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PhD dissertation

Geometric quantisation, the Hitchin-Witten connection and quantum operators in complex Chern-Simons theory

Candidate:
Alessandro Malusà

Advisor:
Prof. Jørgen Ellegaard Andersen

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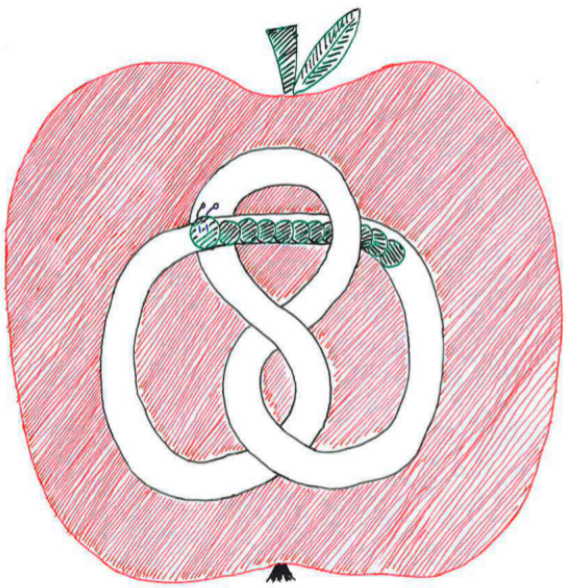

Alessandro Malusà

Dedicated to my parents
and grandparents.

Dedicato ai miei nonni,
alla mia mamma e al mio papà.

Abstract

The present thesis is the result of my three-year PhD studies at the Centre for Quantum Geometry of Moduli Spaces under the supervision of Jørgen Ellegaard Andersen. The main focus is on Chern-Simons theory for the complex group $\mathrm{SL}(n, \mathbb{C})$, as approached via geometric quantisation following Witten's original proposal [Wit91]. The corresponding picture for the compact group $\mathrm{SU}(n)$ was first proposed by Hitchin in [Hit90] and Axelrod, Della Pietra and Witten in [ADPW91], and it was widely studied and understood over the years. This viewpoint is based on a procedure which depends on an extrinsic parameter and, rather than a single Hilbert space, it results in a whole bundle of quantisations over the space of parameters. The different fibres of the bundle are related by the holonomy of a connection, known as the Hitchin connection, which gives isomorphisms between the fibres up to projective factors. From this one can obtain projective representations of the mapping class groups of surfaces. There are other sources of such representations, as proposed by Witten via path integral methods in [Wit89] later formalised rigorously by Reshetikhin and Turaev [RT90, RT91] into the Witten-Reshetikhin-Turaev TQFT. The projective representations coming from this viewpoint were later shown to be equivalent to those from the context of the Hitchin connection [AU07b, AU07a, AU12, AU15, Las98], also involving conformal field theories. One famous result in the field is Andersen's proof of the asymptotic faithfulness of these representations [And06], which employs the tools provided by Toeplitz deformation quantisation. In turn, this scheme is based on the asymptotic properties of certain quantum operators [Sch96, Sch00] and their relations with the Hitchin connection [And12].

This thesis fits in the analogous framework for $\mathrm{SL}(n, \mathbb{C})$ proposed by Witten in [Wit91] and later continued by Andersen and Gammelgaard in [AG14]. This approach defines a bundle of quantisations as in the case of $\mathrm{SU}(n)$, together with a projectively flat connection, the Hitchin-Witten connection. The backbone of the present work is the problem of the quantisation of observables in this setting. Following the chronological development of the research material presented here, the first problem we will address is that of studying the asymptotic properties of certain simple

operators. This will lead to the definition of a formal Hitchin-Witten connection, analogous to the one obtained for $\mathrm{SU}(n)$ in [And12]. Related to this are the problems of finding a formal trivialisation and a deformation quantisation analogous to the one obtained vie Toeplitz operators. After setting these in the general framework and finding some partial results, we specialise to the case of a surface of genus 1, in which some significant technical simplifications give access to more complete solutions. This will lead to an explicit quantisation of certain operators, analogous to those involved in Gukov and Garoufalidis's AJ conjecture [Gar04, Guk05] for the coloured Jones polynomial. We conclude this thesis illustrating a joint paper with Andersen [AM17], in which we use said operators to extend the conjecture to the invariants produced by the Teichmüller TQFT by Andersen and Kashaev [AK14b, AM16b]. This claim relies on an explicit trivialisation of the Hitchin-Witten connection for this case, together with an isomorphism (specific to this particular case) between the quantum Hilbert spaces arising from the two different viewpoints. The conjecture is also checked explicitly for the first two hyperbolic knots.

Dansk Résumé

Den foreliggende afhandling er resultatet af mit tre-årige PhD studie ved Center for Kvante Geometri af Moduli Rum under vejledning af Jørgen Ellegaard Andersen. Hovedfokus er på Chern-Simons teori med kompleks gauge gruppe $\mathrm{SL}(n,\mathbb{C})$, og tilgangen vil være baseret på geometrisk kvantisering jævnfør Witten's oprindelige forslag [Wit91]. Det tilsvarende motiv med kompakt gauge gruppe $\mathrm{SU}(n)$ blev først foreslået af Hitchin i [Hit90] og Axelrod, Della Pietra og Witten i [ADPW91], og har sidenhen været genstand for mange studier igennem årene, hvorigennem vores forståelse for denne situation er blevet øget betragteligt. Tilgangen til geometrisk kvantisering af Chern-Simons teori med kompakt gauge gruppe har været baseret på en procedure, som afhænger af en ekstrinsisk parameter, og resulterer ikke i et enkelt Hilbert rum, men derimod et vektorbundt hvis fibre svarer til kvantisering relativt til et valg af denne parameter. Dette vektorbundt kaldes Verlinde bundtet. De forskellige Hilbert rum opnået på denne vis er isomorfe igennem isomorfier opnået ved parallel transport langs en konnektion i dette bundt, kendt som Hitchin konnektionen, og disse isomorfier er kompatible op til projektive faktorer. Afbildningsklassegruppen for en flade virker på Verlinde bundtet, og idet Hitchin konnektionen bevarer denne virkning opnår man en projektiv repræsentation, kaldet kvanterepræsentationerne. Der er andre kilder til disse repræsentationer, som blev foreslået af Witten i [Wit89] igennem brug af Feynman felt-integraler, og senere formaliseret på matematisk stringent vis af Reshetikhin og Turaev til den berømte

Witten-Reshetikhin-Turaev TQFT [RT90, RT91]. De projektive repræsentationer opnået på denne vis, er ækvivalent til kvanterepræsentationerne opnået igennem Hitchin-konnektionen, hvilket blev etablerede igennem en række arbejder [AU07b, AU07a, AU12, AU15, Las98], som også involverer konform feltteori. Vi vil i denne sammenhæng fremhæve et berømt resultat af Andersen, som viser, at kvanterepræsentationerne er asymptotisk tro [And06], hvilket bevises igennem brug af Toeplitz deformations kvantisering. Dette er baseret på de asymptotiske egenskaber af særlige kvante-operatorer [Sch96, Sch00], og deres relationer til Hitchin-konnektionen [And12]

Denne afhandling tager udgangspunkt i den analoge situation, hvor vi arbejder med kompleks gauge gruppe $\mathrm{SL}(n, \mathbb{C})$, som foreslået af Witten i [Wit91] og videreudviklet af Andersen og Gammelgaard i [AG14]. Denne tilgang definerer igen et vektorbundt af forskellige kvantiseringer som i $\mathrm{SU}(n)$-tilfældet, sammen med en projektiv flad konnektion kaldet Hitchin-Witten konnektionen. Fundamentet for det foreliggende arbejde er problemet vedrørende kvantisering af observable i denne sammenhæng. Jævnfør den kronologiske fremgangsmåde i præsentationen af det relevante forskningsmateriale her gennemgået, vil det første problem vi adresserer være studiet af de asymptotiske egenskaber af udvalgte simple observable. Dette vil lede til definitionen af en formel Hitchin-Witten konnektion, analogt til $\mathrm{SU}(n)$-situationen som i [And12]. Relateret til dette er problemet vedrørende eksistensen af en formel trivialisering samt en deformerings kvantisering analog til den opnået igennem Toeplitz operatorer. Efter at have sat disse problemer ind i en generel ramme og givet en opremsning af visse partielle resultater, vil vi koncentrere os om tilfældet, hvor vi har at gøre med en flade af genus 1. I denne situation vil visse tekniske problemer forsvinde, hvilket giver anledning til nogle langt mere komplette løsninger. Dette vil lede til en eksplicit kvantisering af visse operatorer, analogt til dem involveret i Gukov og Garoufaldis's AJ formodning [Gar04] for det farvede Jones polynomium. Afhandlingen konkluderes med en gennemgang af en fælles artikel med Andersen [AM17], i hvilken vi bruger ovennævnte operatorer til at udvide AJ formodningen til invarianterne associeret til Teichmüller TQFT'erne som blev udviklet af Andersen og Kashaev [AK14b, AM16b]. Denne udvidelse er baseret på en eksplicit trivialisering af Hitchin-Witten konnektionen i denne sammenhæng, sammen med en isomorfi imellem de relevante kvante Hilbert rum som opstår igennem de to forskellige perspektiver - geometrisk kvantisering og Teichmüller teori. Den udvidede AJ formodning verificeres for de første to hyperbolske knuder.

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This thesis is the culmination of three terrific years of work at the Centre for Quantum Geometry of Moduli Spaces, and not only. For this I want to thank my supervisor, Jørgen Ellegaard Andersen, who guided and directed me through many problems and difficulties. This work would have been impossible without his guidance and advice, especially at those times in which, as is unavoidable during a PhD, I was stuck. It was him who gave me this amazing possibility, and the resources to make this an incredible experience, and for this I am deeply grateful.

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Contents

Introduction

The central theme of this thesis is quantum Chern-Simons theory for the complex group $\mathrm{SL}(n,\mathbb{C})$, which we study aiming to extend to this case some of the known results for $\mathrm{SU}(n)$. The theory for this compact group has been widely studied and understood over the last three decades, starting with Witten's famous paper [Wit89]. The previous year, Atiyah had raised the question of how to interpret the coloured Jones polynomial in terms of intrinsic 3-dimensional topology, as opposed to its original definition via planar diagrams. In his paper, Witten argues that the invariant enjoys a list of properties predicted by means of path-integration techniques for the quantum Chern-Simons theory. These ideas were formalised by Reshetikhin and Turaev [RT90, RT91, Tur10] from the rigorous viewpoint of axiomatic TQFTs, resulting among other things in a family of projective representations of the mapping class groups of surfaces parametrised by an integer k, called the level of the theory. This picture, called the Witten-Reshetikhin-Turaev TQFT, is based on the representation theory of the quantum group $U_q(\mathfrak{sl}(2,\mathbb{C}))$ at a root of unity q. A similar approach was later proposed in terms of skein theory in [BHMV92, BHMV95, Bla00].

The theory has been formulated in other different ways through the years; our main focus will be on the approach via geometric quantisation as considered by Hitchin in [Hit90] and Axelrod, Della Pietra and Witten in [ADPW91]. Given a closed, oriented, smooth surface Σ, one may consider its moduli space of flat connections, which is naturally endowed with part of the structure needed for running this procedure. The missing data may be obtained by introducing an auxiliary complex structure on Σ; the procedure results then in a bundle of quantum Hilbert spaces over the Teichmüller space, called the Verlinde bundle. It is proposed in the cited works that one identifies these Hilbert spaces, up to projective factors, by means of a family of isomorphisms arising as the parallel transport operators of a projectively flat connection, called the Hitchin connection.

As well as the combinatorial approach, the picture of geometric quantisation defines a family of projective representations of the mapping class groups of surfaces, labelled by the quantum parameter $k \in \mathbb{Z}_{>0}$. This family is in fact equivalent to the one coming from the Witten-Reshetikhin-Turaev TQFT, as follows from the results of a series of papers of Andersen and Ueno [AU07b, AU07a, AU12, AU15] in combination with [Las98]. The result of these works is a chain of isomorphisms connecting the two viewpoints, passing through the conformal field theory approach [TUY89, BK01]. An important property of these representations is their asymptotic faithfulness, proven by Andersen in [And06] confirming a conjecture of Turaev. The proof uses the deformation quantisation obtained by Bordeman, Karabegov, Meinrenken and Schlichenmaier [BMS94, Sch98, Sch00, KS01] via the asymptotic properties of Toeplitz operators. As in geometric quantisation, the construction gives a bundle of deformation quantisations, on which Andersen defines a formal analogue of the Hitchin connection [And12], obtained again via asymptotic arguments.

Related to the quantum $SU(2)$-Chern-Simons theory is Garoufalidis's AJ conjecture [Gar04]. Briefly, the exterior of a knot inside a 3-manifold can be seen as a bordism of a topological surface of genus 1, to which corresponds a Lagrangian inside the $SU(2)$-character variety of the surface. This defines a knot invariant, the A-polynomial, as the equation cutting out said Lagrangian, which can be seen in Chern-Simons theory as a constraint equation. It is an interesting problem to define a quantisation of this invariant, which should be an operator annihilating the partition function of the theory. With this motivation, Garoufalidis defined an $\widehat{A}$ as a generator of the ideal annihilating the coloured Jones polynomial in an appropriate algebra, and conjectured that this object reproduces A in the relevant limit. The conjecture has been proven for certain classes of knots [GS10, LT15, GL16], but is still open in general.

The quantisation of Chern-Simons theory for the non-compact group $SL(n, \mathbb{C})$ is considered by Witten in [Wit91] using techniques analogous to those employed in the geometric quantisation version of the theory for $SU(n)$. He considers the moduli space of flat $SL(2, \mathbb{C})$-connections on a surface and runs the quantisation process by using the extrinsic data of a complex structure on it. This results again in a bundle of quantum Hilbert spaces over the Teichmüller space, on which he introduces the projectively flat Hitchin-Witten connection. This study is further extended in [AG14].

Another problem in quantum Chern-Simons with non-compact group is that of defining a partition function, addressed in several works in terms of formal, perturbative methods, for example [Hik01, Hik07, Dim13, DGLZ09, DFM11]. Other approaches use combinatorial techniques by considering triangulations on 3-manifolds. This is the case in the theory of quantum hyperbolic invariants of Baseilhac and Benedetti, see e.g. [BB04, BB07], and the infinite-rank Teichmüller TQFT proposed by Andersen and Kashaev [AK14a, AK14b, AM16b]. All of these approaches involve various adaptations and extensions of Faddeev's quantum dilogarithm.

It is an open and extremely interesting problem to understand the interplay between the various incarnations of the theory. As of now, there is no clear connection between the combinatorial pictures and the viewpoint of geometric quantisation. A partial result in this direction is given by the Weil-Gel'fand-Zak transform, relating the quantum Hilbert space obtained by Witten to the result of the Teichmüller TQFT in genus 1.

Contents of this thesis

The central theme of the research material presented in this thesis is the quantum $\mathrm{SL}(n, \mathbb{C})$-Chern-Simons theory, particularly with the approach of geometric quantisation as in [Wit91]. The aim of most of this work is to extend to this situation known results and ideas from the $\mathrm{SU}(n)$-theory, with a particular attention for the case of a surface of genus 1. This is because several simplifications occur in this specific case, including for instance the existence of natural and manageable coordinates on the moduli spaces and the flatness of the Hitchin-Witten connection. In addition, the torus has a special interest in knot theory, being the boundary of tubular neighbourhoods of knots and hence of their exterior.

We start by considering a problem analogous to those addressed in the context of the formal Hitchin connection [And12] and Toeplitz deformation quantisation [Sch00]. The focal points of that settings are certain asymptotic expansions of the so-called Toeplitz operators and their Hitchin covariant derivative. In the case of $\mathrm{SL}(n, \mathbb{C})$ there is no close analogue of the Toeplitz operators, but it still make sense to pose similar questions about asymptotic expansions for another class of operators.

Let $t = k + is$ be a complex parameter, called the level of the theory, and suppose that k is a positive integer. Call $\mathcal{M}$ the moduli space of flat $\mathrm{SU}(n)$-connections over a closed, oriented and connected smooth surface Σ. The quantum Hilbert space is identified with $\mathrm{L}^2(\mathcal{M}, \mathscr{L}^k)$, the space of square-summable sections of the k-th tensor power of an appropriate complex line bundle. However, the quantisation process, and therefore the identification, depends on a parameter σ in the Teichmüller space $\mathcal{T}$. This dependence is measured by the Hitchin-Witten connection $\tilde{\boldsymbol{\nabla}}$ on $\mathrm{L}^2(\mathcal{M}, \mathscr{L}^k) \times \mathcal{T}$, which takes the form

$$\tilde{\boldsymbol{\nabla}} = \nabla^{\mathrm{Tr}} + \frac{1}{2t} b - \frac{1}{2\bar{t}} \bar{b} + \mathrm{d}_{\mathcal{T}} F$$

for appropriate operator-valued 1-forms b, $\bar{b}$ and $\mathrm{d}_{\mathcal{T}} F$ on $\mathcal{T}$. For a fixed value of k, we consider the algebra $\mathscr{A}_k = \mathrm{D}_k[[s^{-1}]]$ of formal power series in s^{-1} with finite-order differential operators on $\mathscr{L}^k$ as coefficients. By expanding $1/t$ and $1/\bar{t}$ as power series in s we find our first result.

Theorem 1. *There exists a unique formal Hitchin-Witten connection*

$$\tilde{\mathcal{D}} = \nabla^{Tr} + \sum_{l=0}^{\infty} \tilde{\mathcal{D}}^{(l)}$$

characterised by the property that, for every vector V on $\mathcal{T}$, differential operator $D \in \mathrm{D}_k$ and positive integer L, one has

$$\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_V D - V[D] - \sum_{l=0}^{L} \tilde{\mathcal{D}}^{(l)}_V D = o\left(s^{-L}\right) \quad \textit{for } s \to \infty,$$

where $\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}$ is the connection induced by $\tilde{\boldsymbol{\nabla}}$ on the endomorphism bundle, and the convergence holds in a sense specified in Section 3.2.

Next we address the problem of finding a trivialisation for this formal connection, i.e. a map sending every differential operator $D \in \mathrm{D}_k$ to a formal sum

$$\mathcal{R}(D) = \sum_{l=0}^{\infty} \mathcal{R}^{(l)}(D)$$

such that $\mathcal{R}^{(0)}(D) = D$ and $\tilde{\mathcal{D}}\mathcal{R}(D) = 0$. Written explicitly, this condition boils down to the recursive relation

$$\mathrm{d}^F_{\mathcal{T}} \mathcal{R}^{(l)}(D) = \frac{1}{2k} \sum_{n=1}^{l} (ik)^n \left[b - (-1)^n \bar{b}, \mathcal{R}^{(l-n)}(D)\right], \tag{1}$$

for every non-negative integer l, where $\mathrm{d}^F_{\mathcal{T}}$ is a suitably defined twisted exterior differential. It is apparent from the equation that an obstruction to the existence of a solution comes in general from the differential of the right-hand side. However, we prove the following statement.

Theorem 2. *Suppose that $D = \mathcal{R}^{(0)}(D), \dots, \mathcal{R}^{(l-1)}(D)$ are given, and that they satisfy the first l steps of the recursion. Then the right-hand side in (1) is closed:*

$$\mathrm{d}^F_{\mathcal{T}} \sum_{n=1}^{l} (ik)^n \left[b - (-1)^n \bar{b}, \mathcal{R}^{(l-n)}(D)\right] = 0\,.$$

In the case when $D = \mathcal{C}_f$ is the operator of multiplication by a smooth function f, independent on the Teichmüller parameter, we find a first-order solution $\mathcal{R}^{(1)}(\mathcal{C}_f)$, for which moreover

$$\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}\left(\mathcal{C}_f + \mathcal{R}^{(1)}(\mathcal{C}_f)s^{-1}\right) = o(s^{-1})\,.$$

In order to illustrate our motivation for restricting to the case when k is fixed, we briefly discuss an analogous recursion for the formal parameters $1/t$ and $1/\bar{t}$. We conclude that the cohomological obstruction arising in that situation does not vanish in general, not even for $\mathcal{C}_f$.

We then consider the specific situation of a surface Σ of genus 1, in which case we find a sequence of operators $A^{(l)}(D)$ satisfying a recursive relation similar to the desired one for the $\mathcal{R}^{(l)}(D)$s. This motivates us to look for a solution of the original recursion of the form

$$\mathcal{R}^{(l)}(D) = \sum_{r=0}^{l} \alpha_r^{(l)} A^{(l-r)}(D)\,,$$

which leads to our next result.

Theorem 3. *There exist infinitely many solution of the form above, and any two resulting trivialisations of the formal Hitchin-Witten connection are related to one another by the multiplication by a power series in s^{-1} with scalar coefficients. Moreover, we obtain a particular solution of this kind by manipulating the trivialisation of the Hitchin-Witten connection presented in Section 2.5.3.*

We next consider the problem of defining quantum operators associated to functions on the moduli space $\mathcal{M}_\mathbb{C}$ of flat $\mathrm{SL}(n,\mathbb{C})$-connections. We approach this by considering a correspondence between differential operators on $\mathscr{L}^k$ and totally symmetric tensor fields on $\mathcal{M}$. Said correspondence is valid in the presence of a linear connection on the moduli space, which can be obtained given the choice of a Riemann surface structure on Σ as the Levi-Civita connection of the corresponding metric. On the other hand, tensor fields on the moduli space correspond to polynomial functions on the cotangent bundle $\mathrm{T}^*\mathcal{M}$. One may then attempt to associate operators to functions on the $\mathrm{SL}(2,\mathbb{C})$ moduli space by defining a map $\mathrm{T}^*\mathcal{M} \to \mathcal{M}\mathbb{C}$, and then using the pull-back and the correspondences above.

One way to achieve a map as above uses the viewpoint of holomorphic and Higgs bundles by means of the various maps between the moduli spaces involved. This gives a map as desired, which moreover is an embedding reaching an open dense subset of $\mathcal{M}_\mathbb{C}$. We study this again in the case of a surface of genus 1; after specifying a notion of smoothness at the singular points in terms of certain covers of the moduli spaces, we reach the following conclusion.

Theorem 4. *When the genus of Σ is 1, the resulting map defines a diffeomorphism between $\mathrm{T}^*\mathcal{M}$ and $\mathcal{M}$ which does not depend on the choice of a Teichmüller parameter σ. The operators obtained via the construction above, also independent of σ, satisfy the Dirac quantisation condition in s^{-1} for every fixed value of k. Moreover, the correspondence can be reversed to obtain a product $\star$ on $\mathscr{A}_k$, the fibre of the bundle on which the formal Hitchin-Witten connection is defined. Using the trivialisation of Theorem 3 above, this $\star$ can be made Hitchin-Witten covariantly constant.*

We also propose a different way to map $\mathrm{T}^*\mathcal{M}$ to $\mathcal{M}_\mathbb{C}$, adapted to the structure used in geometric quantisation. While the construction depends on the choice of $\sigma \in \mathcal{T}$ in an essential way, it is natural to study the construction above in an attempt to find Hitchin-Witten covariantly constant operators. We observe that the result of this construction agrees with the quantum operators of geometric quantisation when these are defined. With this motivation, we consider again the case of genus 1 for $\mathrm{SL}(2, \mathbb{C})$, with the following result.

Theorem 5. *Call U and V the logarithmic coordinates on $\mathcal{M}_{\mathbb{C}}$, corresponding to the holonomy functions along the meridian and longitude on the torus. Geometric quantisation defines for these functions quantum operators $\widehat{U}$ and $\widehat{V}$ which are normal and Hitchin-Witten covariantly constant. Their exponentials form a q-commutative pair for q a suitable function of k and s.*

This leads to the final part of this dissertation, covering the material of [AM17], in which we consider the relation between these operators and the invariant $J_{M,K}^{(\mathrm{b},N)}$ defined by Andersen and Kashaev's Teichmüller TQFT. Said invariant belongs to a space which is related to the quantum Hilbert space of geometric quantisation by means of the so-called Weil-Gel'fand-Zak transform. It takes the meaning of the partition function of the theory, and it is conjectured to enjoy a list of properties compatible with those of the coloured Jones polynomial, verified for the first two hyperbolic knots 4_1 and 5_2. We consider the exponentiated operators of the previous theorem, remove the dependence on the Teichmüller parameter using the trivialisation of the Hitchin-Witten connection, and then conjugate by the Weil-Gel'fand-Zak transform. Because the operators are q-commutative, they define a representation of the algebra considered in the setting of the AJ conjecture, and one may study the ideal annihilating $J_{M,K}^{(\mathrm{b},N)}$. Following the lines of [Gar04], we define the $\widehat{A}^{\mathbb{C}}$-polynomial as a preferred element of this ideal, and formulate the following conjecture.

Conjecture 1. *For K a hyperbolic knot in a closed, oriented 3-manifold M, the $\widehat{A}^{\mathbb{C}}$-polynomial essentially agrees with the $\widehat{A}$ from [Gar04] and reproduces the classical A-polynomial in the relevant limit.*

Using the definition of $J_{M,K}^{(\mathrm{b},N)}$ in the case of the figure-eight knot and of 5_2, we compute their $\widehat{A}^{\mathbb{C}}$-polynomials by means of an elimination procedure, which finally leads to the last result of this work.

Theorem 6. *Conjecture 1 is true for the first two hyperbolic knots in S^3.*

Plan of the exposition

Our discussion begins in Chapter 1 with a review of the fundamental aspects of geometric quantisation [Woo92] necessary for the following discussion. With some basic motivation from physics, we introduce the main structures and constructions involved, and discuss the process for real and Kähler polarisations. We then proceed to describe the construction of the Hitchin and Hitchin-Witten connections in the context of [And12, AG14].

In Chapter 2 we focus on Chern-Simons theory. We give a brief discussion of the general properties of the classical theory, as in [Fre95], in order to set the stage for the structures required for geometric quantisation. We then discuss the construction and main properties of the moduli spaces of flat $\mathrm{SU}(n)$- and $\mathrm{SL}(n, \mathbb{C})$-connections, emphasising the construction of a pre-quantum line bundle out of the classical data. After a detour into the correspondences of flat connections with representations of the fundamental group, and holomorphic and Higgs bundles, we present the polarisations used in [Hit90, ADPW91, Wit91] for the full quantisation. Following [AG14] closely, we argue that all the requirements for the definition and projective flatness of the connections are met, as implied by a collection of various works on the moduli spaces. To the case of a surface of genus 1, which is special in many ways, we dedicate the final part of the chapter, computing the Hitchin-Witten connection in coordinates and showing an explicit trivialisation.

Chapter 3 is where the research material of this thesis starts. We open the chapter with a review of the main definitions of Toeplitz quantisation, and of the results of [And12] for $\mathrm{SU}(n)$, which we aim to extend to the case of $\mathrm{SL}(2, \mathbb{C})$. We start the discussion by showing the existence and uniqueness of a formal connection in s^{-1} approximating the Hitchin-Witten connection as $s \to \infty$, in the same sense as for the formal Hitchin connection of the cited work. Next we study the existence of a formal trivialisation, which boils down to a recursion of differential equations, the existence of a solution to which is equivalent to the exactness of a certain operator-valued differential form. We show that said form is closed, and further motivate our restriction to the parameter s by showing that in the analogous recursion in t this is not the case. We find an explicit solution for the first step of the recursion in general, and use this partial result to guess the form of a full trivialisation in genus 1, which turns the recursion of differential equations into an algebraic one. After showing that the solutions of these form are parametrised by the space of all complex-valued sequences, we determine the precise relation between them. We conclude the chapter by finding a particular one by expanding the exact trivialisation of the Hitchin-Witten connection.

In Chapter 4 we propose a geometric construction for defining differential operators associated to functions of a special kind on $\mathcal{M}_{\mathbb{C}}$. This is based on a chain of correspondences involving differential operator, tensor fields on a smooth manifolds, and polynomial functions on its cotangent bundle. If an identification of $\mathrm{T}^* \mathcal{M}$ with (an open dense of) $\mathcal{M}_{\mathbb{C}}$ is given, one may run the procedure and obtain the desired operators. We propose first an embedding as above by means of the Narasimhan-Seshadri and Hitchin-Kobayashi correspondences, using a known embedding at the level of the moduli spaces of holomorphic and Higgs bundles. We study the details of the construction in genus 1 and establish that the resulting operators satisfy a Dirac quantisation condition in the imaginary part of the quantum parameter. We also use this correspondence to define a deformed product on a suitable algebra, resulting a structure close to a deformation quantisation. Finally, we present another map $T^* \mathcal{M} \to \mathcal{M}_{\mathbb{C}}$ and argue that the operators arising from this for linear functions correspond to the pre-quantum operators of geometric quantisation.

Finally, we present in Chapter 5 the content of [AM17], joint with Andersen, in which we formulate the AJ conjecture for the quantum Teichmüller theory. The first paragraphs are dedicated to the introduction of the main properties of the quantum dilogarithm, the original AJ conjecture, and the Teichmüller TQFT. The logarithmic coordinates are linear along the polarisation used for geometric quantisation, a condition which implies that their quantum operators are well defined. Moreover, said operators are also Hitchin-Witten covariantly constant, and we check that their exponential is well defined. The resulting operators are q-commutative for a suitable q, and using the trivialisation of the Hitchin-Witten connection and the Weil-Gel'fand-Zak transform we define their action on the partition function of the Teichmüller TQFT. Finally we consider, inside the algebra generated by these operators, the ideal annihilating the invariant and choose a preferred element in it. After formulating our conjecture, stating that this element should agree with the $\widehat{A}$ of [Gar04], we use the explicit expression of the invariant for the first two hyperbolic knots to show that the claim holds for these two cases.

Chapter 1

Geometric quantisation

As an informal motivation, geometric quantisation is a mathematical construction aiming to reproduce, in the greatest possible generality, the constructions employed in canonical quantisation. As an input, this procedure should take a classical theory, encoded in terms of a symplectic manifold, and assign to it a Hilbert space together with self-adjoint operators on it associated to smooth functions on the manifold. The relations between such operators should be governed by the Dirac condition, stating that the commutator of two quantum operators should be proportional to the operator coming from the Poisson bracket of the starting functions.

The construction goes by two steps, discussed in detail in [Woo92]. The first, called pre-quantisation, defines a Hilbert space and self-adjoint operators, satisfying the Dirac condition, but the resulting space is too large compared to the result expected from canonical quantisation in some basic cases. This part is discussed in Section 1.1. The aim of the second step is to reduce this space by means of some geometric data on the underlying manifold, called a polarisation and presented in Section 1.2. As will be shown along the discussion, the classical picture does not always come naturally with such a data and there are several ways of carrying out the reduction, possibly leading to sensibly different quantisation schemes.

It may happen, in certain situations, that instead of a single preferred polarisation, the classical picture gives a family of equally valid choices of such an object. This includes, as main example, the geometric quantisation approach to Chern-Simons theory introduced by Hitchin [Hit90] and Axelrod, Della Pietra and Witten [ADPW91] for $\mathrm{SU}(n)$, and Witten [Wit91] for $\mathrm{SL}(n,\mathbb{C})$. If this is the case, and if the polarisations are parametrised by a smooth manifold, one may run geometric quantisation for each of them and obtain a bundle of Hilbert spaces. It is then natural to look for a connection on this bundle, so as to use its parallel transport to identify the results of the various quantisations. Ideally, if the connection is flat and the manifold parametrising the polarisations simply connected, the holonomy defines a path-independent identification of any two such quantisations. Unfortunately, there are deep reasons why this need not happen in general, related to the so-called no-go theorems, which essentially obstruct the existence of a functorial quantisation scheme. However, one may still ask for the connection to be *projectively* flat, i.e. having as curvature an ordinary 2-form times the identity, as this implies that the holonomy is path-independent up to projective factors. This is the role played by the Hitchin and Hitchin-Witten connections in the works mentioned above and studied in a more general and abstract context by Andersen and Gammelgaard in [AG14]. The discussion of these connections will be the content of the last part of the present chapter, Section 1.3.

1.1 Classical setting and pre-quantisation

As already mentioned, the input of geometric quantisation is a classical theory in Hamiltonian mechanics, whose natural mathematical setting is that of symplectic geometry.

Let (M, ω) be a symplectic manifold, i.e. a smooth manifold equipped with a closed, non-degenerate 2-form. For every function $f \colon M \to \mathbb{R}$ denote by X_f the Hamiltonian vector field defined implicitly by

$$X_f \cdot \omega + \mathrm{d}f = 0 \,.$$

This defines a Poisson bracket as

$$\{f, g\} := \omega(X_f, X_g) \,.$$

Hamiltonian vector fields may be thought of as infinitesimal generators of symmetries of (M, ω), as it is easily checked that $\mathcal{L}_{X_f}\omega = 0$ for every smooth function f. The Poisson bracket of two functions measures the non-commutativity of their respective flows, as is shown by

$$\left[X_f, X_g\right] = X_{\{f,g\}} \,.$$

These flows are of crucial importance in Hamiltonian mechanics, as they implement the time evolution via a function H encoding the dynamics. Moreover, they play a central role in studying symmetries of Hamiltonian systems.

Given a symplectic manifold (M, ω), the aim of geometric quantisation is to define a quantum Hilbert space and a quantisation $\widehat{f}$ of each function f as a self-adjoint operator, satisfying the following three conditions:

- to the constant function c should correspond the central operator $c\mathbb{1}$;
- the correspondence $f \mapsto \widehat{f}$ should be linear;

- if f and g are any two smooth functions, then the Dirac quantisation condition should hold:

$$[\widehat{f}, \widehat{g}] = -i\hbar \widehat{\{f, g\}},$$

where $\hbar$ is a positive real number called the quantum parameter.

One way to obtain operators satisfying these three conditions is to consider a complex line bundle $\mathscr{L} \to M$ together with a connection ∇ with curvature

$$F_\nabla = -\frac{i}{\hbar}\omega \tag{1.1}$$

and define operators acting on sections of $\mathscr{L}$ as

$$\widehat{f} := f - i\hbar \nabla_{X_f},$$

where f is identified with the operator of multiplication by the function. The condition on the curvature ensures that for every f and g one has

$$\begin{aligned}\left[\widehat{f}, \widehat{g}\right] &= -i\hbar\left[f, \nabla_{X_g}\right] - i\hbar\left[g, \nabla_{X_f}\right] + i\hbar\omega(X_f, X_g) - \hbar^2 \nabla_{[X_f, X_g]} = \\ &= -i\hbar\Big(\{f, g\} - i\hbar\nabla_{X_{\{f,g\}}}\Big) = -i\hbar\widehat{\{f, g\}}.\end{aligned}$$

If, moreover, $\mathscr{L}$ comes with a Hermitian structure, then it makes sense to restrict to square-summable sections, which form a Hilbert space. Finally, if the connection preserves the metric then $\widehat{f}$ is also self-adjoint, satisfying all the requirements above. This motivates the following definition.

Definition 1.1.1. *Consider on a symplectic manifold (M, ω) a triple $(\mathscr{L}, h, \nabla)$, with $\mathscr{L} \to M$ a complex line bundle, h a Hermitian structure on it, and ∇ a metric connection on $\mathscr{L}$. This data is called a pre-quantum line bundle if the curvature of ∇ is proportional to ω as in (1.1).*

Notice that a pre-quantum line bundle may not be given in general; it is not even guaranteed that such an object exists on a given symplectic manifold, and even if it does it need not be unique. In fact, the constraint on the curvature imposes that $(2\pi\hbar)^{-1}\omega$ represents the Chern class of $\mathscr{L}$, forcing it to be an integral class. Therefore, the set of values of $\hbar$ for which a pre-quantum line bundle exists is $\mathbb{R}_{>0}$ if $[\omega] = 0$, and otherwise either empty or discrete. In the last case one may put $\hbar = 1/k$ with $k \in \mathbb{Z}_{>0}$, up to normalising ω. Moreover, one can also prove that, when this is the case, such bundles are classified by the cohomology group $H^1(M, \mathrm{U}(1))$. This justifies the following definition.

Definition 1.1.2. *The symplectic manifold (M, ω) is called pre-quantisable if $(2\pi)^{-1}\omega$ represents an integral cohomology class.*

From now on we shall assume that (M, ω) is pre-quantisable with a fixed pre-quantum line bundle $(\mathscr{L}, h, \nabla)$ for $k = 1$, denoted for brevity as $\mathscr{L}$. A bundle $\mathscr{L}^k$ for generic k may then be obtained as the k-th tensor power of the given one. We shall refer to the parameter k as the level of the theory.

1.2 Polarisations

Unfortunately, the construction above fails to reproduce the expected result in the case when M is $\mathrm{T}^* \mathbb{R}^n$ with the natural co-tangent bundle symplectic structure. In this situation, the quantum Hilbert space should be that of complex L^2 functions on $\mathbb{R}^n$, and the quantum operators associated to the coordinates q^μ and co-tangent variables p_μ should be

$$\widehat{q^\mu} \colon \psi \mapsto q^\mu \psi \,, \qquad \widehat{p_\mu} \colon \psi \mapsto -i\hbar \frac{\partial}{\partial q^\mu} f \,.$$

The construction presented above, instead, would give as a Hilbert space that of L^2 sections of a line bundle over the whole $\mathrm{T}^* \mathbb{R}^n$, and besides the operators take a form different from the expected. In a sense, pre-quantisation results in a space of sections depending on twice as many variables as it should. Notice, given that the space is contractible, that there exists a unique pre-quantum line bundle, and as a bundle it is the trivial one. Up to gauge transformations, the pre-quantum connection may be represented as

$$\nabla = \mathrm{d} - \frac{i}{\hbar} p_\mu \, \mathrm{d} q^\mu \,.$$

Suppose that we consider sections of the pre-quantum line bundle which are covariantly constant along the fibres of $\mathrm{T}^* \mathbb{R}^n$, so they are completely determined by their values on $\mathbb{R}^n$ (seen as the zero section). Since $X_{q^\mu} = \partial / \partial p_\mu$, if ψ is such a section then the first-order part of $\widehat{q^\mu}$ annihilates it, and the pre-quantum operator of q^μ acts simply as a multiplication, as expected. On the other hand, p_μ restricts to zero on $\mathbb{R}^n$, so the pre-quantum operator associated to this function acts on ψ as the derivative along q^μ, up to the factor $-i\hbar$.

This suggests that, in the general case, one considers a set of constraints imposing that sections should be covariantly constant along n independent directions, where $2n = \dim M$. In other words, one should introduce a half-dimensional distribution P on M and restrict to sections satisfying $\nabla_X \psi = 0$ whenever $X \in P$. In full generality one may allow these constraints to be complex, in which case P should be a distribution in the complexified tangent bundle. This leads to the following definition.

Definition 1.2.1. *By a polarisation on the symplectic manifold (M, ω) one means a distribution $P \subseteq \mathrm{T}_{\mathbb{C}} M = \mathrm{T} M \otimes \mathbb{C}$, such that the following are satisfied:*

- *Lagrangianity: P is half-dimensional and* $\omega(X,Y) = 0$ *for every* $X, Y \in P$*;*
- *Involutivity: if X and Y are fields on M tangent to P, then so is* $[X,Y]$*;*
- *The rank* $\mathrm{rk}(P \cap \overline{P})$ *is constant over M.*

If P is a polarisation, smooth sections of $\mathscr{L}$ *and smooth functions are called polarised when they are (covariantly) constant along the directions of P.*

Having introduced the notion of a polarised section, several problems arise. First of all, the condition on the rank of $P \cap \overline{P}$ ensures that this is (the complexification of) a real distribution on M, while the involutivity implies by the Frobenius theorem that it is integrable, so it defines a foliation of M. The condition for sections to be polarised implies that their norm is constant along the leaves, and when these are non-compact there are no non-trivial polarised square-summable sections. More importantly, the pre-quantum operators may not in general preserve the polarisation condition. Indeed, suppose that f is a smooth function, ψ a polarised section: in order for $\widehat{f}\psi$ to be of the same kind one needs the following to hold for every vector field X in P:

$$\begin{aligned}
0 &= \nabla_X\Big(\Big(f - \frac{i}{k}\nabla_{X_f}\Big)\psi\Big) = \\
&= X[f]\psi - \frac{i}{k}\nabla_X\nabla_{X_f}\psi = \\
&= X[f] - \frac{i}{k}\nabla_{[X,X_f]}\psi - \omega(X,X_f)\psi = -\frac{i}{k}\nabla_{[X,X_f]}\psi\,.
\end{aligned}$$

Here we used the requirement on the curvature of ∇, the condition $\nabla_X\psi = 0$ and, in the last passage, the definition of X_f. In order for this to hold for f fixed and for every X and ψ, one needs for the commutator $[X, X_f]$ to be in P. Geometrically, this condition is equivalent to that the Hamiltonian flow of f should preserve P globally. This is obtained for instance when f is polarised, as in this case X_f is tangent to P. In general, however, this restriction on f can be extremely severe.

In order to carry out the quantisation in full, one needs to adjust the construction according to the specific situation, and the result may vary significantly depending e.g. on the topology of the problem and on the kind of polarisation used. We shall now proceed to discuss some general aspects of the two opposite situations where $P \cap \overline{P}$ is either 0 or P.

1.2.1 Complex polarisations and Kähler quantisation

Consider the case when $P \cap \overline{P} = 0$. Being half-dimensional and with trivial intersection, these two distributions are in direct sum. In particular, any real vector $X \in \mathrm{T}M$ can be split uniquely as a sum $X = X' + X''$ with

$X' \in \overline{P}$ and $X'' \in P$, and it follows from the uniqueness that $\overline{X'} = X''$. Moreover, there is a unique endomorphism J of $\mathrm{T}_{\mathbb{C}} M$ which restricts as the multiplication by $-i$ on P and by i on $\overline{P}$. Clearly $J^2 = -\mathbb{1}$, and J fixes the real tangent bundle since for every real X one has

$$\overline{JX} = \overline{iX' - iX''} = -iX'' + iX' = JX\,.$$

Therefore, J is an almost complex structure on M, for which P and $\overline{P}$ are the anti-holomorphic and holomorphic tangent bundles, respectively. By assumption, P is involutive, which means that J is integrable and defines a complex structure on M. Moreover, for every $X, Y \in \mathrm{T}\,M$ one has by Lagrangianity

$$\begin{aligned}\omega(JX, JY) &= \omega(iX' - iX'', iY' - iY'') = \\ &= -i^2\Big(\omega(X', Y'') + \omega(X'', Y')\Big) = \omega(X, Y)\,.\end{aligned}$$

If moreover the paring $\omega\cdot J$ is positive definite, one obtains a Kähler structure on M. Notice that the argument can be run backwards to show that a Kähler structure on M induces a polarisation, defined as the $(0,1)$-tangent bundle.

Definition 1.2.2. *A complex polarisation with $P \cap \overline{P} = 0$ is called positive, or Kähler, if the pairing $\omega\cdot J$ is a Riemannian metric.*

Once M is endowed with a Kähler polarisation, the pre-quantum line bundle $\mathscr{L}$ becomes a holomorphic bundle, with $\overline{\partial}$-operator given by the $(0,1)$-part of ∇. It follows immediately from the definitions that the polarised sections with respect to a Kähler polarisation are the holomorphic ones. Therefore, one may define as the quantum Hilbert space the closed sub-space

$$H^{(k)} := H^0(M, \mathscr{L}^k) \cap \mathrm{L}^2(M, \mathscr{L}^k)\,.$$

When M is compact, this space is finite-dimensional.

In general, it might be the case that no non-trivial pre-quantum operators descend to this space. However, since $H^{(k)}$ sits inside $\mathrm{L}^2(M, \mathscr{L}^k)$ as a closed sub-space, a projection π onto it is defined. One may then associate an endomorphism of the quantum Hilbert space to every smooth function f on M as

$$\pi \circ \widehat{f}\,.$$

Essentially, this amounts to forcing the output of the operator to be holomorphic by taking its best approximation in L^2-norm. Objects of this kind are called Toeplitz operators, and will be the focus of further discussion in Chapter 3.

1.2.2 Real polarisations

Consider now the complementary case in which $P = \overline{P}$. As already mentioned, this implies that P is the complexification of a real, involutive, Lagrangian distribution of M. By the Frobenius theorem this implies that P is realised geometrically as the complexified tangent bundle of a foliation on M in Lagrangian sub-manifolds. In this case a linear connection can be defined on each leaf via

$$(\nabla_X Y)\cdot\omega = \mathcal{L}_X(Y\cdot\omega)\,.$$

When X and Y lie in P, this defines indeed a covariant derivative, and it turns out that this connection is flat. Moreover, this introduces the notion of geodesics on the leaves, and one may say that they are geodesically complete if every such curve can be extended to one parametrised by $\mathbb{R}$. Under this assumption, the universal cover of each leaf admits a flat metric, thus gaining the structure of a Euclidean affine space. While the metric is arbitrary, the affine structure is completely determined by the flat connection.

As we observed before, it is often the case with this kind of polarisation that the leaves have infinite volume, which implies that no non-zero polarised section can be square-summable. This is precisely the situation of $\mathrm{T}^* \mathbb{R}^n$, and in general when M is the co-tangent bundle of some manifold Q. The following can also be proved (see [Woo92]): suppose P is a real polarisation with complete, simply connected leaves, and $Q \subseteq M$ a Lagrangian sub-manifold intersecting each leaf at one point, and transversely. Then there exists an identification of M with $\mathrm{T}^* Q$, under which Q sits inside M as the zero-section. Nonetheless, if Q is not Lagrangian one may still use the properties of P to realise M as a vector bundle over Q. In this scenario, one may identify the smooth, polarised sections on M with the unconstrained ones on Q. Indeed, any such ψ on Q can be uniquely extended to a polarised one by parallel transport, given that the curvature of $\mathscr{L}^k$ vanishes along the leaves, which are moreover simply connected. If Q has a volume form, it makes sense to define an L^2 pairing for polarised section on M by restriction on Q, and use this to define a quantum Hilbert space. Otherwise, it is possible under certain conditions to twist $\mathscr{L}^k$ with a so-called line bundle δ_P of half-forms. Once this structure is defined, the point-wise pairing of polarised sections gives a top form on Q rather than a function, and one may proceed and make sense of square-summability. This approach does not in fact need for Q to be embedded in M, and it may also be attempted in the more general case when P admits a smooth, orientable reduction Q.

In the following we shall run geometric quantisation using a real polarisation with complete, simply connected leaves intersecting a *symplectic* sub-manifold at one point each and transversely. In this case, the reduction does have a natural volume form, and, in order to make sense of square-summability, no other action is required than restricting to the submanifold.

As for the quantisation of functions, it was mentioned earlier that an operator $\widehat{f}$ preserves the space of polarised sections if and only if P is globally invariant under the flow of f. In the case at hand, having a real polarisation with complete and simply connected leaves, this condition is in fact equivalent to f having a linear restriction on each leaf with respect to its natural affine structure. It is worth mentioning that it is often the case that Hamiltonians are instead quadratic along the polarisation, and in order to obtain a quantisation some *ad hoc* adaptation, specific to the individual cases, is needed. We shall not enter the discussion of this problem, since it will not be necessary for the work done in the following chapters.

1.3 Hitchin and Hitchin-Witten connections

As anticipated above, we shall now consider the situation of a symplectic manifold (M, ω) coming with a pre-quantum line bundle $\mathscr{L}$ and a family of Kähler polarisations parametrised by a smooth manifold.

1.3.1 Smooth families of Kähler structures

Definition 1.3.1. *Let $\mathcal{T}$ be a manifold, $J \colon \mathcal{T} \to \Gamma(M, \mathrm{End}(\mathrm{T}\, M))$ a map such that, for every $\sigma \in \mathcal{T}$, J_σ defines an integrable almost complex structure on M defining a positive Kähler metric. Suppose moreover that, for every point $m \in M$, the evaluation of J at m defines a smooth map $\mathcal{T} \to \mathrm{End}(\mathrm{T}_m\, M)$, and that $V[J]$ is a smooth tensor field on M for every vector V on $\mathcal{T}$. Then J is called a smooth family of Kähler structures on (M, ω) parametrised by $\mathcal{T}$.*

We shall now fix $\mathcal{T}$ and J, and assume for simplicity that $\mathcal{T}$ is connected and simply connected. In order to avoid confusion, we shall use the notation d to mean the exterior differential on M, and $\mathrm{d}_{\mathcal{T}}$ for that on $\mathcal{T}$.

