\mathscr{M}}dk_{\mathscr{N}}dk_{\mathscr{R}}\,\psi^{(m+r)}(k_{\mathscr{M}},k_{\mathscr{R}})\phi^{(n+r)}(k_{\mathscr{N}},k_{\mathscr{R}}) \\
&\qquad\qquad\cdot((|\omega^3|\#|\omega^2|)\#|\omega^1|)^{(r)}_{m,n}(k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}}) \\
&\quad=\langle\psi,\mathrm{Op}((|\omega^3|\#|\omega^2|)\#|\omega^1|)\phi\rangle<\infty.
\end{aligned}$$

by what we have already shown. We are therefore justified in envoking the Tonelli-Fubini Theorem.

Now we see that first decomposing $\mathscr{M}$ into two sets $\mathscr{M}', \mathscr{M}_1$ and then decomposing $\mathscr{M}'$ into two sets $\mathscr{M}_3, \mathscr{M}_2$ corresponds exactly to directly decomposing $\mathscr{M}$ into three sets $\mathscr{M}_3, \mathscr{M}_2, \mathscr{M}_1$. Additionally, decomposing the set $\mathscr{A} \sqcup \mathscr{P}_1$ into two sets $\mathscr{N}_3', \mathscr{N}_2'$ corresponds exactly to decomposing $\mathscr{A}$ into two sets

$$\mathscr{N}_3 = \mathscr{A} \cap \mathscr{N}_3', \qquad \mathscr{N}_2 = \mathscr{A} \cap \mathscr{N}_2'$$

and decomposing $\mathscr{P}_1$ into two sets

$$\mathscr{L}_{31} = \mathscr{P}_1 \cap \mathscr{N}_3', \qquad \mathscr{L}_{21} = \mathscr{P}_1 \cap \mathscr{N}_2'.$$

That is,

$$\begin{aligned}
&((\omega^3 \# \omega^2) \# \omega^1)^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) \\
&= \sum_{\substack{p_1, l_{32} \in \mathbb{N}_0 \\ \mathscr{M}_3 \sqcup \mathscr{M}_2 \sqcup \mathscr{M}_1 = \mathscr{M} \\ \mathscr{A} \sqcup \mathscr{N}_1 = \mathscr{N} \\ \mathscr{N}_3 \sqcup \mathscr{N}_2 = \mathscr{A} \\ \mathscr{L}_{31} \sqcup \mathscr{L}_{21} = \mathscr{P}_1}} \frac{m_3!(n_3+l_{32}+l_{31})!(m_2+l_{32})!(n_2+l_{21})!(m_1+l_{31}+l_{21})!n_1!}{m!n!p_1!l_{32}!} \\
&\quad \cdot \int dk_{\mathscr{P}_1} dk_{\mathscr{L}_{32}}\, \omega^{3(m_2+m_1+r)}_{m_3, n_3+l_{32}+l_{31}}(k_{\mathscr{M}_3}, k_{\mathscr{N}_3 \sqcup \mathscr{L}_{32} \sqcup \mathscr{L}_{31}}, k_{\mathscr{M}_2 \sqcup \mathscr{M}_1 \sqcup \mathscr{R}}) \\
&\qquad \cdot \omega^{2(n_3+l_{31}+m_1+r)}_{m_2+l_{32}, n_2+l_{21}}(k_{\mathscr{M}_2 \sqcup \mathscr{L}_{32}}, k_{\mathscr{N}_2 \sqcup \mathscr{L}_{21}}, k_{\mathscr{N}_3 \sqcup \mathscr{L}_{31} \sqcup \mathscr{M}_1 \sqcup \mathscr{R}}) \\
&\qquad \cdot \omega^{1(n_3+n_2+r)}_{m_1+l_{31}+l_{21}, n_1}(k_{\mathscr{M}_1 \sqcup \mathscr{L}_{31} \sqcup \mathscr{L}_{21}}, k_{\mathscr{N}_1}, k_{\mathscr{N}_3 \sqcup \mathscr{N}_2 \sqcup \mathscr{R}}).
\end{aligned}$$

Once again, we may directly decompose $\mathscr{N}$ into the three sets $\mathscr{N}_3, \mathscr{N}_2, \mathscr{N}_1$, thereby removing $\mathscr{A}$ from the summation. If we then envoke the Tonelli-Fubini Theorem for integrable functions to make the substitution

$$\sum_{l_{32}, l_{31}, l_{21} \in \mathbb{N}_0} \sum_{\substack{p_1 \in \mathbb{N}_0 \\ \mathscr{L}_{31} \sqcup \mathscr{L}_{21} = \mathscr{P}_1}} \frac{1}{p_1!} \int dk_{\mathscr{P}_1} dk_{\mathscr{L}_{32}} \to \sum_{l=0}^{\infty} \sum_{\substack{l_{32}, l_{31}, l_{21} \in \mathbb{N}_0 \\ l_{32}+l_{31}+l_{21}=l}} \frac{1}{l_{21}! l_{31}!} \int dk_{\mathscr{L}},$$

followed by the substitution

$$\sum_{l=0}^{\infty} \sum_{\substack{l_{32}, l_{31}, l_{21} \in \mathbb{N}_0 \\ l_{32}+l_{31}+l_{21}=l}} \frac{1}{l_{32}! l_{31}! l_{21}!} \int dk_{\mathscr{L}} \to \sum_{l=0}^{\infty} \sum_{\mathscr{L}_{32} \sqcup \mathscr{L}_{31} \sqcup \mathscr{L}_{21} = \mathscr{L}} \frac{1}{l!} \int dk_{\mathscr{L}},$$

we arrive at the conclusion

$$\begin{aligned}
&((\omega^3 \# \omega^2) \# \omega^1)^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) \\
&= \sum_{\substack{l \in \mathbb{N}_0 \\ \mathscr{M}_3 \sqcup \mathscr{M}_2 \sqcup \mathscr{M}_1 = \mathscr{M} \\ \mathscr{N}_3 \sqcup \mathscr{N}_2 \sqcup \mathscr{N}_1 = \mathscr{N} \\ \mathscr{L}_{32} \sqcup \mathscr{L}_{31} \sqcup \mathscr{L}_{21} = \mathscr{L}}} \frac{m_3!(n_3+l_{32}+l_{31})!(m_2+l_{32})!(n_2+l_{21})!(m_1+l_{31}+l_{21})!n_1!}{m!n!l!} \\
&\quad \cdot \int dk_{\mathscr{L}}\, \omega^{3(m_2+m_1+r)}_{m_3, n_3+l_{32}+l_{31}}(k_{\mathscr{M}_3}, k_{\mathscr{N}_3 \sqcup \mathscr{L}_{32} \sqcup \mathscr{L}_{31}}, k_{\mathscr{M}_2 \sqcup \mathscr{M}_1 \sqcup \mathscr{R}}) \\
&\qquad \cdot \omega^{2(m_1+n_3+r+l_{31})}_{m_2+l_{32}, n_2+l_{21}}(k_{\mathscr{M}_2 \sqcup \mathscr{L}_{32}}, k_{\mathscr{N}_2 \sqcup \mathscr{L}_{21}}, k_{\mathscr{M}_1 \sqcup \mathscr{N}_3 \sqcup \mathscr{R} \sqcup \mathscr{L}_{31}}) \\
&\qquad \cdot \omega^{1(n_3+n_2+r)}_{m_1+l_{31}+l_{21}, n_1}(k_{\mathscr{M}_1 \sqcup \mathscr{L}_{31} \sqcup \mathscr{L}_{21}}, k_{\mathscr{N}_1}, k_{\mathscr{N}_3 \sqcup \mathscr{N}_2 \sqcup \mathscr{R}}).
\end{aligned}$$

Inspection reveals that this formula coincides with the formula of the theorem for $J = 3$.

Finally, we derive the given formula for $\omega^J \# \cdots \# \omega^1$ inductively, by defining $\omega^J \# \cdots \# \omega^1 = \omega^J \# (\omega^{J-1} \# \cdots \# \omega^1)$. Notice that due to how we put the parenthesis, even the case $J = 3$ has not yet been covered at this point in the proof.

It is straightforward to check that the formula given in the statement of the theorem for arbitrary $J \geqslant 2$ agrees with the formula already derived for the special case $J = 2$. Assume therefore that the formula holds true for fixed $J \geqslant 2$, then we prove that it holds true for $J+1$. Note that the first step, i.e. when we go from $J = 2$ to $J+1 = 3$, thereby computing $\omega^3 \# (\omega^2 \# \omega^1)$, takes care of the proof of associativity of the product, since we already computed $(\omega^3 \# \omega^2) \# \omega^1$ and observed that the result coincided with the formula given in the statement of the theorem.

Fix $\omega^{J+1}, \dots, \omega^1 \in \mathcal{W}$ and let $\tilde{\omega} = (\omega^J \# \cdots \# \omega^1)$ be the symbol defined by our induction hypothesis. Then we may employ our formula for $J = 2$ to compute $\omega = \omega^{J+1} \# \tilde{\omega}$, i.e.

$$\begin{aligned}
&\omega^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) \\
&\quad = \sum_{\substack{q_{J+1} \in \mathbb{N}_0 \\ \mathscr{M}_{J+1} \sqcup \mathscr{A} = \mathscr{M} \\ \mathscr{N}_{J+1} \sqcup \mathscr{N}' = \mathscr{N}}} \frac{m_{J+1}!(n_{J+1} + q_{J+1})!(a + q_{J+1})!n'!}{m!n!q_{J+1}!} \\
&\qquad \cdot \int dk_{\mathscr{Q}_{J+1}} \, \omega^{\{J+1\}(a+r)}_{m_{J+1}, n_{J+1}+q_{J+1}}(k_{\mathscr{M}_{J+1}}, k_{\mathscr{N}_{J+1} \sqcup \mathscr{Q}_{J+1}}, k_{\mathscr{A} \sqcup \mathscr{R}}) \\
&\qquad\qquad \cdot \tilde{\omega}^{(n_{J+1}+r)}_{a+q_{J+1}, n'}(k_{\mathscr{A} \sqcup \mathscr{Q}_{J+1}}, k_{\mathscr{N}'}, k_{\mathscr{N}_{J+1} \sqcup \mathscr{R}}).
\end{aligned}$$

Noting that

$$\begin{aligned}
&\frac{\prod_{j=1}^J (m_j + p_j)!(n_j + q_j)!}{m!n!l!} \\
&\qquad = \frac{\prod_{j=1}^J m_j! n_j!}{m!n!} \left[\prod_{j=1}^J \binom{m_j + p_j}{p_j} \binom{n_j + q_j}{q_j} \right] \frac{\prod_{j=1}^J p_j! q_j!}{l!},
\end{aligned}$$

we have from our induction hypothesis the formula

$$\begin{aligned}
&\tilde{\omega}^{(r')}_{m',n'}(k_{\mathscr{M}'}, k_{\mathscr{N}'}, k_{\mathscr{R}'}) \\
&\quad = \sum_{l'=0}^{\infty} \sum_{\substack{\mathscr{M}'_J \sqcup \ldots \sqcup \mathscr{M}'_1 = \mathscr{M}' \\ \mathscr{N}_J \sqcup \ldots \sqcup \mathscr{N}_1 = \mathscr{N}' \\ \bigsqcup_{J \geqslant k > i \geqslant 1} \mathscr{L}_{ki} = \mathscr{L}'}} \frac{\prod_{j=1}^J (m'_j + p'_j)!(n_j + q_j)!}{m'!n'!l'!} \\
&\qquad \cdot \left[\int dk_{\mathscr{L}'} \right] \prod_{j=1}^J \omega^{j(r'_j + s'_j)}_{m'_j + p'_j, n_j + q_j}(k_{\mathscr{M}'_j \sqcup \mathscr{P}'_j}, k_{\mathscr{N}_j \sqcup \mathscr{Q}_j}, k_{\mathscr{R}'_j \sqcup \mathscr{S}'_j}),
\end{aligned}$$

where, for $1 \leqslant j \leqslant J$, $\mathscr{P}'_j, \mathscr{Q}_j$, respectively, is given as in formula (I) for $\mathscr{P}_j, \mathscr{Q}_j$, respectively, $\mathscr{R}'_j$ is given as in formula (II) for $\mathscr{R}_j$, with $\mathscr{M}_j$ replaced by $\mathscr{M}'_j$, and $\mathscr{S}'_j$ is given as in formula (III) for $\mathscr{S}_j$. All in all,

$$\begin{aligned}
&\omega^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) \\
&\quad = \sum_{\substack{q_{J+1}, l' \in \mathbb{N}_0 \\ \mathscr{M}_{J+1} \sqcup \mathscr{A} = \mathscr{M} \\ \mathscr{N}_{J+1} \sqcup \mathscr{N}' = \mathscr{N} \\ \mathscr{M}'_J \sqcup \ldots \sqcup \mathscr{M}'_1 = \mathscr{A} \sqcup \mathscr{Q}_{J+1} \\ \mathscr{N}_J \sqcup \ldots \sqcup \mathscr{N}_1 = \mathscr{N}' \\ \bigsqcup_{J \geqslant k > i \geqslant 1} \mathscr{L}_{ki} = \mathscr{L}'}} \frac{m_{J+1}!(n_{J+1} + q_{J+1})!(a + q_{J+1})!n'!}{m!n!q_{J+1}!} \\
&\qquad \cdot \frac{\prod_{j=1}^{J} (m'_j + p'_j)!(n_j + q_j)!}{(a + q_{J+1})!n'!l'!} \\
&\qquad \cdot \int dk_{\mathscr{Q}_{J+1}}\, \omega^{\{J+1\}(a+r)}_{m_{J+1}, n_{J+1} + q_{J+1}}(k_{\mathscr{M}_{J+1}}, k_{\mathscr{N}_{J+1} \sqcup \mathscr{Q}_{J+1}}, k_{\mathscr{A} \sqcup \mathscr{R}}) \\
&\qquad \cdot \int dk_{\mathscr{L}'} \prod_{j=1}^{J} \omega^{j(r'_j + s'_j)}_{m'_j + p'_j, n_j + q_j}(k_{\mathscr{M}'_j \sqcup \mathscr{P}'_j}, k_{\mathscr{N}_j \sqcup \mathscr{Q}_j}, k_{\mathscr{R}'_j \sqcup \mathscr{S}'_j}).
\end{aligned}$$

Firstly, instead of first splitting the set $\mathscr{N}$ into two sets $\mathscr{N}_{J+1}$ and $\mathscr{N}'$, and then splitting $\mathscr{N}'$ into J sets $\mathscr{N}_J, \ldots, \mathscr{N}_1$, we may directly split $\mathscr{N}$ into $J+1$ sets $\mathscr{N}_{J+1}, \ldots, \mathscr{N}_1$. Secondly, splitting the set $\mathscr{A} \sqcup \mathscr{Q}_{J+1}$ into J sets $\mathscr{M}'_J, \ldots, \mathscr{M}'_1$ corresponds precisely to splitting the set $\mathscr{A}$ into J sets $\mathscr{M}_J, \ldots, \mathscr{M}_1$ and splitting the set $\mathscr{Q}_{J+1}$ into J sets $\mathscr{L}_{J+1,J}, \ldots, \mathscr{L}_{J+1,1}$. The correspondence is given by

$$\mathscr{M}_j = \mathscr{M}'_j \cap \mathscr{A}, \quad \mathscr{L}_{J+1,j} = \mathscr{M}'_j \cap \mathscr{Q}_{J+1}$$

in one direction and by $\mathscr{M}'_j = \mathscr{M}_j \sqcup \mathscr{L}_{J+1,j}$ in the other direction. We note that

$$\begin{aligned}
\mathscr{P}_j &= \mathscr{P}'_j \sqcup \mathscr{L}_{J+1,j}, & \mathscr{M}'_j &= \mathscr{M}_j \sqcup \mathscr{L}_{J+1,j}, \\
\mathscr{R}'_j &= \mathscr{R}_j \sqcup \bigsqcup_{j > i \geqslant 1} \mathscr{L}_{J+1,i}, & \mathscr{S}_j &= \mathscr{S}'_j \sqcup \bigsqcup_{j > i \geqslant 1} \mathscr{L}_{J+1,i},
\end{aligned}$$

where, for $1 \leqslant j \leqslant J+1$, $\mathscr{P}_j, \mathscr{Q}_j$ is given by formula (I), $\mathscr{R}_j$ is given by formula (II), and $\mathscr{S}_j$ is given by formula (III). In particular,

$$\mathscr{M}'_j \sqcup \mathscr{P}'_j = \mathscr{M}_j \sqcup \mathscr{P}_j, \quad \mathscr{R}'_j \sqcup \mathscr{S}'_j = \mathscr{R}_j \sqcup \mathscr{S}_j,$$

These remarks reveal that

$$\begin{aligned}
&\omega^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) \\
&\quad = \sum_{\substack{q_{J+1}, l' \in \mathbb{N}_0 \\ \mathscr{M}_{J+1} \sqcup \ldots \sqcup \mathscr{M}_1 = \mathscr{M} \\ \mathscr{N}_{J+1} \sqcup \ldots \sqcup \mathscr{N}_1 = \mathscr{N} \\ \mathscr{L}_{J+1,J} \sqcup \ldots \sqcup \mathscr{L}_{J+1,1} = \mathscr{Q}_{J+1} \\ \bigsqcup_{J \geqslant k > i \geqslant 1} \mathscr{L}_{ki} = \mathscr{L}'}} \frac{\prod_{j=1}^{J+1} (m_j + p_j)!(n_j + q_j)!}{m!n!q_{J+1}!l'!} \\
&\qquad \cdot \int dk_{\mathscr{Q}_{J+1}}\, \omega^{\{J+1\}(r_{J+1})}_{m_{J+1}, n_{J+1} + q_{J+1}}(k_{\mathscr{M}_{J+1}}, k_{\mathscr{N}_{J+1} \sqcup \mathscr{Q}_{J+1}}, k_{\mathscr{R}_{J+1}}) \\
&\qquad \cdot \int dk_{\mathscr{L}'} \prod_{j=1}^{J} \omega^{j(r_j + s_j)}_{m_j + p_j, n_j + q_j}(k_{\mathscr{M}_j \sqcup \mathscr{P}_j}, k_{\mathscr{N}_j \sqcup \mathscr{Q}_j}, k_{\mathscr{R}_j \sqcup \mathscr{S}_j}).
\end{aligned}$$

In this expression, we make the substitution

$$\sum_{\substack{l',q_{J+1}\in\mathbb{N}_0\\ \mathscr{L}_{J+1,J}\sqcup\ldots\sqcup\mathscr{L}_{J+1,1}=\mathscr{Q}_{J+1}\\ \bigsqcup_{J\geqslant k>i\geqslant 1}\mathscr{L}_{ki}=\mathscr{L}'}} \frac{1}{l'!q_{J+1}!}\int dk_{\mathscr{Q}_{J+1}}dk_{\mathscr{L}'} \to \sum_{l=0}^{\infty}\sum_{\bigsqcup_{J+1\geqslant k>i\geqslant 1}\mathscr{L}_{ki}=\mathscr{L}}\frac{1}{l!}\int dk_{\mathscr{L}},$$

and thus arrive at the end of our induction step. Here, we once again invoked the Tonelli-Fubini Theorem for integrable functions. □

Combining Theorem 2.22 and Theorem 2.26, we obtain the following corollary.

Corollary 2.30. *The space of bounded creation/annihilation symbols $\mathcal{W}$ with semi-norm $\|\omega\| = \|\omega\|_{\mathcal{W}}$ and product $\omega^2\omega^1 = \omega^2\#\omega^1$ is mapped to a Banach algebra under the quotient mapping defined by $\omega^2 \sim \omega^1 \Leftrightarrow \|\omega^2-\omega^1\|_{\mathcal{W}} = 0$.*

2.4.2 Summable Creation/Annihilation Symbols

Throughout this subsubsection, X denotes a metric space.

Definition 2.31. Let $\omega : X \to \mathcal{W}$ be a mapping into the space of bounded creation/annihilation symbols. We put

$$\omega^{(r)}_{m,n}(k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}};x) := \omega(x)^{(r)}_{m,n}(k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}}),$$

and define symbol valued symbols

$$\mathfrak{w}^{(r)}_{m,n}(k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}};x)^{(w)}_{u,v}(k_{\mathscr{U}},k_{\mathscr{V}},k_{\mathscr{W}}) := \omega^{(r+w)}_{m+u,n+v}(k_{\mathscr{M}\sqcup\mathscr{U}},k_{\mathscr{N}\sqcup\mathscr{V}},k_{\mathscr{R}\sqcup\mathscr{W}};x).$$

Any $\omega \in \mathcal{W}$ may be considered a mapping $\omega : X \to \mathcal{W}$ with $X = \{\star\}$ the one-point set. If $\omega \in \mathcal{W}$, we also use the notation $\mathfrak{w}^{(r)}_{m,n}(k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}}) = \mathfrak{w}^{(r)}_{m,n}(k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}};\star)$.

Definition 2.32. For $\omega \in \mathcal{W}$, we define

$$\|\omega\|_{\mathcal{W}^\Sigma} := \sum_{u,v\in\mathbb{N}_0} 4^{u+v}\|\omega_{u,v}\|_{\mathcal{W}},$$

and denote by $\mathcal{W}^\Sigma$ the Banach space of all $\omega \in \mathcal{W}$ for which the number $\|\omega\|_{\mathcal{W}^\Sigma}$ is finite, where we, in the standard way, identify ω^1 with ω^2 if we have $\|\omega^1-\omega^2\|_{\mathcal{W}^\Sigma} = 0$.

For $\eta > 0$, the space of summable creation/annihilation symbols, denoted $\mathcal{W}^{\Sigma,X}_\eta$, is the space of all mappings $\omega : X \to \mathcal{W}$ which additionally satisfy boundedness of the norm

$$\|\omega\|_{\mathcal{W}^{\Sigma,X}_\eta} := \sum_{m,n\in\mathbb{N}_0}\eta^{-m-n}\sup_{r\in\mathbb{N}_0}\sup_{k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}},x}\|\mathfrak{w}^{(r)}_{m,n}(k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}};x)\|_{\mathcal{W}^\Sigma}.$$

Finally, we let $\mathcal{W}^\Sigma_\eta = \mathcal{W}^{\Sigma,X}_\eta$, with $X = \{\star\}$ the one-point set.

Remark 2.33. It follows from Lemma 2.22 that if $(\omega_l)_{l\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{W}^\Sigma$, then there is $\omega \in \mathcal{W}$ such that $\omega_l \to \omega$ in $\mathcal{W}$ (where we abuse notation in the standard way, and also denote by ω_l an arbitrarily chosen respresentative of ω_l), and it follows from Lemma 2.21 and Fatou's Lemma for sums that we have

$$\|\omega\|_{\mathcal{W}^\Sigma} \leqslant \liminf_{l\to\infty} \|\omega_l\|_{\mathcal{W}^\Sigma} \quad \text{and} \quad \|\omega_j - \omega\|_{\mathcal{W}^\Sigma} \leqslant \liminf_{l\to\infty} \|\omega_j - \omega_l\|_{\mathcal{W}^\Sigma}.$$

Thus, $\mathcal{W}^\Sigma$ is indeed a Banach space.

Remark 2.34. We have introduced the parameter $\eta > 0$ in order to obtain pointwise control over the symbol ω. Explicitly, we have

$$|\omega_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x)| \leqslant |(\mathfrak{w}_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x))_{0,0}^{(0)}| \leqslant \eta^{m+n} \|\omega_{m,n}\|_{\mathcal{W}_\eta^{\Sigma,X}}.$$

Remark 2.35. For $\omega \in \mathcal{W}_\eta^{\Sigma,X}$, we have

$$\|\mathrm{Op}(\omega(x))\| \leqslant \|\omega(x)\|_{\mathcal{W}} \leqslant \|\omega(x)\|_{\mathcal{W}^\Sigma} \leqslant \|\omega\|_{\mathcal{W}_\eta^{\Sigma,X}}.$$

In this sense, the parameter η which provides us with pointwise control over ω is not related to the control of the operator norm. In previous presentations, a similar parameter was often introduced, and the condition $\eta < 1$ imposed in order to obtain nice control over the operator norm and the norm of products of creation/annihilation symbols, but the condition $\eta < 1$ is unnecessary in the framework discussed here. This is one way in which the norm defined here differs essentially from the norms previously studied in the literature.