Since J is smooth one can differentiate the identity $J^2 = -\mathbb{1}$ along any vector V on $\mathcal{T}$ and find

$$0 = V[J]J + JV[J] \,.$$

It follows from this that $V[J]$ exchanges types, i.e. it maps $\mathrm{T}^{(1,0)}\, M$ to $\mathrm{T}^{(0,1)}\, M$, and vice-versa. In particular, since the symplectic form and its inverse $\tilde{\omega}$ are of type $(1,1)$, the tensor field $\tilde{G}(V) := V[J] \cdot \tilde{\omega}$ has no $(1,1)$ part, and it may be split into its $(2,0)$ and $(0,2)$ components:

$$\tilde{G}(V) = G(V) + \overline{G}(V) \,.$$

It is a consequence of the definition of the Kähler metric $g = \omega \cdot J$ and that ω does not depend on σ that $\tilde{G}$ expresses (the opposite of) the variation of the inverse $\tilde{g}$:

$$V[\tilde{g}] = -V[J \cdot \tilde{\omega}] = -\tilde{G}(V)\,.$$

Notice that, since the metric depends on σ, so does in general the induced Levi-Civita connection ∇. If X and Y are σ-independent vector fields on M, and f is a smooth function, one has

$$V[\nabla]_X(fY) = V\Big[\nabla_X(fY)\Big] = V\Big[X[f]Y\Big] + V\Big[f\nabla_X Y\Big] = fV[\nabla]_X Y\,.$$

Therefore, $V[\nabla]$ is $\mathcal{C}^\infty(M)$-linear in Y, while for X the same conclusion is immediate. It follows that $V[\nabla]$ is in fact a tensor field, and it can be argued that it is traceless. As a consequence, the following relation holds for the derivative of the Laplace operator for functions and sections of $\mathscr{L}$

$$V[\Delta] = V\Big[\nabla_\mu \tilde{g}^{\mu\nu}\nabla_\nu\Big] = V[\nabla]^\mu_{\mu\lambda}\tilde{g}^{\lambda\nu}\nabla_\nu - \nabla_\mu \tilde{G}^{\mu\nu}(V)\nabla_\nu = \Delta_{\tilde{G}(V)}\,,$$

where $\Delta_{\tilde{G}(V)}$ is as in Definition A.1.2.

Suppose now that $\mathcal{T}$ is a *complex* manifold. Each of these variations in V may be seen as tensor-valued 1-forms on $\mathcal{T}$, in the sense specified in Section A.3, so they may be split into their $(1,0)$- and $(0,1)$-part as such.

Definition 1.3.2. *The family J is called holomorphic if G and $\overline{G}$, as 1-forms on $\mathcal{T}$, are of type $(1,0)$ and $(0,1)$, respectively. J is called rigid if moreover $G(V)$ is a holomorphic tensor field on M for every $V \in \mathrm{T}\mathcal{T}$.*

Another crucial concept in the definition of the Hitchin and Hitchin-Witten connections is that of a Ricci potential. Given a Kähler manifold, the contraction $J\cdot\mathrm{Ric}$ defines a 2-form ρ called the Ricci form. Since $i\rho$ is the curvature of the characteristic bundle, Chern-Weil theory shows that the rescaled form $\rho/2\pi$ represents the first Chern class of (M,ω). Under the assumption that this class is also represented by $\lambda\omega/2\pi$ for some λ (necessarily an integer if M is pre-quantisable) $\rho - \lambda\omega$ is a real, exact 2-form on M. If moreover M is compact, the global $i\partial\overline{\partial}$-lemma ensures the existence of a real potential F:

$$-i\partial\overline{\partial}F = \rho - \lambda\omega\,.$$

In this case, F is unique up to addition of a constant, so one may fix its mean value to be 0 and call F *the* Ricci potential.

1.3.2 Definition and fundamental properties of the connections

Suppose now that J is a smooth family of Kähler structures on M. For every level k and $\sigma \in \mathcal{T}$, this defines a space of holomorphic sections $H^{(k)}_\sigma$

as the result of geometric quantisation. The idea underlying the Hitchin connection is that of organising these spaces into a vector bundle and introducing a covariant derivative to measure how the quantisations for different values of σ depend on the parameter. Notice that all the quantum spaces sit inside $\mathrm{L}^2(M, \mathscr{L}^k)$, so one may consider the trivial infinite-rank vector bundle $\mathcal{H}^{(k)} = \mathrm{L}^2(M, \mathscr{L}^k) \times \mathcal{T} \to \mathcal{T}$ and define

$$H^{(k)} := \left\{ (\psi, \sigma) \in \mathcal{H}^{(k)} \;\middle|\; \psi \in H^{(k)}_\sigma \right\} .$$

Definition 1.3.3. *Let u denote a 1-form on $\mathcal{T}$, valued in differential operators of finite order on $\mathscr{L}^k$, and consider on $\mathcal{H}^{(k)}$ a connection of the form*

$$\boldsymbol{\nabla} = \nabla^{Tr} + u \, ,$$

where ∇^{Tr} is the trivial connection. $\boldsymbol{\nabla}$ is called a Hitchin connection if it preserves the subset $H^{(k)}$.

Suppose now that J is holomorphic and rigid, and that it furthermore admits a Ricci potential F which depends smoothly on σ. Consider now the two 1-form on $\mathcal{T}$ taking values in finite-order differential operators on $\mathcal{C}^\infty(M, \mathscr{L}^k)$ given by

$$\begin{aligned} b(V) &= \Delta_{G(V)} + 2\nabla_{G(V)\cdot \mathrm{d}F} - 2V'[F] \, , \\ \overline{b}(V) &= \Delta_{\overline{G}(V)} + 2\nabla_{\overline{G}(V)\cdot \mathrm{d}F} - 2V''[F] \, . \end{aligned}$$

Notice that, if a Hitchin connection exists, the subset $H^{(k)} \subseteq \mathcal{H}^{(k)}$ is automatically promoted to a sub-bundle.

The following theorem, first obtained in the already cited works [Hit90, ADPW91, Wit91] in the context of Chern-Simons theory, is the main result of [AG14] in this extended context.

Theorem 1.3.1. *Let (M, ω) be a symplectic manifold with a pre-quantum line bundle $(\mathscr{L}, h, \nabla)$ and a rigid family of Kähler structures parametrised by a complex manifold $\mathcal{T}$. Suppose, moreover, that the first Betti number of M vanishes, while its first Chern class $c_1(M, \omega)$ is represented by $\lambda\omega/2\pi$. In addition, assume that for all of the complex structures on M, every holomorphic function on M is constant, and that a Ricci potential exists. Fix a positive integer k and a complex parameter $t = k + is$ with s real, and consider the two connections*

$$\boldsymbol{\nabla} := \nabla^{Tr} + \frac{1}{4k + 2\lambda} b + \partial'_{\mathcal{T}} F \, , \tag{1.2}$$

$$\tilde{\boldsymbol{\nabla}} := \nabla^{Tr} + \frac{1}{2t} b - \frac{1}{2\bar{t}} \overline{b} + \mathrm{d}_{\mathcal{T}} F \tag{1.3}$$

on $\mathcal{H}^{(k)} = \mathrm{L}^2(M, \mathscr{L}^k) \times \mathcal{T} \to \mathcal{T}$. Then one has:

- ∇ *is a Hitchin connection;*
- *If none of the complex structures on M admits non-trivial holomorphic vector fields, then both connections are projectively flat.*

Definition 1.3.4. *The two above are called* the *Hitchin and Hitchin-Witten connection.*

As discussed in the next chapter, the motivation for the Hitchin-Witten connection comes from a situation where M arises as a symplectic subspace of a larger manifold $M_{\mathbb{C}}$. In that context, M intersects at one point each leaf of a family of real polarisations parametrised again by a smooth manifold $\mathcal{T}$. In that case, each of the quantum Hilbert spaces is identified with the whole $L^2(M, \mathscr{L}^k)$, and the Hitchin-Witten connection is introduced as the right tool for identifying the various quantisations.

The projective flatness of the connections is a crucial property. Indeed, given their role of giving isomorphisms between the various quantum Hilbert spaces by parallel transport, it is fundamental to control the dependence of the holonomy on the specific path. In the case when $\mathcal{T}$ is simply connected, this dependence is measured by the curvature precisely, and projective flatness ensures that the ambiguity on the path is by projective factors.

As a final remark, we also want to consider the situation in which a discrete group Γ acts simultaneously on M and $\mathcal{T}$ in such a way that J is equivariant. When this is the case, it follows from the definitions that the Hitchin and Hitchin-Witten connections are Γ-invariant. This observation will be central in the setting outlined in the next chapter, where Γ will be the mapping class group Mod of the surface Σ on which we study Chern-Simons theory. The projective flatness of the connection and its invariance under Mod will define an important family of projective representations of the group.

Chapter 2

Chern-Simons theory

In this chapter we review the general setting of Chern-Simons theory and its approach via geometric quantisation. In Section 2.1 we consider its classical formulation in terms of a Lagrangian density and the constructions required for defining an action for the theory. We continue, in Section 2.2, with a description of the moduli spaces of flat connections over a surface and the structure supported by it, in particular that of a pre-quantised symplectic (singular) manifold. The content of these first two sections is described more extensively in [Fre95], which we follow.

Some of the constructions involved in the quantisation process rely on some relations of the above-mentioned moduli spaces with those of representations of the fundamental group, holomorphic bundles and Higgs bundles. In Section 2.3 we take a detour into this subject and give a schematic description of the moduli spaces involved and how the correspondences are defined.

Finally, Section 2.4 is dedicated to the discussion of the geometric quantisation approach to the theory. We also present the structure defining the Hitchin and Hitchin-Witten connection in this setting, which is the natural one in which they were initially defined.

2.1 Classical Chern-Simons theory

The Chern-Simons theory is a 3-dimensional gauge field theory. This means that, being set on a 3-manifold M, its object of study is a connection on a G-bundle up to gauge transformations, i.e. isomorphisms covering the identity on M. In the physical language, this is often referred to as a gauge field. Given a closed, oriented curve $\gamma\colon [0,1] \to M$ and a representation ρ of G, one may trace the holonomy of a connection along γ in the representation and consider the result as a function of the field. These holonomy functions make the natural quantities to measure in studying such a theory.

Let A denote a connection on a principal bundle P, and suppose that A_U is the connection form in a trivialisation on $U \subseteq M$. If $\langle \cdot \mid \cdot \rangle$ is a real-valued, positive and non-degenerate pairing on $\mathfrak{g} = \mathfrak{lie}(G)$, assumed symmetric and compatible with the Lie algebra structure, the classical theory is defined locally by the Lagrangian density

$$\mathcal{L}_{\mathrm{CS}}(A_U) = \left\langle A_U \wedge \mathrm{d}A_U + \frac{1}{3} A_U \wedge [A_U \wedge A_U] \right\rangle . \tag{2.1}$$

This expression makes sense for any Lie group G, and its role is that of providing, through a certain formal manipulation, a differential equation which determines the classical solutions of the theory: the Euler-Lagrange equation. As already mentioned, this Lagrangian is defined only locally, and it is subject to the choice of a representative of the connection; however, the resulting Euler-Lagrange equation reads explicitly as

$$\mathrm{d}A + \frac{1}{2} [A_U \wedge A_U] = 0 .$$

The left-hand side expresses the curvature F of the connection, which implies that all the equations arising from different choices of A agree and can be extended to the single, global equation $F = 0$. In other words, the classical solutions of the Chern-Simons theory are the flat G-connections.

Another common way to encode the dynamics of a physical theory is the so-called principle of stationary action. The idea is that a real- or complex-valued functional S, called the action, should be defined on the space of all possible configurations of the fields, so that the classical solutions of the theory are the stationary points of S. This functional is often obtained by integration of a Lagrangian density, provided that this makes sense, and the condition on the action is equivalent in this case to the Euler-Lagrange equations. In a sense, this viewpoint has a more global flavour, in that it deals with entire fields at once rather than point-wise.

From a physical perspective, the formulation of a classical theory via stationary action rather than the Euler-Lagrange equations carries a considerable advantage in view of quantisation, due to the so-called path integral approach. As suggested by Feynman, the expectation value of an observable A in a quantum theory should be given by the sum of the values of A over all classical configurations, weighted with a phase given by the action precisely. Although being effectively predictive with stunning precision, this principle is mathematically problematic in many, deep ways, and it will not be the object of study of the present thesis. However, it does play a role in understanding the language and motivations of many problems in modern physics, and has anticipated important ideas that were later developed in an independent, purely mathematical way. To mention one, Witten's original work [Wit89] on the interpretation of the coloured

Jones polynomial as the partition function of the Chern-Simons theory is based on path integral arguments. These ideas, however, motivated the combinatorial approach proposed by Reshetikhin and Turaev [RT90, RT91] in terms of TQFT's, which gives a completely mathematical argument independent of path-integrals.

The above considerations motivated to look for a stationary-action formulation of the classical Chern-Simons theory. When M is compact and oriented, one could try to define an action by integrating $\mathcal{L}_{\mathrm{CS}}(A)$. This fails immediately, due to the dependence of the Lagrangian on a gauge choice: the expression defining the density was originally intended as a 3-form on the total space of the G-bundle supporting A. In order for A to be defined everywhere on M one would need to have a global section s of said bundle, which is not always guaranteed to exist, and the integral may still depend on s when it does.

Suppose now that G is compact, (semi-)simple and simply connected, and that moreover $\langle \theta \wedge [\theta \wedge \theta] \rangle / 6$ is an integral cohomology class on G, θ the Maurer-Cartan form. Such pairings are all obtained by normalising the Killing form appropriately, and hence they are parametrised by a positive integer k. We shall then fix $\langle \cdot \mid \cdot \rangle$ to be the Killing form corresponding to $k = 1$ and simply write $k \langle \cdot \mid \cdot \rangle$ in the general case. One may regard k as a parameter of the theory, called the level.

It follows from the homotopy properties of G that every G-bundle P admits a global section if the base has dimension 3 or less, so that every connection has a global representative. Therefore, for every gauge choice, the Lagrangian is also a global 3-form and it may be integrated over M. Moreover, a gauge transformation affects the integral by addition of a term which is completely determined, up to integers, by the data on $\Sigma = \partial M$. More precisely, if two representatives A and A' are related by a gauge transformation $g \colon M \to G$, then one has

$$\int_M \mathcal{L}_{\mathrm{CS}}(A') = \int_M \mathcal{L}_{\mathrm{CS}}(A) + \int_\Sigma \left\langle \mathrm{Ad}_{g^{-1}}(A) \wedge \theta_g \right\rangle - \frac{1}{6} \int_M \left\langle \theta_g \wedge [\theta_g \wedge \theta_g] \right\rangle . \tag{2.2}$$

Here $\theta_g = g^{-1}\,\mathrm{d}g$ is the pull-back of the Maurer-Cartan form via g. If M is closed, the second integral on the right-hand side vanishes, while the last is a pairing in $\mathbb{Z}$-cohomology, hence an integer. Therefore, the gauge-dependence can be completely eliminated by taking an exponential. If instead Σ is non-empty, the argument may be refined to show that the last contribution is in fact a function of the boundary data up to integers, called the Wess-Zumino-Witten functional. Therefore, the gauge ambiguity enters the exponentiated integral through a phase $c_\Sigma(\partial A, \partial g)$ which depends only on the restrictions $\partial A := A|_\Sigma$ and $\partial g := g|_\Sigma$.

Suppose now that a pair $(\Sigma, [A])$ is fixed, where Σ is a closed oriented smooth surface and $[A]$ is an isomorphism class of connections on a principal bundle $P \to \Sigma$. This kind of data can be regarded as a natural boundary condition for 3-dimensional Chern-Simons theory. Any representative A of the connection and gauge transformation $g\colon \Sigma \to G$ can be extended to a bordism M of Σ, so one can make sense of the phase $c_\Sigma(A, g)$ in this general setting. As it turns out, these phases satisfy an appropriate cocycle condition required for the existence of an abstract Hermitian line $\mathscr{L}_{(\Sigma,[A])}$ such that:

- every choice of section $s\colon \Sigma \to P$, and hence of a representative A, induces a metric-preserving isomorphism $\varphi_s\colon \mathscr{L}_{(\Sigma,[A])} \to \mathbb{C}$;
- for every such section and every gauge transformation $g\colon \Sigma \to G$, one has that $\varphi_{g\cdot s} = c_\Sigma(A, g) \cdot \varphi_s$.

This defines a functor from the category of pairs $(\Sigma, [A])$, where the morphisms are pairs consisting of an orientation-preserving diffeomorphism and a compatible isomorphism of connection. Moreover, if $(\Sigma, [A])$ and $(\Sigma', [A'])$ are two such pairs, and $-\Sigma$ denotes the surface Σ with reversed orientation, it follows from the properties of integrals that

$$\mathscr{L}_{(\Sigma\sqcup\Sigma',[A\sqcup A'])} = \mathscr{L}_{(\Sigma,[A])} \otimes \mathscr{L}_{(\Sigma',[A'])}\,, \qquad \mathscr{L}_{(-\Sigma,[A])} = \mathscr{L}^*_{(\Sigma,[A])}\,.$$

Fixed a boundary condition $(\Sigma, [A_0])$, equation (2.2) may be viewed as the statement that the exponentiated integral of the Lagrangian density defines an element of $\mathscr{L}_{(\Sigma,[A]_0)}$:

$$\exp(2\pi i S[A]) := \exp\left(2\pi i \int_M \mathcal{L}_{\mathrm{CS}}(A)\right) \in \mathscr{L}_{(\Sigma,[A_0])}\,.$$

Regarding this as a functional in $[A]$, one can make sense of a point being critical, and since this condition is unaffected by the exponential one may argue that it is still equivalent to the Euler-Lagrange equation. We may then call this the action of the level-1 Chern-Simons theory with the given boundary condition, thus giving a stationary-action description of it. Similarly, the level-k action $\exp(2\pi i k S[A])$ may be defined as an element of the k-th tensor power $\mathscr{L}^{\otimes k}_{(\Sigma,[\partial A])}$. Although the choice of a different level does not affect the classical theory, it will be important when it comes to quantisation.

As the compactness of G plays a central role in this picture, it is natural to ask what can be done when it is suppressed. The situation is essentially analogous in the case of a complex group $G_\mathbb{C}$ containing as a real form a compact, (semi-)simple and simply connected subgroup G, onto which it deformation retracts. This is the case, for instance, for $\mathrm{SL}(n, \mathbb{C}) \supseteq \mathrm{SU}(n)$. Since the compactness of G entered the previous discussion through the homotopy theory of the group, the same properties hold in the case at hand. The most relevant change in this case is that all objects, and in particular the Killing form $\langle \cdot \,|\, \cdot \rangle$, are now complex. In order for the action to be unitary, the Lagrangian density needs to be real-valued, which in turn requires that the bilinear pairing used in its definition is real. To this end, one may use the real part of any non-degenerate complex-bilinear pairing, which is then of the form $\mathrm{Re}(t \langle \cdot \,|\, \cdot \rangle)$ for some complex parameter $t = k + is$, also called the level for the complex theory. The integrality condition on $\langle \theta \wedge [\theta \wedge \theta] \rangle /6$ requires for k to be an integer, while s is allowed to be an arbitrary real number. The level-t Lagrangian reads then as

$$\mathcal{L}^{(t)}_{\mathrm{CCS}}(A) = \mathrm{Re}\left(t \left\langle A \wedge \mathrm{d}A + \frac{1}{3} A \wedge [A \wedge A] \right\rangle \right).$$

This Lagrangian is formally of the same kind as (2.1), so it produces the same Euler-Lagrange equation. Moreover, the same constructions as in the compact case may now be carried out to define Hermitian lines $\mathscr{L}^{(t)}_{(\Sigma,[A])}$ and level-t exponential actions for every t.

2.2 Moduli spaces of flat connections

In general, if X is a smooth manifold and G a Lie group, one may attempt to define a moduli space $\mathcal{M} = \mathcal{M}_{X,G}$ of flat G-connections over X, often called the de Rham moduli space. The problem of defining these moduli spaces and their structure precisely is in general a delicate matter, and beyond the purpose of this work; instead, we shall give a schematic overview of the general ideas.

Motivated by the previous section we shall only consider the case when every such bundle admits a section, and fix $P = G \times X \to X$ the trivial bundle. One may then attempt to realise the moduli space set-theoretically as the quotient

$$\mathcal{M} := \mathcal{F} \big/ \mathcal{G} .$$

where $\mathcal{F}$, sitting inside the space $\mathcal{A}$ all connections on P, consists of the flat ones, while $\mathcal{G}$ denotes the gauge group. Concretely, $\mathcal{A}$ may be thought of as the space $\Omega^1(X, \mathfrak{g})$ of $\mathfrak{g}$-valued 1-forms on X, while $\mathcal{G}$ consists of the maps $g\colon X \to G$ and acts as

$$g \cdot A = g^{-1} A g + g^{-1} \,\mathrm{d}g .$$

The subset $\mathcal{F}$ is cut by the equation $F_A = 0$, namely

$$\mathrm{d}A + \frac{1}{2} A \wedge A = 0 .$$

Given its structure of an affine space, one can easily make sense of tangent vectors on $\mathcal{A}$, again as elements of $\Omega^1(X, \mathfrak{g})$. Vectors tangent to $\mathcal{F}$ may also be defined, by linearising the flatness equation, which gives

$$\mathrm{d}\eta + [A \wedge \eta] = 0 .$$

The action of $\mathcal{G}$ can be linearised to induce one of its Lie algebra, the space of smooth maps $\dot{g} \colon X \to \mathfrak{g}$, giving fields on $\mathcal{A}$ expressed by

$$\mathrm{d}\dot{g} + [A, \dot{g}] .$$

The tangent space at a smooth point $[A] \in \mathcal{M}$ should then be given by the quotient of these two, which can be recognised as the twisted cohomology

$$\mathrm{T}_{[A]}\, \mathcal{M} \simeq H^1_A(X, \mathfrak{g}) .$$

Suppose now that $f \colon X \to X'$ is a smooth map. This defines a pull-back $f^* \colon \mathcal{A}^{X'} \to \mathcal{A}^X$ which preserves flatness and factors through the quotient by the gauge group, thus defining a map $\mathcal{M}_{X',G} \to \mathcal{M}_{X,G}$.

2.2.1 Smooth surfaces

We now specialise to the case of a closed, oriented, smooth surface Σ of genus $g > 1$, and we shall consider the groups $\mathrm{SU}(n)$ and $\mathrm{SL}(n, \mathbb{C})$. They are both semi-simple and simply connected, and while $\mathrm{SU}(n)$ is compact, $\mathrm{SL}(n, \mathbb{C})$ contains it as a real form, and deformation retracts onto it. Their normalised Killing form is expressed by

$$\langle X \,|\, Y \rangle = -\frac{1}{8\pi^2} \operatorname{tr}(XY) . \tag{2.3}$$

It should be mentioned at this point, that the case of a surface of genus $g = 0$ is trivial, as simple connectedness implies that all flat connections are gauge-equivalent. If $g = 1$, instead, the construction can still be carried out with a few minor differences; this will be illustrated explicitly in Section 2.5.

The compact group $\mathrm{SU}(n)$ In the case of $\mathrm{SU}(n)$, the picture sketched above succeeds in defining a moduli space $\mathcal{M} = \mathcal{M}_{\Sigma,\mathrm{SU}(n)}$ with the structure of a compact stratified manifold. This is discussed e.g. in [Gol84], in relation to other incarnations of this moduli space to be discussed further below. However, it is not important for our purpose here to make a precise sense of this statement; to us, this will mean that $\mathcal{M}$ is a smooth manifold in a extended sense that allows singularities of a controlled kind. The smooth part is an open dense consisting of the gauge classes of irreducible connections, i.e. those admitting no non-trivial sub-bundles preserved by the connection.

Recall that to every gauge class $[A]$ of connections on Σ is associated, in a functorial way, a Hermitian line, discussed in Section 2.1; having fixed the surface, we shall denote it by $\mathscr{L}_{[A]}$. Moreover, this line comes with a specified isomorphism with $\mathbb{C}$ for every trivialisation of the underlying principal bundle, which in our case is specified by assumption. Moreover, the phases $c_\Sigma(A,g)$ associated to the gauge transformations define a lift of the action of $\mathcal{G}$ to $\mathcal{A}\times\mathbb{C}$. The stabilizer of each point acts trivially on the corresponding fibre [Fre95], so the quotient under the lifted action defines a Hermitian line bundle $\mathscr{L}\to\mathcal{M}$ whose fibres are by construction the abstract lines $\mathscr{L}_{[A]}$.

On this bundle is also defined a metric connection ∇, constructed as follows. A smooth curve $\gamma\colon [0,1]\to\mathcal{A}$ may be thought of as a connection A on the product $M=\Sigma\times[0,1]$, which is a compact, oriented 3-manifold with boundary $-\Sigma\sqcup\Sigma$. By the properties of the functor $\mathscr{L}$, the Hermitian line associated to $(\partial M,[\partial A])$ can be identified with

$$\mathscr{L}_{(\partial M,[\partial A])}\simeq \operatorname{Hom}\Big(\mathscr{L}_{(\Sigma,[A_0])},\mathscr{L}_{(\Sigma,[A_1])}\Big)$$

Therefore, the Chern-Simons action associated to A on M defines a linear map between the fibres at the extrema of the path, and because the action is unitary, this map preserves the metric. These maps enjoy the properties necessary for defining a parallel transport, and hence a connection, which is expressed at a point $A\in\mathcal{A}$ by $\mathrm{d}-i\theta_A$, with

$$\theta_A(\eta)=2\pi\int_\Sigma\langle A\wedge\eta\rangle\ .$$

One can also make sense of the curvature of this connection being given by $-i\omega$, where

$$\omega(\eta_1,\eta_2)=4\pi\int_\Sigma\langle\eta_1\wedge\eta_2\rangle\ .$$

As is immediately checked, the non-degeneracy of $\langle\cdot\,|\,\cdot\rangle$ implies that of ω, which is called the Atiyah-Bott form. It follows from the intrinsic properties of the functor $\mathscr{L}$ that this construction is compatible with the action of the gauge group, thus descending to $\mathcal{M}$. More precisely, this defines a symplectic form ω, also called the Atiyah-Bott form, and the structure of a pre-quantum line bundle on $\mathscr{L}\to\mathcal{M}$. As is easily argued, introducing a level k amounts to taking the k-th tensor power

$$\mathscr{L}^k:=\mathscr{L}^{\otimes k}$$

as a line bundle with induced Hermitian structure and connection, in which case the curvature is $-ik\omega$. In the calculations we shall often drop the superscript (k) in the connection and write simply ∇.

It is interesting to mention that, having defined a symplectic form on $\mathcal{A}$, the moduli space can also be constructed in terms of an infinite-dimensional analogue of symplectic reduction. Indeed, the action of the gauge group on $\mathcal{A}$ is Hamiltonian with respect to ω, and one can show that the moment map is essentially given by the curvature. Therefore, the quotient induced by symplectic reduction would be precisely $\mathcal{F}/\mathcal{G}$, which is how $\mathcal{M}$ was intuitively defined.

Consider now the case where a bordism $\partial M = \Sigma$ is given, M a compact, oriented 3-manifold. The inclusion $\Sigma \to M$ induces a pull-back map $\iota^*\colon \mathcal{M}_M \to \mathcal{M}_\Sigma$ at the level of the moduli spaces. One may regard the image of this map as the subset of $\mathcal{M}_\Sigma$ of points representing connections on Σ admitting an extension to M. The Chern-Simons action defines a section $\exp(2\pi i S)$ of $\mathscr{L}|_{\iota^*(\mathcal{M}_M)}$; this turns out to be covariantly constant, proving in particular that the curvature of $\mathscr{L}$ vanishes on this image. Together with an argument on the dimensions, this concludes that $\iota^*(\mathcal{M}_M)$ is Lagrangian in $\mathcal{M}_\Sigma$.

The complex group $\mathrm{SL}(n, \mathbb{C})$ The case of $\mathrm{SL}(n, \mathbb{C})$ needs some extra care, as a stability condition is required. A connection A is called reductive if the vector bundle $\Sigma \times \mathbb{C}^n$ splits as a direct sum of sub-bundles, each preserved by A and such that the restriction is irreducible. A non-compact moduli space $\mathcal{M}_{\Sigma, \mathrm{SL}(n,\mathbb{C})} = \mathcal{M}_\mathbb{C}$ can be obtained as the space of all reductive flat connections up to gauge equivalence. As in the case of $\mathrm{SU}(n)$, the smooth locus is an open dense which corresponds to the irreducible connections.

The constructions illustrated for the case of $\mathrm{SU}(n)$ extend in the same way to $\mathrm{SL}(n, \mathbb{C})$. This defines a Hermitian line bundle $\mathscr{L}^{(t)} \to \mathcal{M}_\mathbb{C}$ for every value of the level t, coming with a metric-preserving connection $\nabla = \nabla^{(t)}$. The Atiyah-Bott form can be defined in this case by the same expression as for $\mathrm{SU}(n)$, although it is now a complex-valued form $\omega_\mathbb{C}$. However, the pre-quantum condition is still satisfied in the form

$$F_{\nabla^{(t)}} = -i \operatorname{Re}(t\omega_\mathbb{C}) .$$

In this case, since $\mathrm{SL}(2, \mathbb{C})$ is a complex group, the moduli space has a natural complex structure, in the form of a tensor J consisting simply of the multiplication by i.

Notice that an $\mathrm{SU}(n)$-connection is always reductive, and that while a $\mathrm{SU}(n)$-connection is also an $\mathrm{SL}(n, \mathbb{C})$-connection, the gauge-equivalence for two such objects is equivalent for the two groups. Therefore, one obtains an embedding $\mathcal{M} \to \mathcal{M}_\mathbb{C}$, and it can be checked that pre-quantum line bundle and symplectic form on $\mathcal{M}_\mathbb{C}$ restrict via this map to those on $\mathcal{M}$. In particular, this makes $\mathcal{M}$ into a symplectic subspace of $\mathcal{M}$.

Punctured surfaces and smoothness of the moduli spaces We shall now consider the case when a Σ is punctured at a point p enclosed by a small loop γ. If d is a positive integer, one may consider connections with prescribed holonomy $e^{2\pi i d/n}\mathbb{1}$ along γ. Because this is a central element in both $\mathrm{SU}(n)$ and $\mathrm{SL}(n, \mathbb{C})$, the condition is well-posed and gauge-invariant. Using some care to adapt $\mathcal{A}$ and $\mathcal{G}$ to this situation, one can define a moduli space of flat connections subject to this constraint [DW97]. When n and d are co-prime, this space has the remarkable property of being smooth, while having all the structure described above. These spaces may be seen as symplectic leaves in the moduli space of flat connections on the complement of the puncture [FR93, FR97].

2.3 Correspondences with other moduli spaces

Riemann-Hilbert Let π be a finitely presentable group, G a Lie group, and consider the set of homomorphisms $\rho\colon \pi \to G$. The group G acts on this by conjugation, and one may consider the quotient

$$\mathrm{Hom}(\pi, G)/G\,.$$

Suppose that a presentation of π is given:

$$\pi = \langle x_1, \dots, x_n \,|\, w_1, \dots, w_m \rangle\ .$$

From this one gets an identification of $\mathrm{Hom}(\pi, G)$ with the subset of G^n cut out by the words w_j. If G is an algebraic group, the words induce regular maps $G^n \to G$, and under this identification, thus making $\mathrm{Hom}(\pi, G)$ into a sub-variety. Under the suitable hypotheses on G, one may then consider the relevant stability condition and define the quotient as a quasi-projective variety via geometric invariant theory (GIT); for an introduction to this construction for quasi-projective varieties see e.g. [New78]. As is easily seen, the choice of a presentation is inessential, as changing it induces an isomorphism. The resulting object is called the G-character variety of π.

When π arises as the fundamental group of a connected smooth manifold X, the Riemann-Hilbert correspondence identifies conjugacy classes of representations with gauge classes of flat G-connections over X. In the case of a smooth surface as in the previous section and $G = \mathrm{SL}(n, \mathbb{C})$, the relevant stability condition corresponds to the irreducibility, while the semi-stable representations are the reductive ones [Gol84]. The arising character variety is often referred to as the Betti moduli space $\mathcal{M}^B$ of Σ. From the Riemann-Hilbert correspondence one obtains a map $\mathcal{M}^B \to \mathcal{M}_{\mathbb{C}}$ which in the appropriate sense is an isomorphism. The $\mathrm{SU}(n)$-moduli space $\mathcal{M}$ corresponds via this map to a subset of $\mathcal{M}^B$ consisting of the

$\mathrm{SU}(n)$-character variety. This space carries the structure of a real analytic space.

As for the case of the de Rham moduli space, one may also consider the situation of a punctured surface and representations with prescribed behaviour along a small loop around it, with similar conclusions. In particular, when the rank n and the order d of the holonomy around the puncture are co-prime, all spaces are smooth and the maps are diffeomorphisms in the ordinary sense.

Narasimhan-Seshadri Another widely studied moduli space is that of semi-stable holomorphic vector bundles of prescribed rank and degree over a smooth Riemann surface. If Σ is endowed with such a structure, and $E \to \Sigma$ is a vector bundle, the slope is defined as

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}\,.$$

A holomorphic bundle E over Σ is called stable if, whenever $F \subseteq E$ is a proper holomorphic sub-bundle, one has $\mu(F) < \mu(E)$, semi-stable if the weak inequality holds. Assuming that the genus of the surface is $g > 1$, for every rank r and degree d the isomorphism classes of semi-stable holomorphic bundles over Σ with trivial determinant form a moduli space $\mathcal{M}^{\mathrm{Vec}}_{r,d}$. This space has the structure of a projective variety, and its smooth locus is the open dense subset consisting of the stable bundles. When r and d are co-prime, it can be shown that every semi-stable bundle is in fact stable, and so the moduli space is smooth.

Suppose now that A is a flat $\mathrm{SU}(n)$-connection over Σ. The $(0,1)$-part of A defines then a $\bar{\partial}$-operator on $\Sigma \times \mathbb{C}^n$, hence a holomorphic bundle over Σ of degree 0, rank n and trivial determinant. This can be used to define a map $\mathcal{M} \to \mathcal{M}^{\mathrm{Vec}}_{0,n}$, called the Narasimhan-Seshadri correspondence, which is also an isomorphism in the suitable sense. In a similar way, one obtains a map from the moduli space of flat connections with prescribed holonomy $e^{2\pi i d/n}\mathbb{1}$ around a puncture to $\mathcal{M}^{\mathrm{Vec}}_{n,d}$. When $\mathrm{GCD}(n,d) = 1$, both objects are smooth and the map defines a diffeomorphism in the usual sense. The result was first obtained by Narasimhan and Seshadri [NS65] in terms of representations of the fundamental group, and later on by Donaldson [Don83] in terms of connections of constant central curvature.

Hitchin-Kobayashi The last important correspondence we wish to include in this discussion is the one between flat $\mathrm{SL}(n,\mathbb{C})$-connections and Higgs bundles. These objects were first introduced by Hitchin in his famous work [Hit87], where the self-duality equations arise from a 2-dimensional reduction of the Yang-Mills equations.

Given a Riemann surface as above, by a Higgs bundle on Σ one means a pair (E, Φ) consisting of a holomorphic bundle E and a and a Higgs field

$$\Phi \in H^0(\Sigma, K \otimes \mathrm{End}_0(E)) \,.$$

Here End_0 denotes the (holomorphic) bundle of traceless endomorphisms of E. Such a pair is called stable if the slope stability is verified for every sub-bundle preserved by the Higgs field, polystable if E splits as a direct sum of stable sub-bundles of the same slope. Notice that this allows the existence of stable Higgs bundles with underlying unstable holomorphic bundle. Again, the isomorphism classes of polystable Higgs bundles of fixed rank, degree and determinant form a moduli space $\mathcal{M}^{\mathrm{Hit}}_{r,d}$, called the Hitchin moduli space, smooth if $\mathrm{GCD}(r, d) = 1$. One remarkable property of these moduli spaces is that they support a natural hyper-Kähler structure.

A correspondence between polystable Higgs bundles with trivial determinant and flat reductive $\mathrm{SL}(n, \mathbb{C})$-connections is provided by the non-Abelian Hodge theory. Briefly, suppose a Higgs bundle (E, Φ) is given, $\bar\partial$ the holomorphic structure on E, and suppose that a metric h is defined on E, which defines a field Φ^* adjoint to Φ. In an attempt to find a flat $\mathrm{SL}(n, \mathbb{C})$-connection, one may call ∇^h the Chern connection and consider

$$A = \nabla^h + \Phi + \Phi^* \,, \tag{2.4}$$

and explicitly write the curvature

$$F_A = F_{\nabla^h} + [\Phi \wedge \Phi^*] + \bar\partial\Phi + \partial\Phi^* \,.$$

Together with the condition of Φ being holomorphic, this decouples into the self-duality equations (Hitchin's equations) [Hit87]:

$$\begin{cases} F_{\nabla^h} + [\Phi \wedge \Phi^*] = 0 \\ \bar\partial\Phi = 0 \,. \end{cases}$$

This may be seen as a condition on h so that (2.4) is flat. In fact, the works of Hitchin and Simpson [Hit87, Sim92] show that such a metric exists if and only if the Higgs bundle is polystable.

Vice versa, suppose one starts with such a connection A and a Hermitian structure h. This allows one to split A into the sum of a unitary connection ∇^h and a Hermitian 1-form H, and each part can be further separated in $(1, 0)$ and $(0, 1)$ components:

$$\nabla^h = \partial + \bar\partial \,, \qquad H = \Phi + \Phi^* \,.$$

Again, the data of $\bar\partial$ and Φ constitutes a Higgs bundle if and only if the Hitchin equations are satisfied. The problem can be studied in terms of stationary points of a certain functional, and a metric h satisfying the desired conditions exists if and only if A is reductive [Don87, Cor88]. This establishes an identification of the moduli spaces of flat connections and Higgs bundles.

The picture is analogous for the case of flat connections of prescribed holonomy around a puncture and Higgs bundles of fixed rank and degree.

2.4 Geometric quantisation in Chern-Simons theory

We shall now focus again on Chern-Simons theory, summarising the last section of [AG14]. As discussed in Section 2.2, the sets of solutions of the theory for $\mathrm{SU}(n)$ and $\mathrm{SL}(n, \mathbb{C})$ on a 3-manifold can be organised into appropriate moduli spaces. However, in the case of a surface Σ, the resulting moduli spaces have much richer structures, arising as a (singular) symplectic manifolds with pre-quantum data obtained directly from the constructions of the classical theory. More precisely, this data is given as a level-k pre-quantum line bundle $\mathscr{L}^k$ on $\mathcal{M}$ for every $k \in \mathbb{Z}_{>0}$, and a $\mathscr{L}^{(t)}$ on $\mathcal{M}_\mathbb{C}$ for every admissible level t of the $\mathrm{SL}(n, \mathbb{C})$-theory. This setting can be used to start geometric quantisation, as proposed by Hitchin [Hit90] and Axelrod, Della Pietra and Witten [ADPW91] for $\mathrm{SU}(n)$ and by Witten [Wit91] for $\mathrm{SL}(n, \mathbb{C})$. Intuitively, one may think of this as changing the viewpoint from a 3-dimensional theory with boundary conditions to a 2-dimensional one, with a constraint that the fields should extend to a given bordism.

As discussed in Chapter 1, in order to carry out the full construction of geometric quantisation one needs the additional data of a polarisation on the relevant moduli space. In order to obtain this data, the choice of an auxiliary structure on Σ is needed, namely that of a Riemann surface. As discussed below, such a choice induces a Kähler structure on each of the relevant moduli spaces, from which one can define a polarisation. Since the choice of a Riemann surface structure is extrinsic for Chern-Simons theory, one may regard the result on each moduli space as a smooth family of Kähler structures parametrised by the Teichmüller space $\mathcal{T}$ of Σ. The resulting setting satisfies all the requirements listed in Theorem 1.3.1, thus defining projectively flat Hitchin and Hitchin-Witten connections, which agree with those defined in the works cited earlier.

We shall now proceed to illustrate the constructions in the two different cases of $\mathrm{SU}(n)$ and $\mathrm{SL}(n, \mathbb{C})$. For the rest of this section we shall assume that Σ is a closed, oriented, smooth surface of genus at least 2. We will denote by $\mathcal{M}$ and $\mathcal{M}_\mathbb{C}$ the respective moduli spaces of flat connections with prescribed holonomy $e^{2\pi i d/n}$ around a puncture p, without explicitly referring to n or d. Moreover, we will refer to the smooth parts of the moduli spaces as $\mathcal{M}^s$ and $\mathcal{M}^s_\mathbb{C}$, although for the remainder of this work we shall not always stress the superscript.

2.4.1 The family of Kähler structures on $\mathcal{M}$

If a Riemann surface structure is fixed on Σ, a metric comes with it, and hence a Hodge star-operator $*\colon \Omega^1(\Sigma) \to \Omega^1(\Sigma)$, defined by

$$\varphi_1 \wedge *\varphi_2 = \langle \varphi_1 | \varphi_2 \rangle \,\mathrm{dvol}\ .$$

A metric can also be defined for $\mathfrak{su}(n)$-valued 1-forms by using the Killing form, thus extending $*$ to these forms. Recall that the tangent space of $\mathcal{M}$ at a smooth point $[A]$ can be identified with the appropriate cohomology group:

$$\mathrm{T}_{[A]}\,\mathcal{M} \simeq H^1_A(\Sigma, \mathfrak{su}(n))\,.$$

In turn, this space can be realised via Hodge theory with that of holomorphic forms, which is preserved by $*$. Since this operator squares to $-\mathbb{1}$, it defines an almost complex structure J on $\mathcal{M}^s$ as $[A]$ ranges through it. It follows from the definitions that the contraction $\omega \cdot J$ with the Atiyah-Bott form defines the L^2 pairing of harmonic forms:

$$\omega(\eta_1, J\eta_2) = 4\pi \int_\Sigma \langle \eta_1 \wedge *\eta_2 \rangle = 4\pi \int_\Sigma \langle \eta_1 \,|\, \eta_2 \rangle \,\mathrm{dvol}\ .$$

Here the right-most bracket denotes the point-wise pairing defined by the metric on Σ, while $\langle \cdot \wedge \cdot \rangle$ is the wedge product traced with the Killing form. Because this pairing is positive-definite, it gives a Riemannian metric on $\mathcal{M}^s$. In order to obtain a Kähler structure, it remains to argue that J is integrable. This can be done by means of the Narasimhan-Seshadri correspondence, which maps J to the complex structure on the moduli space of holomorphic bundles, which is naturally an algebraic variety.

As it turns out, J depends on the Riemann surface structure on Σ only through its isotopy class. Therefore, it can be seen as a family of Kähler structures parametrised by the Teichmüller space $\mathcal{T}$ of Σ, which is a contractible complex manifold. One may then run Kähler quantisation for each $\sigma \in \mathcal{T}$ and obtain $\mathcal{H}^{(k)}$ and $H^{(k)}$ as in Section 1.3.2. As follows from Hitchin's work [Hit90], this family is holomorphic and rigid, and moreover $\mathcal{M}^s$ supports no non-constant holomorphic functions. The space $\mathcal{M}^s$ is also simply connected, with $H^2(\mathcal{M}^s, \mathbb{Z}) \simeq \mathbb{Z}$ and first Chern class represented by $\lambda\omega/2\pi$ with $\lambda = \mathrm{GCD}(n, d)$ [AB83, DW97, AHJ+17, ND89]. Moreover, a Ricci potential for these moduli spaces is found in [ZT90]. Altogether, this shows the existence of a Hitchin connection as in Theorem 1.3.1. Finally, excluding the special case of $n = g = 2$ and d even, there are no holomorphic vector fields on $\mathcal{M}^s$ [NR75, Hit90], so the connection is projectively flat.

The content of this paragraph can be summarised in the following statement.

Theorem 2.4.1. *Consider the level-k Kähler quantisation on the moduli space $\mathcal{M}^s$ of irreducible flat $\mathrm{SU}(n)$-connection with holonomy $e^{2\pi i d/n}$ around a puncture, excluding the case of $n = g = 2$ and d even. This defines a bundle of quantum Hilbert spaces over the Teichmüller space $\mathcal{T}$, which supports a projectively flat Hitchin connection $\boldsymbol{\nabla}$ as in (1.2).*

This is commonly referred to as the Verlinde bundle. As anticipated in Section 1.3.2 and in the introduction, this bundle supports an action of the mapping class group Mod of the surface, which fixes the connection. This defines quantum projective representations of Mod, equivalent to those obtained in the combinatorial approach [Wit89, RT90, RT91], whose asymptotic faithfulness is proven in [And06].

2.4.2 Geometric quantisation for the complex group $\mathrm{SL}(n,\mathbb{C})$

The construction of a Kähler structure on $\mathcal{M}^s_{\mathbb{C}}$ goes along the same lines as for $\mathcal{M}$. Given a Riemann surface structure $\sigma \in \mathcal{T}$, the tensor J is defined on this space by

$$J\eta := -J\overline{\eta}\,.$$

Clearly, J is still an almost complex structure, and its integrability can be shown by comparison with the corresponding moduli space of Higgs bundles. Moreover, this J restricts to the one considered on $\mathcal{M}^s$ via its embedding. By construction, this tensor is anti-linear with respect to the natural complex structure on $\mathcal{M}^s_{\mathbb{C}}$ induced by the fact that the group $\mathrm{SL}(n,\mathbb{C})$, and hence its Lie algebra, is complex itself. Together with the L^2-pairing and the Atiyah-Bott form, this makes $\mathcal{M}^s_{\mathbb{C}}$ into a Hyper-Kähler manifold. A Kähler quantisation may then be attempted, but we will follow Witten [Wit91] instead, and proceed using a real polarisation, also based on the choice of $\sigma \in \mathcal{T}$.

Recall that, at a smooth point $[A] \in \mathcal{M}$, the tangent space $\mathrm{T}_{[A]}\,\mathcal{M}$ is identified with $H^1_A(\Sigma,\mathfrak{su}(n))$, and similarly $\mathrm{T}_{[A]}\,\mathcal{M}_{\mathbb{C}} = H^1_A(\Sigma,\mathfrak{sl}(2,\mathbb{C}))$. Since $\mathfrak{sl}(n,\mathbb{C})$ is naturally a complex vector space, once a Riemann surface structure on Σ is chosen it makes sense to split 1-forms valued in this Lie algebra into their holomorphic and anti-holomorphic parts. This induces a decomposition

$$H^1_A(\Sigma,\mathfrak{sl}(2,\mathbb{C})) = H^{1,0}_A(\Sigma,\mathfrak{sl}(2,\mathbb{C})) \oplus H^{0,1}_A(\Sigma,\mathfrak{sl}(2,\mathbb{C}))\,.$$

We stress that, while this does not involve the almost complex structure J nor the complexification of $\mathrm{T}\,\mathcal{M}^s_{\mathbb{C}}$, this still depends crucially on the choice of $\sigma \in \mathcal{T}$. Because the wedge product of two $(1,0)$- or two $(0,1)$-forms vanishes in complex dimension 1, each of these spaces is isotropic, and for dimensional reasons they are both Lagrangian. Moreover, if A is a flat connection, and η is a d_A-closed $\mathfrak{sl}(n,\mathbb{C})$-valued 1-form, the curvature of $A+\eta$

is $[\eta \wedge \eta]$, which vanishes if η is of type $(1,0)$. For $[A] \in \mathcal{M}^s_{\mathbb{C}}$ given, with a fixed representative A, the set of classes $[A+\eta]$ with η as above forms a subset of $\mathcal{M}_{\mathbb{C}}$ containing $[A]$ and tangent to $H^{1,0}_A(\Sigma, \mathfrak{sl}(2,\mathbb{C}))$. Therefore, the distribution $H^{1,0}_\bullet(\Sigma, \mathfrak{sl}(2,\mathbb{C}))$ is integrable, hence a polarisation.

Suppose now that A is an irreducible $\mathrm{SU}(n)$-connection. The tangent space to $\mathcal{M}$ at $[A]$ is the subset of $H^1_A(\Sigma, \mathfrak{sl}(n,\mathbb{C})$ consisting of the classes represented by a real form, which is transverse to both summands of the decomposition. Witten suggests that each leaf intersects the symplectic sub-manifold $\mathcal{M}^s$ at one point exactly, in which case we recover the situation anticipated in Section 1.2.2 since the intersection is transverse.

One may now fix a level $t = k + is$ and consider the space of polarised sections of $\mathscr{L}^{(t)}$ on $\mathcal{M}^s_{\mathbb{C}}$. By the argument above, these correspond to all the smooth sections of $\mathscr{L}^{(t)}|_{\mathcal{M}_s}$; using the volume form of this symplectic subspace one may define an L^2-pairing and take a completion. Because this restriction is precisely $\mathscr{L}^k$, the resulting Hilbert space is identified with $\mathrm{L}^2(\mathcal{M}, \mathscr{L}^k)$, regardless of the choice of σ. However, Witten argues in [Wit91] that this identification is of little physical relevance. Instead, one should again consider the collection of the various spaces as a bundle of quantisations over $\mathcal{T}$ and study the dependence on σ via the relevant connection. After the identification, the bundle can be described as $\mathrm{L}^2(\mathcal{M}, \mathscr{L}^k) \times \mathcal{T}$. Although this is independent of s, we shall denote this as $\mathcal{H}(t)$ rather than $\mathcal{H}^{(k)}$ to stress that we regard this as the bundle of full quantisations of the complex theory. As already argued in the discussion of the $\mathrm{SU}(n)$-theory, all the hypotheses of Theorem 1.3.1 are satisfied, and one obtains the following conclusion.

Theorem 2.4.2. *The smooth part of the moduli space of flat* $\mathrm{SL}(n,\mathbb{C})$*-connections with prescribed holonomy around a fixed puncture has a natural family of real polarisations parametrised by the Teichmüller space* $\mathcal{T}$*. This defines a bundle* $\mathcal{H}^{(t)}$ *of level-*t *quantum Hilbert spaces over* $\mathcal{T}$*, each identified with* $\mathrm{L}^2(\mathcal{M}, \mathscr{L}^k)$*, with a projectively flat connection* $\tilde{\boldsymbol{\nabla}}$*, as in* (1.3).

2.5 A special case: genus 1

In the previous sections it was often stressed that the surface whose moduli spaces we quantise should have genus $g > 1$. As was observed, the sphere has trivial character variety, since the surface is simply connected. This leaves out the case of a torus, which is special in several ways; we shall discuss it now. We stress that we are only considering the case without punctures on the surface.

The construction of the Hitchin and Hitchin-Witten connections and the proof of their projective flatness involve many properties of the moduli spaces of the surfaces with $g \geq 2$, while some of these are not verified for the torus. For instance, the fundamental group of this surface is Abelian, and therefore admits no irreducible $\mathrm{SU}(n)$- or $\mathrm{SL}(n,\mathbb{C})$-representations, while its character varieties still have a non-empty smooth locus. This means that there is no correspondence between irreducible representations (or flat connections) and smooth points on the moduli spaces. Moreover, it is not true that the character varieties for $\mathrm{SU}(n)$ and $\mathrm{SL}(n,\mathbb{C})$ have simply connected smooth locus in this case, as they admit an explicit cover of degree $n!$.

In spite of the peculiarities of this case, the constructions needed for the geometric quantisation and the Hitchin and Hitchin-Witten connections can still be carried out on a surface of genus 1. As a matter of fact, the procedures can be described explicitly in this case by means of natural holonomy coordinates, and many important differential equations appearing in the theory reduce to pure linear algebra in this case. Specifically, the Kähler metric has constant coefficients in these natural coordinates, which makes it flat; this implies that the Ricci potential vanishes, unlike in higher genus. From this it can be deduced that the Hitchin and Hitchin-Witten connections are also flat.

Besides the remarkable simplifications arising in genus 1 compared to the others, there is another motivation to focusing on this surface. Some of the problems of interest of Chern-Simons theory are related to knots and their invariants. A clear example of this is the motivation of Witten's original formulation of the theory to give a 3-dimensional interpretation of the coloured Jones polynomial. The genus 1 surface arises as the common boundary of a tubular neighbourhood and the exterior of a knot in a 3-manifold. This is, for example, the first step towards the definition of the A-polynomial, which will be discussed in greater detail in Chapter 5.

In this section we present the above-mentioned coordinates on the moduli spaces of flat $\mathrm{SU}(n)$- and $\mathrm{SL}(n,\mathbb{C})$-connections. Since the case of arbitrary n is completely analogous to $n = 2$, we shall briefly mention the general situation and then restrict for simplicity to case of $n = 2$. We describe explicitly all the ingredients involved in the quantisation of Chern-Simons theory in this case, and show the details about the Hitchin-Witten connection together with a trivialisation, as proposed in [Wit91].

2.5.1 The surface and its moduli space

Consider Σ as the quotient of $\mathbb{R}^2$ by the lattice $2\pi\cdot(\mathbb{Z}\times\mathbb{Z})$, regarded as a smooth surface, and use 1-periodic coordinates x and y. Letting G denote either $\mathrm{SU}(n)$ or $\mathrm{SL}(n,\mathbb{C})$, let $\mathbb{T}^n = (S^1)^n$ and $\mathbb{T}^{2n}_{\mathbb{C}} = (\mathbb{C}^*)^{2n}$. By the Riemann-Hilbert correspondence, a class of flat G-connections on Σ is completely determined by the class of its holonomy in $\mathrm{Hom}(\pi_1(\Sigma), G)/G$. Any reductive G-representation of the Abelian group $\pi_1(\Sigma)$ takes values in a maximal torus, identified with either $\mathbb{T}^n$ or $\mathbb{T}^n_{\mathbb{C}}$, and is therefore conjugated to one in diagonal matrices. The full quotient is obtained by further dividing by the Weyl group W, which acts on each diagonal matrix by permutations of its entries. This gives an identification

$$\mathcal{M} \simeq \mathbb{T}^{2n} \Big/ \sim , \qquad \mathcal{M}_{\mathbb{C}} \simeq \mathbb{T}^{2n}_{\mathbb{C}} \Big/ \sim ,$$

The elements of $\mathbb{T}^{2n}$ and $\mathbb{T}^{2n}_{\mathbb{C}}$ descending to smooth points are those with trivial stabiliser, i.e. with all distinct entries. On the subset consisting of these points the action of W is free and properly discontinuous, thus giving a $n!$-degree cover as anticipated. Notice that the specific choice of a maximal torus in G is inessential, and the resulting covers of the moduli spaces are natural up to diffeomorphism.

Having smooth, ramified covers of the moduli spaces arising naturally from the character variety picture, one can make good sense of smooth functions on the moduli spaces as Weyl-invariant objects on the covers. The same goes for tensor fields and sections of the Chern-Simons line bundle. This is the meaning that we will always have in mind when talking about smoothness on these moduli spaces.

One can then define logarithmic coordinates on $\mathcal{M}$ by choosing a basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{su}(n)$ of diagonal matrices, and more precisely of the lattice defining $\mathbb{T}^n \subseteq \mathrm{SU}(n)$. Denoting by E^{μ}_{ν} the elements of the standard basis of the space of $n \times n$ matrices, one can define such a basis $(T_1, \dots, T_{n-1})$ as

$$T_\mu = 2\pi i \left(\sum_{j=1}^{\mu} E^j_j - \mu E^{\mu+1}_{\mu+1} \right) .$$

In other words, T_μ has ones in the first μ diagonal entries and a $-\mu$ in the $(\mu+1)$-th. This ensures that T_μ is traceless, and moreover, by an easy calculation, we have:

$$\left\langle T_\mu \,\middle|\, T_\nu \right\rangle = \frac{n(n+1)}{2} \delta_{\mu\nu} .$$

This induces 1-periodic real coordinates $(\mathbf{u}, \mathbf{v}) = (u^\mu, v^\nu)$ on $\mathcal{M}$ and complex coordinates $(\mathbf{U}, \mathbf{V}) = (U^\mu, V^\nu)$ on $\mathcal{M}_{\mathbb{C}}$. These coordinates correspond to the connection forms

$$A_{(\mathbf{u},\mathbf{v})} = u^\mu T_\mu \, \mathrm{d}x + v^\nu T_\nu \, \mathrm{d}y , \qquad A_{(\mathbf{U},\mathbf{V})} = U^\mu T_\mu \, \mathrm{d}x + V^\nu T_\nu \, \mathrm{d}y , \tag{2.5}$$

where summation over repeated indices is understood. In the identification above, one has correspondences

$$\frac{\partial}{\partial u^\mu} \rightsquigarrow T_\mu \, \mathrm{d}x \qquad \text{and} \qquad \frac{\partial}{\partial v^\nu} \rightsquigarrow T_\nu \, \mathrm{d}y .$$

Since the T_μ's are orthogonal to one another, it follows from the definitions that the Atiyah-Bott form has non-zero coefficients only for $\mathrm{d}u^\mu \wedge \mathrm{d}v^\nu$ with $\mu = \nu$. Also, by the definition of the almost complex structure as the Hodge star, J fixes the span of any pair of such vectors. Therefore, the whole picture is essentially diagonalised in these coordinates. For this reason, we shall now continue the discussion for $n = 2$ and drop all the indices in the coordinates.

For $n = 2$, the action of the Weyl group $W = \mathbb{Z}/2\mathbb{Z}$ identifies a pair $(e^{2\pi i u}, e^{2\pi i v})$ with $(e^{-2\pi i u}, e^{-2\pi i v})$, and the fixed points of the action are the four corresponding to $(\pm 1, \pm 1)$. It can be argued that $\mathcal{M} = \mathbb{T}^2/\sim$ is a sphere with four singular points.

In the following we will use the notation

$$T = \begin{pmatrix} 2\pi i & 0 \\ 0 & -2\pi i \end{pmatrix},$$

and besides (U, V), which we call the logarithmic coordinates, it will also be convenient to consider on $\mathcal{M}$ the exponential coordinates $m = e^{2\pi i U}$ and $\ell = e^{2\pi i V}$.

The Atiyah-Bott form is completely determined by the value on these two vectors fields

$$\omega\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = -\frac{1}{2\pi}\int_\Sigma (\operatorname{tr} T^2)\, \mathrm{d}x \wedge \mathrm{d}y = 4\pi\,.$$

Therefore, the form is expressed in coordinates as

$$\omega = 4\pi\, \mathrm{d}u \wedge \mathrm{d}v\,,$$

and the analogous argument on $\mathcal{M}_\mathbb{C}$ gives

$$\omega_\mathbb{C} = 4\pi\, \mathrm{d}U \wedge \mathrm{d}V\,.$$

For a description of the Chern-Simons line bundles, it is enough to consider that on $\mathcal{M}_\mathbb{C}$, as the one on $\mathcal{M}$ is obtained by restriction. One may use the identification (2.5) to see the universal cover of $\mathbb{T}^2_\mathbb{C}$ as a subsets of $\mathcal{F}_{\mathrm{SL}(2,\mathbb{C})}$. The stabiliser of this in the gauge group is generated by three elements, which act on the trivial bundle as

$$g_U \colon \big((U, V), \psi\big) \mapsto \Big((U+1, V), e^{-2\pi i \operatorname{Re}((k+is)V)}\psi\Big),$$

$$g_V \colon \big((U, V), \psi\big) \mapsto \Big((U, V+1), e^{2\pi i \operatorname{Re}((k+is)U)}\psi\Big),$$

$$g_- \colon \big((U, V), \psi\big) \mapsto \big((-U, -V), \psi\big)\,.$$

Recalling that the level-t Chern-Simons connection form on $\mathcal{A}_{\mathrm{SL}(2,\mathbb{C}))}$ is given by

$$\theta_A^{(k+is)}(\eta) = -2\pi \int_\Sigma \mathrm{Re}\Big((k+is)\, \langle A \wedge \eta \rangle \Big) = \frac{1}{4\pi} \int_\Sigma \mathrm{Re}\Big((k+is)\, \mathrm{tr}(A \wedge \eta) \Big)\,,$$

on our subset we find explicitly

$$\theta_{(U,V)}^{(k+is)} = 2\pi\, \mathrm{Re}\Big((k+is) \Big(V\, \mathrm{d}U - U\, \mathrm{d}V \Big) \Big)\,.$$

On $\mathcal{M}$ we find

$$\theta_{u,v}^{(k)} = 2\pi k \Big(v\, \mathrm{d}u - u\, \mathrm{d}v \Big)\,.$$

2.5.2 The family of Kähler structures

For a complex number σ in the upper half plane $\mathbb{H} \subseteq \mathbb{C}$ consider the map $z_\sigma : \mathbb{R}^2 \to \mathbb{C}$ sending (x, y) to $x - \sigma^{-1} y$. The natural Kähler structure on $\mathbb{C}$ pulls back to one on $\mathbb{R}^2$ which is compatible with the action of $\mathbb{Z}^2$, thus defining one on Σ with $z_\sigma = x - \sigma^{-1} y$ as holomorphic coordinate. This produces a bijection between $\mathbb{H}$ and the Teichmüller space $\mathcal{T}$, whose complex structure makes this into a biholomorphism.

The volume coming from this structure equals the area of the parallelogram spanned by 1 and $-\sigma^{-1}$ in $\mathbb{C}$, which equals $\mathrm{Im}\,\sigma / |\sigma|^2$. It is convenient to rescale the metric by this factor, so as to obtain a torus of unitary volume. The resulting metric μ, together with the one on co-vectors, can be explicitly expressed by the matrices

$$\mu = \frac{i}{\sigma - \overline{\sigma}} \begin{pmatrix} 2\sigma\overline{\sigma} & -(\sigma + \overline{\sigma}) \\ -(\sigma + \overline{\sigma}) & 2 \end{pmatrix}\,, \qquad \tilde{\mu} = \frac{i}{\sigma - \overline{\sigma}} \begin{pmatrix} 2 & \sigma + \overline{\sigma} \\ \sigma + \overline{\sigma} & 2\sigma\overline{\sigma} \end{pmatrix}\,.$$

By construction, the volume form is just $\mathrm{d}x \wedge \mathrm{d}y$. Using the defining relation of the Hodge star, it is straightforward to check that

$$\mathrm{d}x \wedge * \,\mathrm{d}x = \frac{2i}{\sigma - \overline{\sigma}}\, \mathrm{d}x \wedge \mathrm{d}y\,, \qquad \mathrm{d}y \wedge * \,\mathrm{d}x = \frac{i(\sigma + \overline{\sigma})}{\sigma - \overline{\sigma}}\, \mathrm{d}x \wedge \mathrm{d}y\,,$$

$$\mathrm{d}x \wedge * \,\mathrm{d}y = \frac{i(\sigma + \overline{\sigma})}{\sigma - \overline{\sigma}}\, \mathrm{d}x \wedge \mathrm{d}y\,, \qquad \mathrm{d}y \wedge * \,\mathrm{d}y = \frac{2i\sigma\overline{\sigma}}{\sigma - \overline{\sigma}}\, \mathrm{d}x \wedge \mathrm{d}y\,,$$

from which

$$* \,\mathrm{d}x = -\frac{\sigma + \overline{\sigma}}{\sigma - \overline{\sigma}} i\, \mathrm{d}x + \frac{2}{\sigma - \overline{\sigma}} i\, \mathrm{d}y\,,$$

$$* \,\mathrm{d}y = -\frac{2\sigma\overline{\sigma}}{\sigma - \overline{\sigma}} i\, \mathrm{d}x + \frac{\sigma + \overline{\sigma}}{\sigma - \overline{\sigma}} i\, \mathrm{d}y\,.$$

Recall that the almost complex structure on $\mathcal{M}$ is defined by the action of the Hodge star on the harmonic representatives. In coordinates (u, v), the chosen representatives for the tangent vectors are easily seen to be harmonic, so the matrix corresponding to $\eta \mapsto *\eta$ is

$$J = \frac{i}{\sigma - \overline{\sigma}} \begin{pmatrix} -(\sigma + \overline{\sigma}) & -2\sigma\overline{\sigma} \\ 2 & \sigma + \overline{\sigma} \end{pmatrix} .$$

Now $g = \omega \cdot J$ is a (positive) Riemannian metric, with matrix

$$g = \frac{4\pi i}{\sigma - \overline{\sigma}} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} -(\sigma + \overline{\sigma}) & -2\sigma\overline{\sigma} \\ 2 & \sigma + \overline{\sigma} \end{pmatrix} = \frac{4\pi i}{\sigma - \overline{\sigma}} \begin{pmatrix} 2 & \sigma + \overline{\sigma} \\ \sigma + \overline{\sigma} & 2\sigma\overline{\sigma} \end{pmatrix} .$$

Because the coefficients of the metric are constant in (u, v), the Christoffel symbols vanish in these coordinates, so the Levi-Civita connection is trivial and in particular flat. As a consequence, the Ricci form is zero, and hence so is the first Chern class of $(\mathcal{M}, \omega)$. The Ricci potential is then well-defined, and it vanishes.

We now look for a vector of type $(1, 0)$, which can be found as

$$X = \frac{1}{2} \left(\frac{\partial}{\partial u} - iJ \frac{\partial}{\partial u} \right) = \frac{1}{\sigma - \overline{\sigma}} \left(-\overline{\sigma} \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) .$$

It follows immediately that the complex function $w = u + \sigma v$ satisfies $X(w) = 1$, $\overline{X}(w) = 0$, so w is a holomorphic coordinate, and

$$\frac{\partial}{\partial w} = X = \frac{1}{\sigma - \overline{\sigma}} \left(-\overline{\sigma} \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) , \qquad \frac{\partial}{\partial \overline{w}} = \overline{X} = \frac{1}{\sigma - \overline{\sigma}} \left(\sigma \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right) .$$

Written explicitly, the differential forms corresponding to these vectors are

$$\frac{\partial}{\partial w} \rightsquigarrow -\frac{i\overline{\sigma}T}{\sigma - \overline{\sigma}} \mathrm{d}\overline{z} , \qquad \frac{\partial}{\partial \overline{w}} \rightsquigarrow \frac{i\sigma T}{\sigma - \overline{\sigma}} \mathrm{d}z .$$

Notice that these expressions can be thought of as defining not only vectors on the complexified tangent bundle of $\mathcal{M}$, but also on the ordinary non-complexified tangent bundle of $\mathcal{M}_{\mathbb{C}}$. In fact, the same arguments as for the discussion of the structure on $\mathcal{M}$ show that $\partial/\partial w$ and $\partial/\partial \overline{w}$ span $\overline{P}$ and P, respectively, as complex vector bundles.

It is convenient to write down the metric and symplectic form in the complex coordinates. Because as tensors they are both of type $(1, 1)$, they are completely determined by

$$\begin{aligned} \omega \left(\frac{\partial}{\partial w}, \frac{\partial}{\partial \overline{w}} \right) &= \frac{1}{4} \omega \left(\frac{\partial}{\partial u} - iJ \frac{\partial}{\partial u}, \frac{\partial}{\partial u} + iJ \frac{\partial}{\partial u} \right) = \\ &= \frac{i}{2} \omega \left(\frac{\partial}{\partial u}, J \frac{\partial}{\partial u} \right) = -\frac{4\pi}{\sigma - \overline{\sigma}} \end{aligned}$$

and

$$g\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial \overline{w}}\right) = -i\omega\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial \overline{w}}\right) = \frac{4\pi i}{\sigma - \overline{\sigma}}.$$

In particular, the Christoffel symbols vanish in these coordinates as well, while on the pre-quantum line bundle one has

$$\left[\nabla_w^{(k)}, \nabla_{\overline{w}}^{(k)}\right] = \frac{4\pi i k}{\sigma - \overline{\sigma}}.$$

It also follows from these computations that the Laplace operator can be expressed as

$$\Delta = -i\frac{\sigma - \overline{\sigma}}{4\pi}\left(\nabla_w \nabla_{\overline{w}} + \nabla_{\overline{w}} \nabla_w\right).$$

The variation of J is easily calculated:

$$\begin{aligned}\frac{\partial}{\partial \sigma} J =& -\frac{i}{(\sigma - \overline{\sigma})^2}\begin{pmatrix} -(\sigma + \overline{\sigma}) & -2\sigma\overline{\sigma} \\ 2 & \sigma + \overline{\sigma} \end{pmatrix} + \frac{i}{\sigma - \overline{\sigma}}\begin{pmatrix} -1 & -2\overline{\sigma} \\ & 1 \end{pmatrix} \\ =& \frac{2i}{(\sigma - \overline{\sigma})^2}\begin{pmatrix} \overline{\sigma} & \overline{\sigma}^2 \\ -1 & -\overline{\sigma} \end{pmatrix}.\end{aligned}$$

In a completely analogous way one finds

$$\frac{\partial}{\partial \overline{\sigma}} J = \frac{2i}{(\sigma - \overline{\sigma})^2}\begin{pmatrix} -\sigma & -\sigma^2 \\ 1 & \sigma \end{pmatrix}.$$

The tensor $\tilde{G}$ can now be calculated:

$$\begin{aligned}\tilde{G}\left(\frac{\partial}{\partial \sigma}\right) = \left(\frac{\partial}{\partial \sigma} J\right) \cdot \tilde{\omega} =& \frac{2i}{(\sigma - \overline{\sigma})^2}\begin{pmatrix} \overline{\sigma} & \overline{\sigma}^2 \\ -1 & -\overline{\sigma} \end{pmatrix} \frac{1}{4\pi}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \\ =& \frac{i}{2\pi(\sigma - \overline{\sigma})^2}\begin{pmatrix} \overline{\sigma}^2 & -\overline{\sigma} \\ -\overline{\sigma} & 1 \end{pmatrix}\end{aligned}$$

and

$$\tilde{G}\left(\frac{\partial}{\partial \overline{\sigma}}\right) = \frac{i}{2\pi(\sigma - \overline{\sigma})^2}\begin{pmatrix} -\sigma^2 & \sigma \\ \sigma & -1 \end{pmatrix}.$$

The coefficients in these matrices are easily compared to those in ∂_w and $\partial_{\overline{w}}$, so one obtains

$$\tilde{G} = \frac{i}{2\pi}\left(\frac{\partial}{\partial w} \otimes \frac{\partial}{\partial w} \otimes d\sigma - \frac{\partial}{\partial \overline{w}} \otimes \frac{\partial}{\partial \overline{w}} \otimes d\overline{\sigma}\right).$$

This expression makes the splitting of $\tilde{G}$ into its various parts very transparent, and on a side note it also shows that the family of Kähler structures is holomorphic and rigid.

2.5.3 The Hitchin-Witten connection

The discussion up to this point shows that all the ingredients necessary for defining the Hitchin-Witten connection are defined in this special case. In particular, we stress that since the first Chern class vanishes due to flatness of the Kähler metric, the condition for $c_1(\mathcal{M}, \omega)$ to be represented by $\lambda\omega/2\pi$ is trivially satisfied for $\lambda = 0$. Moreover, one can choose the constant function 0 as Ricci potential.

Recall the general definition of the connection:

$$\tilde{\boldsymbol{\nabla}} = \nabla^{\mathrm{Tr}} + \frac{1}{2t} b - \frac{1}{2\bar{t}} \bar{b} + \mathrm{d}F\,.$$

The last term gives no contribution, and more terms disappear in b and $\bar{b}$:

$$b = \Delta_G + 2\nabla_{G\cdot dF} - 2\lambda \partial_{\mathcal{T}} F = \Delta_G$$

$$\bar{b} = \Delta_{\overline{G}} + 2\nabla_{\overline{G}\cdot dF} - 2\lambda \bar{\partial}_{\mathcal{T}} F = \Delta_{\overline{G}}\,.$$

Finally, we have

$$\tilde{\boldsymbol{\nabla}} = \nabla^{\mathrm{Tr}} + \frac{1}{2t}\Delta_G - \frac{1}{2\bar{t}}\Delta_{\overline{G}}\,.$$

We shall now prove the following useful fact, suggested by Witten in [Wit91], which we shall rely on in the following chapters.

Theorem 2.5.1. *The Hitchin-Witten connection for $\mathcal{M}$ is gauge-equivalent to the trivial connection, and the equivalence is realised as*

$$\exp\Big(r\Delta\Big)\,\tilde{\boldsymbol{\nabla}}\,\exp\Big(-r\Delta\Big) = \nabla^{Tr}\,,$$

where $r \in \mathbb{C}$ is such that

$$e^{4rN} = -\frac{\bar{t}}{t}\,. \tag{2.6}$$

Proof. Since $\left|\bar{t}/t\right| = 1$, r is purely imaginary. That the gauge transformation $\exp(r\Delta)$ is unitary follows from this and that Δ is a self-adjoint operator.

In order to prove the statement it is enough to show that

$$\exp\left(-r\Delta\right)\frac{\partial}{\partial\sigma}\Big[\exp\left(r\Delta\right)\Big] = \frac{1}{2t} b\left(\frac{\partial}{\partial\sigma}\right),$$

$$\exp\left(-r\Delta\right)\frac{\partial}{\partial\bar{\sigma}}\Big[\exp\left(r\Delta\right)\Big] = -\frac{1}{2\bar{t}} \bar{b}\left(\frac{\partial}{\partial\bar{\sigma}}\right).$$

As will be apparent momentarily, the usual formula for the derivative of the exponential does no apply in this situation, because Δ does not commute with its derivatives. Instead, we will compute the derivative by using the definition of the exponential as a strongly convergent power series. Let now ψ be a smooth section of $\mathscr{L}^k$, and consider the following as a formal series

$$\sum_{n=0}^{\infty} \frac{(r\Delta)^n}{n!}\psi\,.$$

Since the Laplace operator depends on σ as a polynomial divided by $\sigma - \overline{\sigma}$, the above can be viewed point-wise as a Laurent series in the real and imaginary parts of σ. If the series converges point-wise on $\mathbb{T}^2$ at a given σ, then so does it on an open neighbourhood of σ. This is indeed the case when ψ is an eigenvector for the Laplace operator at σ. One can then differentiate term-by-term to determine how the derivative of $\exp(r\Delta)$ acts on ψ, and because Δ is diagonalisable this determines the operator.

We now proceed by deriving the exponential series of $r\Delta$ term-wise along $\partial/\partial\sigma$. Recall that the derivative of Δ is given by $-\Delta_{\tilde{G}}$, so

$$\frac{\partial}{\partial\sigma}\Delta = -\Delta_{\tilde{G}(\frac{\partial}{\partial\sigma})} = -\frac{i}{2\pi}\nabla_w\nabla_w = -b\left(\frac{\partial}{\partial\sigma}\right).$$

It is useful to compute the commutator

$$\begin{aligned}\left[\frac{\partial}{\partial\sigma}\Delta, \Delta\right] &= -\frac{\sigma-\overline{\sigma}}{8\pi^2}\left[\nabla_w\nabla_w, \nabla_w\nabla_{\overline{w}} + \nabla_{\overline{w}}\nabla_w\right] = \\ &= -\frac{\sigma-\overline{\sigma}}{2\pi^2}\left[\nabla_w, \nabla_{\overline{w}}\right]\nabla_w\nabla_w = -\frac{2ki}{\pi}\nabla_w\nabla_w = 4k\frac{\partial}{\partial\sigma}\Delta\,,\end{aligned}$$

and it is checked by induction that furthermore

$$\left[\frac{\partial}{\partial\sigma}\Delta, \Delta^n\right] = \sum_{l=1}^{n}\binom{n}{l}(4k)^l\Delta^{n-l}\frac{\partial}{\partial\sigma}\Delta\,.$$

This can be used to compute the derivative of Δ^n:

$$\begin{aligned}\frac{\partial}{\partial\sigma}\left(\Delta^n\right) &= \sum_{j=1}^{n}\Delta^{n-j}\frac{\partial\Delta}{\partial\sigma}\Delta^{j-1} = \\ &= n\Delta^{n-1}\frac{\partial\Delta}{\partial\sigma} + \sum_{j=1}^{n}\sum_{l=1}^{j-1}\binom{j-1}{l}(4k)^l\Delta^{n-l-1}\frac{\partial\Delta}{\partial\sigma}\,.\end{aligned}$$

One can now exchange the sums and use the identity

$$\sum_{j=l+1}^{n}\binom{j-1}{l} = \binom{n}{l+1},$$

and after incorporating the single term on the left one finds

$$\frac{\partial}{\partial\sigma}\left(\Delta^n\right) = \sum_{l=0}^{n-1}\binom{n}{l+1}(4k)^l\Delta^{n-l-1}\frac{\partial\Delta}{\partial\sigma}\,.$$

We now apply this to the derivative of the exponential series

$$\frac{\partial}{\partial\sigma}\exp\Big(r\Delta\Big) = \sum_{n=1}^{\infty}\sum_{l=0}^{n-1}\binom{n}{l+1}\frac{(4k)^l r^n}{n!}\Delta^{n-l-1}\frac{\partial\Delta}{\partial\sigma}\,.$$

We now change l with $n-l-1$, switch the integrals and further change n with $n+l$ to find

$$\begin{aligned}\frac{\partial}{\partial\sigma}\exp\Big(r\Delta\Big) &= \sum_{l=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n!}\frac{1}{l!}(4k)^{n-1}r^{n+l}\Delta^l\frac{\partial\Delta}{\partial\sigma} = \\ &= \left(\frac{1}{4k}\sum_{n=1}^{\infty}\frac{(4kr)^n}{n!}\right)\sum_{l=0}^{\infty}\frac{(r\Delta)^l}{l!}\frac{\partial\Delta}{\partial\sigma}\,.\end{aligned}$$

The sum on the left is the exponential series at $4Nr$, but starting at $n=1$, so it gives

$$\frac{e^{4kr}-1}{4k} = -\frac{1}{4k}\left(\frac{\bar{t}}{t}+1\right) = -\frac{1}{2t}$$

Finally, this gives

$$\frac{\partial}{\partial\sigma}\exp\Big(r\Delta\Big) = -\frac{1}{2t}\exp\Big(r\Delta\Big)\frac{\partial\Delta}{\partial\sigma} = \frac{1}{2t}\exp\Big(r\Delta\Big)b\left(\frac{\partial}{\partial\sigma}\right).$$

This concludes the proof for $\frac{\partial}{\partial\sigma}$. The argument for $\frac{\partial}{\partial\bar{\sigma}}$ is analogous. □

Chapter 3

Asymptotic properties of the Hitchin-Witten connection

3.1 Introduction

3.1.1 Background and motivations

Several interesting results were produced in the context of $\mathrm{SU}(n)$-Chern-Simons theory and Kähler quantisation by studying its asymptotic properties in the limit $k \to \infty$. We shall present here some of them as a motivation for our work in the complex theory.

Consider first of all the situation of geometric quantisation on a compact Kähler manifold (M, ω) equipped with a pre-quantum line bundle $(\mathscr{L}, h, \nabla)$. As discussed in Section 1.2.1, the resulting quantum Hilbert space $H^{(k)} = H^0(M, \mathscr{L}^k)$ is that of holomorphic sections of the line bundle. While the pre-quantum operators $P_f^{(k)}$ do not preserve this space in general, the quantisation of observables is achieved by means of the Toeplitz operators. As square-summable sections the holomorphic ones form a closed subspace, so there exists a projection $\pi^{(k)} \colon \mathrm{L}^2(M, \mathscr{L}^k) \to H^{(k)}$. Any operator acting on the former space may be forced to take values in the latter by composing with said projection; the Toeplitz operator associated to a function f is defined by

$$T_f^{(k)} \psi = \pi^{(k)}(f\psi).$$

As studied by Schlichenmaier et al., the composition of two Toeplitz operators is not in general of the same type, and neither does their commutator satisfy the Dirac quantisation condition. However, the study of these operators as k goes to infinity reveal that, under certain conditions on M, the relation can be retrieved in an asymptotic sense. Even more, one can obtain a star-product, as in the following definition.

Definition 3.1.1. *Let $(P, \{\cdot, \cdot\})$ be a Poisson manifold, i.e. a smooth manifold equipped with a bilinear antisymmetric pairing between smooth functions acting as a derivation on each argument and satisfying the Jacobi identity. By a star-product one means an associative, bilinear product*

$$\star \colon \mathcal{C}^\infty(P, \mathbb{C})[[\hbar]] \times \mathcal{C}^\infty(P, \mathbb{C})[[\hbar]] \to \mathcal{C}^\infty(P, \mathbb{C})[[\hbar]] ,$$

where $\hbar$ is a formal parameter, such that for every $f, g \in \mathcal{C}^\infty(P, \mathbb{C})$ one has

$$f \star g \equiv fg \pmod{\hbar} ,$$
$$f \star g - g \star f \equiv -i\hbar\{f, g\} \pmod{\hbar^2} .$$

A star-product essentially deforms the usual product on the algebra of smooth functions, introducing corrections at every order in the quantum parameter, into one that verifies the Dirac condition at the first order. This sets the stage for the precise formulation of the following result.

Theorem 3.1.1 (Schlichenmaier [Sch96, Sch00], Karabegov and Schlichenmaier [KS01])**.** *Suppose that (M, ω) is a compact Kähler manifold equipped with a pre-quantum line bundle $(\mathscr{L}, h, \nabla)$. Then there exists a unique star-product*

$$f \star^{BT} g = \sum_{l=0}^{\infty} c^{(l)}(f, g)\hbar^l$$

on M such that for every pair of functions f and g and for every positive integer L one has

$$\left\| T_f^{(k)} T_g^{(k)} - \sum_{l=0}^{L} T^{(k)}_{c^{(l)}(f,g)} \right\|_{\mathrm{L}^2} = o(k^{-L}) .$$

Moreover, $c^{(l)}$ acts as a differential operator in each of its arguments.

In other words, the theorem states that the product of two Toeplitz operators, though not being of the same kind in itself, it has an all order asymptotic expansion as a power series in k^{-1} with Toeplitz operators as coefficients. Moreover, at the zero-th order one has $T_f^{(k)} T_g^{(k)} \sim T_{fg}^{(k)}$, while at the first one finds the Dirac relation, and the whole expansion defines a deformation quantisation. The star-product $\star^{\mathrm{BT}}$ is called the Berezin-Toeplitz deformation quantisation.

One may then consider the case of a symplectic manifold (M, ω) with a rigid family of Kähler structures parametrised by a smooth manifold $\mathcal{T}$. Under the hypotheses of Theorem 1.3.1, the arising quantum Hilbert spaces form a vector bundle $H^{(k)}$ over $\mathcal{T}$ with the projectively flat Hitchin connection. Ideally, the quantum operators associated to functions on M should be covariantly constant with respect to $\boldsymbol{\nabla}$ (or rather its associated endomorphism connection); this is in fact not the case for Toeplitz operators. However, Andersen proves in [And12] that this condition holds in an asymptotic sense as follows.

Theorem 3.1.2 (Andersen [And12]). *Consider the setting of Theorem 1.3.1, and suppose that f is a smooth function on $M \times \mathcal{T}$, and V is a vector field on $\mathcal{T}$. Then there exists a unique sequence of functions $\mathcal{D}_V^l(f)$ on $M \times \mathcal{T}$ such that for every positive integer L one has*

$$\left\| \nabla_V^{\mathrm{End}} T_{f,\sigma}^{(k)} - T_{V[f],\sigma}^{(k)} - \sum_{l=0}^{L} T_{\mathcal{D}_V^l(f),\sigma}^{(k)} \left(\frac{1}{4k - 2\lambda} \right)^l \right\|_{L^2} = o(k^{-L}) \tag{3.1}$$

uniformly over compact subsets of $\mathcal{T}$. Moreover, each $\mathcal{D}_V^l$ acts on f as a differential operator.

As the asymptotic expansion of the product of Toeplitz operators can be used to form a star-product, so can these coefficients be arranged into what is called a formal connection. More precisely, Schlichenmaier's construction defines a deformation quantisation on M for every choice of $\sigma \in \mathcal{T}$. Up to rearranging the coefficients, the asymptotic expansion may be turned into one in $(4k + 2\lambda)^{-1}$ rather than k^{-1}. The result can be thought of as a bundle of deformation quantisations $\star_\sigma^{\mathrm{BT}}$ over $\mathcal{T}$, supported on the trivial bundle $\mathcal{C}^\infty(M, \mathbb{C})[[\hbar]] \times \mathcal{T} \to \mathcal{T}$. Since, by uniqueness, the $\mathcal{D}_V^l$'s are linear in V, one may think of them as 1-forms on $\mathcal{T}$ valued in differential operators on $\mathcal{C}^\infty(M, \mathbb{C})$. Altogether, their formal sum, weighted with powers of the parameter $\hbar$, may be thought of as a connection form on $\mathcal{C}^\infty(M, \mathbb{C})[[\hbar]] \times \mathcal{T}$. This is called the formal Hitchin connection D: the content of the theorem is that this of all the formal connections is the closest analogue of the Hitchin connection.

As it turns out, for every vector V tangent to $\mathcal{T}$ at σ, $\mathcal{D}_V$ is a derivation for $\star_\sigma^{\mathrm{BT}}$. Since the purpose of the formal Hitchin connection is to measure how $\star_\sigma^{\mathrm{BT}}$ depends on σ, it is natural to study its holonomy and to ask whether it can be trivialised in any good sense. It follows from the projective flatness of ∇ that $\mathcal{D}$ is flat; if moreover $\mathcal{T}$ is simply connected the holonomy of the formal Hitchin connection is trivial. When this is the case, one may consider the space of all $\mathcal{D}$-covariantly constant sections of $\mathcal{C}^\infty(M, \mathbb{C})[[\hbar]] \times \mathcal{T}$, which comes with a star-product since $\mathcal{D}$ acts by derivations of $\star^{\mathrm{BT}}$. The last step needed for obtaining a $\mathcal{T}$-independent deformation quantisation is an identification between this space and $\mathcal{C}^\infty(M, \mathbb{C})[[\hbar]]$. This is obtained via a formal trivialisation, defined as follows.

Definition 3.1.2. *By a formal trivialisation of the formal Hitchin connection one means an infinite formal sum*

$$P = \sum_{l=0}^{\infty} P^l \hbar^l ,$$

where P^l is a differential operator on $\mathcal{C}^\infty(M, \mathbb{C})$ depending smoothly on σ, such that $P^0 = \mathbb{1}$ and, for every σ-independent function $f \in \mathcal{C}^\infty(M, \mathbb{C})$ and vector V

on $\mathcal{T}$, one has

$$\mathcal{D}_V\Big(P(f)\Big) = 0 \,.$$

Such a trivialisation may be regarded as a σ-dependent perturbation of functions on M making the Hitchin covariant derivative of their Toeplitz operator decay as arbitrary powers of k^{-1}.

Theorem 3.1.3 (Andersen [And12])**.** *If the holonomy of $\mathcal{D}$ is trivial, then there exists a formal trivialisation P.*

It should be stressed that, if Γ is a group acting simultaneously on M and $\mathcal{T}$, in such a way that the family of Kähler structures is equivariant, then the construction above is Γ-invariant.

This whole construction can be applied to the smooth part of the moduli space of flat $\mathrm{SU}(n)$-connections over a closed oriented surface Σ of genus $g \geq 2$. As discussed in Section 2.4.1, these spaces satisfy all the hypotheses for the existence and projective flatness of the Hitchin connection, which was in fact considered for the first time in this setting. Since the Teichmüller space is contractible, the condition on the triviality of the holonomy of $\mathcal{D}$ is satisfied, and one gets a deformation quantisation of the $\mathrm{SU}(n)$-Chern-Simons theory on Σ.

One of the most remarkable applications of the picture sketched above is the asymptotic faithfulness of the quantum projective representations of the mapping class group Mod_Σ, proven by Andersen in [And06]. These representations are obtained in this framework by considering the simultaneous action of Mod_Σ on $\mathcal{T}$ and $\mathcal{M}$, and hence on the quantum Hilbert space. Indeed, it follows from the definitions that the family of Kähler structures is equivariant with respect to these actions. The action on $\mathcal{M}$ induces by pull-back one on the level-k Verlinde bundle, which by equivariance of J covers the action on $\mathcal{T}$. Again, it is apparent from the definitions that the Hitchin connection is invariant under this action. Therefore, a projective representation of Mod_Σ can be obtained on each fibre by acting on the bundle first, and then going back to the original fibre using the holonomy of $\boldsymbol{\nabla}$. Clearly, all the representations on the different fibres are identified by means of parallel transport.

Another interesting aspect of the asymptotic properties of Toeplitz operators arises in relation between the approaches to Chern-Simons theory via geometric quantisation and Reshetikhin-Turaev TQFT. The data of a simple closed curve $\gamma \subseteq \Sigma$ and a representation ρ of $\mathrm{SU}(n)$ defines a quantum operator in each of these two viewpoints. On the one hand, the holonomy function $h_{\gamma,\rho}$ determines a Toeplitz operator, call it $T^{(k)}_{\gamma,\rho}$. On the other hand, γ determines a framed knot in $\Sigma \times [0,1]$; once decorated with ρ, this defines an operator in the Reshetikhin-Turaev TQFT. These two operators can be compared using the chain of isomorphisms of [AU07b, AU07a, AU12, AU15, Las98], and it was proven by Andersen in [And10] that their difference vanishes for $k \to \infty$.

3.1.2 Goals and results of this chapter

In this chapter, we consider the Hitchin-Witten connection, and we address the problem of studying its asymptotic properties in analogy to the work done for the Hitchin connection. Unlike in the case of Kähler quantisation, if f is a smooth function (say the holonomy along a curve γ in a representation ρ) and ψ is a quantum state, then so is $f\psi$. Therefore, there is no analogue in this case of the Toeplitz operators; one may define a curve operator $\mathcal{C}_f$ as the multiplication by f:

$$\mathcal{C}_f(\psi) = f\psi\,.$$

The terminology is motivated by the ideas of [And10]. The initial goal of this work was that of finding perturbations of a curve operator as Laurent polynomials in t and $\bar{t}$ making its Hitchin-Witten covariant derivative decay faster than arbitrary powers of $|t|^{-1}$.

It is not to be expected in general that such an operator has a covariantly constant asymptotic expansion in operators of the same kind, as will be apparent from the computations in the next sections. As a matter of fact, in the quantisation of Chern-Simons theory, which is our main motivation, $\mathrm{L}^2(\mathcal{M}, \mathscr{L}^k)$ arises as the space of polarised sections over a larger manifold $\mathcal{M}_{\mathbb{C}}$. Accordingly, the relevant Poisson algebra to be quantised is that of functions on $\mathcal{M}_{\mathbb{C}}$ rather than $\mathcal{M}$, and it is natural to expect the appearance of operators of arbitrary (finite) order. However, in this work we shall consider the problem of the asymptotic properties of these zero-order operators, as a particular case.

Part of this problem is discussed in the qualifying dissertation [Mal16], attempting first of all a formal approach. The key idea is that, if a Laurent series in the quantum parameter is given, which is annihilated by the Hitchin-Witten connection term by term, the covariant derivative of its L-th truncation is a Laurent polynomial of order $L+1$. This gives rise to a recursion of differential equations, coming with a cohomological obstruction which, in general, does not vanish. One should then look for solutions up to terms of higher degree in the quantum parameter, and include such remainders in the subsequent steps of the recursion.

Alternatively, one can use a different approach by fixing the value of k and looking for an expansion in s instead of t. First of all, this makes the power counting more straightforward: since the quantum Hilbert spaces depend heavily in k, it is delicate to consider formal Laurent polynomials or series in t with differential operators as coefficients. If, instead, only s is allowed to vary, one can consistently make sense of the algebra $\mathscr{A}_k$ of

Laurent power series with such coefficients. As in the case of the Hitchin connection, we consider the trivial bundle $\mathscr{A}_k \times \mathcal{T}$ and we find a unique formal Hitchin-Witten connection satisfying an analogue of (3.1).

Theorem 3.1.4. *There exists a unique formal Hitchin-Witten connection $\tilde{\mathcal{D}}$ on $\mathscr{A}_k \times \mathcal{T}$ such that its L-th truncation approximates $\tilde{\boldsymbol{\nabla}}$ up to $o(s^{-L})$.*

The statement will be formulated more precisely in the next sections, as well as the meaning of the asymptotic convergence.

The problem of finding a formally covariantly constant perturbation of a curve operator as a power series in s is now well posed. Again, the condition for such a series to be covariantly constant translates into a recursion of differential equations, which also comes with a cohomological obstruction.

Theorem 3.1.5. *The cohomological obstructions to the existence of solutions of this recursion vanish.*

This established, we find an explicit solution $\mathcal{R}^{(1)}(f)$ of the first step of the recursion for the curve operator $\mathcal{C}_f$. Being obtained explicitly as a combination of the various tensors defined by the Kähler structure, this solution is manifestly invariant under any group action on M and $\mathcal{T}$ making J equivariant. This concludes the part of work overlapping with [Mal16].

In the case of the moduli space $\mathcal{M}_{\mathbb{C}}$ of flat $\mathrm{SL}(n,\mathbb{C})$-connection over a surface of genus 1, the above-mentioned solution may be written as

$$\mathcal{R}^{(1)}(f) = [a, \mathcal{C}_f] ,$$

where a is a multiple of the Laplace operator and, in this context, a primitive of the 1-form $b + \overline{b}$. It is then natural to consider iterations of ad_a to define new operators $A^{(l)}(f)$ of increasing order and study their relations. As it turns out, these operators satisfy a relation similar to the recursion of interest, so one may look for solutions $\mathcal{R}^{(l)}(f)$ as linear combinations of the $A^{(l)}(f)$'s. This converts the recursion of differential equations into a numeric one, which can be separated into two different sub-systems involving the same variables. This would hint that the problem is overdetermined; instead, each sub-system can be put in a triangular form to show that they are in fact redundant.

Theorem 3.1.6. *There exists a unique solution of the numeric recursion for every choice of an initial condition, of the form of an infinite sequence of complex numbers. The resulting formal trivialisations, are related to one another by the multiplication by a formal Laurent series in s^{-1}. Moreover, a particular solution of this form can be obtained as an expansion of the explicit trivialisation of the Hitchin-Witten connection of Theorem 2.5.1.*

3.2 Meaning of the convergence of the asymptotic expansions

The asymptotic results in [Sch00, And12] are phrased in terms of the L^2 operator norm, which makes good sense in the context of the finite-dimensional spaces of holomorphic sections. In the situation at hand, however, we need to consider the space of all smooth sections, on which differential operators are typically unbounded with respect to the L^2 norm.

One natural way around this would be to use Sobolev norms, but some care is needed, as their definition for sections of smooth bundles relies in general on various choices, in an essential way. As a matter of fact, the resulting norms are equivalent but different, so what is intrinsically well defined is just the Sobolev topology alone. However, in the case when the asymptotic limit is taken for $s \to \infty$ while keeping k fixed, one can pick one Sobolev norm $\|\cdot\|_W$ and work with it. We do not discuss here how to make precise sense of convergence in Sobolev norm if k is allowed to change.

Alternatively, one can use strong convergence and say that an operator D decays at a given rate if the L^2 norm of $D\psi$ does, for every smooth section ψ with $\|\psi\| = 1$. Of course this approach still requires that the bundle, hence k, is fixed, and it does not define a norm for the operators. On the other hand, it has the advantage of carrying an intrinsic meaning in terms of the Hilbert space structure relevant for geometric quantisation.

Another way to phrase the matters of convergence is via the symbols of differential operators, see Section A.1. If D is a finite-order differential operator acting on $\mathscr{L}^k$, its total symbol $\sigma(D)$ is a formal (i.e. not necessarily homogeneous) tensor field on $\mathcal{M}$, whose L^2 norm is well defined via the Riemannian metric. Since the correspondence between differential operators and totally symmetric tensor fields is a bijection, this gives a norm $\|\cdot\|_T$ on the space of operators, which makes sense independently on t.

Suppose now that k is fixed, and that an operator D depends on s as a Laurent polynomial, i.e. D has the form

$$D = \sum_{n=N_0}^{N_1} D^{(n)} s^{-n} ,$$

where each $D^{(n)}$ is independent of s. One may then argue that each of the norms considered above is bounded by $C|s|^{-N_0}$ for some positive real constant C. Whenever this is the case, we shall write

$$D = o\left(|s|^{-\alpha}\right) \quad \text{for } s \to \infty,$$

for every $\alpha < N_0$, without any further reference to the norm. In a similar fashion, if D depends on t and its symbol can be expressed as a Laurent polynomial in t and $\bar{t}$, then $\|D\|_T$ is also bounded by a power of $|t|$; we shall write

$$D = o\left(|t|^{-\alpha}\right) \quad \text{for } t \to \infty.$$

In view of the formal approach used in the next section, and in order to make the above statements even more precise, it is convenient to introduce the following graded algebras.

Definition 3.2.1. *For each fixed $k \in \mathbb{Z}_{>0}$ let D_k be the algebra of finite-order differential operators on $\mathcal{C}^\infty(\mathcal{M}, \mathscr{L}^k)$, and call*

$$\mathscr{A}_k := \mathrm{D}_k[[s^{-1}]]$$

regarded as a graded algebra.

As endomorphisms of $\mathscr{A}_k$ we shall only consider those coming as formal power series with coefficients in $\mathrm{End}(\mathrm{D}_k)$. In the setting of the formal Hitchin connection, all the transformations of Toeplitz operators are obtained by acting on the function defining them as differential operators. Similarly, we shall require that the coefficients of the endomorphisms of $\mathscr{A}_k$ should act as differential operators on their symbols.