Remark 2.36. Another way in which the summable creation/annihilation symbols differ from spaces commonly studied in the literature is that we do not impose the support condition, which is the condition

$$\begin{aligned}
&\omega_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) \\
&\quad = 1_{\{d\Gamma(\varepsilon)^{(m+r)}<1\}}(k_{\mathscr{M}}, k_{\mathscr{R}}) 1_{\{d\Gamma(\varepsilon)(n+r)<1\}}(k_{\mathscr{N}}, k_{\mathscr{R}}) \omega_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}).
\end{aligned}$$

Lemma 2.37. *For each $x \in X$, let $(\omega_l(x))_{l\in\mathbb{N}}$ be a sequence in $\mathcal{W}^0$. If we have pointwise convergence $(\omega_l)_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x) \to \omega_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x)$ for all $m, n, r \in \mathbb{N}_0$, then we have $\|\omega\|_{\mathcal{W}_\eta^{\Sigma,X}} \leqslant \liminf_{l\to\infty} \|\omega_l\|_{\mathcal{W}_\eta^{\Sigma,X}}$.*

Proof. We have, for fixed m, n, r, u, v, w, $(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x)$, the pointwise convergence $(\mathfrak{w}_l)_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x)_{u,v}^{(w)} \to \mathfrak{w}_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x)_{u,v}^{(w)}$. Then Proposition 2.3 ensures that the convergence also happens locally in measure. Lemma 2.21 therefore ensures that

$$\|\mathfrak{w}_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x)_{u,v}\|_{\mathcal{W}} \leqslant \liminf_{l\to\infty} \|(\mathfrak{w}_l)_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x)_{u,v}\|_{\mathcal{W}},$$

and Fatou's Lemma for sums ensures that $\|\omega\|_{\mathcal{W}_\eta^{\Sigma,X}} \leqslant \liminf_{l\to\infty} \|\omega_l\|_{\mathcal{W}_\eta^{\Sigma,X}}$. □

Theorem 2.38. *$\mathcal{W}_\eta^{\Sigma,X}$ is a Banach space.*

Proof. Let $(\omega_l)_{l\in\mathbb{N}}$ be a Cauchy sequence. From Remark 2.34, we see that $((\omega_l)^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}, x))_{l\in\mathbb{N}}$ is also Cauchy. Denoting the corresponding limit point by $\omega^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}, x)$, it follows from Lemma 2.37 that

$$\|\omega\|_{\mathcal{W}_\eta^{\Sigma,X}} \leqslant \liminf_{l\to\infty} \|\omega_l\|_{\mathcal{W}_\eta^{\Sigma,X}}, \qquad \|\omega - \omega_l\|_{\mathcal{W}_\eta^{\Sigma,X}} \leqslant \liminf_{j\to\infty} \|\omega_j - \omega_l\|_{\mathcal{W}_\eta^{\Sigma,X}}.$$

□

For the next theorem, recall the notation introduced in Remark 2.29.

Theorem 2.39. *Let $\omega^1, \dots, \omega^J \in \mathcal{W}$ and let $\omega = \omega^J \# \cdots \# \omega^1 \in \mathcal{W}$. Then*

$$\begin{aligned}
&\mathfrak{w}^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}})^{(w)}_{u,v} \\
&\quad = \sum_{\substack{\mathscr{M}_J \sqcup \ldots \sqcup \mathscr{M}_1 = \mathscr{M} \\ \mathscr{N}_J \sqcup \ldots \sqcup \mathscr{N}_1 = \mathscr{N} \\ u_J + \ldots + u_1 = u \\ v_J + \ldots + v_1 = v \\ p_J, \ldots p_1 \in \mathbb{N}_0 \\ q_J, \ldots q_1 \in \mathbb{N}_0}} \left[\frac{\prod_{j=1}^J m_j! n_j! \binom{m_j+u_j}{u_j}\binom{n_j+v_j}{v_j}}{m! n! \binom{m+u}{u}\binom{n+v}{v}} \right] \left[\prod_{j=1}^J \frac{\binom{m_j+u_j+p_j}{p_j}\binom{n_j+v_j+q_j}{q_j}}{\binom{u_j+p_j}{p_j}\binom{v_j+q_j}{q_j}} \right] \\
&\qquad \cdot \mathop{\#}_{\bar{u},\bar{p},\bar{v},\bar{q}}^{J} [\mathfrak{w}^{J(r_J)}_{m_J,n_J}(k_{\mathscr{M}_J}, k_{\mathscr{N}_J}, k_{\mathscr{R}_J}), \dots, \mathfrak{w}^{j(r_1)}_{m_1,n_1}(k_{\mathscr{M}_1}, k_{\mathscr{N}_1}, k_{\mathscr{R}_1})]^{(w)}_{u,v},
\end{aligned}$$

with $\mathscr{R}_j$ given by formula (II)*. In particular, $\|\omega\|_{\mathcal{W}_{2\eta}^{\Sigma,X}} \leqslant 4^{J-1} \prod_{j=1}^J \|\omega^j\|_{\mathcal{W}_\eta^{\Sigma,X}}$.*

Remark 2.40. Introduce the notation

$$\tilde{W}^{(r)}_{m,u,n,v}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) := \mathrm{Op}(\mathfrak{w}^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}})_{u,v}),$$

which in the notation of Remark 2.16 may be written as

$$\begin{aligned}
&\tilde{W}^{(r)}_{m,u,n,v}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) \\
&\quad = \int dk_{\mathscr{U}} dk_{\mathscr{V}}\, a^*(k_{\mathscr{U}}) v^{\otimes u}(k_{\mathscr{U}}) \mathfrak{w}^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}})_{u,v}(k_{\mathscr{U}}, k_{\mathscr{V}}) \bar{v}^{\otimes v}(k_{\mathscr{V}}) a(k_{\mathscr{V}}).
\end{aligned}$$

If $\Omega = (1, 0, 0, \dots) \in \mathcal{F}_{\mathrm{sym}}$ denotes the vacuum vector, then we have

$$\begin{aligned}
&\mathop{\#}_{\bar{0},\bar{p},\bar{0},\bar{q}}^{J} [\mathfrak{w}^{J(r_J)}_{m_J,n_J}(k_{\mathscr{M}_J}, k_{\mathscr{N}_J}, k_{\mathscr{R}_J}), \dots, \mathfrak{w}^{1(r_1)}_{m_1,n_1}(k_{\mathscr{M}_1}, k_{\mathscr{N}_1}, k_{\mathscr{R}_1})]^{(0)}_{0,0} \\
&\quad = \langle \Omega, \prod_{j=1}^J W^{j(r_j)}_{m_j,p_j,n_j,q_j}(k_{\mathscr{M}_j}, k_{\mathscr{N}_j}, k_{\mathscr{R}_j}) \Omega \rangle,
\end{aligned}$$

where $\prod_{j=1}^J a_j = a_J \cdots a_1$. Then the case $u = v = w = 0$ of the theorem reveals that ω is the symmetrization

$$\omega^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}) := \frac{1}{m! n!} \sum_{\sigma \in S_m, \tau \in S_n} \hat{\omega}^{(r)}_{m,n}(k_{\sigma \bar{\mathscr{M}}}, k_{\tau \bar{\mathscr{N}}}, k_{\bar{\mathscr{R}}})$$

of the function (which is already symmetric in $k_{\bar{\mathscr{R}}}$)

$$\hat{\omega}^{(r)}_{m,n}(k_{\bar{\mathscr{M}}}, k_{\bar{\mathscr{N}}}, k_{\bar{\mathscr{R}}}) := \sum_{\substack{m_1+\ldots+m_J=m\\ n_1+\ldots+n_J=n\\ p_1,\ldots,p_J\in\mathbb{N}_0\\ q_1,\ldots,q_J\in\mathbb{N}_0}} \prod_{j=1}^{J} \binom{m_j+p_j}{p_j}\binom{n_j+q_j}{q_j} \cdot \langle \Omega, \prod_{j=1}^{J} \tilde{W}^{j(r_j)}_{m_j,p_j,n_j,q_j}(k_{\bar{\mathscr{M}}_j}, k_{\bar{\mathscr{N}}_j}, k_{\bar{\mathscr{R}}_j})\Omega\rangle,$$

where $\bar{\mathscr{M}} = (\bar{\mathscr{M}}_J, \ldots, \bar{\mathscr{M}}_1)$, $\bar{\mathscr{N}} = (\bar{\mathscr{N}}_J, \ldots, \bar{\mathscr{N}}_1)$ and

$$\bar{\mathscr{R}}_j = (\bar{\mathscr{N}}_J, \ldots, \bar{\mathscr{N}}_{j+1}, \bar{\mathscr{M}}_{j-1}, \bar{\mathscr{M}}_1, \bar{\mathscr{R}}).$$

Thus, the theorem presents a direct generalization of a result due to Bach et al. [4, Theorem A.4].

Proof. Due to Theorem 2.26 and Remark 2.29, we have the identity

$$\begin{aligned}
&\mathfrak{w}^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}})^{(w)}_{u,v}(k_{\mathscr{U}}, k_{\mathscr{V}}, k_{\mathscr{W}}) \\
&= \sum_{\substack{l\in\mathbb{N}_0\\ \mathscr{M}_J\sqcup\ldots\sqcup\mathscr{M}_1=\mathscr{M}\\ \mathscr{N}_J\sqcup\ldots\sqcup\mathscr{N}_1=\mathscr{N}\\ \mathscr{U}_J\sqcup\ldots\sqcup\mathscr{U}_1=\mathscr{U}\\ \mathscr{V}_J\sqcup\ldots\sqcup\mathscr{V}_1=\mathscr{V}\\ \mathscr{P}_J\sqcup\ldots\sqcup\mathscr{P}_1=\mathscr{L}\\ \mathscr{Q}_J\sqcup\ldots\sqcup\mathscr{Q}_1=\mathscr{L}\\ \mathscr{P}_k\cap\mathscr{Q}_i=\emptyset,k\geqslant i}} \left[\frac{\prod_{j=1}^{J}(m_j+u_j)!(n_j+v_j)!\binom{m_j+u_j+p_j}{p_j}\binom{n_j+v_j+q_j}{q_j}p_j!q_j!}{(m+u)!(n+v)!l!}\right] \\
&\quad \cdot \left[\int dk_{\mathscr{L}}\right] |v^{\otimes l}(k_{\mathscr{L}})|^2 \tilde{\omega}^{(\bar{r},\bar{w},\bar{s})}_{\bar{m},\bar{u},\bar{p},\bar{n},\bar{v},\bar{q}}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{U}}, k_{\mathscr{V}}, k_{\mathscr{L}})
\end{aligned}$$

with the placeholder notation

$$\begin{aligned}
&\tilde{\omega}^{(\bar{r},\bar{w},\bar{s})}_{\bar{m},\bar{u},\bar{p},\bar{n},\bar{v},\bar{q}}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{U}}, k_{\mathscr{V}}, k_{\mathscr{L}}) \\
&\quad = \prod_{j=1}^{J} \omega^{j(r_j+w_j+s_j)}_{m_j+u_j+p_j,n_j+v_j+q_j}(k_{\mathscr{M}_j\sqcup\mathscr{U}_j\sqcup\mathscr{P}_j}, k_{\mathscr{N}_j\sqcup\mathscr{V}_j\sqcup\mathscr{Q}_j}, k_{\mathscr{R}_j\sqcup\mathscr{W}_j\sqcup\mathscr{S}_j}),
\end{aligned}$$

and $\mathscr{R}_j$ given by (II), $\mathscr{S}_j$ given by (IV), and

$$\mathscr{W}_j = \left(\bigsqcup_{J\geqslant k>j} \mathscr{V}_k\right) \sqcup \left(\bigsqcup_{j>i\geqslant 1} \mathscr{U}_i\right) \sqcup \mathscr{W}.$$

The first formula of the theorem now follows from another application of Remark 2.29.

It remains to derive the inequality. Recall first that we have the inequality $\prod_{j=1}^{J}\binom{m_j+u_j}{u_j} \leqslant \binom{m+u}{u}$, as follows from Vandermonde's identity. Furthermore, we have $\frac{\binom{m+u+p}{p}}{\binom{u+p}{p}} = \frac{(m+u+p)!u!}{(m+u)!(u+p)!} \leqslant \binom{m+p}{p}$. In order to see that this last inequality holds true, note that if $u \geqslant 1$, then

$$\begin{aligned}
&\frac{(m+u-1+p)!(u-1)!}{(m+u-1)!(u-1+p)!} - \frac{(m+u+p)!u!}{(m+u)!(u+p)!} \\
&\quad = \frac{(m+u-1+p)!(u-1)!}{(m+u)!(u+p)!}\left[(m+u)(u+p)-(m+u+p)u\right] \\
&\quad = \frac{(m+u-1+p)!(u-1)!}{(m+u)!(u+p)!} mp \geqslant 0.
\end{aligned}$$

We conclude that

$$\begin{aligned}
&\langle \psi^{(u+w)}, \mathrm{Op}_+(\mathfrak{w}^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x)^{(w)}_{u,v})\phi^{(v+w)}\rangle \\
&\quad \leqslant \sum_{\substack{\mathscr{M}_J \sqcup \ldots \sqcup \mathscr{M}_1 = \mathscr{M} \\ \mathscr{N}_J \sqcup \ldots \sqcup \mathscr{N}_1 = \mathscr{N} \\ u_J + \ldots + u_1 = u \\ v_J + \ldots + v_1 = v \\ p_{J-1}, \ldots, p_1 \in \mathbb{N}_0 \\ q_J, \ldots, q_2 \in \mathbb{N}_0}} \left[\frac{1}{m!n!}\prod_{j=1}^{J} m_j! n_j! \binom{m_j+p_j}{p_j}\binom{n_j+p_j}{p_j}\right] \\
&\qquad \cdot \prod_{j=1}^{J} \|\mathfrak{w}^{j(r_j)}_{m_j,n_j}(k_{\mathscr{M}_j}, k_{\mathscr{N}_j}, k_{\mathscr{R}_j}; x)_{u_j+p_j, v_j+q_j}\|_{\mathcal{W}}.
\end{aligned}$$

Recalling the bound $\binom{m+p}{p} \leqslant 2^{m+p}$, we find

$$\begin{aligned}
&\|\mathfrak{w}^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x)\|_{\mathcal{W}^{\Sigma}} \\
&\quad \leqslant \sum_{\substack{\mathscr{M}_J \sqcup \ldots \sqcup \mathscr{M}_1 = \mathscr{M} \\ \mathscr{N}_J \sqcup \ldots \sqcup \mathscr{N}_1 = \mathscr{N}}} \frac{4^{J-1} 2^{m+n} \prod_{j=1}^{J} m_j! n_j!}{m!n!} \prod_{j=1}^{J} \|\mathfrak{w}^{j(r_j)}_{m_j,n_j}(k_{\mathscr{M}_j}, k_{\mathscr{N}_j}, k_{\mathscr{R}_j}; x)\|_{\mathcal{W}^{\Sigma}},
\end{aligned}$$

which implies the inequality $\|\omega\|_{\mathcal{W}^{\Sigma,X}_{2\eta}} \leqslant 4^{J-1}\prod_{j=1}^{J}\|\omega^j\|_{\mathcal{W}^{\Sigma,X}_{\eta}}$. □

2.4.3 Smooth Creation/Annihilation Symbols

In various applications, it is useful to impose various regularity conditions on the space of symbols. In the following examples, we discuss simple types of regularity conditions. We will use the notation $\mathcal{W}^{\partial}_{\eta}$ for a subspace of $\mathcal{W}^{\Sigma,X}_{\eta}$ (possibly with a stronger norm) corresponding to some extra regularity condition, and refer to the space $\mathcal{W}^{\partial}_{\eta}$ as the space of smooth creation/annihilation symbols.

Example 2.41. Consider the subspace $\mathcal{W}^{\partial}_{\eta} \subseteq \mathcal{W}^{\Sigma,X}_{\eta}$ consisting of all $\omega \in \mathcal{W}^{\Sigma,X}_{\eta}$ such that $\omega : X \to \mathcal{W}^{\Sigma}_{\eta}$ is continuous. If $(\omega_l)_{l\in\mathbb{N}}$ is a sequence in $\mathcal{W}^{\partial}_{\eta}$ and we have $\omega_l \to \omega$ in $\mathcal{W}^{\Sigma,X}_{\eta}$, then we have $\omega_l(x) \to \omega(x) \in \mathcal{W}^{\Sigma}_{\eta}$ uniformly, which ensures that $\omega \in \mathcal{W}^{\partial}_{\eta}$. Thus, $\mathcal{W}^{\partial}_{\eta}$ is a closed subspace of $\mathcal{W}^{\Sigma,X}_{\eta}$.

Example 2.42. Suppose $X = \mathcal{O} \subseteq \mathbb{C}^\nu$ is an open set. Let $\mathcal{W}_\eta^\partial$ denote the space of all $\omega \in \mathcal{W}_\eta^{\Sigma,X}$ such that the mapping $\omega : \mathcal{O} \to \mathcal{W}_\eta^\Sigma$ is analytic. Now, if $(\omega_l)_{l\in\mathbb{N}}$ is a sequence in $\mathcal{W}_\eta^\partial$ and $\omega_l \to \omega$ in $\mathcal{W}_\eta^{\Sigma,X}$, it follows that $\omega_l(x) \to \omega(x)$ uniformly, and therefore that ω is analytic. Thus, $\mathcal{W}_\eta^\partial$ is a closed subspace of $\mathcal{W}_\eta^{\Sigma,X}$.

Since the map $(\omega^J, \dots, \omega^1) \mapsto \omega^J \# \cdots \# \omega^1$ is multilinear, we find from Theorem 2.39, if $\omega^J, \dots, \omega^1 \in \mathcal{W}_\eta^\partial$, that $\omega^J \# \cdots \# \omega^1 \in \mathcal{W}_{2\eta}^\partial$.

2.5 Fröhlich Polaron Model

In this subsection, we apply the framework of bounded creation/annihilation symbols to the spectral analysis of the Fröhlich polaron model. Specifically, we compute a certain term which appears in relation to a Feshbach reduction, reducing the Fröhlich polaron model to a generalized Friedrichs model [25]. The argument is a refinement of a method employed by Minlos [28] and Angelescu et al. [2].

2.5.1 Smooth Creation/Annihilation Symbols

The following class of smooth creation/annihilation symbols is useful in relation to the Fröhlich polaron model. Suppose $\mathcal{M} = \mathbb{R}^d$ and let $X = \mathcal{O} \subseteq \mathbb{R}^\nu$ be an open subset. Let $\mathcal{W}_\eta^\partial \subseteq \mathcal{W}_\eta^{\Sigma,X}$ be the subspace consisting of all $\omega \in \mathcal{W}_\eta^{\Sigma,X}$ such that, for $m, n, r \in \mathbb{N}_0$, the mappings

$$(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x) \mapsto \mathfrak{w}_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x) \in \mathcal{W}^\Sigma \tag{48}$$

have two continuous Frechét derivatives, endowed with the norm

$$\|\omega\|_{\mathcal{W}_\eta^\partial} = \sum_{m,n\in\mathbb{N}_0} \eta^{-m-n} \sup_{r\in\mathbb{N}_0} \max_{|\alpha|\leqslant 2} \sup_{k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}, x} \|\partial^\alpha \mathfrak{w}_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x)\|_{\mathcal{W}^\Sigma},$$

where $\alpha \in \mathbb{N}_0^{d(m+n+r)+\nu}$.

Letting $(\omega_l)_{l\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{W}_\eta^\partial$, Theorem 2.38 implies existence of $\omega \in \mathcal{W}_\eta^{\Sigma,X}$ such that $\omega_l \to \omega$ in $\mathcal{W}_\eta^{\Sigma,X}$. Furthermore, it is a standard result that $\mathfrak{w}_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x)$ is twice continuously differentiable, and that we have

$$\partial^\alpha (\mathfrak{w}_l)_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x) \to \partial^\alpha \mathfrak{w}_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x)$$

in $\mathcal{W}^\Sigma$. A combination of Lemma 2.21 and Theorem 2.22 ensures that

$$\|\partial^\alpha \mathfrak{w}_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x)\|_{\mathcal{W}^\Sigma} \leqslant \liminf_{l\to\infty} \|\partial^\alpha (\mathfrak{w}_l)_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x)\|_{\mathcal{W}^\Sigma},$$

so that we find $\|\omega\|_{\mathcal{W}^\partial_\eta} \leqslant \liminf_{l\to\infty}\|\omega_l\|_{\mathcal{W}^\partial_\eta}$, and therefore $\omega \in \mathcal{W}^\partial_\eta$, and similarly $\omega_l \to \omega$ in $\mathcal{W}^\partial_\eta$. We conclude that $\mathcal{W}^\partial_\eta$ is complete.

If $\omega^J, \ldots, \omega^1 \in \mathcal{W}^\partial_\eta$, we find from Theorem 2.39, since $\#^J_{\bar u,\bar p,\bar v,\bar q}$ is multilinear, that $\omega^J\#\cdots\#\omega^1 \in \mathcal{W}^\partial_{2\eta}$, and

$$\|\omega^J\#\cdots\#\omega^1\|_{\mathcal{W}^\partial_{2\eta}} \leqslant J^2 4^{J-1}\prod_{j=1}^J \|\omega^j\|_{\mathcal{W}^\partial_\eta} \leqslant 2\cdot 8^{J-1}\prod_{j=1}^J\|\omega^j\|_{\mathcal{W}^\partial_\eta}. \tag{49}$$

It can be quite laborious to verify explicitly that a symbol $\omega \in \mathcal{W}^{\Sigma,X}_\eta$ lies in $\mathcal{W}^\partial_\eta$. For the class of smooth symbolds $\mathcal{W}^\partial_\eta$ considered in this example, there is, however, a simple set of conditions ensuring that $\omega \in \mathcal{W}^\partial_\eta$, which we will now describe.

Definition 2.43. Fix $m, n, r \in \mathbb{N}_0$. Let $f^{(w)}_{u,v} : \mathbb{R}^{(m+n+r)d+2+d} \times \mathbb{R}^{(u+v+w)d} \to \mathbb{C}$ be a triple sequence of functions. With the notation $k = (k_\mathscr{M}, k_\mathscr{N}, k_\mathscr{R}, x)$, $q = (k_\mathscr{U}, k_\mathscr{V}, k_\mathscr{W})$, suppose $f^{(w)}_{u,v}$ is three times continuously differentiable in k, and denote partial derivatives with respect to k by ∂^α_1. Suppose that there are constants $C, \delta > 0$ (independent from u, v, w) such that

1. If $|k - k'| < \delta$, then $|f^{(w)}_{u,v}(k', q)| \leqslant C|f^{(w)}_{u,v}(k, q)|$.

2. If $|\alpha| \leqslant 3$, then $|\partial^\alpha_1 f^{(w)}_{u,v}(k, q)| \leqslant C|f^{(w)}_{u,v}(k, q)|$.

Then we say that the family $(f^{(w)}_{u,v})_{u,v,w\in\mathbb{N}_0}$ is self-controlling.

Remark 2.44. It is straightforward to show that if $h^{(w)}_{u,v}, g^{(w)}_{u,v}$ are two self-controlling families, then $h^{(w)}_{u,v}\cdot g^{(w)}_{u,v}$ is a new self-controlling family, and if $h^{(w)}_{u,v}, g^{(w)}_{u,v} > 0$, then $\sqrt{h^{(w)}_{u,v}}$, $1/h^{(w)}_{u,v}$ and $h^{(w)}_{u,v} + g^{(w)}_{u,v}$ are also self-controlling families. In our application to the Fröhlich polaron model, we will consider combinations of the form

$$f^{(w)}_{u,v} = 1/\sqrt{h^{(w)}_{u,v}\cdot g^{(w)}_{u,v}}$$

with $h^{(w)}_{u,v}, g^{(w)}_{u,v}$ two self-controlling families.

Lemma 2.45. *Fix $\omega \in \mathcal{W}^{\Sigma,X}_\eta$ and $m, n, r \in \mathbb{N}_0$ and define a family of functions*

$$f^{(w)}_{u,v}((k_\mathscr{M}, k_\mathscr{N}, k_\mathscr{R}, x), (k_\mathscr{U}, k_\mathscr{V}, k_\mathscr{W})) := \omega^{(r+w)}_{m+u,n+v}(k_{\mathscr{M}\sqcup\mathscr{U}}, k_{\mathscr{N}\sqcup\mathscr{V}}, k_{\mathscr{R}\sqcup\mathscr{W}}; x).$$

If $(f^{(w)}_{u,v})_{u,v,w\in\mathbb{N}_0}$ is self-controlling, then $\omega \in \mathcal{W}^\partial_\eta$.