The analogous notion in the case when k is also allowed to vary is complicated by the fact that this parameter also enters the curvature of the pre-quantum connection, and hence the commutators of operators. This is another motivation for using symbols rather than the differential operators themselves. Consider then the algebra $\mathscr{S}$ of formal, totally symmetric, contra-variant tensor fields on M, with the symmetrised tensor product. Thinking of k, $|t|$ and $\bar{t}^{-1}$ as independent variables, consider the vector space $\mathscr{S}[k][[t^{-1}, \bar{t}^{-1}]]$ with grading given by $\deg(t^{-1}) = \deg(\bar{t}^{-1}) = 1$ and $\deg(k) = -1$.

Definition 3.2.2. *Let $\mathscr{A}$ denote the graded algebra*

$$\mathscr{A} := \mathscr{S}[k][[t^{-1}, \bar{t}^{-1}]] \Big/ (a)\,,$$

where a denotes the homogeneous element

$$a = \frac{1}{t} + \frac{1}{\bar{t}} - \frac{2k}{t\bar{t}}\,.$$

3.3 The formal approach

3.3.1 The formal Hitchin-Witten connection

The Hitchin-Witten covariant derivative of an operator D which depends smoothly on σ reads as

$$\tilde{\boldsymbol{\nabla}}^{\mathrm{End}} D = \mathrm{d}^F_{\mathcal{T}} D + \frac{1}{2t}[b, D] - \frac{1}{2\bar{t}}[\bar{b}, D]\,.$$

Here $\mathrm{d}_{\mathcal{T}}^F$ denotes the exterior differential on $\mathcal{T}$ twisted with the Ricci potential, which acts on operators as

$$\left(\mathrm{d}_{\mathcal{T}}^F D\right)(V) = \nabla_V^F(D) = V[D] + \left[\mathcal{C}_{V[F]}, D\right].$$

We now study the existence of a formal connection on $\mathscr{A}_k \times \mathcal{T}$ reproducing this covariant derivative asymptotically. The following statement is a more precise formulation of Theorem 3.1.4.

Definition 3.3.1. *By a formal connection on $\mathscr{A}_k \times \mathcal{T}$ we mean a sum*

$$\tilde{\mathcal{D}} = \nabla^F + \sum_{l=0}^{\infty} \tilde{\mathcal{D}}^{(l)} s^{-l}, \tag{3.2}$$

where each $\tilde{\mathcal{D}}^{(l)}$ is a 1-form on $\mathcal{T}$ with values in $\mathrm{End}(\mathrm{D}_k)$.

Theorem 3.3.1. *There exists a unique formal connection $\tilde{\mathcal{D}}$ on $\mathscr{A}_k \times \mathcal{T}$ such that, for any positive integer L, any vector V on $\mathcal{T}$ and any operator $D \in \mathrm{D}_k$, one has*

$$\tilde{\boldsymbol{\nabla}}_V^{\mathrm{End}}(D) - \nabla_V^F(D) - \sum_{l=1}^{L} \tilde{\mathcal{D}}_V^{(l)}(D) s^{-l} = o\left(|s|^{-L}\right). \tag{3.3}$$

in the sense specified in Section 3.2. Moreover, the formal connection $\tilde{\mathcal{D}}$ is flat.

Proof. First of all, consider the Taylor expansions of t^{-1} and $\bar{t}^{-1}$ in s at $s = \infty$:

$$\frac{1}{t} = \frac{1}{k+is} = -\frac{1}{k}\sum_{n=1}^{\infty}\left(\frac{ik}{s}\right)^n, \qquad \frac{1}{\bar{t}} = \frac{1}{k-is} = -\frac{1}{k}\sum_{n=1}^{\infty}\left(-\frac{ik}{s}\right)^n. \tag{3.4}$$

These converge for $|s| > k$; in particular the error of each L-th truncated sum decays faster than $|s|^{-L}$ for $s \to \infty$, since k is fixed. Basing on this, we choose

$$\tilde{\mathcal{D}}^{(l)}(D) := -\frac{(ik)^l}{2k}\left[b - (-1)^l \bar{b}, D\right].$$

As a consequence of the convergence of (3.4), for any positive integer L one can notice that for any norm $\|\cdot\|$ one has

$$\left\| \tilde{\boldsymbol{\nabla}}_V^{\mathrm{End}}(D) - \nabla_V^F(D) - \sum_{l=1}^{L} \tilde{\mathcal{D}}_V^{(l)} s^{-l}(D) \right\| \leq$$

$$\leq \left\| \left(\frac{1}{2t} + \frac{1}{2k}\sum_{l=1}^{L}\left(\frac{ik}{s}\right)^l\right)[b(V), D] \right\| + \left\| \left(\frac{1}{2\bar{t}} + \frac{1}{2k}\sum_{l=1}^{L}\left(-\frac{ik}{s}\right)^l\right)\left[\bar{b}(V), D\right] \right\| =$$

$$= o\left(|s|^{-L}\right)\|[b, D]\| + o\left(|s|^{-L}\right)\left\|\left[\bar{b}, D\right]\right\| = o\left(|s|^{-L}\right).$$

This goes verbatim for the case when all the operators are applied to a smooth section ψ, proving the existence. The uniqueness is implied by the condition (3.3). Indeed, if two such formal connections $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}}'$ are given, their 0-order terms agree by assumption. Suppose, on the other hand, that $\tilde{\mathcal{D}}^{(L)} \neq \tilde{\mathcal{D}}'^{(L)}$ for some L, which we assume to be minimum. Then for every $D \in \mathrm{D}_k$ one has:

$$\left\| \left(\tilde{\mathcal{D}}^{(L)} - \tilde{\mathcal{D}}'^{(L)} \right)(D) \right\| s^{-L} \leq \left\| \left(\tilde{\nabla}^{\mathrm{End}}_V - \nabla^F_V - \sum_{l=1}^{L} \tilde{\mathcal{D}}^{(l)}_V s^{-l} \right)(D) \right\| + \\ + \left\| \left(\tilde{\nabla}^{\mathrm{End}}_V - \nabla^F_V - \sum_{l=1}^{L} \tilde{\mathcal{D}}'^{(l)}_V s^{-l} \right)(D) \right\| .$$

The last expression decays faster than s^{-L}, but since $\left\| \left(\tilde{\mathcal{D}}^{(L)} - \tilde{\mathcal{D}}'^{(L)} \right)(D) \right\|$ does not depend on s, it has to be zero for every D, which contradicts $\tilde{\mathcal{D}}^{(L)} \neq \tilde{\mathcal{D}}'^{(L)}$.

Flatness can be proven in a similar fashion. Indeed, the curvature of $\tilde{\mathcal{D}}$ is expressed by

$$\sum_{l=1}^{\infty} \left(\mathrm{d}^F_{\mathcal{T}} \tilde{\mathcal{D}}^{(l)} + \frac{1}{2} \sum_{n+m=l} \left[\tilde{\mathcal{D}}^{(n)} \wedge \tilde{\mathcal{D}}^{(m)} \right] \right) s^{-l} .$$

More explicitly, the l-th coefficient of its action on operators is given by the commutator with

$$-\frac{(ik)^l}{2k} \mathrm{d}^F_{\mathcal{T}} \left(b - (-1)^l \overline{b} \right) + \frac{(ik)^l}{8k^2} \sum_{n+m=l} \left[\left(b - (-1)^n \overline{b} \right) \wedge \left(b - (-1)^m \overline{b} \right) \right] .$$

Since $[b \wedge b]$ and $[\overline{b} \wedge \overline{b}]$ take values in central differential operators (see Proposition 4.6 in [AG14]), the whole curvature is

$$\sum_{l=1}^{\infty} \left(-\frac{(ik)^l}{2k} \mathrm{d}^F_{\mathcal{T}} \left(b - (-1)^l \overline{b} \right) + \frac{(ik)^l}{4k^2} \sum_{n+m=l} (-1)^n \left[b \wedge \overline{b} \right] \right) s^{-l} \tag{3.5}$$

For comparison, the curvature of $\tilde{\nabla}^{\mathrm{End}}$ is given by the commutator with

$$\mathrm{d}^F_{\mathcal{T}} \left(\frac{1}{2t} b - \frac{1}{2\overline{t}} \overline{b} \right) - \frac{1}{4|t|^2} [b \wedge \overline{b}] \tag{3.6}$$

It can be seen that the coefficients in (3.5) give the Laurent expansions of those in (3.6) at $s = \infty$. Similar arguments to those used to prove the existence of $\tilde{\mathcal{D}}$, combined with the flatness of $\tilde{\nabla}^{\mathrm{End}}$, imply that for every positive integer L and every operator D one has

$$\left[\sum_{l=1}^{L} \left(\frac{(ik)^l}{2k} \mathrm{d}^F_{\mathcal{T}} \left(b - (-1)^l \overline{b} \right) - \frac{(ik)^l}{4k^2} \sum_{n+m=l} (-1)^n \left[b \wedge \overline{b} \right] \right) s^{-l}, D \right] = o(s^{-L}).$$

On the other hand, this expression depends on s as a Laurent polynomial of order at most L, hence it vanishes. Therefore, all the truncations of the curvature of $\tilde{\mathcal{D}}$ are zero, which proves the flatness. □

Of course the same result may also be proven by direct application of the algebraic relations found in [AG14] in the process of proving the projective flatness of the Hitchin and Hitchin-Witten connection.

We emphasise that the symbols of $\tilde{\mathcal{D}}_V^{(l)}(D)$ depend as differential operators on those of D, so this connection acts on the fibres by endomorphisms of $\mathscr{A}_k$ of the kind described after Definition 3.2.1.

Definition 3.3.2. *We shall refer as the Hitchin-Witten connection, and use the notation $\tilde{\mathcal{D}}$, to the one defined by Theorem 3.1.4, explicitly given by*

$$\tilde{\mathcal{D}}(D) = \nabla^F(D) - \frac{1}{2k}\sum_{l=1}^{\infty}(ik)^l\left[b-(-1)^l\bar{b}, D\right]s^{-l}. \tag{3.7}$$

3.3.2 The recursion of differential equations

We now consider an operator D depending smoothly on σ, and look for an asymptotically $\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}$-parallel expansion

$$\mathcal{R}(D) := \sum_{l=0}^{\infty}\mathcal{R}^{(l)}(D)s^{-l}, \tag{3.8}$$

where $\mathcal{R}^{(0)}(D)$ should be D. As usual, the asymptotic requirement is that, for every vector V on $\mathcal{T}$ and every positive integer L, the covariant derivative of the L-th truncation of the series should decay faster than s^{-L}:

$$\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}\left(\sum_{l=0}^{L}\mathcal{R}^{(l)}(D)s^{-l}\right) = o(s^{-L}). \tag{3.9}$$

Notice that such an expansion cannot exist unless $\mathrm{d}^F_{\mathcal{T}}D = 0$. Indeed, this is the only contribution to the covariant derivative of degree 0 in s, so it cannot be counter-balanced by any others. We shall then assume from now on that this condition is indeed satisfied. Notice that, in the case of a curve operator $\mathcal{C}_f$, this is equivalent to f being independent of σ.

We set the problem in terms of the formal Hitchin-Witten connection by using the following fact.

Lemma 3.3.2. *The asymptotic condition (3.9) is satisfied if and only if, as a formal power series, (3.8) is covariantly constant with respect to $\tilde{\mathcal{D}}$.*

Proof. Written explicitly, the formal covariant derivative of a power series as in (3.8) reads

$$\tilde{\mathcal{D}}\Big(\mathcal{R}(D)\Big) = \sum_{l=0}^{\infty}\left(\mathrm{d}^F_{\mathcal{T}}\mathcal{R}^{(l)}(D) + \sum_{n=1}^{l}\tilde{\mathcal{D}}^{(n)}\Big(\mathcal{R}^{(l-n)}(D)\Big)\right)s^{-l}. \tag{3.10}$$

Let now V and L be fixed. By the defining property of the Hitchin-Witten connection, one has

$$\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_V\left(\sum_{l=0}^{L}\mathcal{R}^{(l)}(D)s^{-l}\right) = \nabla^F_V\left(\sum_{l=0}^{L}\mathcal{R}^{(l)}(D)s^{-l}\right)+ \\ + \sum_{n=1}^{L}\tilde{\mathcal{D}}^{(n)}_V\left(\sum_{l=0}^{L}\mathcal{R}^{(l)}(D)s^{-l}\right)s^{-n} + o(s^{-L})\,.$$

All the terms of degree higher than L in s^{-1} on the right-hand side may be absorbed in $o(s^{-L})$ and disregarded. After rearranging the others by gathering the terms with the same degree, one obtains:

$$\sum_{l=0}^{L}\left(\nabla^F_V\mathcal{R}^{(l)}(D) + \sum_{n=1}^{l}\tilde{\mathcal{D}}^{(l)}_V\left(\mathcal{R}^{(l-n)}(D)\right)\right)s^{-l} + o(s^{-L})\,.$$

By comparison with (3.10), this sum expresses the truncation of the formal Hitchin-Witten covariant derivative of $\mathcal{R}(D)$. The assertion follows. □

Spelling out the definition of $\tilde{\mathcal{D}}$ in (3.10), the condition for the series to be covariantly constant becomes

$$\sum_{l=0}^{\infty}\mathrm{d}^F_{\mathcal{T}}\left(\mathcal{R}^{(l)}(D)\right)s^{-l} = \frac{1}{2k}\sum_{n=1}^{\infty}\sum_{l=0}^{\infty}\left(\frac{ik}{s}\right)^n s^{-l}\left[b-(-1)^n\bar{b},\mathcal{R}^{(l)}(D)\right].$$

By collecting the coefficients of each power of s^{-1} on the right-hand side one can finally reduce this to:

$$\mathrm{d}^F_{\mathcal{T}}\mathcal{R}^{(l)}(D) = \frac{1}{2k}\sum_{n=1}^{l}(ik)^n\left[b-(-1)^n\bar{b},\mathcal{R}^{(l-n)}(D)\right]\,. \tag{3.11}$$

This implies in particular that the right-hand side should be an exact form with respect to $\mathrm{d}^F_{\mathcal{T}}$, which gives a necessary condition on the solution of the first l steps in order for the next one to exist. An obstruction comes then from the differential of the right-hand side; the next result, which is a re-phrasing of Theorem 3.3.3, shows that this obstruction vanishes.

Theorem 3.3.3. *Suppose that l steps of the recursion have been solved, giving an $\mathcal{R}^{(n)}(D)$ for every $n \leq l$. Then:*

$$\mathrm{d}^F_{\mathcal{T}}\left(\sum_{n=1}^{l}(ik)^n\left[b-(-1)^n\bar{b},\mathcal{R}^{(l-n)}(D)\right]\right) = 0$$

Proof. We proceed by using the properties of the differential to expand it as

$$\mathrm{d}^F_{\mathcal{T}}\left(\sum_{n=1}^{l}(ik)^n\left[b-(-1)^n\overline{b},\mathcal{R}^{(l-n)}(D)\right]\right)=$$

$$=\sum_{n=1}^{l}(ik)^n\left(\left[\mathrm{d}^F_{\mathcal{T}}\left(b-(-1)^n\overline{b}\right),\mathcal{R}^{(l-n)}(D)\right]+\left[\left(b-(-1)^n\overline{b}\right)\wedge\mathrm{d}^F_{\mathcal{T}}\mathcal{R}^{(l-n)}(D)\right]\right). \tag{3.12}$$

Using the recursive relation on the second term in the parentheses, replacing $j=m-n$ and then carefully exchanging the sums one obtains:

$$\frac{1}{2k}\sum_{n=1}^{l}\sum_{j=1}^{l-n}(ik)^{n+j}\left[\left(b-(-1)^n\overline{b}\right)\wedge\left[\left(b-(-1)^j\overline{b}\right)\wedge\mathcal{R}^{(l-n-j)}(D)\right]\right]=$$

$$=\frac{1}{2k}\sum_{n=1}^{l}\sum_{m=n+1}^{l}(ik)^m\left[\left(b-(-1)^n\overline{b}\right)\wedge\left[\left(b-(-1)^{m-n}\overline{b}\right)\wedge\mathcal{R}^{(l-m)}(D)\right]\right]=$$

$$=\frac{1}{2k}\sum_{m=2}^{l}\sum_{n=1}^{m-1}(ik)^m\left[\left(b-(-1)^n\overline{b}\right)\wedge\left[\left(b-(-1)^{m-n}\overline{b}\right)\wedge\mathcal{R}^{(l-m)}(D)\right]\right]$$

Except for the factors, each term of this sum can be expanded as:

$$\left[b\wedge\left[b\wedge\mathcal{R}^{(l-m)}(D)\right]\right]+(-1)^m\left[\overline{b}\wedge\left[\overline{b}\wedge\mathcal{R}^{(l-m)}(D)\right]\right]+$$
$$-(-1)^n\left((-1)^m\left[b\wedge\left[\overline{b}\wedge\mathcal{R}^{(l-m)}(D)\right]\right]+\left[\overline{b}\wedge\left[b\wedge\mathcal{R}^{(l-m)}(D)\right]\right]\right) \tag{3.13}$$

By the analogue of the Jacobi identity presented in Lemma A.2.1, the following expression vanishes:

$$\left[b\wedge\left[b\wedge\mathcal{R}^{(l-m)}(D)\right]\right]-\left[b\wedge\left[\mathcal{R}^{(l-m)}(D)\wedge b\right]\right]+\left[\mathcal{R}^{(l-m)}(D)\wedge[b\wedge b]\right].$$

While the right-most term is zero due to the centrality of $[b\wedge b]$, the other two actually sum together, showing that the first term in (3.13) vanishes. It can be argued in the same way that neither the second gives any contribution. Due to the factor $(-1)^n$, the sum over n of the remaining two terms yields 0 whenever the summation range has even length, i.e. when m is odd. On the other hand, for even m the Jacobi identity gives:

$$\left[b\wedge\left[\overline{b}\wedge\mathcal{R}^{(l-m)}(D)\right]\right]+\left[\overline{b}\wedge\left[b\wedge\mathcal{R}^{(l-m)}(D)\right]\right]=-\left[\left[b\wedge\overline{b}\right]\wedge\mathcal{R}^{(l-m)}(D)\right].$$

From this we conclude that the second part ot the sum in (3.12) equals

$$-\frac{1}{2k}\sum_{0<2r\leq l}(ik)^{2r}\left[\left[b\wedge\overline{b}\right]\wedge\mathcal{R}^{(l-2r)}(D)\right]. \tag{3.14}$$

On the other hand, the remaining part of (3.12) can be handled by comparison with the equation expressing the flatness of $\tilde{\mathcal{D}}$. Applying the curvature to $\mathcal{R}^{(l-n)}(D)$ and taking the part of order n in s^{-1} gives

$$\left[d^F_{\mathcal{T}}(b-(-1)^n\bar{b})+\frac{1}{4k}\sum_{m=1}^{n-1}\left[\left(b-(-1)^m\bar{b}\right)\wedge\left(b-(-1)^{n-m}\bar{b}\right)\right],\mathcal{R}^{(l-n)}(D)\right].$$

Because this vanishes, using again that $[b\wedge b]$ and $[\bar{b}\wedge\bar{b}]$ are central one obtains that the first part of (3.12) is

$$\frac{1}{4k}\sum_{n=1}^{l}\sum_{m=1}^{n-1}(ik)^n(-1)^m\left(1+(-1)^n\right)\left[\left[b\wedge\bar{b}\right],\mathcal{R}^{(l-n)}(D)\right].$$

As before, the sum over m gives 0 whenever $n-1$ is even, leaving

$$\frac{1}{2k}\sum_{0<2r\leq l}(ik)^{2r}\left[\left[b\wedge\bar{b}\right],\mathcal{R}^{(l-n)}(D)\right].$$

The proof is concluded by comparing with (3.14). □

3.3.3 First step of the recursion for curve operators

For $l=1$ and $D=\mathcal{C}_f$, the recursion (3.11) reads

$$\mathrm{d}^F_{\mathcal{T}}\mathcal{R}^{(1)}(f)=\frac{i}{2}\left[b+\bar{b},\mathcal{C}_f\right]. \tag{3.15}$$

Identifying a function with its curve operators for notational convenience, and recalling the definition of b and $\bar{b}$, one has

$$b+\bar{b}=\Delta_{\tilde{G}}+2\nabla_{\tilde{G}\cdot\mathrm{d}F}-2\lambda\mathrm{d}_{\mathcal{T}}F.$$

Notice that the first an last terms are both exact, with primitives $-\Delta$ and $-2\lambda F$, respectively. However, the last term does not contribute to the commutator in (3.15), which becomes

$$\frac{i}{2}\left[b+\bar{b},\mathcal{C}_f\right]=\Delta_{\tilde{G}}f+2\nabla_{\tilde{G}\cdot\mathrm{d}f}+2\,\mathrm{d}f\cdot\tilde{G}\cdot\mathrm{d}F.$$

Since f does not depend on σ, the first term is clearly $\mathrm{d}_{\mathcal{T}}$-exact, with primitive $-\Delta f$; being central as a differential operator, this is also a primitive for $\mathrm{d}^F_{\mathcal{T}}$. The second term can be written as $-2\mathrm{d}_{\mathcal{T}}\nabla_{\tilde{g}\cdot\mathrm{d}f}$, while on the other hand

$$-2\left[\mathrm{d}_{\mathcal{T}}F,\nabla_{\tilde{g}\cdot\mathrm{d}f}\right]=2\,\mathrm{d}f\cdot\tilde{g}\cdot\mathrm{d}\mathrm{d}_{\mathcal{T}}F=2\mathrm{d}_{\mathcal{T}}\left(\mathrm{d}f\cdot\tilde{g}\cdot\mathrm{d}F\right)+2\,\mathrm{d}f\cdot\tilde{G}\cdot\mathrm{d}F.$$

This way one obtains the missing term, up to an exact correction. All in all, we have found that

$$-\mathrm{d}^F_{\mathcal{T}}\left(2\nabla_{\tilde{g}\cdot\mathrm{d}f}+2\,\mathrm{d}f\cdot\tilde{g}\cdot\mathrm{d}F+\Delta f\right)=\left[\Delta_{\tilde{G}}+2\nabla_{\tilde{G}\cdot\mathrm{d}F}-2\lambda\mathrm{d}_{\mathcal{T}}F,f\right],$$

thus proving the following statement.

Theorem 3.3.4. *For a curve operator $\mathcal{C}_f$, the first step of the recursion has a solution expressed as*

$$\mathcal{R}^{(1)} = -\frac{i}{2}\left(2\nabla_{\tilde{g}\cdot \mathrm{d}f} + \mathcal{C}_{2\,\mathrm{d}f\cdot\tilde{g}\cdot\mathrm{d}F+\Delta f}\right). \tag{3.16}$$

Moreover, this expression is Γ-invariant for any action of a group Γ on M and $\mathcal{T}$ making the Kähler structure equivariant.

3.3.4 Recursion in t and $\bar{t}$ and cohomological obstruction

We include this section to show where the complications arise in the analogous formal approach in the full parameter t, in addition to the technical ones discussed in Section 3.2.

We now consider $\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}$ as a formal connection, being manifestly a Laurent polynomial in t and $\bar{t}$. We address the problem of finding a perturbation $P(f)$ of $\mathcal{C}_f$ in such a way that $\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}P(f) \equiv 0$ formally. Explicitly, the equation we are interested in reads, for every vector V on $\mathcal{T}$, as

$$\sum_{l=0}^{\infty} V\left[P^{(l)}(f)\right] + \sum_{l=0}^{\infty}\left[\mathcal{C}_{V[F]}, P^{(l)}(f)\right] + \sum_{l=0}^{\infty}\left[\frac{1}{2t}b(V) - \frac{1}{2\bar{t}}\bar{b}(V), P^{(l)}(f)\right] = 0.$$

Putting for convenience $P^{(-1)}(f) = 0$, by separating this degree-by-degree one obtains the recursive relation for every $l \geq 0$:

$$\mathrm{d}^F_{\mathcal{T}} P^{(l+1)}(f) = -\left[\frac{1}{2t}b - \frac{1}{2\bar{t}}\bar{b}, P^{(l)}(f)\right]. \tag{3.17}$$

Again, a necessary condition for the existence of a solution $P^{(l+1)}(f)$ is for right-hand side to be $\mathrm{d}^F_{\mathcal{T}}$-closed. However, we its differential is given by the following formula.

Proposition 3.3.5. *If the recursive relation is satisfied for $0 = P^{(-1)}(f)$, $\mathcal{C}_f = P^{(0)}(f), \ldots, P^{(l)}(f)$, then the differential of the right-hand side of (3.17) is:*

$$\mathrm{d}^F_{\mathcal{T}}\left[\frac{1}{2t}b - \frac{1}{2\bar{t}}\bar{b}, P^{(l)}(f)\right] = \frac{1}{4|t|^2}\left[\left[b \wedge \bar{b}\right], P^{(l)}(f)\right]. \tag{3.18}$$

Proof. Let for convenience

$$\tilde{b} = \frac{1}{2t}b - \frac{1}{2\bar{t}}\bar{b}.$$

Following the lines of the proof of Theorem 3.3.3, we proceed by direct computation of the twisted differential:

$$\mathrm{d}^F_{\mathcal{T}}\left[\tilde{b}, P^{(l)}(f)\right] = \left[\left(\mathrm{d}^F_{\mathcal{T}}\tilde{b}\right) \wedge P^{(l)}(f)\right] - \left[\tilde{b} \wedge \left(\mathrm{d}^F_{\mathcal{T}} P^{(l)}(f)\right)\right]. \tag{3.19}$$

Using the recursive relation, the second term can be written as:

$$-\left[\tilde{b}\wedge \mathrm{d}^F_{\mathcal{T}}P^{(l)}(f)\right] = \left[\tilde{b}\wedge\left[\tilde{b}\wedge P^{(l-1)}(f)\right]\right].$$

The analogue of the Jacobi identity gives:

$$-\left[\tilde{b}\wedge\left[\tilde{b}\wedge P^{(l-1)}(f)\right]\right]+\left[\tilde{b}\wedge\left[P^{(l-1)}(f)\wedge\tilde{b}\right]\right]-\left[P^{(l-1)}(f)\wedge\left[\tilde{b}\wedge\tilde{b}\right]\right]=0\,.$$

After the due rearrangements, this implies that

$$-\left[\tilde{b}\wedge \mathrm{d}^F_{\mathcal{T}}P^{(l)}(f)\right] = \frac{1}{2}\left[\left[\tilde{b}\wedge\tilde{b}\right],P^{(l-1)}(f)\right].$$

Recalling the definition of $\tilde{b}$, one can write

$$\left[\tilde{b}\wedge\tilde{b}\right] = \frac{1}{4t^2}\left[b\wedge b\right] - \frac{1}{2|t|^2}\left[b\wedge\bar{b}\right] + \frac{1}{4\bar{t}^2}\left[\bar{b}\wedge\bar{b}\right].$$

However, due to the centrality of $[b\wedge b]$ and $[\bar{b}\wedge\bar{b}]$, one finds

$$-\left[\tilde{b}\wedge \mathrm{d}^F_{\mathcal{T}}P^{(l)}(f)\right] = -\frac{1}{4|t|^2}\left[\left[b\wedge\bar{b}\right]\wedge P^{(l-1)}(f)\right]. \tag{3.20}$$

By induction, for positive l this represents the obstruction to the existence of $P^{(l)}(f)$, so it vanishes, while for $l=0$ one has simply $P^{(-1)}(f)=0$. This proves that the second term on the right-hand side of (3.19) gives no contribution.

As for the other term, we shall compute it by comparison with the explicit expression for the curvature of $\tilde{\nabla}^{\mathrm{End}}$ applied to $P^{(l)}(f)$. By the flatness of ∇^F, one can use (A.5) to obtain:

$$0 = \left[\mathrm{d}^F_{\mathcal{T}}\tilde{b} + \frac{1}{2}\left[\tilde{b}\wedge\tilde{b}\right],P^{(l)}(f)\right].$$

From this we conclude:

$$\mathrm{d}^F_{\mathcal{T}}\left[\tilde{b},P^{(l)}(f)\right] = -\frac{1}{2}\left[\left[\tilde{b}\wedge\tilde{b}\right]\wedge P^{(l)}(f)\right].$$

Using again the expression for $[\tilde{b}\wedge\tilde{b}]$ we find the desired result. □

The expression (3.18) does not vanish in general. Indeed, for V and W vector fields on $\mathcal{T}$, the principal symbol of $\left[b\wedge\bar{b}\right](V,W)$ is $-i4k\Theta(V,W)$

(see [AG14], Proposition 4.9). In particular, it follows that the symbols of $\left[[b(V),\bar{b}(W)],\mathcal{C}_f\right]$ are

$$\sigma_1\left[[b(V),\bar{b}(W)],\mathcal{C}_f\right] = -8ikdf\cdot\Theta(V,W),$$
$$\sigma_0\left[[b(V),\bar{b}(W)],\mathcal{C}_f\right] = -4ik\Big(\Delta_{\Theta(V,W)}f + 2df\cdot\Theta(V,W)\cdot dF\Big).$$

This shows that the recursion (3.17) has no solution in general, even for $l = 0$, and the formal approach in the full parameter t fails even at the level of operators of order 0.

3.4 Solution of the recursion in genus 1

We now consider the situation arising on (the smooth part of) the moduli space $\mathcal{M}$ of flat $\mathrm{SU}(n)$-connections over a closed oriented smooth surface of genus 1, discussed in detail in Section 2.5. In that case, the moduli space has natural coordinates in which the Kähler metric g_σ has constant coefficients for every $\sigma \in \mathcal{T}$, thus giving a trivial Levi-Civita connection. Therefore, the Ricci potential vanishes as a consequence of the flatness of g_σ, which in turn implies that of the Hitchin-Witten connection. These properties, very specific to this case, constitute an important simplification that allows one to solve the recursion (3.11) completely.

First of all, notice that $b+\bar{b} = \Delta_{\tilde{G}} = -\mathrm{d}_{\mathcal{T}}\Delta$ is in this case an exact form, and moreover $\mathrm{d}^F_{\mathcal{T}}$ can be simply replaced with $\mathrm{d}_{\mathcal{T}}$. A solution to the first step of the recursion (3.15) is then easily found as a commutator with the primitive of $b+\bar{b}$:

$$\mathrm{d}_{\mathcal{T}}\left[-\frac{i}{2}\Delta, D\right] = \frac{i}{2}\left[b+\bar{b}, D\right].$$

Notice that, for a curve operator $\mathcal{C}_f$, this is indeed the solution $\mathcal{R}^{(1)}(f)$ found in (3.16):

$$\left[-\frac{i}{2}\Delta,\mathcal{C}_f\right] = -\frac{i}{2}\Big(\mathcal{C}_{\Delta f} + 2\nabla_{\tilde{g}\cdot\mathrm{d}f}\Big).$$

This hints that one may look for a solution of the next steps by iterating the commutator with this operator. Since Δ is defined directly from the Kähler metric, it follows that a solution constructed this way is automatically Γ-invariant for any Γ as usual.

Definition 3.4.1. *Call a the primitive considered above:*

$$a := -\frac{i}{2}\Delta.$$

Also, for every non-negative integer l and σ-independent $D \in \mathrm{D}_k$ let $A^{(l)}(D)$ denote the following operator:

$$A^{(l)}(D) := \frac{\mathrm{ad}_a^l(D)}{l!} .$$

When $D = \mathcal{C}_f$ for some function f, we shall use the short-hand $A^{(l)}(f)$ in place of $A^{(l)}(\mathcal{C}_f)$. In this notation, one has that $\mathcal{R}^{(1)}(f) = A^{(1)}(f)$.

Lemma 3.4.1. *For every positive integer l one has*

$$\left[a^l, b \pm \overline{b}\right] = \sum_{n=1}^{l} (2ik)^n \binom{l}{n} \left(b \pm (-1)^n \overline{b}\right) a^{l-n} .$$

Proof. We prove the statement working by induction on l.

For $l = 1$, notice that all the symbols of $\left[a, b \pm \overline{b}\right]$ vanish (see e.g. Lemma A.1.1), as a consequence of that $\nabla \tilde{g} = \nabla \tilde{G} = 0$, except possibly for the second

$$\begin{aligned} \sigma_2 \left[a, b \pm \overline{b}\right] =& 4ik \cdot \left(-\frac{i}{2}\right) \mathcal{S}\left((G \pm \overline{G}) \cdot \omega \cdot \tilde{g}\right) = \\ =& 2k\mathcal{S}((G \pm \overline{G}) \cdot J) . \end{aligned}$$

Because G is of type $(2, 0)$, the contraction $G \cdot J$ gives iG, and similarly $\overline{G} \cdot J = -i\overline{G}$. Therefore we find

$$\sigma_2 \left[a, b \pm \overline{b}\right] = 2ki(G \mp \overline{G}) = 2ik\sigma_2\left(b \mp \overline{b}\right) .$$

Since the only non-vanishing symbol of $b \mp \overline{b}$ is the second, this concludes the proof of the base step.

Assuming that the statement is true for l, one checks that

$$\begin{aligned} \left[a^{l+1}, b \pm \overline{b}\right] =& a \left[a^l, b \pm \overline{b}\right] + \left[a, b \pm \overline{b}\right] a^l \\ =& a \sum_{n=1}^{l} (2ik)^n \binom{l}{n} \left(b \pm (-1)^n \overline{b}\right) a^{l-n} - 2ik(b \mp \overline{b}) a^l = \\ =& \sum_{n=1}^{l} (2ik)^n \binom{l}{n} \left(b \pm (-1)^n \overline{b}\right) a^{l-n+1} - 2ik(b \mp \overline{b}) a^l \\ &+ \sum_{n=1}^{l} (2ik)^n \binom{l}{n} \left[a, b \pm (-1)^n \overline{b}\right] a^{l-n} = \\ =& \sum_{n=1}^{l} (2ik)^n \binom{l}{n} \left(b \pm (-1)^n \overline{b}\right) a^{l-n+1} - 2ik(b \mp \overline{b}) a^l \\ &+ \sum_{n=1}^{l} (2ik)^{n+1} \binom{l}{n} \left(b \pm (-1)^{n+1} \overline{b}\right) a^{l-n} . \end{aligned}$$

After shifting the summation index, the second sum becomes

$$\sum_{n=2}^{l+1}(2ik)^n\binom{l}{n-1}\left(b\pm(-1)^n\overline{b}\right)a^{l+1-n}.$$

Separating the terms corresponding to $n = 1$ and $n = l+1$ in the two sums and putting them together yields

$$\left[a^{l+1}, b\pm\overline{b}\right] = \sum_{n=2}^{l}(2ik)^n\left[\binom{l}{n}+\binom{l}{n-1}\right]\left(b\pm(-1)^n\overline{b}\right)a^{l-n+1}+$$
$$+(2ik)^{l+1}\left(b\pm(-1)^{l+1}\overline{b}\right)+2ikl(b\mp\overline{b})a^l+2ik(b\mp\overline{b})a^l$$

Using the properties of the binomials, it is now easy to recognise the summand as the expected one. The three separated terms form together the summands for $n = 1$ and $n = l+1$, and re-including them in the sum gives the result. □

This can now be used for calculating the differential of a^l.

Lemma 3.4.2. *For every non-negative l one has:*

$$\mathrm{d}_{\mathcal{T}}a^l = \frac{1}{4k}\sum_{n=1}^{l}\binom{l}{n}(2ik)^n\left(b-(-1)^n\overline{b}\right)a^{l-n}.$$

Proof. By direct computation, using the Leibniz rule and the commutators calculated above:

$$\begin{aligned}
\mathrm{d}_{\mathcal{T}}a^l &= \frac{i}{2}\sum_{m=0}^{l-1}a^{l-m-1}(b+\overline{b})a^m = \\
&= \frac{i}{2}\sum_{m=0}^{l-1}\left((b+\overline{b})a^{l-1}+\left[a^{l-m-1}, b+\overline{b}\right]a^m\right) = \\
&= \frac{il}{2}(b+\overline{b})a^{l-1}+\sum_{m=0}^{l-1}\sum_{n=1}^{l-m-1}(2ik)^n\binom{l-m-1}{n}\left(b+(-1)^n\overline{b}\right)a^{l-n-1},
\end{aligned}$$

and by exchanging the sums and shifting m one finds

$$\frac{il}{2}(b+\overline{b})a^{l-1}+\frac{i}{2}\sum_{n=1}^{l-1}\left(\sum_{m=1}^{l-n}\binom{l-m}{n}\right)(2ik)^n\left(b+(-1)^n\overline{b}\right)a^{l-n-1}.$$

The sum over m is determined by the following identity for binomial coefficients:

$$\sum_{r=n}^{l-1}\binom{r}{n} = \binom{l}{n+1}.$$

After shifting the index n by 1, one can then write

$$\mathrm{d}_{\mathcal{T}} a^l = \frac{il}{2}(b+\bar{b})a^{l-1} + \frac{1}{4k}\sum_{n=2}^{l}\binom{l}{n}(2ik)^n\left(b-(-1)^n\bar{b}\right)a^{l-n} .$$

Finally, the first term can be recognised as the summand for $n = 1$, and included in the sum to give the wanted result. □

Notice now that a may be regarded as a 0-form valued in differential operators, as well as b and $\bar{b}$ are 1-forms with the same values. It follows from the analogue of the Jacobi identity in A.2.1 that, when acting on 0-forms, the following rule applies:

$$[\mathrm{ad}_a, \mathrm{ad}_{b\pm\bar{b}}] = \mathrm{ad}_{[a,b\pm\bar{b}]} = 2ik\,\mathrm{ad}_{b\mp\bar{b}} .$$

It follows that the same algebraic relations used in the proofs of the last two lemmas hold for ad_a as for a. By incorporating the relevant factorials, this leads to the following statement.

Proposition 3.4.3. *The operators $A^{(l)}(D)$ satisfy the following relation:*

$$\mathrm{d}^F_{\mathcal{T}} A^{(l)}(D) = \sum_{n=1}^{l}\frac{(2ik)^n}{4kn!}\left[b-(-1)^n\bar{b}, A^{(l-n)}(D)\right] .$$

From this point on we shall often simplify the notation by leaving D unspecified in $\mathcal{R}^{(l)}$ and $A^{(l)}$. Given the analogy of this equation with the recursion of interest (3.11), it is natural to look for $\mathcal{R}^{(l)}$ as a linear combination of the $A^{(n)}$'s:

$$\mathcal{R}^{(l)} = \sum_{r=0}^{l}\alpha_r^{(l)}A^{(l-r)} . \tag{3.21}$$

The condition that $\mathcal{R}^{(0)} = f$ is equivalent to $\alpha_0^{(0)} = 1$.

Substituting (3.21) in the right-hand side of the recursion (3.11) gives

$$\sum_{n=1}^{l}\sum_{r=0}^{l-n}\frac{(ik)^n}{2k}\alpha_r^{(l-n)}\left[b-(-1)^n\bar{b}, A^{(l-n-r)}\right] =$$
$$= \sum_{r=0}^{l-1}\sum_{n=1}^{l-r}\frac{(ik)^n}{2k}\alpha_r^{(l-n)}\left[b-(-1)^n\bar{b}, A^{(l-n-r)}\right] .$$

Now substitute $n = m - r$ and find:

$$\sum_{r=0}^{l}\sum_{m=r+1}^{l}\frac{(ik)^{m-r}}{2k}\alpha_r^{(l-m+r)}\left[b-(-1)^{m-r}\bar{b}, A^{(l-m)}\right] =$$
$$= \sum_{m=1}^{l}\left(\sum_{r=0}^{m-1}\frac{(ik)^{m-r}}{2k}\alpha_r^{(l-m+r)}\right)\left[b-(-1)^{m-r}\bar{b}, A^{(l-m)}\right] .$$

Making similar use of the known differential of the $A^{(l)}$'s, the left-hand side of (3.11) reads:

$$\sum_{r=0}^{l} \alpha_r^{(l)} d^{\mathcal{T}} A^{(l-r)} = \sum_{r=0}^{l} \alpha_r^{(l)} \sum_{n=1}^{l-r} \frac{(2ik)^n}{4kn!} \left[b - (-1)^n \bar{b}, A^{(l-r-n)} \right] =$$
$$= \sum_{r=0}^{l} \sum_{m=r+1}^{l} \frac{(2ik)^{m-r}}{4k(m-r)!} \alpha_r^{(l)} \left[b - (-1)^{m-r} \bar{b}, A^{(l-m)} \right] =$$
$$= \sum_{m=1}^{l} \left(\sum_{r=0}^{m-1} \frac{(2ik)^{m-r}}{4k(m-r)!} \alpha_r^{(l)} \right) \left[b - (-1)^{m-r} \bar{b}, A^{(l-m)} \right] .$$

The recursion translates then into these last two expressions being equal. By separating holomorphic and anti-holomorphic types as forms on $\mathcal{T}$, each equation may be split into

$$\sum_{m=1}^{l} \left(\sum_{r=0}^{m-1} \frac{(ik)^{m-r}}{2k} \alpha_r^{(l-m+r)} - \sum_{r=0}^{m-1} \frac{(2ik)^{m-r}}{4k(m-r)!} \alpha_r^{(l)} \right) \left[b, A^{(l-m)} \right] = 0 ,$$
$$\sum_{m=1}^{l} \left(\sum_{r=0}^{m-1} \frac{(-ik)^{m-r}}{2k} \alpha_r^{(l-m+r)} - \sum_{r=0}^{m-1} \frac{(-2ik)^{m-r}}{4k(m-r)!} \alpha_r^{(l)} \right) \left[\bar{b}, A^{(l-m)} \right] = 0 .$$

One may now observe by induction that, in the case of a curve operator $\mathcal{C}_f$, the top symbol of $A^{(l)}(f)$ essentially consists of the l-th derivatives of f. Generically, the $A^{(l)}(f)$'s form a family of differential operators of increasing order, hence linearly independent. Therefore, the above equations hold for every differential operator D if and only if each of the summands in m vanishes. To summarise, we have established the following fact.

Proposition 3.4.4. *Assuming that every $\mathcal{R}^{(l)}(f)$ is of the form (3.21), the recursion (3.11) is equivalent to the system of equations*

$$E^{\pm}_{m,l} = 0$$

for all pairs of positive integers $m \leq l$, where

$$E^{\pm}_{m,l} = \sum_{r=0}^{m-1} \left(\frac{(\pm 2ik)^{m-r}}{2(m-r)!} \alpha_r^{(l)} - (\pm ik)^{m-r} \alpha_r^{(l-m+r)} \right) . \tag{3.22}$$

We shall now study the equivalent system

$$E^{\pm}_{m,l} \mp ik E^{\pm}_{m-1,l-1} = 0 \qquad 1 \leq m \leq l , \tag{3.23}$$

where it is understood that $E^{\pm}_{0,l} = 0$ for all l. This way, all the terms with $\alpha_r^{(l-m+r)}$ disappear, except for the one corresponding to $r = m - 1$. It is now convenient to replace $r = \rho - 1$ and collect the coefficients with fixed l into the vectors

$$X^{(l)} = \left(\alpha^{(l)}_{\rho-1}\right)_{1\le\rho\le l+1}, \qquad \tilde{X}^{(l)} = \left(\alpha^{(l)}_{\rho-1}\right)_{1\le\rho\le l}. \tag{3.24}$$

In this notation, the equation reads

$$\sum_{\rho=1}^{m} \frac{(\pm 2ik)^{m-\rho+1}}{2(m-\rho+1)!} X^{(l)}_{\rho} = \pm ik \sum_{\rho=1}^{m-1} \frac{(\pm 2ik)^{m-\rho}}{2(m-\rho)!} X^{(l-1)}_{\rho} \pm ik X^{(l-1)}_{m}.$$

Notice that, as ρ ranges between 1 and l, the coefficients $X^{(l)}_{\rho}$ are the entries of the vector $\tilde{X}^{(l)}$, while the $X^{(l-1)}_{\rho}$'s are those of $X^{(l)}$. The equation may then be seen as a linear relation

$$L^{(l)}_{\pm} \tilde{X}^{(l)} = R^{(l)}_{\pm} X^{(l-1)},$$

where $L^{(l)}_{\pm}$ and $R^{(l)}_{\pm}$ are square matrices of size l. Notice that the entry (m, ρ) in each of these vanishes whenever $\rho > m$, which makes them lower-triangular. Moreover, each entry depends only on the difference $m - \rho$, which means that they are polynomials in the standard nilpotent matrix

$$N = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{pmatrix}.$$

More precisely, one may write

$$L^{(l)}_{\pm} = \sum_{n=0}^{\infty} \frac{(\pm 2ik)^{n+1}}{2(n+1)!} N^n, \qquad R^{(l)}_{\pm} = \pm ik\mathbb{1} \pm ik \sum_{n=1}^{\infty} \frac{(\pm 2ik)^n}{2n!} N^n.$$

Both matrices are invertible, being triangular with no zeroes on the diagonal; in particular, the inverse of $L^{(l)}_{\pm}$ determines $\tilde{X}^{(l)}$ in terms of $X^{(l-1)}$.

As is easily seen, the sums expressing the matrices are in fact the Taylor series of the analytic functions

$$\sum_{n=0}^{\infty} \frac{(\pm 2ik)^{n+1}}{2(n+1)!} z^n = \frac{e^{\pm 2ikz} - 1}{2z},$$
$$\pm ik\Big(1 + \sum_{n=1}^{\infty} \frac{(\pm 2ik)^n}{2n!} N^n\Big) = \pm ik \frac{e^{2ikz} + 1}{2}.$$

Since the Taylor series of a product is the formal product of the Taylor series of the factors, one may turn the problem from matrices to holomorphic functions. Indeed, if φ is a holomorphic function in a neighbourhood of $0 \in \mathbb{C}$, it makes sense to write $\varphi(N)$ to mean the evaluation at N of the Taylor series of φ at 0, because the sum terminates. Therefore, one may finally conclude that the system of equations (3.23) for even m is equivalent to

$$\tilde{X}^{(l)} = \pm ik \frac{e^{\pm 2ikN} + 1}{e^{\pm 2ikN} - 1} N X^{(l-1)} = \pm ik \tanh(\pm ikN) N X^{(l-1)} . \qquad (3.25)$$

Since the hyperbolic function tanh is odd, the signs may finally be disregarded: the two sets of equations corresponding to the signs are in fact equivalent.

This essentially concludes the discussion of the solutions of the numeric recursion, which is summarised as the following reformulation of Theorem 3.1.6.

Theorem 3.4.5. *There exists a unique solution of the numeric recursion for every choice of the coefficients $\alpha_l^{(l)}$ for $l \geq 1$.*

Proof. We have proved that the system is equivalent to the vanishing of $E^{\pm}_{m,l} \mp ikE^{\pm}_{m-1,l-1} = 0$ for every positive odd $m \leq l$. In turn, this set of equations is equivalent to a triangular one expressing $\tilde{X}^{(l)}$ in terms of $X^{(l-1)}$. This leaves as free variables precisely the last entry of each $X^{(l)}$, which corresponds to $\alpha_l^{(l)}$ as claimed. □

In spite of this ambiguity, it should be noted that the solution is essentially unique. Indeed, consider the particular trivialisation $\mathcal{R}_0$ corresponding to $\alpha_l^{(l)} = 0$ for every $l > 0$, and the $\mathcal{R}_l$ with $\alpha_l^{(l)} = 1$ for some specific value of l. By linearity, the difference $\mathcal{R}_0 - \mathcal{R}_l$ is a solution of the recursion starting with $\alpha_0^{(0)} = 0$ instead of 1, and is therefore determined by the same expressions. It is then immediate to check that this difference takes the form $s^{-l}\mathcal{R}_0$, and by linearity one obtains the following statement.

Proposition 3.4.6. *The solution $\mathcal{R}_\alpha$ corresponding to a sequence of coefficients $\alpha_l^{(l)}$ is related to $\mathcal{R}_0$ as above by the relation*

$$\mathcal{R}_\alpha = \sum_{l=0}^{\infty} \alpha_l^{(l)} s^{-l} \mathcal{R}_0 .$$

In other words, the various solutions differ by a factor of an invertible power series in s^{-1}.

3.4.1 Example: a solution from the trivialisation of $\tilde{\nabla}$

A particular solution of the recursion can be obtained in the case at hand as an expansion of the explicit trivialisation of the Hitchin-Witten connection, which we discuss in 2.5.1. Indeed, recall that this is given by

$$\exp(r\Delta)\, \tilde{\nabla} \exp(-r\Delta) = \nabla^{\mathrm{tr}} ,$$

where the parameter $r \in \mathbb{C}$ is implicitly determined (up to addition of a constant) by

$$e^{4rk} = -\frac{k - is}{k + is}\,.$$

In fact, this was deduced from the formal relation

$$\mathrm{d}_{\mathcal{T}} \sum_{n=0}^{\infty} \frac{(r\Delta)^n}{n!} = \sum_{n=0}^{\infty} \frac{(r\Delta)^n}{n!} \left(\frac{1}{2t} b - \frac{1}{2\bar{t}} \bar{b} \right), \tag{3.26}$$

which was derived as a mere consequence of the properties of Δ and the assumption on r. Here the derivative of the series takes the meaning of the formal sum of the derivatives of the individual terms. The usual algebraic identities, such as the Leibniz rule for the product, are still valid for this formal derivative.

Suppose now that $D \in \mathrm{D}_k$ is a σ-independent finite-order differential operator. One can use induction on the terms on the right-hand side to prove the following formal analogue of the Baker-Campbell-Hausdorff formula:

$$\left(\sum_{n=0}^{\infty} \frac{(-r\Delta)^n}{n!} \right) D \left(\sum_{n=0}^{\infty} \frac{(-r\Delta)^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{(r\,\mathrm{ad}_{\Delta})^n(D)}{n!}\,. \tag{3.27}$$

From the explicit trivialisation of $\tilde{\nabla}$ and the fact that D is independent on Σ it follows that $e^{-r\Delta} D e^{r\Delta}$ is Hitchin-Witten covariantly constant. This can be derived purely from algebraic manipulations involving the explicit gauge transformation of the trivial connection into $\tilde{\nabla}$. After replacing every exponential with its formal power series, the same algebraic manipulation, together with (3.27), leads to the following relation:

$$\mathrm{d}_{\mathcal{T}} \left(\sum_{n=0}^{\infty} \frac{(r\,\mathrm{ad}_{\Delta})^n(D)}{n!} \right) = - \left[\frac{1}{2t} b - \frac{1}{2\bar{t}} \bar{b}, \sum_{n=0}^{\infty} \frac{(r\,\mathrm{ad}_{\Delta})^n(D)}{n!} \right]. \tag{3.28}$$

Aside from the matters of convergence, one may regard this as a Taylor expansion of the exact relation expressing the parallelism of $e^{-r\Delta} D e^{r\Delta}$.

We now wish to switch all the parameters to s. We claim that r can be expanded as a power series in s^{-1} as s goes to ∞. Indeed, notice first of all that $|\bar{t}/t| = 1$, and this ratio can only take the value -1 when $s = 0$. On the other hand, in the limit for $s \to \infty$ with k fixed, the ratio goes to 1. Therefore, for $s \neq 0$ one may use the determination of the logarithm on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ with $\log(1) = 0$ to express r as

$$r = \frac{1}{4k} \log\left(-\frac{k - is}{k + is} \right).$$

This can indeed be expanded as a formal power series ρ in s^{-1}, and since for $s \to \infty$ we have $r = 0$, the zero-order term vanishes. Explicitly, it can be checked that this series reads

$$\rho = \frac{1}{2k} \sum_{n=0}^{\infty} \frac{(ik)^{2n+1}}{2n+1} s^{-2n-1} . \tag{3.29}$$

In particular, it makes sense to evaluate at ρ any power series in r to obtain a new one in s^{-1}, so the following expression defines a well-defined section of $\mathscr{A}_k \times \mathcal{T}$ for every $D \in \mathrm{D}_k$:

$$\sum_{n=0}^{\infty} \frac{(\rho \operatorname{ad}_\Delta)^n (D)}{n!} = \sum_{n=0}^{\infty} \frac{1}{(2k)^n n!} \left(\sum_{l=0}^{\infty} \frac{(ik)^{2l+1}}{2l+1} \operatorname{ad}_\Delta s^{-2l-1} \right)^n (D) . \tag{3.30}$$

Furthermore, it follows from (3.28) that the formally analogous relation holds when r, r^{-1} and $\bar{t}^{-1}$ are replaced by their Taylor series:

$$\mathrm{d}_{\mathcal{T}} \left(\sum_{n=0}^{\infty} \frac{(\rho \operatorname{ad}_\Delta)^n (D)}{n!} \right) = \left[\sum_{l=0}^{\infty} \frac{(ik)^l}{2k} \left(b - (-1)^l \bar{b} \right), \sum_{n=0}^{\infty} \frac{(\rho \operatorname{ad}_\Delta)^n (D)}{n!} \right] .$$

In other words, (3.30) is a solution for the recursion and a $\tilde{\mathcal{D}}$-flat section of $\mathscr{A}_k \times \mathcal{T}$.

Notice also that this solution is of the form of (3.21). Indeed, it is apparent from the sum that every term proportional to $\operatorname{ad}^n(D) s^{-l}$ has $l \geq n$. Therefore, collecting all the terms with the same power of s one finds precisely sums of the form

$$\sum_{n=0}^{l} \lambda_n^{(l)} \operatorname{ad}_\Delta^{l-n}(D) s^{-l} ,$$

and one can choose $\alpha_n^{(l)} = \lambda_n^{(l)} / (l-n)!$. As a final observation, thinking of this sum as a power series in ad_Δ with coefficients in formal sums in s^{-1}, the 0-th coefficient is 1. On the other hand, this can also be realised as

$$1 = \sum_{n=0}^{\infty} \lambda_n^n s^{-l} . \tag{3.31}$$

Therefore, this solution corresponds to $\alpha_l^{(l)} = 0$ for every $l > 0$.

In summary, the content of this section proves the following statement.

Theorem 3.4.7. *The recursion of differential equations (3.11) on the moduli space of flat $\mathrm{SL}(n, \mathbb{C})$-connections has infinitely many solution of the form (3.21). Two such solutions are obtained from one another via multiplication by an invertible power series in s^{-1}. Moreover, a particular solution of this form is obtained as the formal Taylor expansion of the trivialisation of the Hitchin-Witten connection discussed in 2.5.1, as s goes to ∞.*

Chapter 4

Quantum operators from geometric constructions

4.1 Introduction and summary

In the previous chapter we considered the task of extending to $\mathrm{SL}(n,\mathbb{C})$ the ideas of [And12], by defining a formal Hitchin-Witten connection and looking for a trivialisation. Tied to this is the problem of defining a deformation quantisation on the relevant moduli space, which in the case of $\mathrm{SU}(n)$ is achieved via Toeplitz quantisation [Sch10]. This is done by defining a star-product for each point of the Teichmüller space $\mathcal{T}$ and identifying the resulting non-commutative algebras via the holonomy of the formal connection, which returns a rather abstract algebra. The role of the trivialisation may then be seen as that of transforming each concrete element of the Poisson algebra into one for which the star-product is defined.

The main goal of this chapter is to complete the analogous picture in the setting obtained in the previous one. Recall that the bundle on which we considered the formal Hitchin-Witten connection has as fibre the space $\mathscr{A}_k$ of power series in s^{-1} with coefficients in differential operators acting on the pre-quantum line bundle. This space is naturally a non-commutative algebra, so one may attempt an approach similar to the idea discussed above, namely by associating a parallel section of the formal connection to every smooth function on $\mathcal{M}_{\mathbb{C}}$.

We propose to do this using the correspondence between differential operators on $\mathscr{L}^k$ and tensor fields on $\mathcal{M}$, as presented in Section A.1 in the appendix. In turn, these tensor fields may be identified with functions on $\mathrm{T}^* \mathcal{M}$ which restrict to polynomials on the fibres. The association of differential operators to a particular class of functions on $\mathcal{M}_{\mathbb{C}}$ can then be obtained given an identification of (an open dense inside) this space with $\mathrm{T}^* \mathcal{M}$. We consider two such maps, and discuss the construction again in the case when the genus of Σ is 1.

The ideal outcome of this construction would be an operator which is either σ-independent, or Hitchin-Witten parallel; the two scenarios are related by the trivialisation of the Hitchin-Witten connection. The correspondence between tensors on $\mathcal{M}$ and differential operators depends on σ only through the Levi-Civita connection, which in genus 1 is always the trivial one. Therefore, in this particular case the Teichmüller parameter enters only through the map $\mathrm{T}^* \mathcal{M} \to \mathcal{M}_{\mathbb{C}}$.

The first construction we propose is presented in Section 4.2. This uses an embedding of $\mathrm{T}^* \mathcal{M}^{\mathrm{Vec}}$ into $\mathcal{M}^{\mathrm{Hit}}$ [Hit90], in combination with the Narasimhan-Seshadri and Hitchin-Kobayashi correspondences. A fundamental ingredient of this construction is an identification of co-vectors on $\mathcal{M}^{\mathrm{Vec}}$ with Higgs fields, obtained via Serre duality.

Specialising to the case of genus 1 and choosing an appropriate normalisation of the duality pairing, we study the resulting map in coordinates and find the following statement.

Lemma 4.1.1. *Consider the coordinates of Section 2.5, i.e.* $(\mathbf{u}, \mathbf{v})$ *and their duals* $(\mathbf{u}^*, \mathbf{v}^*)$ *on* $\mathrm{T}^* \mathcal{M}$*, and* $(\mathbf{U}, \mathbf{V})$ *on* $\mathcal{M}_{\mathbb{C}}$*. Then the map above sends* $(\mathbf{u}, \mathbf{v}, \mathbf{u}^*, \mathbf{v}^*)$ *to* $(\mathbf{U}, \mathbf{V})$ *with*

$$\mathbf{U} = \mathbf{u} - \frac{i}{4\pi}\mathbf{v}\,, \qquad \mathbf{V} = \mathbf{v} + \frac{i}{4\pi}\mathbf{u}\,.$$

This correspondence is clearly a diffeomorphism at the level of the ramified covers $\mathbb{T}^{2n}$ and $\mathbb{T}^{2n}_{\mathbb{C}}$. Moreover, since this is Weyl-equivariant it defines the desired bijection between smooth function on $\mathcal{M}_{\mathbb{C}}$ and tensor fields on $\mathcal{M}$, smoothness being intended in the sense specified in Section 2.5.

We continue our discussion in Section 4.3 by studying the Poisson bracket of functions and the commutator of operators in terms of the associated tensor fields, and argue the following two lemmas.

Lemma 4.1.2. *If* $g = 1$ *and the ranks of the totally symmetric tensor fields* T *and* S *are* r_T *and* r_S *respectively, the Poisson bracket between* P_T *and* P_S *at the level* t *is a polynomial function represented by the formal tensor*

$$\frac{k}{|t|^2}\Big(\mathcal{S}\Big(\nabla T\cdot\widetilde{\omega}\cdot\nabla S\Big) + r_T r_S \mathcal{S}\Big(T\cdot\omega\cdot S\Big)\Big) + \frac{s}{|t|^2}\mathcal{S}\Big(r_T T\cdot\nabla S - r_S S\cdot\nabla T\Big)\,.$$

Lemma 4.1.3. *Suppose* $g = 1$*, and let* T *and* S *be two totally symmetric tensor fields* T *and* S *of ranks* r_T *and* r_S*, and call for simplicity* $r = r_T + r_S$*. Then the top symbols of the commutator of the corresponding differential operators are*

$$\begin{aligned}
\sigma_{r-1}\left[\nabla^{r_T}_T, \nabla^{r_S}_S\right] &= r_T\mathcal{S}(T\cdot\nabla S) - r_S\mathcal{S}(S\cdot\nabla T)\,,\\
\sigma_{r-2}\left[\nabla^{r_T}_T, \nabla^{r_S}_S\right] &= \binom{r_T}{2}\mathcal{S}(T\cdot\cdot\nabla^2 S) - \binom{r_S}{2}\mathcal{S}(S\cdot\cdot\nabla^2 T) - ikr_T r_S\mathcal{S}(T\cdot\omega\cdot S)\,,
\end{aligned}$$

were the double dot $\cdot\cdot$ *means contraction of two indices.*

This sets the stage for considering our original problem of defining a star-product for functions on $\mathcal{M}_{\mathbb{C}}$. As is apparent from the previous lemmas, the term proportional to s in the Poisson bracket also appears in the commutator of operators, and so does part of the term in k. However, the other term does not find a match in the symbols of the commutators. Bearing this in mind, and following the spirit of the previous chapter, we fix k and introduce a version of the correspondences above weighted with a function of s in order to associate operators to functions, and backwards. Combining this construction with the composition of differential operators, this defines a non-commutative product $\star$ between power series in s^{-1} with functions of polynomial type as coefficients.

Theorem 4.1.4. *For $g = 1$ and every k fixed, $\star$ defines an associative non-commutative product which deforms the point-wise multiplication, and satisfies the Dirac quantisation at the first order as $s \to \infty$. This product is made covariantly constant with respect to an analogue of the formal Hitchin-Witten connection by means of the corresponding trivialisation.*

Notice that this does not, strictly speaking, define a deformation quantisation. In fact, the Poisson structure itself depends on the parameter s, so it makes little sense to even raise the question.

Finally, we consider a different way to map $\mathrm{T}^* \mathcal{M}$ to $\mathcal{M}_{\mathbb{C}}$, based on the polarisation used for the geometric quantisation. As such, this map depends on the Teichmüller parameter in an essential way, so one may use it in an attempt to define Hitchin-Witten covariantly constant operators. In the final part of the chapter we briefly discuss this map and argue that, for functions which are linear along the leaves, the operators coming from this construction agree with those of geometric quantisation.

4.2 Embedding via non-Abelian Hodge theory

One way to obtain a map of $\mathrm{T}^* \mathcal{M}$ into $\mathcal{M}_{\mathbb{C}}$ is by means of the identifications with the moduli spaces of holomorphic and Higgs bundles. Indeed, an embedding of $\mathrm{T}^* \mathcal{M}^{\mathrm{Vec}}_{(r,d)}$ into $\mathcal{M}^{\mathrm{Hit}}_{(r,d)}$ was proposed by Hitchin [Hit87]. We shall give here the construction of the map in the case of $g > 1$, and then adapt the ideas to the torus using the explicit coordinates.

Let A be a flat $\mathrm{SU}(n)$-connection on Σ representing a smooth point of $\mathcal{M}$, and consider the tangent space $\mathrm{T}_{[A]} \mathcal{M}$, which is isomorphic to $H^1_A(\Sigma, \mathfrak{su}(n))$. Recall that $H^*_A(\Sigma, \mathfrak{su}(n))$ denotes the cohomology of $\mathfrak{su}(n)$-valued forms with respect to the exterior differential $\mathrm{d}_A = \mathrm{d} + \mathrm{ad}_A$ induced by A. This space sits inside its complexification

$$H^1_A(\Sigma, \mathfrak{su}(n)) \subseteq H^1_A(\Sigma, \mathfrak{su}(n)) \otimes \mathbb{C} \simeq H^1_A(\Sigma, \mathfrak{sl}(n, \mathbb{C}))\,.$$

Once a point $\sigma \in \mathcal{T}$ is chosen, the complex structure on $\mathcal{M}$ induces a splitting of this space into holomorphic and anti-holomorphic part, each isomorphic to $H^1_A(\Sigma, \mathfrak{su}(n))$ through the corresponding projection. Holomorphic and anti-holomorphic vectors correspond to forms on Σ_σ of type $(0,1)$ and $(1,0)$ respectively—notice that the types on Σ and $\mathcal{M}$ are exchanged. This means that $\mathrm{T}'_{[A]}\,\mathcal{M}$ may be regarded as the first cohomology group of $\Sigma \times \mathfrak{sl}(n,\mathbb{C})$ as a holomorphic bundle.

In terms of the moduli space of vector bundles, to $[A]$ corresponds a holomorphic bundle $E = E_{A,\Sigma}$ via the Narasimhan-Seshadri correspondence. This is basically $\Sigma_\sigma \times \mathbb{C}^n$ with the holomorphic structure $\overline{\partial}_A$ defined as the $(0,1)$ part of $\nabla^A = \mathrm{d}+A$. The bundle $\mathrm{End}_0(E)$ of traceless endomorphisms of E has $\Sigma \times \mathfrak{sl}(n,\mathbb{C})$ as underlying (topological) vector bundle, with holomorphic structure given by the $(0,1)$ part of $\mathrm{d}+\mathrm{ad}_A$. The argument above results then in the isomorphisms

$$\mathrm{T}_{[A]}\,\mathcal{M} \simeq \mathrm{T}'_{[A]}\,\mathcal{M} \simeq H^1(\Sigma_\sigma, \mathrm{End}_0(E_{A,\sigma}))\,.$$

Our interest, however, is for the *co*-tangent space $\mathrm{T}^*_{[A]}\,\mathcal{M}$. Real linear functionals on $\mathrm{T}_{[A]}\,\mathcal{M}$ can be identified with complex ones on the holomorphic tangent space, simply by $\mathbb{C}$-linear extension to $\mathrm{T}_{[A]}\,\mathcal{M} \times \mathbb{C}$ and then restriction. After taking duals, the chain of isomorphisms above can be continued with Serre duality:

$$\mathrm{T}^*_{[A]}\,\mathcal{M} \simeq \Big(H^1(\Sigma_\sigma, \mathrm{End}_0(E_{A,\sigma}))\Big)^* \simeq H^0(\Sigma_\sigma, K \otimes \mathrm{End}_0(E_{A,\sigma}))\,. \tag{4.1}$$

Here we use the Killing form to identify $\mathrm{End}_0(E_{A,\sigma})$ with its dual. The right-most space is that of traceless holomorphic sections of $K \otimes \mathrm{End}(E)$, i.e. of Higgs fields on E. The conclusion of this argument is then that the data of a co-tangent vector is equivalent to that of a Higgs field Φ. The collection of all these isomorphisms produces an embedding of the whole cotangent bundle $\mathrm{T}^*\,\mathcal{M}$ into the moduli space of Higgs bundle. The image of this map is the open dense consisting of the polystable Higgs fields with underlying semi-stable vector bundle.

Recall now from Section 2.3 the correspondence between flat $\mathrm{SL}(n,\mathbb{C})$-connections and Higgs bundles given by non-Abelian Hodge theory. In either direction, the map is obtained by looking first for a suitable Hermitian structure h on the bundle, and then using it to construct the desired object. In our situation, however, we start with the data of a *unitary* flat connection A and a vector in $\mathrm{T}^*_{[A]}\,\mathcal{M}$, equivalent to a Higgs field, and want to obtain a flat $\mathrm{SL}(n,\mathbb{C})$ connection. In other words, a Hermitian structure is part of the initial data as implicitly defined by A, which gives all the ingredients needed. In the language used above, $\partial + \overline{\partial}$ is just the covariant derivative ∇^A, so finally the desired connection is

$$A^{\mathbb{C}}_\Phi = A + \Phi + \Phi^*\,.$$

Notice that this correspondence, as well as its whole construction depends in an essential way on the choice of σ.

4.2.1 The embedding in genus 1

In the case of genus 1, recall from (2.5) the coordinates $(\mathbf{u}, \mathbf{v})$ on $\mathcal{M}$ and $(\mathbf{U}, \mathbf{V})$ on $\mathcal{M}_{\mathbb{C}}$ corresponding to connections

$$A_{(\mathbf{u},\mathbf{v})} = u^\mu T_\mu \, \mathrm{d}x + v^\nu T_\nu \, \mathrm{d}y \,, \qquad A_{(\mathbf{U},\mathbf{V})} = U^\mu T_\mu \, \mathrm{d}x + V^\nu T_\nu \, \mathrm{d}y \,.$$

Also, recall the correspondence between vectors on $\mathcal{M}$ and forms on Σ

$$\frac{\partial}{\partial u^\mu} \rightsquigarrow T_\mu \, \mathrm{d}x \,, \qquad \frac{\partial}{\partial v^\nu} \rightsquigarrow T_\nu \, \mathrm{d}y \,.$$

We shall use $\mathbf{u}^*$ and $\mathbf{v}^*$ as dual coordinates on the co-tangent spaces.

Suppose now that $\sigma \in \mathcal{T}$ is fixed, so the holomorphic coordinate $z = x - \sigma^{-1} y$ is induced on Σ. This allows one to write

$$\mathrm{d}x = \frac{\sigma}{\sigma - \overline{\sigma}} \mathrm{d}z - \frac{\overline{\sigma}}{\sigma - \overline{\sigma}} \mathrm{d}\overline{z} \,, \qquad \mathrm{d}y = \frac{\sigma\overline{\sigma}}{\sigma - \overline{\sigma}} \mathrm{d}z - \frac{\sigma\overline{\sigma}}{\sigma - \overline{\sigma}} \mathrm{d}\overline{z} \,.$$

In particular, the $(0,1)$ part of $A_{(\mathbf{u},\mathbf{v})}$ is given by

$$A^{0,1}_{(\mathbf{u},\mathbf{v})} = -\frac{\overline{\sigma}}{\sigma - \overline{\sigma}} \Big(u^\mu T_\mu + v^\nu T_\nu \Big) \, \mathrm{d}\overline{z} \,.$$

If φ is a diagonal element in $\mathfrak{sl}(n, \mathbb{C})$, then of course it commutes with the coefficient of $A^{0,1}_{(\mathbf{u},\mathbf{v})}$, so that $\Phi = \varphi dz$ is holomorphic, hence a Higgs field. For dimensional reasons, this gives a complete description of all the elements of $H^0(\Sigma_\sigma, K \otimes \mathrm{End}_0(E_{A,\sigma}))$.

Since we are going to use it explicitly, we wish to recall the details of the Serre duality pairing. In general, given two holomorphic bundles F_1 and F_2 on a complex manifold X, one can define pairings

$$H^j(X, F_1) \otimes H^l(X, F_1) \to H^{j+l}(X, F_1 \otimes F_2) \,.$$

In terms of differential forms, this is just the wedge product. In our situation, $X = \Sigma$ and the bundles are $F_1 = K \otimes \mathrm{End}_0(E)$ and $F_2 = \mathrm{End}_0(E)$. The Killing form gives a pairing $F_1 \otimes F_2 \to K$, which results in a map

$$H^0(\Sigma, K \otimes \mathrm{End}_0(E)) \otimes H^1(\Sigma, \mathrm{End}_0(E)) \to H^1(\Sigma, K) \,.$$

The elements of $H^1(\Sigma, K)$ are represented by $(0,1)$-forms with values in $(1,0)$-forms, which can be thought of as $(1,1)$-forms and hence be mapped to $\mathbb{C}$ by integration. This fixes the pairing up to a a normalisation α, which we shall set to $4\pi i$. If $\varphi\,\mathrm{d}z$ is a Higgs field and $\eta\,\mathrm{d}\bar{z}$ represents a class in $H^1(\Sigma, \mathrm{End}_0(E))$, the pairing reads

$$\left\langle \varphi dz \,\middle|\, \eta d\bar{z} \right\rangle = 4\pi i \int_\Sigma \kappa(\varphi, \eta)\, \mathrm{d}z \wedge \mathrm{d}\bar{z} = -4\pi i \frac{\sigma - \overline{\sigma}}{\sigma\overline{\sigma}} \kappa(\varphi, \eta)\,.$$

To summarise up to this point, the map that we are considering works as follows. Consider a *real* co-vector on $\mathcal{M}$, i.e. a real linear functional on the tangent space on $\mathcal{M}$ at a fixed point. This may be extended by $\mathbb{C}$-linearity to the complexified tangent space, and then restricted to the holomorphic part, whose elements are represented by forms $\eta\,\mathrm{d}\bar{z}$ on Σ with constant η. There exists then a unique $\varphi\,\mathrm{d}z$ for which the Serre pairing $\left\langle \varphi\,\mathrm{d}z \,\middle|\, \eta\,\mathrm{d}\bar{z} \right\rangle$ gives the desired functional for every η. This will be the Higgs field associated to the starting functional.

For the functionals $\mathrm{d}u^\mu$ and $\mathrm{d}v^\nu$ and $\eta = T_j$ we find

$$\delta^\mu_j = \mathrm{d}u^\mu\Big(T_j\Big(\mathrm{d}x - \frac{1}{\overline{\sigma}}\,\mathrm{d}y\Big)\Big) = \Big\langle \varphi\,\mathrm{d}z \,\Big|\, T_j\,\mathrm{d}\bar{z}\Big\rangle = -4\pi i \frac{\sigma - \overline{\sigma}}{\sigma\overline{\sigma}} \kappa(\varphi, T_j)\,,$$

$$-\frac{1}{\overline{\sigma}}\delta^\nu_j = \mathrm{d}v^\mu\Big(T_j\Big(\mathrm{d}x - \frac{1}{\overline{\sigma}}\,\mathrm{d}y\Big)\Big) = \Big\langle \varphi\,\mathrm{d}z \,\Big|\, T_j d\bar{z}\Big\rangle = -4\pi i \frac{\sigma - \overline{\sigma}}{\sigma\overline{\sigma}} \kappa(\varphi, T_j)\,.$$

Therefore, the Higgs fields corresponding to $\mathrm{d}u^\mu$ and $\mathrm{d}v^\nu$ are

$$\mathrm{d}u^\mu \rightsquigarrow \frac{i\sigma\overline{\sigma}}{4\pi(\sigma - \overline{\sigma})} T^\mu\,\mathrm{d}z\,, \qquad \mathrm{d}v^\nu \rightsquigarrow -\frac{i\sigma}{4\pi(\sigma - \overline{\sigma})} T^\nu\,\mathrm{d}z\,,$$

where the T^μ's are dual to the T_μ's; explicitly

$$T^\mu = \frac{2}{\mu(\mu+1)} T_\mu\,.$$

For the sake of the embedding we are only interested in (twice) the real part of each of these Higgs fields. Bearing in mind that T^j is anti-self-adjoint and $\sigma - \overline{\sigma}$ is purely imaginary, this gives respectively

$$\begin{aligned} \frac{i\sigma\overline{\sigma}}{4\pi(\sigma - \overline{\sigma})} T_\mu(\mathrm{d}z - \mathrm{d}\bar{z}) &= \frac{i}{4\pi} T^\mu\,\mathrm{d}y\,, \\ -\frac{i}{4\pi(\sigma - \overline{\sigma})} T^\mu(\sigma\,\mathrm{d}z - \overline{\sigma}\,\mathrm{d}\bar{z}) &= -\frac{i}{4\pi} T^\mu\,\mathrm{d}x\,. \end{aligned} \tag{4.2}$$

The map finally reads

$$\begin{aligned} (\mathbf{u}, \mathbf{v}, \mathbf{u}^*, \mathbf{v}^*) \longmapsto \qquad & A_{(\mathbf{u},\mathbf{v})} - \frac{i}{4\pi}(v^*_\nu T^\nu\,\mathrm{d}x - u^*_\mu T^\mu\,\mathrm{d}y) = \\ &= \Big(u^\mu T_\mu - \frac{i}{4\pi} v^*_\nu T^\nu\Big)\,\mathrm{d}x + \Big(v^\nu T_\nu + \frac{i}{4\pi} u^*_\mu T^\mu\Big)\,\mathrm{d}y\,. \end{aligned} \tag{4.3}$$

Here summation over repeated indices is understood, even when they are both lower. This gives a σ-independent map $\iota : \mathrm{T}^* \mathcal{M} \to \mathcal{M}_{\mathbb{C}}$ which is clearly injective, and whose image consists of the points with coordinates $(\mathbf{U}, \mathbf{V})$ without repetitions in their real parts. Notice moreover that, if $u^\mu T_\mu + \alpha v^*_\nu T_\nu$ has repeated entries (modulo 2π), then so does $u^\mu T_\mu$, and analogously for the coefficient in dy. Therefore, if $(\mathbf{u}, \mathbf{v})$ represents a smooth point in $\mathcal{M}$, then $(\mathbf{u}, \mathbf{v}, \mathbf{u}^*, \mathbf{v}^*)$ is mapped to a smooth point. In conclusion, this map gives an embedding of $\mathrm{T}^* \mathcal{M}^s$ into $\mathcal{M}^s_{\mathbb{C}}$ reaching an open dense. However, the expression defining the embedding lifts to the covers of $\mathcal{M}$ and $\mathcal{M}_{\mathbb{C}}$ and defines an explicit Weyl-equivariant diffeomorphism $\mathrm{T}^* \mathbb{T}^{2n} \simeq \mathbb{T}^{2n}_{\mathbb{C}}$. Consequently, the pull-back gives a σ-independent identification between functions on the two spaces that are invariant under the action of the Weyl group.

4.3 Quantisation of functions

4.3.1 Functions of polynomial type and tensor fields

The discussion above provides an identification of functions on $\mathcal{M}_{\mathbb{C}}$ and $\mathrm{T}^* \mathcal{M}$. The correspondence is, moreover, independent on the Teichmüller parameter.

Definition 4.3.1. *Let f be a smooth function on $\mathcal{M}_{\mathbb{C}}$. We shall call it of polynomial type if its pull-back on* $\mathrm{T}^* \mathcal{M}$ *can be written as a polynomial in the co-tangent coordinates with coefficients varying smoothly on $\mathcal{M}$. We call* Pol *the algebra of such functions.*

Notice that the above definition is well posed without reference to any coordinate system, since it always makes good sense to talk about polynomial functions on a vector space. In fact, homogeneous polynomials of degree d on a vector space V can be identified with the elements of the d-th symmetric power of V^*. When V is a cotangent space on a smooth manifold, its dual is the tangent space, and the d-th symmetric power consists of the totally symmetric d-contra-variant tensors. In this language, functions of homogeneous polynomial type of degree d on $\mathcal{M}_{\mathbb{C}}$ correspond to totally symmetric tensor fields on $\mathcal{M}$ of rank d. In fact, a function of polynomial type can be associated to any (not necessarily symmetric) tensor field T, but the result depends only on the symmetric part $\mathcal{S}(T)$ of the field. More in general, this correspondence extends to one between functions of polynomial type on $\mathcal{M}_{\mathbb{C}}$ and formal sums of symmetric tensor fields on $\mathcal{M}$.

Definition 4.3.2. *If f is a function of polynomial type, we denote by T_f the formal tensor associated to it. Vice-versa, the function corresponding to a formal tensor field T will be denoted as P_T.*

We shall now study the Poisson bracket of functions on $\mathcal{M}_{\mathbb{C}}$. For the purpose of the quantisation we are interested in the symplectic structure

$$\omega_t = \frac{t\omega_{\mathbb{C}} + \bar{t}\overline{\omega_{\mathbb{C}}}}{2} = k\,\mathrm{Re}(\omega_{\mathbb{C}}) - s\,\mathrm{Im}(\omega_{\mathbb{C}})$$

In order to study the corresponding bracket $\{\cdot,\cdot\}_t$ it is best to have a description of the pairing in terms of tensor fields on $\mathcal{M}$ and coordinates on this space. This can be achieved by pulling back the Atiyah-Bott form of $\mathcal{M}_{\mathbb{C}}$ to $\mathrm{T}^*\mathcal{M}$, still denoting by $\omega_{\mathbb{C}}$ the pull back. The only coefficients that we need to compute in order for $\omega_{\mathbb{C}}$ to be determined are

$$\begin{aligned} \omega_{\mathbb{C}}\left(\frac{\partial}{\partial u^\mu}, \frac{\partial}{\partial v^\nu}\right) &= 4\pi \int_\Sigma \kappa(T_\mu, T_\nu)\,\mathrm{d}x \wedge \mathrm{d}y = 2\pi\mu(\mu+1)\delta_{\mu\nu}\,, \\ \omega_{\mathbb{C}}\left(\frac{\partial}{\partial u^*_\mu}, \frac{\partial}{\partial u^\nu}\right) &= 4\pi \int_\Sigma \kappa\Big(\frac{i}{4\pi}T^\mu, T_\nu\Big)\,\mathrm{d}y \wedge \mathrm{d}x = -i\delta^\nu_\mu\,, \end{aligned} \tag{4.4}$$

and their analogues

$$\omega_{\mathbb{C}}\left(\frac{\partial}{\partial v^*_\mu}, \frac{\partial}{\partial v^\nu}\right) = -i\delta^\mu_\nu\,, \qquad \omega_{\mathbb{C}}\left(\frac{\partial}{\partial v^*_\mu}, \frac{\partial}{\partial u^*_\nu}\right) = \frac{1}{2\pi\mu(\mu+1)}\delta^{\mu\nu}\,. \tag{4.5}$$

This symplectic form is in fact a linear combination of two natural ones on $\mathrm{T}^*\mathcal{M}$. One is of course the symplectic structure as a co-tangent bundle, for which the only non-zero terms are

$$\omega_{\mathrm{T}^*}\left(\frac{\partial}{\partial u^*_\mu}, \frac{\partial}{\partial u^\nu}\right) = \delta^\mu_\nu = \omega_{\mathrm{T}^*}\left(\frac{\partial}{\partial v^*_\mu}, \frac{\partial}{\partial v^\nu}\right). \tag{4.6}$$

The second form comes in fact from that on $\mathcal{M}$. First of all, one can consider the projection $\pi\colon \mathrm{T}^*\mathcal{M} \to \mathcal{M}$ and pull ω back to a clearly degenerate 2-form. On the other hand, the inverse $\widetilde{\omega}$ of this form defines a symplectic structure on each co-tangent space on $\mathcal{M}$. However, to promote this to a pairing on the tangent bundle of $\mathrm{T}^*\mathcal{M}$, seen as a manifold itself, one also needs a projection onto its vertical part. On the other hand, recall that every $\sigma \in \mathcal{T}$ defines a metric, and hence a Levi-Civita connection on $\mathrm{T}^*\mathbb{T}^2$ which, as we discussed, is independent of σ and trivial in our coordinates. This finally gives two differential forms ω_H and ω_V on the co-tangent space, whose only non-vanishing entries are

$$\omega_H\left(\frac{\partial}{\partial u^\mu}, \frac{\partial}{\partial v^\nu}\right) = 2\pi\mu(\mu+1)\delta_{\mu\nu}\,, \qquad \omega_V\left(\frac{\partial}{\partial u^*_\mu}, \frac{\partial}{\partial v^*_\nu}\right) = \frac{1}{2\pi\mu(\mu+1)}\delta^{\mu\nu}\,,$$

where the second equality is obtained by inverting the matrix which represents ω. Although these forms are both degenerate, their sum is immediately verified to be a symplectic structure, and we will use the notation

$\widetilde{\omega}_H$ and $\widetilde{\omega}_V$ to mean the two blocks of the inverse. Together with (4.4), (4.5) and (4.6), this shows that

$$\omega_{\mathbb{C}} = \omega_H + \omega_V - i\omega_{\mathrm{T}^*}\,,$$

which allows to conclude that

$$\omega_t = k(\omega_H + \omega_V) + s\omega_{\mathrm{T}^*}\,.$$

The inverse of this form can be determined by representing it as a block matrix corresponding to the splitting in horizontal and vertical part under their identification with vectors and co-vectors on $\mathcal{M}$

$$\begin{pmatrix} k\omega & -s\mathbb{1} \\ s\mathbb{1} & k\widetilde{\omega} \end{pmatrix}^{-1} = \frac{1}{|t|^2}\begin{pmatrix} k\widetilde{\omega} & s\mathbb{1} \\ -s\mathbb{1} & k\omega \end{pmatrix}\,.$$

Therefore, the bi-vector inverse to ω_t is determined by

$$\widetilde{\omega}_t(\mathrm{d}u^\mu, \mathrm{d}v^\nu) = \frac{k}{|t|^2}\widetilde{\omega}(\mathrm{d}u^\mu, \mathrm{d}v^\nu)\,, \qquad \widetilde{\omega}_t(\mathrm{d}u^*_\mu, \mathrm{d}v^*_\nu) = \frac{k}{|t|^2}\omega\left(\frac{\partial}{\partial u^\mu}, \frac{\partial}{\partial v^\nu}\right),$$

$$\widetilde{\omega}_t(\mathrm{d}u^\mu, \mathrm{d}u^*_\nu) = \frac{s}{|t|^2}\delta^\mu_\nu = \widetilde{\omega}_t(\mathrm{d}v^\mu, \mathrm{d}v^*_\nu)\,.$$

We shall now study the Poisson bracket of two functions of polynomial type by considering the contributions corresponding to the three parts of $\widetilde{\omega}_t$ separately. To simplify the notation, rename the coordinates on $\mathcal{M}$ as q^μ and the dual ones as p_μ, so that $\boldsymbol{q} = (\mathbf{u}, \mathbf{v})$ and $\boldsymbol{p} = (\mathbf{u}^*, \mathbf{v}^*)$. If $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r)$ is a multi-index, we denote by $p_{\boldsymbol{\mu}}$ the monomial

$$p_{\boldsymbol{\mu}} = p_{\mu_1} \cdot \ldots \cdot p_{\mu_r}$$

Also, we will use the notation $\widehat{\boldsymbol{\mu}}_j$ to mean the multi-index obtained from $\boldsymbol{\mu}$ by removing its j-th entry.

If T is a totally symmetric tensor field on $\mathcal{M}$ of rank r_T, the polynomial function associated to it can then be expressed in coordinates as

$$P_T(\boldsymbol{p}, \boldsymbol{q}) = T^{\boldsymbol{\mu}}(\boldsymbol{q})p_{\boldsymbol{\mu}}\,.$$

A straightforward check shows that its differential is

$$\mathrm{d}P_T = \frac{\partial T^{\boldsymbol{\mu}}}{\partial q^\nu} p_{\boldsymbol{\mu}}\,\mathrm{d}q^\nu + \sum_{j=1}^{r_T} T^{\boldsymbol{\mu}} p_{\widehat{\boldsymbol{\mu}}_j}\,\mathrm{d}p_{\mu_j}\,.$$

Since the Levi-Civita connection is trivial in these coordinates, the left term can be written in a more intrinsic way as a covariant derivative. As for the term on the right, one can use that T is symmetric to replace the sum with a factor r_T

$$\mathrm{d}P_T = \nabla_\nu T^{\boldsymbol{\mu}} p_{\boldsymbol{\mu}} \,\mathrm{d}q^\nu + r_T T^{\boldsymbol{\mu}} p_{\widehat{\boldsymbol{\mu}}_1} \,\mathrm{d}p_{\mu_1} \,.$$

Let S be another totally symmetric tensor field on $\mathcal{M}$, of rank r_S. The contribution to $\{P_T, P_S\}_t$ corresponding to $\widetilde{\omega}_H$ only depends on the coefficients of the $\mathrm{d}q^\nu$'s, and it gives

$$\begin{aligned}\{P_T, P_S\}_H &= \nabla_\nu T^{\boldsymbol{\mu}} \nabla_\lambda S^{\boldsymbol{\rho}} p_{\boldsymbol{\mu}} p_{\boldsymbol{\rho}} \widetilde{\omega}_H \left(\mathrm{d}q^\nu, \mathrm{d}q^\lambda\right) = \\ &= (\nabla T \cdot \widetilde{\omega}_H \cdot \nabla S)^{\boldsymbol{\mu}\boldsymbol{\rho}} p_{\boldsymbol{\mu}} p_{\boldsymbol{\rho}} = \\ &= P_{\mathcal{S}(\nabla T \cdot \widetilde{\omega} \cdot \nabla S)} \,.\end{aligned}$$

Similarly, the term coming from $\widetilde{\omega}_V$ is

$$\begin{aligned}\{P_T, P_S\}_V &= r_T r_S T^{\boldsymbol{\mu}} S^{\boldsymbol{\rho}} p_{\widehat{\boldsymbol{\mu}}_1} p_{\widehat{\boldsymbol{\rho}}_1} \widetilde{\omega}_V \left(\mathrm{d}p_{\mu_1}, \mathrm{d}p_{\rho_1}\right) = \\ &= r_T r_S T^{\boldsymbol{\mu}} S^{\boldsymbol{\rho}} p_{\widehat{\boldsymbol{\mu}}_1} p_{\widehat{\boldsymbol{\rho}}_1} \omega_{\mu_1 \rho_1} = \\ &= r_T r_S (T \cdot \omega \cdot S)^{\widehat{\boldsymbol{\mu}}_1 \widehat{\boldsymbol{\rho}}_1} p_{\widehat{\boldsymbol{\mu}}_1} p_{\widehat{\boldsymbol{\rho}}_1} = \\ &= r_T r_S P_{\mathcal{S}(T \cdot \omega \cdot S)} \,.\end{aligned}$$

Finally, the term from $\widetilde{\omega}_{\mathrm{T}^*}$ gives instead

$$\begin{aligned}\{P_T, P_S\}_{\mathrm{T}^*} &= r_S (\nabla_\nu T^{\boldsymbol{\mu}}) S^{\boldsymbol{\rho}} p_{\boldsymbol{\mu}} p_{\widehat{\boldsymbol{\rho}}_1} \widetilde{\omega}_{\mathrm{T}^*} \left(\mathrm{d}q^\nu, \mathrm{d}p_{\rho_1}\right) + \\ &\quad + r_T T^{\boldsymbol{\mu}} (\nabla_\lambda S^{\boldsymbol{\rho}}) p_{\widehat{\boldsymbol{\mu}}_1} p_{\boldsymbol{\rho}} \widetilde{\omega}_{\mathrm{T}^*} \left(\mathrm{d}p_{\mu_1}, \mathrm{d}q^\lambda\right) = \\ &= r_S (\nabla_\nu T^{\boldsymbol{\mu}}) S^{\boldsymbol{\rho}} p_{\boldsymbol{\mu}} p_{\widehat{\boldsymbol{\rho}}_1} \delta^\nu_{\rho_1} - r_T T^{\boldsymbol{\mu}} (\nabla_\lambda S^{\boldsymbol{\rho}}) p_{\widehat{\boldsymbol{\mu}}_1} p_{\boldsymbol{\rho}} \delta^\lambda_{\mu_1} = \\ &= (r_S S \cdot \nabla T)^{\boldsymbol{\mu} \widehat{\boldsymbol{\rho}}_1} p_{\boldsymbol{\mu}} p_{\widehat{\boldsymbol{\rho}}_1} - (r_T T \cdot \nabla S)^{\widehat{\boldsymbol{\mu}}_1 \boldsymbol{\rho}} p_{\widehat{\boldsymbol{\mu}}_1} p_{\boldsymbol{\rho}} = \\ &= r_S P_{\mathcal{S}(S \cdot \nabla T)} - r_T P_{\mathcal{S}(T \cdot \nabla S)} \,.\end{aligned}$$

After including the needed weights, this calculation gives the result of Lemma 4.1.2 for the tensor associated to the Poisson commutator:

$$\frac{k}{|t|^2}\left(\mathcal{S}\left(\nabla T \cdot \widetilde{\omega} \cdot \nabla S\right) + r_T r_S \mathcal{S}\left(T \cdot \omega \cdot S\right)\right), + \frac{s}{|t|^2}\mathcal{S}\left(r_T T \cdot \nabla S - r_S S \cdot \nabla T\right).$$

4.3.2 Tensor fields and differential operators

As discussed in Appendix A.1, totally symmetric tensor fields define differential operators on $\mathscr{L}^{(t)}$, provided a linear connection on $\mathcal{M}$ is defined. We have observed that a choice of $\sigma \in \mathcal{T}$ induces a metric whose Levi-Civita connection does not depend on σ, although the metric does.

We now proceed to study the composition and commutator of two such operators, after introducing some more notation. If $\boldsymbol{\mu}$ is a multi-index, we denote by $l(\boldsymbol{\mu})$ its length. Let $\boldsymbol{\mu}$ be a multi-index of length l and $(\boldsymbol{\mu}^1, \boldsymbol{\mu}^2)$ an ordered pair of multi-indices $\boldsymbol{\mu}^1 = (\mu_{j_1^1}, \ldots, \mu_{j_{l-d}^1})$ and $\boldsymbol{\mu}^2 = (\mu_{j_1^2}, \ldots, \mu_{j_l^2})$. We call such a pair a partition of $\boldsymbol{\mu}$ if $\{ j_1^1 \leq \cdots \leq j_{l-d}^1 \}$ and $\{ j_1^2 \leq \cdots \leq j_l^2 \}$ form a partition of $\{ 1, \ldots, l \}$. In the following, unless otherwise specified, the appearance of a multi-index and a partition of it, one in lower and one upper position, will mean that summation is taken over all values of the multi-index and all its partitions.

Suppose now that T and S are two smooth tensor fields on $\mathcal{M}$, of respective rank r_T and r_S. As before, we express the associated operators in the coordinates $\boldsymbol{q}$ and $\boldsymbol{p}$

$$\nabla_T^{r_T} = T^{\boldsymbol{\mu}} \nabla_{\boldsymbol{\mu}}, \qquad \nabla_S^{r_S} = S^{\boldsymbol{\rho}} \nabla_{\boldsymbol{\rho}}.$$

Iterated use of the Leibniz rule shows that he composition of these two operators acts on a a smooth section ψ of $\mathscr{L}^{(t)}$ as

$$\left(\nabla_T^{r_T} \circ \nabla_S^{r_S} \right) \psi = T^{\boldsymbol{\mu}} \nabla_{\boldsymbol{\mu}} \left(S^{\boldsymbol{\rho}} \nabla_{\boldsymbol{\rho}} \psi \right) = T^{\boldsymbol{\mu}} (\nabla_{\boldsymbol{\mu}^1} S)^{\boldsymbol{\rho}} \nabla_{\boldsymbol{\mu}^2} \nabla_{\boldsymbol{\rho}} \psi. \tag{4.7}$$

We now wish to study the symbols of this operator. Each term in the sum gives a contribution of order $l(\boldsymbol{\mu}^2) + l(\boldsymbol{\rho}) = r - l(\boldsymbol{\mu}^1)$ in ψ, therefore only the terms with $l(\boldsymbol{\mu}^1) \leq d$ contribute to the $(r-d)$-th symbol of the composition. The only term contributing to the r-rh symbol is that coming from $l(\boldsymbol{\mu}^1) = 0$, which gives

$$\sigma_{r_T + r_S} \left(\nabla_T^{r_T} \circ \nabla_S^{r_S} \right) = \mathcal{S}(T \otimes S).$$

All the hypotheses of lemma A.1.2 are verified in this context, so it can be applied to $T \otimes S$ to show that this term does not contribute to the $(r-1)$-th symbol, which is then

$$\sigma_{r_T + r_S - 1} \left(\nabla_T^{r_T} \circ \nabla_S^{r_S} \right) = \sum_{l(\boldsymbol{\mu}^1)=1} \mathcal{S}\left(T^{\boldsymbol{\mu}} \nabla_{\boldsymbol{\mu}^1} S^{\boldsymbol{\rho}} \right) = r_T \mathcal{S}(T \cdot \nabla S).$$

Lemma A.1.2 further implies that the contribution of $T \otimes S$ to the next symbol is $-ik\frac{r_T r_S}{2}\mathcal{S}(T \cdot \omega \cdot S)$. Moreover, $T \cdot \nabla S$ is also of the kind considered by the lemma, being symmetric in the first $r_T - 1$ entries and the last r_S, so no contribution to the $(r-2)$-th symbol arises from the terms with $l(\boldsymbol{\mu}^1) = 1$. Finally, from $l(\boldsymbol{\mu}^1) = 2$ we find

$$\sum_{l(\boldsymbol{\mu}^1)=2} \mathcal{S}\left(T^{\boldsymbol{\mu}} \nabla_{\boldsymbol{\mu}^1}^2 S^{\boldsymbol{\rho}} \right) = \binom{r_T}{2} \mathcal{S}(T \cdot\cdot \nabla^2 S).$$

This argument proves the result of Lemma 4.1.3 by exchanging T and S and taking the difference. Notice that the operation of symmetrising $S \cdot \omega \cdot T$ acts only on the indices remaining after the contraction, and in particular this term is opposite to $\mathcal{S}(S \cdot \omega \cdot T)$. Therefore, the operation above doubles this term, thus giving the wanted expression.

4.3.3 Functions of polynomial type and differential operators

A combination of the two correspondences established above leads to one between functions of polynomial type on $\mathcal{M}_{\mathbb{C}}$ and sums of differential operators acting on $\mathcal{C}^\infty(\mathcal{M}, \mathscr{L}^k)$. Let $k \in \mathbb{Z}_{>0}$ be fixed, set for convenience $\hbar = s/|t|^2$, and for $f \in \mathrm{Pol}$ homogeneous of degree r let

$$\widehat{f} = (-i\hbar)^r \nabla^r_{T_f} \, .$$

Thinking of this expression as a formal polynomial in $\hbar$, the map extends by $\mathbb{C}$-linearity to a map from $\mathrm{Pol}[\hbar]$ to $\mathrm{D}_k[\hbar]$. We may now consider the sub-algebra $\mathrm{D}_k[\hbar]_\leq$ consisting of the polynomials $\sum D_r \hbar^r$ such that each D_r has order at most r. It is clear that the map above takes values in this sub-algebra, and it has an inverse

$$\sigma^\hbar(D) = \sum_{r=1}^{\infty} \sigma^\hbar_r(D) \, , \quad \text{where} \quad \sigma^\hbar_r(D) = (-i\hbar)^{-r} \sigma_r(D) \, .$$

Clearly, for any two functions $f, g \in \mathrm{Pol}$ the composition $\widehat{f} \circ \widehat{g}$ is an element of $\mathrm{D}_k[\hbar]_\leq$ of degree equal to the sum of those of f and g. This argument allows one to define a non-commutative product $\star$ for elements of $\mathrm{Pol}[h]$:

$$f \star g := P_{\sigma^\hbar(\widehat{f} \circ \widehat{g})} \, . \tag{4.8}$$

Notice that, if f and g are homogeneous and the degree of their product is r, the coefficient of a given power $\hbar^d$ is given by the $r-d$-th symbol of $\widehat{f} \circ \widehat{g}$. Indeed, this composition defines a homogeneous element of $\mathrm{D}_k[\hbar]$ of degree r in $\hbar$; in the weighted symbol, the part of order $r-d$ gets a factor $(-i\hbar)^{d-r}$, thus leaving a residual $\hbar^d$. Moreover, the presence of additional factors of k due to the curvature of the pre-quantum line bundle is determined by Lemma A.1.2. In particular, the degree of the $(r-d)$-th (un-weighted) symbol in k is at most $d/2$.