Proof. We find, if $|k'-k|<\delta$, that

$$\begin{aligned}
&|f_{u,v}^{(w)}(k',q)-f_{u,v}^{(w)}(k,q)-\sum_{|\alpha|=1}\partial_1^\alpha f_{u,v}^{(w)}(k,q)(k'-k)^\alpha| \\
&\quad\leqslant \sum_{|\alpha|=1}\sup_{0\leqslant t\leqslant 1}|\partial_1^\alpha f_{u,v}^{(w)}(tk'+(1-t)k,q)-\partial_1^\alpha f_{u,v}^{(w)}(k,q)||k'-k| \\
&\quad\leqslant \sum_{|\alpha|=1}\sum_{|\beta|=1}\sup_{0\leqslant t\leqslant 1}|\partial_1^\alpha\partial_1^\beta f_{u,v}^{(w)}(tk'+(1-t)k,q)||k'-k|^2 \\
&\quad\leqslant ((m+n+r+1)d+2)^2 C\sup_{0\leqslant t\leqslant 1}|f_{u,v}^{(w)}(tk'+(1-t)k,q)||k'-k|^2 \\
&\quad\leqslant ((m+n+r+1)d+2)^2 C^2|f_{u,v}^{(w)}(k,q)||k'-k|^2.
\end{aligned}$$

If we define, for $|\alpha|=1$,

$$(\partial^\alpha \mathfrak{w}_{m,n}^{(r)}(k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}},x))_{u,v}^{(w)}(k_{\mathscr{U}},k_{\mathscr{V}},k_{\mathscr{W}}):=\partial_1^\alpha f_{u,v}^{(w)}(k,q) \tag{50}$$

and put $C'':=((m+n+r+1)d+2)^2C^2$, we conclude, if $|k'-k|<\delta$, that

$$\begin{aligned}
&\|\mathfrak{w}_{m,n}^{(r)}(k')-\mathfrak{w}_{m,n}^{(r)}(k)-\sum_{|\alpha|=1}\partial^\alpha\mathfrak{w}_{m,n}^{(r)}(k)(k'-k)^\alpha\|_{\mathcal{W}^\Sigma} \\
&\quad\leqslant |k'-k|^2C''\|\mathfrak{w}_{m,n}^{(r)}(k)\|_{\mathcal{W}^\Sigma}\leqslant |k'-k|^2C''\|\omega_{m,n}^{(r)}\|_{\mathcal{W}_\eta^{\Sigma,X}},
\end{aligned}$$

which ensures that $\mathfrak{w}_{m,n}^{(r)}$ is differentiable in the Frechét sense. The second order of differentiability and continuity of the second derivatives are obtained similarly. We note that formula (50) holds true whenever $|\alpha|\leqslant 2$. Finally, we find

$$\begin{aligned}
\|\omega\|_{\mathcal{W}_\eta^\partial} &= \sum_{m,n\in\mathbb{N}_0}\eta^{-m-n}\sup_{r\in\mathbb{N}_0}\max_{|\alpha|\leqslant 2}\sup_{k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}},x}\|\partial^\alpha\mathfrak{w}_{m,n}^{(r)}(k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}};x)\|_{\mathcal{W}^\Sigma} \\
&\leqslant C\sum_{m,n\in\mathbb{N}_0}\eta^{-m-n}\sup_{r\in\mathbb{N}_0}\sup_{k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}},x}\|\mathfrak{w}_{m,n}^{(r)}(k_{\mathscr{M}},k_{\mathscr{N}},k_{\mathscr{R}};x)\|_{\mathcal{W}^\Sigma} \\
&= C\|\omega\|_{\mathcal{W}_\eta^{\Sigma,X}}<\infty,
\end{aligned}$$

and therefore $\omega\in\mathcal{W}_\eta^{\Sigma,X}$. □

2.5.2 The Feshbach Operator

Recall that for the application to the Fröhlich polaron model, one considers the Birman-Schwinger operator $B_{g,E,\xi}=g(b_{E,\xi}+b_{E,\xi}^*)$, where

$$\begin{aligned}
(b_{E,\xi}\psi)^{(n)}(k_1,\dots,k_n) &:= \int\frac{dk_{n+1}\,(n+1)^{1/2}\bar{v}(k_{n+1})\psi^{(n+1)}(k_1,\dots,k_{n+1})}{U_{E,\xi}^{(n)}(k_1,\dots,k_n)U_{E,\xi}^{(n+1)}(k_1,\dots,k_{n+1})}, \\
(b_{E,\xi}^*\psi)^{(n)}(k_1,\dots,k_n) &:= \sum_{j=1}^n\frac{n^{-1/2}v(k_j)\psi^{(n-1)}(k_1,\dots,\hat{k}_j,\dots,k_n)}{U_{E,\xi}^{(n)}(k_1,\dots,k_n)U_{E,\xi}^{(n-1)}(k_1,\dots,\hat{k}_j,\dots,k_n)},
\end{aligned}$$

and the notation $\hat{k}_j$ indicates that the variable k_j is omitted. Here, we denote by $U_{E,\xi}^{(n)}$ the function

$$U_{E,\xi}^{(n)}(k_1,\ldots,k_n) = (\frac{1}{2}|\xi - k_1 - \ldots - k_n|^2 + n - E)^{1/2}, \quad \text{for } n \geqslant 1,$$

while $U_{E,\xi}^{(0)} = |\frac{1}{2}|\xi|^2 - E|^{1/2}$ if $\frac{1}{2}|\xi|^2 \neq E$, and $U_{E,\xi}^{(0)} = 1$ if $\frac{1}{2}|\xi|^2 = E$. Furthermore, we denote by $P_{\geqslant 2}$ the orthogonal projection onto $\mathcal{F}_{\text{sym}}^{(\geqslant 2)} := \bigoplus_{n=2}^{\infty} \mathcal{F}_{\text{sym}}^{(n)}$ and by $\bar{B}_{g,E,\xi} := P_{\geqslant 2} B_{g,E,\xi} \restriction_{\mathcal{F}_{\text{sym}}^{(\geqslant 2)}}$.

Recalling Definition 1.15 and Corollary 1.17, the operator $1 - \bar{B}_{g,E,\xi}$ is invertible when $g < 1/6L_{-1}$, and we may therefore consider the Feshbach map of $J_{E,\xi} - B_{g,E,\xi}$ with respect to $P_{\leqslant 1}$. The Feshbach operator, $F_{g,E,\xi}$, acts in the direct sum space $\mathcal{F}_{\text{sym}}^{(\leqslant 1)} = \mathcal{F}_{\text{sym}}^{(0)} \oplus \mathcal{F}_{\text{sym}}^{(1)}$ as a block operator matrix

$$F_{g,E,\xi} := \begin{bmatrix} \operatorname{sgn}(\frac{1}{2}|\xi|^2 - E) & -g b_{E,\xi} \\ -g b_{E,\xi}^* & 1 - g^2 b_{E,\xi}(1 - \bar{B}_{g,E,\xi})^{-1} b_{E,\xi}^* \end{bmatrix}.$$

In this subsection, we wish to determine the form of the operator

$$g^2 P^{(1)} b_{E,\xi} P^{(\geqslant 2)} (1 - \bar{B}_{g,E,\xi})^{-1} P^{(\geqslant 2)} b_{E,\xi}^* P^{(1)} = \sum_{J=2}^{\infty} g^2 P^{(1)} b_{E,\xi} \bar{B}_{g,E,\xi}^{J-2} b_{E,\xi}^* P^{(1)},$$

along with some of its properties. Here, $P^{(1)}$ denotes the orthogonal projection in $\mathcal{F}_{\text{sym}}$ onto $\mathcal{F}_{\text{sym}}^{(1)}$.

We first define $X = (-g_0, g_0) \times (-\infty, 1) \times \mathbb{R}^d$ for some $g_0 > 0$ to be chosen later. We let $\bar{\omega} = \bar{\omega}_{1,0} + \bar{\omega}_{0,1}$, where, for $r \geqslant 2$,

$$\bar{\omega}_{0,1}^{(r)}(\star, k_{\mathscr{N}}, k_{\mathscr{R}}; g, E, \xi) := \frac{g}{U_{E,\xi}^{(r)}(k_{\mathscr{R}}) U_{E,\xi}^{(r+1)}(k_{\mathscr{N} \sqcup \mathscr{R}})},$$
$$\bar{\omega}_{1,0}^{(r)}(k_{\mathscr{M}}, \star, k_{\mathscr{R}}; g, E, \xi) := \frac{g}{U_{E,\xi}^{(r+1)}(k_{\mathscr{M} \sqcup \mathscr{R}}) U_{E,\xi}^{(r)}(k_{\mathscr{R}})}.$$

Furthermore, for fixed $E' < 1$, we let

$$b_{0,1}^{(1)}(\star, k_{\mathscr{N}}, k_{\mathscr{R}}; g, E, \xi \mid E') := \frac{1}{U_{E',\xi}^{(1)}(k_{\mathscr{R}}) U_{E,\xi}^{(2)}(k_{\mathscr{N} \sqcup \mathscr{R}})},$$
$$b_{1,0}^{(1)}(k_{\mathscr{M}}, \star, k_{\mathscr{R}}; g, E, \xi \mid E') := \frac{1}{U_{E,\xi}^{(2)}(k_{\mathscr{M} \sqcup \mathscr{R}}) U_{E',\xi}^{(1)}(k_{\mathscr{R}})}.$$

According to the proof of Lemma 1.12, $\bar{\omega}(g, E, \xi), b(g, E, \xi \mid E') \in \mathcal{W}$, and we have

$$\begin{aligned} P^{(\geqslant 2)}\bar{B}_{g,E,\xi}P^{(\geqslant 2)} &= \mathrm{Op}(\bar{\omega}(g,E,\xi)), \\ P^{(1)}b_{E,\xi} &= \mathrm{Op}(b_{0,1}(g,E,\xi \mid E)), \\ b^*_{E,\xi}P^{(1)} &= \mathrm{Op}(b_{1,0}(g,E,\xi \mid E)). \end{aligned}$$

It therefore follows from Theorem 2.26 that $g^2P^{(1)}b_{E,\xi}\bar{B}^{J-2}_{g,E,\xi}b^*_{E,\xi}P^{(1)}$ is a bounded creation/annihilation operator with symbol $\omega_J(g,E,\xi \mid E)$, where

$$\omega_J(g,E,\xi \mid E') := g^2 b_{0,1}(g,E,\xi \mid E')\#\bar{\omega}(g,E,\xi)^{\#(J-2)}\#b_{1,0}(g,E,\xi \mid E'),$$

Furthermore, $\omega_J(g,E,\xi \mid E') = 0$ unless $J \in 2\mathbb{N}$, and $\omega_{2J}(g,E,\xi \mid E')$ only has two nonvanishing terms, one with $r = 1, m = n = 0$ and one with $r = 0, m = n = 1$ and it follows from Theorem 2.22 that

$$\omega(g,E,\xi \mid E') = \sum_{J=1}^{\infty} \omega_{2J}(g,E,\xi \mid E') \in \mathcal{W}. \tag{51}$$

In fact,

$$\begin{aligned} \omega^{(1)}_{0,0}(k;E,g,\xi \mid E') &= \frac{g^2\tilde{M}_{g,E,\xi}(k)}{\frac{1}{2}|\xi-k|^2+1-E'} \\ \omega^{(0)}_{1,1}(k_1,k_2;E,g,\xi \mid E') &= \frac{g^2C_{g,E,\xi}(k_1,k_2)}{U^{(1)}_{E',\xi}(k_1)U^{(1)}_{E',\xi}(k_2)}, \end{aligned}$$

with

$$\tilde{M}_{g,E,\xi}(k) := \sum_{\substack{l\in\mathbb{N} \\ \bigsqcup_{2l\geqslant k>i\geqslant 1}\mathscr{L}_{ki}=\mathscr{L}}} \frac{g^{2(l-1)}}{(2l)!}\int dk_{\mathscr{L}}\,|v^{\otimes l}(k_{\mathscr{L}})|^2\prod_{j=2}^{2l}\frac{1}{T^{(d_j)}_{E,\xi}(k_{\mathscr{D}_j})},$$

where

$$\mathscr{D}_j = \mathscr{R} \sqcup \bigsqcup_{2l\geqslant k>j\geqslant i\geqslant 1}\mathscr{L}_{ki},$$

and the sum is further restricted to run only over those integers which satisfy

$$d_j \geqslant 2, \quad p_j + q_j = 1, \quad p_J = 0, \quad q_1 = 0,$$

where $p_j = \sum_{J\geqslant k>j} l_{kj}, \quad q_j = \sum_{j>i\geqslant 1} l_{ji}$. Similarly, we have

$$C_{g,E,\xi}(k_{\mathscr{M}},k_{\mathscr{N}}) := \sum_{\substack{l\in\mathbb{N}_0 \\ \mathscr{M}_J\sqcup\ldots\sqcup\mathscr{M}_1=\mathscr{M} \\ \mathscr{N}_J\sqcup\ldots\sqcup\mathscr{N}_1=\mathscr{N} \\ \bigsqcup_{2l\geqslant k>i\geqslant 1}\mathscr{L}_{ki}=\mathscr{L}}} \frac{g^{2l}}{(2l)!}\left[\int dk_{\mathscr{L}}\right]|v^{\otimes l}(k_{\mathscr{L}})|^2\cdot\prod_{j=1}^{2l-1}\frac{1}{T^{(d_j)}_{E,\xi}(k_{\mathscr{D}_j})},$$

with $m = n = 1$ and

$$\mathscr{D}_j = \left[\bigsqcup_{J \geqslant k > j \geqslant i \geqslant 1} \mathscr{L}_{ki} \right] \sqcup \left[\bigsqcup_{J \geqslant k > j} \mathscr{N}_k \right] \sqcup \left[\bigsqcup_{j \geqslant i \geqslant 1} \mathscr{M}_i \right],$$

and the sum is further restricted to run only over those integers which satisfy

$$d_j \geqslant 2, \quad m_j + p_j + n_j + q_j = 1, \quad m_J + p_J = 0, \quad n_1 + q_1 = 0,$$

where $p_j = \sum_{J \geqslant k > j} l_{kj}$, $q_j = \sum_{j > i \geqslant 1} l_{ji}$.

We now observe that $(k; g, E, \xi) \mapsto \tilde{M}_{g,E,\xi}$ and $(k_1, k_2; g, E, \xi) \mapsto C_{g,E,\xi}$ are both twofold continuously differentiable. In fact, the symbols $\bar{\omega}$ and b (for fixed $E' > 0$) are constructed in accordance with Remark 2.44, so due to Lemma 2.45, it suffices to show that they are in $\mathcal{W}_\eta^{\Sigma,X}$. We provide the argument only for $\bar{\omega}$, since the argument for b is very similar.

In order to see that $\bar{\omega} \in \mathcal{W}_\eta^{\Sigma,X}$, note first that

$$\begin{aligned} &\bar{\omega}_{m+u,n+v}^{(r+w)}(k_{\mathscr{M} \sqcup \mathscr{U}}, k_{\mathscr{N} \sqcup \mathscr{V}}, k_{\mathscr{R} \sqcup \mathscr{W}}; (g, E, \xi)) \\ &\quad \leqslant \bar{\omega}_{m+u,n+v}^{(w)}(k_{\mathscr{M} \sqcup \mathscr{U}}, k_{\mathscr{N} \sqcup \mathscr{V}}, k_{\mathscr{W}}; (g, E, \xi - \sum_{j \in \mathscr{R}} k_j)), \end{aligned}$$

which implies

$$\begin{aligned} &\sup_{r \in \mathbb{N}} \sup_{k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}, x} \| \bar{\mathfrak{w}}_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; x) \|_{\mathcal{W}^\Sigma} \\ &\quad \leqslant \sup_{k_{\mathscr{M}}, k_{\mathscr{N}}, x} \| \bar{\mathfrak{w}}_{m,n}^{(0)}(k_{\mathscr{M}}, k_{\mathscr{N}}, \star; x) \|_{\mathcal{W}^\Sigma}. \end{aligned}$$

But then we find

$$\begin{aligned} \|\bar{\omega}\|_{\mathcal{W}_\eta^{\Sigma,X}} &\leqslant \sup_{x \in X} \| \bar{\mathfrak{w}}_{0,0}^{(0)}(\star, \star, \star; x) \|_{\mathcal{W}^\Sigma} \\ &\quad + \eta^{-1} \sup_{k \in \mathbb{R}^d, x \in X} \| \bar{\mathfrak{w}}_{0,1}^{(0)}(\star, k, \star; x) \|_{\mathcal{W}^\Sigma} \\ &\quad + \eta^{-1} \sup_{k \in \mathbb{R}^d, x \in X} \| \bar{\mathfrak{w}}_{1,0}^{(0)}(k, \star, \star; x) \|_{\mathcal{W}^\Sigma} \\ &\leqslant \sup_{g,E,\xi} \| \bar{B}_{g,E,\xi} \| + \frac{2g_0}{\eta} \leqslant 6 g_0 L_{-1} + \frac{2g_0}{\eta}. \end{aligned}$$

We are now ready for the

Proof of Theorem 1.20. According to formula (49), we can pick g_0 sufficiently small that the series (51) converges. It follows, for fixed $E' < 1$, that $(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; g, E, \xi) \mapsto \omega(g, E, \xi \mid E')$ is twice continuously differentiable, which implies that $(k; g, E, \xi) \mapsto \tilde{M}_{g,E,\xi}$ and $(k_1, k_2; g, E, \xi) \mapsto C_{g,E,\xi}$ are both twice continuously differentiable.

Note that the mapping $(E, k) \mapsto -[T^{(1)}_{E,\xi}(k) - g^2 \tilde{M}_{g,E,\xi}(k)]$ is jointly continuous, and strictly increasing in E. Furthermore, since it is twice continuously differentiable, it is easy to check that it is strictly convex if $|g|$ is sufficiently small and that it is also unboundedly increasing for large $|k|$. Thus, the function

$$E \mapsto \lambda_{g,E,\xi} := -\min_{k \in \mathbb{R}^d} [T^{(1)}_{E,\xi}(k) - g^2 \tilde{M}_{g,E,\xi}(k)]$$

is well-defined, continuous and strictly increasing. Define

$$M_{g,E,\xi}(k) := T^{(1)}_{E,\xi}(k) - g^2 \tilde{M}_{g,E,\xi} + \lambda.$$

Since $M_{g,E,\xi}(k)$ is strictly convex, any minimum point is necessarily unique, so by the arguments above, it has a unique minimum point, $k_{\min}$. Putting $f(t) = M_{g,E,\xi}(tk + (1-t)k_{\min})$, we find $f(0) = f'(0) = 0$, so if we denote by $[H_k]_{ji} = \frac{\partial^2 M_{g,E,\xi}(k)}{\partial k_j \partial k_i}$ the Hessian of $M_{g,E,\xi}$ at $k \in \mathbb{R}^d$, Taylor's formula with remainder ensures that

$$M_{g,E,\xi}(k) = \int_0^1 \langle k - k_{\min}, H_{tk+(1-t)k_{\min}}(k - k_{\min}) \rangle (1-t) dt.$$

Noting that $1/2 \leqslant H_k \leqslant 2$ in the sense of quadratic forms, we conclude that we have $\frac{1}{4}|k - k_{\min}|^2 \leqslant M(k) \leqslant |k - k_{\min}|^2$.

It follows by construction that if $E < 1$, then $F_{g,E,\xi}$ is the Birman-Schwinger operator corresponding to the choice of operators

$$T := \operatorname{diag}(\frac{1}{2}|\xi|^2 - E, T^{(1)}_{E,\xi}), H := A_g$$

at $\lambda := \lambda_{g,E,\xi}$. □

We may also give the

Proof of Lemma 1.24. We first consider the monotonicity and continuity assertions for $E \mapsto \lambda_{g,E,\xi}$, along with the existence and uniqueness of E_{ess}. Noting that the mapping $(E, k) \mapsto -[T^{(1)}_{E,\xi}(k) - g^2 \tilde{M}_{g,E,\xi}(k)]$ is jointly continuous, and strictly increasing in E, the function

$$E \mapsto \lambda_{g,E,\xi} := -\min_{k \in \mathbb{R}^d} [T^{(1)}_{E,\xi}(k) - g^2 \tilde{M}_{g,E,\xi}(k)]$$

is continuous and strictly increasing. For $g = 0$, we have $\lambda_{0,E,\xi} = E - 1$, so $E_{\text{ess}} = 1$ is the unique number $E \leqslant 1$ such that $\lambda_{0,E,\xi} = 0$. If $g \neq 0$, then

$$\lambda_{g,1,\xi} = -\min_{k \in \mathbb{R}^d} [\frac{1}{2}|\xi - k|^2 - g^2 \tilde{M}_{g,1,\xi}(k)] \geqslant g^2 \tilde{M}_{g,1,\xi}(\xi) > 0.$$

On the other hand, if we pick E sufficiently negative that $E \leqslant g^2 \tilde{M}_{g,E,\xi}(k)$ for all $k \in \mathbb{R}^d$, then $\lambda_{g,E,\xi} \leqslant -1$. By continuity, there is a unique $E_{\mathrm{ess}} < 1$ such that $\lambda_{g,E_{\mathrm{ess}},\xi} = 0$.

It remains to obtain the expansions in g. Evaluation in the minimum point $k_{\min}$ of the derivative of the function $M_{g,E,\xi}$ yields

$$k_{\min} = \xi + g^2 \int \frac{dq\,|v(q)|^2(\xi - k_{\min} - q)}{[\frac{1}{2}|\xi - k_{\min} - q|^2 + 2 - E]^2} + g^4(\nabla \tilde{M}^{(2)}_{g,E,\xi})(k).$$

Noting that, since v is reflection symmetric,

$$\int \frac{dq\,|v(q)|^2(\xi - k_{\min} - q)}{[\frac{1}{2}|\xi - k_{\min} - q|^2 + 2 - E]^2} = \int \frac{dq\,|v(q)|^2 q}{[\frac{1}{2}|q|^2 + 2 - E]^2} + O(g^2) = O(g^2),$$

the expansion of $k_{\min}$ follows.

Second, evaluation in the minimum point $k_{\min}$ of the function $M_{g,E_{\mathrm{ess}},\xi}$ yields, noting that $\lambda = 0$,

$$E_{\mathrm{ess}} = \frac{1}{2}|\xi - k_{\min}|^2 + 1 - g^2 \int \frac{dq\,|v(q)|^2}{\frac{1}{2}|\xi - k_{\min} - q|^2 + 2 - E_{\mathrm{ess}}} + g^4 \tilde{M}^{(2)}_{g,E,\xi}(k).$$

Since

$$\frac{1}{2}|\xi - k_{\min}|^2 + 1 - \int \frac{dq\,g^2|v(q)|^2}{\frac{1}{2}|\xi - k_{\min} - q|^2 + 2 - E_{\mathrm{ess}}} = 1 - g^2 \int \frac{dq\,|v(q)|^2}{\frac{1}{2}|q|^2 + 1} + O(g^4),$$

the expansion of E_{ess} follows.

Third, evaluation of $M_{g,E,\xi}$ in the minimum point $k_{\min}$ yields

$$0 = \frac{1}{2}|\xi - k_{\min}|^2 + 1 - E + \lambda - g^2 \int \frac{dq\,|v(q)|^2}{\frac{1}{2}|\xi - k_{\min} - q|^2 + 2 - E} - g^4 \tilde{M}^{(2)}_{g,E,\xi}(k).$$

Using the expansion of E_{ess}, we obtain the expansion of λ.

Similarly, the expansions of M and e_0 follow from the expansion of λ. □

2.6 Spin Boson Model

Throughout this subsection, $\mathcal{M} = \mathbb{R}^3$ and $\varepsilon(k) = |k|$. We will give a very brief overview of selected results on the spin boson model and then give an outline of the remainder of this subsection. We denote by $v \in L^2(\mathbb{R}^3)$ the coupling function of the spin boson model. In order to ensure self-adjointedness of the spin boson Hamiltonian, we impose the condition $\varepsilon^{-\frac{1}{2}} v \in L^2(\mathbb{R}^3)$.

Over the years, many papers have been published on the spin boson model, and we will not attempt to give a thorough historical review. We are mainly interested in the application of the operator theoretic renormalization group method to the spin boson model, and our choice of works mentioned below reflects this. For further references, see those contained in the works referenced below.

Bach et al. [4] introduced the BFS operator theoretic renormalization group method, and applied the technique to a model of an atom interacting with a massless boson field. The renormalization map consisted of a Feshbach projection combined with a suitable rescaling. Non-differentiability of the spectral projection $1_{[0,1]}(H)$ of their Hamiltonian H, made the analysis technically difficult.

The difficulties related to non-differentiability were overcome by Bach et al. [3], with the introduction of the smooth Feshbach map, which allows one to replace the sharp projections of the Feshbach method by a smooth partition of unity. They demonstrated that the operator theoretic renormalization analysis is vastly simplified from the technical point of view if one makes use of the smooth Feshbach map instead of the Feshbach map. They applied the new framework to the infrared regular spin boson model, and proved that if one imposes the condition $\varepsilon^{-1-\alpha}v \in L^2(\mathbb{R}^3)$, with $\alpha > 0$, then there is a ground state in the weak coupling regime.

Later, Hasler and Herbst [20] gave a treatment of the infrared singular spin boson model, imposing the condition $\varepsilon^{\frac{1}{2}}v \in L^\infty(\mathbb{R}^3)$. Note that this condition is weaker than the condition $\varepsilon^{-1}v \in L^2(\mathbb{R}^3)$, but it is strong enough to ensure $\varepsilon^{-\frac{1}{2}-\alpha}v \in L^2(\mathbb{R}^3)$ for all $\alpha \in [0, 1/2)$. For their proof, it is essential that the interaction of the spin boson model has no diagonal terms. It was remarked upon by Hasler and Herbst [20, Remark 2.2], that they expected that, with a different choice of norm, one would be able to relax their condition $\varepsilon^{\frac{1}{2}}v \in L^\infty(\mathbb{R}^3)$ to the weaker condition $\varepsilon^{-\frac{1}{2}-\alpha}v \in L^2(\mathbb{R}^3)$ for some $\alpha > 0$. The method employed by Hasler and Herbst was a modification of the operator theoretic renormalization group method based on the smooth Feshbach method of Bach et al..

In this subsection, we will pick a norm which allows us to treat the case $v \in L^2(\mathbb{R}^3), \varepsilon^{-\frac{1}{2}-\alpha}v \in L^2(\mathbb{R}^3), \alpha > 0$, thereby verifying the expectation of Hasler and Herbst. For a concise treatment of the smooth Feshbach map, see the paper by Griesemer and Hasler [17].

Abstractly, the analysis we present is essentially the same as the one given by Hasler and Herbst. Technically, the presentations differ slightly, since we apply the framework developed in Subsection 2.4. In the first subsubsection below, we carry out a preparatory Feshbach reduction in accordance with the procedure of Hasler and Herbst. The second subsubsection below deals with the contraction property of the renormalization map.

In the final chapter of the thesis, an abstract account of the spectral theoretic renormalization group is given, which, coupled with the contraction property that the renormalization map is shown to have in this subsection, suffices to give a proof of existence and uniqueness of the ground state eigenvalue of the spin boson model in the weak coupling regime.