For every fixed s, one obtains a non-commutative product on Pol by setting $\hbar = s/|t|^2$. Notice that, while k enters the product through the curvature of the pre-quantum line bundle, the $\star$ depends on s only via the parameter $\hbar$.

We now proceed to check the first part of Theorem 4.1.4.

Proposition 4.3.1. *The product $f \star g$ above is associative and it gives the usual point-wise multiplication in the limit when s goes to ∞. It also satisfies the following first-order Dirac condition:*

$$f \star g - g \star f + i\{f, g\}_t = o(s^{-1}) \quad s \to \infty \, .$$

Notice that this is indeed a Dirac condition, because the Poisson structure itself contains the quantum parameter and decays as s^{-1} in the limit $s \to \infty$.

Proof. The associativity of $\star$ is an immediate consequence of that of the action of differential operators. As mentioned above, for homogeneous f and g the coefficient of the d-th power of $\hbar$ in $f \star g$ is determined by the $(r-d)$-th symbol of $\nabla_{T_f} \circ \nabla_{T_g}$. The top symbol of this composition is simply $\mathcal{S}(T \otimes S)$, which implies that

$$f \star g = fg + \hbar R$$

for some remainder R. But of course $\hbar = O(s^{-1})$ in the limit for $s \to \infty$, which proves that $\star$ reproduces the point-wise multiplication. The statement extends by linearity to arbitrary functions of polynomial type.

The Dirac condition is argued similarly, by comparing Lemmas 4.1.2 and 4.1.3. Indeed, the only term of degree 1 in $\hbar$ in the commutator $f \star g - g \star f$ for homogeneous f and g is given by the top symbol

$$\frac{s}{|t|^2}\left(r_{T_f}\mathcal{S}\left(T_f \cdot \nabla T_g\right) - r_{T_g}\mathcal{S}\left(T_g \cdot \nabla T_f\right)\right).$$

On the other hand, this term appears in the tensor expressing the Poisson bracket, together with two others proportional to $k/|t|^2$, which decay faster than s^{-1}. □

The product $\star$ can also be used to define a formal one on $\mathrm{Pol}[[s^{-1}]]$ by expanding $\hbar$ as a power series in s^{-1}:

$$\hbar = \frac{s}{|t|^2} = \sum_{n=0}^{\infty} (-k)^{2n} s^{-2n-1}.$$

Notice that this algebra has a natural filtration given by the degree of the coefficients as polynomial functions, and that the product is filtered. The proposition above implies that this defines a formal associative product, commutative modulo s^{-1} and satisfying the Dirac condition

$$f \star g - g \star f - \{f, g\}_t \equiv 0 \pmod{s^{-2}}.$$

Using the weighted correspondences between operators and functions of polynomial type, one can pull the formal Hitchin-Witten connection back to $\mathrm{Pol}[[s^{-1}]] \times \mathcal{T}$. Since the formal connection on $\mathscr{A}_k \times \mathcal{T}$ acts by derivations with respect to the composition of operators, so does the new one. Finally, to the formal trivialisation $\mathcal{R}$ discussed in the previous chapter corresponds one on this bundle, and because it acts essentially via conjugation by an invertible power series, it also preserves the products. This concludes the argument and proves Theorem 4.1.4.

4.4 A map from the real polarisation

Recall that a fixed $\sigma \in \mathcal{T}$ defines a real polarisation P on $\mathcal{M}_{\mathbb{C}}$, which following Witten we have considered for the geometric quantisation of Chern-Simons theory. This is defined by identifying each tangent space with $H^1_A(\Sigma, \mathfrak{sl}(2,\mathbb{C}))$ and taking as P its subspace consisting of classes represented by $(1,0)$-forms on Σ. This space is Lagrangian, and dual to its conjugate $\overline{P}$, consisting of $(0,1)$-forms, via the symplectic structure. On the other hand, if A is a unitary connection then $\overline{P}$ is naturally identified with the holomorphic tangent space of $\mathcal{M}$ at $[A]$ via the structure induced by σ. Thinking of P and $\overline{P}$ as sub-bundles of $\mathrm{T}\,\mathcal{M}_{\mathbb{C}}$, one may restrict them to $\mathcal{M}$, and the resulting objects may be identified with the holomorphic cotangent and tangent bundles of $\mathcal{M}$, respectively. These, in turn, can be identified with the real bundles with their complex structures coming from σ.

The geometry of real polarisations defines moreover a flat connection on each leaf, thus a notion of geodesics. Using this one may define an exponential map from each tangent or cotangent space on $\mathcal{M}$ valued in the leaf through the base point. Assuming that the leaves are all complete and simply connected, this map defines embeddings of $\mathrm{T}\,\mathcal{M}$ and $\mathrm{T}^*\,\mathcal{M}$ into $\mathcal{M}_{\mathbb{C}}$. In the case of genus 1, it is easily seen via its explicit coordinates that this is indeed satisfied, and the image is an open dense subset.

Suppose now that a function f is given on the part of $\mathcal{M}_{\mathbb{C}}$ reached by this map, and that it is of linear type. As seen on $\mathrm{T}^*\,\mathcal{M}$, this means that there exists a vector field T on $\mathcal{M}$ such that for every point $[A]$ and covector η at $[A]$, one has $f([A],\eta) = \eta(T)$. Thinking of T and η as vectors on $\overline{P}$ and P respectively, this pairing reads $\omega_{\mathbb{C}}(\eta, T)$. Therefore, since f is linear along P, the vector field T, as seen on P, satisfies the condition determining the Hamiltonian field of f, at least along the directions of P. But this means that the actual Hamiltonian field differs from T by a vector tangent to P itself, because this space is Lagrangian. On the other hand, the quantum Hilbert space consists of the polarised sections of the prequantum line bundle, which are by definition not sensitive to the addition of vector fields tangent to P. Therefore, the first-order part of the desired quantum operator is indeed represented by ∇_T, up to weights. The part of order zero, on the other hand, vanishes because f restricts to zero on $\mathcal{M}$. This concludes the argument that the quantum operators of linear functions are represented by those constructed via the polarisation in the way analogous to that of the previous sections.

This evidence motivates one to study the Hitchin-Witten derivative of these operators in an attempt to find a covariantly constant quantisation. This will be the starting point of the next chapter, where we use the approach of the operators from geometric quantisation for genus 1 and $n = 2$.

Chapter 5

The AJ conjecture for the Teichmüller TQFT

The content of this chapter is essentially that of the joint paper [AM17] with Andersen.

5.1 Overview and summary of the results

We consider the level-N Andersen-Kashaev invariant $J_{M,K}^{(\mathrm{b},N)}$, which is a minor transform of the Teichmüller TQFT partition function for a knot K sitting inside a closed, oriented 3-manifold M, where b is a unitary complex quantum parameter. The level $N = 1$ Teichmüller TQFT was introduced by Andersen and Kashaev in [AK14a], and then extended to arbitrary (odd) N in [AK14b] and further detailed in [AM16b]. As was mentioned in the introduction, one of the most interesting problems in the current development of the theory is the relation between the combinatorial approaches and the one via geometric quantisation. In the specific situation of a genus 1 surface, however, an identification between the vector spaces coming from the two different approaches is provided by the so called Weil-Gel'fand-Zak transform.

The invariant $J_{M,K}^{(\mathrm{b},N)}$ consists of a complex-valued function on $\mathbb{A}_N := \mathbb{R} \oplus \mathbb{Z}/N\mathbb{Z}$ which has a natural meromorphic extension to $\mathbb{A}_N^{\mathbb{C}} := \mathbb{C} \oplus \mathbb{Z}/N\mathbb{Z}$. Its definition is based on a multiple integral involving the level-N quantum dilogarithm [AK14b], an extension of Faddeev's function to $\mathbb{A}_N$ satisfying an adaptation of the same difference equation. In [AK14a, AM16b], the invariant was conjectured to enjoy certain properties analogous to those expected for the coloured Jones polynomial, thus making it into an $\mathrm{SL}(2,\mathbb{C})$ analogous of the $\mathrm{SU}(2)$ invariant. The statement was checked for the first two hyperbolic knots by explicit computation of the invariant, using the properties of the quantum dilogarithm. The final expression was found to agree for $N = 1$ with the partition function of the quantum $\mathrm{SL}(2,\mathbb{C})$-Chern-Simons theory derived in the physics literature [Hik01, Hik07, Dim13, DGLZ09, DFM11]. In some of the cited works, Faddeev's equation for the quantum dilogarithm is used for showing that said partition function is annihilated by some version of the quantum $\widehat{A}$-polynomial from Garoufalidis' original AJ-conjecture [Gar04].

In the present chapter we address the problem of the quantisation of the observables of the $\mathrm{SL}(2,\mathbb{C})$-Chern-Simons theory in genus one, with a specific interest for the A-polynomial of a knot. In analogy with [Gar04], we search for q-commutative operators $\widehat{m}_{\mathbf{x}}$ and $\widehat{\ell}_{\mathbf{x}}$ acting on functions on $\mathbb{A}_N$, associated to the holonomy functions m and ℓ on the $\mathrm{SL}(2,\mathbb{C})$-character variety of the surface. The starting point of this process will be geometric quantisation, but for the sake of the correspondence with the Teichmüller TQFT it is convenient to work on the *cover* $\mathbb{T}^2$ of $\mathcal{M}$. This will require a slight change of some of the notations used in the rest of the thesis, including the level of the theory. For every value of the Teichmüller parameter $\sigma \in \mathcal{T}$, we find that the pre-quantum operators associated to the *logarithmic* holonomy coordinates U and V preserve the polarisation. Therefore, they can be promoted to quantum operators $\widehat{U}_\sigma$ and $\widehat{V}_\sigma$, which moreover turn out to be normal, thus admitting well-defined exponentials $\widehat{m}_\sigma$ and $\widehat{\ell}_\sigma$.

Theorem 5.1.1. *The quantum operators $\widehat{U}_\sigma$ and $\widehat{V}_\sigma$, and hence $\widehat{m}_\sigma$ and $\widehat{\ell}_\sigma$, are covariantly constant with respect to the Hitchin-Witten connection.*

Combining the explicit trivialisation of the Hitchin-Witten connection with the Weil-Gel'Fand-Zak transform, we obtain σ-independent operators $\widehat{m}_{\mathbf{x}}$ and $\widehat{\ell}_{\mathbf{x}}$ acting on functions on $\mathbb{A}_N^{\mathbb{C}}$.

Theorem 5.1.2. *Let $t = N + iS$ be fixed, and put*

$$\mathrm{b} = -ie^{2rN}, \quad \text{where} \quad e^{4rN} = -\frac{\bar{t}}{t}.$$

Then the operators $\widehat{m}$ and $\widehat{\ell}$ as above act on functions on $\mathbb{A}_N^{\mathbb{C}}$ via the Weil-Gel'fand-Zak transform as

$$\widehat{m}_{\mathbf{x}} \colon f(x,n) \mapsto e^{-2\pi \frac{\mathrm{b}x}{\sqrt{N}}} e^{2\pi i \frac{n}{N}} f(x,n),$$
$$\widehat{\ell}_{\mathbf{x}} \colon f(x,n) \mapsto f\left(x - \frac{i\mathrm{b}}{\sqrt{N}}, n+1\right).$$

Moreover, the operators are q-commutative, i.e.

$$\widehat{\ell}_{\mathbf{x}} \widehat{m}_{\mathbf{x}} = q \widehat{m}_{\mathbf{x}} \widehat{\ell}_{\mathbf{x}},$$

with

$$q = \exp\left(2\pi i \frac{\mathrm{b}^2 + 1}{N}\right) = e^{4\pi i/t}.$$

We then consider the algebra $\mathcal{A}$ generated by these operators, and sitting inside this the left ideal $\mathcal{I}(J^{(\mathrm{b},N)}_{M,K})$ annihilating the invariant. Following the lines of [Gar04], we define the $\widehat{A}^{\mathbb{C}}$-polynomial as a preferred element of this ideal. Using the explicit expression of the invariant for 4_1 and 5_2 inside S^3, we compute the $\widehat{A}^{\mathbb{C}}$-polynomial for these knots, and check the following conjecture for these two cases.

Conjecture 5.1.1. *Let $K \subseteq M$ be a knot inside a closed, oriented 3-manifold, with hyperbolic complement. Then the non-commutative polynomial $\widehat{A}^{\mathbb{C}}_K$ agrees with $\widehat{A}_K$ up to a right factor, linear in $\widehat{m}_{\mathbf{x}}$, and it reproduces the classical A-polynomial in the sense of the original AJ-conjecture.*

Besides its intrinsic interest in relation to the original AJ-conjecture, we also find in this statement a further indication concerning the role of the Weil-Gel'fand-Zak transform in connecting geometric quantisation and Teichmüller TQFT in genus one.

Theorem 5.1.3. *Conjecture 5.1.1 holds true for the figure-eight knot 4_1, and 5_2.*

We shall re-formulate the statements more precisely later in the discussion; the proof of the result goes as follows. For either knot, the Andersen-Kashaev invariant is a function on $\mathbb{A}_N$ defined as an integral in $d\mathbf{y}$ of an appropriate function of two variables $\mathbf{x}, \mathbf{y}$ along $\mathbb{A}_N$. The integrand is obtained as a combination of quantum dilogarithms and Gaussian functions, and it has a natural meromorphic extension to the whole $(\mathbb{A}^{\mathbb{C}}_N)^2$. In order to understand the ideal, we study first the polynomials in $\widehat{m}_{\mathbf{x}}$, $\widehat{m}_{\mathbf{y}}$, $\widehat{\ell}_{\mathbf{x}}$ and $\widehat{\ell}_{\mathbf{y}}$ annihilating the integrand. We start by considering the convergence properties of the integral, so as to ensure that it can be extended to the whole $\mathbb{A}^{\mathbb{C}}_N$. For parameters a and ε in a suitable domain, we find open regions $R_{a,\varepsilon}$ on which the meromorphic extension of the invariant is given by integration along appropriate contours $\Gamma_{a,\varepsilon}$. These regions exhaust the whole $\mathbb{A}^{\mathbb{C}}_N$ as the parameters range in their domain, thus giving explicit expressions for the invariant at every given point. Next, we use Lemma 5.2.2 to express the action of $\widehat{\ell}_{\mathbf{x}}$ and $\widehat{\ell}_{\mathbf{y}}$ on the integrand in terms of $\widehat{m}_{\mathbf{x}}$ and $\widehat{m}_{\mathbf{y}}$, thus giving two operators with the desired property. With the help of a computer, we run reduction in $\widehat{m}_{\mathbf{y}}$, and since the action of $\widehat{\ell}_{\mathbf{y}}$ inside an integral is equivalent to a shift in the integration contour we can evaluate the resulting polynomial at $\widehat{\ell}_{\mathbf{y}} = 1$. Using some care, one can move the operator outside the integral and finally find an element of $\mathcal{I}_{\mathrm{loc}}(J^{(\mathrm{b},N)}_{M,K})$, which we then check to be the desired generator $\widehat{A}^{\mathbb{C}}_{t,K}$.

This paper is organised as follows. In section 5.2 we overview the background material we refer to throughout the the rest of the chapter. This includes generalities on the level-N quantum dilogarithm and the Weil-Gel'fand-Zak transform, the Teichmüller TQFT, and the original AJ-conjecture. In section 5.3 we run the geometric quantisation machinery to obtain the desired operators. First, we use the standard definition of the pre-quantum operators to quantise the logarithmic holonomy functions corresponding to the meridian and longitude on the torus. Next, we check that the operators are compatible with the chosen polarisation, thus descending to *quantum* operators which are Hitchin-Witten covariantly constant. We then show that the operators can be consistently exponentiated, whence we define quantum operators for the exponential holonomy eigenvalues, which in fact generate the algebra of regular functions on the character variety. Finally, we present the explicit trivialisation of the Hitchin-Witten connection and use it to make the operators independent on the Teichmüller parameter. We conclude the section by determining the action of the operators on functions on $\mathbb{A}_N$ via by the Weil-Gel'fand-Zak transform. In section 5.4, we explicitly carry out the procedure described above to determine the $\widehat{A}^{\mathbb{C}}$-polynomial for the first two hyperbolic knots.

5.2 Introduction

5.2.1 $\mathbb{A}_N$ and the level-N quantum dilogarithm

Definition 5.2.1. *For every positive integer N, let $\mathbb{A}_N$ be the locally compact Abelian group $\mathbb{R} \oplus \mathbb{Z}/N\mathbb{Z}$ endowed with the nomalised Haar measure $\mathrm{d}(x,n)$ defined by*

$$\int_{\mathbb{A}_N} f(x,n)\,\mathrm{d}(x,n) := \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \int_{\mathbb{R}} f(x,n)\,\mathrm{d}x\,. \tag{5.1}$$

We denote by $\mathcal{S}(\mathbb{A}_N, \mathbb{C})$ the space of Schwartz class functions on $\mathbb{A}_N$, i.e. functions $f(x,n)$ on $\mathbb{A}_N$ which restrict to Schwartz class functions on $\mathbb{R}$ for every n. We shall denote $\mathbb{C} \oplus \mathbb{Z}/N\mathbb{Z}$ by $\mathbb{A}_N^{\mathbb{C}}$.

Clearly $\mathcal{S}(\mathbb{A}_N, \mathbb{C})$ sits inside the space $\mathrm{L}^2(\mathbb{A}_N, \mathbb{C})$ of square-summable functions, as a dense subspace. We shall often use the notation $\mathbf{x} = (x,n)$; moreover, if $\lambda \in \mathbb{C}$ we shall write $\mathbf{x} + \lambda$ as a short-hand for $(x+\lambda, n)$.

As in [AK14b], we use the following notation for the Fourier kernels on $\mathbb{A}_N$:

$$\big\langle (x,n),(y,m) \big\rangle = e^{2\pi i x y} e^{-2\pi i n m/N}\,.$$

It is straightforward to check that

$$\langle -\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle^{-1} = \langle \mathbf{x}, -\mathbf{y} \rangle\,, \qquad \langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle\,.$$

Also, the Gaussian function on $\mathbb{A}_N$ is denoted by

$$\big\langle (x,n) \big\rangle = e^{\pi i x^2} e^{-\pi i n(n+N)/N}\,.$$

Clearly, the Gaussian is symmetric, i.e. $\langle \mathbf{x} \rangle = \langle -\mathbf{x} \rangle$.

Fix now b, a complex unitary parameter with $\mathrm{Re}(\mathrm{b}) > 0$ and $\mathrm{Im}(\mathrm{b}) \geq 0$, and introduce constants c_{b} and q defined by

$$c_{\mathrm{b}} := \frac{i(\mathrm{b}+\mathrm{b}^{-1})}{2} = i\,\mathrm{Re}(\mathrm{b})\,, \qquad q^{\frac{1}{2}} := -e^{\pi i \frac{b^2+1}{N}} = \left\langle\!\left(\frac{i\mathrm{b}}{\sqrt{N}}, -1\right)\!\right\rangle^{-1}.$$

We shall summarise here the fundamental properties of the level N quantum dilogarithm which are relevant for this work. For the precise definition and further details see for example [AK14b, AM16b]. For N a positive *odd* integer, the quantum dilogarithm D_{b} at level N and quantum parameter b is a meromorphic function defined on $\mathbb{A}_N^{\mathbb{C}}$. One of its fundamental properties is that it solves the Faddeev difference equations

$$\mathrm{D}_{\mathrm{b}}\left(x \pm \frac{i\mathrm{b}}{\sqrt{N}}, n \pm 1\right) = \left(1 - e^{\pm\frac{\mathrm{b}^2+1}{N}} e^{2\pi \frac{\mathrm{b}}{\sqrt{N}} x} e^{2\pi i \frac{n}{N}}\right)^{\mp 1} \mathrm{D}_{\mathrm{b}}(x,n)\,,$$

$$\mathrm{D}_{\mathrm{b}}\left(x \pm \frac{i\overline{\mathrm{b}}}{\sqrt{N}}, n \mp 1\right) = \left(1 - e^{\pm\frac{\overline{\mathrm{b}}^2+1}{N}} e^{2\pi \frac{\overline{\mathrm{b}}}{\sqrt{N}} x} e^{-2\pi i \frac{n}{N}}\right)^{\mp 1} \mathrm{D}_{\mathrm{b}}(x,n)\,.$$

It also satisfies the inversion relation

$$\mathrm{D}_{\mathrm{b}}(\mathbf{x})\mathrm{D}_{\mathrm{b}}(-\mathbf{x}) = \zeta_{N,\mathrm{inv}}^{-1} \langle \mathbf{x} \rangle\,, \qquad \zeta_{N,\mathrm{inv}} = e^{\pi i (N + 2c_{\mathrm{b}}^2 N^{-1})/6}\,.$$

The zeroes of this function are located at the points

$$\left(-\frac{c_{\mathrm{b}} + i\alpha \mathrm{b} + i\beta\overline{\mathrm{b}}}{\sqrt{N}}, \beta - \alpha\right) \quad \text{for } \alpha, \beta \in \mathbb{Z}_{\geq 0}\,.$$

By the inversion relation, the poles occur at the opposites of these points.

The following lemma is particularly relevant for studying the convergence of integrals involving the quantum dilogarithm.

Lemma 5.2.1. *For $n \in \mathbb{Z}/N\mathbb{Z}$ fixed, the quantum dilogarithm has the following asymptotic behaviour for $x \to \infty$:*

$$\mathrm{D}_{\mathrm{b}}(x,n) \approx \begin{cases} 1 & \text{on } |\arg(x)| > \frac{\pi}{2} + \arg(\mathrm{b}), \\ \zeta_{N,\mathrm{inv}}^{-1} \langle \mathbf{x} \rangle & \text{on } |\arg(x)| < \frac{\pi}{2} - \arg(\mathrm{b}). \end{cases}$$

Furthermore, the dilogarithm enjoys the following unitarity property

$$\overline{\mathrm{D}_{\mathrm{b}}(x,n)}\mathrm{D}_{\mathrm{b}}(\overline{x},n) = 1\,.$$

Proof. The asymptotic behaviour of D_{b} for $N = 1$ and the unitarity relation for all N are stated in [Mar16]. Calling Φ_{b} the level 1 dilogarithm, the asymptotics in the general case can be checked using the relation

$$\mathrm{D}_{\mathrm{b}}(x,n) = \prod_{j=0}^{N-1} \Phi_{\mathrm{b}}\left(\frac{x}{\sqrt{N}} + \frac{N-1}{N} c_{\mathrm{b}} - i\mathrm{b}^{-1}\frac{j}{N} - i\mathrm{b}\left\{\frac{j+n}{N}\right\}\right)\,.$$

Here $\{\cdot\}$ denotes the fractional part of a real number. In the limit for $x \to \infty$ on the sector $|\arg(x)| > \pi/2 + \arg \mathsf{b}$, one can use the statement for $N = 1$ to see that each factor in the product is asymptotic to 1, hence so is the whole product. The other limit is implied by this, together with the inversion formula. □

It is convenient for the following discussion to change the notation according to [AM16b]. We shall call

$$\varphi_{\mathsf{b}}(x,n) := \mathrm{D}_{\mathsf{b}}(x,-n)\,.$$

The zeroes and poles of φ_{b} occur at the points $\mathbf{p}_{\alpha,\beta}$ and $-\mathbf{p}_{\alpha,\beta}$ respectively, for $\alpha, \beta \in \mathbb{Z}_{\geq 0}$, where

$$\mathbf{p}_{\alpha,\beta} := \left(-\frac{c_{\mathsf{b}} + i\alpha\mathsf{b} + i\beta\overline{\mathsf{b}}}{\sqrt{N}}, \alpha - \beta\right).$$

We shall often use T to refer to the infinite closed triangle

$$T = \left\{ x \in \mathbb{C} : x \text{ lies below } -\frac{c_{\mathsf{b}}}{\sqrt{N}} + i\mathbb{R}\mathsf{b} \text{ and } -\frac{c_{\mathsf{b}}}{\sqrt{N}} + i\mathbb{R}\overline{\mathsf{b}} \right\}.$$

In particular, the zeroes and poles of $\varphi_{\mathsf{b}}(x,n)$ for n fixed occur only for $x \in T$ and $x \in -T$ respectively. Lemma 5.2.1 holds unchanged for φ_{b} in place of D_{b}.

Definition 5.2.2. *If $k \in \mathbb{Z}_{>0}$ and $\boldsymbol{\mu} : (\mathbb{A}_N^{\mathbb{C}})^k \to \mathbb{A}_N^{\mathbb{C}}$ is a $\mathbb{Z}$-linear function, let us denote by $\widehat{m}_{\boldsymbol{\mu}}$ the operator acting on complex-valued functions on $(\mathbb{A}_N^{\mathbb{C}})^k$ as*

$$\widehat{m}_{\boldsymbol{\mu}} f := \left\langle \boldsymbol{\mu}, \left(\frac{i\mathsf{b}}{\sqrt{N}}, -1\right) \right\rangle f\,.$$

Moreover, $\widehat{\ell}_{\mathbf{x}}$ is the following operator acting on $\mathbb{C}$-valued functions on $\mathbb{A}_N^{\mathbb{C}}$ by

$$\widehat{\ell}_{\mathbf{x}} f(x,n) := f\left(x - \frac{i\mathsf{b}}{\sqrt{N}}, n+1\right).$$

It is clear that the operator $\widehat{m}_{\mu}$ always restricts to functions defined on $\mathbb{A}_N^k$, and that in particular for $\boldsymbol{\mu} = \mathbf{x}$ one gets explicitly

$$\widehat{m}_{\mathbf{x}} f(\mathbf{x}) = e^{-2\pi \frac{\mathsf{b}x}{\sqrt{N}}} e^{2\pi i \frac{n}{N}} f(\mathbf{x})$$

On the other hand, the condition for b to have positive real part implies that $i\mathsf{b}$ is never real, so strictly speaking $\widehat{\ell}_{\mathbf{x}}$ is only defined for functions on $\mathbb{A}_N^{\mathbb{C}}$. However, every analytic function $f\colon \mathbb{A}_N \to \mathbb{C}$ with infinite radius of convergence has a unique holomorphic extension to $\mathbb{A}_N^{\mathbb{C}}$. One can then make sense of the action of $\widehat{\ell}_{\mathbf{x}}$ by applying the shift to the extended function and then restricting back to $\mathbb{A}_N$. The set of square-summable such functions is dense in $L^2(\mathbb{A}_N)$, so $\widehat{\ell}_{\mathbf{x}}$ is a densely defined operator on this space.

The following lemma is an immediate consequence of the definitions and Faddeev's difference equation.

Lemma 5.2.2. *The operator $\widehat{\ell}_{\mathbf{x}}$ acts on the Gaussian as*

$$\widehat{\ell}_{\mathbf{x}} \langle \mathbf{x} \rangle = q^{-\frac{1}{2}} \widehat{m}_{\mathbf{x}}^{-1} \langle \mathbf{x} \rangle \ ,$$

and on the quantum dilogarithm as

$$\widehat{\ell}_{\mathbf{x}} \varphi_{\mathrm{b}}(\mathbf{x}) = \Big(1 + q^{-\frac{1}{2}} \widehat{m}_{\mathbf{x}}^{-1}\Big) \varphi_{\mathrm{b}}(\mathbf{x}) \,,$$
$$\widehat{\ell}_{\mathbf{x}}^{-1} \varphi_{\mathrm{b}}(\mathbf{x}) = \Big(1 + q^{\frac{1}{2}} \widehat{m}_{\mathbf{x}}^{-1}\Big)^{-1} \varphi_{\mathrm{b}}(\mathbf{x}) \,.$$

Moreover, the following commutation relation holds

$$\widehat{\ell}_{\mathbf{x}} \widehat{m}_{\mathbf{x}} = q \widehat{m}_{\mathbf{x}} \widehat{\ell}_{\mathbf{x}} \,.$$

5.2.2 Garoufalidis's original AJ-conjceture

Following the notation of [Gar04], we consider the q-commutative algebra

$$\mathcal{A} := \mathbb{Z}[q^{\pm 1}]\langle Q, E \rangle \Big/ \Big(EQ - qQE\Big) \,,$$

One can also make sense of inverting polynomials in Q by considering the algebra of rational functions $\mathbb{Q}(q, Q)$ and embedding the above into

$$\mathcal{A}_{\mathrm{loc}} := \left\{ \sum_{k=0}^{l} a_k(q, Q) E^k \colon l \in \mathbb{Z}_{\geq 0},\ a_k \in \mathbb{Q}(q, Q) \right\} \tag{5.2}$$

with product given by

$$\Big(a(q, Q)E^k\Big) \cdot \Big(b(q, Q)E^h\Big) := a(q, Q) b(q, q^k Q) E^{k+h} \,.$$

In the above mentioned work, Garoufalidis considers the space

$$\mathcal{F} = \{\, f : \mathbb{N} \to \mathbb{Q}(q) \,\}$$

and the action of $\mathcal{A}$ and $\mathcal{A}_{\mathrm{loc}}$ determined by

$$Qf(n) = q^n f(n) \,, \qquad Ef(n) = f(n+1) \,.$$

It is immediately checked that this defines indeed an algebra representation. For a given element $f \in \mathcal{F}$ one can consider the sets of annihilators of f, which are in fact left ideals:

$$\mathcal{I}_{\mathrm{loc}}(f) = \{\, p(Q,E) \in \mathcal{A}_{\mathrm{loc}} \colon p(Q,E)f = 0 \,\} \,, \qquad \mathcal{I}(f) = \mathcal{I}_{\mathrm{loc}}(f) \cap \mathcal{A} \,.$$

Since every ideal in $\mathcal{A}_{\mathrm{loc}}$ is principal, such a function defines uniquely a generator $\widehat{A}_{q,f}$ of $\mathcal{I}_{\mathrm{loc}}(f)$ of minimal degree in E and co-prime coefficients in $\mathbb{Z}[q,Q]$. The AJ conjecture, as formulated in [Gar04], states then the following.

Conjecture 5.2.1 (AJ Conjecture)**.** *Let K be a knot in S^3 and $J_K : \mathbb{N} \to \mathbb{Z}[q^{\pm 1}]$ its coloured Jones function, and abbreviate $\widehat{A}_{q,J_K}$ as $\widehat{A}_{q,K}$. Then, up to a left factor, $\widehat{A}_{q,K}(Q,E)$ returns the classical A-polynomial of K when evaluated at $q = 1$, $Q = m^2$ and $E = \ell$.*

Given $f \in \mathcal{F}$ one may also study the problem of finding $p \in \mathcal{A}$ such that

$$p(Q,E)f \in \mathbb{Q}(q^n, q)\,,$$

and define the non-homogeneous $\widehat{A}_q^{\mathrm{nh}}$-polynomial of f as the minimal degree solution with co-prime coefficients in $\mathbb{Z}[q,Q]$. Clearly, if $\widehat{A}_q^{\mathrm{nh}}(f)f = B(q^n,q)$, then one can divide both sides on the left by $B(Q,q)$ in $\mathcal{A}_{\mathrm{loc}}$ to find an operator sending f to a constant, so

$$\Big(E-1\Big)\left(\frac{1}{B(Q,q)}\widehat{A}_q^{\mathrm{nh}}(f)\right)f = 0\,.$$

This can be used to retrieve the $\widehat{A}_q$-polynomial of f and state an AJ-conjecture for the non-homogeneous $\widehat{A}_q^{\mathrm{nh}}$-polynomial.

The problem of the non-homogeneous recursion is addressed in [GS10] for the specific class of twist knots K_p. The figure-eight knot 4_1 and 5_2 correspond to K_p for $p = -1$ and $p = 2$, respectively, and in the cited work the homogeneous $\widehat{A}_q^{\mathrm{nh}}$-polynomials for these knots are explicitly found to be

$$\begin{aligned} \widehat{A}^{\mathrm{nh}}_{q,4_1} &= q^2Q^2\Big(q^2Q-1\Big)\Big(qQ^2-1\Big)E^2 \\ &\quad -\Big(qQ-1\Big)\Big(q^4Q^4 - q^3Q^3 - q(q^2+1)Q^2 - qQ + 1\Big)E \\ &\quad + q^2Q^2\Big(Q-1\Big)\Big(q^3Q^2-1\Big)\,, \end{aligned} \tag{5.3}$$

$$\begin{aligned}\widehat{A}^{\mathrm{nh}}_{q,5_2} = &\left(q^3Q-1\right)\left(qQ^2-1\right)\left(q^2Q^2-1\right)E^3\\ &+q\left(q^2Q-1\right)\left(qQ^2-1\right)\left(q^4Q^2-1\right)\\ &\cdot\left(q^9Q^5-q^7Q^4-q^4(q^3-q^2-q+1)Q^3+q^2(q^3+1)Q^2+2q^2Q-1\right)E^2\\ &-q^5Q^2\left(qQ-1\right)\left(q^2Q^2-1\right)\left(q^5Q^2-1\right)\\ &\cdot\left(q^6Q^5-2q^5Q^4-q^2(q^3+1)Q^3+q(q^3-q^2-q+1)Q^2+qQ-1\right)E\\ &+q^9Q^7\left(Q-1\right)\left(q^4Q^2-1\right)\left(q^5Q^2-1\right).\end{aligned} \tag{5.4}$$

Notice how a factor (q^jQ-1) appears next to each E^j; by the commutation relation, each of these can be turned into $(Q-1)$ and taken to the right. These non-commutative polynomials can be explicitly compared to their classical counterparts

$$A_{4_1}(m,\ell) = m^4\ell^2 - \left(m^8-m^6-2m^4-m^2+1\right)\ell + m^4\,, \tag{5.5}$$

$$\begin{aligned}A_{5_2}(m,\ell) = \ell^3 + &\left(m^{10}-m^8+2m^4+2m^2-1\right)\ell^2\\ &-m^4\left(m^{10}-2m^8-2m^6+m^2-1\right)\ell+m^{14}\,.\end{aligned} \tag{5.6}$$

These polynomials are known to be irreducible [HS04].

5.2.3 The Andersen-Kashaev theory

The Andersen-Kashaev theory defines an infinite-dimensional TQFT $\mathcal{Z}$ from quantum Teichmüller theory, which in particular defines an invariant $\mathcal{Z}(M,K)$ for every hyperbolic knot K inside a closed, oriented 3-manifold M. The following conjecture was stated first in [AK14a], and then generalised to the present form in [AM16b].

Conjecture 5.2.2 ([AK14a, AM16b]). *Let M be a closed oriented compact 3-manifold. For any hyperbolic knot $K \subseteq M$, there exists a two-parameter (b,N) family of smooth functions $J^{(\mathsf{b},N)}_{M,K}(\mathbf{x})$ on $\mathbb{A}_N = \mathbb{R}\times\mathbb{Z}/N\mathbb{Z}$ which enjoys the following properties:*

1. *For any fully balanced shaped ideal triangulation X of the complement of K in M, there exist a gauge-invariant real linear combination of dihedral angles λ, and a (gauge-dependent) real quadratic polynomial of dihedral angles ϕ, such that*

$$\mathcal{Z}^{(N)}_{\mathsf{b}}(X) = e^{ic_{\mathsf{b}}^2\phi}\int_{\mathbb{A}_N} J^{(\mathsf{b},N)}_{M,K}(\mathbf{x})e^{i\lambda c_{\mathsf{b}}x}\,\mathrm{d}\mathbf{x}\,.$$

2. *For any vertex shaped H-triangulation Y of the pair (M, K), there exists a real quadratic polynomial of dihedral angles φ such that*

$$\lim_{\omega_Y \to \tau} \mathrm{D_b}\left(c_{\mathrm{b}} \frac{\omega_Y(K) - \pi}{\pi\sqrt{N}}, 0 \right) \mathcal{Z}_{\mathrm{b}}^{(N)}(Y) = e^{i c_{\mathrm{b}}^2 \phi - i \frac{\pi N}{12}} J_{M,K}^{(\mathrm{b},N)}(0,0)\,.$$

3. *The hyperbolic volume of the complement of K in M is recovered as the limit*

$$\lim_{\mathrm{b}\to 0} 2\pi \mathrm{b}^2 N \log \left| J_{M,K}^{(\mathrm{b},N)}(0,0) \right| = -\operatorname{Vol}(M \setminus K)\,.$$

In the same works, the conjecture is proven for the knots 4_1 and 5_2 in S^3, in which cases $J_{M,K}^{(\mathrm{b},N)}$ is found to be

$$J_{S^3,4_1}^{(\mathrm{b},N)}(\mathbf{x}) = e^{4\pi i \frac{c_{\mathrm{b}} x}{\sqrt{N}}} \chi_{4_1}(\mathbf{x})\,, \qquad \chi_{4_1}(\mathbf{x}) = \int_{\mathbb{A}_N} \frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y}) \langle \mathbf{x}-\mathbf{y} \rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y}) \langle \mathbf{y} \rangle^{-2}}\, \mathrm{d}\mathbf{y}\,,$$

$$J_{S^3,5_2}^{(\mathrm{b},N)}(\mathbf{x}) = e^{2\pi i \frac{c_{\mathrm{b}} x}{\sqrt{N}}} \chi_{5_2}(\mathbf{x})\,, \qquad \chi_{5_2}(\mathbf{x}) = \int_{\mathbb{A}_N} \frac{\langle \mathbf{y} \rangle \langle \mathbf{x} \rangle^{-1}}{\varphi_{\mathrm{b}}(\mathbf{y}+\mathbf{x}) \varphi_{\mathrm{b}}(\mathbf{y}) \varphi_{\mathrm{b}}(\mathbf{y}-\mathbf{x})}\, \mathrm{d}\mathbf{y}\,.$$

Remark 5.2.3. *The expressions above differ from those in [AK14a, AM16b] by an exponential factor. Because λ is a linear combination of the dihedral angles, which are constrained by a linear, non-homogeneous relation, this extra factor can be re-absorbed into it. As is easily checked, this does not affect the asymptotic properties of the invariant, nor the validity of Conjecture 5.2.2. The role of this factor will be discussed further later on.*

In the case of level $N = 1$, these expressions agree with those found in the literature for the partition functions of Chern-Simons theory, obtained via formal, perturbative methods [Hik01, Hik07, Dim13, DGLZ09, DFM11]. Said function is expected to be annihilated by some version of the $\hat{A}$-polynomial; for instance, in [Dim13] this is argued to be the case for the trefoil and figure-eight knot in S^3.

5.3 Operators from geometric quantisation on $\mathbb{T}^2_{\mathbb{C}}$

We open this section by remarking that, compared to the rest of the thesis, there are some changes of notation throughout this chapter. Namely, we reverse the orientation of the surface Σ, and we move our focus from the moduli spaces $\mathcal{M}$ and $\mathcal{M}_{\mathbb{C}}$ to their covers $\mathbb{T}^2$ and $\mathbb{T}^2_{\mathbb{C}}$. Because these are two-sheeted covers, the integrality condition on k is slightly relaxed, and we shall use a new parameter N with $k = 2N$. The new quantum parameter will be denoted as $t = N + iS$.

Here is a brief summary of our new conventions. The coordinates (u,v), (U,V) and (m,ℓ), as in 2.5, are now thought of as functions on $\mathbb{T}^2$ and $\mathbb{T}^2_{\mathbb{C}}$. The symplectic forms will be

$$\omega = -2\pi\, \mathrm{d}u \wedge \mathrm{d}v\,; \qquad \omega_{\mathbb{C}} = -2\pi\, \mathrm{d}U \wedge \mathrm{d}V\,.$$

As a level-t pre-quantum line bundle we shall use $\mathscr{L}^{(t)}$ defined by the quasi-periodicity conditions and connection form

$$\psi(U+1,V) = e^{-\pi i \operatorname{Re}(tV)}\psi(U,V)\,,$$
$$\psi(U,V+1) = e^{\pi i \operatorname{Re}(tU)}\psi(U,V)\,,$$
$$\theta^{(t)}_{(U,V)} = \pi \operatorname{Re}\left(t\Big(V\,\mathrm{d}U - U\,\mathrm{d}V\Big)\right).$$

The almost complex structure on $\mathbb{T}^2$ associated to $\sigma \in \mathcal{T}$ is represented in the logarithmic coordinates by

$$J := \frac{i}{\sigma-\overline{\sigma}}\begin{pmatrix} -(\sigma+\overline{\sigma}) & 2\sigma\overline{\sigma} \\ -2 & \sigma+\overline{\sigma}\end{pmatrix}, \tag{5.7}$$

and correspondingly the metric is

$$g = \frac{2\pi i}{\sigma-\overline{\sigma}}\begin{pmatrix} 2 & -(\sigma+\overline{\sigma}) \\ -(\sigma+\overline{\sigma}) & 2\sigma\overline{\sigma}\end{pmatrix}.$$

The holomorphic and anti-holomorphic vectors are given by

$$\begin{aligned}\frac{\partial}{\partial\overline{w}} &:= \frac{\mathbb{1}+iJ}{2}\frac{\partial}{\partial u} = \frac{1}{\sigma-\overline{\sigma}}\left(\sigma\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right) \in P\,,\\ \frac{\partial}{\partial w} &:= \frac{\mathbb{1}-iJ}{2}\frac{\partial}{\partial u} = -\frac{1}{\sigma-\overline{\sigma}}\left(\overline{\sigma}\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right) \in \overline{P}\,,\end{aligned} \tag{5.8}$$

and we may think of the polarisation and its dual purely as the spans of these fields. Although they correspond to a holomorphic and an anti-holomorphic coordinates $w = u - \overline{\sigma}v$ and $u - \sigma v$, we stress that we think of these as real vectors on $\mathbb{T}^2_{\mathbb{C}}$. In these conventions, we have the commutator

$$\Big[\nabla_w, \nabla_{\overline{w}}\Big] = \frac{2N\pi i}{\sigma-\overline{\sigma}}\,. \tag{5.9}$$

The Laplace operator can be expressed in terms of w and $\overline{w}$ as

$$\Delta = -i\frac{\sigma-\overline{\sigma}}{2\pi}\Big(\nabla_w\nabla_{\overline{w}} + \nabla_{\overline{w}}\nabla_w\Big)\,.$$

For the variation of the metric we have

$$\tilde{G} = \frac{i}{\pi}\left(\frac{\partial}{\partial w}\otimes\frac{\partial}{\partial w}\otimes d\sigma - \frac{\partial}{\partial\overline{w}}\otimes\frac{\partial}{\partial\overline{w}}\otimes d\overline{\sigma}\right),$$

and finally the Hitchin-Witten connection reads

$$\tilde{\boldsymbol{\nabla}}_V = \nabla^{\mathrm{Tr}}_V + \frac{iV'}{2\pi t}\nabla_w\nabla_w + \frac{iV''}{2\pi\overline{t}}\nabla_{\overline{w}}\nabla_{\overline{w}}\,. \tag{5.10}$$

5.3.1 The quantum operators on $\mathcal{H}_\sigma^{(t)}$

We shall now study the level-t pre-quantum operators associated to the logarithmic coordinates on $\mathbb{T}^2_\mathbb{C}$. Notice that, since U and V are multi-valued functions, these operators are not well defined in principle, unless a fixed branch is specified.

We now fix the level t and $\sigma \in \mathcal{T}$, and introduce coordinates adapted to the polarisation. By the properties of real polarisations, on each leaf is defined a linear connection, which in turn gives an exponential map identifying the spaces tangent to P with its leaves. Using $\frac{\partial}{\partial \overline{w}}$ as a generator of P at each point, this induces a complex coordinate $\xi + i\eta$ on every leaf. The coordinates on $\mathbb{T}^2$, restricted to $0 \leq u, v < 1$, can be uniquely extended to real functions u_σ and v_σ on $\mathbb{T}^2_\mathbb{C}$ which are constant along P_σ. Although these functions are non-continuous, their differentials are well defined on the whole $\mathbb{T}^2_\mathbb{C}$, and hence so is their Hamiltonian vector field. Together with ξ and η these form global real coordinates on the complex torus, and it follows from the definitions that one may write U and V as

$$U = u_\sigma + \frac{\sigma}{\sigma - \overline{\sigma}}\Big(\xi + i\eta\Big), \qquad V = v_\sigma + \frac{1}{\sigma - \overline{\sigma}}\Big(\xi + i\eta\Big).$$

Notice that u_σ enters (U, V) only through $u = \mathrm{Re}(U)$, so

$$\frac{\partial}{\partial u_\sigma}\mathrm{Im}(U) = \frac{\partial}{\partial u_\sigma}\mathrm{Re}(V) = \frac{\partial}{\partial u_\sigma}\mathrm{Im}(V) = 0.$$

From this, and the corresponding argument for v_σ, one can conclude that

$$\frac{\partial}{\partial u_\sigma} = \frac{\partial}{\partial u}, \qquad \frac{\partial}{\partial v_\sigma} = \frac{\partial}{\partial v}.$$

In these coordinates, the complex symplectic form may be written as

$$\omega_\mathbb{C} = -2\pi\, \mathrm{d}u_\sigma \wedge \mathrm{d}v_\sigma - \frac{2\pi}{\sigma - \overline{\sigma}}\,\mathrm{d}\Big(u_\sigma + \sigma v_\sigma\Big) \wedge \mathrm{d}\Big(\xi + i\eta\Big).$$

It is clear from this that all the Poisson brackets of the coordinates with respect to ω_t are constant on $\mathbb{T}^2_\mathbb{C}$. Since every $f \in \mathcal{C}^\infty(\mathbb{T}^2_\mathbb{C})$ constant on the leaves is a function of u_σ and v_σ, this is enough to conclude that the bracket of any such function with ξ or η is of the same kind. In conclusion, all four coordinates are affine linear on the leaves, so their pre-quantum operators preserve the space of polarised sections of $\mathscr{L}^{(t)}$, and can be promoted to quantum operators.

Theorem 5.3.1. *The quantum operators $\widehat{U}_\sigma$ and $\widehat{V}_\sigma$ act on the smooth sections of $\mathscr{L}^{(N)}$ over $\mathbb{T}^2$ as*

$$\widehat{U}_\sigma = u - \frac{i\sigma}{\pi t}\nabla_w, \qquad \widehat{V}_\sigma = v - \frac{i}{\pi t}\nabla_w.$$

Proof. By linearity, $\widehat{U}_\sigma$ and $\widehat{V}_\sigma$ are determined once the quantum operators associated to the coordinates u_σ, v_σ, ξ and η are known.

Recall that the pre-quantum operator of a smooth, real function $f \in \mathcal{C}^\infty(\mathbb{T}^2_{\mathbb{C}})$ is defined on sections of $\mathscr{L}^{(t)}$ as

$$\widehat{f} := f - i\nabla_{H_f^{(t)}},$$

where the Hamiltonian vector field $H_f^{(t)}$ of f relative to ω_t is characterised by

$$X[f] = \omega_t\left(X, H_f^{(t)}\right) \quad \text{for every } X \in \mathrm{T}\,\mathbb{T}^2_{\mathbb{C}}.$$

If a vector field $\tilde{H}$ satisfies this condition for every X tangent to P, then $\tilde{H} - H_f^{(t)}$ is orthogonal to P, and hence tangent to it. Therefore, the covariant derivatives along $\tilde{H}$ and $H_f^{(t)}$ act in the same way on polarised sections, so the quantum operator $\widehat{f_\sigma}$ may as well be defined using $\tilde{H}$ in place of $H_f^{(t)}$. We shall look for such a $\tilde{H}$ as a linear combination of $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ for each of the coordinates.

Since u_σ and v_σ are constant along P, their Hamiltonian vector fields are tangent to the polarisation. As a consequence, the first-order part of their quantum operators vanish, leading to

$$\widehat{u_\sigma} = u_\sigma, \qquad \widehat{v_\sigma} = v_\sigma.$$

For real functions α and β, and $\xi, \eta \in \mathbb{R}$ fixed, one can compute

$$\begin{aligned}\omega_t\left((\xi + i\eta)\frac{\partial}{\partial \overline{w}}, \alpha\frac{\partial}{\partial u} + \beta\frac{\partial}{\partial v}\right) &= 2\pi\,\mathrm{Re}\left(t(\xi + i\eta)\frac{\alpha - \sigma\beta}{\sigma - \overline{\sigma}}\right) = \\ &= 2\pi\left(\mathrm{Re}\left(t\frac{\alpha - \sigma\beta}{\sigma - \overline{\sigma}}\right)\xi - \mathrm{Im}\left(t\frac{\alpha - \sigma\beta}{\sigma - \overline{\sigma}}\right)\eta\right).\end{aligned}$$

In order to determine $\widehat{\xi_\sigma}$, we look for real functions α and β so that the above equals ξ for every ξ and η. This is equivalent to

$$2\pi t\frac{\alpha - \sigma\beta}{\sigma - \overline{\sigma}} = 1.$$

A simple algebraic manipulation leads to

$$\alpha = -\frac{1}{2\pi}\left(\frac{\overline{\sigma}}{t} + \frac{\sigma}{\overline{t}}\right), \qquad \beta = -\frac{1}{2\pi}\left(\frac{1}{t} + \frac{1}{\overline{t}}\right).$$

Therefore we have that

$$\begin{aligned}\widehat{\xi_\sigma} &= \xi + \frac{i}{2\pi t}\left(\overline{\sigma}\nabla_u + \nabla_v\right) + \frac{i}{2\pi\overline{t}}\left(\sigma\nabla_u + \nabla_v\right) = \\ &= \xi - \frac{i(\sigma - \overline{\sigma})}{2\pi t}\nabla_w + \frac{i(\sigma - \overline{\sigma})}{2\pi\overline{t}}\nabla_{\overline{w}}.\end{aligned}$$

The operator $\widehat{\eta}_\sigma$ is obtained in the same way: we look for α and β so that

$$2\pi t \frac{\alpha\sigma\beta}{\sigma - \overline{\sigma}} = -i\,,$$

which gives

$$\alpha = \frac{i}{2\pi}\left(\frac{\overline{\sigma}}{t} - \frac{\sigma}{\overline{t}}\right) \qquad \beta = \frac{i}{2\pi}\left(\frac{1}{t} - \frac{1}{\overline{t}}\right)\,.$$

From this we find that

$$\widehat{\eta}_\sigma = \eta + \frac{1}{2\pi t}\left(\overline{\sigma}\nabla_u + \nabla_v\right) - \frac{1}{2\pi\overline{t}}\left(\sigma\nabla_u + \nabla_v\right) = \eta - \frac{\sigma - \overline{\sigma}}{2\pi t}\nabla_w - \frac{\sigma - \overline{\sigma}}{2\pi\overline{t}}\nabla_{\overline{w}}\,.$$

Putting everything together, the quantum operators $\widehat{U}_\sigma$ and $\widehat{V}_\sigma$ act on polarised sections as

$$\widehat{U}_\sigma = U - \frac{i\sigma}{\pi t}\nabla_w\,, \qquad \widehat{V}_\sigma = V - \frac{i}{\pi t}\nabla_w\,.$$

The action on sections on $\mathbb{T}^2$ is obtained by restricting U and V to u and v. Since ∇_w is a linear combination of ∇_u and ∇_v, and since $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ are tangent to $\mathbb{T}^2$, this operator is unchanged under the restriction. This concludes the proof. $\square$

Having established this, it is no longer necessary to consider $\mathbb{T}^2_{\mathbb{C}}$, and we may focus our attention to the picture on $\mathbb{T}^2$ instead.

We now wish to define quantum operators for the exponential coordinates m and ℓ, to which end we rely on the spectral theorem for normal operators, see e.g. [Con94]. A densely defined operator N on a separable Hilbert space is called normal if it is closed and $NN^\dagger = N^\dagger N$, where $N^\dagger$ denotes the adjoint of N and the identity includes the equality of the domains. For instance, if X is a measured space of σ-finite measure, f a complex-valued measurable function on X, then the multiplication by f defines a normal operator on $\mathrm{L}^2(X,\mathbb{C})$. As a matter of fact, the theorem states that every normal operator is unitarily equivalent to one of this kind. One remarkable consequence of this is that the exponential series of a normal operator is strongly convergent on a suitable dense subspace.

Lemma 5.3.2. *The quantum operators $\widehat{U}_\sigma, \widehat{V}_\sigma \in \mathcal{H}^{(t)}_\sigma$ are normal.*

Proof. If X is any possibly complex vector field on $\mathbb{T}^2$, then ∇_X is defined at least on the dense subspace consisting of smooth sections. By integration by parts, the adjoint of ∇_X is $-\nabla_{\overline{X}}$, so every such operator arises as the adjoint of a densely defined one, which ensure that they are closed. Moreover, it is easily seen that for every smooth section ψ one has

$$\left\|\nabla_X\psi\right\| \leq \left\|\nabla_{\mathrm{Re}(X)}\psi\right\| + \left\|\nabla_{\mathrm{Im}(X)}\psi\right\| \leq 2\left\|-\nabla_{\overline{X}}\psi\right\|$$

Therefore, if a sequence ψ_n is convergent and $-\nabla_{\overline{X}}\psi_n$ is Cauchy, then so is $\nabla_X\psi_n$, hence $\mathrm{dom}(\nabla_X^\dagger) \subseteq \mathrm{dom}(\nabla_X)$. The same argument with X and $-\overline{X}$ exchange shows the equality of the domains. It follows from general, standard arguments that covariant derivatives are operators on $\mathrm{L}^2(\mathbb{T}^2, \mathscr{L}^{(N)})$ defined on the same dense domains as their adjoints. On the other hand, in our convention the functions u and v range through $[0, 1)$, so their multiplication operators on $\mathrm{L}^2(\mathbb{T}^2, \mathscr{L}^{(N)})$ are everywhere defined and bounded. From this it follows hat $\widehat{U}_\sigma$ and $\widehat{V}_\sigma$ are also closed, and have the same dense domain as ∇_w. Moreover, it is immediate to check that $\widehat{U}_\sigma$ and $\widehat{V}_\sigma$ also have the same image as their adjoints, so

$$\mathrm{dom}\left(\widehat{U}_\sigma\widehat{U}_\sigma^\dagger\right) = \mathrm{dom}\left(\widehat{U}_\sigma^\dagger\widehat{U}_\sigma\right), \qquad \mathrm{dom}\left(\widehat{V}_\sigma\widehat{V}_\sigma^\dagger\right) = \mathrm{dom}\left(\widehat{V}_\sigma^\dagger\widehat{V}_\sigma\right).$$

We now proceed to checking the commutation relation for $\widehat{U}_\sigma$ and $\widehat{U}_\sigma^\dagger$ by direct computation

$$\begin{aligned}\left[\widehat{U}_\sigma, \widehat{U}_\sigma^\dagger\right] &= \left[u - \frac{i\sigma}{\pi t}\nabla_w, u - \frac{i\overline{\sigma}}{\pi\bar{t}}\nabla_{\overline{w}}\right] = \\ &= \frac{i\sigma}{\pi t} \cdot \frac{\overline{\sigma}}{\sigma - \overline{\sigma}} + \frac{i\overline{\sigma}}{\pi\bar{t}} \cdot \frac{\sigma}{\sigma - \overline{\sigma}} - \frac{\sigma\overline{\sigma}}{\pi^2 t\bar{t}} \cdot \frac{2N\pi i}{\sigma - \overline{\sigma}} = 0\,.\end{aligned}$$

Analogously we find that

$$\begin{aligned}\left[\widehat{V}_\sigma, \widehat{V}_\sigma^\dagger\right] &= \left[v - \frac{i}{\pi t}\nabla_w, v - \frac{i}{\pi\bar{t}}\nabla_{\overline{w}}\right] = \\ &= \frac{i}{\pi t} \cdot \frac{\overline{\sigma}}{\sigma - \overline{\sigma}} + \frac{i}{\pi\bar{t}} \cdot \frac{\sigma}{\sigma - \overline{\sigma}} - \frac{1}{\pi^2 t\bar{t}} \cdot \frac{2N\pi i}{\sigma - \overline{\sigma}} = 0\,.\end{aligned}$$

□

It is easily seen that for every $\lambda \in \mathbb{C}$ and every normal operator N on a Hilbert space λN is also normal. The lemma ensures then that the following is well posed.

Definition 5.3.1. *We define quantum operators associated to m and ℓ on $\mathcal{H}_\sigma^{(t)}$ as*

$$\widehat{m}_\sigma = \exp\left(2\pi i\widehat{U}_\sigma\right), \qquad \widehat{\ell}_\sigma = \exp\left(2\pi i\widehat{V}_\sigma\right).$$

The next result shows that the quantum operators found above are compatible with the identification of the various Hilbert space via parallel transport.

We now proceed to prove that the operators are covariantly constant.

Proof of Theorem 5.1.1. We need to show that $\widehat{U}_\sigma$ and $\widehat{V}_\sigma$ are covariantly constant with respect to the connection $\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}$ induced on $\mathrm{End}(\mathcal{H}^{(t)})$ by $\tilde{\boldsymbol{\nabla}}$.

The covariant derivative of an endomorphisms E of $\mathcal{H}$ along a direction V on $\mathcal{T}$ is given by its commutator with $\tilde{\boldsymbol{\nabla}}_V$

$$\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_V(E) = \left[\tilde{\boldsymbol{\nabla}}_V, E\right].$$

It is a consequence of (5.9) that the commutator of the covariant derivatives along ∂_w and $\partial_{\overline{w}}$ is

$$\left[\nabla_w, \nabla_{\overline{w}}\right] = -iN\omega\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial \overline{w}}\right) = \frac{2N\pi i}{\sigma - \overline{\sigma}}.$$

We start with the Hitchin-Witten derivatives of the multiplication by u

$$\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_\sigma u = \frac{\partial}{\partial\sigma}u + \frac{i}{2\pi t}\left[\nabla_w\nabla_w, u\right] = -\frac{i\overline{\sigma}}{\pi t(\sigma-\overline{\sigma})}\nabla_w\,,$$
$$\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\overline{\sigma}} u = \frac{\partial}{\partial\overline{\sigma}}u + \frac{i}{2\pi \overline{t}}\left[\nabla_{\overline{w}}\nabla_{\overline{w}}, u\right] = \frac{i\sigma}{\pi \overline{t}(\sigma-\overline{\sigma})}\nabla_{\overline{w}}\,.$$

Analogously for v

$$\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_\sigma v = \frac{\partial}{\partial\sigma}v + \frac{i}{2\pi t}\left[\nabla_w\nabla_w, v\right] = -\frac{i}{\pi t(\sigma-\overline{\sigma})}\nabla_w\,,$$
$$\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\overline{\sigma}} v = \frac{\partial}{\partial\overline{\sigma}}v + \frac{i}{2\pi \overline{t}}\left[\nabla_{\overline{w}}\nabla_{\overline{w}}, v\right] = \frac{i}{\pi \overline{t}(\sigma-\overline{\sigma})}\nabla_{\overline{w}}\,.$$

Consider next the variation of ∇_w in σ

$$\begin{aligned}\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_\sigma(\nabla_w) &= \frac{\partial}{\partial\sigma}(\nabla_w) + \frac{i}{2\pi t}\left[\nabla_w\nabla_w, \nabla_w\right] = -\frac{\partial}{\partial\sigma}\left(\frac{\overline{\sigma}\nabla_u + \nabla_v}{\sigma-\overline{\sigma}}\right) = \\ &= \frac{\overline{\sigma}\nabla_u + \nabla_v}{(\sigma-\overline{\sigma})^2} = -\frac{1}{\sigma-\overline{\sigma}}\nabla_w\,.\end{aligned}$$

The variation in $\overline{\sigma}$ is more involved

$$\begin{aligned}\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\overline{\sigma}}(\nabla_w) &= \frac{\partial}{\partial\overline{\sigma}}(\nabla_w) + \frac{i}{2\pi \overline{t}}\left[\nabla_{\overline{w}}\nabla_{\overline{w}}, \nabla_w\right] = \\ &= -\frac{\partial}{\partial\overline{\sigma}}\left(\frac{\overline{\sigma}\nabla_u + \nabla_v}{\sigma-\overline{\sigma}}\right) + \frac{i}{\pi\overline{t}}\left[\nabla_{\overline{w}}, \nabla_w\right]\nabla_{\overline{w}} = \\ &= -\frac{\overline{\sigma}\nabla_u + \nabla_v}{(\sigma-\overline{\sigma})^2} - \frac{1}{\sigma-\overline{\sigma}}\nabla_u + \frac{2N}{\overline{t}(\sigma-\overline{\sigma})}\nabla_{\overline{w}} = \\ &= -\frac{1}{\overline{t}(\sigma-\overline{\sigma})}\left(\overline{t}\frac{\sigma\nabla_u + \nabla_v}{\sigma-\overline{\sigma}} - 2N\nabla_{\overline{w}}\right) = \frac{t}{\overline{t}(\sigma-\overline{\sigma})}\nabla_{\overline{w}}\,.\end{aligned}$$

Altogether, this shows the covariant derivative of $\widehat{U}_\sigma$ with respect to σ

is

$$\begin{aligned}\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\sigma}(\widehat{U}_\sigma) &= \tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\sigma}\left(u - \frac{i\sigma}{\pi t}\nabla_w\right) = \\ &= \tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\sigma}(u) - \frac{i}{\pi t}\nabla_w - \frac{i\sigma}{\pi t}\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\sigma}(\nabla_w) = \\ &= -\frac{i}{\pi t}\left(\frac{\overline{\sigma}}{\sigma-\overline{\sigma}} + 1 - \frac{\sigma}{\sigma - \overline{\sigma}}\right)\nabla_w = 0\,.\end{aligned}$$

In $\overline{\sigma}$ we have that

$$\begin{aligned}\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\overline{\sigma}}(\widehat{U}_\sigma) &= \tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\overline{\sigma}}(u) - \frac{i\sigma}{\pi t}\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\overline{\sigma}}(\nabla_w) = \\ &= \frac{i\sigma}{\pi\bar{t}(\sigma-\overline{\sigma})}\nabla_{\overline{w}} - \frac{i\sigma}{\pi t}\frac{t}{\bar{t}(\sigma-\overline{\sigma})}\nabla_{\overline{w}} = 0\,.\end{aligned}$$

The derivatives of $\widehat{V}_\sigma$ are completely analogous. □

5.3.2 Trivialisation of $\tilde{\boldsymbol{\nabla}}$ and σ-independent operators

It was established in the previous paragraph that the quantum operators found there are independent on the Teichmüller parameter in the sense of the Hitchin-Witten connection. Our next goal is to remove the dependence on σ explicitly.

Definition 5.3.2. *We define the σ-independent quantum operators associated to U, V, m and ℓ to be*

$$\widehat{U} = \exp\Big(r\Delta\Big)\widehat{U}_\sigma\Big(-r\Delta\Big)\,, \qquad \widehat{V} = \exp\Big(r\Delta\Big)\widehat{V}_\sigma\Big(-r\Delta\Big)\,,$$
$$\widehat{m} = \exp\Big(r\Delta\Big)\widehat{m}_\sigma\Big(-r\Delta\Big)\,, \qquad \widehat{\ell} = \exp\Big(r\Delta\Big)\widehat{\ell}_\sigma\Big(-r\Delta\Big)\,.$$

Theorem 5.3.3. *The σ-independent operators are expressed as*

$$\widehat{U} = u - i\frac{e^{2rN}-1}{2N\pi}\nabla_v \qquad \widehat{V} = v + i\frac{e^{2rN}-1}{2N\pi}\nabla_u\,,$$
$$\widehat{m} = \exp\Big(2\pi i\widehat{U}\Big) = e^{2\pi i u}\exp\left(\frac{e^{2rN}-1}{N}\nabla_v\right),$$
$$\widehat{\ell} = \exp\Big(2\pi i\widehat{V}\Big) = e^{2\pi i v}\exp\left(-\frac{e^{2rN}-1}{N}\nabla_u\right).$$

We wish to stress how these operators are manifestly independent of the Teichmüller parameter.

Proof. We shall make use of the following formula:

$$\mathrm{Ad}_{\exp(r\Delta)} = \exp\Big(r\,\mathrm{ad}_\Delta\Big)\,.$$

According to this, in order to conjugate the operators by $\exp(r\Delta)$ it is enough to understand their commutator with Δ.

We start by computing the commutator of Δ with u

$$\Big[\Delta, u\Big] = -i\frac{\sigma-\overline{\sigma}}{\pi}\left(\frac{\sigma}{\sigma-\overline{\sigma}}\nabla_w - \frac{\overline{\sigma}}{\sigma-\overline{\sigma}}\nabla_{\overline{w}}\right) = -\frac{i}{\pi}\Big(\sigma\nabla_w - \overline{\sigma}\nabla_{\overline{w}}\Big)\,.$$

Similarly we have

$$\Big[\Delta, v\Big] = -i\frac{\sigma-\overline{\sigma}}{\pi}\left(\frac{1}{\sigma-\overline{\sigma}}\nabla_w - \frac{1}{\sigma-\overline{\sigma}}\nabla_{\overline{w}}\right) = -\frac{i}{\pi}\Big(\nabla_w - \nabla_{\overline{w}}\Big)\,.$$

The action of ad^n_Δ on u and v is then determined by that on ∇_w and $\nabla_{\overline{w}}$. This is easily determined by the following relations

$$\begin{aligned}\Big[\Delta, \nabla_w\Big] &= -\frac{\sigma-\overline{\sigma}}{\pi}\frac{2N\pi}{\sigma-\overline{\sigma}}\nabla_w = -2N\nabla_w\,,\\ \Big[\Delta, \nabla_{\overline{w}}\Big] &= \frac{\sigma-\overline{\sigma}}{\pi}\frac{2N\pi}{\sigma-\overline{\sigma}}\nabla_{\overline{w}} = 2N\nabla_{\overline{w}}\,.\end{aligned}$$

Given this, we can move on to computing the conjugation of the operators. For the multiplication by u we find

$$\begin{aligned}\exp\Big(r\Delta\Big)u\exp\Big(-r\Delta\Big) &= \sum_{n=0}^{\infty}\frac{r^n}{n!}\,\mathrm{ad}^n_\Delta(u) = \\ &= u - \frac{i}{\pi}\sum_{n=1}^{\infty}\frac{r^n}{n!}\,\mathrm{ad}^n_\Delta\Big(\sigma\nabla_w - \overline{\sigma}\nabla_{\overline{w}}\Big) = \\ &= u + \frac{i\sigma\Big(e^{-2rN}-1\Big)}{2N\pi}\nabla_w + \frac{i\overline{\sigma}\Big(e^{2rN}-1\Big)}{2N\pi}\nabla_{\overline{w}}\,.\end{aligned}$$

Similarly, the multiplication by v becomes

$$\begin{aligned}\exp\Big(r\Delta\Big)v\exp\Big(-r\Delta\Big) &= \sum_{n=0}^{\infty}\frac{r^n}{n!}\,\mathrm{ad}^n_\Delta(v) = \\ &= v - \frac{i}{\pi}\sum_{n=1}^{\infty}\frac{r^n}{n!}\,\mathrm{ad}^n_\Delta\Big(\nabla_w - \nabla_{\overline{w}}\Big) = \\ &= v + \frac{i\Big(e^{-2rN}-1\Big)}{2N\pi}\nabla_w + \frac{i\Big(e^{2rN}-1\Big)}{2N\pi}\nabla_{\overline{w}}\,.\end{aligned}$$

Since ∇_w is a $(-2N)$-eigenvector for ad_Δ, we simply have

$$\exp\Big(r\Delta\Big)\nabla_w \exp\Big(-r\Delta\Big) = e^{-2rN}\nabla_w\,.$$

Putting the pieces together we finally find

$$\begin{aligned}\widehat{U} &= u + \frac{i\sigma\Big(e^{-2rN}-1\Big)}{2N\pi}\nabla_w + \frac{i\overline{\sigma}\Big(e^{2rN}-1\Big)}{2N\pi}\nabla_{\overline{w}} - \frac{i\sigma e^{-2rN}}{\pi t}\nabla_w = \\ &= u - \frac{i\sigma\Big(\overline{t}e^{-2rN}+t\Big)}{2Nt\pi}\nabla_w + \frac{i\overline{\sigma}\Big(e^{2rN}-1\Big)}{2N\pi}\nabla_{\overline{w}} = \\ &= u + \frac{i\sigma\Big(e^{2rN}-1\Big)}{2N\pi}\nabla_w + \frac{i\overline{\sigma}\Big(e^{2rN}-1\Big)}{2N\pi}\nabla_{\overline{w}} = u - i\frac{e^{2rN}-1}{2N\pi}\nabla_v\,.\end{aligned}$$

Here we used the relation defining r, and that

$$\sigma\nabla_w + \overline{\sigma}\nabla_{\overline{w}} = \frac{1}{\sigma-\overline{\sigma}}\Big(-\sigma\overline{\sigma}\nabla_u - \sigma\nabla_v + \sigma\overline{\sigma}\nabla_u + \overline{\sigma}\nabla_v\Big) = -\nabla_v\,.$$

In complete analogy one has

$$\begin{aligned}\widehat{V} &= v + \frac{i\Big(e^{-2rN}-1\Big)}{2N\pi}\nabla_w + \frac{i\Big(e^{2rN}-1\Big)}{2N\pi}\nabla_{\overline{w}} - \frac{ie^{-2rN}}{\pi t}\nabla_w = \\ &= v - \frac{i\Big(\overline{t}e^{-2rN}+t\Big)}{2Nt\pi}\nabla_w + \frac{i\Big(e^{2rN}-1\Big)}{2N\pi}\nabla_{\overline{w}} = \\ &= v + \frac{i\Big(e^{2rN}-1\Big)}{2N\pi}\nabla_w + \frac{i\Big(e^{2rN}-1\Big)}{2N\pi}\nabla_{\overline{w}} = v + i\frac{e^{2rN}-1}{2N\pi}\nabla_u\,.\end{aligned}$$

In the last step we used that

$$\nabla_w + \nabla_{\overline{w}} = \frac{1}{\sigma-\overline{\sigma}}\Big(-\overline{\sigma}\nabla_u - \nabla_v + \sigma\nabla_u + \nabla_v\Big) = \nabla_u\,.$$

The relation for $\widehat{m}$ and $\widehat{\ell}$ follow from the fact that taking the exponential of an operator commutes with conjugating it by a unitary map. The splitting is a consequence of the Baker-Campbell-Hausdorff formula, which can be applied here since u commutes with ∇_v, and v with ∇_u. □

5.3.3 The Weil-Gel′fand-Zak transform

Lemma 5.3.4. *The spaces $\mathcal{C}^\infty(\mathbb{T}^2,\mathcal{L}^N)$ and $\mathcal{S}(\mathbb{A}_N,\mathbb{C})$ are isometric via the transformation $W^{(N)}:\mathcal{S}(\mathbb{A}_N,\mathbb{C})\to\mathcal{C}^\infty(\mathbb{T}^2,\mathcal{L}^N)$ expressed by*

$$W^{(N)}\Big(f(x,n)\Big) = s(u,v) := e^{i\pi Nuv}\sum_{m\in\mathbb{Z}} f\bigg(\sqrt{N}u + \frac{m}{\sqrt{N}}, -m\bigg)e^{2\pi imv}\,.$$

Since this is an isometry, it extends in a natural way to the L^2-completions of the two spaces.

The above map is called the Weil-Gel'fand-Zak transform. We shall now study the conjugation of the quantum operators on $\mathcal{H}^{(t)}$ by the Weil-Gel'fand-Zak transform.

Lemma 5.3.5. *The following relations hold for every* $f \in \mathcal{S}(\mathbb{A}_N, \mathbb{C})$

$$\begin{aligned}
\nabla_u W^{(N)}\Big(f(\mathbf{x})\Big) &= W^{(N)}\Big(\sqrt{N} f'(\mathbf{x})\Big)\,, \\
\nabla_v W^{(N)}\Big(f(\mathbf{x})\Big) &= W^{(N)}\Big(2\pi i \sqrt{N} x f(\mathbf{x})\Big)\,, \\
e^{2\pi i u} W^{(N)}\Big(f(\mathbf{x})\Big) &= W^{(N)}\Big(e^{2\pi i \frac{x}{\sqrt{N}}} e^{2\pi i \frac{n}{N}} f(\mathbf{x})\Big)\,, \\
e^{2\pi i v} W^{(N)}\Big(f(\mathbf{x})\Big) &= W^{(N)}\left(f\left(x - \frac{1}{\sqrt{N}}, n+1\right)\right).
\end{aligned}$$

Proof. This follows directly from computations, by applying the definitions. Notice that the derivatives commute with the infinite sum due to the properties of Schwartz class functions

$$\begin{aligned}
\nabla_u W^{(N)}\Big(f(\mathbf{x})\Big) &= \frac{\partial}{\partial u} W^{(N)}\Big(f(\mathbf{x})\Big) - i\pi N v W^{(N)}\Big(f(\mathbf{x})\Big) = \\
&= e^{i\pi N u v} \sum_{m \in \mathbb{Z}} \frac{\partial}{\partial u} f\left(\sqrt{N} u + \frac{m}{\sqrt{N}}, -m\right) e^{2\pi i m v} + \\
&\quad + i\pi N v W^{(N)}\Big(f\Big) - i\pi N v W^{(N)}\Big(f\Big) = \\
&= e^{i\pi N u v} \sum_{m \in \mathbb{Z}} \sqrt{N} f'\left(\sqrt{N} u + \frac{m}{\sqrt{N}}, -m\right) e^{2\pi i m v}\,.
\end{aligned}$$

The second computation is similar

$$\begin{aligned}
\nabla_v W^{(N)}\Big(f(\mathbf{x})\Big) &= \frac{\partial}{\partial v} W^{(N)}\Big(f(\mathbf{x})\Big) + N\pi i u W^{(N)}\Big(f(\mathbf{x})\Big) = \\
&= N\pi i u e^{N\pi i u v} \sum_{m \in \mathbb{Z}} f\left(\sqrt{N} u + \frac{m}{\sqrt{N}}, -m\right) e^{2\pi i m v} + \\
&\quad 2m\pi i e^{N\pi i u v} \sum_{m \in \mathbb{Z}} f\left(\sqrt{N} u + \frac{m}{\sqrt{N}}, -m\right) e^{2\pi i m v} + \\
&\quad + N\pi i u W^{(N)}\Big(f(x,n)\Big) = \\
&= 2\pi i \sqrt{N} e^{N\pi i u v} \sum_{m \in \mathbb{Z}} \left(\sqrt{N} u + \frac{m}{\sqrt{N}}\right) f\left(\sqrt{N} u + \frac{m}{\sqrt{N}}, -m\right) e^{2\pi i m v}\,.
\end{aligned}$$

The crucial passage in the other two computations is a change of variable in the sums

$$
\begin{aligned}
e^{2\pi i u} W^{(N)}\Big(f(\mathbf{x})\Big) &= e^{i\pi N u v} \sum_{m\in\mathbb{Z}} e^{2\pi i u} f\left(\sqrt{N}u + \frac{m}{\sqrt{N}}, -m\right) e^{2\pi i m v} = \\
&= e^{i\pi N u v} \sum_{m\in\mathbb{Z}} e^{2\pi i (u+\frac{m}{N})} e^{-2\pi i \frac{m}{N}} f\left(\sqrt{N}u + \frac{m}{\sqrt{N}}, -m\right) e^{2\pi i m v} .
\end{aligned}
$$

For the last relation we have that

$$
\begin{aligned}
e^{2\pi i v} W^{(N)}\Big(f(\mathbf{x})\Big) &= e^{i\pi N u v} \sum_{m\in\mathbb{Z}} f\left(\sqrt{N}u + \frac{m}{\sqrt{N}}, -m\right) e^{2\pi i (m+1) v} = \\
&= e^{N\pi i u v} \sum_{m\in\mathbb{Z}} f\left(\sqrt{N}u + \frac{m-1}{\sqrt{N}}, -m+1\right) e^{-2\pi i m v} .
\end{aligned}
$$

This concludes the proof. □

Proof of Theorem 5.1.2. We need to check that, for the given values of the quantum parameters, the Weil-Gel'fand-Zak transform induced the correspondence

$$
\widehat{m} \longmapsto \widehat{m}_{\mathbf{x}} , \qquad \widehat{\ell} \longmapsto \widehat{\ell}_{\mathbf{x}} .
$$

We proceed again by direct computation

$$
\begin{aligned}
\widehat{m} W^{(N)}\Big(f(\mathbf{x})\Big) &= e^{2\pi i u} \exp\left(\frac{e^{2Nr}-1}{N}\nabla_v\right) W^{(N)}\Big(f(\mathbf{x})\Big) = \\
&= e^{2\pi i u} W^{(N)}\left(\exp\left(2\pi i \frac{e^{2Nr}-1}{\sqrt{N}} x\right) f(\mathbf{x})\right) = \\
&= W^{(N)}\left(\exp\left(2\pi i \frac{e^{2Nr}}{\sqrt{N}} x\right) e^{2\pi i \frac{n}{N}} f(\mathbf{x})\right) .
\end{aligned}
$$

For the next computation we shall use the fact that, if λ is a complex parameter, the exponential of $\lambda \frac{d}{dx}$ in $L^2(\mathbb{R}, \mathbb{C})$ is the shift by λ, as discussed immediately after Definition 5.2.2

$$
\begin{aligned}
\widehat{\ell} W^{(N)}\Big(f(\mathbf{x})\Big) &= e^{2\pi i v} \exp\left(-\frac{e^{2Nr}-1}{N}\nabla_u\right) W^{(N)}\Big(f(\mathbf{x})\Big) = \\
&= e^{2\pi i v} W^{(N)}\left(f\left(x - \frac{e^{2Nr}-1}{\sqrt{N}}, n\right)\right) = \\
&= W^{(N)}\left(f\left(x - \frac{e^{2Nr}}{\sqrt{N}}, n+1\right)\right) .
\end{aligned}
$$

As for the q-commutativity, the following relation can easily be checked directly from the expression of the operators, or deduced from Dirac's relation

$$\left[\widehat{U}, \widehat{V}\right] = -i\{U, V\} = -i \cdot \frac{-1}{\pi t} = \frac{i}{\pi t}\,.$$

The desired result follows from this and the Baker-Campbell-Hausdorff formula. □

5.4 Operators annihilating the Andersen-Kashaev invariant

Throughout this section we will always assume that N is an *odd* positive integer.

For a fixed level $t = N + iS$ and b and q as above there is an action of the algebra $\mathcal{A}_{\mathrm{loc}}$ from (5.2) on the space of meromorphic functions on $\mathbb{A}_N^{\mathbb{C}}$ by

$$Q \mapsto \widehat{m}_{\mathbf{x}}\,, \qquad E \mapsto \widehat{\ell}_{\mathbf{x}}\,.$$

As before, if f is a meromorphic function it makes sense to consider its annihilating left ideals $\mathcal{I}(f)$ and $\mathcal{I}_{\mathrm{loc}}(f)$ in $\mathcal{A}_{\mathrm{loc}}$ and $\mathcal{A}$, respectively:

$$\mathcal{I}_{\mathrm{loc}}(f) = \left\{\, p \in \mathcal{A}_{\mathrm{loc}} \colon p(\widehat{m}_{\mathbf{x}}, \widehat{\ell}_{\mathbf{x}})f = 0 \,\right\}, \qquad \mathcal{I}(f) = \mathcal{I}_{\mathrm{loc}}(f) \cap \mathcal{A}\,.$$

Notice moreover that, if p_1 is a polynomial in $\widehat{m}_{\mathbf{x}}$ alone and j a non-negative integer, then

$$p_1 \widehat{\ell}_{\mathbf{x}}^{\,j} p_2 \in \mathcal{I}(f) \implies p_2 \in \mathcal{I}(f)\,.$$

We extend the notation to the case of a function f on $\mathbb{A}_N$ having a unique meromorphic extension to $\mathbb{A}_N^{\mathbb{C}}$.

Definition 5.4.1. *Suppose that* $f = J_{M,K}^{(\mathrm{b},N)}$ *for a hyperbolic knot* K *in a closed, oriented compact* 3*-manifold* M*, for which Conjecture 5.2.2 holds. We define the non-commutative polynomial* $\widehat{A}_{t,(M,K)}^{\mathbb{C}}$ *as the unique element of* $\mathcal{I}_{loc}(J_{M,K}^{(\mathrm{b},N)})$ *of lowest degree in* $\widehat{\ell}_{\mathbf{x}}$ *with integral and co-prime coefficients.*

We are now ready to rephrase Theorem 5.1.3 more precisely.

Theorem 5.4.1. *For* $K \subseteq S^3$ *the figure-eight knot* 4_1 *or* 5_2*, we have that*

$$\widehat{A}_{t,K}^{\mathbb{C}}(\widehat{m}_{\mathbf{x}}, \widehat{\ell}_{\mathbf{x}}) \cdot \left(\widehat{m}_{\mathbf{x}} - 1\right) = \widehat{A}_{q,K}(\widehat{m}_{\mathbf{x}}, \widehat{\ell}_{\mathbf{x}})\,.$$

In the semi-classical limit $t \to \infty$*, we have that*

$$\left(m^4 - 1\right) \cdot \widehat{A}_{\infty,K}^{\mathbb{C}}(m^2, \ell) = A_K(m, \ell)\,.$$

We shall dedicate the rest of the chapter to the proof of this statement.

5.4.1 The figure-eight knot 4_1

The invariant and its holomorphic extension

Recall the formula for $\chi_{4_1}^{(\mathrm{b},N)}(\mathbf{x})$ for $\mathbf{x} \in \mathbb{A}_N \subseteq \mathbb{C} \times \mathbb{Z}/N\mathbb{Z}$ (see e.g. [AM16b])

$$\chi_{4_1}^{(\mathrm{b},N)}(\mathbf{x}) := \int_{\mathbb{A}_N} \frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})\langle \mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y})\langle \mathbf{y}\rangle^{-2}}\, d\mathbf{y}\,. \tag{5.11}$$

We shall often omit the superscript (b, N). It follows from lemma 5.2.1 that, for $\mathbf{x}$ and $\mathbf{y}$ real, the integrand has constant absolute value 1, so the integral is not absolutely convergent. However, suppose $y = \eta + i\varepsilon\frac{\mathrm{b}}{\sqrt{N}}$ for real η and ε. One may write

$$\begin{aligned} \left|\langle \mathbf{y}\rangle\right| &= e^{-\pi \operatorname{Im}(y^2)} = e^{\varepsilon^2 \frac{\operatorname{Im}(\mathrm{b}^2)}{N}} e^{-2\pi\eta\varepsilon\frac{\operatorname{Re}(\mathrm{b})}{\sqrt{N}}}\,, \\ \left|\langle \mathbf{x}-\mathbf{y}\rangle\right| &= e^{-\pi \operatorname{Im}((x-y)^2)} = e^{\varepsilon^2 \frac{\operatorname{Im}(\mathrm{b}^2)}{N}} e^{-2\pi(\eta-x)\varepsilon\frac{\operatorname{Re}(\mathrm{b})}{\sqrt{N}}}\,. \end{aligned}$$

Then the same lemma shows that, for appropriate smooth functions C_- and C_+ of $\mathbf{x}$ and ε alone, one has

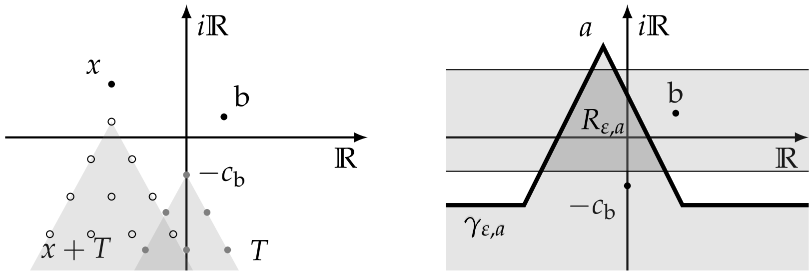
$$\left|\frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})\,\langle \mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y})\,\langle \mathbf{y}\rangle^{-2}}\right| \approx \begin{cases} \left|\dfrac{\langle \mathbf{x}-\mathbf{y}\rangle^{-1}}{\langle \mathbf{y}\rangle^{-2}}\right| = C_-(\mathbf{x},\varepsilon)e^{-2\pi\eta\varepsilon\frac{\operatorname{Re}(\mathrm{b})}{\sqrt{N}}} & \eta \to -\infty\,, \\ \left|\dfrac{\langle \mathbf{x}-\mathbf{y}\rangle^{-2}}{\langle \mathbf{y}\rangle^{-1}}\right| = C_+(\mathbf{x},\varepsilon)e^{4\pi\eta\varepsilon\frac{\operatorname{Re}(\mathrm{b})}{\sqrt{N}}} & \eta \to +\infty\,. \end{cases}$$

Therefore, the integrand decays exponentially in η for every fixed $\varepsilon < 0$, and the integral converges absolutely on $\mathbb{A}_N + i\varepsilon\mathrm{b}/\sqrt{N}$, provided that this contour does not cross any poles.

Recall that the zeroes of $\varphi_{\mathrm{b}}(\mathbf{y})$ only occur for y in the lower infinite triangle T, and similarly the poles of $\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})$ have $y - x \in T$, or equivalently $y \in T + x$. Therefore, since $c_{\mathrm{b}} = i\operatorname{Re}(\mathrm{b})$, the integral over the contour $\mathbb{A}_N + i\varepsilon\mathrm{b}/\sqrt{N}$ is absolutely convergent for $-1 < \varepsilon < 0$. Moreover, given the exponential decay of the integrand at infinity, the residue theorem and dominated convergence together show that, in the limit for $\varepsilon \to 0$, one recovers the improper integral over $\mathbb{A}_N$. In fact, the same value is obtained for each $\mathbf{x}$ if the contour is pushed up within a compact region of $\mathbb{C} \times \mathbb{Z}/N\mathbb{Z}$. In other words, one may use *any* contour Γ which goes along $\mathbb{A}_N + i\varepsilon\mathrm{b}/\sqrt{N}$ near ∞, provided that

- $\varepsilon < 0$,
- all the poles of the integrand lie below Γ.

In order to study the action of the operators on χ_{4_1}, and in particular the shift, we need to understand its holomorphic extension to $\mathbb{C} \times \mathbb{Z}/N\mathbb{Z}$.



(a) For fixed (x, n) and m, the poles of the integrand occur inside the infinite triangles.

(b) The integrand decays exponentially for x in the strip, and the poles lie below $\gamma_{\varepsilon,a}$ if $-c_{\mathrm{b}}$ and x do.

Figure 5.1: The distribution of the poles of the integrand and the contour are illustrated for $N = 1$. The situation is analogous for higher N, up to rescaling c_{b} by $\sqrt{N}$ and replicating the picture N times.

To this end, we shall fix a contour Γ as above and determine for which values of $\mathbf{x}$ the integral converges absolutely. If $x = \xi + i\lambda\mathrm{b}/\sqrt{N}$ and $y = \eta + i\varepsilon/\sqrt{N}$ for $\xi, \eta, \varepsilon, \lambda \in \mathbb{R}$, then one has

$$\left|\langle \mathbf{x} - \mathbf{y} \rangle\right| = e^{-\pi \operatorname{Im}((x-y)^2)} = e^{(\lambda-\varepsilon)^2 \operatorname{Im}(\mathrm{b}^2)} e^{-2\pi(\eta-\xi)(\varepsilon-\lambda)\frac{\operatorname{Re}(\mathrm{b})}{\sqrt{N}}} .$$

The behaviour of the integrand at infinity is then given by

$$\left| \frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y}) \langle \mathbf{x}-\mathbf{y} \rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y}) \langle \mathbf{y} \rangle^{-2}} \right| \approx \begin{cases} \dfrac{\langle \mathbf{x}-\mathbf{y} \rangle^{-1}}{\langle \mathbf{y} \rangle^{-2}} = C_-(x,\varepsilon) e^{-2\pi\eta(\varepsilon+\lambda)\frac{\operatorname{Re}(\mathrm{b})}{\sqrt{N}}} & \eta \to -\infty , \\ \dfrac{\langle \mathbf{x}-\mathbf{y} \rangle^{-2}}{\langle \mathbf{y} \rangle^{-1}} = C_+(x,\varepsilon) e^{2\pi\eta(\varepsilon-2\lambda)\frac{\operatorname{Re}(\mathrm{b})}{\sqrt{N}}} & \eta \to +\infty . \end{cases}$$

Therefore, the convergence condition near ∞ is equivalent to

$$\frac{\varepsilon}{2} < \lambda < -\varepsilon .$$

Fix now $\varepsilon < 0$ and $a \in -2c_{\mathrm{b}}/\sqrt{N} - T$, and consider the contour $\gamma_{\varepsilon,a}$ in $\mathbb{C}$ which deviates from $\mathbb{R} + i\varepsilon\mathrm{b}/\sqrt{N}$ along $a + i\mathbb{R}\overline{\mathrm{b}}$ and $a + i\mathbb{R}\mathrm{b}$. The condition on a ensures that the tip of T, and hence the whole triangle, lies below this contour. Similarly, $\gamma_{\varepsilon,a}$ avoids $T + x$ if and only if it stays above its tip $x - N^{1/2}c_{\mathrm{b}}$, which is the case if e.g. $x \in T + a$. If Γ is chosen to be $\gamma_{\varepsilon,a}$ in each component of $\mathbb{C} \times \mathbb{Z}/N\mathbb{Z}$, the integral converges absolutely and defines a holomorphic function at least on the region $R_{\varepsilon,a} \times \mathbb{Z}/N\mathbb{Z}$, where

$$R_{\varepsilon,a} := \left\{ x = \xi + i\lambda \frac{\mathrm{b}}{\sqrt{N}} \in T + a \, , \frac{\varepsilon}{2} < \lambda < -\varepsilon \right\} .$$

The resulting function agrees with the one defined by the improper integral over $\mathbb{A}_N$ on $R_{\varepsilon,a} \cap \mathbb{R}$, which is non-empty since it always contains a neighbourhood of 0. Also, as ε and a vary on the allowed domains, these regions cover the whole complex plane, thus giving the full holomorphic extension of χ_{4_1} to $\mathbb{C} \times \mathbb{Z}/N\mathbb{Z}$.

Operators annihilating the integrand

We now approach the problem of studying the operators annihilating the integrand of (5.11), which we shall call for convenience I_{4_1}. Following the plan illustrated above, we use Lemma 5.2.2 to see that $\widehat{\ell}_{\mathbf{x}}$ acts on it as

$$\widehat{\ell}_{\mathbf{x}}(I_{4_1}) = \left(1 + q^{-\frac{1}{2}}\widehat{m}_{\mathbf{x}}^{-1}\widehat{m}_{\mathbf{y}}\right) q\widehat{m}_{\mathbf{x}}^{2}\widehat{m}_{\mathbf{y}}^{-2}\frac{\varphi_{\mathsf{b}}(\mathbf{x}-\mathbf{y})\langle\mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathsf{b}}(\mathbf{y})\langle\mathbf{y}\rangle^{-2}}\,.$$

A simple manipulation translates this into

$$\left(\widehat{m}_{\mathbf{y}}^{2}\widehat{\ell}_{\mathbf{x}} - q^{\frac{1}{2}}\left(q^{\frac{1}{2}}\widehat{m}_{\mathbf{x}} + \widehat{m}_{\mathbf{y}}\right)\widehat{m}_{\mathbf{x}}\right)\frac{\varphi_{\mathsf{b}}(\mathbf{x}-\mathbf{y})\langle\mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathsf{b}}(\mathbf{y})\langle\mathbf{y}\rangle^{-2}} = 0\,.$$

Similarly, the action of $\widehat{\ell}_{\mathbf{y}}$ gives

$$\widehat{\ell}_{\mathbf{y}}(I_{4_1}) = \left(1 + q^{\frac{1}{2}}\widehat{m}_{\mathbf{x}}^{-1}\widehat{m}_{\mathbf{y}}\right)^{-1}\left(1 + q^{-\frac{1}{2}}\widehat{m}_{\mathbf{y}}^{-1}\right)^{-1}\widehat{m}_{\mathbf{x}}^{-2}\frac{\varphi_{\mathsf{b}}(\mathbf{x}-\mathbf{y})\langle\mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathsf{b}}(\mathbf{y})\langle\mathbf{y}\rangle^{-2}}\,,$$

hence

$$\left(\left(\widehat{m}_{\mathbf{x}} + q^{\frac{1}{2}}\widehat{m}_{\mathbf{y}}\right)\left(q^{\frac{1}{2}}\widehat{m}_{\mathbf{y}} + 1\right)\widehat{m}_{\mathbf{x}}\widehat{\ell}_{\mathbf{y}} - q^{\frac{1}{2}}\widehat{m}_{\mathbf{y}}\right)\frac{\varphi_{\mathsf{b}}(\mathbf{x}-\mathbf{y})\langle\mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathsf{b}}(\mathbf{y})\langle\mathbf{y}\rangle^{-2}} = 0\,.$$

We can then start elimination in $\widehat{m}_{\mathbf{y}}$ on

$$\begin{aligned} g_1 &= \widehat{\ell}_{\mathbf{x}}\widehat{m}_{\mathbf{y}}^{2} - q^{\frac{1}{2}}\widehat{m}_{\mathbf{x}}\widehat{m}_{\mathbf{y}} - q\widehat{m}_{\mathbf{x}}^{2}\,,\\ g_2 &= \widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}\widehat{m}_{\mathbf{y}}^{2} + q^{\frac{1}{2}}\left(\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^{2} + \widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}} - q\right)\widehat{m}_{\mathbf{y}} + q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^{2}\,. \end{aligned}$$

We run the operation on a computer, and find

$$q^{\frac{9}{2}}\widehat{m}_{\mathbf{x}}^{2}\widehat{A} = qa_1g_1 - a_2g_2\,,$$

where a_1 is given by

$$\begin{aligned} a_1 = {} & \widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}\left(\left(q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^{2} - 1\right)\left(q^{3}\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^{2} + q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}} - 1\right)\widehat{\ell}_{\mathbf{x}} + q^{2}\widehat{\ell}_{\mathbf{y}}\left(q^{3}\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^{2} - 1\right)\widehat{m}_{\mathbf{x}}^{2}\right)\widehat{m}_{\mathbf{y}}\\ & + q^{\frac{1}{2}}\left(q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^{2} - 1\right)\\ & \cdot\left(q^{5}\widehat{\ell}_{\mathbf{y}}^{2}\widehat{m}_{\mathbf{x}}^{4} + q^{3}\widehat{\ell}_{\mathbf{y}}^{2}\widehat{m}_{\mathbf{x}}^{3} + q\widehat{\ell}_{\mathbf{y}}^{2}\widehat{m}_{\mathbf{x}}^{2} - q^{2}(q+1)\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^{2} - (q+1)\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}} + 1\right)\widehat{\ell}_{\mathbf{x}}\\ & + q^{\frac{5}{2}}\widehat{\ell}_{\mathbf{y}}\left(\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}} - q\right)\left(q^{3}\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^{2} - 1\right)\widehat{m}_{\mathbf{x}}^{2}\,, \end{aligned}$$

and a_2 is

$$
\begin{aligned}
a_2 = &\left(\left(q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2-1\right)\left(q^3\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2+q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}-1\right)\widehat{\ell}_{\mathbf{x}}^2+q^3\widehat{\ell}_{\mathbf{y}}\left(q^3\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2-1\right)\widehat{m}_{\mathbf{x}}^2\widehat{\ell}_{\mathbf{x}}\right)\widehat{m}_{\mathbf{y}}\\
&-q^{\frac{5}{2}}\widehat{\ell}_{\mathbf{y}}\left(q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2-1\right)\widehat{m}_{\mathbf{x}}^2\widehat{\ell}_{\mathbf{x}}^2\\
&-q^{\frac{3}{2}}\left(q^4(q+1)\widehat{\ell}_{\mathbf{y}}^2\widehat{m}_{\mathbf{x}}^4+q^2\widehat{\ell}_{\mathbf{y}}^2\widehat{m}_{\mathbf{x}}^3-q(q^2+q+1)\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2-q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}+1\right)\widehat{m}_{\mathbf{x}}\widehat{\ell}_{\mathbf{x}}\\
&-q^{\frac{7}{2}}\widehat{\ell}_{\mathbf{y}}\left(q^3\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2-1\right)\widehat{m}_{\mathbf{x}}^3
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{A} = &\; q^3\widehat{\ell}_{\mathbf{y}}^2\left(q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2-1\right)\widehat{m}_{\mathbf{x}}^2\widehat{\ell}_{\mathbf{x}}^2\\
&-\left(q^2\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2-1\right)\left(q^4\widehat{\ell}_{\mathbf{y}}^2\widehat{m}_{\mathbf{x}}^4-q^3\widehat{\ell}_{\mathbf{y}}^2\widehat{m}_{\mathbf{x}}^3-q(q^2+1)\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2-q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}+1\right)\widehat{\ell}_{\mathbf{x}}\\
&+q\widehat{\ell}_{\mathbf{y}}\left(q^3\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2-1\right)\widehat{m}_{\mathbf{x}}^2 .
\end{aligned}
\tag{5.12}
$$

It is apparent from this, (5.3) and (5.5) that this non-commutative polynomial, evaluated at $\widehat{\ell}_{\mathbf{y}} = 1$, verifies the statement of Theorem 5.4.1 up to rescaling $\widehat{\ell}_{\mathbf{x}}$ by q, and an overall factor q.