2.6.1 Smooth Creation/Annihilation Symbols

Throughout the subsection, we will consider the following class of smooth creation/annihilation symbols. Let $X = [0, \infty) \times \tilde{X}$, with $\tilde{X} = \mathcal{O} \subseteq \mathbb{C}^\nu$ open. Consider the subspace $\mathcal{W}_\eta^\partial \subseteq \mathcal{W}_\eta^{\Sigma,X}$ consisting of all $\omega \in \mathcal{W}_\eta^{\Sigma,X}$ such that:

$$\omega_{m,n}^{(r)}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; t, \tilde{x}) = \omega_{m,n}^{(0)}(k_{\mathscr{M}}, k_{\mathscr{N}}; t + d\Gamma(\varepsilon)^{(r)}(k_{\mathscr{R}}), \tilde{x}), \tag{52}$$

such that the mapping $x \mapsto \omega(x) \in \mathcal{W}_\eta^\Sigma$ is continuous, such that (for fixed $\tilde{x} \in \mathcal{O}$) the mapping $[0, \infty) \ni t \mapsto \omega(t, \tilde{x}) \in \mathcal{W}_\eta^\Sigma$ has one continuous Frechét derivative with a one-sided derivative at $t = 0$, and such that (for fixed $t \geqslant 0$) the mapping $\mathcal{O} \ni \tilde{x} \mapsto \omega(t, \tilde{x}) \in \mathcal{W}_\eta^\Sigma$ is analytic. We impose additionally finiteness of the norm

$$\|\omega\|_{\mathcal{W}_\eta^\partial} = \|\omega\|_{\mathcal{W}_\eta^{\Sigma,X}} + \|\partial_t \omega\|_{\mathcal{W}_\eta^{\Sigma,X}},$$

and note that it is a standard result that $\mathcal{W}_\eta^\partial$ is complete with respect to this norm. Furthermore, it follows as in Example 2.42 that, if $\omega^J, \dots, \omega^1 \in \mathcal{W}_\eta^\partial$, then $\omega^J \# \cdots \# \omega^1 \in \mathcal{W}_{2\eta}^\partial$, with $\|\omega^J \# \cdots \# \omega^1\|_{\mathcal{W}_{2\eta}^\partial} \leqslant 4^{J-1} \prod_{j=1}^J \|\omega^j\|_{\mathcal{W}_\eta^\partial}$, and we have the Leibniz product rule

$$\partial_t[\omega^2 \# \omega^1] = [\partial_t \omega^2] \# \omega^1 + \omega^2 \# [\partial_t \omega^1]. \tag{53}$$

Due to equation (52), it is consistent to write

$$\mathrm{Op}(\omega_{0,0}(0, \tilde{x})) = \omega_{0,0}^{(0)}(d\Gamma(\varepsilon), \tilde{x}), \tag{54}$$

where the right hand side is defined in terms of the spectral calculus. Furthermore, we have

$$\|\omega_{0,0}\|_{\mathcal{W}_\eta^\partial} = \sup_{t,\tilde{x}} |\omega_{0,0}^{(0)}(t, \tilde{x})| + \sup_{t,\tilde{x}} |\partial_t \omega_{0,0}^{(0)}(t, \tilde{x})|. \tag{55}$$

2.6.2 Preparatory Feshbach Reduction

Let $\mathcal{M} = \mathbb{R}^3$. The spin boson model is defined in the space $\mathcal{F}_{\mathrm{sym}} \oplus \mathcal{F}_{\mathrm{sym}}$ by the Hamiltonian

$$H_g^{init} - E = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} d\Gamma(\varepsilon) & 0 \\ 0 & d\Gamma(\varepsilon) \end{bmatrix} - E + g \begin{bmatrix} 0 & \Phi(v) \\ \Phi(v) & 0 \end{bmatrix},$$

where $v \in L^2(\mathbb{R}^3)$, $\Phi(v) = a(v) + a^*(v)$ is the field operator and $\varepsilon(k) = |k|$. We assume that $\|(1 \vee \varepsilon^{-\frac{1}{2}-\alpha})v\|_{L^2(\mathcal{M})} = 1$. Here, v is fixed in accordance with remark 2.7, $g \in \mathbb{C}$ is a dimensionless coupling constant and $E \in \mathbb{C}$ is a spectral parameter. We have $\Phi(v) = \mathrm{Op}(\omega)$, where $\omega^{(r)}_{1,0}(k, k_{\mathscr{R}}) = \omega^{(r)}_{0,1}(k, k_{\mathscr{R}}) = 1$, is a standard creation/annihilation operator in the sense of Subsection 2.3. That is, $\omega \in \mathcal{W}^{st}_{1,0} \oplus \mathcal{W}^{st}_{0,1}$, and $\|w\|_{\mathcal{W}^{st}} \leqslant 2\|(1 \vee \varepsilon^{-\frac{1}{2}})v\|_{L^2(\mathcal{M})}$. We find from Theorem 2.18 that H_g is self-adjoint on the domain $\mathcal{D}(d\Gamma(\varepsilon)) \oplus \mathcal{D}(d\Gamma(\varepsilon))$.

If $|E| < 1/2$, we may consider the Feshbach projection to the second factor. We find the effective Hamiltonian, acting in $\mathcal{F}_{\mathrm{sym}}$,

$$\tilde{H}_{g,E} = d\Gamma(\varepsilon) - E - g^2\Phi(v)(2 + d\Gamma(\varepsilon) - E)^{-1}\Phi(v),$$

Defining standard creation/annihilation symbols, in the sense of Subsection 2.3, by

$$\begin{aligned}\tilde{\omega}^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; t, g, E) &= \tilde{\omega}^{(0)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}; t + d\Gamma(\varepsilon)^{(r)}(k_{\mathscr{R}}), g, E),\\ \tilde{\omega}^{(0)}_{0,0}(t, g, E) &= g^2 \left[\int \frac{dk|v(k)|^2}{2 + t + \varepsilon(k) - E} - \int \frac{dk|v(k)|^2}{2 + \varepsilon(k) - E}\right],\\ \tilde{\omega}^{(0)}_{1,1}(k, q; t, g, E) &= \frac{g^2}{2 + t - E} + \frac{g^2}{2 + t + \varepsilon(k) + \varepsilon(q) - E},\\ \tilde{\omega}^{(0)}_{2,0}(k, q; t, g, E) &= \frac{1}{2}\left[\frac{g^2}{2 + t + \varepsilon(k) - E} + \frac{g^2}{2 + t + \varepsilon(q) - E}\right],\end{aligned}$$

and $\tilde{\omega}^{(0)}_{0,2}(k, q; t, g, E) = \tilde{\omega}^{(0)}_{2,0}(k, q; t, g, E)$, we have, by Theorem 2.49 the formula

$$\begin{aligned}\tilde{H}_{g,E} &= d\Gamma(\varepsilon) - \lambda_{g,E} - \mathrm{Op}(\tilde{\omega}(0, g, E)),\\ \lambda_{g,E} &= E + g^2 \int \frac{dk|v(k)|^2}{2 + \varepsilon(k) - E},\end{aligned}$$

on $\mathcal{D}_{fin}$, and the extension to $\mathcal{D}(d\Gamma(\varepsilon))$ is uniquely determined by continuity with respect to the graph norm, according to Theorem 2.18. We have

$$\begin{aligned}&\tilde{\omega}^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; t, g, E)\\ &\leqslant \frac{g^2(m+n)!}{m!n!}[1 + d\Gamma(1 \wedge \varepsilon)^{(m+r)}(k_{\mathscr{M} \sqcup \mathscr{R}})]^{-\frac{m}{2}}[1 + d\Gamma(1 \wedge \varepsilon)^{(n+r)}(k_{\mathscr{N} \sqcup \mathscr{R}})]^{-\frac{n}{2}}.\end{aligned}$$

Therefore, by Theorem 2.18, we find $\tilde{\omega}(t,g,E) \in \mathcal{W}$. Specifically,

$$\|\tilde{\omega}(t,g,E)_{0,0}\|_{\mathcal{W}} = \sup_{s\geqslant 0}|\tilde{\omega}^{(0)}_{0,0}(s+t,g,E)| \leqslant g^2\|\varepsilon^{-\frac{1}{2}}v\|^2_{L^2(\mathcal{M})} \leqslant g^2,$$
$$\|(\tilde{\omega}(t,g,E)_{m,n})_{m+n=2}\|_{\mathcal{W}} \leqslant 4g^2\|(1\vee\varepsilon^{-\frac{1}{2}})v\|^2_{L^2(\mathcal{M})} \leqslant 4g^2,$$

where the second bound is a consequence of Theorem 2.18. In fact, we find that $\tilde{\omega} \in \mathcal{W}^{\Sigma}_{\eta}$ for any $\eta > 0$, with the bound

$$\|(\tilde{\omega}(t,g,E)_{m,n})_{m+n=2}\|_{\mathcal{W}^{\Sigma}_{\eta}} \leqslant \sum_{m+n+u+v=2} g^2 \frac{(m+u+n+v)!}{(m+u)!(n+v)!}\frac{4^{u+v}}{\eta^{m+n}}. \tag{56}$$

We now observe that, since $\partial_E\lambda_{g,E} = 1 + O(g^2)$, the mapping $\lambda_g(E) = \lambda_{g,E}$ is bi-holomorphic from the set $\lambda_g^{-1}(\mathbb{D}_{1/4})$ to the set $\mathbb{D}_{1/4}$. More precisely,

Lemma 2.46. *Let $\lambda : \mathbb{D}_{1/2} \to \mathbb{C}$ be holomorphic and let $0 \leqslant c < 1$. Suppose*

$$|\lambda(0)| + \sup_{E\in\mathbb{D}_{1/2}}|\lambda(E) - E| \leqslant c2^{-4}. \tag{57}$$

Then the mapping $\lambda\restriction_{\lambda^{-1}(\mathbb{D}_{1/4})} : \lambda^{-1}(\mathbb{D}_{1/4}) \to \mathbb{D}_{1/4}$ is biholomorphic.

Proof. when $|E| \leqslant 5/16$, it follows from Cauchy's estimate, where we use a contour of radius $3/8$, that $|\lambda'(E) - 1| \leqslant c$. It follows that the real part of λ is injective in $\mathbb{D}_{5/16}$. Furthermore, if $E \in \lambda^{-1}(\mathbb{D}_{1/4})$, then we have

$$1/4 > |\lambda(E)| \geqslant |E| - |\lambda(E) - E| > |E| - 2^{-4},$$

which implies $E \in \mathbb{D}_{5/16}$. We conclude that $\lambda\restriction_{\lambda^{-1}(\mathbb{D}_{1/4})}$ is injective.

We will now argue that $\lambda\restriction_{\lambda^{-1}(\mathbb{D}_{1/4})} : \lambda^{-1}(\mathbb{D}_{1/4}) \to \mathbb{D}_{1/4}$ is surjective, so pick $E_0 \in \mathbb{D}_{1/4}$. Consider the auxiliary function $g : \mathbb{D}_{5/16} \to \mathbb{C}$ given by $g(E) = E + [E_0 - \lambda(E)]$, which satisfies $|g'(E)| = |1 - \lambda'(E)| \leqslant c$. If we can show that there is $t \in (0,1)$ such that g maps $\bar{\mathbb{D}}_{t5/16}$ into $\bar{\mathbb{D}}_{t5/16}$, we conclude that g is a contraction on $\bar{\mathbb{D}}_{t5/16}$, and therefore, according to the Banach fixed point theorem, that g has a unique fixed point, say E'. But $g(E') = E'$ if and only if $\lambda_\rho(E) = E_0$, so this implies that $\lambda_\rho\restriction_{\lambda_\rho^{-1}(\mathbb{D}_{1/4})}$ is onto.

It remains to show that g maps $\bar{\mathbb{D}}_{t5/16}$ into $\bar{\mathbb{D}}_{t5/16}$ for some $t \in (0,1)$. But this is true for any $t \geqslant 3/5$, as follows from the inequality

$$|g(E)| \leqslant |g(E) - g(0)| + |g(0)| \leqslant |E| \sup_{E\in\mathbb{D}_{1/2}}|\lambda(E) - E| + |E_0 + \lambda(0)|$$
$$\leqslant \frac{5t}{16}\left(\sup_{E\in\mathbb{D}_{1/2}}|\lambda(E) - E| + \frac{4}{5t} + |\lambda(0)|\right) \leqslant \frac{5t}{16}\cdot\frac{69}{80t},$$

which in turn is consequence of inequality (57). Thus, when $t \geqslant 69/80$, we conclude that g maps $\bar{\mathbb{D}}_{t5/16}$ into $\bar{\mathbb{D}}_{t5/16}$. Since any bijective holomorphic function is biholomorphic, this finishes the proof. □

Defining $\omega(t,g,E) = \tilde{\omega}(t,g,\lambda_g^{-1}(E))$ and $H_{g,E} = \tilde{H}_{g,\lambda_g^{-1}(E)}$, we then find

$$
\begin{aligned}
H_{g,E} &= d\Gamma(\varepsilon) - E - \mathrm{Op}(\omega(0,g,E)) \\
&= d\Gamma(\varepsilon) - \omega_{0,0}^{(0)}(d\Gamma(\varepsilon),g,E)) - E - \sum_{m+n=2} \mathrm{Op}(\omega_{m,n}(0,g,E)) \\
&= T_{g,E}(d\Gamma(\varepsilon)) - W_{g,E}, \\
T_{g,E}(t) &= t - \omega_{0,0}^{(0)}(t,g,E) - E, \\
W_{g,E} &= \sum_{m+n=2} \mathrm{Op}(\omega_{m,n}(0,g,E)).
\end{aligned}
$$

where we recall formula (54). We see that E is an eigenvalue of H_g^{init} if and only if 0 is an eigenvalue of $H_{g,\lambda_g(E)}$. If we put $X = [0,\infty) \times \tilde{X}$, with $\tilde{X} = \{(g,E) \in \mathbb{C}^2 \,|\, |E| < 1/4\}$, then we find $\omega \in \mathcal{W}_\eta^{\Sigma,X}$ and now it is not difficult to realize that $\omega \in \mathcal{W}_\eta^{\partial}$. We have the bound $\|\omega\|_{\mathcal{W}_\eta^{\partial}} \leqslant 2\|\omega\|_{\mathcal{W}_\eta^{\Sigma,X}}$, with $\|\omega\|_{\mathcal{W}_\eta^{\Sigma,X}}$ bounded by the right hand side of (56). Whenever convenient, we will write $T_{g,E} = T_{g,E}(d\Gamma(\varepsilon))$, identifying the function $T_{g,E}$ with the operator it induces by spectral calculus. We now want to argue that ω lies in a certain set which will allow us to set up an iterated Grushin problem for $H_{g,E}$, and in order to do this, we introduce some notation.

Definition 2.47. Fix $\eta = 1/4$ and $\rho \in (0,1)$ such that $\max(\rho,\rho^\alpha) = 2^{-8}$. Let $\theta \in C^1(\mathbb{R})$ be a function such that $0 \leqslant \theta \leqslant 1$, $\theta(t) = 1$ if $0 \leqslant t \leqslant 3/4$, $\theta(t) = 0$ if $1 \leqslant t$ and $-16/\pi \leqslant \theta'(t) \leqslant 0$. Define functions $\chi(t) = \sin(\frac{\pi}{2}\theta(t))$, $\bar{\chi}(t) = \cos(\frac{\pi}{2}\theta(t))$, and denote also $\chi = \chi(d\Gamma(\varepsilon)), \bar{\chi} = \bar{\chi}(d\Gamma(\varepsilon))$. Finally, let $\chi_\rho(t) = \chi(t/\rho), \bar{\chi}_\rho(t) = \bar{\chi}(t/\rho)$ and put $\chi_\rho = \chi_\rho(d\Gamma(\varepsilon)), \bar{\chi}_\rho = \bar{\chi}_\rho(d\Gamma(\varepsilon))$.

Define a unitary operator U_ρ in $\mathcal{F}_{\mathrm{sym}}$ by

$$(\mathcal{U}_\rho\psi)^{(n)}(k_1,\dots,k_n) = \rho^{-\frac{d}{2}}\psi^{(n)}(\rho^{-1}k_1,\dots,\rho^{-1}k_n).$$

and set $\mathscr{X} = \mathscr{Y} = \mathcal{F}_{\mathrm{sym}}$, $\mathscr{D} = \mathcal{D}(d\Gamma(\varepsilon))$ along with

$$
\begin{aligned}
\chi_1 &= \mathcal{U}_\rho^*\chi_\rho = \chi\mathcal{U}_\rho^*, & \chi_2 &= \chi_\rho\mathcal{U}_\rho = \mathcal{U}_\rho\chi, \\
\bar{\chi}_1 &= \mathcal{U}_\rho^*\bar{\chi}_\rho = \bar{\chi}\mathcal{U}_\rho^*, & \bar{\chi}_2 &= \bar{\chi}_\rho\mathcal{U}_\rho = \mathcal{U}_\rho\bar{\chi}, \\
\mathscr{W} &= \mathrm{ran}\, 1_{[3/4,\infty)}(d\Gamma(\varepsilon)), & \mathscr{V} &= \mathrm{ran}\, 1_{[3\rho/4,\infty)}(d\Gamma(\varepsilon)).
\end{aligned}
$$

Let $S_{g,E} = \mathcal{U}_\rho^* T_{g,E}\mathcal{U}_\rho$ and define

$$
\begin{aligned}
\bar{T}_{g,E} &= T_{g,E}\restriction_{\mathscr{V}} : \mathscr{V} \cap \mathscr{D} \to \mathscr{V}, \\
\bar{H}_{g,E} &= (S_{g,E} - \bar{\chi}_1 W \bar{\chi}_2)\restriction_{\mathscr{W}} : \mathscr{W} \cap \mathscr{D} \to \mathscr{W}
\end{aligned}
$$

Making the dependence on v explicit, in accordance with Remark 2.7, we finally define $B(\delta, \epsilon, u)$ to consist of those $w \in \mathcal{W}^{\partial}_{1/4}(u)$ which have an even number of creations and annihilations, i.e. $w = (w_{m,n})_{m+n \in 2\mathbb{N}}$, and additionally fulfill

$$\sup_{t,g,E} |\partial_t w^{(0)}_{0,0}(t, g, E)| \leqslant \delta, \qquad \|(w_{m,n})_{m+n\in 2\mathbb{N}}\|_{\mathcal{W}^{\partial}_{1/4}} \leqslant \epsilon,$$
$$w^{(0)}_{0,0}(0, g, E) = 0, \qquad \|(1 \vee \varepsilon^{-1-\alpha})u\|_{L^2(\mathbb{R}^d)} \leqslant 1.$$

Here, we recall formula (55).

Having introduced all of the relevant definitions, we see that we have $\omega \in B(2g^2, 2^{10}g^2, v)$. Furthermore, we have in $\mathscr{V}$ the relation

$$d\Gamma(\varepsilon) - E = \frac{1}{3} d\Gamma(\varepsilon) + \frac{2}{3} d\Gamma(\varepsilon) - E \geqslant \frac{1}{3} d\Gamma(\varepsilon)$$

if $|E| \leqslant \rho/2$, so $\bar{d\Gamma}(\varepsilon) - E = (d\Gamma(\varepsilon) - E)\restriction_{\mathscr{V}}$ is invertible. We conclude that, if $\delta \leqslant \rho/8$, then

$$\bar{T}_{g,E} = [1 - \omega^{(0)}_{0,0}(d\Gamma(\varepsilon), g, E))(\bar{d\Gamma}(\varepsilon) - E)^{-1}](\bar{d\Gamma}(\varepsilon) - E)$$

is invertible. Furthermore, since we have $\|\bar{\chi}_1 W_{g,E} \bar{\chi}_2\|_{op} \leqslant \epsilon$, we find similarly that

$$\bar{H}_{g,E} = [1 - \bar{\chi}_1 W_{g,E} \bar{\chi}_2 \bar{S}^{-1}_{g,E}] \bar{S}_{g,E}$$

is invertible if $\epsilon \leqslant \rho/16$. Thus, whenever $|E| \leqslant \rho/2$, $\delta \leqslant \rho/8$ and $\epsilon \leqslant \rho/16$, we find that $\bar{T}_{g,E}$ and $\bar{H}_{g,E}$ are bijective, and

$$\|\bar{\chi}_1 \bar{T}^{-1}_{g,E} \bar{\chi}_2\|_{op} \leqslant \frac{16}{\rho}, \quad \|\bar{\chi}_2 \bar{H}^{-1}_{g,E} \bar{\chi}_1\|_{op} \leqslant \frac{32}{\rho}. \tag{58}$$

2.6.3 Contraction Property

Throughout this subsubsection, we will write $\omega_{\geqslant 2} = (\omega_{m,n})_{m+n\in 2\mathbb{N}}$ whenever $\omega \in \mathcal{W}$.

Theorem 2.48. *Suppose $\omega \in B(\delta, \epsilon, v)$ with $\delta \leqslant 2^{-3}\rho$, $\epsilon \leqslant \rho^2 2^{-11}$ and $|E| \leqslant \rho/2$. Then $(H_{g,E}, T_{g,E}, S_{g,E}, \chi_1, \bar{\chi}_1, \chi_2, \bar{\chi}_2, \mathscr{U}, \mathscr{V})$ is Feshbach data, so we may define the Feshbach map*

$$F_{g,E} = S_{g,E} - \chi_1 W_{g,E} \chi_2 - \chi_1 W_{g,E} \bar{\chi}_2 \bar{H}^{-1}_{g,E} \bar{\chi}_1 W_{g,E} \chi_2.$$

Defining $\lambda_\rho : \mathbb{D}_{\rho/2} \to \mathbb{C}$ by $\lambda_\rho(E) = \rho^{-1}(E + \langle \Omega, W_{g,E} \bar{\chi}_2 \bar{H}^{-1}_{g,E} \bar{\chi}_1 W_{g,E} \Omega \rangle)$, we find that the mapping $\lambda_\rho \restriction_{\lambda_\rho^{-1}(\mathbb{D}_{1/4})} : \lambda_\rho^{-1}(\mathbb{D}_{1/4}) \to \mathbb{D}_{1/4}$ is biholomorphic, *so we may consider the renormalization map* $R_{g,E} = \frac{1}{\rho} F_{g,\lambda_\rho^{-1}(E)}$. *We find* $\omega' \in B(\delta + \frac{\epsilon}{2}, \frac{\epsilon}{2}, v')$ *such that*

$$R_{g,E} = d\Gamma(\varepsilon) - E - \mathrm{Op}(\omega'(0, g, E)).$$

Proof. We already argued that $(H_{g,E}, T_{g,E}, S_{g,E}, \chi_1, \bar{\chi}_1, \chi_2, \bar{\chi}_2, \mathscr{U}, \mathscr{V})$ is Feshbach data in the previous subsubsection, so we turn to the statement about the function λ_ρ. We find from equation (58) that

$$|\langle \Omega, W_{g,E} \bar{\chi}_2 \bar{H}_{g,E}^{-1} \bar{\chi}_1 W_{g,E} \Omega \rangle| \leqslant \frac{32\epsilon^2}{\rho} \leqslant 2^{-8}\rho,$$

We may conclude from Lemma 2.46 that the mapping $\lambda : \mathbb{D}_{1/2} \to \mathbb{C}$ given by $\lambda(E) = \lambda_\rho(\rho E)$ induces a biholomorphic map $\lambda\restriction_{\lambda^{-1}(\mathbb{D}_{1/4})} : \lambda^{-1}(\mathbb{D}_{1/4}) \to \mathbb{D}_{1/4}$. But this is equivalent to $\lambda_\rho\restriction_{\lambda_\rho^{-1}(\mathbb{D}_{\rho/4})} : \lambda_\rho^{-1}(\mathbb{D}_{\rho/4}) \to \mathbb{D}_{1/4}$ being biholomorphic, which was what we wanted.

Next, we observe that

$$\begin{aligned} &S_{g,E} - \langle \Omega, W_{g,E} \bar{\chi}_2 \bar{H}_{g,E}^{-1} \bar{\chi}_1 W_{g,E} \Omega \rangle \\ &\quad = \rho d\Gamma(\varepsilon) - \omega_{0,0}^{(0)}(\rho d\Gamma(\varepsilon), g, E) - \rho\lambda_\rho(E), \end{aligned}$$

so

$$\begin{aligned} F_{g,\lambda_\rho^{-1}(E)} &= \rho d\Gamma(\varepsilon) - \omega_{0,0}^{(0)}(\rho d\Gamma(\varepsilon), g, \lambda_\rho^{-1}(E)) - \rho E \\ &\qquad - [\chi_1 \tilde{W}_{g,\lambda_\rho^{-1}(E)} \chi_2 - \langle \Omega, \tilde{W}_{g,\lambda_\rho^{-1}(E)} \Omega \rangle], \\ \tilde{W}_{g,\lambda_\rho^{-1}(E)} &:= W_{g,\lambda_\rho^{-1}(E)} + W_{g,\lambda_\rho^{-1}(E)} \bar{\chi}_2 \bar{H}_{g,\lambda_\rho^{-1}(E)}^{-1} \bar{\chi}_1 W_{g,\lambda_\rho^{-1}(E)}. \end{aligned}$$

We will now find $\tilde{\omega} \in \mathcal{W}_{2\eta}^{\partial}$ such that $\mathrm{Op}(\tilde{\omega}(0, g, E)) = \tilde{W}_{g,\lambda_\rho^{-1}(E)}$, where

$$\begin{aligned} \|\tilde{\omega}\|_{\mathcal{W}_{2\eta}^{\Sigma,X}} &\leqslant 2\epsilon, & \|\partial_t \tilde{\omega}\|_{\mathcal{W}_{2\eta}^{\Sigma,X}} &\leqslant 3 \cdot 2^3 \rho^{-1}\epsilon, \\ \|\tilde{\omega}_{0,0}\|_{\mathcal{W}_{\eta}^{\Sigma,X}} &\leqslant \rho\epsilon, & \|\partial_t \tilde{\omega}_{0,0}\|_{\mathcal{W}_{\eta}^{\Sigma,X}} &\leqslant 2^{-3}\epsilon. \end{aligned}$$

Note first that

$$\bar{\chi}_2 \bar{H}_{g,E}^{-1} \bar{\chi}_1 = \frac{\bar{\chi}_\rho^2}{T_{g,E} - E} \sum_{J=0}^{\infty} \left(W_{g,E} \frac{\bar{\chi}_\rho^2}{T_{g,E}} \right)^J,$$

which implies

$$W_{g,E} + W_{g,E} \bar{\chi}_2 \bar{H}_{g,E}^{-1} \bar{\chi}_1 W_{g,E} = \sum_{J=1}^{\infty} \left(W_{g,E} \frac{\bar{\chi}_\rho^2}{T_{g,E}} \right)^{J-1} W_{g,E}.$$

We will take the notational liberty of identifying the direct sum of multiplication operators $\frac{\bar{\chi}_\rho^2}{T_{g,E}}$ with its symbol, i.e.

$$[\frac{\bar{\chi}_\rho^2}{T_{g,E}}]_{0,0}^{(r)}(k_{\mathscr{R}};t) := \frac{\bar{\chi}_\rho^2(t + d\Gamma(\varepsilon)^{(r)}(k_{\mathscr{R}}))}{T_{g,E}(t + d\Gamma(\varepsilon)^{(r)}(k_{\mathscr{R}}))}.$$

From the definition of $\bar{\chi}$, we have

$$\|\frac{\bar{\chi}_\rho^2}{T_{g,E}}\|_{\mathcal{W}_\eta^{\Sigma,X}} \leqslant \frac{4}{3\rho}, \quad \|\partial_t \frac{\bar{\chi}_\rho^2}{T_{g,E}}\|_{\mathcal{W}_\eta^{\Sigma,X}} \leqslant \frac{16\delta}{9\rho^2} + \frac{8\cdot 4}{3\rho} \leqslant \frac{98}{9\rho} \leqslant \frac{11}{\rho}.$$

Then we find from Theorem 2.39 that

$$\|[\omega_{\geqslant 2} \# \frac{\bar{\chi}_\rho^2}{T_{g,E}}]^{\# J-1} \# \omega_{\geqslant 2}\|_{\mathcal{W}_{2\eta}^{\Sigma,X}} \leqslant \left[4\frac{4}{3\rho}\epsilon\right]^{J-1} \epsilon \leqslant \rho^{J-1}\epsilon,$$

so Theorem 2.38 ensures that $\tilde{\omega} := \sum_{J=1}^\infty [\omega_{\geqslant 2} \# \frac{\bar{\chi}_\rho^2}{T_{g,E}}]^{\# J-1} \# \omega_{\geqslant 2}$ is well defined and satisfies

$$\mathrm{Op}(\tilde{\omega}) = \sum_{J=1}^\infty \left(W_{g,E}\frac{\bar{\chi}_\rho^2}{T_{g,E}-E}\right)^{J-1} W_{g,E}, \quad \|\tilde{\omega}\|_{\mathcal{W}_{2\eta}^{\Sigma,X}} \leqslant 2\epsilon.$$

Furthermore, according to formula (53), we have

$$\begin{aligned}
\|\partial_t\tilde{\omega}\|_{\mathcal{W}_\eta^{\Sigma,X}} &\leqslant \sum_{J=1}^\infty \epsilon^J (J-1)4^{J-1} \|\partial_t \frac{\bar{\chi}_\rho^2}{T_{g,E}}\|_{\mathcal{W}_\eta^{\Sigma,X}} \|\frac{\bar{\chi}_\rho^2}{T_{g,E}}\|_{\mathcal{W}_\eta^{\Sigma,X}}^{J-2} \\
&\quad + \sum_{J=1}^\infty \epsilon^J J 4^{J-1} \|\frac{\bar{\chi}_\rho^2}{T_{g,E}}\|_{\mathcal{W}_\eta^{\Sigma,X}}^{J-1} \\
&\leqslant \sum_{J=1}^\infty \epsilon^J J 4^{J-1} \frac{2^4}{\rho}(\frac{4}{3\rho})^{J-2} = \frac{12\epsilon}{\rho}\sum_{J=1}^\infty J(\frac{16\epsilon}{3\rho})^{J-1} \leqslant \frac{2^4\epsilon}{\rho}.
\end{aligned}$$

Finally, noting that $\tilde{\omega}_{0,0} = \sum_{J=2}^\infty ([\omega_{\geqslant 2} \# \frac{\bar{\chi}_\rho^2}{T_{g,E}}]^{\# J-1} \# \omega_{\geqslant 2})_{0,0}$, we find similarly

$$\|\tilde{\omega}_{0,0}\|_{\mathcal{W}_\eta^{\Sigma,X}} \leqslant \epsilon \sum_{J=2}^\infty (\frac{16\epsilon}{3\rho})^{J-1} \leqslant 2\epsilon\frac{16\epsilon}{3\rho} \leqslant 2^{-7}\rho\epsilon,$$

$$\|\partial_t\tilde{\omega}_{0,0}\|_{\mathcal{W}_\eta^{\Sigma,X}} \leqslant \frac{12\epsilon}{\rho}\sum_{J=2}^\infty J(\frac{16\epsilon}{3\rho})^{J-1} \leqslant \frac{12\epsilon}{\rho}\frac{64\epsilon}{3\rho} \leqslant 2^{-3}\epsilon.$$

We now observe that

$$\chi_1 \,\mathrm{Op}(\tilde{\omega})\chi_2 = \chi\mathcal{U}_\rho^* \,\mathrm{Op}^v(\tilde{\omega})\mathcal{U}_\rho\chi_2 = \mathrm{Op}^{v_\rho}(\chi \# \tilde{\omega}_\rho \# \chi),$$

where we put $v_\rho(k) := \rho^{1-\alpha} 1_{[0,1]}(|k|) v(\rho k)$ and

$$(\tilde{\omega}_\rho)^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; t, g, E) := \rho^{(\frac{1}{2}+\alpha)(m+n)} \tilde{\omega}^{(r)}_{m,n}(\rho k_{\mathscr{M}}, \rho k_{\mathscr{N}}, \rho k_{\mathscr{R}}; \rho t, g, E)$$

and made the dependence of Op on v explicit. Note that

$$\|\varepsilon^{-\frac{1}{2}-\alpha} v_\rho\|^2 = \int_{|k|\leqslant 1} \frac{dk\, \rho^3 |v(\rho k)|^2}{\varepsilon(\rho k)^{1+2\alpha}} = \int_{|k|\leqslant \rho} \frac{dk |v(k)|^2}{\varepsilon(k)^{1+2\alpha}} \leqslant \|\varepsilon^{-\frac{1}{2}-\alpha} v\|^2 \leqslant 1,$$

and that we have, according to Theorem 2.18, the inequality

$$\begin{aligned} \|[\tilde{\mathfrak{w}}^{(r)}_{m,n}(k_{\mathscr{M}}, k_{\mathscr{N}}, k_{\mathscr{R}}; t, g, E)]_{u,v}\|_{\mathcal{W}} &\leqslant \|\tilde{\omega}_{m+u,n+v}\|_{L^\infty} \|(1 \vee \epsilon^{-\frac{1}{2}}) v\|^{u+v} \\ &\leqslant \eta^{m+n+u+v} \|\tilde{\omega}\|_{\mathcal{W}^\partial_\eta}. \end{aligned}$$

Recalling that $\eta = 1/4$ and $\rho^\alpha \leqslant 2^{-8}$, we find

$$\begin{aligned} \|[\tilde{\omega}_\rho]_{\geqslant 2}\|_{\mathcal{W}^{\Sigma,X}_\eta} &\leqslant \sum_{m+n+u+v \in 2\mathbb{N}} (2\rho^{\frac{1}{2}+\alpha})^{m+n} (4\eta \cdot 2\rho^{\frac{1}{2}+\alpha})^{u+v} \frac{\|\tilde{\omega}_{m+u,n+v}\|_{L^\infty}}{(2\eta)^{m+n+u+v}} \\ &\leqslant \|\tilde{\omega}_{\geqslant 2}\|_{\mathcal{W}^{\Sigma,X}_{2\eta}} \sum_{m+n+u+v \in 2\mathbb{N}} (2\rho^{\frac{1}{2}+\alpha})^{m+n+u+v} \\ &\leqslant \frac{\rho}{2^7} \|\tilde{\omega}_{\geqslant 2}\|_{\mathcal{W}^{\Sigma,X}_{2\eta}} \leqslant \frac{\rho\epsilon}{2^6} \end{aligned}$$

where we used that

$$\begin{aligned} \sum_{m+n+u+v \in 2\mathbb{N}} (2\rho^{\frac{1}{2}+\alpha})^{m+n+u+v} &\leqslant \sum_{l=1}^\infty (4\rho^{1+2\alpha})^l (2l)^3 \leqslant \sum_{l=1}^\infty 8(32\rho^{1+2\alpha})^l \\ &= \frac{2^8 \rho^{1+2\alpha}}{1 - 2^5 \rho^{1+2\alpha}} \leqslant 2^9 \rho^{1+2\alpha} \leqslant \frac{\rho}{2^7}. \end{aligned}$$

Similarly, we find $\|[\partial_t \tilde{\omega}_\rho]_{\geqslant 2}\|_{\mathcal{W}^{\Sigma,X}_\eta} \leqslant 2^{-7}\rho \|([\partial_t \tilde{\omega}_\rho]_{\geqslant 2}\|_{\mathcal{W}^{\Sigma,X}_{2\eta}} \leqslant 2^{-3}\rho\epsilon$, along with

$$\|[\tilde{\omega}_\rho]_{0,0}\|_{\mathcal{W}^{\Sigma,X}_\eta} \leqslant 2^{-7}\rho\epsilon, \qquad \|[\partial_t \tilde{\omega}_\rho]_{0,0}\|_{\mathcal{W}^{\Sigma,X}_\eta} \leqslant \rho \|\partial_t \tilde{\omega}_{0,0}\|_{\mathcal{W}^{\Sigma,X}_\eta} \leqslant 2^{-3}\rho\epsilon.$$

Putting everything together, we define $\omega'_{m,n} := \rho^{-1}(\chi \# \tilde{\omega}_\rho \# \chi)_{m,n}$ for $m + n \in 2\mathbb{N}$, and

$$\begin{aligned} \omega'_{0,0} := \rho^{-1} \omega^{(0)}_{0,0}(\rho t, g, \lambda_\rho^{-1}(E)) &- \rho^{-1}((\chi \# \tilde{\omega}_\rho(t, g, \lambda_\rho^{-1}(E)) \# \chi)_{0,0} \\ &- \rho^{-1} \langle \Omega, W_{g,\lambda_\rho^{-1}(E)} \bar{\chi}_2 \bar{H}^{-1}_{g,\lambda_\rho^{-1}(E)} \bar{\chi}_1 W_{g,\lambda_\rho^{-1}(E)} \Omega \rangle). \end{aligned}$$

With these definitions, we have $R_{g,E} = d\Gamma(\varepsilon) - E - \mathrm{Op}(\omega')$, and we find

$$\|\omega'_{\geqslant 2}\|_{\mathcal{W}^\partial_\eta} \leqslant \rho^{-1} \left[16 \|[\tilde{\omega}_\rho]_{\geqslant 2}\|_{\mathcal{W}^{\Sigma,X}_\eta} + \|[\partial_t \tilde{\omega}_\rho]_{\geqslant 2}\|_{\mathcal{W}^{\Sigma,X}_\eta} \right] \leqslant [2^{-2} + 2^{-3}]\epsilon \leqslant \epsilon/2.$$

We now want to argue that $\|\partial_t \omega'_{0,0}\|_{\mathcal{W}^{\Sigma}_{\eta}} \leqslant \delta + \epsilon/2$. Note first that

$$\omega'^{(r)}_{0,0}(t,g,E) = \frac{[\omega_\rho]^{(r)}_{0,0}(t,g,E) + \chi^{(r)}(t)[\tilde{\omega}_\rho]^{(r)}_{0,0}(t,g,E)\chi^{(r)}(t) - [\tilde{\omega}_\rho]^{(0)}_{0,0}(0,g,E)}{\rho}.$$

where $\|\partial_t[\omega_\rho]_{0,0}\|_{\mathcal{W}^{\partial}_{\eta}} = \rho\|\partial_t\omega_{0,0}\|_{\mathcal{W}^{\partial}_{\eta}} \leqslant \rho\delta$ by assumption. Furthermore, according to formula (53), we have

$$\|\partial_t(\chi \,\#[\tilde{\omega}_\rho]_{0,0} \,\#\, \chi)\|_{\mathcal{W}^{\Sigma,x}_{\eta}} \leqslant 2^4\|[\tilde{\omega}_\rho]_{0,0}\|_{\mathcal{W}^{\Sigma,x}_{\eta}} + \|[\partial_t\tilde{\omega}_\rho]_{0,0}\|_{\mathcal{W}^{\Sigma,x}_{\eta}} \leqslant 2^{-2}\rho\epsilon,$$

and therefore in particular $\|\partial_t\omega'_{0,0}\|_{\mathcal{W}^{\Sigma}_{\eta}} \leqslant \delta + \epsilon/2$. This was what we wanted. □

2.7 Appendix

2.7.1 Standard Version of Wick's Theorem

It is sometimes useful to have Wick's Theorem available for standard creation/annihilation symbols. We observe that if $\omega \in \mathcal{W}^{st}$, then $\mathcal{R}(\mathrm{Op}(\omega)) \subseteq \mathcal{D}(\mathrm{Op}(\omega))$, so one may ask if the composition of finitely many standard creation/annihilation operators $\mathrm{Op}(\omega^J)\cdots\mathrm{Op}(\omega^1)$ is again a standard creation/annihilation operator, i.e. if there is $\omega \in \mathcal{W}^{st}$ such that $\mathrm{Op}(\omega) = \mathrm{Op}(\omega^J)\cdots\mathrm{Op}(\omega^1)$.

Note that the standard creation/annihilation operator $\mathrm{Op}(\omega^J)\cdots\mathrm{Op}(\omega^1)$ is uniquely determined by the action of the operators $\mathrm{Op}(\omega^J)\cdots\mathrm{Op}(\omega^1)P_l$ as l ranges over all integers, where P_l is the projection onto $\mathcal{F}^{(l)}_{\mathrm{sym}}$. However, any standard creation/annihilation symbol is turned into a bounded creation/annihilation symbol when restricted to a subspace of finite particle number. That is, if we have $\omega \in \bigoplus_{m+n\leqslant h}\mathcal{W}^{st}_{m,n}$, and $P_{\leqslant l}$ is the projection onto $\bigoplus_{k=0}^{l}\mathcal{F}^{(k)}_{\mathrm{sym}}$, then we have the identity $\mathrm{Op}(\omega)P_l = \mathrm{Op}(\omega_{\leqslant l})$, where

$$(\omega_{\leqslant l})^{(r)}_{m,n} = \begin{cases} \omega^{(r)}_{m,n}, & \text{if } n+r \leqslant l, m+n \leqslant h \\ 0, & \text{otherwise} \end{cases}$$

But the conditions $m+n \leqslant h$ and $n+r \leqslant l$ force only finitely many components of $\omega_{\leqslant l}$ to be non-zero. In fact, any non-zero component must have $r \leqslant l, m \leqslant h, n \leqslant h$, and therefore $\|\omega_{\leqslant l}\|_{\mathcal{W}} \leqslant (h+l)!\|\omega\|_{\mathcal{W}^{st}}$. Furthermore, $\mathrm{Op}(\omega)P_{\leqslant l} = P_{\leqslant l+h}\,\mathrm{Op}(\omega)P_l$.

In conclusion, if we put

$$(\omega_l)^{(r)}_{m,n} = \begin{cases} \omega^{(r)}_{m,n}, & \text{if } n+r = l, m+n \leqslant h \\ 0, & \text{otherwise ,} \end{cases}$$

and we have $\omega^J, \ldots, \omega^1 \in \bigoplus_{m+n\leqslant h} \mathcal{W}^{st}_{m,n}$, we may compute the action of the standard creation/annihilation operator $\mathrm{Op}(\omega^J)\cdots\mathrm{Op}(\omega^1)$ by computing the action of $\mathrm{Op}(\omega^J_{\leqslant l+(j-1)h})\cdots\mathrm{Op}(\omega^1_l)$ as l ranges over all integers. But this problem now has a solution in terms of Theorem 2.26, and all that remains to be checked is that the resulting symbol is in fact a standard creation/annihilation symbol.

Theorem 2.49. *If $\omega^J, \ldots, \omega^1 \in \mathcal{W}^{st}$, we may define $\omega^J \# \cdots \# \omega^1 \in \mathcal{W}^{st}$ by the formula given in Theorem 2.26. The mapping $\# : \mathcal{W}^{st} \times \mathcal{W}^{st} \to \mathcal{W}^{st}$ is an associative product, and we have $\mathrm{Op}(\omega^2 \# \omega^1) = \mathrm{Op}(\omega^2)\,\mathrm{Op}(\omega^1)$.*

Proof. In light of Theorem 2.26, it remains only to check that $\omega \in \mathcal{W}^{st}$, where we put $\omega = \omega^J \# \cdots \# \omega^1$. For this, it suffices to assume $\omega^1, \ldots, \omega^J \geqslant 0$. Define $U_J = I$ to be the identity, and, for $1 \leqslant j \leqslant J-1$, define

$$U_j : \mathcal{F}^{(m_j+p_j+r_j+s_j)} \to \mathcal{F}^{(n_{j+1}+q_{j+1}+r_{j+1}+s_{j+1})}$$

to be the unitary permutation operator given by

$$(U_j\psi)(k_{\mathscr{N}_{j+1}}, k_{\mathscr{Q}_{j+1}}, k_{\mathscr{R}_{j+1}}, k_{\mathscr{S}_{j+1}}) = \psi(k_{\mathscr{M}_j}, k_{\mathscr{P}_j}, k_{\mathscr{R}_j}, k_{\mathscr{S}_j}),$$

Note that U_j depends on the partitions

$$\mathscr{M}_J \sqcup \ldots \sqcup \mathscr{M}_1 = \mathscr{M}, \qquad \mathscr{N}_J \sqcup \ldots \sqcup \mathscr{N}_1 = \mathscr{N},$$
$$\mathscr{P}_J \sqcup \ldots \sqcup \mathscr{P}_1 = \mathscr{L}, \qquad \mathscr{Q}_J \sqcup \ldots \sqcup \mathscr{Q}_1 = \mathscr{L},$$

where $\mathscr{P}_k \cap \mathscr{Q}_i = \emptyset, k \geqslant i$.

Using Remark 2.29, we have, with the notation $\prod_{j=1}^J a_j = a_J \cdots a_1$, the inquality

$$\mathrm{Op}'(\gamma_{m,n}\omega^{(r)}_{m,n}) \leqslant \sum_{\substack{l\in\mathbb{N}_0 \\ \mathscr{M}_J\sqcup\ldots\sqcup\mathscr{M}_1=\mathscr{M} \\ \mathscr{N}_J\sqcup\ldots\sqcup\mathscr{N}_1=\mathscr{N} \\ \mathscr{P}_J\sqcup\ldots\sqcup\mathscr{P}_1=\mathscr{L} \\ \mathscr{Q}_J\sqcup\ldots\sqcup\mathscr{Q}_1=\mathscr{L} \\ \mathscr{P}_k\cap\mathscr{Q}_i=\emptyset,k\geqslant i}} \left[\frac{1}{m!n!(l!)^2}\prod_{j=1}^J m_j!n_j!p_j!q_j!\binom{m_j+p_j}{p_j}\binom{n_j+q_j}{q_j}\right] \cdot l! \left[\prod_{j=1}^J U_j\,\mathrm{Op}'(\gamma_{m_j+p_j,n_j+q_j}\omega^{j(r_j+s_j)}_{m_j+p_j,n_j+q_j})\right].$$

Here, $A \leqslant B$ should be understood in the sense $A\phi^{(n+r)} \leqslant B\phi^{(n+r)}$ for all $\phi^{(n+r)} \geqslant 0$. We have put $\gamma = 1 \vee \varepsilon^{-\frac{1}{2}}$ and

$$r_j = \sum_{j>i\geqslant 1} m_i + \sum_{J\geqslant k>j} n_k + r, \quad s_j = \sum_{i=1}^{j-1}(p_i - q_i).$$

In conclusion,

$$\|\omega_{m,n}\|_{\mathcal{W}^{st}_{m,n}} \leqslant \sum_{\substack{m_J+\ldots+m_1=m\\ n_J+\ldots+n_1=n\\ p_J,\ldots,p_1\in\mathbb{N}_0\\ q_J,\ldots,q_1\in\mathbb{N}_0}} \left[\prod_{j=1}^{J}\binom{m_j+p_j}{p_j}\binom{n_j+q_j}{q_j}\right]\left(\sum_{j=1}^{J}p_j\right)! \cdot \prod_{j=1}^{J}\|\omega^j_{m_j+p_j,n_j+q_j}\|_{\mathcal{W}^{st}_{m_j+p_j,n_j+q_j}}.$$

There is $h \in \mathbb{N}_0$ such that, for all $1 \leqslant j \leqslant J$, we have $m_j + p_j + n_j + q_j \leqslant h$ for all non-zero components of ω^j, so this bound suffices in order to ensure that $\omega \in \mathcal{W}^{st}$. □

3 Schur Complements, Feshbach Maps, Grushin Problems, and Spectral Renormalization

Jonas Dahlbæk and Oliver Matte

Abstract

It is shown that the smooth Feshbach method can be posed as a Grushin problem, and that the spectral theoretic renormalization group can be phrased in the language of iterated Grushin problems. Based on this, an abstrat account of the spectral theoretic renormalization group is given.

3.1 Overview

We review some standard tools related to the inversion of block operator matrices as relevant for applying spectral theoretic renormalization group techniques. In Subsection 3.2 we start with the classical Schur complement theorem [36] which essentially results from applying the Gauss algorithm to a two-by-two block operator matrix. The Feshbach projection method [7] is an example of the Schur complement theorem; see Corollary 3.2. In Subsection 3.3 we infer another theorem on the inversion of block operator matrices from the Schur complement theorem that gives rise to a general strategy in spectral analysis which is commonly referred to as the Grushin problem method, mainly following J. Sjöstrand's nomenclature; see [39] and the references given there. We shall demonstrate in Subsection 3.4 that the smooth Feshbach method introduced in [3] and generalized in [17] can be seen as a special case of a Grushin problem, which might reveal a handy framework for the method. After that, in Subsection 3.5, we consider iterated Grushin problems following the presentation in [39]. While all preceding results are pure linear algebra, the corollary on inductive applications of infinitely many Grushin problems in Subsection 3.5 will actually be the first point in this appendix, where norms are introduced on the involved vector spaces. In the final Subsection 3.6 we explain how the spectral renormalization group strategy based on the smooth Feshbach map [3] fits into the framework of iterated Grushin problems.

The following presentation is mainly motivated by [3, 17, 39].

3.2 Schur complements

To recall the Schur complement theorem [36] we let $\mathscr{X}_1, \mathscr{X}_2, \mathscr{Y}_1, \mathscr{Y}_2$ denote real or complex vector spaces and suppose that

$$A : \mathscr{X}_1 \to \mathscr{Y}_1, \quad B : \mathscr{X}_2 \to \mathscr{Y}_1,$$
$$C : \mathscr{X}_1 \to \mathscr{Y}_2, \quad D : \mathscr{X}_2 \to \mathscr{Y}_2,$$

are linear operators with A being *bijective*. We define the operator block matrix

$$M := \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathscr{X}_1 \oplus \mathscr{X}_2 \longrightarrow \mathscr{Y}_1 \oplus \mathscr{Y}_2. \tag{59}$$

We further introduce the canonical projections $\pi_{\mathscr{Z}_i} : \mathscr{Z}_i \oplus \mathscr{Z}_2 \to \mathscr{Z}_i$ and injections $\iota_{\mathscr{Z}_i} : \mathscr{Z}_i \to \mathscr{Z}_1 \oplus \mathscr{Z}_2$, $\mathscr{Z}_i \in \{\mathscr{X}_i, \mathscr{Y}_i\}$, $i = 1, 2$, and write

$$Q(B) := \begin{pmatrix} -A^{-1}B \\ \mathbb{1}_{\mathscr{X}_2} \end{pmatrix}, \qquad Q^\sharp(C) := \begin{pmatrix} -CA^{-1} & \mathbb{1}_{\mathscr{Y}_2} \end{pmatrix}.$$

Finally, we introduce the *Schur complement* of A in M by

$$S := D - CA^{-1}B : \mathscr{X}_2 \longrightarrow \mathscr{Y}_2.$$

The first formula given in Part (4) of the following theorem is known as the (Schur-)Banachiewicz inversion formula; see, e.g., [34] for a historical discussion of early developments related to the Schur complement and references.