Theorem 5.4.2. *The complex non-commutative $\widehat{A}^{\mathbb{C}}$-polynomial for the figure-eight knot is given by the evaluation $\widehat{\ell}_{\mathbf{y}} = 1$, i.e.*

$$\widehat{A}^{\mathbb{C}}_{t,4_1}(\widehat{m}_{\mathbf{x}},\widehat{\ell}_{\mathbf{x}}) = q^{-1}\widehat{A}(\widehat{m}_{\mathbf{x}},q^{-1}\widehat{\ell}_{\mathbf{x}},1)\,.$$

Proof. Since $q^{\frac{9}{2}}\widehat{m}_{\mathbf{x}}^2\widehat{A}$ is a left combination of g_1 and g_2, it annihilates the integrand defining χ_{4_1}, hence so does $\widehat{A}$ itself. For ε and a as usual, $\rho < \varepsilon$ a non-negative integer, call

$$\Gamma^{(\rho)}_{\varepsilon,a} := \Gamma_{\varepsilon,a} - i\rho\mathrm{b}/\sqrt{N} = \Gamma_{\varepsilon-\rho,a-i\frac{\rho\mathrm{b}}{\sqrt{N}}}\,.$$

Of course $\widehat{m}_{\mathbf{x}}$ and $\widehat{\ell}_{\mathbf{x}}$ commute with taking integrals in $d\mathbf{y}$, so for any λ, μ and every $\mathbf{x} \in \mathbb{A}_N^{\mathbb{C}}$ one may write

$$\int_{\Gamma_{\varepsilon,a}} \widehat{\ell}_{\mathbf{y}}^{\rho}\widehat{m}_{\mathbf{x}}^{\mu}\widehat{\ell}_{\mathbf{x}}^{\lambda}\frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})\langle\mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y})\langle\mathbf{y}\rangle^{-2}}\,d\mathbf{y} = \widehat{m}_{\mathbf{x}}^{\mu}\widehat{\ell}_{\mathbf{x}}^{\lambda}\int_{\Gamma^{(\rho)}_{\varepsilon,a}}\frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})\langle\mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y})\langle\mathbf{y}\rangle^{-2}}\,d\mathbf{y}\,.$$

The right-hand side gives $\widehat{m}_{\mathbf{x}}^{\mu}\widehat{\ell}_{\mathbf{x}}^{\lambda}\chi_{4_1}(\mathbf{x})$ for

$$\mathbf{x} \in R_{\varepsilon-\rho,a-i\frac{\rho\mathrm{b}}{\sqrt{N}}} + \left(i\frac{\lambda\mathrm{b}}{\sqrt{N}},-1\right).$$

If ε and $|a|$ are big enough, the intersection of these domains as $0 \le \rho \le 3$ and $0 \le \lambda \le 2$ is non-empty, and in fact these sets cover all $\mathbb{A}_N^{\mathbb{C}}$ for ε and a going to ∞. Applying this point-wise in $\mathbf{x}$ and monomial by monomial in $\widehat{A}$, this shows that this polynomial, evaluated at $\widehat{\ell}_{\mathbf{y}} = 1$, annihilates χ_{4_1}.

On the other hand, recall that

$$J^{(\mathrm{b},N)}_{S^3,4_1}(\mathbf{x}) = e^{4\pi i \frac{c_{\mathrm{b}} x}{\sqrt{N}}} \chi_{4_1}(\mathbf{x})\,.$$

As is easily checked, this implies that $\widehat{A}(\widehat{m}_{\mathbf{x}}, q^{-1}\widehat{\ell}_{\mathbf{x}}, 1)$ belongs to $\mathcal{I}_{\mathrm{loc}}(4_1)$, so there exists $p \in \mathcal{A}_{\mathrm{loc}}$ such that

$$\widehat{A}(\widehat{m}_{\mathbf{x}}, q^{-1}\widehat{\ell}_{\mathbf{x}}, 1) = p(\widehat{m}_{\mathbf{x}}, \widehat{\ell}_{\mathbf{x}}) \cdot \widehat{A}^{\mathbb{C}}_{t,4_1}(\widehat{m}_{\mathbf{x}}, \widehat{\ell}_{\mathbf{x}})\,.$$

However, for $q = 1$ the left-hand side gives $(m^4 - 1)A_{4_1}$, so the expression on the right-hand side gives a factorisation of this polynomial. On the other hand, the classical A-polynomial is irreducible, which implies that either p or $\widehat{A}^{\mathbb{C}}_{t,4_1}$ is a polynomial in $\widehat{m}_{\mathbf{x}}$ alone. If this were the case for $\widehat{A}^{\mathbb{C}}_{t,4_1}$, this would mean that $\chi_{4_1} = 0$, which is not the case as, for instance, this would contradict its known asymptotic properties. This means that $q^{-1}\widehat{A}(\widehat{m}_{\mathbf{x}}, q^{-1}\widehat{\ell}_{\mathbf{x}}, 1)$ is proportional to $\widehat{A}^{\mathbb{C}}_{t,4_1}$, and since its coefficients satisfy the required conditions the conclusion follows. □

This completes the argument for the figure-eight knot.

5.4.2 The knot 5_2

For the knot 5_2, the function $\chi^{(N)}_{5_2} = \chi_{5_2}$ on $\mathbb{A}_N$ is given by the integral

$$\chi_{5_2}(\mathbf{x}) := \int_{\mathbb{A}_N} \frac{\langle \mathbf{y} \rangle \langle \mathbf{x} \rangle^{-1}}{\varphi_{\mathrm{b}}(\mathbf{y}+\mathbf{x})\varphi_{\mathrm{b}}(\mathbf{y})\varphi_{\mathrm{b}}(\mathbf{y}-\mathbf{x})}\, d\mathbf{y}\,. \tag{5.13}$$

As in the case of the figure-eight knot, we need first of all to discuss the convergence of the integral. Calling $\Phi(\mathbf{x}, \mathbf{y})$ the integrand, for $\mathbf{x}$ fixed and $\mathbf{y}$ of the usual form we have the following asymptotic behaviour:

$$|\Phi(\mathbf{x},\mathbf{y})| \approx \begin{cases} \left| \langle \mathbf{y} \rangle \langle \mathbf{x} \rangle^{-1} \right| = C_-(\mathbf{x}, \varepsilon) e^{-2\pi\eta\varepsilon \frac{\mathrm{Re(b)}}{\sqrt{N}}} & \eta \to -\infty\,, \\ \left| \dfrac{1}{\langle \mathbf{y}+\mathbf{x} \rangle \langle \mathbf{y}-\mathbf{x} \rangle} \right| = \left| \dfrac{1}{\langle \mathbf{x} \rangle^2 \langle \mathbf{y} \rangle^2} \right| = C_+(\mathbf{x}, \varepsilon) e^{4\pi\eta\varepsilon \frac{\mathrm{Re(b)}}{\sqrt{N}}} & \eta \to +\infty\,. \end{cases}$$

Independently on $\mathbf{x}$, the integrand decays exponentially as $\eta \to \infty$ as long as ε is positive. Moreover, for fixed $\mathbf{x}$ the poles of Φ lie inside of the region

$$T \cup (T + \mathbf{x}) \cup (T - \mathbf{x})\,.$$

Using $\Gamma_{\varepsilon,a}$ as above, the integral is absolutely convergent if e.g. both $\mathbf{x}$ and $-\mathbf{x}$ lie below the contour, which is to say that $\mathbf{x}$ lies below $\Gamma_{\varepsilon,a}$ and above $-\Gamma_{\varepsilon,a}$. This defines a holomorphic function in the region R_a delimited by the four lines $\pm a + i\mathbb{R}\mathrm{b}$ and $\pm a + i\mathbb{R}\overline{\mathrm{b}}$

$$R_a := \left\{ (x,n) : x \in \left(a + \frac{c_{\mathrm{b}}}{\sqrt{N}} + T\right) \cap \left(-a - \frac{c_{\mathrm{b}}}{\sqrt{N}} - T\right) \right\}$$

Again, by a combination of the residue theorem and dominated convergence, the result is independent on the choice of the path, and the limit $\varepsilon \to 0$ with $a = 0$ gives the improper integral along $\mathbb{A}_N$ for $\mathbf{x}$ real. This proves that χ_{5_2} can be extended to a holomorphic function over the whole $\mathbb{C} \times \mathbb{Z}/N\mathbb{Z}$. We want to stress that, since R_a depends on a alone, the specific choice of $\varepsilon < 0$ has no influence on the region of convergence of the integral.

Operators annihilating the integrand

Following the lines of the case of the figure-eight knot, we use Lemma 5.2.2 to determine the action of $\widehat{\ell}_{\mathbf{x}}$ and $\widehat{\ell}_{\mathbf{y}}$ on the integrand, which we call I_{5_2}. The first operator acts as

$$\widehat{\ell}_{\mathbf{x}} I_{5_2} = q^{\frac{1}{2}} \widehat{m}_{\mathbf{x}} \left(1 + q^{-\frac{1}{2}} \widehat{m}_{\mathbf{x}}^{-1} \widehat{m}_{\mathbf{y}}^{-1}\right)^{-1} \left(1 + q^{\frac{1}{2}} \widehat{m}_{\mathbf{x}} \widehat{m}_{\mathbf{y}}^{-1}\right),$$

while the second gives

$$\widehat{\ell}_{\mathbf{y}} I_{5_2} = q^{-\frac{1}{2}} \widehat{m}_{\mathbf{y}}^{-1} \left(1 + q^{-\frac{1}{2}} \widehat{m}_{\mathbf{x}}^{-1} \widehat{m}_{\mathbf{y}}^{-1}\right)^{-1} \left(1 + q^{-\frac{1}{2}} \widehat{m}_{\mathbf{y}}^{-1}\right)^{-1} \left(1 + q^{-\frac{1}{2}} \widehat{m}_{\mathbf{y}}^{-1} \widehat{m}_{\mathbf{x}}\right)^{-1}.$$

From this we get the operators

$$\begin{aligned} g_1 &= q^{\frac{1}{2}} \left(\widehat{m}_{\mathbf{x}} \widehat{\ell}_{\mathbf{x}} - q^{\frac{1}{2}} \widehat{m}_{\mathbf{x}}^2\right) \widehat{m}_{\mathbf{y}} + \widehat{\ell}_{\mathbf{x}} - q^{\frac{3}{2}} \widehat{m}_{\mathbf{x}}^3, \\ g_2 &= \widehat{\ell}_{\mathbf{y}} \widehat{m}_{\mathbf{x}} \widehat{m}_{\mathbf{y}}^3 + q^{\frac{1}{2}} \left(\widehat{\ell}_{\mathbf{y}} \widehat{m}_{\mathbf{x}}^2 + \widehat{\ell}_{\mathbf{y}} \widehat{m}_{\mathbf{x}} - q^2 \widehat{m}_{\mathbf{x}} + \widehat{\ell}_{\mathbf{y}}\right) \widehat{m}_{\mathbf{y}}^2 \\ &\quad + q \widehat{\ell}_{\mathbf{y}} \left(\widehat{m}_{\mathbf{x}}^2 + \widehat{m}_{\mathbf{x}} + 1\right) \widehat{m}_{\mathbf{y}} + q^{\frac{3}{2}} \widehat{\ell}_{\mathbf{y}} \widehat{m}_{\mathbf{x}}. \end{aligned}$$

Again, running elimination in $\widehat{m}_{\mathbf{y}}$ returns

$$\widehat{A} = a_1 g_1 + \widehat{m}_{\mathbf{x}} g_2,$$

where a_1 is expressed as

$$
\begin{aligned}
a_1 = & -q^{\frac{1}{2}}\left(q\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^2\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^4\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}^2+1\right)\widehat{\ell}_{\mathsf{x}}^2-q\left(q\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^5\widehat{m}_{\mathsf{x}}^2-1\right)\\
& \cdot\left(q^6\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}^4-q^5\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}^3-q^5\widehat{m}_{\mathsf{x}}^3-q^2(q^2+1)\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}^2-q^2\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}-q^2\widehat{m}_{\mathsf{x}}+\widehat{\ell}_{\mathsf{y}}\right)\widehat{\ell}_{\mathsf{x}}\\
& -q^{\frac{9}{2}}\widehat{m}_{\mathsf{x}}^2\left(q^2\widehat{m}_{\mathsf{x}}^2+\widehat{\ell}_{\mathsf{y}}\right)\left(q^4\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^5\widehat{m}_{\mathsf{x}}^2-1\right)\\
& \Bigg(-q^{\frac{1}{2}}\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}\left(q\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^2\widehat{m}_{\mathsf{x}}^2-1\right)\widehat{\ell}_{\mathsf{x}}^2+q^2(q+1)\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}^2\left(q\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^5\widehat{m}_{\mathsf{x}}^2-1\right)\widehat{\ell}_{\mathsf{x}}\\
& -q^{\frac{7}{2}}\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}^3\left(q^4\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^5\widehat{m}_{\mathsf{x}}^2-1\right)\Bigg)\widehat{m}_{\mathsf{y}}^2\\
& +\Bigg(-q\widehat{m}_{\mathsf{x}}\left(q\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^2\widehat{m}_{\mathsf{x}}-1\right)\left(q^3\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}+\widehat{\ell}_{\mathsf{y}}-q^2\right)\widehat{\ell}_{\mathsf{x}}^2\\
& +q^{\frac{5}{2}}\widehat{m}_{\mathsf{x}}\left(q\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^5\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^3\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}^2+(q+1)\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}-q^2(q+1)\widehat{m}_{\mathsf{x}}+\widehat{\ell}_{\mathsf{y}}\right)\widehat{\ell}_{\mathsf{x}}\\
& -q^8\widehat{m}_{\mathsf{x}}^2\left(q^4\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^5\widehat{m}_{\mathsf{x}}^2-1\right)\left(\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}-q^2\widehat{m}_{\mathsf{x}}+\widehat{\ell}_{\mathsf{y}}\right)\Bigg)\widehat{m}_{\mathsf{y}}\,,
\end{aligned}
$$

and a_2 is

$$
\begin{aligned}
a_2 = & \left(q\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^2\widehat{m}_{\mathsf{x}}^2-1\right)\widehat{\ell}_{\mathsf{x}}^3-q^{\frac{3}{2}}\widehat{m}_{\mathsf{x}}\left(q^2+q+1\right)\left(q\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^4\widehat{m}_{\mathsf{x}}^2-1\right)\widehat{\ell}_{\mathsf{x}}^2\\
& +q^3\widehat{m}_{\mathsf{x}}^2\left(q^2+q+1\right)\left(q^2\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^5\widehat{m}_{\mathsf{x}}^2-1\right)\widehat{\ell}_{\mathsf{x}}\\
& -q^{\frac{9}{2}}\widehat{m}_{\mathsf{x}}^3\left(q^4\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^5\widehat{m}_{\mathsf{x}}^2-1\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{A} = & -q^{\frac{1}{2}}\left(q\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^2\widehat{m}_{\mathsf{x}}^2-1\right)\widehat{\ell}_{\mathsf{x}}^3\\
& +q\left(q\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^4\widehat{m}_{\mathsf{x}}^2-1\right)\Bigg(q^9\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}^5-q^7\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}^4-q^4(q^3+1)\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}^3\\
& +q^5(q+1)\widehat{m}_{\mathsf{x}}^3+q^2(q^3+1)\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}^2+q^2(\widehat{\ell}_{\mathsf{y}}+1)\widehat{m}_{\mathsf{x}}-\widehat{\ell}_{\mathsf{y}}\Bigg)\widehat{\ell}_{\mathsf{x}}^2\\
& +q^{\frac{9}{2}}\widehat{m}_{\mathsf{x}}^2\left(q^2\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^5\widehat{m}_{\mathsf{x}}^2-1\right)\Bigg(q^6\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}^5-q^5(\widehat{\ell}_{\mathsf{y}}+1)\widehat{m}_{\mathsf{x}}^4\\
& -q^2(q^3+1)\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}^3+q(q^3\widehat{\ell}_{\mathsf{y}}-q^2-q+\widehat{\ell}_{\mathsf{y}})\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}^2+q\widehat{\ell}_{\mathsf{y}}\widehat{m}_{\mathsf{x}}-\widehat{\ell}_{\mathsf{y}}\Bigg)\widehat{\ell}_{\mathsf{x}}\\
& +q^8\widehat{m}_{\mathsf{x}}^7\left(q^4\widehat{m}_{\mathsf{x}}^2-1\right)\left(q^5\widehat{m}_{\mathsf{x}}^2-1\right).
\end{aligned}
$$

As for the figure-eight knot, for $\widehat{\ell}_{\mathbf{y}} = 1$ we obtain a non-commutative polynomial in $\widehat{m}_{\mathbf{x}}$ and $\widehat{\ell}_{\mathbf{x}}$ which, up to a global factor in q and a rescaling of $\widehat{\ell}_{\mathbf{x}}$ by $-q^{\frac{1}{2}}$, verifies Theorem 5.4.1. With few slight adaptations, the same argument as for the previous case finally shows the following.

Theorem 5.4.3. *The complex non-commutative $\widehat{A}^{\mathbb{C}}$-polynomial for the knot 5_2 is given by the evaluation*

$$\widehat{A}^{\mathbb{C}}_{t,4_1}(\widehat{m}_{\mathbf{x}}, \widehat{\ell}_{\mathbf{x}}) = q^{-1}\widehat{A}(\widehat{m}_{\mathbf{x}}, -q^{-\frac{1}{2}}\widehat{\ell}_{\mathbf{x}}, 1)\,.$$

Chapter 6

Further perspectives

The problems discussed in this dissertation leave a number of questions open for further study.

Asymptotic properties of curve operators in higher genus In the study of the existence of a trivialisation of the formal Hitchin-Witten connection, a crucial use is made of a primitive of the operator-valued 1-form $b + \overline{b}$ on the Teichmüller space. The exactness of this form is strongly related to the flatness of the Hitchin-Witten connection, which is specific to genus 1. In the general case, instead, the connection is only *projectively* flat, and the (twisted) differential of $b + \overline{b}$ takes values in non-vanishing central operators. It is conceivable, however, that for higher genera there exists a *projective* potential, i.e. a primitive up to central corrections. Since in the solution we propose the various forms appear only through commutators, this would be enough to introduce a recursion similar to that used in genus 1, possibly leading to analogous results.

Full Dirac condition for the quantum operators Another possible research line is related to our construction of differential operators using the embedding of $\mathrm{T}^* \mathcal{M}$ into $\mathcal{M}_{\mathbb{C}}$ via the Narasimhan-Seshadri and Hitchin-Kobayashi correspondences. As we discussed in Chapter 4, the resulting operators for a surface of genus 1 satisfy a Dirac quantisation condition only in the imaginary part s of the quantum parameter. However, Lemmas 4.1.2 and 4.1.3 show that the Poisson bracket of two functions includes a term, proportional to k, which does not appear in the commutator of their operators. It would be interesting to study whether this extra term can be included by modifying the operators. One possibility would be to use the variation ∇T of the tensor field associated to a polynomial function, contracted with the inverse $\tilde{\omega}$ of the symplectic form on $\mathcal{M}$. In a good sense, this would be compatible with the definition of the pre-quantum operators of geometric quantisation. Indeed, their first-order term, $-\frac{i}{k}\nabla_{X_f}$, is obtained precisely via the contraction of $\mathrm{d}f$ with $\tilde{\omega}$. The goal of this construction would then be to match precisely the terms of lowest decay in the expression appearing in the Dirac quantisation condition. This would finally set the ground for an approach analogous to deformation quantisation with complex Poisson structure and quantum parameter.

Quantum operators from geometric constructions in higher genus Unlike in the case of genus 1, the Levi-Civita connection on $\mathcal{M}$ induced by the Riemann surface structure σ on Σ strictly depends on this parameter in general. On the other hand, it is also to be expected that the embedding of $\mathrm{T}^* \mathcal{M}$ in $\mathcal{M}_{\mathbb{C}}$ should depend on it in a crucial way. It would be interesting to understand the properties of the operators arising in the general case from the construction carried out in genus 1, especially in relation to the dependence on the Teichmüller parameter.

AJ conjecture for other knots In our discussion of the AJ conjecture for the Teichmüller TQFT, we argue the statement for the knots 4_1 and 5_2 by means of an elimination process. As an interesting natural continuation of this work, one could use the same approach for other hyperbolic knots, once an expression for $J_{M,K}^{(\mathrm{b},N)}$ is given. It should be noted that the complexity of the computation seems to grow significantly even for the knot 6_1, so it is to be expected that more sophisticated computational methods might be needed.

Appendix A

Miscellanea on differential operators

A.1 Symbols and useful algebraic relations

Throughout this section let E be a vector bundle, equipped with a connection ∇, over a smooth manifold M endowed with a Riemannian metric g and Levi-Civita connection also called ∇. We shall assume for simplicity that E is a complex line bundle, although this requirement is inessential. In the following we shall use bold Greek letter to denote multi-indices, i.e. finite ordered sequences of indices, and call $l(\boldsymbol{\mu})$ the length of a multi-index $\boldsymbol{\mu}$. Also, we use Einstein's convention for repeated indices, meaning that a sum over all possible values of a multi-index is intended when it appears twice in an expression, once upper and once lower.

Following [Wel08], differential operators of order at most n acting on smooth sections of E can be defined as maps $D\colon \mathcal{C}^\infty(M,E) \to \mathcal{C}^\infty(M,E)$ which can be described in local coordinates and trivialisations as

$$D(\psi) = \sum_{l(\boldsymbol{\mu})\leq n} T^{\boldsymbol{\mu}} \partial_{\boldsymbol{\mu}} \psi \,,$$

where $\partial_{\boldsymbol{\mu}}$ denotes the subsequent derivative along the directions indicated by the multi-index. As is easily checked, when this is the case the coefficients $T^{\boldsymbol{\mu}}$ with $l(\boldsymbol{\mu}) = n$ transform as a totally symmetric contra-variant tensor field on M, called the n-th symbol of D and denoted $\sigma_n(D)$. Clearly, there is at most one value of n for which $\sigma_n(D)$ is well-defined and non-zero; when this is the case, it is also called the principal symbol. Moreover, $\sigma_n(D) = 0$ for every n if and only if $D = 0$. While this object, which encodes the top-order part of the operator, is defined whether or not a connection on E or a metric on M are defined, there is no natural way in general to make sense of the lower-order terms.

In the situation at hand there is a consistent way of defining them. Indeed, one can associate a differential operator ∇^n_T to any n-contra-variant tensor field on M, using the following procedure. Let ψ be a section of E. Then $\nabla\psi$ is well defined as a section of $\mathrm{T}^*M \otimes E$, which also comes with a connection; by iteration one can define $\nabla^n\psi$ as a section of $(\mathrm{T}^*M)^{\otimes n} \otimes E$. It makes then sense to contract the indices of T with those of $\nabla^n\psi$, thus obtaining a new section of E. The result is tensorial in T and differential of order at most n in ψ, as claimed, with n-th symbol $\mathcal{S}(T)$, the totally symmetric part of T. In coordinates, it is written as

$$\nabla^n_T\psi = T^{\mu_1\cdots\mu_n}\nabla_{\mu_1}\dots\nabla_{\mu_n}\psi\,.$$

Notice that, given n vector fields $X_1,\dots,X_n$ on M, the differential operator $\nabla^n_{X_1\otimes\cdots\otimes X_n}$ is *not* the same as the iterated derivative $\nabla_{X_1}\dots\nabla_{X_n}$. In fact, the former is tensorial in each vector field, while the latter is tensorial in X_1 alone and differential of order $j-1$ in X_j.

One can think of this construction as a section of σ_n: if T is totally symmetric, then the symbol of ∇^n_T is T itself. On the other hand, given a differential operator D of order n, $\nabla^n_{\sigma_n(D)}$ may very well be different from D. However, these two operators have the same principal symbol, so their difference is an operator of order at most $n-1$. This motivates the following definition.

Definition A.1.1. *If D is a differential operator of finite order $m > n$ on E, its n-th symbol is defined recursively as*

$$\sigma_n(D) := \sigma_n\left(D - \sum_{j=m+1}^{n} \nabla^j_{\sigma_j(D)}\right).$$

We shall call the total symbol of D the formal sum of all its symbols.

It follows from the definition that if D has order m then

$$D = \sum_{n=0}^{m} \nabla^n_{\sigma_n(D)}\,.$$

Consequently, such an operator is completely determined by its symbols. This is particularly useful in calculations involving compositions and commutators of differential operators, as it allows to separate homogeneous parts and work on them individually. A simple but rather lengthy calculation proves the following result, presented in [AG14] as Lemma 3.1.

We include the following lemma, used in Chapter 4.

Lemma A.1.1. *Suppose that M is a Kähler manifold with symplectic structure ω, and that E is a pre-quantum line bundle for (M,ω), and call* Riem *the Riemann curvature tensor on M. Let X be a tangent vector field on $\mathcal{M}$, B a symmetric bi-vector field. Call ∇^2_B the differential operator described in coordinates* *as* $B^{\mu\nu}\nabla_\mu\nabla_\nu$*. Then the commutator* $[\nabla^2_B, \nabla_X]$ *is completely determined by its symbols:*

$$\sigma_2\left[\nabla^2_B, \nabla_X\right] = 2\mathcal{S}\left(B\cdot\nabla X\right) - \nabla_X B\,; \tag{A.1}$$

$$\sigma_1\left[\nabla^2_B, \nabla_X\right] = \nabla^2_B X - 2iB\cdot\omega\cdot X + \mathrm{Riem}(X\otimes B)\,; \tag{A.2}$$

$$\sigma_0\left[\nabla^2_B, \nabla_X\right] = -i\omega(B\cdot\nabla X)\,. \tag{A.3}$$

Here $\mathcal{S}$ *denotes symmetrisation.*

Definition A.1.2. *For* B *a bi-vector field on* M*, define*

$$\Delta_B = \nabla^2_B + \nabla_{\delta B}\,.$$

In coordinates, Δ_B can be written as

$$\Delta_B = B^{\mu\nu}\nabla_\mu\nabla_\nu + (\nabla_\mu B^{\mu\nu})\nabla_\nu = \nabla_\mu B^{\mu\nu}\nabla_\nu\,.$$

As a particular case, for $B = \tilde{g}$ one obtains the usual Laplace operator, as a consequence of the parallelism of g.

Lemma A.1.2. *Suppose as before that* M *is Kähler with symplectic structure* ω*, that* E *is a pre-quantum line bundle and moreover that the Levi-Civita connection of* g *is flat. Let* T *be a tensor field on* M *of rank* $r = r_1 + r_2$*, and assume further that it is totally symmetric in the first* r_1 *and last* r_2 *indices. Then for* $0 \leq 2n \leq r$ *there are coefficients* $c_{r,n}(r_1, r_2)$ *so that the symbols of* ∇^r_T *are given by:*

$$\sigma_{r-2n}(\nabla^r_T) = c_{r,n}(r_1, r_2)\mathcal{S}\Big(\omega^n(T)\Big)\,,$$
$$\sigma_{r-2n-1}(\nabla^r_T) = 0\,.$$

Here $\omega(T) = \omega_{\mu_1\mu_r}T^{\boldsymbol{\mu}}$*, and* $\omega^n(T)$ *is defined inductively. Furthermore:*

$$c_{r,0}(r_1, r_2) = 1\,, \qquad c_{r,1}(r_1, r_2) = \frac{r_1 r_2}{2}\,.$$

Proof. The first equation is just an expression of the general fact that the principal symbol of ∇^r_T is the symmetrisation of T. For the lower symbols, consider:

$$\nabla^r_T - \nabla^r_{\mathcal{S}(T)} = T^{\boldsymbol{\mu}}\nabla^r_{\boldsymbol{\mu}} - \frac{1}{r!}\sum_{\pi\in\Pi(r)} T^{\pi^{-1}(\boldsymbol{\mu})}\nabla^r_{\boldsymbol{\mu}} = \frac{1}{r!}\sum_{\pi\in\Pi(r)}\Big(T^{\boldsymbol{\mu}} - T^{\pi^{-1}(\boldsymbol{\mu})}\Big)\nabla^r_{\boldsymbol{\mu}}\,.$$

Here $\Pi(r)$ denotes the group of permutations of r elements, and a permutation π applied to a multi-index $\boldsymbol{\mu}$ of length r means the multi-index obtained from $\boldsymbol{\mu}$ by permuting its entries according to π. More precisely, π will move the j-th component of $\boldsymbol{\mu}$ to the $\pi(j)$-th position, so

$(\pi(\boldsymbol{\mu}))_j = \boldsymbol{\mu}_{\pi^{-1}(j)}$. Sending $\boldsymbol{\mu}$ to $\pi(\boldsymbol{\mu})$ defines a left action of $\Pi(r)$ on the multi-indices, and hence a right action on (not necessarily symmetric) tensor fields.

Notice that inside $\Pi(r)$ sits $\Pi(r_1) \times \Pi(r_2)$ as the subgroup fixing the partition of r elements into the first r_1 and the last r_2. For every $1 \leq j \leq r$ and $0 \leq l \leq r - j$ call $\gamma_{j,l}$ be the cyclic permutation $(j \;\; j{+}1 \;\; \dots \;\; j{+}l)$.

Lemma A.1.3. *Any permutation of $r = r_1 + r_2$ elements has a unique decomposition of the form*

$$\gamma_{1,l_1}^{-1} \circ \cdots \circ \gamma_{r_1,l_{r_1}}^{-1} \circ (\pi_1, \pi_2) \tag{A.4}$$

with $0 \leq l_1 \leq \cdots \leq l_{r_1} \leq r_2$ and $(\pi_1, \pi_2) \in \Pi(r_1) \times \Pi(r_2) \subseteq \Pi(r)$.

Proof. Suppose now that π is any permutation. For each $j = 1, \dots, r_1$ define recursively

$$\begin{cases} n_1 := \min\{\pi(x) : 1 \leq x \leq r_1\}\,, \\ n_{j+1} := \min\{\pi(x) : 1 \leq x \leq r_1 \text{ and } \pi(x) > n_j\},. \end{cases}$$

In other words, n_j is the j-th occurrence in $(1, 2, \dots, r)$ of an element $\pi(x)$ with x coming from the first r_1 elements. Define further a permutation $\pi_1 \in \Pi(r_1)$ implicitly by $\pi_1^{-1}(j) = \pi^{-1}(n_j)$. Intuitively, this is the permutation that rearranges the ordering of $(1, \dots, r_1)$ in the same way as π. One can define in a similar way a permutation $\pi_2 \in \Pi(r_2)$ producing the same re-ordering of the last r_2 elements as π. Finally, let $l_j := n_j - j$: it is immediately seen that l_j counts the number of elements y with $1 \leq y \leq r_2$ and $\pi(r_1 + y) < m_j$, so $l_j \leq r_2$ for each j. Furthermore, $l_j \leq l_{j+1}$, so one has indeed that $0 \leq l_1 \leq \cdots \leq l_{r_1} \leq r_2$.

Consider now the composition $\gamma_{1,l_1} \circ \pi$. The range $\{1, \dots, n_1\}$ contains exactly one element coming from $\{1, \dots, r_1\}$ through π, while all the others come from $\{r_1 + 1, \dots, r\}$. The effect of γ_{1,l_1} is then that of moving n_1 to the first position and shift forward all the smaller elements while keeping all the rest fixed. Therefore, the composition sends $\pi^{-1}(n_1)$ to 1 and keeps the same re-arrangements of $\{1, \dots, r_1\}$ and $\{r_1 + 1, \dots, r\}$. Inductively, it is easily proven that $\gamma_{j,l_j} \circ \cdots \circ \gamma_{1,l_1} \circ \pi$ keeps the same partial re-arrangements as π, while hitting $\{1, \dots, j\}$ with some of the first r_1 elements. Finally, one may write

$$\gamma_{r_1,l_{r_1}} \circ \cdots \circ \gamma_{1,l_1} \circ \pi = (\pi_1, \pi_2)\,.$$

This concludes the argument for the existence.

As for uniqueness, a basic combinatorial argument shows that $0 \leq l_1 \leq \ldots \leq l_r \leq r_2$ can be chosen in $\binom{r_1+r_2}{r_2}$ ways. On the other hand, (π_1, π_2) can be chosen in $r_1! \cdot r_2!$ ways, so that there exist exactly $r!$ distinct expressions of the form of (A.4). This concludes the proof. □

Now pick a permutation π, and decompose it as in (A.4). For notational convenience call $\Gamma_j := \gamma_{r_1,l_{r_1}} \circ \cdots \circ \gamma_{j,l_j}$ for $1 \leq j \leq r_1$ and $\Gamma_{r_1+1} := \mathbb{1}$, and consider

$$\left(T^{\boldsymbol{\mu}} - T^{\pi^{-1}(\boldsymbol{\mu})}\right) \nabla^r_{\boldsymbol{\mu}} = \left(T^{\boldsymbol{\mu}} - T^{(\pi_1^{-1},\pi_2^{-1})\left(\Gamma_1(\boldsymbol{\mu})\right)}\right) \nabla^r_{\boldsymbol{\mu}} .$$

Since the action of $\Pi(r)$ on tensors is on the right, the factor (π_1^{-1}, π_2^{-1}) is acting first, and because T is symmetric in the first r_1 and last r_2 indices this is irrelevant and can be dropped, so

$$\left(T^{\boldsymbol{\mu}} - T^{\pi^{-1}(\boldsymbol{\mu})}\right) \nabla^r_{\boldsymbol{\mu}} = \left(T^{\boldsymbol{\mu}} - T^{\Gamma_1(\boldsymbol{\mu})}\right) \nabla^r_{\boldsymbol{\mu}} = T^{\boldsymbol{\mu}} \left(\nabla^r_{\boldsymbol{\mu}} - \nabla^r_{\Gamma_1^{-1}(\boldsymbol{\mu})}\right) .$$

The action of the cycle γ^{-1}_{j,l_j} consists of "dragging" j to the position n_j, and subsequent application of such cycles as in Γ_j brings each $j' \leq j$ to the position $n_{j'}$. We may then write the above expression as a telescoping sum of contributions coming from subsequent applications of cycles

$$\sum_{j=1}^{r_1} T^{\boldsymbol{\mu}} \left(\nabla^r_{\Gamma^{-1}_{j+1}(\boldsymbol{\mu})} - \nabla^r_{\Gamma^{-1}_j(\boldsymbol{\mu})}\right) \nabla^r_{\boldsymbol{\mu}} = \sum_{j=1}^{r_1} T^{\boldsymbol{\mu}} \left(\nabla^r_{\Gamma^{-1}_{j+1}(\boldsymbol{\mu})} - \nabla^r_{\gamma^{-1}_{j,l_j}\left(\Gamma^{-1}_{j+1}(\boldsymbol{\mu})\right)}\right) \nabla^r_{\boldsymbol{\mu}} .$$

Analogously, we can further expand the sum by viewing each cycle as a composition of transpositions: $\gamma^{-1}_{j,l_j} = (j{+}l_j \;\; j{+}l_j{-}1) \dots (j{+}1 \;\; j)$. In particular for every $1 \leq h \leq l_j$ we can write $\gamma^{-1}_{j,h} = (j{+}h \;\; j{+}h{-}1)\gamma^{-1}_{j,h-1}$, so we obtain the difference $\nabla^r_T - \nabla^{r-2}_{\sigma_{r-2}(\nabla^r_T)}$

$$\frac{1}{r!} \sum_{\pi \in \Pi} \sum_{j=1}^{r_1} \sum_{h=1}^{l_j} T^{\boldsymbol{\mu}} \left(\nabla^r_{\gamma^{-1}_{j,h-1}\left(\Gamma^{-1}_{j+1}(\boldsymbol{\mu})\right)} - \nabla^r_{(j+h \;\; j+h-1)\left(\gamma^{-1}_{j,h-1}\left(\Gamma^{-1}_{j+1}(\boldsymbol{\mu})\right)\right)}\right) =$$

$$= \frac{1}{r!} \sum_{\pi \in \Pi} \sum_{j=1}^{r_1} \sum_{h=1}^{l_j} T^{\boldsymbol{\mu}} \left(\nabla^{j+h-2} \omega \nabla^{r-j-h}\right)_{\gamma^{-1}_{j,h-1}\left(\Gamma^{-1}_{j+1}(\boldsymbol{\mu})\right)} .$$

Here we used that, since the Levi-Civita connection is flat, the only relevant curvature is ω. Furthermore, because $\nabla\omega = 0$ we can pull it out of the derivatives. The expression becomes then one involving only $r - 2$ derivatives, which is enough to conclude that $\sigma_{r-1}(\nabla^r_T) = 0$. Inductively, one can find a similar expression for $\nabla^r_T - \sum_{m=0}^{n} \nabla^{r-2m}_{\sigma_{r-2m}(\nabla^r_T)}$: at each step, two occurrences of ∇ are exchanged with a factor ω. This shows that the $(r - 2n - 1)$-th symbol vanishes, while the $(r - 2n - 2)$-th is the symmetrisation of such an expression. Determining the coefficient $c_{r,n}(r_1, r_2)$ is then a matter of counting the non-vanishing contributions.

In order to determine $\sigma_{r-2}(\nabla_T^r)$, or equivalently $c_{r,2}(r_1, r_2)$, we need to understand how T is pared with ω in the above expression. The indices of ω are those in positions $j+h-1$ and $j+h$ in $\gamma_{j,h-1}^{-1}\left(\Gamma_{j+1}^{-1}(\boldsymbol{\mu})\right)$, which are obtained by applying the inverse of the permutation to them. By construction, $\gamma_{j,h-1}$ takes $j+h-1$ to j and fixes $j+h$, while Γ_{j+1} fixes j and every element $j' \leq j$. The contribution of π to the symbol is then

$$\frac{1}{r!}\sigma_{r-2}\left(\left(T^{\boldsymbol{\mu}} - T^{\pi^{-1}(\boldsymbol{\mu})}\right)\nabla_{\boldsymbol{\mu}}^r\right) = \frac{1}{r!}\sum_{j=1}^{r_1}\sum_{h=1}^{l_j}\mathcal{S}\left(T^{\boldsymbol{\mu}}\omega_{\mu_j,\mu_{\Gamma_{j+1}(j+h)}}\right).$$

Because of the symmetries of T and the anti-symmetricity of ω, all the terms with $\Gamma_{j+1}(j+h) \leq r_1$ are zero, given that $j \leq r_1$. Therefore, each summand equals $\frac{1}{r!}\mathcal{S}(T^{\boldsymbol{\mu}}\omega_{\mu_1,\mu_r})$ if $\Gamma_{j+1}(j+h) > r_1$, and 0 otherwise. Said differently, the sum over h counts the number of elements $j < x \leq n_j$ with $\Gamma_{j+1}(x) > r_1$ or, equivalently, the number of elements $y > r_1$ for which $j < \Gamma_{j+1}^{-1}(y) \leq \Gamma_j^{-1}(j)$. Here the first inequality is redundant when $y > r_1$, again because of the properties of these permutations. However, using once more the fact that every $\gamma_{j'}^{-1}$ fixes every $y > n_{j'}$, one can easily see that $\Gamma_{j+1}^{-1}(y) \leq \Gamma_j^{-1}(j)$ is equivalent to $\Gamma_1^{-1}(y) < \Gamma_1^{-1}(j)$. In conclusion, the whole sum over j counts the number of all pairs $j \leq r_1 < y$ satisfying this last inequality, and an easy induction (starting from r_1 backwards) shows that each j carries a contribution of l_j. Finally we have

$$\frac{1}{r!}\sigma_{r-2}\left(\left(T^{\boldsymbol{\mu}} - T^{\pi^{-1}(\boldsymbol{\mu})}\right)\nabla_{\boldsymbol{\mu}}^r\right) = \frac{1}{r!}\mathcal{S}\left(T^{\boldsymbol{\mu}}\omega_{\mu_1,\mu_r}\right)\sum_{j=1}^{r_1} l_j.$$

To conclude we need to sum over all $\pi \in \Pi(r)$. As already observed, each sequence $0 \leq l_1 \leq \cdots \leq l_{r_1} \leq r_2$ occurs exactly $r_1! \cdot r_2!$ times, so the sum can be written as

$$\sigma_2(\nabla_T^r) = \binom{r}{r_1}^{-1}\mathcal{S}\left(T^{\boldsymbol{\mu}}\omega_{\mu_1,\mu_r}\right)\sum_{l_{r_1}=0}^{r_2}\sum_{l_{r_1-1}=0}^{l_{r_1}}\cdots\sum_{l_1=0}^{l_2}\sum_{j=1}^{r_1} l_j.$$

It remains to check that the sum on the right equals $\frac{r_1 r_2}{2}\binom{r}{r_1}$, which can be done by induction. Notice first that if either r_1 or r_2 is zero the whole sum is. If instead they are both positive, the sum corresponding to $r_2 - 1$ in place of r_2 is given by the sum of the terms with $l_{r_1} = r_2$, which gives

$$\sum_{l_{r_1-1}=0}^{r_2}\cdots\sum_{l_1=0}^{l_2}\left(\sum_{j=1}^{r_1-1} l_j + r_2\right) = \sum_{l_{r_1-1}=0}^{r_2}\cdots\sum_{l_1=0}^{l_2}\sum_{j=1}^{r_1-1} l_j + \sum_{l_{r_1-1}=0}^{r_2}\cdots\sum_{l_1=0}^{l_2}\sum_{j=1}^{r_1-1} r_2.$$

The first sum in the right-hand side is the one corresponding to $r_1 - 1$ in place of r_1, while the second one gives r_2 times the number of choices of

$0 \le l_1 \le \cdots \le l_{r_1-1} \le r_2$, i.e. $r_2\binom{r-1}{r_1-1}$. Notice that this quantity can also be written as $r_1\binom{r-1}{r_1}$. Therefore the desired sum, as a function of r_1 and r_2, solves the recursion

$$x_{r_1,r_2} = x_{r_1-1,r_2} + x_{r_1,r_2-1} + r_2\binom{r_1+r_2-1}{r_1-1}.$$

with initial conditions $x_{0,r_2} = x_{r_1,0} = 0$. Clearly this recursion has a unique solution, so we just need to check that $\frac{r_1 r_2}{2}\binom{r}{r_1}$ is also a solution. The initial condition is of course satisfied, so we just need to check the following:

$$\begin{aligned}
&\frac{(r_1-1)r_2}{2}\binom{r-1}{r_1-1} + \frac{r_1(r_2-1)}{2}\binom{r-1}{r_1} + r_2\binom{r-1}{r_1-1} = \\
=&\frac{(r_1-1)r_2}{2}\binom{r-1}{r_1-1} + \frac{r_1(r_2-1)}{2}\binom{r-1}{r_1} + \frac{1}{2}r_2\binom{r-1}{r_1-1} + \frac{1}{2}r_1\binom{r-1}{r_1} = \\
=&\frac{r_1 r_2}{2}\binom{r-1}{r_1-1} + \frac{r_1 r_2}{2}\binom{r-1}{r_1} = \frac{r_1 r_2}{2}\binom{r}{r_1}.
\end{aligned}$$

This concludes the proof. □

A.2 Twisted exterior differentials

Suppose again that $E \to M$ is a vector bundle over a smooth manifold, coming with a connection ∇. One can then define an exterior differential d^∇ for E-valued forms on M by anti-symmetrising the covariant derivative. Formally, d^∇ is defined in the same way as the usual de Rham exterior differential by using ∇ in order to take derivatives. Similarly, ∇^{End} induces a de Rham differential (which we shall also denoted by d^∇) on $\mathrm{End}(E)$-valued differential forms.

We stress that, in general, it is not guaranteed that these twisted differentials square to zero; indeed the square of d^∇ is proportional to the curvature F_∇. Therefore, the condition for d^∇ to square to zero is equivalent to the flatness of ∇. When this is the case, one may consider the complex $(\omega^\bullet(M,E), \mathrm{d}^\nabla)$ of E-valued forms with this exterior differential, and define its cohomology:

$$H^n_A(M,E) = H^n\left(\omega^\bullet(M,E), \mathrm{d}^\nabla\right).$$

The case of $\mathrm{End}(E)$ is formally analogous.

The commutator of endomorphisms induces a wedge product $[\cdot \wedge \cdot]$ of $\mathrm{End}(E)$-valued forms, which we shall often denote as $[\cdot, \cdot]$ when one of the two arguments has degree 0. This product enjoys a list of properties analogous to those for Lie algebra-valued differential forms, see e.g. [CS74].

Lemma A.2.1. *Let φ, ψ and ρ be* End(E)*-valued differential forms of rank a, b and c respectively. Then the following hold:*

$$d^\nabla[\varphi \wedge \psi] = \Big[(d^\nabla \varphi) \wedge \psi\Big] + (-1)^a \Big[\varphi \wedge d^\nabla \psi\Big]\,,$$
$$[\varphi \wedge \psi] = -(-1)^{ab}[\psi \wedge \varphi]\,,$$
$$(-1)^{ac}\Big[\varphi \wedge [\psi \wedge \rho]\Big] + (-1)^{ba}\Big[\psi \wedge [\rho \wedge \varphi]\Big] + (-1)^{cb}\Big[\rho \wedge [\varphi \wedge \psi]\Big] = 0\,.$$

The first two equations are proven following the steps of their analogous for ordinary differential form, the extra sign in the second coming from anti-symmetricity of the commutator. The last follows from the Jacobi identity, with some care for the signs when rearranging the arguments of the forms. If in a local trivialisation ∇ is expressed as $\mathrm{d}+A$, the twisted exterior differential of an End(E)-valued form η can be written in terms of this product as

$$\mathrm{d}^\nabla \eta = \mathrm{d}\eta + [A \wedge \eta]\,.$$

As a final comment, we also briefly mention the case when a connection is given, and a new one is defined as $\nabla + \eta$,where η is an End(E)-valued 1-form. By a straightforward application of the usual formula for the curvature in local trivialisation it follows that

$$F_{\nabla+\eta} = F_\nabla + \mathrm{d}^\nabla \eta + \frac{1}{2}[\eta \wedge \eta]\,. \tag{A.5}$$

A.3 Differential forms with values in sections and differential operators

Throughout this thesis we often refer to functions and differential forms taking values in infinite-dimensional vector spaces, and one might ask how to make sense of smoothness for such objects. However, the targets of these functions and forms are always of one of two kinds: either that of sections of some bundle, or that of differential operators on them, and smoothness can be defined as follows. Let $\mathcal{T}$ and M be smooth manifolds, E a vector bundle over M, and call $\mathrm{D}_k(E)$ the space of finite-order differential operators on E.

Definition A.3.1. *We say that a map $\psi\colon \mathcal{T} \to \mathcal{C}^\infty(M, E)$ is smooth if it is as a section of the pull-back bundle of E on $M \times \mathcal{T}$. In this case, every vector V tangent to $\mathcal{T}$ defines a new smooth section over M as the point-wise derivative $V[\psi]$. We also say that $D\colon \mathcal{T} \to \mathrm{D}_k(E)$ is smooth if, for every $\psi \in \mathcal{C}^\infty(M, E)$, the map $D\psi\colon \mathcal{T} \to \mathcal{C}^\infty(M, E)$ is smooth. We define differential forms valued in these spaces in the usual way using this notion of smoothness. Also, we define the derivative $V[D]$ of an operator D which depends smoothly on $\mathcal{T}$ along a vector V as the operator acting on smooth sections as*

$$V[D]\psi := V\Big[D\psi\Big] - D(V[\psi])\,. \tag{A.6}$$

Notice that, if A is an operator-valued 1-form, one may regard it as a connection and induce a covariant derivative for functions with values in $\mathcal{C}^\infty(M, E)$ and $\mathrm{D}_k(E)$:

$$\nabla^A_V \psi = V[\psi] + A(V)\psi\,, \qquad \left(\nabla^A_V D\right)\psi = \nabla^A_V(D\psi) - D\nabla^A_V \psi\,.$$

Finally, using the commutator on $\mathrm{D}_k(E)$ as a Lie algebra structure, the considerations of the previous section apply to the case of operator-valued forms.

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