Theorem 3.1 (Schur complement theorem)**.** *Under the assumptions of the preceding paragraphs the following assertions hold true:*

(1) *The maps*

$$Q(B)\restriction_{\ker S} : \ker S \to \ker M \quad \textit{and} \quad \pi_{\mathscr{X}_2}\restriction_{\ker M} : \ker M \to \ker S$$

are mutually inverse bijections.

(2) $Q^\sharp(C)$ *maps* $\operatorname{ran} M$ *into* $\operatorname{ran} S$, $\iota_{\mathscr{Y}_2}$ *maps* $\operatorname{ran} S$ *into* $\operatorname{ran} M$, *and the induced maps*

$$\widehat{\iota_{\mathscr{Y}_2}} : \operatorname{coker} S \longrightarrow \operatorname{coker} M, \quad y_2 + \operatorname{ran} S \longmapsto \iota_{\mathscr{Y}_2} y_2 + \operatorname{ran} M,$$
$$\widehat{Q^\sharp(C)} : \operatorname{coker} M \longrightarrow \operatorname{coker} S, \quad y + \operatorname{ran} M \longmapsto Q^\sharp(C)y + \operatorname{ran} S,$$

between $\operatorname{coker} M := (\mathscr{Y}_1 \oplus \mathscr{Y}_2)\big/\operatorname{ran} M$ *and* $\operatorname{coker} S := \mathscr{Y}_2\big/\operatorname{ran} S$ *are mutually inverse bijections.*

(3) *M is bijective, if and only if S is bijective. In the affirmative case*

$$M^{-1} = \iota_{\mathscr{X}_1} A^{-1} \pi_{\mathscr{Y}_1} + Q(B) S^{-1} Q^\sharp(C), \qquad \pi_{\mathscr{X}_2} M^{-1} \iota_{\mathscr{Y}_2} = S^{-1}.$$

Proof. Applying a block operator analogue of Gauss' algorithm we obtain

$$\underbrace{\begin{pmatrix} \mathbb{1}_{\mathscr{Y}_1} & 0 \\ -CA^{-1} & \mathbb{1}_{\mathscr{Y}_2} \end{pmatrix}}_{=:R^\sharp(C)} M \underbrace{\begin{pmatrix} \mathbb{1}_{\mathscr{X}_1} & -A^{-1}B \\ 0 & \mathbb{1}_{\mathscr{X}_2} \end{pmatrix}}_{=:R(B)} = \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix}. \tag{60}$$

Here $R(B)$ is a bijection on $\mathscr{X}_1 \oplus \mathscr{X}_2$ and $R^\sharp(C)$ is a bijection on $\mathscr{Y}_1 \oplus \mathscr{Y}_2$; in fact, their inverses are given by $R(-B)$ and $R^\sharp(-C)$, respectively. Moreover, the resulting identities

$$\begin{aligned} &MQ(B) = R^\sharp(-C)\iota_{\mathscr{Y}_2} S, \quad \pi_{\mathscr{Y}_2} R^\sharp(C) M = S\pi_{\mathscr{X}_2} R(-B) = S\pi_{\mathscr{X}_2}, \\ &\iota_{\mathscr{X}_1} A^{-1} \pi_{\mathscr{Y}_1} R^\sharp(C) M = \iota_{\mathscr{X}_1} \pi_{\mathscr{X}_1} R(-B) = \mathbb{1}_{\mathscr{X}_1 \oplus \mathscr{X}_2} - Q(B)\pi_{\mathscr{X}_2}, \end{aligned} \tag{61}$$

and the trivial relation $\pi_{\mathscr{X}_2} Q(B) = \mathbb{1}_{\mathscr{X}_2}$ imply Part (1). If M and S are injective, then the formulas asserted in Part (3) are obtained by solving (60) for M^{-1}. Part (2) follows from the immediate $Q^\sharp(C)\iota_{\mathscr{Y}_2} = \mathbb{1}_{\mathscr{Y}_2}$, the bijectivity of $R(B)$, and

$$\begin{aligned} &MQ(B) = \iota_{\mathscr{Y}_2} S, \quad Q^\sharp(C) M R(B) = S\pi_{\mathscr{X}_2}, \\ &MR(B)\iota_{\mathscr{X}_1} A^{-1} \pi_{\mathscr{Y}_1} = R^\sharp(-C)\iota_{\mathscr{Y}_1} \pi_{\mathscr{Y}_1} = \mathbb{1}_{\mathscr{Y}_1 \oplus \mathscr{Y}_2} - \iota_{\mathscr{Y}_2} Q^\sharp(C). \end{aligned}$$

Here the first relation follows trivially from the first one in (61) and the other two can again be read off from (60). □

The previous theorem can be applied to the spectral analysis of a Hamilton operator H in case a good guess for a projection P onto an approximate spectral subspace is at hand [7]:

Corollary 3.2 (Feshbach projection method)**.** *Let $\mathscr{X}$ be a vector space, $\mathrm{dom}(H) \subset \mathscr{X}$ a subspace, and let $H : \mathrm{dom}(H) \to \mathscr{X}$ be a linear operator. Assume that $P : \mathscr{X} \to \mathscr{X}$ is a projection such that $P\,\mathrm{dom}(H) \subset \mathrm{dom}(H)$. Set $\overline{P} := \mathbb{1}_{\mathscr{X}} - P$ and assume further that $\overline{H} := \overline{P}H\restriction_{\mathrm{ran}\,\overline{P}}$ is bijective from $\overline{P}\,\mathrm{dom}(H)$ onto $\mathrm{ran}\,\overline{P}$. Finally, define the* Feshbach operator *on the domain $\mathrm{dom}(F) = P\,\mathrm{dom}(H)$ by*

$$F := PH\restriction_{\mathrm{ran}\,P} - PH\overline{H}^{-1}\overline{P}H\restriction_{\mathrm{ran}\,P}.$$

Then the following maps are mutually inverse bijections,

$$(P - \overline{H}^{-1}\overline{P}HP)\restriction_{\ker F}\colon \ker F \longmapsto \ker H, \qquad P\restriction_{\ker H}\colon \ker H \longmapsto \ker F.$$

Proof. Choose $\mathscr{X}_1 := \overline{P}\,\mathrm{dom}(H)$, $\mathscr{X}_2 := P\,\mathrm{dom}(H)$, $\mathscr{Y}_1 := \overline{P}\mathscr{X}$, and $\mathscr{Y}_2 := P\mathscr{X}$ and $A := \overline{H}$, $B := \overline{P}H\restriction_{\operatorname{ran} P}$, $C := PH\restriction_{\operatorname{ran}\overline{P}}$, $D := PH\restriction_{\operatorname{ran} P}$ in Theorem 3.1. Then $F = S$ is the Schur complement of $\overline{H}$ in $H = M$. □

In fact, the analogue of the Schur-Banachiewicz formula is usually stated as a part of the Feshbach projection method as well. We leave its straightforward translation into the setting of Corollary 3.2 to the reader.

Often the projection P in Corollary 3.2 is a spectral projection of a self-adjoint operator T in a Hilbert space and $H - T$ is a small perturbation of T in the operator or form sense. A detailed formulation of the Feshbach projection method taylor-made for this situation can be found in [4]; see also Example 3.5(2) below.

3.3 Grushin problems

In a Grushin problem one *assumes* invertibility of a block matrix M as in (59) and exploits a *resulting criterion* for the invertibility of A to analyze the *given* operator A. The entries B, C, and D of M stem from some clever, problem-dependent ansatz. Often, $D = 0$, B is the projection onto a space spanned by approximate eigenvectors of A for the eigenvalue 0, and C is the map dual to B. For a discussion of various examples we refer to [39].

We again consider a situation as in the beginning of the previous subsection, but with no assumptions other than linearity imposed on A.

Theorem 3.3 (Grushin problem)**.** *Let M be a block operator matrix of the form* (59)*, assume that $M : \mathscr{X}_1 \oplus \mathscr{X}_2 \to \mathscr{Y}_1 \oplus \mathscr{Y}_2$ is bijective, and write the block matrix decomposition of $M^{-1} : \mathscr{Y}_1 \oplus \mathscr{Y}_2 \to \mathscr{X}_1 \oplus \mathscr{X}_2$ as*

$$M^{-1} =: \mathcal{E} =: \begin{pmatrix} E & E_2 \\ E_1 & E_{12} \end{pmatrix}. \tag{62}$$

Then the following assertions hold true:

(1) *The maps*

$$C\restriction_{\ker A}\colon \ker A \longrightarrow \ker E_{12} \quad \textit{and} \quad E_2\restriction_{\ker E_{12}}\colon \ker E_{12} \longrightarrow \ker A$$

are mutually inverse bijections.

(2) *E_1 maps $\operatorname{ran} A$ into $\operatorname{ran} E_{12}$, B maps $\operatorname{ran} E_{12}$ into $\operatorname{ran} A$, and the induced maps*

$$\widehat{E}_1 : \operatorname{coker} A \longrightarrow \operatorname{coker} E_{12} \quad \textit{and} \quad \widehat{B} : \operatorname{coker} E_{12} \longrightarrow \operatorname{coker} A$$

are mutually inverse bijections.

(3) *$A : \mathscr{X}_1 \to \mathscr{Y}_1$ is bijective, if and only if $E_{12} : \mathscr{Y}_2 \to \mathscr{X}_2$ is bijective. In the affirmative case $E_{12}^{-1} = S$ is the Schur complement of A in M and $A^{-1} = E - E_2 E_{12}^{-1} E_1$.*

Proof. We consider the larger block operator matrix

$$\hat{M} := \begin{pmatrix} A & B & 0 \\ C & D & -\mathbb{1}_{\mathscr{Y}_2} \\ 0 & -\mathbb{1}_{\mathscr{X}_2} & 0 \end{pmatrix} : \mathscr{X}_1 \oplus \mathscr{X}_2 \oplus \mathscr{Y}_2 \to \mathscr{Y}_1 \oplus \mathscr{Y}_2 \oplus \mathscr{X}_2.$$

Here the sub-block $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is bijective by assumption and $\hat{D} := \left(\begin{smallmatrix} D & -\mathbb{1}_{\mathscr{Y}_2} \\ -\mathbb{1}_{\mathscr{X}_2} & 0 \end{smallmatrix}\right)$ has the obvious left and right inverse $\hat{D}^{-1} = \left(\begin{smallmatrix} 0 & -\mathbb{1}_{\mathscr{X}_2} \\ -\mathbb{1}_{\mathscr{Y}_2} & -D \end{smallmatrix}\right)$. In view of (62), the Schur complement of M in $\hat{M}$ is $-E_{12}$. The Schur complement of $\hat{D}$ in $\hat{M}$ is A. Hence, by Theorem 3.1, the maps

$$\begin{pmatrix} \mathbb{1}_{\mathscr{X}_1} \\ 0 \\ C\restriction_{\ker A} \end{pmatrix} : \ker A \longrightarrow \ker \hat{M} \quad \text{and} \quad \pi_{\mathscr{X}_1} : \ker \hat{M} \longrightarrow \ker A$$

are mutually inverse bijections, and so are

$$\begin{pmatrix} E_2\restriction_{\ker E_{12}} \\ 0 \\ \mathbb{1}_{\mathscr{Y}_2} \end{pmatrix} : \ker E_{12} \longrightarrow \ker \hat{M} \quad \text{and} \quad \pi_{\mathscr{Y}_2} : \ker \hat{M} \longrightarrow \ker E_{12}.$$

This implies Part (1), and Part (2) can be proved in a similar fashion. If A and E_{12} are bijective, then we can employ the Schur-Banachiewicz formula in Theorem 3.1(4) to both M and $\mathcal{E}$ to find that $E_{12} = S^{-1}$ and $A^{-1} = E - E_1 E_{12}^{-1} E_2$. □

Remark 3.4. The Schur complement theorem and the above theorem on Grushin problems are equivalent. For we derived Theorem 3.3 from Theorem 3.1 in the preceding proof, and to go in the opposite direction we only have to observe that, for bijective $A : \mathscr{X}_1 \to \mathscr{Y}_1$, the block operator matrices

$$\widetilde{M} := \begin{pmatrix} S & -CA^{-1} & \mathbb{1}_{\mathscr{Y}_2} \\ A^{-1}B & A^{-1} & 0 \\ -\mathbb{1}_{\mathscr{X}_2} & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & -\mathbb{1}_{\mathscr{X}_2} \\ 0 & A & B \\ \mathbb{1}_{\mathscr{Y}_2} & C & D \end{pmatrix} \tag{63}$$

are left and right inverses of each other. Then Parts (1) and (2) of Theorem 3.3 imply the corresponding parts of Theorem 3.1, and the Schur-Banachiewicz formula follows upon computing the Schur complement of S in $\widetilde{M}$.

The statements of Theorem 3.3 can alternatively be read off from the eight relations expressing that $M\mathcal{E}$ and $\mathcal{E}M$ are block identity matrices; see [39].

For a better understanding of the smooth Feshbach method introduced in the next subsection we demonstrate how the Feshbach projection method fits into the framework of Grushin problems [39]:

Example 3.5. Under the hypotheses of Corollary 3.2 the following holds:

(1) Define F as in the statement of that corollary. Then, as a special case of (63), the block matrix

$$\begin{pmatrix} F & P - PH\overline{PH}^{-1}\overline{P} \\ -P + \overline{H}^{-1}\overline{P}HP & \overline{PH}^{-1}\overline{P} \end{pmatrix},$$

which maps $(P\,\mathrm{dom}(H)) \oplus \mathscr{X}$ into $(P\mathscr{X}) \oplus \mathrm{dom}(H)$ is invertible with inverse $\begin{pmatrix} 0 & -P \\ P & H \end{pmatrix}$. Applying Theorem 3.3 we re-obtain the assertion of Corollary 3.2.

(2) In some applications of the Feshbach projection method the fact that P is a projection may cause technical issues [4]. Hence, it is desirable to have the freedom to choose more general operators in $\mathscr{X}$ replacing P and $\overline{P}$. Since no distinguished subspace like $P\mathscr{X}$ might then be available anymore, it is natural to start from the formulation of the Feshbach projection method as a Grushin problem and try to extend the matrices appearing in Part (1) to the full spaces $\mathrm{dom}(H) \oplus \mathscr{X}$ and $\mathscr{X} \oplus \mathrm{dom}(H)$, respectively, before generalizing P.

To do so, let us assume, in addition to the hypotheses of Corollary 3.2, that $H = T + W$, where $T, W : \mathrm{dom}(T) := \mathrm{dom}(H) \to \mathscr{X}$ and T is reduced by $P\mathscr{X}$ in the sense that $TP \subset TP$ and $T\overline{P} \subset \overline{P}T$. Then $PT\overline{P} = \overline{P}TP = 0$ on $\mathrm{dom}(T)$. We further suppose that $\overline{T} := T\restriction_{\overline{P}\,\mathrm{dom}(T)} : \overline{P}\,\mathrm{dom}(T) \to \overline{P}\mathscr{X}$ is bijective. Adding the mutually inverse blocks $\overline{T}$ and $\overline{T}^{-1}$ as direct summands to the two matrices appearing in Part (1), we then observe that

$$\begin{pmatrix} \overline{PT}^{-1}\overline{P} & -P \\ P & H \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} F' & P - PW\overline{PH}^{-1}\overline{P} \\ -P + \overline{H}^{-1}\overline{P}WP & \overline{PH}^{-1}\overline{P} \end{pmatrix} \tag{64}$$

are left and right inverses to each other, where the extended Feshbach operator $F' : \mathrm{dom}(T) \to \mathscr{X}$ is given by

$$F' := T + PWP - PW\overline{PH}^{-1}\overline{P}WP.$$

From Theorem 3.3 we now obtain the assertion of Corollary 3.2 with F replaced by F'.

3.4 The smooth Feshbach method

As anticipated in Example 3.5(2) the smooth Feshbach method has been introduced in [3] as a tool to overcome technical issues caused by the choice of a sharp spectral projection P in the application of the Feshbach projection method in [4]. It is a generalization of Example 3.5(2). In its original formulations [3, 17] the projections P and $\overline{P}$ are replaced by two linear operators χ and $\overline{\chi}$ satisfying $\chi\overline{\chi} = \overline{\chi}\chi$ and $\chi^2 + \overline{\chi}^2 = 1$. In important examples, χ and $\overline{\chi}$ are given by a smooth partition of unity on the spectrum of some self-adjoint operator, which explains the nomenclature "smooth Feshbach method".

We shall consider a slightly generalized setting:

Definition 3.6. Let $\mathscr{X}, \mathscr{Y}$ be real or complex vector spaces, $\mathrm{dom}(T) \subset \mathscr{X}$ and $\mathrm{dom}(S) \subset \mathscr{Y}$ subspaces, and let $H, T : \mathrm{dom}(T) \to \mathscr{X}$, $S : \mathrm{dom}(S) \to \mathscr{Y}$, $\chi_1, \overline{\chi}_1 : \mathscr{X} \to \mathscr{Y}$, and $\chi_2, \overline{\chi}_2 : \mathscr{Y} \to \mathscr{X}$ be linear maps satisfying

$$\chi_2\overline{\chi}_1 = \overline{\chi}_2\chi_1, \quad \chi_2\chi_1 + \overline{\chi}_2\overline{\chi}_1 = \mathbb{1}_{\mathscr{X}}, \tag{65}$$

$$\chi_1\overline{\chi}_2 = \overline{\chi}_1\chi_2, \quad \chi_1\chi_2 + \overline{\chi}_1\overline{\chi}_2 = \mathbb{1}_{\mathscr{Y}}, \tag{66}$$

$$\chi_1 T \subset S\chi_1, \quad \overline{\chi}_1 T \subset S\overline{\chi}_1, \quad \chi_2 S \subset T\chi_2, \quad \overline{\chi}_2 S \subset T\overline{\chi}_2. \tag{67}$$

This includes the assumptions $\chi_1 \,\mathrm{dom}(T) \subset \mathrm{dom}(S)$, $\overline{\chi}_1 \,\mathrm{dom}(T) \subset \mathrm{dom}(S)$ and $\chi_2 \,\mathrm{dom}(S) \subset \mathrm{dom}(T)$, $\overline{\chi}_2 \,\mathrm{dom}(S) \subset \mathrm{dom}(T)$. Assume in addition that $\mathscr{V} \subset \mathscr{X}$, $\mathscr{W} \subset \mathscr{Y}$ are subspaces such that $\mathrm{ran}\,\overline{\chi}_2 \subset \mathscr{V}$, $\mathrm{ran}\,\overline{\chi}_1 \subset \mathscr{W}$, $\chi_1\mathscr{V} \subset \mathscr{W}$, $\chi_2\mathscr{W} \subset \mathscr{V}$, T maps $\mathrm{dom}(T) \cap \mathscr{V}$ into $\mathscr{V}$, and S maps $\mathrm{dom}(S) \cap \mathscr{W}$ into $\mathscr{W}$. Then we abbreviate

$$\overline{T} := T\restriction_{\mathscr{V}} : \mathscr{V} \cap \mathrm{dom}(T) \to \mathscr{V}, \quad \overline{S} := S\restriction_{\mathscr{W}} : \mathrm{dom}(S) \cap \mathscr{W} \to \mathscr{W},$$
$$\overline{H} := (S + \overline{\chi}_1 W \overline{\chi}_2)\restriction_{\mathscr{W}} : \mathscr{W} \cap \mathrm{dom}(S) \to \mathscr{W},$$

with $W := H - T$. *In case $\overline{H}$ is bijective from $\mathscr{W} \cap \mathrm{dom}(S)$ to $\mathscr{W}$*, we further introduce the associated *smooth Feshbach operator*,

$$F := S + \chi_1 W \chi_2 - \chi_1 W \overline{\chi}_2 \overline{H}^{-1} \overline{\chi}_1 W \chi_2.$$

which is well-defined on the domain $\mathrm{dom}(S)$, as well as the operators

$$Q := (\chi_2 - \overline{\chi}_2 \overline{H}^{-1} \overline{\chi}_1 W \chi_2)\restriction_{\mathrm{dom}(S)} : \mathrm{dom}(S) \to \mathrm{dom}(T),$$
$$Q^\sharp := \chi_1 - \chi_1 W \overline{\chi}_2 \overline{H}^{-1} \overline{\chi}_1 : \mathscr{X} \to \mathscr{Y}.$$

In the case $W = 0$ with bijective $\overline{T}$ and $\overline{S}$, analogues of the matrices in (64) are (after some block permutations) now given by

$$\mathfrak{T} := \begin{pmatrix} T & -\chi_2 \\ -\chi_1 & -\overline{\chi}_1 \overline{T}^{-1} \overline{\chi}_2 \end{pmatrix}, \quad \mathfrak{T}^{-1} = \begin{pmatrix} \overline{\chi}_2 \overline{S}^{-1} \overline{\chi}_1 & -\chi_2 \\ -\chi_1 & -S \end{pmatrix}, \tag{68}$$

where $\mathfrak{T} : \mathrm{dom}(T) \oplus \mathscr{Y} \to \mathscr{X} \oplus \mathrm{dom}(S)$. Here the second relation in (68) can easily be verified with the help of (65)–(67) and the definitions of $\overline{T}$ and $\overline{S}$. To see what happens when we add the perturbation W we first prove a lemma:

Lemma 3.7. *In the situation of Definition 3.6, assume that $\overline{S} : \mathrm{dom}(S) \cap \mathscr{W} \to \mathscr{W}$ is bijective. Then $X := \mathbb{1}_{\mathscr{X}} + W\overline{\chi}_2\overline{S}^{-1}\overline{\chi}_1 : \mathscr{X} \to \mathscr{X}$ is bijective, if and only if $\overline{H} : \mathscr{W} \cap \mathrm{dom}(S) \to \mathscr{W}$ is bijective. In the affirmative case*

$$X^{-1} = \mathbb{1}_{\mathscr{X}} - W\overline{\chi}_2\overline{H}^{-1}\overline{\chi}_1. \tag{69}$$

Proof. On the one hand, X is the Schur complement of $\overline{S}$ in

$$N := \begin{pmatrix} \overline{S} & \overline{\chi}_1 \\ -W\overline{\chi}_2 & \mathbb{1}_{\mathscr{X}} \end{pmatrix} : (\mathscr{W} \cap \mathrm{dom}(S)) \oplus \mathscr{X} \longmapsto \mathscr{W} \oplus \mathscr{X}.$$

On the other hand, $\overline{H}$ is the Schur complement of $\mathbb{1}_{\mathscr{X}}$ in N. Since both $\overline{S}$ and $\mathbb{1}_{\mathscr{X}}$ are bijective, Theorem 3.1 thus implies that X is bijective, if and only if N is bijective, and that this is true, if and only if $\overline{H}$ is bijective. In the affirmative case X^{-1} is given by the lower right block of N^{-1}. Computing this block by applying the Schur-Banachiewicz formula to $\mathbb{1}_{\mathscr{X}}$ and its Schur complement $\overline{H}$, we arrive at (69). □

The assertion in the next theorem that M and $\mathcal{E}$ are inverses is equivalent to the validity of eight algebraic relations. In the case $\chi_1 = \chi_2$, $\overline{\chi}_1 = \overline{\chi}_2$, two of them go back to [3], another four appear in [17], while the remaining two seem to be new. The proof of the next theorem differs from the arguments used in [3, 17].

Theorem 3.8 (Smooth Feshbach method)**.** *Let $\rho > 0$. In the situation of Definition 3.6, assume in addition that $\overline{T}$, $\overline{S}$, and $\overline{H}$ are bijective. Then the operator block matrix*

$$M := \begin{pmatrix} H & -\chi_2 \\ -\rho\chi_1 & -\rho\overline{\chi}_1\overline{T}^{-1}\overline{\chi}_2 \end{pmatrix} : \mathrm{dom}(T) \oplus \mathscr{Y} \longrightarrow \mathscr{X} \oplus \mathrm{dom}(S) \tag{70}$$

is bijective, and the block decomposition of its inverse, denoted $\mathcal{E}$, reads

$$\mathcal{E} = \begin{pmatrix} \overline{\chi}_2\overline{H}^{-1}\overline{\chi}_1 & -\rho^{-1}Q \\ -Q^\sharp & -\rho^{-1}F \end{pmatrix} : \mathscr{X} \oplus \mathrm{dom}(S) \longrightarrow \mathrm{dom}(T) \oplus \mathscr{Y}. \tag{71}$$

As a consequence the following statements hold true:

(1) *The maps*

$$\chi_1\restriction_{\ker H}\colon \ker H \longrightarrow \ker F \quad \textit{and} \quad Q\restriction_{\ker F} \longrightarrow \ker H$$

are mutually inverse bijections.

(2) $Q^\sharp$ *maps* $\operatorname{ran} H$ *into* $\operatorname{ran} F$, χ_2 *maps* $\operatorname{ran} F$ *into* $\operatorname{ran} H$, *and the induced maps*

$$\widehat{Q}^\sharp : \operatorname{coker} H \longrightarrow \operatorname{coker} F \quad \textit{and} \quad \widehat{\chi}_2 : \operatorname{coker} F \longrightarrow \operatorname{coker} H$$

are mutually inverse bijections.

(3) $H : \operatorname{dom}(T) \to \mathscr{X}$ *is bijective, if and only if* $F : \operatorname{dom}(S) \to \mathscr{Y}$ *is bijective, and in the affirmative case*

$$F^{-1} = \overline{\chi}_1 \overline{T}^{-1} \overline{\chi}_2 + \chi_1 H^{-1} \chi_2, \quad H^{-1} = \overline{\chi}_2 \overline{H}^{-1} \overline{\chi}_1 + Q F^{-1} Q^\sharp.$$

Proof. If the first assertion holds true, i.e., if (71) is the inverse of (70), then Statements (1)–(3) immediately follow from Theorem 3.3.

To verify that (71) is indeed the inverse of (70) it suffices to treat the case $\rho = 1$. Then we observe that M is the Schur complement of $\mathbb{1}_{\mathscr{X}\oplus\mathscr{X}}$ in

$$\mathfrak{M} := \begin{pmatrix} \mathbb{1}_{\mathscr{X}\oplus\mathscr{X}} & \begin{matrix} -W & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} \mathbb{1}_{\mathscr{X}} & 0 \\ 0 & 0 \end{matrix} & \mathfrak{T} \end{pmatrix},$$

which maps $\mathscr{X} \oplus \mathscr{X} \oplus \operatorname{dom}(T) \oplus \mathscr{Y}$ into $\mathscr{X} \oplus \mathscr{X} \oplus \mathscr{X} \oplus \operatorname{dom}(S)$. In view of (68) the Schur complement of $\mathfrak{T}$ in $\mathfrak{M}$ is

$$Y := \mathbb{1}_{\mathscr{X}\oplus\mathscr{X}} + \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} \mathfrak{T}^{-1} \begin{pmatrix} \mathbb{1}_{\mathscr{X}} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{\mathscr{X}} + W\overline{\chi}_2 \overline{S}^{-1} \overline{\chi}_1 & 0 \\ 0 & \mathbb{1}_{\mathscr{X}} \end{pmatrix}.$$

By Lemma 3.7, Y is bijective with $Y^{-1} = \left(\begin{smallmatrix} X^{-1} & 0 \\ 0 & \mathbb{1}_{\mathscr{X}} \end{smallmatrix}\right)$. Since $\mathfrak{T}$ is bijective, Theorem 3.1 now implies that $\mathfrak{M}$ is bijective. Since $\mathbb{1}_{\mathscr{X}\oplus\mathscr{X}}$ is bijective, we further conclude that M is bijective and that $\mathcal{E} = M^{-1}$ is given by the lower right block of $\mathfrak{M}^{-1}$. Computing the latter block by applying the Schur-Banachiewicz formula to $\mathfrak{T}$ and its Schur complement Y, we find

$$\begin{aligned} \mathcal{E} &= \mathfrak{T}^{-1} + \mathfrak{T}^{-1} \begin{pmatrix} \mathbb{1}_{\mathscr{X}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X^{-1} & 0 \\ 0 & \mathbb{1}_{\mathscr{X}} \end{pmatrix} \begin{pmatrix} -W & 0 \\ 0 & 0 \end{pmatrix} \mathfrak{T}^{-1} \\ &= \mathfrak{T}^{-1} - \begin{pmatrix} \overline{\chi}_2 \overline{S}^{-1} \overline{\chi}_1 X^{-1} W \overline{\chi}_2 \overline{S}^{-1} \overline{\chi}_1 & -\overline{\chi}_2 \overline{S}^{-1} \overline{\chi}_1 X^{-1} W \chi_2 \\ -\chi_1 X^{-1} W \overline{\chi}_2 \overline{S}^{-1} \overline{\chi}_1 & \chi_1 X^{-1} W \chi_2 \end{pmatrix}. \end{aligned} \tag{72}$$

Here we again used the second relation in (68). On account of (69),

$$\begin{aligned} X^{-1}W\overline{\chi}_2\overline{S}^{-1}\overline{\chi}_1 &= W\overline{\chi}_2\overline{S}^{-1}\overline{\chi}_1 - W\overline{\chi}_2\overline{H}^{-1}\overline{\chi}_1 W\overline{\chi}_2\overline{S}^{-1}\overline{\chi}_1 \\ &= W\overline{\chi}_2\overline{S}^{-1}\overline{\chi}_1 - W\overline{\chi}_2\overline{H}^{-1}(\overline{H}-\overline{S})\overline{S}^{-1}\overline{\chi}_1 \\ &= W\overline{\chi}_2\overline{H}^{-1}\overline{\chi}_1, \end{aligned} \tag{73}$$

which further implies

$$\begin{aligned} \overline{\chi}_2\overline{S}^{-1}\overline{\chi}_1 X^{-1}W\overline{\chi}_2\overline{S}^{-1}\overline{\chi}_1 &= \overline{\chi}_2\overline{S}^{-1}\overline{\chi}_1 W\overline{\chi}_2\overline{H}^{-1}\overline{\chi}_1 \\ &= \overline{\chi}_2\overline{S}^{-1}(\overline{H}-\overline{S})\overline{H}^{-1}\overline{\chi}_1 \\ &= \overline{\chi}_2\overline{S}^{-1}\overline{\chi}_1 - \overline{\chi}_2\overline{H}^{-1}\overline{\chi}_1. \end{aligned} \tag{74}$$

Analogously to (73) we finally deduce that

$$\overline{\chi}_2\overline{S}^{-1}\overline{\chi}_1 X^{-1}W = \overline{\chi}_2\overline{H}^{-1}\overline{\chi}_1 W. \tag{75}$$

Inserting (69) into the lower right block of the last matrix in (72) and rewriting its remaining blocks by means of (73)–(75), we arrive at (71). □

3.5 Iterated Grushin problems

As a next step towards the spectral theoretic renormalization group we discuss iterated Grushin problems in this subsection. We learned the next theorem (in the case $D^{(1)} = 0$ and $D^{(2)} = 0$) on a second order Grushin problem from [39, Prop. 3.4]. Combining it with a straightforward induction argument, we shall obtain the subsequent theorem on a successive iteration of Grushin problems. A particularly virtuous application of higher order Grushin problems can be found in [38].

Theorem 3.9. *Let $\mathscr{X}_i, \mathscr{Y}_i$, $i = 1, 2, 3$ be real or complex vector spaces, assume that the block operator matrix*

$$M^{(1)} = \begin{pmatrix} A & B^{(1)} \\ C^{(1)} & D^{(1)} \end{pmatrix} : \mathscr{X}_1 \oplus \mathscr{X}_2 \longrightarrow \mathscr{Y}_1 \oplus \mathscr{Y}_2 \tag{76}$$

is bijective, and denote its inverse as

$$(M^{(1)})^{-1} =: \mathcal{E}_1 =: \begin{pmatrix} E^{(1)} & E_2^{(1)} \\ E_1^{(1)} & E_{12}^{(1)} \end{pmatrix} : \mathscr{Y}_1 \oplus \mathscr{Y}_2 \longrightarrow \mathscr{X}_1 \oplus \mathscr{X}_2. \tag{77}$$

Assume further that

$$M^{(2)} := \begin{pmatrix} -E_{12}^{(1)} & B^{(2)} \\ C^{(2)} & D^{(2)} \end{pmatrix} : \mathscr{Y}_2 \oplus \mathscr{X}_3 \longrightarrow \mathscr{X}_2 \oplus \mathscr{Y}_3$$

is bijective with inverse

$$(M^{(2)})^{-1} =: \mathcal{E}^{(2)} =: \begin{pmatrix} E^{(2)} & E_2^{(2)} \\ E_1^{(2)} & E_{12}^{(2)} \end{pmatrix} : \mathscr{X}_2 \oplus \mathscr{Y}_3 \longrightarrow \mathscr{Y}_2 \oplus \mathscr{X}_3.$$

Then

$$\mathcal{M}_2 := \begin{pmatrix} A & B^{(1)}B^{(2)} \\ C^{(2)}C^{(1)} & D^{(2)} + C^{(2)}D^{(1)}B^{(2)} \end{pmatrix} : \mathscr{X}_1 \oplus \mathscr{X}_3 \longrightarrow \mathscr{Y}_1 \oplus \mathscr{Y}_3 \tag{78}$$

is bijective with inverse

$$\mathcal{E}_2 := \begin{pmatrix} E^{(1)} + E_2^{(1)}E^{(2)}E_1^{(1)} & E_2^{(1)}E_2^{(2)} \\ E_1^{(2)}E_1^{(1)} & E_{12}^{(2)} \end{pmatrix}. \tag{79}$$

Proof. Of course, it would be a pure matter of patience to directly verify by block matrix multiplication that $\mathcal{E}_2$ is a left and right inverse of $\mathcal{M}_2$. It is, however, more convenient to argue as follows: Consider the block operator matrices

$$J_\pm := \begin{pmatrix} \mathbb{1}_{\mathscr{Y}_1} & 0 & 0 \\ \pm E_1^{(1)} & \mathbb{1}_{\mathscr{X}_2} & 0 \\ 0 & 0 & \mathbb{1}_{\mathscr{Y}_3} \end{pmatrix}, \quad P_{\mathscr{Z}} := \begin{pmatrix} 0 & \mathbb{1}_{\mathscr{X}_2} & 0 \\ \mathbb{1}_{\mathscr{Z}_1} & 0 & 0 \\ 0 & 0 & \mathbb{1}_{\mathscr{Z}_3} \end{pmatrix}, \quad \mathscr{Z} \in \{\mathscr{X}, \mathscr{Y}\},$$

and enlarge $M^{(1)}$ and $M^{(2)}$ in a canonical fashion to bijective maps $M^{(1)} \oplus \mathbb{1}_{\mathscr{X}_3} : \mathscr{X}_1 \oplus \mathscr{X}_2 \oplus \mathscr{X}_3 \to \mathscr{Y}_1 \oplus \mathscr{Y}_2 \oplus \mathscr{X}_3$ and $\mathbb{1}_{\mathscr{Y}_1} \oplus M^{(2)} : \mathscr{Y}_1 \oplus \mathscr{Y}_2 \oplus \mathscr{X}_3 \to \mathscr{Y}_1 \oplus \mathscr{X}_2 \oplus \mathscr{Y}_3$ which can be composed. The bijective map J_- has been chosen in such a way that $E_{12}^{(1)}$ is cancelled for in the product

$$\hat{M} := P_{\mathscr{Y}} J_-(\mathbb{1}_{\mathscr{Y}_1} \oplus M^{(2)})(M^{(1)} \oplus \mathbb{1}_{\mathscr{X}_3})P_{\mathscr{X}} = \begin{pmatrix} -\mathbb{1}_{\mathscr{X}_2} & 0 & B^{(2)} \\ B^{(1)} & A & 0 \\ C^{(2)}D^{(1)} & C^{(2)}C^{(1)} & D^{(2)} \end{pmatrix}$$

by virtue of the relations $E_1^{(1)}A + E_{12}^{(1)}C^{(1)} = 0$ and $E^{(1)}B^{(1)} + E_{12}^{(1)}D^{(1)} = \mathbb{1}_{\mathscr{X}_2}$. Obviously, $\hat{M}$ is bijective with inverse $\hat{M}^{-1} = P_{\mathscr{X}}^{-1}(\mathcal{E}_1 \oplus \mathbb{1}_{\mathscr{X}_3})(\mathbb{1}_{\mathscr{Y}_1} \oplus \mathcal{E}^{(2)})J_+P_{\mathscr{Y}}^{-1}$. Since $\mathcal{M}_2$ is the Schur complement of $-\mathbb{1}_{\mathscr{X}_2}$ in $\hat{M}$, we conclude from Theorem 3.1 that $\mathcal{M}_2$ is bijective and that its inverse is given by the lower right two by two block in $\hat{M}^{-1}$, which is given by (79). □

In what follows, every empty product of operators equals the identity operator in the respective vector space by definition; e.g., $\prod_{j=n+1}^{n} X_j := \mathbb{1}$, $\prod_{j=1}^{0} X_j := \mathbb{1}$.

Theorem 3.10. *Let $\mathscr{X}_n, \mathscr{Y}_n$, $n \in \mathbb{N}$, be real or complex vector spaces. Again we consider a bijective block matrix $M^{(1)}$ as in* (76) *with inverse* (77)*. Furthermore, we assume that we are given a sequence of bijective block matrices*

$$M^{(n)} := \begin{pmatrix} -E_{12}^{(n-1)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} : \mathscr{Y}_n \oplus \mathscr{X}_{n+1} \longrightarrow \mathscr{X}_n \oplus \mathscr{Y}_{n+1}, \quad n \in \mathbb{N},\, n \geqslant 2,$$

with inverses

$$(M^{(n)})^{-1} =: \mathcal{E}^{(n)} =: \begin{pmatrix} E^{(n)} & E_2^{(n)} \\ E_1^{(n)} & E_{12}^{(n)} \end{pmatrix} : \mathscr{X}_n \oplus \mathscr{Y}_{n+1} \longrightarrow \mathscr{Y}_n \oplus \mathscr{X}_{n+1}.$$

Then, for any $n \in \mathbb{N}$, the block operator matrix $\mathcal{M}_n : \mathscr{X}_1 \oplus \mathscr{X}_{n+1} \longrightarrow \mathscr{Y}_1 \oplus \mathscr{Y}_{n+1}$ given by

$$\mathcal{M}_n := \begin{pmatrix} A & \prod_{\ell=1}^n B^{(\ell)} \\ \prod_{\ell=1}^n C^{(n+1-\ell)} & \sum_{\ell=1}^n \{\prod_{i=1}^{n-\ell} C^{(n+1-i)}\} D^{(\ell)} \prod_{j=\ell+1}^n B^{(j)} \end{pmatrix}$$

is bijective with inverse

$$\mathcal{E}_n := \begin{pmatrix} \sum_{\ell=1}^n \{\prod_{i=1}^{\ell-1} E_2^{(i)}\} E^{(\ell)} \prod_{j=1}^{\ell-1} E_1^{(\ell-j)} & \prod_{\ell=1}^n E_2^{(\ell)} \\ \prod_{\ell=1}^n E_1^{(n+1-\ell)} & E_{12}^{(n)} \end{pmatrix}. \tag{80}$$

Proof. By assumption the assertion is true for $n = 1$ and, by Theorem 3.9, it is true for $n = 2$.

So assume that the assertion holds true for some $n \in \mathbb{N}$, $n \geqslant 2$. Then we can apply Theorem 3.9 to the matrix $\mathcal{M}_n$ with inverse $\mathcal{E}_n$ given by (80) and the matrix $M^{(n+1)}$ with its inverse $\mathcal{E}^{(n+1)}$. Computing $\mathcal{M}_{n+1}$ and $\mathcal{E}_{n+1}$ according to (78) and (79), we may then verify the assertion for $n + 1$. □

3.6 Iterated smooth Feshbach reductions

Next, we employ Theorem 3.8 and Theorem 3.9 to study a repeated application of the smooth Feshbach projection method. In the two lemmas and the example at the end of this subsection we provide conditions under which conclusions can be drawn from letting the number of iteration steps go to infinity. Essentially, the content of this subsection is part of the general strategy of the spectral renormalization group analysis based on the smooth Feshbach map as introduced in [3]. More precisely, we shall consider the general situation encountered *after* a successful adjustment of the spectral parameter along the iteration steps as described in the succeeding subsection. We added some minor abstractions and variations, however, and tried to isolate model-independent aspects. (For instance, a reduced subspace in which the Feshbach operators are bounded appears only in Lemma 3.12 below (the space $\mathscr{C}$), while earlier works put some emphasis on viewing the renormalization group as a fixpoint problem on a space of bounded operators; see in particular [3, 14].)

It also seems that the simple proof of Lemma 3.11 below has not been noticed before. This lemma reveals that, thanks to their general structure, infinite iterations of smooth Feshbach reductions are amenable to control the degeneracy of eigenvalues. A similar observation has been made in [22], under closely related but slightly different assumptions and by means of a different proof.

Let $\rho \in (0,1)$, $\mathscr{X}$ be a real or complex vector space, $\mathscr{D}, \mathscr{V}, \mathscr{W} \subset \mathscr{X}$ subspaces, and

$$H_n, T_n, S_n : \mathscr{D} \to \mathscr{X}, \quad \chi_{n,1}, \chi_{n,2}, \overline{\chi}_{n,1}, \overline{\chi}_{n,2} : \mathscr{X} \to \mathscr{X}, \tag{81}$$

be linear operators, for every $n \in \mathbb{N}$. Assume that, for every $n \in \mathbb{N}$, the data in (81) together with $\mathscr{V}$ and $\mathscr{W}$ fulfills all hypotheses postulated in Definition 3.6. Define the operators W_n, $\overline{T}_n$, $\overline{H}_n$, F_n, Q_n, and $Q_n^\sharp$ according to Definition 3.6 applied to the data in (81). Finally, assume that $\rho H_{n+1} = F_n$ is the smooth Feshbach operator associated with the data in (81), for every $n \in \mathbb{N}$. Then a combination of Theorem 3.8 (applied for each n) and Theorem 3.10 (with $\mathscr{X}_1 = \mathscr{Y}_n = \mathscr{D}$ and $\mathscr{Y}_1 = \mathscr{X}_n = \mathscr{X}$, $n \geqslant 2$) shows that, for every $n \in \mathbb{N}$, the block operator matrix

$$\mathcal{M}_n := \begin{pmatrix} H_1 & -\chi_{1,2} \cdots \chi_{n,2} \\ -\rho^n \chi_{n,1} \cdots \chi_{1,1} & D_n \end{pmatrix} : \mathscr{D} \oplus \mathscr{X} \longrightarrow \mathscr{X} \oplus \mathscr{D} \tag{82}$$

is bijective with inverse given by

$$\mathcal{E}_n := \begin{pmatrix} E_n & -\rho^{-n} Q_1 \ldots Q_n \\ -Q_n^\sharp \ldots Q_1^\sharp & -H_{n+1} \end{pmatrix} : \mathscr{X} \oplus \mathscr{D} \longrightarrow \mathscr{D} \oplus \mathscr{X}, \tag{83}$$

where

$$D_n := -\rho \overline{\chi}_{n,1} \overline{T}_n^{-1} \overline{\chi}_{n,2} - \sum_{\ell=1}^{n-1} \rho^{n-\ell+1} \chi_{n,1} \cdots \chi_{\ell+1,1} \overline{\chi}_{\ell,1} \overline{T}_\ell^{-1} \overline{\chi}_{\ell,2} \chi_{\ell+1,2} \cdots \chi_{n,2},$$

$$E_n := \sum_{\ell=1}^{n} \rho^{1-\ell} Q_1 \ldots Q_{\ell-1} \overline{\chi}_{\ell,2} \overline{H}_\ell^{-1} \overline{\chi}_{\ell,1} Q_{\ell-1}^\sharp \ldots Q_1^\sharp.$$

We further conclude from Theorem 3.10 and Theorem 3.3 that $H_1 : \mathscr{D} \to \mathscr{X}$ is bijective, if and only if some (and hence all) H_{n+1} are bijective, and that

$$\chi_{n,1} \cdots \chi_{1,1} \restriction_{\ker H_1} : \quad \ker H_1 \longrightarrow \ker H_{n+1}, \tag{84}$$

$$Q_1 \ldots Q_n \restriction_{\ker H_{n+1}} : \ker H_{n+1} \longrightarrow \ker H_1, \tag{85}$$

are mutually inverse bijections, for every $n \in \mathbb{N}$.

Next, we note conditions permitting to draw conclusions in the limit $n \to \infty$. Criteria to check these somewhat implicit conditions are given in the example below.

Lemma 3.11. *In the situation considered in the preceding paragraphs, assume in addition that $\mathscr{X}$ is a Banach space, that H_1 is a closed operator in $\mathscr{X}$, and that $\chi_{n,i}, \overline{\chi}_{n,i} \in \mathscr{B}(\mathscr{X})$, $n \in \mathbb{N}$, $i \in \{1,2\}$, all have norm $\leqslant 1$. Define an operator H_∞ in $\mathscr{X}$ by $\mathrm{dom}(H_\infty) := \{\phi \in \mathscr{D} : \lim_{n\to\infty} H_n\phi \,\text{exists}\}$ and $H_\infty\psi := \lim_{n\to\infty} H_n\psi$, $\psi \in \mathrm{dom}(H_\infty)$. Assume that $D_n \in \mathscr{B}(\mathscr{X})$ are uniformly bounded in $n \in \mathbb{N}$ and that the strong limit $C_\infty := \text{s-}\lim_{n\to\infty}\chi_{n,1}\dots\chi_{1,1}$ exists. Finally, assume that, for every $\psi \in \ker H_\infty$, the limit $Q_\infty\psi := \lim_{n\to\infty} Q_1 \dots Q_n\psi$ exists. Then Q_∞ maps $\ker H_\infty$ injectively into $\ker H_1$ and $C_\infty Q_\infty = \mathbb{1}_{\ker H_\infty}$.*

Proof. Since $\mathcal{E}_n$ in (83) is the inverse of $\mathcal{M}_n$ in (82),

$$H_1Q_1 \dots Q_n = -\rho^n\chi_{1,2}\dots\chi_{n,2}H_{n+1} \quad \text{on } \mathscr{D}.$$

Let $\psi \in \ker H_\infty$. Then the previous relation shows that $H_1Q_1 \dots Q_n\psi \to 0$ (recall $0 < \rho < 1$) and by assumption $Q_1 \dots Q_n\psi \to Q_\infty\psi$, as $n \to \infty$. Since H_1 is closed, this implies $Q_\infty\psi \in \ker H_1$. Using again that $\mathcal{E}_n$ is the inverse of $\mathcal{M}_n$, we further observe that

$$C_\infty Q_\infty\psi = \lim_{n\to\infty} \chi_{n,1}\dots\chi_{1,1}Q_1\dots Q_n\psi = \psi + \lim_{n\to\infty} D_nH_{n+1}\psi = \psi.$$

Here we also employed that $\sup_n \|D_n\| < \infty$ and $H_n\psi \to 0$, $n \to \infty$. □

Lemma 3.12. *In addition to the hypotheses of Lemma 3.11 assume that the closure in $\mathscr{X}$ of $\bigcup_{n\in\mathbb{N}} \mathrm{ran}\,\chi_{n,1}$, call it $\mathscr{C}$, satisfies $\mathscr{C} \subset \mathscr{D}$, that $H_n\restriction_{\mathscr{C}} \in \mathscr{B}(\mathscr{C}, \mathscr{X})$, for all $n \in \mathbb{N}$, and that the sequence $(H_n\restriction_{\mathscr{C}})_{n\in\mathbb{N}}$ has a strong limit in $\mathscr{B}(\mathscr{C}, \mathscr{X})$, denoted $H_\infty\restriction_{\mathscr{C}}$. Furthermore, assume that every Q_n, $n \in \mathbb{N}$, is bounded and that $\sup_n \|Q_1 \dots Q_n\| < \infty$. (Here we always equip $\mathscr{D}$ with the norm on $\mathscr{X}$.) Then C_∞ maps $\ker H_1$ injectively into $\ker H_\infty$ and $Q_\infty C_\infty = \mathbb{1}_{\ker H_1}$.*

By our definitions, $H_\infty\restriction_{\mathscr{C}}$ is indeed the restriction to $\mathscr{C}$ of the operator H_∞ appearing in Lemma 3.11. The relations $C_\infty Q_\infty = \mathbb{1}_{\ker H_\infty}$ and $\mathrm{ran}\, C_\infty \subset \mathscr{C}$ proved in Lemma 3.11 further show that $\ker H_\infty \subset \mathscr{C}$.

Proof. Let $\phi \in \ker H_1$. In view of the uniform boundedness principle applied to $(H_n\restriction_{\mathscr{C}})_{n\in\mathbb{N}}$ and (84) we then obtain

$$H_\infty\restriction_{\mathscr{C}} C_\infty\phi = \lim_{n\to\infty} H_{n+1}\chi_{n,1}\dots\chi_{1,1}\phi = 0.$$

Thus, $C_\infty\phi \in \ker H_\infty$. Let $\varepsilon > 0$. Then, by definition of Q_∞, we find some $n_0 \in \mathbb{N}$ such that $\|Q_\infty C_\infty\phi - Q_1 \dots Q_n C_\infty\phi\| < \varepsilon/2$, for all $n \geqslant n_0$. Let $c > 0$ be an upper bound on the norms $\|Q_1 \dots Q_n\|$, $n \in \mathbb{N}$. Then, by definition of C_∞, we find some $m_0 \in \mathbb{N}$ such that $\|C_\infty\phi - \chi_{m,1} \dots \chi_{1,1}\phi\| < \varepsilon/2c$, for all $m \geqslant m_0$. Hence,

$$\|Q_\infty C_\infty\phi - Q_1 \dots Q_n\chi_{m,1} \dots \chi_{1,1}\phi\| < \varepsilon, \quad n \geqslant n_0,\, m \geqslant m_0.$$

Since the maps in (84) and (85) are inverse to each other, we know, however, that $Q_1 \dots Q_n\chi_{n,1} \dots \chi_{1,1}\phi = \phi$, for all $n \in \mathbb{N}$. Considering indices $n = m \geqslant \max\{n_0, m_0\}$ in the previous estimate we conclude that $\|Q_\infty C_\infty\phi - \phi\| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $Q_\infty C_\infty\phi = \phi$. □

The arguments in Part (2) of the next example appeared in [3]:

Example 3.13. Under the hypotheses of Lemma 3.11 the following holds true:

(1) Assume that $\|\overline{\chi}_{n,1}\overline{T}_n^{-1}\overline{\chi}_{n,2}\| \leqslant c/\rho$, for all $n \in \mathbb{N}$ and some $c \in (0,\infty)$. Then

$$\|D_n\| \leqslant c\sum_{\ell=0}^{n-1}\rho^\ell \leqslant \frac{c}{1-\rho}, \quad n \in \mathbb{N}.$$

(2) Assume that $\|\overline{\chi}_{n,2}\overline{H}_n^{-1}\overline{\chi}_{n,1}W_n\chi_{n,2}\| \leqslant c_n$, $n \in \mathbb{N}$, where the constants $c_n \in [0,\infty)$ are summable, $\Sigma := \sum_{n=1}^\infty c_n < \infty$. Then $\|Q_n\| \leqslant 1 + c_n$ and

$$\|Q_1 \dots Q_n\| \leqslant \prod_{\ell=1}^n (1 + c_\ell) \leqslant \prod_{\ell=1}^n e^{c_\ell} \leqslant e^\Sigma < \infty, \quad n \in \mathbb{N}.$$

Next, assume in addition that $\chi_{n,2}\ker H_\infty = \ker H_\infty$, for every $n \in \mathbb{N}$. Let $\psi \in \ker H_\infty$. We shall show that $Q_\infty\psi := \lim_{n\to\infty} Q_1 \dots Q_n\psi$ exists. In fact, the right hand side of

$$\|Q_1 \dots Q_n(1 - Q_{n+1})\psi\| \leqslant e^\Sigma\|(\chi_{n+1,2} - Q_{n+1})\psi\| \leqslant e^\Sigma c_{n+1}\|\psi\|$$

is summable with respect to $n \in \mathbb{N}$. Therefore, the existence of $Q_\infty\psi$ follows from the Weierstrass test and telescopic summations.

3.7 Spectral renormalization group scheme

Suppose we wish to employ iterated smooth Feshbach reductions as described in the previous subsection to prove that some semi-bounded Hamiltonian H_0 has an eigenvalue at the bottom of its spectrum, $E_0 := \inf \sigma(H_0)$. Then $H_1 = H_0 - E_0$ would be the right choice for the initial operator in the iterated smooth Feshbach analysis. The relation $\ker H_\infty \neq \{0\}$ would then imply the existence of some non-zero vector ψ in the domain of H_0 with $H_0\psi = E_0\psi$. Of course, the precise value of E_0 is usually unknown in applications and its computation or characterization is part of a given problem. However, we might at least have a good idea of where E_0 should approximately be located like, for instance, in a perturbative situation where H_0 is close to an operator whose spectral properties are well understood. Let us thus assume in what follows that, perhaps after an appropriate energy shift, we expect E_0 to be located in the disc $\mathbb{D}_{r\rho} = \{z \in \mathbb{C} : |z| < r\rho\}$, for some $\rho \in (0,1)$ and $r > 0$. Then all we can do is to consider iterated smooth Feshbach reductions involving an additional free spectral parameter $E \in \mathbb{D}_{r\rho}$ and to hope that, along the iteration process, we gather sufficient information to uniquely characterize E_0 in terms of relevant quantities.

Proceeding in the manner just described, we might find the following objects in the n-th iteration step, where $n \in \mathbb{N}$, $\mathscr{X}$ is a vector space, $\mathscr{D} \subset \mathscr{X}$ a subspace, and $\mathcal{O}(\mathbb{D}_{r\rho})$ denotes the set of holomorphic functions on $\mathbb{D}_{r\rho}$:

(a) A set $\mathscr{H}_n$ of linear operators from $\mathscr{D}$ to $\mathscr{X}$ and a class $\mathscr{M}_n$ of maps from $\mathbb{D}_{r\rho}$ into $\mathscr{H}_n$ such that $h[\Theta] = h \circ \Theta \in \mathscr{M}_n$, for all $h \in \mathscr{M}_n$ and $\Theta \in \mathcal{O}(\mathbb{D}_{r\rho})$ with $\Theta(\mathbb{D}_{r\rho}) \subset \mathbb{D}_{r\rho}$.

(b) A map $\mathscr{R}_n : \mathscr{H}_n \times \mathbb{D}_{r\rho} \to \mathscr{H}_n \times \mathbb{C}$ of the form

$$\mathscr{R}_n(H,E) = (R_n(H,E), \lambda_n(H,E)), \quad \lambda_n(H,E) := \rho^{-1}(E - \gamma_n(H,E)),$$

where $\mathbb{D}_{r\rho} \ni E \mapsto R_n(h[E],E)$ is in $\mathscr{M}_{n+1}$ and $\mathbb{D}_{r\rho} \ni E \mapsto \gamma_n(h[E],E)$ is holomorphic, for every $h \in \mathscr{M}_n$.

In particular, we obtain induced maps $\widetilde{R}_n : \mathscr{M}_n \to \mathscr{M}_{n+1}$ and $\tilde{\lambda}_n : \mathscr{M}_n \to \mathcal{O}(\mathbb{D}_{r\rho})$,

$$\widetilde{R}_n(h)[E] := R_n(h[E],E), \quad \tilde{\lambda}_n(h)[E] := \lambda_n(h[E],E), \quad h \in \mathscr{M}_n,\ E \in \mathbb{D}_{r\rho}.$$

For example, we might have $R_n(H,E) + \lambda_n(H,E) = \rho^{-1}F_n(H,E)$, where $F_n(H,E)$ is the smooth Feshbach operator associated with some Grushin problem for $H - E$. In practice, we often find uniform bounds

$$\sup_{E \in \mathbb{D}_{r\rho}} \sup_{H \in \mathscr{H}_n} |\gamma_n(H,E)| \leqslant a_n \in [0,\infty), \quad n \in \mathbb{N}. \tag{86}$$

Obviously, the question arises whether, for a given $H \in \mathscr{H}_1$, there is any $E \in \mathbb{D}_{r\rho}$ such that $(\mathscr{R}_n \circ \cdots \circ \mathscr{R}_1)(H,E)$ is defined for all $n \in \mathbb{N}$. To answer this question it is convenient to exploit the holomorphy of the maps $\mathbb{D}_{r\rho} \ni E \mapsto \gamma_n(h[E],E)$ with $h \in \mathscr{M}_n$. Notice, however, that the map

$E \mapsto R_{n+1}(R_n(h[E], E), \lambda_n(h[E], E))$ does in general not belong to $\mathscr{M}_{n+1}$, because its domain is given by

$$\mathscr{U}_n(h) := \{E \in \mathbb{D}_{r\rho} : \lambda_n(h[E], E) \in \mathbb{D}_{r\rho}\},$$

which might be a proper subset of $\mathbb{D}_{r\rho}$. To remedy this we may use the following result, which is essentially due to [3]:

Lemma 3.14. *In the situation described in the preceding paragraphs, let* $\rho \in (0, 1)$ *and* $r, \epsilon > 0$ *be related in such a way that* $r\epsilon < (r - r_1)^2$ *and* $\epsilon < r_1\rho - r\rho^2$*, for some* $r_1 \in (r\rho, r)$*. Assume that the numbers in* (86) *satisfy* $a_n \leqslant \epsilon$*,* $n \in \mathbb{N}$*. Then* $\mathscr{U}_n(h) \subset \mathbb{D}_{r_1\rho}$ *and* $\tilde{\lambda}_n(h)$ *is biholomorphic from* $\mathscr{U}_n(h)$ *onto* $\mathbb{D}_{r\rho}$*, for all* $n \in \mathbb{N}$ *and* $h \in \mathscr{M}_n$*.*

Proof. The assertion follows from Part (1) of Theorem 3.17 below. □

Under the conditions of the preceding lemma we know in particular that

$$\hat{R}_n(h) := \widetilde{R}_n(h) \circ \tilde{\lambda}_n(h)^{-1} \in \mathscr{M}_{n+1}, \quad h \in \mathscr{M}_n,\ n \in \mathbb{N}.$$

The composition with $\tilde{\lambda}_n(h)^{-1}$ is referred to as a renormalization of the spectral parameter, since the second component of $\mathscr{R}_n(h[\tilde{\lambda}_n(h)^{-1}[E]], \tilde{\lambda}_n(h)^{-1}[E])$ is just E. Moreover, $\hat{R}_n : \mathscr{M}_n \to \mathscr{M}_{n+1}$ is called spectral renormalization group map at step n, at least in more specialized situations arising in mathematical quantum field theory. While the latter objects do not define an actual group, all compositions $\hat{R}_n \circ \cdots \circ \hat{R}_1 : \mathscr{M}_1 \to \mathscr{M}_{n+1}$ are well-defined by construction.

Lemma 3.15. *Under the conditions of Lemma 3.14, let* $H \in \mathscr{H}_1$*,* $E \in \mathbb{C}$*, and* $n \in \mathbb{N}$*. Set* $h_0[z] := H$*,* $z \in \mathbb{D}_{r\rho}$*, and* $h_n := (\hat{R}_n \circ \cdots \circ \hat{R}_1)(h_0)$*, and define* $\lambda_{H,n} \in \mathcal{O}(\mathbb{D}_{r\rho})$ *by* $\lambda_{H,n}(z) := \tilde{\lambda}_n(h_n)[z]$*, for all* $z \in \mathbb{D}_{r\rho}$*. Then* $(H, E) \in \mathrm{dom}(\mathscr{R}_n \circ \cdots \circ \mathscr{R}_1)$*, if and only if* $E \in \mathrm{dom}(\lambda_{H,n} \circ \cdots \circ \lambda_{H,1})$*. In the affirmative case,*

$$(\mathscr{R}_n \circ \cdots \circ \mathscr{R}_1)(H, E) = \Big(h_n(\lambda_{H,n} \circ \cdots \circ \lambda_{H,1}(E)), \lambda_{H,n} \circ \cdots \circ \lambda_{H,1}(E)\Big).$$

Proof. The assertion can be proved by induction and turns out to be a tautological consequence of our definitions, of course. □

The notation introduced in the preceding lemma is more convenient than the one given in (a) and (b) when it comes to proofing the following main result of this subsection, which again is essentially due to [3]:

Theorem 3.16. *Under the conditions of Lemma 3.14, assume in addition that $a_n \to 0$, as $n \to \infty$. Pick some $H \in \mathscr{H}_1$. Then there exists precisely one $E_H \in \mathbb{C}$ such that (H, E_H) is contained in the intersection of the domains of definition of all compositions $\mathscr{R}_n \circ \cdots \circ \mathscr{R}_1$, $n \in \mathbb{N}$. If $E_{H,n} := (\lambda_{H,n} \circ \cdots \circ \lambda_{H,1})(E)$, $n \in \mathbb{N}$, where we use the notation of Lemma 3.15, then*

$$\lim_{n\to\infty} E_{H,n} = 0 \quad \textit{and} \quad E_H = \sum_{n=0}^{\infty} \rho^n \gamma_{H,n+1}(E_{H,n}),$$

with $\gamma_{H,n}(z) := \gamma_n(h_{n-1}[z], z)$, $z \in \mathbb{D}_{r\rho}$, $n \in \mathbb{N}$.

Proof. Take Lems. 3.14 and 3.15 into account and apply Part (2) of Theorem 3.17 below to the functions $\gamma_{H,n} \in \mathcal{O}(\mathbb{D}_{r\rho})$, $n \in \mathbb{N}$. □

The statement of the next theorem and its proof are slight elaborations of corresponding results in [3]:

Theorem 3.17. *Let $\rho \in (0,1)$ and $r, \epsilon > 0$ be related in such a way that $r\epsilon < (r - r_1)^2$ and $\epsilon < r_1\rho - r\rho^2$, for some $r_1 \in (r\rho, r)$. Let $a_n \in [0, \epsilon]$, $n \in \mathbb{N}$. Assume that, for every $n \in \mathbb{N}$, we are given a holomorphic map $\mathbb{D}_{r\rho} \ni z \mapsto \gamma_n(z) \in \mathbb{C}$ with $\sup_{\mathbb{D}_{r\rho}} |\gamma_n| \leqslant a_n$. Define $\lambda_n(z) := \rho^{-1}(z - \gamma_n(z))$, $z \in \mathbb{D}_{r\rho}$. Then the following holds true:*

(1) *Let $n \in \mathbb{N}$ and set $\mathscr{U}_n := \lambda_n^{-1}(\mathbb{D}_{r\rho})$. Then $\mathscr{U}_n \subset \mathbb{D}_{r_1\rho}$ and $\Lambda_n := \lambda_n\restriction_{\mathscr{U}_n}$ is biholomorphic from $\mathscr{U}_n$ onto $\mathbb{D}_{r\rho}$.*

(2) *Assume in addition that $a_n \to 0$, as $n \to \infty$. Then there is precisely one point e_0 contained in the intersection of the domains of all compositions $\lambda_n \circ \cdots \circ \lambda_1$, $n \in \mathbb{N}$. If $e_n := (\lambda_n \circ \cdots \circ \lambda_1)(e_0)$, then*

$$\lim_{n\to\infty} e_n = 0 \quad \textit{and} \quad e_n = \sum_{\ell=0}^{\infty} \rho^\ell \gamma_{n+\ell+1}(e_{n+\ell}), \quad n \in \mathbb{N}_0. \tag{87}$$

Proof. Step 1. To prove Part (1), we employ the Cauchy estimate

$$|\gamma_n'(z)| \leqslant \inf_{r_2 \in (r_1, r)} \frac{r_2}{(r_2 - r_1)^2} \max_{|\zeta| = r_2} |\gamma_n(\zeta)| \leqslant \frac{a_n r}{(r - r_1)^2} =: b_n,\ z \in \overline{\mathbb{D}}_{r_1\rho}. \tag{88}$$

Here $b_n < 1$, whence λ_n is locally biholomorphic from $\mathbb{D}_{r_1\rho}$ onto the domain $\lambda_n(\mathbb{D}_{r_1\rho})$. In fact, $\rho|\lambda_n'| \geqslant 1 - b_n > 0$ on $\overline{\mathbb{D}}_{r_1\rho}$. Let $z_0 \in \mathbb{D}_{r\rho}$ and put $g(z) := z - \rho z_0 - \rho\lambda_n(z) = \gamma_n(z) - \rho z_0$, $z \in \mathbb{D}_{r\rho}$. Pick $r_0 \in (0, r_1)$ such that $\epsilon \leqslant r_0\rho - r\rho^2$. Then $g(\overline{\mathbb{D}}_{r_0\rho}) \subset \overline{\mathbb{D}}_{r_0\rho}$ since $|g| \leqslant \epsilon + r\rho^2 \leqslant r_0\rho$. On account of (88), g thus is a contraction on $\overline{\mathbb{D}}_{r_0\rho}$ and, by the Banach fixpoint theorem, it has a unique fixpoint $z_* \in \overline{\mathbb{D}}_{r_0\rho}$. Since $r_0 < r_1$ can be chosen arbitrarily close to r_1, we see that z_* is actually the only fixpoint of g in $\mathbb{D}_{r_1\rho}$. Since any $z' \in \mathbb{D}_{r_1\rho}$ is a fixpoint of g, if and only if $\lambda_n(z') = z_0$, this shows that λ_n maps $\mathbb{D}_{r_1\rho} \cap \lambda_n^{-1}(\mathbb{D}_{r\rho})$ biholomorphically onto $\mathbb{D}_{r\rho}$. Since $\epsilon/\rho < r_1 - r\rho$, the membership $z \in \mathbb{D}_{r\rho} \setminus \mathbb{D}_{r_1\rho}$ entails, however, $|\lambda_n(z)| \geqslant (|z| - \epsilon)/\rho \geqslant r\rho$, i.e., $\mathscr{U}_n = \mathbb{D}_{r_1\rho} \cap \lambda_n^{-1}(\mathbb{D}_{r\rho})$.

Step 2. According to Part (1), each Λ_n^{-1} maps $\overline{\mathbb{D}}_{r_1\rho}$ continuously into $\mathscr{U}_n \subset \mathbb{D}_{r_1\rho}$. Consequently, the sets $K_n := (\Lambda_1^{-1} \circ \cdots \circ \Lambda_n^{-1})(\overline{\mathbb{D}}_{r_1\rho})$ are compact, non-empty, and satisfy $K_{n+1} \subset K_n$, for all $n \in \mathbb{N}_0$. Therefore, $K_\infty := \bigcap_{n=1}^\infty K_n \neq \emptyset$. Let us observe for later reference that Step 1 also implies $K_n = (\lambda_1^{-1} \circ \cdots \circ \lambda_n^{-1})(\overline{\mathbb{D}}_{r_1\rho})$, for all $n \in \mathbb{N}$.

Step 3. We fix some $e_0 \in K_\infty$ and assume that $a_n \to 0$, $n \to \infty$, in the rest of this proof. We set $e_n := (\lambda_n \circ \cdots \circ \lambda_1)(e_0) \in \overline{\mathbb{D}}_{r_1\rho}$, $n \in \mathbb{N}$. Then $\rho e_{n+1} = e_n - \gamma_{n+1}(e_n)$, $n \in \mathbb{N}$, which implies

$$e_n = \rho^m e_{n+m} + \sum_{\ell=0}^{m-1} \rho^\ell \gamma_{n+\ell+1}(e_{n+\ell}), \quad m \in \mathbb{N},\, n \in \mathbb{N}_0.$$

Since $|e_n| \leqslant r_1\rho^{m+1} + \sup_{\ell>n} a_\ell/(1-\rho)$, this proves the relations in (87).

Step 4. Next, we show that $K_\infty = \{e_0\}$.

Suppose for contradiction that there exists $e_0' \in K_\infty \setminus \{e_0\}$ and set $e_n' := (\lambda_n \circ \cdots \circ \lambda_1)(e_0') \in \overline{\mathbb{D}}_{r_1\rho}$, $n \in \mathbb{N}$. In view of (88), $|(\lambda_n^{-1})'| \leqslant \rho/(1-b_n)$ on $\lambda_n(\overline{\mathbb{D}}_{r_1\rho})$, for all $n \in \mathbb{N}$. Since the line segment connecting e_n and e_n' lies in $\overline{\mathbb{D}}_{r_1\rho}$ and $\overline{\mathbb{D}}_{r_1\rho} \subset \mathbb{D}_{r\rho} = \lambda_n(\mathscr{U}_n) \subset \lambda_n(\mathbb{D}_{r_1\rho})$, we obtain

$$|e_{n-1} - e_{n-1}'| = |\lambda_n^{-1}(e_n) - \lambda_n^{-1}(e_n')| \leqslant \frac{\rho}{1-b_n}|e_n - e_n'|, \quad n \in \mathbb{N}.$$

Since $b_n \to 0$, we further find $\rho_1 \in (\rho, 1)$ and $n_0 \in \mathbb{N}$ such that $\rho/(1-b_n) \leqslant \rho_1$, for all $n \geqslant n_0$. Therefore, there exists a constant $C > 0$ such that $|e_0 - e_0'| \leqslant C\rho_1^{n-n_0}|e_n - e_n'| \leqslant r_1\rho C\rho_1^{n-n_0} \to 0$, $n \to \infty$; a contradiction!

Step 5. Finally, let e_0'' be any point in the domain of definition of all compositions $\lambda_n \circ \cdots \circ \lambda_1$, $n \in \mathbb{N}$, and set $e_n'' := (\lambda_n \circ \cdots \circ \lambda_1)(e_0'')$, $n \in \mathbb{N}$. By Step 1, $e_{n-1}'' \in \lambda_n^{-1}(\mathbb{D}_{r\rho}) = \mathscr{U}_n \subset \overline{\mathbb{D}}_{r_1\rho}$, for all $n \in \mathbb{N}$. The last remark of Step 2 now implies $e_0'' \in K_n$, for all $n \in \mathbb{N}$, thus $e_0'' = e_0$ by Step 4. $\square$

References

[1] E. R. Akchurin, *Spectral properties of the generalized Friedrichs model*, Theor. Math. Phys. **163** (2010), 414–428.

[2] N. Angelescu, R. A. Minlos and V. A. Zagrebnov, *Lower spectral branches of a particle coupled to a Bose field*, Rev. Math. Phys. **17** (2005), 1111–1142.

[3] V. Bach, T. Chen, J. Fröhlich and I. M. Sigal, *Smooth Feshbach map and operator-theoretic renormalization group methods*, J. Funct. Anal. **203** (2003), 44–92.

[4] V. Bach, J. Fröhlich and I. M. Sigal, *Renormalization group analysis of spectral problems in quantum field theory*, Adv. Math. **137** (1998), 205–298.

[5] M. S. Birman, *On the spectrum of singular boundary-value problems*, Mat. Sb. (N.S.) **55(97)** (1961), 125–174.

[6] M. S. Birman, *The spectrum of singular boundary problems*, Amer. Math. Soc. Transl. Ser. 2 **53** (1966), 23–80.

[7] H. Feshbach, *Unified theory of nuclear reactions*, Ann. Phys. **5** (1958), 357–390.

[8] R. L. Frank, A. Laptev and O. Safronov, *On the number of eigenvalues of Schrödinger operators with complex potentials*, J. London Math. Soc. **94** (2016), 377–390.

[9] R. L. Frank and B. Schlein, *Dynamics of a strongly coupled polaron*, Lett. Math. Phys. **104** (2014), 911–929.

[10] K. O. Friedrichs, *Über die spektralzerlegung eines integraloperators*, Math. Ann. **115** (1938), 249–272.

[11] K. O. Friedrichs, *On the perturbation of continuous spectra*, Comm. Pure Appl. Math. **1** (1948), 361–406.

[12] H. Fröhlich, *Electrons in lattice fields*, Adv. Phys. **3** (1954), 325–361.

[13] J. Fröhlich, *Existence of dressed one electron states in a class of persistent models*, Fortschr. Phys. **22** (1974), 159–198.

[14] J. Fröhlich, M. Griesemer and I. M. Sigal, *On spectral renormalization group*, Rev. Math. Phys. **21** (2009), 511–548.

[15] B. Gerlach and H. Löwen, *Analytical properties of polaron systems or: Do polaronic phase transitions exist or not?*, Rev. Mod. Phys. **63** (1991), 63–90.

[16] F. Gesztesy, H. Holden and R. Nichols, *On factorizations of analytic operator-valued functions and eigenvalue multiplicity questions*, Integr. Equ. Oper. Theory **82** (2015), 61–94.

[17] M. Griesemer and D. Hasler, *On the smooth Feshbach-Schur map*, J. Funct. Anal. **254** (2008), 2329–2335.

[18] M. Griesemer and A. Wünsch, *Self-adjointness and domain of the Fröhlich Hamiltonian*, J. Math. Phys. **57** (2016), 021902.

[19] E. P. Gross, *Particle-like solutions in field theory*, Ann. Phys. **19** (1962), 219–233.

[20] D. Hasler and I. Herbst, *Ground states in the spin boson model*, Ann. Henri Poincaré **12** (2011), 621–677.

[21] D. Hasler and I. Herbst, *Smoothness and analyticity of perturbation expansions in qed*, Adv. Math. **228** (2011), 3249 – 3299.

[22] D. Hasler and I. Herbst, *Uniqueness of the ground state in the Feshbach renormalization analysis*, Lett. Math. Phys. **100** (2012), 171–180.

[23] E. V. Haynsworth, *Determination of the inertia of a partitioned Hermitian matrix*, Linear Algebra Appl. **1** (1968), 73–81.

[24] I. A. Ikromov and F. Sharipov, *On the discrete spectrum of the non-analytic matrix-valued Friedrichs model*, Funct. Anal. Appl. **32** (1998), 49–51.

[25] S. N. Lakaev, *Some spectral properties of the generalized Friedrichs model*, J. Sov. Math. **45** (1989), 1540–1563.

[26] T. D. Lee, F. E. Low and D. Pines, *The motion of slow electrons in a polar crystal*, Phys. Rev. **90** (1953), 297–302.

[27] E. H. Lieb and L. E. Thomas, *Exact ground state energy of the strong-coupling polaron*, Comm. Math. Phys. **183** (1997), 511–519.

[28] R. A. Minlos, *Lower branch of the spectrum of a fermion interacting with a bosonic gas (polaron)*, Theor. Math. Phys. **92** (1992), 869–878.

[29] T. Miyao, *Polaron with at most one phonon in the weak coupling limit*, Monatsh. Math. **157** (2009), 365–378.

[30] T. Miyao and H. Spohn, *The bipolaron in the strong coupling limit*, Ann. Henri Poincaré **8** (2007), 1333–1370.

[31] J. S. Møller, *The polaron revisited*, Rev. Math. Phys. **18** (2006), 485–517.

[32] E. Nelson, *Interaction of nonrelativistic particles with a quantized scalar field*, J. Math. Phys. **5** (1964), 1190–1197.

[33] F. M. Peeters, W. Xiaoguang and J. T. Devreese, *Ground-state energy of a polaron in n dimensions*, Phys. Rev. B **33** (1986), 3926–3934.

[34] S. Puntanen and G. P. H. Styan, *Historical Introduction: Issai Schur and the Early Development of the Schur Complement*, Springer US, Boston, MA (2005), 1–16.

[35] A. Pushnitski, *The Birman–Schwinger principle on the essential spectrum*, J. Funct. Anal. **261** (2011), 2053–2081.

[36] I. Schur, *Über potenzreihen, die im innern des einheitskreises beschränkt sind.*, J. Reine Angew. Math. **147** (1917), 205–232.

[37] J. Schwinger, *On the bound states of a given potential*, Proc. Natl. Acad. Sci. U.S.A. **47** (1961), 122–129.

[38] J. Sjöstrand, *Complete asymptotics for correlations of Laplace integrals in the semi-classical limit*, Mém. Soc. Math. France (N.S.) **83** (2000), 1–104.

[39] J. Sjöstrand and M. Zworski, *Elementary linear algebra for advanced spectral problems*, Ann. Inst. Fourier **57** (2007), 2095–2141.

[40] H. Spohn, *The polaron at large total momentum*, J. Phys. A: Math. Gen. **21** (1988), 1199–1211.